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STUDY OF PULSED RADAR SPECTRA

STUDY OF PULSED RADAR SPECTRA

BY

ROGER H. Y. TO, B.Eng. (UNIV. OF ALBERTA)

A Thesis
Submitted to the School of Graduate Studies
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Master of Engineering

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ABSTRACT

In a pulsed radar system, due to the existence of incidental frequency and amplitude modulation, the spectrum of the transmitted RF pulse may deviate markedly from the ideal form. It is the purpose of this thesis to study the spectra of pulsed RF signals with different forms of incidental frequency modulation and amplitude modulation.

The contribution of this thesis may be summarized as follows:

(1) A phenomenological approach to the analysis of pulsed RF signals containing incidental modulation has been developed using the Fast Fourier transform method.

(2) A fairly general analytical formula has been developed for the pulsed RF spectra, using the convolution integral.

(3) A real-time experimental simulation of pulsed signals with incidental modulation has been successfully designed to measure the pulsed RF spectra.

Close agreements between the results of these three approaches has been demonstrated.

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It is with pleasure that I acknowledge those whose encouragement and assistance made this thesis possible.

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CHAPTER 1

INTRODUCTION

1-1 MOTIVATION

The spectrum of a signal is an important indication of the transmitter performance in a radar or communication system. For example, for a rectangular pulse of width T and unit-amplitude, the spectrum is of the form

$$\left|\frac{\sin(x)}{x}\right|$$

where $x=\pi fT$

$f=$ frequency

$T=$ pulse width

In a pulsed radar system, the transmitted RF pulse should ideally be of constant amplitude and constant frequency. However, due to the existence of incidental frequency and amplitude modulation, the spectrum may deviate markedly from the ideal. The incidental modulation characteristics are different for different transmitters. It is the purpose of this thesis to study the spectra of various kinds of pulsed RF signals that have incidental frequency modulation and incidental amplitude modulation.

1-2 APPROACHES

Three approaches are taken in the study.

(a) FFT Approach : Because of the tedious mathematics involved in taking the Fourier transform of a pulsed signal, even for a simple modulating waveform, the first approach taken is the phenomenological approach. A numerical computation technique known as the Fast Fourier transform (FFT) is used to compute the spectra of a few pulses with simple modulating waveforms. By changing the parameters of the modulating pulse, insight is developed into the resulting spectra.

(b) Analytical Approach : Various Fourier transform techniques are used to try to obtain a closed form mathematical solution to the Fourier transform of a pulsed signal.

(c) Measurement : This approach involves measuring the spectrum using a spectrum analyzer. This provides a check on the numerical or analytical result.

1-3 APPLICATION

In radar signal processing, a magnetron is used to generate the pulsed RF signal. A high voltage pulse is fed as an input to the magnetron and ideally a constant frequency and constant magnitude pulse is emitted at the output. This frequency is the resonant frequency of the magnetron. The high voltage pulse is usually very short in time duration, in the order of a micro-second or so. In a practical system, we have a finite rise and fall time for the resonation of the magnetron, and consequently, the output will not be a perfectly constant amplitude and constant frequency pulse. In order to fully understand the spectrum of the pulse produced by a magnetron, the resonating characteristics should be measured. The question is what kind of frequency and amplitude response can be expected from the magnetron as a result of a certain given wave-shape of an input voltage pulse and a given operating environment. Indeed, much research has to be done in this particular area. This resonant characteristic partly depends on the pushing figure and pulling figure of the magnetron used. The pushing figure is the instantaneous frequency change in MHz. per ampere change in peak DC current at constant load. It may be obtained by measuring the frequency shift on a spectrum analyzer as the pulse current is changed by a known amount. The pulling figure denotes the variation of the resonant frequency with the variation of the load to the magnetron. We thus find that different magnetrons, operating under different conditions, produce totally different incidental frequency modulation and incidental amplitude modulation.

Sometimes it may be desirable to purposely create a non-symmetric spectrum by introducing frequency modulation in the pulse.

The sidelobes of one side of the spectrum would be lowered as compared with the symmetric $\left|\frac{\sin (x)}{x}\right|$ spectrum. More channels may be allocated within a fixed frequency band, because the inter-channel interference is reduced.

1-4 STRUCTURE OF THE THESIS

The study of pulse spectra is described in four seperate chapters. Chapter 2 is on some general properties of pulse spectra. Chapter 3 is on the phenomenological approach of analysis using the Fast Fourier transform technique. Four different shapes of modulating pulses are considered. These are the trapezoidal, exponential, sinusoidal and modified gaussian wave-shape.

Chapter 4 is on the analytical approach to the problem. Two ways of computing the Fourier transform are described. Section 4-1 describes the method using the convolution techniques. A general formula is developed that can compute the spectrum analytically. The requirements for using this method are the analytical formula of the Fourier transform of the amplitude modulating pulse, and the Fourier series representation of the frequency modulating pulse. Section 4-2 describes the method using direct integration. Only a limited number of cases are considered using this method, because of the mathematical complexity. The tables for Fresnel Integral and Bessel function have to be referred to for some of the formulae contained in chapter 4.

Chapter 5 is on the description of the experiment used to analyze the pulse spectra. Two modulating waveforms are acquired simultaneously from the CDC 1700 digital computer. They are used as the amplitude modulating and the frequency modulating pulses, respectively. The spectrum is measured using a spectrum analyzer. The experimental results appear to agree very well with the theoretical results.

CHAPTER 2

PROPERTIES OF PULSED RF SPECTRA

In this chapter some pertinent properties of pulse spectra will be considered. These properties hold true for general modulating pulses. However we shall begin by briefly reviewing the pre-envelope function representation of a modulated signal.

2-1 THE PRE-ENVELOPE SIGNAL

The pre-envelope function representation of an amplitude and frequency modulated pulse can be written as [1]

$$p(t) = A(t)\, e^{j2\pi[f_c t + \int_0^t g(x)dx]} \qquad (2.1)$$

where $p(t)$ = pre-envelope function or complex waveform representation of the signal

$A(t)$ = amplitude modulation

$g(t)$ = frequency modulation

f_c = carrier frequency

T = pulse duration

Taking the Fourier transform of the pre-envelope function, one obtains,

$$S(f) = \int_{-\infty}^{\infty} A(t)\, e^{j2\pi[f_c t + \int_0^t g(x)dx]}\, e^{-j2\pi ft}\, dt$$

$$S(f) = \int_{-\infty}^{\infty} A(t)\, e^{j2\pi \int_0^t g(x)dx}\, e^{-j2\pi f' t}\, dt \qquad (2.2)$$

where $f' = f - f_c$

We have assumed that the bandwidth of the spectrum is small compared to f_c .

Equation (2.2) implies that the analysis of the pulsed signal can be done using a baseband signal with amplitude modulation $A(t)$, and frequency modulation $g(t)$. Effectively this is equivalent to shifting the origin of the spectrum graph from the point $f=0$. to $f=f_c$.

2-2 PROPERTIES OF PULSED RF SPECTRA

The following properties can be deduced from looking at the Fourier transform of the signal. The waveshape of the modulating pulse can be of any form.

Property A

A signal with frequency modulation $g_1(t)$ and amplitude modulation $A_1(t)$ would have the same spectrum as another signal with $g_2(t)$ and $A_2(t)$ as the frequency modulation and amplitude modulation respectively, with

$$g_1(t) = g_2(T-t) \qquad (2.3)$$

$$A_1(t) = A_2(T-t) \tag{2.4}$$

where T= pulse period (Refer to Fig. 2.1)

This property will be used extensively in the subsequent sections of the analysis.

Proof

We have to prove : $|S_1(f)| = |S_2(f)|$

$$\text{where} \quad S_1(f) = \int_{-\infty}^{\infty} A_1(t)\, e^{j2\pi\int_0^t g_1(x)dx}\, e^{-j2\pi ft}\, dt \tag{2.5}$$

$$S_2(f) = \int_{-\infty}^{\infty} A_2(t)\, e^{j2\pi\int_0^t g_2(x)dx}\, e^{-j2\pi ft}\, dt \tag{2.6}$$

Now let $v = -x+T$

Hence

$$S_2(f) = \int_{-\infty}^{\infty} A_2(t)\, e^{j2\pi \int_{T-t}^{T} g_2(T-v)dv}\, e^{-j2\pi ft}\, dt$$

$$= \phi_1 \int_{-\infty}^{\infty} A_2(t)\, e^{-j2\pi\int_0^{T-t} g_1(v)dv}\, e^{-j2\pi ft}\, dt \tag{2.7}$$

$$\text{where} \quad \phi_1 = e^{j2\pi\int_0^T g_2(T-v)dv} \tag{2.8}$$

Next let $u = T-t$

$$\text{Hence} \quad S_2(f) = \phi_2 \int_{-\infty}^{\infty} A_2(T-u)\, e^{-j2\pi\int_0^u g_1(v)dv}\, e^{j2\pi fu}\, du$$

$$\text{where} \quad \phi_2 = \phi_1 e^{-j2\pi fT} \tag{2.9}$$

Fig. 2.1

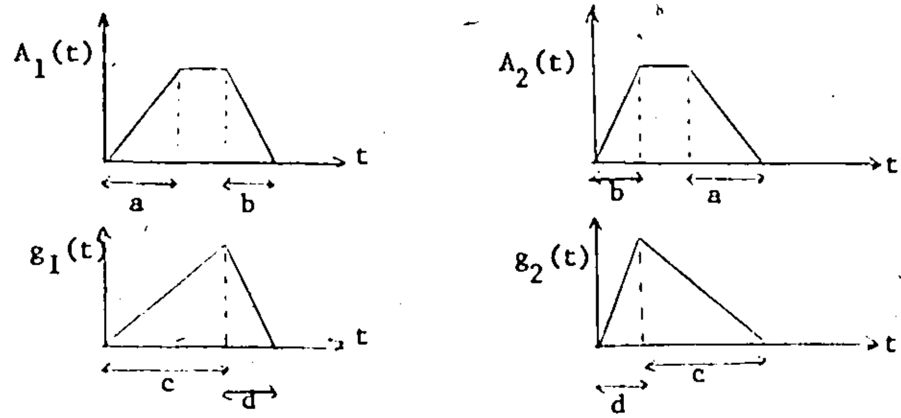


Fig. 2.2

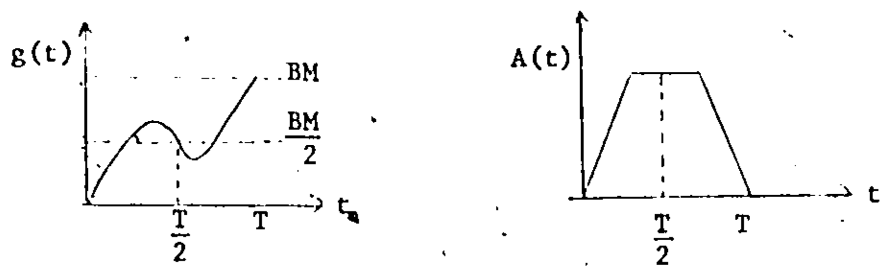
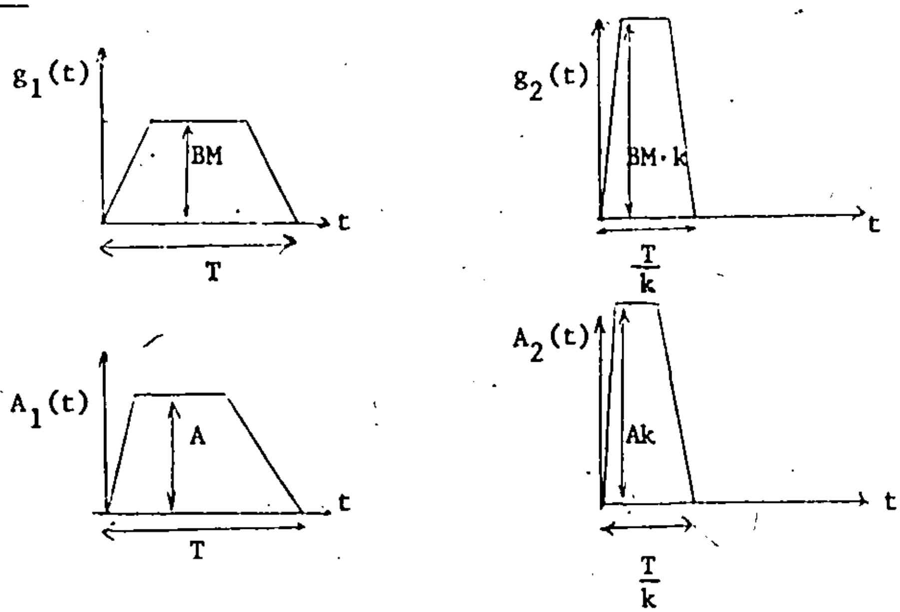


Fig. 2.3



Thus, $|S_2(f)| = \left| \int_{-\infty}^{\infty} A_1(u)\, e^{-j2\pi \int_0^u g_1(v)dv}\, e^{j2\pi f u}\, du \right|$

Now $|S_2(f)| = |\text{Conjugate of } S_2(f)|$

$$= \left| \int_{-\infty}^{\infty} A_1(u)\, e^{j2\pi \int_0^u g_1(v)dv}\, e^{-j2\pi f u}\, du \right|$$

$$= |S_1(f)| \qquad \text{Q.E.D.}$$

Property B

When the frequency modulation is odd symmetric about the centre of the frequency deviation BM/2 , and the amplitude modulation is even symmetric about the middle of the pulse T/2 , the spectrum will be symmetric about the frequency f_c+BM/2. This is only a sufficient condition for a spectrum to be symmetric. An example is given in Fig. 2.2.

Proof

Let AM and FM of the signal be represented by A(t) and g(t) respectively, and

$$A(t+\frac{T}{2}) = B(t) \tag{2.10}$$

$$g(t+\frac{T}{2}) = h(t) \tag{2.11}$$

$$B(t) = B(-t) \tag{2.12}$$

$$h(t) - \frac{BM}{2} = -[h(-t) - \frac{BM}{2}] \tag{2.13}$$

Here we assume the signal is at baseband.

$$\text{Hence} \quad S(f) = \int_{-\infty}^{\infty} A(t)\, e^{j2\pi\int_{0}^{t} g(x)dx}\, e^{-j2\pi ft}\, dt \qquad (2.14)$$

Now let $t=u+t_o$ and $x=v+t_o$.

Hence

$$S(f) = e^{-j2\pi ft_o}\, e^{j2\pi\int_{t_o}^{0} g(v+t_o)dv} \int_{-\infty}^{\infty} A(u+t_o) e^{j2\pi\int_{0}^{u} g(v+t_o)dv}\, e^{-j2\pi fu}\, du$$

$$\text{and} \quad |S(f)| = \left|\int_{-\infty}^{\infty} A(u+t_o)\, e^{j2\pi\int_{0}^{u} g(v+t_o)dv}\, e^{-j2\pi fu}\, du\right|$$

$$= \left|\int_{-\infty}^{\infty} A(t+t_o)\, e^{j2\pi\int_{0}^{t} g(x+t_o)dx}\, e^{-j2\pi ft}\, dt\right| \qquad (2.15)$$

This proves time shifting a signal with a fixed delay does not alter the magnitude of the Fourier transform. Now if $t_o = T/2$ and substitute eqt. (2.10) and (2.11) in (2.15) , we have

$$|S(f)| = \left|\int_{-\infty}^{\infty} B(t)\, e^{j2\pi\int_{0}^{t} h(x)dx}\, e^{-j2\pi ft}\, dt\right|$$

$$|S(f+\frac{BM}{2})| = \left|\int_{-\infty}^{\infty} B(t)\, e^{j2\pi\int_{0}^{t}[h(x)-\frac{BM}{2}]dx}\, e^{-j2\pi ft}\, dt\right|$$

$$|S(-f+\frac{BM}{2})| = \left|\int_{-\infty}^{\infty} B(t)\, e^{j2\pi\int_{0}^{t}[h(x)-\frac{BM}{2}]dx}\, e^{j2\pi ft}\, dt\right|$$

Now let $u=-t$ and $v=-x$

$$\text{Hence} \quad |S(-f+\frac{BM}{2})| = \left|\int_{-\infty}^{\infty} B(-u)\, e^{-j2\pi\int_{0}^{u}[h(-v)-\frac{BM}{2}]\,dv}\, e^{-j2\pi fu}\, du\right|$$

$$= \left|\int_{-\infty}^{\infty} B(u)\, e^{j2\pi\int_{0}^{u}[h(v)-\frac{BM}{2}]\,dv}\, e^{-j2\pi fu}\, du\right|$$

This proves $\left|S(-f+\frac{BM}{2})\right| = \left|S(f+\frac{BM}{2})\right|$

Thus, $|S(f)|$ is symmetric about $\frac{BM}{2}$. Q.E.D.

Property C

The pulse with $g_1(t)$ and $A_1(t)$ as the frequency modulation and amplitude modulation respectively, will have the same spectrum waveform as the pulse with $g_2(t)$ and $A_2(t)$ as the frequency modulation and amplitude modulation, with

$$A_1(t) = \frac{1}{k} A_2(u) \tag{2.16}$$

$$g_1(t) = \frac{1}{k} g_2(u) \tag{2.17}$$

$$u = \frac{t}{k} \tag{2.18}$$

k = constant

$A_1(t) = 0$ for $t \leq 0$ and $T < t$

$A_2(t) = 0$ for $t \leq 0$ and $\frac{T}{k} < t$

T = pulse duration

An example is given in Fig. 2.3.

The frequency scale should be expanded by a facter of K in order to obtain the same scale. Using this property, T can be normalized to be unity. This leaves only the maximum deviation (BM or ΔF) to be the parameter that changes the T·ΔF product.

Proof

To show : $S_1(f) = S_2(f, k)$

where
$$S_1(f) = \int_{-\infty}^{\infty} A_1(t)\, e^{j2\pi \int_0^t g_1(x)dx}\, e^{-j2\pi ft}\, dt \qquad (2.19)$$

$$S_2(f) = \int_{-\infty}^{\infty} A_2(t)\, e^{j2\pi \int_0^t g_2(x)dx}\, e^{-j2\pi ft}\, dt \qquad (2.20)$$

Substitute eqt. (2.16), (2.17) and (2.18) into (2.19).

Hence
$$S_1(f) = \int_{-\infty}^{\infty} \frac{1}{k} A_2(u)\, e^{j2\pi \int_0^{u} \frac{1}{k} g_2(u)du \cdot k}\, e^{-j2\pi fku}\, k\, du$$

$$= \int_{-\infty}^{\infty} A_2(u)\, e^{j2\pi \int_0^u g_2(u)du}\, e^{-j2\pi fku}\, du$$

$$S_1(f/k) = \int_{-\infty}^{\infty} A_2(u)\, e^{j2\pi \int_0^u g_2(u)du}\, e^{-j2\pi fu}\, du$$

$$= S_2(f)$$

or $\quad S_1(f) = S_2(k \cdot f)$ Q.E.D.

Property D

The total area under the spectrum representing the square root of the total power, remains unchanged for any frequency modulating pulse, as long as the amplitude modulating pulse remains constant. This means that the total average power is invariant under the change of frequency modulation so long as the amplitude modulation is kept fixed.

Proof

To show :

$$E = \int_{-\infty}^{\infty} |S(f)|^2 \, df$$

= constant for a fixed A(t)

where E = total power of the signal

S(f) = Fourier transform of signal s(t)

$$= \int_{-\infty}^{\infty} s(t) \, e^{-j2\pi ft} \, dt$$

$$s(t) = A(t) \, e^{j2\pi \int_0^t g(x)dx}$$

A(t) = amplitude modulation

g(t) = frequency modulation

Hence

$$E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(u) \, e^{-j2\pi fu} \, du \int_{-\infty}^{\infty} s^*(v) \, e^{+j2\pi fv} \, dv \, df$$

where $s^*(t)$ denotes complex conjugate of s(t)

Now

$$E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(u) \, s^*(v) \, du \, dv \int_{-\infty}^{\infty} e^{j2\pi f(v-u)} \, df$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(u) \, s^*(v) \, du \, dv \, \delta(u-v)$$

$$= \int_{-\infty}^{\infty} |s(u)|^2 \, du$$

$$= \int_{-\infty}^{\infty} [A(t)]^2 \, dt$$

= constant for a fixed A(t) Q.E.D.

CHAPTER 3

SPECTRAL ANALYSIS USING THE FAST FOURIER TRANSFORM METHOD

The Fast Fourier transform (FFT) method [2] provides the fastest numerical technique that is available for spectral analysis. In this chapter the FFT method is used in the analysis of pulsed signal spectra . The waveforms of the modulating pulse used are of trapezoidal, exponential, sinusoidal and modified gaussian wave shapes.

NOTE

As depicted in the diagrams to follow, the pulse waveforms on the top are the modulating pulses. The left one is the frequency modulating pulse and the right one is the amplitude modulating pulse. Different spectrum graphs are stacked on top of each other to contrast the change in shape of the spectrum after a parameter is incremented. The parameter changed is printed on the side of each spectrum.

3-1 TRAPEZOIDAL MODULATING PULSE

The mathematical representation for a trapezoidal amplitude modulating pulse is :

$$A(t) = 0 \qquad \text{for} \quad t<0 \ , \ T<t$$

$$A(t) = \frac{t}{A2} \qquad 0<t<A2$$

$$= 1 \qquad A2 \leq t \leq T-B2$$

$$= \frac{T-t}{B2} \qquad T-B2<t<T \qquad (3.1)$$

where A2 = rise time of the pulse

B2 = fall time of the pulse

T = pulse width normalized to be 1

The mathematical representation for a trapezoidal frequency modulating pulse is :

$$g(t) = 0 \qquad t \leq 0, \quad T \leq t$$

$$= BM \frac{t}{A1} \qquad 0<t<A1$$

$$= BM \qquad A1 \leq t \leq T-B1$$

$$= BM \frac{(T-t)}{B1} \qquad T-B1<t<T \qquad (3.2)$$

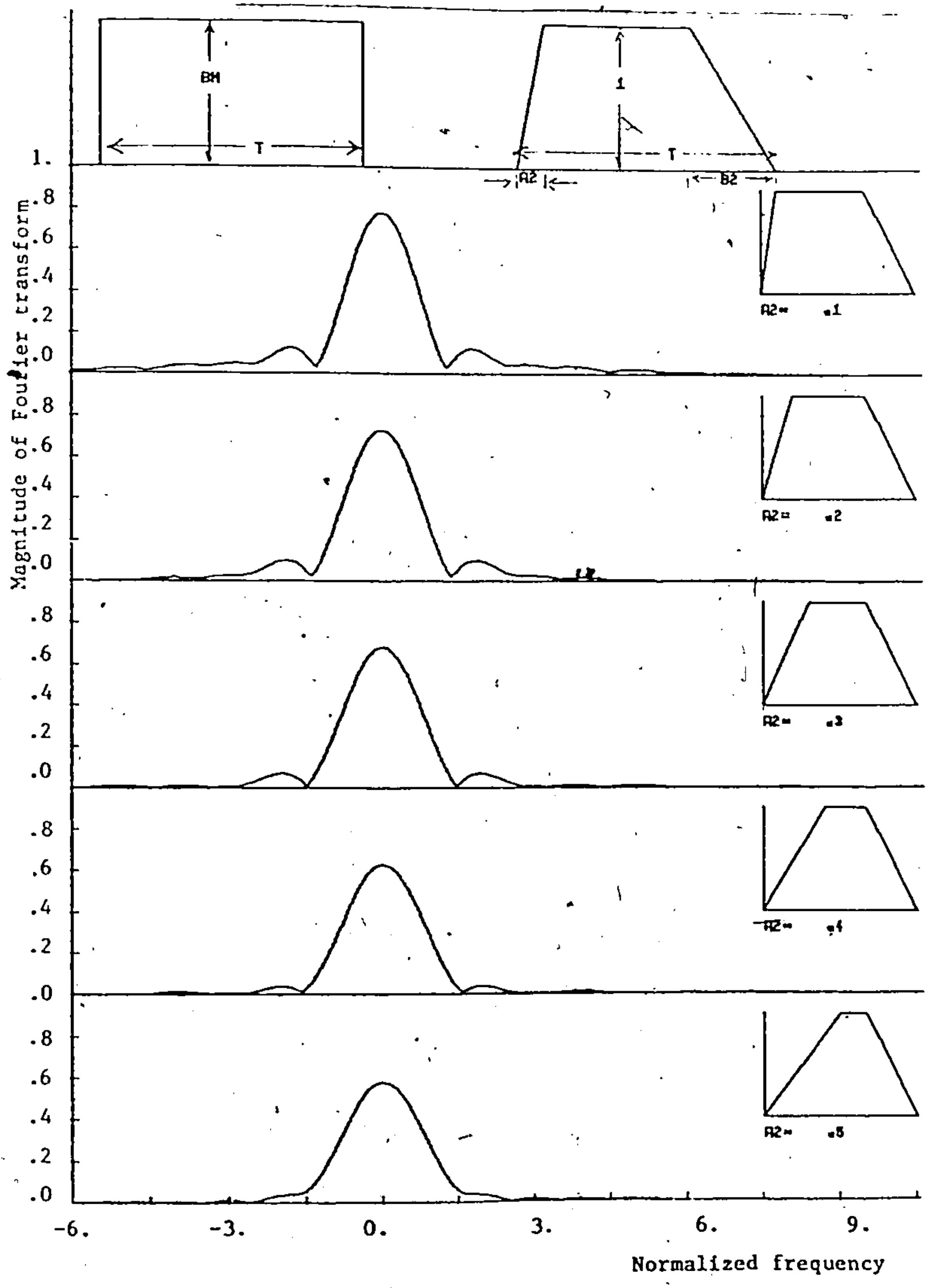
where BM = maximum frequency deviation

A1 = rise time of the pulse

B1 = fall time of the pulse

In Fig. 3.1, the pulse has trapezoidal amplitude modulation and no frequency modulation (BM=0). The resulting spectrum is symmetric no matter how the parameter A2, the rise time, is varied. The rest of the parameters remain constant with the values as shown on the figure. It can be observed that when A2 increases, the side-lobe levels decrease at an increasing rate. The main-lobe also

Fig. 3.1 Rect. FM and Trap. AM, T=1, B2=0.33



decreases but at a slower rate . The effect of spectrum broadening is symmetric on both sides of the RF frequency.

In Fig. 3.2, 3.3 and 3.4, the signal considered is frequency modulated by a trapezoidal pulse and amplitude modulated by a rectangular pulse. The only parameter that is varied in each figure is the rise time A1 of the frequency modulating pulse. It is observed that the spectrum becomes increasingly asymmetrical with increasing A1. We see that the main lobe decreases in height while the lower side lobes increase. Also, the position of the main lobe shifts to a lower frequency. When the frequency modulating pulse becomes more asymmetrical, one can also observe, both in Fig. 3.2 and 3.3, that the local minima at the lower frequency range of the spectrum rise from the zero point to some non-zero value. This phenomenon becomes more pronounced when the frequency modulating pulse becomes more asymmetrical. It should be noted that the area under the spectrum appears to have a constant value. Also the cases, when B1=0.1 with A1=0.5 on Fig.3.2, and B1=0.5 with A1=0.1 on Fig. 3.3, do actually give rise to the same spectrum. These two examples verify properties D and A, respectively.

