NATIONAL LIBRARY
OTTAWA

BIBLIOTHÈQUE NATIONALE
OTTAWA

CANADA

NAME OF AUTHOR....Kathryn Anne Murphy....................

TITLE OF THESIS...."TRANSFINITE CARDINAL ARITHMETIC"........

....

....

UNIVERSITY........McMaster University....................

DEGREE FOR WHICH THESIS WAS PRESENTED..M.A................

YEAR THIS DEGREE GRANTED........1975....................

Permission is hereby granted to THE NATIONAL LIBRARY OF CANADA to microfilm this thesis and to lend or sell copies of the film.

The author reserves other publication rights, and neither the thesis nor extensive extracts from it may be printed or otherwise reproduced without the author's written permission.

(Signed)..Kathryn Murphy........

PERMANENT ADDRESS:

c/o Dept. of Philosophy,
McMaster University
Hamilton, Ontario.

DATED..May 20..............19 75

NL-91 (10-68)

TRANSFINITE CARDINAL ARITHMETIC

By

KATHRYN ANNE MURPHY, B.A.

A Thesis

Submitted to the School of Graduate Studies

in Partial Fulfilment of the Requirements

For the Degree

Master of Arts

McMaster University

(May) 1975

TRANSFINITE CARDINAL ARITHMETIC

The establishing of the principles of mathematics and the natural sciences is the responsibility of metaphysics... The general theory of sets ... belongs entirely to metaphysics. You can easily convince yourself of this by testing the categories of cardinal number and ordinal type, these fundamental concepts of set theory, with respect to the degree of their generality, and also notice that the reasoning about them is quite pure, so that fancy has no room for play.

This is in no way changed by the pictures which I, like all metaphysicians, sometimes make use of to explain metaphysical concepts. Nor does the fact that my work appears in mathematical journals affect its metaphysical character and content.

(From a letter dated February 2, 1896 from Cantor to Father Thomas Esser. Translated by H. Meschkowski.[3])

MASTER OF ARTS (1975) McMASTER UNIVERSITY

(Philosophy) Hamilton, Ontario

TITLE: Transfinite Cardinal Arithmetic

AUTHOR: Kathryn Anne Murphy, B.A. (Reading University)

SUPERVISOR: Dr. D.L. Hitchcock

NUMBER OF PAGES: v, 94

TO

ROGER

PREFACE

Hilbert once wrote:

> From time immemorial, the infinite has stirred men's emotions more than any other question. Hardly any other idea has stimulated the mind so fruitfully. Yet, no other concept needs clarification more than it does[1].

This thesis was written in accordance with the above sentiments. I hope that it has contributed in some way to that "clarification" of which Hilbert speaks.

I owe the inspiration for this thesis, and indeed for my interest in the work of Georg Cantor, to the teaching and writings of Professor Stephan Körner. In February, 1973 I attended a course of extramural lectures given by Professor Körner at the University of Bristol. The lecture concerning infinity I found to be extraordinarily stimulating. It was, for me, what Maslow would call, a "peak experience". The combination of such wonderful subject matter and such a wise and exciting teacher proved irresistible: it has guided my studies ever since.

I also wish to thank Dr. Radner and Dr. Hitchcock for helping me to carry out my project. Dr. Hitchcock has been especially helpful in his scrupulous criticism of certain technical details.

---

[1]David Hilbert, "On the Infinite" (1925) in Paul Benacerraf and Hilary Putnam, ed., Philosophy of Mathematics: Selected Readings (New Jersey, 1964)., p.4.

CONTENTS

Chapter One: INTRODUCTION

In this thesis I propose to examine the foundations of transfinite cardinal arithmetic. I intend to present a "formal" version of the "naive" account of transfinite cardinal arithmetic given by Georg Cantor in 1895. To this end I shall present a formal axiomatization of the particular number system (Chapter II). Having thus obtained an uninterpreted calculus, I shall provide in Chapter III a model for this calculus. This model or interpretation will be based on Cantor's own account of transfinite cardinal arithmetic. In Chapters IV and V I shall discuss some of the arithmetical and algebraic properties of this particular number system. In Chapter VI I shall attempt to prove the consistency of the axiomatized version of transfinite cardinal arithmetic; the proof which I shall provide will be a relative rather than an absolute consistency proof. Finally, in Chapter VII, I shall discuss the ontological status of transfinite cardinal numbers, whether and in what sense they exist.

Cantor's "naive" account of transfinite cardinal arithmetic is to be found in the article entitled "Beiträge zur Begründung der transfiniten Mengenlehre"[1] which is signed "Halle,

---

[1] Translated as "Contributions to the Founding of the Theory of Transfinite Sets".

March 1895". It appeared in the journal Mathematische Annalen[2] in 1895. Together with another article possessing the same title[3] it has been translated into English by P.E.B. Jourdain and appears in the book entitled Contributions to the Founding of the Theory of Transfinite Numbers (New York, 1955).

My reasons for selecting the 1895 article to serve as the basis for a formal axiomatization of transfinite cardinal arithmetic are as follows: (1) The text is readily available in translation. (2) It constitutes excellent source material since it is Cantor's most comprehensive and definitive treatment of the subject. It should be noted, however, that most of the important ideas in this article (except the definition of continuity in Section 11) had been previously published in a piecemeal fashion by Cantor. To trace the historical development of these ideas is beyond the scope of this thesis.

Nevertheless, I shall make a cursory reference to three other publications of Cantor. The first of these is a monograph entitled Grundlagen einer Allgemeinen Mannichfaltigkeitslehre[4]

---

[2]Vol. xlvi (1895), 481-512.

[3]Published in Mathematische Annalen, xlix (1897), 207-246.

[4]Translated as "The Foundations of a General Theory of Sets".

(Leipzig, 1883) which consists of an elaborated account of the fifth part of "Ueber Unendliche Lineare Punktmannichfaltigkeiten"[5] published in Mathematische Annalen, xxi (1883), 545-591. This has been translated into French and appears in Acta Mathematica, ii (1883), 381-408.

The second source is a letter from Cantor to F. Goldscheider, dated "Halle, June, 1886". This letter has been translated by Herbert Meschkowski and appears in Chapter IX of his book entitled Ways of Thought of Great Mathematicians (San Francisco, 1964).

The third source is a letter from Cantor to Dedekind, dated "Halle, June, 1899". This letter was first published by Zermelo in Gesammelte Abhandlungen (Berlin, 1932). It has been translated by Stefan Bauer-Mengelberg and Jean van Heijenoort, and can be found in From Frege to Gödel: A Source Book in Mathematical Logic 1879-1931, ed. Jean van Heijenoort (Cambridge, Massachusetts, 1967), 113-117.

In conclusion, let me restate the aim of this thesis: it is to examine the foundations of transfinite cardinal arith-

---

[5]Translated as "On Infinite Linear Sets of Points".

metic. In making this examination I hope to accomplish four things: (1) Firstly, there is the (partly) historical task of making clear some of the presuppositions in Cantor's work. (2) Secondly, there is the (albeit elementary) mathematical task of drawing conclusions about the system oftransfinite cardinal arithmetic unrecognized by Cantor himself. (3) Thirdly, there is the logical task of making a formal axiomatization of this particular branch of mathematics, this last being undertaken in accordance with the "Hilbert programme". (4) Fourthly, there is the philosophical task of considering whether transfinite cardinal numbers exist according to Cantor's theory of the existence of mathematical entities.

Chapter II: THE SYNTAX OF THE THEORY OF ALEPHS[1]

In this chapter I shall present a formalization of the theory of alephs, thus obtaining an uninterpreted calculus or logistic system which I shall call A_o. There are four principal stages in this procedure:

(1) The specification of the primitive symbols of A_o.

(2) The definitions of "well-formed terms" and "well-defined formulae" of A_0 by certain formation rules.

(3) The specification of the axioms of A_o. These axioms are divided into two categories, i.e. "logical" and "non-logical" axioms.

(4) The specification of the rules of inference for A_o.

A. Primitive Symbols

(i) Individual Variables

Small roman letters from the end of the alphabet, i.e. x, y, z, and also any of these followed by one or more occurrences of the symbol $|$, are used for individual variables. Thus x, $x_{|}$, $x_{||}$, $x_{|||}$, y, $y_{|}$, $y_{||}$, $y_{|||}$, etc. are used for individual variables.

---

[1]Throughout this thesis I shall follow Cantor's procedure in using the terms "transfinite cardinal number" and "aleph" interchangeably.

(ii) Individual Constants

There is one individual constant, i.e. 0.

(iii) Functional Constants

There is one unary and two binary functional constants. The unary functional constant is ′ , and the two binary functional constants are + and · .

(iv) Predicate Constants

There are two binary predicate constants, i.e. < and = .

B. Improper Symbols

(i) Logical Constants

There are two logical constants, i.e. ∼ and the connective → .

(ii) Operators

There is one operator, i.e. the universal quantifier ∀ .

(iii) Auxiliary Symbols

These consist of parentheses ( ) .

(2) Formation Rules

Note on Metamathematical Symbols

(i) Small roman letters from the beginning of the alphabet, i.e. a, b, c, are used as syntactical variables whose range is the individual variables and individual constant.

(ii) Capital roman letters from the beginning of the alphabet, i.e. A, B, C, are used as syntactical variables whose range is the well-formed formulae.

(iii) The syntactical variables E a, F a, G a, etc. are used, whose range is the well-formed formulae containing one or more occurrences of individual variables, and such that, for any individual variable, the well-formed formulae contain at least one free occurrence of the variable.

(A) <u>Definition of "Well-formed Term"</u>

1. 0 is a term.

2. The individual variables x, y, z are terms.

3. If a is a term, so is a′.

4 & 5. If a and b are terms, so are (a + b) and (a . b).

6. Only those expressions defined by 1-5 are terms.

(B) Definition of "Well-formed Formula"

1. If a and b are terms, then $a = b$ is a wff[2].

2. If a and b are terms, then $a < b$ is a wff.

3.& 4. If A and B are wffs, then $\sim A$ and $(A \rightarrow B)$ are wffs.

5. If A is a wff and a is an individual variable, then $\forall a\, A$ is a wff.

6. The only wffs are those given by 1-5.

Note on Definitions

The list of definitions is divided into two categories: the first category deals with definitions of logical symbols; the second category deals with definitions of non-logical symbols.

Category A:

(i) $(A \vee B) =_{df} (\sim A \rightarrow B)$

(ii) $(A \wedge B) =_{df} \sim (\sim A \vee \sim B)$

(iii) $(A \leftrightarrow B) =_{df} ((A \rightarrow B) \wedge (B \rightarrow A))$

---

[2]Hereafter we shall emvloy "wff" as an abbreviation for a well-formed formula.

(iv) $a \neq b =_{df} \sim a = b$

(v) $\exists a\, F a =_{df} \sim \forall a \sim F a$

Category B

(i) $a \nless b =_{df} \sim a < b$

(ii) $a > b =_{df} b < a$

(iii) $a \leq b =_{df} (a < b \vee a = b)$

(iv) $a \geq b =_{df} (a > b \vee a = b)$

(3) Axioms

The axioms of A_o are divided into two categories:

(A) Logical Axioms (i.e. axioms 1-7).

(B) Non-logical Axioms (i.e. axioms 8-21).

(A) Logical Axioms

The underlying logic of A_o is the predicate calculus of the first order, with identity. We have the following logical axiom schemata of A_o:

1. $((A \vee A) \rightarrow A)$

2. $(A \rightarrow (A \vee B))$

3. $((A \vee B) \rightarrow (B \vee A))$

4. $((A \rightarrow B) \rightarrow ((C \vee A) \rightarrow (C \vee B)))$

5. $(\forall a\, F a \rightarrow F b)$

6. $a = a$

7. $(a = b \rightarrow (F a \rightarrow F b)$

Axiom schemata 1-4 are axiom schemata for the propositional calculus. Axiom schema 5 is an additional axiom schema for the predicate calculus of first order[3]. Axiom schemata 6 and 7 are axiom schemata for identity.

---

[3]Axioms 1-5 for the first order predicate calculus are taken from Principles of Mathematical Logic, p.67, by D. Hilbert and W. Ackermann (New York, 1950).

(B) Non-logical Axioms

8. $(x' = y' \rightarrow x = y)$

9. $x' \neq 0$

10. $\forall a (((F a \rightarrow F a') \wedge \forall a (\sim \exists b\, a = b' \rightarrow F a)) \rightarrow F a)$[4]

11. $(x \neq y \rightarrow (x < y \vee y < x))$

12. $(x < y \rightarrow y \nless x)$

13. $((x < y \wedge y < z) \rightarrow x < z)$

14. $(a = b' \longleftrightarrow \forall x (x \leqslant b \vee a \leqslant x))$

15. $(x + (y + z)) = ((x + y) + z)$[5]

16. $(x + y) = (y + x)$

---

[4] 10 is an axiom schema rather than an axiom.

[5] Axioms 15 and 18 are not independent of the other axioms.

