ON SOME CLASSES OF BALANCED GRAPH DESIGNS

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BY

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SCOPE AND CONTENT:

Necessary and sufficient conditions for the existence of special types of balanced graph designs, namely, balanced bipartite designs, resolvable balanced bipartite designs, balanced tripartite designs and balanced circuit designs, are investigated. Given a set of parameters, the necessary conditions for the existence of a design with these parameters are obtained, and to find out whether these conditions are also sufficient, we attempt to construct a collection of base blocks which generate the design. When such a collection cannot be found, we proceed to prove that the particular design does not exist. Several infinite series of designs have also been constructed.

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INTRODUCTION

The term balanced graph design was first introduced by Hell and Rosa in [11]. If G is a graph with k vertices (and no isolated vertices), then a block is said to be a G-block if its adjacency matrix is equivalent to the adjacency matrix of the graph G. A balanced graph design (briefly BGD) is an arrangement of v elements into b G-blocks such that each G-block contains k distinct elements, each element occurs in exactly r G-blocks and any two distinct elements are linked in exactly λ G-blocks (Definition 1.3).

BGD's are in fact generalization of balanced incomplete block designs (briefly BIBD's). In a BIBD, the adjacency matrices of the blocks are those of complete graphs, that is, graphs in which there is one and only one edge joining each pair of distinct vertices. By varying the graphs, we get different classes of BGD's.

Several classes of BGD's have been investigated, for example, balanced path designs by Hell and Rosa [11], Hung and Mendelsohn [14], balanced circuit designs by Kotzig [15], Rosa [19], Rosa and Huang [22] and the most commonly studied ones, which have been mentioned already, BIBD's. The classes of BGD's which we consider are balanced bipartite designs, BBD's (Chapter II), resolvable balanced bipartite designs, RBBD's (Chapter III), balanced tripartite designs, BTD's (Chapter IV) and balanced circuit designs,BCD's (Chapter V).

We search for the necessary conditions for the existence of a BGD of a particular class with a given set of parameters and then investigate whether these conditons are also sufficient by constructing a design with these parameters or by proving that a design with these parameters cannot exist. Since Theorem 1.15 implies that a BGD exists for $\lambda = q\lambda'$, $q \geq 1$ whenever a BGD with $\lambda = \lambda'$ exists (keeping v and k fixed), we need to consider BGD's with minimal λ only.

We obtain the necessary and sufficient conditions for the existence of a BBD $(v,k,1;\ k_1)$ where $k_1 < k_2$ or $k_1 = k_2 \equiv 0 \pmod 2$, a BBD $(v,k,2;\ k_1)$ where $k_1 < k_2$ or $k_1 = k_2 \not\equiv 2 \pmod 4$ and a BBD $(v,k,\lambda;\ 1)$. For $k_1 = k_2 \equiv 1 \pmod 2$, we are able to construct BBD's $(v,k,1;\ k_1)$ for all $v \equiv 1 \pmod{2k_1^2}$ but not for $v \equiv k_1^2 + 1 \pmod{2k_1^2}$ except when $k_1 = 1$ or 3. Similarly, for $k_1 = k_2 \equiv 2 \pmod 4$, we are able to construct BBD's $(v,k,2;\ k_1)$ for all $v \equiv 1 \pmod{k_1^2}$ but not for $v \equiv (3/4)k_1^2 + 1 \pmod{k_1^2}$ except when $k_1 = 2$. However, an infinite series of BBD's $(v,k,\lambda;\ 2)$ where $v \geq 2k - 3$ is odd, is constructed and the necessary and sufficient conditions for the existence of a BBD $(v,k,\lambda;\ k_1)$ where $3 \leq k \leq 6$, $1 \leq k_1 \leq [\frac{1}{2}k]$ are found.

