Flexible Multiple Description Lattice Vector Quantizer with General Number of Descriptions

FLEXIBLE MULTIPLE DESCRIPTION LATTICE VECTOR QUANTIZER WITH GENERAL NUMBER OF DESCRIPTIONS

BY

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A THESIS

SUBMITTED TO THE DEPARTMENT OF ELECTRICAL & COMPUTER ENGINEERING

AND THE SCHOOL OF GRADUATE STUDIES

OF MCMASTER UNIVERSITY

IN PARTIAL FULFILMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF APPLIED SCIENCE

Master of Applied Science (2014)	McMaster University
(Electrical & Computer Engineering)	Hamilton, Ontario, Canada

TITLE:	Flexible Multiple Description Lattice Vector Quantizer with General Number of Descriptions
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SUPERVISOR:	Dr. Sorina Dumitrescu
NUMBER OF PAGES:	ix, 72

To my family

Abstract

This thesis addresses the design of multiple description lattice vector quantizer (MDLVQ) with a general number L of descriptions, $L \geq 3$. In the previous work on MDLVQ with $L \geq 3$, once the central and side lattice codebooks are fixed, the decoding quality is determined for all numbers k of received descriptions. Therefore, it is not possible to achieve tradeoffs between the quality of reconstruction for different values of $k, 1 \leq k \leq L-1$.

In order to overcome the above drawback, we propose two flexible MDLVQ schemes for $L \geq 3$. Our first design employs a different reconstruction method than in prior work and a heuristic index assignment algorithm, which uses $L-2$ parameters to control the distortions for $2 \leq k \leq L-1$. Experimental results for the cases $L=3$ and $L=4$ show that significant tradeoffs are achieved by controlling the parameters mentioned above.

Our second design is based on a structured index assignment. We start with the case $L=3$ and then generalize the index assignment to any $L \geq 3$. The structured index assignment is able to control the tradeoff by adjusting the sizes of some $L-1$ subsets of side lattice points. Another important contribution of the thesis is the derivation of analytical expressions of the distortions for the structured index assignment, under the high resolution assumption. These expressions show that a wide range of distortion values can be achieved.

Acknowledgements

First and foremost, I would like to thank my supervisor Dr. Sorina Dumitrescu, for all the patient guidance, encouragement and advice she has provided throughout my study. It has been an honor to be her student and I will never forget all the help and support she has given me in the past two years.

For this thesis, I would like to thank my reading committee members, Dr. Jun Chen and Dr. Dongmei Zhao, for their time, interest and valuable input.

A special thanks goes out to all of my friends who made Hamilton my home.

Finally, I would like to express my grateful thanks to my parents, Linfang Gao and Heqin Zhou, and my sister, Yunyue Gao for their unconditional love and support.

Notation and abbreviations

MDC: multiple description coding

MDVQ: multiple description vector quantizer

MDLVQ: multiple description lattice vector quantizer

MDSQ: multiple description scalar quantizer

pdf: probability density function

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Chapter 1

Introduction

1.1 Multiple Description Coding (MDC)

A multiple description coder consists of L encoders for some $L \geq 2$, each encoder generating a separate description of the signal. Each description is sent to the destination over a separate channel, which either transmits the whole description correctly or breaks down. The decoder is able to reconstruct the signal to some quality from any subset of received descriptions, while the fidelity of the reconstruction generally increases with the number of received descriptions. Figure 1.1 illustrates the block diagram of an MDC system for $L = 3$.

For the potential application in modern communication systems, MDC has greatly attracted the attention of researchers in the past two decades. During communications over the Internet for instance, a packet may be discarded by the routers for a variety of reasons, such as congestion. With MDC the transmitter sends multiple descriptions as separate packets, each subset of which would lead to a reconstruction of certain fidelity. Thus MDC enables graceful recovery from losses when retransmission is not an option due to stringent delay constraints or to avoid channel congestion.

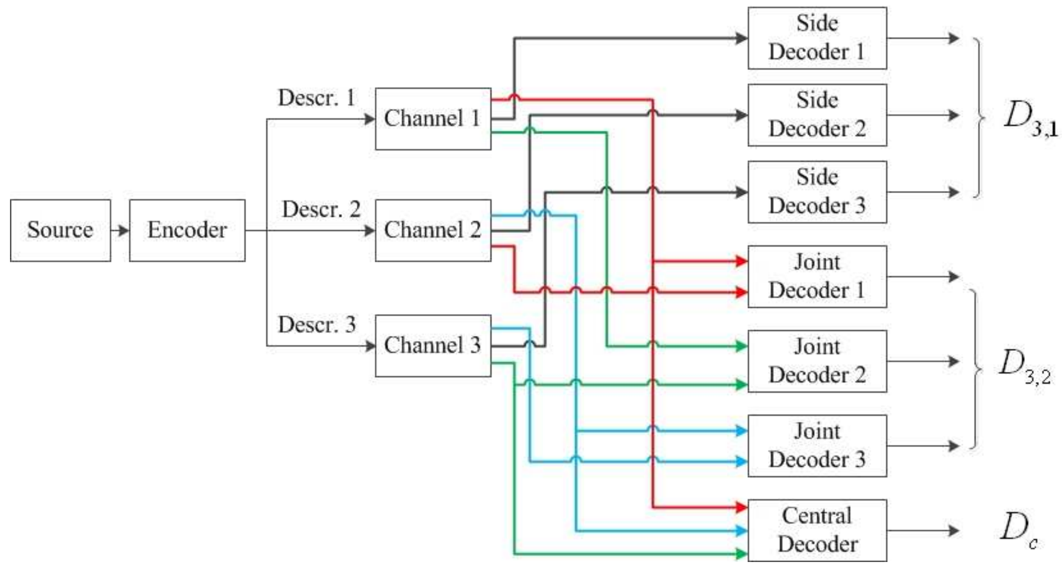


Figure 1.1: Block diagram of an MDC system with $L = 3$.

MDC can be generated by a variety of techniques. One of them is quantization. In (Vaishampayan, 1993), the author proposed a practical multiple description scalar quantizer (MDSQ) and addressed the problem of index assignment design. The MDSQ design was further addressed in (Berger-Wolf and Reingold, 2002; Tian and Hemami, 2004a,b; Dumitrescu and Wu, 2005, 2007; Muresan and Effros, 2008; Dumitrescu and Wu, 2009). Multiple description vector quantizer (MDVQ) was addressed in (Fleming *et al.*, 2004) and multiple description lattice vector quantizer (MDLVQ) was first introduced in (Servetto *et al.*, 1999; Vaishampayan *et al.*, 2001) for the case $L = 2$. Other work on MDLVQ can be found in (Kelner *et al.*, 2000; Diggavi *et al.*, 2002; Goyal *et al.*, 2002; Frank-Dayan and Zamir, 2002; Chen *et al.*, 2006; Ostergaard *et al.*, 2006; Huang and Wu, 2006; Liu and Zhu, 2009; Ostergaard *et al.*, 2010; Zhang *et al.*, 2011, 2012a). Beside quantization, multiple description codes can be generated using correlated transforms (Wang *et al.*, 2001; Goyal and Kovacevic, 2001), domain partitioning (Jiang and Ortega, 1999; Subbalakshmi and Somasundaram, 2002; Bajic and Woods, 2003; Akyol *et al.*, 2007; Tillo *et al.*, 2010), unequal erasure protection

(Puri and Ramchandran, 1999; Mohr *et al.*, 2000; Dumitrescu *et al.*, 2004; Thie and Taubman, 2005; Dumitrescu *et al.*, 2007, 2010), low density generator matrix codes (Zhang *et al.*, 2012b; Chen *et al.*, 2012), etc. Additionally, information theoretical results on the problem of multiple descriptions were discussed in (Ozarow, 1980; Wolf *et al.*, 1980; Ahlswede, 1985; Zhang and Berger, 1987, 1995; Venkataramani *et al.*, 2003; Pradhan *et al.*, 2004; Puri *et al.*, 2005; Tian and Chen, 2010; Wang *et al.*, 2011).

Most of the existing practical MDC frameworks are designed for the case of two descriptions. Moreover, the existing MDC schemes for general L have limited mechanisms to control the tradeoff between the quality of reconstruction when different numbers of descriptions are received. Therefore, they have limited performance. In this thesis we address the design of MDLVQ scheme with high flexibility in adjusting this tradeoff. The next section briefly reviews related work on MDLVQ. Finally, section 1.3 presents the contribution and organization of the thesis.

1.2 Multiple Description Lattice Vector Quantizer

An n-dimensional MDLVQ is an MDC consisting of a so-called central lattice $\Lambda_c \subset \mathbf{R}^n$, a so-called side lattice Λ_s, which is a sublattice of Λ_c and an injective mapping $\alpha : \Lambda_c \rightarrow \Lambda_s^L$ (termed index assignment), which assigns to each central lattice point an L-tuple of side lattice points. For each $\lambda_c \in \Lambda_c$, we will denote by $\alpha_i(\lambda_c)$ the i-th component of the L-tuple $\alpha(\lambda_c)$. The encoder of the MDLVQ quantizes the input vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ to the closest central lattice point λ_c and outputs $\alpha_i(\lambda_c)$ as the i-th description, for $1 \leq i \leq L$. When all descriptions $\lambda_1, \cdots, \lambda_L$ are received at the destination the decoder is able to uniquely identify the corresponding central lattice point and uses it as the source reconstruction. When only one description λ_i

is received, the decoder uses it as the source reconstruction. Notice that when $L = 2$, only the above mentioned situations are possible at the decoder. On the other hand, when $L \geq 3$ and the number of received description is $k, 1 < k < L$, then a decoding method has to be specified based on the received descriptions.

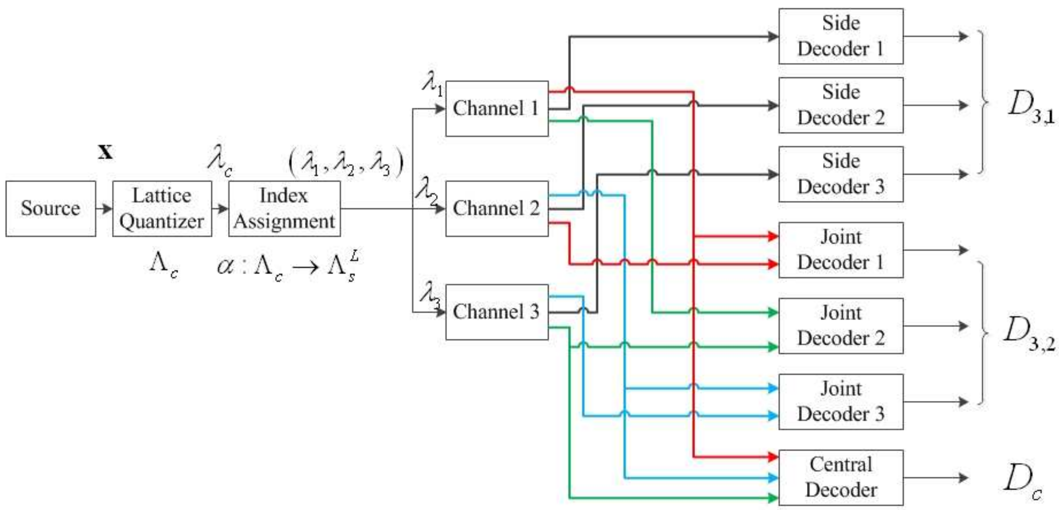


Figure 1.2: Block diagram of an MDLVQ system with $L = 3$.

Recently, a number of MDLVQ schemes have been proposed. In (Servetto *et al.*, 1999; Vaishampayan *et al.*, 2001), the authors defined the structure of the MDLVQ with two symmetric descriptions and presented guiding principles for the design of the index assignment. Additionally, the analytical expressions of the distortions at high resolution were derived. The asymmetric case was addressed in (Diggavi *et al.*, 2002) by Diggavi, Sloane and Vaishampayan.

By modifying the encoding rule of (Servetto *et al.*, 1999), the authors of (Kelner *et al.*, 2000) proposed an index assignment scheme which improves the performance for the case when the description loss probability is known and an extension to more than two descriptions is briefly mentioned. In (Goyal *et al.*, 2002), the authors pointed out that the MDLVQ structure proposed in (Servetto *et al.*, 1999; Vaishampayan *et al.*, 2001) inherently optimizes the case when both descriptions are received. Moreover, by replacing the central lattice with a nonlattice codebook, additional tradeoffs between the central and side distortion were achieved.

A systematic study of the MDLVQ for $L > 2$ descriptions is performed in (Ostergaard *et al.*, 2006). The authors of (Ostergaard *et al.*, 2006) use the arithmetic average of the received descriptions to reconstruct the source, when the number of received descriptions is $k, 2 \leq k \leq L-1$. They propose an algorithm for optimizing the index assignment and derive asymptotical expressions of the distortions for different numbers of received descriptions. The authors of (Huang and Wu, 2006) propose a simpler and faster index assignment algorithm and prove its optimality for $L = 2$ with any N, and for $L > 2$ when $N \to \infty$, where N is the index of the side lattice with respect to the central lattice. They also derive asymptotical expressions of the distortions. The work (Liu and Zhu, 2009) proposes an improvement to the index assignment of (Huang and Wu, 2006) for finite N. The work (Zhang *et al.*, 2011) uses a simpler method to analyze the asymptotical performance of the MDLVQ of (Huang and Wu, 2006). A multiple description quantizer with translated lattice codebooks and the associated optimal index assignment are discussed in (Zhang *et al.*, 2012a). In (Ostergaard *et al.*, 2010), the asymmetric MDLVQ with $L \geq 2$, which uses the weighted average of the received descriptions as reconstruction, is investigated.

1.3 Contribution and Organization of the Thesis

The MDLVQ framework for $L \geq 3$ considered in prior work is able to achieve tradeoffs between the reconstruction quality when all descriptions are received versus the case when only a subset of them are received, by varying the value of N for fixed rate R of individual description. However, it is not possible to achieve tradeoffs between the decoding quality for different numbers k, $1 \leq k \leq L-1$, of received descriptions.

In this thesis we propose two flexible MDLVQ systems for $L \geq 3$, which are able to adjust the decoding quality for various values of $k, 1 \leq k \leq L-1$. The first scheme uses a different reconstruction method and a heuristic index assignment algorithm, which employs $L-2$ parameters to control the reconstruction quality for $2 \leq k \leq L-1$. Experimental results for the cases $L = 3$ and $L = 4$ show that significant tradeoffs are achieved by controlling the parameters mentioned above.

The second scheme is based on a structured index assignment with a simple mechanism to control the tradeoffs between the distortions when different numbers of descriptions are received. Asymptotical expressions of the distortions for $1 \leq k \leq L$ are derived under the high resolution assumption. These expressions show that a wide range of distortion values can be achieved when $k < L$ even if N and R are fixed.

We point out that the MDLVQ framework based on the heuristic index assignment and preliminary version of the structured index assignment for $L = 3$ were presented in the conference paper (Gao and Dumitrescu, 2014). Additionally, a paper containing the results of chapters 3 and 4 is currently under review for possible publication in IEEE Transactions on Communications.

The thesis is organized as follows. Chapter 2 introduces the definitions and notations of MDLVQ and reviews a previous MDLVQ design. In chapter 3, a heuristic index assignment for a flexible MDLVQ scheme is proposed and experimental results for the cases $L = 3$ and $L = 4$ are given. Chapter 4 presents the proposed structured index assignment for $L = 3$ and shows the derivation of the distortion expressions at high resolution. In chapter 5, the structured index assignment proposed in chapter 4 is extended to any $L, L \geq 3$ and the asymptotic analysis under high resolution is performed. Chapter 6 concludes the thesis.

Chapter 2

Preliminaries

This chapter presents definitions, notations and general results related to MDLVQ. The following section introduces the definitions and notations of MDLVQ. Section 2.2 reviews the rate computation and preliminary results needed for the distortion computation. Finally, a previous MDLVQ design is briefly reviewed in section 2.3.

2.1 MDLVQ Definitions and Notations

2.1.1 Central Lattice and Side Lattice

The central lattice Λ_c is a discrete set of points

$$\Lambda_c \triangleq \{\lambda_c \in \mathbf{R}^n : \lambda_c = \mathbf{z}G_c, \mathbf{z} \in \mathbb{Z}^n\}, \tag{2.1}$$

where G_c is the n-by-n generator matrix of the lattice Λ_c. The Gram matrix of Λ_c is

$$M_c = G_c {G_c}^T \tag{2.2}$$

where $\mathcal{G}^T$ denotes the transpose of matrix $\mathcal{G}$.

The volume of a Voronoi region of Λ_c with Gram matrix M_c is (Conway and Sloane, 1998)

$$\nu_c = \sqrt{|M_c|} \tag{2.3}$$

where $|\mathcal{M}|$ is the determinant of matrix $\mathcal{M}$.

The side lattice Λ_s is a sublattice of Λ_c obtained by applying a rotation, scaling and reflection to Λ_c, i.e., $G_s = cG_cU$ where G_s is the generator matrix of Λ_s, c is a scalar and U is an orthogonal matrix of determinant 1. In other words, Λ_s is geometrically similar to Λ_c. The index number N of Λ_s is defined as

$$N = \frac{\nu_s}{\nu_c} = c^n, \tag{2.4}$$

where ν_s denotes the volume of a Voronoi region of Λ_s.

Let us define the inner product of n-dimensional vectors $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$ as

$$\langle \mathbf{x}, \mathbf{y} \rangle \triangleq \frac{1}{n} \sum_{i=1}^{n} x_i y_i. \tag{2.5}$$

Additionally, define the l_2-norm as $\|\mathbf{x}\| \triangleq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.