These phenomena can be explained by considering the distribution of energy in the modulated pulse. When A1 increases, a deceased portion of the modulated pulse is maintained at the constant

Fig. 3.2 Trap. FM and Rect. AM, T=1, BM=3.5, B1=0.1

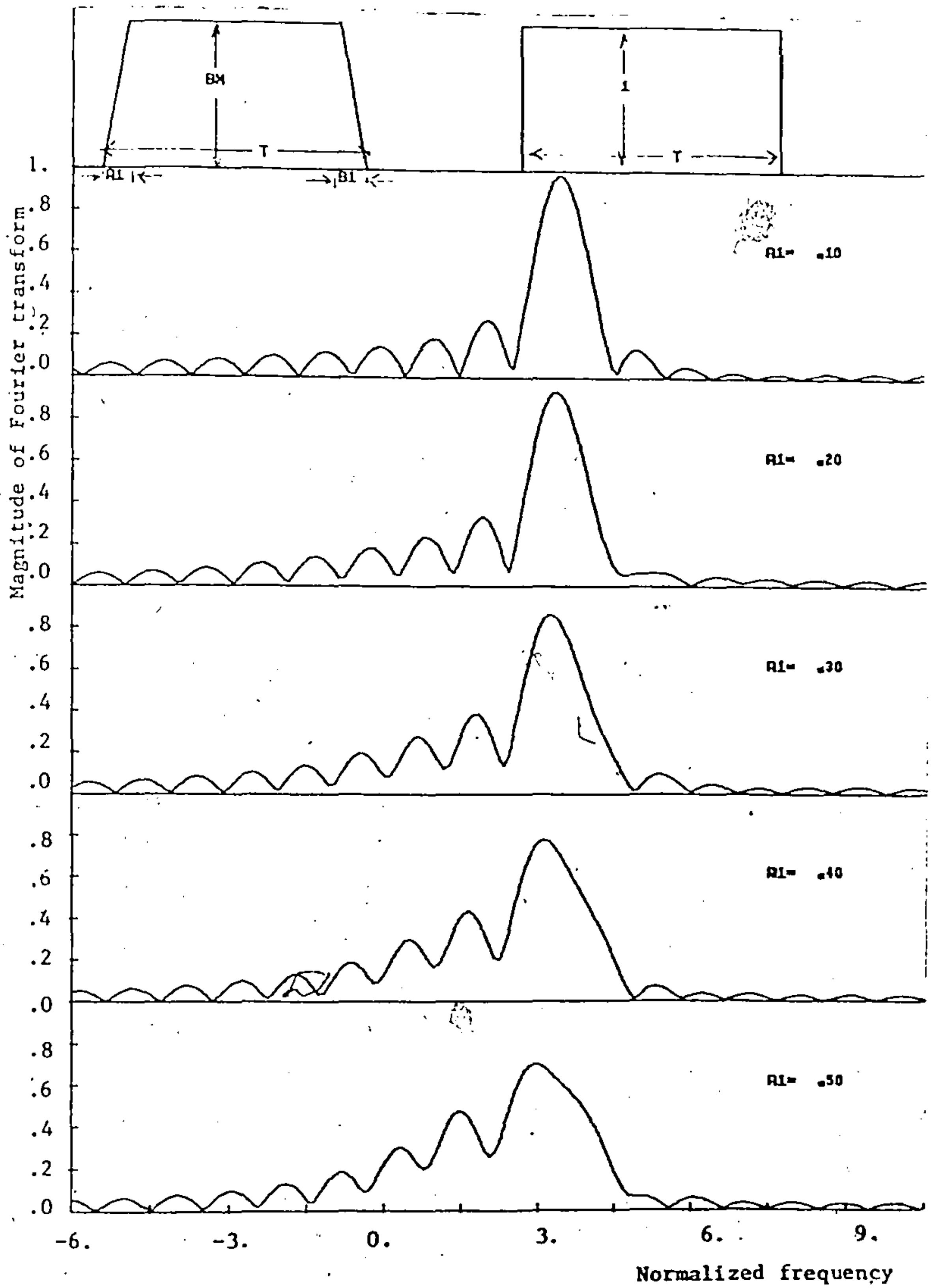


Fig. 3.3 Trap. FM and Rect. AM, T=1,BM=3.5,B1=0.5

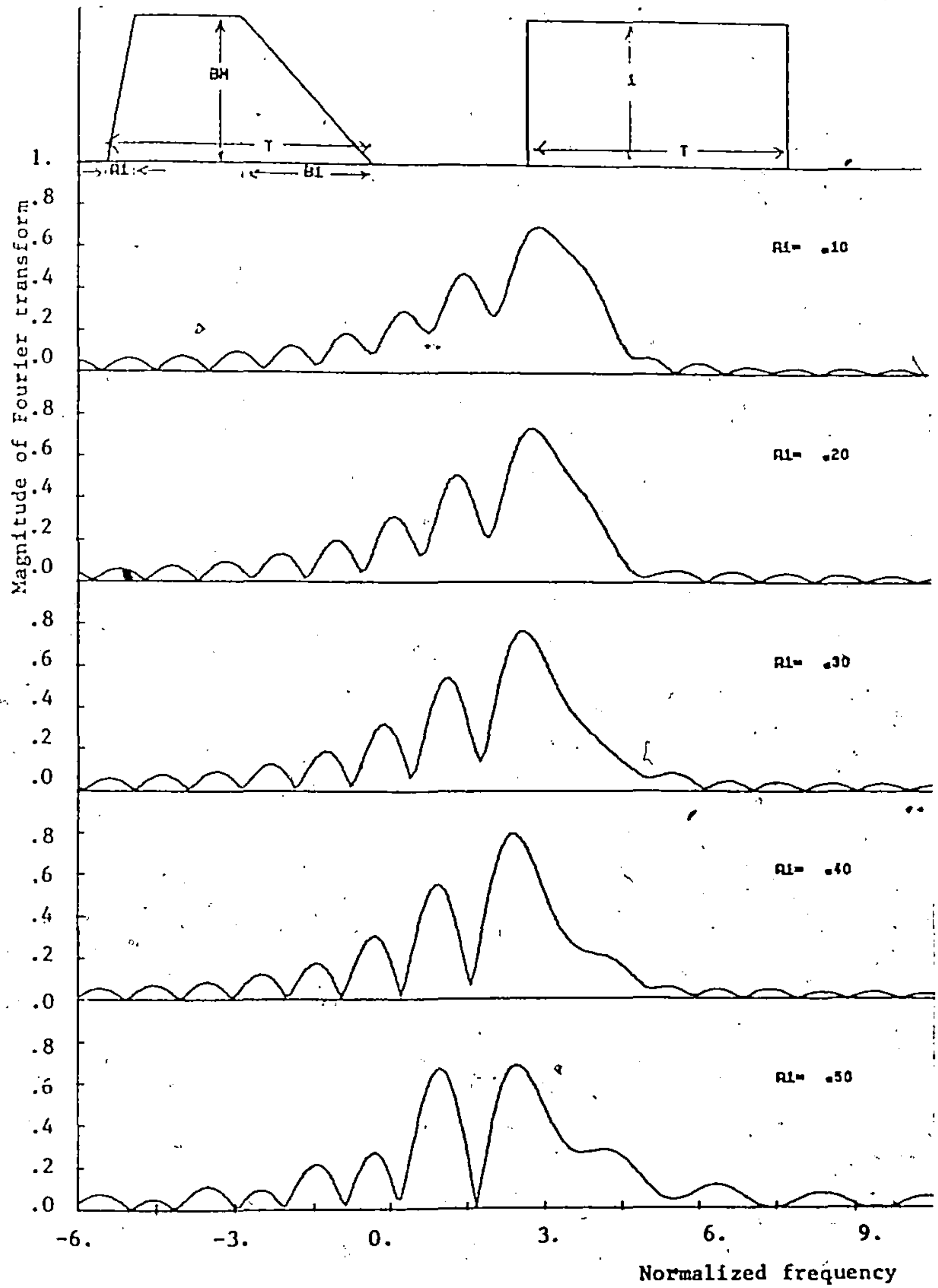
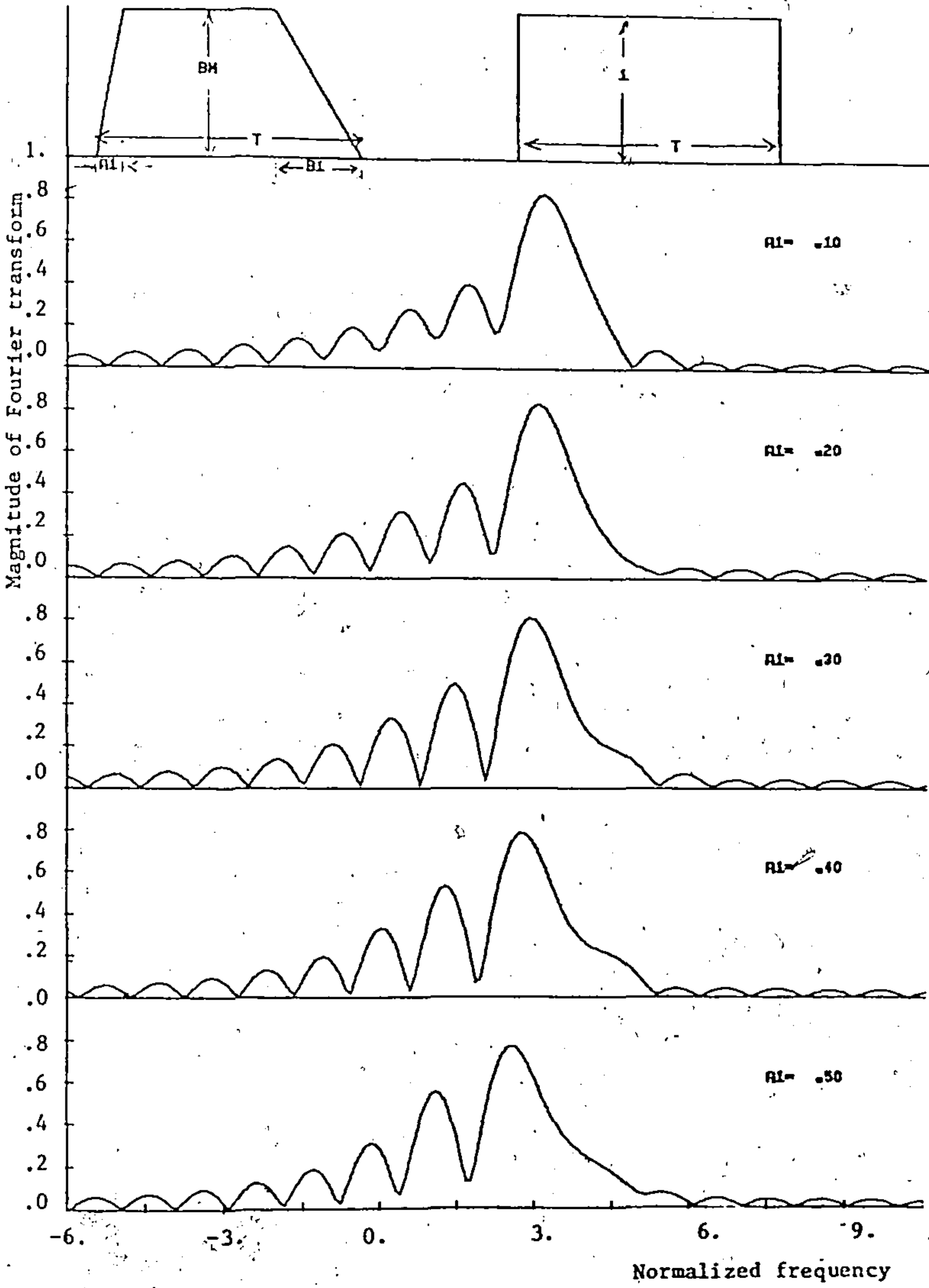


Fig. 3.4 Trap. FM and Rect. AM, T=1,BM=3.5,B1=0.33



frequency BM. Accordingly, the main-lobe of the spectrum, which is situated near the maximum frequency deviation BM=3.5, decreases in magnitude. The reduction in the energy of the main-lobe is compensated by an increase in the level (and therefore energy) of the lower side-lobe. The rise in height of the local minima as the frequency modulating pulse becomes more asymmetrical, will be explained in section 4.2, using the analytical approach.

In Figs. 3.5,3.6 and 3.7, three different trapezoidal pulses are used to modulate the frequency and amplitude of the signal. The only parameter that is varied in each figure is A2, the rise time of the amplitude modulating pulse. The same phenomena that were observed previously can also be seen in these figures. However, we observe that as A2 increases, the lower side-lobes decrease in height at a faster rate than the main-lobe. This is due to the increasing attenuation, on the rising edge of the modulated pulse. Comparing the top spectra in Figs. 3.5 to 3.7 which correspond to maintaining the rise time A2 constant at 0.1 and increasing the fall time B2, we observe that the height of the lower side-lobe decreases at a faster rate than the main-lobe.

When we compare the spectra in Figs. 3.5 to 3.7 with those in Figs. 3.2 to 3.4, we observe that the main-lobe positions, in the spectra of the first set of figures, are located closer to the peak

Fig. 3.5 Trap. FM and Trap. AM, T=1,BM=3.5,A1=0.17,B1=0.33,B2=0.1

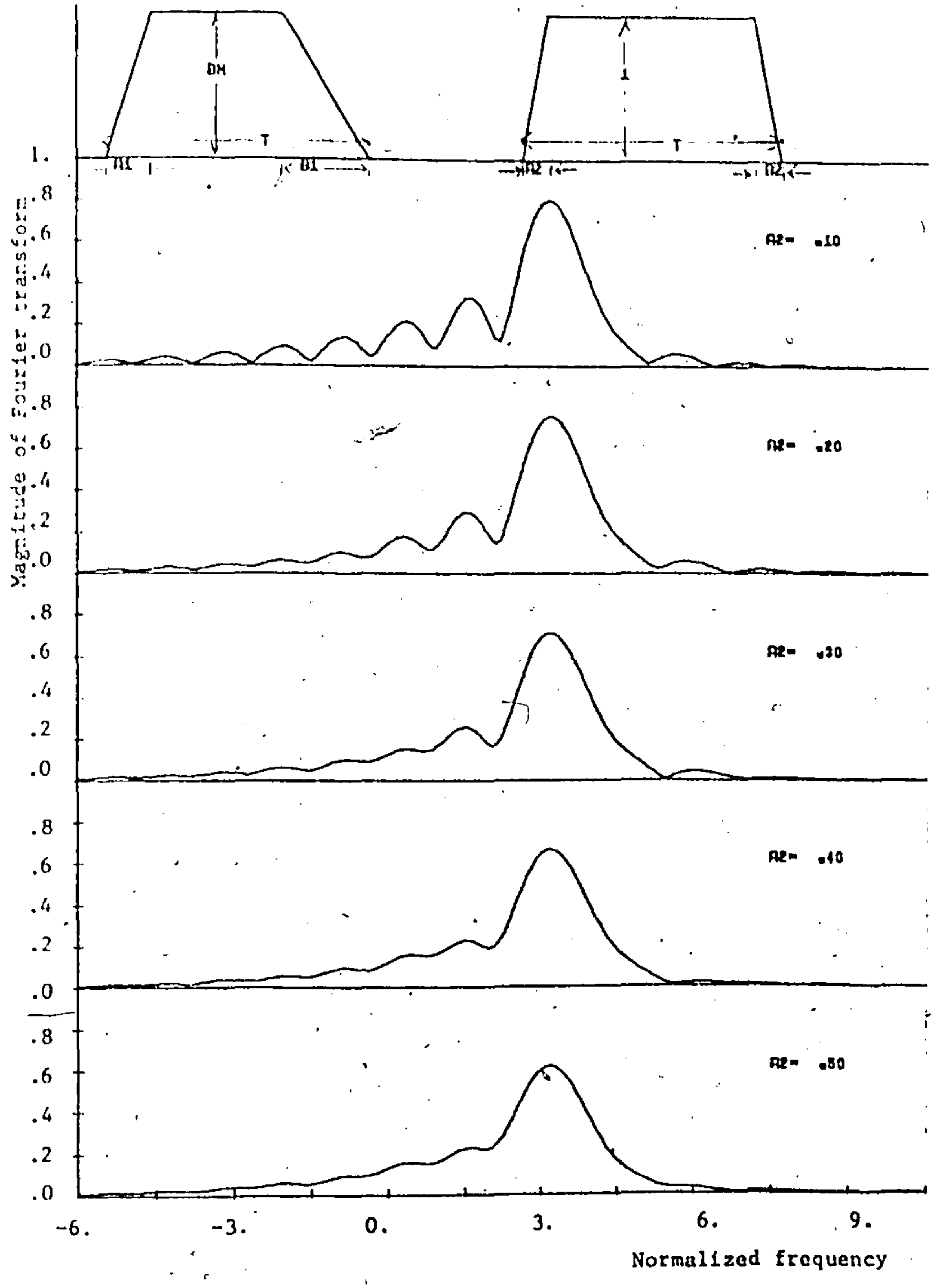


Fig. 3.6 Trap. FM and Trap. AM, T=1,BM=3.5,A1=0.17,B1=0.33,B2=0.33

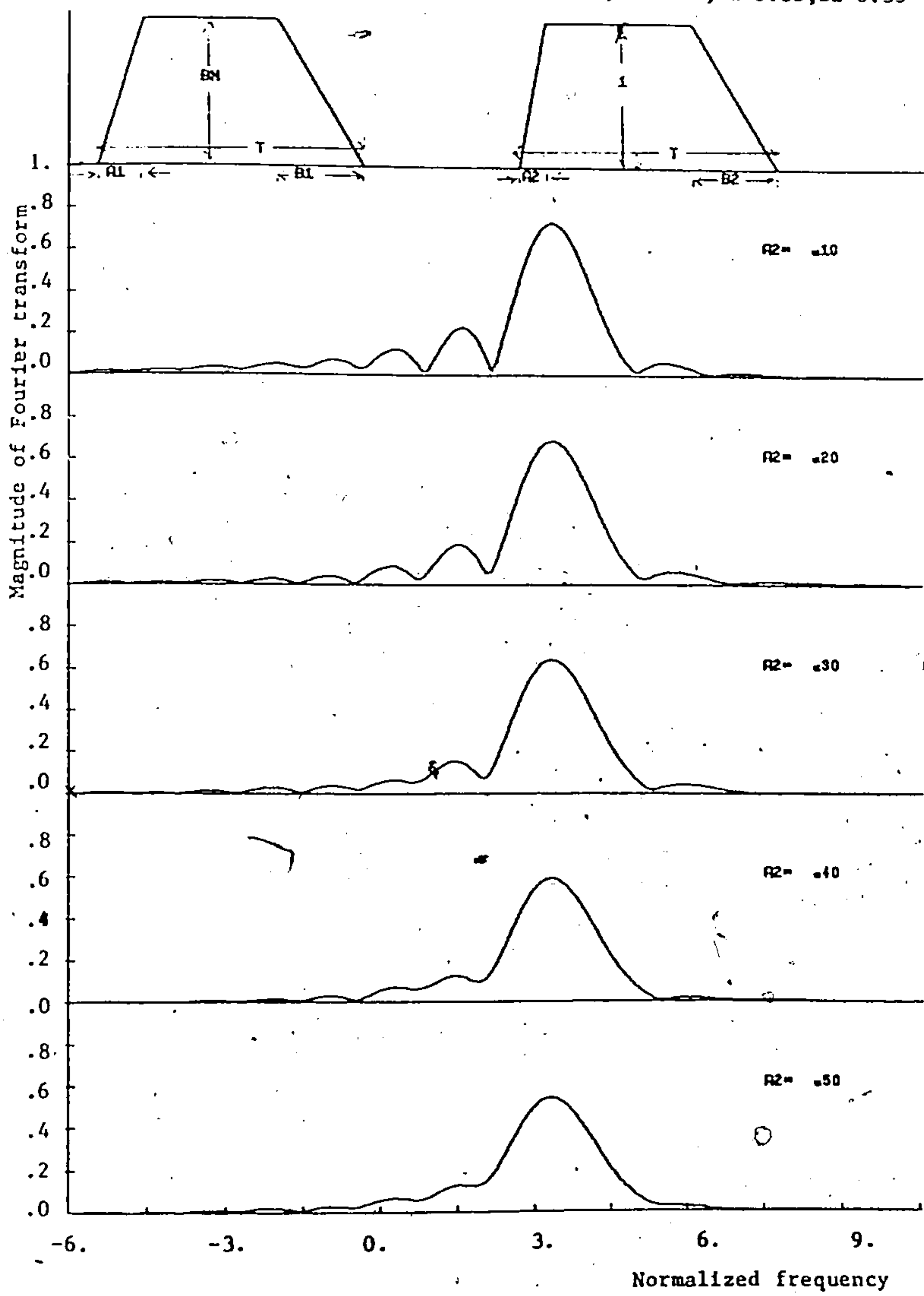
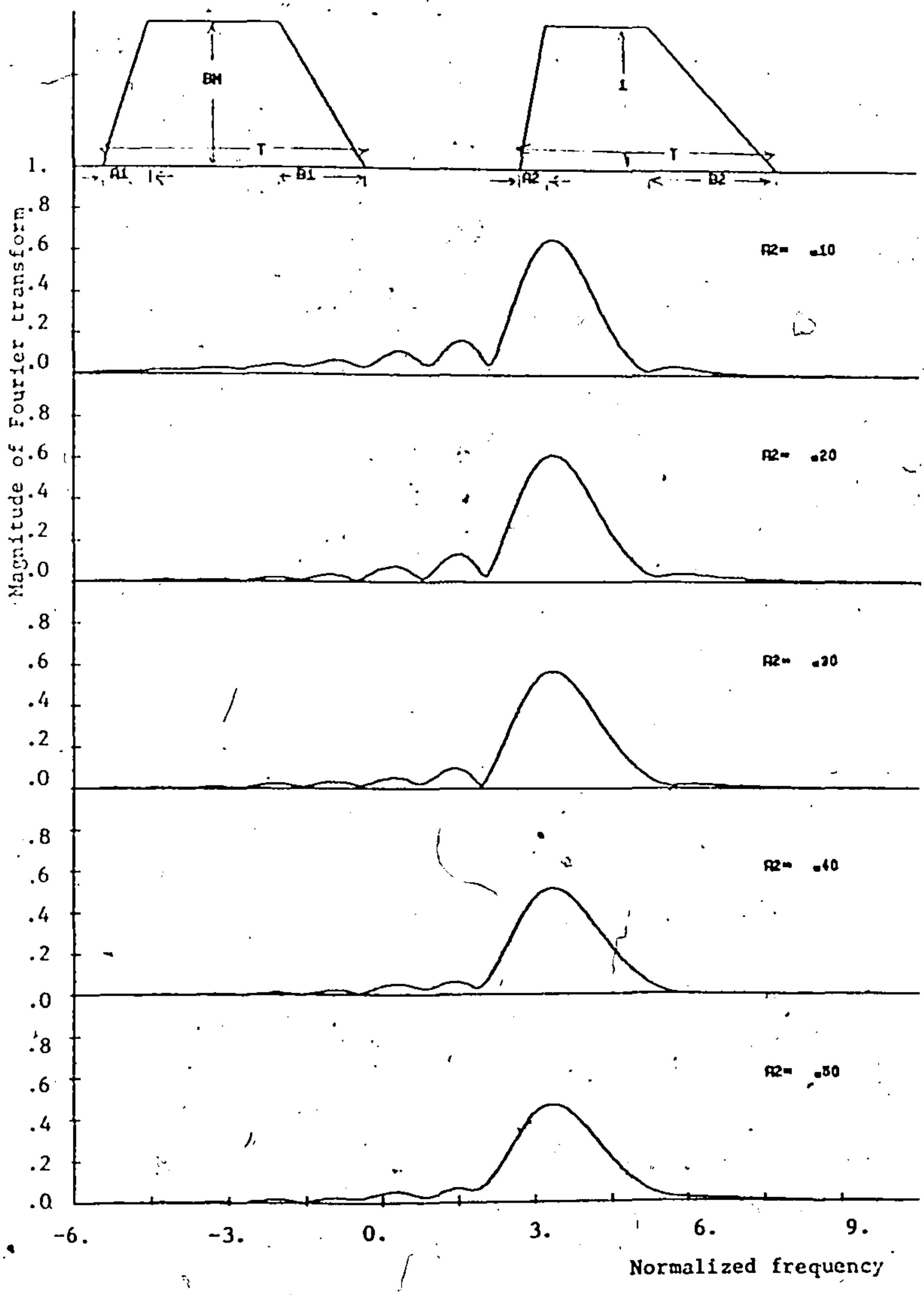


Fig. 3.7 Trap. FM and Trap. AM, T=1,BM=3.5,A1=0.17,B1=0.33,B2=0.5



frequency deviation than the spectra in the second set of figures. The side-lobe levels also decreases significiently.

In Figs. 3.8 and 3.9, the signal is frequency modulated by a trapezoidal pulse. However, the signal in Fig. 3.8 is amplitude modulated by a rectangular pulse, as distinct from the trapezoidal pulse used in Fig. 3.9. The varying parameter in both figures is the maximum frequency deviation. In both cases, we observe the phenomenon of spectrum broadening. However, because of the linear attenuation in the rising and falling edges of the amplitude modulating pulse in Fig. 3.9, the side-lobe levels in Fig. 3.9 are lower than the corresponding one in Fig. 3.8. This phenomenon of spectrum broadening will be treated analytically in section 4.2.

In Fig. 3.10, the amplitude modulating pulse is kept fixed with a rectangular waveform. In Fig. 3.11, the amplitude modulating pulse is of the same shape as the frequency modulating pulse. In both cases, the waveshape of the frequency modulating pulse is varied from a symmetric one about T/2 to an asymmetric one about T/2. The corresponding spectra, as shown in these figures, demonstrate that the levels of the local minima of the spectra increase when the frequency modulating pulse or the amplitude modulating pulse or both

