STUDIES ON CATEGORIES OF UNIVERSAL TOPOLOGICAL ALGEBRAS

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UNIVERSAL TOPOLOGICAL ALGEBRAS

By

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SCOPE AND CONTENTS: In this thesis we study categories of universal topological algebras and topological partially ordered sets. Topologically algebraic functors are introduced. It is shown that most of categories usually studied in general topology, universal algebras and universal topological algebras can be covered by this concept. In addition, we consider extensive subcategories of various categories of Hausdorff topological algebras and continuous homomorphisms. In particular, we obtain a method to construct extensive subcategories which is applied to categories of topological partially ordered sets and topological lattices.

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INTRODUCTION

Recently there has been a growing interest in universal topological algebras [2, 3, 10, 11, 12] which is an obvious result of the strong enthusiasm on universal algebras. Not only does it allow one to generalize the old topological algebras, namely topological groups, topological rings et al., but it offers the possibility for study of their individual structures as whole.

It is known that category theory provides a convenient language to every modern mathematics. In particular, one can compare different structures with the concept of functors and the universal mapping property [6] can be precisely comprehended through the adjunctions.

Our purpose in writing this work is to present categorical properties of universal topological algebras.

To simplify the language in this thesis, universal topological algebras of any type will be called topological algebras.

This paper is divided into five chapters.

Chapter 0 contains those definitions and results from category theory and universal algebras which will pave the way for the further development of the present thesis.

In particular, we list definitions of limits, adjunctions, (E, M)-categories, universal algebras and topological algebras. Also included are discussions of adjoint functor theorems, categories of universal algebras of fixed type and homomorphisms, free algebras and equational classes.

In Chapter I, we introduce topologically algebraic functors. Most of the categories usually studied in algebra - such as the categories SGrp of semigroups, Mon of monoids, Grp of groups, Ring of rings and Lat of lattices and others - have many properties in common. So do the categories studied in general topology - such as the categories Top of topological spaces, Unif of uniform spaces, Haus of Hausdorff spaces and others. Because of these, there have been numerous attempts [19, 20, 22, 36, 37] to study them simultaneously and to abstract the essence of their common properties via general concept. It has been known that those facts are essentially due to some common properties of the corresponding underlying set functors into the category Set of sets and maps. Comprehensive bibliography of papers in this field can be found in [20].

In the direction of categories of algebras, algebraic categories (see §32 and §38 in [22]) have been defined among others and in the direction of categories of topological spaces, topological functors have been introduced by H. Herrlich [19]. It is known that each one doesn't cover the other. Moreover the categories of topological algebras can't be covered by either one of those concepts. We attempt to introduce a new concept which covers not only the both cases but also categories studied in topological algebras.

Extending Herrlich's idea [19, 20], that for any family $(A_i)_{i \in I}$ of topological spaces and a family $(f_i: X \longrightarrow A_i)_{i \in I}$ of set maps, one can endow the initial structure [6, 7] on X with respect to $(f_i)_{i \in I}$, we reach our goal.

We define a topologically algebraic functor as follows: a functor $U: \underline{A} \longrightarrow \underline{B}$ is called topologically algebraic if for any family $(A_i)_{i \in I}$ of $\underline{A}$-objects and a $\underline{B}$-source $(B, B \xrightarrow{s_i} UA_i)_{i \in I}$, there is a U-initial source $(A, A \xrightarrow{\bar{s}_i} A_i)_{i \in I}$ and a morphism $h: B \longrightarrow UA$ which U-generates A such that $(U\bar{s}_i)h = s_i$ $(i \in I)$.

Then it is shown that every topologically algebraic functor is faithful and has a left adjoint. Moreover for a topologically algebraic functor $U: \underline{A} \longrightarrow \underline{B}$, whenever $\underline{B}$ is complete (resp. cocomplete), then so is $\underline{A}$. Those limits and colimits can be constructed from the data in $\underline{B}$. The most striking part is that every topological functor is topologically algebraic and that the underlying set functor of an algebraic category is also topologically algebraic. Furthermore, for any class $\underline{A}$ of universal algebras of fixed type which is closed under the operations S (isomorphic images of subalgebras of members of the class) and P (isomorphic images of products of members of the class) and any full subcategory T of Top (resp. Unif) which is productive and hereditary, the category TA of topological (resp. uniform) algebras whose underlying algebras belong to A and whose underlying spaces belong to T, and (resp. uniformly) continuous homomorphisms is topologically algebraic over the underlying set functor. For topological algebras, we have two more related functors, namely the underlying algebra functor and the underlying space functor. For the same category TA as the above, those functors are also topologically algebraic. We note that every equational class is closed under the operations S and P.

Finally it is shown that a concrete category (A, U) is algebraic if and only if U is topologically algebraic, U preserves regular epimorphisms and any morphism e with Ue being onto is U-coinitial. This shows that the most essential properties in categories of universal algebras of fixed type and homomorphisms are those that they are (epi, mono-sources)-categories, that every mono-source is initial and that onto homomorphisms are coinitial.

Chapter II deals with extensions in topological algebras, in particular those with the universal mapping property or extensive subcategories.

S. S. Hong has introduced [23, 24, 25] the concept of extensive operators on a subcategory of Haus (HUnif) which enables him to construct new extensive subcategories from well known ones. Extending his idea to topological algebras, it is shown that the same machinery works in categories of Hausdorff topological algebras and that our result covers his case. In particular, whenever a coreflective subcategory C of Top is (resp. finitely) productive, we can associate with C a new extensive subcategory from an extensive subcategory of any hereditary category of (resp. finitary) Hausdorff topological algebras and continuous homomorphisms.

Using G_k-sets [25], we construct new extensive subcategories in various categories of topological groups.

Chapter III is devoted to categories of topological partially ordered sets. Since Ward has introduced a topology - we will call it W-topology - on a partially ordered set which makes the order continuous in his sense, W-topology has been believed to be a pertinent comparable topological structure on a partially ordered set. Since our setting is mainly based on universal algebras, partially ordered sets are considered as partial universal algebras, i.e., the partially ordered set with the join (or meet) defined on the set of all pairs of comparable elements. In this thesis, topological partial algebras in this setting are called topological partially ordered sets. We observe that every algebra topology on a partially ordered set need not be a W-topology. However, it is shown that the category

WPOS of partially ordered sets with W-topologies and continuous isotone maps is reflective in the category TPOS of topological partially ordered sets and continuous isotone maps. Observing that the underlying set functors of TPOS and WPOS are topologically algebraic, the basic categorical properties of those categories are directly deduced.

We introduce the cocept of o-completely regular filters on a completely regular ordered space in order to characterize compact ordered spaces. It is known that the category COMPOS of compact ordered spaces is extensive in the category CROS of completely regular ordered spaces and continuous isotone maps. It is shown that the COMPOS-reflection of a completely regular ordered space X is precisely the strict extension [1] of X with all maximal o-completely regular filters as the filter trace.

We define k-compact ordered spaces for every infinite cardinal k, which give rise to a chain of extensive subcategories of CROS, analogous to the chain given by the category of k-compact spaces [16], for the different cardinals k. We note that $\aleph_1$-compact spaces are precisely realcompact spaces. However it is shown that every $\aleph_1$-compact ordered space need not be an R-compact ordered space and that the category of k-compact ordered spaces is simple.

Finally, in Chapter IV, we deal with the category TLAT of topological lattices and continuous homomorphisms and subcategories of TLAT. Observing that the subcategory determined by locally convex topological lattices is bireflective in TLAT, we show that every compact topological lattice is locally convex. For the unit interval I, the subcategory I-COMPTL determined by I-compact topological lattices is obviously extensive in the category CRTL of completely regular topological lattices and continuous homomorphisms. Again introducing 1-completely regular filters, for any L ∈ CRTL, the reflection is given by the strict extension of L with all maximal 1-completely regular filters on L as the filter trace. By the exactly same way as that in Chapter III, we introduce k-compact topological lattices which enjoy the same role as k-compact ordered spaces.

Furthermore, every completely regular ordered space (resp. topological lattice) X is k-compact if and only if it is k-complete with respect to the uniform structure generated by the set of continuous homomorphisms into I.

Finally, we observe that every coreflective subcategory of TLAT is bijection-coreflective. Hence we are able to associate a new reflective subcategory of a hereditary subcategory A of TLAT with every coreflective subcategory of TLAT and an extensive subcategory of A.

CHAPTER 0

PRELIMINARIES

This chapter is a collection of basic definitions and results which will be needed in the ensuing chapters.

Section 1: Categories.

1.1 Definition A source in a category $\underline{C}$ is a pair $(X, (s_i)_{i \in I})$, where X is a $\underline{C}$-object, I is a class, and $(s_i : X \longrightarrow X_i)_{i \in I}$ is a family of $\underline{C}$-morphisms each with domain X. The source $(X, (s_i)_{i \in I})$ will be also denoted by $(X, X \xrightarrow{s_i} X_i)_I$.

Dual notion: sink in $\underline{C}$.

1.2 Definition (1) Let $\underline{I}$ be a small category and $\underline{C}$ a category. Then a functor $D: \underline{I} \longrightarrow \underline{C}$ is called a diagram in $\underline{C}$ over $\underline{I}$. A lower bound of the diagram D is a source $(L, L \xrightarrow{k_i} D(i))_{i \in \underline{I}}$ in $\underline{C}$ such that for any morphism $f: i \longrightarrow j$ in $\underline{I}$, $D(f)k_i = k_j$.

A lower bound $(L, L \xrightarrow{k_i} D(i))_{i \in \underline{I}}$ of D is called a limit of D if for any lower bound $(L', L' \xrightarrow{k'_i} D(i))_{i \in \underline{I}}$ of D, there exists a unique morphism $g: L' \longrightarrow L$ in $\underline{C}$ such that

$k_i g = k'_i$ for each $i \in \underline{I}$.

(2) If every diagram in $\underline{C}$ over $\underline{I}$ has a limit, then $\underline{C}$ is said to be <u>$\underline{I}$-complete</u> (or to have $\underline{I}$-limits). If $\underline{C}$ is $\underline{I}$-complete for every small category $\underline{I}$, then $\underline{C}$ is called <u>complete</u>.

Dual notions: <u>upper bounds</u> of a diagram; <u>colimits</u>; <u>$\underline{I}$-cocomplete</u>; <u>cocomplete</u>.

1.3 <u>Proposition</u> ~~Any limit of a diagram~~ $D: \underline{I} \longrightarrow \underline{C}$ is an extremal mono-source.

1.4 <u>Theorem</u> Let $\underline{C}$ be a category. Then the following are equivalent:

1) $\underline{C}$ is complete.
2) $\underline{C}$ has products and pullbacks.
3) $\underline{C}$ has products and equalizers.
4) $\underline{C}$ has products and inverse images.

1.5 <u>Definition</u> Let $G: \underline{A} \longrightarrow \underline{B}$ be a functor and B an object of $\underline{B}$. A pair (u, A) with $A \in \underline{A}$ and $u: B \longrightarrow GA$ is called a <u>universal map</u> for B with respect to G (or a <u>G-universal map</u> for B) provided that for each $A' \in \underline{A}$ and each $f: B \longrightarrow GA'$ there exists a unique $\underline{A}$-morphism $\bar{f}: A \longrightarrow A'$ with $(G\bar{f})u = f$.

1.6 <u>Definition</u> Let $\underline{A}$ and $\underline{B}$ be categories. An <u>adjunction</u> from $\underline{B}$ to $\underline{A}$ is a triple (F, G, φ), where F and

G are functors

$$\underline{B} \underset{G}{\overset{F}{\rightleftarrows}} \underline{A},$$

while φ is a map which assigns to each pair of objects $B \in \underline{B}$ and $A \in \underline{A}$ a bijection $\varphi_{B,A}: \underline{A}(FB, A) \longrightarrow \underline{B}(B, GA)$ which is natural in B and A.

In this case, the functor F is said to be a <u>left adjoint</u> for G, while G is called a <u>right adjoint</u> for F.

1.7 <u>Theorem</u> Let (F, G, φ) be an adjunction from $\underline{B}$ to $\underline{A}$. Then

1) G preserves limits.

2) F preserves colimits.

3) There exists a natural transformation $\eta: 1_{\underline{B}} \longrightarrow GF$ and a natural transformation $\varepsilon: FG \longrightarrow 1_{\underline{A}}$.

1.8 <u>Definition</u> For an adjunction (F, G, φ) from $\underline{B}$ to $\underline{A}$ and the natural transformation $\eta = (\eta_B)_{B \in \underline{B}}$ (resp. $\varepsilon = (\varepsilon_A)_{A \in \underline{A}}$) determined by the above theorem, η_B (resp. ε_A) will be called the <u>front</u> (resp. <u>back</u>) <u>adjunction</u> for each $B \in \underline{B}$ (resp. $A \in \underline{A}$)

1.9 <u>Theorem</u> Let $G: \underline{A} \longrightarrow \underline{B}$ be a functor.

1) If each $B \in \underline{B}$ has a G-universal map (η_B, A_B), then G has a left adjoint F such that $FB = A_B$ and η_B is the front adjunction for B.

2) Conversely, if G has a left adjoint F, then (η_B, FB)

is a G-universal map for each $B \in \underline{B}$, where $\eta_B: B \longrightarrow GFB$ is the front adjunction for B.

1.10 <u>Definition</u> Let $G: \underline{A} \longrightarrow \underline{B}$ be a functor and let B be a $\underline{B}$-object. A set-indexed family $(u_i, A_i)_I$, where each A_i is an $\underline{A}$-object and each $u_i: B \longrightarrow GA_i$ is a $\underline{B}$-morphism is called a <u>G-solution set</u> for B provided that for each $\underline{A}$-object A' and each morphism $f: B \longrightarrow GA'$, there exists some $i \in I$ and some morphism $f': A_i \longrightarrow A'$ such that $f = (Gf')u_i$.

1.11 <u>Theorem</u> Let $\underline{A}$ be complete and $G: \underline{A} \longrightarrow \underline{B}$. Then G has a left adjoint if and only if

1) G preserves limits, and

2) Each $\underline{B}$-object has a G-solution set.

1.12 <u>Theorem</u> Suppose that $\underline{A}$ is well-powered, complete, and has a coseparator C. Then for each functor $G: \underline{A} \longrightarrow \underline{B}$ the following are equivalent:

1) G has a left adjoint.

2) G preserves limits.

1.13 <u>Definition</u> Let $\underline{A}$ be a subcategory of $\underline{B}$ with embedding functor $E: \underline{A} \longrightarrow \underline{B}$:

1) An E-universal map (r_B, A_B) for a $\underline{B}$-object B is called an <u>$\underline{A}$-reflection</u> of B.

2) $\underline{A}$ is called <u>reflective</u> in $\underline{B}$ or a <u>reflective subcategory</u> of $\underline{B}$ if and only if there exists an $\underline{A}$-reflection for

each B-object; i.e., if and only if E has a left adjoint, R: B $\longrightarrow$ A. In this case, R is called a *reflector* for A.

3) If E is a class of B-morphisms, then A is called *E-reflective* in B provided that for each B-object B there exists an A-reflection (r_B, A_B) such that each $r_B \in$ E. For the case that E is the class of all epimorphisms (resp. monomorphisms) of B we say that A is *epireflective* (resp. *monoreflective*) in B.

Dual notions: *A-coreflection* of B; *coreflective* in B (or a *coreflective subcategory* of B); *coreflector* for A; *E-coreflective* in B.

1.14 *Theorem* Every reflective subcategory of a complete (resp. cocomplete) category is complete (resp. cocomplete). For each small category I, a full, isomorphism closed reflective subcategory A of B is closed under the formation of I-limits in B.

In the following, let E be a class of epimorphisms in a category which is closed under composition with isomorphisms and let M (resp. M_0) be a class of sources (resp. monomorphisms) in a category which is closed under composition with isomorphisms.

1.15 *Definition* A category C is called an *(E, M)-category* provided that

1) any source $(X, X \xrightarrow{s_i} X_i)_I$ has an $(\underline{E}, \underline{M})$-factorization, i.e., there is a morphism $e\colon X \longrightarrow Y$ in $\underline{E}$ and a source $(Y, Y \xrightarrow{m_i} X_i)_I$ in $\underline{M}$ such that

$$X \xrightarrow{s_i} X_i = X \xrightarrow{e} Y \xrightarrow{m_i} X_i \text{ for each } i \in I, \text{ and}$$

2) For commutative diagrams in $\underline{C}$

$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ f\downarrow & & \downarrow g_i \\ Z & \xrightarrow[m_i]{} & Y_i \end{array} \quad (i \in I) \text{ with } e \in \underline{E} \text{ and}$$

$(Z, Z \xrightarrow{m_i} Y_i)_I \in \underline{M}$, there exists a morphism $k\colon Y \longrightarrow Z$ that makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ f\downarrow & \swarrow k & \downarrow g_i \\ Z & \xrightarrow[m_i]{} & Y_i \end{array}$$

commute for each $i \in I$.

Similarly one can define an $(\underline{E}, \underline{M}_0)$-category.

1.16 Proposition If $\underline{C}$ is an $(\underline{E}, \underline{M})$-category, then every $(\underline{E}, \underline{M})$-factorization is unique up to isomorphism, i.e., for any source $(X, X \xrightarrow{s_i} Y_i)_I$ and its $(\underline{E}, \underline{M})$-factorizations $X \xrightarrow{s_i} Y_i = X \xrightarrow{e} Y \xrightarrow{m_i} Y_i = X \xrightarrow{e'} Y' \xrightarrow{m_i'} Y_i$, there exists an isomorphism $h\colon Y \longrightarrow Y'$ such that the following diagram

$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ {\scriptstyle e'}\downarrow & {\scriptstyle h}\swarrow & \downarrow{\scriptstyle m_i} \\ Y' & \xrightarrow{m_i'} & Y_i \end{array}$$

commutes for each $i \in I$.

1.17 <u>Proposition</u> A category $\underline{C}$ is an $(\underline{E}, \underline{M}_0)$-category if and only if every $(\underline{E}, \underline{M}_0)$-factorization is unique up to isomorphism and both $\underline{E}$ and $\underline{M}_0$ are closed under composition.

1.18 <u>Theorem</u> If $\underline{B}$ is an $\underline{E}$-co-(well-powered) $(\underline{E}, \underline{M}_0)$-category that has products and $\underline{A}$ is a full isomorphism closed subcategory of $\underline{B}$, then $\underline{A}$ is $\underline{E}$-reflective in $\underline{B}$ if and only if $\underline{A}$ is closed under the formation of products and $\underline{M}_0$-subobjects in $\underline{B}$.

<u>Section 2</u>: <u>Universal algebras</u>.

2.1 <u>Definition</u> (1) A map $f: X^\lambda \longrightarrow X$, where X is any set and λ any ordinal number, is called an <u>operation</u> on X. In particular, λ is called the <u>arity</u> of f, and f is called a <u>λ-ary operation</u> on X.

(2) A <u>type</u> τ of universal algebras is a family $\tau = (\lambda_i)_{i \in I}$ of ordinals, indexed by a set I. Given a type $\tau = (\lambda_i)_{i \in I}$, a <u>universal algebra of type τ</u> is a pair $A = (X, f)$, where X is a set and $f = (f_i)_{i \in I}$ a family of

operations on X, each f_i of arity λ_i.

For such A, X is called the underlying set of A, f the algebra structure of A, and f_i the ith operation of A.

In this thesis, universal algebras will be called algebras in order to simplify the language.

(3) If all λ_i in $\tau = (\lambda_i)_{i \in I}$ are finite, τ is called finitary type of algebras. Algebras of finitary type are refered to as finitary algebras.

In what follows, the underlying set and ith operation of an algebra A will be denoted by $|A|$ and f_i^A respectively.

2.2 Definition Let A and B be algebras of a fixed type $\tau = (\lambda_i)_{i \in I}$. A homomorphism from A to B is a map $h : |A| \longrightarrow |B|$ such that $hf_i^A = f_i^B h^{\lambda_i}$, i.e., the diagram

$$\begin{array}{ccc} |A|^{\lambda_i} & \xrightarrow{\;h^{\lambda_i}\;} & |B|^{\lambda_i} \\ \downarrow{\scriptstyle f_i^A} & & \downarrow{\scriptstyle f_i^B} \\ |A| & \xrightarrow{\;h\;} & |B| \end{array}$$

commutes, for each $i \in I$.

It is obvious that homomorphisms are closed under composition and every identity map of an algebra is also a homomorphism. Hence the algebras of a given type τ together with their homomorphisms constitute a category, which will be denoted by $\underline{A(\tau)}$.

2.3 Theorem Let $f: A \longrightarrow B$ be an onto homomorphism in $\underline{A(\tau)}$ and C an object of $\underline{A(\tau)}$. A map $h: |B| \longrightarrow |C|$ is a homomorphism if and only if hf is a homomorphism.

2.4 Definition (1) For any algebra A and any subset X of A, the subalgebra $S_A X$ of A generated by X is the smallest subalgebra of A containing X.

(2) If $A = S_A X$, one calls X a generating set, or set of generators, of A.

2.5 Proposition Let X be a generating set of $A \in \underline{A(\tau)}$. For any pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$ of morphisms in $\underline{A(\tau)}$, $f = g$ if and only if their restrictions to X are same.

2.6 Proposition Let $f: A \longrightarrow B$ be a morphism in the category $\underline{A(\tau)}$. Then

1) f is one-one if and only if f is a monomorphism if and only if f is a regular monomorphism.

2) f is onto if and only if f is an epimorphism if and only if f is a regular epimorphism.

2.7 Definition A congruence on an algebra A is a subset $\theta \subseteq A \times A$ which is both an equivalence relation on $|A|$ and (the underlying set of) a subalgebra of $A \times A$.

2.8 Theorem The category $\underline{A(\tau)}$ is an (epi, mono)-

category. In particular, for any morphism $h: A \longrightarrow B$, $A \xrightarrow{h} B = A \xrightarrow{e} A/\ker h \xrightarrow{m} B$ is an (epi, mono)-factorization and $A/\ker h$ is isomorphic with $h(A)$.

Notation: For any class $\underline{C}$ of algebras of a fixed type τ, let

(1) $P\underline{C}$ be the class of all algebras isomorphic to some products $\prod C_\alpha$ where all $C_\alpha \in \underline{C}$;

(2) $S\underline{C}$ the class of all algebras isomorphic to a subalgebra of some $C \in \underline{C}$; and

(3) $H\underline{C}$ the class of all algebras isomorphic to a quotient of some $C \in \underline{C}$.

2.9 Definition For a set X and a class $\underline{C} \subseteq \underline{A(\tau)}$, an algebra A of type τ is called $\underline{C}$-free over X, if X is a generating set of A such that any map $X \longrightarrow |B|$, $B \in \underline{C}$, extends to a homomorphism $A \longrightarrow B$. In particular, an $\underline{A(\tau)}$-free algebra over a set will be called absolutely free.

2.10 Theorem If $\underline{C} = SP\underline{C}$, equivalently $\underline{C}$ is closed under the operations S and P, and $\underline{C}$ is non-trivial, i.e., contains algebras with at least two elements, there exists, for any set X, a $\underline{C}$-free algebra over X in $\underline{C}$.

2.11 Corollary For any set X, there exists an absolutely free algebra over X, and any algebra $A \in \underline{A(\tau)}$ is a homomorphic image of an absolutely free algebra.

2.12 Definition An infinite cardinal α is called regular if for every cofinal subset S of α considered as the initial ordinal, the cardinal of S is equal to α.

Notation: For a type $\tau = (\lambda_i)_{i \in I}$ of algebras, the smallest regular infinite cardinal such that $|\lambda_i| < \alpha_\tau$ for all $i \in I$ will be denoted by α_τ.

2.13 Definition Let α be an ordinal and τ a type of algebras. An equation in α variables is a pair $(t, s) \in F_\tau(\alpha) \times F_\tau(\alpha)$, where $F_\tau(\alpha)$ is an $A(\tau)$-free algebra over α.

An equation $(t, s) \in F_\tau(\alpha) \times F_\tau(\alpha)$ is said to be valid in an algebra A of type τ if $h(t) = h(s)$ for all $h: F_\tau(\alpha) \longrightarrow A$ in $A(\tau)$.

2.14 Theorem For every equation $(t, s) \in (F_\tau(\alpha))^2$, there exists $(\bar{t}, \bar{s}) \in F_\tau(\alpha_\tau) \times F_\tau(\alpha_\tau)$ such that for every algebra A of type τ, (t, s) is valid in A if and only if $(\bar{t}, \bar{s})$ is valid in A.

Notation: For $\Sigma \subseteq F_\tau(\alpha_\tau) \times F_\tau(\alpha_\tau)$, let $m(\Sigma)$ be the class of all $A \in A(\tau)$ in which all $(t, s) \in \Sigma$ are valid.

For a class C of algebras of type τ, let eq(C) be the set of all equations which are valid in all $A \in$ C.

2.15 Definition A class C of algebras of type τ is called an equational class if C = m(eq(C)).

2.16 Theorem (G. Birkhoff) A class $\underline{C}$ of algebras of type τ is equational if and only if $\underline{C}$ = HSP$\underline{C}$, i.e., if and only if $\underline{C}$ is closed under the operations H, S, and P.

2.17 Definition A topological (resp. uniform) algebra of type $\tau = (\lambda_i)_{i \in I}$ is a triple $A = (X, (f_i)_{i \in I}, \underline{O})$ in which $(X, (f_i)_{i \in I})$ is an algebra of type τ and $\underline{O}$ is a topology (resp. uniform structure) on X such that $f_i : (X, \underline{O})^{\lambda_i} \longrightarrow (X, \underline{O})$ is (resp. uniformly) continuous for each $i \in I$.