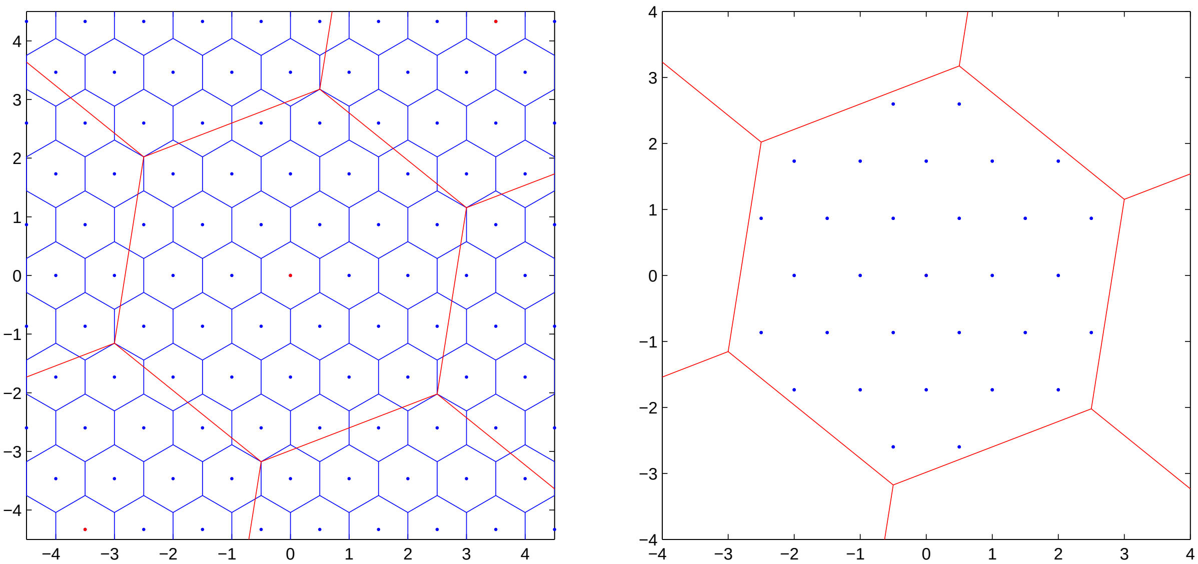
Let $\hat{V}_s(\lambda)$ denote the Voronoi region of a side point $\lambda, \lambda \in \Lambda_s$. It is defined as

$$\hat{V}_s(\lambda) \triangleq \{\mathbf{x} \in \mathbf{R}^n : \|\mathbf{x} - \lambda\| \leq \|\mathbf{x} - \lambda'\|, \forall \lambda' \in \Lambda_s\}. \tag{2.6}$$

We define a discrete Voronoi cell for each side point $\lambda, \lambda \in \Lambda_s$ as

$$V_s(\lambda) \triangleq \{\lambda_c \in \Lambda_c : \|\lambda_c - \lambda\| \leq \|\lambda_c - \lambda'\|, \forall \lambda' \in \Lambda_s\}. \tag{2.7}$$

For simplicity, we assume that the sublattice Λ_s is clean, i.e., there is no central



(a) (b)

Figure 2.1: Figure 2.1a shows the Voronoi regions of Λ_c and Λ_s. The small blue hexagons are the Voronoi regions of the central lattice Λ_c and the big red hexagons are the Voronoi regions of the side lattice Λ_s. Figure 2.1b shows the discrete Voronoi set of $\mathbf{0}$, i.e., $V_s(\mathbf{0})$. The central lattice points $\lambda_c \in V_s(\mathbf{0})$ are marked by $\cdot$.

lattice point lying on the boundary of $\hat{V}_s(\lambda), \lambda \in \Lambda_s$. If Λ_s is clean, then the index N equals to the number of central lattice points within $\hat{V}_s(\lambda), \lambda \in \Lambda_s$. Figure 2.1 illustrates an example of central and side lattices and their Voronoi regions.

2.1.2 System Setting

We consider an information source which generates a sequence of independent and identically distributed random variables with probability density function (pdf) f. The source is segmented into n-dimensional vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. The pdf of the vectors $\mathbf{x}$ is denoted by $f(\mathbf{x})$, where

$$f(\mathbf{x}) = \prod_{j=1}^{n} f(x_j). \tag{2.8}$$

The process of multiple description lattice vector quantization generally involves two steps. First, the vector $\mathbf{x}$ is quantized to the nearest central lattice point λ_c in the central lattice Λ_c. Let $P(\lambda_c)$ denote the probability of quantizing $\mathbf{x}$ to λ_c and let $\mathrm{Q}(\mathbf{x}) = \lambda_c$ denote the quantization mapping. Second, the central lattice point λ_c is mapped to L descriptions by employing an index assignment scheme. Let α be the injective function which maps $\lambda_c \in \Lambda_c$ to an L-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_L) \in \Lambda_s^L$. Recall that if the receiver receives all the descriptions, it can reconstruct $\mathbf{x}$ to the central lattice point with the inverse labeling function α^{-1}, i.e., $\alpha^{-1}(\lambda_1, \lambda_2, \ldots, \lambda_L) = \lambda_c$. When only one description $\lambda_i, i = 1, 2, ..., L$ is received, the receiver reconstructs the transmitted vector as λ_i. Recall that $(\alpha_1, \alpha_2, ..., \alpha_L)$ are the component functions of α, i.e.,

$$\alpha_i(\lambda_c) = \lambda_i, 1 \le i \le L. \tag{2.9}$$

In the previous work, when a subset of the L descriptions is received, the reconstruction is the average of the received descriptions. The flexible MDLVQ schemes that we propose will use different reconstruction rules. We use the symbol $*$ to represent a missing description at the decoder. For instance, for the case $L = 3$, if the triple $(\lambda_1, \lambda_2, \lambda_3)$ is transmitted, and the second description is lost, then we say that the triple $(\lambda_1, *, \lambda_3)$ is received. We use L-bit sequences $\mathbf{b} = (b_1, b_2, \cdots, b_L) \in \{0, 1\}^L$ to represent various patterns of received descriptions, where $b_i = 1$ means that description i is received, while $b_i = 0$ means that description i is lost. For $\lambda_c \in \Lambda_c$ and $\mathbf{b} \in \{0, 1\}^L$ let us denote by $y_{\mathbf{b}}(\lambda_c)$ the reconstruction of λ_c when the pattern of received descriptions is $\mathbf{b}$. For $\mathbf{b} \in \{0,1\}^L$, let $H(\mathbf{b})$ denote the Hamming weight of $\mathbf{b}$, which equals the number of 1's in $\mathbf{b}$. Therefore, $H(\mathbf{b})$ equals the number of received descriptions corresponding to pattern $\mathbf{b}$.

2.2 Rate and Distortion Computation

In this section we will analyze the rate and distortion of the MDLVQ described in the previous section.

As in prior work on MDLVQ we will use the high resolution assumption in order to simplify the derivation of the rate and distortion expressions. In other words, we will derive an approximation of the rate and distortion, which becomes accurate as $\nu_c \to 0$ and $\nu_s \to 0$. Additionally, we will assume that $N \to \infty$. Thus, we assume that ν_c is small enough so that the pdf is approximately uniform over each Voronoi region of Λ_c. This implies that each central lattice point is approximately equal to the centroid of its Voronoi region.

Further, we assume that the index assignment α satisfies the *shift-invariance* property, i.e.,

$$\alpha_i(\lambda_c + \lambda) = \alpha_i(\lambda_c) + \lambda, \forall \lambda_c \in \Lambda_c, \lambda \in \Lambda_s, 1 \leq i \leq L.$$

Moreover, since α is shift-invariant it makes sense to assume that the decoder mapping is shift invariant as well, i.e.,

$$y_{\mathbf{b}}(\lambda_c + \lambda) = y_{\mathbf{b}}(\lambda_c) + \lambda, \tag{2.10}$$

for all $\lambda_c \in \Lambda_c$, $\lambda \in \Lambda_s$ and $\mathbf{b} \in \{0,1\}^L$.

2.2.1 Rate Computation

Let R_c denote the dimension-normalized entropy of the central quantizer and let R_i denote the dimension-normalized entropy rate of side description i, i.e., $R_c = \mathcal{H}\left(Q\left(\mathbf{x}\right)\right)$, $R_i = \mathcal{H}\left(\alpha_i\left(Q\left(\mathbf{x}\right)\right)\right), 1 \leq i \leq L$, where $\mathcal{H}\left(\cdot\right)$ denotes the entropy of a dimension-normalized discrete random variable. Then the central rate R_c can be expressed as follows (Vaishampayan *et al.*, 2001)

$$\begin{aligned} R_c &= \mathcal{H}\left(Q\left(\mathbf{x}\right)\right) \\ &= -\frac{1}{n}\sum_{\lambda_c\in\Lambda_c}\int_{\hat{V}_c(\lambda_c)} f(\mathbf{x})d\mathbf{x}\,\log_2\int_{\hat{V}_c(\lambda_c)} f(\mathbf{x})d\mathbf{x} \\ &\approx -\frac{1}{n}\sum_{\lambda_c\in\Lambda_c}\int_{\hat{V}_c(\lambda_c)} f(\mathbf{x})d\mathbf{x}\,\log_2\left(f(\mathbf{x})\nu_c\right) \\ &\approx h\left(f\right) - \frac{1}{n}\log_2\left(\nu_c\right), \end{aligned} \tag{2.11}$$

where $\hat{V}_c(\lambda_c)$ denotes the Voronoi region of $\lambda_c, \lambda_c \in \Lambda_c$, with respect to the central lattice, i.e.,

$$\hat{V}_c\left(\lambda_c\right) \triangleq \{\mathbf{x}\in\mathbf{R}^n : \|\mathbf{x}-\lambda_c\| \leq \|\mathbf{x}-\lambda'_c\|, \lambda'_c \in \Lambda_c\}$$

and $h(f)$ denotes the differential entropy of f, i.e., $h(f) = -\int_{\mathbf{R}} f(x)\log_2 f(x)dx$.

To evaluate R_i, under the high-resolution assumption, $f(\mathbf{x})$ is also considered as a constant over each Voronoi region of Λ_s, i.e., $f(\mathbf{x}) \approx f(\lambda), \mathbf{x} \in \hat{V}_s\left(\lambda\right)$. Additionally, the shift invariance property ensures that the cardinality of $\alpha_i^{-1}\left(\lambda_s\right)$ is N, i.e., $\left|\alpha_i^{-1}\left(\lambda_s\right)\right| = N$, for every $1 \leq i \leq L, \lambda_s \in \Lambda_s$ and we further assume that the pdf of the source vectors is uniform over each set $\alpha_i^{-1}\left(\lambda_s\right), 1 \leq i \leq L, \lambda_s \in \Lambda_s$. Based on the above observation, one can obtain (Vaishampayan *et al.*, 2001)

$$\begin{aligned}
R_i &= -\frac{1}{n}\sum_{\lambda\in\Lambda_s}\left(\sum_{\lambda_c\in\alpha_i^{-1}(\lambda)}\int_{\hat{V}_c(\lambda_c)} f(\mathbf{x})\,d\mathbf{x}\right)\log_2\left(\sum_{\lambda_c\in\alpha_i^{-1}(\lambda)}\int_{\hat{V}_c(\lambda_c)} f(\mathbf{x})\,d\mathbf{x}\right)\\
&\approx -\frac{1}{n}\sum_{\lambda\in\Lambda_s}\left(\sum_{\lambda_c\in\alpha_i^{-1}(\lambda)}\int_{\hat{V}_c(\lambda)} f(\mathbf{x})\,d\mathbf{x}\right)\log_2\left(f(\lambda)\,N\nu_c\right)\\
&\approx -\frac{1}{n}\sum_{\lambda\in\Lambda_s}\left(\sum_{\lambda_c\in\alpha_i^{-1}(\lambda)}\int_{\hat{V}_c(\lambda)} f(\mathbf{x})\log_2\left(f(\lambda)\right)d\mathbf{x}\right)-\frac{1}{n}\log_2\left(N\nu_c\right)\\
&\approx h(f)-\frac{1}{n}\log_2\left(N\nu_c\right).
\end{aligned} \tag{2.12}$$

It follows that the rates of all descriptions are equal. Let R denote their common value. It further follows that

$$N\nu_c \approx 2^{n(h(f)-R)}. \tag{2.13}$$

It is clear that R and R_c have to satisfy the following conditions

$$R \leq R_c \leq LR. \tag{2.14}$$

Let us write now as in (Zhang *et al.*, 2011)

$$R_c = R(1+\rho(L-1)). \tag{2.15}$$

Then conditions (2.14) are equivalent to $0 \leq \rho \leq 1$. Further, relations (2.11), (2.13) and (2.15) imply that

$$N \approx 2^{n\rho(L-1)R}. \tag{2.16}$$

According to relations (2.11), (2.13) and (2.16), for fixed ρ satisfying $0 < \rho \leq 1$, when R increases to ∞, one has $\nu_c \to 0$, $\nu_s \to 0$ and $N \to \infty$.

Remark 1 . *Recall that to obtain relation (2.12), the pdf of the source vectors are assumed to be approximately uniform over each set* $\alpha_i^{-1}(\lambda_s)$. *A sufficient condition for the latter to hold as* $R \to \infty$, *is that the volume of the convex closure of the set* $\cup_{\lambda_c \in \alpha_i^{-1}(\lambda_s)} \hat{V}_c(\lambda_c)$ *to approach* 0 *for all* i *and* λ_s. *Assume that the largest such volume equals* $N\nu_c \times N^\gamma$, *for some value* γ. *According to relations (2.13) and (2.16),* $N\nu_c \times N^\gamma \to 0$ *holds if and only if*

$$\rho < \frac{1}{\gamma(L-1)}. \tag{2.17}$$

2.2.2 Distortion Computation

We consider the mean square error as the distortion measure.

Let D_c be the central distortion, i.e.,

$$D_c = \frac{1}{n} \sum_{\lambda_c \in \Lambda_c} \int_{\hat{V}_c(\lambda_c)} \|\mathbf{x} - \lambda_c\|^2 f(\mathbf{x}) d\mathbf{x}.$$

Under the high-resolution assumptions, we can rewrite the central distortion as (Gersho, 1979)

$$D_c \approx G(\Lambda_c) \nu_c^{2/n}, \tag{2.18}$$

where

$$G(\Lambda_c) \triangleq \frac{\int_{V_c(\mathbf{0})} \|\mathbf{x}\|^2 d\mathbf{x}}{\nu_c^{1+\frac{2}{n}}}$$

is the normalized second moment of a Voronoi region of the central lattice Λ_c (Conway and Sloane, 1998).

Next we will evaluate the distortion when a subset of the L descriptions is received.

Note that the distortion of the source reconstruction corresponding to pattern $\mathbf{b}$ is

$$D_{\mathbf{b}} = \frac{1}{n}\sum_{\lambda_c\in\Lambda_c}\int_{\hat{V}_c(\lambda_c)}\|\mathbf{x}-y_{\mathbf{b}}(\lambda_c)\|^2 f(\mathbf{x})d\mathbf{x}. \tag{2.19}$$

Further, the distortion when k of L, $1\le k\le L$, descriptions are received, is defined as

$$D_{L,k} \triangleq \frac{1}{\binom{L}{k}}\sum_{\substack{\mathbf{b}\in\{0,1\}^L\\ H(\mathbf{b})=k}} D_{\mathbf{b}}. \tag{2.20}$$

Notice that $D_{L,L}=D_c$.

By plugging (2.19) into (2.20), one obtains

$$D_{L,k}=\frac{1}{n}\frac{1}{\binom{L}{k}}\sum_{\substack{\mathbf{b}\in\{0,1\}^L\\ H(\mathbf{b})=k}}\sum_{\lambda_c\in\Lambda_c}\int_{\hat{V}_c(\lambda_c)}\|\mathbf{x}-y_{\mathbf{b}}\left(\lambda_c\right)\|^2 f\left(\mathbf{x}\right)d\mathbf{x}. \tag{2.21}$$

Under the high resolution assumption, one has (Vaishampayan *et al.*, 2001; Ostergaard *et al.*, 2006; Huang and Wu, 2006)

$$\begin{aligned} D_{L,k} &= \frac{1}{n}\frac{1}{\binom{L}{k}} \sum_{\substack{\mathbf{b}\in\{0,1\}^L \\ H(\mathbf{b})=k}} \sum_{\lambda_c\in\Lambda_c} \int_{\hat{V}_c(\lambda_c)} \|\mathbf{x}-\lambda_c+\lambda_c-y_{\mathbf{b}}(\lambda_c)\|^2 f(\mathbf{x})d\mathbf{x} \\ &= \frac{1}{n}\frac{1}{\binom{L}{k}} \sum_{\substack{\mathbf{b}\in\{0,1\}^L \\ H(\mathbf{b})=k}} \sum_{\lambda_c\in\Lambda_c} \int_{\hat{V}_c(\lambda_c)} \Big(\|\mathbf{x}-\lambda_c\|^2+\|\lambda_c-y_{\mathbf{b}}(\lambda_c)\|^2 \\ &\quad +2\left\langle \mathbf{x}-\lambda_c, \lambda_c-y_{\mathbf{b}}(\lambda_c)\right\rangle\Big) f(\mathbf{x})d\mathbf{x} \\ &\approx \frac{1}{n}\frac{1}{\binom{L}{k}} \sum_{\substack{\mathbf{b}\in\{0,1\}^L \\ H(\mathbf{b})=k}} \sum_{\lambda_c\in\Lambda_c} \left(\int_{\hat{V}_c(\lambda_c)} \|\mathbf{x}-\lambda_c\|^2 f(\mathbf{x})d\mathbf{x} + \int_{\hat{V}_c(\lambda_c)} \|\lambda_c-y_{\mathbf{b}}(\lambda_c)\|^2 f(\mathbf{x})d\mathbf{x}\right) \\ &\approx D_c + \frac{1}{n}\frac{1}{\binom{L}{k}} \sum_{\substack{\mathbf{b}\in\{0,1\}^L \\ H(\mathbf{b})=k}} \sum_{\lambda_c\in\Lambda_c} \int_{\hat{V}_c(\lambda_c)} \|\lambda_c-y_{\mathbf{b}}(\lambda_c)\|^2 f(\mathbf{x})d\mathbf{x} \\ &\approx D_c + \frac{1}{n}\frac{1}{\binom{L}{k}} \sum_{\substack{\mathbf{b}\in\{0,1\}^L \\ H(\mathbf{b})=k}} \sum_{\lambda_c\in\Lambda_c} \|\lambda_c-y_{\mathbf{b}}(\lambda_c)\|^2 \int_{\hat{V}_c(\lambda_c)} f(\mathbf{x})d\mathbf{x}. \end{aligned} \tag{2.22}$$