17. $\forall x \forall y (x \geq y \rightarrow (x + y) = x)$

18. $(x . (y . z)) = ((x . y) . z)$[5]

19. $(x . y) = (y . x)$

20. $\forall x \forall y (x \geq y \rightarrow (x . y) = x)$

21. $(x . (y + z)) = ((x . y) + (x . z))$

22. $\forall x \exists y ((y > x \wedge \sim \exists z\, y = z') \wedge \sim \exists z (z < y \wedge (x < z \wedge \sim \exists w\, z = w'))$

(4) Rules of Inference

There are two rules of inference for A_o. The first rule belongs specifically to the propositional calculus. The second rule belongs to the predicate calculus.

(i) The Rule of Modus Ponens

If A and B are wffs, given A and (A→B), we may infer B.

$$\frac{A, (A \rightarrow B)}{B}$$

(ii) The Rule of Consequent Universalization

If $(A \rightarrow B\ a)$ is any wff such that the variable a occurs free in B a but does not occur at all in A, and if the variable b is either a itself or else some variable different from a that does not occur free in A or at all in B a, we may pass to the formula $(A \rightarrow \forall b\ B\ b)$.

$$\frac{(A \rightarrow B\ a)}{(A \rightarrow \forall b\ B\ b)}$$

Chapter III: THE SEMANTICS OF THE THEORY OF ALEPHS

In this chapter I shall provide an "interpretation" or "model" of the system A_o, thereby turning it into an interpreted calculus. This will be called the "principal interpretation" of the system A_o. The interpretation of the non-logical primitive symbols and axioms will be based (as I have explained in the introduction) on Cantor's theory of alephs. I shall divide the interpretation into two parts, concerning (1) Primitive Symbols and (2) Axioms.

(1) The Interpretation of the Primitive Symbols of A_o

A. Proper Symbols

(i) Individual Variables

The range of the individual variables x, y, z, etc., is the set of alephs or transfinite cardinal numbers. These are denoted by the first letter of the Hebrew alphabet, together with an ordinal integral subscript. The subscript may be either a finite ordinal, e.g. $\aleph_o, \aleph_1, \aleph_2, \aleph_3, \ldots, \aleph_n$, or a transfinite ordinal, e.g. $\aleph_\omega, \aleph_{\omega+1}, \aleph_{\omega+2}, \ldots, \aleph_{2\omega}, \ldots$, etc.

(ii) Individual Constants

The range of the one individual constant 0 is the first member of the sequence of alephs, i.e. $\aleph_0$.

The interpretations (i) and (ii) may be found in (Cantor, 1895, section 1, pp.85-86, and section 6, pp.103-110)[1].

Four definitions are involved here, i.e. the definitions of "cardinal number", of "set", of "aleph-zero" (or the first transfinite cardinal number), and of the succeeding "alephs".

Cantor defines cardinal number[2] or power (Mächtigkeit) in terms of sets and acts of "double abstraction" by the mind. Any set M possesses both a particular nature and a particular order. If we disregard its nature, we arrive at its ordinal type, $\overline{M}$; if we disregard both its nature and its order, we arrive at its cardinality or power, $\overline{\overline{M}}$. Thus Cantor says (Cantor, 1895,

---

[1] Regarding citations of primary source material from Cantor's work, I shall use the following convention:- I shall cite Cantor's name, followed by the date of the work in question, followed by the relevant section and page numbers. The appropriate edition may then be located in the Bibliography, A(1) where the date of the work will be written in square brackets.

[2] This definition applies both to finite and transfinite cardinal numbers (Cantor, 1895, section 5, pp.97-8).

section 1, p.86):

> Every aggregate M has a definite "power", which we will call its "cardinal number". We will call by the name "power" or "cardinal number" of M the general concept which, by means of our active faculty of thought, arises from the aggregate M when we make abstraction of the nature of its various elements m and of the order in which they are given. We denote the result of this double act of abstraction, the cardinal number or power of M by $\overline{\overline{M}}$.

Cantor's definition of set or aggregate runs as follows (Cantor, 1895, section 1, p.85):

> By an "aggregate" (Menge) we are to understand any collection into a whole (Zusammenfassung zu einem Ganzen) M of definite and separate objects m of our intuition or our thought.

Cantor defines aleph-zero as the cardinal number of the totality of finite cardinal numbers (Cantor, 1895, section 6, pp.103-4):

> The first example of a transfinite aggregate is given by the totality of finite cardinal numbers ; we call its cardinal number (1) "aleph-zero" and denote it by $\aleph_0$; thus we define
>
> $$(1)\ \aleph_0 = \overline{\overline{\{\nu\}}}\ .$$

A more significant definition of $\aleph_0$ is in terms of what Cantor calls "number classes". It is more significant in that Cantor defines succeeding alephs in this way. The notion of "number class" rests upon that of ordinal numbers. Thus Cantor says (Cantor, 1897, section 14, p.159):

> ... to one and the same transfinite cardinal number belong an infinity of ordinal numbers which form a unitary and connected system. We will call this system the "number-class Z ($\mathfrak{a}$)".

In the same paragraph Cantor goes on to say:

> For in this connexion we understand by "the first number-class" the totality $\{\nu\}$ of finite ordinal numbers.

Aleph-zero may then be defined as the cardinality of the first number class which is composed of the totality of finite ordinal numbers. Cantor then goes on to define the second number-class. Thus he says (ibid., section 15, p.160):

> The second number-class Z ($\aleph_0$) is the totality $\{\alpha\}$ of ordinal types[3] α of well-ordered aggregates of the cardinal number $\aleph_0$.

---

[3]Cantor defines an "ordinal type" as follows (Cantor, 1895, section 7, pp.111-112):

> Every ordered aggregate M has a definite "ordinal type", or more shortly a definite "type", which we will denote by
>
> (2) $\overline{M}$.

By this we understand the general concept which results from M if we only abstract from the nature of the elements m, and retain the order of precedence among them. We may add that this definition differs from that of an ordinal number in that it is stipulated that the latter be types of "well-ordered aggregates"

In (ibid., section 16, p.173) Cantor proves as a theorem that

> The power of the second number-class {a} is the second greatest transfinite cardinal number Aleph-one.

and adds:

> In the second number-class Z ($\aleph_0$) we possess, consequently, the natural representative for the second greatest transfinite cardinal number Aleph-one.

(ibid., p.115). Every ordinal number is an ordinal type but not vice versa.

Cantor defines a simply ordered aggregate as follows (ibid., p.110):

> We call an aggregate M "simply ordered" if a definite "order of precedence" rules over its elements m, so that, of every 2 elements m_1 and m_2, one takes the "lower" and the other the "higher" rank.

He defines a well-ordered aggregate as follows (Cantor, 1897, section 12, p.137):

> We call a simply ordered aggregate F(§ 7) "well-ordered" if its elements f ascend in a definite succession from a lowest f_1 in such a way that:
> 1. There is in F an element f_1 which is lowest in rank.
> 2. If F' is any part of F and if F has one or many elements of higher rank than all elements of F', then there is an element f' of F which follows immediately after the totality F', so that no elements in rank between f' and F' occur in F.

In the letter of 1899, Cantor generalizes this to show that any aleph $\aleph_\nu$ may be defined as the cardinality of the number-class Z $(\aleph_{\nu-1})$. Furthermore, he says:

> We see that this process of formation of the alephs and of the number classes of the system Ω that correspond to them is absolutely limitless.

Thus, the sequence of number classes Z $(\aleph_0)$, Z $(\aleph_1)$, Z $(\aleph_2)$, ..., Z $(\aleph_\nu)$ (or, as Cantor denotes them in 1899: Ω_0,[4] Ω_1, Ω_2, ..., Ω_ν) may be regarded as "similar" to the sequence of alephs $\aleph_0$, $\aleph_1$, $\aleph_2$, ..., $\aleph_\nu$. Hence, we may say that there exists a one-to-one correspondence between alephs and number-classes.

---

[4] In 1899 Cantor used the symbol Ω to denote the system of all ordinal numbers (both finite and transfinite). He appended the subscripts Ω_0, Ω_1, Ω_2, etc. to denote the successive transfinite "segments"* of the system Ω which he called "number classes".

*In (Cantor, 1897, section 13, p.141) Cantor writes:

> If f is any element of the well-ordered aggregate F which is different from the initial element f_1, then we will call the aggregate A of all elements of F which precede f a "segment (Abschnitt) of F".

However, as Cantor says (Cantor, 1895, section 6, p.109):

> But even the unlimited sequence of cardinal numbers
>
> $\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\nu, \ldots$
>
> does not exhaust the conception of transfinite cardinal number. We will prove the existence of a cardinal number which we denote by $\aleph_\omega$ and which shows itself to be the next greater to all the numbers $\aleph_\nu$; out of it proceeds in the same way as $\aleph_1$ out of $\aleph_0$ a next greater $\aleph_{\omega+1}$, and so on, without end.

Unfortunately, we have no direct primary source material concerning the way in which Cantor defines alephs whose subscripts are transfinite ordinals. However, I think that a definition of $\aleph_\omega$ (and alephs with even greater ordinal subscripts) may be derived from suggestions made by Cantor. In 1883 he expounded two "Principles of Formation" by which one could construct the system of ordinals. He permitted these two principles to be extended to the system of cardinal numbers (Cantor, 1895, section 6, p.109).

The second principle describes a process of defining a "limit" number. In this case, the "limit" number is $\aleph_0$ and is defined as the limit of the totality of finite numbers (Cantor, 1883, section 11):

> Étant donné une succession quelconque déterminée de nombres entiers réels définis, parmi lesquels il n'y en a pas qui sont plus grands que tous les autres, on pose, en s'appuyant sur ce deuxième principe de formation, un nouveau nombre que l'on regarde comme la limite des premiers, c.à.d.

> qui est défini comme étant immédiatement
> supérieur à tous ces nombres.

Analogously, we may define $\aleph_\omega$ as the limit of the alephs $\aleph_0$, $\aleph_1$, $\aleph_2$, ..., $\aleph_\nu$, ... We may define $\aleph_{2\omega}$ as the limit of the alephs $\aleph_{\omega+1}$, $\aleph_{\omega+2}$, ..., $\aleph_{\omega+\nu}$, and so on.

(iii) <u>Functional Constants</u>

(α) The unary functional constant ' ' ' is defined as the "successor of". A detailed interpretation of the successor function has been made on pp.16-21. There we defined the successor of each aleph as the cardinality of its corresponding number class. Thus, in general, the successor of $\aleph_\nu$ is $\overline{\overline{Z(\aleph_\nu)}}$.

(β) The binary functional constant ' + ' designates the operation of addition.

Cantor defines the operation of addition in terms of the union of disjoint sets. He defines the union of sets as follows (Cantor, 1895, section 1, pp.85-6):

> We denote the uniting of many aggregates M, N, P ..., which have no common elements, into a single aggregate by
> (2) (M, N, P, ...).
> The elements of this aggregate are, therefore, the elements of M, of N, of P, ..., taken together.

He then goes on (Cantor, 1895, section 3, p.91):

> The union of two aggregates M and N which have no common elements was denoted in § 1, (2), by (M, N). We call it the "union-aggregate (Vereinigungsmenge) of M and N."

> If M' and N' are two other aggregates without common elements, and if $M \sim M'$ and $N \sim N'$, we saw that we have
> $(M, N) \sim (M', N')$.
> Hence the cardinal number of (M, N) only depends upon the cardinal numbers $M = \mathfrak{a}$ and $N = \mathfrak{b}$.
> This leads to the definition of the sum of $\mathfrak{a}$ and $\mathfrak{b}$. We put
> (1) $\mathfrak{a} + \mathfrak{b} = \overline{\overline{(M, N)}}$.

(γ) The binary functional constant ' . ' designates the operation of multiplication.

Cantor defines the operation of multiplication in terms of the "cartesian product" of two sets. He defines the cartesian product, or as he calls it, the "aggregate of bindings of two sets", as follows (Cantor, 1895, section 3, p.92):

> Any element m of an aggregate M can be thought to be bound up with any element n of another aggregate N so as to form a new element (m, n); we denote by (M. N) the aggregate of all these bindings (m, n), and call it the "aggregate of bindings (Verbindungsmenge) of M and N."
> Thus
> (4) $(M . N) = \{(m, n)\}$.

He then goes on (Cantor, 1895, section 3, p. 92):

> We see that the power of $(M . N)$ only depends on the powers $\overline{\overline{M}} = \mathfrak{a}$ and $\overline{\overline{N}} = \mathfrak{b}$; for, if we replace the aggregates M and N by the aggregates
> $M' = \{m'\}$ and $N' = \{n'\}$
> respectively equivalent to them, and consider m, m' and n, n' as corresponding elements, then the

aggregate $(M' . N') = \{(m', n')\}$
is brought into a reciprocal and univocal correspondence with $(M . N)$ by regarding (m, n) and (m', n') as corresponding elements. Thus

(5) $(M' . N') \sim (M . N)$.

