CONSTRUCTION OF HOLOGRAPHIC DUALS FOR QUANTUM FIELD THEORIES WITH GLOBAL SYMMETRIES FROM QUANTUM RENORMALIZATION GROUP

CONSTRUCTION OF HOLOGRAPHIC DUALS FOR QUANTUM FIELD THEORIES WITH GLOBAL SYMMETRIES FROM QUANTUM RENORMALIZATION GROUP

By GRIGORY BEDNIK, SPECIALIST IN PHYSICS

A Thesis Submitted to the School of Graduate Studies in Partial Fulfilment of the Requirements for the Degree Master of Sciences

McMaster University MASTER OF SCIENCES (2014) Hamilton, Ontario (Physics)

TITLE: Construction of holographic duals for quantum field theories with global symmetries from quantum renormalization group. AUTHOR: Grigory Bednik, Specialist (McMaster University) SUPERVISOR: Professor Sung-Sik Lee, 41 pages

We present a method of quantum renormalization group, which makes it possible to construct a bulk theory for a general conformal field theory in the context of anti-de Sitter/conformal field theory correspondence. We demonstrate that within this method it is possible to construct scalar field theory in anti-de Sitter space. We also demonstrate that from a conformal field theory possessing global symmetry, it is possible to construct non-abelian gauge theory in anti-de Sitter space.

Acknowledgements

I would like to thank my supervisor and collaborator Dr. Sung-Sik Lee for giving very interesting research projects, lively discussions, detailed feedback on my work, shaping my physical thinking. I also would like to thank my collaborator Joao Penedones and the members of my supervisory committee: Sung-Sik Lee, Cliff Burgess and Itay Yavin. I would like to thank my teachers in McMaster: Bruce Gaulin, Sung-Sik Lee, Edward Taylor, Takashi Imai for giving general knowledge of Condensed Matter Physics and answering my numerous questions. I would like to thank my teaching supervisors: Karen Hughes, Alex Vorobyov, Frank Hayes, Reza Nejat and administrative members of the department: Rosemary McNeice, Mara Esposto, Tina Stewart for all help during my work in the last two years. I would like to thank my fellow students: Peter Lunts and Shouvik Sur for numerous scientific discussions.

Contents

1 Introduction **1**

2 AdS/CFT correspondence. Conventional picture. **3**
2.1 Scalar field in AdS . . . 4
2.2 Free gauge vector field . . . 6
2.3 Non-abelian gauge field in AdS . . . 8
2.3.1 Bulk-to-bulk propagator . . . 9
2.3.2 Three-point function . . . 11
2.3.3 Four-point function . . . 12

3 Quantum RG for CFT with one scalar operator **17**
3.1 Scalar field in the bulk. . . . 20
3.2 CFT correlation functions . . . 22

4 Quantum RG for conserved current **25**
4.1 Non-abelian field in the bulk . . . 29
4.2 Current correlation functions . . . 31
4.2.1 Ward identities . . . 34
4.2.2 Two-point function . . . 34
4.2.3 Three-point function . . . 35
4.3 Four-point function . . . 36

5 Summary **39**

List of Figures

2.1 Witten diagram for the three-point function. 12

2.2 Contribution to the four-point function from the quartic vertex. 14

2.3 Contribution to the four-point function from the cubic vertices via the s-channel. 14

2.4 Contribution to the four-point function from the cubic vertices via the t-channel. 15

2.5 Contribution to the four-point function from the cubic vertices via the u-channel. 15

4.1 The diagrams responsible for generation of the cubic and quartic term entering $(F^a_{\mu\nu})^2$. The source A^a_μ is represented by the external lines. The internal lines represent high-energy modes of the fermions. 30

List of Symbols

AdS anti-de Sitter space

CFT conformal field theory

RG renormalization group

g_{MN} AdS metric, see Eq. 2.4

x boundary/parallel to the boundary coordinate

y radial bulk coordinate

D dimension of the boundary space

$D+1$ dimension of the bulk space (AdS)

$\alpha, \beta, \ldots$ matrix indices of the fields

$a, b, c \ldots$ flavor indices

f^{abc} structure constant of the symmetry group

$t^a_{\alpha\beta}$ generators of the symmetry group

$\mu, \nu \ldots$ transverse indices in the bulk

$i, j \ldots$ boundary indices

$M, N \ldots$ $(D+1)$ - dimensional indices in the bulk

a Regularized AdS boundary; equivalently RG parameter (e.g. inverse energy cut-off)

Declaration of academic achievement

This work was done in collaboration with Sung-Sik Lee and Joao Penedones. The bulk-to-bulk propagator for gauge field in momentum representation presented in Sec. 2.3.1 is new and obtained by the author. The four-point function dual to non-abelian gauge field (see Sec. 2.3.3) is also obtained by the author. The method presented in Chapter 3 is proposed by Sung-Sik Lee and Joao Penedones. The method presented in Sec. 4 is developed by the author based on the ideas proposed by Sung-Sik Lee.

Chapter 1

Introduction

The concept of AdS/CFT correspondence is one of the major achievements in physics at the close of the twentieth century. This idea ([1], see also [2] for review) conjectures the duality between string theories in anti-de Sitter space (also referred as bulk space) and conformal theories (CFTs) in its boundary. A notable example is correspondence between IIB string theory in $AdS_5 \times S^5$ space and $\mathcal{N} = 4$ Super-Yang-Mills theory (SYM) in four dimensional flat space. Within this correspondence, the coupling parameter λ in the SYM theory [1] is related to the radius of curvature R of AdS measured in the units of string length l_s as: $\frac{R^4}{l_s^4} \sim g_{YM}^2 N$. Thus the limit of strong coupling (i.e. $\lambda \gg 1$) on the CFT side corresponds to the limit of small string length on the AdS side, which implies that the string theory can be approxiamted by supergravity, i.e. a series of fields. Each field in the bulk space is dual to an operator in the CFT, and scaling dimension of the latter is related to mass of the bulk field. Furthermore, in the limit of large number of colors N, the string theory can be treated as semiclassical. Thus the AdS/CFT correspondence makes it possible to use semiclassical field theory in anti-de Sitter space to obtain correlation functions of its dual strongly coupled CFT in the large N limit [3, 4], hence making the AdS/CFT correspondence an attractive tool for many applications, including condensed matter systems (see e.g. [5]). A conventional approach is the following: one takes qualitative

[1] The coupling strength in SYM is characterized by parameter $\lambda = g_{YM}^2 N$, where g_{YM} is the coupling constant entering the SYM action, and N is the number of colors. To see this, one has to consider e.g. one-loop correction to any propagator. In this case, the factor g_{YM}^2 corresponds to the number of external vortices, and N results from the number of degrees of freedom entering the internal loop.

properties of the system of interest (e.g. type of order parameter, symmetries etc.), writes AdS theory satisfying them, and then calculates CFT correlation functions. However, from applicational point of view, it is of interest to consider inverse problem, i.e. to design a systematic way of constructing bulk theories from general CFT. Solving this puzzle is complicated for a number of reasons, for instance, due to the fact that a general boundary field theory is dual to infinitely many fields in the bulk [6].

One approach to tackle this problem is quantum renormalization group scheme. It is known that short distance cutoff in the CFT corresponds to the distance apart from the boundary in the AdS. For this reason, it is natural to to attempt constructing bulk theory by considering RG mapping in the CFT. In the case of matrix model as an example of the CFT, it was shown [7] that by introducing new auxiliary fields at every step of the RG procedure, it is possible to write the resulting quantum correction to the CFT action in terms of only single-trace operators. The presence of this additional transformation constitutes the difference of quantum RG scheme from conventional RG. In the work [7] it was also shown that this method can be implemented to construct holographic dual to a given CFT, so that the auxiliary fields constitute the bulk theory. In this manuscript, we demonstrate that this method can be implemented to construct AdS scalar and vector fields from a CFT.

Our work is organized as follows. In Chap. 2 we briefly review the AdS/CFT correspondence (for more detailed review, see e.g. [2]). We calculate two-point functions dual to scalar field in AdS and two-, three-, four-point functions dual to non-abelian gauge theory in AdS. In Chap. 3, we illustrate our quantum RG method in the simplest case, where the quantum correction to the action depends only on one scalar operator. We construct the bulk action, which proves to be scalar field in AdS. We also demonstrate that our quantum correction to the action, written solely based on our assumptions of small field-strength and long wavelength, leads to the conventional CFT two-point function dual to the scalar. In Chap. 4 we consider the case, where the quantum correction to the action is a functional of conserved current of the CFT. We demonstrate within our scheme, that such model is dual to bulk gauge theory, which is consistent with the conventional AdS/CFT. We also demonstrate that two-, three-, and four-point functions are correctly reproduced. We summarize our results and discuss possible future directions in Chap. 5.

Chapter 2

AdS/CFT correspondence. Conventional picture.

In this section we briefly summarize the conventional picture of AdS/CFT correspondence. As we mentioned in the introduction, the AdS/CFT correspondence conjectures that $\mathcal{N}=4$ Super-Yang-Mills field in the limit of large number of colors N is dual to semiclassical IIB string theory in $AdS_5 \times S^5$. However, it is known, that in low energy limit, the string theory is reduced to a supergravity. Moreover, since the space S^5 (five-dimensional sphere) is compact, its field modes are discrete, and therefore, in the low energy limit, one can consider only its lowest lying modes, or, equivalently, an effective field theory in five-dimensional anti-de Sitter (AdS) space.

The AdS/CFT correspondence is formulated as follows. One writes partition function for a field Φ in the $(D+1)$-dimensional anti-de Sitter space

$$Z = \int D\Phi e^{-S[\Phi]}, \tag{2.1}$$

assuming that the value of the field at the boundary is fixed and equal to a given function $\Phi \mid_{boundary} = \Phi_0$, henceforth referred as a source. Then the AdS/CFT duality prescribes that the partition function, as a functional of the boundary field Φ_0, is equal to generating function of a D-dimensional CFT:

$$\langle e^{\int \Phi_0 \mathcal{O}} \rangle = Z[\Phi_0]. \tag{2.2}$$

Here $\mathcal{O}$ is a CFT operator dual to the bulk field Φ.

The power of this relation is in the fact, that the limit of strong coupling and large N on the CFT side correspond to the semiclassical limit on the AdS side (see [2] for more details). In that case, the relation (2.2) can be simplified using the saddle-point approximation:

$$\langle e^{\int \Phi_0 \mathcal{O}} \rangle = e^{-S[\Phi[\Phi_0]]}, \tag{2.3}$$

where $S[\Phi[\Phi_0]]$ is the action computed on a classical solution of the field Φ with the boundary value Φ_0.

The correlation functions in the strongly coupled CFT can be computed by performing semiclassical calculations in the bulk theory. In the next parts of this section we will demonstrate these calculations in the cases of scalar and vector fields in AdS. In our calculations, we will use metric in AdS in the following form:

$$ds^2 = \frac{dy^2 + \sum_i dx_i^2}{y^2}. \tag{2.4}$$

Here y is so-called radial coordinate, which equals to zero at the boundary and x_i are transverse coordinates, which act both at the bulk and the boundary. We take the dimension of the boundary space (i.e. the range of i) to be an arbitrary number D. Finally, we note, that our expression for the metric (2.4), does not include curvature of the AdS space. This does not lead to loss of generality, because in our work we do not consider dynamical gravity, which could affect its value.