Wilson [25] proved that the necessary and sufficient condition for the existence of a RBBD $(v,3,1;\ 1)$ is $v \equiv 9 \pmod{12}$; we show that a RBBD $(v,3,\lambda;\ 1)$ exists if and only if

$$v \equiv 9 \pmod{12} \text{ for } \lambda \equiv 1 \pmod 2$$
$$v \equiv 3 \pmod 6 \text{ for } \lambda \equiv 2 \pmod 4$$
$$v \equiv 0 \pmod 3 \text{ for } \lambda \equiv 0 \pmod 4.$$

It is well-known (see for example Theorem 9.1 of [10] which is formulated in graph-theoretical language), that a RBBD $(v,2,\lambda; 1)$ always exists for each $v \equiv 0 \pmod 2$, $\lambda \geq 1$; we are able to prove that the necessary conditions are also sufficient for the existence of RBBD's $(v,2n,\lambda; n)$ for $2 \leq n \leq 4$. We also show that if a RBBD $(2nt, 2n, ns; n)$ exists for $n,t,s \geq 1$, then a RBBD $(2nqt, 2n, pns; n)$ also exists for $p,q \geq 1$.

Necessary and sufficient conditions for the existence of some BTD's are also obtained, for example, BTD $(v,k,1; 1,1)$, k odd, and BTD $(v,4,\lambda; 1,1)$. For some other sets of parameters, we construct several infinite (but not all possible) series of BTD's. Finally, we find the necessary and sufficient conditions for the existence of a BCD (v,k,λ) for $3 \leq k \leq 6$.

CHAPTER I: Basic Concepts

§1.1. Preliminaries

A central problem of Combinatorial Theory is that of arranging elements into a specified number of sets called blocks, so that the ith element appears in r_i blocks, the jth block contains k_j distinct elements, and so that pairs, triples, and similar groups of elements occur a specified number of times. Such an arrangement is called an incidence system.

Balanced incomplete block designs (briefly BIBD's) are examples of incidence systems. They were first introduced in experimental studies by F. Yates in 1936 [26], and are useful in the proper designing of experiments, particularly in agricultural experiments or laboratory and technological processes in which the nature of the processes or the apparatus available imposes a definite limit on the number of treatments allowable in a group [26].

Definition 1.1. A BIBD with parameters b, v, r, k and λ, denoted by BIBD (b,v,r,k,λ), is an arrangement of v elements into b blocks such that each block contains k distinct elements, each element occurs in r blocks and each pair of distinct elements occur together in exactly λ blocks.

There are several generalizations of BIBD's, for instance, pairwise balanced designs (briefly PBD's) which are incidence systems, partially balanced incomplete block designs (briefly PBIBD's) which are not incidence systems, (for definitions see [16]), and tactical configurations which are incidence systems in which the r_i's and k_j's are constants. However, BIBD's can be generalized in an entirely different way by imposing a graph structure on the elements of a block by specifying whether two elements in the block are linked or unlinked. This generalization first arose in the work of Hell and Rosa on paths [11] and they call the generalized design balanced graph design. Before we discuss the details of this generalization, several graph-theoretical definitions are given [10].

Definition 1.2. A graph G consists of a set V(G) of v elements which are called vertices and a set E(G) of e elements which are called edges. An edge f of G has associated with it an unordered pair of distinct vertices, say x_i and x_j, and f is said to join x_i and x_j.

A pair of vertices x_i and x_j could be joined by several edges. A graph G is said to be complete if every pair of vertices in V(G) is joined by exactly one edge. Denote by $\langle v,q \rangle$ the graph with v vertices, each pair of which is joined by q edges. An n-vertex clique G' of the graph $\langle v,q \rangle$ is a complete graph on n (where $n \le v$) vertices with $V(G') \subseteq V(G)$.

The adjacency matrix M(G) of a graph G with v vertices is a v by v symmetric matrix with $M(G) = \| m_{ij} \|$ where $m_{ii} = 0$ and $m_{ij} = 1$ if the vertices x_i and x_j of G are joined by at least one edge and $m_{ij} = 0$ otherwise.