We now define the product . by the equation

(6) $\mathfrak{a} \cdot \mathfrak{b} = \overline{\overline{(M . N)}}$.

(iv) Predicate Constants

(α) The binary predicate constant '$<$' designates the relation "less than". In other words, it designates the set of well-ordered pairs in which the first element is less than the second element. Consider a wff of the form $a < b$ where a and b are syntactical variables whose range is the individual variables and individual constant of the system A_0. Two cases are presented:

(1) If $a < b$ contains no occurrences of individual variables, then $a < b$ is true if the aleph assigned by the interpretation of a is less than the aleph assigned to b, and otherwise false.

(2) If $a < b$ contains occurrences of variables, then $a < b$ is satisfied by those assignments Γ of values to the variables for which, the aleph assigned to a_Γ is less than the aleph b_Γ, and $a < b$ is not satisfied if those assignments Δ of value a_Δ are not less than the assignments Δ of value b_Δ. In this case $a < b$ is true if it is satisfied by all assignments of values to the variables, and false if it is satisfied by no assignments of values to the variables.

Cantor defines the relation '$<$' between cardinal numbers (both finite and transfinite) in terms of sets. If we define the cardinal numbers a and b as follows:

$$a =_{df} \overline{\overline{M}} \quad \text{and} \quad b =_{df} \overline{\overline{N}},$$

then $a < b$ if M is a proper subset of N.

Thus Cantor says (Cantor, 1895, section 2, pp.89-91):

> If for two aggregates M and N with the cardinal numbers $\mathfrak{a} = \overline{\overline{M}}$ and $\mathfrak{b} = \overline{\overline{N}}$, both the conditions:
> (a) There is no part of M which is equivalent to N,
> (b) There is a part N_1 of N, such that $N_1 \sim M$,
> are fulfilled, it is obvious that these conditions still hold if in them M and N are replaced by two equivalent aggregates M′ and N′. Thus they express a definite relation of the cardinal numbers $\mathfrak{a}$ and $\mathfrak{b}$ to one another...
>
> We express the relation of $\mathfrak{a}$ to $\mathfrak{b}$ characterized by (a) and (b) by saying: $\mathfrak{a}$ is "less"" than $\mathfrak{b}$ or $\mathfrak{b}$ is "greater" than $\mathfrak{a}$; in signs
> (1) $\mathfrak{a} < \mathfrak{b}$ or $\mathfrak{b} > \mathfrak{a}$.

(β) The binary predicate constant ' = ' designates the relation "is identical with". In other words, it designates the set of all ordered pairs in which the first element is the same as the second element. Consider a wff of the form a = b where a and b are syntactical variables whose range is the individual variables and individual constant of the system A_o. Two cases are presented:

(1) If $a = b$ contains no occurrences of individual variables, then $a = b$ is true if the aleph assigned by the interpretation of a is the same as the aleph assigned to b, and otherwise false.

(2) If $a = b$ contains occurrences of variables, then $a = b$ is satisfied by those assignments Γ of values to the variables for which the aleph assigned to a_{Γ} is the same as the aleph b_{Γ}, and $a = b$ is not satisfied if those assignments Δ of value a_{Δ} are not the same as those assignments Δ of value b_{Δ}. In this case $a = b$ is true if it is satisfied by all assignments of values to the variables, and false if it is satisfied by no assignments of values to the variables.

For Cantor, it is a theorem[5] that identity of "equality" between cardinal numbers (both finite and transfinite) holds if the sets (of which they are the power) are equivalent (aequivalent).

If we define the cardinal numbers a and b as follows:

$$a =_{df} \overline{\overline{M}} \quad \text{and} \quad b =_{df} \overline{\overline{N}},$$

then $a = b$ if $M \sim N$. Two sets are equivalent if their elements can be put into a one-to-one correspondence.

---

[5]It should be noted that what occurs as a theorem for Cantor occurs as part of the semantics of the principal interpretation of A_o.

Thus Cantor says (Cantor, 1895, section 1, pp.87-8):

> Of fundamental importance is the theorem that two aggregates M and N have the same cardinal number if, and only if they are equivalent: thus,
>
> (7) from $M \sim N$ we get $\bar{\bar{M}} = \bar{\bar{N}}$,
>
> and
>
> (8) from $\bar{\bar{M}} = \bar{\bar{N}}$ we get $M \sim N$.
>
> Thus the equivalence of aggregates forms the necessary and sufficient condition for the equality of their cardinal numbers.

B. Improper Symbols

(i) Logical Constants

(α) The logical constant '$\sim$' is to be interpreted as the negation sign, and is to be read as "not". If A is a wff with no free occurrences of variables, then $\sim$ A is false if A is true and $\sim$ A is true if A is false. If A is a wff with free occurrences of variables, then $\sim$ A is not satisfied by those assignments of values to the free variables which satisfy A and is satisfied by those assignments of values to the free variables which do not satisfy A.

(β) The connective '$\rightarrow$' is to be interpreted as the conditional sign and is to be read as "If... then...". If A and B are wffs with no free occurrences of variables, then $(A \rightarrow B)$ is true whenever A is false or B is true. If A and B are wffs with free occurrences of variables, then $(A \rightarrow B)$ is satisfied by those assignments of values to the free variables which do not satisfy

A or which do satisfy B ; (A → B) is not satisfied by any other assignment of values to the free variable.

(ii) Operators

The operator '∀' is to be interpreted as the universal quantifier and is to be read as "For all...". Let a be an individual variable and A a wff. For a given system of values of the free variables of ∀aA, the value of ∀aA is true if A is satisfied by every assignment of values to a, and ∀aA is false if A is not satisfied by at least one assignment of values to a.

(2) The Interpretation of the Axioms of A_o

To circumvent unnecessary repetition, I shall only provide an interpretation of the non-logical axioms of A_o.

Axiom 8

"If two alephs have the same successor then the two alephs are identical."

Together with the successor function Axiom 8 assures that the sequence of alephs is infinite. Thus by application of the successor function, for every aleph there is a next greater aleph, and by Axiom 8, this aleph cannot be one of those already defined (for if it were, two alephs might have the same successor). Thus Cantor says (Cantor, 1899):

> We see that the process of formation of alephs... is absolutely limitless.

Again we find (Cantor, 1899):

> The system of all alephs, when ordered according to magnitude, forms a sequence that is similar to the system Ω (i.e. the system of all ordinals) and therefore inconsistent[3], or absolutely infinite.

Axiom 9

"Aleph zero is not the successor of any aleph."

Cantor proves as a theorem that aleph-zero is the least transfinite cardinal number. Thus he says (Cantor, 1895, section 6, p.104):

> ... $\aleph_0$ is the least transfinite cardinal number. If α is any transfinite cardinal number different from $\aleph_0$, then
>
> (4) $\aleph_0 < \alpha$ ..

---

[3]N.B., By an inconsistent set, Cantor means (1899)

> a multiplicity such that the assumption that all of its elements 'are together' leads to a contradiction.

Axiom 10 -

"Any property which belongs to the successor of every aleph which has that property and to every aleph which has no immediate predecessors, belongs to all alephs."

One could argue that Cantor defines $\aleph_0$ as a "non-inductive" number. Thus he says (Cantor, 1895, section 5, p.99):

> Every cardinal number except 1 is the sum of the immediately preceding one and 1.

However, this does not apply to $\aleph_0$ since $\aleph_0 + 1 = \aleph_0$. Does this imply that the principle of transfinite induction has no significance? This I must contest. Again I shall cite the passage referred to earlier in connection with the definition of $\aleph_0$, $\aleph_1$, ..., $\aleph_\nu$, ..., etc. (Cantor, 1895, section 6, p.109):

> Out of $\aleph_0$ proceeds, by a definite law, the next greater cardinal number $\aleph_1$, out of this by the same law, the next greater $\aleph_2$, and so on. But even the unlimited sequence of cardinal numbers
>
> $$\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\nu, \ldots$$
>
> does not exhaust the conception of transfinite cardinal number. We will prove the existence of a cardinal number which we denote by $\aleph_\omega$ and which shows itself to be the next greater to all the numbers $\aleph_\nu$; out of it proceeds in the same way as $\aleph_1$ out of $\aleph_0$ a next greater $\aleph_{\omega+1}$, and so on, without end.

Now it is this "definite law" which, I contend, constitutes the basis for transfinite induction. As we have already seen (pp.16-19), the immediate successor of any aleph is defined not as $\aleph_{\nu} + 1$ but for any $\aleph_{\nu}$ its successor is $Z(\aleph_{\nu})$. Then, the question arises, do we have a suitable basis for transfinite induction with regard to alephs which have no immediate predecessors? Cantor does, however, have a method for generating such alephs or "limit" numbers, i.e. the transfinite analogue of the second principle of formation (referred to earlier). As $\aleph_0$ is posited as the "limit" of all the finite numbers, so, for instance, $\aleph_{\omega}$ is posited as the "limit" of all the alephs, $\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_{\nu}, \ldots$ Again, $\aleph_{2\omega}$ is posited as the "limit" of all the alephs $\aleph_{\omega+1}$, $\aleph_{\omega+2}, \ldots, \aleph_{\omega+\nu}, \ldots$

Axiom 11

"If a and b are any two distinct alephs, then either a is less than b or b is less than a."

This axiom is known as the law of trichotomy. For Cantor it was a theorem which he believed held for cardinal numbers in general (Cantor, 1895. section 2, p.90):

> If $\mathfrak{a}$ and $\mathfrak{b}$ are any two cardinal numbers, then either $\mathfrak{a} = \mathfrak{b}$ or $\mathfrak{a} > \mathfrak{b}$ or $\mathfrak{a} < \mathfrak{b}$.

Cantor was unable to prove the theorem that of these three relations between cardinal numbers, one must necessarily hold. However, he did prove the weaker theorem that at most one of the three relations must hold between any two cardinals.

Axiom 12

"If a and b are any two alephs and a is less than b, then b cannot be less than a."

Cantor makes a clear statement of the asymmetry of '$<$' when speaking of cardinals in general. He says (Cantor, 1895, section 2, pp.89-90):

> The relation of $\mathfrak{a}$ to $\mathfrak{b}$ is such that it makes impossible the same relation of $\mathfrak{b}$ to $\mathfrak{a}$.

Axiom 13

"If a, b and c are any three alephs, and if a is less than b and b is less than c, then a is less than c."

Cantor makes a clear statement of the transitivity of '$<$' when, speaking of cardinals in general, he says (ibid., p.90):

> If $\mathfrak{a} < \mathfrak{b}$ and $\mathfrak{b} < \mathfrak{c}$, then we also have $\mathfrak{a} < \mathfrak{c}$.

Axiom 14

"An aleph a is the successor of b if and only if every aleph c is either less than or equal to b, or greater than or equal to a."

This axiom, together with the use of the successor function as a primitive symbol, implies that the sequence of alephs is consecutive. Thus Cantor says (Cantor, 1899):

> $\aleph_0$ means the cardinality of the sets "denumerable" in the usual sense, $\aleph_1$ is the next greater cardinal number, $\aleph_2$ is the next greater still, and so on...
> $\aleph_1$ is not only distinct from $\aleph_0$, but it is the next greater aleph, for we can prove that there is no cardinal number between $\aleph_0$ and $\aleph_1$.

Axiom 15

"For any three alephs a, b, c the result of adding b and c to a is the same as the result of adding c to a and b."

Speaking of cardinal numbers in general, Cantor states the law of associativity for addition (Cantor, 1895, section 3, p.92):

> For any three cardinal numbers $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, we have
> $$(3)\quad \mathfrak{a} + (\mathfrak{b} + \mathfrak{c}) = (\mathfrak{a} + \mathfrak{b}) + \mathfrak{c}.$$

Axiom 16

"For any two alephs a and b, the result of adding b to a is the same as the result of adding a to b."

Speaking of cardinal numbers in general, Cantor states the law of commutativity (Cantor, 1895, section 3, p.92):

> Since in the conception of power, we abstract from the order of the elements, we conclude at once that
>
> (2) $\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}$.

Axiom 17

"For any two alephs a and b, if a is greater than or equal to b, then the sum of a and b equals a."

Thus Cantor gives the following example (Cantor, 1895, section 6, p.106):

$$\aleph_0 + \aleph_0 = \aleph_0.$$

Again, he says (Cantor, 1886):

> For finite cardinals it is easily seen that in the equation
>
> $a + a' = b$
>
> b is never equal to either of the summands a and a'. For actually infinite cardinals, however, it is easily proved that the last theorem does not hold. For example, if a is any actually infinite cardinal,
>
> $1 + a = a$
>
> $a + a = a \cdot 2 = a$

Axiom 18

"For any three alephs a, b, and c, the result of multiplying a by b and c is the same as the result of multiplying a and b by c."

Speaking of cardinal numbers in general, Cantor states the law of associativity for multiplication (Cantor, 1895, section 3,

p. 93):

$$(10)\quad \mathfrak{a} \cdot (\mathfrak{b} \cdot \mathfrak{c}) = (\mathfrak{a} \cdot \mathfrak{b}) \cdot \mathfrak{c}$$

It should be noted that Cantor regards this as a theorem.