2.1 Scalar field in AdS

To illustrate AdS/CFT correspondence, we take a specific theory of field Φ defined in the anti-de Sitter space, write its partition function (2.1) as a functional of the boundary value of the field Φ_0 and treat it as CFT generating function. The simplest example of a field theory in the anti-de Sitter space is a massless scalar field, defined by the action:

$$S = \frac{1}{2}\int dx dy \sqrt{g} g^{MN} \partial_M \phi \partial_N \phi, \tag{2.5}$$

where g_{MN} is the metric defined in accordance to the Eq. (2.4). For convenience, we regularize our boundary, i.e. we assume, that it is located at small $y = a$, where a corresponds to inverse energy cut-off in CFT. In addition, we treat the boundary directions in momentum representation. The semiclassical field ϕ in the bulk can be expressed in terms of the boundary field ϕ_0 as:

$$\phi(k, y) = \mathtt{g}(k, y, a)\phi_0(k, a), \tag{2.6}$$

where $\mathtt{g}(k, y, a)$ is bulk-to-boundary propagator, which is computed by solving the equation of motion

$$y^{D-1}\partial_y\left(\frac{\partial_y\phi}{y^{D-1}}\right) - k^2\phi = 0.$$

Since this is second-order equation, we have to specify two boundary conditions to find the unique solution. From the Eq. (2.6) it follows that the boundary condition at $y = a$ has the form $\mathtt{g}(k, y = a, a) = 1$. In addition, the expression for $\mathtt{g}(k, y, a)$ should be decreasing at $y \to \infty$, otherwise it would correspond to additional source at the IR boundary. The solution satisfying these boundary conditions has the form [8]:

$$\mathtt{g}(k, y, a) = \left(\frac{y}{a}\right)^{\frac{D}{2}} \frac{K_{D/2}(ky)}{K_{D/2}(ka)}, \tag{2.7}$$

where $K_{D/2}(ky)$ is the modified Bessel function.

To obtain the semiclassical equation for the CFT generating function, one has to express the action (2.5) in terms of the boundary field ϕ_0, by using the Eqs. (2.6, 2.7) and to substitute the former into the partition function (see Eq. 2.3). After completing these steps the generating function takes the form:

$$Z = \exp\left\{\frac{1}{2}\int\frac{dk}{(2\pi)^D}\phi_0(k)\phi_0(-k)\frac{1}{a^{D-1}}\frac{\partial_a\left(a^{D/2}K_{D/2}(ka)\right)}{a^{D/2}K_{D/2}(ka)}\right\}.$$

The CFT two-point function is extracted from the above expression by taking variation over the source ϕ_0, and is equal:

$$G_2(k, a) = \frac{1}{a^{D-1}}\frac{\partial_a\left(a^{D/2}K_{D/2}(ka)\right)}{a^{D/2}K_{D/2}(ka)}.$$

One can generalize this argument to the case of massive scalar in AdS, defined by action:

$$S = \frac{1}{2}\int dxdy\sqrt{g}\left(g^{MN}\partial_M\phi\partial_N\phi + m^2\phi^2\right) - \frac{C}{2}\int dx\sqrt{\gamma}\phi^2\ |_{y=a}, \tag{2.8}$$

where for generality we included UV boundary term with γ being AdS metric computed at the boundary, i.e. at $y = a$.

The situation here is different from the massless case, because the classical solution does not have well-defined boundary limit. Indeed, the field depends on the radial coordinate y as $\phi \sim y^{\frac{D}{2}-\nu}$, where $\nu = \sqrt{m^2 + \left(\frac{D}{2}\right)^2}$. For this reason, we write the classical field as a functional of a boundary function $\mathcal{J}$, such that near the boundary:

$$\phi(k, a) = a^{\frac{D}{2}-\nu}\mathcal{J}(k).$$

and consider the partition function (2.1) as a functional of $\mathcal{J}$. The classical solution for the field in the bulk, which decreases at $y \to \infty$, is written as:

$$\phi(k, y) = \left(\frac{y}{a}\right)^{\frac{D}{2}} \frac{K_\nu(ky)}{K_\nu(ka)} a^{\frac{D}{2}-\nu}\mathcal{J}(k).$$

As in the previous case, we denoted by $K_\nu(ky)$ the modified Bessel function of order ν.

By substituting this expression into the action (2.8) and taking its double variation over $\mathcal{J}$, one can calculate the two-point function, which becomes equal to [8]:

$$G_2(k, a) = \frac{1}{a^{2\nu}} \left(\frac{a\partial_a \left(a^{\frac{D}{2}} K_\nu(ka) \right)}{a^{\frac{D}{2}} K_\nu(ka)} + \Delta - \frac{D}{2} + C \right). \tag{2.9}$$

In the long wavelength limit, it scales as $G_2(k, a) \sim k^{2\nu}$, which implies that the dimension of the CFT operator $\mathcal{O}(x)$ is equal to $\Delta = \frac{D}{2} + \nu$ or

$$\Delta = \frac{D}{2} + \sqrt{\left(\frac{D}{2}\right)^2 + m^2}. \tag{2.10}$$

2.2 Free gauge vector field

Massless vector field A_M in AdS is considered in direct analogy to the scalar field. For simplicity, we will focus on the abelian gauge theory in this section. The CFT generating function is equal to the AdS partition function, expressed in terms of the boundary value of the field A_μ, which we denote by a_μ:

$$Z[a_\mu] = \langle e^{\int a_\mu O_\mu} \rangle. \tag{2.11}$$

In contrast to the scalar case, the massless vector field has an important property of gauge invariance, i.e. invariance of the action under transformations $A_M(x,y) \to A_M(x,y) + \partial_M \theta(x,y)$ (for simplicity, in this section we consider only the case of abelian gauge group.). If one applies them to the generating function (2.11), it is easy to see that the CFT operator O_μ satisfies the conservation law:

$$\partial_\mu O_\mu = 0,$$

which implies that O_μ is an operator of conserved current in the boundary theory.

Now let us compute the two-point function of the operators O_μ. We write the action for the vector field in AdS in the following form:

$$S = \frac{1}{2} \int dx dy \sqrt{g} g^{MK} g^{NL} F_{MN} F_{KL}. \tag{2.12}$$

If we choose the gauge $A_y = 0$, the equations of motion in the momentum representation become:

$$\begin{aligned} &\partial_y \left(\frac{\partial_y A_\mu^a}{y^{d-3}} \right) - \frac{1}{y^{d-3}} k^2 \Pi_{\mu\nu} A_\nu^a = 0, \\ &k_\mu \partial_y A_\mu = 0. \end{aligned} \tag{2.13}$$

In analogy to the scalar case, the field A_μ is expressed in terms of its boundary value a_i through bulk-to-boundary propagator $\mathsf{g}_{\mu i}$ as:

$$A_\mu(k,y) = \mathsf{g}_{\mu i}(k,y,a) a_i(k,a). \tag{2.14}$$

The last equation implies that the bulk-to-boundary propagator has to satisfy the boundary condition $\mathsf{g}_{\mu i}(k, y = a, a) = \delta_{\mu i}$ at the UV boundary. In addition, the bulk field and consequently the bulk-to-boundary propagator have to be decreasing at the IR boundary. To find the explicit form of $\mathsf{g}_{\mu i}$, it is convenient to decompose it into transverse and tangential parts as $\mathsf{g}_{\mu i}(k,y,a) = \mathsf{g}^{trans}(k,y,a)\Pi_{\mu i} + \mathsf{g}^{tang}(k,y,a)\frac{k_\mu k_i}{k^2}$. After applying this ansatz to the equations (2.13), we can obtain decoupled equations for $\mathsf{g}^{trans}(k,y,a)$ and $\mathsf{g}^{tang}(k,y,a)$. After combining their solutions, we obtain an answer:

$$\mathsf{g}_{\mu i}(k) = \frac{\tilde{K}(k,y)}{\tilde{K}(k,a)}\Pi_{\mu i} + \frac{k_\mu k_i}{k^2}, \tag{2.15}$$

where for shortness we have introduced $\tilde{K}(k,y) = y^{\frac{D}{2}-1}K_{\frac{D}{2}-1}(ky)$ (as in the scalar case, here $K_{\frac{D}{2}-1}(ky)$ is modified Bessel function) and denoted by $\Pi_{\mu i}$ the projection operator $\Pi_{\mu i} = \delta_{\mu i} - \frac{k_\mu k_i}{k^2}$.

By using the semiclassical solution for A_μ defined by the Eqs. (2.14, 2.15), we can compute the partition function and the CFT two-point correlator exactly in the same way as it was done in the case of scalar field. The calculation leads to the following answer:

$$G_{\mu\nu}(k,a) = \frac{2}{a^{D-3}}\frac{\partial_a\tilde{K}(k,a)}{\tilde{K}(k,a)}\Pi_{\mu\nu}(k). \tag{2.16}$$

As one can see, this expression satisfies the conservation law, as predicted.

2.3 Non-abelian gauge field in AdS

In this section we consider non-abelian vector gauge field in AdS defined by the action:

$$S = \frac{1}{2}\int dxdy\sqrt{g}g^{MK}g^{NL}F^a_{MN}F^a_{KL}, \tag{2.17}$$

where we use the notation:

$$F^a_{MN} = \partial_{[M}A^a_{N]} + \frac{e}{N}f^{abc}A^b_M A^c_N.$$

Here f^{abc} is the structure constant of the non-abelian gauge group. The coupling constant is denoted by $\frac{e}{N}$, to ensure that it has an order of $1/N$, which follows from the fact that the action is of order N^2 and the field A^a_μ is of order N (see the beginning of Chap. 2).

The presence of interaction terms in the action (2.17) leads to non-trivial connected many-point CFT correlators. To find them, we consider the bulk partition function

$$Z[a^a_\mu] = \int DAe^{-S[A^a_M, a^a_\mu]} \tag{2.18}$$

as a functional of the fixed boundary value a_μ^a of the field. We decompose the bulk field entering the action into the classical expression and its quantum correction: $A_M^a = A_M^{(cl)a} + \delta A_M^a$. By the classical expression we mean here the solution of the classical equations of motion for the non-interacting (i.e. quadratic) part of the action with the boundary value a_μ^a, i.e. the Eq. (2.14). The quantum correction vanishes at the boundary because the field is fixed there. After this decomposition, we can change the integration variables in Eq. (2.18) to δA_M^a and write the non-interacting part of the action as a sum of the on-shell non-interacting action and non-interacting action as a functional of the quantum correction δA_M^a. Thus the partition function can be rewritten as:

$$Z[a_\mu^a] = \int D\delta A e^{-S^{(non-int)}[A_M^{(cl)a}[a_\mu^a]] - S^{(non-int)}[\delta A_M^a] - S^{(int)}[A_M^{(cl)a} + \delta A_M^a]}. \quad (2.19)$$

After that we can expand the exponent of the interacting part of the action and compute correlation functions by using Witten diagrams [4], analogous to the well-known Feynman diagrams. The Witten diagrams for each correlator include external vertices describing operators entering the correlators, internal vertices referring to interaction terms entering the action, external lines denoting bulk-to-boundary propagators and internal lines denoting bulk-to-bulk propagators. We start our analysis from calculating the latter, and after that we use the result to compute CFT three- and four-point functions.

2.3.1 Bulk-to-bulk propagator

The bulk-to-bulk propagator is defined as two-point correlator of the bulk field (not to be confused with CFT correlator) with zero boundary value:

$$\langle A_M^a A_N^b \rangle = \int DA e^{-S^{(non-int)}[A_M^a]} A_M^a A_N^b \mid_{A_M^a(y=a)=0} .$$

It can be rewritten using the notations from the previous section by making change of variables $A_M^a \to \delta A_M^a$.

To compute the bulk-to-bulk propagator, we have to consider partition function for the non-interacting action, deformed by a bulk source:

$$Z = \int DA e^{-S^{(deformed)}} \quad (2.20)$$

$$S^{(deformed)} = \frac{1}{2}\int dxdy\sqrt{g} g^{MP} g^{NQ} F_{MN} F_{PQ} - \int dxdy J_M A_M$$

and to compute on-shell value of the latter.

In analogy to the case of the bulk-to-boundary propagator, we are working in the gauge $A_y = 0$ and in the momentum representation (in coordinate representation this problem was considered in [9]).