Fig. 3.8 Trap. FM and Rect. AM, T=1,A1=0.17,B1=0.33

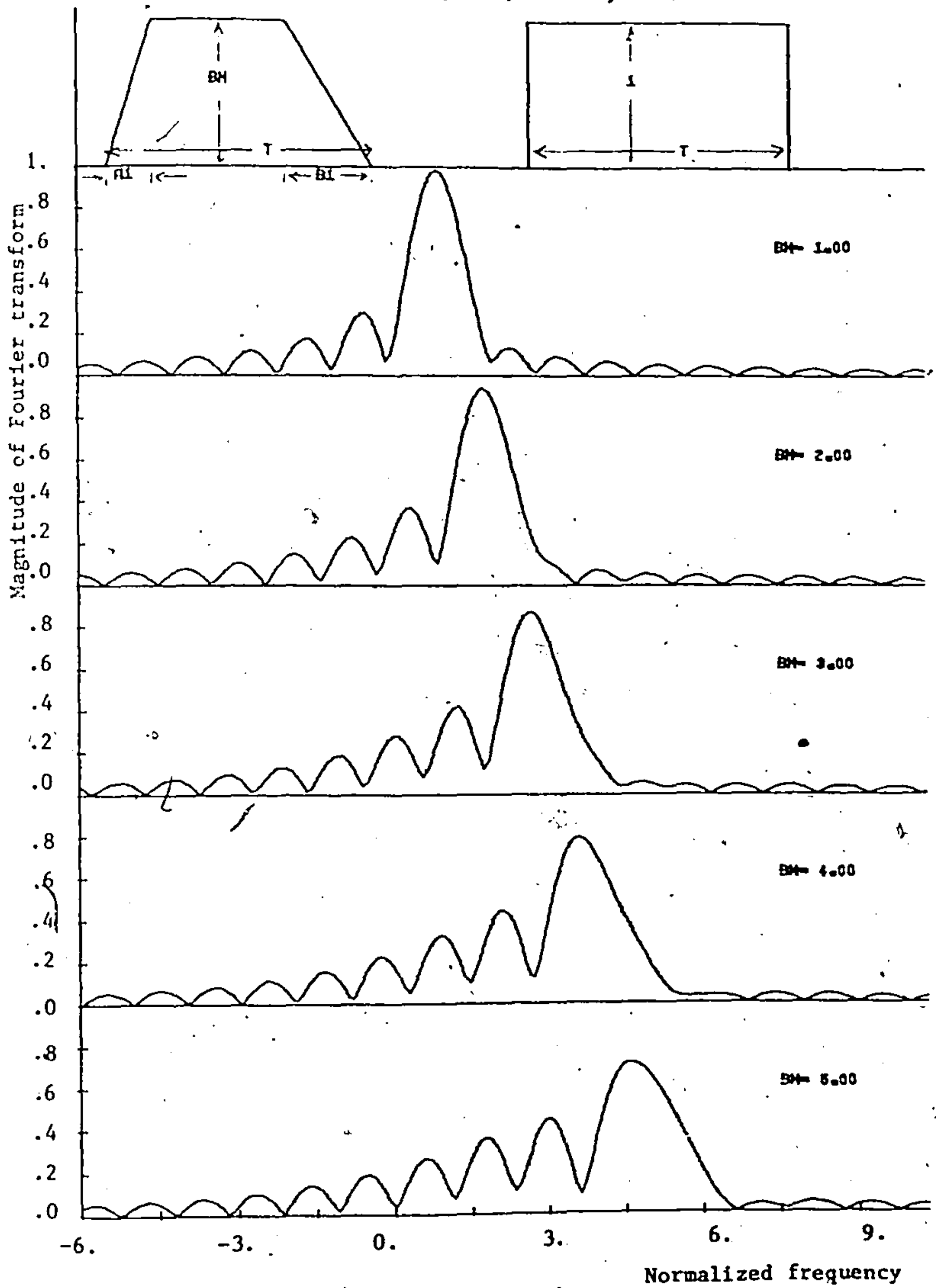


Fig. 3.9 Trap. FM and Trap. AM, T=1,A1=0.17,B1=0.33,A2=0.17,B2=0.33

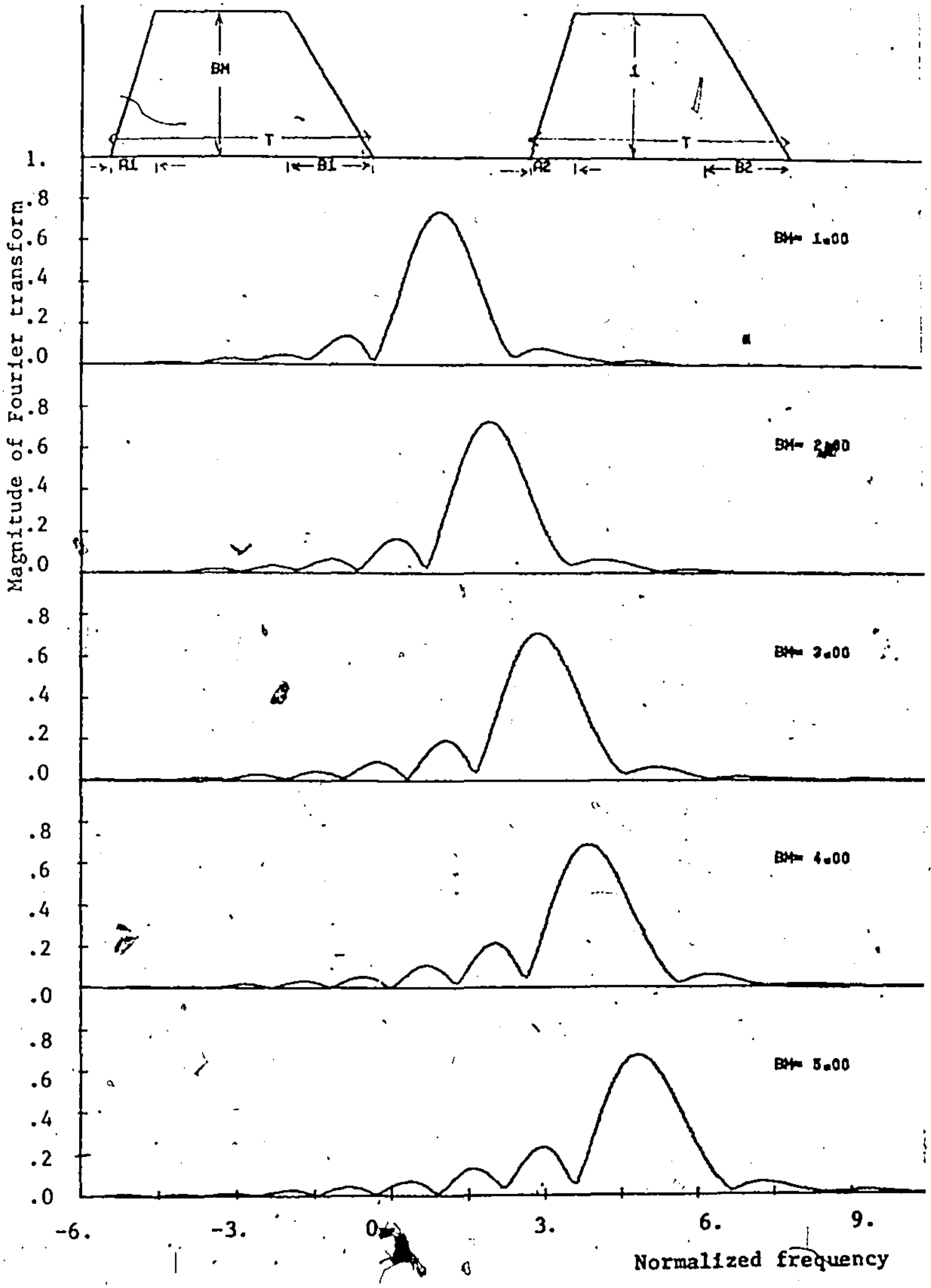


Fig. 3.10 Trap. FM and Rect. AM, T=1, BM=3.5

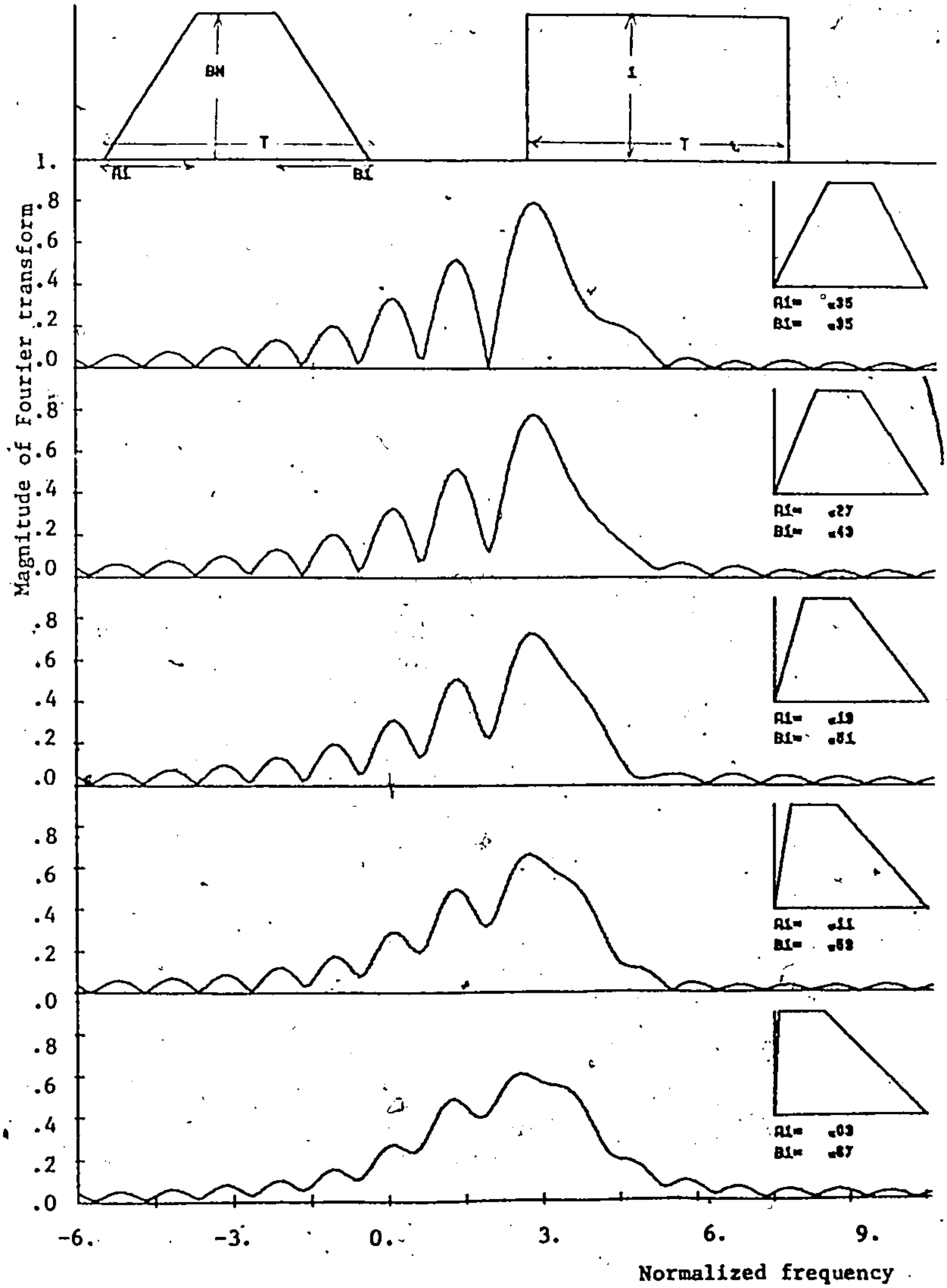
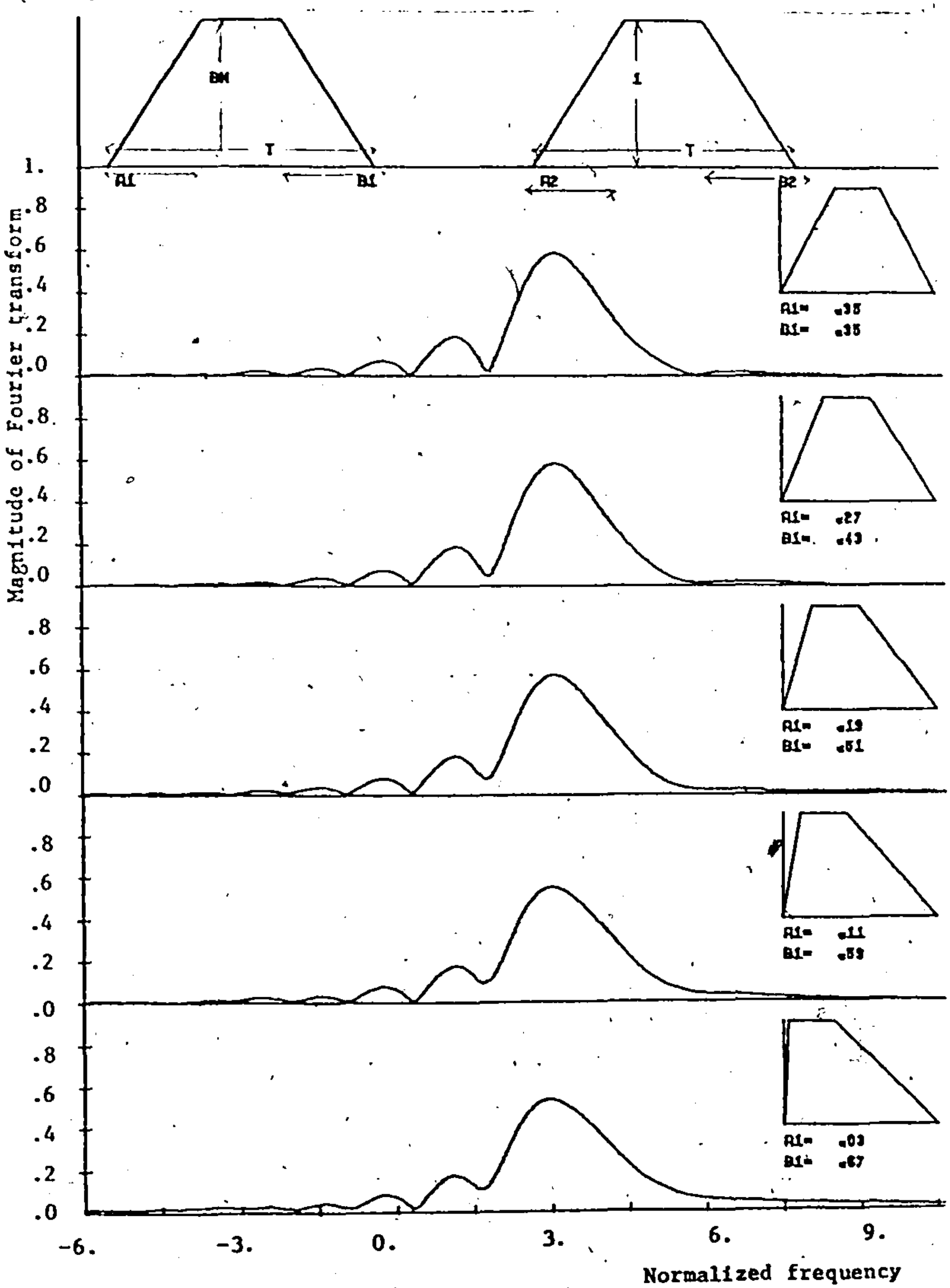


Fig. 3.11 Trap. FM and Trap. AM, T=1,BM=3.5,A1=A2,B1=B2



become more asymmetrical with respect to the mid-point (T/2) of the pulse.

3-3 EXPONENTIAL MODULATING PULSE

The mathematical representation of an exponential amplitude modulating pulse is as follows :

$$A(t) = 0 \qquad \text{when } t<0,\ T<t$$

$$= \frac{1-e^{-C2 \cdot t}}{1-e^{-C2 \cdot A2}} \qquad 0 \leq t \leq A2$$

$$= 1 \qquad A2<t<T-B2$$

$$= \frac{1-e^{-K2(T-t)}}{1-e^{-K2 \cdot B2}} \qquad T-B2 \leq t \leq T \qquad (3.3)$$

where A2 = rise time

B2 = fall time

T = pulse width

C2 = exponential rise constant

K2 = exponential fall constant

Similarly the mathematical representation of an exponential frequency modulating pulse is :

$$g(t) = 0 \qquad \text{when } t<0,\ T<t$$

$$= BM\,\frac{1-e^{-C1 \cdot t}}{1-e^{-C1 \cdot A1}} \qquad 0 \leq t \leq A1$$

$$= BM \qquad A1<t<T-B1$$

Fig. 3.12 Rect. FM and Exp. AM, T=1,A2=B2=0.45,K2=10

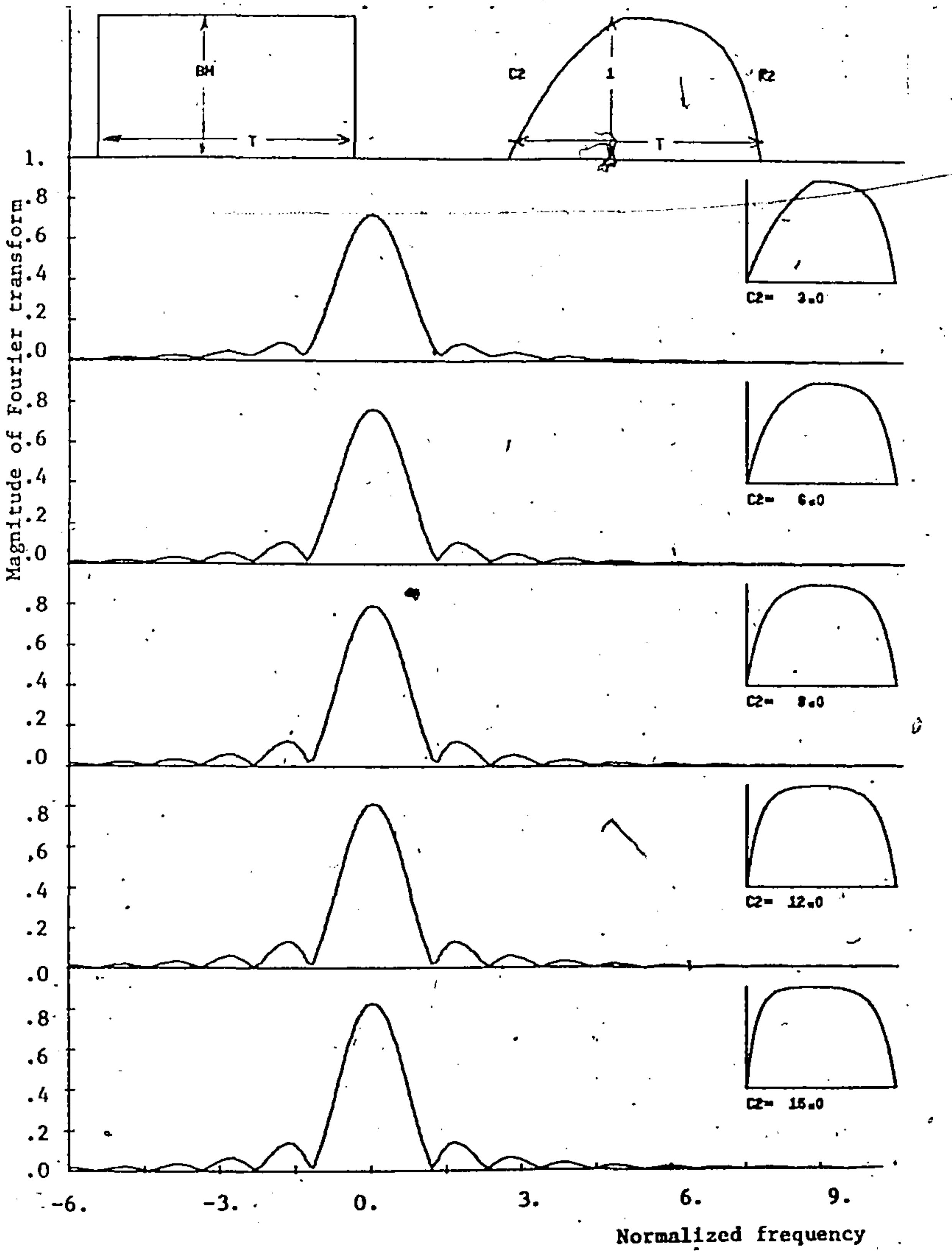


Fig. 3.13 Exp. FM and Rect. AM, T=1,BM=3.5,A1=0.5,B1=0.5,C1=K1

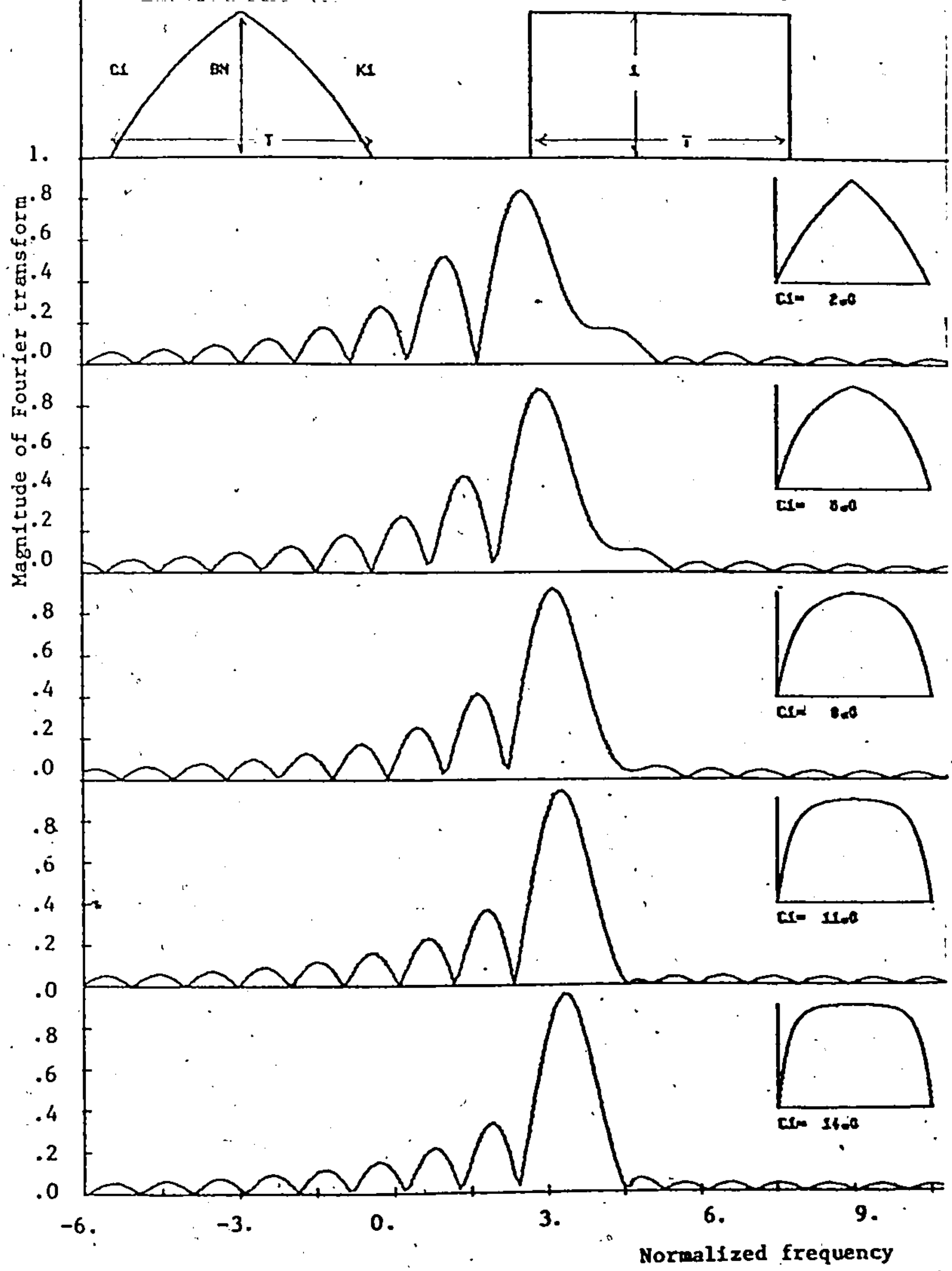


Fig. 3.14 Exp. FM and Exp. AM, T=1,BM=3.5,A1=A2=B1=B2=0.5,C1=K1=2,C2=K2

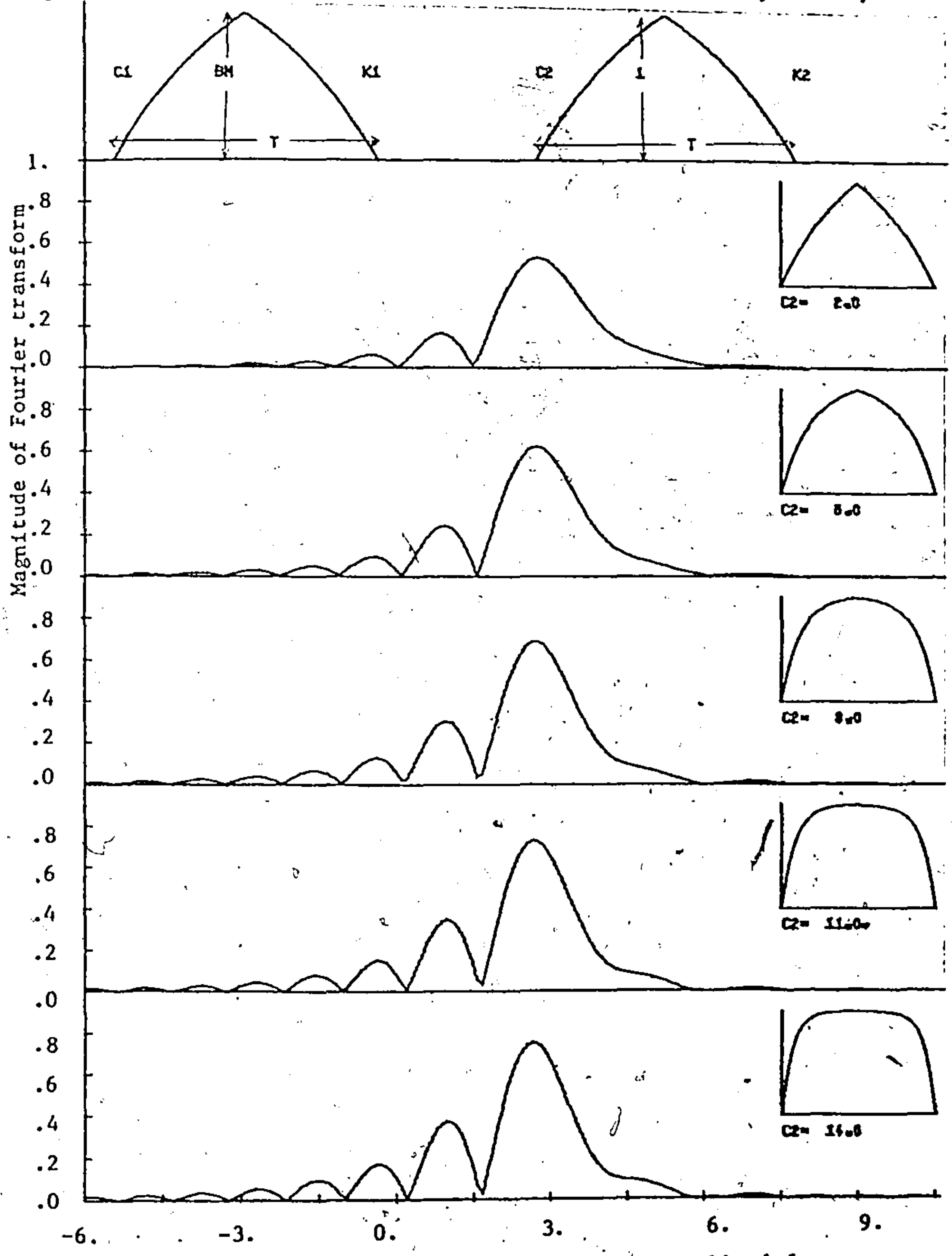


Fig. 3.15 Exp. FM and Exp. AM, T=1,BM=3.5,A1=A2=B1=B2=0.5,C1=K1=11,C2=K2

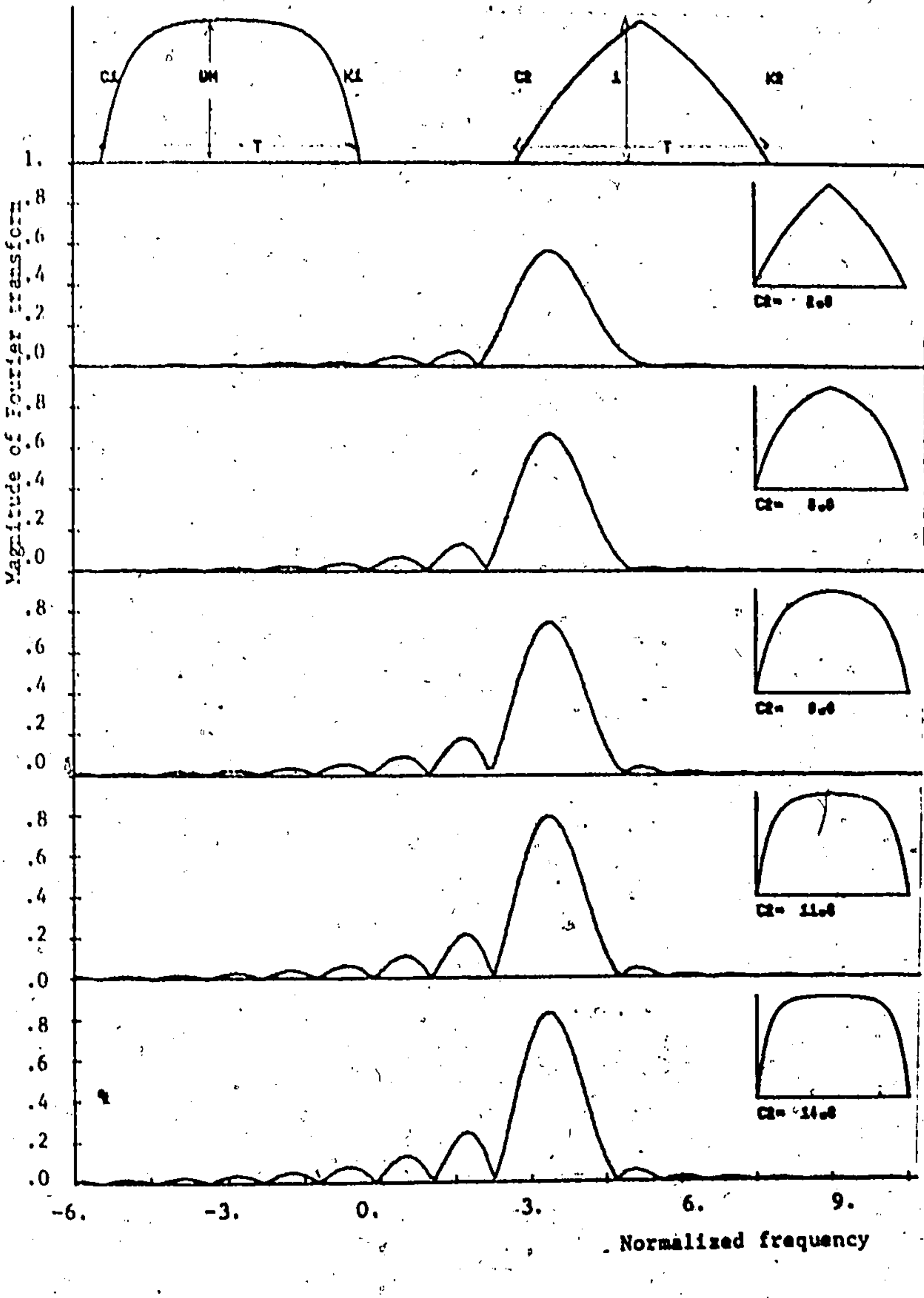


Fig. 3.16 Exp. FM and Rect. AM, T=1,A1=B1=0.33,C1=10,K1=6

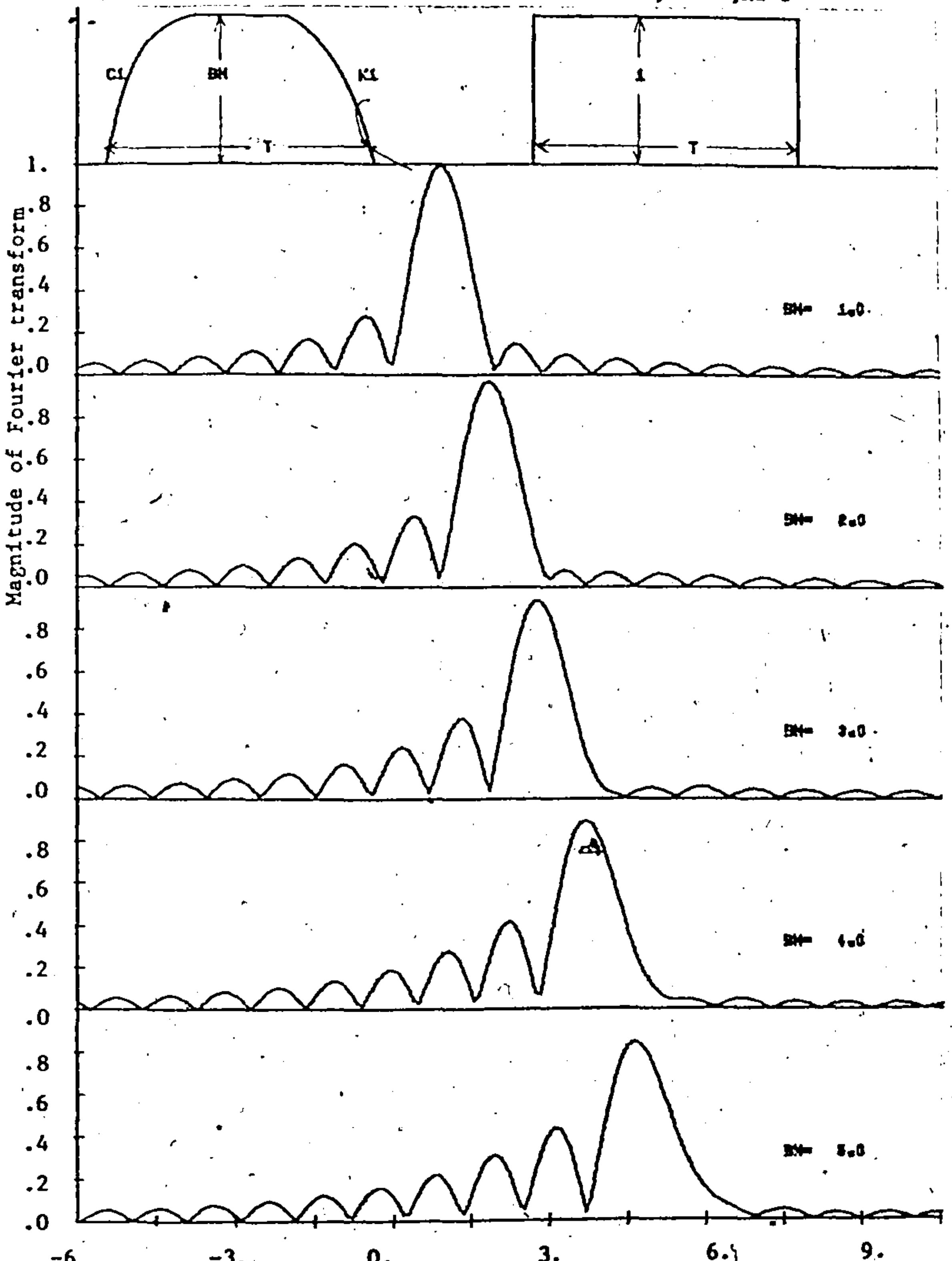
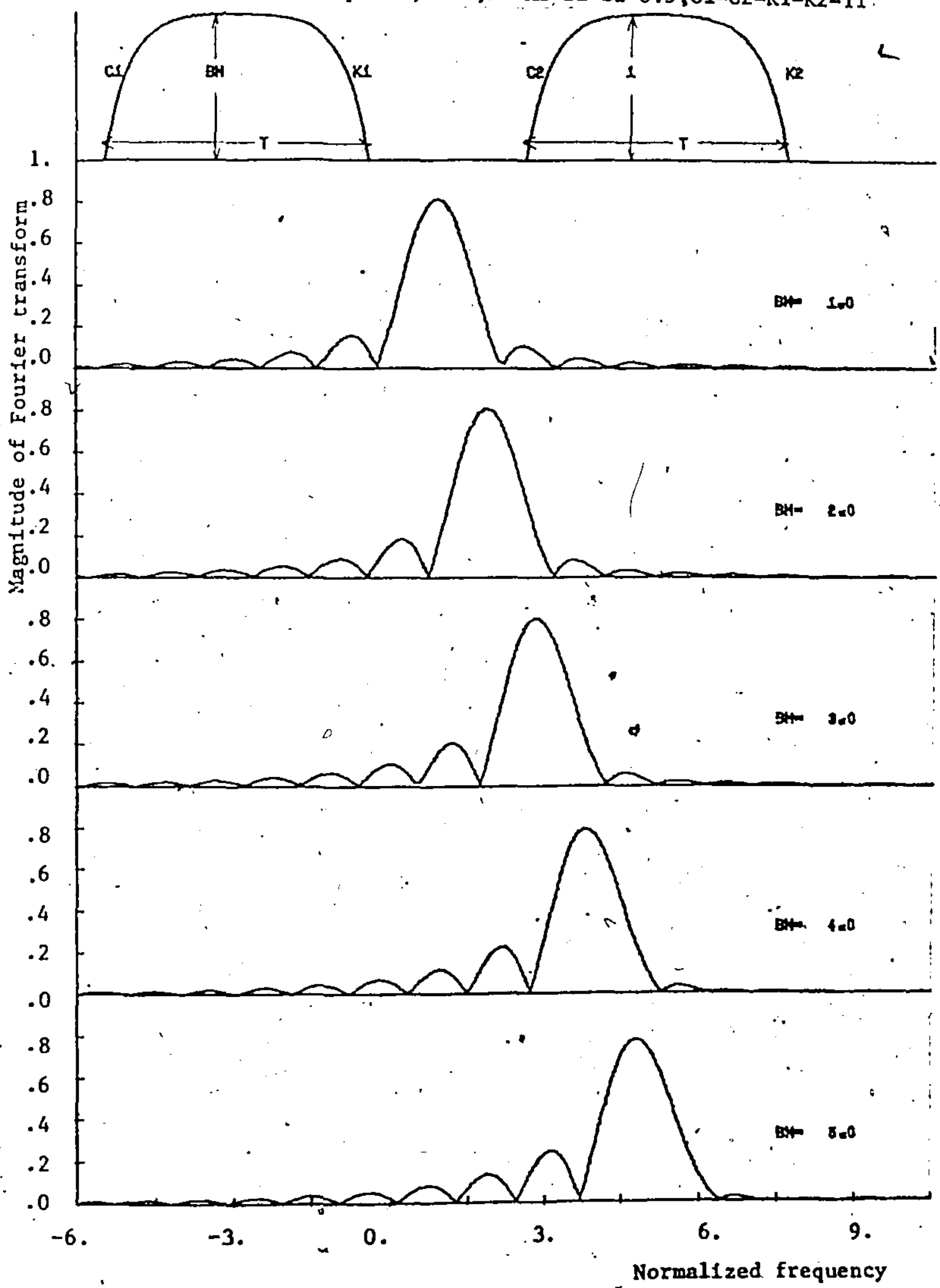


Fig. 3.17 Exp. FM and Exp. AM, T=1,A1=A2=B1=B2=0.5,C1=C2=K1=K2=11



$$g(t) = BM \frac{1-e^{-C1\cdot(T-t)}}{1-e^{-C1\cdot B1}} \qquad \text{when} \quad T-B1\leq t\leq T \qquad (3.4)$$

where A1 = rise time

B1 = fall time

C1 = exponential rise constant

K1 = exponential fall constant

In Figs. 3.12 to 3.17, the signals are modulated by exponential pulses and rectangular pulses as shown on the top of each figure. The various kinds of phenomena that were observed in Figs. 3.1 to 3.11 on signals modulated by a trapezoidal pulse are also observed in Figs. 3.12 to 3.17.