Axiom 19

"For any two alephs, a and b, the result of multiplying a by b is the same as the result of multiplying b by a."

Speaking of cardinals in general, Cantor states the commutative law for multiplication (Cantor, 1895, section 3, p.93):

$$\mathfrak{a} \cdot \mathfrak{b} = \mathfrak{b} \cdot \mathfrak{a}$$

It should be noted that Cantor regards this as a theorem.

Axiom 20

"For any two alephs a and b, if a is greater than or equal to b, then the product of a and b equals a."

Thus Cantor gives the following example (Cantor, 1895, section 6, p.106):

$$\aleph_0 \cdot \aleph_0 = \aleph_0$$

Axiom 21

"For any three alephs a, b and c, the result of multiplying a by the sum of b and c is the same as the result of adding the product of a and b to the product of a and c."

Speaking of cardinals in general, Cantor states the law of distribution (Cantor, 1895, section 3, p.93):

$$(11) \quad \mathfrak{a}(\mathfrak{b}+\mathfrak{c}) = \mathfrak{a}\mathfrak{b}+\mathfrak{a}\mathfrak{c}$$

It should be noted that Cantor regards this as a theorem.

Axiom 22

"For any aleph x, there is a least aleph y which is greater than x but has no immediate predecessor."

This axiom provides for the existence of "limit" alephs, i.e. alephs which have no immediate predecessors. It is clear from the following quotation that Cantor maintained the existence of such alephs (Cantor, 1895, section 6, p.109):

> But even the unlimited sequence of cardinal numbers
>
> $\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\nu, \ldots$
>
> does not exhaust the conception of transfinite cardinal number. We will prove the existence of a cardinal number which we denote by $\aleph_\omega$ and which shows itself to be next greater to all the numbers $\aleph_\nu$.

It should be noted that, although this axiom allows us to generate such alephs as $\aleph_\omega$, $\aleph_{2\omega}$, $\aleph_{n\omega}$, it does not allow us to generate $\aleph_{\omega^2}$. For although this aleph is a "limit" aleph in the sense that it has no immediate predecessors, yet it is not the least aleph in a sequence which has this property. Similar remarks apply a fortiori to alephs of greater complexity such as

$$\aleph_{\omega^3}, \aleph_{\omega^n}, \aleph_{\omega^\omega}, \aleph_{\omega^{\omega^\omega}}, \text{etc.}$$

It is clear, from these remarks, that we cannot generate the entire set of alephs from the principal interpretation of A_0, but only a subset of them. Thus the principal interpretation is incomplete.

Chapter IV: SOME ARITHMETICAL PROPERTIES OF THE THEORY OF ALEPHS

In this chapter I shall draw some arithmetical conclusions from the "principal interpretation" of A_0 given in Chapter III. The chapter will be divided into two parts, giving (1) a discussion of the four elementary operations of addition, multiplication, subtraction and division with respect to the theory of alephs, and (2) an account of the laws of monotony for the addition of alephs.

(1)

(α) Addition

Axiom 17 expresses some of the most peculiar facts concerning the addition of alephs. First, every aleph may be characterized as a "neutral" element for the set of its successors under the operation of addition. To clarify this, I shall define a neutral element in general algebraic terms. Let G be any set and 0 a binary operation on G. Any element c of G which satisfies

$$(c \, 0 \, x) = (x \, 0 \, c) = x \quad \text{for all } x \text{ in } g$$

is called a neutral element for the operation 0[1]. Thus it is clear

---

[1] See J.A. Green, Sets and Groups, (London, 1971), p.40.

that any aleph a acts as a neutral element in the following situation:

$$(a + b) = b \quad \text{where} \quad b > a$$

For instance, $\aleph_0 + \aleph_1 = \aleph_1$; or again, $\aleph_0 + \aleph_0 = \aleph_0$.

Secondly, every aleph may be characterized as an "annihilator" element for the set of its predecessors under the operation of addition. Again, I shall define an "annihilator" element in general algebraic terms. Let G be any set and 0 a binary operation on G. Any element n of G which satisfies

$$(n \, 0 \, x) = (x \, 0 \, n) = n \quad \text{for all } x \text{ in } G$$

is called an annihilator element in the following situation:

$$(a + b) = a \quad \text{where} \quad a > b$$

For instance, $\aleph_1 + \aleph_0 = \aleph_1$; or again, $\aleph_1 + \aleph_1 = \aleph_1$.

Thirdly, it is clear that in the case where

$$(a + b) = a \quad \text{where} \quad a = b$$

that a may be described as either a neutral or an annihilator element. In other words, we may say that every aleph may also be characterized as an "<u>idempotent</u>" element for the operation 0. Again, I shall define an idempotent element in general algebraic terms. Let G be any set and 0 a binary operation on G. Any element d of the set G which satisfies

$$(d \, 0 \, d) = d$$

is called an "idempotent" element for the operation 0. Thus we see that for any aleph a,

$$(a + a) = a ;$$

e.g. $\aleph_2 + \aleph_2 = \aleph_2$.

It is interesting to note that the idempotency[2] of transfinite cardinal addition gives it an algebraic structure which is closer to the algebra of propositions and of sets than to that of ordinary finite cardinal addition. Thus in the algebra of propositions we see that conjunction is idempotent and, similarly, in the algebra of sets, we see that union is idempotent. Thus for any proposition p $(p \wedge p) = p$; for any set K, $K \cup K = K$. However, for any finite cardinal number n (except 0) it is not the case that $(n + n) = n$.

In conclusion, we may say that every aleph is a neutral element for the set of its successors under addition and every aleph is an annihilator element for the set of its predecessors under addition. Every aleph meanwhile is idempotent with respect to itself under addition. The only neutral element for the operation of addition on the entire set of alephs is $\aleph_0$. Moreover, since $\aleph_0$ has no predecessors, it is the only aleph which cannot act as an annihilator element with respect to a subset of the set of alephs. There is no annihilator element for addition on the set

---

[2]We speak of the "idempotency" of an operation in a derived sense: an operation is "idempotent" when each of its arguments is an idempotent element.

of alephs since there is no greatest aleph. Finally, it should be added that the operation of addition on the set of alephs is closed.

(β) Multiplication

Like Axiom 17, Axiom 20 expresses some peculiar facts concerning the arithmetic of alephs. In this case, the facts relate to multiplication. First, every aleph may be characterized as a "neutral" element for the set of its successors under the operation of multiplication. Any aleph a acts as a neutral element in the following situation:

$$(a . b) = b \quad \text{where} \quad b \geq a$$

For instance, $\aleph_0 . \aleph_1 = \aleph_1$; or again, $\aleph_0 . \aleph_0 = \aleph_0$

Secondly, every aleph may be characterized as an "annihilator" element for the set of its predecessors under the operation of multiplication. Any aleph a acts as an annihilator element in the following situation:

$$(a . b) = a \quad \text{where} \quad a \geq b$$

For instance, $\aleph_1 . \aleph_0 = \aleph_1$; or again, $\aleph_1 . \aleph_1 = \aleph_1$

It is clear that in the case where

$$(a . b) = a \quad \text{where} \quad a = b$$

that a may be characterized as an "idempotent" element, i.e.

$$(a . a) = a$$

For instance, $\aleph_2 . \aleph_2 = \aleph_2$

In conclusion, we may say that every aleph is a neutral element for the set of its successors under multiplication and every aleph is an annihilator element for the set of its predecessors under multiplication. Every aleph, meanwhile, is idempotent with respect to itself under multiplication. The only neutral element for the operation of multiplication on the entire set of alephs is $\aleph_0$. Moreover, since $\aleph_0$ has no predecessors, it is the only aleph which cannot act as an annihilator element with respect to a subset of the set of alephs. There is no annihilator element for multiplication on the set of alephs since there is no greatest aleph. Finally it should be added that the operation of multiplication on the set of alephs is closed.

N.B. A few further remarks must be made concerning the operations of addition and multiplication. These remarks will be further simplified in Chapter VI. Although, as we saw in Chapter III, these operations are semantically differentiated, they always produce the same results, e.g.

$$\aleph_3 + \aleph_2 = \aleph_3 \cdot \aleph_2 = \aleph_3$$

In other words, they are extensionally identical but intensionally different.

(γ) Subtraction

In his account of transfinite cardinal arithmetic, Cantor makes no mention of the operation of subtraction. For this reason, I have not made provision for it either in the primitive symbols, the definitions or the axioms of the principal interpretation of A_0. I shall, however, discuss the difficulty of introducing such an operation.

One could define subtraction as the "inverse" operation of addition, i.e.

$$(a - b) =_{df} \iota c (b + c) = a$$

However, it can be shown in general that there is more than one aleph c such that b + c = a, when, for instance, b = a. Put $a = b = \aleph_3$. Then $\aleph_3 + \aleph_0 = \aleph_3$, $\aleph_3 + \aleph_1 = \aleph_3$, $\aleph_3 + \aleph_2 = \aleph_3$.

It is also clear that equations of the type (a - b) = c where $b > a$ cannot be solved in the system of alephs. In order to solve such an equation as

$$\aleph_1 - \aleph_3 = c$$

an extension of the number concept is necessary. However, Cantor does not consider the possibility of introducing "negative alephs".

(δ) Division

As with the operation of subtraction, Cantor makes no mention of the operation of division in his account of transfinite cardinal arithmetic. Again, I shall discuss the difficulty of introducing such an operation. One could define division as the "inverse" operation of multiplication, i.e.

$$(a \div b) = \iota c (b \cdot c) = a$$

However, it can be shown in general that there is more than one aleph c such that $(b \cdot c) = a$, when, for instance, $b = a$. Put $a = b = \aleph_4$. Then $\aleph_4 \cdot \aleph_0 = \aleph_4$; $\aleph_4 \cdot \aleph_1 = \aleph_4$, $\aleph_4 \cdot \aleph_2 = \aleph_4$, $\aleph_4 \cdot \aleph_3 = \aleph_4$.

It is also clear that equations of the type $(a \div b) = c$ where $b > a$ cannot be solved within the system of alephs. In order to solve an equation such as $\aleph_1 \div \aleph_3 = c$, an extension of the number concept is necessary. However, Cantor does not consider the possibility of introducing "transfinite rationals".

(2)

In the second part of this chapter I shall give an account of the laws of monotony for transfinite addition, with respect to the two relations ' = ' and ' < '.

(i) The first law of monotony with respect to the relation ' = ' runs as follows:

"<u>For any alephs a, b and c, $(a = b \rightarrow (a + c) = (b + c))$</u>"

The first law of transfinite monotony is a theorem in the system of alephs. Here is a sketch of the required proof:

Assume $a = b$.

Then $c < a$ or $c = a$ or $c > a$ (Axiom 11).

If $c < a$, then $c < b$ (Axiom 7);

$(c + a) = a$ and $(c + b) = b$ (Axiom 17);

hence $(c + a) = (c + b)$.

If $c = a$, then $c = b$, $(c + a) = c$ and $(c + b) = c$;

hence $(a + c) = (b + c)$.

If $c > a$, then $c > b$, $(a + c) = c$ and $(b + c) = c$;

hence $(a + c) = (b + c)$.

Hence in any case $(a + c) = (b + c)$.

We see here that the law of monotony for ordinary finite cardinal arithmetic (i.e. for any integers a, b, c, $(a = b \rightarrow (a + c) = (b + c))$ holds absolutely.

(2) The second law of transfinite monotony with respect to the relation '<'[3] runs as follows:

"For any alephs a, b and c, such that $c < b$, $(a < b \rightarrow (a + c) < (b + c))$"

The second law of transfinite monotony is a theorem in the system of alephs. Here is a sketch of the required proof:

Assume $a < b$ and $c < b$.

Either $a \leq c$ or $c < a$ (Axiom 11).

If $a \leq c$, $(a + c) = c$ and hence $(a + c) < b$.

If $c < a$, $(a + c) = a$ and hence $(a + c) < b$.

---

[3]Since, in Chapter II, we defined $a > b$ as $b < a$, the following law also holds for the relation '>' (with suitable change of sign). Similarly, it holds (again with suitable change of sign) for the fourth law of transfinite monotony.

Since $c < b$, $(b + c) = b$.

Hence $(a + c) < (b + c)$.

We see here that the second law of monotony for ordinary finite cardinal addition (i.e. for any integers a, b and c, $(a < b \rightarrow (a + c) < (b + c))$ holds under certain restricted conditions, i.e. when $c < b$. It does not hold, however, where $c > b$, i.e. where c acts as an annihilator element for b. Thus, if $a = \aleph_0$, $b = \aleph_1$ and $c = \aleph_2$, then it is not the case that

$$\aleph_0 < \aleph_1 \rightarrow \aleph_0 + \aleph_2 < \aleph_1 + \aleph_2 .$$

Again, the law does not hold where $c = b$, i.e. where c acts as an idempotent element for b. Thus:

If $a = \aleph_0$, $b = \aleph_1$, and $c = \aleph_1$

then $a < b$ $(\aleph_0 < \aleph_1)$

but $(a + c) \nless (b + c)$;

$(a + c) = \aleph_0 + \aleph_1 = \aleph_1$

and $(b + c) = \aleph_1 + \aleph_1 = \aleph_1$,

so that in this case $(a + c) = (b + c)$.