The action (2.21) leads to the equations:

$$\partial_y \left(\frac{\partial_y A_\mu(k)}{y^{D-3}} \right) - k^2 \Pi_{\mu\nu} \frac{A_\nu(k)}{y^{D-3}} = -\frac{J_\mu(k)}{2}, \tag{2.21}$$

$$\frac{ik_\mu}{y^{D-3}} \partial_y A_\mu(k) = \frac{J_y}{2}, \tag{2.22}$$

which have to be supplemented with Dirichlet boundary conditions, because, as we mentioned, the fluctuation of A_μ is zero at the boundary. Furthermore, from these equations, it follows that the bulk source is also a conserved current, i.e. $\partial_y J_y + ik_\mu J_\mu = 0$, which means that only its transverse components (i.e. J_μ) can be considered as independent. In other words, the solution for A_μ can be expressed in terms of only transverse components J_ν. This condition, together with the gauge choice $A_y = 0$ implies that only transverse components of the Green function $G^{B-B}_{\mu\nu}$ are non-zero, i.e. the solution of the equations (2.21, 2.22) can be written as

$$A_\mu(k, y) = \int dy' G^{B-B}_{\mu\nu}(k, y, y') J_\nu(k, y').$$

If we substitute this equation into the action (2.21), it can be rewritten as

$$S^{(deformed)} = \int \frac{dk}{(2\pi)^D} dy J_\mu G_{\mu\nu} J_\nu.$$

After that we can substitute $S^{(deformed)}$ into the partition function (2.20) and take its double variation over the source J_μ. By setting J_μ to zero, we can see that the bulk-to-bulk propagator is actually equal to the Green function (we choose the normalization conditions in such way that the factors of $(2\pi)^D$ vanish in the final answer).

To find the explicit form the Green function, it is convenient to decompose the fields into normal and tangential parts, relative to the momentum vector k:

$$G^{B-B}_{\mu\nu} = G^\perp \Pi_{\mu\nu} + G^\parallel \frac{k_\mu k_\nu}{k^2}, \tag{2.23}$$

$$A_\mu = A^\perp_\mu + ik_\mu \xi,$$

$$J_\mu = J^\perp_\mu + ik_\mu J.$$

After this decomposition and separation of the corresponding terms in the equations (2.21, 2.22), we can obtain equations for the components of the propagator:

$$\partial_y \left(\frac{\partial_y G^{\perp}}{y^{D-3}} \right) - \frac{k^2 G^{\perp}}{y^{D-3}} = -\frac{\delta(y-y')}{2},$$
$$\partial_y \left(\frac{\partial_y G^{\parallel}}{y^{D-3}} \right) = -\frac{\delta(y-y')}{2}.$$

The solution satisfying Dirichlet boundary conditions at $y = a$ and finite at $y \to \infty$ has the following form:

$$\begin{aligned} G^{\perp} = \quad & -\frac{1}{2}\left(\frac{\tilde{I}(k,a)}{\tilde{K}(k,a)} \tilde{K}(k,y)\tilde{K}(k,y') \right. \\ & \left. -\tilde{I}(k,y)\tilde{K}(k,y')\theta(y'-y) - \tilde{K}(k,y)\tilde{I}(k,y')\theta(y-y') \right), \\ G^{\parallel} = \quad & -\frac{1}{2}\left(\frac{(a^{D-2} - y^{D-2})}{D-2}\theta(y'-y) + \frac{(a^{D-2} - y'^{D-2})}{D-2}\theta(y-y') \right). \end{aligned}$$

The full answer for the bulk-to-bulk propagator is obtained by combining these expressions with the Eq. (2.23).

2.3.2 Three-point function

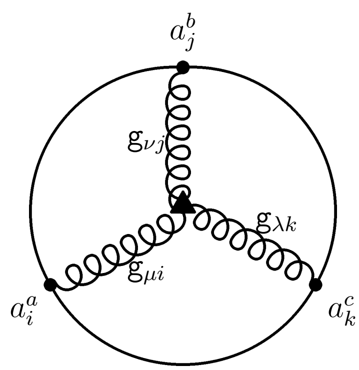
In this section we compute the connected three-point point function dual to the non-abelian gauge theory defined by the action (2.17) (this problem was previously studied in [10]). As we mentioned, the calculation is performed using the Witten diagrams. More precisely, we have to expand $e^{-S^{(int)}}$ entering the Eq. (2.19) in powers of e and take its triple variation over a_μ^a. In the leading order, the only contribution to the connected three-point function results from the cubic term in the action (2.17). Thus the relevant contribution to the partition function (2.19) is

$$-\int D\delta A e^{-S^{(non-int)}[\delta A_\mu^a]} \int \frac{dxdy}{y^{D-3}} \frac{e}{N} f^{abc} \partial_{[M} A_{N]}^{(cl)a} A_M^{(cl)b} A_N^{(cl)c}.$$

In this equation we can perform Fourier transformation of the field and substitute its expression defined by the Eqs. (2.14, 2.15). After that we take its variation over the boundary value a_μ^a and divide it by the normalization factor $\int D\delta A e^{-S^{(non-int)}[\delta A_\mu^a]}$, thus obtaining the answer:

$$G_{ijk}^{abc}(-k_1,-k_2,-k_3) = \qquad (2.24)$$
$$-2ief^{abc}\int_a^\infty \frac{dy}{y^{D-3}}\left((k_3-k_2)_\mu\delta_{\nu\lambda}+\text{cyclic}\right)\mathsf{g}_{\mu i}(k_1)\mathsf{g}_{\nu j}(k_2)\mathsf{g}_{\lambda k}(k_3).$$

Schematically, it can be represented by the Witten diagram shown in Fig. 2.1.



$$\blacktriangle \qquad = -2ief^{abc}\frac{1}{y^{D-3}}\left((k_3-k_2)_\mu\delta_{\nu\lambda}+\text{cyclic}\right).$$

Figure 2.1: Witten diagram for the three-point function.

2.3.3 Four-point function

In this section we compute the connected four-point function, from the non-abelian gauge theory in *AdS* (see Eq. 2.17). In the leading order, it has contributions from linear power of the quartic term and quadratic power of the qubic term entering the action (2.17). The relevant part of the partition function is written as

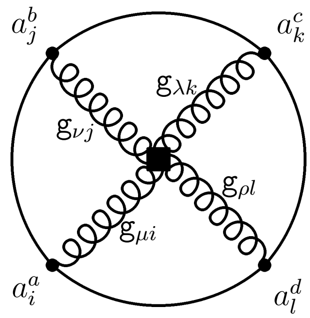
$$\int D\delta A e^{-S^{(non-int)}[A_\mu^{(cl)a}[a_\mu^a]]-S^{(non-int)}[\delta A_\mu^a]}$$
$$\times\left\{\frac{1}{2}\left(\int\frac{dxdy}{y^{D-3}}\frac{e}{N}f^{abc}\partial_{[M}A_{N]}^a A_M^b A_N^c\right)^2\right.$$

$$- \int \frac{dxdy}{2y^{D-3}} \left(\frac{e}{N}\right)^2 f^{abc} f^{ade} A^b_M A^c_N A^d_M A^e_N \Big\} \Big|_{A^a_M = A^{(cl)a}_M + \delta A^a_M} .$$

To extract the four-point function, we have to transform this expression into momentum representation, take its quartic variation over the boundary value of the field a^a_μ and set the latter to zero. In addition, we select only terms represented by connected Witten diagrams. In particular, there is one connected diagram resulting from the quartic term (see Fig. 2.2) and three diagrams resulting from the cubic term (see Fig. 2.3, 2.4, 2.5) representing three possible channels. The answer for the connected four-point function has the following form:

$$\begin{aligned}
&G^{abcd}_{ijkl} = -2e^2 f^{abe} f^{cde} \delta_{\mu[\lambda}\delta_{\rho]\nu} \int_a^\infty \frac{dy}{y^{D-3}} \mathrm{g}_{\mu i}(k_1)\mathrm{g}_{\nu j}(k_2)\mathrm{g}_{\lambda k}(k_3)\mathrm{g}_{\rho l}(k_4) \\
&+ \int_a^\infty dy_1 \frac{2ie}{y_1^{D-3}} f^{abe} \left((k_3+k_4)_{[\mu}\delta_{\nu]\sigma} + k_{1_{[\nu}}\delta_{\sigma]\mu} + k_{2_{[\sigma}}\delta_{\mu]\nu}\right) \mathrm{g}_{\mu i}(k_1)\mathrm{g}_{\nu j}(k_2) \\
&\times \int_a^\infty dy_1 \frac{2ie}{y_2^{D-3}} f^{cde} \left((k_1+k_2)_{[\lambda}\delta_{\rho]\tau} + k_{3_{[\rho}}\delta_{\tau]\lambda} + k_{4_{[\tau}}\delta_{\lambda]\rho}\right) \mathrm{g}_{\lambda k}(k_3)\mathrm{g}_{\rho l}(k_4) \\
&\times G^{B-B}_{\sigma\tau}(k_1+k_2) \\
&\qquad + \begin{pmatrix} k_2 \leftrightarrow k_3 \\ j \leftrightarrow k \\ \nu \leftrightarrow \lambda \\ b \leftrightarrow c \end{pmatrix} + \begin{pmatrix} k_3 \leftrightarrow k_4 \\ k \leftrightarrow l \\ \lambda \leftrightarrow \rho \\ c \leftrightarrow d \end{pmatrix} .
\end{aligned} \tag{2.25}$$

Here, to simplify the equation, we have decomposed the expression into three channels both for the quartic and cubic terms.



$$\blacksquare \qquad = -2e^2 f^{abe} f^{cde} \delta_{\mu[\lambda}\delta_{\rho]\nu} \frac{1}{y^{D-3}} + \begin{pmatrix} \nu \leftrightarrow \lambda \\ b \leftrightarrow c \end{pmatrix} + \begin{pmatrix} \lambda \leftrightarrow \rho \\ c \leftrightarrow d \end{pmatrix}.$$

Figure 2.2: Contribution to the four-point function from the quartic vertex.

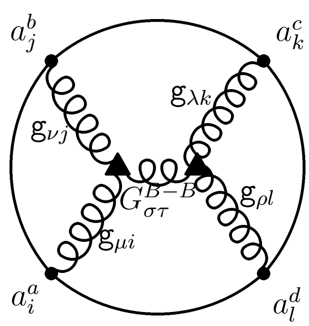


Figure 2.3: Contribution to the four-point function from the cubic vertices via the s-channel.

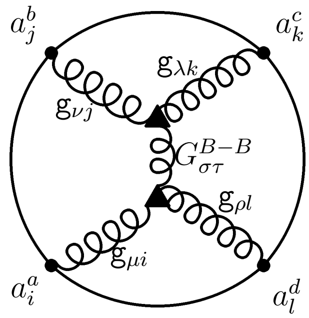


Figure 2.4: Contribution to the four-point function from the cubic vertices via the t-channel.

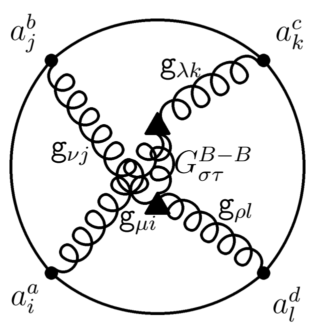


Figure 2.5: Contribution to the four-point function from the cubic vertices via the u-channel.