3-4 SINUSOIDAL MODULATING PULSE

Mathematical representation for a sinusoidal amplitude modulating pulse is as follows :

$$A(t) = 0 \qquad \text{when} \quad t\leq 0, \quad T\leq t$$
$$= \sin(\pi t/T) \qquad 0\leq t\leq T \qquad (3.5)$$

Similarly, the mathematical representation for a sinusoidal frequency modulating pulse is :

$$g(t) = 0 \qquad \text{when} \quad t<0, \quad T<t$$
$$= \sin(\pi t/T) \qquad 0\leq t\leq T \qquad (3.6)$$

Fig. 3.18 Sin. FM and Rect. AM, T=1

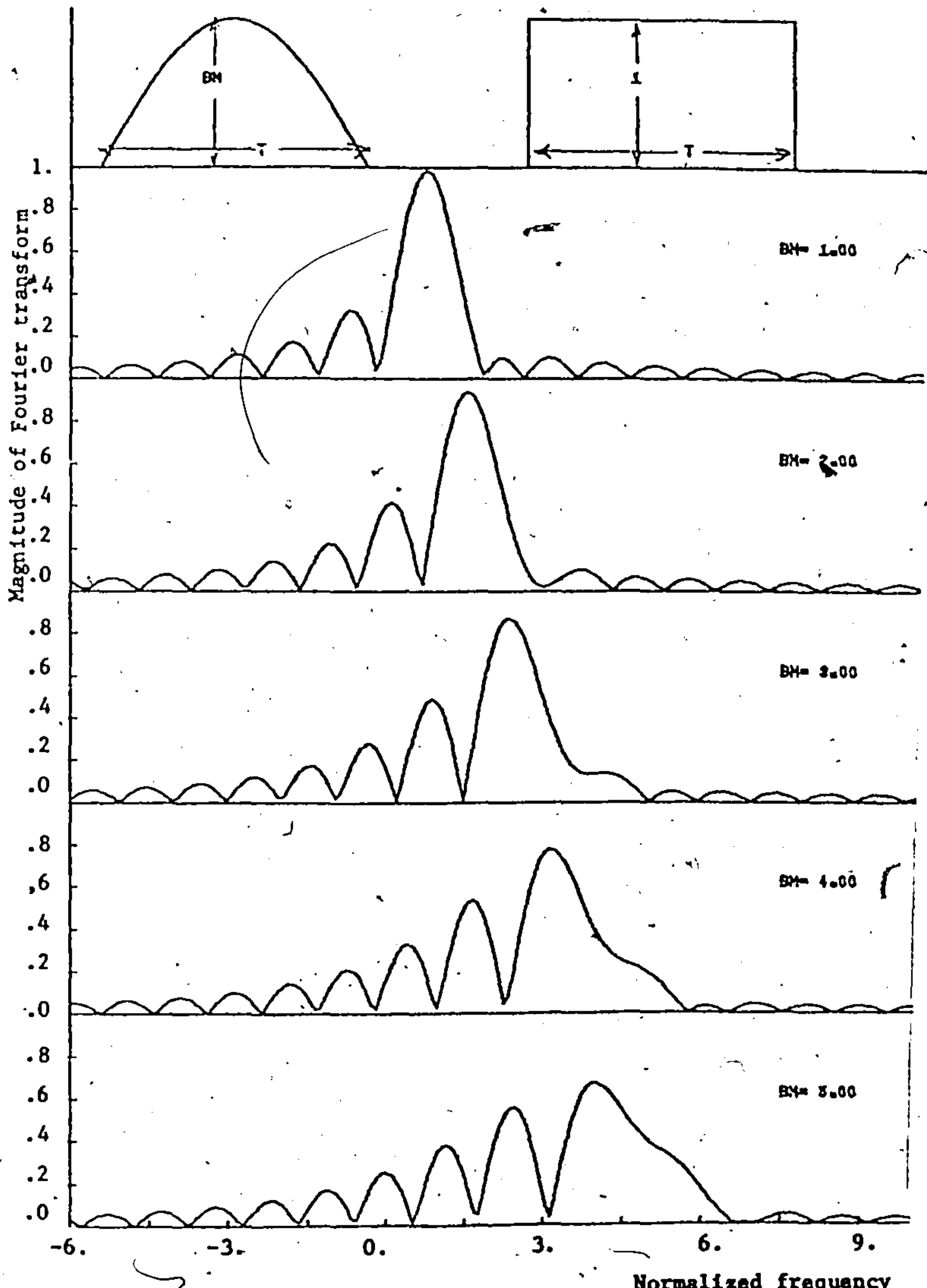
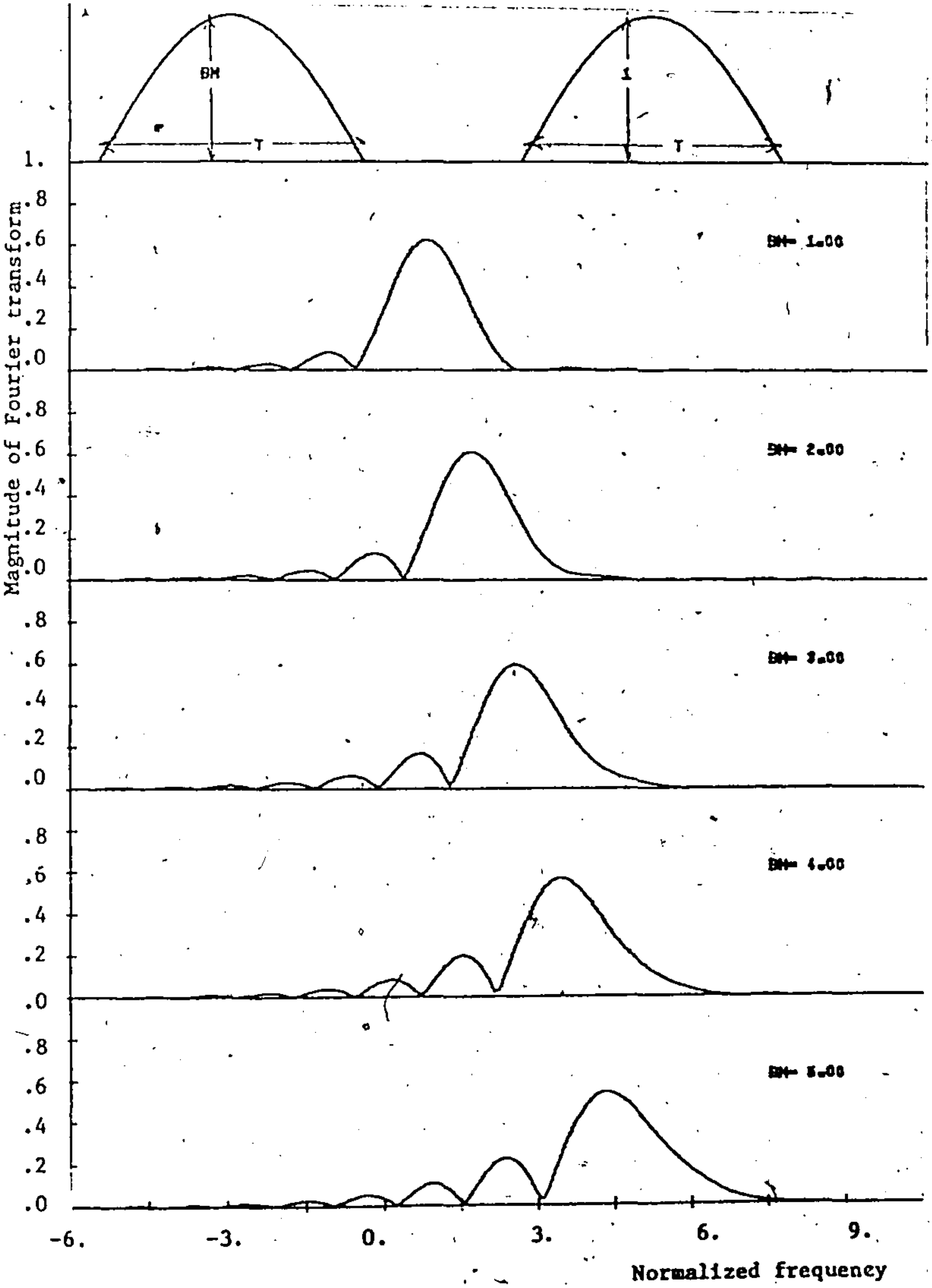


Fig. 3.19 Sin. FM and Sin. AM, T=1



where T = pulse width normalized to unity.

BM = maximum frequency deviation.

In Figs. 3.18 and 3.19, the pulses are modulated in frequency by a sinusoidal pulse. In Fig. 3.18, the amplitude modulating pulse is of rectangular waveshape, whereas in Fig. 3.19, the amplitude modulating pulse is of sinusoidal waveshape. The varying parameter is the maximum frequency deviation BM. When BM increases, the phenomenon of spectrum broadening, similar to that observed previously, do reoccur. However in Figs. 3.18 and 3.19, we observe that the main-lobe is located further away from the maximum frequency deviation. The phenomena of increasing side-lobes as BM increases, and the lowering of side-lobes with the application of the sinusoidal amplitude modulation can also be observed here.

3-5 MODIFIED GAUSSIAN MODULATING PULSE

The mathematical representation of the modified gaussian amplitude modulating pulse is as follows :

$$A(t) = 0 \qquad \text{when} \quad t \leq 0,\ T \leq t$$

$$= \frac{e^{-K2\left(t-\frac{T}{2}\right)^2} - e^{-K2\frac{T^2}{4}}}{1 - e^{-K2\frac{T^2}{4}}} \qquad 0 \leq t \leq T \qquad (3.7)$$

where K2 = the constant that determines the shape of the pulse.

The maximum amplitude of the pulse is normalized to unity.

Similarly, the mathematical representation of the modified gaussian frequency modulating pulse is

$$g(t) = 0 \qquad \text{when} \quad t<0,\ T<t$$

$$= BM \frac{e^{-K1(t-\frac{T}{2})^2} - e^{-K1\frac{T^2}{4}}}{1 - e^{-K1\frac{T^2}{4}}} \qquad 0\leq t\leq T \qquad (3.8)$$

where K1 = the constant that determines the shape of the pulse

BM = the maximum frequency deviation.

In Figs. 3.20 to 3.24, the pulse is frequency modulated by a modified gaussian pulse, and amplitude modulated by either a rectangular pulse or another modified gaussian pulse. The gaussian pulse is modified such that the pulse has zero magnitude when t=0 and t=T. The various phenomena that were mentioned previously can be observed here too. However, we now see that the lower side-lobe becomes comparable in height with the main lobe or even larger in height. Such a behaviour occurs in Fig. 3.21, when the frequency modulating pulse behaves more like an impulse (when K1>25 in Fig. 3.21), and in Fig. 3.23, when the maximum frequency deviation is large. The phenomenon of shifting in location of the main-lobe to the lower frequency, when the rise time and fall time of the frequency modulating pulse increases, can also be observed in Figs. 3.20 to 3.24. This phenomenon is, however, more pronounced here, because the rise time and fall time are quite long compared with those used in the other frequency modulating pulses considered earlier. The

Fig. 3.20 Rect. FM and Modified Gaussian AM, T=1

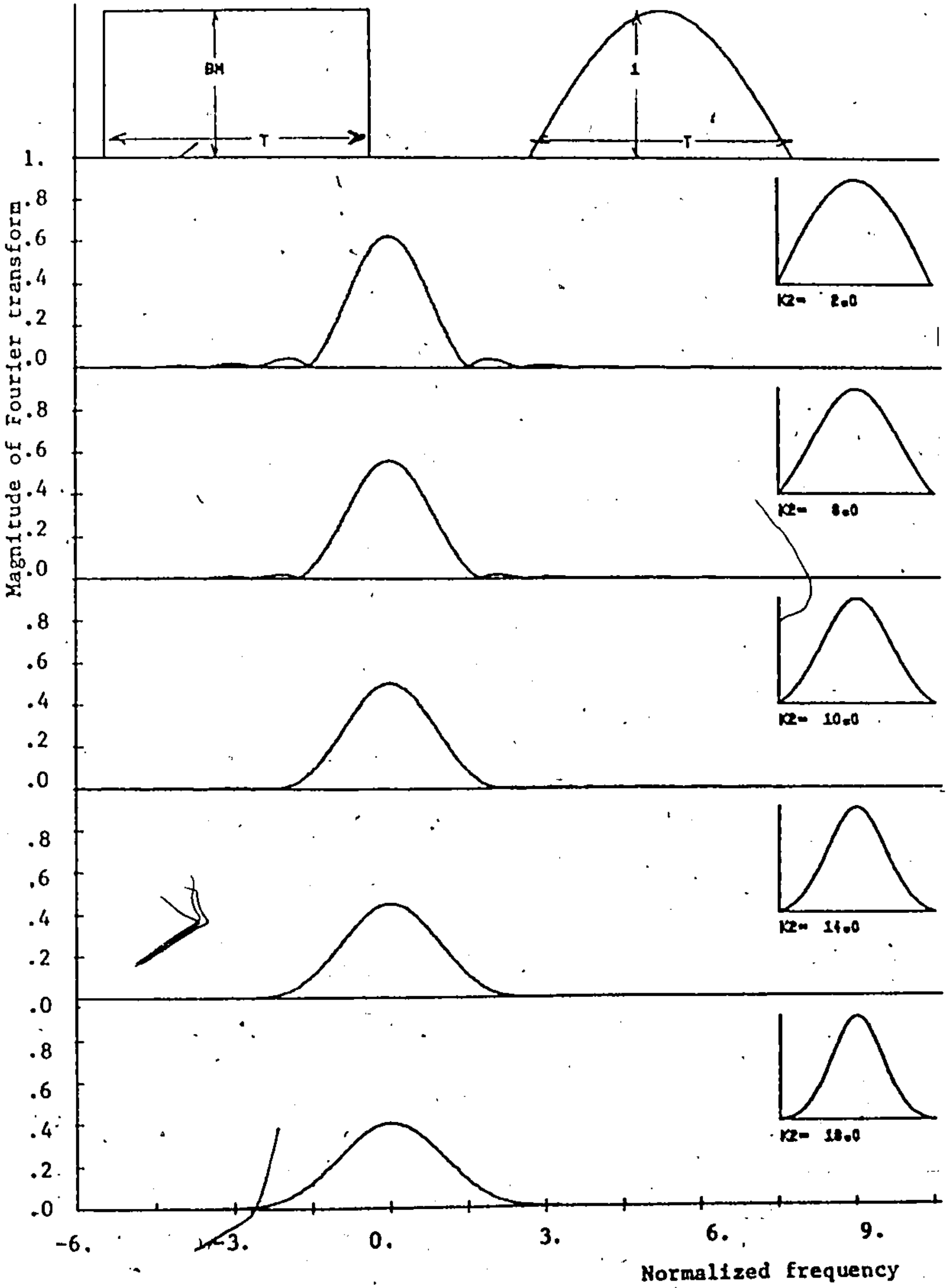


Fig. 3.21 Modified Gaussian FM and Rect. AM, T=1,BM=3.5

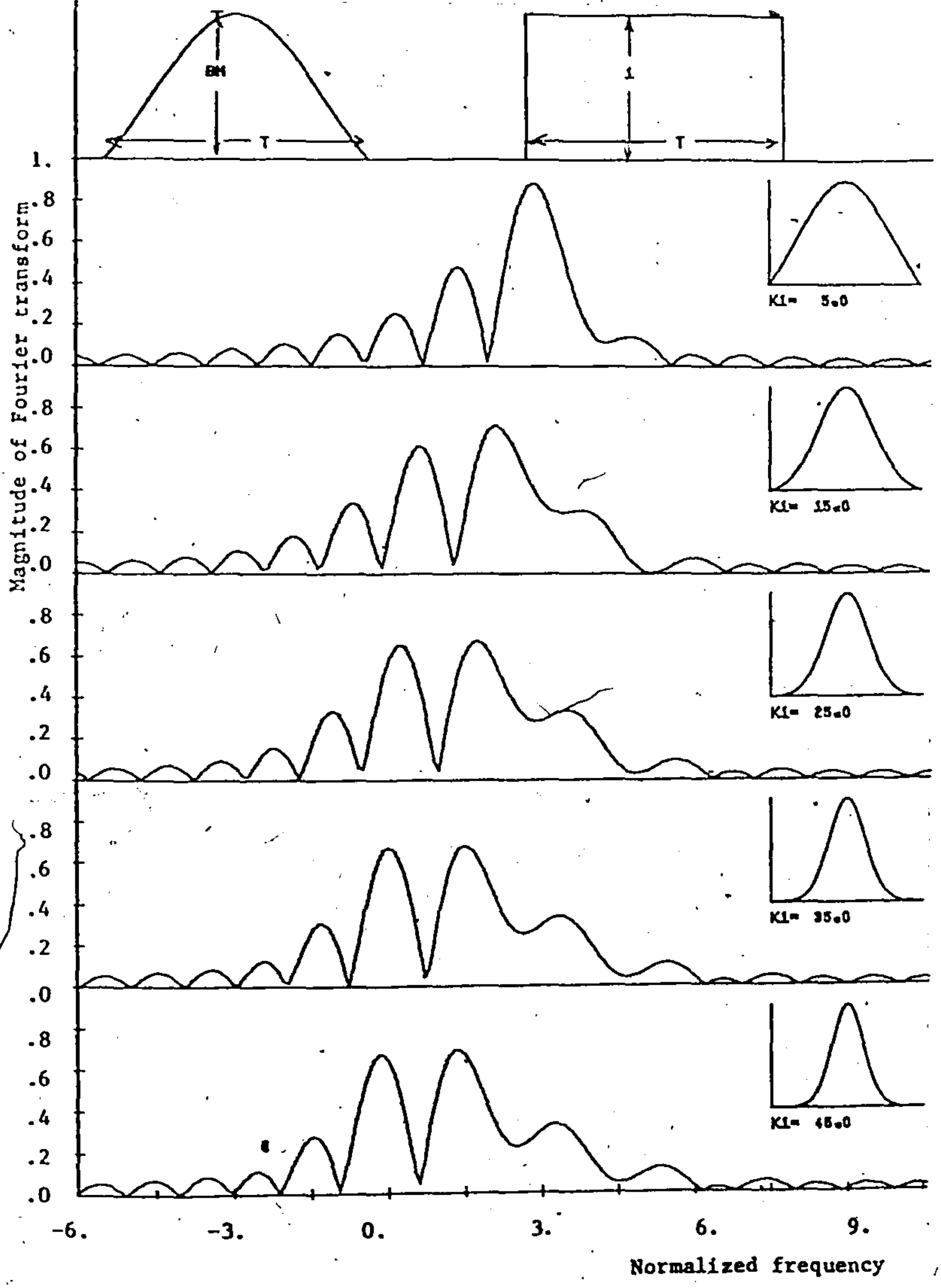


Fig. 3.22 Modified Gaussian-FM and AM, T=1,BM=3.5,K1=10

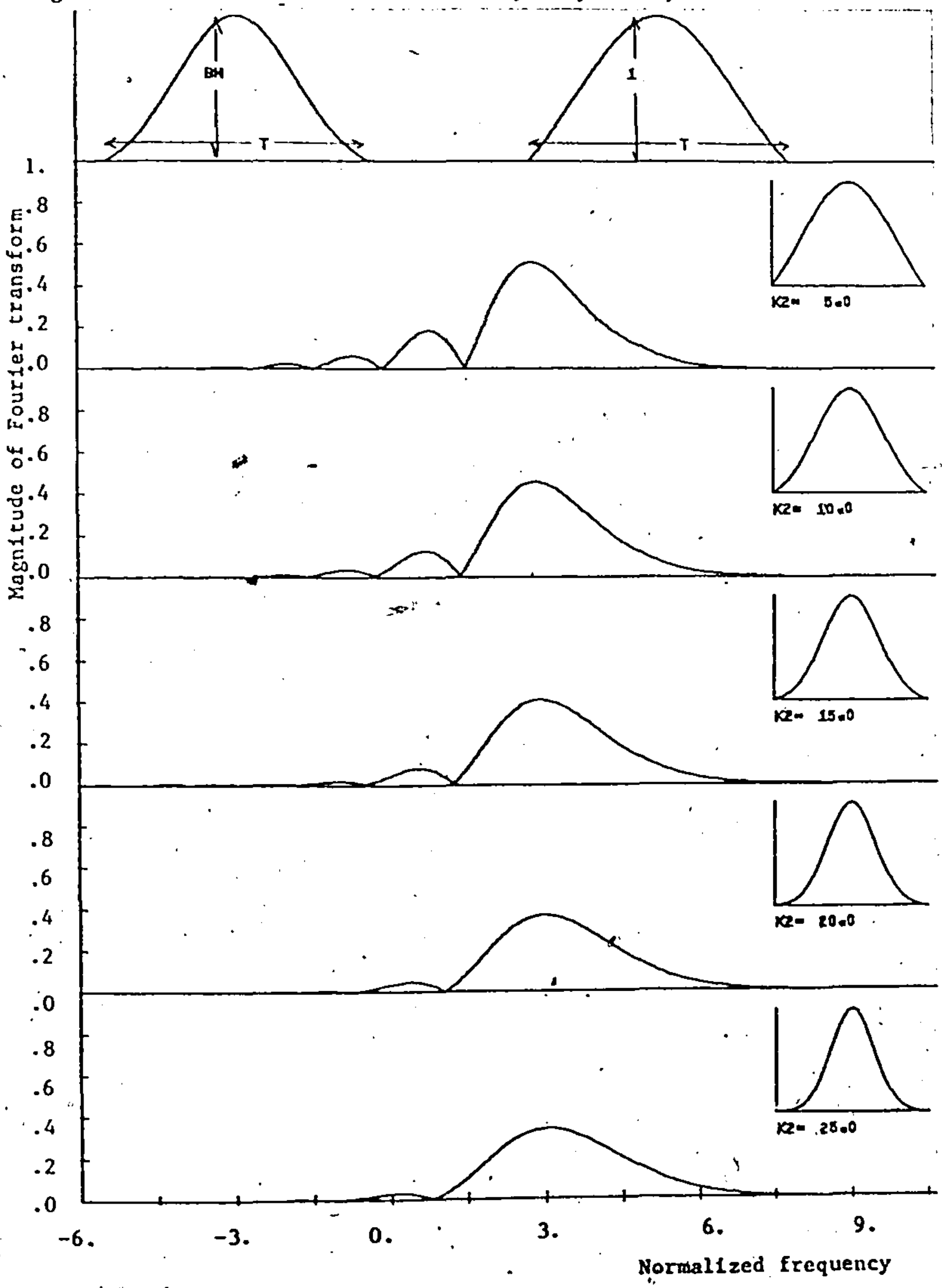


Fig. 3.23 Modified Gaussian FM and Rect. AM, T=1,K1=10

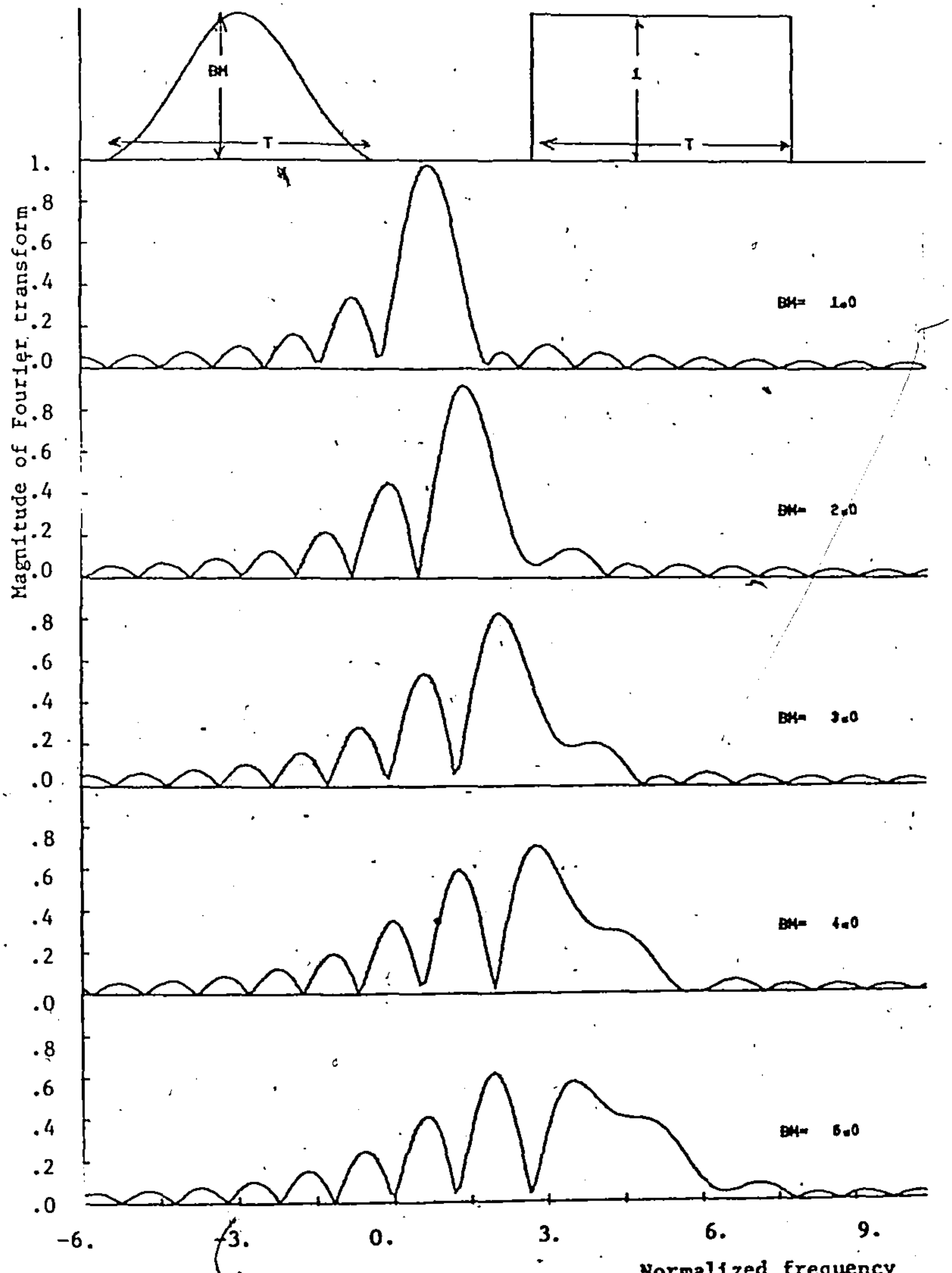
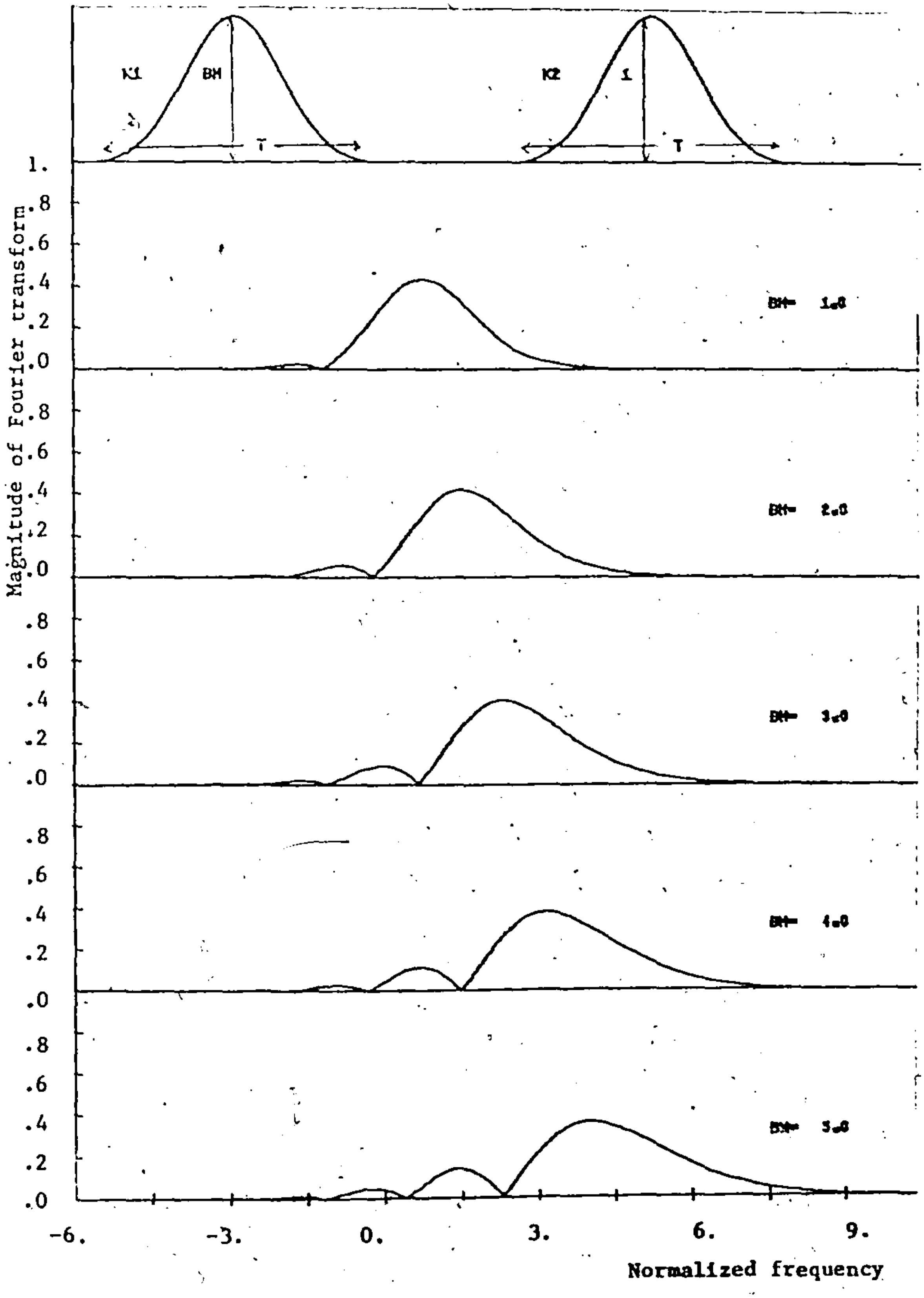


Fig. 3.24 Modified Gaussian FM and AM, T=1,K1=K2=15



spectrum also can be observed to have a wide main lobe when the amplitude modulating pulse has a narrow half power pulse width. This is because of the fact that a narrower pulse in the time domain corresponds to a wide spectrum in the frequency domain.