(iii) The third law of transfinite monotony runs as follows:

"<u>For any alephs a, b and c, such that $c \leq a$ and $c \leq b$, $((a + c) = (b + c) \rightarrow a = b)$</u>"

This is a restricted version of the converse of Law 1. The third law of transfinite monotony is a theorem in the system of alephs.

Here is a sketch of the rquired proof:

Assume $c \leq a$, $c \leq b$, $(a + c) = (b + c)$, but $a \neq b$.

Then $a < b$ or $b < a$.

If $a < b$, (by the second law) $(a + c) < (b + c)$, contradicting our assumption.

If $b < a$, (by the second law) $(b + c) < (a + c)$, contradicting our assumption.

Hence $a = b$.

We see here that the third law of monotony for ordinary finite cardinal addition[4] (i.e. for any integers a, b and c, $((a + c) = (b + c) \rightarrow a = b))$ holds under certain restricted conditions, i.e. where $c \leq a$ and $c \leq b$. However, the third law does not hold where $c > a$ or $c > b$. For instance, let us examine two cases:

(1) If $c > a$, then it is not the case that

$$((a + c) = (b + c) \rightarrow a = b).$$

Thus, if $a = \mathcal{N}_0$, $b = \mathcal{N}_2$, and $c = \mathcal{N}_2$, then

$$\mathcal{N}_0 + \mathcal{N}_2 = \mathcal{N}_2 + \mathcal{N}_2 \nrightarrow \mathcal{N}_0 = \mathcal{N}_2.$$

(2) If $c > b$, then it is not the case that

$$((a + c) = (b + c) \rightarrow a = b).$$

Thus, if $a = \mathcal{N}_3$, $b = \mathcal{N}_2$, and $c = \mathcal{N}_3$, then

$$\mathcal{N}_3 + \mathcal{N}_3 = \mathcal{N}_2 + \mathcal{N}_3 \nrightarrow \mathcal{N}_3 = \mathcal{N}_2.$$

---

[4]Sometimes called the "law of cancellation".

It should be noted that the third law holds trivially (because the antecedent is false) in the cases where:

(1) $c > a, \quad b > c.$

(2) $c > b, \quad a > c.$

(iv) The fourth law of transfinite monotony runs as follows:

"For any alephs a, b and c $((a + c) < (b + c) \rightarrow a < b)$"

This is the converse of Law 2 without the restriction that $c < b$. The fourth law of transfinite monotony is a theorem in the system of alephs. Here is a sketch of the required proof:

Assume $(a + c) < (b + c)$ but $a \nless b$.

Then $a = b$ or $b < a$.

If $a = b$, then by the first law $(a = b \rightarrow (a + c) = (b + c))$, contradicting our assumption.

If $b < a$, then, by Axiom 11, either $c \leq a$ or $a < c$;

if $c \leq a$, then $(a + c) = a$ (Axiom 17)

so that $b < (a + c)$ (Axiom 7), $b < (b + c)$ (Axiom 13), $(b + c) = c$ (Axiom 17) and $(b + c) \leq (a + c)$ (Axiom 7), contradicting our assumption;

if $a < c$ then $b < c$ (Axiom 13), so that $(a + c) = (b + c) = c$ contradicting our assumption.

Hence $a < b$.

We see here that the fourth law of monotony for ordinary finite cardinal addition (i.e., for any integers a, b and c, $((a + c) < (b + c) \rightarrow a < b))$ holds absolutely.

Note on Multiplication

The four laws of transfinite monotony with respect to ' = ' and '<' (also, by definition, with respect to ' > '), apply absolutely to transfinite multiplication. This is connected with the fact cited in Chapter III that, regarding the set of alephs, the operations of addition and multiplication are extensionally identical. It is interesting to note that although the transfinite laws of monotony for multiplication correspond exactly to the transfinite laws of monotony for addition, this is not so in the case of ordinary finite cardinal arithmetic. Let us compare ordinary addition and multiplication. In the case of addition we find that if $a < b$, then $(a + c) < (b + c)$. In the case of multiplication we find that if $a < b$, then $(a \cdot c) < (b \cdot c)$ provided that $c \neq 0$. We see that 0 acts as an annihilator element in the case of ordinary multiplication, whereas it acts as a neutral element in the case of ordinary addition. There is no such asymmetry in the case of transfinite cardinal arithmetic due to the extensional identity of transfinite addition and multiplication. Hence transfinite cardinal addition and multiplication are similar in that one does not contain a restriction (concerning the laws of monotony) which is not found in the other. For this reason the full account of the laws of transfinite monotony for addition can be rendered true for multiplication simply by change of sign.

Chapter V: THE ALGEBRAIC STRUCTURE OF THE THEORY OF ALEPHS

In this chapter I shall discuss the set of alephs in terms of its algebraic structure. To begin with, the set of alephs, together with the operation of addition (both being listed amongst the primitive symbols of the principal interpretation of A_o), constitutes a "gruppoid". This is one of the most elementary and general algebraic structures. We may define a gruppoid as follows:

> A gruppoid is a pair (G, O) where G is a non-empty set and O a binary operation on the set[1].

Thus we have the set of alephs, which I shall call A, and a binary operation on this set, i.e. ' + ', making the pair (A, +). It may be added that the set of alephs is an "absolutely infinite" gruppoid[2].

---

[1] See J.A. Green, Sets and Groups, (London 1971), p.41.

[2] Thus Cantor says (Cantor, 1899):

> The system ת of all alephs is similar to the system Ω [i.e. the system of all ordinal numbers] and therefore likewise inconsistent, or absolutely infinite.

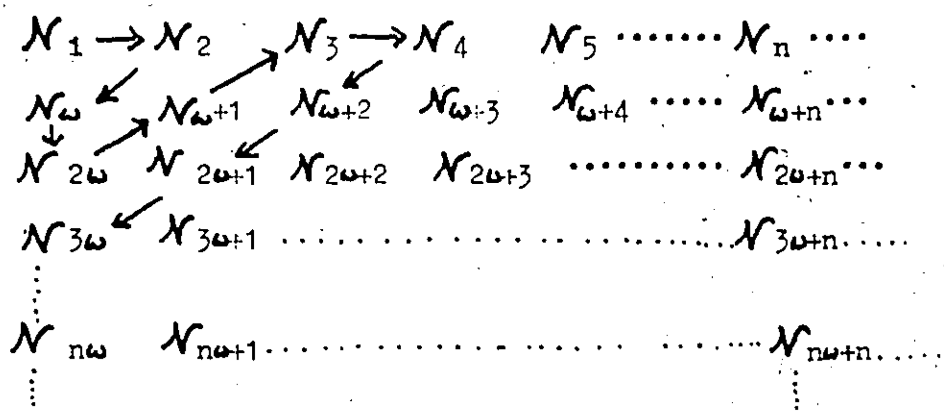
Axiom 15 characterizes the set of alephs as a "semi-group" or "monoid". This particular algebraic structure is somewhat less elementary and more specific than that described above. We

---

Cantor defines inconsistent or absolutely infinite sets as follows (Cantor, ibid.):

> For a multiplicity can be such that the assumption that all of its elements "are together" leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as "one finished thing". Such multiplicities I call absolutely infinite or inconsistent multiplicities.

It should be noted, however, that, as we saw in Chapter III, the principal interpretation of A_o only allows us to generate a subset of the set of alephs. This subset is, in fact, denumerably infinite and hence a "consistent multiplicity". It is denumerably infinite in the sense that it can be enumerated (like the rational numbers) by Cantor's "diagonal procedure":



It can thus be put into a one-to-one correspondence with the natural numbers and is thus denumerably infinite.

may define a "semi-group" as follows:

> A semi-group is a gruppoid (A, +) whose operation is associative.

Furthermore, Axiom 16 characterizes the set of alephs as an "abelian" semi-group insofar as it states the law of commutativity for addition.

The question may now be asked: Can the set of alephs be characterized as an infinite abelian "group" ? The answer is in the negative. This can be easily demonstrated by reference to the inverse operation of subtraction (discussed in Chapter IV).

To begin with, however, I shall first state the necessary properties of a group:

> A group is a semi-group (G, O) which satisfies the additional conditions:
>
> (i) G has a unique neutral element e, i.e.
>
> $$\forall x \quad \iota e \quad (x + e) = x$$
>
> (ii) Every element of G has a unique inverse element, i.e. for each x in G there is an element -x in G, called the inverse of x, such that
>
> $$\forall x \quad \iota -x \quad (x + (-x)) = e$$

The set of alephs does, in fact, satisfy condition (i) insofar as aleph zero constitutes a unique neutral element for the entire set of alephs.

The set of alephs, however, fails to satisfy condition (ii). It is not the case that for every aleph a there is an inverse aleph $-a$ such that $(a + (-a)) = e$ (where $e = \aleph_0$). As we saw in the discussion of the operation of subtraction, it is not significant to speak of "inverse" elements in this context. The set of alephs is comparable to the set of positive integers in that we cannot solve the following type of equation in either system:

$$(a - b) = c \quad \text{where } b > a$$

In the case of the positive integers we may solve this equation by introducing negative integers. However, no such extension of the set of alephs has been made: there are no "negative alephs" and hence no "inverse" alephs. Moreover, even if negative alephs were to be introduced so that we could solve equations of the form

$$(a - b) = c \quad \text{where } b > a$$

there would still remain the problem that $a - b$ has in general several values where $a = b$ (as was shown in the discussion of subtraction in Chapter IV).

I conclude, therefore, that the set of alephs fails to satisfy the conditions necessary to characterize it as a group.

A Note on "One-Element" Groups

Finally, it may be noted in connection with this discussion of groups that although we cannot characterize the set of alephs as a group, yet we may represent each aleph as a "one-element" group. Thus, taking any aleph a and the operation of addition we see that the five conditions of an abelian group are fulfilled:

(i) We have a pair (A, +) where A is a non-empty set (i.e. the singleton $\{a\}$ and ' + ' is a binary operation on A.

(ii) The operation of addition is associative, i.e.

$$((a + a) + a) = (a + (a + a)).$$

(iii) A has a unique neutral element a, i.e.

$$(a + a) = a.$$

This follows from Axiom 17.

(iv) Every element of A has an inverse, i.e. for every a there is an element -a in A called the inverse of a, such that

$$(a + (-a)) = a.$$

(v) The operation of addition is commutative, i.e.

$$(a + a) = (a + a)$$

I shall now consider the set of alephs, together with the binary operations of multiplication (both being listed amongst the primitive symbols of the principal interpretation of A_o). The pair (A, .) constitutes a "gruppoid". It must be noted that here we are describing a different gruppoid from that described at the beginning of this section. Although both algebraic structures are concerned with the same set, i.e. the set of alephs, they represent the set under distinct operations. Again, there is an absolutely infinite gruppoid.

Axiom 18 characterizes the set of alephs as a "semi-group" insofar as it states the law of associativity for multiplication. Furthermore, Axiom 19 characterizes the set of alephs as an abelian semi-group insofar as it states the law of commutativity.

The question may now be asked: Can the set of alephs be characterized as an infinite abelian "group" ? This question is obviously parallel to that concerning the set of alephs under the operation of addition. Again, the answer is in the negative.

First, it should be noted that, like the additive semi-group, the multiplicative semi-group fulfills the first condition of being a group, i.e. the set of alephs possesses a unique element e such that

$$\forall x \iota e (x . e) = x.$$

Once again, this element is $\aleph_0$. However, as in the case of the additive semi-group, the multiplicative semi-group fails to fulfill condition (ii), i.e. that for every aleph a there is an inverse aleph $\hat{a}$ such that

$$(a . \hat{a}) = \aleph_0$$

($\aleph_0$ being the unique neutral element for the set of alephs under multiplication), This is connected with the difficulty of defining an inverse operation, i.e. division, upon the set of alephs.

The set of alephs is comparable to the set of integers in that we cannot define the following type of equation in either system:

$$(a \div b) = c \quad \text{where } b > a.$$

In the case of the integers, we may solve this equation by introducing rational numbers. However, no such extension of the set of alephs has been made: there are no "transfinite rationals" and hence no "inverse" alephs. I conclude, therefore, that again the set of alephs fails to satisfy the conditions necessary to characterize it as a group.

A Note on Multiplicative "One-element" Groups

Because of the identity regarding the extensionality of the two binary operations ' + ' and ' . ', the remarks on additive one-element groups apply (with suitable change of sign) to multiplicative "one-element" groups.