Chapter 3

Quantum RG for CFT with one scalar operator

In this chapter we demonstrate a way of constructing AdS dual to a given CFT, via quantum renormalization group (this method was developed in [7]). We deform a CFT by source $\mathcal{J}(x)$ and introduce regularization parameter a, which in the simplest case can be a short distance cut-off[1]. Thus, the CFT partition function has the form:

$$Z[\mathcal{J};a) = \int D\phi \; e^{-S[\phi;a)+\int dx \mathcal{J}(x)\mathcal{O}(x)}, \tag{3.1}$$

where ϕ is the fundamental field and $\mathcal{O}$ is an operator, constructed from ϕ. and having the lowest dimension Δ. As an example, this model may refer to a matrix model [7] with large rank N of the matrix fields (analogously to the large number of colors) with ϕ referring to the fundamental matrix field and $\mathcal{O}$ referring to its single-trace operator. Following this example, we assume that the operator $\mathcal{O}$, as well as the source $\mathcal{J}$ are of order N.

Now we consider renormalization group mapping of our theory into a theory with a new value of the regularization parameter $a+dy$. In the case of a being short distance (or equivalently, inverse energy) cut-off, this mapping refers to integrating out high-energy modes. Under this mapping, the action

[1] In this work we do not discuss the precise nature of our regularization. However, we assume that the partition function becomes zero in the limit $a \to \infty$. This assumption makes it possible to construct the dual theory in pure AdS space, otherwise the dual space might contain singularities.

acquires a quantum correction δS, and the partition function becomes:

$$Z = \int D\phi \; e^{-S[\phi;a+dy)+\int dx \mathcal{J}(x)\mathcal{O}(x)-\delta S[\mathcal{J},\mathcal{O};a)} \,, \tag{3.2}$$

The correction δS is of order dy, since infinitely small change in the regularization parameter a results infinitely small change of the action. In general, δS contains infinite number of operators composed from the field ϕ and its derivatives. However, we make an assumption that δS can be considered as a function of only one operator $\mathcal{O}$. This assumption is true if all other operators are suppressed by having higher dimensions. We leave the question about specific examples of such theories and regularizations for future work.

In [7] it was shown, that, in the case of matrix model, δS contains only single- and double-trace operators. In analogy to that work, we assume here, that δS can be expressed as a quadratic function of the operator $\mathcal{O}$. The essence of the quantum renormalization group procedure is that we can remove the non-linear in $\mathcal{O}$ terms entering δS, by introducing new auxiliary fields $P(x)$ and $J(x)$, thus rewriting the partition function as:

$$Z = \int DPDJD\phi \; e^{-S[\phi;a')+i\int J(P-\mathcal{O})+\int \mathcal{J}P-\delta S[\mathcal{J},P;a)}. \tag{3.3}$$

By integrating out $J(x)$ and $P(x)$ explicitly, one can verify, that the new partition function (3.3) is equivalent to its previous expression (3.2). However, we would like the new dynamic source $J(x)$ to couple the operator $\mathcal{O}(x)$ in the same way as it does in the initial expression (3.1). For this reason, we perform Wick rotation of the auxiliary fields J and P:

$$J(x) \to iJ(x), \quad P(x) \to -iP(x).$$

This transformation changes the partition function into the following form:

$$Z = \int DPDJD\phi \; e^{-S[\phi;a')+\int J\mathcal{O}+i\int(J-\mathcal{J})P-\delta S[\mathcal{J},P;a)}. \tag{3.4}$$

As we mentioned previously, we assume here that δS is a quadratic function of $\mathcal{O}$, which explicitly can be written as:

$$\begin{aligned} \delta S[\mathcal{J},P;a) \;\; = \;\; & dy \left[B_0[\mathcal{J};a) + i\int dx B_1[\mathcal{J};a,x)P(x) \right. \\ & \left. -\frac{1}{2}\int dx_1 dx_2 B_2[\mathcal{J};a,x_1,x_2)P(x_1)P(x_2) \right]. \end{aligned} \tag{3.5}$$

One can check, that if δS has this form, then the integration of J and P in the Eq. (3.4) leads to the previous expression for the partition function (see Eq. 3.2).

For the further calculations it is convenient to introduce

$$H[\mathcal{J}, P; a) = \frac{\delta S[\mathcal{J}, -iP; a)}{da} \tag{3.6}$$

no comma and to rename our variables as:

$$j^{(0)} = \mathcal{J}, \qquad j^{(1)} = J, \qquad p^{(1)} = P.$$

After these changes, the partition function (3.4) can be rewritten as:

$$Z[j^{(0)}; a) = \int Dj^{(1)} Dp^{(1)} e^{i\int p^{(1)}(j^{(1)} - j^{(0)}) - daH[j^{(0)}, p^{(1)}; a)} Z[j^{(1)}; a + dy).$$

The last equation actually describes the partition function $Z[j^{(0)}; a)$ as a functional of the partition function with the shifted regularization $Z[j^{(1)}; a + dy)$ and dynamical source $j^{(1)}$. If we iterate, we can rewrite the initial partition function as:

$$Z[j^{(0)}; a) = \int \prod_{k=1}^{M} Dj^{(k)} Dp^{(k)} e^{i\int \sum_{k=1}^{M} p^{(k)}(j^{(k)} - j^{(k-1)}) - \sum_{k=1}^{M} \delta S[j^{(k-1)}, -ip^{(k)}; a + (k-1)da)} \times Z[j^{(M)}; a + Mda).$$

Finally we can take limit $M \to \infty$ with $Mda = y^* - a$ being constant. In this case, the set of fields $j^{(k)}(x)$ is replaced by one y-dependent field $j(y, x)$, such that $j(y = a + kdy, x) = j^{(k)}(x)$. Analogously, the set of $p^{(k)}(x)$ is replaced by $p(y, x)$. After these steps, the partition function is rewritten as:

$$Z[\mathcal{J}; a) = \int DjDp \;\; e^{\int_a^{y^*} dy \left(i\int dx\, p\partial_y j - H[j, p; y) \right)} Z[j(y^*, x); y^*), \tag{3.7}$$

where at the boundary $j(a, x) = \mathcal{J}(x)$. In such a way, we have converted the original D-dimensional path integral (3.1) into a $(D + 1)$-dimensional path integral (3.7). In other words we have constructed $(D+1)$-dimensional theory dual to a given D-dimensional theory defined by (3.1).

To finalize our construction, we make an assumption that the bulk space can be extended by taking limit $y^* \to \infty$, which implies that all correlation functions vanish at $a = y^* \to \infty$. Also we assume, that in this limit, we can neglect the contribution from $Z[j(y^*, x); y^*)$, which, from the bulk perspective, means that the action does not contain IR boundary terms.

3.1 Scalar field in the bulk.

In the previous section we showed that under a number of assumptions, D-dimensional CFT defined by the partition function (3.1) is equivalent to $(D+1)$-dimensional theory defined by the partition function (3.7). In this section we show that under a few new assumptions, this field theory can describe scalar field in AdS.

To find out the nature of the bulk field, one has to answer the question: what is the general form of the quantum correction to the action δS, which emerges under RG mapping? To answer this question we first note that, in conventional AdS/CFT picture, CFT correlation functions are obtained by treating the source $\mathcal{J}(x)$ as infinitesimal. For this reason, we can expand δS by assuming the fields j and p are small and have the same order.[2]. In addition, we perform derivative expansion of the action, because we want to focus on the large distance behavior of the correlation functions. Our third assumption deals with the fact, that the bulk action is analytic functional of the field and local in both transverse and radial directions. Finally, we assume that the action is invariant under Poincaré symmetry, $\mathbb{Z}_2$ symmetry (i.e. $\mathcal{O} \to -\mathcal{O}$) and the scaling symmetry:

$$Z[\mathcal{J}; a] = Z[\mathcal{J}'; \lambda a], \qquad j'(x) = \lambda^{\Delta - D} j(x/\lambda), \quad p'(x) = \lambda^{-\Delta} p(x/\lambda).$$

Under all these assumptions, we can write the coefficients entering δS (defined by the Eq. (3.5)) as:

$$\begin{aligned}
B_0[j; y] &= \frac{1}{y^{D+1}} \int dx \Big[N^2 b_0^{(0)} + y^{2(D-\Delta)} \Big(b_0^{(1)} j^2 + b_0^{(2)} y^2 (\partial_x j)^2 + \ldots \Big) \\
&\qquad\qquad + O(j^4/N^2) \Big] , \\
B_1[j; y, x] &= \frac{i}{y} b_1^{(0)} j(y,x) + \ldots + O(j^3/N^2), \\
B_2[j; y, x_1, x_2] &= y^{2\Delta - D - 1} \Big(b_2^{(0)} + \ldots \Big) \delta(x_1 - x_2) + O(j^2/N^2),
\end{aligned} \tag{3.8}$$

where $b_n^{(j)}$ are real constants of order N^0.

[2]Strictly speaking, we are interested in the bulk action of order N^2, where, in the case of matrix model, N refers to the rank of matrix fields. Therefore, we assume that j and p are proportional to large N, so that we can make expansion over $1/N$.

By substituting these coefficients into the Eq. (3.5) and then into Eqs. (3.6) and (3.7), we obtain the partition function of our bulk theory. If we integrate out the field p, we obtain the partition function containing the action of the following form:

$$I[j] = \int_a^\infty dy B_0[j;y) + \int_a^\infty dydx \frac{y^{D+1-2\Delta}}{2b_2^{(0)}} \left(\partial_y j(y,x) + iB_1[j;y,x)\right)^2 + O(j^4/N^2).$$

If we also make change of variables $j(y,x) = y^{\Delta-D}\phi\,(y,x)$ and neglect higher order terms, our bulk action simplifies to

$$I = \int_a^\infty \frac{dydx}{y^{D+1}} \left\{ N^2 b_0^{(0)} + b_0^{(1)}\phi^2 + b_0^{(2)} y^2 (\partial_x \phi)^2 + \frac{y^2}{2b_2^{(0)}} \left(\partial_y \phi + \frac{(\Delta - D - b_1^{(0)})\phi}{y} \right)^2 \right\} \tag{3.9}$$

From this expression we can see, that our bulk theory describes scalar field in AdS. However, we would like to transform the action into canonical form, and we perform it in two steps. First, we rescale the field ϕ and the radial coordinate y (together with a, y^*, p etc.) to eliminate the non-trivial coefficients from the kinetic term. Next, we expand the bracket, containing $\partial_y\phi$, in the Eq. (3.9) and separate an emerging total derivative term. Thus, we get the canonical action for scalar field in AdS, supplemented with UV boundary term:

$$I = \frac{1}{2}\int_a^\infty dydx\sqrt{g}\left[E + m^2\phi^2 + g^{MN}\partial_M\phi\partial_N\phi +\right] + \frac{(\Delta - D - b_1^{(0)})}{2} \int \frac{dx}{y^D}\phi^2 \,|_{y=a}^{y^*} , \tag{3.10}$$

In this equation, m is a mass of the bulk field, E is its vacuum energy [3] which, in terms of the coefficients b, are expressed as:

$$m^2 = 2b_2^{(0)} b_0^{(1)} + (\Delta - D - b_1^{(1)})(\Delta - b_1^{(0)}),$$
$$E = \left(2b_2^{(0)} b_0^{(2)}\right)^{D/2} N^2 b_0^{(0)}$$

[3] We expect that in the full theory (that includes all operators) the vacuum energy should be related to the cosmological constant of the bulk space. However it is not the case in this model, because we do not consider operators leading to gravitational physics in the bulk, e.g. energy-momentum tensor.

and g is the Poincaré AdS metric defined by the Eq. (2.4). As we mentioned previously, our equations do not include AdS curvature. The action written in terms of our dimensionless metric is related to a physical action by rescaling of the fields and its parameters (mass, vacuum energy etc.) by a factor depending on the AdS curvature. However, in our model we cannot find it, because when we construct the bulk theory, we do not take into account dynamical gravity, which is essential to find the properties of the bulk space.