Note that the third relation in (2.22) holds because λ_c approximately equals the centroid of $\hat{V}_c(\lambda_c)$, thus we have

$$\begin{aligned} &\int_{\hat{V}_c(\lambda_c)} \left\langle \mathbf{x}-\lambda_c, \lambda_c-y_{\mathbf{b}}(\lambda_c)\right\rangle f(\mathbf{x})d\mathbf{x} \\ =&\ \left\langle \int_{\hat{V}_c(\lambda_c)} (\mathbf{x}-\lambda_c) f(\mathbf{x})d\mathbf{x}, \lambda_c-y_{\mathbf{b}}(\lambda_c)\right\rangle \\ \approx&\ 0. \end{aligned}$$

Using the shift-invariance property (2.10) along with the high resolution assumption, the second term in (2.22) can be simplified as follows

$$\begin{aligned}
&\frac{1}{n}\frac{1}{\binom{L}{k}} \sum_{\substack{\mathbf{b}\in\{0,1\}^L \\ H(\mathbf{b})=k}} \sum_{\lambda_c\in\Lambda_c} \|\lambda_c - y_{\mathbf{b}}(\lambda_c)\|^2 \int_{\hat{V}_c(\lambda_c)} f(\mathbf{x})d\mathbf{x} \\
\approx\ &\frac{1}{n}\frac{1}{\binom{L}{k}} \sum_{\substack{\mathbf{b}\in\{0,1\}^L \\ H(\mathbf{b})=k}} \sum_{\lambda_s\in\Lambda_s} \sum_{\lambda_c\in V_s(\lambda_s)} \|\lambda_c - y_{\mathbf{b}}(\lambda_c)\|^2 f(\lambda_s) \int_{\hat{V}_c(\lambda_c)} d\mathbf{x} \\
=\ &\frac{1}{n}\frac{1}{\binom{L}{k}} \sum_{\substack{\mathbf{b}\in\{0,1\}^L \\ H(\mathbf{b})=k}} \sum_{\lambda_s\in\Lambda_s} \sum_{\lambda'_c\in V_s(\mathbf{0})} \|\lambda'_c + \lambda_s - y_{\mathbf{b}}(\lambda'_c + \lambda_s)\|^2 f(\lambda_s)\nu_c \\
=\ &\frac{1}{nN}\frac{1}{\binom{L}{k}} \sum_{\substack{\mathbf{b}\in\{0,1\}^L \\ H(\mathbf{b})=k}} \sum_{\lambda_c\in V_s(\mathbf{0})} \|\lambda_c - y_{\mathbf{b}}(\lambda_c)\|^2 \sum_{\lambda_s\in\Lambda_s} f(\lambda_s)N\nu_c \\
=\ &\frac{1}{nN}\frac{1}{\binom{L}{k}} \sum_{\substack{\mathbf{b}\in\{0,1\}^L \\ H(\mathbf{b})=k}} \sum_{\lambda_c\in V_s(\mathbf{0})} \|\lambda_c - y_{\mathbf{b}}(\lambda_c)\|^2 \qquad (2.23)
\end{aligned}$$

where the second last equality holds in view of the shift-invariance property of the decoder.

According to relation (2.22), (2.23) can be rewritten as (Vaishampayan *et al.*, 2001; Ostergaard *et al.*, 2006; Huang and Wu, 2006)

$$D_{L,k} \approx D_c + \frac{1}{nN}\frac{1}{\binom{L}{k}} \sum_{\lambda_c\in V_s(\mathbf{0})} \sum_{\substack{\mathbf{b}\in\{0,1\}^L \\ H(\mathbf{b})=k}} \|\lambda_c - y_{\mathbf{b}}(\lambda_c)\|^2. \qquad (2.24)$$

2.3 Previous MDLVQ Scheme for $L >= 3$

In this section we will briefly review a previous design of MDLVQ (Huang and Wu, 2006) which is closely related to the heuristic index assignment scheme we propose in the next chapter.

Denote the centroid of the L-tuple that is associated with $\lambda_c, \lambda_c \in \Lambda_c$ as

$$\mu_s(\lambda_c) = \frac{\sum_{i=1}^{L} \alpha_i(\lambda_c)}{L}. \tag{2.25}$$

Additionally, for any L-tuple $(\lambda_1, \cdots, \lambda_L) \in \Lambda_s^L$ define its *spread* as (Ostergaard *et al.*, 2006)

$$sp(\lambda_1, \cdots, \lambda_L) = \sum_{i=1}^{L} \|\lambda_i - \frac{1}{L}\sum_{j=1}^{L} \lambda_j\|^2. \tag{2.26}$$

In prior work (Servetto *et al.*, 1999; Vaishampayan *et al.*, 2001; Ostergaard *et al.*, 2006; Huang and Wu, 2006; Zhang *et al.*, 2011), when a subset of the L-tuple is received, the reconstruction is the average of the received descriptions, i.e.,

$$y_{\mathbf{b}}(\lambda_c) = \frac{1}{H(\mathbf{b})} \sum_{i=1}^{L} \alpha_i(\lambda_c)\, b_i, \tag{2.27}$$

where $\mathbf{b}$ is the pattern of received descriptions. Then the decoding rule used in prior work implies that for all $1 \leq k \leq L-1$, (Ostergaard *et al.*, 2006)

$$\begin{aligned} D_{L,k} \approx D_c \;&+\; \frac{1}{nN} \sum_{\lambda_c \in V_s(\mathbf{0})} \|\lambda_c - \mu_s(\lambda_c)\|^2 \\ &+\; \frac{1}{nN} \frac{L-k}{Lk(L-1)} \sum_{\lambda_c \in V_s(\mathbf{0})} sp(\alpha(\lambda_c)). \end{aligned} \tag{2.28}$$

A key factor in the index assignment design proposed in (Huang and Wu, 2006) is the so-called L-fraction lattice $\Lambda_{s/L}$, defined as

$$\Lambda_{s/L} \triangleq \frac{1}{L}\Lambda_s = \left\{ \lambda_{s/L} \in \mathbf{R}^n : \lambda_{s/L} = \frac{\mathbf{k}}{L} G_s, \mathbf{k} \in \mathbb{Z}^n \right\},$$

where G_s is the generator matrix of Λ_s.

Define the discrete Voronoi cell for each L-fraction lattice point as (Huang and

Wu, 2006)

$$V_{s/L}\left(\lambda_{s/L}\right) \triangleq \{\lambda_c : \left\|\lambda_c - \lambda_{s/L}\right\| \leq \left\|\lambda_c - \lambda'_{s/L}\right\|, \forall \lambda'_{s/L} \in \Lambda_{s/L}\}.$$

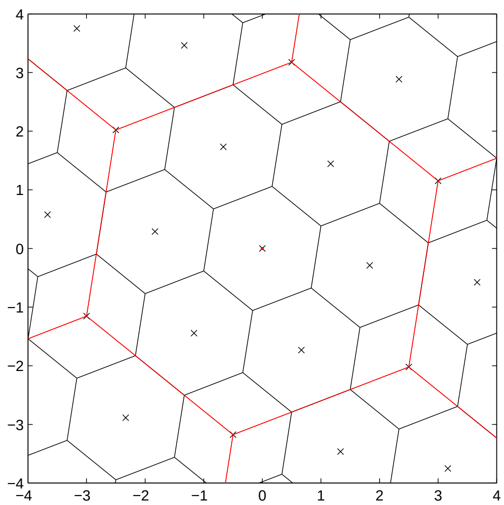


Figure 2.2: Λ_s and $\Lambda_{s/3}$. The small black hexagons are the Voronoi regions of $\Lambda_{s/3}$ and the big red hexagons are the Voronoi regions of Λ_s. Points of $\Lambda_{s/3}$ are marked by $\times$.

Further, for each $\lambda_{s/L} \in \Lambda_{s/L}$ consider

$$\mathcal{T}\left(\lambda_{s/L}\right) \triangleq \left\{(\lambda_1, \lambda_2, ..., \lambda_L) \in \Lambda_s^L : \frac{1}{L} \sum_{1 \leq j \leq L} \lambda_j = \lambda_{s/L}\right\}$$

to be the set containing all L-tuples of side lattice points whose average equals $\lambda_{s/L}$. It is shown in (Huang and Wu, 2006) that by assigning the central lattice points in $V_{s/L}\left(\lambda_{s/L}\right)$ the L-tuples in $\mathcal{T}(\lambda_{s/L})$ of smallest spread minimizes $D_{L,k}$ for all $1 \leq k \leq L-1$, as $N \rightarrow \infty$.

Since $D_{L,L} = D_c$, according to (2.18), one has

$$D_{L,L} \approx G(\Lambda_c)\nu_c^{\frac{2}{n}}.$$

Additionally, the asymptotic analysis in (Zhang *et al.*, 2011) leads, for $1 \leq k \leq L-1$, to

$$D_{L,k} \approx \frac{L-k}{k} L^{-\frac{L}{L-1}} (N\nu_c)^{\frac{2}{n}} G(S_{Ln-n}) N^{\frac{2}{(L-1)n}} \tag{2.29}$$

as $N \to \infty$, where

$$G(S_n) = \frac{1}{(n+2)\pi} \Gamma\left(\frac{n}{2}+1\right)^{\frac{2}{n}} \tag{2.30}$$

is the normalized second moment of a sphere in n dimensions.

Chapter 3

Flexible MDLVQ with Heuristic Index Assignment

Notice that in the previous design of MDLVQ once Λ_c and Λ_s have been chosen the system's performance is determined. Interestingly, the index assignment used in evaluating the performance minimizes the distortion (as $N \to \infty$) for every possible $k, 1 \leq k \leq L-1$. The fact that it is possible to optimize simultaneously the distortions for all values of k is contrary to our intuition that there should be tradeoffs between these distortions. The first insight towards solving this conflict is that the optimality of the index assignment is a result of the particular way the decoder is designed. The second insight is that the decoding rule is natural for the side decoders, i.e., when $k = 1$, while for $2 \leq k \leq L-1$, it seems to rather be dictated by convenience. These observations lead to the conclusion that distortion $D_{L,1}$ is, indeed, the smallest possible, but there should be room for further decreasing $D_{L,k}$, for $2 \leq k \leq L-1$, if $D_{L,1}$ is allowed to increase. Nevertheless, for this to be possible, the decoding mapping for $2 \leq k \leq L-1$ has to be changed.

In this chapter we present a flexible MDLVQ with a new decoding rule and a heuristic index assignment. The proposed MDLVQ system is introduced in section 3.1, while section 3.2 presents experimental results for $L = 3$ and $L = 4$.

3.1 Heuristic Index Assignment Algorithm

In order to introduce more flexibility into the system we will consider a different decoding mapping for $2 \leq k \leq L-1$, as follows. For any L-tuple $(\xi_1, \xi_2, \cdots, \xi_L) \in \Big(\Lambda_s \cup \{*\}\Big)^L$ that has at least one component equal to $*$ and at least two other components in Λ_s, we compute the reconstruction value as the arithmetic average of the central lattice points in the set $\alpha^{-1}(\xi_1, \xi_2, \cdots, \xi_L)$, where

$$\alpha^{-1}(\xi_1, \xi_2, \cdots, \xi_L) \triangleq \{\lambda_c \in \Lambda_c : \alpha_i(\lambda_c) = \xi_i \text{ for all } i, 1 \leq i \leq L, \text{ with } \xi_i \in \Lambda_s\}.$$

Notice that this decoding rule is optimal assuming that the pdf is uniform over the set $\alpha^{-1}(\xi_1, \xi_2, \cdots, \xi_L)$. Further, in order to control the performance when k, $2 \leq k \leq L-1$, descriptions are received, we introduce $L-2$ parameters: δ_k, $2 \leq k \leq L-1$, satisfying the inequalities: $\delta_2 \geq \delta_3 \geq \cdots \geq \delta_{L-1} \geq 0$ and impose the following condition.

Condition A. For any $k, 2 \leq k \leq L-1$, and any L-tuple $(\xi_1, \xi_2, \cdots, \xi_L) \in \Big(\Lambda_s \cup \{*\}\Big)^L$ having exactly $L-k$ components equal to $*$, one must have $\|\lambda_c - \lambda_c'\| \leq \delta_k$ for any $\lambda_c, \lambda_c' \in \alpha^{-1}(\xi_1, \xi_2, \cdots, \xi_L)$.

It is easy to see that Condition A guarantees that

$$D_{L,k} \lessapprox D_c + \delta_k^2, \quad \forall k, 2 \leq k \leq L-1.$$

Finally, when designing the index assignment α we attempt to minimize $D_{L,1}$ while ensuring that Condition A is satisfied. Note that the decoding rule for $k = 1$ is the same as in previous work, therefore the value of $D_{L,1}$ can be still computed using equation (2.28) for $k = 1$, leading to

$$D_{L,1} \approx D_c + \frac{1}{nN} \sum_{\lambda_c \in V_s(\mathbf{0})} \|\lambda_c - \mu_s(\lambda_c)\|^2 + \frac{1}{nNL} \sum_{\lambda_c \in V_s(\mathbf{0})} sp(\alpha(\lambda_c)). \quad (3.1)$$

The proposed index assignment algorithm first selects L^n L-fraction lattice points $\lambda_{s/L} \in \Lambda_{s/L}$ situated in the Voronoi region of $\mathbf{0}$ with respect to the side lattice Λ_s, such that the difference of any two such points is not in Λ_s. Let $\mathcal{F}$ denote this set and let $\mathcal{C}$ denote the union of central lattice points in the Voronoi region of $\lambda_{s/L} \in \mathcal{F}$, i.e.,

$$\mathcal{C} = \cup_{\lambda_{s/L} \in \mathcal{F}} V_{s/L}(\lambda_{s/L}). \quad (3.2)$$

Notice that it is sufficient to specify the index assignment for the central lattice points in $\mathcal{C}$ and then extend it to Λ_c via shifting. Guided by (3.1), for each $\lambda_{s/L} \in \mathcal{F}$ we assign the central lattice points in $V_{s/L}(\lambda_{s/L})$ to L-tuples in $\mathcal{T}(\lambda_{s/L})$, to minimize the first sum in (3.1). However, we can no longer select the L-tuples of smallest spread to be assigned as in (Huang and Wu, 2006), since this would lead to violations of Condition A. Having in mind the need to keep the last sum in (3.1) as small as possible we proceed in a greedy manner as described next.

The algorithm maintains a list $\mathcal{T}$ of candidate tuples to be assigned. The assignment is built up gradually such that at every moment Condition A to be satisfied for the assignment obtained by extending via shifting the partial assignment built so far. The set $\mathcal{T}$ is initialized to $\mathcal{T} = \cup_{\lambda_{s/L} \in \mathcal{F}} \mathcal{T}(\lambda_{s/L})$. At each iteration the L-tuple of smallest spread from the set $\mathcal{T}$ is selected as the current candidate to be assigned. Let $(\lambda_1, \cdots, \lambda_L)$ denote this L-tuple. Next the value $\lambda_{s/L} = \frac{\sum i{=}1^L \lambda_i}{L}$ is determined and for each λ_c in $V_{s/L}(\lambda_{s/L})$ it is tested whether assigning the L-tuple to λ_c preserves Condition A for the assignment extended by shifting. Specifically, it is checked whether λ_c satisfies Condition B, stated next.

Condition B. Given $\lambda_c \in \Lambda_c$, we say that λ_c satisfies Condition B if and only if for every $\lambda'_c \in \mathcal{C}$ assigned so far and every k, $2 \le k \le L-1$, such that there are $s_k \in \Lambda_s$ and k different positions $i_1, i_2, \cdots, i_k \in \{1, \cdots, L\}$ with the property that $\alpha_{i_j}(\lambda'_c) = \alpha_{i_j}(\lambda_c) + s_k$, $1 \le j \le k$, the inequality $\|\lambda'_c - s_k - \lambda_c\| \le \delta_k$ holds.

If a point λ_c satisfying Condition B is found then the L-tuple is assigned to it and removed from the list $\mathcal{T}$. Otherwise the L-tuple is simply removed from the list $\mathcal{T}$ without being assigned. When all central lattice points from the set $V_{s/L}(\lambda_{s/L})$, for some $\lambda_{s/L}$, are assigned, all L-tuples in $\mathcal{T}(\lambda_{s/L}) \cap \mathcal{T}$ are removed from $\mathcal{T}$. Finally, the algorithm stops when $\mathcal{T}$ becomes empty.

We point out that when $\delta_k = \infty$ for all $2 \le k \le L-1$, the algorithm produces an index assignment as in (Huang and Wu, 2006).