The graph $G = \langle v,q \rangle$ is said to be decomposable into ℓ edge-disjoint n-vertex cliques G_i, $1 \le i \le \ell$, if $\bigcup_{i=1}^{\ell} \{V(G_i)\} = v(G)$, $\bigcup_{i=1}^{\ell} \{E(G_i)\} = E(G)$ and $E(G_i) \cap E(G_j) = \phi$ for $i \neq j$. It is easy to see that the construction of a BIBD (b,v,r,k,λ) corresponds to a decomposition of a graph $\langle v,\lambda \rangle$ into b edge-disjoint k-vertex cliques. The elements of the design correspond to the vertices of the graph and the blocks to the k-vertex cliques. Hence r is the number of cliques in which an arbitrary vertex is contained. The terms vertices and elements will be used interchangeably and described by the same notation; similarly for the terms cliques and blocks (see Table 1). Furthermore, the adjacency matrix of a block is defined to be the adjacency matrix of the corresponding clique.

There is a property of a BIBD which is also present in the generalizations given earlier, namely, that the adjacency matrices of all the blocks are that of complete graphs. Hence it is reasonable to assume that the BIBD's can also be generalized in a different way, that is, by assigning different adjacency matrices to the blocks. These matrices are k by k symmetric (0,1)-matrices. If $M(B) = \|m_{ij}\|$ is an adjacency matrix of a block B, then two elements x_i and x_j are said to be linked in B if $m_{ij} = 1$.

Definition 1.3. Let G be a graph with k vertices and no isolated vertices and let M(G), a k by k symmetric (0,1)-matrix, be its adjacency matrix, then a block B is called G-block if its adjacency matrix M(B) is equivalent to M(G), that is, $M(G) = N^{-1}M(B)N$ for some permutation matrix N. A balanced graph design (briefly balanced G-design) with parameters b,v,r b,v,r,k, λ (and G), is an arrangement of v elements into b G-blocks such that every G-block contains k distinct elements, each element occurs in exactly r G-blocks and any two distinct elements are linked in exactly λ G-blocks.

If $M(G) = J-I$, where J is the k by k matrix with all entries 1 and I is the k by k identity matrix, then the corresponding balanced G-design is a BIBD. G-blocks and balanced G-designs were first introduced in [11]. Several types of balanced G-designs have been investigated, for instance, balanced P-designs (briefly BPD's) ([11], [14]), where P is a k by k matrix with $P = \|p_{ij}\|$ and

$$p_{ij} = 1 \text{ for } i < k \text{ and } j = i + 1 \text{ or } i > 1 \text{ and } j = i - 1,$$

$p_{ij} = 0$ otherwise (that is, P is the adjacency matrix of a path); <u>balanced C-designs</u> (briefly BCD's) ([15], [19], [22]), where C is a k by k matrix with $C = \|c_{ij}\|$ and $c_{ij} = 1$ for $i < k$ and $j = i + 1$ or $i > 1$, $j = i - 1$, or $i = 1$, $j = k$ or $i = k$, $j = 1$, and $c_{ij} = 0$ otherwise (that is, C is the adjacency matrix of a circuit). The type of designs which are the main concern of this thesis are as follows (cf. [12], [13]).

<u>Definition 1.4.</u> A <u>balanced bipartite design</u> (briefly BBD) with parameters b, v, r, k, λ and k_1, k_2 with $k_1 + k_2 = k$, is a balanced G-design, where G is the complete graph K_{k_1,k_2}, that is,

$$M(G) = \begin{Vmatrix} 0 & J \\ J^T & 0 \end{Vmatrix},$$

where J is the k_1 by k_2 matrix with all the entries being 1 and J^T denotes the transpose of J.

Graphical representation of a block in a BBD is shown in Figure 1. A natural generalization of BBD's would be:

<u>Definition 1.5.</u> A <u>balanced tripartite design</u> (briefly BTD) with parameters $b, v, r, k, \lambda, k_1, k_2$ and k_3 with $k_1 + k_2 + k_3 = k$, is a balanced G-design where G is the complete tripartite graph K_{k_1,k_2,k_3}, that is

$$N(G) = \begin{vmatrix} 0 & J_1 & J_2 \\ J_1^T & 0 & J_3 \\ J_2^T & J_3^T & 0 \end{vmatrix}$$

where J_1, J_2, J_3 is respectively a k_1 by k_2, k_1 by k_3, k_2 by k_3 matrix with all the entries being one, and J_1^T, J_2^T and J_3^T are their transposes respectively.