3-6 SUMMARY OF THE PHENOMENA OBSERVED

The various phenomena exhibited by the spectra in Figs. 3.1 to 3.24 may be summarized as follows :

(1) The spectra for RF pulses without frequency modulation are always symmetrical. The attenuation at the rising and falling edges of the pulse lowers the side-lobe levels at a faster rate than the main-lobe level. (see Figs. 3.1, 3.12 and 3.20)

(2) An RF signal that is frequency modulated by a pulse with a rising and falling edge has an asymmetric spectrum. The degree of this asymmetry becomes more pronounced as the rising edge or falling edge or both increase in duration. (see Figs. 3.2 to 3.4)

(3) When the RF pulse with frequency modulation is amplitude modulated by a trapezoidal, exponential, sinusoidal or modified gaussian pulse, the side-lobe levels are reduced, and the main-lobe is broadened. (see Figs. 3.5 to 3.7 , Figs. 3.14 to 3.15, Figs. 3.19 and 3.22)

(4) When the frequency modulating and amplitude modulating pulses are both symmetric about the mid-point of the pulse, the local minima of the spectra at the lower frequency range are at zero. (see Fig. 3.2, 3.3, 3.10, 3.13 to 3.15 and 3.21 to 3.24)

(5) When the frequency modulating or amplitude modulating pulse becomes asymmetric about the mid-point of the pulse, the local minima of the pulse rise in level while the main-lobe level decreases. (see Figs. 3.2, 3.3 and 3.10)

(6) As the maximum frequency deviation increases the spectrum broadens to the right, while the lower side-lobes increase in level. (see Figs. 3.8, 3.9, 3.16, 3.17, 3.18, 3.23 and 3.24)

(7) When the frequency modulating pulse is narrow, the lower side-lobe levels tend to be as large as the main-lobe or even larger. (see Figs. 3.21)

CHAPTER 4

ANALYTICAL APPROACH

In this chapter, the analytical approach is used in the analysis of pulsed RF spectra. The mathematical technique applied to the spectral analysis of pulsed signals was described in the paper by Cumming [3], in which the pulsed signal with AM/PM conversion is considered. Cumming considered signals with a symmetric gaussian envelope. The incidental frequency modulation is odd symmetric about the carrier frequency. This means the spectra considered are symmetric as stated in property B in chaper 2. However, the only case of interest in [3] is when the phase deviation is smaller than unity and it is plausible to use the series representation of $e^{j\phi(t)}$.

i.e. $e^{j\phi(t)} \approx 1 + j\phi(t)$

A recent paper by Brookner and Bonneau [4] compared this method with the discrete Fourier transform technique. The result was quite satisfactory when maximum phase deviation is less than or equal to unity. For the case of large phase deviation this method will no longer hold true.

In this chapter two analytical techniques are considered. No restriction is imposed on the magnitude of the maximum phase deviation or frequency deviation.

4-1 CONVOLUTION APPROACH

The convolution theorem [5] may be stated as follows :

$$F\left[\int_{-\infty}^{\infty} x(v)\, y(t-v)\, dv\right] = X(f)\, Y(f) \tag{4.1}$$

or

$$F\,[x(t)\, y(t)] = \int_{-\infty}^{\infty} X(v)\, Y(f-v)\, dv \tag{4.2}$$

where F denotes the "Fourier transform of",

and $X(f) = F\,[x(t)]$

$Y(f) = F\,[y(t)]$

Now from eqt. (2.2), the Fourier transform of a pulsed signal, with amplitude modulation A(t) and Frequency modulation g(t), is

$$S(f) = \int_{-\infty}^{\infty} A(t)\, e^{j2\pi\int_0^t g(x)dx}\, e^{-j2\pi ft}\, dt \tag{4.3}$$

Define $M(f) = F\,[A(t)]$ (4.4)

and T = pulse width

Next, we use the well known property that time shifting a pulsed signal would neither alter its power spectrum nor the magnitude of its Fourier transform Thus, without loss of generality we shall assume A(t) and g(t) to have already been time shifted by $-T/2$. Now expressing g(t) as a periodic waveform p(t) as follows :

$$p(t+nT) = g(t) \qquad \text{for} \quad -\frac{T}{2} \le t \le \frac{T}{2} \tag{4.5}$$

where $n = 0, \pm 1, \pm 2, \ldots\ldots$

The Fourier series representation of p(t) is

$$p(t) = \sum_{n=o}^{\infty} a_n \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right) \qquad (4.6)$$

where a_n and b_n are Fourier series coefficients. Now because

$$A(t)\, e^{j2\pi \int_o^t g(x)dx} = A(t)\, e^{j2\pi \int_o^t p(x)dx} \qquad (4.7)$$

$$\left(A(t) = 0 \text{ for } t \geq \frac{T}{2} \text{ and } t \leq -\frac{T}{2} \right)$$

Therefore we only have to find the Fourier transform of

$$e^{j2\pi \int_o^t p(x)dx}$$

and then convolve it with F [A(t)] to obtain the desired spectrum S(f).

Substituting eqt. (4.6) into $e^{j2\pi \int_o^t p(x)dx}$, we obtain

$$e^{j2\pi \int_o^t p(x)dx} = e^{j2\pi \int_o^t \left\{ \sum_{n=0}^{\infty} a_n \cos\left(\frac{2\pi nx}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nx}{T}\right) \right\} dx}$$

Hence $e^{j2\pi \int_o^t p(x)\, dx}$

$$= e^{j2\pi a_o t}\; e^{jTa_1 \sin\left(\frac{2\pi t}{T}\right)}\; e^{jT\frac{a_2}{2} \sin\left(\frac{4\pi t}{T}\right)}\; e^{jT\frac{a_3}{3} \sin\left(\frac{6\pi t}{T}\right)} \cdots\cdots$$

$$e^{jT \sum_{n=1}^{\infty} \frac{b_n}{n}}\; e^{jT(-b_1) \sin\left(\frac{2\pi t}{T} + \frac{\pi}{2}\right)}\; e^{jT(-\frac{b_2}{2}) \sin\left(\frac{4\pi t}{T} + \frac{\pi}{2}\right)} \cdots\cdots$$

Next, we note that $$e^{jz\sin(\theta)} = \sum_{n=-\infty}^{\infty} J_k(z)\, e^{jk\theta} \qquad (4.8)$$

where $J_k(z)$ is Bessel function of the first kind of k^{th} order. (Appendix A)

As an approximation, only three terms† of the Fourier series will be considered. Hence

$$e^{j2\pi \int_o^t p(x)dx}$$

$$= e^{jT\sum_{n=1}^{\infty}\frac{b_n}{n}}\, e^{j2\pi a_o t} \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{k_3=-\infty}^{\infty}\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\sum_{m_3=-\infty}^{\infty} J_{k_1}(z_1)\, J_{k_2}(z_2)$$

$$J_{k_3}(z_3)\, J_{m_1}(u_1)\, J_{m_2}(u_2)\, J_{m_3}(u_3)\, e^{j\frac{\pi}{2}(m_1+m_2+m_3)}$$

$$e^{j2\pi\frac{t}{T}(k_1+2k_2+3k_3+m_1+2m_2+3m_3)} \qquad (4.9)$$

$$\text{where } z_n = T\frac{a_n}{n} \quad \text{and} \quad u_n = -T\frac{b_n}{n} \qquad (4.10)$$

---

† The reason this approximation holds is that modulation index is small.

Taking the Fourier transform of (4.9), we obtain

$$F\left[e^{j2\pi \int_o^t p(x)dx} \right]$$

$$= e^{jT \sum_{n=1}^{\infty} \frac{b_n}{n}} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{k_3=-\infty}^{\infty} \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \sum_{m_3=-\infty}^{\infty} J_{k_1}(z_1)\, J_{k_2}(z_2)$$

$$J_{k_3}(z_3)\, J_{m_1}(u_1)\, J_{m_2}(u_2)\, J_{m_3}(u_3)\, e^{j\frac{\pi}{2}(m_1+m_2+m_3)}$$

$$\delta\left(f'' - \frac{k_1+2k_2+3k_3+m_1+2m_2+3m_3}{T}\right) \qquad (4.11)$$

where $f'' = f - a_o$

From the convolution property and eqt. (4.11), we have

$$S(f) = F\,[A(t)] * F\left[e^{j2\pi \int_o^t p(x)dx}\right]$$

where * denotes the convolution.

$$S(f) = e^{jT\sum_{n=1}^{\infty}\frac{b_n}{n}} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{k_3=-\infty}^{\infty} \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \sum_{m_3=-\infty}^{\infty} J_{k_1}(z_1)\, J_{k_2}(z_2)$$

$$J_{k_3}(z_3)\, J_{m_1}(u_1)\, J_{m_2}(u_2)\, J_{m_3}(u_3)\, e^{j\frac{\pi}{2}(m_1+m_2+m_3)}$$

$$F\,[A(t)] \;*\; \delta\left(f'' - \frac{k_1+2k_2+3k_3+m_1+2m_2+3m_3}{T}\right) \qquad (4.12)$$

This is a general formula for computing the Fourier transform S(f) of a pulsed signal with amplitude modulation A(t) and frequency modulation g(t). Now S(f) can be real or imaginary depending on the terms

$$e^{j\frac{\pi}{2}(m_1+m_2+m_3)} \quad \text{and} \quad F\,[A(t)]$$

where m_1 , m_2 and m_3 etc. are indices of the summations generated from the terms

$$b_n \sin\left(\frac{2n\pi t}{T}\right)$$

of the Fourier series representation of g(t). This formula can be simplified further if A(t) and g(t) are symmetric about the mid-point of the pulse. Then F[A(t)] is real and $b_n = 0$, i.e. $u_n = 0$. These are due to the initial conditions that A(t) and g(t) are time shifted by $-\frac{T}{2}$, making A(t) and g(t) now symmetric about t=0.

Hence, eqt. (4.12) can be re-written as

$$F\,[A(t)\, e^{j2\pi \int_0^t p(x)dx}]$$

$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{k_3=-\infty}^{\infty} J_{k_1}(z_1)\, J_{k_2}(z_2)\, J_{k_3}(z_3)\, F[A(t)] * \delta(f'' - \frac{k_1+2k_2+3k_3}{T}) \tag{4.13}$$

In the following, two examples are shown demonstrating the use of this convolution method.

<u>Example (1)</u>

$$A(t) = \text{Rect}(t) = 1 \qquad \text{for} \quad -\frac{T}{2} \le t \le \frac{T}{2} \tag{4.14}$$

$$= 0 \qquad t < -\frac{T}{2},\ t > \frac{T}{2}$$

$$g(t) = BM\cos(\frac{t}{T}) \qquad \text{for} \quad -\frac{T}{2} \le t \le \frac{T}{2} \tag{4.15}$$

$$= 0 \qquad t < -\frac{T}{2},\ t > \frac{T}{2}$$

$$F\,[A(t)] = T\,\frac{\sin(\pi f T)}{\pi f T} = T\,\mathrm{sinc}(\pi f T) \tag{4.16}$$

From eqt. (4.7)

$$p(t) = BM\cos(\pi\frac{t}{T})$$

Hence

$$e^{j2\pi\int_0^t p(x)dx} = e^{j2T\cdot BM\,\sin(\pi\frac{t}{T})}$$

$$= \sum_{k=-\infty}^{\infty} J_k(2T\cdot BM)\, e^{j\frac{k\pi t}{T}}$$

The latter equality follows from eqt. (4.8). Now using the convolution method, we obtain

$$F\,[A(t)\, e^{j2\pi\int_0^t p(x)dx}]$$

$$= \sum_{k=-\infty}^{\infty} J_k(2T\cdot BM)\; T\,\mathrm{sinc}(\pi f T) * \delta(f-\frac{k}{2T})$$

$$= T\sum_{k=-\infty}^{\infty} J_k(2T\cdot BM)\;\mathrm{sinc}[\pi T(f-\frac{k}{2T})] \tag{4.17}$$

In Fig. 4.1 and 4.2, the pulse spectra for two different values of BM are plotted using eqt. (4.17). The spectrum obtained using the FFT method is also plotted on each graph to compare the result.

Example 2

The amplitude modulating pulse A(t) is Rect (t) as defined in eqt. (4.14). The frequency modulating pulse g(t) is a trapezoidal

Fig. 4.1 Sin. FM and Rect. AM, T=1, BM=1.

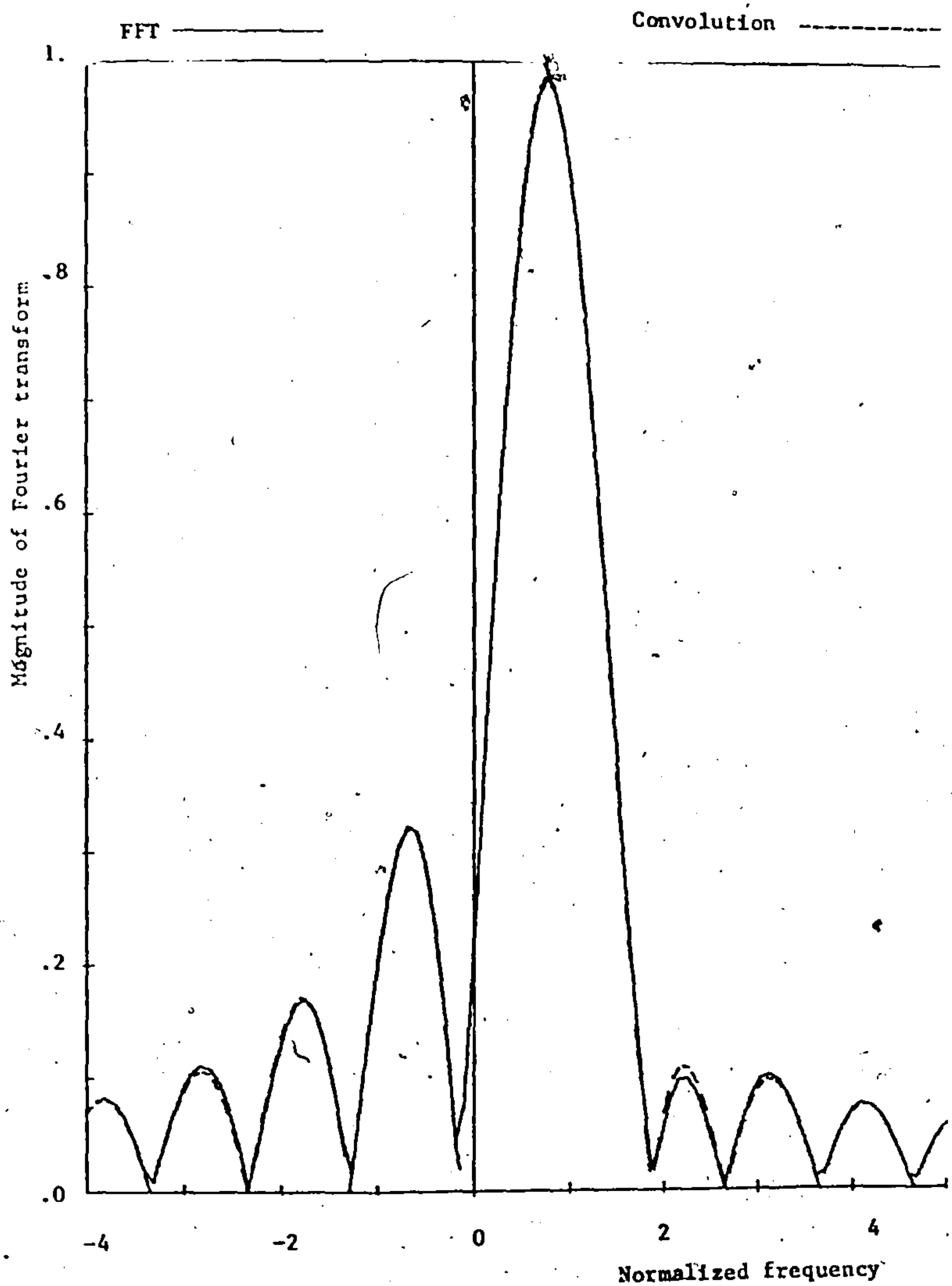
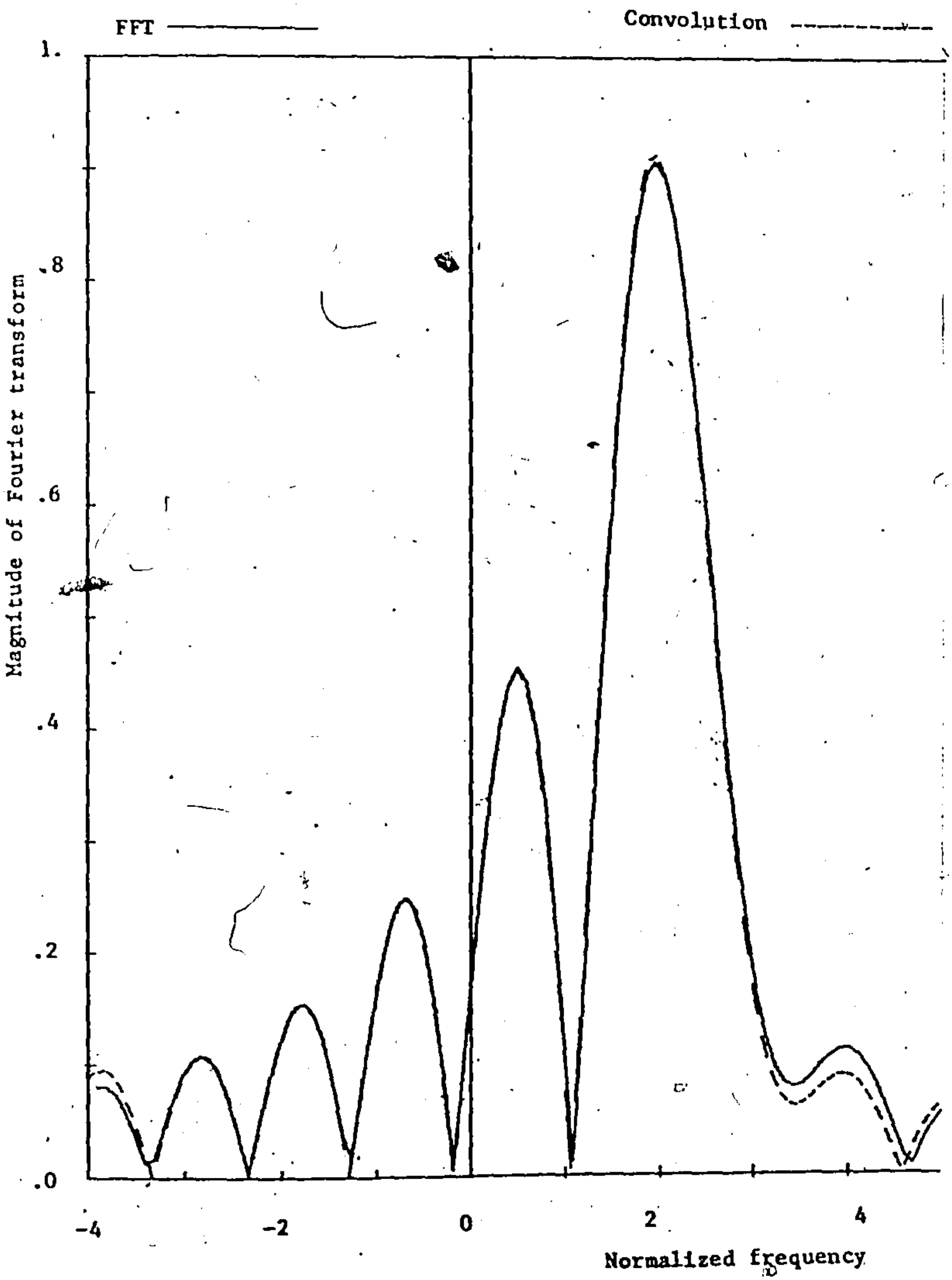


Fig. 4.2 Sin. FM and Rect. AM, T=1, BM=2.5.



pulse as defined in eqt. (3.2). However, g(t) is time shifted by $-\frac{T}{2}$ now. From eqt. (4.5), p(t) can be expressed in a Fourier series form

$$p(t) = BM(1-a) + 2BM\cdot a \sum_{n=1}^{\infty} (-1)^{n+1} \operatorname{sinc}^2(n\pi a) \cos\left(\frac{2\pi nt}{T}\right) \quad (4.18)$$

where BM = maximum frequency deviation.

a = rise time = fall time of the FM pulse.

Retaining only three terms in the Fourier series, as an approximation, we have from eqt. (4.10)

$$z_1 = T(2BM\cdot a)\ \operatorname{sinc}^2(\pi a) \quad (4.19)$$

$$z_2 = -T(2BM\cdot a)\ \operatorname{sinc}^2(2\pi a)/2 \quad (4.20)$$

$$z_3 = T(2BM\cdot a)\ \operatorname{sinc}^2(3\pi a)/3 \quad (4.21)$$

Using eqt. (4.13), we get the Fourier transform

$$S(f) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{k_3=-\infty}^{\infty} J_{k_1}(z_1)\, J_{k_2}(z_2)\, J_{k_3}(z_3)\, T \operatorname{sinc}\left[\pi T\left(f'' - \frac{k_1+2k_2+3k_3}{T}\right)\right] \quad (4.22)$$

where $f'' = f - BM(1-a)$.

In Fig. 4.3 and 4.4, the pulse spectra for two different values of BM are plotted using eqt. (4.22). The spectrum obtained using the FFT method is also plotted on each graph to compare the results.

As shown in Figs. 4.1 to 4.4, the spectra obtained using the

Fig. 4.3 Trap. FM and Rect. AM, T=1, BM=1, A1=B1=0.1.

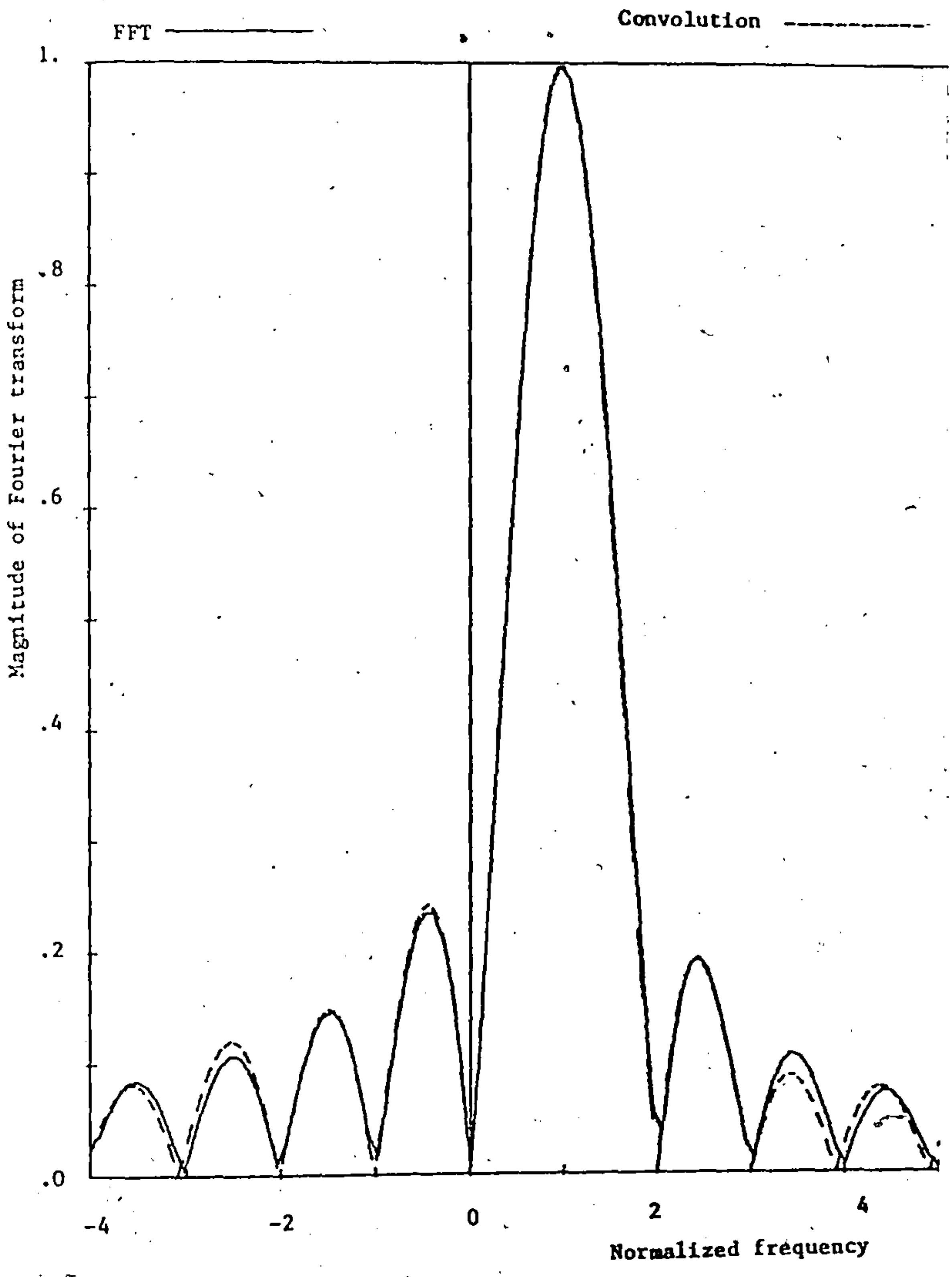
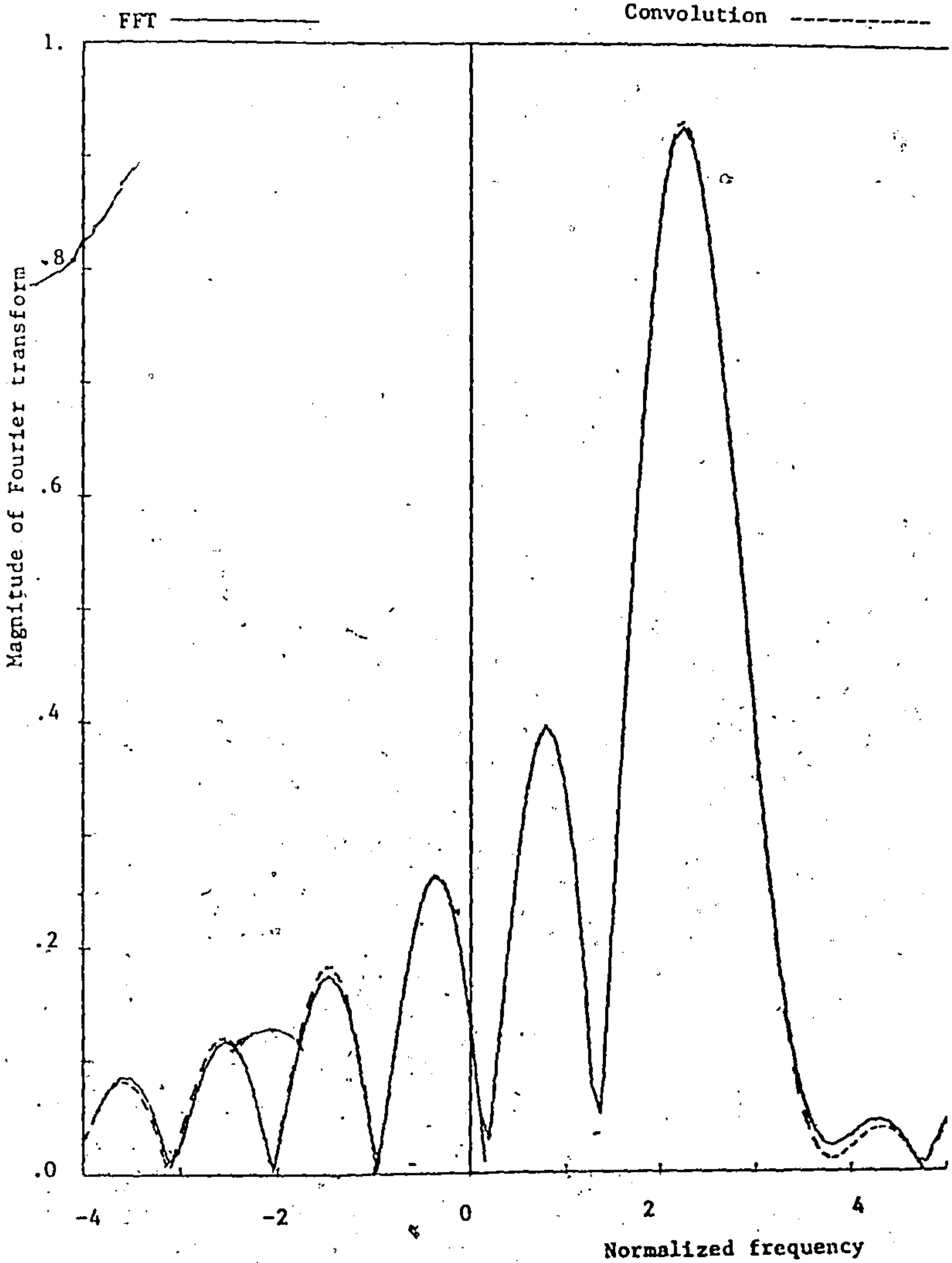


Fig. 4.4 Trap. FM and Rect. AM, T=1, BM=2.5, A1=B1=0.25



convolution method agree well with those using the FFT method. The analytical approach considered in this section is good for any kind of frequency and amplitude modulated pulsed signal, provided that the Fourier transform of A(t) and the analytical form of g(t) are known.

Using this method, we can explain some of the phenomena that occur in pulsed RF spectra :

(a) When A(t) and g(t) are symmetric about the mid-point of the pulse, their corresponding Fourier transforms are real, and the local minima of the magnitude of the spectrum are zero.

(b) When A(t) or g(t) become asymmetric, the Fourier transform becomes complex as shown in eqn. (4.12). The spectrum would then be the combination of the real and imaginary parts of its Fourier transform. As the deviation from symmetry of A(t) and g(t) becomes more pronounced, the magnitude of the imaginary part of the spectrum increases. Consequently, the local minima of the spectrum are raised above zero level.

(c) When the product BM·T increases, the Bessel functions of higher order cannot be neglected in the computation of the pulse spectra. The inclusion of these extra terms of Bessel functions results in the broadening of the spectrum.

4-2 DIRECT INTEGRATION

The Fourier transform of a signal $s_i(t)$ is defined by

$$S_i(f) = \int_{-\infty}^{\infty} s_i(t) e^{-j2\pi f t}\, dt$$

In this section we shall evaluate $S_1(t)$ using direct integration.

(1) Trapezoidal AM pulse.