3.1.1 An Example

In this subsection, we will present an example for the proposed heuristic index assignment algorithm. For simplicity, we choose the A_2 lattice with generator matrix

$$G_c = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \tag{3.3}$$

as the central lattice. Note that the associated Gram matrix is (Conway and Sloane, 1998)

$$M_c = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}. \tag{3.4}$$

The volume of the fundamental region of Λ_c is (Conway and Sloane, 1998)

$$\nu_c = \sqrt{|M_c|} = \frac{\sqrt{3}}{2}. \tag{3.5}$$

We choose a sublattice of Λ_c with generator matrix

$$G_s = \begin{pmatrix} \frac{11}{2} & \frac{\sqrt{3}}{2} \\ -2 & -3\sqrt{3} \end{pmatrix} \tag{3.6}$$

as the side lattice Λ_s with corresponding Gram matrix (Conway and Sloane, 1998)

$$M_s = \begin{pmatrix} 31 & -\frac{31}{2} \\ -\frac{31}{2} & 31 \end{pmatrix}. \tag{3.7}$$

The volume of the fundamental region of Λ_s is (Conway and Sloane, 1998)

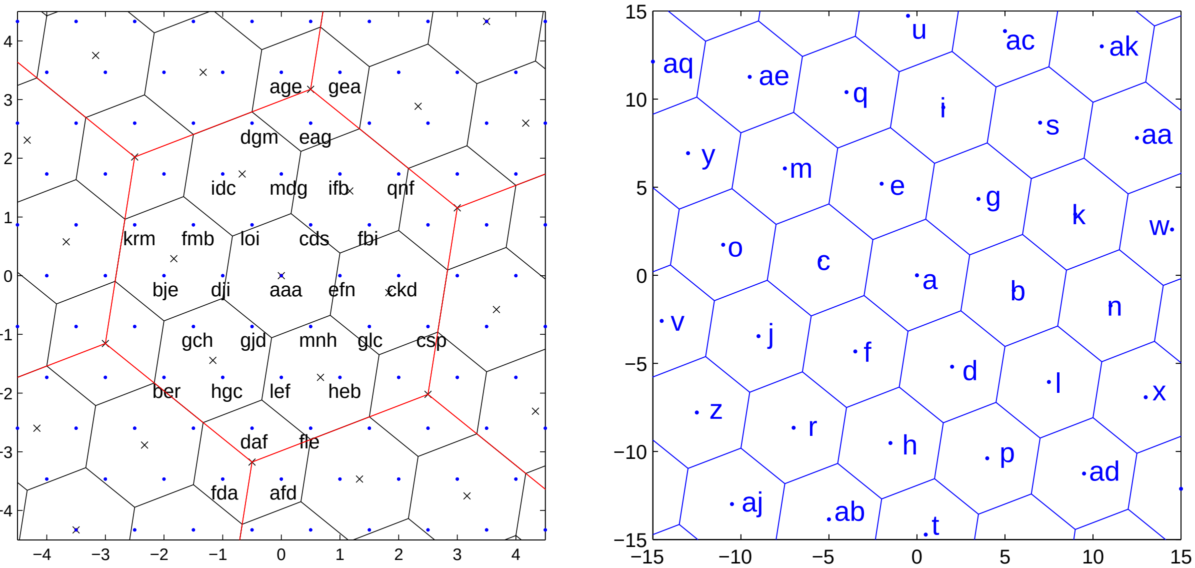
$$\nu_s = \sqrt{|M_s|} = \frac{31\sqrt{3}}{2}. \tag{3.8}$$

With (3.5) and (3.8), one can compute the index as $N = \nu_s/\nu_c = 31$.

The index assignment obtained for this example and $\delta_2 = 0$ is shown in Figure 3.1.

3.2 Experimental Result

In this section, we assess empirically the performance of the proposed index assignment algorithm for $L = 3$ and $L = 4$. In both cases $n = 2$ and the central lattice Λ_c is the hexagonal lattice A_2 with generator matrix given in (3.3).



(a) (b)

Figure 3.1: Three description index assignment for the A_2 lattice with $N = 31$ and $\delta_2 = 0$. Points of Λ_c and $\Lambda_{s/3}$ are marked by $\cdot$ and $\times$ respectively. In Figure 3.1a the small black hexagons are the Voronoi regions of $\Lambda_{s/3}$ and the big red hexagons are the Voronoi regions of Λ_s. Each lower case letter represents a side lattice point. The side lattice points and their corresponding lower case letter representations are shown in Figure 3.1b on the right.

3.2.1 Three Descriptions

For $L = 3$ we consider a clean sublattice Λ_s of Λ_c with index $N = 307$. We use various values for the parameter δ_2, starting at 0 and incrementing by 2τ up to 24τ, then incrementing by 10τ, where τ is the squared distance between two neighboring central lattice points, i.e., $\tau = 1$. In our experiments incrementing δ_2 beyond 24τ did not produce any change in performance.

Figure 3.2 plots $D_{3,1}$ versus $D_{3,2}$ for the specified values of δ_2. The rightmost point in the plot is achieved for the largest δ_2. Its corresponding pair of distortions $(D_{3,2}, D_{3,1})$ is the same as in (Huang and Wu, 2006). We see that as δ_2 decreases towards 0, the value of $D_{3,2}$ decreases towards D_c, while the value of $D_{3,1}$ increases.

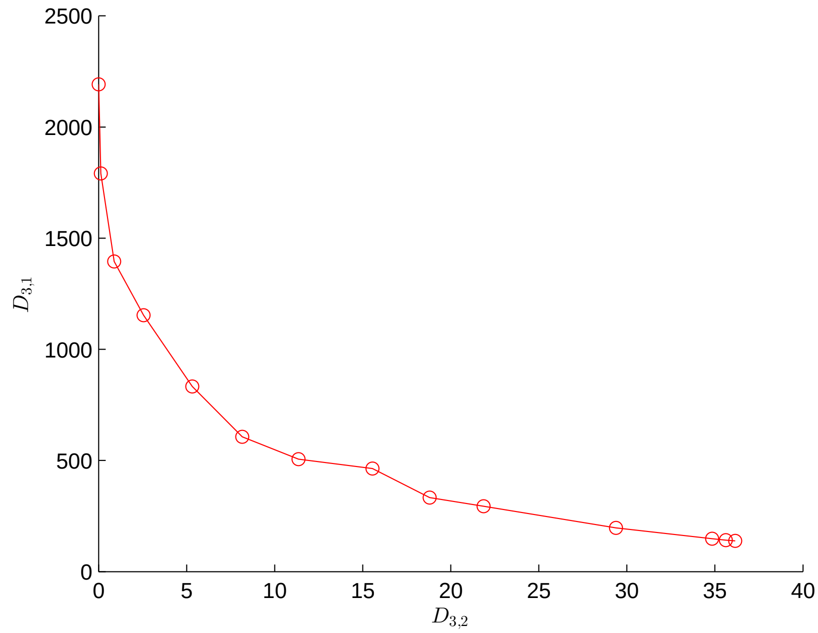


Figure 3.2: The value of $D_{3,1}$ versus $D_{3,2}$ for the hexagonal lattice A_2 with $N = 307$ and various values of δ_2.

3.2.2 Four Descriptions

For $L = 4$ we consider $N = 703$. The results are presented in Tables 3.1-3.6.

From Table 3.1 we can see that if we let δ_3 go to infinity, with the increase of δ_2, $D_{4,2}$ and $D_{4,3}$ increase while $D_{4,1}$ decreases. In Table 3.2 and Table 3.3, we fix δ_3 and let δ_2 range from 20 to infinity. We can find that $D_{4,3}$ only varies in a small range and as δ_2 becomes larger, $D_{4,2}$ increases while $D_{4,1}$ decreases. Similarly, in Table 3.4-3.6, we fix δ_2 and the results show that we achieve the goal of adjusting $D_{4,1}$ and $D_{4,3}$. So from the simulation results, we can see that we can achieve the tradeoffs between all the side distortions by controlling $\delta_k, k = 2, 3$.

Table 3.1

δ_2	δ_3	$D_{4,1}$	$D_{4,2}$	$D_{4,3}$
20	∞	2914.68	26.93	7.18
25	∞	1569.43	40.07	9.96
30	∞	801.68	52.43	14.69
35	∞	228.43	66.50	21.06
∞	∞	198.68	64.78	22.47

Table 3.2

δ_2	δ_3	$D_{4,1}$	$D_{4,2}$	$D_{4,3}$
20	10	3025.06	25.88	3.15
25	10	2736.06	36.20	3.09
30	10	1974.56	49.68	3.21
35	10	1356.56	67.51	3.77
40	10	888.81	89.68	4.25
∞	10	391.68	114.53	5.71

Table 3.3

δ_2	δ_3	$D_{4,1}$	$D_{4,2}$	$D_{4,3}$
20	0	5256.18	28.14	0.07
30	0	3659.56	48.20	0.07
40	0	2282.93	88.79	0.07
50	0	1077.43	161.79	0.07
60	0	684.31	187.83	0.07
∞	0	666.93	189.28	0.07

Table 3.4

δ_2	δ_3	$D_{4,1}$	$D_{4,2}$	$D_{4,3}$
∞	0	666.93	189.28	0.07
∞	5	530.18	150.20	0.90
∞	10	391.68	114.53	5.71
∞	15	262.18	78.42	16.24
∞	∞	198.68	64.78	22.47

Table 3.5

δ_2	δ_3	$D_{4,1}$	$D_{4,2}$	$D_{4,3}$
30	0	3659.56	48.20	0.07
30	5	2820.56	50.50	0.49
30	10	1974.56	49.68	3.21
30	15	1089.31	49.99	9.21
30	∞	801.68	52.43	14.69

Table 3.6

δ_2	δ_3	$D_{4,1}$	$D_{4,2}$	$D_{4,3}$
20	0	5256.18	28.14	0.07
20	4	5037.68	23.97	0.25
20	8	3218.31	25.57	1.84
20	10	3025.06	25.88	3.15
20	∞	2914.68	26.93	7.18

Chapter 4

Flexible MDLVQ with Structured Index Assignment for $L = 3$

The index assignment proposed in Chapter 3 lacks structure and therefore it is difficult to analyze theoretically. In this section we propose a structured index assignment for a flexible MDLVQ in the case $L = 3$ and derive its asymptotical performance. Our results show that the proposed scheme can achieve a wide range of values for $D_{3,1}$ and $D_{3,2}$ when Λ_c and Λ_s are fixed. We first develop an index assignment ensuring $D_{3,2} = D_c$ in section 4.1. Then proceed to the case $D_{3,2} > D_c$ in section 4.2.

4.1 Case $D_2 = D_c$

Consider the set $\mathcal{A}_0$ consisting of the N side lattice points that are closest to $\mathbf{0}$. Further, let $\mathcal{T}_0$ denote the set of triples $(\lambda, \mathbf{0}, -\lambda)$ with $\lambda \in \mathcal{A}_0$. The central lattice points in $V_s(\mathbf{0})$ are assigned to the triples in $\mathcal{T}_0$ in an one-to-one manner. Further, the index assignment is extended via shifting to all points in Λ_c. Thus, the central lattice points in $V_s(v)$, for $v \in \Lambda_s$ are assigned triples of the form $(\lambda + v, v, -\lambda + v)$.

It is easy to see that any two components of any assigned triple $(\lambda_1, \lambda_2, \lambda_3)$ uniquely identify the corresponding central lattice point $\lambda_c = \alpha^{-1}(\lambda_1, \lambda_2, \lambda_3)$. Therefore, one has

$$D_{3,2} = D_c \approx G(\Lambda_c)\nu_c^{\frac{2}{n}}.$$

In order to proceed with the derivation of $D_{3,1}$ we need the following result, which follows from (Vaishampayan *et al.*, 2001).

Proposition. Let Λ' be a lattice in $\mathbf{R}^n$ with ν' denoting the volume of its fundamental region, and let N_0 be a positive integer. Denote by $S(N_0, \Lambda')$ the sum of the squared distances to $\mathbf{0}$ of the N_0 lattice points that are closest to $\mathbf{0}$. Additionally, let $r\left(N_0, \Lambda'\right)$ denote the radius of the smallest sphere centered in $\mathbf{0}$, whose convex closure[1] contains the N_0 lattice points that are close to $\mathbf{0}$. Then one has

$$\begin{aligned} S(N_0, \Lambda') &= N_0^{\frac{n+2}{n}}(\nu')^{\frac{2}{n}} nG(S_n)(1 + o(1)), \\ r\left(N_0, \Lambda'\right) &= \left(N_0\nu'\right)^{\frac{1}{n}}\sqrt{(n+2)\,G\left(S_n\right)}\left(1 + o\left(1\right)\right), \end{aligned} \tag{4.1}$$

as $N_0 \to \infty$.

Using (2.28) with $k = 1$, one obtains

$$D_{3,1} \approx D_c + \frac{1}{nN}\sum_{\lambda_c \in V_s(\mathbf{0})} \|\lambda_c\|^2 + \frac{2}{3nN}\sum_{\lambda \in \mathcal{A}_0} \|\lambda\|^2. \tag{4.2}$$

We will first prove that

$$D_c + \frac{1}{nN}\sum_{\lambda_c \in V_s(\mathbf{0})} \|\lambda_c\|^2 \approx G(\Lambda_s)(N\nu_c)^{\frac{2}{n}}. \tag{4.3}$$

[1]We point out that the convex closure of a sphere equals the union between the sphere and its interior.

Let D_s denote the distortion of a quantizer having Λ_s as a codebook. Then, according to (Gersho, 1979), as ν_s approaches 0, one has

$$D_s \approx \frac{\int_{\hat{V}_s(\mathbf{0})} \mathbf{x}^2 d\mathbf{x}}{nN\nu_c} = G(\Lambda_s)(N\nu_c)^{\frac{2}{n}},$$

where $\hat{V}_s(\mathbf{0})$ denotes the Voronoi region of $\mathbf{0}$ with respect to the side lattice. The above relation further implies that

$$\begin{aligned} D_s &\approx \frac{\sum_{\lambda_c \in V_s(\mathbf{0})} \int_{\hat{V}_c(\lambda_c)} \|\mathbf{x} - \lambda_c + \lambda_c\|^2 d\mathbf{x}}{nN\nu_c} \\ &\approx \frac{\sum_{\lambda_c \in V_s(\mathbf{0})} \int_{\hat{V}_c(\lambda_c)} \|\mathbf{x} - \lambda_c\|^2 d\mathbf{x} + \|\lambda_c\|^2 \nu_c}{nN\nu_c} \\ &= \frac{N \int_{\hat{V}_c(\mathbf{0})} \|\mathbf{x}\|^2 d\mathbf{x}}{nN\nu_c} + \frac{\sum_{\lambda_c \in V_s(\mathbf{0})} \|\lambda_c\|^2}{nN}, \end{aligned}$$

which proves (4.3) since $D_c \approx \frac{\int_{\hat{V}_c(\mathbf{0})} \|\mathbf{x}\|^2 d\mathbf{x}}{n\nu_c}$ (Gersho, 1979).

Further, notice that $\sum\limits_{\lambda \in \mathcal{A}_0} \|\lambda\|^2 = S(N, \Lambda_s)$ and based on (4.1), (4.2) and (4.3), one obtains the following

$$D_{3,1} \approx G(\Lambda_s)(N\nu_c)^{\frac{2}{n}} + \frac{2}{3} G\left(S_n\right) N^{\frac{2}{n}} (N\nu_c)^{\frac{2}{n}}.$$

Notice that as $N \to \infty$ the last term becomes dominant, therefore we may write

$$D_{3,1} \approx \frac{2}{3} G\left(S_n\right) N^{\frac{2}{n}} (N\nu_c)^{\frac{2}{n}}. \tag{4.4}$$

Let us now express $D_{3,1}$, $D_{3,2}$ and $D_{3,3}$ in terms of R. Using relations (2.13) and

(2.16), one obtains

$$D_{3,1} \approx \frac{2}{3} G(S_n)\, 2^{2(h(f)-R(1-2\rho))}, \tag{4.5}$$

$$D_{3,2} = D_{3,3} \approx G(\Lambda_c) 2^{2(h(f)-R(1+2\rho))}, \tag{4.6}$$

as $R \to \infty$.

Notice that the above equations hold only if ρ satisfies relation (2.17). Let us first evaluate γ in relation (2.17). For $\lambda_s \in \Lambda_s$, the set $\alpha_i^{-1}(\lambda_s)$ is most spread out if $i = 1$ or $i = 3$. Therefore it is sufficient to analyze the case $i = 3$. It can be easily seen that the set $\alpha_3^{-1}(\lambda_s)$ contains a central lattice point from each Voronoi region $\hat{V}_s(v)$, with $v \in \lambda_s + \mathcal{A}_0$, where $\lambda_s + \mathcal{A}_0 \triangleq \{\lambda_s + \lambda : \lambda \in \mathcal{A}_0\}$. Then we may approximate the volume of the convex closure of $\cup_{\lambda_c \in \alpha_i^{-1}(\lambda_s)} \hat{V}_c(\lambda_c)$ by the volume of the set $\cup_{v \in \lambda_s + \mathcal{A}_0} \hat{V}_s(v)$, which equals $N\nu_s$. Then the value of γ in relation (2.17) is $\gamma = 1$, and relation (2.17) is equivalent to $\rho < \frac{1}{2}$.

4.2 Case $D_2 > D_c$

Let $\mu > 0$ and let $\mathcal{U}$ denote the set of side lattice points in the convex closure of the n dimensional sphere of radius μ, centered at $\mathbf{0}$. Let $\mathcal{A}$ denote the set of the closest $\lceil \frac{N}{|\mathcal{U}|} \rceil$ side lattice points to $\mathbf{0}$. Assume that $1 < |\mathcal{U}| < \sqrt{N}$. Then we can write $|\mathcal{U}| = N^{\frac{\beta}{2}}$ for some $\beta \in (0, 1)$. If β is kept fixed, then letting $N \to \infty$ leads to $|\mathcal{U}| \to \infty$. Additionally, as $N \to \infty$, one has $|\mathcal{A}||\mathcal{U}| = N$. It follows that $|\mathcal{A}| = N^{1-\frac{\beta}{2}}$.

Consider now a mapping $\varphi : \mathcal{U} \to \Lambda_s$ which assigns to each $u \in \mathcal{U}$ a side lattice point denoted by $\varphi(u)$ such that $\frac{u}{6} \in \hat{V}_s(-\varphi(u))$, or, in other words, $\frac{u}{6} + \varphi(u) \in \hat{V}_s(\mathbf{0})$. Define the set $\mathcal{T}(\mathbf{0}) \triangleq \{(\lambda + \varphi(u), \varphi(u), -\lambda + u + \varphi(u)) : \lambda \in \mathcal{A}, u \in \mathcal{U}\}$. It can be easily verified that for any two distinct ordered pairs $(\lambda, u), (\lambda', u') \in \Lambda_s^2$, one has

$$(\lambda + \varphi(u), \varphi(u), -\lambda + u + \varphi(u)) \neq (\lambda' + \varphi(u'), \varphi(u'), -\lambda' + u' + \varphi(u')).$$

It follows that $|\mathcal{T}(\mathbf{0})| = |\mathcal{A}||\mathcal{U}| = N$. Then the central lattice points in $V_s(\mathbf{0})$ are assigned triples from the set $\mathcal{T}(\mathbf{0})$. To perform the assignment the set $V_s(\mathbf{0})$ is first partitioned into $|\mathcal{U}|$ subsets $\mathcal{L}_u$, $u \in \mathcal{U}$, such that $\lfloor \frac{N}{|\mathcal{U}|} \rfloor \leq |\mathcal{L}_u| \leq \lceil \frac{N}{|\mathcal{U}|} \rceil$ and $\mathcal{L}_u$ contains at most one point λ_c for which $-\lambda_c$ is not in $\mathcal{L}_u$. A moment of thought reveals that such a partition is possible since $V_s(\mathbf{0})$ is symmetric with respect to $\mathbf{0}$. The latter condition ensures that

$$\left\| \sum_{\lambda_c \in \mathcal{L}_u} \lambda_c \right\| \leq \ell, \tag{4.7}$$

as $N \to \infty$, for each $u \in \mathcal{U}$, where ℓ is the covering radius of Λ_s, i.e.,

$$\ell = \max\{\|\mathbf{x}\| : \mathbf{x} \in \hat{V}_s(\mathbf{0})\}. \tag{4.8}$$

Finally, for each $u \in \mathcal{U}$ the central lattice points in each $\mathcal{L}_u$ are assigned triples of the form $(\lambda + \varphi(u), \varphi(u), -\lambda + u + \varphi(u))$, where $\lambda \in \mathcal{A}$. Further, the index assignment is extended to Λ_c using shifting.