In concluding, I shall contend that the set of alephs may be characterized as a commutative "semi-field". Some explanation is needed of this neologism. A "field", more especially, a commutative field, consists of a non-empty set F with two binary operations 'S' and 'P' defined on it. Moreover, it satisfies the following conditions:

(i) $(x\ S\ (y\ S\ z)) = ((x\ S\ y)\ S\ z)$

(ii) $(x\ S\ y) = (y\ S\ x)$

(iii) F has a unique neutral element e such that

$$\forall x \in e \ (x\ S\ e) = x$$

(iv) Every element x of F has an inverse, i.e. for each x of F there is an element -x called the inverse of x such that

$$((-x)\ S\ x) = e$$

(v) $(x\ P\ (y\ P\ z)) = ((x\ P\ y)\ P\ z)$

(vi) $(x\ P\ y) = (y\ P\ x)$

(vii) F has a unique neutral element f such that

$$\forall x \in f \ (x\ P\ f) = x$$

(viii) Every x of F has an inverse, i.e. for each x of F there is an element called the inverse of x such that

$$(x\ P\ \hat{x}) = f$$

(ix) $(xP(ySz)) = ((xPy)S(xPz))$.

The conditions for a commutative field consist of the conditions for two abelian groups together with condition (ix), connecting the two binary operations. By extension, we might say that the conditions for a commutative "semi-field" consist of the conditions for two abelian semi-groups, together with an axiom connecting the two operations (i.e. conditions (i)-(iii), (v)-(vii) and (ix)). As we have seen, the set of alephs may be characterized as two abelian semi-groups. If we add to this the distributive law (i.e. Axiom 21), we may characterize the set of alephs as a commutative semi-field.

Chapter VI: THE CONSISTENCY OF THE THEORY OF ALEPHS

In this chapter I shall consider the consistency of the logistic system A_o, and that of the principal interpretation. The notion of consistency may be defined thus:

> A deductive theory is called consistent or non-contradictory if no two asserted statements of this theory contradict each other, or, in other words, if of any two contradictory sentences at least one cannot be proved[1].

According to Hilbert[2], there are two kinds of consistency proofs:

(1) Relative Consistency Proofs

We may establish a one-to-one correspondence between a formalized theory and another theory which we believe to be consistent. The consistency of our first theory will then be demonstrated. We will say that if the second theory is consistent, so is the first: the first is consistent relative to the second.

---

[1] A. Tarski, Introduction to Logic (New York, 1965), p.135.

[2] Hilbert's account is to be found in vol.1 of the Grundlagen der Mathematik by D. Hilbert and P. Bernays (2 vols., Berlin, 1934 and 1939).

(2) Absolute Consistency Proofs

Such proofs attempt to demonstrate the consistency of a theory without assuming the consistency of a second theory.

In dealing with the consistency of the logistic system A_o and its principal interpretation, I shall attempt to show two things:

(A) That of the "Logical Axioms" (i.e. Axioms 1-7) which constitute an axiomatization of the predicate calculus of the first order, with identity, Axioms 1-5 can be shown to be consistent by an absolute consistency proof.

(B) That the "Logical Axioms" 6-7 and the "Non-Logical Axioms" (i.e. Axioms 8-21) can be shown to be consistent by a relative consistency proof.

(A)

In presenting an absolute consistency proof for Axioms 1-5, I shall follow Hilbert's procedure[3]. Thus in order to show that two formulae S and $\sim$S are not both derivable from a system Q

---

[3]For an exposition in English of Hilbert's proof, see G.T. Kneebone, Mathematical Logic and the Foundations of Mathematics (London, 1965), c.7, section 6; and E. Nagel & J. Newman, Gödel's Proof (New York, 1973), cs. 2 & 3.

we proceed as follows:

We first select some property of the formulae of Q which satisfy the following three conditions:

(i) The property must belong to all the axioms.

(ii) The property must be 'hereditary' under the rules of inference, and must therefore be inherited by every derived formula.

(iii) The property must not belong to every formula that can be constructed in Q, i.e. we must exhibit at least one formula that does not have the property. In other words, we must seek to discover some formula which is not a theorem; for if a system is not consistent, then any formula whatever can be derived from the axioms[4].

---

[4]This may be demonstrated using the theorem $(A \rightarrow (\sim A \rightarrow B))$. This theorem may be derived from Axiom 1 $(A \rightarrow (A \vee B))$. As we saw in Chapter II, the wff $(A \vee B)$ may be defined as $(\sim A \rightarrow B)$. Thus we may define $(A \rightarrow (\sim A \rightarrow B))$ as $(A \rightarrow (A \vee B))$. Thus, given the theorem $(A \rightarrow (\sim A \rightarrow B))$ and the rule of modus ponens, we may proceed as follows:

First, suppose that some formula S and its contradictory $\sim$ S were deducible from the axioms. By substituting S for A

Now let us apply the procedure to our absolute consistency proof of Axioms 1-5. For the sake of brevity, I shall refer to the system derived from (and including) these Axioms 1-5 as the system A_1. First we select a property of the formulae of A_1 which satisfies the above conditions (i)-(iii). The property chosen is that of being "tautologous". Now it can be shown by use of truth-tables that (i) Axioms 1-5 are tautologous, and (ii) that this property is hereditary in that it is inherited by every derived formula.

Let us first consider Axioms 1-4, i.e. those of the propositional calculus. These can all be shown by means of truth-

---

in the theorem cited above and applying modus ponens twice, the formula B is deducible. Thus we first obtain $(S \rightarrow (\sim S \rightarrow B))$. From this, together with S which is asserted to be demonstrable, we obtain by modus ponens $(\sim S \rightarrow B)$. Finally, since $\sim S$ is also asserted to be demonstrable, using modus ponens once more, we obtain B. But if the formula B is demonstrable, it follows that by substituting any formula whatsoever for B, any formula whatsoever is deducible from the axioms. Thus if some formula S and its contradictory $\sim S$ were deducible from the axioms, every formula would be deducible. In other words, if Q is inconsistent, then every formula is a theorem.

tables to be tautologous. The same proof is also applicable to Axiom 5 (i.e. the axiom of the predicate calculus of first order). Here we delete the individual variables and quantifiers and treat the predicate variables as if they were propositional variables; let us call the result of performing this operation on a formula the elementary propositional form associated with that formula. Thus Axiom 5 becomes $(A \rightarrow A)$, which can again be shown, by truth-tables to be tautologous[5]. Thus we see that, by means of truth-tables, we can show that the chosen property of being "tautologous"

---

[5]It should be noted that Axiom 5 is not strictly speaking tautologous, but its associated elementary propositional form is. This condition appears to be sufficient to prove the consistency of the system. For if the system is inconsistent there is at least one pair of contradictory propositions derivable in it, and hence the conjunction of that pair is derivable. But the conjunction of a pair of contradictory propositions is equivalent to a conjunction of the form $(P \wedge \sim P)$, whose associated elementary form is not tautologous.

Hence, if the associated elementary propositional form of every formula derivable in the system is tautologous, then the system is consistent.

belongs to all the axioms of A_1. Thus it satisfies condition (i). Again, using truth-tables, we can demonstrate that this property belongs to all the derived formulae of A_1 (or their associated elementary propositional form). Thus it satisfies condition (ii).

Finally, we can construct a formula of A_1 which does not have the property of being tautologous and is thus not a theorem of A_1. For instance, take the formula $A \vee B$ which is a formula but not a theorem of A_1 since it is not tautologous. This can, once again, be demonstrated by means of truth-tables. Thus the property of being tautologous satisfies condition (iii), i.e. it does not belong to every formula of A_1. Thus, in accordance with Hilbert's procedure, we have given an absolute consistency proof of A_1 and consequently of the Axioms 1-5.

(B)

I shall now give a relative consistency proof of the Logical Axioms 6 and 7 and the Axioms 8-22 of the logistic system A_0 and thereby of its principal interpretation. For the sake of brevity we will designate the system derived from these axioms (including the axioms themselves) as A_2. I shall now attempt to show that the system A_2 is consistent relative to another system which I shall call B. I shall establish a one-to-one correspondence between A_0, its principal interpretation (of which A_2 forms a part) and the system B. Let us first consider this correspondence with regard to the "primitive symbols" of each system:

Proper Symbols	A_0	A_2	B
Individual Variables	x, y, z, etc.	Set of alephs	$0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots$ $1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16} \ldots$ The set of rational dyads (i.e. a subset of the set of rationals): $\{x: (\exists w)(\exists y)(\exists z)$ (w, y and z are natural numbers $\wedge\ x = \frac{w}{2^y} \wedge z = \frac{w+1}{2^y})\}$
Individual Constant	0	$\aleph_0$	Zero
Functional Constants	′	Successor function,	Successor function, x' : $x + \frac{1}{2}$ (the least integer greater than x−x), i.e. $\left(\frac{w}{2^y}\right)' = \frac{2w+1}{2^{y+1}}$
	+	Sign of addition	the maximum of x and y: $+(x,y)$, $\cdot(x,y)$
	·	Sign of multiplication	
Predicate Constants	<	less than	less than
	=	sign of equality	sign of equality
Improper Symbols	~	not	not
	→	if...then...	if...then...
	∀	universal quantifier	universal quantifier
	( )	marks of punctuation	marks of punctuation

We can now interpret the axioms 6–22 of B as follows:

Axiom 6

"Every rational dyad is identical to itself."

Axiom 7

"If a rational dyad a is identical with a rational dyad b, then any property belonging to a, belongs to b."

Axiom 8

"No two rational dyads have the same successor."

Axiom 9

"0 is not the successor of any rational dyad."

Axiom 10

"Any property which belongs to the successor of every rational dyad which has that property and to every rational dyad which has no immediate predecessors, belongs to all rational dyads."

Axiom 11

"If m and n are any two distinct rational dyads then either m is less than n or n is less than m."

Axiom 12

"If m and n are any two rational dyads and m is less than n, then n cannot be less than m."

Axiom 13

"If m, n and p are any three rational dyads and if m is less than n and n is less than p, then m is less than p."

Axiom 14

"A rational dyad m is the successor of n if and only if every rational dyad p is either less than or equal to n, or greater than or equal to m."

Axiom 15

"For any three rational dyads m, n, p, the maximum of m and the maximum of n and p is the same as the maximum of the maximum of m and n and p."

$$\max (m, \max (n, p) = \max (\max(m, n), p)$$

Axiom 16

"For any two rational dyads m and n, the maximum of m and n is the same as the maximum of n and m."

$$\max (m, n) = \max (n, m)$$

Axiom 17

"For any rational dyads m and n, if m is greater than or equal to n, then the maximum of m and n is m."

$$(m \geq n \rightarrow \max (m, n) = m)$$

Axiom 18

"For any three rational dyads m, n and p, the maximum of m and the maximum of n and p is the same as the maximum of the maximum of m and n and p."

$$\max(m, \max(n, p)) = \max(\max(m, n), p)$$

Axiom 19

"For any two rational dyads m and n, the maximum of m and n is the same as the maximum of n and m."

$$\max(m, n) = \max(n, m)$$

Axiom 20

"For any two rational dyads m and n, if m is greater than or equal to n, then the maximum of m and n is m."

$$(m \geq n \rightarrow \max(m, n) = m)$$

Axiom 21

"For any three rational dyads m, n and p, the maximum of m and the maximum of n and p is the same as the maximum of the maximum of m and n and the maximum of m and p."

$$\max(m, \max(n, p)) = \max(\max(m, n), \max(m, p))$$

Axiom 22

"For any rational dyad m, there is a least rational dyad n which is greater than m but has no immediate predecessor."

If the system B is consistent, then it follows (due to the one-to-one correspondence between A_o, A_2 and B) that A_o and A_2 are consistent relative to B. Now, we may ask: Is B consistent? Unfortunately, this very question shows the inadequacy of the proof by relative consistency from one system to another. All we can say is that if the arithmetic of the particular subset of rationals, under the operation "maximum of" described in B, is consistent, then so are A_o and A_2.

Chapter VII: THE ONTOLOGICAL STATUS OF ALEPHS

In this final chapter I wish to discuss the ontological status accorded to alephs by Cantor. In the current literature it is now quite commonplace to view mathematical entities in general from one of three standpoints: these are the formalist, intuitionist and logicist positions[1]. For synoptic and clarificatory purposes, I shall begin by stating Quine's outline of the three positions. I shall then examine the Cantorian corpus with a view to establishing Cantor's general "philosophy of mathematics" and the particular ontological status of alephs.

It should first be noted that for Quine (and indeed for most writers on the philosophy of mathematics), the formalist-intuitionist-logicist controversy is essentially a recrudescence of the mediaeval controversy concerning universals, i.e. the nominalist-conceptualist-realist dispute. Thus the formalist position, whose leading proponent in this century has been David Hilbert, is associated with mediaeval nominalism. Quine characterizes formalism as follows:

---

[1]See inter alia W.v.O. Quine, From a Logical Point of View (New York, 1963); S. Körner, The Philosophy of Mathematics (New York, 1962); S. Barker, Philosophy of Mathematics (New Jersey, 1966); M. Black, The Nature of Mathematics (London, 1933).