If $s_1(t)$ is a pulse with trapezoidal envelope as in eqt. (3.1), then

$$S_1(f) = \frac{-A}{4\pi^2 f^2 T}\left[\frac{1}{a}\left(1-e^{-j2\pi faT}\right) - \frac{1}{b}\left(e^{-j2\pi fT(1-b)} - e^{-j2\pi fT}\right)\right] \tag{4.23}$$

(Appendix B)

where a = rise time of AM pulse. (A2)

b = fall time of AM pulse. (B2)

A comparison with the spectrum obtained using the FFT method is shown in Fig. 4.5.

(2) Exponential AM pulse

If $s_2(t)$ is a pulse with an exponential amplitude modulation as in eqt.(3.3), then

$$S_2(f) = A_1 \cdot t_1 e^{j\pi ft_1} \operatorname{sinc}(\pi ft_1) + A_1 \frac{e^{-t_1(c_2+j\pi f)} - 1}{c_2 + j2\pi f}$$

$$+ A \frac{e^{-j2\pi ft_2} - e^{-j2\pi ft_1}}{-j2\pi f} + A_2 \frac{e^{-j2\pi fT} - e^{-j2\pi ft_2}}{-j2\pi f}$$

$$- A_2 \frac{e^{-j2\pi fT} - e^{-j2\pi ft_2} e^{-k_2(T-t_2)}}{k_2 - j2\pi f} \tag{4.24}$$

where $A_1 = \dfrac{A}{1- e^{-c_2 t_1}}$ and $A_2 = \dfrac{A}{1- e^{-k_2(T-t_2)}}$

(Proof in Appendix C)

t_1 = rise time of pulse. (A2)

$T - t_2$ = fall time of pulse. (B2)

A comparison with the spectrum obtained using the FFT method is shown in Fig. 4.6.

(3) Sinusoidal AM pulse.

If $s_3(t)$ is a pulse with sinusoidal amplitude modulation, as shown in eqt. (3.5), then

$$S_3(f) = \frac{AT}{2}\{\mathrm{sinc}[\pi(f-\frac{1}{2T})T] + \mathrm{sinc}[\pi(f+\frac{1}{2T})T]\} \qquad (4.25)$$

(4) Trapezoidal FM pulse.

If $s_4(t)$ is a pulse with trapezoidal frequency modulation, as shown in eqt. (3.2), then

$$S(f) = A\{\sqrt{\frac{aT}{2B}}\; e^{-j\frac{\pi aTf^2}{B}}\; I[\sqrt{\frac{2aT}{B}}(B-f), -\sqrt{\frac{2aT}{B}}f)] +$$

$$\sqrt{\frac{bT}{2B}}\; e^{j\frac{\pi bT}{B}(f-\frac{B}{b})^2}\; e^{j2\pi BT(1-\frac{a}{2}-\frac{b}{2}-\frac{1}{2b})}\; I[\sqrt{\frac{2bT}{B}}f, \sqrt{\frac{2bT}{B}}(f-B)]$$

$$-\frac{1}{j2\pi f''}\; e^{-j\pi aBT}\; [e^{-j2\pi(1-b)Tf''} - e^{-j2\pi f''Ta}]\} \qquad (4.26)$$

where B = maximum frequency deviation

a = rise time of the frequency modulating pulse. (A1)

b = fall time of the frequency modulating pulse. (B1)

$II(x,y)$ = complex conjugate of $I(x,y)$

T = pulse width

$f'' = f - B$

and

$$I(x,y) = \int_y^x e^{j\frac{\pi}{2}t^2}\,dt$$

$$= \left[\int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt - \int_0^y \cos\left(\frac{\pi}{2}t^2\right)dt\right] + j\left[\int_0^x \sin\left(\frac{\pi}{2}t^2\right)dt - \int_0^y \sin\left(\frac{\pi}{2}t^2\right)dt\right] \quad (4.27)$$

Each of the integrals in eqt.(4.27) is recognized as a Fresnel integral; tables of this integral are given in [6]. Proof of this result is given in Appendix D. A comparison with the spectrum obtained using the FFT method is shown in Fig. 4.7.

(5) A special case of Combined Amplitude Modulation and Frequency Modulation

If $s_5(t)$ is a pulse with frequency and amplitude both modulated by a pulse with waveform $g(t)$, then

$$S(f) = \frac{A}{B}\left\{\frac{1}{j2\pi}\left[e^{j2\pi B\int_0^T g(x)dx}\, e^{-j2\pi fT} - 1\right] + f\, F\left[Rect(t)\, e^{j2\pi B\int_0^t g(x)dx}\right]\right\} \quad (4.28)$$

where B = maximum frequency deviation

$Rect\ (t) = 1$ for $0 \le t \le T$

$= 0$ $t<0\ ,\ t>T$

For a proof, see Appendix E.

Fig. 4.5 Trap. AM and Rect. FM, T=1, A2=0.1, B2=0.3.

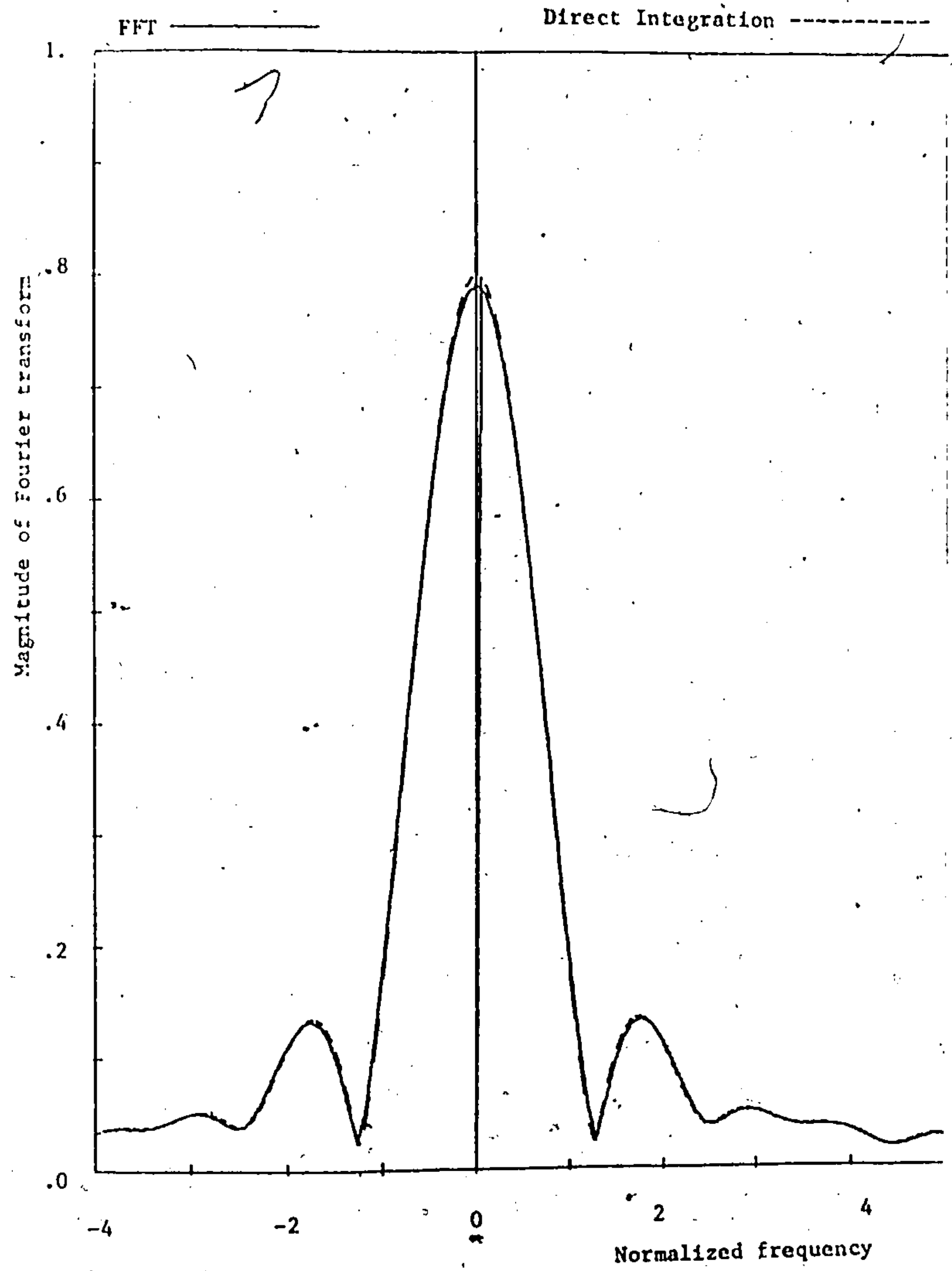


Fig. 4.6 Exp. AM and Rect. FM, T=1, A2=0.1, B2=0.3, C2=10, K2=20.

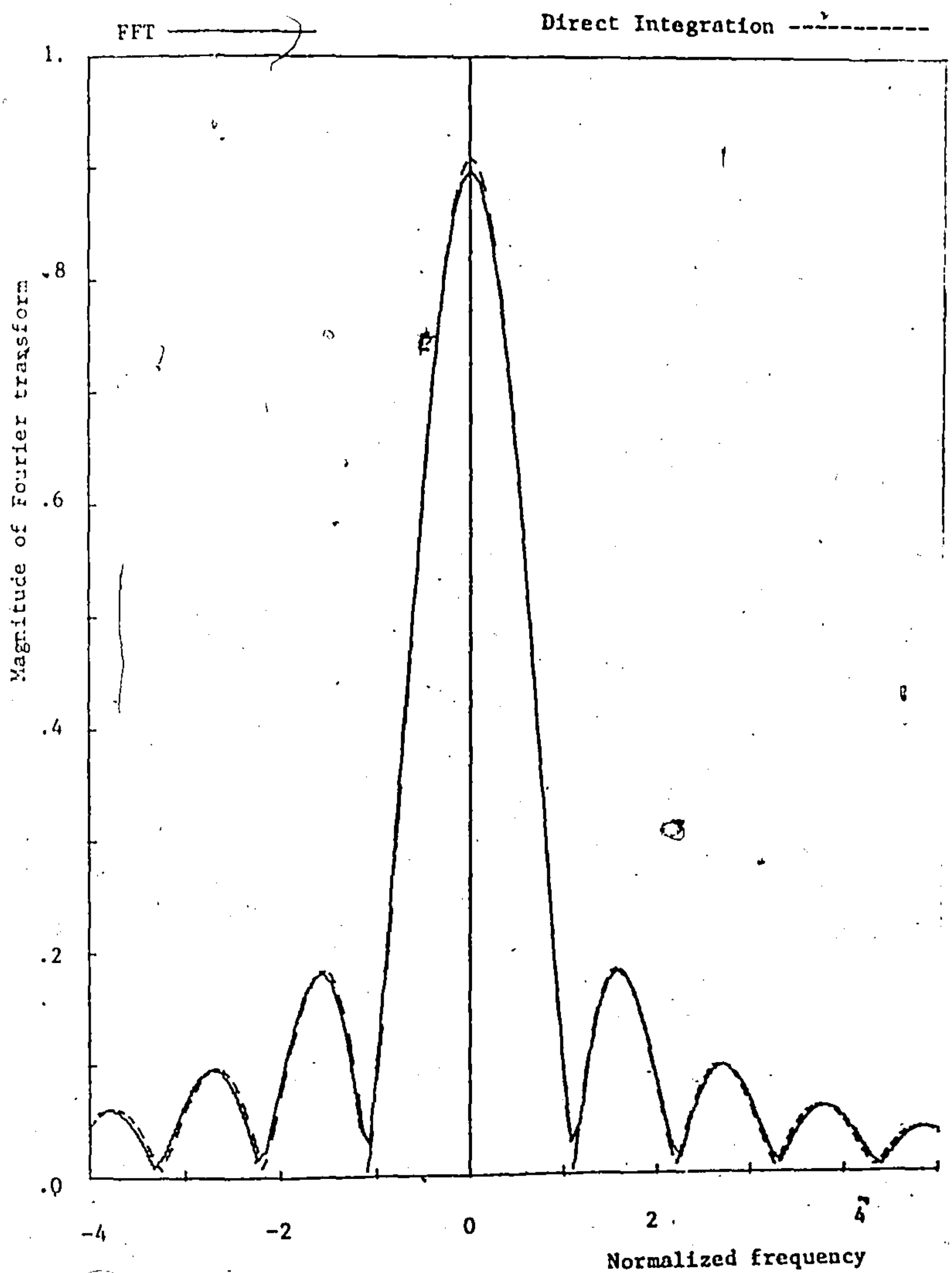


Fig. 4.7 Trap. FM and Rect. AM, T=1, BM=3, A1=0.1, B1=0.3.

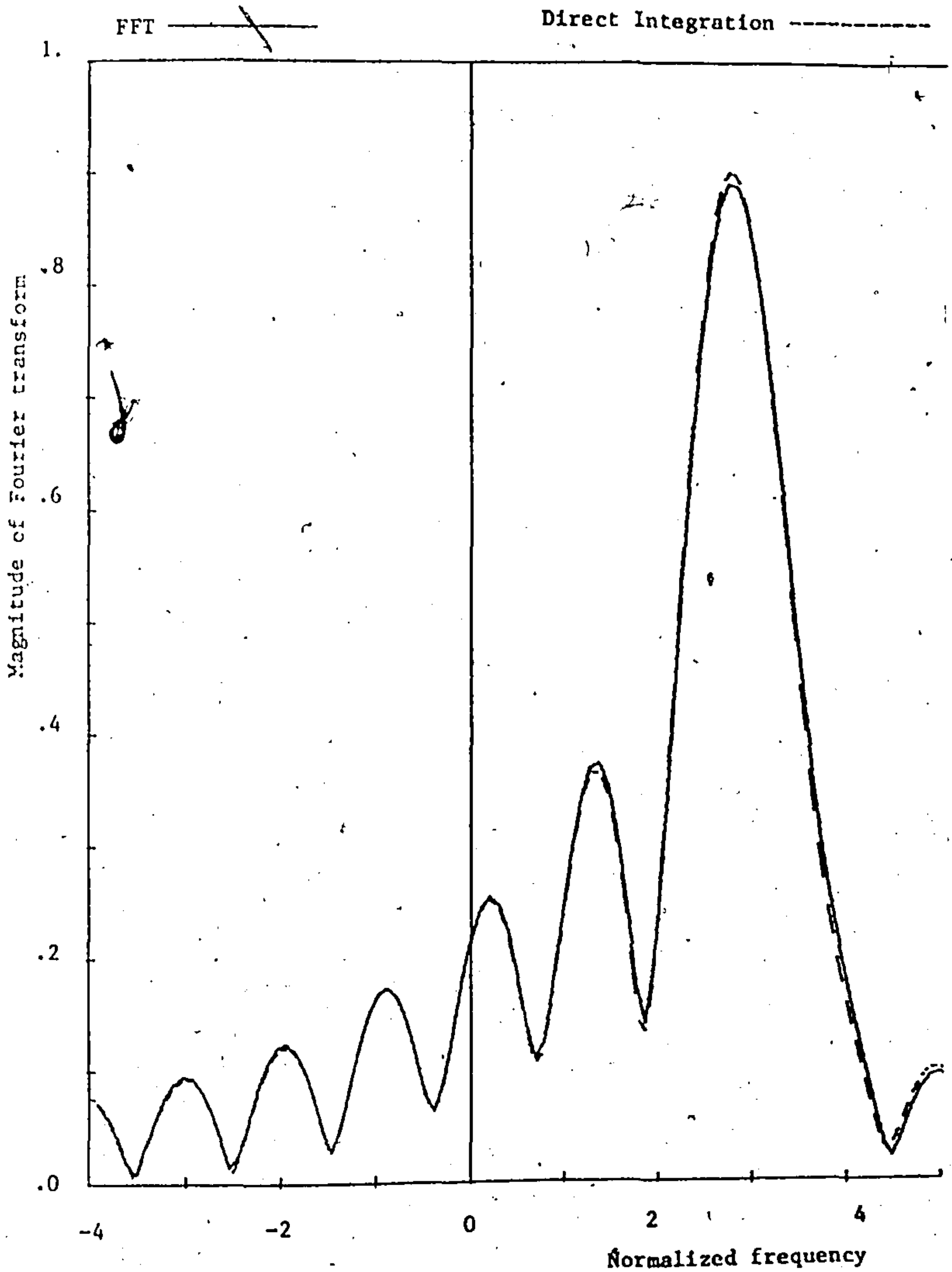
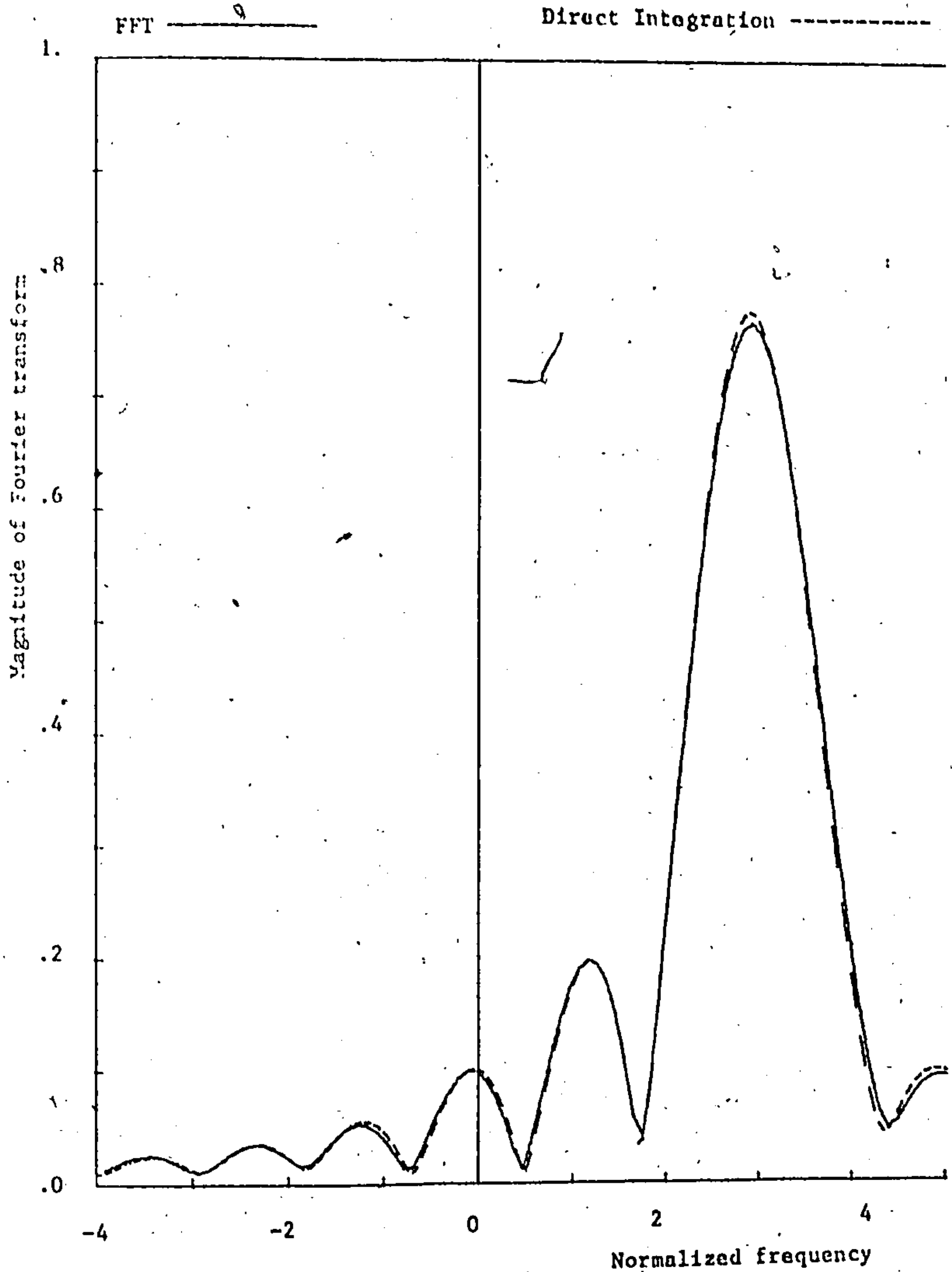


Fig. 4.8 Trap. AM and Trap. FM, T=1, BM=3, A1=A2=0.1, B1=B2=0.3.



The spectrum of an RF pulse that is both frequency and amplitude modulated by the same trapezoidal waveform is plotted in Fig. 4.8, using eqt. (4.28). The result of using the FFT method is also included in Fig. 4.8 for comparison.

As shown in Fig. 4.5 to 4.8, the spectra obtained using the analytical method described in this section agree well with the corresponding spectra obtained using the FFT method. The analytical approach is manageable when there is only amplitude modulation in the RF pulse. The mathematics becomes rather cumbersome when there is frequency modulation in the signal.

CHAPTER 5

EXPERIMENTAL APPROACH

In this chapter, the experimental approach is taken to investigate the spectra of pulsed RF signals under varying conditions.The experimental set up is as shown in Fig. 5.1. Two pulses, each of 1 msec. in duration, are generated simultaneously from a CDC 1700 digital computer. Each pulse is constructed from 50 digital samples of a pre-determined waveform. The pulse repetition period is 10 msec. One of these pulses is used as a frequency modulating pulse to a HP 8660B synthesized signal generator, which is set in FM mode. This equipment controls the setting of the value for the maximum frequency deviation of the pulse. The unmodulated carrier frequency is arbitrarily chosen to be 1 MHz. The output of this signal generator forms a continuous, frequency modulated waveform. This signal is in turn multiplied by the amplitude modulating pulse to give the desired modulated pulse. Fig. 5.2 shows the circuit design of the analog multiplier using the MC1596G Monolithic Balanced Modulator-Demodulator unit. The output of the multiplier is fed into the HP 8553B spectrum analyzer to display its spectrum. Because the analyzer has a low input impedance, a buffer stage, with circuit design as shown in Fig. 5.3, was built and connected between the multiplier and the spectrum analyzer.

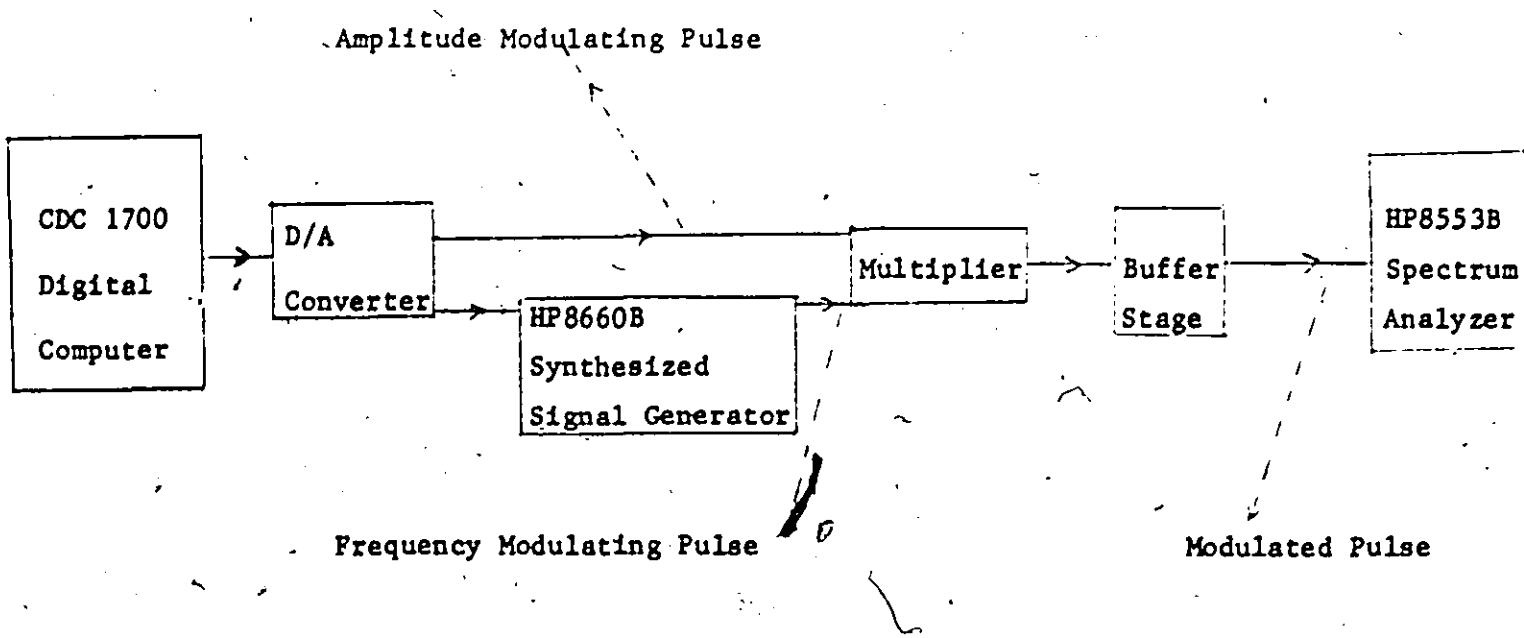


Fig. 5.1 Block Diagram of Experimental Simulation.

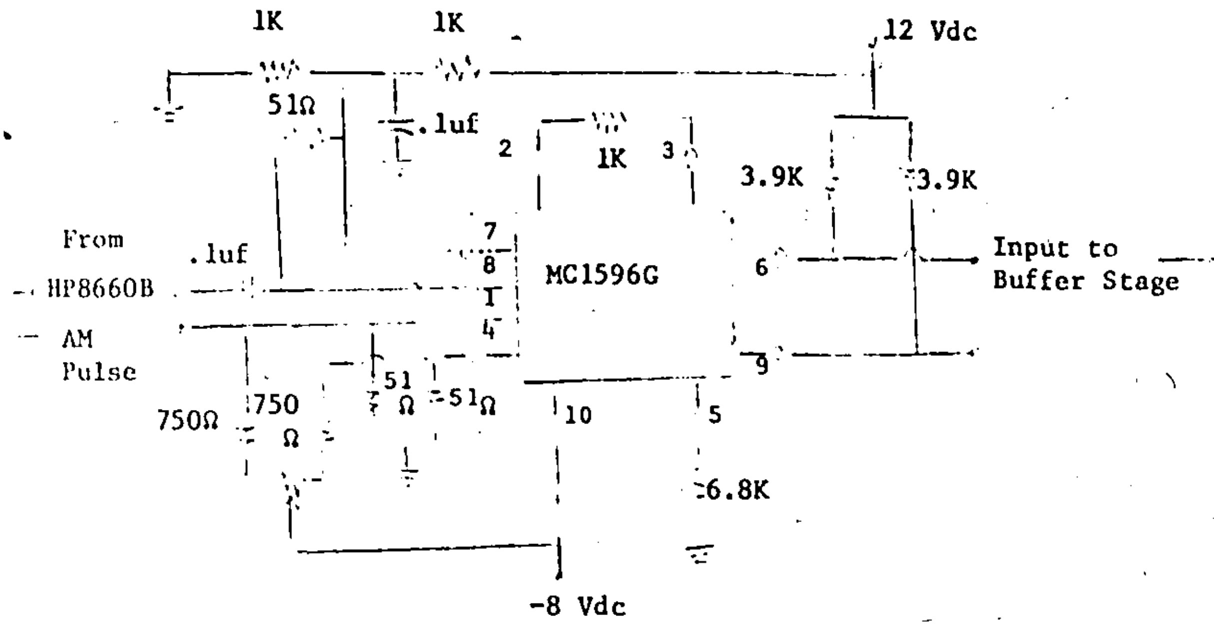


Fig. 5.2 Circuit Diagram of the Analog Multiplier.

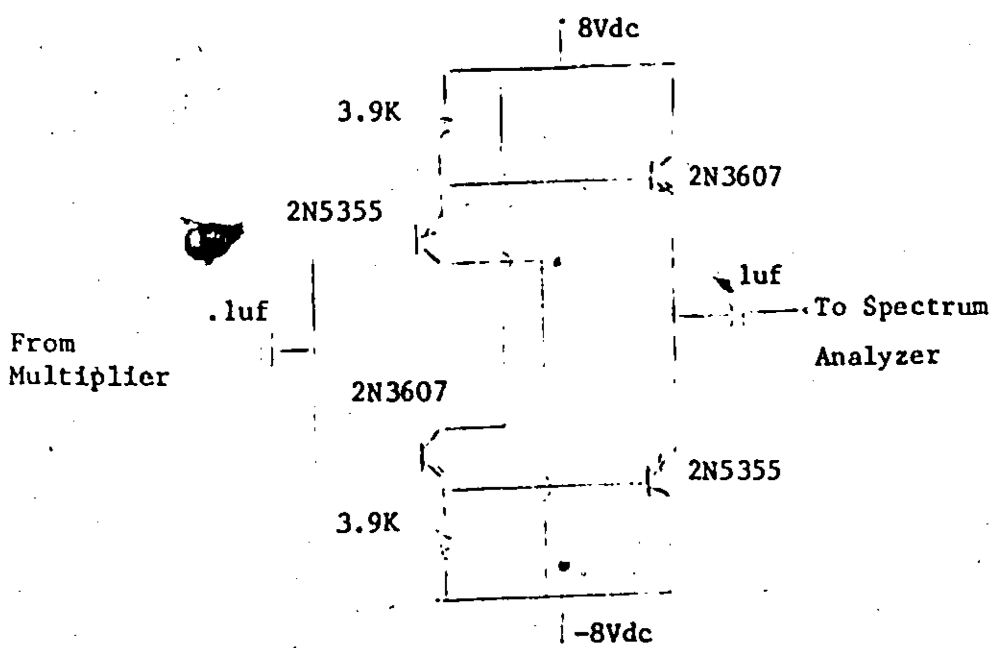


Fig. 5.3 Circuit Diagram of the Buffer Stage.

The experimental results are depicted in Figs. 5.4 to 5.35. From Figs. 5.4 to 5.31, the pulses and their spectra, considered in the experiment, are used to verify the spectra that were derived theoretically in chapters 2 and 3. In Figs. 5.32 to 5.35, the modulating pulses are chosen to satisfy the sufficient condition necessary for property B discussed in chapter 2. We see that the pulse spectra are symmetric, as expected.

Comparison of Figs. 5.4 to 5.31 with Figs. 3.1 to 3.24 indicates that the spectra obtained experimentally agree rather well with the corresponding results obtained previously in chapter 3 and 4. This is illustrated by the four tables below comparing the spectra, obtained using the FFT approach and the experimental approach, for four arbitrarily chosen modulated pulsed signals.

Table 1

Pulse : Rectangular AM and Trapezoidal FM.

BM=1, AI=0.17, B1=0.33.

	FFT Approach (Fig. 3.8)	Experimental Approach (Fig. 5.6)
Amplitude of main-lobe	1.	1.
Amplitude of 1st lower side-lobe	0.28	0.30
Amplitude of 2nd lower side-lobe	0.17	0.17

Table 2

Pulse : Rectangular AM and Exponential FM.

BM=3.5, A1=B1=0.5, C1=K1=2.