Let us now derive $D_{3,2}$. To simplify the analysis we assume the following suboptimal decoding method. For every $(\lambda_1, \lambda_2, \lambda_3) \in \Lambda_s^3$, let $\eta(\lambda_1, \lambda_2, *) \triangleq \lambda_2$, $\eta(*, \lambda_2, \lambda_3) \triangleq \lambda_2$ and $\eta(\lambda_1, *, \lambda_3) \triangleq \frac{\lambda_1 + \lambda_3}{2}$, where η denotes the decoding mapping. It is easy to see that this decoder is shift invariant. Then, according to (2.24), the following holds

$$D_{3,2} \approx D_c + \frac{1}{3nN} \sum_{\lambda_c \in V_s(\mathbf{0})} \left(\|\lambda_c - y_{110}(\lambda_c)\|^2 + \|\lambda_c - y_{011}(\lambda_c)\|^2 + \|\lambda_c - y_{101}(\lambda_c)\|^2 \right). \tag{4.9}$$

Recall that for $\lambda_c \in \mathcal{L}_u$, one has $\alpha(\lambda_c) = (\lambda + \varphi(u), \varphi(u), -\lambda + u + \varphi(u))$, for some $\lambda \in \mathcal{A}$. Then $y_{110}(\lambda_c) = y_{011}(\lambda_c) = \varphi(u)$ and $y_{101}(\lambda_c) = \frac{u}{2} + \varphi(u)$. It follows that

$$\frac{y_{110}(\lambda_c) + y_{011}(\lambda_c) + y_{101}(\lambda_c)}{3} = \frac{u}{6} + \varphi(u).$$

Relation (4.9) and Lemma 1 in the appendix imply that

$$\begin{aligned} D_{3,2} &\approx D_c + \frac{1}{3nN} \sum_{u \in \mathcal{U}} \sum_{\lambda_c \in \mathcal{L}_u} \left(3\left\| \lambda_c - \left(\frac{u}{6} + \varphi(u) \right) \right\|^2 + \frac{1}{6} \|u\|^2 \right) \\ &= D_c + \frac{1}{3nN} \sum_{u \in \mathcal{U}} \sum_{\lambda_c \in \mathcal{L}_u} \left(3\|\lambda_c\|^2 + 3\left\| \frac{u}{6} + \varphi(u) \right\|^2 - 6 \left\langle \lambda_c, \frac{u}{6} + \varphi(u) \right\rangle + \frac{1}{6} \|u\|^2 \right) \\ &= D_c + \frac{1}{nN} \sum_{\lambda_c \in V_s(\mathbf{0})} \|\lambda_c\|^2 + \frac{1}{18nN} \sum_{u \in \mathcal{U}} \sum_{\lambda_c \in \mathcal{L}_u} \|u\|^2 + \frac{1}{nN} \sum_{u \in \mathcal{U}} \sum_{\lambda_c \in \mathcal{L}_u} \left\| \frac{u}{6} + \varphi(u) \right\|^2 \\ &\quad - \frac{2}{nN} \sum_{u \in \mathcal{U}} \left\langle \sum_{\lambda_c \in \mathcal{L}_u} \lambda_c, \frac{u}{6} + \varphi(u) \right\rangle. \end{aligned} \tag{4.10}$$

To rewrite the third term in the last relation in (4.10) we use the proposition in section 4.1 and obtain that

$$\begin{aligned} & \frac{1}{18nN} \sum_{u \in \mathcal{U}} \sum_{\lambda_c \in \mathcal{L}_u} \|u\|^2 \\ = \; & \frac{1}{18nN} |\mathcal{A}| \, S(|\mathcal{U}|, \Lambda_s) \\ \approx \; & \frac{1}{18} G(S_n) (N\nu_c)^{\frac{2}{n}} |\mathcal{U}|^{\frac{2}{n}} = \frac{1}{18} G(S_n) (N\nu_c)^{\frac{2}{n}} N^{\frac{\beta}{n}}. \end{aligned} \tag{4.11}$$

Plugging (4.3) and (4.11) in (4.10) leads to

$$D_{3,2} \approx G(\Lambda_s)(N\nu_c)^{\frac{2}{n}} + \frac{1}{18} G(S_n) (N\nu_c)^{\frac{2}{n}} N^{\frac{\beta}{n}} + T, \tag{4.12}$$

where

$$T \triangleq \frac{1}{nN}\sum_{u\in\mathcal{U}}\sum_{\lambda_c\in\mathcal{L}_u}\left\|\frac{u}{6}+\varphi(u)\right\|^2 - \frac{2}{nN}\sum_{u\in\mathcal{U}}\left\langle\sum_{\lambda_c\in\mathcal{L}_u}\lambda_c, \frac{u}{6}+\varphi(u)\right\rangle.$$

Next we will show that the second term dominates in (4.12) when $N \to \infty$. For this we will derive an upper bound for T. Recall that $\frac{u}{6}+\varphi(u) \in \hat{V}_s(\mathbf{0})$. Therefore, one has

$$\left\|\frac{u}{6}+\varphi(u)\right\| \leq \ell. \tag{4.13}$$

Further, using the Cauchy-Schwarz inequality along with (4.7) and (4.13) one obtains

$$\left|\left\langle\sum_{\lambda_c\in\mathcal{L}_u}\lambda_c, \frac{u}{6}+\varphi(u)\right\rangle\right| \leq \left\|\sum_{\lambda_c\in\mathcal{L}_u}\lambda_c\right\|\left\|\frac{u}{6}+\varphi(u)\right\| \leq \ell^2. \tag{4.14}$$

Relations (4.13) and (4.14) imply that

$$T \leq \frac{|\mathcal{U}||\mathcal{A}|}{nN}\ell^2 + 2\frac{|\mathcal{U}|}{nN}\ell^2 \approx \frac{\ell^2}{n}\left(1+\frac{2}{N^{1-\frac{\beta}{2}}}\right) = k(\Lambda_s)\left(N\nu_c\right)^{\frac{2}{n}}\left(1+\frac{2}{N^{1-\frac{\beta}{2}}}\right), \tag{4.15}$$

where

$$k(\Lambda_s) \triangleq \frac{\ell^2}{n\nu_s^{\frac{2}{n}}} \tag{4.16}$$

is a constant that does not change as N increases (it does not change if Λ_s is scaled). Finally, using (4.15) we conclude that the second term in (4.12) is dominant as $N \to \infty$, therefore one obtains

$$D_{3,2} \approx \frac{1}{18}G(S_n)\left(N\nu_c\right)^{\frac{2}{n}}N^{\frac{\beta}{n}}. \tag{4.17}$$

Let us now evaluate $D_{3,1}$. According to (3.1) one has

$$D_{3,1} \approx D_c + \frac{1}{nN} \sum_{\lambda_c \in V_s(\mathbf{0})} \|\lambda_c - \mu_s(\lambda_c)\|^2 + \frac{1}{3nN} \sum_{\lambda_c \in V_s(\mathbf{0})} \sum_{j=1}^{3} \|\alpha_j(\lambda_c) - \mu_s(\lambda_c)\|^2. \quad (4.18)$$

We will first evaluate the last summation in (4.18). Note that for $\lambda_c \in \mathcal{L}_u$, one has

$$\mu_s(\lambda_c) = \frac{u}{3} + \varphi(u).$$

Then the following equalities hold

$$\begin{aligned} & \frac{1}{3nN} \sum_{\lambda_c \in V_s(\mathbf{0})} \sum_{j=1}^{3} \|\alpha_j(\lambda_c) - \mu_s(\lambda_c)\|^2 \\ = \; & \frac{1}{3nN} \sum_{u \in \mathcal{U}} \sum_{\lambda \in \mathcal{A}} \left(\left\|\lambda - \frac{u}{3}\right\|^2 + \left\|\frac{u}{3}\right\|^2 + \left\|\lambda - \frac{2u}{3}\right\|^2 \right) \\ = \; & \frac{1}{3nN} \sum_{u \in \mathcal{U}} \sum_{\lambda \in \mathcal{A}} \left(2\|\lambda\|^2 + \frac{2}{3}\|u\|^2 - 2\langle \lambda, u \rangle \right) \\ = \; & \frac{1}{3nN} \left(2|\mathcal{U}| \sum_{\lambda \in \mathcal{A}} \|\lambda\|^2 + \frac{2}{3}|\mathcal{A}| \sum_{u \in \mathcal{U}} \|u\|^2 \right) \\ = \; & \frac{1}{3nN} \left(2|\mathcal{U}| S(|\mathcal{A}|, \Lambda_s) + \frac{2}{3}|\mathcal{A}| S(|\mathcal{U}|, \Lambda_s) \right) \\ = \; & G(S_n)(N\nu_c)^{\frac{2}{n}} \left(\frac{2}{3}|\mathcal{A}|^{\frac{2}{n}} + \frac{2}{9}|\mathcal{U}|^{\frac{2}{n}} \right) \\ = \; & G(S_n)(N\nu_c)^{\frac{2}{n}} \left(\frac{2}{3} N^{\frac{2}{n}(1-\frac{\beta}{2})} + \frac{2}{9} N^{\frac{\beta}{n}} \right), \end{aligned} \quad (4.19)$$

where the second last equality follows from the proposition in section 4.1, while the forth last equality relies on the fact that $\sum_{u \in \mathcal{U}} \sum_{\lambda \in \mathcal{A}} \langle \lambda, u \rangle = \sum_{\lambda \in \mathcal{A}} \langle \lambda, \sum_{u \in \mathcal{U}} u \rangle = 0$ because $\mathcal{U}$ is symmetric with respect to the origin.

For the remaining portion of (4.18) we will derive an upper bound as follows.

$$\begin{aligned}
& D_c + \frac{1}{nN} \sum_{\lambda_c \in V_s(\mathbf{0})} \|\lambda_c - \mu_s(\lambda_c)\|^2 \\
= \; & D_c + \frac{1}{nN} \sum_{u \in \mathcal{U}} \sum_{\lambda_c \in \mathcal{L}_u} \left\| \lambda_c - \left(\frac{u}{3} + \varphi(u) \right) \right\|^2 \\
= \; & D_c + \frac{1}{nN} \sum_{u \in \mathcal{U}} \sum_{\lambda_c \in \mathcal{L}_u} \left(\|\lambda_c\|^2 + \left\| \frac{u}{3} + \varphi(u) \right\|^2 - 2 \left\langle \lambda_c, \frac{u}{3} + \varphi(u) \right\rangle \right) \\
\leq \; & D_c + \frac{1}{nN} \left(\sum_{\lambda_c \in V_s(\mathbf{0})} \|\lambda_c\|^2 + |\mathcal{A}| \sum_{u \in \mathcal{U}} \left\| \frac{u}{3} + \varphi(u) \right\|^2 + 2 \sum_{u \in \mathcal{U}} \left| \left\langle \sum_{\lambda_c \in \mathcal{L}_u} \lambda_c, \frac{u}{3} + \varphi(u) \right\rangle \right| \right).
\end{aligned} \tag{4.20}$$

Using the inequality $\|\mathbf{y}_1 + \mathbf{y}_2\|^2 \leq 2(\|\mathbf{y}_1\|^2 + \|\mathbf{y}_2\|^2)$ and (4.13) one gets

$$\begin{aligned}
\left\| \frac{u}{3} + \varphi(u) \right\|^2 &= \left\| \frac{u}{6} + \left(\frac{u}{6} + \varphi(u) \right) \right\|^2 \\
&\leq 2 \left\| \frac{u}{6} \right\|^2 + 2 \left\| \frac{u}{6} + \varphi(u) \right\|^2 \\
&\leq 2 \left(\left\| \frac{u}{6} \right\|^2 + \ell^2 \right).
\end{aligned} \tag{4.21}$$

Applying further (4.7), (4.21) and the inequality $2|\langle \mathbf{y}_1, \mathbf{y}_2 \rangle| \leq \|\mathbf{y}_1\|^2 + \|\mathbf{y}_2\|^2$, for $\mathbf{y}_1, \mathbf{y}_2 \in \mathbf{R}^n$, one obtains that

$$2 \left| \left\langle \sum_{\lambda_c \in \mathcal{L}_u} \lambda_c, \frac{u}{3} + \varphi(u) \right\rangle \right| \leq \left\| \sum_{\lambda_c \in \mathcal{L}_u} \lambda_c \right\|^2 + \left\| \frac{u}{3} + \varphi(u) \right\|^2 \leq 2 \left\| \frac{u}{6} \right\|^2 + 3\ell^2. \tag{4.22}$$

Relations (4.20), (4.21) and (4.22) imply that

$$
\begin{aligned}
& D_c + \frac{1}{nN} \sum_{\lambda_c \in V_s(\mathbf{0})} \|\lambda_c - \mu_s(\lambda_c)\|^2 \\
\leq \; & D_c + \frac{1}{nN} \sum_{\lambda_c \in V_s(\mathbf{0})} \|\lambda_c\|^2 + \frac{|\mathcal{A}|+1}{18nN} S(|\mathcal{U}|, \Lambda_s) + \frac{\ell^2}{nN}\left(2|\mathcal{A}||\mathcal{U}| + 3|\mathcal{U}|\right) \\
\approx \; & (N\nu_c)^{\frac{2}{n}} \left(G(\Lambda_s) + \frac{G(S_n)}{18}|\mathcal{U}|^{\frac{2}{n}} + \frac{G(S_n)}{18|\mathcal{A}|}|\mathcal{U}|^{\frac{2}{n}} + k(\Lambda_s)\left(2 + \frac{3}{|\mathcal{A}|}\right)\right) \\
\approx \; & \frac{G(S_n)}{18}(N\nu_c)^{\frac{2}{n}}|\mathcal{U}|^{\frac{2}{n}} \\
= \; & \frac{G(S_n)}{18}(N\nu_c)^{\frac{2}{n}} N^{\frac{\beta}{n}},
\end{aligned}
$$

where the third last relation follows from (4.1) and (4.16), while the second last relation is obtained by keeping only the dominant term as $N \to \infty$. Corroborating with (4.18) and (4.19) and keeping only the dominant term in the expression of $D_{3,1}$ leads to the conclusion that when $\beta \in (0,1)$ and $N \to \infty$, the following holds

$$
D_{3,1} \approx \frac{2}{3} G(S_n)(N\nu_c)^{\frac{2}{n}} N^{\frac{2}{n}(1-\frac{\beta}{2})}. \tag{4.23}
$$

Relations (4.17) and (4.23) imply that, as β varies in the interval $(0,1)$, the product $D_{3,1}D_{3,2}$ is constant achieving

$$
D_{3,1}D_{3,2} \approx \frac{1}{27}(G(S_n))^2 (N\nu_c)^{\frac{4}{n}} N^{\frac{2}{n}}. \tag{4.24}
$$

On the other hand, according to (2.29), the distortion product for the MDLVQ in (Huang and Wu, 2006) is

$$
D_{3,1}D_{3,2} \approx \frac{1}{27}(G(S_{2n}))^2 (N\nu_c)^{\frac{4}{n}} N^{\frac{2}{n}}. \tag{4.25}
$$

It can be seen from (4.24) and (4.25) that the proposed index assignment achieves the same distortion product $D_{3,1}D_{3,2}$ as in (Huang and Wu, 2006), as n goes to ∞, since $\lim_{n\to\infty} G(S_{2n}) = \lim_{n\to\infty} G(S_n) = \frac{1}{2\pi e}$. However, the MDLVQ scheme of (Huang and Wu, 2006) can only achieve one distortion pair $(D_{3,1}, D_{3,2})$ with the ratio $D_{3,1}/D_{3,2} = 4$, while the proposed scheme can achieve a wide range of pairs $(D_{3,1}, D_{3,2})$ with ratios $D_{3,1}/D_{3,2} = 12N^{\frac{2}{n}(1-\beta)}$, for all $\beta \in (0, 1)$.