A BBD with $k_1 = k_2 = k/2$ can be considered a weakening of a balanced weight design (briefly BWD or alternatively a tournament design as in [3] and [4]). In a BWD, in addition to the conditions of Definition 1.4, any two distinct elements must occur together but not linked in exactly λ' G-blocks.

A G-block will henceforth be called simply a block.

Definition 1.6. Two balanced G-designs D_1 and D_2 are said to be isomorphic if there is a one-to-one mapping A of elements of D_1 onto elements of D_2 such that, if $B = \{a_1, a_2, \ldots, a_k\}$ is a block in D_1, then $A(B) = \{A(a_1), A(a_2), \ldots, A(a_k)\}$ is a block in D_2, and a_i, a_j are linked in block B if and only if $A(a_i)$ and $A(a_j)$ are linked in block A(B). If $D_1 = D_2$, then the mapping A is called an automorphism of D_1.

In a graph G with k vertices, two vertices x_i and x_j are said to be in the same similarity class if there exists an automorphism A of G (cf. [10]) such that $A(x_i) = x_j$. Hence the vertices of G can be partitioned into similarity classes $\mathcal{C}_1'$, $\mathcal{C}_2'$, ..., $\mathcal{C}_p'$, $1 \le p \le k$. Consider a balanced G-design associated with the graph G; since the adjacency matrix of each block of the design is equivalent to that of G, the set of elements of each block is partitioned into subsets $\mathcal{C}_1$, $\mathcal{C}_2$, ..., $\mathcal{C}_p$, where there is a natural correspondence between $\mathcal{C}_i'$ and $\mathcal{C}_i$, $1 \leq i \leq p$. Therefore these subsets are again called <u>similarity classes</u>.

Consider an element x_i, if $r_s(x_i)$ denotes the number of blocks in which the element x_i belongs to the class $\mathcal{C}_s$, then

$$(1.1) \qquad \sum_{s=1}^{p} r_s(x_i) = r,$$

which is true for all i. We can now impose another condition on a balanced G-design;

<u>Definition 1.7</u>. A balanced G-design is said to be a <u>strongly balanced G-design</u> if, for $1 \leq s \leq p$, $r_s(x_i)$ is independent of the element x_i.

Most of the designs considered in this thesis are strongly balanced G-designs. In fact, any balanced G-design with p = 1 or 2 is a strongly balanced G-design. The proof will be given in the next section.

<u>Defintion 1.8</u>. A balanced G-design is said to be <u>resolvable</u> if the b blocks may be divided into r sets, F_i, i = 1,2,...,r, each with b/r blocks, such that each F_i contains all the v elements in the design. These r sets are called <u>complete replications</u> (or <u>factors</u> in graph-theoretical terminology and <u>parallel classes</u> in geometrical terminology).

§1.2. Elementary Relations.

In order to exlude trivial cases in any balanced G-design, k is assumed to be greater than two. By counting the total number of incidences in two ways, one gets the equality

$$(1.2) \qquad bk = vr.$$

Let e be the number of links in a block, as there are b blocks in a design, the total number of links in the design is eb; on the other hand, since there are altogether $\binom{v}{2}$ pairs of distinct elements and each pair are linked λ times, the total number of links is $\lambda\binom{v}{2}$, hence we have

(1.3) $$\lambda\binom{v}{2} = eb.$$

Consider an arbitrary element x_i, let g_i be the number of elements to which x_i is linked in the b blocks, then

(1.4) $$g = \lambda(v - 1),$$

as each element is linked to every other element λ times.