3.2 CFT correlation functions

In the previous sections we defined our CFT by the partition function (3.1) and constructed its holographic dual theory, by assuming that the quantum correction to the action δS emerging from RG mapping, has the simplest possible form defined by the Eq. (3.5). In this section we show that this form of δS is closely related to the form of CFT correlation functions. In particular, we demonstrate that the form of δS is sufficient to compute all correlation functions, and, in the case of the form of δS given by Eqs. (3.5) and (3.8), the correlation functions prove to be precisely the same, as the ones calculated from the bulk theory (3.7) via the conventional AdS/CFT techniques. In other words, we will show, that the form of δS fixes the correlation functions, which is consistent with the conventional AdS/CFT dictionary described in Chap. 2.

Without loss of generality, we can limit our considerations to connected correlation functions. To study them, it is convenient to introduce their generating functional:

$$\begin{aligned} W[\mathcal{J};a) &= \log Z[\mathcal{J};a) \\ &= \sum_{\substack{n=0 \\ \text{even}}}^{\infty} \frac{1}{n!} \int dx_1 \dots dx_n \; N^{2-n} G_n(x_1,\dots,x_n)\mathcal{J}(x_1)\dots\mathcal{J}(x_n). \end{aligned}$$

As we mentioned previously, we assume that our source $\mathcal{J}$ is proportional to N, and our generating functional W has an order of N^2. From these facts we can conclude, that n-point function has an order of N^{2-n}, and for this reason we denote it by $N^{2-n}G_n(x_1,\dots,x_n)$ in order to make the unknown part $G_n(x_1,\dots,x_n)$ be of order of N^0. Strictly speaking, one has to expand the n-point functions in powers of $1/N^2$, but in this work we limit our analysis only to the leading order.

An equation which relates δS with the correlation functions is obtained in the following way. The partition function (3.2) can be rewritten as:

$$Z[\mathcal{J};a) = e^{-\delta S[\mathcal{J},\frac{\delta}{\delta\mathcal{J}};a)} Z[\mathcal{J};a+dy),$$

or equivalently,

$$\left(\delta S\left[\mathcal{J},\frac{\delta}{\delta\mathcal{J}};a\right) - dy\frac{d}{da}\right) Z[\mathcal{J};a) = 0.$$

As a next step, we replace the partition function Z by the connected generating function W, expand the equation in powers of $1/N^2$ and separate its terms having equal powers of $\mathcal{J}$. In this section we consider only terms, quadratic in $\mathcal{J}$. After assuming that the one-point function (i.e. vacuum expectation) vanishes, and performing Fourier transformation over the boundary coordinates,[4] we obtain the equation:

$$\frac{1}{2}a\frac{d}{da}G_2(k) = a^{D-2\Delta}\left(b_0^{(1)} + b_0^{(2)}a^2k^2\right) - b_1^{(0)}G_2(k) - \frac{1}{2}a^{2\Delta-D}b_2^{(0)}\left[G_2(k)\right]^2,$$

which can be treated as conventional Riccati equation (see e.g.[11]). Its solution decreasing at $a \to \infty$ (we use this boundary condition because we have assumed that our bulk theory acts in pure AdS without singularities) has the form:

$$G_2(k) = \frac{a^{-2\nu}}{b_2^{(0)}}\left[\nu - b_1^{(0)} + a\frac{d}{da}\log u\left(ak\gamma\right)\right], \tag{3.11}$$

where $\nu = \Delta - \frac{D}{2}$ and $\gamma^2/2 = b_0^{(2)}b_2^{(0)}$. Here $u(t)$ is a solution of the modified Bessel differential equation

$$\left[a^2\partial_a^2 + a\partial_a - \left(\alpha^2 + k^2a^2\right)\right]u(t) = 0$$

with

$$\alpha^2 = 2b_0^{(1)}b_2^{(0)} + (b_1^{(0)} - \nu)^2\ . \tag{3.12}$$

[4]Our conventions for the Fourier transformation are:

$$\mathcal{J}(x) = \int \frac{dk}{(2\pi)^D}e^{-i\vec{k}\cdot\vec{x}}\mathcal{J}(k),$$
$$G_2(x_1,x_2) = \int \frac{dk}{(2\pi)^D}e^{i\vec{k}\cdot(\vec{x_1}-\vec{x_2})}G_2(k).$$

Without loss of generality, we can limit our analysis to the coefficients taken from the rescaled action (3.10), i.e. obeying the relations:

$$\begin{aligned} b_0^{(2)} &= \frac{1}{2}, \\ b_2^{(0)} &= 1, \\ \frac{m^2}{2} &= b_0^{(1)} + \frac{(\Delta - D - b_1^{(0)})(\Delta - b_1^{(0)})}{2b_2^{(0)}}. \end{aligned} \tag{3.13}$$

After their substitution into the Eq. (3.11) and applying the boundary conditions, we can rewrite the Green function as:

$$G_2(k) = a^{-2\nu}\left[\nu - b_1^{(0)} + \frac{a\partial_a(K_\alpha(ka))}{K_\alpha(ka)}\right]. \tag{3.14}$$

Its long wavelength asymptotics has the form:

$$G_2^{(0)}(k) \sim a^{-2\nu}\left[(\text{analytic in } k) + (a^2k^2)^\alpha + \ldots\right].$$

Now let us take into account that, as we mentioned previously, dimension of the operator $\mathcal{O}(x)$ is equal to Δ. This fact implies, that the two-point function (in momentum representation) should scale as $k^{2\nu}$. In addition, since the two-point function is well-defined and nonzero in the limit $a \to 0$, its leading non-analytic term has to be independent on a. Thus, it follows that $\alpha = \nu$ and therefore the obtained two-point function (3.14) is coincident with its conventional expression (2.9). Furthermore, the last equation, together with the Eqs. (3.12, 3.13) leads to relation:

$$m^2 = \Delta(\Delta - D),$$

which is also equivalent to the conventional relation (2.10).

Chapter 4

Quantum RG for conserved current

In this chapter we demonstrate that, by using the quantum RG scheme, it is possible to construct AdS vector gauge field in the same way, as we constructed scalar field in Chap. 3. As we mentioned previously, in conventional AdS/CFT, bulk gauge field is dual to conserved current operator in CFT. Since the current operator is present in theories possessing global symmetry, we assume that our CFT is invariant under global symmetry transformations of the fundamental field: [1]

$$\phi_\alpha \to \phi_\alpha + i\frac{e}{N} t^c_{\alpha\beta}\theta^c\phi_\beta. \tag{4.1}$$

Here $t^c_{\alpha\beta}$ are generators of the symmetry group and θ^c are parameters of the transformations. We also assume that this invariance holds in the regularized CFT.

The current $\mathcal{O}^c_\mu$ corresponding to this symmetry is defined through spatially dependent symmetry transformations:

$$S[(\delta_{\alpha\beta} + i\frac{e}{N} t^c_{\alpha\beta}\theta^c(x))\phi_\beta] - S[\phi_\alpha] = \int dx \partial_\mu\theta^c(x)\mathcal{O}^c_\mu[\phi]. \tag{4.2}$$

From this definition, it follows that the current has dimension $D-1$. It forms a non-trivial representation of the Poincare and the flavor groups, and therefore the source (which we denote by $\mathcal{A}^a_\mu$), has to be in the same representation of these groups. We also assume that the deformed theory is

[1] Here we choose our coupling constant to be equal to $\frac{e}{N}$, as it was done in Sec. 2.3

"gauge invariant", [2] i.e. invariant under spatially dependent transformations (4.1) supplemented with the transformations of the source

$$\mathcal{A}^a_\mu \to \mathcal{A}^a_\mu + D_\mu \theta^a, \tag{4.3}$$

where $D_\mu \theta^a = \partial_\mu \theta^a + \frac{e}{N} f^{abc} \mathcal{A}^b_\mu \theta^c$, and f^{abc} is structure constant of the flavor group. The last assumption makes it possible to write the current as

$$\mathcal{O}^a_\mu = -\frac{\delta S[\phi, A^a_\mu]}{\delta A^a_\mu}. \tag{4.4}$$

One can demonstrate, that in the limit $A^a_\mu \to 0$, this definition is equivalent to the Eq. (4.2). One can check explicitly, that under symmetry transformations, the current transforms as:

$$\mathcal{O}^a_\mu[\phi_\alpha + i\frac{e}{N} t^a_{\alpha\beta} \theta^a \phi_\beta] = \mathcal{O}^a_\mu[\phi_\alpha] + \frac{e}{N} f^{abc} \mathcal{O}^b_\mu \theta^c. \tag{4.5}$$

If we assume that the deformed action is linear with $\mathcal{A}^a_\mu$ (or equivalently, the current does not depend on the source), the regularized CFT partition function is written in the form:

$$Z[A^a_\mu; a) = \int D\phi e^{-S[\phi;a) + \int dx \mathcal{A}^a_\mu \mathcal{O}^a_\mu}, \tag{4.6}$$

which is analogous to the partition function with a scalar operator (3.1).

Now let us perform RG mapping in the same way, as it was done in Chap. 3. We get a new expression for the partition function:

$$Z = \int D\phi \; e^{-S[\phi; a+dy) + \int dx \mathcal{A}^a_\mu(x) \mathcal{O}^a_\mu(x) - \delta S[\mathcal{A}^a_\mu, \mathcal{O}^a_\mu]}. \tag{4.7}$$

As we mentioned previously, the partition function is "gauge invariant", i.e. invariant under the transformations defined by the Eqs. (4.5 , 4.3). In the same way, we can show "gauge invariance" of the combination $-S + \int dx \mathcal{A}\mathcal{O}$ in the new partition function (4.7). Therefore, we deduce that the quantum correction δS is also "gauge invariant", and can be written as a functional of $\mathcal{F}^a_{\mu\nu}, \mathcal{O}^a_\mu$, where $\mathcal{F}^a_{\mu\nu}$ is a "gauge invariant" combination of $\mathcal{A}^a_\mu$, defined as: $\mathcal{F}^a_{\mu\nu} = \partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu + \frac{e}{N} f^{abc} \mathcal{A}^b_\mu \mathcal{A}^c_\nu$.

[2] Here we write the term "gauge invariance" in quotes because of its indirect meaning. In particular, $\mathcal{A}^a_\mu$ here is a source but not a physical gauge field, therefore its transformation is not real gauge transformation.

As a next step, we introduce auxiliary fields in analogy to Chap. 3, and thus rewrite the partition function as:

$$Z = \int D\phi DA^{(1)} DP^{(1)} e^{\int dx A_\mu^{(1)a} O^{\mu a} + i\int dx (A_\mu^{(1)a} - \mathcal{A}_\mu^a) P_\mu^{(1)a} - S[\phi; a+dy) - \delta S[\mathcal{F}_{\mu\nu}^a, -iP_\mu^{(1)a}]} \tag{4.8}$$

From the Eq. (4.4) it follows that the current $\mathcal{O}_\mu^a$ does not depend on the source $\mathcal{A}_\mu^a$. In principle one can consider more general case, where the current depends on the source, and the source is introduced into the action in gauge invariant way. [3] In this case, one can also perform the procedure of RG mapping as:

$$Z = \int D\phi e^{-S[\phi, A_\mu^a; a)} = \int D\phi e^{-S[\phi, A_\mu^a; a+dy) - \delta S[\mathcal{F}_{\mu\nu}^a, -\frac{\delta S}{\delta \mathcal{A}_\mu^a}]}$$

and inclusion of the auxiliary fields as,

$$Z = \int D\phi DA^{(1)} DP^{(1)} e^{i\int dx (A_\mu^{(1)a} - \mathcal{A}_\mu^a) P_\mu^{(1)a}} e^{-\delta S[\mathcal{F}_{\mu\nu}^a, -\frac{\delta S}{\delta \mathcal{A}_\mu^a}]} e^{-S[\phi, A_\mu^{(1)a}; a+dy)}.$$

In the leading order in $1/N$, we can replace $\frac{\delta S}{\delta \mathcal{A}_\mu^a}$ in the last equation with the variation operator $\frac{\delta}{\delta \mathcal{A}_\mu^a}$ acting on e^{-S}. After that we can replace it by $-iP_\mu^{a(1)}$ and thus get an action having the same form as (4.8) (here $S[\phi, A_\mu^{(1)a}; a+dy)$ is equivalent to $S[\phi; a+dy) - \int dx A_\mu^{(1)a} O_\mu^a$ considered previously).