	FFT Approach (Fig. 3.13)	Experimental Approach (Fig. 5.15)
Amplitude of main-lobe	0.84	0.84
Amplitude of 1st lower side-lobe	0.50	0.51
Amplitude of 2nd lower side-lobe.	0.27	0.27

Table 3

Pulse : Sinusoidal AM and Sinusoidal FM.

BM=3.

	FFT Approach (Fig. 3.19)	Experimental Approach (Fig. 5.24)
Amplitude of main-lobe	0.60	0.59
Amplitude of 1st lower side-lobe	0.15	0.15
Amplitude of 2nd lower side-lobe	0.05	0.05

Table 4

Pulse : Rectangular AM and Modified Gaussian FM.

BM=3.5, K1=25.

	FFT Approach (Fig. 3.21)	Experimental Approach (Fig. 5.27)
Amplitude of main-lobe	0.66	0.69
Amplitude of 1st lower side-lobe	0.63	0.63
Amplitude of 2nd lower side-lobe	0.32	0.31

EXPERIMENTAL MEASUREMENT.

Digital computer CDC 1700 setting :

D/A Converter setting (before connected to circuit)

Peak voltage of amplitude modulating pulse	= 10 volt.
Peak voltage of frequency modulating pulse	= 5 volt.
Pulse repeated period	= 10 msec.
pulse width	= 1 msec.

Frequency Synthesizer setting :

Centre frequency	= 1 MHz.
Modulation mode	= FM
R.M.S. voltage of continuous wave output	= 40 mV.

Spectrum Analyzer setting :

Display mode	= linear
Centre frequency	= 1 MHz.
Bandwidth	= 0.1 KHz.
Scan width	= 1 KHz.
Input attenuation	= 0.
Scan time per division	= 0.5 sec.

Figures on pulses

Upper trace : Modulated pulse signal

Vertical scale	= 0.5 volt./cm.
Horizontal scale	= 0.2 msec./cm.

Lower trace : Frequency modulating pulse

Vertical scale	= 1 volt./cm.
Horizontal scale	= 0.2 msec./cm.

Figures on spectra

Display mode	= linear
Centre frequency	= 1 MHz.
Vertical scale	= 0.125 unit/division
Horizontal scale (Figs. 5.4 to 5.27)	= 1 KHz./division
(Figs. 5.28 to 5.35)	= 2 KHz./division

Fig. 5.4

Ideal spectrum obtained in experiment.

Pulse with constant amplitude and frequency.

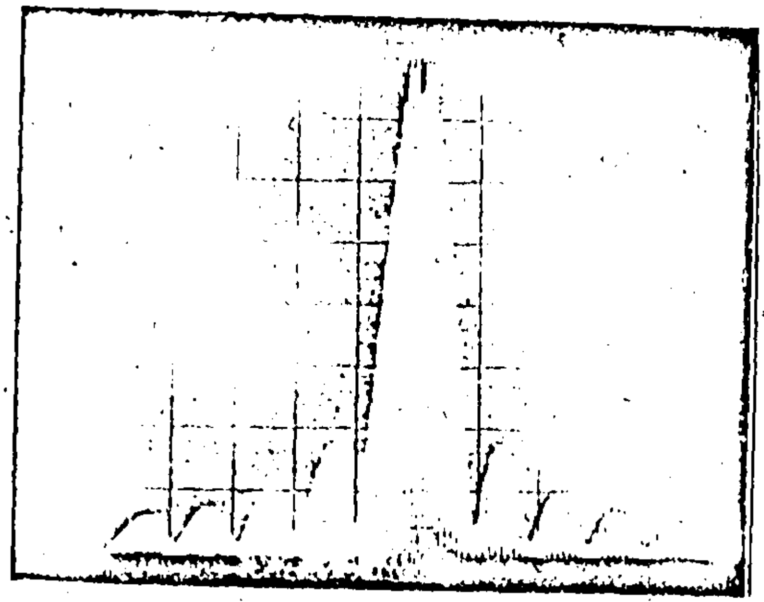


Fig. 5.5

Pulse : constant amplitude and Trapezoidal FM.

A1= 0.17, B1=0.33.

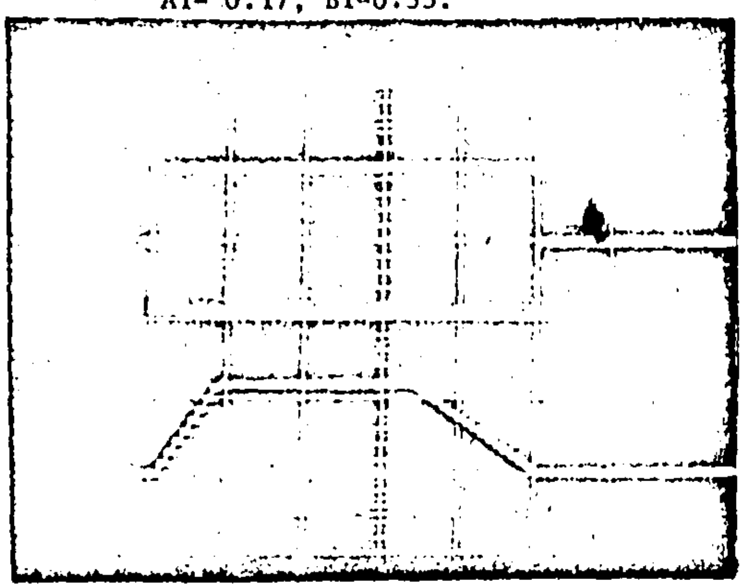


Fig. 5.6

Spectrum Pulse : as shown in Fig. 5.5

Max. frequency deviation BM = 1 KHz.

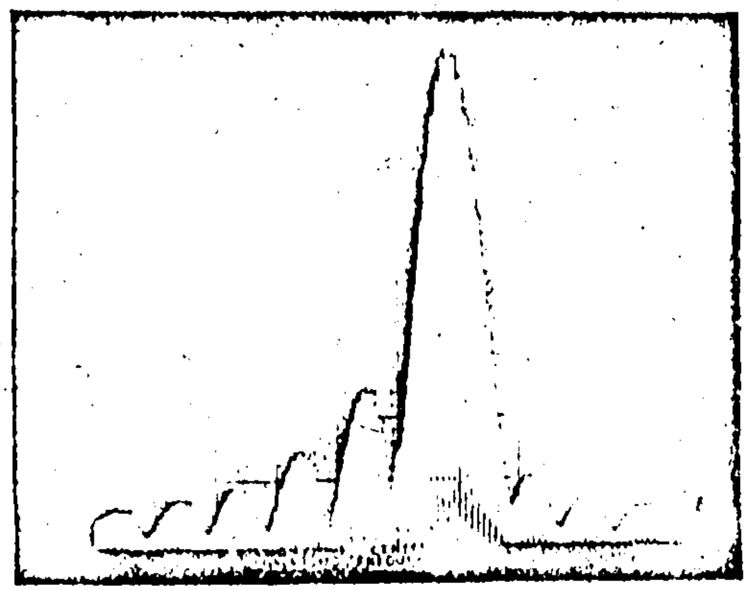


Fig. 5.7

Spectrum Pulse : as shown in Fig. 5.5

Max. frequency deviation BM = 4 KHz.

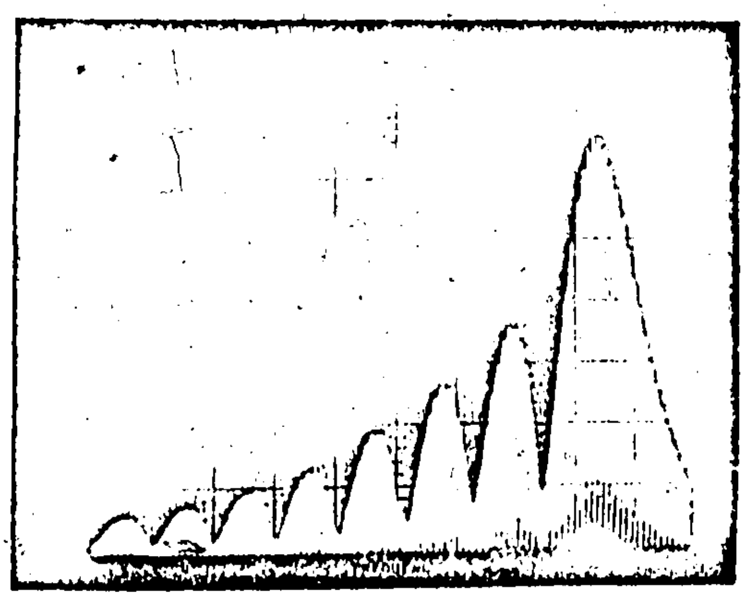


Fig. 5.8

Pulse : Trapezoidal AM and constant FM.

A2=0.17, B2=0.33.

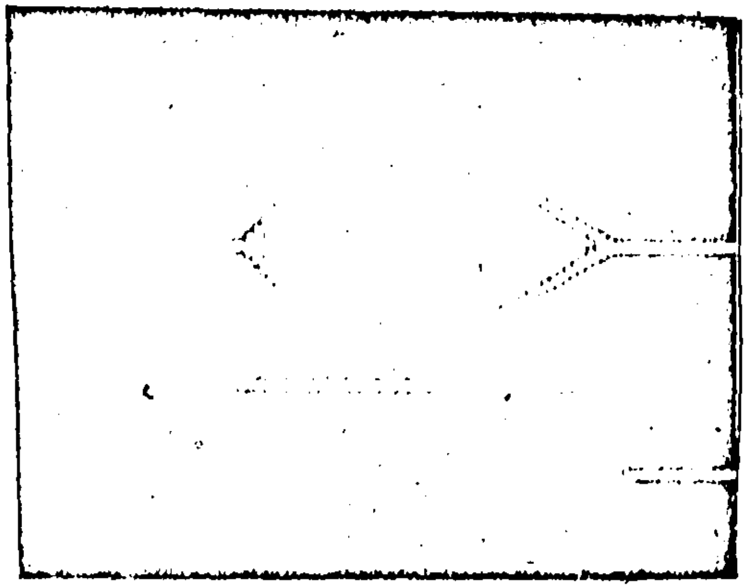
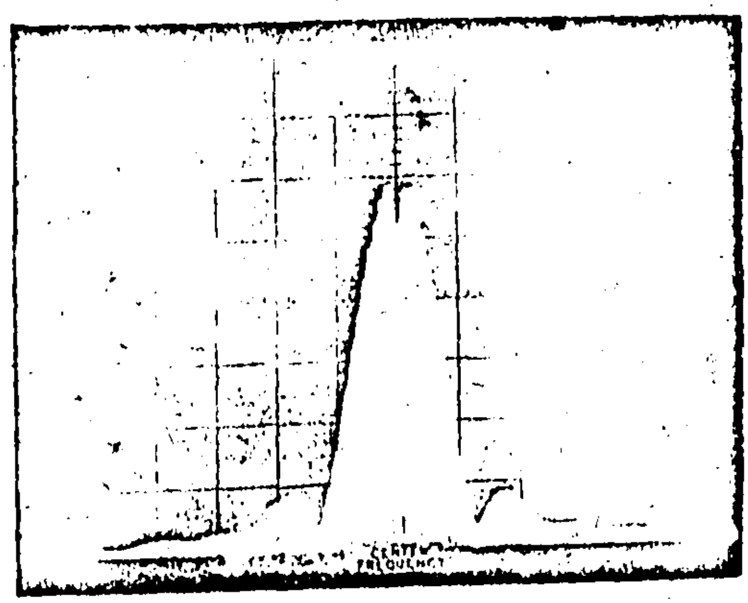


Fig. 5.9

Spectrum

Pulse : as shown in Fig. 5.8

Max. frequency deviation BM= 0.



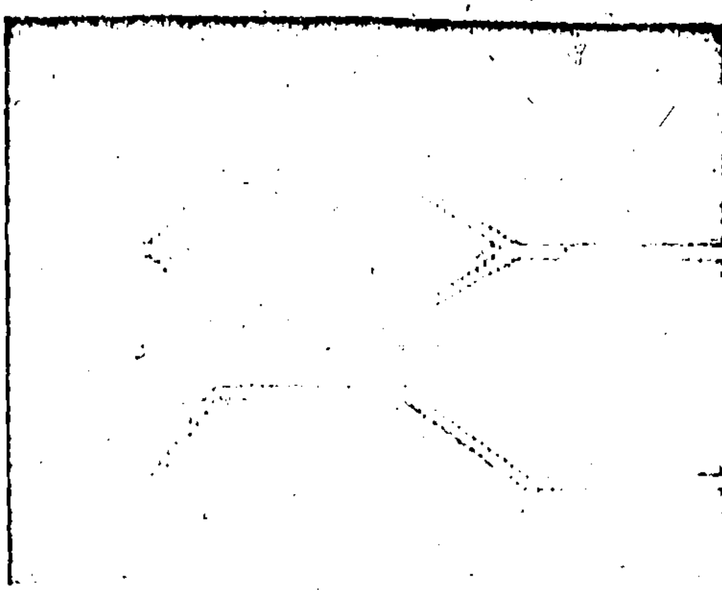


Fig. 5.10

Pulse:

Trapezoidal AM and

Trapezoidal FM

A1=0.17, B1=0.33,

A2=0.17, B2=0.33.

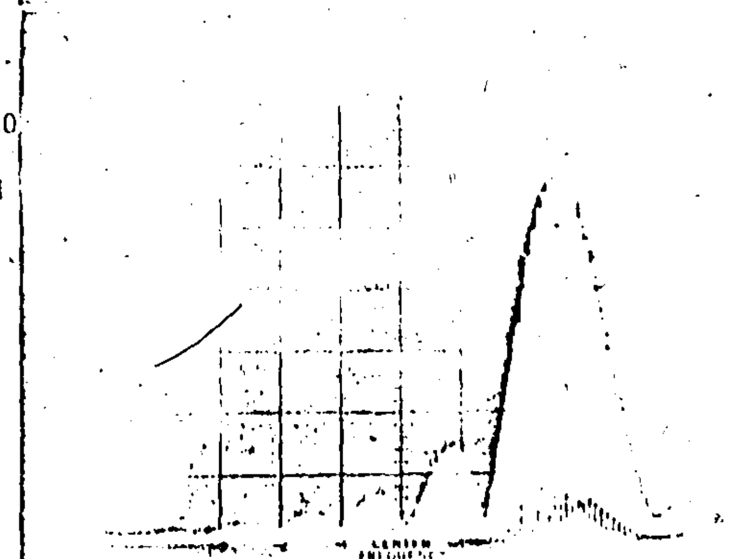


Fig. 5.11

Spectrum

Pulse: as shown in Fig. 5.10

Max. Frequency deviation BM

= 3 KHz.

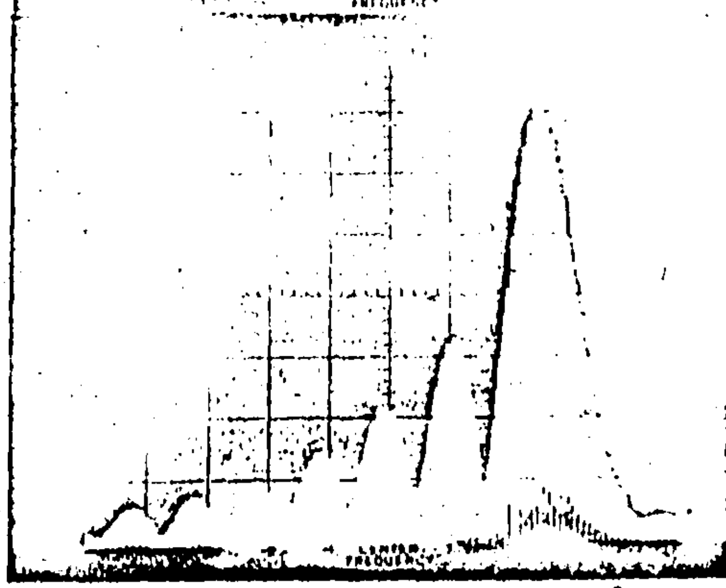


Fig. 5.12

Spectrum

Pulse:

Rectangular AM and same

FM pulse as shown in

Fig. 5.10

Max. frequency deviation BM

= 3 KHz.

Fig. 5.10

Pulse:

Trapezoidal AM and

Trapezoidal FM

A1=0.17, B1=0.33,

A2=0.17, B2=0.33.

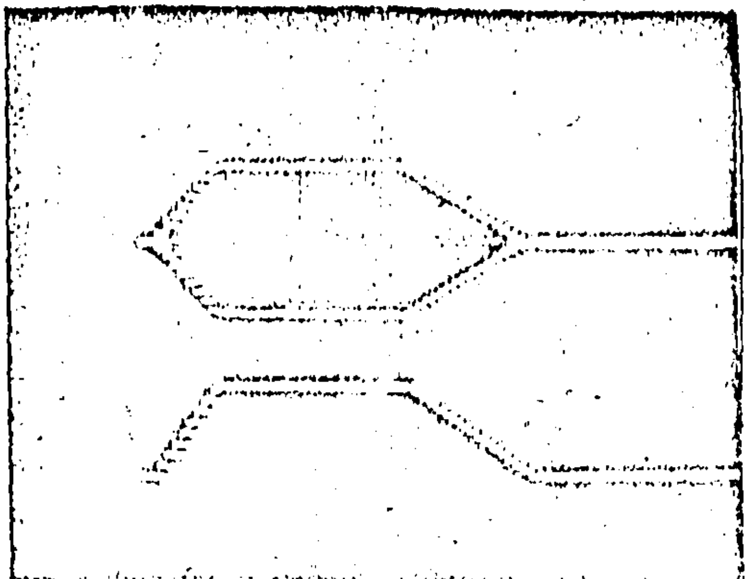


Fig. 5.11

Spectrum

Pulse: as shown in Fig. 5.10

Max. Frequency deviation BM

= 3 KHz.

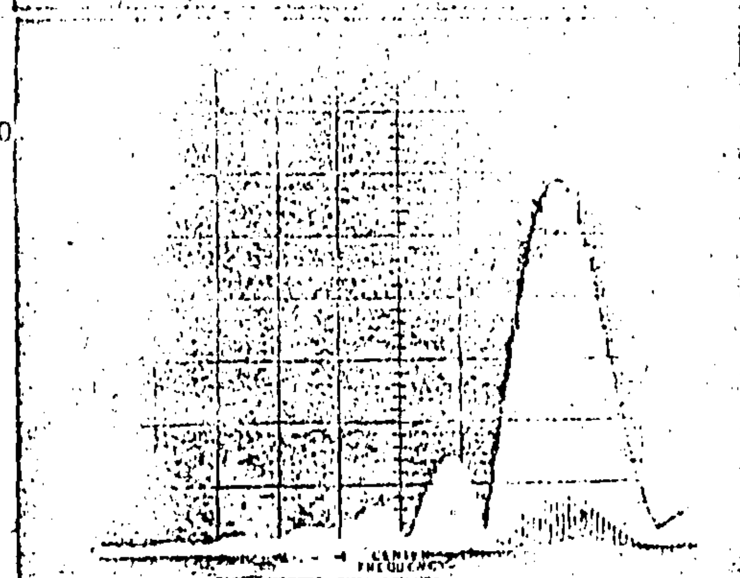


Fig. 5.12

Spectrum

Pulse:

Rectangular AM and same

FM pulse as shown in

Fig. 5.10

Max. frequency deviation BM

= 3 KHz.

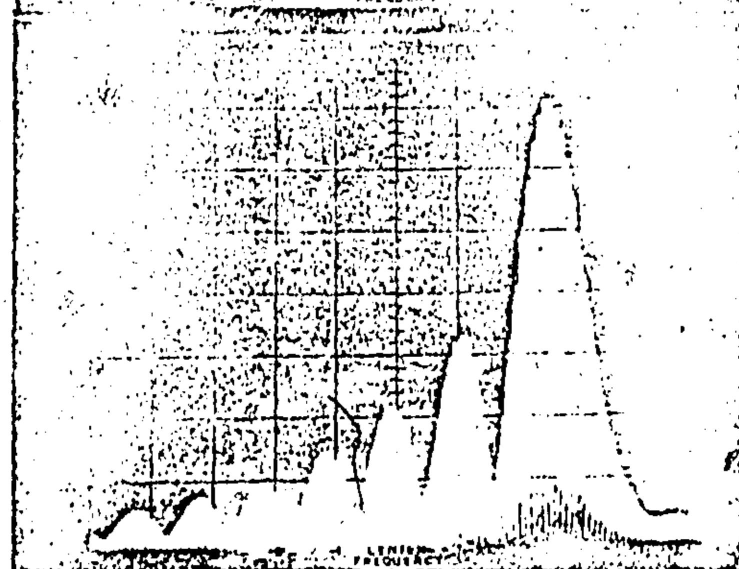


Fig. 5.13

Pulse : Rectangular AM and Trapzoidal FM.

A1=0.40, B1=0.50.

Fig 5.14

Spectrum

Pulse : as shown in Fig. 5.13.

Max. frequency deviation BM=3.5 KHz.

Fig. 5.15

Pulse : Rectangular AM and Exponential FM.

A1=0.5, B1=0.5, C1=2, K1=2.

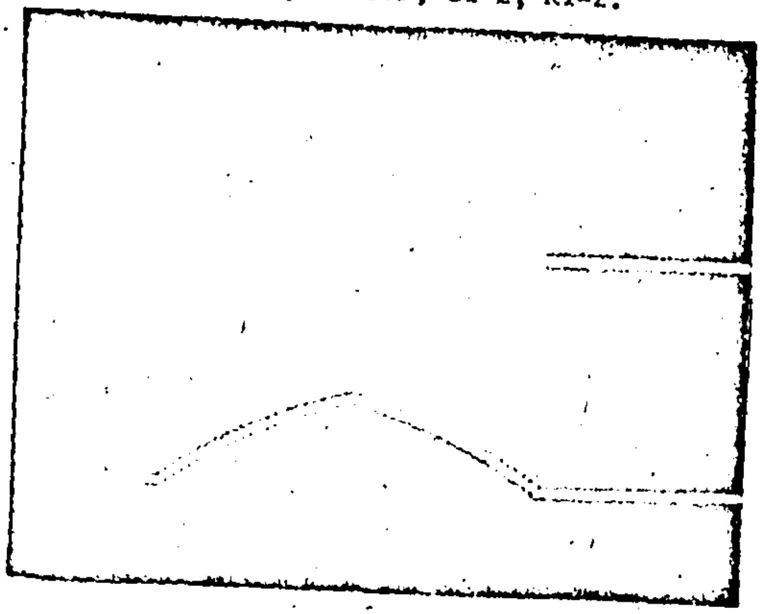


Fig. 5.16

Spectrum

Pulse : as shown in Fig. 5.15

Max. frequency deviation BM= 3.5 KHz.

Fig. 5.17

Pulse:

Exponential AM and

Exponential FM.

A1=B1=A2=B2= 0.5

C1=K1= 11, C2=K2= 2.

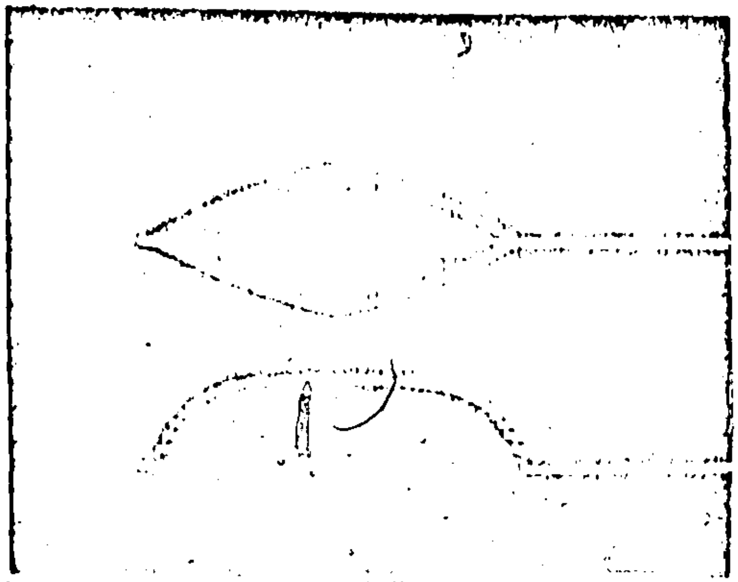


Fig. 5.18

Spectrum

Pulse : as shown in Fig. 5.17

Max. frequency deviation BM=0.

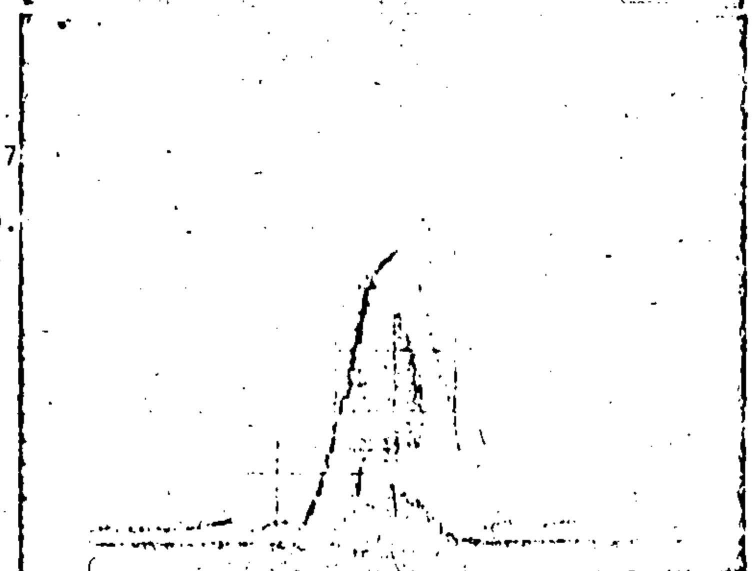


Fig. 5.19

Spectrum

Pulse : as shown in Fig. 5.17

Max. frequency deviation BM

= 3.5 KHz.

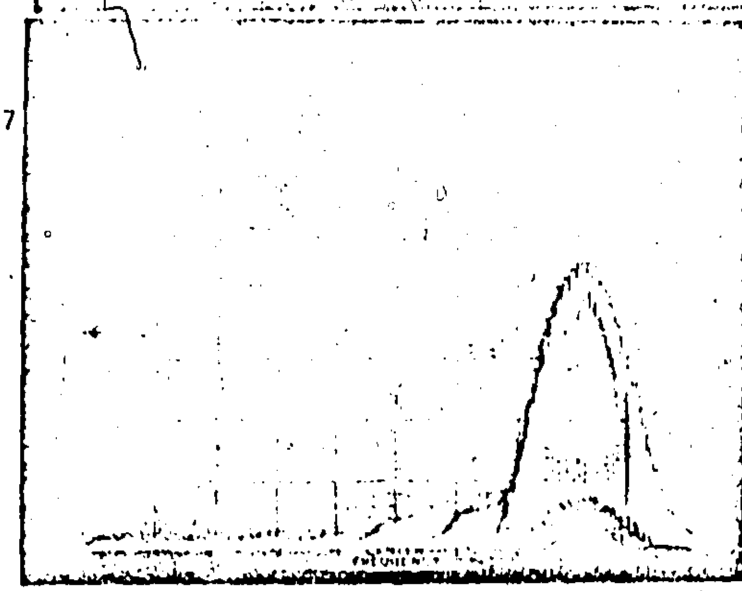


Fig. 5.20

Pulse: Exponential AM and Exponential FM.

A1=B1=A2=B2= 0.5

C1=K1= 11, C2=K2= 14.

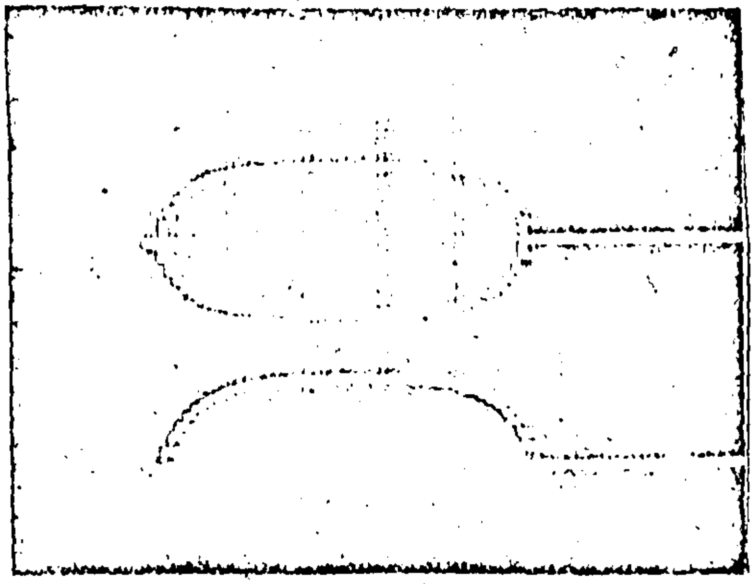


Fig. 5.21

Spectrum

Pulse: as shown in Fig. 5.20

Max. frequency deviation BM = 3.5 KHz.

Fig. 5.22

Pulse: Sinusoidal AM and Sinusoidal FM

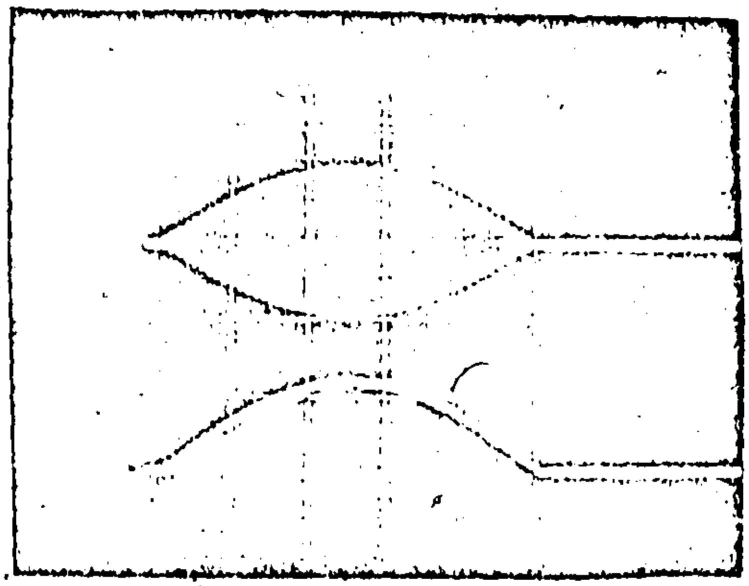


Fig. 5.23

Spectrum

Pulse: as shown in Fig. 5.22

Max. frequency deviation BM = 0.