Let us now express the distortions in terms of the rate R. First let us see for what values of ρ inequality (2.17) is satisfied. For this we need to determine the value of γ in (2.17). It is easy to see that for fixed $\lambda_s \in \Lambda$, the set $\alpha_i^{-1}(\lambda_s)$ is most spread out for $i = 3$. Notice that the set $\alpha_3^{-1}(\lambda_s)$ contains central lattice points from each set $\hat{V}_s(v)$, with $v = \lambda_s + \lambda - u - \varphi(u)$, for some $\lambda \in \mathcal{A}$ and $u \in \mathcal{U}$. Recall that $\frac{u}{6} + \varphi(u) \in \hat{V}_s(\mathbf{0})$. It follows that, as $N \to \infty$, one has $\alpha_3^{-1}(\lambda_s) \subset \cup_{v\in\mathcal{A}-\mathcal{U}} \hat{V}_s(\lambda_s + v)$, where $\mathcal{A} - \mathcal{U} \triangleq \{\lambda - u : \lambda \in \mathcal{A}, u \in \mathcal{U}\}$. Then we may approximate the volume of the convex closure of $\cup_{\lambda_c\in\alpha_3^{-1}(\lambda_s)} \hat{V}_c(\lambda_c)$ by the volume of the set $\cup_{v\in\lambda_s+\mathcal{A}-\mathcal{U}} \hat{V}_s(v)$, which can be approximated, in turn, by the volume of a sphere of radius

$$r_3 = r(|\mathcal{A}|, \Lambda_s) + r(|\mathcal{U}|, \Lambda_s).$$

Using further relation (4.1), the volume of the n-dimensional sphere of radius r_3 given above becomes

$$vol = \left(\frac{r_3}{\sqrt{(n+2)G(S_n)}}\right)^n \approx \nu_s \left(N^{\frac{1}{n}(1-\frac{\beta}{2})} + N^{\frac{1}{n}\frac{\beta}{2}}\right)^n.$$

When $0 < \beta < 1$, one has $vol \approx \nu_s N^{1-\frac{\beta}{2}}$. Then the value of γ in (2.17) is $\gamma = 1 - \frac{\beta}{2}$, and relation (2.17) becomes

$$\rho < \frac{1}{2-\beta}.$$

By replacing $N\nu_c$, respectively N, from (2.13), respectively (2.16), in (4.23) and in (4.17) one obtains that, for $0 < \beta < 1$, $0 < \rho < \frac{1}{2-\beta}$, and $R \to \infty$, the following relations hold

$$D_{3,1} \approx \frac{2}{3} G(S_n) 2^{2\left(h(f) - R\left(1 - 2\rho\left(1 - \frac{\beta}{2}\right)\right)\right)},$$

$$D_{3,2} \approx \frac{1}{18} G(S_n) 2^{2(h(f) - R(1 - \rho\beta))}.$$

Chapter 5

Flexible MDLVQ with Structured Index Assignment for General L

In the previous chapter we presented a flexible MDLVQ with a structured index assignment for $L = 3$. In this chapter we will generalize the structured index assignment to any $L \geq 3$.

The following section 5.1 introduces the index assignment scheme. Section 5.2 presents the derivation of the distortion expressions at high resolution. The construction of the coefficients involved in the index assignment is discussed in Section 5.3.

5.1 System Setting

Let $\mathcal{U}_i, 1 \leq i \leq L-1$, denote the set of side lattice points in the n-dimensional sphere of radius μ_i centred at $\mathbf{0}$ for some $\mu_1 > \mu_2 > ... > \mu_{L-1} > 0$, such that

$$\prod_{i=1}^{L-1} |\mathcal{U}_i| = N. \tag{5.1}$$

Then we can write $|\mathcal{U}_i| = N^{\beta_i}, 1 \leq i \leq L-1$, with $1 > \beta_1 > \beta_2 > ... > \beta_{L-1} > 0$ and $\sum_{i=1}^{L-1} \beta_i = 1$. We will show the construction of a practical index assignment scheme which is able to achieve the tradeoffs between side distortions by changing the value of $|\mathcal{U}_i|, 1 \leq i \leq L-1$. In other words, we want to control the distortion when k descriptions are received by controlling the value of $|\mathcal{U}_k|$. For this we first construct an L-by-$(L-1)$ matrix A_L with elements in $\mathbb{Z}$,

$$A_L = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,L-1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,L-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{L,1} & a_{L,2} & \cdots & a_{L,L-1} \end{pmatrix}.$$

For every $(\lambda_1, \lambda_2, ..., \lambda_L) \in \Lambda_s^L$, let $\varphi(\lambda_1, \lambda_2, ..., \lambda_L)$ denote a shift vector in Λ_s which will be discussed shortly.

Define now the set

$$\mathcal{T}_L \triangleq \left\{ (\lambda_1, \lambda_2, ..., \lambda_L) + \varphi(\lambda_1, \lambda_2, ..., \lambda_L) : \lambda_i = \sum_{j=1}^{L-1} a_{i,j} u_j, u_j \in \mathcal{U}_j, 1 \leq j \leq L-1 \right\},$$

where

$$\begin{aligned} & (\lambda_1, \lambda_2, ..., \lambda_L) + \varphi(\lambda_1, \lambda_2, ..., \lambda_L) \\ \triangleq \; & (\lambda_1 + \varphi(\lambda_1, \lambda_2, ..., \lambda_L), \lambda_2 + \varphi(\lambda_1, \lambda_2, ..., \lambda_L), ..., \lambda_L + \varphi(\lambda_1, \lambda_2, ..., \lambda_L)). \end{aligned}$$

Notice that the L-tuples $(\lambda_1, \lambda_2, ..., \lambda_L) + \varphi(\lambda_1, ..., \lambda_L)$ in $\mathcal{T}_L$ can be written as

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_L \end{pmatrix} + \begin{pmatrix} \varphi(\lambda_1, ..., \lambda_L) \\ \varphi(\lambda_1, ..., \lambda_L) \\ \vdots \\ \varphi(\lambda_1, ..., \lambda_L) \end{pmatrix} = A_L \times \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{L-1} \end{pmatrix} + \begin{pmatrix} \varphi(\lambda_1, ..., \lambda_L) \\ \varphi(\lambda_1, ..., \lambda_L) \\ \vdots \\ \varphi(\lambda_1, ..., \lambda_L) \end{pmatrix}.$$

Denote now the matrix

$$\bar{A}_L \triangleq \begin{pmatrix} & 1 \\ A_L & \vdots \\ & 1 \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,L-1} & 1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,L-1} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{L,1} & a_{L,2} & \cdots & a_{L,L-1} & 1 \end{pmatrix}. \tag{5.2}$$

We impose the condition that matrix $\bar{A}_L$ has rank L in order to ensure an one-to-one correspondence between the $(L-1)$-tuples in $\mathcal{U}_1 \times \mathcal{U}_2 \times ... \times \mathcal{U}_{L-1}$ and the L-tuples in $\mathcal{T}_L$. This condition and relation (5.1) imply that $|\mathcal{T}_L| = N$. In the sequel, we will use the notation $\psi(u_1, u_2, \cdots, u_{L-1})$ instead of $\varphi(\lambda_1, \lambda_2, \cdots, \lambda_L)$.

Finally, the index assignment α is constructed such that to assign each $\lambda_c \in V_s(\mathbf{0})$ with an L-tuple from the set $\mathcal{T}_L$ in a one-to-one manner. Further, the index assignment is extended to Λ_c via shifting.

For any pattern $\mathbf{b} = (b_1, b_2, ..., b_L) \in \{0,1\}^L$ with $1 \leq H(\mathbf{b}) < L$, the reconstruction value when the pattern of received descriptions is $\mathbf{b}$ is defined as

$$y_{\mathbf{b}}(\lambda_c) = \sum_{i=1}^{L} \omega(\mathbf{b}, i)\, b_i \alpha_i(\lambda_c), \tag{5.3}$$

where $\omega(\mathbf{b}, i) \in \mathbf{R}$ and $\alpha_i(\lambda_c)$ is defined in (2.9). The choice of coefficients $\omega(\mathbf{b}, i)$ will be discussed in section 5.3.

Then equation (5.3) can be rewritten as

$$y_{\mathbf{b}}(\lambda_c) = \sum_{i=1}^{L} \omega(\mathbf{b}, i)\, b_i \left(\sum_{j=1}^{L-1} a_{i,j} u_j(\lambda_c) + \psi(\bar{u}(\lambda_c)) \right), \tag{5.4}$$

where $\bar{u}(\lambda_c) = (u_1(\lambda_c), u_2(\lambda_c), ..., u_{L-1}(\lambda_c))$ is the unique $(L-1)$-tuple in $\mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_{L-1}$ such that $(\alpha(\lambda_c))^T = A_L \cdot (\bar{u}(\lambda_c))^T + \psi(\bar{u}(\lambda_c))$.

To ensure that the decoder mapping of our index assignment satisfies the *shift-invariance* property, i.e.,

$$y_{\mathbf{b}}(\lambda_c + \lambda) = y_{\mathbf{b}}(\lambda_c) + \lambda, \lambda \in \Lambda_s,$$

the following condition should be satisfied

$$\sum_{i=1}^{L} \omega(\mathbf{b}, i)\, b_i = 1, \text{ for any pattern } \mathbf{b} \text{ with } 1 \leq H(\mathbf{b}) \leq L-1. \tag{5.5}$$

To achieve the goal of using $|\mathcal{U}_k|$ to control $D_{L,k}, 1 \leq k \leq L-1$, we further impose the following condition

$$\sum_{i=1}^{L} \omega(\mathbf{b}, i)\, b_i a_{i,j} = 0, 1 \leq j \leq k-1, \tag{5.6}$$

for each pattern $\mathbf{b}$ with $H(\mathbf{b}) = k, 2 \leq k \leq L-1$. Under (5.6), the reconstruction when k descriptions are received is independent of $u_i \in \mathcal{U}_i, 1 \leq i \leq k-1$. Therefore, $D_{L,k}$ is now only determined by $u_i \in \mathcal{U}_i, k \leq i \leq L-1$.

We will discuss in detail the construction of A_L and the choices of the decoding coefficients $\omega(\mathbf{b}, i), 1 \leq H(\mathbf{b}) \leq L-1, 1 \leq i \leq L$, such that conditions (5.5) and (5.6) to be satisfied in section 5.3.

Based on equations (5.5) and (5.6), relation (5.4) can be simplified as

$$y_{\mathbf{b}}\left(\lambda_c\right) = \sum_{j=k}^{L-1} \sum_{i=1}^{L} \omega\left(\mathbf{b}, i\right) b_i a_{i,j} u_j\left(\lambda_c\right) + \psi\left(\bar{u}\left(\lambda_c\right)\right). \tag{5.7}$$

5.2 Distortion Evaluation

In this section we will present the derivation of the expression of $D_{L,k}$ at high resolution, for $1 \leq k \leq L-1$. Note that $D_{L,L}$ is given by (2.18).

For every $\mathbf{b} \in \{0,1\}^L, H(\mathbf{b}) \leq L-1$, and $1 \leq j \leq L-1$, let

$$B\left(\mathbf{b}, j\right) \triangleq \sum_{i=1}^{L} \omega\left(\mathbf{b}, i\right) b_i a_{i,j}.$$

Then equation (5.7) can be rewritten as

$$y_{\mathbf{b}}\left(\lambda_c\right) = \sum_{j=k}^{L-1} B\left(\mathbf{b}, j\right) u_j\left(\lambda_c\right) + \psi\left(\bar{u}\left(\lambda_c\right)\right). \tag{5.8}$$

For $1 \leq k \leq L-1$ and $\lambda_c \in \Lambda_c$, let $\mu_{s,k}\left(\lambda_c\right)$ denote the average of the reconstructions $y_{\mathbf{b}}\left(\lambda_c\right)$, for $\mathbf{b} \in \{0,1\}^L, H\left(\mathbf{b}\right) = k$, i.e.,

$$\mu_{s,k}\left(\lambda_c\right) = \frac{1}{\binom{L}{k}} \sum_{\substack{\mathbf{b}\in\{0,1\}^L \\ H(\mathbf{b})=k}} y_{\mathbf{b}}\left(\lambda_c\right).$$

According to (5.8), we have

$$\begin{aligned}\mu_{s,k}(\lambda_c) &= \frac{1}{\binom{L}{k}} \sum_{\substack{\mathbf{b}\in\{0,1\}^L \\ H(\mathbf{b})=k}} \sum_{j=k}^{L-1} B(\mathbf{b},j)\, u_j(\lambda_c) + \psi(\bar{u}(\lambda_c)) \\ &= \sum_{j=k}^{L-1} \frac{1}{\binom{L}{k}} \sum_{\substack{\mathbf{b}\in\{0,1\}^L \\ H(\mathbf{b})=k}} B(\mathbf{b},j)\, u_j(\lambda_c) + \psi(\bar{u}(\lambda_c)).\end{aligned} \tag{5.9}$$

Denote now

$$C_{k,j} = \frac{1}{\binom{L}{k}} \sum_{\substack{\mathbf{b}\in\{0,1\}^L \\ H(\mathbf{b})=k}} B(\mathbf{b},j), \tag{5.10}$$

for every $1 \le k \le L-1$ and $k \le j \le L-1$. Then relation (5.9) can be rewritten as

$$\mu_{s,k}(\lambda_c) = \sum_{j=k}^{L-1} C_{k,j} u_j(\lambda_c) + \psi(\bar{u}(\lambda_c)). \tag{5.11}$$

To make the distortion when $L-1$ descriptions are received small, we let $\mu_{s,L-1}(\lambda_c)$ be in the same Voronoi region of Λ_s as $\lambda = 0$, i.e., $\hat{V}_s(\mathbf{0})$. Thus, $\psi(\bar{u}(\lambda_c))$ is chosen such that $\mu_{s,L-1}(\lambda_c)$ to be in $\hat{V}_s(\mathbf{0})$. Further, by denoting the first term in (5.11) as $\mu'_{s,k}(\lambda_c)$, we have

$$\mu_{s,k}(\lambda_c) = \mu'_{s,k}(\lambda_c) + \psi(\bar{u}(\lambda_c)). \tag{5.12}$$

Further, the fact that $\mu_{s,L-1}(\lambda_c) \in \hat{V}_s(\mathbf{0})$ implies that

$$\left\| {\mu'}_{s,L-1}(\lambda_c) + \psi(\bar{u}(\lambda_c)) \right\| \le \ell, \tag{5.13}$$

where ℓ is the covering radius of Λ_s defined in (4.8).

Now let us evaluate the distortion when k out of L descriptions are received, for

$k \le L-1$. Based on (2.20), we have

$$D_{L,k} \approx D_c + \frac{1}{nN}\frac{1}{\binom{L}{k}} \sum_{\lambda_c \in V_s(\mathbf{0})} \sum_{\substack{\mathbf{b}\in\{0,1\}^L \\ H(\mathbf{b})=k}} \left\|\lambda_c - y_{\mathbf{b}}\left(\lambda_c\right)\right\|^2 . \tag{5.14}$$

According to Lemma 1 in the appendix, we have

$$\begin{aligned} &\sum_{\substack{\mathbf{b}\in\{0,1\}^L \\ H(\mathbf{b})=k}} \left\|\lambda_c - y_{\mathbf{b}}\left(\lambda_c\right)\right\|^2 \\ &= \binom{L}{k}\left\|\lambda_c - \mu_{s,k}\left(\lambda_c\right)\right\|^2 + \sum_{\substack{\mathbf{b}\in\{0,1\}^L \\ H(\mathbf{b})=k}} \left\|\mu_{s,k}\left(\lambda_c\right) - y_{\mathbf{b}}\left(\lambda_c\right)\right\|^2. \end{aligned} \tag{5.15}$$

Then relation (5.14) becomes

$$D_{L,k} \approx D_c + \frac{1}{nN} \sum_{\lambda_c \in V_s(\mathbf{0})} \left\|\lambda_c - \mu_{s,k}\left(\lambda_c\right)\right\|^2 + \frac{1}{nN}\frac{1}{\binom{L}{k}} \sum_{\lambda_c \in V_s(\mathbf{0})} \sum_{\substack{\mathbf{b}\in\{0,1\}^L \\ H(\mathbf{b})=k}} \left\|\mu_{s,k}\left(\lambda_c\right) - y_{\mathbf{b}}\left(\lambda_c\right)\right\|^2. \tag{5.16}$$

Next we will evaluate the second term in the above expression.

$$\begin{aligned} & \frac{1}{nN} \sum_{\lambda_c \in V_s(\mathbf{0})} \left\|\lambda_c - \mu_{s,k}\left(\lambda_c\right)\right\|^2 \\ = & \frac{1}{nN} \sum_{\lambda_c \in V_s(\mathbf{0})} \left\|\lambda_c - {\mu'}_{s,k}\left(\lambda_c\right) - \psi\left(\bar{u}\left(\lambda_c\right)\right) - {\mu'}_{s,L-1}\left(\lambda_c\right) + {\mu'}_{s,L-1}\left(\lambda_c\right)\right\|^2 \\ = & \frac{1}{nN} \sum_{\lambda_c \in V_s(\mathbf{0})} \left\|\lambda_c - \left(\psi\left(\bar{u}\left(\lambda_c\right)\right) + {\mu'}_{s,L-1}\left(\lambda_c\right)\right) + \left({\mu'}_{s,L-1}\left(\lambda_c\right) - {\mu'}_{s,k}\left(\lambda_c\right)\right)\right\|^2 \\ = & \frac{1}{nN} \sum_{\lambda_c \in V_s(\mathbf{0})} \Big(\left\|\lambda_c - \left(\psi\left(\bar{u}\left(\lambda_c\right)\right) + {\mu'}_{s,L-1}\left(\lambda_c\right)\right)\right\|^2 + \left\|{\mu'}_{s,L-1}\left(\lambda_c\right) - {\mu'}_{s,k}\left(\lambda_c\right)\right\|^2 \\ & +2\left\langle \lambda_c - \left(\psi\left(\bar{u}\left(\lambda_c\right)\right) + {\mu'}_{s,L-1}\left(\lambda_c\right)\right), {\mu'}_{s,L-1}\left(\lambda_c\right) - {\mu'}_{s,k}\left(\lambda_c\right)\right\rangle \Big) \end{aligned} \tag{5.17}$$

where the first equation follows from relation (5.12).