To obtain the bulk theory, we can, in principle, iterate the steps of RG mapping and introducing auxiliary fields, following the steps discussed in Chap. 3. However, if we complete it, we obtain a theory in which the number of spatial components of A_μ is equal to the number of boundary dimensions, and thus smaller, than the number of bulk dimensions. Indeed, one can check that such theory is not invariant under gauge transformations with radially dependent parameters. Actually, such theory corresponds to a gauge theory with a fixed gauge. To obtain a bulk theory possessing the full gauge invariance, we perform an infinitesimal gauge transformation of the fundamental field ϕ and the auxiliary field $A^{(n)}$ at each step (which is numerated by n).

[3] In this case the action does not have the form of (4.6). A well-known example is the case, where ϕ is scalar field, and $\mathcal{A}_\mu^a$ is introduced by replacing all spatial derivatives with covariant.

Under this transformation, the deformed action $-S[\phi, A_\mu^{(n)a}, a+ndy)$ remains invariant, and only the term $\int dx(A^{(n)} - A^{(n-1)})P^{(n)}$ varies . Particularly, if $A^{(n)}$ transforms as:

$$A_\mu^{(n)a} \to A_\mu^{(n)a} + D_\mu \theta^{(n)a},$$

the non-invariant term transforms as:

$$\int dx(A_\mu^{(n)a} - A_\mu^{(n-1)a})P_\mu^{(n)a}$$
$$\to \int dx \left(A_\mu^{(n)a} + \partial_\mu \theta^{(n)a} + \frac{e}{N} f^{abc} A_\mu^{(n)b} \theta^{(n)c} - A_\mu^{(n-1)a} \right) P_\mu^{(n)a}.$$

Following the reasoning, performed in the case of scalar (see Chap. 3), we are going to replace $A_\mu^{(n)a} - A_\mu^{(n-1)a}$ by $dy\partial_y A_\mu^a$. For this reason, it is convenient to take θ of order dy and rewrite it as:

$$\theta^{(n)a} = -dy A_y^{(n)a}.$$

As a result, the considered term changes under the gauge transformation as:

$$\int dx(A_\mu^{(n)a} - A_\mu^{(n-1)a})P_\mu^{(n)a} \tag{4.9}$$
$$\to dy \int dx \left(\frac{A_\mu^{(n)a} - A_\mu^{(n-1)a}}{dy} - \partial_\mu A_y^{(n)a} + \frac{e}{N} f^{abc} A_y^{(n)b} A_\mu^{(n)c} \right) P_\mu^{(n)a}.$$

If we iterate these steps M times and take the limit $M \to \infty$, $Mdy = y^* - a = const$, the set of auxiliary fields $A_\mu^{(n)a}(x), P_\mu^{a(n)}(x)$ is replaced by radially dependent fields $A_\mu^a(x,y), P_\mu^a(x,y)$, and the combination of M terms defined by (4.9) is replaced by $\int dy dx P_\mu^a F_{y\mu}^a$, where $F_{y\mu}^a = \partial_y A_\mu - \partial_\mu A_y + \frac{e}{N} f^{abc} A_y^b A_\mu^c$. Finally, we rewrite our partition function as integral over all possible gauge transformations made at every step, divided by the corresponding gauge volume. Thus, we obtain:

$$Z_{bulk} = \frac{1}{V_{gauge}} \int D\phi DA_\mu^a DP_\mu^a DA_y^a e^{i\int_a^{y^*} dy dx P_\mu^a F_{y\mu}^a - \int \delta S[F_{\mu\nu}^a, P_\mu^a]} Z[A_\mu^a; y^*]. \tag{4.10}$$

Here we assume that, as in Chap. 3, we can extend $y^* \to \infty$ and drop the IR contribution. One can see explicitly, that the action possesses the symmetry under gauge transformations:

$$A_M^a(x,y) \to A_M^a(x,y) + D_M \theta^a(x,y),$$
$$P_\mu^a(x,y) \to P_\mu^a(x,y) + \frac{e}{N} f^{abc} P_\mu^b \theta^c(x,y),$$

provided that they leave the boundary values of A^a_μ invariant, i.e. $\theta^c(x, y = a) = 0$.

4.1 Non-abelian field in the bulk

In this section we show that, in the limit of small field strength and long wavelength, the bulk theory of vector gauge field (4.10), obtained via quantum RG method, describes the conventional non-abelian gauge field in AdS. Our reasoning will be similar to the case of scalar field, studied in section 3.1.

To figure out the specific structure of the bulk action, we return to the step of RG mapping and consider the structure of the quantum correction δS, introduced in the Eq. (4.7). In analogy to the scalar case, we can expand δS in powers of the auxiliary field P^a_μ, and by considering an example of matrix model, we can claim, that at most quadratic powers of P^a_μ are present. Thus, assuming locality, we can write the quantum correction to the action as:

$$\delta S[A^a_\mu, P^a_\mu; a) = \tag{4.11}$$
$$dy\left(B[A^a_\mu; a) + i\int dx B^a_\mu[A^a_\mu; a, x)P^a_\mu(x) - \frac{1}{2}\int dx B^{ab}_{\mu\nu}[A^a_\mu; a, x)P^a_\mu(x)P^b_\nu(x)\right).$$

Here we use the long wavelength approximation, i.e. neglect the derivatives of P^a_μ. In addition, we work in the limit of small field strength, which allows us to consider only the lowest powers of A^a_μ entering the coefficients B. After taking into account the gauge and translational invariance of δS, we conclude that B should be proportional to $(F^a_{\mu\nu})^2$ [4], and the coefficient of the quadratic term should have the form $B^{ab}_{\mu\nu}(x, y) = B_2(y)\delta^{ab}\delta_{\mu\nu}$. The coefficient of the linear term may have a form $B^a_\mu = B_1(y)A^a_\mu$, but from the requirement, that $\mathcal{O}^a_\mu$ is a physical current of the "low energy" theory (4.7), it follows that B_1 vanishes.[5] From the fact that δS is dimensionless and P^a_μ

[4] In $D = 4$ the expression for B can also contain θ-term $\epsilon^{\mu\nu\lambda\rho}F^a_{\mu\nu}F^a_{\lambda\rho}$. However, this term is a total derivative, and, after integrating over the space, it becomes equal to an integer number. We can neglect it by assuming that A^a_μ is a small excitation over the topologically trivial vacuum, i.e. vacuum with zero field.

[5] In the initial theory 4.6, the fact that $\mathcal{O}^a_\mu$ is a current, leads to Ward identities, e.g.

$$\langle D_\mu O^c_\mu(x)\phi_\beta(x_1)\phi_\gamma(x_2)\rangle = -iet^c_{\beta\beta'}\delta(x-x_1)\langle\phi_{\beta'}(x_1)\phi_\gamma(x_2)\rangle - iet^c_{\gamma\gamma'}\delta(x-x_2)\langle\phi_\beta(x_1)\phi_{\gamma'}(x_2)\rangle$$

. The condition, that this relation holds in the coarse-grained theory (4.7), leads to the has the same dimension as the current, i.e. $D-1$, it follows that the term B is proportional to $1/y^{D-3}$, whereas B_2 is proportional to y^{D-3}. After the appropriate rescaling of the fields A^a_μ, P^a_μ and the radial coordinate y , we obtain the following expressions for the coefficients: [6]

$$B = \int dx \frac{F^2_{\mu\nu}}{2y^{D-3}}, \qquad B_2 = \frac{y^{D-3}}{2}. \tag{4.12}$$

If we substitute them into the bulk partition function (4.10), it takes the form:

$$Z = \int DP^a_\mu DA^a_M e^{i\int_a^\infty dydx P^a_\mu F^a_{y\mu} - \int_a^\infty dydx \left(\frac{y^{D-3}(P^a_\mu)^2}{4} + \frac{(F^a_{\mu\nu})^2}{2y^{D-3}}\right)}.$$

In the last equation we can integrate out the momentum P^a_μ, and thus rewrite the partition function as:

$$Z = \int DA^a_M e^{-\int_a^\infty \frac{dydx}{y^{D-3}}\left((F^a_{y\mu})^2 + \frac{(F^a_{\mu\nu})^2}{2}\right)}. \tag{4.13}$$

This equation demonstrates that our bulk theory takes the form of gauge theory in AdS.

Figure 4.1: The diagrams responsible for generation of the cubic and quartic term entering $(F^a_{\mu\nu})^2$. The source A^a_μ is represented by the external lines. The internal lines represent high-energy modes of the fermions.

absence of the term B^a_μ.

[6]In principle it can be surprising that the RG mapping leads to non-linear powers of the source A^a_μ entering the quantum correction to the action. However this can be easily understood by considering an example of Dirac action as fundamental theory. In this case, the cubic and quartic powers of A^a_μ are generated via diagrams represented in the Fig. 4.1.

4.2 Current correlation functions

In the previous section, we showed the duality between current operator and vector field. In this section, we will demonstrate this duality in terms of the relation between the quantum correction to the action δS and correlation functions, similarly to the case of scalar field studied in Sec. 3.2. To derive this relation, we compare the expressions for the CFT partition function (4.6) and (4.7), obtain an equation connecting $Z[A^a_\mu; a)$ and $Z[A^a_\mu; a + dy)$ and write it in infinitesimal form:

$$\left(\delta S\left[A^a_\mu, \frac{\delta}{\delta A^a_\mu}; a\right) - dy\frac{d}{da}\right) Z[A^a_\mu; a) = 0. \tag{4.14}$$

After that, we introduce the generating function for connected correlators $W[\mathcal{A}^a_\mu; a) = \log Z[\mathcal{A}^a_\mu; a)$ and substitute it into the equation. After taking into account, that the quantum correction to the action δS has the form (4.11) with zero linear term, we can rewrite our equation as:

$$\frac{d}{da}W[A^a_\mu; a) = B - \int \frac{dx}{2}B_2\left(\left(\frac{\delta W[A^a_\mu; a)}{\delta A^a_\mu}\right)^2 + \frac{\delta^2 W[A^a_\mu; a)}{(\delta A^a_\mu)^2}\right). \tag{4.15}$$

The generating function W can be expanded as:

$$\begin{aligned} &W[\mathcal{A}^a_\mu; a) = \log Z[\mathcal{A}^a_\mu; a) \\ &= \sum_{n=2}^{\infty} \frac{1}{n!}\int dx_1 \ldots dx_n\ N^{2-n} G^{a_1\ldots a_n}_{\mu_1\ldots\mu_n}(x_1, \ldots, x_n)\mathcal{A}^{a_1}_{\mu_1}(x_1)\ldots\mathcal{A}^{a_n}_{\mu_n}(x_n), \end{aligned} \tag{4.16}$$

where, as we mentioned in Sec. 3.2, we denote the correlation functions by $N^{2-n}G^{a_1\ldots a_n}_{\mu_1\ldots\mu_n}(x_1, \ldots, x_n)$ to make the expansion over $1/N$ more convenient. In addition, to simplify the calculations, we do not take into account the zero-point function (i.e. vacuum energy) and assume that the one-point function (i.e. vacuum expectation value) is zero.