For such A, X is called the underlying set of A, $(X, (f_i)_{i \in I})$ the underlying algebra, and $(X, \underline{O})$ the underlying space of A.

2.18 Definition A topology (resp. uniform structure) $\underline{O}$ on an algebra $(X, (f_i)_{i \in I})$ of type τ is called an algebra topology (resp. uniform structure) if $(X, (f_i)_I, \underline{O})$ becomes a topological (resp. uniform) algebra.

CHAPTER I

TOPOLOGICALLY ALGEBRAIC FUNCTORS

Section 1: Topologically algebraic functors.

1.1 Definition Let $\underline{A}$ and $\underline{B}$ be categories and $U: \underline{A} \longrightarrow \underline{B}$ a functor. A source $(A, A \xrightarrow{s_i} A_i)_{i \in I}$ in $\underline{A}$ is called U-initial if for any source $(A', A' \xrightarrow{t_i} A_i)_{i \in I}$ in $\underline{A}$ and any $\underline{B}$-morphism $h: UA' \longrightarrow UA$ with $(Us_i)h = Ut_i$ for each $i \in I$, there exists a unique $\underline{A}$-morphism $\bar{h}: A' \longrightarrow A$ with $U\bar{h} = h$ and $s_i\bar{h} = t_i$ for each $i \in I$.

Dual notion: coinitial sink.

Examples: (1) In the category Top (resp. Unif) of topological (resp. uniform) spaces and (resp. uniformly) continuous maps, for the underlying set functor U: Top $\longrightarrow$ Set (resp. Unif $\longrightarrow$ Set), a source $(A, A \xrightarrow{s_i} A_i)_I$ is U-initial if and only if A has the initial topology (resp. uniform structure) with respect to $(s_i)_I$.

(2) By Theorem 2.3 in Chap. 0, every onto morphism in $\underline{A(\tau)}$ is U-coinitial for the underlying set functor $U: \underline{A(\tau)} \longrightarrow$ Set.

1.2 Remark For a functor $U: \underline{A} \longrightarrow \underline{B}$, U-initial sources are closed under composition with isomorphisms.

1.3 Definition Let $U: \underline{A} \longrightarrow \underline{B}$ be a functor and $A \in \underline{A}$. A $\underline{B}$-morphism $e: B \longrightarrow UA$ is said to U-generate A provided that whenever $A \overset{r}{\underset{s}{\rightrightarrows}} A'$ are $\underline{A}$-morphisms such that $(Ur)e = (Us)e$, then $r = s$. In this case, e will be also called a U-generating morphism.

Examples: (1) Let $E: \underline{Haus} \longrightarrow \underline{Top}$ be the embedding functor where Haus is the category of Hausdorff spaces and continuous maps. A morphism $f: X \longrightarrow EY$ in Top is E-generating if and only if $f(X)$ is dense in EY.

(2) For the underlying set functor $U: \underline{A(\tau)} \longrightarrow \underline{Set}$, a morphism $f: X \longrightarrow UA$ U-generates A if $f(X)$ generates A.

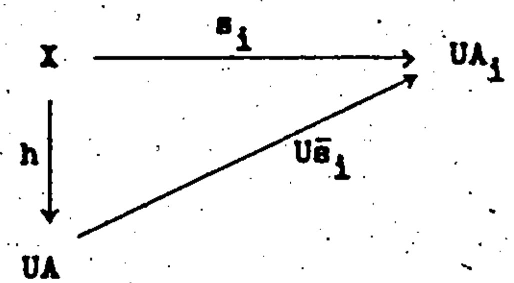
(3) Every front adjunction $\eta_B: B \longrightarrow GFB$ in an adjunction (F, G, φ) from $\underline{B}$ to $\underline{A}$ is a G-generating morphism.

1.4 Remark For a functor $U: \underline{A} \longrightarrow \underline{B}$,

(1) if g is a U-generating morphism and e is an epimorphism, then ge and (Ue)g are again U-generating whenever they are defined.

(2) if $e: B \longrightarrow UA$ is an epimorphism and U is faithful, then e U-generates A.

1.5 Definition A functor $U: \underline{A} \longrightarrow \underline{B}$ is called topologically algebraic (resp. topological) if for each family $(A_i)_{i \in I}$ of A-objects and each source $(X, X \xrightarrow{s_i} UA_i)_I$ in B there exists a U-initial source $(A, A \xrightarrow{\bar{s}_i} A_i)_I$ in A and a U-generating morphism (resp. isomorphism) $h: X \longrightarrow UA$ such that the diagram



commutes for each $i \in I$. In this case, the category A is also called topologically algebraic (resp. topological), or topologically algebraic (resp. topological) over U and B.

The definitions of topological functors together with the following (E, M) -topological functors are due to H. Herrlich ([19]).

1.6 <u>Definition</u> Let B be an (E, M)-category and $U: A \longrightarrow B$ a functor. U is called <u>(E, M)-topologically algebraic</u> (resp. <u>(E, M)-topological</u>) if for each family $(A_i)_{i \in I}$ of A-objects and each source $(X, X \xrightarrow{s_i} UA_i)_I$ in M there exists a U-initial source $(A, A \xrightarrow{\bar{s}_i} A_i)_I$ and a U-generating morphism (resp. isomorphism) $h: X \longrightarrow UA$ with $(U\bar{s}_i)\, h = s_i$ for each $i \in I$.

1.7 <u>Remark</u> (1) <u>Top</u> and <u>Unif</u> are topological over the underlying set functors.

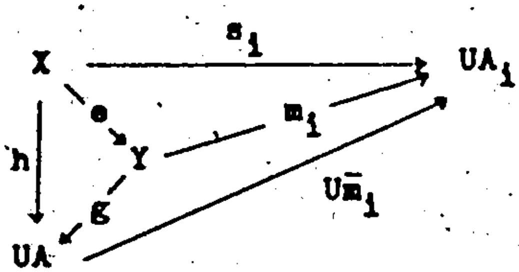
(2) Topologically algebraic functors are exactly (all isomorphisms, all sources)-topologically algebraic functors.

(3) If $\underline{B}$ is an $(\underline{E}, \underline{M})$-category and $U: \underline{A} \longrightarrow \underline{B}$ is an $(\underline{E}, \underline{M})$-topological functor then U is an $(\underline{E}, \underline{M})$-topologically algebraic functor, for every $(\underline{E}, \underline{M})$-topological functor is faithful (see [19], or Theorem 1.11.).

(4) For an $(\underline{E}, \underline{M})$-category $\underline{B}$, a functor $U: \underline{A} \longrightarrow \underline{B}$ is $(\underline{E}, \underline{M})$-topologically algebraic if and only if U is topologically algebraic.

<u>Proof</u>: $(\Longleftarrow)$ It is immediate from the definitions.

$(\Longrightarrow)$ For a family $(A_i)_{i \in I}$ of $\underline{A}$-objects and a $\underline{B}$-source $(X, X \xrightarrow{s_i} UA_i)_I$, let $X \xrightarrow{s_i} UA_i = X \xrightarrow{e} Y \xrightarrow{m_i} UA_i$ $(i \in I)$ be the $(\underline{E}, \underline{M})$-factorization. Since U is $(\underline{E}, \underline{M})$-topologically algebraic and $(Y, (m_i)_I) \in \underline{M}$, there exists a U-initial source $(A, A \xrightarrow{\bar{m}_i} A_i)_I$ and a U-generating morphism $g: Y \longrightarrow UA$ such that $(U\bar{m}_i)g = m_i$ for each i. Let $h = ge$. Then h is a U-generating morphism and the diagram



commutes for each $i \in I$.

(5) Every $(\underline{E}, \underline{M})$-topological functor is topologically

algebraic.

1.8 Theorem A topologically algebraic functor $U: \underline{A} \longrightarrow \underline{B}$ has a left adjoint.

Proof: For a $\underline{B}$-object B, let $(B, B \xrightarrow{s_i} UA_i)_{i \in I}$ be the family of all morphisms with $A_i \in \underline{A}$. Then there exists a U-initial source $(FB, FB \xrightarrow{\bar{s}_i} A_i)_I$ in $\underline{A}$ and a U-generating morphism $h_B: B \longrightarrow UFB$ such that $(U\bar{s}_i)h_B = s_i$ for each $i \in I$. For any $\underline{A}$-object A and any morphism $B \xrightarrow{f} UA$, there exists some $i \in I$ such that $A = A_i$ and $f = s_i$. Let $\bar{f} = \bar{s}_i$. Then $(U\bar{f})h_B = f$ and the uniqueness of $\bar{f}$ follows from the fact that h_B is a U-generating morphism. Hence B has the U-universal map (h_B, FB), so that U has a left adjoint.

1.9 Corollary A topologically algebraic functor preserves limits, monomorphisms and mono-sources.

Proof: It is well known that every functor which has a left adjoint preserves limits and monomorphisms. Let $U: \underline{A} \longrightarrow \underline{B}$ be a functor with a left adjoint F and let $(X, (m_i)_I)$ be a mono-source in $\underline{A}$. Suppose $B \overset{u}{\underset{v}{\rightrightarrows}} UX$ are $\underline{B}$-morphisms with $(Um_i)u = (Um_i)v$ for each $i \in I$. Let $\eta_B: B \longrightarrow UFB$ be the front adjunction for B. Then there exist $\underline{A}$-morphisms $FB \overset{\bar{u}}{\underset{\bar{v}}{\rightrightarrows}} X$ such that $(U\bar{u})\,\eta_B = u$ and $(U\bar{v})\,\eta_B = v$. Hence $(Um_i)(U\bar{u})\,\eta_B = (Um_i)(U\bar{v})\,\eta_B$; $m_i\bar{u} = m_i\bar{v}$ for each i. Thus $\bar{u} = \bar{v}$ and hence $u = v$.

1.10 Corollary Let $\underline{A}$ be a subcategory of a category $\underline{B}$ and $E: \underline{A} \longrightarrow \underline{B}$ the embedding functor. Then

(1) if E is topologically algebraic, $\underline{A}$ is reflective in $\underline{B}$.

(2) if $\underline{A}$ is full reflective in $\underline{B}$, E is topologically algebraic.

Proof: (1) It is immediate from Theorem 1.8.

(2) For each family $(A_i)_{i \in I}$ of $\underline{A}$-objects and each source $(B, B \xrightarrow{s_i} EA_i)_I$ in $\underline{B}$, let (r_B, RB) be an $\underline{A}$-reflection of B. Then there exists a unique morphism $\bar{s}_i : RB \longrightarrow A_i$ in $\underline{A}$ such that $(E\bar{s}_i)r_B = s_i$ for each $i \in I$. Obviously r_B is an E-generating morphism and $(RB, (\bar{s}_i)_I)$ is E-initial, for $\underline{A}$ is full.

1.11 Theorem A topologically algebraic functor is faithful.

Proof: Let $U: \underline{A} \longrightarrow \underline{B}$ be a topologically algebraic functor. Suppose $A \overset{r}{\underset{s}{\rightrightarrows}} A'$ are $\underline{A}$-morphisms such that $r \neq s$ and $Ur = Us$ $(= g)$. Let I be a proper class and $(UA, UA \xrightarrow{s_i} UA_i)_I$ be a source in $\underline{B}$, where $A_i = A'$, $s_i = g$ for each $i \in I$. Then there exists a U-initial source $(C, C \xrightarrow{\bar{s}_i} A_i)_I$ in $\underline{A}$ and a U-generating morphism $h: UA \longrightarrow UC$ such that $(U\bar{s}_i)h = s_i$ for each $i \in I$. Take i_1 and i_2 in I such that $i_1 \neq i_2$. Let $(A, A \xrightarrow{t_i} A_i)_I$ be a source in $\underline{A}$, where $t_{i_1} = r$ and $t_i = s$ for each $i \neq i_1$. Since $(U\bar{s}_i)h =$

Ut_i, there exists a unique morphism $\bar{h}: A \longrightarrow C$ in $\underline{A}$ such that $U\bar{h} = h$ and $\bar{s}_i\bar{h} = t_i$, so that $\bar{s}_{i_1} \neq \bar{s}_{i_2}$. Hence $\underline{A}(C, A')$ is a proper class. This contradicts to the definition of a category.

1.12 <u>Corollary</u> Every topologically algebraic functor reflects monomorphisms and mono-sources.

1.13 <u>Corollary</u> Every topologically algebraic category over <u>Set</u> is a concrete category.

1.14 <u>Theorem</u> Let $U: \underline{A} \longrightarrow \underline{B}$ be a topologically algebraic functor, $\underline{R}$ a full reflective subcategory of $\underline{A}$, and (r_A, rA) the $\underline{R}$-reflection of A for each $\underline{A}$-object A. If Ur_A has a right inverse for each $\underline{A}$-object A then $S: \underline{R} \longrightarrow \underline{B}$ is a topologically algebraic functor, where S is the restriction of U to $\underline{R}$.

<u>Proof</u>: Let $(A_i)_{i \in I}$ be a family of $\underline{R}$-objects and $(X, X \xrightarrow{s_i} SA_i)_I$ a source in $\underline{B}$. Since $X \xrightarrow{s_i} SA_i = X \xrightarrow{s_i} UA_i$ for each i, there exists a U-initial source $(A, A \xrightarrow{\bar{s}_i} A_i)_I$ and a U-generating morphism $h: X \longrightarrow UA$ such that $(U\bar{s}_i)h = s_i$ for each $i \in I$. Then there exists a unique $h_i: rA \longrightarrow A_i$ in $\underline{R}$ such that $\bar{s}_i = h_i r_A$ for each $i \in I$. $(Ur_A)h$ is an S-generating morphism. Indeed, if $(Su)(Ur_A)h = (Sv)(Ur_A)h$ for a pair of $\underline{R}$-morphisms $rA \overset{u}{\underset{v}{\rightrightarrows}} B$, then $(Uu)(Ur_A)h = (Uv)(Ur_A)h$; $ur_A = vr_A$, which implies $u = v$.

We wish to show that $(rA, (h_i)_I)$ is an S-initial source.

Let $(A', A' \xrightarrow{k_i} A_i)_I$ be a source in $\underline{R}$ and $k: SA' \longrightarrow SrA$ a $\underline{B}$-morphism with $(Sh_i)k = Sk_i$ for each $i \in I$. Let w be the right inverse of Ur_A. Then $(U\bar{s}_i)wk = (Uh_i)(Ur_A)wk = (Uh_i)k$ $= (Sh_i)k = Uk_i$ for each $i \in I$. Since $(A, (\bar{s}_i)_I)$ is a U-initial source, there exists a unique $\bar{k}: A' \longrightarrow A$ with $U\bar{k} = wk$ and $\bar{s}_i\bar{k} = k_i$ for each $i \in I$. Let $\bar{\bar{k}} = r_A\bar{k}: A' \longrightarrow rA$. Then $\bar{\bar{k}}$ is a morphism of $\underline{R}$, for $\underline{R}$ is full, and $S\bar{\bar{k}} = U\bar{\bar{k}} =$ $(Ur_A)(U\bar{k}) = (Ur_A)wk = k$ and $h_i\bar{\bar{k}} = h_i r_A \bar{k} = \bar{s}_i\bar{k} = k_i$ for each $i \in I$. Since U is faithful by Theorem 1.11, S is also faithful. Hence the morphism $\bar{\bar{k}}$ with $S\bar{\bar{k}} = k$ is unique. This completes the proof.

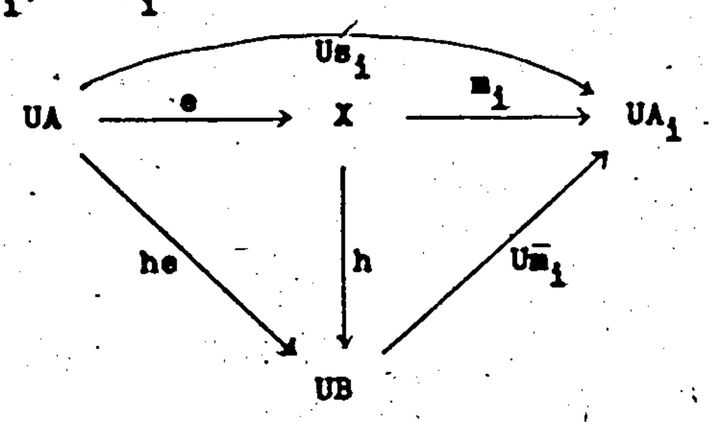
Since epimorphisms in Top or Unif are exactly onto morphisms, i.e., images of epimorphisms under the underlying set functor of Top or Unif have right inverses, we immediately have the following.

1.15 Corollary Every epireflective subcategory of Top or Unif is topologically algebraic over the underlying set functor.

Section 2: Limits and colimits in topologically algebraic categories.

2.1 Lemma If $U: \underline{A} \longrightarrow \underline{B}$ is a topologically algebraic functor and $(A, A \xrightarrow{s_i} A_i)_I$ is an extremal mono-source in $\underline{A}$, then $(UA, UA \xrightarrow{Us_i} UA_i)_I$ is an extremal mono-source and the source $(A, (s_i)_I)$ is U-initial.

Proof: Since a topologically algebraic functor preserves mono-sources by Corollary 1.9, it is enough to show that $(UA, (Us_i)_I)$ satisfies the extremal condition and $(A, (s_i)_I)$ is U-initial. Suppose that $UA \xrightarrow{Us_i} UA_i = UA \xrightarrow{e} X \xrightarrow{m_i} UA_i$ for each $i \in I$ and e is an epimorphism. For the source $(X, (m_i)_I)$, there is a U-initial source $(B, B \xrightarrow{\bar{m}_i} A_i)_I$ in $\underline{A}$ and a U-generating morphism $h: X \longrightarrow UB$ such that $(U\bar{m}_i)h = m_i$ for each $i \in I$. Since the diagram



commutes for each $i \in I$, there exists a unique morphism $k: A \longrightarrow B$ with $Uk = he$ and $\bar{m}_i k = s_i$ for each $i \in I$. We wish to show that k is an isomorphism. For a pair of $\underline{A}$-morphisms $B \underset{s}{\overset{r}{\rightrightarrows}} B'$ with $rk = sk$, $(Ur)(Uk) = (Us)(Uk)$, i.e., $(Ur)he = (Us)he$. Since he is a U-generating morphism, $r = s$. Hence k is an epimorphism. By the fact that $(A, (s_i)_I)$ is an extremal mono-source, k must be an isomorphism. Hence $(Uk)^{-1}he = 1$, so that e is an isomorphism. Moreover $(A, A \xrightarrow{s_i} A_i)_I = (A, A \xrightarrow{k} B \xrightarrow{\bar{m}_i} A_i)_I$ is a U-initial source by Remark 1.2.

2.2 <u>Theorem</u> Let $D: \underline{I} \longrightarrow \underline{A}$ be a diagram and let $(L, L \xrightarrow{k_i} D(i))_{i \in \underline{I}}$ be a source in $\underline{A}$. If a functor $U: \underline{A} \longrightarrow \underline{B}$ is topologically algebraic then the following are equivalent:

(1) $(L, (k_i)_{\underline{I}})$ is a limit of D.

(2) $(UL, (Uk_i)_{\underline{I}})$ is a limit of UD and $(L, (k_i)_{\underline{I}})$ is U-initial.

<u>Proof:</u> (1) $\Rightarrow$ (2) Since every limit of a diagram is an extremal mono-source, $(L, (k_i)_I)$ is a U-initial source by the above lemma. By Corollary 1.9, $(UL, (Uk_i)_I)$ is a limit of UD.

(2) $\Rightarrow$ (1) Since U is faithful, $(L, (k_i)_{\underline{I}})$ is a lower bound of D. Let $(L', L' \xrightarrow{k_i'} D(i))_{i \in \underline{I}}$ be a lower bound of D. Since $(UL', (Uk_i')_{\underline{I}})$ is a lower bound of UD, there exists a unique morphism $h: UL' \longrightarrow UL$ with $(Uk_i)h = Uk_i'$. Since $(L, (k_i)_{\underline{I}})$ is U-initial, there exists a unique morphism $\bar{h}: L' \longrightarrow L$ such that $U\bar{h} = h$ and $k_i\bar{h} = k_i'$.

Suppose that $h': L' \longrightarrow L$ is a morphism with $k_i h' = k_i'$. Then $(Uk_i)(Uh') = Uk_i'$. From the uniqueness of h with $(Uk_i)h = Uk_i'$, $Uh' = h = U\bar{h}$. Since U is faithful, $h' = \bar{h}$. This completes the proof.

2.3 Theorem If $U: \underline{A} \longrightarrow \underline{B}$ is a topologically algebraic functor and $\underline{B}$ is a complete category, then $\underline{A}$ is also complete.

Proof: Let $D: \underline{I} \longrightarrow \underline{A}$ be a diagram and $(L, L \xrightarrow{k_i} UD(i))_{i \in \underline{I}}$ a limit of UD. Then there exists a U-initial source $(\bar{L}, \bar{L} \xrightarrow{\bar{k}_i} D(i))_{i \in \underline{I}}$ and a U-generating morphism $h: L \longrightarrow U\bar{L}$ such that $(U\bar{k}_i)h = k_i$ for each $i \in \underline{I}$. Since $(UD(f))k_i = k_j$ for each $f: i \longrightarrow j$ in $\underline{I}$, and h is a U-generating morphism, $(U\bar{L}, (U\bar{k}_i)_{\underline{I}})$ is a lower bound of UD. Hence there exists a unique morphism $k: U\bar{L} \longrightarrow L$ with $k_i k = U\bar{k}_i$. Since $k_i kh = (U\bar{k}_i)h = k_i$ for each $i \in \underline{I}$, $kh = 1_L$. Since $(U\bar{k}_i)hk = k_i k = U\bar{k}_i$ and $(\bar{L}, (\bar{k}_i)_I)$ is a U-initial source, there exists a unique morphism $w: \bar{L} \longrightarrow \bar{L}$ with $Uw = hk$ and $\bar{k}_i w = \bar{k}_i$. Since $(Uw)h = hkh = h = (U1_{\bar{L}})h$ and since h is a U-generating morphism, $w = 1_{\bar{L}}$. Hence $hk = Uw = U1_{\bar{L}} = 1_{U\bar{L}}$. Consequently h is an isomorphism and $(U\bar{L}, (U\bar{k}_i)_{\underline{I}})$ is a limit of UD. By Theorem 2.2, $(\bar{L}, (\bar{k}_i)_I)$ is a limit of D.

2.4 Lemma If $U: \underline{A} \longrightarrow \underline{B}$ is a topologically algebraic functor then for each family $(A_i)_{i \in I}$ of $\underline{A}$-objects and each sink $(UA_i \xrightarrow{s_i} B, B)_I$ in $\underline{B}$, there exists a sink

$(A_i \xrightarrow{\bar{s}_i} A,\ A)_I$ in $\underline{A}$ and a U-generating morphism $e\colon B \longrightarrow UA$ such that $U\bar{s}_i = es_i$ for each $i \in I$ and such that the following holds: for each sink $(A_i \xrightarrow{t_i} A',\ A')_I$ in $\underline{A}$ and each $\underline{B}$-morphism $g\colon B \longrightarrow UA'$ with $Ut_i = gs_i$ for each $i \in I$, there exists an $\underline{A}$-morphism $k\colon A \longrightarrow A'$ with $g = (Uk)e$.

Proof: Let $(g_j,\ (A_i \xrightarrow{g_{ij}} B_j,\ B_j)_{i \in I})_{j \in J}$ be the family of all pairs, where $(A_i \xrightarrow{g_{ij}} B_j,\ B_j)_I$ is a sink in $\underline{A}$ and $g_j\colon B \longrightarrow UB_j$ is a morphism such that $Ug_{ij} = g_j s_i$. For the source $(B,\ B \xrightarrow{g_j} UB_j)_{j \in J}$ there exists a U-initial source $(A,\ A \xrightarrow{\bar{g}_j} B_j)_J$ and a U-generating morphism $e\colon B \longrightarrow UA$ such that $(U\bar{g}_j)e = g_j$ for each $j \in J$. Since $Ug_{ij} = (U\bar{g}_j)es_i$ for each i and j, there exists a unique morphism $\bar{s}_i\colon A_i \longrightarrow A$ with $U\bar{s}_i = es_i$ and $\bar{g}_j\bar{s}_i = g_{ij}$ for each $i \in I$ and $j \in J$. Let $(A_i \xrightarrow{t_i} A',\ A')_I$ be a sink in $\underline{A}$ and $g\colon B \longrightarrow UA'$ with $Ut_i = gs_i$ for each $i \in I$. Then there exists some $j \in J$ with $A' = B_j$, $g_{ij} = t_i$ for each i and $g = g_j\colon B \longrightarrow UB_j$. Let $k = \bar{g}_j\colon A \longrightarrow A'$. Clearly $(Uk)e = g_j = g$. This completes the proof.

2.5 Theorem If $U\colon \underline{A} \longrightarrow \underline{B}$ is a topologically algebraic functor and $\underline{B}$ is cocomplete then $\underline{A}$ is also cocomplete.