Based on relation (5.13) and the fact that $\|\lambda_c\| \leq \ell$, for $\lambda_c \in V_s(\mathbf{0})$, the first term inside the parenthese in the last expression in (5.17) can be bounded as follows

$$\left\|\lambda_c - \left(\psi\left(\bar{u}\left(\lambda_c\right)\right) + {\mu'}_{s,L-1}\left(\lambda_c\right)\right)\right\| \leq 2\ell. \tag{5.18}$$

According to relations (5.11) and (5.12), the second term inside the parentheses in the last expression in (5.17) can be simplified as follows

$$\begin{aligned} \left\|{\mu'}_{s,L-1}\left(\lambda_c\right) - {\mu'}_{s,k}\left(\lambda_c\right)\right\|^2 &= \left\|C_{L-1,L-1}u_{L-1}\left(\lambda_c\right) - \sum_{j=k}^{L-1} C_{k,j}u_j\left(\lambda_c\right)\right\|^2 \\ &= \left\|\sum_{j=k}^{L-1} E_{k,j}u_j\left(\lambda_c\right)\right\|^2, \end{aligned} \tag{5.19}$$

where

$$E_{k,j} = \begin{cases} -C_{k,j}, & \text{for } 1 \leq k < L-1, k \leq j \leq L-1, \\ 0, & \text{for } k = j = L-1. \end{cases} \tag{5.20}$$

Further, using the Cauchy-Schwarz inequality along with (5.18) and (5.19), the third term inside the parentheses in the last expression in (5.17) can be bounded as

$$\begin{aligned} &\left|\left\langle \lambda_c - \left(\psi\left(\bar{u}\left(\lambda_c\right)\right) + {\mu'}_{s,L-1}\left(\lambda_c\right)\right), {\mu'}_{s,L-1}\left(\lambda_c\right) - {\mu'}_{s,k}\left(\lambda_c\right)\right\rangle\right| \\ &\leq \left\|\lambda_c - \left(\psi\left(\bar{u}\left(\lambda_c\right)\right) + {\mu'}_{s,L-1}\left(\lambda_c\right)\right)\right\| \left\|{\mu'}_{s,L-1}\left(\lambda_c\right) - {\mu'}_{s,k}\left(\lambda_c\right)\right\| \\ &\leq 2\ell \left\|{\mu'}_{s,L-1}\left(\lambda_c\right) - {\mu'}_{s,k}\left(\lambda_c\right)\right\| = 2\ell \left\|\sum_{j=k}^{L-1} E_{k,j}u_j\left(\lambda_c\right)\right\|. \end{aligned} \tag{5.21}$$

Plugging relations (5.18), (5.19) and (5.21) in (5.17) leads to

$$\begin{aligned} & \frac{1}{nN}\sum_{\lambda_c\in V_s(\mathbf{0})}\|\lambda_c-\mu_{s,k}(\lambda_c)\|^2 \\ \leq\ & \frac{1}{nN}\sum_{\lambda_c\in V_s(\mathbf{0})}\left(4\ell^2+4\ell\left\|\sum_{j=k}^{L-1}E_{k,j}u_j(\lambda_c)\right\|+\left\|\sum_{j=k}^{L-1}E_{k,j}u_j(\lambda_c)\right\|^2\right). \end{aligned} \quad (5.22)$$

Further, using the fact that $2ab \leq a^2+b^2, a, b \in \mathbf{R}$, we have

$$4\ell\left\|\sum_{j=k}^{L-1}E_{k,j}u_j(\lambda_c)\right\| \leq 4\ell^2+\left\|\sum_{j=k}^{L-1}E_{k,j}u_j(\lambda_c)\right\|^2.$$

Then relation (5.22) can be rewritten as

$$\frac{1}{nN}\sum_{\lambda_c\in V_s(\mathbf{0})}\|\lambda_c-\mu_{s,k}(\lambda_c)\|^2 \leq \frac{1}{nN}\sum_{\lambda_c\in V_s(\mathbf{0})}\left(8\ell^2+2\left\|\sum_{j=k}^{L-1}E_{k,j}u_j(\lambda_c)\right\|^2\right). \quad (5.23)$$

By keeping only the dominant term on the right hand side of (5.23), one obtains the following approximation as $N \to \infty$

$$\frac{1}{nN}\sum_{\lambda_c\in V_s(\mathbf{0})}\|\lambda_c-\mu_{s,k}(\lambda_c)\|^2 \approx \begin{cases} \frac{2}{nN}\sum_{\lambda_c\in V_s(\mathbf{0})}\left\|\sum_{j=k}^{L-1}E_{k,j}u_j(\lambda_c)\right\|^2, & 1 \leq k < L-1, \\ \frac{8\ell^2}{n}, & k = L-1. \end{cases} \quad (5.24)$$

For $1 \leq k < L-1$, using further Lemma 2 from appendix A, one has

$$\frac{1}{nN}\sum_{\lambda_c\in V_s(\mathbf{0})}\|\lambda_c-\mu_{s,k}(\lambda_c)\|^2 \approx 2(N\nu_c)^{\frac{2}{n}}G(S_n)E_{k,k}^2 N^{\frac{2\beta_k}{n}}, \quad (5.25)$$

as $N \to \infty$, provided that $E_{k,k} \neq 0$.

Relations (5.8) and (5.11) imply that

$$\begin{aligned}
&\frac{1}{N}\sum_{\lambda_c\in V_s(\mathbf{0})}\sum_{\substack{\mathbf{b}\in\{0,1\}^L\\ H(\mathbf{b})=k}}\left\|\mu_{s,k}\left(\lambda_c\right)-y_{\mathbf{b}}\left(\lambda_c\right)\right\|^2\\
=&\ \frac{1}{N}\sum_{\lambda_c\in V_s(\mathbf{0})}\sum_{\substack{\mathbf{b}\in\{0,1\}^L\\ H(\mathbf{b})=k}}\left\|\sum_{j=k}^{L-1}C_{k,j}u_j\left(\lambda_c\right)-\sum_{j=k}^{L-1}B\left(\mathbf{b},j\right)u_j\left(\lambda_c\right)\right\|^2\\
=&\ \frac{1}{N}\sum_{\substack{\mathbf{b}\in\{0,1\}^L\\ H(\mathbf{b})=k}}\sum_{\lambda_c\in V_s(\mathbf{0})}\left\|\sum_{j=k}^{L-1}\left(C_{k,j}-B\left(\mathbf{b},j\right)\right)u_j\left(\lambda_c\right)\right\|^2\\
\approx&\ \sum_{\substack{\mathbf{b}\in\{0,1\}^L\\ H(\mathbf{b})=k}}\left(C_{k,k}-B\left(\mathbf{b},k\right)\right)^2\left(N\nu_c\right)^{\frac{2}{n}}nG\left(S_n\right)N^{\frac{2\beta_k}{n}} \qquad (5.26)
\end{aligned}$$

as $N\to\infty$, provided that $\sum_{\substack{\mathbf{b}\in\{0,1\}^L\\ H(\mathbf{b})=k}}\left(C_{k,k}-B\left(\mathbf{b},k\right)\right)^2\neq 0$, or $k=L-1$, where the last relation follows from Lemma 2 in the appendix.

According to equations (5.16), (5.25) and (5.26), for $1\leq k<L-1$, and $N\to\infty$, one obtains

$$\begin{aligned}
D_{L,k}\ &\approx\ D_c+\left(2E_{k,k}^2+\frac{1}{\binom{L}{k}}\sum_{\substack{\mathbf{b}\in\{0,1\}^L\\ H(\mathbf{b})=k}}\left(C_{k,k}-B\left(\mathbf{b},k\right)\right)^2\right)\left(N\nu_c\right)^{\frac{2}{n}}G\left(S_n\right)N^{\frac{2\beta_k}{n}}\\
&\approx\ \left(2E_{k,k}^2+\frac{1}{\binom{L}{k}}\sum_{\substack{\mathbf{b}\in\{0,1\}^L\\ H(\mathbf{b})=k}}\left(C_{k,k}-B\left(\mathbf{b},k\right)\right)^2\right)\left(N\nu_c\right)^{\frac{2}{n}}G\left(S_n\right)N^{\frac{2\beta_k}{n}}.
\end{aligned}$$

Further, based on equations (2.13) and (2.16), for $1\leq k<L-1$ one has

$$D_{L,k}\approx\left(2E_{k,k}^2+\frac{1}{\binom{L}{k}}\sum_{\substack{\mathbf{b}\in\{0,1\}^L\\ H(\mathbf{b})=k}}\left(C_{k,k}-B\left(\mathbf{b},k\right)\right)^2\right)G\left(S_n\right)2^{2(h(f)-(1-\rho\beta_k(L-1))R)}. \qquad (5.27)$$

For $k = L-1$, according to relations (5.24) and (5.26) one obtains

$$D_{L,L-1} \approx D_c + 8\ell^2 + \frac{1}{L} \sum_{\substack{\mathbf{b}\in\{0,1\}^L \\ H(\mathbf{b})=L-1}} \left(C_{L-1,L-1} - B\left(\mathbf{b}, L-1\right)\right)^2 \left(N\nu_c\right)^{\frac{2}{n}} G\left(S_n\right) N^{\frac{2\beta_{L-1}}{n}}. \tag{5.28}$$

Further, notice that if $\sum_{\substack{\mathbf{b}\in\{0,1\}^L \\ H(\mathbf{b})=L-1}} \left(C_{L-1,L-1} - B\left(\mathbf{b}, L-1\right)\right)^2 \neq 0$, then

$$D_{L,L-1} \approx \frac{1}{L} \sum_{\substack{\mathbf{b}\in\{0,1\}^L \\ H(\mathbf{b})=L-1}} \left(C_{L-1,L-1} - B\left(\mathbf{b}, L-1\right)\right)^2 (N\nu_c)^{\frac{2}{n}} G\left(S_n\right) N^{\frac{2\beta_{L-1}}{n}},$$

while if $\sum_{\substack{\mathbf{b}\in\{0,1\}^L \\ H(\mathbf{b})=L-1}} \left(C_{L-1,L-1} - B\left(\mathbf{b}, L-1\right)\right)^2 = 0$, then

$$D_{L,L-1} \approx D_c + 8\ell^2.$$

Remark 2 . *The above equations hold only if ρ satisfies relation (2.17). Let us first evaluate γ in relation (2.17). Let $\lambda_s \in \Lambda_s$ and $1 \leq i \leq L$. Let $\lambda_s = \alpha_i(\lambda_c + v)$, where $\lambda_c \in \hat{V}_s(\mathbf{0})$. Then using relations (5.10), (5.11) and (5.12), one has*

$$\begin{aligned} v &= \lambda_s - \sum_{j=1}^{L-1} a_{i,j} u_j\left(\lambda_c\right) - \mu'_{s,L-1} - \psi\left(\bar{u}\left(\lambda_c\right)\right) + \mu'_{s,L-1} \\ &= \lambda_s - \sum_{j=1}^{L-1} a_{i,j} u_j\left(\lambda_c\right) + C_{L-1,L-1} u_{L-1}\left(\lambda_c\right) - \mu'_{s,L-1} - \psi\left(\bar{u}\left(\lambda_c\right)\right) \\ &= \lambda_s + \sum_{j=1}^{L-1} F_{i,j} u_j\left(\lambda_c\right) - \left({\mu'}_{s,L-1} + \psi\left(\bar{u}\left(\lambda_c\right)\right)\right) \end{aligned} \tag{5.29}$$

where

$$F_{i,j} = \begin{cases} -a_{i,j}, & 1 \leq j < L-1 \\ C_{L-1,L-1} - a_{i,L-1}, & j = L-1 \end{cases}. \tag{5.30}$$

It follows that, as $N \to \infty$*, one has* $\alpha_i^{-1}(\lambda_s) \subset \cup_{v \in F_{i,1}\mathcal{U}_1 + \cdots + F_{i,L-1}\mathcal{U}_{L-1}} \hat{V}_s(\lambda_s + v)$*, where* $F_{i,k}\mathcal{U}_k + F_{i,j}\mathcal{U}_j \triangleq \{F_{i,k}u_k + F_{i,j}u_j : u_k \in \mathcal{U}_k, u_j \in \mathcal{U}_j\}$*. Then we may approximate the volume of the convex closure of* $\cup_{\lambda_c \in \alpha_i^{-1}(\lambda_s)} \hat{V}_c(\lambda_c)$ *by the volume of the set* $\cup_{v \in \lambda_s - F_{i,1}\mathcal{U}_1 - \cdots - F_{i,L-1}\mathcal{U}_{L-1}} \hat{V}_s(v)$*, which can be approximated, in turn, by the volume of a sphere of radius*

$$r = \sum_{j=1}^{L-1} F_{i,j} r\left(|\mathcal{U}_j|, \Lambda_s\right).$$

Using further Proposition in section 4.1, the volume of the n*-dimensional sphere of radius* r *given above becomes*

$$vol = \left(\frac{r}{\sqrt{(n+2)G(S_n)}}\right)^n \approx \nu_s \left(\sum_{j=1}^{L-1} F_{i,j} N^{\frac{\beta_j}{n}}\right)^n.$$

Because $1 > \beta_1 > \cdots > \beta_{L-1} > 0$*, one has* $vol \approx \nu_s N^{\beta_{j(i)} + \log_N F_{i,j(i)}}$ *where* $j(i)$ *is the smallest* j *such that* $F_{i,j} \neq 0$*. The maximum volume is obtained when* $j(i) = 1$*. Note that there must exist an* i *such that* $j(i) = 1$ *since* A_L *has maximum rank. Then the value of* γ *in relation (2.17) is* $\gamma = \beta_1$ *as* $N \to \infty$*, and relation (2.17) becomes*

$$\rho < \frac{1}{\beta_1 (L-1)}.$$

Remark 3 . *The structured index assignment scheme we proposed in section 4.2 is a special case of the index assignment from this chapter. Additionally, we can obtain* *the same results as (4.23) and (4.17) by applying the formulae (5.27) and (5.28).*

To see this, notice that the index assignment from section 4.2 *corresponds to*

$$A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$$

and Table 5.1.

Table 5.1

$H(\mathbf{b})=1$	$H(\mathbf{b})=2$		
$\omega\,(001,3)=1$	$\omega\,(011,2)=\frac{1}{2}$	$\omega\,(101,1)=1$	$\omega\,(110,1)=1$
$\omega\,(010,2)=1$	$\omega\,(011,3)=\frac{1}{2}$	$\omega\,(101,3)=0$	$\omega\,(110,2)=0$
$\omega\,(100,1)=1$			

Further by applying the formulas (5.27) and (5.28), one has

$$\begin{aligned} D_{3,1} &\approx \left(2E_{1,1}^2 + \frac{1}{3}\sum_{\substack{\mathbf{b}\in\{0,1\}^3 \\ H(\mathbf{b})=1}} \left(C_{1,1} - B\left(\mathbf{b},1\right)\right)^2\right) \left(N\nu_c\right)^{\frac{2}{n}} G\left(S_n\right) N^{\frac{2\beta_1}{n}} \\ &= \frac{2}{3}\left(N\nu_c\right)^{\frac{2}{n}} G\left(S_n\right) N^{\frac{2\beta_1}{n}} \\ &= \frac{2}{3} G\left(S_n\right) 2^{2(h(f)-(1-2\rho\beta_1)R)} \end{aligned}$$

and

$$\begin{aligned} D_{3,2} &\approx \frac{1}{3}\sum_{\substack{\mathbf{b}\in\{0,1\}^3 \\ H(\mathbf{b})=2}} \left(C_{2,2} - B\left(\mathbf{b},2\right)\right)^2 \left(N\nu_c\right)^{\frac{2}{n}} G\left(S_n\right) N^{\frac{2\beta_2}{n}} \\ &= \frac{1}{18}\left(N\nu_c\right)^{\frac{2}{n}} G\left(S_n\right) N^{\frac{2\beta_2}{n}} \\ &= \frac{1}{18} G\left(S_n\right) 2^{2(h(f)-(1-2\rho\beta_2)R)}. \end{aligned}$$

It can be seen that the results we obtain above are equivalent to relations (4.23) and (4.17).

5.3 Construction of A_L and Choice of Decoder Coefficients

In this section, we will present a construction of the matrix A_L and of the decoder coefficients $\omega(\mathbf{b}, i)$ satisfying conditions (5.5) and (5.6), based on induction.

For convenience, we construct A_L such that to satisfy the following constraints

$$a_{i,j} = 0, j \geq i, \tag{5.31}$$

$$a_{i,i-1} = 1, 1 < i < L. \tag{5.32}$$

Note that the above relations ensure that $\bar{A}_L$ has full rank where $\bar{A}_L$ is given in (5.2).

Consider now an arbitrary pattern $\mathbf{b} \in \{0,1\}^L$ and let $1 \leq z_1 < z_2 < \cdots < z_k \leq L$, denote its nonzero positions, i.e., $b_{z_i} \neq 0$, $1 \leq i \leq k$, where $k = H(\mathbf{b})$. Then conditions (5.5) and (5.6) can be expressed in the form of the following matrix equation

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_{z_1,1} & a_{z_2,1} & \cdots & a_{z_k,1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{z_1,k-1} & a_{z_2,k-1} & \cdots & a_{z_k,k-1} \end{pmatrix} \times \begin{pmatrix} \omega(\mathbf{b}, z_1) \\ \omega(\mathbf{b}, z_2) \\ \vdots \\ \omega(\mathbf{b}, z_k) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{5.33}$$

When $H(\mathbf{b}) = 1$, clearly choosing $\omega(\mathbf{b}, z_1) = 1$ satisfies the above relation. Let $\mathcal{C}_{L,\mathbf{b}}$ denote the k-by-k matrix in (5.33) and let $\Omega_{\mathbf{b}}$ denote the column vector containing the decoder coefficients in (5.33). Let us now identify the relationship between A_L

and $\mathcal{C}_{L,\mathbf{b}}$.

We define the L-by-L matrix

$$\tilde{A}_L = \begin{pmatrix} 1 \quad 1 \quad \cdots \quad 1 \\ A_L^T \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & a_{3,1} & \cdots & a_{L,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{L,L-2} \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{5.34}$$

Then $\mathcal{C}_{L,\mathbf{b}}$ can be rewritten as

$$\mathcal{C}_{L,\mathbf{b}} = \left(\mathbf{c}_{z_1}, \mathbf{c}_{z_2}, \cdots, \mathbf{c}_{z_k}\right),$$

where $\mathbf{c}_{z_i}, 1 \leq z_i \leq k$ is a column vector consisting of the first k elements of z_i-th column of $\tilde{A}_L$.

From (5.33), we can see that if $|C_{L,\mathbf{b}}| \neq 0$, i.e., $C_{L,\mathbf{b}}$ is of full rank, then $\Omega_{\mathbf{b}}$ is uniquely determined by (5.33). This implies that there exist decoder coefficients such that conditions (5.5) and (5.6) are satisfied for pattern $\mathbf{b}$. Therefore, in order to ensure that it is possible to satisfy conditions (5.5) and (5.6) for all patterns $\mathbf{b}$, it is sufficient to construct matrix A_L obeying Condition C stated below.