If we substitute the expansion (4.16) into the equation (4.15), use the explicit form of coefficients B and B_2 defined by Eq. (4.12) and decompose it into a series of equations containing equal powers of A^a_μ, we get the following set of equations:

$$\int \frac{d}{da}G^{ab}_{\mu\nu}(x_1, x_2) \cdot A^a_\mu(x_1) \cdot A^b_\nu(x_2)$$

$$
\begin{aligned}
&= \int \frac{dx}{y^{D-3}} \left(\partial_{[\mu} A^a_{\nu]}(x)\right)^2 - \frac{y^{D-3}}{2} \int \Big(G^{ac}_{\mu\tau}(x_1, x_3) \cdot G^{bc}_{\nu\tau}(x_2, x_3) \cdot A^a_\mu(x_1) \cdot A^b_\nu(x_2) \\
&+ \frac{G^{abcd}_{\mu\mu\lambda\rho}(x_1, x_1, x_3, x_4) \cdot A^c_\lambda(x_3) \cdot A^d_\rho(x_4)}{2N^2} \Bigg), \\
&\int \frac{d}{da} G^{abc}_{\mu\nu\lambda}(x_1, x_2, x_3) \cdot A^a_\mu(x_1) \cdot A^b_\nu(x_2) \cdot A^c_\lambda(x_3) \\
&= \int \frac{dx}{y^{D-3}} 6 e f^{abc} \partial_{[\mu} A^a_{\nu]}(x) A^b_\mu(x) A^c_\nu(x) \\
&- \frac{y^{D-3}}{2} \int dx \Big(3 G^{ad}_{\mu\tau}(x_1, x_4) \cdot G^{bcd}_{\nu\lambda\tau}(x_2, x_3, x_4) \cdot A^a_\mu(x_1) \cdot A^b_\nu(x_2) \cdot A^c_\lambda(x_3) \\
&+ \frac{G^{aacde}_{\mu\mu\lambda\rho\sigma}(x_1, x_1, x_3, x_4, x_5) \cdot A^c_\lambda(x_3) \cdot A^d_\rho(x_4) \cdot A^e_\sigma(x_5)}{2N^2} \Bigg), \\
&\int \frac{d}{da} G^{abcd}_{\mu\nu\lambda\rho}(x_1, x_2, x_3, x_4) \cdot A^a_\mu(x_1) \cdot A^b_\nu(x_2) \cdot A^c_\lambda(x_3) \cdot A^d_\rho(x_4) \\
&= 12 \int \frac{dx}{y^{D-3}} e^2 f^{abe} f^{ade} A^b_\mu(x) A^c_\nu(x) A^d_\mu(x) A^e_\nu(x) - \int \frac{y^{D-3}}{2} \times \\
&\Bigg(4 G^{ae}_{\mu\tau}(x_1, x_5) \cdot G^{bcde}_{\nu\lambda\rho\tau}(x_2, x_3, x_4, x_5) \cdot A^a_\mu(x_1) \cdot A^b_\nu(x_2) \cdot A^c_\lambda(x_3) \cdot A^d_\rho(x_4) \\
&+ 3 G^{abe}_{\mu\nu\tau}(x_1, x_2, x_5) \cdot G^{cde}_{\lambda\rho\tau}(x_3, x_4, x_5) \cdot A^a_\mu(x_1) \cdot A^b_\nu(x_2) \cdot A^c_\lambda(x_3) \cdot A^d_\rho(x_4) \\
&+ \frac{G^{aacdef}_{\mu\mu\lambda\rho\sigma\tau}(x_1, x_1, x_3, x_4, x_5, x_6) \cdot A^c_\lambda(x_3) \cdot A^d_\rho(x_4) \cdot A^e_\sigma(x_5) \cdot A^f_\tau(x_6)}{2N^2} \Bigg).
\end{aligned}
$$

Here the symbol of dot $(\cdot)$ refers to integration over all coordinates entering the correlators and the fields. After we take the leading order in $1/N$, transfer into the momentum space [7] and eliminate the fields, we obtain the following equations for two-, three- and four-point functions:

$$\frac{d}{dy}G^{ab}_{\mu\nu}(-k) = \frac{2k^2}{y^{D-3}}\Pi_{\mu\nu}(k)\delta^{ab} - \frac{y^{D-3}}{2}G^{ca}_{\lambda\mu}(k)G^{cb}_{\lambda\nu}(-k), \tag{4.17}$$

$$\frac{d}{dy}G^{abc}_{\mu\nu\lambda}(k_1,k_2,k_3) = -\frac{2ief^{abc}}{y^{D-3}}\left((k_3)_\mu\delta_{\nu\lambda} - (k_3)_\nu\delta_{\mu\lambda} + \text{(cyclic)}\right)$$
$$-\frac{y^{D-3}}{2}\left(G^{dc}_{\rho\lambda}(y,-k_3)G^{dab}_{\rho\mu\nu}(k_3,k_1,k_2) + \text{(cyclic)}\right),$$
$$\frac{d}{dy}G^{abcd}_{\mu\nu\lambda\rho}(k_1,k_2,k_3,k_4) = \frac{2e^2}{y^{D-3}} \times \big(f^{abe}f^{cde}(\delta_{\mu\lambda}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\lambda})$$
$$+f^{ace}f^{bdc}(\delta_{\mu\nu}\delta_{\lambda\rho} - \delta_{\mu\rho}\delta_{\nu\lambda}) + f^{ade}f^{bce}(\delta_{\mu\nu}\delta_{\lambda\rho} - \delta_{\mu\lambda}\delta_{\nu\rho})\big)$$
$$-\frac{y^{D-3}}{2}\left(G^{ea}_{\sigma\mu}(k_1)G^{ebcd}_{\sigma\nu\lambda\rho}(k_1,k_2,k_3,k_4) + \text{cyclic}\right)$$
$$-\frac{y^{D-3}}{2}\Big(G^{eab}_{\sigma\mu\nu}(k_3+k_4,k_1,k_2)G^{ecd}_{\sigma\lambda\rho}(k_1+k_2,k_3,k_4)$$
$$+G^{eac}_{\sigma\mu\lambda}(k_2+k_4,k_1,k_3)G^{ebd}_{\sigma\nu\rho}(k_1+k_3,k_2,k_4)$$
$$+G^{ead}_{\sigma\mu\rho}(k_2+k_3,k_1,k_4)G^{ebc}_{\sigma\nu\lambda}(k_1+k_4,k_2,k_3)\Big).$$

In the next subsections we solve these equations. For this, we use the Ward identities for the correlation functions, which we derive in the next subsection.

[7]Our conventions for the Fourier transformation have the following form:

$$\mathcal{A}^a_\mu(x) = \int \frac{dk}{(2\pi)^d}e^{-i\vec{k}\cdot\vec{x}}\mathcal{A}^a_\mu(k),$$
$$G^{ab}_{\mu\nu}(x_1,x_2) = \int \frac{dk}{(2\pi)^D}e^{i\vec{k}\cdot(\vec{x_1}-\vec{x_2})}G^{ab}_{\mu\nu}(k),$$
$$G^{a_1\ldots a_n}_{\mu_1\ldots\mu_n}(x_1,\ldots,x_n) = \int \frac{dk_1\ldots dk_n}{(2\pi)^{nd}}e^{i\sum_{r=1}^n i\vec{k_r}\cdot\vec{x_r}}G^{a_1\ldots a_n}_{\mu_1\ldots\mu_n}(k_1,\ldots,k_n)(2\pi)^D\delta(\sum_r k_r)$$

for $n \geq 3$

4.2.1 Ward identities

As we discussed previously, our CFT has the global symmetry (4.1) of the fundanemtal field. Moreover, the boundary theory is invariant under the gauge transformations, provided, the source transforms according to the Eq. (4.3). Therefore, the generating function W is also invariant under these transformations. Particularly, if we perform gauge transformation of the source entering the generating function, its variation over the gauge function has to vanish. The equations describing this invariance, constitute Ward identities. In the case, when the one-point function is zero, they have the following form:

$$k_\mu G^{ab}_{\mu\nu}(y,k) = 0, \tag{4.18}$$

$$k_{3_\lambda} G^{abc}_{\mu\nu\lambda}(k_1,k_2,k_3) = -ief^{abc}\left(G_{\mu\nu}(k_1) - G_{\mu\nu}(k_2)\right), \tag{4.19}$$

$$\begin{aligned} k_4^\rho G^{abcd}_{\mu\nu\lambda\rho}(k_1,k_2,k_3,k_4) = -ie\big(&f^{ecd}G^{abe}_{\mu\nu\lambda}(k_1,k_2,k_3+k_4) \\ &+f^{ebd}G^{aec}_{\mu\nu\lambda}(k_1,k_2+k_4,k_3) + f^{ead}G^{ebc}_{\mu\nu\lambda}(k_1+k_4,k_2,k_3)\big)\,. \end{aligned} \tag{4.20}$$

We will use them to solve the equations (4.17 - 4.17).

4.2.2 Two-point function

From the Ward identity for the two-point function (4.18), we can conclude that the latter has to be proportional to projection operator. In addition, the equation for the two-point function (4.17) has trivial color structure, therefore we assume, that the color indices enter two-point function as:

$$G^{ab}_{\mu\nu}(k) = G(k)\delta^{ab}\Pi_{\mu\nu}(k).$$

If we substitute it into (4.17), all tensor structures cancel out, and the equation becomes:

$$\frac{d}{dy}G(-k) = \frac{2k^2}{y^{D-3}} - \frac{y^{D-3}}{2}G(k)G(-k).$$

The solution that decays at $a \to \infty$ is analogous to the case of scalar field, and has the form:

$$G^{ab}_{\mu\nu}(y) = \frac{2}{y^{D-3}}\frac{\partial_y \tilde{K}(k,y)}{\tilde{K}(k,y)}\delta^{ab}\Pi_{\mu\nu}(k), \tag{4.21}$$

where to simplify the notations we introduced

$$\tilde{K}(k,y) = y^{\frac{D}{2}-1} K_{\frac{D}{2}-1}(ky),$$

where, as usually, K is the modified Bessel function. One can see that this expression for $G^{ab}_{\mu\nu}$ exactly matches the conventional expression for the two-point function (2.16).

4.2.3 Three-point function

In this section, we consider the equation for the three-point function (4.17). One can check explicitly that this equation is consistent with the Ward identity (4.19). In particular, if the Eq. (4.17) is contracted with k_{3_λ}, then the equation for $k_{3_\lambda} G^{abc}_{\mu\nu\lambda}$ has the right-hand side of the Ward identity (4.19) as a solution.

To simplify the equation (4.17), we expand the projection operators in it, and apply the Ward identities to all combinations of the three-point function $G^{abc}_{\mu\nu\lambda}$ contracted with any momentum k. Thus we come to the equation:

$$\begin{aligned}
&\frac{d}{dy} G^{abc}_{\mu\nu\lambda}(k_1,k_2,k_3) = \\
&-\frac{2ief^{abc}}{y^{D-3}}\left(k_{3_\mu}\delta_{\nu\lambda} - k_{3_\nu}\delta_{\mu\lambda}\right) - \frac{y^{D-3}}{2} G(-k_1) G^{abc}_{\mu\nu\lambda}(k_1,k_2,k_3) \\
&-\frac{iey^{D-3}f^{abc}}{2}\left(\frac{k_{3_\lambda}}{k_3^2} G(k_3)(G_{\mu\nu}(k_1) - G_{\mu\nu}(k_2))\right) + \text{cyclic},
\end{aligned} \tag{4.22}$$

which, after substitution of the explicit form for the two-point function (4.21), can be rewritten as:

$$\begin{aligned}
&\frac{d}{dy}\left(\prod_r \tilde{K}(k_r,y) G^{abc}_{\mu\nu\lambda}(k_1,k_2,k_3)\right) = \\
&-\frac{2ief^{abc}}{y^{D-3}}\left(k_{3_\mu}\delta_{\nu\lambda} - k_{3_\nu}\delta_{\mu\lambda}\right) \prod_r \tilde{K}(k_r,y) \\
&-\frac{2ief^{abc}}{y^{D-3}} \frac{k_{1_\mu}}{k_1^2} \partial_y \tilde{K}(k_1,y) \Big(\tilde{K}(k_3,y)\partial_y \tilde{K}(k_2,y)\Pi_{\nu\lambda}(k_2) \\
&\quad - \tilde{K}(k_2,y)\partial_y \tilde{K}(k_3,y)\Pi_{\nu\lambda}(k_3)\Big) + \text{cyclic}.
\end{aligned}$$

Its general solution is obtained by direct integration and has the form as following:

$$G^{abc}_{\mu\nu\lambda}(k_1,k_2,k_3) = \tag{4.23}$$
$$\frac{2ief^{abc}}{\prod_r \tilde{K}(k_r,a)} \int_a^{y^*} \frac{dy}{y^{D-3}} \Bigg((k_{2_\lambda} - k_{1_\lambda})\delta_{\mu\nu} \prod_r \tilde{K}(k_r,y) + \frac{k_{1_\mu}}{k_1^2}\partial_y \tilde{K}(k_1,y)$$
$$\times \Big(\tilde{K}(k_3,y)\partial_y\tilde{K}(k_2,y)\Pi_{\nu\lambda}(k_2) - \tilde{K}(k_2,y)\partial_y\tilde{K}(k_3,y)\Pi_{\nu\lambda}(k_3) \Big) + \text{cyclic} \Bigg) .$$

In this equation, y^* is an arbitrary constant of integration. If we take into account the boundary condition, which states that $G^{abc}_{\mu\nu\lambda}$ vanishes at the IR boundary, we come to conclusion that y^* should be equal to the value of radial coordinate at the IR boundary. Particularly, if the IR boundary is located at $y = \infty$, then y^* should also be infinite.