Proof: Let $D\colon \underline{I} \longrightarrow \underline{A}$ be a diagram and let $(UD(i) \xrightarrow{n_i} L,\ L)_{i \in \underline{I}}$ be a colimit of UD. By the above lemma there exists a sink $(D(i) \xrightarrow{\bar{n}_i} A, A)_{\underline{I}}$ in $\underline{A}$ and a U-generating morphism $e: L \longrightarrow UA$ such that $U\bar{n}_i = en_i$ for each $i \in \underline{I}$ and the last condition in the lemma holds. We wish to show that $((\bar{n}_i)_{\underline{I}}, A)$ is a colimit of D. For any $f: i \longrightarrow j$ in $\underline{I}$, $n_i = n_j UD(f)$. Hence $en_i = en_j UD(f)$, i.e., $U\bar{n}_i = (U\bar{n}_j)UD(f)$. From the faithfulness of U, $\bar{n}_i = \bar{n}_j D(f)$, i.e., $((\bar{n}_i)_{\underline{I}}, A)$ is an upper bound of D. Let $(D(i) \xrightarrow{t_i} A', A')_{i \in \underline{I}}$ be an upper bound of D. Then clearly $((Ut_i)_{\underline{I}}, UA')$ is an upper bound of UD; there exists a unique morphism $g: L \longrightarrow UA'$ with $gn_i = Ut_i$ for each $i \in \underline{I}$. By the above lemma, there exists a morphism $k: A \longrightarrow A'$ with $g = (Uk)e$. Since $(Uk)(U\bar{n}_i) = (Uk)en_i = gn_i = Ut_i$, $k\bar{n}_i = t_i$ for each $i \in \underline{I}$. Suppose $k': A \longrightarrow A'$ is a morphism with $k'\bar{n}_i = t_i$ for each $i \in \underline{I}$. Then $Ut_i = (Uk')(U\bar{n}_i) = (Uk')en_i$. From the uniqueness of g, $(Uk')e = g = (Uk)e$. Since e is an U-generating morphism, $k' = k$. Hence $((\bar{n}_i)_{\underline{I}}, A)$ is a colimit of D.

2.6 <u>Corollary</u> Every topologically algebraic category over <u>Set</u> is complete and cocomplete.

Section 3: Categories of topological algebras and continuous homomorphisms.

Throughout this section except in Lemma 3.3, every subcategory of a category will be assumed to be full and isomorphism closed.

3.1 Proposition The category $\underline{A(\tau)}$ of algebras of type τ and homomorphisms is an (epi, mono-sources)-category.

Proof: Let $(A, A \xrightarrow{s_i} A_i)_I$ be a source in $\underline{A(\tau)}$. Let $e_i : A \longrightarrow A/\ker s_i$ and $e : A \longrightarrow A/\bigcap \ker s_i$ be quotient maps. For each $i \in I$, there exists a one-one homomorphism $m_i : A/\ker s_i \longrightarrow A_i$ such that $s_i = m_i e_i$, and a homomorphism $h_i : A/\bigcap \ker s_i \longrightarrow A/\ker s_i$ such that $h_i e = e_i$. Let $g_i = m_i h_i$ for each $i \in I$. It is obvious that $(g_i)_I$ separates the points. Hence $A \xrightarrow{s_i} A_i = A \xrightarrow{e} A/\bigcap \ker s_i \xrightarrow{g_i} A_i$ is an (epi, mono-sources)-factorization.

Let e be an epimorphism and $(C, C \xrightarrow{m_i} F_i)_I$ a mono-source such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g_i} \\ C & \xrightarrow[m_i]{} & F_i, \end{array}$$

commutes for each $i \in I$. Since epimorphisms in $\underline{A(\tau)}$ are exactly onto morphisms, every mono-source in $\underline{A(\tau)}$ is a mono-source in $\underline{\text{Set}}$ as set maps, and $\underline{\text{Set}}$ is an (epi, mono-sources)-category, there exists $h: B \longrightarrow C$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle f}\downarrow & \overset{h}{\swarrow} & \downarrow{\scriptstyle g_i} \\ C & \xrightarrow[m_i]{} & P_i, \end{array}$$

commutes. By Theorem 2.3 in Chap. 0, and the fact that $he = f$, h is a homomorphism from B to C. This completes the proof.

3.2 <u>Proposition</u> A subcategory $\underline{A}$ of $\underline{A}(\tau)$ is epireflective in $\underline{A}(\tau)$ if and only if $\underline{A}$ is closed under the operations S and P.

<u>Proof</u>: It is immediate from Theorem 1.18 in Chap. 0, and the fact that $\underline{A}(\tau)$ is a co-(well-powered) (epi, mono)-category and has products.

The following lemma is due to H. Herrlich [19].

3.3 <u>Lemma</u> For an isomorphism closed subcategory $\underline{A}$ of an $(\underline{E}, \underline{M})$-category $\underline{B}$, the following are equivalent:

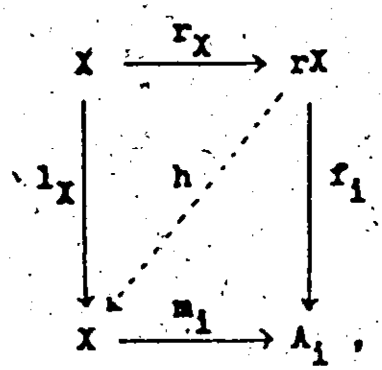
(1) If $(X, X \xrightarrow{m_i} A_i)_I \in \underline{M}$, and $(A_i)_I$ a family of $\underline{A}$-objects, then X and every m_i belong to $\underline{A}$.

(2) $\underline{A}$ is a full and $\underline{E}$-reflective subcategory of $\underline{B}$.

<u>Proof</u>: (1) $\Longrightarrow$ (2) If $f: A \longrightarrow A'$ is a $\underline{B}$-morphism between $\underline{A}$-objects then $(A, (f, 1_A))$ belongs to $\underline{M}$, which implies that $f \in \underline{A}$. Hence $\underline{A}$ is full. For any $\underline{B}$-object X let $(f_i, A_i)_{i \in I}$ be the family of all pairs with $A_i \in \underline{A}$ and

$f_i: X \longrightarrow A_i$. Let $X \xrightarrow{f_i} A_i = X \xrightarrow{e} Y \xrightarrow{m_i} A_i$ $(i \in I)$ be an $(\underline{E}, \underline{M})$-factorization. Then Y and every m_i belong to $\underline{A}$ and consequently (e, Y) is an $\underline{A}$-reflection of X.

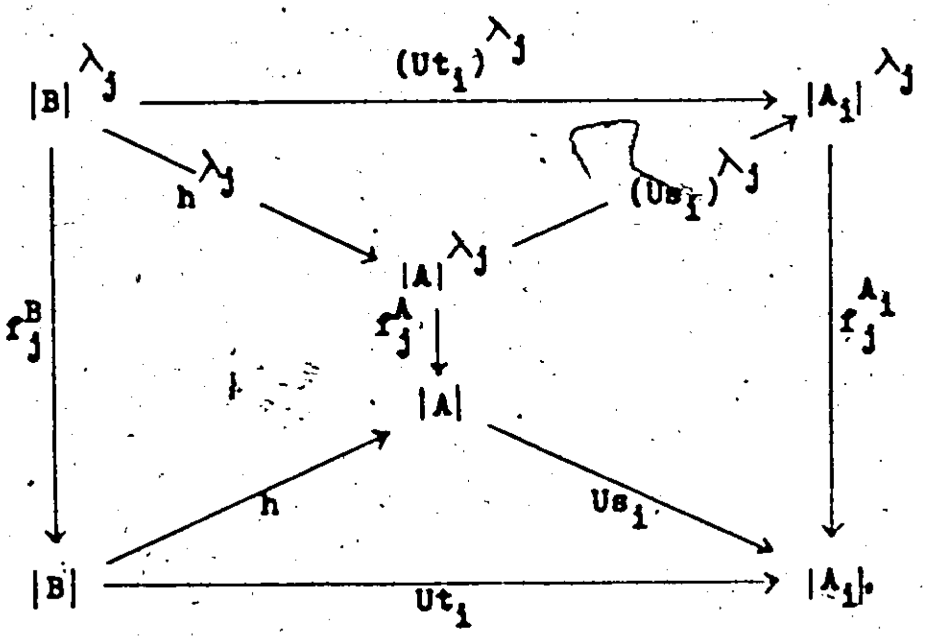
(2) $\Rightarrow$ (1) Suppose that $(X, X \xrightarrow{m_i} A_i)_I$ belongs to $\underline{M}$ and all A_i belong to $\underline{A}$. Let (r_X, rX) be the $\underline{A}$-reflection of $X \in \underline{B}$. There exists $f_i: rX \longrightarrow A_i$ with $f_i r_X = m_i$ for each $i \in I$. Then there exists a morphism $h: rX \longrightarrow X$ such that the diagram



commutes. Hence r_X is an isomorphism and so $X \in \underline{A}$. Since $\underline{A}$ is full, every m_i also belongs to $\underline{A}$.

3.4 Lemma Let $\underline{A}$ be a category of algebras of type τ and homomorphisms and $U: \underline{A} \longrightarrow \underline{Set}$ the underlying set functor. If $(A, A \xrightarrow{s_i} A_i)_I$ is a source in $\underline{A}$ such that $(UA, (Us_i)_I)$ is a mono-source, then $(A, (s_i)_I)$ is U-initial.

Proof: Let $(B, B \xrightarrow{t_i} A_i)_I$ be a source in $\underline{A}$ and $h: UB \longrightarrow UA$ is a morphism with $(Us_i)h = Ut_i$ for each $i \in I$. Let f_j^X denote the jth operation of an algebra X of type τ. For any $\lambda_j \in \tau$ and any $i \in I$, in the following diagram

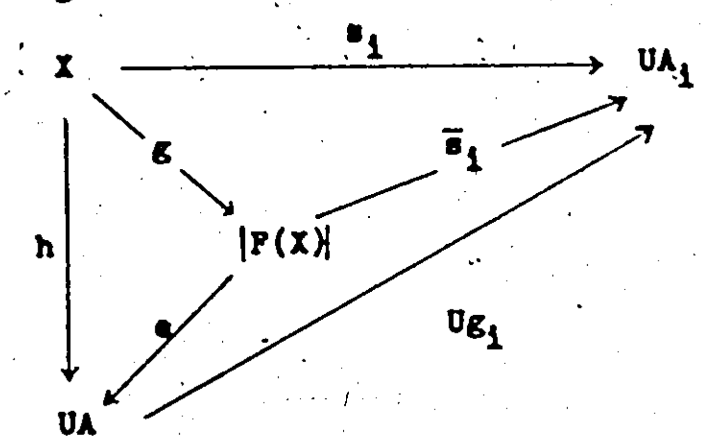


the outer rectangle and the right trapezoid commute, for t_i and s_i are homomorphisms. Since $(UA, (Us_i)_I)$ is a mono-source, the left trapezoid also commutes; h is a homomorphism. Thus h is a morphism of A. This completes the proof.

3.5 Theorem Let A be a category of algebras of type τ and homomorphisms. If A is closed under the operations S and P then the underlying set functor $U: \underline{A} \longrightarrow \underline{Set}$ is topologically algebraic.

Proof: Let $(A_i)_{i \in I}$ be a family of A-objects and $(X, X \xrightarrow{s_i} UA_i)_I$ a source in Set. Let $F(X)$ be the $\underline{A(\tau)}$-free algebra over X and $X \xrightarrow{g} |F(X)|$ be the natural embedding. Then there exists $\bar{s}_i: F(X) \longrightarrow A_i$ in $\underline{A(\tau)}$ such that $\bar{s}_i g = s_i$ for each $i \in I$. Let $F(X) \xrightarrow{\bar{s}_i} A_i = F(X) \xrightarrow{e} A \xrightarrow{g_i} A_i$ be an (epi, mono-sources)-factorization in $\underline{A(\tau)}$. Note that A is an epireflective subcategory of

$\underline{A}(\tau)$ by Proposition 3.2. It follows from the fact that $(A, (g_i)_I)$ is a mono-source and $A_i \in \underline{A}$ for all $i \in I$ that A and all g_i belong to $\underline{A}$. Moreover, by Lemma 3.4, $(A, (g_i)_I)$ is a U-initial source, and the morphism $h: X \longrightarrow UA$ defined by $X \xrightarrow{g} |F(X)| \xrightarrow{e} UA$ is a U-generating morphism. Finally the diagram



commutes. This completes the proof.

Since every equational class is closed under the operations H, S and P, the following is immediate:

3.6 __Corollary__ Every subcategory of $\underline{A}(\tau)$ determined by an equational class is topologically algebraic over the underlying set functor.

3.7 __Corollary__ If $\underline{A}$ is a subcategory of $A(\tau)$ which is closed under the operations S and P, then $\underline{A}$ is complete and cocomplete, and the underlying set functor $U: \underline{A} \longrightarrow \underline{Set}$ has a left adjoint, preserves limits, monomorphisms and mono-sources, and reflects mono-sources.

<u>Notation</u>: For a subcategory <u>T</u> of <u>Top</u> (resp. <u>Unif</u>) and a category <u>A</u> of algebras of a fixed type and homomorphisms, the category which is as follows:

objects: topological (resp. uniform) algebras whose underlying algebras belong to <u>A</u> and underlying spaces belong to <u>T</u>;

morphisms: all (resp. uniformly) continuous homomorphisms between them, will be denoted by <u>TA</u>.

3.8 <u>Lemma</u> Let $(A_i)_{i \in I}$ be a family of topological (or uniform) algebras of type $\tau = (\lambda_j)_{j \in J}$, and let A be an algebra of type τ. Then the initial topology (resp. uniform structure) on $|A|$ with respect to a source $(A, A \xrightarrow{s_i} A_i)_I$ in $\underline{A(\tau)}$ is again an algebra topology (resp. uniform structure) on A.

<u>Proof</u>: Since the diagram

$$\begin{array}{ccc} |A|^{\lambda_j} & \xrightarrow{s_i^{\lambda_j}} & |A_i|^{\lambda_j} \\ \Big\downarrow f_j^A & & \Big\downarrow f_j^{A_i} \\ |A| & \xrightarrow[s_i]{} & |A_i| \end{array},$$

commutes for each $i \in I$ and each $j \in J$, and $f_j^{A_i} s_i^{\lambda_j}$ is (resp. uniformly) continuous for each $i \in I$, f_j^A is also (resp. uniformly) continuous for each $j \in J$, where A is endowed with the initial (resp. uniform structure) topology.

3.9 Theorem If $\underline{A}$ is a category of algebras of fixed type and homomorphisms which is closed under the operations S and P and $\underline{T}$ is an epireflective subcategory of Top (resp. Unif) (equivalently, $\underline{T}$ is productive and hereditary), then the underlying set functor $U: \underline{TA} \longrightarrow \underline{Set}$ is topologically algebraic.

Proof: It is obvious that $\underline{TA} \xrightarrow{U} \underline{Set} = \underline{TA} \xrightarrow{U_1} \underline{A} \xrightarrow{U_2} \underline{Set}$, where U_1 is the underlying algebra functor and U_2 is the underlying set functor from $\underline{A}$. For a family $(A_i)_{i \in I}$ of $\underline{TA}$-objects and a source $(X, X \xrightarrow{s_i} UA_i)_I$ in $\underline{Set}$, there exists a U_2-initial source $(A, A \xrightarrow{\bar{s}_i} U_1A_i)_I$ and a U_2-generating morphism $h: X \longrightarrow U_2A$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad s_i \quad} & UA_i = U_2(U_1A_i) \\ {\scriptstyle h}\downarrow & \nearrow {\scriptstyle U_2\bar{s}_i} & \\ U_2A & & \end{array}$$

commutes for each $i \in I$. Here we note that $(U_2A, (U_2\bar{s}_i)_I)$ is a mono-source in $\underline{Set}$. By Lemma 3.8, A with the initial topology (resp. uniform structure) with respect to $(U_2\bar{s}_i)_I$, is a topological (resp. uniform) algebra, which is denoted again by A. Moreover, using Lemma 3.3 together with the fact that $(U_2A, (U_2\bar{s}_i))$ is a mono-source, it follows that A belongs to $\underline{TA}$ and $\bar{s}_i$ becomes a $\underline{TA}$-morphism for each $i \in I$. Obviously, $(A, (\bar{s}_i)_I)$ is a U-initial source in $\underline{TA}$ and h is U-generating such that $U(\bar{s}_i)h = s_i$ for each $i \in I$.

3.10 Corollary If A is a subcategory of A(τ) which is closed under the operations S and P, and if T is an epireflective subcategory of Top or Unif, then the category TA is complete and cocomplete, and the underlying set functor U: TA $\longrightarrow$ Set has a left adjoint, preserves limits, monomorphisms and mono-sources, and reflects mono-sources.

3.11 Corollary For every subcategory A of A(τ) determined by an equational class and any epireflective subcategory T of Top or Unif, the category TA is topologically algebraic over the underlying set functor.

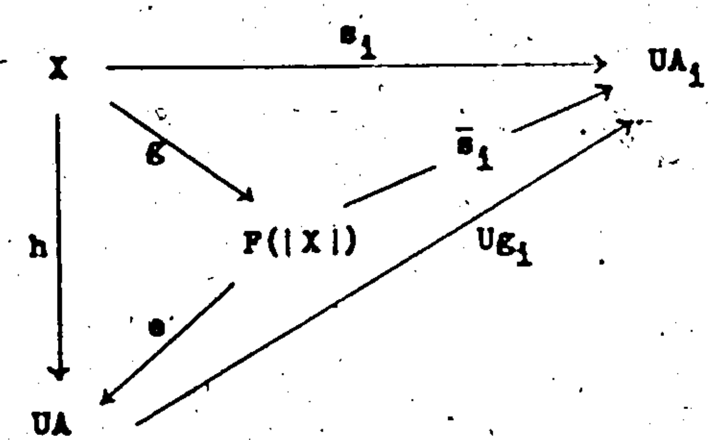
The following corollary has appeared in 1.15. But it is also a direct consequence of Theorem 3.9.

3.12 Corollary For an epireflective subcategory T of Top or Unif, the underlying set functor U: T $\longrightarrow$ Set is topologically algebraic.

Proof: It follows from the fact that T = TA, where A is the category A(τ) with $\tau = \phi$.

3.13 Theorem For a category A of algebras of fixed type and homomorphisms which is closed under the operations S and P and an epireflective subcategory T of Top (resp. Unif), the underlying space functor U: TA $\longrightarrow$ Top (resp. Unif) (or TA $\longrightarrow$ T) is topologically algebraic.

<u>Proof</u>: Let $(A_i)_{i \in I}$ be a family of <u>TA</u>-objects and $(X, X \xrightarrow{s_i} UA_i)_I$ a source in <u>Top</u> (resp. <u>Unif</u>, or <u>T</u>). Let $|X|$ denote the underlying set of X. Let $F(|X|)$ be the <u>$A(\tau)$</u>-free algebra over $|X|$, and $g: |X| \longrightarrow |F(|X|)|$ the natural embedding. Then there exists a unique $\bar{s}_i: F(|X|) \longrightarrow A_i$ in <u>$A(\tau)$</u> such that $\bar{s}_i g = s_i$ for each i, where A_i is, of course, understood to be the underlying algebra of A_i. Let $\bar{s}_i = F(|X|) \xrightarrow{e} A \xrightarrow{g_i} A_i$ be the (epi, mono-sources)-factorization in <u>$A(\tau)$</u>. Since $(A, (g_i)_I)$ is a mono-source and all A_i belong to <u>A</u>, A also belongs to <u>A</u>. Using Lemma 3.8 and Lemma 3.3, A with the initial topology (resp. uniform structure) with respect to $(g_i)_I$ is a <u>TA</u>-object, which will be again denoted by A. Furthermore g_i becomes a <u>TA</u>-morphism for each i and the source $(A, (g_i)_I)$ in <u>TA</u> is U-initial. Let $h: |X| \longrightarrow UA = |X| \xrightarrow{g} |F(|X|)| \xrightarrow{e} UA$. Then $h(|X|)$ generates A, for e is onto. Since the diagram



commutes for each i and A is endowed with the initial topology (resp. uniform structure), $h: X \longrightarrow UA$ is also (resp.

uniformly) continuous, i.e. h is a U-generating morphism in $\underline{T}$. This completes the proof.

3.14 Corollary For an epireflective subcategory $\underline{T}$ of $\underline{Top}$ or $\underline{Unif}$ and a subcategory $\underline{A}$ of $\underline{A(\tau)}$ which is closed under the operations S and P, the underlying space functor $U: \underline{TA} \longrightarrow \underline{T}$ has a left adjoint.

3.15 Corollary For an epireflective subcategory $\underline{T}$ of $\underline{Top}$ or $\underline{Unif}$ and a subcategory $\underline{A}$ of $\underline{A(\tau)}$ determined by an equational class, the category $\underline{TA}$ is topologically algebraic over the underlying space functor.

3.16 Theorem For a category $\underline{A}$ of algebras of fixed type and homomorphisms which is closed under the operations S and P, and for an epireflective subcategory $\underline{T}$ of $\underline{Top}$ (resp. $\underline{Unif}$), the underlying algebra functor $U: \underline{TA} \longrightarrow \underline{A}$ is topologically algebraic.

Proof: For a family $(A_i)_{i \in I}$ of $\underline{TA}$-objects and a source $(X, X \xrightarrow{s_i} UA_i)_I$ in $\underline{A}$, let $X \xrightarrow{s_i} UA_i =$ $X \xrightarrow{e} A \xrightarrow{m_i} UA_i$ $(i \in I)$ be the (epi, mono-sources)-factorization in $\underline{A(\tau)}$. By Proposition 3.2 and Lemma 3.3, A belongs to $\underline{A}$. Again using Lemma 3.8 and Lemma 3.3, A with the initial topology (resp. uniform structure) with respect to $(m_i)_I$ is a $\underline{TA}$-object. Let's denote it again by A. Then $m_i: A \longrightarrow A_i$ becomes a $\underline{TA}$-morphism for each $i \in I$.

Since $(A, A \xrightarrow{m_i} UA_i)_I$ is a mono-source in $\underline{A(\tau)}$ and A is endowed with the initial topology (resp. uniform structure), the source $(A, A \xrightarrow{m_i} A_i)$ in $\underline{TA}$ is U-initial. Since e is an epimorphism in $\underline{A}$ and U is faithful, e is U-generating. Finally $s_i = (Um_i)e$ for each $i \in I$. This completes the proof.

3.17 Corollary For an epireflective subcategory $\underline{T}$ of Top or Unif and a subcategory $\underline{A}$ of $\underline{A(\tau)}$ which is closed under the operations S and P, the underlying algebra functor $U: \underline{TA} \longrightarrow \underline{A}$ has a left adjoint.

3.18 Corollary For an epireflective subcategory $\underline{T}$ of Top or Unif and a subcategory $\underline{A}$ of $\underline{A(\tau)}$ determined by an equational class, the category $\underline{TA}$ is topologically algebraic over the underlying algebra functor.

Section 4: Algebraic categories.

4.1 Definition A concrete category (A, U); i.e., U is a faithful functor from A to Set, is said to be algebraic if it satisfies the following three conditions:

(1) A has coequalizers.

(2) U has a left adjoint.

(3) U preserves and reflects regular epimorphisms.

4.2 Proposition Any mono-source in an algebraic category (A, U) is U-initial.

Proof: Since U has a left adjoint, U preserves mono-sources. Hence it is immediate from Proposition 32.8 in [22].

4.3 Proposition An algebraic category is well-powered, regular co-(well-powered) and a (regular epi, mono)-category.

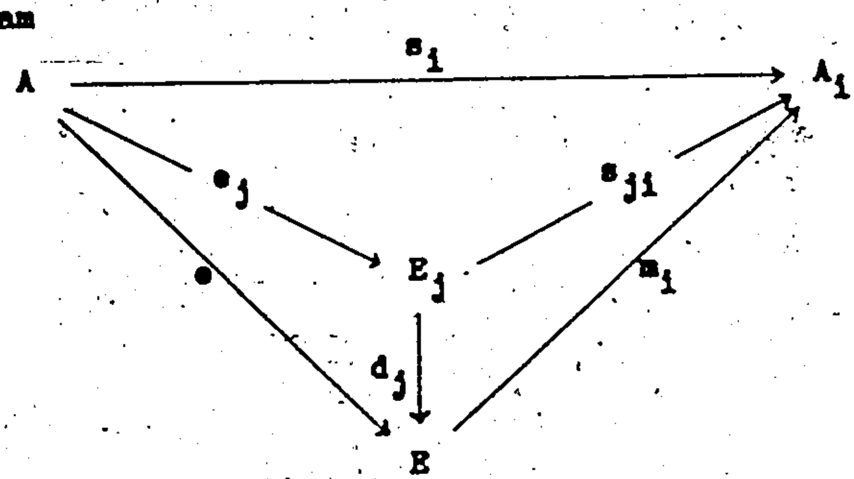
Proof: It can be found in [22].

4.4 Remark From Proposition 4.3, it is immediate that in an algebraic category, a morphism is a regular epimorphism if and only if it is an extremal epimorphism. Hence every algebraic category is extremally co-(well-powered).

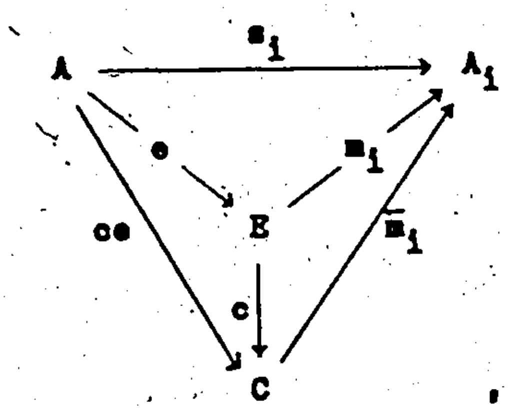
4.5 Proposition Every algebraic category is an (extremal epi, mono-sources)-category.