> The formalist keeps classical mathematics as a play of insignificant notations. This play of notations can still be of utility - whatever utility it has already shown itself to have as a crutch for physicists and technologists[2].

Again, intuitionism, whose leading proponent has been L.E.J. Brouwer, is associated with mediaeval conceptualism. As mediaeval conceptualists hold "that there are universals but they are mind-made"[3], so intuitionists believe that mathematical entities are not reducible to notation, but exist in the mind. Finally, logicism, whose leading proponent has been Russell, is associated with mediaeval realism. According to Quine, realism

> is the Platonic doctrine that universals or abstract entities have being independently of the mind; the mind may discover but cannot create them[4].

Having outlined these three basic positions concerning the ontological status of mathematical entities, I now wish to consider which viewpoint was held by Cantor regarding alephs. However, before I begin this investigation, I must issue two caveats:

(i) Although the tripartite division outlined above is commonplace

---

[2] Quine, 1963, p.15.

[3] Ibid., p.14

[4] Ibid., p.14

in the current literature, it is overly simplistic. For instance, A.A. Fraenkel, Y. Bar-Hillel and A. Levy add a fourth division to what they term "Platonism", "neo-nominalism" and "neo-conceptualism", i.e. the "anti-ontological view". The leading proponent of this view (which appears not to have any mediaeval forbears) is Carnap[5]. According to this view, the question of whether mathematical entities exist or not is a "pseudo-question". Carnap argues that there are two types of "existence" questions regarding mathematical entities. First, there are "internal questions". We may ask whether some type of mathematical entity exists, but only within the framework of a certain theory which we have already accepted. Secondly, there are "external questions" regarding the framework as a whole. These questions are not properly ontological since they reduce to the "acceptability" of the framework or theory. Such "acceptability" in turn devolves on such pragmatic questions as the utility and fecundity of the theory and on aesthetic considerations such as simplicity. Thus, for Carnap, the question "Do alephs exist?" is a pseudo-question. We must construe it either as an internal question of some previously accepted theory or as an external

---

[5]Rudolf Carnap, "Empricism, Semantics, and Ontology" reprinted in Philosophy of Mathematics: Selected Readings, ed. Paul Benacerraf & Hilary Putnam (New Jersey, 1964).

question which involves pragmatic and aesthetic considerations.

(ii) Next, it should be stressed that Quine's equation of nominalist-formalist, conceptualist-intuitionist and realist-logicist is again overly simplistic. Indeed, it can be downright misleading. For although some logicists such as Frege are also realists in that they accept mathematical entities as mind-independent and which we discover rather than invent, other logicists such as Russell cannot so easily be classified as realists. Thus, although I must agree with Quine that in Principia Mathematica Russell "condones the use of bound variables to refer to abstract entities known and unknown, specifiable or unspecifiable, indiscriminately"[6], yet I see no need to draw any conclusions from these regarding Russell's ontological commitments. It is quite compatible with the logicist programme to hold a nominalist position: to view classical mathematics as a body of tautologies.

With these two caveats in mind, let us now approach Cantor. Let us begin with the question: Is there any evidence to suggest that Cantor was a formalist? The evidence is slight but deserves consideration. There are two passages within the Cantorian corpus which have a formalist flavour:

---

[6] Quine, 1963, p.14.

(a) We may regard the whole numbers as 'actual' in so far as they, on the ground of definitions, take a perfectly determined place in our understanding, are clearly distinguished from all other constituents of our thought, stand in definite relations to them, and thus modify, in a definite way, the substance of our mind[7].

(b) Mathematics is, in its development, quite free, and only subject to the self-evident condition that its conceptions are both free from contradiction in themselves and stand in fixed relations, arranged by definitions, to previously formed and tested conceptions. In particular, in the introduction of new numbers, it is only obligatory to give such definitions of them as will afford them such a definiteness, and, under certain circumstances, such a relation to the older numbers, as permits them to be distinguished from one another in given cases. As soon as a number satisfies all these conditions, it can and must be considered as existent and real in mathematics. In this I see the grounds on which we must regard the rational, irrational and complex numbers as just as existent as the positive integers[8].

---

[7] Trans. P.E.B. Jourdain in (Cantor, 1883(a), section 8, pp.545-591).

[8] Ibid., section 8 (Trans. P.E.B. Jourdain).

Both passages are formalistic in the sense that they appear to imply that freedom from contradiction and consistency are necessary and sufficient conditions for ascribing "existence" to mathematical entities. This is consonant with Hilbert's view that if we can show the consistency of a formal system, then this is a necessary and sufficient condition for the ascription of existence to the entities implicitly defined by the axioms of that system. Thus, as early as 1904, Hilbert says:

> Having thus established a certain property[9] for the axioms adopted here, we recognize that they never lead to any contradiction at all, and therefore we speak of the thought-objects defined by means of them... as consistent notions or operations, or as consistently existing[10].

There are, however, many objections to viewing Cantor as a formalist:

(i) In dealing with "consisteny", Hilbert is dealing with a metamathematical property of formal systems. Cantor, however, had no

---

[9]Hilbert is here referring to the property of "homogeneity" or tautologousness.

[10]David Hilbert, "On the Foundations of Logic and Arithmetic", Jean van Heijenoort, ed., <u>From Frege to Gödel, A source Book in Mathematical Logic, 1879-1931</u> (Cambridge, Massachusetts, 1967).

notion of a formal system. As I stated in Chapter I, Cantor's work on transfinite cardinal arithmetic was "naive" rather than "formal". Thus Cantor's notion of "consistency" is so much less exact than that of Hilbert that it would seem inappropriate to compare them.

(ii) It has been the principal aim of this thesis to formalize Cantor's work. However, as I have shown in Chapter VI, I have been unable to establish an absolute consistency proof for the system of alephs. Thus, even if we agreed that Cantor possessed an embryonic notion of the metamathematical property of consistency, his system has not so far been shown to be consistent.

(iii) We can see quite clearly from passages (a) and (b) that Cantor has none of the formalist tendency to identify mathematical entities with the notation for such entities. There is no indication that he would subscribe to the Hilbertian dictum:

> The subject matter of mathematics is ... the concrete symbols themselves whose structure is immediately clear and recognisable[11].

---

[11]David Hilbert, "On the Infinite" (1925) in Paul Benacerraf & Hilary Putnam, ed., Philsophy of Mathematics: Selected Readings (New Jersey, 1964).

Or again,

> These numerical symbols which are themselves our subject matter have no significance in themselves[12].

Nor can such an identification of sign and thing signified be found anywhere else in his work. Moreover, in passage (a) there appears to be a definite reference to mathematical entities as existing not merely on paper but in our mind. In fact, they "modify, in a definite way, the substance of our mind." I conclude, therefore, that the evidence for Cantor's view of mathematical entities as formalistic is very inadequate.

We may now ask the question: Is there any evidence to show that Cantor was an intuitionist? Did Cantor regard mathematical entities in general and alephs in particular as mind-made entities which we invent? Let us consider those passages which might encourage us to view Cantor as a forerunner of Brouwer.

First, there appears the statement from passage (b) cited above which runs ". . . mathematics is, in its development, quite free." This seems to presage Heyting's later remarks:

> The Intuitionist mathematician proposes to do mathematics as a natural

---

[12]Ibid., p.143.

> function of his intellect, as a free, vital activity of thought. For him, mathematics is a production of the human mind[13].

In saying that the development of mathematics is "free", Cantor appears to be speaking in the intuitionist spirit.

Again, we see another resemblance betweeen Cantor's view of mathematical entities and that of the intuitionists, in that Cantor frequently makes reference to the notion of mind in his definitions. Thus, in his famous definition of a set, he says:

> By an 'aggregate' we are to understand any collection into a whole M of definite and separate objects m of our intuition or our thought[14].

Interestingly enough we find a reference to the (intuionistic) notion of creating rather than the (realistic) notion of discovering numbers in his definition of the first transfinite ordinal number:

> If there is defined any definite succession of real integers, of which there is no greatest, on the basis of this second principle[15] a new number is created which

---

[13]Arend Heyting, "The Intuitionist Foundations of Mathematics" in Paul Benacerraf & Hilary Putnam, eds., Philosophy of Mathematics: Selected Readings (New Jersey, 1964).

[14](Cantor, 1895, section 1, p.85).

[15]This refers to the second principle of formation.

> is defined as the next greater number to them all[16].

Again, Cantor says:

> It is even permissible to think of the newly created number as the limit to which the numbers strive[17].

Such passages are definitely conceptualist as opposed to realist in tone: in their talk of "creation of" rather than of "discovering" certain numbers and in their reference to the activity of the mind.

Finally, we may cite our strongest evidence for regarding Cantor as a prospective intuitionist. In his definition of cardinal number, he says:

> We will call by the name 'power' or 'cardinal number' of M the general concept which, by means of our active faculty of thought, arises ~~from the~~ aggregate M when we make abstraction of the nature of its various elements m and of the order in which they are given [18].

Thus it appears that cardinal number is not something which exists independently of the mind but is instead the result of a double

---

[16]Translated and quoted by P.E.B. Jourdain in his introduction of Cantor, 1895, p.57.

[17]Ibid., pp.56-7

[18]Ibid., section I, p.86.

act of abstraction performed by the mind. Again Cantor says:

> Since every single element m, if we abstract from its nature, becomes a 'unit', the cardinal number $\overline{\overline{M}}$ is a definite aggregate composed of units, and this number has existence in our mind as an intellectual image or projection of the given aggregate M[19].

Here we seem to have incontrovertible evidence that a cardinal number (both finite and infinite) is an "intellectual image" for Cantor. We now recall Heyting's remark:

> Even if they should be independent of individual acts of thought, mathematical objects are by their very nature dependent on human thought. Their existence is guaranteed only in so far as they can be determined by thought[20].

So far, it seems that we have very good evidence for believing that Cantor held a conceptualist view with regard to cardinal number (both finite and transfinite). The evidence is clearly much stronger than that for characterizing him as a formalist. However, apart from the very explicit platonistic statements which I shall soon cite, there are other reasons to believe that the conceptualism is paler than at first sight. We may argue, for instance, that Cantor's references to "mind" in his definitions are inessential

---

[19] Ibid., section 1, p.86.

[20] A. Heyting in "the Intuitionist Foundations of Mathematics" (1964, p.42)

insofar as they can be omitted without any formal loss. It is quite clear that his definition of a set would lose nothing in the deletion of references to cognitive processes. Again, it may be argued that although we arrive at the notion of cardinal number by a double act of abstraction, this is viewed by Cantor only as a description of certain subjective cognitive processes which happen to occur in our mind in the formation of the concept. To suppose that such processes "define" a cardinal number is akin to the fallacy of "psychologism". I think that these objections are valid and that Cantor need not be classed as an intuitionist.

Finally, I come to consider the position known variously as logicism, realism or platonism. For the sake of convenience, I shall refer to the position hereafter as "platonism". It is my contention that the evidence is overwhelmingly in favour of our ascribing this position to Cantor. Thus I hope to show that for Cantor, mathematical entities and alephs in particular exist independently of the mind. To begin with, let us attempt some further clarification as to what it is to be a platonist, with regard to mathematical entities. Platonism appears to involve two aspects:

(i) The Ontological Aspect.
As already stated, a platonist believes that mathematical entities somehow "exist" independently of the human mind.

(ii) The Epistemological Aspect.

A platonist believes that we "discover" rather than "create" such entities. We find a beautiful statement of this in Frege:

> The geographer does not create a sea when he draws border lines and says: The part of the surface of the ocean delimited by these lines, I am going to call the Yellow Sea; and no more can the mathematician really create anything by his act of definition[21].

Again, Frege says:

> . . . even the mathematician cannot create things at will, any more than the geographer can; he too can only discover what is there and give it a name[22].

I shall now cite the evidence for Cantorian platonism. First, I shall cite passages from Cantor's work and letters; secondly, passages from the work ~~of his contemporary~~, Frege; thirdly, I shall refer to certain late nineteenth century controversies which support the view of Cantor qua platonist:

---

[21] Gottlob Frege, Grundgesetze der Arithmetik, vol.i, trans. P. Geach & Max Black in Translations from the Philosophical Writings of Gottlob Frege (Oxford, 1970), p.145.

[22] Gottlob Frege, The Foundations of Arithmetic, trans. J.L. Austin (Illinois,1968), pp.107-8.

Q(1)

(α) In discussing the integers, Cantor holds them to be "actual" insofar as the "take a perfectly determined place in our understanding." However, he goes on:

> We may ascribe actuality to them in so far as they must be held to be an expression or an image (Abbild) of processes and relations in the outer world, as distinguished from the intellect[23].