Thus, we have obtained the expression for $G^{abc}_{\mu\nu\lambda}$ as a solution of the auxiliary equation (4.22). One can verify that this expression for $G^{abc}_{\mu\nu\lambda}$ satisfies the Ward identity (4.19), and therefore is also a solution of the initial equation (4.17).

One can check that the three-point function obtained from conventional AdS/CFT (see Eq. 2.25) also satisfies the initial equation (4.17) and has the same behavior at the boundaries. From this it follows, that our expression for the three-point function (4.23) is consistent with the conventional AdS/CFT.

4.3 Four-point function

The equation for the four-point function (4.17) is solved in the same way as the equation for three-point function considered in the previous section. We expand the projectors in the two-point functions, and substitute the Ward identities (4.20) for combinations of $G^{abcd}_{\mu\nu\lambda\rho}$ contracted with any k, thus obtaining the equation:

$$\frac{d}{dy} G^{abcd}_{\mu\nu\lambda\rho}(k_1,k_2,k_3,k_4) = -\sum_r \frac{\partial_y \tilde{K}(k_r,y)}{\tilde{K}(k_r,y)} G^{abcd}_{\mu\nu\lambda\rho}(k_1,k_2,k_3,k_4)$$
$$+ \frac{2e^2}{y^{D-3}} \times \big(f^{abe}f^{cde}(\delta_{\mu\lambda}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\lambda})$$
$$+ f^{ace}f^{bdc}(\delta_{\mu\nu}\delta_{\lambda\rho} - \delta_{\mu\rho}\delta_{\nu\lambda}) + f^{ade}f^{bce}(\delta_{\mu\nu}\delta_{\lambda\rho} - \delta_{\mu\lambda}\delta_{\nu\rho}) \big)$$

$$
\begin{aligned}
&-\frac{y^{D-3}}{2}\big(G^{eab}_{\sigma\mu\nu}(k_3+k_4,k_1,k_2)G^{ecd}_{\sigma\lambda\rho}(k_1+k_2,k_3,k_4)\\
&\qquad+G^{eac}_{\sigma\mu\lambda}(k_2+k_4,k_1,k_3)G^{ebd}_{\sigma\nu\rho}(k_1+k_3,k_2,k_4)\\
&\qquad+G^{ead}_{\sigma\mu\rho}(k_2+k_3,k_1,k_4)G^{ebc}_{\sigma\nu\lambda}(k_1+k_4,k_2,k_3)\Big)\\
&-\frac{iey^{D-3}}{2}\Big\{G(k_1)\frac{k_{1_\mu}}{k_1^2}\big(f^{eda}G^{bce}_{\nu\lambda\rho}(k_2,k_3,k_4+k_1)\\
&\quad+f^{eca}G^{bed}_{\nu\lambda\rho}(k_2,k_3+k_1,k_4)+f^{eba}G^{ecd}_{\nu\lambda\rho}(k_2+k_1,k_3,k_4)\big)+\text{cyclic}\Big\},
\end{aligned}
$$

which can be rewritten as:

$$
\begin{aligned}
&\frac{d}{dy}\left(\prod_r\tilde{K}(k_r,y)G^{abcd}_{\mu\nu\lambda\rho}(k_1,k_2,k_3,k_4)\right)=\\
&+\prod_r\tilde{K}(k_r,y)\times\Bigg(\frac{2e^2}{y^{D-3}}\big(f^{abe}f^{cde}(\delta_{\mu\lambda}\delta_{\nu\rho}-\delta_{\mu\rho}\delta_{\nu\lambda})\\
&\qquad+f^{ace}f^{bdc}(\delta_{\mu\nu}\delta_{\lambda\rho}-\delta_{\mu\rho}\delta_{\nu\lambda})+f^{ade}f^{bce}(\delta_{\mu\nu}\delta_{\lambda\rho}-\delta_{\mu\lambda}\delta_{\nu\rho})\big)\\
&-\frac{y^{D-3}}{2}\big(G^{eab}_{\sigma\mu\nu}(k_3+k_4,k_1,k_2)G^{ecd}_{\sigma\lambda\rho}(k_1+k_2,k_3,k_4)\\
&\qquad+G^{eac}_{\sigma\mu\lambda}(k_2+k_4,k_1,k_3)G^{ebd}_{\sigma\nu\rho}(k_1+k_3,k_2,k_4)\\
&\qquad+G^{ead}_{\sigma\mu\rho}(k_2+k_3,k_1,k_4)G^{ebc}_{\sigma\nu\lambda}(k_1+k_4,k_2,k_3)\Big)\\
&-\frac{iey^{D-3}}{2}\Big\{G(k_1)\frac{k_{1_\mu}}{k_1^2}\big(f^{eda}G^{bce}_{\nu\lambda\rho}(k_2,k_3,k_4+k_1)\\
&\quad+f^{eca}G^{bed}_{\nu\lambda\rho}(k_2,k_3+k_1,k_4)+f^{eba}G^{ecd}_{\nu\lambda\rho}(k_2+k_1,k_3,k_4)\big)+\text{cyclic}\Big\}\Bigg).
\end{aligned}
$$

Its general solution is:

$$
\begin{aligned}
&G^{abcd}_{\mu\nu\lambda\rho}(k_1,k_2,k_3,k_4)= \qquad (4.24)\\
&-\frac{1}{\prod_r\tilde{K}(k_r,a)}\int_a^{y^*}dy\prod_r\tilde{K}(k_r,y)\Bigg(\frac{2e^2}{y^{D-3}}\times\big(f^{abe}f^{cde}(\delta_{\mu\lambda}\delta_{\nu\rho}-\delta_{\mu\rho}\delta_{\nu\lambda})\\
&\qquad+f^{ace}f^{bde}(\delta_{\mu\nu}\delta_{\lambda\rho}-\delta_{\lambda\nu}\delta_{\mu\rho})+f^{ade}f^{bce}(\delta_{\mu\nu}\delta_{\lambda\rho}-\delta_{\nu\rho}\delta_{\mu\lambda})\big)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{iey^{D-3}}{2}\left(\frac{k_{1_\mu}}{k_1^2}G(k_1)\,\big(f^{eda}G^{bce}_{\nu\lambda\rho}(k_2,k_3,k_4+k_1)\right.\\
&+f^{eca}G^{bed}_{\nu\lambda\rho}(k_2,k_3+k_1,k_4)+f^{eba}G^{ecd}_{\nu\lambda\rho}(k_2+k_1,k_3,k_4)\big)\ \text{+cyclic})\\
&-\frac{y^{D-3}}{2}\,\big(G^{eab}_{\sigma\mu\nu}(k_3+k_4,k_1,k_2)G^{ecd}_{\sigma\lambda\rho}(k_1+k_2,k_3,k_4)\\
&\qquad+G^{eac}_{\sigma\mu\lambda}(k_2+k_4,k_1,k_3)G^{ebd}_{\sigma\nu\rho}(k_1+k_3,k_2,k_4)\\
&\qquad\quad+G^{ead}_{\sigma\mu\rho}(k_2+k_3,k_1,k_4)G^{ebc}_{\sigma\nu\lambda}(y,k_1+k_4,k_2,k_3)\big)\ )\,.
\end{aligned}
$$

As in the case of the three-point function, the expression contains an arbitrary constant y^*, and the condition that $G^{abcd}_{\mu\nu\lambda\rho}$ vanishes at the IR boundary, forces y^* to be equal to the radial coordinate of the latter. Also, in analogy to the case of the three-point function, the assumption of IR boundary being at infinite value of the radial coordinate y, results $y^* = \infty$. Finally, the answer for $G^{abcd}_{\mu\nu\lambda\rho}$ satisfies the Ward identity (4.20) and thus it is the solution of the initial equation (4.17).

One can check that the four-point function (2.25) obtained from conventional AdS/CFT also satisfies the equation (4.17) and has the same boundary behavior, as our solution (4.24). Therefore, our answer is consistent with the conventional AdS/CFT dictionary.

Chapter 5

Summary

In this work we demonstrated that the quantum RG scheme can be applied to construct holographic duals for regularized CFTs. We showed that within this approach, it is possible to construct standard AdS actions for scalar and vector fields from a conformal theory. In particular we showed that global symmetry in the boundary theory becomes a gauge symmetry in the bulk. This was explicitly shown in the case where the quantum correction to the action (resulting from the RG mapping) contains only current operator. We also showed that the CFT correlators in the resulting theory have the same form as predicted by conventional AdS/CFT, and their dependence on the regularization parameter corresponds to conventional regularization in the AdS/CFT, i.e. placing the boundary at finite value of the radial coordinate.

For the future work, it would be interesting to apply this method to specific theories and to find an explicit form of regularization and RG mapping, which satisfies our assumptions. Finding such regularization in the theory of $\mathcal{N} = 4$ Super-Yang-Mills may allow to derive AdS supergravity from the field theory and thus to prove the AdS/CFT conjecture. Also, it is of interest to apply this method to other known theories, as it may lead to construction of new simple models of bulk/boundary duality. An example of boundary theory, which is worth studying, includes Kondo model (see e.g. [12]), describing interaction between conduction electrons in a solid and an impurity spin.

Bibliography

[1] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [hep-th/9711200].

[2] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. **323**, 183 (2000) [hep-th/9905111].

[3] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998) [hep-th/9802109].

[4] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998) [hep-th/9802150].

[5] S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, Phys. Rev. Lett. **101**, 031601 (2008) [arXiv:0803.3295 [hep-th]].

[6] R. Sundrum, Phys. Rev. D **86**, 085025 (2012) [arXiv:1106.4501 [hep-th]].

[7] S. -S. Lee, JHEP **1210**, 160 (2012) [arXiv:1204.1780 [hep-th]].

[8] D. Z. Freedman, S. D. Mathur, A. Matusis and L. Rastelli, Nucl. Phys. B **546**, 96 (1999) [hep-th/9804058].

[9] H. Liu and A. A. Tseytlin, Phys. Rev. D **59**, 086002 (1999) [hep-th/9807097].

[10] G. Chalmers, H. Nastase, K. Schalm and R. Siebelink, Nucl. Phys. B **540**, 247 (1999) [hep-th/9805105].

[11] http://en.wikipedia.org/wiki/Riccati_equation

[12] Coleman, P., eprint arXiv:cond-mat/0612006, 2006