Proof: Let (A, U) be an algebraic category and let

$(A, A \xrightarrow{s_i} A_i)_{i \in I}$ be a source in $\underline{A}$. Let $(A \xrightarrow{e_j} E_j)_{j \in J}$ be the class of all extremal epimorphisms such that there exists a source $(E_j, E_j \xrightarrow{s_{ji}} A_i)_{i \in I}$ with $A \xrightarrow{s_i} A_i =$ $A \xrightarrow{e_j} E_j \xrightarrow{s_{ji}} A_i$ for each $i \in I$. Since $\underline{A}$ is extremally co-(well-powered), there exists a subset K of J such that $(A \xrightarrow{e_k} E_k)_{k \in K}$ is a representative set of $(A \xrightarrow{e_j} E_j)_{j \in J}$. Since every algebraic category is cocomplete (Theorem 32.14, [22]), the family $(e_k, E_k)_{k \in K}$ has a cointersection (e, E) that is a quotient object of A. In fact, (e, E) is obviously a "cointersection" of $(e_j, E_j)_J$. For each $i \in I$, there exists a unique morphism $m_i: E \longrightarrow A_i$ such that for each $j \in J$ the diagram

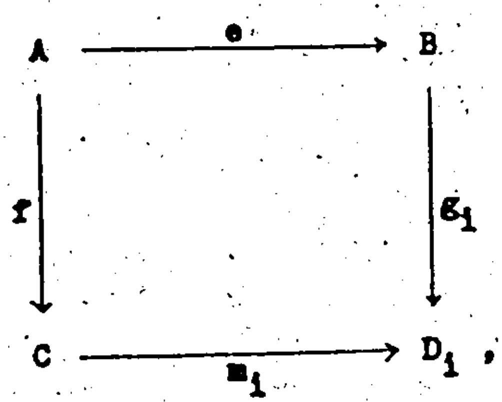


commutes. Now we wish to show that $(E, E \xrightarrow{m_i} A_i)_{i \in I}$ is a mono-source. Suppose $P \underset{v}{\overset{u}{\rightrightarrows}} E$ are $\underline{A}$-morphisms with $m_i u =$ $m_i v$ for each $i \in I$. Let $(E \xrightarrow{c} C, C)$ be a coequalizer of (u, v). Since $m_i u = m_i v$, there exists a unique morphism $\bar{m}_i: C \longrightarrow A_i$ with $\bar{m}_i c = m_i$. Since $\underline{A}$ is an (extremal epi, mono)-category, extremal epimorphisms in $\underline{A}$ are closed under composition. Thus the morphism ce is again an extremal epi-morphism. Moreover the diagram



commutes for each i. Hence there exists some $j_0 \in J$ such that $C = E_{j_0}$ and $ce = e_{j_0}$. Since $d_{j_0}(ce) = d_{j_0}e_{j_0} = e$, $d_{j_0}c = 1$, which implies that c is an isomorphism. Thus u = v. Therefore $A \xrightarrow{s_i} A_i = A \xrightarrow{e} E \xrightarrow{m_i} A_i$ $(i \in I)$ is an (extremal epi, mono-sources)-factorization.

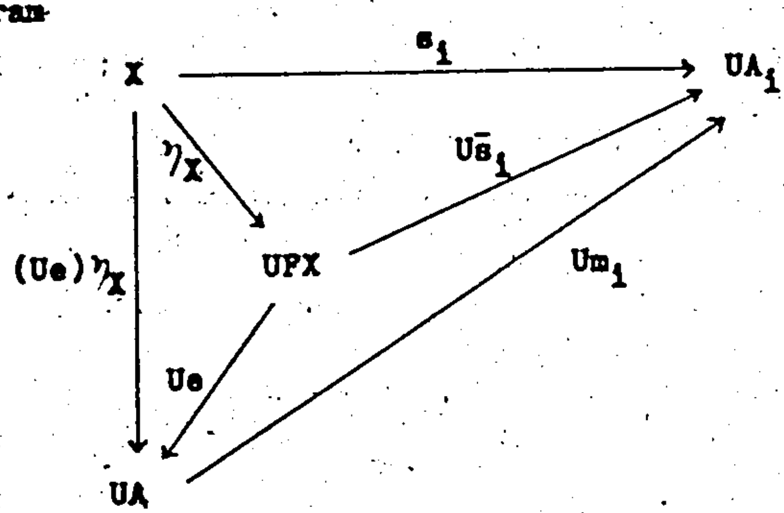
Suppose that the diagram



commutes for each $i \in I$, where e is an extremal epimorphism and $(C, (m_i)_I)$ is a mono-source. Then there exists a pair $E \overset{u}{\underset{v}{\rightrightarrows}} A$ of A-morphisms such that e = coequ(u,v). Since $m_i fu = g_i eu = g_i ev = m_i fv$, fu = fv. Hence there exists a unique morphism $h\colon B \longrightarrow \hat{C}$ with $he = f$. This completes the proof.

4.6 <u>Theorem</u> If $(\underline{A}, U)$ is an algebraic category, then $U\colon \underline{A} \longrightarrow \underline{Set}$ is a topologically algebraic functor.

<u>Proof</u>: Let $(A_i)_{i \in I}$ be a family of $\underline{A}$-objects and $(X, X \xrightarrow{s_i} UA_i)_{i \in I}$ a source in $\underline{Set}$. Let F be a left adjoint of U and $\eta_X\colon X \longrightarrow UFX$ the front adjunction for X. Then there exists a unique $\underline{A}$-morphism $\bar{s}_i\colon FX \longrightarrow A_i$ such that $(U\bar{s}_i)\,\eta_X = s_i$ for each $i \in I$. Let $FX \xrightarrow{\bar{s}_i} A_i = FX \xrightarrow{e} A \xrightarrow{m_i} A_i$ be the (extremal epi, mono-sources)-factorization. By Proposition 4.2, $(A, (m_i)_I)$ is a U-initial source, and $(Ue)\,\eta_X$ is a U-generating morphism. Furthermore, the diagram



commutes. This completes the proof.

4.7 <u>Definition</u> A Hausdorff space is said to be <u>zero-dimensional</u> if it has a basis consisting of sets which are both open and closed.

The category of (resp. zero-dimensional) compact Hausdorff spaces and continuous maps will be denoted by Comp (resp. ZComp). It is well known that Comp and ZComp are algebraic categories with respect to the underlying set functors. Hence we have the following:

4.8 Corollary The categories Comp and ZComp are topologically algebraic over the underlying set functors.

4.9 Theorem A concrete category (A, U) is algebraic if and only if it satisfies the following three conditions:

(1) U is topologically algebraic.

(2) U preserves regular epimorphisms.

(3) A morphism e in A is U-coinitial provided that Ue is onto.

Proof: ($\Longrightarrow$) It is immediate from Theorem 4.6, Definition 4.1, and Proposition 32.7 in [22].

($\Longleftarrow$) Since every topologically algebraic functor has a left adjoint and a topologically algebraic category over Set is cocomplete, it is enough to show that the functor U reflects regular epimorphisms. Let $e: A \longrightarrow C$ be an A-morphism such that Ue is a regular epimorphism in Set. Let (u, v) be a pair of morphisms from X to UA such that (Ue, UC) is the coequalizer of (u,v). Then there exists a U-generating morphism $h: X \longrightarrow UB$ and a U-initial source

$(B, B \overset{\bar{u}}{\underset{\bar{v}}{\rightrightarrows}} A)$ such that $(U\bar{u})h = u$ and $(U\bar{v})h = v$. We wish to show that (e, C) is the coequalizer of the pair $(\bar{u}, \bar{v})$. Since $(Ue)(U\bar{u})h = (Ue)u = (Ue)v = (Ue)(U\bar{u})h$, $e\bar{u} = e\bar{v}$, for h is U-generating. Suppose $d: A \longrightarrow D$ is an A-morphism with $d\bar{u} = d\bar{v}$. Since $(Ud)u = (Ud)(U\bar{u})h = (Ud)(U\bar{v})h = (Ud)v$, there exists a unique morphism $g: UC \longrightarrow UD$ in Set with $g(Ue) = Ud$. Since regular epimorphisms in Set are exactly onto morphisms, e is U-coinitial, so that there exists a unique morphism $\bar{g}: C \longrightarrow D$ with $\bar{g}e = d$ and $U\bar{g} = g$. The uniqueness of $\bar{g}$ with $\bar{g}e = d$ comes from the fact that U is faithful. This completes the proof.

Since a full subcategory A of the category $A(\tau)$, which is closed under the operations S and P, is epireflective in $A(\tau)$, every regular epimorphism in A is onto, and every onto morphism in a full subcategory B of the category $A(\tau)$ is U-coinitial for the underlying set functor $U: B \longrightarrow$ Set, the following is immediate from Theorem 4.9 and Theorem 3.5.

4.10 Corollary If a full subcategory A of the category $A(\tau)$ is closed under the operations S and P, then (A, U) is algebraic for the underlying set functor $U: A \longrightarrow$ Set.

CHAPTER II

EXTENSIONS IN TOPOLOGICAL ALGEBRAS

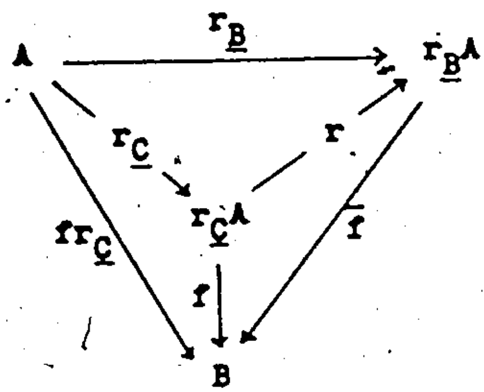
Throughout this chapter, every subcategory of a category will be assumed to be full and isomorphism closed.

Section 1: Extensive subcategories.

1.1 Definition Let $\underline{A}$ be a category of Hausdorff topological algebras of fixed type and continuous homomorphisms. A subcategory $\underline{B}$ of $\underline{A}$ is said to be an extensive subcategory of $\underline{A}$ if it is a reflective subcategory such that the $\underline{B}$-reflection $r_A: A \longrightarrow rA$ is a dense embedding for each $A \in \underline{A}$.

1.2 Theorem Let $\underline{A}$ be a category of Hausdorff topological algebras of fixed type and continuous homomorphisms and $\underline{B}$ an extensive subcategory of $\underline{K}$. Then any reflective subcategory $\underline{C}$ of $\underline{A}$ containing $\underline{B}$ is also an extensive subcategory of $\underline{A}$.

Proof: For any $\underline{A}$-object A, let $r_{\underline{B}}: A \longrightarrow r_{\underline{B}}A$ and $r_{\underline{C}}: A \longrightarrow r_{\underline{C}}A$ be reflections of A with respect to $\underline{B}$ and $\underline{C}$, respectively. Since $\underline{B}$ is contained in $\underline{C}$, there exists a unique morphism $r: r_{\underline{C}}A \longrightarrow r_{\underline{B}}A$ with $rr_{\underline{C}} = r_{\underline{B}}$. Since $r_{\underline{B}}$ is an embedding, $r_{\underline{C}}$ is also an embedding. We wish to show that r is a $\underline{B}$-reflection of $r_{\underline{C}}A$. Indeed, for any $\underline{B}$-object B and any $\underline{A}$-morphism $f: r_{\underline{C}}A \longrightarrow B$, there exists a unique morphism $\bar{f}: r_{\underline{B}}A \longrightarrow B$ such that the outer triangle in the diagram



commutes. Then $\bar{f}r = f$ and the uniqueness of $\bar{f}$ follow from the fact that $r_{\underline{C}}$ is a reflection. Hence r is the $\underline{B}$-reflection of $r_{\underline{C}}A$, so that r is also a dense embedding. Now we can conclude that $r_{\underline{C}}$ is dense. For a non-empty open set U in $r_{\underline{C}}A$, there exists an open set V in $r_{\underline{B}}A$ with $r(U) = V \cap r(r_{\underline{C}}A)$. Since $rr_{\underline{C}}(A) = r_{\underline{B}}(A)$ is dense in $r_{\underline{B}}A$, $V \cap rr_{\underline{C}}(A) \neq \phi$. Let x be an element of A such that $rr_{\underline{C}}(x) \in V \cap rr_{\underline{C}}(A)$. Then $r_{\underline{C}}(x)$ belongs to U, which implies $U \cap r_{\underline{C}}(A) \neq \phi$. Hence $r_{\underline{C}}$ is a dense embedding.

1.3 <u>Definition</u> Let $\underline{A}$ be a category of Hausdorff topological algebras of fixed type and continuous homomorphisms. An operator ℓ which associates with every pair (A, S), where A is an $\underline{A}$-object and S is a subalgebra of A, a subalgebra

$\ell_A S$ of A is said to be an <u>extensive operator</u> on $\underline{A}$, if ℓ satisfies the following two conditions:

(1) If S is a subalgebra of A, then $S \subseteq \ell_A S \subseteq cl_A S$, where cl_A denotes the closure operator on the underlying space of A.

(2) If $f: A \longrightarrow B$ is an $\underline{A}$-morphism and S is a subalgebra of A then $f(\ell_A S) \subseteq \ell_B f(S)$.

An extensive operator ℓ is said to be <u>idempotent</u> if ℓ satisfies the following:

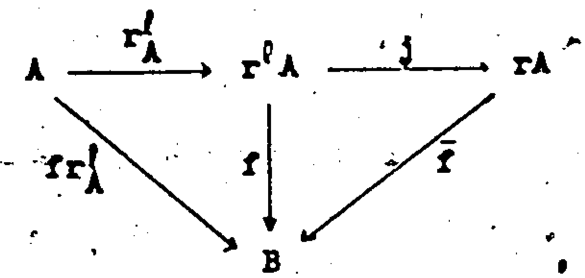
(3) If S is a subalgebra of an $\underline{A}$-object A, then $\ell_A(\ell_A S) = \ell_A S$.

A subalgebra S of A is said to be ℓ-closed in A if $\ell_A S = S$.

1.4 <u>Theorem</u> Let $\underline{B}$ be an extensive subcategory of a hereditary category $\underline{A}$ of Hausdorff topological algebras of fixed type and continuous homomorphisms and let ℓ be an idempotent extensive operator on $\underline{B}$. Then the subcategory $\underline{B}_\ell$ determined by those objects in $\underline{A}$ which are ℓ-closed in their $\underline{B}$-reflection spaces, is also an extensive subcategory of $\underline{A}$.

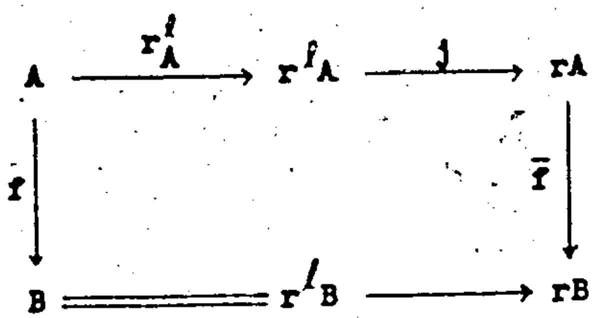
<u>Proof</u>: For any $\underline{A}$-object A, let $r_A: A \longrightarrow rA$ be the $\underline{B}$-reflection of A such that A is a dense subalgebra of rA and r_A is the natural embedding. Let $r'A$ be the subalgebra of rA with $\ell_{rA}A$ as its underlying set. Since $\underline{A}$ is hereditary, $r'A$ belongs to $\underline{A}$. It is easy to show that $r'A$ belongs to $\underline{B}_\ell$.

Indeed, let $r^{\ell}_A\colon A \longrightarrow r^{\ell}A$ and $j\colon r^{\ell}A \longrightarrow rA$ be the natural embeddings respectively. We claim that j is the $\underline{B}$-reflection of $r^{\ell}A$. Because, for any $\underline{B}$-object B and for any $\underline{A}$-morphism $f\colon r^{\ell}A \longrightarrow B$, there is a unique $\underline{B}$-morphism $\bar{f}\colon rA \longrightarrow B$ such that the outer triangle in the diagram



commutes. Hence $\bar{f}j = f$ and the uniqueness of $\bar{f}$ follow from the fact that r^{ℓ}_A and j are dense embeddings. Since $r^{\ell}A$ is ℓ-closed in its $\underline{B}$-reflection space rA, $r^{\ell}A$ belongs to $\underline{B}_{\ell}$.

Now, we can conclude that $r^{\ell}_A\colon A \longrightarrow r^{\ell}A$ is the $\underline{B}_{\ell}$-reflection. For any $\underline{B}_{\ell}$-object B and any $\underline{A}$-morphism $f\colon A \longrightarrow B$, there exists a unique $\underline{B}$-morphism $\bar{f}\colon rA \longrightarrow rB$ such that the diagram



commutes. Since $\bar{f}(r^{\ell}A) = \bar{f}(\ell_{rA}A) \subseteq \ell_{rB}\bar{f}(A) \subseteq \ell_{rB}B = B$, the map $\bar{\bar{f}}\colon r^{\ell}A \longrightarrow B$ which is a restriction and corestriction of $\bar{f}j$ to $r^{\ell}A$ and B respectively, is a continuous homomorphism, and $\bar{\bar{f}}r^{\ell}_A = f$. Noting that r^{ℓ}_A is a dense embedding, $\bar{\bar{f}}$ with

$\bar{\bar{f}} r_A^{\ell} = f$ is unique. This completes the proof.

Every subcategory T of the category Haus of Hausdorff spaces and continuous maps, may be considered as a category of Hausdorff topological algebras and continuous homomorphisms, namely TA(τ) with $\tau = \phi$. In the category TA(τ) ($\tau = \phi$), every subset of $X \in$ TA(τ) is obviously a subalgebra of X. Hence every idempotent extensive operator in the category T = TA(τ) ($\tau = \phi$) is same as that introduced by S. S. Hong (see [24]). Hence we immediately have the following.

1.5' Corollary Let B be an extensive subcategory of a hereditary category A of Hausdorff spaces and continuous maps, and let ℓ be an idempotent extensive operator on B. Then the subcategory B_{ℓ} determined by those objects in A which are ℓ-closed in their B-reflection spaces, is also an extensive subcategory of A.

Section 2: k-complete topological algebras.

The following definition is due to H. Herrlich [18].

2.1 Definition An operator ℓ which associates with every pair (X, A), where X is a topological space and A is a subset of X, a subset $\ell_X A$ of X is said to be an idempotent limit operator if ℓ satisfies the following four conditions:

(1) If A is a subset of X, then $A \subseteq \ell_X A \subseteq cl_X A$, where cl_X denotes the closure operator on X.

(2) If A and B are subsets of X then $\ell_X(A \cup B) = \ell_X A \cup \ell_X B$.

(3) If $f: X \longrightarrow Y$ is a continuous map and A is a subset of X then $f(\ell_X A) \subseteq \ell_Y f(A)$.

(4) If A is a subset of X then $\ell_X(\ell_X A) = \ell_X A$.

A subset A of a topological space X is said to be ℓ-closed if $\ell_X A = A$.

2.2 Remark By Propositions and Theorem 1 in [18], every coreflective subcategory C of the category Top generates an idempotent limit operator $\ell(\underline{C})$ and for any topological space X, $\ell(\underline{C})_X$ is precisely the closure operator on the C-coreflection space of X.

Every idempotent limit operator ℓ generates a coreflective subcategory $\underline{C}(\ell)$ of Top and $\underline{C}(\ell)$ is determined by topological spaces in which any ℓ-closed subset is closed.

2.3 <u>Proposition</u> Let ℓ be an idempotent limit operator. If the coreflective subcategory $\underline{C}(\ell)$ of $\underline{Top}$ generated by ℓ is (resp. finitely) productive, then ℓ induces an idempotent extensive operator on any category of (resp. finitary) Hausdorff topological algebras and continuous homomorphisms.

<u>Proof</u>: Let $\underline{A}$ be a category of (resp. finitary) Hausdorff topological algebras and continuous homomorphisms.

For an $\underline{A}$-object $(A, (f_i)_{i \in I}, \underline{O})$, let $1_A: (A, \underline{O}_\ell) \longrightarrow (A, \underline{O})$ be the $\underline{C}(\ell)$-coreflection of $(A, \underline{O})$. For any $i \in I$, since $(A, \underline{O}_\ell)^{\lambda_i} \in \underline{C}(\ell)$, there exists a unique continuous map $g: (A, \underline{O}_\ell)^{\lambda_i} \longrightarrow (A, \underline{O}_\ell)$ such that the diagram

$$\begin{array}{ccc} (A, \underline{O}_\ell)^{\lambda_i} & \xrightarrow{\;1_A^{\lambda_i}\;} & (A, \underline{O})^{\lambda_i} \\ \Big\downarrow g & & \Big\downarrow f_i \\ (A, \underline{O}_\ell) & \xrightarrow[\;1_A\;]{} & (A, \underline{O}) \end{array} \quad ,$$

commutes. Obviously g must be equal to f_i as set maps. Hence $\underline{O}_\ell$ is an algebra topology of A. For any subalgebra S of A, let $\bar{\ell}_A S = \ell_A S$. Since the closure of a subalgebra of a topological algebra is again a subalgebra, $\bar{\ell}$ is an idempotent extensive operator on $\underline{A}$.

2.4 <u>Definition</u> Let k be an infinite cardinal, and let X be a topological space. A subset of X is said to be a

G_k-set if it is an intersection of fewer than k open subsets of X.

It is clear that the G_k-sets of a topological space (X, $\underline{O}$) form a basis for a topology on X. We denote the new topology by $\underline{O}_k$. Since the inverse image of a G_k-set under a continuous map is also a G_k-set, the closure operator ℓ_X^k on (X, $\underline{O}_k$) gives rise to an idempotent limit operator ℓ^k = $(\ell_X^k)_{X \in \underline{Top}}$.

A subset S of a topological space X will be called k-closed if $\ell_X^k S = S$.

2.5 Proposition For any category $\underline{A}$ of finitary Hausdorff topological algebras and continuous homomorphisms, ℓ^k induces an idempotent extensive operator on $\underline{A}$.

Proof: It is enough to show that the coreflective subcategory $\underline{C}(\ell^k)$ of $\underline{Top}$, which is generated by ℓ^k, is finitely productive. We note that a topological space X belongs to $\underline{C}(\ell^k)$ if and only if every G_k-set of X is again open. It is obvious that every singleton space belongs to $\underline{C}(\ell^k)$. For any pair X and Y in $\underline{C}(\ell^k)$, suppose that G is a G_k-set in the product space $X \times Y$. Let $(x, y) \in G = \cap\{G_i \mid i \in I \text{ and } |I| < k\}$. Then there exists an open neighborhood U_i (resp. V_i) of x (resp. y) such that $U_i \times V_i$ is contained in G_i. The set $U = \cap U_i$ (resp. $V = \cap V_i$) $(i \in I)$ is a G_k-set in X (resp. Y); hence open in X (resp.

Y). Consequently, $(x, y) \in U \times V \subseteq G$, which implies that G is open in $X \times Y$. This completes the proof.

It is well known that every topological group has two uniform structures which are compatible with its group topology: for a topological group G, the uniform structure generated by $\{V^* \mid V^* = \{(x, y) \in G \times G \mid yx^{-1} \in V$ and $V \in \underline{O}(e)\}$, which will be called the right uniform structure on G, and the uniform structure generated by $\{{}^*V \mid {}^*V = \{(x, y) \in G \times G \mid x^{-1}y \in V$ and $V \in \underline{O}(e)\}$, which will be called the left uniform structure on G, where $\underline{O}(e)$ denotes the neighborhood filter of the unit e of G.

A topological group G with the right (resp. left) uniform structure will be denoted by G^* (resp. ${}^*G$).

In the following, every topological algebra will be assumed to be Hausdorff.

2.6 Definition A topological group G is said to be complete if G^* (equivalently, ${}^*G$) is complete.

A topological group G is called completable if every Cauchy filter in G^* is also a Cauchy filter in ${}^*G$.

The following theorem is well known and the proof can be found in [7].

2.7 Theorem Every completable topological group is isomorphic with a dense subgroup of a complete topological group.

Notation: We will denote the category of complete topological groups and continuous homomorphisms by *CGRP* and the category of completable topological groups and continuous homomorphisms by R*CGRP*.

Since every subgroup of a complete topological group is completable, the following is immediate from the above theorem.

2.8 *Corollary* A topological group is completable if and only if it is isomorphic with a subgroup of a complete topological group. Moreover, the category R*CGRP* is complete and hereditary.

Using Theorem 2.7 and Proposition 5, § 3, Chap. III ([7])~~, we have the following~~, which is again well known.

2.9 *Theorem* The category *CGRP* is extensive in R*CGRP*.

The following definition is due to M. Hušek [28].

2.10 *Definition* Let k be an infinite cardinal. A Hausdorff uniform space X is said to be *k-complete* if any Cauchy filter with the k-intersection property on X is convergent.

2.11 *Definition* Let k be an infinite cardinal. A completable topological group G is said to be *k-complete* if G^* is k-complete; equivalently if $^*G$ is k-complete.

2.12 <u>Definition</u> Let X be a dense subspace of a topological space Y, i.e. Y an extension of X. For each point $y \in Y$, $T(y) = \{V \cap X \mid V = \text{open neighborhood of } y \text{ in } Y\}$ will be called the <u>trace filter</u> of y on X. And the family $(T(y))_{y \in Y}$ will be called the <u>filter trace</u> of Y on X.

The following proposition is due to S. S. Hong [25].

2.13 <u>Proposition</u> Let X be a dense subspace of a topological space Y, and let k be an infinite cardinal. Then X is k-closed in Y if and only if any point of Y whose trace filter on X has the k-intersection property, belongs to X.

<u>Proof</u>: Suppose that X is k-closed in Y. Take $y \in Y$ whose trace filter has the k-intersection property. For any family $(G_i)_{i \in I}$ of open neighborhoods of y with $|I| < k$, $G_i \cap X \in T(y)$ for each $i \in I$, so that $\cap G_i \cap X \neq \phi$. Hence $y \in \ell_Y^k X = X$.