Consider again the similarity classes $\mathcal{C}_s$, $1 \leq s \leq p \leq k$, and the relation

(1.1) $$\sum_{s=1}^{p} r_s(x_i) = r, \text{ for all } i,$$

where $r_s(x_i)$ is the number of blocks in which the element x_i belongs to the class $\mathcal{C}_s$. Let the number of elements in each class $\mathcal{C}_s$ be denoted by k_s, then

(1.5) $$\sum_{s=1}^{p} k_s = k.$$

If the number p of the similarily classes equals one (as it is in the cases of BIBD and BCD, for instance), then $r_1(x_i) = r$ and the design is a strongly balanced G-design; consequently the equalities (1.1) and (1.5) are redundant. If $p = 2$, let ℓ_1 or ℓ_2 be the number of elements linked to an element of $\mathcal{C}_1$ or $\mathcal{C}_2$ respectively in a block, then

(1.6) $$\begin{cases} e = \tfrac{1}{2}(\ell_1 k_1 + \ell_2 k_2), \\ \text{and} \quad g = r_1(x_i)\ell_1 + r_2(x_i)\ell_2 \text{ for an arbitrary element } x_i. \end{cases}$$

Taking into account that conditions (1.3) and (1.6) imply

(1.7) $$b = \frac{\lambda v(v-1)}{\ell_1 k_1 + \ell_2 k_2}$$

and conditions (1.2), (1.6) imply

(1.8) $$r = \frac{\lambda k(v-1)}{\ell_1 k_1 + \ell_2 k_2}$$

we obtain, from conditions (1.4) and (1.1)

(1.9) $$r_1(x_i) = \frac{\lambda k_1(v-1)}{\ell_1 k_1 + \ell_2 k_2}, \quad r_2(x_i) = \frac{\lambda k_2(v-1)}{\ell_1 k_1 + \ell_2 k_2}$$

Thus r_1 and r_2 do not depend on the particular element x_i, and so the design is a strongly balanced G-design.

Equalities (1.1) - (1.5) which imply that the parameters b and r can be expressed through v, k, λ and $\bigcup_{s=1}^{p-1} \{k_s\}$, are the necessary conditions for existence of a balanced G-design. A value of v which satisfies the necessary conditions will be called an <u>admissible value</u>.

With some parameters fixed, it will be proved in this thesis, that in some cases the necessary conditions are also sufficient. This is done by exhibiting a balanced G-design with the given parameters.

§1.3. Methods of Construction.

The methods for constructing balanced G-designs are either recursive which involves building a design from a smaller one or direct. Direct methods used in the literature usually employ finite fields, groups et cetera; but they are applicable only for special values of parameters (for example when v is a prime power). Therefore, direct methods based on combinatorical principles are favoured in the constructions in this thesis.

Definition 1.9. Let A be an automorphism of a balanced G-design D with v elements. Two elements x_i and x_j are said to be in the same orbit of elements if $A^t(x_i) = x_j$ for some $t \geq 1$; similarly, two blocks B_m and B_n are in the same orbit of blocks if $A^s(B_m) = B_n$ for some $s \geq 1$.

The property of being in the same orbit is an equivalence relation, so the elements and the blocks of a design D are partitioned into disjoint orbits by an automorphism A of D. An orbit of elements (or of blocks) can be represented by any one of its elements (or blocks) which will be called a base element (or base block). Hence a collection of base blocks, denoted by $\mathcal{B}$, with one base block from each orbit of blocks, determines the whole design D with the given automorphism.

Definition 1.10. A base block B has order m if m is the smallest positive integer such that $A^m(B) = B$.

The following lemma is an immediate consequence of the last definition:

Lemma 1.11. If m is the order of a base block B, then m satisfies the conditions

(1.10) $$1 \leq m \leq v, \quad mk \geq v \quad \text{and} \quad m|v.$$

Most of the designs constructed in this thesis are cyclic:

Definition 1.12. A balanced G-design D with v elements is said to be cyclic if it has an automorphism C consisting of a single cycle of length v (cf. [7]). C is called a cyclic automorphism.

The elements of a design D will be the residues modulo v in the range (0, v-1), unless otherwise stated, and the cyclic automorphism C, if it exists, will be $C = (0\ 1\ 2\ \ldots\ (v-1))$.