Condition C: $|C_{L,\mathbf{b}}| \neq 0$, for all $\mathbf{b} \in \{0,1\}^L$ with $2 \leq H(\mathbf{b}) \leq L-1$.

Next we will show that it is possible to construct A_L satisfying Condition C, using induction over L.

For the base case, $L = 3$, let

$$A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

It can be easily verified that A_3 satisfies conditions (5.31), (5.32) and Condition C.

For the induction step, assume that the matrix

$$A_{L-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ a_{3,1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{L-1,1} & a_{L-1,2} & \cdots & 1 \end{pmatrix}$$

satisfies Condition C. To obtain A_L we can simply add a new row and column to the bottom, respectively to the right of A_{L-1} as follows,

$$A_L = \begin{pmatrix} & & & 0 \\ & A_{L-1} & & 0 \\ & & & \vdots \\ a_{L,1} & a_{L,2} & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ a_{3,1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{L-1,1} & a_{L-1,2} & \cdots & 1 & 0 \\ a_{L,1} & a_{L,2} & \cdots & a_{L,L-2} & 1 \end{pmatrix}.$$

Then one has

$$\tilde{A}_L = \begin{pmatrix} & & & & 1 \\ & \tilde{A}_{L-1} & & & a_{L,1} \\ & & & & \vdots \\ & & & & a_{L,L-2} \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & a_{3,1} & \cdots & a_{L,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{L,L-2} \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

We will prove next that it is possible to find the integers $a_{L,1}, \ldots, a_{L,L-2}$ such that A_L given above obeys Condition C.

Notice that for any pattern $\mathbf{b}, H(\mathbf{b}) = k$, with $b_L = 0$, the newly added elements $a_{L,j}, 1 \leq j \leq L-2$, will not appear in $\mathcal{C}_{L,\mathbf{b}}$. Moreover, if $H(\mathbf{b}) = L-1$ with $b_L = 0$, then $\mathcal{C}_{L,\mathbf{b}} = \tilde{A}_{L-1}$, thus $|\mathcal{C}_{L,\mathbf{b}}| = 1 \neq 0$. On the other hand, if $H(\mathbf{b}) < L-1$ with $b_L = 0$, let $\mathbf{b}' \in \{0,1\}^{L-1}$ be the pattern obtained from $\mathbf{b}$ by removing the last component. Then $\mathcal{C}_{L,\mathbf{b}} = \mathcal{C}_{L-1,\mathbf{b}'}$ and $|\mathcal{C}_{L-1,\mathbf{b}'}| \neq 0$ according to the induction hypothesis.

According to the above discussion, in order to ensure that Condition C is satisfied for matrix A_L. We only need to consider patterns $\mathbf{b}$ with $b_L = 1$. We will determine $a_{L,j}, 1 \leq j \leq L-2$, progressively by considering such patterns with increasing $H(\mathbf{b})$. Let us start with $a_{L,1}$. This value is chosen such that $|\mathcal{C}_{L,\mathbf{b}}| \neq 0$ for all patterns $\mathbf{b}$ with $b_L = 1$ and $H(\mathbf{b}) = 2$. For this, one must have $a_{L,1} \neq a_{j,1}, 1 \leq j \leq L-1$. Clearly, such $a_{L,1}$ exists.

Consider now some i, $2 \leq i \leq L-2$, and assume that we have determined $a_{L,1}, \ldots, a_{L,i-1}$ such that $|\mathcal{C}_{L,\mathbf{b}}| \neq 0$ for all patterns $\mathbf{b}$ with $b_L = 1$ and $H(\mathbf{b}) \leq i$. Consider a pattern $\mathbf{b}$ with $b_L = 1$ and $H(\mathbf{b}) = i+1$. Notice the last column of $\mathcal{C}_{L,\mathbf{b}}$ is $(1, a_{L,1}, \ldots, a_{L,i})^T$, while all the other elements are from $\tilde{A}_{L-1}$. Using Laplace's formula $|\mathcal{C}_{L,\mathbf{b}}|$ can be expressed as

$$|C_{L,\mathbf{b}}| = (-1)^{L+i+1} a_{L,i} \left|C_{L-1,\mathbf{b}'}\right| + \sum_{j=1}^{i-1} (-1)^{L+j+1} a_{L,j} \left|\mathcal{M}_{L,j+1}\right| + (-1)^{L+1} \left|\mathcal{M}_{L,1}\right|,$$

where $\mathbf{b} = (\mathbf{b}', b_L)$ and $|\mathcal{M}_{L,j+1}|$ is the determinant of the $(L-1)$-by-$(L-1)$ matrix obtained by removing the L-th column and the j-th row of $\tilde{A}_L$. Since A_{L-1} satisfies Condition C, one has $|\mathcal{C}_{L-1,\mathbf{b}'}| \neq 0$. Therefore there is exactly one choice for $a_{L,i}$ which makes $|\mathcal{C}_{L,\mathbf{b}}| = 0$. Since the set $\mathbb{Z}$ is infinite, it follows that it is possible to find $a_{L,i}$ in $\mathbb{Z}$ such that $|\mathcal{C}_{L,\mathbf{b}}| \neq 0$ for all patterns $\mathbf{b}$ with $b_L = 1$ and $H(\mathbf{b}) = i + 1$. With this observation, the proof by induction is concluded.

5.3.1 Example for $L = 4$

In this subsection we will present an example for $L = 4$.

Based on the discussion in the previous section, we can obtain A_4:

$$A_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}.$$

Based on the above matrix and equations (5.5) and (5.6), we can compute the weights when a specific pattern $\mathbf{b}$ is received and the results are listed in Table 5.2-Table 5.4.

Table 5.2

$H(\mathbf{b})=1$	Weights	Reconstruction $y_{\mathbf{b}}(\lambda_c)$
$\mathbf{b}=1000$	$\omega(\mathbf{b},1)=1$	$\alpha_1(\lambda_c)$
$\mathbf{b}=0100$	$\omega(\mathbf{b},2)=1$	$\alpha_2(\lambda_c)$
$\mathbf{b}=0010$	$\omega(\mathbf{b},3)=1$	$\alpha_3(\lambda_c)$
$\mathbf{b}=0001$	$\omega(\mathbf{b},4)=1$	$\alpha_4(\lambda_c)$

Table 5.3

$H(\mathbf{b})=2$	Weights	Reconstruction $y_{\mathbf{b}}(\lambda_c)$
$\mathbf{b}=1100$	$\omega(\mathbf{b},1)=1,\omega(\mathbf{b},2)=0$	$\alpha_1(\lambda_c)$
$\mathbf{b}=1001$	$\omega(\mathbf{b},1)=1,\omega(\mathbf{b},4)=0$	$\alpha_1(\lambda_c)$
$\mathbf{b}=1010$	$\omega(\mathbf{b},1)=1,\omega(\mathbf{b},3)=0$	$\alpha_1(\lambda_c)$
$\mathbf{b}=0011$	$\omega(\mathbf{b},3)=\frac{2}{3},\omega(\mathbf{b},4)=\frac{1}{3}$	$\frac{2\alpha_3(\lambda_c)+\alpha_4(\lambda_c)}{3}$
$\mathbf{b}=0110$	$\omega(\mathbf{b},2)=\frac{1}{2},\omega(\mathbf{b},3)=\frac{1}{2}$	$\frac{\alpha_2(\lambda_c)+\alpha_3(\lambda_c)}{2}$
$\mathbf{b}=0101$	$\omega(\mathbf{b},2)=2,\omega(\mathbf{b},4)=-1$	$2\alpha_2(\lambda_c)-\alpha_4(\lambda_c)$

Table 5.4

$H(\mathbf{b})=3$	Weights	Reconstruction $y_{\mathbf{b}}$
$\mathbf{b}=1110$	$\omega(\mathbf{b},1)=1,\omega(\mathbf{b},2)=0,\omega(\mathbf{b},3)=0$	$\alpha_1(\lambda_c)$
$\mathbf{b}=1101$	$\omega(\mathbf{b},1)=1,\omega(\mathbf{b},2)=0,\omega(\mathbf{b},4)=0$	$\alpha_1(\lambda_c)$
$\mathbf{b}=1011$	$\omega(\mathbf{b},1)=1,\omega(\mathbf{b},3)=0,\omega(\mathbf{b},4)=0$	$\alpha_1(\lambda_c)$
$\mathbf{b}=0111$	$\omega(\mathbf{b},2)=1,\omega(\mathbf{b},3)=\frac{1}{3},\omega(\mathbf{b},4)=-\frac{1}{3}$	$\frac{3\alpha_2(\lambda_c)+\alpha_3(\lambda_c)-\alpha_4(\lambda_c)}{3}$

Using relations (5.27) and (5.28), we obtain

$$
\begin{aligned}
D_{4,1} &\approx \frac{7}{4}\left(N\nu_c\right)^{\frac{2}{n}} G\left(S_n\right) N^{\frac{2\beta_1}{n}} = \frac{7}{4} G\left(S_n\right) 2^{2(h(f)-(1-3\rho\beta_1)R)},\\
D_{4,2} &\approx \frac{55}{144}\left(N\nu_c\right)^{\frac{2}{n}} G\left(S_n\right) N^{\frac{2\beta_2}{n}} = \frac{55}{144} G\left(S_n\right) 2^{2(h(f)-(1-3\rho\beta_2)R)},\\
D_{4,3} &\approx \frac{1}{48}\left(N\nu_c\right)^{\frac{2}{n}} G\left(S_n\right) N^{\frac{2\beta_3}{n}} = \frac{1}{48} G\left(S_n\right) 2^{2(h(f)-(1-3\rho\beta_3)R)}.
\end{aligned}
$$

Chapter 6

Conclusion and Future Work

In the previous work on multiple description lattice vector quantizers (MDLVQ) for $L \geq 3$ descriptions, once the central and side lattice are fixed, it is not possible to adjust the decoding quality when the number of received descriptions is higher than 1, but lower than L.

This thesis proposes two flexible MDLVQ schemes that overcome the aforementioned shortcoming. In our first design, a different reconstruction method is adopted and a heuristic index assignment algorithm is developed, which uses $L-2$ parameters to control the distortions when k descriptions are received, for $2 \leq k \leq L-1$. In our second design, structured index assignments, amenable to theoretical analysis, are proposed for the cases $L = 3$ and $L \geq 3$ respectively. Additionally, by deriving the asymptotical expressions of the distortions at high resolution, we show that a wide range of values can be achieved for the distortions when different numbers of descriptions are received.

Future research efforts could be directed to solving the following issues:

1. Derive a tighter upper bound for the second term in relation (5.16). The upper bound we obtained in (5.23) is a loose upper bound and it is possible to make it

tighter.

2. Construct A_L in section 5.3 such that the distortion when receiving some k, $1 \leq k \leq L-1$ of L descriptions, i.e., $D_{L,k}$, $1 \leq k \leq L-1$, is optimized.

3. In the structured index assignment scheme, design the one-to-one correspondence between the L-tuples in $\mathcal{T}_L$ and the central lattice points in $V_s(\mathbf{0})$ such that $D_{L,k}$, for some k, $1 \leq k \leq L-1$, is minimized when N is finite,.

Appendix A

Appendix

Lemma 1: Let $\mathbf{y}, \mathbf{y}_i \in \mathbf{R}^n, 1 \leq i \leq m$ and $\overline{\mathbf{y}}$ be the average of $\mathbf{y}_i, 1 \leq i \leq m$, i.e., $\overline{\mathbf{y}} = \frac{1}{m}\sum_{i=1}^{m} \mathbf{y}_i$. Then

$$\sum_{i=1}^{m} \|\mathbf{y} - \mathbf{y}_i\|^2 = m\,\|\mathbf{y} - \bar{\mathbf{y}}\|^2 + \sum_{i=1}^{m} \|\bar{\mathbf{y}} - \mathbf{y}_i\|^2.$$

Proof:

$$\begin{aligned}
& \sum_{i=1}^{m} \|\mathbf{y} - \bar{\mathbf{y}} + \bar{\mathbf{y}} - \mathbf{y}_i\|^2 \\
= & \sum_{i=1}^{m} \|\mathbf{y} - \bar{\mathbf{y}}\|^2 + \sum_{i=1}^{m} \|\bar{\mathbf{y}} - \mathbf{y}_i\|^2 + 2\sum_{i=1}^{m} \langle \mathbf{y} - \bar{\mathbf{y}}, \bar{\mathbf{y}} - \mathbf{y}_i \rangle \\
= & m\|\mathbf{y} - \bar{\mathbf{y}}\|^2 + \sum_{i=1}^{m} \|\bar{\mathbf{y}} - \mathbf{y}_i\|^2 + 2\left\langle \mathbf{y} - \bar{\mathbf{y}}, m\bar{\mathbf{y}} - \sum_{i=1}^{m} \mathbf{y}_i \right\rangle
\end{aligned}$$

The definition of $\bar{\mathbf{y}}$ implies that the last term in the above expression is $\mathbf{0}$.

Lemma 2: Consider the sets $\mathcal{U}_j$, $1 \leq j \leq L-1$, and the index assignment α defined in section 5.1. Recall that $\prod_{i=1}^{L-1} |\mathcal{U}_i| = N$ and $\mathcal{U}_i = N^{\beta_i}$, where $1 > \beta_1 > \beta_2 > \cdots > \beta_{L-1} > 0$. Let $\nu_j \in \mathbf{R}$, for $1 \leq j \leq L-1$. Then for any $1 \leq k \leq L-1$ such that

$\nu_k \neq 0$, one has

$$\frac{1}{N}\sum_{\lambda_c \in V_s(\mathbf{0})} \left\| \sum_{j=k}^{L-1} \nu_j u_j(\lambda_c) \right\|^2 \approx (N\nu_c)^{\frac{2}{n}} nG(S_n)\, \nu_k^2 N^{\frac{2\beta_k}{n}}$$

as $N \to \infty$.

Proof:

$$\begin{aligned}
&\frac{1}{N}\sum_{\lambda_c \in V_s(\mathbf{0})} \left\| \sum_{j=k}^{L-1} \nu_j u_j(\lambda_c) \right\|^2 \\
= \;& \frac{1}{N}\sum_{u_1 \in \mathcal{U}_1}\sum_{u_2 \in \mathcal{U}_2}\cdots\sum_{u_{L-1} \in \mathcal{U}_{L-1}} \left\| \sum_{j=k}^{L-1} \nu_j u_j \right\|^2 \\
= \;& \frac{1}{N}\prod_{i=1}^{k-1}|\mathcal{U}_i| \sum_{u_k \in \mathcal{U}_k}\sum_{u_{k+1} \in \mathcal{U}_{k+1}}\cdots\sum_{u_{L-1} \in \mathcal{U}_{L-1}} \left\| \sum_{j=k}^{L-1} \nu_j u_j \right\|^2 \\
= \;& \frac{1}{N}\prod_{i=1}^{k-1}|\mathcal{U}_i| \sum_{u_k \in \mathcal{U}_k}\sum_{u_{k+1} \in \mathcal{U}_{k+1}}\cdots\sum_{u_{L-1} \in \mathcal{U}_{L-1}} \left(\sum_{j=k}^{L-1} \|\nu_j u_j\|^2 + 2\sum_{s=k}^{L-2}\sum_{t=s+1}^{L-1} \langle \nu_s u_s, \nu_t u_t \rangle \right) \\
= \;& \frac{1}{N}\prod_{i=1}^{k-1}|\mathcal{U}_i| \sum_{j=k}^{L-1}\sum_{u_k \in \mathcal{U}_k}\sum_{u_{k+1} \in \mathcal{U}_{k+1}}\cdots\sum_{u_{L-1} \in \mathcal{U}_{L-1}} \|\nu_j u_j\|^2 \\
= \;& \frac{1}{N}\prod_{i=1}^{k-1}|\mathcal{U}_i| \sum_{j=k}^{L-1}\prod_{\substack{s=k \\ s\neq j}}^{L-1}|\mathcal{U}_s| \sum_{u_j \in \mathcal{U}_j} \|\nu_j u_j\|^2 \\
= \;& \sum_{j=k}^{L-1}\frac{1}{|\mathcal{U}_j|}\sum_{u_j \in \mathcal{U}_j} \|\nu_j u_j\|^2 \approx \sum_{j=k}^{L-1} (N\nu_c)^{\frac{2}{n}} nG(S_n)\, \nu_j^2 |\mathcal{U}_j|^{\frac{2}{n}} \\
\approx \;& (N\nu_c)^{\frac{2}{n}} nG(S_n)\, \nu_k^2 N^{\frac{2\beta_k}{n}} \qquad \text{(A.1)}
\end{aligned}$$

where the forth equality holds because

$$
\begin{aligned}
&\frac{1}{N}\prod_{i=1}^{k-1}|\mathcal{U}_i|\sum_{u_k\in\mathcal{U}_k}\sum_{u_{k+1}\in U_{k+1}}\cdots\sum_{u_{L-1}\in U_{L-1}}\sum_{s=k}^{L-2}\sum_{t=s+1}^{L-1}\langle\nu_s u_s,\nu_t u_t\rangle\\
&\frac{1}{N}\prod_{i=1}^{k-1}|\mathcal{U}_i|\sum_{s=k}^{L-2}\sum_{t=s+1}^{L-1}\sum_{u_k\in\mathcal{U}_k}\sum_{u_{k+1}\in\mathcal{U}_{k+1}}\cdots\sum_{u_{L-1}\in\mathcal{U}_{L-1}}\langle\nu_s u_s,\nu_t u_t\rangle\\
&=\sum_{s=k}^{L-2}\sum_{t=s+1}^{L-1}\frac{1}{|\mathcal{U}_s|\,|\mathcal{U}_t|}\left\langle\sum_{u_s\in\mathcal{U}_s}\nu_s u_s,\sum_{u_t\in\mathcal{U}_t}\nu_t u_t\right\rangle=0.
\end{aligned}
$$

The last relation is true because $\mathcal{U}_s$ is symmetric with regard to $\mathbf{0}$, therefore $\sum_{u_s\in\mathcal{U}_s} u_s = 0$, $1 \le s \le L-1$. Additionally, the second last relation in (A.1) follows based on (4.1), while the last relation is obtained by keeping only the dominant term as $N \to \infty$.

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