Conversely, take $y \in \ell_Y^k X$. For any subfamily $(G_i)_{i \in I}$ of $T(y)$ with $|I| < k$, there exists an open neighborhood V_i of y for each $i \in I$ such that $V_i \cap X = G_i$. Since $\cap V_i$ is a G_k-set, $\cap V_i \cap X \neq \phi$, i.e. $\cap G_i \neq \phi$. Hence T(y) has the k-intersection property. Thus $y \in X$.

2.14 <u>Definition</u> The minimal elements (by the inclusion relation) of the set of all Cauchy filters on a uniform space X are called minimal Cauchy filters on X.

Recall that for a Hausdorff uniform space X its completion cX is given as follows: its underlying set is the set of all minimal Cauchy filters on X and its uniform structure is generated by $\{\tilde{V} \mid V:$ symmetric entourage on $X\}$, where $\tilde{V}$ is the set of all pairs (ξ, η) of minimal Cauchy filters such that there is a set M in $\xi \cap \eta$ which is a V-small set (see [7]).

In what follows, we identify each point of X with its neighborhood filter, so that X is a subspace of cX.

Using the fact that each minimal Cauchy filter ξ is generated by $\{V(F) \mid V:$ symmetric entourage on $X, F \in \xi\}$, it is easy to show that the trace filter of $\xi \in cX$ on X generates ξ itself. Moreover, for any Cauchy filter η, there is a unique minimal Cauchy filter which is coarser than η. Hence by Proposition 2.13, we have the following:

2.15 Lemma A Hausdorff uniform space X is k-complete if and only if it is k-closed in cX.

In particular, a completable topological group is k-complete if and only if it is k-closed in its completion.

2.16 Theorem Let k be an infinite cardinal. The subcategory kCGRP of RCGRP determined by all k-complete topological groups is extensive in RCGRP.

Proof: It is immediate from Proposition 2.5, Lemma 2.15 and Theorem 1.4.

Remark: CGRP = $\aleph_0$CGRP and kCGRP $\supseteq$ tCGRP for infinite cardinals k and t with k $\geq$ t.

2.17 Corollary The category kCGRP is complete.

Since for any abelian topological group G, G* and *G are same, every abelian topological group is completable. The following is immediate from Theorem 2.16.

2.18 Theorem Let k be an infinite cardinal. The subcategory determined by k-complete abelian topological groups is extensive in the category of abelian topological groups and continuous homomorphisms.

2.19 Definition A topological group G is said to be totally bounded provided that for any neighborhood V of the unit e, there is a finite subset F of G with G = VF.

Remark: (1) It is known that for a totally bounded topological group G, G* = *G. Hence every totally bounded topological group is completable.

(2) A topological group G is totally bounded if and only if G* (and equivalently *G) is a totally bounded uniform space.

Using the above remark and the fact that a uniform space is totally bounded if and only if its completion is compact, a topological group is totally bounded if and only if it is isomorphic with a subgroup of a compact topological group. Furthermore, we have the following from Theorem 2.9.

2.20 Theorem The subcategory CompGRP determined by compact topological groups is extensive in the category tbGRP of totally bounded topological groups and continuous homomorphisms.

The following definition is due to H. Herrlich [16].

2.21 Definition A completely regular space X is called k-compact for an infinite cardinal k if every z-ultrafilter with the k-intersection property on X is fixed.

By Theorem 1.18 in Chap. 0 and Theorem 1.2, one has:

2.22 Theorem The subcategory kCompGRP determined by k-compact topological groups is extensive in tbGRP.

Remark: (1) Since a k-closed subspace of a compact space is k-compact (see [25]), a k-complete object of tbGRP is k-compact. However, it is an open question that any object of kCompGRP is k-complete.

(2) Since for any Hausdorff space $(X, \mathcal{O})$, $\mathcal{O}_k$ is discrete for some infinite cardinal k, every completable (resp. totally bounded) topological group is k-complete (resp. k-compact) for some infinite cardinal k.

(3) It is a well known convention that whenever we speak

of the uniform structure of a topological ring, it is the uniform structure of its additive group. In particular, a topological ring A is said to be complete if the additive group of A is complete. Moreover, it is known that the subcategory determined by complete topological rings is extensive in the category TR of topological rings and continuous homomorphisms (see [7]). By the exactly same argument as that in the above, we can conclude that the subcategory determined by k-complete topological rings is extensive in TR and that the subcategory determined by k-compact topological rings is extensive in the category of totally bounded topological rings and continuous homomorphisms.

Also one can conclude the same results for the category of topological A-modules for a topological ring A and continuous homomorphisms.

Example: (1) Let Q be the topological additive group (resp. ring) of rational numbers with the usual topology. Since the real line (R, $\underline{O}$) with the usual topology $\underline{O}$ is the completion of Q and (R, $\underline{O}_{\aleph_1}$) is discrete, Q is $\aleph_1$-complete but not complete.

(2) Let $T = \{e^{2\pi ix} \mid 0 \leq x < 1\}$ be the one-dimensional circle group and $T_q = \{e^{2\pi ix} \mid 0 \leq x < 1 \text{ and } x \in Q\}$ the subgroup of T. Then one can easily conclude that T_q is totally bounded and that T_q is $\aleph_1$-compact but not compact.

CHAPTER III

TOPOLOGICAL PARTIALLY ORDERED SETS

Throughout this chapter, we assume that every topological space is a Hausdorff space.

Section I: Topological partially ordered sets.

1.1 Definition A type of partial algebras is a family $\tau = (\lambda_j)_{j \in J}$ of ordinals, indexed by a set J.

A topological partial algebra of type τ is a quadruple $A = (X, (f_j)_{j \in J}, (A_j)_{j \in J}, \underline{O})$ in which

(1) $(X, (f_j)_{j \in J}, (A_j)_{j \in J})$ is a partial algebra of type τ, i.e., $A_j \subseteq X^{\lambda_j}$ and $f_j : A_j \longrightarrow X$ is a map for each $j \in J$, and

(2) $(X, \underline{O})$ is a topological space such that $f_j : A_j \longrightarrow X$ is continuous for each $j \in J$, where A_j is the subspace of the product space X^{λ_j}.

A topology on a partial algebra satisfying the continuity of operations is said to be an algebra topology.

If $A_j = X^{\lambda_j}$ for each j, then $(X, (f_j)_J, (A_j)_J, \underline{O})$ becomes a topological algebra.

1.2 <u>Definition</u> Let $A = (X, (f_j^A)_{j \in J}, (A_j)_{j \in J})$ and $B = (Y, (f_j^B)_{j \in J}, (B_j)_{j \in J})$ be partial algebras of fixed type $\tau = (\lambda_j)_{j \in J}$. A set map $h: X \longrightarrow Y$ is said to be a homomorphism from A to B if $h^{\lambda_j}(A_j) \subseteq B_j$ and $hf_j^A = f_j^B h^{\lambda_j}$ for each $j \in J$, where h^{λ_j} is, of course, understood as the restriction of h^{λ_j} to A_j.

<u>Remark</u> For a partially ordered set $(P, \leqslant)$, let G be the graph of the order $\leqslant$, i.e., $G = \{(x, y) \mid x \leqslant y,\ x, y \in P\}$ and let $f_v : G \longrightarrow P$ be a map defined by $f_v((x, y)) = y$. Then for any topology $\underset{\sim}{O}$ on P, $(P, f_v, G, \underset{\sim}{O})$ is a topological partial algebra, since f_v is the restriction of the projection.

However, if we take $G \cup G^{-1}$ as the domain of f_v defined by $f_v((x, y)) = y$ when $(x, y) \in G$ and $f_v((x, y)) = x$ when $(x, y) \in G^{-1}$, every topology on P need not be an algebra topology.

<u>Example</u> Let $P = \{1/n \mid n \in \mathbb{N}\}$ with the usual order, where $\mathbb{N}$ is the set of natural numbers. Let $\underset{\sim}{O}$ be the topology generated by $\{\{1\} \cup V_k \mid V_k = \{1/n \mid n > k\},\ k \in \mathbb{N}\} \cup \{\{1/n\} \mid n \geqslant 2\}$. Then clearly $(P, \underset{\sim}{O})$ is a Hausdorff space. Considering the neighborhood $\{1\} \cup V_3$ of 1, one can easily conclude that the map $f_v : G \cup G^{-1} \longrightarrow P$ is not continuous at $(1/2, 1)$.

In what follows, every partially ordered set $(P, \leqslant)$ will be also considered as a partial algebra $(P, f_v, G \cup G^{-1})$, where G is the graph of the order $\leq$ and $f_v : G \cup G^{-1} \longrightarrow P$ is a map defined by the join. Then it is obvious that homomorphisms between two partially ordered sets are exactly isotone maps.

The category of topological partially ordered sets (as topological partial algebras) and continuous homomorphisms will be denoted by TPOS.

The following definition is due to L. E. Ward, Jr [35].

1.3 Definition Let (X, O) be a topological space endowed with a partial order $\leq$. The partial order $\leq$ is called continuous provided that whenever $x \nleq y$ in X, there are open sets U and V, $x \in U$ and $y \in V$, such that if $u \in U$ and $v \in V$, then $u \nleq v$.

In this case, the topology O on X will be called W-topology on (X, $\leq$) and (X, $\leq$, O) will be called a W-topological partially ordered set.

The category of W-topological partially ordered sets and continuous isotone maps will be denoted by WPOS.

1.4 Proposition Every W-topology on a partially ordered set (P, $\leq$) is an algebra topology on the partial algebra (P, f_v, $G \cup G^{-1}$).

Proof: Let O be a W-topology on (P, $\leq$).

For $(x, y) \in G \cup G^{-1}$, either $x \leq y$ or $y \leq x$. We may assume that $x \leq y$ and $x \neq y$. It follows from the continuity of the order $\leq$ and $y \not\leq x$ that there are two open sets U and V, $y \in U$ and $x \in V$, such that $U \not\leq V$, i.e., $a \in U$ and $b \in V$ imply that $a \not\leq b$. We note that $f_v((x, y)) = y$. For a neighborhood W of y, take the neighborhood $O_y = W \cap U$ of y and $O_x = V$. Let $(s, t) \in O_x \times O_y \cap (G \cup G^{-1})$. Then it is easy to show that $f_v((s, t)) \in W$. Indeed, suppose $t < s$. Then it contradicts to $U \not\leq V$. Hence $s \leq t$, i.e. $f_v((s, t)) = t \in W$. Hence f_v is continuous.

1.5 <u>Remark</u> Every algebra topology on a partially ordered set need not be a W-topology, i.e., <u>WPOS</u> $\subsetneq$ <u>TPOS</u>.

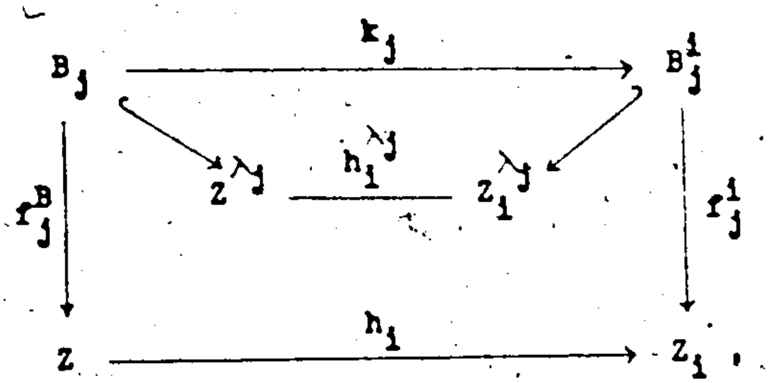
<u>Proof</u>: Let $P = \{(0, 0)\} \cup \{(0, -1/n) \mid n \in \mathbb{N}\} \cup \{(1, 0)\} \cup \{(1, 1/n) \mid n \in \mathbb{N}\}$ with a relation $\leq$ defined as follows: $(0, -1/n) \leq (0, 0)$, $(0, -1/n) \leq (1, 0)$, $(0, -1/k) \leq (1, 1/n)$ for all $n, k \in \mathbb{N}$, $(1, 0) \leq (1, 1/n)$ for all n, $(0, -1/n) \leq (0, -1/k)$ for $n \leq k$, $(1, 1/n) \leq (1, 1/k)$ for $k \leq n$, $(x, y) \leq (x, y)$ for all $(x, y) \in P$. Clearly $(P, \leq)$ is a partially ordered set.

Let $\underline{O}$ be the topology on P generated by $\{\{(0, 0)\} \cup V_k \mid V_k = \{(1, 1/n) \mid n > k \text{ and } k \in \mathbb{N}\}\} \cup \{\{(1, 0)\} \cup V_{-k} \mid V_{-k} = \{(0, -1/n) \mid n > k \text{ and } k \in \mathbb{N}\}\} \cup \{\{(x, y)\} \mid (x, y) \neq (0, 0) \text{ and } (x, y) \neq (1, 0)\}$. Then it is easy to show that $(P, \leq, \underline{O})$ is a topological partial algebra. We note that $(1, 0) \not\leq (0, 0)$. For any basic neighborhood $N(0, 0) = \{(0, 0)\} \cup V_k$ of $(0, 0)$ and any basic open neighborhood $N(1, 0) = \{(1, 0)\} \cup V_{-k}$, of $(1, 0)$, every element of V_k is bigger than any element of V_{-k}, : $N(1, 0) \lneq N(0, 0)$ does not hold. Hence $\underline{0}$ is not a W-topology.

1.6 Theorem The underlying set functors from TPOS and WPOS are topologically algebraic.

Proof: Let U be the underlying set functor from TPOS (resp. WPOS). For each family $(A_i)_{i \in I}$ of TPOS-objects (resp. WPOS-objects) and a source $(X, X \xrightarrow{s_i} UA_i)_I$ in Set, we have the (epi, mono-sources)-factorization $X \xrightarrow{s_i} UA_i = X \xrightarrow{e} Y \xrightarrow{m_i} UA_i$ $(i \in I)$, for Set is an (epi, mono-sources)-category. Let $A = (Y, \leq, \underline{0})$, where $\underline{0}$ is the initial topology on Y with respect to $(m_i)_I$ and $\leq$ is an order on Y defined by $x \leq y$ if and only if $m_i(x) \leq m_i(y)$ for each $i \in I$. We will show that $A \in$ TPOS (resp. WPOS).

Regarding $A \in$ TPOS: We claim that for any family $(B_i)_{i \in I}$ of topological partial algebras of fixed type $\tau = (\lambda_j)_{j \in J}$ and a partial algebra B of type τ, the initial topology on B with respect to a point-separating source $(B, B \xrightarrow{h_i} B_i)_I$ of homomorphisms is again an algebra topology on B. Indeed, let $B = (Z, (f_j^B)_J, (B_j)_J)$, $B_i = (Z_i, (f_j^i), (B_j^i)_J, \underline{0}_i)$ for each $i \in I$ and let $\underline{0}$ be the initial topology with respect to $(h_i)_I$. Then obviously $(Z, \underline{0})$ is a Hausdorff space. Let j be an element of J. Since the diagram



commutes, where k_j is the restriction and corestriction of $h_i^{\lambda_j}$ to B_j and B_j^i respectively, and $h_i f_j^B = f_j^i k_j$ is continuous for each $i \in I$, f_j^B must be continuous.

In particular, it follows from the above claim that the topology $\underline{O}$ is an algebra topology on Y. Hence $A = (Y, \leq, \underline{O})$ belongs to TPOS.

Regarding $A \in$ WPOS: suppose $x \not\leq y$ in $A = (Y, \leq, \underline{O})$; there exists an $i \in I$ such that $m_i(x) \not\leq m_i(y)$. Hence there exist open sets N_x^i and N_y^i in A_i such that $m_i(x) \in N_x^i$, $m_i(y) \in N_y^i$ and $N_x^i \not\leq N_y^i$. Let $N_x = m_i^{-1}(N_x^i)$ and $N_y = m_i^{-1}(N_x^i)$. Clearly N_x and N_y are open neighborhoods of x and y respectively with $N_x \not\leq N_y$. Hence $A \in$ WPOS.

Let $A \xrightarrow{\bar{m}_i} A_i = m_i$ as set map. Then it is obvious that $(A, (\bar{m}_i)_I)$ is a U-initial source in TPOS (resp. WPOS), $e: X \longrightarrow UA$ is a U-generating morphism and $(U\bar{m}_i)e = s_i$ for each $i \in I$. This completes the proof.

1.7 Corollary The categories TPOS and WPOS are complete and cocomplete. The underlying set functors from TPOS and WPOS have left adjoints.

In the following, the cardinal of a set X will be denoted by $\bar{\bar{X}}$.

1.8 <u>Theorem</u> The category <u>WPOS</u> is a reflective subcategory of the category <u>TPOS</u>.

<u>Proof</u>: Let E: <u>WPOS</u> $\longrightarrow$ <u>TPOS</u> be the embedding functor. Obviously E preserves limits. For any $P = (X, \leq, \underline{O}) \in$ <u>TPOS</u>, let $\mathcal{J} = \{(u_i, A_i) \mid A_i = (X_i, \leq, \underline{O}_i) \in$ <u>WPOS</u>, $\bar{\bar{X}}_i \leq \bar{\bar{X}}$, $u_i : P \longrightarrow EA_i\}$. Then $\bar{\bar{\mathcal{J}}} \leq 2^{\bar{\bar{X}}} \times 2^{\bar{\bar{X}} \times \bar{\bar{X}}} \times 2^{2^{\bar{\bar{X}}}} \times \bar{\bar{X}}^{\bar{\bar{X}}}$, i.e., $\mathcal{J}$ is a set. For any $Q = (Y, \leq, \underline{O}')\in$ <u>WPOS</u> and any morphism $f: P \longrightarrow EQ$, let $A = (f(P), \leq, \underline{O}' | f(P))$ be the subspace of Q. It is obvious that $A \in$ <u>WPOS</u> and $(g, A) \in \mathcal{J}$, where g is the corestriction of f to f(P). Let e be the natural embedding of A into Q. Then the diagram

$$\begin{array}{ccc} P & \xrightarrow{\;g\;} & EA \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle Ee} & \\ EQ & & \end{array},$$

commutes. Hence $\mathcal{J}$ is an E-solution set for P. Hence E has a left adjoint, i.e., <u>WPOS</u> is a reflective subcategory of <u>TPOS</u>.

Every Hausdorff space may be considered as a topological partially ordered set with the discrete order.

1.9 <u>Proposition</u> The category <u>Haus</u> of Hausdorff spaces and continuous maps is a coreflective subcategory of TPOS (and WPOS).

Proof: For an object $(X, \underline{O})$ of Top, let $A_X =$ $(X, =_X, \underline{O})$, where $=_X$ is the discrete order on X, i.e., $=_X$ is the diagonal of X. It is obvious that for any $(X, \underline{O}) \in$ TOP, A_X belongs to WPOS if and only if $(X, \underline{O}) \in$ Haus and that a map $f\colon X \longrightarrow Y$ between Hausdorff spaces is continuous if and only if the same map $f\colon A_X \longrightarrow A_Y$ is a morphism in WPOS. Identifying X and A_X, we may assume that Haus is a subcategory of WPOS.

For an object $A = (X, \leq, \underline{O})$ in TPOS (resp. WPOS), let A_H be $A_{(X, \underline{O})}$ and $c\colon A_H \longrightarrow A$ be the morphism defined by the identity map of X. Then $c\colon A_H \longrightarrow A$ is the coreflection map of A. Because, for any Haus-object $(Y, \underline{O}') = A_{(Y, \underline{O}')}$ and a morphism $f\colon A_{(Y, \underline{O}')} \longrightarrow A$, the diagram

$$\begin{array}{ccc} A_H & \xrightarrow{\quad c \quad} & A \\ & {}_{f}\nwarrow \quad \nearrow_{f} & \\ & A_{(Y, \underline{O}')} & \end{array},$$

commutes.

Section 2: Completely regular ordered spaces.

2.1 Definition A W-topological partially ordered set $(X, \leq, \underline{O})$ will be called completely separated provided that whenever $x \nleq y$ in X, there exists a continuous isotone map $f: X \longrightarrow [0, 1]$ such that $f(x) > f(y)$, where $[0, 1]$ is the unit interval with the usual order and topology.

The following definition is due to L. Nachbin [32].

2.2 Definition A completely separated W-topological partially ordered set $(X, \leq, \underline{O})$ is said to be a completely regular ordered space if for any point x in X and for any open neighborhood V of x there exists a continuous isotone map $f: X \longrightarrow I$ and a continuous decreasing map $g: X \longrightarrow I$ such that $f(x) = 1 = g(x)$ and $\complement V \subseteq f^{-1}(0) \cup g^{-1}(0)$, where I denotes the unit interval $[0, 1]$ and $\complement V$ denotes the complement of V in X.

The category of completely regular ordered spaces and continuous isotone maps will be denoted by CROS.

Remark It is immediate from the definition that the category CROS is hereditary.

2.3 Proposition For any completely separated W-topological partially ordered set $(X, \leq, \underline{O})$, the following are equivalent:

(1) $(X, \leq, \underline{O})$ is a completely regular ordered space.

(2) For any point x of X and any open neighborhood V of x, there exist finitely many continuous isotone maps $f_1, \ldots, f_n : X \longrightarrow [-1, 1]$ such that $f_i(x) = 0$ for each $i = 1, \ldots, n$ and $\complement V \subseteq \cup f_i^{-1}(\{-1, 1\})$, where $[-1, 1]$ is endowed with the usual order and topology.

(3) $\{ f^{-1}(0) \mid f \in \hom(X, R)\}$ forms a subbase for closed sets of $\underline{O}$, where $\hom(X, R) = \underline{WPOS}(X, R)$ and R denotes the real line with the usual order and topology.

(4) $(X, \leq, \underline{O})$ is an R-regular ordered space, i.e., it is iseomorphic with a subspace of a power of R.

(5) $(X, \leq, \underline{O})$ is an I-regular ordered space, i.e., it is iseomorphic with a subspace of a power of the unit interval I.

<u>Proof</u>: (1) $\Longrightarrow$ (2) For any point $x \in X$ and any open neighborhood V of x, there exists a continuous isotone map $f : X \longrightarrow I$ and a continuous decreasing map $g : X \longrightarrow I$ such that $f(x) = 1 = g(x)$ and $\complement V \subseteq f^{-1}(0) \cup g^{-1}(0)$. Let $f_1 = f - 1$ and $f_2 = 1 - g$. Then the two continuous isotone maps $f_1, f_2 : X \longrightarrow [-1, 1]$ satisfy the desired conditions.

(2) $\Longrightarrow$ (3) Let F be a closed subset of X and $x \notin F$. Then there exist finitely many continuous isotone maps $f_1, \ldots, f_n : X \longrightarrow [-1, 1] \subset R$ such that $f_i(x) = 0$ for each $i = 1, \ldots, n$ and $F \subseteq \cup f_i^{-1}(\{-1, 1\})$. Since $\{f^{-1}(0) \mid f \in \hom(X, R)\} = \{ g^{-1}(p) \mid g \in \hom(X, R)\}$ for any fixed $p \in R$,

there exists a family $g_1, \ldots, g_n, h_1, \ldots, h_n$ in hom(X, R) such that $g_i^{-1}(0) = f_i^{-1}(1)$ and $h_i^{-1}(0) = f_i^{-1}(-1)$ for i = 1, ..., n. Obviously $x \notin \bigcup \{g_i^{-1}(0) \cup h_i^{-1}(0)\}$ and F is contained in $\bigcup (g_i^{-1}(0) \cup h_i^{-1}(0))$.

(3) $\Longrightarrow$ (4) Let $\varphi : X \longrightarrow R^{\hom(X,\, R)}$ be a map defined by $\varphi(x) = (f(x))_{f \in \hom(X,\, R)}$ for each $x \in X$.

Firstly, since $pr_f\, \varphi = f$ is continuous for each f, where pr_f is the f-th projection, φ is continuous.

Secondly, φ is one-one, because for $x \neq y$ in X, either $x \not\leq y$ or $y \not\leq x$; there exists an $f \in \hom(X, R)$ with $f(x) \neq f(y)$, so that $\varphi(x) \neq \varphi(y)$.

Thirdly we will show that for a closed subset F of X, $\varphi(F) \cap \varphi(X)$ is closed in $\varphi(X)$. Suppose that $z \in X$ such that $\varphi(z) \notin \varphi(F) \cap \varphi(X)$. Then $z \notin F$. Hence there exist finitely many continuous isotone maps $f_1, \ldots, f_n : X \longrightarrow R$ such that $\bigcup f_i^{-1}(0) \supset F$ and $z \notin \bigcup f_i^{-1}(0)$, i.e., $f_i(z) \neq 0$ for i = 1, ..., n. Let $V = \Pi \{V_h : h \in \hom(X, R)\}$, where V_{f_i} is an open neighborhood of $f_i(z)$ with $0 \notin V_{f_i}$ and $V_h = R$ for $h \neq f_i$ (i = 1, ..., n). Then V is an open neighborhood of $\varphi(z)$ and $V \cap \varphi(F) \cap \varphi(X) = \phi$. Hence φ is relatively closed. Finally, it follows from Definition 2.1 that φ is an order isomorphism. Hence X is an R-regular ordered space.

(4) $\Longrightarrow$ (5) It is immediate from the fact that R is iseomorphic with the open interval]0, 1[which is a subspace

of I.

(5) $\Longrightarrow$ (1) Since every compact ordered space is a completely regular ordered space (see [32]), and the category CROS is hereditary, an I-regular ordered space is a completely regular ordered space.