Thus we see that Cantor goes beyond the conceptualist position that the integers exist only in the mind, to the platonistic position that they exist in our minds only as "images" or "copies" of something external to the mind. There seems to be no reason why this should not also apply to transfinite numbers. Now, it should be stated, of course, that by this Cantor did not believe that such images were copies of something in the sensible world - for to maintain this would be perilously near to J.S. Mill's view of mathematics as an empirical science[24]. No, for Cantor, number concepts in our minds are images of some supra-sensible

---

[23] Trans. P.E.B. Jourdain in (Cantor, 1883(a), section 8, pp.545-591).

[24] J.S. Mill, <u>System of Logic</u> (London, 1879), Bk.II, c.VI, sections 1-4; Bk.III, c.XXIV, section 5.

reality as they were for Plato[25].

(β) In 1884 Cantor wrote a letter to A. Schoenflies, in which he said:

> As to everything else (except the art of style and the economy of exposition) this is not my merit; with regard to the contents of my research work I am only a kind of reporter and secretary[26].

Here we see that Cantor is far from regarding his work as the "free, vital activity of thought" that it was for Heyting. As we know from Cantor's religious convictions, it is likely that he regarded himself as God's secretary.

(γ) As a note to his 1895 article Cantor wrote:

> Nor do we give laws to the intellect or to things according to our own judgement, but like faithful scribes, those laws which are born on the voice of nature itself and proclaimed, we take up and describe[27].

---

[25] Plato, <u>The Republic</u>, trans. F.M. Cornford (New York, 1973), Bk.VI 509d-511E.

[26] Quoted in a footnote by A.A. Fraenkel in <u>Abstract Set Theory</u> (Amsterdam, 1968), p.80.

[27] (Cantor, 1895, p.85).

There is some difficulty here regarding the interpretation of the "voice of nature" (naturae voce). However, I think that the most likely interpretation is that Cantor is referring to a supra-sensible world.

(2)

Further evidence for regarding Cantor as a platonist comes from remarks made by his contemporary, Frege. Thus, in discussing Cantor, Frege says: "our number ∞, is as sound as 2 or 3."[28] For Frege, furthermore, all numbers are "self-sufficient objects"[29] which may exist independently of the mind. Thus for Frege (and he seems to imply that Cantor would agree to this) numbers, both finite and transfinite, are to be platonistically conceived.

(3)

Finally, we will look at the various controversies (both mathematical and theological) in which Cantor was embroiled:

---

[28]Gottlob Frege, Foundations of Arithmetic, sections 84-6.

[29]Ibid., section 57.

(α) Mathematical Controversies

It is well-known from biographical sources[30] that Cantor was engaged for many years in an acrimonious dispute with Kronecker. The issue which concerns us here is the ontological status of numbers. Kronecker (a forerunner of intuitionism) held the view that "God made the natural numbers, all the rest is man's handiwork"[31]. All other numbers, according to Kronecker, were less "real" than the natural numbers, and statements about the former could be "reduced" to statements about the latter. Now Cantor clearly rejects this philosophy. In 1883, he not only gave a clear characterization of Kronecker's views, but also stated an equally lucid repudiation of them[32]: for him, the process of reduction was unnecessary since all numbers were equally real. In this controversy we see the platonistic versus the conceptualist viewpoint.

---

[30] I. Grattan-Guinness, "Towards a Biography of Georg Cantor", Annals of Science, vol.27, No.4 (1971), 345-391; E.T. Bell, Men of Mathematics (New York, 1965), c.29.

[31] Not to be found in any of Kronecker's published works, since it comes from an after-dinner speech.

[32] (Cantor, 1883(a), section 8, pp.545-591).

(β) Theological Controversies

Less well-known is Cantor's controversy with the mathematician, Hermite. Paradoxically, Hermite was both a platonist and an anti-Cantorian. His objection to Cantor was the presumption of the latter. Thus Poincaré says:

> Doubtless because of his [Hermite's] religious convictions he considered it a kind of impiety to wish to penetrate a domain which God alone can encompass, without waiting for Him to reveal its mysteries one by one[33].

Thus we see that Hermite assumes Cantor to be a platonist, though a rather 'Faustian' one, who obviously wanted to eat of the tree of knowledge before his time. This arcane "domain" is obviously a reference to Cantor's theory of transfinite arithmetic. However, Hermite need not have taken such objection, since Cantor was far more modest than this. For Cantor, transfinite arithemtic is still only a symbol of the Absolute:

> The absolutely infinite sequence of numbers thus seems to me to be, in a certain sense, a suitable symbol of the Absolute[34].

I conclude, therefore, that Cantor was a platonist in his view of mathematical entities in general and in his view of alephs in

---

[33]H. Poincare, Dernières Pensées (Paris, 1913).

[34]Quoted by P.E.B. Jourdain in his Introduction to (Cantor, 1895, p.62).

particular. Thus, for Cantor, alephs do indeed exist in some transcendent realm independent of human minds.

Finally, we must consider the relation between the axiomatized version of transfinite cardinal arithmetic (as presented in this thesis) and Cantor's platonistic position. Would a platonist consider the axiomatization of a particular number system as a kind of ontological proof for the existence of that number system? We have seen that, for the formalist, a consistent axiomatic system forms a necessary and sufficient condition for the existence of the entities which it implicitly defines. We have no justification for believing the same to be true for the platonist. Rather we might say that for the platonist, the consistency of an axiomatic system is a necessary (but not sufficient) condition for ascribing existence to the entities in question.

Now, we have seen that in the passages cited on p.74 Cantor, at one time, appears to have regarded consistency as both a necessary and sufficient condition for establishing the existence of a particular number system. However, since there may be internally consistent but mutually incompatible number systems, this cannot be a sufficient condition for the platonist. We have seen that Cantor is a platonist and that therefore consistency can only be a necessary condition for him. It may be questioned as to what additional condition a platonist would require in order to establish a proof for the existence of a particular entity. The only suggestion made by Cantor in this respect concerns his reference (p.84) to the "dictation" by God of such mathematical theories. However, besides the inexactitude of such a condition, there remains the problem of how to distinguish between a dictated and a non-dictated theory.

Unfortunately, as we have seen, we have only been able to give a relative consistency proof -- and that only for a part of the system of alephs. Thus we have been unable to establish even a necessary condition for the existence of alephs (which would require an absolute consistency proof). We may draw the following two conclusions: First, since we have been unable to give an absolute consistency proof even of part of the system of alephs, there is a strong suspicion that an absolute consistency proof of the whole system is impossible. Secondly, because of the intractable metaphysical element which attaches to any platonistic proof for the existence of a number system, there seems to be a radical difficulty in substantiating Cantor's claim that alephs exist.

BIBLIOGRAPHY

A. Works Cited

(i) Primary Sources

Cantor, 1883 Cantor, Georg. "Fondements d'Une Theorie Générale Des Ensembles," Acta Mathematica, I (1883), 381-408.

Cantor, 1883(a) Cantor, Georg. "Über unendliche lineare Punktmannigfaltigkeiten", part 5, Mathematische Annalen, XXI (1883), 545-591.

Cantor, 1886 Cantor, Georg. "Letter to F. Goldscheider", in Herbert Meschkowski. Ways of thought of Great Mathematicians, San Francisco: Holden-Day Inc., 1964.

Cantor, 1895; Cantor, 1897 Cantor, Georg. Contributions to the Founding of the Theory of Transfinite Numbers. Trans. with an introduction and notes, by Philip E.B. Jourdain. New York: Dover Publications, inc., 1955.

Cantor, 1899 Cantor, Georg. "Letter to Dedekind," in Jean van Heijenoort, ed., From Frege to Gödel, A source Book in Mathematical Logic, 1879-1931. Cambridge, Massachusetts: Harvard University Press, 1967.

(ii) Secondary Sources

Barker, Stephen. Philosophy of Mathematics. New Jersey: Prentice-Hall, 1964.

Bell, E.T. Men of Mathematics. New York: Simon & Schuster, 1965.

Black, Max. The Nature of Mathematics. London, 1933.

Carnap, Rudolf. "Empiricism, Semantics and Ontology", in Paul Benacerraf and Hilary Putnam, ed., Philosophy of Mathematics: Selected Readings. New Jersey: Prentice-Hall Inc., 1964

Fraenkel, Abraham A. Abstract Set Theory. Amsterdam: North-Holland Publishing Company, 1968.

Frege, Gottlob. The Foundations of Arithmetic. Trans. by J.L. Austin. Illinois: Northwestern University Press, 1968.

Frege, Gottlob. Translations from the Philosophical Writings of Gottlob Frege. Trans. by P. Geach and Max Black. Oxford: Basil Blackwell, 1970.

Grattan-Guiness, I. "Towards a Biography of Georg Cantor", Annals of Science, vol.27, no.4 (1971), 345-391.

Green, J.A. Sets and Groups. London: Routledge & Kegan Paul Ltd., 1971.

Heyting, Arend. "The Intuitionist Foundations of Mathematics", in Paul Benacerraf and Hilary Putnam, ed., Philosophy of Mathematics: Selected Readings. New Jersey: Prentice-Hall, 1964, p.42

Hilbert, David. "On the foundations of Logic and Arithemtic", in

Jean van Heijenoort, ed., From Frege to Gödel, A Source Book in Mathematical Logic, 1879-1931. Cambridge, Massachusetts: Harvard University Press, 1967.

Hilbert, David. "On the Infinite"(1925), in Paul Benacerraf and Hilary Putnam, ed., Philosophy of Mathematics: Selected Readings. New Jersey: Prentice-Hall, 1964.

Hilbert, David, and W. Ackermann. Principles of Mathematical Logic. Trans. from the German by Lewis M. Hammond, George C. Leikie and F. Steinhardt; ed. and with notes by Robert E. Luce. New York: Chelsea, 1950.

Hilbert, David, and P. Bernays. Grundlagen der Mathematik. 2 vols. Berlin: Springer, 1934 & 1939.

Kneebone, G.T. Mathematical Logic and the Foundations of Mathematics, an Introductory Survey. London: D. Van Nostrand Company Limited, 1965.

Körner, Stefan. The Philosophy of Mathematics. New Jersey: Prentice-Hall, 1964.

Mill, J.S. System of Logic. London: Longmans, Green & Co.,1879.

Nagel, Ernest, and James R. Newman. Gödel's Proof. New York: New York University Press, 1973.

Plato. The Republic. Trans. by F.M. Cornford. New York: Oxford University Press, 1973.

Poincaré, Henri. Dernières Pensées. Paris: E. Flammarion, 1913.

Quine, W.v.O. From a Logical Point of View. New York: Harper & Row, 1963.

Tarski, Alfred. Introduction to Logic and to the Methodology of the Deductive Sciences. New York: Oxford University Press, 1965.

B. Other Works

Bernays, Paul. Axiomatic Set Theory. With a historical introduction by Abraham A. Fraenkel. Amsterdam: North-Holland Publishing Company, 1958.

Beth, E.W. The foundations of Mathematics. Amsterdam: North-Holland Publishing Company, 1965.

Cantor, Georg. Gesammelte Abhandlungen mathematischen und philosophischen Inhalts. Edited by E. Zermelo. Berlin, 1932.

Cantor, Georg. "Sur les Ensembles Infinis et Linéaires de Points, I" Acta Mathematica, II (1883), 349-356.

Cantor, Georg. "Sur les Ensembles Infinis et Linéaires de Points, II" Acta Mathematica, II (1883), 357-360.

Cantor, Georg. "Sur les Ensembles Infinis et Linéaires de Points, III" Acta Mathematica, II (1883), 361-371.

Cantor, Georg. "Sur les Ensembles Infinis et Linéaires de Points, IV" Acta Mathematica, II (1883), 372-380.

Cantor, Georg. "Une Contribution à la Theorie des Ensembles," Acta Mathematica, II (1883), 311-328.

Carney, James D. Introduction to Symbolic Logic. New Jersey: Prentice-Hall, Inc., 1970.

Church, Alonzo. Introduction to Mathematical Logic, vol.I. Princeton University Press, 1956.

Fraenkel, A. and Y. Bar-Hillel. Foundations of Set Theory. Amsterdam: North-Holland Publishing Company, 1958.

Gödel, Kurt. "What is Cantor's Continuum Problem?", American Mathematical Monthly, 54 (1947), 515-525.

Jourdain, Philip. E.B. "The Development of the Theory of Transfinite

Numbers, Part 2 - Weierstrass (1840-1880)," Archiv der Mathematik und Physik, 3rd ser., XIV (1909), 289-311.

Kleene, Stephen Cole. Mathematical Logic. New York: John Wiley & Sons, Inc., 1967.

Kneale, William and Martha. The Development of Logic. Oxford: Clarendon Press, 1968.

Russell, Bertrand. Introduction to Mathematical Philosophy. London: George Allen & Unwin Ltd., 1963.

Russell, Bertrand. Our Knowledge of the External World, as a Field for Scientific Method in Philosophy. London: George Allen & Unwin Ltd., 1961.

Russell, Bertrand. The Principles of Mathematics. New York: W.W. Norton & Company, Inc.,n.d.

Wang, Hao. "The Axiomatization of Arithmetic", Journal of Symbolic Logic, XXII (1957), 145-158.