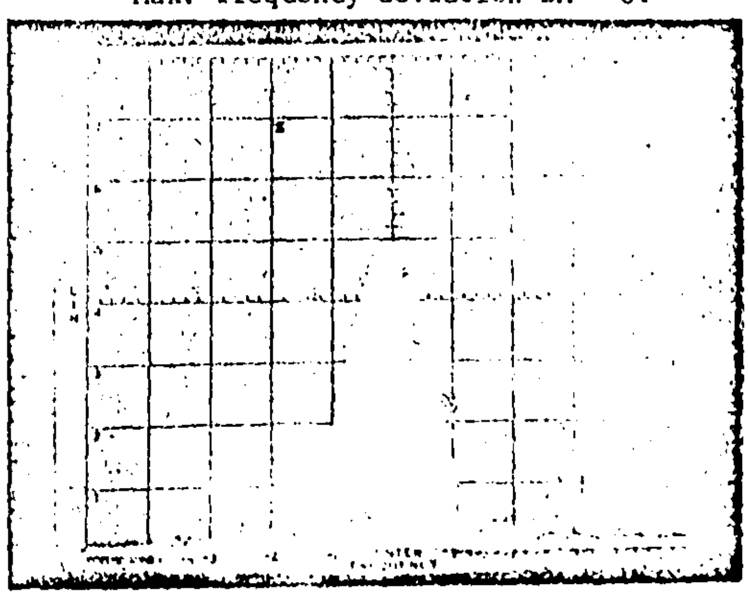


Fig. 5.24

Spectrum

Pulse: as shown in Fig. 5.22

Max. frequency deviation BM= 1 KHz.

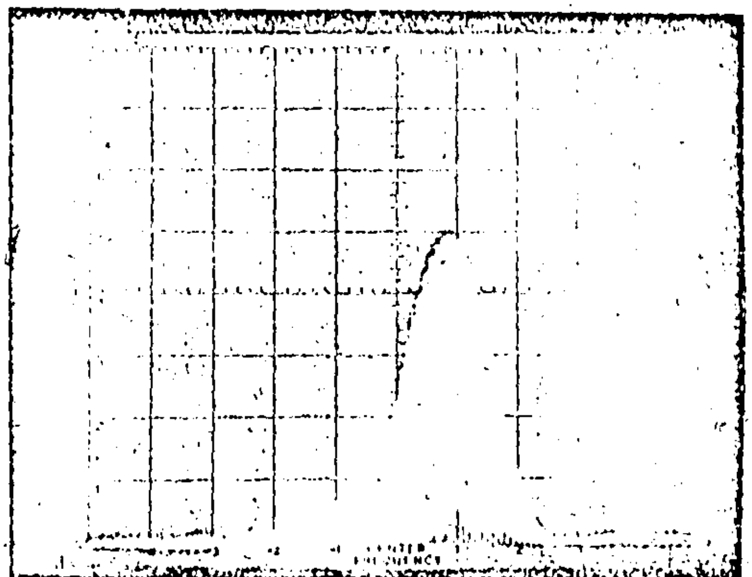


Fig. 5.25

Spectrum

Pulse: as shown in Fig. 5.22

Max. frequency deviation BM= 3 KHz.

Fig. 5.26

<u>Pulse</u>: Rectangular AM and modified Gaussian FM

Kl= 25

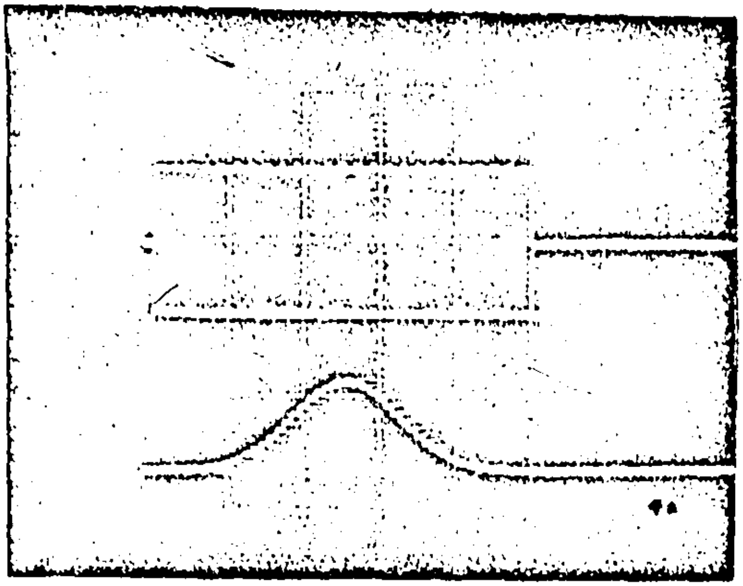


Fig. 5.27

<u>Spectrum</u>

Pulse: as shown in Fig. 5.26

Max. frequency deviation BM = 3.5 KHz.

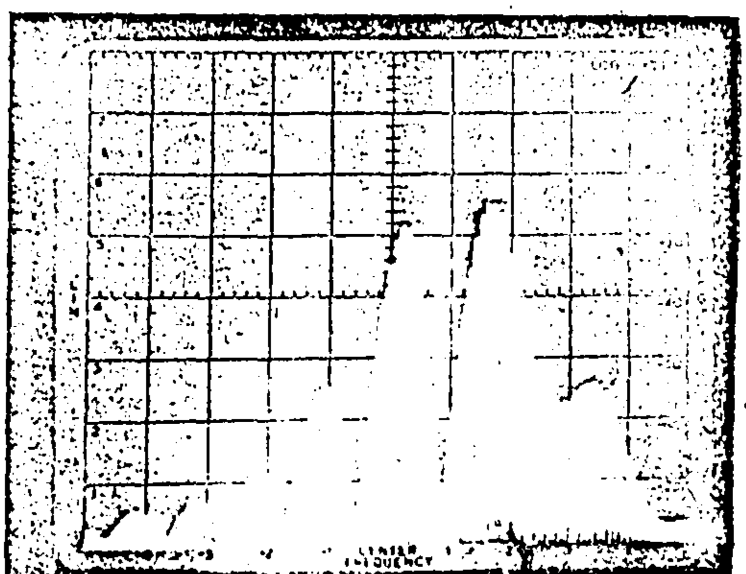


Fig. 5.28

Pulse: Modified Gaussian AM and modified Gaussian FM.

K1=K2= 10

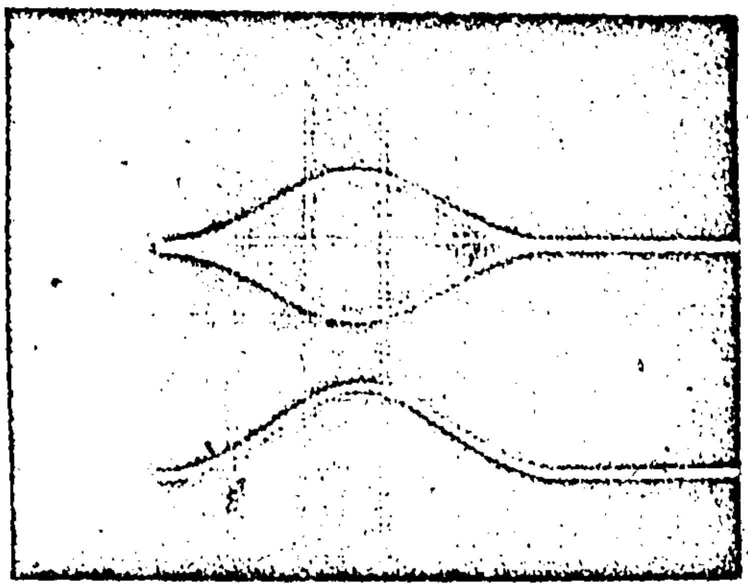


Fig. 5.29

Spectrum

Pulse: as shown in Fig. 5.28

Max. frequency deviation BM=0.

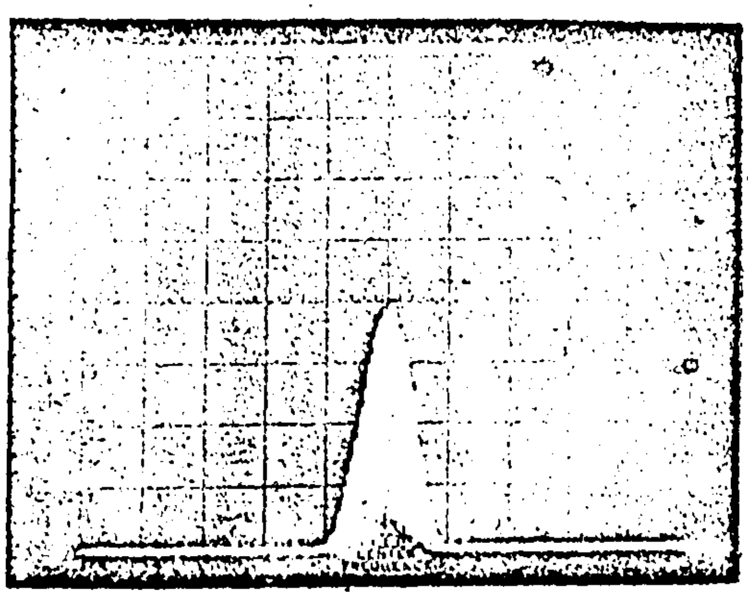


Fig. 5.30

Spectrum

Pulse: as shown in Fig. 5.28

Max. frequency deviation BM= 5 KHz.

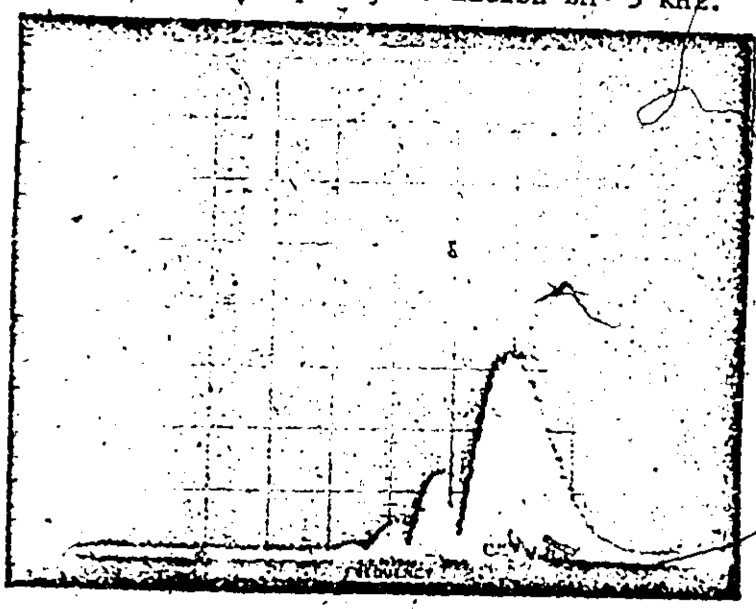


Fig. 5.31

Spectrum

Pulse: Rectangular AM, and same FM as shown in Fig. 5.28.

Max. frequency deviation BM= 5 KHz.

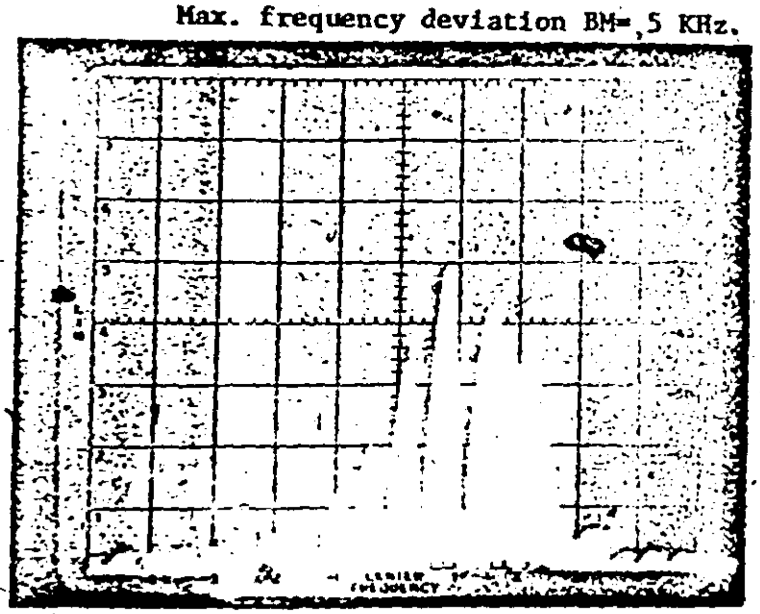


Fig. 5.32

Pulse: AM --- even symmetric about the middle of the pulse T/2.

FM --- odd symmetric about BM/2

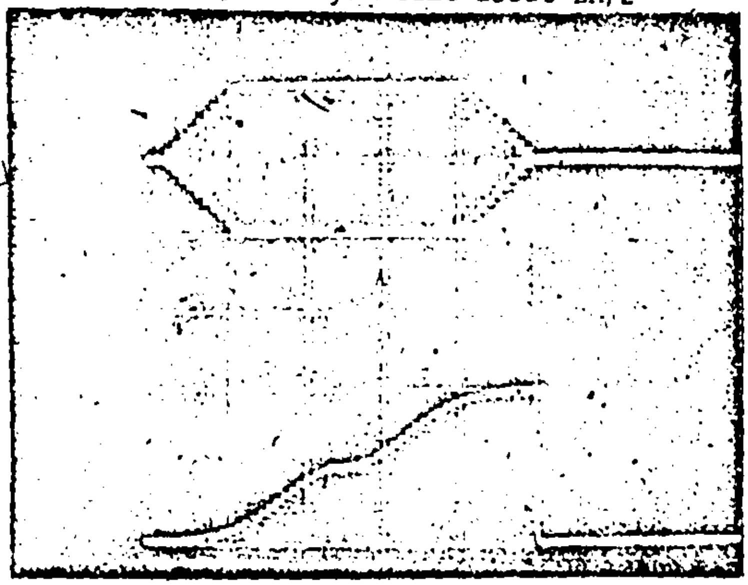


Fig. 5.33

Spectrum: Symmetric about BM/2

Pulse: as shown in Fig. 5.32

Max. frequency deviation BM = 3 KHz.

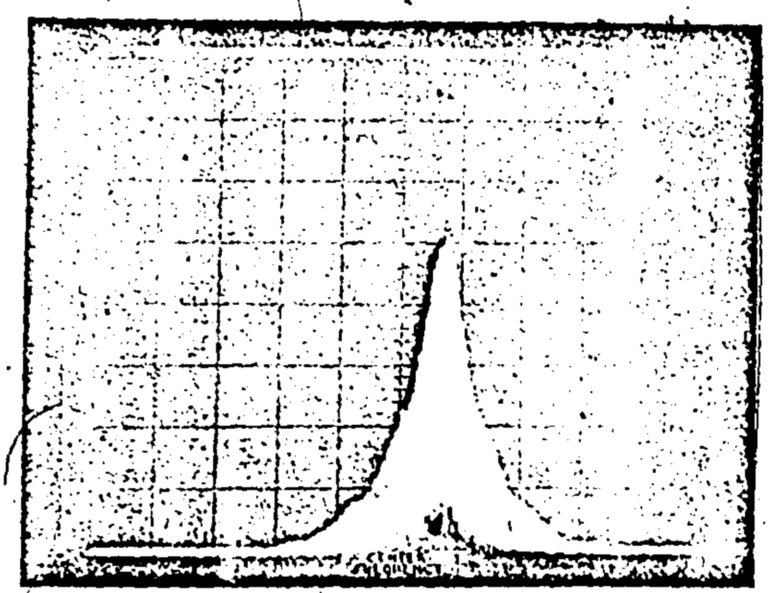


Fig. 5.34

Pulse: AM --- even symmetric about T/2.

FM --- odd symmetric about BM(2/3)

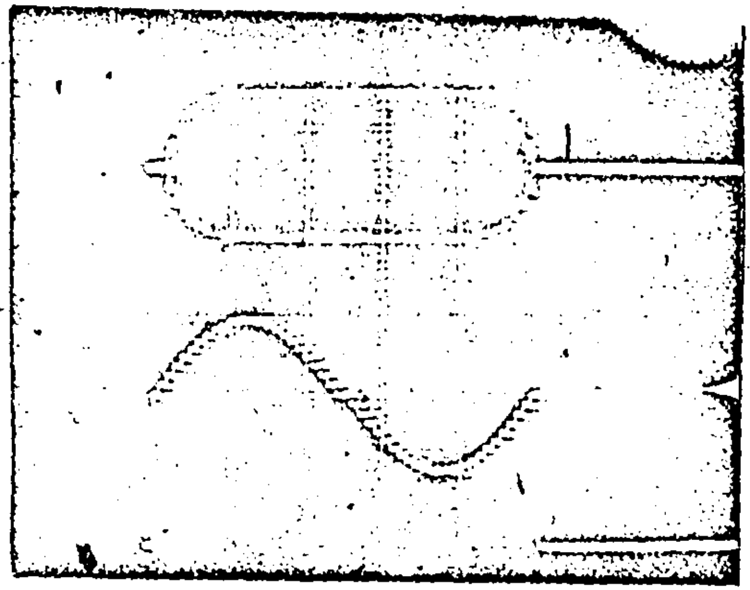
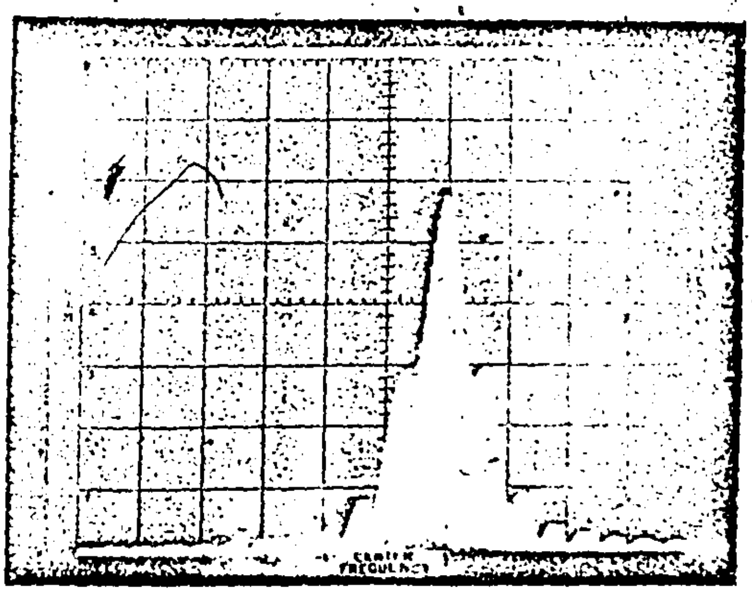


Fig. 5.35

Spectrum: Symmetric about BM(2/3)

Pulse: as shown in Fig. 5.34

Max frequency deviation BM= 3 KHz.



CHAPTER 6.

CONCLUSION AND RECOMMENDATIONS FOR FURTHER RESEARCH

The contributions of this thesis may be summarized as follows :

(1) A phenomenological approach to the analysis of pulsed RF signals containing incidental FM has been developed. This approach was aided by use of the Fast Fourier transform (FFT) technique.

(2) Using the convolution integral, a fairly general analytical formula has been developed that can be used to compute the Fourier transform of a pulse with incidental AM and FM. Also closed form analytical formulae were developed for a few selected simple modulated pulsed RF signals.

(3) A real-time experimental simulation of pulsed signals, using a digital computer, was designed to measure pulsed RF spectra. The experimental approach was used to verify the theoretical results obtained in (1) and (2) above. Good agreement was demonstrated between the theoretical and experimental results.

With regard to the problem of estimating the original complex time domain signal from a knowledge of only its power spectrum, it seems that this knowledge is not sufficient. In general, the phase information of this signal connot be obtained from its power spectrum only.

Several areas for further research are suggested by the results of this thesis; for example :

(1) There may be need for research in the area of measuring the frequency characteristics, the building up and decay of the oscillating frequency in the transmitter. A recent paper by N.S.Nahman [7] suggests some appropriate measuring technique.

(2) It would be highly desirable to develope an optimal amplitude modulating waveform that can reduce the asymmetry of the spectrum of a pulsed RF signal, and decrease the side-lobes level [8].

(3) There is need for further research on the effect of local oscillator frequency pulling upon the observed spectrum of radar signals. This effect can be substantially reduced by adding a well designed isolator in between the mixer and the local oscillator [9].

(4) The travelling-wave-tube amplifier (TWT), used in a radar system, for the purpose of RF amplification, exhibits two kinds of nonlinearities : (a) amplitude modulation to phase modulation (AM-PM) conversion, and (b) nonlinear input-output power characteristic. These can be treated as known phase and amplitude modulations applied to the input signal. If these characteristics can be measured, then by differentiating the resultant phase characteristic with respect to time, we get a frequency modulating waveform. Then we can estimate the spectrum using the methods considered in this thesis.

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APPENDIX

APPENDIX A

Derivation of eqt. (4.8)

$$e^{jz\sin\theta} = \sum_{k=-\infty}^{\infty} J_k(z)\, e^{jk\theta}$$

where $J_k(z)$ is the Bessel function of the first kind of k^{th} order.

Now $$\cos(z\sin\theta) = J_o(z) + 2\sum_{k=1}^{\infty} J_{2k}(z)\cos(2k\theta)$$

$$\sin(z\sin\theta) = 2\sum_{k=0}^{\infty} J_{2k+1}(z)\sin[(2k+1)\theta]$$

$$J_{-n}(z) = (-1)^n J_n(z)$$

$$J_{2n+1}(-z) = -J_{2n+1}(z)$$

and $$J_{2n}(z) = J_{2n}(-z)$$

Hence $$\cos(z\sin\theta) = J_o(z) + 2\sum_{k=1}^{\infty} J_{2k}(z)\cos(2k\theta) + \sum_{k=-\infty}^{\infty} J_{2k+1}(z)\cos[(2k+1)\theta]$$

$$= \sum_{k=-\infty}^{\infty} J_{2k}(z)\cos(2k\theta) + \sum_{k=-\infty}^{\infty} J_{2k+1}(z)\cos[(2k+1)\theta]$$

$$= \sum_{k=-\infty}^{\infty} J_k(z)\cos(k\theta)$$

Similarly $$\sin(z\sin\theta) = 2\sum_{k=0}^{\infty} J_{2k+1}(z)\sin[(2k+1)\theta]$$

$$= 2\sum_{k=0}^{\infty} J_{2k+1}(z)\sin[(2k+1)\theta] + \sum_{k=-\infty}^{\infty} J_{2k}(z)\sin(2k\theta)$$

$$= \sum_{k=-\infty}^{\infty} J_{2k+1}(z)\sin[(2k+1)\theta] + \sum_{k=-\infty}^{\infty} J_{2k}(z)\sin(2k\theta)$$

$$= \sum_{k=-\infty}^{\infty} J_k(z)\sin(k\theta)$$

Hence

$$e^{jz\sin\theta} = \cos(z\sin\theta) + j\sin(z\sin\theta)$$

$$= \sum_{k=-\infty}^{\infty} J_k(z)\cos(k\theta) + j\sum_{k=-\infty}^{\infty} J_k(z)\sin(k\theta)$$

$$= \sum_{k=-\infty}^{\infty} J_k(z)\, e^{jk\theta}$$

Q.E.D.

APPENDIX B

Derivation of eqt. (4.23)

$$s(t) = 0 \qquad \text{for } t \le 0,\ T \le t$$

$$= A\frac{t}{aT} \qquad 0 \le t \le aT$$

$$= A \qquad aT \le t \le T(1-b)$$

$$= A\frac{T-t}{Tb} \qquad T(1-b) \le t \le T$$

where T = pulse width

aT = rise time of the pulse.

bT = fall time of the pulse.

Let $r(t) = \frac{d\, s(t)}{dt}$

Hence

$$r(t) = 0 \qquad \text{for } t \le 0,\ T \le t$$

$$= \frac{A}{aT} \qquad 0 \le t \le aT$$

$$= 0 \qquad aT \le t \le T(1-b)$$

$$= -\frac{A}{Tb} \qquad T(1-b) \le t \le T$$

Now

$$F[r(t)] = \int_o^{aT} \frac{A}{aT} e^{-j2\pi ft}\, dt + \int_{T(1-b)}^{T} -\frac{A}{bT} e^{-j2\pi ft}\, dt$$

$$= \frac{A}{j2\pi fT}\{\frac{1}{a}(1- e^{-j2\pi faT}) - \frac{1}{b}[e^{-j2\pi f(1-b)T} - e^{-j2\pi fT}]\}$$

Because $F[r(t)] = j2\pi f \, F[s(t)]$

Hence

$$F[s(t)] = \frac{-A}{4\pi^2 f^2 T}\{ \frac{1}{a}(1- e^{-j2\pi faT}) - \frac{1}{b}[e^{-j2\pi f(1-b)T} - e^{-j2\pi fT}]\}$$

Q.E.D.

APPENDIX C

Derivation of eqt. (4.24)

$$s(t) = 0 \qquad \text{for } t \le 0,\ T \le t$$

$$= A_1(1-e^{-c_2 t}) \qquad 0 \le t \le t_1$$

$$= A \qquad t_1 < t < t_2$$

$$= A_2[1-e^{-k_2(T-t)}] \qquad t_2 \le t \le T$$

$$\text{where } A_1 = \frac{A}{1-e^{-c_2 t_1}} \quad \text{and} \quad A_2 = \frac{A}{1-e^{-k_2(T-t_2)}}$$

Then $S(f)$

$$= \int_{-\infty}^{\infty} s(t)\, e^{-j2\pi ft}\, dt$$

$$= \int_0^{t_1} A_1(1-e^{-c_2 t})\, e^{-j2\pi ft}\, dt + A\int_{t_1}^{t_2} e^{-j2\pi ft}\, dt + \int_{t_2}^{T} A_2[1-e^{-k_2(T-t)}]e^{-j2\pi ft} dt$$

$$= A_1\int_0^{t_1} e^{-j2\pi ft}\, dt - A_1\int_0^{t_1} e^{-c_2 t-j2\pi ft}\, dt + A\int_{t_1}^{t_2} e^{-j2\pi ft}\, dt +$$

$$A_2\int_{t_2}^{T} e^{-j2\pi ft}\, dt - A_2\int_{t_2}^{T} e^{-k_2(T-t)-j2\pi ft}\, dt$$

After the integration, we have

$$S(f) = A_1 t_1 e^{-j\pi f t_1} \operatorname{sinc}(\pi f t_1) + A_1 \frac{e^{-t_1(c_2+j2\pi f)} - 1}{c_2 + j2\pi f} +$$

$$A \frac{e^{-j2\pi f t_2} - e^{-j2\pi f t_1}}{-j2\pi f} + A_2 \frac{e^{-j2\pi f T} - e^{-j2\pi f t_2}}{-j2\pi f}$$

$$- A_2 \frac{e^{-j2\pi f T} - e^{-j2\pi f t_2 - k_2(T-t_2)}}{k_2 - j2\pi f}$$

Q.E.D.

APPENDIX D

Derivation of eqt. (4.27)

Define $s(t) = 0$ for $t<0,\ T<t$

$$= A\, e^{j2\pi \int_0^t g(x)dx} \qquad 0 \le t \le T$$

where $g(t) = 0$ for $t<0,\ T<t$

$$= B \frac{t}{aT} \qquad 0 \le t \le aT$$

$$= B \qquad aT \le t \le T(1-b)$$

$$= B \frac{T-t}{bT} \qquad T(1-b) \le t \le T$$

and aT = rise time of FM pulse.

bT = fall time of FM pulse.

B = maximum frequency deviation.

T = pulse width.

A = maximum amplitude of modulated pulse.

Hence $S(f)$

$$= \int_{-\infty}^{\infty} s(t)\, e^{-j2\pi ft}\, dt$$

$$= A\int_0^{aT} e^{-j2\pi ft}\, e^{j2\pi\int_0^t B\frac{x}{aT}dx}\, dt + A\int_{aT}^{(1-b)T} e^{-j2\pi ft}\, e^{j2\pi\left[\int_0^{aT} B\frac{x}{aT}dx + \int_{aT}^{t} Bdx\right]}\, dt$$

$$+ A\int_{(1-b)T}^{T} e^{-j2\pi ft}\, e^{j2\pi\left[\int_0^{aT} B\frac{x}{aT}\,dx + \int_{aT}^{T(1-b)} Bdx + \int_{T(1-b)}^{t} B\,\frac{T-x}{bT}\,dx\right]}\, dt$$

$$= A\int_{o}^{aT} e^{-j2\pi(ft - \frac{B}{2aT}t^2)}\,dt + A\int_{aT}^{T(1-b)} e^{-j2\pi(f-B)t}\,e^{-jBaT\pi}\,dt$$

$$+ A\int_{T(1-b)}^{T} e^{-j2\pi(ft-\frac{B}{b}t + \frac{B}{2bT}t^2)}\,e^{j2\pi BT(1-\frac{a}{2}-\frac{b}{2}-\frac{1}{2b})}\,dt$$

$$\}$$

Now complete the square. Then we have $S(f)$

$$= A\{\sqrt{\frac{aT}{2B}}\,e^{-j\frac{\pi aTf^2}{B}}\int_{-\sqrt{\frac{2aT}{B}}f}^{\sqrt{\frac{2aT}{B}}(B-f)} e^{j\frac{\pi}{2}t^2}\,dt$$

$$+ e^{-jBaT\pi}\frac{1}{-j2\pi(f-B)}\left[e^{-j2\pi(1-b)T(f-B)} - e^{-j2\pi Ta(f-B)}\right]$$

$$+ e^{j2\pi BT(1-\frac{a}{2}-\frac{b}{2}-\frac{1}{2b})}\,e^{j\pi T\frac{(fb-B)^2}{Bb}}\sqrt{\frac{bT}{2B}}\int_{\sqrt{\frac{2bT}{B}}(f-B)}^{\sqrt{\frac{2bT}{B}}f} e^{-j\frac{\pi}{2}t^2}\,dt\,\}$$

Q.E.D.

APPENDIX E

Derivation of eqt. (4.28)

$$s(t) = 0 \qquad \text{for} \qquad t<0, \quad t>T$$

$$= A\, g(t)\, e^{j2\pi B\int_0^t g(x)\,dx} \qquad 0 \le t \le T$$

where A = maximum amplitude of modulated pulse

B = maximum frequency deviation

$$g(t) \neq 0 \qquad \text{for} \qquad 0 \le t \le T$$

$$= 0 \qquad t<0, \quad t>T$$

Hence $S(f) = \int_{-\infty}^{\infty} s(t)\, e^{-j2\pi ft}\, dt$

$$= A\int_0^T g(t)\, e^{j2\pi B\int_0^t g(x)dx}\, e^{-j2\pi ft}\, dt$$

$$= \frac{A}{j2\pi B}\int_0^T e^{j2\pi B\int_0^t g(x)dx}\, e^{-j2\pi ft}\, d\,[j2\pi B\int_0^t g(x)dx]$$

$$= \frac{A}{j2\pi B}\; [(e^{j2\pi B\int_0^T g(x)dx}\, e^{-j2\pi fT} - 1) + j2\pi f\int_0^T e^{j2\pi B\int_0^t g(x)dx}\, e^{-j2\pi ft}\, dt]$$

$$= \frac{A}{B}\,\{\frac{1}{j2\pi}\, [e^{j2\pi B\int_0^T g(x)dx}\, e^{-j2\pi fT} - 1] + f\, F[\,Rect(t)\, e^{j2\pi B\int_0^t g(x)dx}\,]\,\}$$

Q.E.D.