Remark It has been shown [12] that OI-Reg $\subseteq$ CROS, where OI-Reg is the category of I-regular ordered spaces and continuous isotone maps. Proposition 2.3 shows that OI-Reg = CROS = OR-Reg, where OR-Reg is the category of R-regular ordered spaces and continuous isotone maps.

2.4 Corollary The category CROS is complete. Moreover CROS is an epireflective subcategory of WPOS.

Proof: It follows from Proposition 2.3 that CROS is productive and has equalizers. Hence CROS is complete.

Let E: CROS $\longrightarrow$ WPOS be the embedding functor. Since a singleton space belongs to CROS, monomorphisms in CROS are exactly one-one morphisms, and hence CROS is well-powered. Obviously I is a coseparator for CROS and E preserves limits. Hence by Theorem 1.12 in Chap. 0, E has a left adjoint, i.e., CROS is a reflective subcategory of WPOS. For any $A \in$ WPOS, let $\varphi: A \longrightarrow I^{hom(A, I)}$ be the morphism defined by $\varphi(x) = (f(x))_{f \in hom(A, I)}$. It is easy to show that the CROS-reflection of A is given by $r: A \longrightarrow rA$, where rA is the subspace of $I^{hom(A, I)}$ whose underlying set is $\varphi(A)$ and r is the corestriction of φ to $\varphi(A)$. Hence CROS is epireflective in WPOS.

2.5 Corollary A W-topological partially ordered set is a completely regular ordered space if and only if it is iseomorphic with a subspace of a compact ordered space.

2.6 Corollary A W-topological partially ordered set is compact if and only if it is iseomorphic with a closed subspace of a power of the unit interval.

Proof: It follows immediately from the fact that every compact subset of a Hausdorff space is closed and every compact ordered space is completely regular.

Section 3: Compact ordered spaces.

It is well known [7] that a completely regular space is compact if and only if every maximal completely regular filter on the space is convergent.

3.1. Definition Let X be a completely regular ordered space. A filter $\mathcal{F}$ on X is said to be o-completely regular if $\mathcal{F}$ has an open base $\mathcal{B}$ satisfying that for each $U \in \mathcal{B}$, there exists a $V \in \mathcal{B}$ with $V \subseteq U$ and there exist finitely many continuous isotone maps $f_1, \ldots, f_n: X \longrightarrow [-1, 1]$ such that $f_i(V) = 0$ for each $i = 1, \ldots, n$ and $\complement U$ is contained in $\bigcup f_i^{-1}(\{-1, 1\})$.

By a maximal o-completely regular filter on X is meant an o-completely regular filter not contained in any other o-completely regular filter.

Remark For every o-completely regular filter, there exists, by Zorn's Lemma, a maximal o-completely regular filter containing it.

3.2 Lemma Let $\mathcal{F}$ be an o-completely regular filter on a completely regular ordered space X. Then $\mathcal{F}$ is a maximal o-completely regular filter if and only if for any pair of open sets U and V with $V \subseteq U$ and finitely many continuous isotone maps $f_1, \ldots, f_n: X \longrightarrow [-1, 1]$ such that $f_i(V) = 0$ for $i = 1, \ldots, n$ and $(\complement U \subseteq \bigcup f_i^{-1}(\{-1, 1\}))$, either $U \in \mathcal{F}$, or $U \notin \mathcal{F}$ and there exists some $F \in \mathcal{F}$ with $F \cap V = \emptyset$.

<u>Proof</u>: ($\Longrightarrow$) Suppose that $U \notin \mathcal{F}$ and $F \cap V \neq \phi$ for all $F \in \mathcal{F}$. Let $\mathcal{B}$ be the open base of $\mathcal{F}$ satisfying the conditions in the definition of an o-completely regular filter. Let $\mathcal{B}' = \mathcal{B} \cup \{f_i^{-1}(]\!]-r_i, r_i[\![) \mid r_i > 0 \text{ and } 1 \leq i \leq n\}$. It follows from $f_i(V) = 0$ for $i = 1, \ldots, n$ that $\mathcal{B}'$ has the finite intersection property. Hence $\mathcal{B}'$ generates a filter $\mathcal{U}$. It is easy to show that $\mathcal{U}$ is an o-completely regular filter and $\mathcal{U} \supseteq \mathcal{F}$. Moreover, $U \notin \mathcal{F}$ but $U \in \mathcal{U}$, which contradicts the fact that $\mathcal{F}$ is a maximal o-completely regular filter.

($\Longleftarrow$) Suppose that $\mathcal{F}$ is not a maximal o-completely regular filter. There exists an o-completely regular filter $\mathcal{U}$ which is strictly finer than $\mathcal{F}$, i.e., there exists a $U \in \mathcal{U}$ such that $U \notin \mathcal{F}$. Then there exists a $V \in \mathcal{U}$ with $V \subseteq U$ and there exist continuous isotone maps $f_i: X \longrightarrow [\![-1, 1]\!]$ $(i = 1, \ldots, n)$ such that $f_i(V) = 0$ and $(U \subseteq \cup f_i^{-1}(\{-1, 1\}))$. Hence there exists an $F \in \mathcal{F}$ such that $F \cap V = \phi$, which is a contradiction.

3.3 <u>Theorem</u> A filter $\mathcal{U}$ on a completely regular ordered space X contains a maximal o-completely regular filter if and only if $f(\mathcal{U})$ is convergent for each continuous isotone map $f: X \longrightarrow [\![-1, 1]\!]$.

<u>Proof</u>: ($\Longrightarrow$) Since a filter containing a convergent filter is again convergent, it is enough to show that for every maximal o-completely regular filter $\mathcal{F}$ and

any continuous isotone map $f: X \longrightarrow [\![-1, 1]\!]$, $f(\mathfrak{F})$ is convergent. Since $[\![-1, 1]\!]$ is compact, $\bigcap_{F \in \mathfrak{F}} \text{cl}f(F) \neq \phi$, where cl is the closure operator on $[\![-1, 1]\!]$. Suppose $a, b \in \bigcap_{F \in \mathfrak{F}} \text{cl}f(F)$ and $a < b$. For each r such that $a < r \leq 1$, suppose $f^{-1}([\![-1, r[\![) \notin \mathfrak{F}$. Then $f^{-1}([\![-1, r'[\![) \subseteq f^{-1}([\![-1, r[\![)$ for $a < r' < r$ and there is a continuous isotone map h from X into $[\![-1, 1]\!]$ such that $h(f^{-1}([\![-1, r'[\![) = 0$ and $\complement(f^{-1}([\![-1, r[\![))$ $\subseteq h^{-1}(\{-1, 1\})$. Hence by Lemma 3.2, there exists an $F \in \mathfrak{F}$ such that $F \cap f^{-1}([\![-1, r'[\![) = \phi$; $f(F) \cap [\![-1, r'[\![= \phi$. Since $a \in [\![-1, r'[\![$, $a \notin \text{cl}f(F)$ which is a contradiction. Hence $f^{-1}([\![-1, r[\![) \in \mathfrak{F}$ for each r with $a < r \leq 1$. Similarly $f^{-1}(]\!]r, 1]\!]) \in \mathfrak{F}$ for each r with $-1 \leq r < b$. Hence for any r with $a < r \leq b$, $f^{-1}([\![-1, r[\![) \cap f^{-1}(]\!]r, 1]\!]) \in \mathfrak{F}$, which is a contradiction. Hence $a = b$.

We wish to show that $f(\mathfrak{F})$ converges to a. Indeed, for any $r > 0$, $f^{-1}([\![-1, a + r[\![) \in \mathfrak{F}$ and $f^{-1}(]\!]a - r, 1]\!]) \in \mathfrak{F}$; $f^{-1}(]\!]a - r, a + r[\![) \in \mathfrak{F}$ for $a \neq 1$ and $a \neq -1$. Hence $f(\mathfrak{F})$ is convergent to a.

($\Longleftarrow$) Let $\mathcal{U}$ be a filter such that for any continuous isotone map $f: X \longrightarrow [\![-1, 1]\!]$ $f(\mathcal{U})$ is convergent. Let $x_f = \lim f(\mathcal{U})$. Then $f^{-1}(\underline{O}(x_f)) \subseteq \mathcal{U}$, where $\underline{O}(x_f)$ is the neighborhood filter of x_f. Hence $\cup\{f^{-1}(\underline{O}(x_f)) \mid f \in \text{hom}(X, [\![-1, 1]\!])\}$ generates a filter. Let $\mathfrak{F} = \vee f^{-1}(\underline{O}(x_f))$ $(f \in \text{hom}(X, [\![-1, 1]\!]))$. It is easy to show that a join of o-completely regular filters is again o-completely regular and that $f^{-1}(\underline{O}(x_f))$ is an o-completely regular filter.

Hence $\mathcal{F}$ is an o-completely regular filter. Suppose that $\mathcal{F}$ is not a maximal o-completely regular filter. By Lemma 3.2, there exists a pair of open sets U, V with $V \subseteq U$ and there exist continuous isotone maps $f_i : X \longrightarrow [\![-1, 1]\!]$ $(i = 1, \ldots, n)$ such that $f_i(V) = 0$ and $(U \subseteq \bigcup f_i^{-1}(\{-1, 1\})$ and $U \notin \mathcal{F}$ and $F \cap V \neq \emptyset$ for any $F \in \mathcal{F}$. Hence 0 belongs to $f_i(F)$ for all $F \in \mathcal{F}$ and $i = 1, \ldots, n$, so that x_{f_i} must be 0. Thus $\bigcap f_i^{-1}(]\!]-1/2, 1/2[\![) \in \mathcal{F}$. Since $\bigcap f_i^{-1}(]\!]-1/2, 1/2[\![) \subseteq U$, $U \in \mathcal{F}$, which is a contradiction. Hence $\mathcal{F}$ is a maximal o-completely regular filter.

<u>Remark</u> For a maximal o-completely regular filter $\mathcal{F}$ on a completely regular ordered space X and a continuous isotone map $f : X \longrightarrow [\![-1, 1]\!]$, let $x_f = \lim f(\mathcal{F})$. Then $\mathcal{F} = \bigvee \{f^{-1}(\underline{O}(x_f)) \mid f \in \hom(X, [\![-1, 1]\!])\}$.

3.4 <u>Corollary</u> Every neighborhood filter of a completely regular ordered space X is a maximal o-completely regular filter.

<u>Proof</u>: For any $x \in X$, let $\underline{O}(x)$ be the neighborhood filter of x. Obviously for any continuous isotone map $f : X \longrightarrow [\![-1, 1]\!]$, $f(\underline{O}(x))$ is convergent. Hence $\underline{O}(x)$ contains a maximal o-completely regular filter by Theorem 3.3.

It is enough to show that $\underline{O}(x)$ is an o-completely regular filter. For any $U \in \underline{O}(x)$, there exist finitely many continuous isotone maps $f_i : X \longrightarrow [\![-1, 1]\!]$, $i = 1, \ldots, n$, such that $f_i(x) = 0$ $(i = 1, \ldots, n)$ and $\complement U \subseteq \bigcup f_i^{-1}(\{-1, 1\})$. Let $V = \bigcap f_i^{-1}(]\!]-1/2, 1/2[\![)$. Then $V \in \underline{O}(x)$ and $V \subseteq U$. Moreover one can easily find continuous isotone maps $h_i: X \longrightarrow [\![-1, 1]\!]$, $i = 1, \ldots, n$, such that $h_i(V) = 0$ and $\complement U \subseteq \bigcup h_i^{-1}(\{-1, 1\})$. This completes the proof.

3.5 <u>Theorem</u> For a completely regular ordered space X, the following are equivalent:

(1) X is compact.

(2) Every o-completely regular filter has a cluster point.

(3) Every maximal o-completely regular filter is convergent.

<u>Proof:</u> (1) $\Longrightarrow$ (2) Trivial.

(2) $\Longrightarrow$ (3) Let $\mathfrak{F}$ be a maximal o-completely regular filter. By (2), $\bigcap_{F \in \mathfrak{F}} \mathrm{cl}F \neq \emptyset$. Let $x \in \bigcap_{F \in \mathfrak{F}} \mathrm{cl}F$. Clearly $\underline{O}(x) \vee \mathfrak{F}$ exists and both $\underline{O}(x)$ and $\mathfrak{F}$ are maximal o-completely regular filters. Hence $\mathfrak{F} = \underline{O}(x)$, so that $\mathfrak{F}$ converges to x.

(3) $\Longrightarrow$ (1) It is enough to show that every ultrafilter on X is convergent. Let $\mathcal{U}$ be an ultrafilter on X. For any continuous isotone map $f: X \longrightarrow [\![-1, 1]\!]$, $f(\mathcal{U})$ is an ultrafilter base on $[\![-1, 1]\!]$. Since $[\![-1, 1]\!]$ is compact, $f(\mathcal{U})$ is convergent. By Theorem 3.3, $\mathcal{U}$ contains a maximal o-completely regular filter $\mathfrak{F}$. Since $\mathfrak{F}$ is convergent, $\mathcal{U}$ is also convergent.

Using Theorem 1.12 in Chap. 0 and Corollary 2.6, one can easily conclude that the subcategory determined by compact ordered spaces is epireflective in CROS.

For the further development, we will characterize the reflection of a completely regular ordered space X by o-completely regular filters on X.

For a completely regular ordered space $(X, \leq, \underset{\sim}{O})$, let $\beta_o X$ be the set of all maximal o-completely regular filters on X, endowed with the topology $\underset{\sim}{O}^*$ generated by $\{U^* \mid U^* = \{\mathfrak{m} \in \beta_o X,\ U \in \mathfrak{m}\},\ U$ is an open set of $X\}$ and a relation $\leq$ defined as follows: $\mathfrak{m} \leq \mathfrak{n}$ in $\beta_o X$ if and only if $\lim f(\mathfrak{m}) \leq \lim f(\mathfrak{n})$ for all $f \in hom(X, [\![-1, 1]\!])$.

It is obvious that $(\beta_o X, \leq)$ is a partially ordered set and that $\{U^* \mid U \in \underset{\sim}{O}\}$ forms a base for $\underset{\sim}{O}^*$.

Let $\beta_o : X \longrightarrow \beta_o X$ be a map defined by $\beta_o(x) = \underset{\sim}{O}(x)$ for $x \in X$. Clearly β_o is one-one, for X is Hausdorff.

Since $\beta_o^{-1}(U^*) = U$ for an open set U of X, β_o is continuous. Since $\beta_o(U) = U^* \cap \beta_o(X)$ for an open set U of X, β_o is relatively open. Thus β_o is a topological embedding.

Furthermore, $\beta_o(X)$ is dense in $\beta_o X$, because for any non-empty open set U of X, taking $x \in U$, $\beta_o(x) = \underset{\sim}{O}(x) \in U^* \cap \beta_o(X) \neq \phi$.

Since for any $x \in X$ and any $f \in hom(X, [\![-1, 1]\!])$, $\lim f(\underset{\sim}{O}(x)) = f(x)$ and X is a completely regular ordered space, it follows that β_o is an order isomorphism.

Consequently the map $\beta_o : X \longrightarrow \beta_o X$ is a dense iseomorphism.

3.6 <u>Lemma</u> The space $\beta_o X$ is a W-topological partially ordered set.

<u>Proof</u>: For any $\pi \nleq \varpi$ in $\beta_o X$, there is an $f \in \hom(X, [\![-1, 1]\!])$ with $\lim f(\varpi) < \lim f(\pi)$. Let r_1 and r_2 be elements of $[\![-1, 1]\!]$ with $\lim f(\varpi) < r_1 < r_2 < \lim f(\pi)$ and let $U = f^{-1}([\![-1, r_1[\![)$ and $V = f^{-1}(]\!]r_2, 1]\!])$. Then it is obvious that U^* (resp. V^*) is an open neighborhood of ϖ (resp. π). Moreover, for any $\varpi' \in U^*$ and $\pi' \in V^*$, $\lim f(\varpi') \leq r_1$ and $\lim f(\pi') \geq r_2$; $U^* \ngtr V^*$.

Since for any $\varpi \neq \pi$ in $\beta_o X$, either $\varpi \nleq \pi$ or $\pi \nleq \varpi$, hence they have disjoint neighborhoods by the above arguments, so that <u>O</u>* is Hausdorff. Hence $(\beta_o X, \leq, \underline{O}^*)$ is an object of <u>WPOS</u>.

3.7 <u>Definition</u> Let <u>A</u> be a category and C an object of <u>A</u>. An <u>A</u>-morphism $f: A \longrightarrow B$ is said to be <u>C-extendable</u> if for any <u>A</u>-morphism $g: A \longrightarrow C$, there is an <u>A</u>-morphism $\bar{g}: B \longrightarrow C$ with $g = \bar{g}f$.

For a class <u>C</u> of <u>A</u>-objects, a morphism is said to be <u>C-extendable</u> provided that for any $C \in \underline{C}$, it is C-extendable.

3.8 <u>Lemma</u> The morphism $\beta_o : X \longrightarrow \beta_o X$ is $[\![-1, 1]\!]$-extendable in <u>WPOS</u>.

<u>Proof</u>: For any $f: X \longrightarrow [\![-1, 1]\!]$ in <u>WPOS</u> and for any

$\mathfrak{M} \in \beta_o X$, define $\bar{f}(\mathfrak{M}) = \lim f(\mathfrak{M})$. It is obvious that $\bar{f}$ is an isotone map. It is enough to show that $\bar{f}$ is continuous. For any $\mathfrak{M} \in \beta_o X$, let $a = \bar{f}(\mathfrak{M}) = \lim f(\mathfrak{M})$ and let U be an open neighborhood of a. Since $[\![-1, 1]\!]$ is regular, there is an open neighborhood V of a with $\mathrm{cl}V \subseteq U$. Then $(f^{-1}(V))^*$ is an open neighborhood of $\mathfrak{M}$ such that $\bar{f}((f^{-1}(V))^*) \subseteq U$, because for any $\mathfrak{N} \in (f^{-1}(V))^*$, $f^{-1}(V) \in \mathfrak{N}$; $\bar{f}(\mathfrak{N}) = \lim f(\mathfrak{N}) \in \mathrm{cl}V \subseteq U$. This completes the proof.

3.9 <u>Theorem</u> The space $\beta_o X$ is a compact ordered space.

<u>Proof</u>: Firstly we will show that $\beta_o X$ is a completely regular ordered space. For any $\mathfrak{M} \nleq \mathfrak{N}$ in $\beta_o X$, there is an $f \in \mathrm{hom}(X, [\![-1, 1]\!])$ such that $\lim f(\mathfrak{N}) < \lim f(\mathfrak{M})$. Let $\bar{f}: \beta_o X \longrightarrow [\![-1, 1]\!]$ be the extension of f determined by Lemma 3.8. Then $\bar{f}(\mathfrak{N}) < \bar{f}(\mathfrak{M})$ and hence $\beta_o X$ is completely separated. Let U^* be a basic open neighborhood of $\mathfrak{M} \in \beta_o X$. Since $U \in \mathfrak{M}$, there is an open set $V \in \mathfrak{M}$ with $V \subseteq U$ and there are continuous isotone maps $f_1, \ldots, f_n: X \longrightarrow [\![-1, 1]\!]$ such that $f_i(V) = 0$ for each i and $\complement U \subseteq \bigcup\{f_i^{-1}(\{-1, 1\}) \mid 1 \leq i \leq n\}$. Let $\bar{f}_i: \beta_o X \longrightarrow [\![-1, 1]\!]$ be the extension of f_i for each i. Then $\bar{f}_i(\mathfrak{M}) = \lim f_i(\mathfrak{M}) = 0$. Furthermore, it is easy to show that $\complement U^* \subseteq \bigcup\{\bar{f}_i^{-1}(\{-1, 1\}) \mid 1 \leq i \leq n\}$. Indeed, for any $\mathfrak{N} \notin \bigcup \bar{f}_i^{-1}(\{-1, 1\})$, $\bar{f}_i(\mathfrak{N}) = \lim f_i(\mathfrak{N})$ is neither -1 nor 1 for each i. For each i = 1, ..., n, let a_i and b_i be elements of $[\![-1, 1]\!]$ with $-1 < a_i < \lim f(\mathfrak{N}) <$

$b_i < 1$. Then $f_i^{-1}(]a_i, b_i[) \in \Pi$ for each i;
$\bigcap \{f_i^{-1}(]a_i, b_i[) \mid 1 \le i \le n\} \in \Pi$. Since $\complement U$ is contained in $\bigcup\{f_i^{-1}(\{-1, 1\}) \mid 1 \le i \le n\}$, U contains $\bigcap f_i^{-1}(]a_i, b_i[)$. Hence $U \in \Pi$; $\Pi \in U^*$.

By Theorem 3.5, it is enough to show that every maximal o-completely regular filter on $\beta_o X$ is convergent. Let $\mathfrak{F}$ be a maximal o-completely regular filter on $\beta_o X$ and let $\mathcal{B} = \{U \mid U$: open in X and $U^* \in \mathfrak{F}\}$. Using the fact that β_o is a $[\![-1, 1]\!]$-extendable extension, we can conclude that $\mathcal{B}$ generates a maximal o-completely regular filter, say $\mathfrak{M}$ and that $\mathfrak{F}$ converges to $\mathfrak{M}$. This completes the proof.

In what follows, we will identify any point $x \in X$ with its neighborhood filter $\underline{O}(x)$, so that X is a dense subspace of $\beta_o X$ and β_o is the natural embedding.

3.10 <u>Remark</u> The trace filter of a point $\mathfrak{M} \in \beta_o X$ on X generates $\mathfrak{M}$ itself. Moreover, $\beta_o X$ is the strict extension (see [1]) of X with all maximal o-completely regular filters on X as the filter trace.

3.11 <u>Lemma</u> Let $h: X \longrightarrow T$ be a dense morphism in <u>WPOS</u>. Then the following are equivalent:

(1) h is $[\![-1, 1]\!]$-extendable.

(2) h is <u>COMPOS</u>-extendable, where <u>COMPOS</u> is the category of compact ordered spaces and continuous isotone maps.

<u>Proof</u>: (2) $\Longrightarrow$ (1) Trivial.

(1) $\Longrightarrow$ (2) For a $K \in$ COMPOS, let $e: K \longrightarrow [\![-1, 1]\!]^{\hom(K, [\![-1, 1]\!])}$ be the unique morphism defined by $pr_f e = f$ for each $f \in \hom(K, [\![-1, 1]\!])$, where pr_f is the f-th projection. Then e is a closed embedding. For any $g \in \hom(X, K)$, and for any $f \in \hom(K, [\![-1, 1]\!])$, there is a morphism $\bar{f}: T \longrightarrow [\![-1, 1]\!]$ such that $\bar{f}h = pr_f eg$. Then there is a unique morphism $\bar{\bar{f}}: T \longrightarrow [\![-1, 1]\!]^{\hom(K, [\![-1, 1]\!])}$ such that the right triangle in the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\ h\ } & T & & \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \bar{\bar{f}}} & \searrow^{\bar{f}} & \\ K & \xrightarrow[\ e\ ]{} & [\![-1, 1]\!]^{\hom(K, [\![-1, 1]\!])} & \xrightarrow[pr_f]{} & [\![-1, 1]\!], \end{array}$$

commutes. Since $\{pr_f \mid f \in \hom(K, [\![-1, 1]\!])\}$ is a mono-source, the left rectangle in the above diagram commutes. Hence $\bar{\bar{f}}(T) = \bar{\bar{f}}(\mathrm{cl}\,h(X)) \subseteq \mathrm{cl}(\bar{\bar{f}}h(X)) \subseteq K$. Let $\bar{g}$ be the corestriction of $\bar{\bar{f}}$ to K, then $\bar{g}$ is a WPOS-morphism and $\bar{g}h = g$.

Combining Lemma 3.8, Theorem 3.9 and Lemma 3.11, we immediately have the following.

3.12 Theorem For any completely regular ordered space X, $\beta_o X$ is a $[\![-1, 1]\!]$-extendable compactification of X. Furthermore, COMPOS is an extensive subcategory of CROS.

3.13 Remark We define a zero-dimensional ordered space $(X, \leq, \underline{O})$ as a W-topological partially ordered set satisfying the following two conditions:

(1) For every point x of X and an open neighborhood V of x, there exist finitely many continuous isotone maps, $f_1, \ldots, f_n : X \longrightarrow \underline{3}$ such that $f_i(x) = 0$ for each i and $V \subseteq \bigcup\{f_i^{-1}(\{-1, 1\}) \mid 1 \leq i \leq n\}$, where $\underline{3}$ is the chain of three elements $\{-1, 0, 1\}$ with the usual order and the discrete topology.

(2) For $x \nleq y$ in X, there exists a continuous isotone map $f: X \longrightarrow \underline{3}$ with $f(y) < f(x)$.

Let ZOS denote the category of zero-dimensional ordered spaces and continuous isotone maps. Then ZOS is an epireflective subcategory of WPOS.

Let ZCOMPOS denote the category of compact zero-dimensional ordered spaces and continuous isotone maps.

Substituting $[-1, 1]$ with $\underline{3}$ in the above arguments, one can conclude that ZCOMPOS is an extensive subcategory of ZOS.

Section 4: k-compact ordered spaces.

For a W-topological partially ordered set $(X, \leq, \underline{O})$ and a subset $A \subseteq X$, the space A endowed with the induced order of $\leq$ on A and the relative topology of $\underline{O}$ on A is obviously a W-topological partially ordered set.

In the following, we assume that every subcategory of a category is full and isomorphism closed.

Let $\underline{A}$ be a category of W-topological partially ordered sets and continuous isotone maps and $\underline{B}$ an extensive subcategory of $\underline{A}$. For any idempotent limit operator ℓ, let $\underline{B}_\ell$ be the subcategory of $\underline{A}$ determined by those objects of $\underline{A}$ which are ℓ-closed in their $\underline{B}$-reflection spaces.

Using the exactly same argument as that in Theorem 1.4 in Chap. II, we have the following:

4.1 Theorem If $\underline{A}$ is hereditary, then $\underline{B}_\ell$ is again an extensive subcategory of $\underline{A}$.

4.2 Definition Let k be an infinite cardinal. A completely regular ordered space X is said to be k-compact if every maximal o-completely regular filter on X with the k-intersection property is convergent.

4.3 Lemma A completely regular ordered space X is k-compact if and only if it is k-closed in $\beta_o X$.

Proof: It is immediate from Definition 4.2, Proposition 2.13 in Chap. II and Remark 3.10.

For an infinite cardinal k, the category of k-compact ordered spaces and continuous isotone maps will be denoted by kCOS.

Using Lemma 4.3 and the fact that COMPOS is extensive in the hereditary category CROS, kCOS = COMPOS_{k}. Hence we have the following:

4.4 Theorem The category kCOS is extensive in CROS.

4.5 Corollary The category kCOS is complete.

Using the same argument as that in Proposition 1.9 in [25], we have the following:

4.6 Proposition A k-closed subspace of a k-compact ordered space is again a k-compact ordered space.

4.7 Definition Let k be an infinite cardinal. A Hausdorff space is said to be k-Lindelöf if every filter with the k-intersection property has a cluster point.

We note that $\aleph_0$-Lindelöf spaces are exactly compact spaces and that $\aleph_1$-Lindelöf spaces are exactly Lindelöf spaces.

4.8 Proposition Every k-Lindelöf completely regular ordered space is a k-compact ordered space.

Proof: Let X be a k-Lindelöf completely regular ordered space. For any maximal o-completely regular filter $\mathfrak{F}$ on X with the k-intersection property, let x be a cluster point of $\mathfrak{F}$. Then $\underline{O}(x) \vee \mathfrak{F}$ exists; $\underline{O}(x) = \mathfrak{F}$. Hence $\mathfrak{F}$ is convergent.

4.9 <u>Theorem</u> If a filter $\mathfrak{F}$ on a completely regular ordered space X contains a maximal o-completely regular filter with the countable intersection property, then $f(\mathfrak{F})$ is convergent for any continuous isotone map $f: X \longrightarrow R$.

<u>Proof</u>: It is enough to show that for any maximal o-completely regular filter $\mathfrak{F}$ with the countable intersection property and a continuous isotone $f: X \longrightarrow R$, $f(\mathfrak{F})$ is convergent. Since $f(\mathfrak{F})$ is a filter base with the countable intersection property and R is Lindelöf, $f(\mathfrak{F})$ has a cluster point. Moreover, by the same argument as that in the proof of Theorem 3.3, one can easily show that $f(\mathfrak{F})$ has only one cluster point, say a, and that $f(\mathfrak{F})$ converges to a.

<u>Remark</u> It is known [26] that for a filter $\mathfrak{F}$ on a completely regular space X, and for any continuous map $f: X \longrightarrow R$, $f(\mathfrak{F})$ is convergent if and only if $\mathfrak{F}$ contains a maximal completely regular filter on X with the countable intersection property. However the converse of the above theorem need not be true (see Remark 4.14).

4.10 <u>Definition</u> For a $P \in$ <u>TPOS</u> (resp. <u>WPOS</u>), an object A in TPOS (resp. WPOS) is said to be P-compact if A is isomorphic with a closed subspace of a power of P.

For a WPOS-object P, the subcategory of WPOS determined by all P-compact objects will be denoted by OP-COMP.

Note that compact ordered spaces are exactly $[-1, 1]$-compact ordered spaces and that OP-COMP is productive and closed hereditary.

4.11 Theorem Every R-compact ordered space is an $\aleph_1$-compact ordered space.

Proof: Since the real line R is Lindelöf, R is an $\aleph_1$-compact ordered space. Since $\aleph_1$COS is productive and closed hereditary, OR-COMP $\subseteq \aleph_1$COS.

It is well known [31] that R-compact topolological spaces are exactly $\aleph_1$-compact topological spaces. But the converse of Theorem 4.11 does not hold (see Remark 4. 14).

4.12 Definition A completely regular ordered space X is said to be o-pseudocompact if f(X) is bounded for each continuous isotone map $f: X \longrightarrow R$.

Remark A completely regular ordered space X is compact if and only if it is an o-pseudocompact, R-compact ordered space.

4.13 Example Let $E = \{-\infty\} \cup R \cup \{\infty\}$ endowed with the usual order and a topology generated by $\{\{-\infty\}\} \cup \{\{\infty\}\} \cup \{\underline{O}(R)\}$, where $\underline{O}(R)$ is the usual topology on R. Then E is not an R-compact ordered space, but an $\aleph_1$-compact ordered space.

Proof: It is obvious that E is a completely regular ordered space. Since E is not compact but o-pseudocompact, by the above remark, X is not R-compact. We note that E is not a pseudocompact topological space.

We can easily show that every maximal o-completely regular filter on R is a maximal o-completely regular filter base on E, and conversely, if $\mathfrak{M}$ is a maximal o-completely regular filter on E then either $\mathfrak{M} = \underline{O}(\infty)$ or $\mathfrak{M} = \underline{O}(-\infty)$ or $\mathfrak{M}|R = \{M \mid M \in \mathfrak{M} \text{ and } M \subset R\}$ is a maximal o-completely regular filter on R. Now we will show that E is $\aleph_1$-compact. Suppose it is not, then there is a free maximal o-completely regular filter $\mathfrak{M}$ with the countable intersection property. Clearly $\mathfrak{M} \neq \underline{O}(-\infty)$ and $\mathfrak{M} \neq \underline{O}(\infty)$. Hence the trace $\mathfrak{M}|R$ of $\mathfrak{M}$ on R is a maximal o-completely regular filter on R. Obviously $\mathfrak{M}|R$ has the countable intersection property; $\mathfrak{M}|R$ must converge to a point $r \in R$. Hence $\mathfrak{M}$ converges to r, which is a contradiction.

4.14 Remark The above example shows that

(1) OR-COMP is a proper subcategory of $\aleph_1$COS, and

(2) the converse of Theorem 4.9 does not hold.

By the exactly same argument as that in Theorem 1.2

in Chap. II, we have the following.

4.15 <u>Remark</u> Let $\underline{A}$ be a category of topological partial algebras of fixed type and continuous homomorphisms, and $\underline{B}$ an extensive subcategory of $\underline{A}$. If a reflective subcategory $\underline{C}$ of $\underline{A}$ contains $\underline{B}$, then $\underline{C}$ is also extensive in $\underline{A}$ and for each $X \in \underline{A}$, the $\underline{B}$-reflection $r_{\underline{B}}: X \longrightarrow r_{\underline{B}}X$ of X has a factorization $X \xrightarrow{r_{\underline{B}}} r_{\underline{B}}X = X \xrightarrow{r_{\underline{C}}} r_{\underline{C}}X \xrightarrow{r} r_{\underline{B}}X$, where $r_{\underline{C}}$ is the $\underline{C}$-reflection of X and r is the $\underline{B}$-reflection of $r_{\underline{C}}X$.

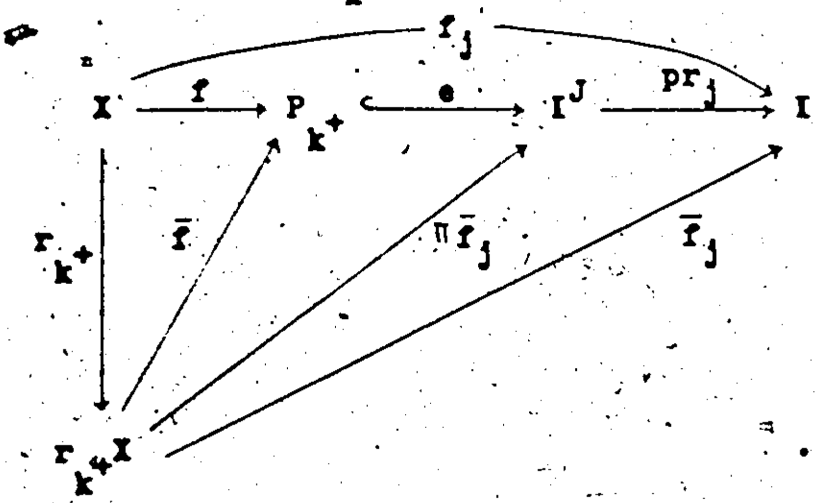
4.16 <u>Lemma</u> For a <u>WPOS</u>-object P, the category <u>OP-COMP</u> is a reflective subcategory of <u>WPOS</u>.

<u>Proof</u>: Since a singleton space belongs to <u>OP-COMP</u>, monomorphisms in <u>OP-COMP</u> are exactly one-one morphisms. Hence <u>OP-COMP</u> is well-powered. Obviously P is a coseparator for <u>OP-COMP</u> and <u>OP-COMP</u> is complete. Hence it follows from Theorem 1.12 in Chap. 0 that <u>OP-COMP</u> is a reflective subcategory of <u>WPOS</u>.

4.17 <u>Theorem</u> Let k be an infinite cardinal. Let $P_{k^+} = I^k - \{(0)\}$, where k^+ is the successor of k, I denotes the interval $[-1, 1]$, (0) is the element of I^k whose every coordinate is 0, and P_{k^+} is a subspace of I^k. Then $\underline{k^+COS} = \underline{OP_{k^+}\text{-}COMP}$, i.e., a completely regular ordered space is k^+-compact if and only if it is iseomorphic with a closed subspace of a power of P_{k^+}.

Proof: ($\Longleftarrow$) Since I satisfies the first axiom of countability, the neighborhood filter $\underline{O}((0))$ of (0) has a base of cardinal less than k^+. Since I^k is a Hausdorff space, $\{(0)\}$ is itself a G_{k^+}-set. Thus P_{k^+} is a k^+-closed subspace of I^k. Clearly I^k is k^+-compact and hence P_{k^+} is also k^+-compact, so that every P_{k^+}-compact ordered space is k^+-compact.

($\Longrightarrow$) Suppose that X is a k^+-compact ordered space which is not P_{k^+}-compact. Let $r_{k^+}: X \longrightarrow r_{k^+}X$ be the $\underline{OP_{k^+}\text{-COMP}}$-reflection of X. Since $\underline{COMPOS} \subseteq \underline{OP_{k^+}\text{-COMP}} \subseteq \underline{CROS}$, by Remark 4.15, $X \xrightarrow{\beta_o} \beta_o X = X \xrightarrow{r_{k^+}} r_{k^+}X \xrightarrow{r} \beta_o X$ and r is an extension. Thus we may assume that $r_{k^+}X$ is a subspace of $\beta_o X$. Since X is not P_{k^+}-compact, there exists a point $x \in r_{k^+}X - X \subseteq \beta_o X - X$. Let T(x) be the trace filter of x on X, then T(x) is a maximal o-completely regular filter. Since X is k^+-compact, there exists a family $(U_i)_{i \in I} \subseteq T(x)$ such that $\cap U_i = \phi$ and $\overline{\overline{I}} = k$. By the maximal o-complete regularity of T(x), there is $V_i \in T(x)$ with $V_i \subseteq U_i (i \in I)$ and there are continuous isotone maps $f^i_m: X \longrightarrow I$, $m = 1, \ldots, n_i$, such that $f^i_m(V_i) = 0$ $(m = 1, \ldots, n_i)$ and $\complement U_i$ is contained in $\cup \{(f^i_m)^{-1}(\{-1, 1\}) \mid 1 \leq m \leq n_i\}$ for each $i \in I$. Let $(f^i_m)^{-1}(0) = Z^i_m \supseteq V_i$, then $V_i \subseteq \cap \{Z^i_m \mid 1 \leq m \leq n_i\} \subseteq U_i$. Hence if we rearrange $\{f^i_m \mid 1 \leq m \leq n_i, i \in I\}$, there is a family $\{f_j: X \longrightarrow I \mid j \in J\}$ of continuous isotone maps such that $\bar{J} = k$, $f_j^{-1}(0) = Z_j \in T(x)$ and $\bigcap\{Z_j \mid j \in J\} = \emptyset$. Let $f: X \longrightarrow I^J$ be a unique continuous isotone map such that $pr_j f = f_j$ for each $j \in J$. Since for any $y \in X$ there is some $j \in J$ with $y \notin Z_j$, $f(X) \subseteq P_{k^+}$. We will denote again the corestriction of f to P_{k^+} by f.



There exists a unique continuous isotone map $\bar{f}: r_{k^+}X \longrightarrow P_{k^+}$ with $\bar{f}r_{k^+} = f$ and there is a unique continuous isotone map $\bar{f}_j: r_{k^+}X \longrightarrow I$ with $\bar{f}_j r_{k^+} = f_j$ for each $j \in J$, and hence there is a unique continuous isotone map $\Pi\bar{f}_j: r_{k^+}X \longrightarrow I^J$ with $pr_j(\Pi\bar{f}_j) = \bar{f}_j$. Since $pr_j(\Pi\bar{f}_j)r_{k^+} = \bar{f}_j r_{k^+} = f_j = pr_j ef = pr_j e\bar{f}r_{k^+}$, where $e: P_{k^+} \longrightarrow I^J$ is the natural embedding, $\Pi\bar{f}_j = e\bar{f}$. It is obvious that $x \in cl_{\beta_0 X} Z_j$ for each j. Hence $x \in cl_{r_{k^+}X} Z_j$ for each j; $\bar{f}_j(x) = 0$ for each $j \in J$, i.e., $\Pi\bar{f}_j(x) = (0)$. Hence $e\bar{f}(x) = (0)$, i.e., $\bar{f}(x) \notin P_{k^+}$, which is a contradiction. This completes the proof.

4.18 Theorem Let k be a limit cardinal with $k > \aleph_0$ and let $P_k = \prod_{t<k} P_{t^+}$. Then kCOS = OP_k-COMP.

Proof: Since tCOS $\subseteq$ t'COS for $t < t'$ and kCOS is productive, P_k is k-compact. Thus OP_k-COMP $\subseteq$ kCOS.

Let X be a k-compact ordered space and $r_k: X \longrightarrow r_kX$ be the OP_k-COMP-reflection of X. Note that by Remark 4.15 and Lemma 4.16, r_k is an embedding and r_kX is a subspace of β_oX. Suppose that there is an $x \in r_kX - X \subseteq \beta_oX - X$. Let T(x) be the trace filter of x on X. Then T(x) is a free maximal o-completely regular filter and hence T(x) does not have the k-intersection property. By the same argument as that in Theorem 4.17, there is a family $\{f_i: X \longrightarrow I \mid i \in I\}$ of continuous isotone maps such that $\bar{\bar{I}} < k$, $f_i^{-1}(0) = Z_i \in T(x)$ and $\cap Z_i = \phi$. Let $\bar{\bar{I}} = t < k$, then there exists a continuous isotone map $f: X \longrightarrow P_{t^+}$ with $pr_i f = f_i$. Since P_{t^+} is P_k-compact, there exists a unique continuous isotone map $\bar{f}: r_kX \longrightarrow P_{t^+}$ with $f = \bar{f}r_k$. However $\bar{f}(x) = (0) \notin P_{t^+}$, which is a contradiction.

Remark Every P_k is o-pseudocompact. Hence $P_{\aleph_1}$ is an $\aleph_1$-compact ordered space but not an R-compact ordered space.

CHAPTER IV

TOPOLOGICAL LATTICES

Section 1: The category of topological lattices and continuous homomorphisms

1.1 Definition A topological lattice is a topological algebra $(L, \vee, \wedge, \underline{O})$ where $(L, \vee, \wedge)$ is a lattice and $(L, \underline{O})$ is a Hausdorff space.

We will denote the category of topological lattices and continuous homomorphisms by TLAT.

1.2 Theorem For the category TLAT, the underlying set functor $U: \underline{TLAT} \longrightarrow \underline{Set}$ is topologically algebraic.

Proof: Actually $\underline{TLAT} = \underline{TA}$, where $\underline{T} = \underline{Haus}$ and $\underline{A}$ is the category of lattices and homomorphisms, which is an equational class. By Corollary 3.15 in Chap. I, U is topologically algebraic.

1.3 Corollary TLAT is complete and cocomplete and the underlying set functor $U: \underline{TLAT} \longrightarrow \underline{Set}$ has a left adjoint.

1.4 Lemma Every algebra topology on a lattice is a

W-topology on the partially ordered set induced by the lattice.

Proof: Suppose that $x \not\leq y$ in a topological lattice L, i.e., $x \wedge y \neq x$. Then there are open neighborhoods U and V of $x \wedge y$ and x respectively such that $U \cap V = \phi$. By the continuity of $\wedge$, there are open neighborhoods U_1 and V_1 of x and y respectively such that $U_1 \wedge V_1 \subseteq U$. Let $W = U_1 \cap V$. Then W is an open neighborhood of x and $W \not\leq V_1$. This completes the proof.

The following lemma is well known (see [8]).

1.5 Lemma Let $\tau = (\lambda_i)_{i \in I}$ be a finitary type of algebras with $\overline{\overline{I}} \leq \aleph_0$ and α a cardianl.

If $A = (X, (f_i)_I)$ is an algebra of type τ and $M \subseteq X$ with $\overline{\overline{M}} < \alpha$ then $\overline{\overline{S_A M}} \leq \aleph_0 + \alpha$, where $S_A M$ is the subalgebra of A generated by M.

1.6 Theorem TLAT is a reflective subcategory of WPOS.

Proof: Obviously, every TLAT-morphism is a WPOS-morphim. Let E: TLAT $\longrightarrow$ WPOS be the embedding functor. Note that E is not full. Obviously E preserves limits. For any WPOS-object $(P, \leq, \underline{O})$, let $\mathcal{S}_0$ be the family of topological lattices whose cardinals are not greater than $\overline{\overline{P}} + \aleph_0$, then clearly $\mathcal{S}_0$ is a set. Let $\mathcal{S} = \{(u_i, A_i) \mid A_i \in \mathcal{S}_0, u_i: P \longrightarrow EA_i$ is a WPOS-morphism, $i \in I\}$. Then $\mathcal{S}$ is again a set. Let L be a topological lattice and $f: P \longrightarrow EL$ be a

WPOS-morphism. Let L_1 be the sublattice of L generated by f(P). By Lemma 1.5, $\overline{\overline{L_1}} \leq \overline{\overline{P}} + \aleph_0$, hence $(g: P \longrightarrow EL_1, L_1)$ belongs to $\mathcal{J}$, where g is the corestriction of f to EL_1. Let $e: L_1 \longrightarrow L$ be the natural embedding, then obviously $(Ee)g = f$. Hence $\mathcal{J}$ is an E-solution set for $(P, \leq, \underset{\sim}{0})$. By Theorem 1.11 in Chap. 0, E has a left adjoint, i.e., TLAT is reflective in WPOS.

1.7 Definition A subset A of a partially ordered set $(X, \leq)$ is called convex provided that whenever $x \leq z \leq y$ and $x, y \in A$, then $z \in A$.

A topological lattice is said to be locally convex if its topology has a base consisting of convex sets.

1.8 Theorem The category LCTLAT of locally convex topological lattices and continuous homomorphisms is a bireflective subcategory of TLAT.

Proof: Let $(L, \vee, \wedge, \underset{\sim}{0})$ be a topological lattice. For any $A \subseteq L$, let $C(A) = (L \vee A) \cap (L \wedge A)$. Then C(A) is convex for any $A \subseteq L$. It is easy to show that whenever U is open, C(U) is again open. Let $\underset{\sim}{0}^c$ be the topology on L generated by $\{C(U) \mid U \in \underset{\sim}{0}\}$. We wish to show that $\underset{\sim}{0}^c$ is a Hausdorff algebra topology on $(L, \vee, \wedge)$. Indeed, for $x \neq y$ in L, there are open neighborhoods G and H of x and y respectively such that $G = G \vee L$, $H = H \wedge L$ and $G \cap H = \phi$. Hence $C(G) \cap C(H) = \phi$, i.e., $(L, \underset{\sim}{0}^c)$ is a Hausdorff space.

Since $U \vee V \subseteq W$ (resp. $U \wedge V \subseteq W$) implies that $C(U) \vee C(V) \subseteq C(U \vee V) \subseteq C(W)$ (resp. $C(U) \wedge C(V) \subseteq C(U \wedge V) \subseteq C(W)$), $\underset{\sim}{O}^c$ is an algebra topology on L.

Since an intersection of convex sets is again convex, $(L, \vee, \wedge, \underline{O}^c)$ is locally convex.

Let $r: (L, \vee, \wedge, \underline{O}) \longrightarrow (L, \vee, \wedge, \underline{O}^c)$ be the map defined by the identity of L. Obviously r is a morphism of TLAT. In order to show that r is the desired reflection map, it is enough to show that for any $(L', \vee, \wedge, \underline{O}') \in$ LCTLAT and any $f: (L, \vee, \wedge, \underline{O}) \longrightarrow (L', \vee, \wedge, \underline{O}')$ in TLAT, the same map $f: (L, \vee, \wedge, \underline{O}^c) \longrightarrow (L', \vee, \wedge, \underline{O}')$ is continuous.

For any $x \in L$ and a convex open neighborhood V of f(x), there is an open neighborhood $U \in \underline{O}(x)$ such that f(U) is contained in V. Since f is a lattice homomorphism, $f(C(U)) \subseteq V$. This completes the proof.

1.9 Corollary The category LCTLAT is complete and cocomplete.

The following corollary is well known ([32], [35]). However, it is immediate from Theorem 1.8.

1.10 Corollary Any compact topological lattice is locally convex.

Proof: It is immediate from the fact that whenever a one-one map from a compact space onto a Hausdorff space is continuous, it is a homeomorphism.

Section 2: Completely regular topological lattices.

2.1 Definition A topological lattice L is said to be completely regular if for any point x of L and an open neighborhood V of x, there exist finitely many continuous homomorphisms $f_1, \ldots, f_n: L \longrightarrow [-1, 1]$ such that $f_i(x) = 0$ for each $i = 1, \ldots, n$ and $L \setminus V \subseteq \bigcup\{f_i^{-1}(\{-1, 1\}) \mid 1 \leq i \leq n\}$.

Remark We note that in the above definition we don't assume that topological lattice is completely separated (compare Definition 2.2 in Chap. III).

The category of completely regular topological lattices and continuous homomorphisms will be denoted by CRTL.

2.2 Definition Let L be a topological lattice. A topological lattice is said to be L-regular (resp. L-compact) if it is iseomorphic with a (resp. closed) sublattice of a power of L.

Notation For topological lattices L and L', we will denote the set of TLAT-morphisms from L to L' by Hom(L, L') which is differentiated from hom(X, X') for the WPOS-morphism set from X to X'.

2.3 Proposition For a topological lattice L, the following are equivalent:

(1) L is a completely regular topological lattice.

(2) $\{f^{-1}(0) \mid f \in \mathrm{Hom}(L, R)\}$ forms a subbase for closed

sets.

(3) L is an R-regular topological lattice.

(4) L is an I-regular topological lattice.

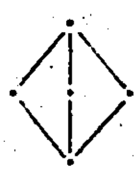
Proof: Proof is similar to that of Proposition 2.3 in Chap. III, except the case of (2) ⟹ (3). In fact, the parametric map $\varphi : L \longrightarrow R^{\mathrm{Hom}(L,\ R)}$ defined by $pr_f \varphi = f$ for each $f \in \mathrm{Hom}(L, R)$, is one-one homomorphism. Hence without using the complete separatedness of the space, we can conclude that φ is a lattice isomorphism onto $\varphi(L)$.

2.4 Corollary The category CRTL is complete. Moreover CRTL is an epireflective subcategory of TLAT.

2.5 Corollary Every completely regular topological lattice is distributive.

2.6 Corollary Every completely regular topological lattice is iseomorphic with a sublattice of a compact topological lattice. But the converse does not hold.

Proof: The first part of this assertion being obvious, it needs only show the second part. Let L be the lattice



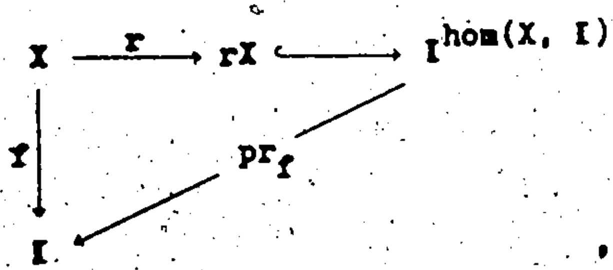
endowed with the discrete topology. Clearly L is a compact topological lattice. Since L is not distributive, L is not completely regular.

2.7 Corollary A topological lattice is compact completely regular if and only if it is I-compact.

2.8 <u>Proposition</u> <u>CRTL</u> is a (non-full) monoreflective subcategory of <u>CROS</u>.

<u>Proof</u>: For any $X \in$ <u>CROS</u>, let $\varphi: X \longrightarrow I^{\hom(X,\, I)}$ be the embedding with $pr_f \varphi = f$ for each $f \in \hom(X, I)$. Let rX be the sublattice of $I^{\hom(X,\, I)}$ generated by $\varphi(X)$. We wish to show that the map $r: X \longrightarrow rX$ defined by $r(x) = \varphi(x)$, is the desired reflection of X.

Firstly we will show that for any continuous isotone map $f: X \longrightarrow I$, there exists a continuous lattice homomorphism $\bar{f}: rX \longrightarrow I$ such that $\bar{f}r = f$. Since the diagram



commutes and clearly pr_f is a lattice homomorphism, if we let $\bar{f}$ ~~be the restriction~~ of pr_f to rX then $\bar{f}$ is a continuous lattice homomorphism and $\bar{f}r = f$.

Secondly we will show that for any $Y \in$ <u>CRTL</u> and any $g: X \longrightarrow Y$ in <u>CROS</u>, there is a unique morphism $\bar{g}: rX \longrightarrow Y$ in <u>CRTL</u> with $\bar{g}r = g$. For any $h \in \mathrm{Hom}(Y, I)$, there exists a continuous lattice homomorphism $\bar{h}: rX \longrightarrow I$ with $pr_h eg = \bar{h}r$, where $e: Y \longrightarrow I^{\mathrm{Hom}(Y,\, I)}$ is the embedding with $pr_h e = h$ for each $h \in \mathrm{Hom}(Y, I)$. Thus there is a continuous lattice homomorphism $k: rX \longrightarrow I^{Hom(Y, I)}$ such that $pr_h k = \bar{h}$ for each $h \in Hom(Y, I)$. Since $k(rX) = k(S_{rX}r(X)) = Skr(X) = Seg(X) \subseteq Y$, where SA (resp. $S_{rX}A$) denotes the sublattice of $I^{Hom(Y, I)}$ (resp. rX) generated by A, the map $\bar{g}: rX \longrightarrow Y$ which is the corestriction of k to Y, is a continuous homomorphism, and $\bar{g}r = g$. It follows from the fact that $r(X)$ generates rX that a <u>TLAT</u>-morphism $\bar{g}$ with $\bar{g}r = g$ is unique. It is obvious that r is a monomorphism. Hence <u>CRTL</u> is monoreflective in <u>CROS</u>.

The category of I-compact topological lattices and continuous homomorphisms will be denoted by <u>I-COMPTL</u>.

2.9 <u>Corollary</u> <u>I-COMPTL</u> is a (non-full) monoreflective subcategory of <u>CROS</u>.

2.10 <u>Lemma</u> Let $(Y_i)_{i \in I}$ be a family of topological lattices whose topologies have bases of open sublattices. For a lattice X, let $f_i: X \longrightarrow Y_i$ be a homomorphism for each $i \in I$, and let $\underline{O}$ be the initial topology on X with respect to $(f_i)_I$. Then X with $\underline{O}$ is a topological lattice which has again a base of open sublattices.

<u>Proof</u>: It follows immediately from the fact that an inverse image of a sublattice under a homomorphism is again a sublattice.

2.11 <u>Lemma</u> Every completely regular topological lattice has a base of open sublattices.

Proof: It follows immediately from Proposition 2.3 and the fact that I has a base of open sublattices.

2.12 Definition A topology on a lattice L is said to be the interval topology if it is generated by taking $\{U(a) = \{x \in L \mid a \leq x\} \mid a \in L\} \cup \{L(a) = \{x \in L \mid x \leq a\} \mid a \in L\}$ as a subbase for the closed sets.

2.13 Remark It is well known that a lattice is complete if and only if its interval topology is compact.

Since an algebra topology on a lattice is finer than its interval topology, every compact topological lattice is complete.

2.14 Theorem For a compact topological lattice $(L, \underline{O})$, let $\mathcal{J}$ be the interval topology of L. Then the following are equivalent:

(1) L is completely regular and hence I-compact.

(2) L is complete, completely distributive and has a base of open sublattices.

(3) L is distributive and $\underline{O} = \mathcal{J}$.

Proof: It follows immediately from Lemma 2.11, Remark 2.13, and Theorem 5, Theorem 6 in [33].

2.15 Definition A filter $\mathcal{F}$ on a completely regular topological lattice L is said to be an l-completely regular

<u>filter</u> if $\mathfrak{F}$ contains an open base $\mathcal{B}$ satisfying that for each $U \in \mathcal{B}$, there exists a $V \in \mathcal{B}$ with $V \subsetneq U$ and there are finitely many continuous homomorphisms $f_1, \ldots, f_n$: $L \longrightarrow [-1, 1]$ such that $f_i(V) = 0$ $(i = 1, \ldots, n)$ and $\complement U \subseteq \bigcup \{f_i^{-1}(\{-1, 1\}) \mid 1 \leq i \leq n\}$.

By a <u>maximal l-completely regular filter</u> on L is meant an l-completely regular filter not contained in any other l-completely regular filter.

<u>Remark</u>: For any l-completely regular filter $\mathfrak{F}$, there is, by Zorn's Lemma, a maximal l-completely regular filter containing $\mathfrak{F}$.

In order to characterize the I-compactifications of completely regular topological lattices, we will need the following four properties of l-completely regular filters. Using the same arguments as those in Section 3, in Chap. III, one can easily prove them. We will omit the proof.

2.16 <u>Lemma</u> Let $\mathfrak{F}$ be an l-completely regular filter on a completely regular topological lattice L. Then $\mathfrak{F}$ is a maximal l-completely regular filter if and only if for any pair of open sets U and V with $V \subsetneq U$ and finitely many continuous homomorphisms $f_1, \ldots, f_n$: $L \longrightarrow [-1, 1]$ such that $f_i(V) = 0$ $(i = 1, \ldots, n)$ and $\complement U \subseteq \bigcup \{f_i^{-1}(\{-1, 1\}) \mid 1 \leq i \leq n\}$, either $U \in \mathfrak{F}$ or $U \notin \mathfrak{F}$ and there exists some $F \in \mathfrak{F}$ with $F \cap V = \emptyset$.

2.17 <u>Theorem</u> A filter $\mathcal{U}$ on a completely regular topological lattice L contains a maximal l-completely regular filter if and only if $f(\mathcal{U})$ is convergent for each continuous homomorphism $f: L \longrightarrow [-1, 1]$.

2.18 <u>Corollary</u> Every neighborhood filter of a completely regular topological lattice is a maximal l-completely regular filter.

2.19 <u>Theorem</u> For a completely regular topological lattice L, the following are equivalent:

(1) L is compact and hence I-compact.

(2) Every l-completely regular filter has a cluster point.

(3) Every maximal l-completely regular filter is convergent.

Section 3: k-compact topological lattices.

For a completely regular topological lattice $(L, \vee_L, \wedge_L, \underline{0})$, let $\beta_1 L$ be the set of all maximal 1-completely regular filters on L endowed with a topology $\underline{0}^*$ generated by $\{U^* \mid U^* = \{\mathfrak{M} \in \beta_1 L,\ U \in \mathfrak{M}\},\ U:\text{ open in } L\}$.

Then it is easy to show that $(\beta_1 L, \underline{0}^*)$ is a Hausdorff space.

For a pair $(\mathfrak{M}, \mathfrak{N})$ of elements of $\beta_1 L$ and any $f \in \mathrm{Hom}(L, I)$, let $x_f = \lim f(\mathfrak{M}) \vee_I \lim f(\mathfrak{N})$, where I is the closed interval $[-1, 1]$ and $\vee_I$ is the join map of I. Let $\mathfrak{M} \vee_L \mathfrak{N} = \{M \vee_L N \mid M \in \mathfrak{M},\ N \in \mathfrak{N}\}$, then obviously $\mathfrak{M} \vee_L \mathfrak{N} = \vee_L(\mathfrak{M} \times \mathfrak{N})$ and it is a filter base on L. Since the diagram

$$\begin{array}{ccc} L \times L & \xrightarrow{f^2} & I \times I \\ \downarrow{\scriptstyle \vee_L} & & \downarrow{\scriptstyle \vee_I} \\ L & \xrightarrow{f} & I \end{array},$$

commutes, $f(\mathfrak{M} \vee_L \mathfrak{N}) = f(\vee_L(\mathfrak{M} \times \mathfrak{N})) = \vee_I(f^2(\mathfrak{M} \times \mathfrak{N})) = \vee_I(f(\mathfrak{M}) \times f(\mathfrak{N}))$ converges to $x_f = \lim f(\mathfrak{M}) \vee_I \lim f(\mathfrak{N})$, for $\vee_I$ is continuous. Hence $\mathfrak{M} \vee_L \mathfrak{N} \supseteq f^{-1}(\underline{0}(x_f))$ for each $f \in \mathrm{Hom}(L, I)$; $\bigcup\{f^{-1}(\underline{0}(x_f)) \mid f \in \mathrm{Hom}(L, I)\}$ generates a filter. Let $\mathfrak{M} \vee \mathfrak{N}$ be the filter generated by the family. Then it is obvious that $\mathfrak{M} \vee \mathfrak{N}$ is a maximal 1-completely regular filter, i.e., $\mathfrak{M} \vee \mathfrak{N} \in \beta_1 L$.

Let $\vee: \beta_1 L \times \beta_1 L \longrightarrow \beta_1 L$ be the map defined by $(\mathfrak{M}, \mathfrak{N}) \longmapsto \mathfrak{M} \vee \mathfrak{N}$. Similarly, one can define the map $\wedge: \beta_1 L \times \beta_1 L \longrightarrow \beta_1 L$; i.e., for any $(\mathfrak{M}, \mathfrak{N}) \in \beta_1 L \times \beta_1 L$, let $y_f = \lim f(\mathfrak{M}) \wedge_I \lim f(\mathfrak{N})$, then $\mathfrak{M} \wedge \mathfrak{N}$ is the filter generated by $\cup \{f^{-1}(\underline{O}(y_f)) \mid f \in \mathrm{Hom}(L, I)\}$, where $\wedge_I$ is the meet map of I.

Using the fact that I is a lattice, it is easy to show that the maps $\vee$ and $\wedge$ satisfy the defining equations of lattices. Hence we can conclude that $(\beta_1 L, \vee, \wedge)$ is a lattice.

Let $\beta_1: L \longrightarrow \beta_1 L$ be a map defined by $\beta_1(x) = \underline{O}(x)$ for $x \in L$. Then obviously β_1 is a topological dense embedding and a lattice homomorphism.

3.1 <u>Lemma</u> $(\beta_1 L, \vee, \wedge, \underline{O}^*)$ is a topological lattice.

<u>Proof</u>: Let $(\mathfrak{M}, \mathfrak{N}) \in (\beta_1 L)^2$ and U^* a basic neighborhood of $\mathfrak{M} \vee \mathfrak{N}$, i.e., $U \in \mathfrak{M} \vee \mathfrak{N}$. For each $f \in \mathrm{Hom}(L, I)$, let $m_f = \lim f(\mathfrak{M})$, $n_f = \lim f(\mathfrak{N})$ and $x_f = m_f \vee_I n_f$. By the definition of the join $\vee$, there are $f_i \in \mathrm{Hom}(L, I)$ and $U_i \in \underline{O}(x_{f_i})$ $(i = 1, \ldots, n)$ such that $\cap \{f_i^{-1}(U_i) \mid 1 \leq i \leq n\} \subseteq U$. For each $i = 1, \ldots, n$, there are $M_i, V_i \in \underline{O}(m_{f_i})$ and $N_i, W_i \in \underline{O}(n_{f_i})$ such that $M_i \vee N_i \subseteq U_i$, $V_i \subseteq \mathrm{cl}V_i \subseteq M_i$ and $W_i \subseteq \mathrm{cl}W_i \subseteq N_i$. Let $V = \cap \{f_i^{-1}(V_i) \mid 1 \leq i \leq n\}$ and $W = \cap \{f_i^{-1}(W_i) \mid 1 \leq i \leq n\}$. Then it is easy

to show that $V^* \vee W^* \subseteq U^*$. Hence $\vee$ is continuous. Similarly one can prove that $\wedge$ is continuous.

3.2 __Lemma__ The morphism $\beta_1 : L \longrightarrow \beta_1 L$ is I-extendable in __TLAT__. Moreover $\beta_1 L$ is compact completely regular.

__Proof:__ For any $f \in \mathrm{Hom}(L, I)$ and $\mathfrak{m} \in \beta_1 L$, let $\bar{f}(\mathfrak{m}) = \lim f(\mathfrak{m})$. By the same argument as that in Lemma 3.8 in Chap. III, $\bar{f}$ is continuous and $\bar{f}\beta_1 = f$.

Recalling the definitions of joins and meets in $\beta_1 L$, it is obvious that $\bar{f}$ is a lattice homomorphism.

By substituting continuous isotone maps in the proof of Theorem 3.9 in Chap. III with continuous homomorphisms and using Theorem 2.19, one can easily show the second part of the assertion. We omit the detail of the proof.

In what follows, we will identify any point $x \in L$ with its neighborhood filter $\underline{O}(x)$, so that L is a subspace of $\beta_1 L$ and β_1 is the natural embedding.

Using the same argument as that in Lemma 3.11 in Chap. III, one has the following.

3.3 __Lemma__ Let $h : L \longrightarrow T$ be a dense morphism in __TLAT__. Then the following are equivalent:

(1) h is I-extendable.

(2) h is __I-COMPTL__-extendable.

Combining Lemma 3.1, 3.2, and 3.3, we have the following.

3.4 Theorem For any completely regular topological lattice L, $\beta_1 : L \longrightarrow \beta_1 L$ is an I-COMPTL-extendable I-compactification. Moreover the category I-COMPTL is an extensive subcategory of CRTL.

3.5 Remark For any $\mathfrak{m} \in \beta_1 L$, the trace filter of $\mathfrak{m}$ on L generates $\mathfrak{m}$ itself. Moreover, $\beta_1 L$ is the strict extension of L with all maximal l-completely regular filters as the filter trace.

3.6 Definition Let k be an infinite cardinal. A completely regular topological lattice L is said to be k-compact if every maximal l-completely regular filter on L with the k-intersection property is convergent.

3.7 Lemma A completely regular topological lattice L is k-compact if and only if it is k-closed in $\beta_1 L$.

Proof: It is immediate from Definition 3.6, Proposition 2.13 in Chap. II and Remark 3.5.

For an infinite cardinal k, the category of k-compact topological lattices and continuous homomorphisms will be denoted by kCTL.

Using Lemma 3.7 and the fact that I-COMPTL is extensive in the hereditary category CRTL and kCTL = $\text{I-COMPTL}_{\ell k}$, we have the following by Theorem 1.4 in Chap. II.

3.8 Theorem The category kCTL is extensive in CRTL.

3.9 Corollary. The category kCTL is complete. Moreover, a k-closed sublattice of a k-compact topological lattice is again a k-compact topological lattice.

Since every k-Lindelöf completely regular topological lattice is a k-compact topological lattice, the following is immediate.

3.10 Proposition Every R-compact topological lattice is an $\aleph_1$-compact topological lattice, i.e., the category R-COMPTL of R-compact topological lattices and continuous homomorphisms is a subcategory of $\aleph_1$CTL.

Remark The same space $E = \{-\infty\} \cup R \cup \{\infty\}$ in Example 4.13 in Chap. III, is not an R-compact topological lattice but an $\aleph_1$-compact topological lattice. Hence R-COMPTL is a proper subcategory of $\aleph_1$CTL.

Let α be an ordinal and $W(\alpha)$ the space of ordinals less than α endowed with the usual order and the interval topology. Then 0 is the isolated point, and for any $\tau > 0$, the set of all open closed intervals $[\![\sigma + 1, \tau]\!] = \{x \mid \sigma < x < \tau + 1\}$ ($\sigma < \tau$) is a family of basic neighborhoods of τ. Evidently, a point is an isolated point if and only if it is not a limit ordinal. It is obvious that $W(\alpha)$ is a topological lattice. Moreover, $W(\alpha)$ is a completely regular topological lattice.

3.11 Theorem Let ω_α be the first ordinal of an infinite cardinal $\aleph_\alpha$. Then the topological lattice $W(\omega_{\alpha+1}) = X_{\alpha+1}$ is an $\aleph_{\alpha+2}$-compact topological lattice but not an $\aleph_{\alpha+1}$-compact topological lattice.

Proof: It is known (see [13]) that the Stone-Čech compactification $\beta X_{\alpha+1}$ is $W(\omega_{\alpha+1}+1) = \{\lambda \mid \lambda:$ ordinal, $\lambda \leq \omega_{\alpha+1}\}$ with the interval topology. Since $\beta_1 X_{\alpha+1}$ is a compactification of $X_{\alpha+1}$, $\beta X_{\alpha+1}$ is homeomorphic with $\beta_1 X_{\alpha+1}$. Hence the trace filter $T(\omega_{\alpha+1})$ is generated by $\{T_\sigma = \{\lambda \mid \lambda > \sigma\} \mid \sigma < \omega_{\alpha+1}\}$. For any continuous homomorphism $f: X_{\alpha+1} \longrightarrow I$ and any $\sigma \in X_{\alpha+1}$, $\lim f(T(\omega_{\alpha+1})) \geq \lim f(\underline{O}(\sigma)) = f(\sigma)$, $\omega_{\alpha+1}$ is bigger than σ in $\beta_1 X_{\alpha+1}$. Hence $\beta_1 X_{\alpha+1}$ is isomorphic with $W(\omega_{\alpha+1}+1)$. Since only free maximal 1-completely regular filter on $X_{\alpha+1}$ is $T(\omega_{\alpha+1})$ which has not the $\aleph_{\alpha+2}$-intersection property but the $\aleph_{\alpha+1}$-intersection property, $X_{\alpha+1}$ is an $\aleph_{\alpha+2}$-compact topological lattice but not an $\aleph_{\alpha+1}$-compact topological lattice.

3.12 Theorem For any limit cardinal k, there is a k-compact topological lattice which is not a t-compact topological lattice for every infinite cardinal $t < k$.

Proof: Let $k = \aleph_\gamma$ and I be the set of all isolated infinite cardinals less than k. And let $X = \prod_{\aleph_{\alpha+1} \in I} W(\omega_{\alpha+1})$.

It is obvious that X is a k-compact topological lattice.

Suppose that X is $\aleph_\alpha$-compact for some $\aleph_\alpha < k$. Then the closed subspace of X which is isomorphic with $W(\omega_{\alpha+1})$ is also $\aleph_\alpha$-compact which is a contradiction to Theorem 3.11.

3.13 __Lemma__ A filter $\mathcal{F}$ on a completely regular topological lattice (resp. ordered space) X is a minimal Cauchy filter with respect to the uniform structure $\mathcal{U}$ generated by Hom(X, I) (resp. hom(X, I)), where I is the closed interval [-1, 1] endowed with the usual uniform structure, if and only if it is a maximal 1-completely regular filter (resp. o-completely regular filter) on X.

__Proof:__ ($\Longrightarrow$) For any $f \in$ Hom(X, I) (resp. $f \in$ hom(X, I)), $f(\mathcal{F})$ is a Cauchy filter on I. Since I is complete, $f(\mathcal{F})$ is convergent. Hence $\mathcal{F}$ contains a maximal 1-completely (resp. o-completely)regular filter, say $\mathcal{G}$. Since for any $f \in$ Hom(X, I) (resp. $f \in$ hom(X, I)), $f(\mathcal{G})$ is convergent, $\mathcal{G}$ is also a Cauchy filter. Hence $\mathcal{G} = \mathcal{F}$.

($\Longleftarrow$) It is obvious that every maximal 1-completely (o-completely) regular filter is a Cauchy filter with respect to $\mathcal{U}$. Since every Cauchy filter contains a minimal Cauchy filter, every maximal 1-completely (o-completely) regular filter is a minimal Cauchy filter with respect to $\mathcal{U}$.

3.14 __Theorem__ For a completely regular topological lattice (resp. ordered space) X, $\beta_1 X$ (resp. $\beta_0 X$) is homeomorphic with the completion of the uniform space X with the uniform structure $\mathcal{U}$ generated by Hom(X, I) (resp. hom(X, I)).

Proof: It is known that the completion of any uniform space Y is the strict extension of Y with all minimal Cauchy filters as the filter trace. Since $\beta_1 X$ (resp. $\beta_0 X$) is the strict extension of X with all maximal 1-completely (resp. o-completely) regular filters on X as the filter trace, using the above lemma, we can conclude that $\beta_1 X$ (resp. $\beta_0 X$) is homeomorphic with the completion of the uniform space (X, $\mathcal{U}$).

3.15 Theorem Let k be an infinite cardinal. A completely regular topological lattice (resp. ordered space) X is k-compact if and only if it is k-complete with respect to the uniform structure on X generated by Hom(X, I) (resp. hom(X, I)).

Proof: X is k-compact if and only if it is k-closed in $\beta_1 X$ (resp. $\beta_0 X$) if and only if it is k-complete with respect to the uniform structure on X generated by Hom(X, I) (resp. hom(X, I)).

Section 4: Coreflective subcategories of TLAT.

In this section, every subcategory of TLAT will be assumed to be full, isomorphism closed and to contain at least one non-empty object.

4.1 **Theorem** Every coreflective subcategory of TLAT is bijection-coreflective in TLAT.

Proof: Let C be a coreflective subcategory of TLAT. Since any constant map from a non-empty topological lattice to a topological lattice is a morphism in TLAT, every non-empty topological lattice is a separator for TLAT. Hence C is bicoreflective in TLAT (see Theorem 13.1.1 [17]), for we assumed that C contains at least one non-empty object.

Now we can conclude that every C-coreflection is a bijection. Since monomorphisms in TLAT are exactly one-one morphisms, it needs only show that every C-coreflection is onto. For any $L \in$ TLAT, let $c: cL \longrightarrow L$ be the C-coreflection of L and let C be a non-empty C-object. For any $x \in L$, let $f_x: C \longrightarrow L$ be a map defined by $f_x(y) = x$ for all $y \in C$. Then obviously f_x is a TLAT-morphism, so that there is a unique C-morphism $\bar{f}_x: C \longrightarrow cL$ with $c\bar{f}_x = f_x$. Since for any $y \in C$, $c(\bar{f}_x(y)) = f_x(y) = x$, c is onto. This completes the proof.

4.2 **Corollary** For a coreflective subcategory C of TLAT and any $L \in$ TLAT, the C-coreflection cL of L and L are isomorphic as lattices.

4.3 Corollary For any coreflective subcategory C of Haus, the category CLAT is a coreflective subcategory of TLAT provided that C is finitely productive, where CLAT is the subcategory of TLAT determined by topological lattices whose underlying spaces belong to C.

Proof: It is well known that C is also bijection-coreflective in Haus. For any L ∈ TLAT, let id: (L, cO) ⟶ (L, O) be the C-coreflection of the underlying space (L, O) of L. Since C is finitely productive, cO is again an algebra topology on L. Hence (L, cO) ∈ CLAT. It is obvious that id: (L, cO) ⟶ (L, O) is the desired CLAT-coreflection of (L, O).

For any coreflective subcategory C of TLAT, we may assume that C-coreflection maps c_L: cL ⟶ L are identities for all L ∈ TLAT.

4.4 Lemma Let C be a coreflective subcategory of TLAT and A a subcategory of TLAT. For an A-object L and a sublattice S of L, let $\ell^C_L S$ be the closure of S in the C-coreflection space cL of L. Then $\ell^C_A = (\ell^C_L)_{L \in A}$ is an idempotent extensive operator on A.

Proof: Since cL is a topological lattice and S is a sublattice, $\ell^C_L S$ is again a sublattice of L. Obviously $S \subseteq \ell^C_L S \subseteq cl_L S$, for the C-coreflection map c_L of L is continuous. For any morphism $f: L \longrightarrow L'$ in $\underline{A}$, we have the following commutative diagram

$$\begin{array}{ccc} cL & \xrightarrow{c_L} & L \\ \Big\downarrow f & & \Big\downarrow f \\ cL' & \xrightarrow[c_{L'}]{} & L' \end{array},$$

where c_L and $c_{L'}$ are $\underline{C}$-coreflection maps of L and L' respectively. For any sublattice S of L, $f(\ell_L^{\underline{C}} S) = f(cl_{cL} S) \subseteq cl_{cL'} f(S) = \ell_{L'}^{\underline{C}} f(S)$. Hence $\ell_{\underline{A}}^{\underline{C}}$ is an extensive operator on $\underline{A}$. Since $\ell_L^{\underline{C}}$ is the closure operator on cL, $\ell_{\underline{A}}^{\underline{C}}$ is an idempotent extensive operator on $\underline{A}$.

4.5 <u>Theorem</u> Let $\underline{A}$ be a hereditary subcategory of <u>TLAT</u> and $\underline{B}$ an extensive subcategory of $\underline{A}$. For any coreflective subcategory $\underline{C}$ of <u>TLAT</u>, let $\underline{B}_{\underline{C}}$ be the subcategory determined by $\{L \in \underline{A} \mid L$ is $\ell_{\underline{B}}^{\underline{C}}$-closed in rL, where rL is a $\underline{B}$-reflection space of $L\} = \{L \in \underline{A} \mid L$ is closed in the $\underline{C}$-coreflection space of the $\underline{B}$-reflection space of $L\}$.
Then $\underline{B}_{\underline{C}}$ is again an extensive subcategory of $\underline{A}$ containing $\underline{B}$.

<u>Proof</u>: It is immediate from Theorem 1.4 in Chap. II and the above Lemma 4.4.

4.6 <u>Corollary</u> Let $\underline{A}$ be a hereditary subcategory of <u>TLAT</u> and $\underline{B}$ an extensive subcategory of $\underline{A}$. Then we have a correspondence Coref(<u>TLAT</u>) $\longrightarrow \mathrm{Ext}_{\underline{A}}(\underline{B})$ defined by $\underline{C} \longmapsto \underline{B}_{\underline{C}}$, where Coref(TLAT) is the class of coreflective subcategories of TLAT and $EXT_A(B)$ is the class of reflective subcategories of A containing B, i.e., every coreflective subcategory of TLAT induces another reflective subcategory of A containing B. Moreover, if $C \subseteq C'$ for C, C' $\in$ Coref(TLAT), then $B_C \supseteq B_{C'}$.

4.7 Corollary Every coreflective subcategory of TLAT induces an extensive subcategory of CRTL containing I-COMPTL.

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