Definition 1.13. Let x_i and x_j be two distinct elements in a base block $B \in \mathcal{B}$, then the difference $x_i - x_j$ is called a pure difference if x_i and x_j are in the same orbit and a mixed difference otherwise.

In a cyclic balanced G-design D, there is only one orbit of elements; hence, all the differences are pure. Instead of pure differences, we will consider edgelengths in a cyclic design, where the edgelength between two elements i and j which are linked is defined to be

$\delta_{ij} = \min(|i - j|, v - |i - j|)$. Hence $1 \leq \delta_{ij} \leq [v/2]$, where [v/2] represents the greatest integer less than or equal to v/2. An integral ℓ, where $1 \leq \ell \leq [v/2]$ is said to occur q times (q a rational number) in a base block B of order m if qv/m edgelengths in B have the value ℓ. The number q is non-negative of course and it is an integer in most cases, but for example, when v is even and $\ell = v/2$, then q is not necessarily an integer.

A cyclic balanced G-design D exists if $w(m_i)$ base blocks, each of order m_i satisfying (1.10), can be constructed with $\sum_i w(m_i)m_i = b$.

Furthermore, every element of the set of edgelengths $E = \{1,2,\ldots, [v/2]\}$ should occur precisely λ times in the $\sum_i w(m_i)$ base blocks, except when v is even, in which case v/2 should occur only $\lambda/2$ times.

We will illustrate the procedure with a simple example of the construction of a cyclic BBD with $v = k = 5$, $k_1 = 2$, $k_3 = 3$ and $\lambda = 6$ (cf. Corollary 2.21). Such a design exists if a collection $\mathcal{B}$ of base blocks can be constructed such that each element of the set of edgelengths, $E = \{1,2\}$, occurs six times in these base blocks.

Let $B_i \in \mathcal{B}$ be of order m_i, then m_i satisfying (1.10) implies that $m_i = 1$ or 5. Assume that $\mathcal{B}$ contains w(1) base blocks of order 1 and w(5) base blocks of order 5, then $w(1) + 5w(5) = b = 10$ and we have three possible cases: (i) $w(1) = 10$, $w(5) = 0$; (ii) $w(1) = 5$, $w(5) = 1$; (iii) $w(1) = 0$, $w(5) = 2$. Let the elements of the design be 0, 1, 2, 3 and 4. Consider a base block B of order 1, that is $C(B) = B$ where $C = (0\ 1\ 2\ 3\ 4)$. Let B be the block which has an adjacency matrix equivalent to that of the graph B* in Figure 1(a) (we will say simply that block B is represented by graph B* in Figure 1(a)). But C(B) is represented by graph C(B)* in Figure 1(b) which is different from B* (for example, vertices 0 and 3 are joined by an edge in B* but not in C(B)*), hence $C(B) \neq B$, and a base block of order 1 cannot exist. We are left with case (iii) only and two base blocks, each of order 5 are needed. Consider graphs B_1^* and B_2^* in Figure 1(c) and blocks B_1, B_2 which are represented by the graphs respectively. The sets of pure differences of blocks B_1 and B_2 are {2,3,4,1,2,3}, {1,3,4,1,1,2} respectively and the sets of edgelengths of B_1 and B_2 are {2,2,1,1,2,2} {1,2,1,1,1,2} respectively. B_1 and B_2 are base blocks under C and they obviously satisfy the required condition that edgelengths 1 and 2 must each occur six times in them. Blocks B_1 and B_2, together with eight other blocks which are obtained by applying C on them successively, constitute the required design.

The procedure given above is essentially R.C. Boses method of symmetrically repeated differences [2]. A simple observation results in the following theorem.

Theorem 1.14. A cyclic balanced G-design is a strongly balanced G-design.

The construction of a balanced G-design with automorphism A which is non-cyclic is similar to that of a cyclic design with pure and mixed differences replacing edgelengths. The details involved will be given later when the occasion arises.

Since a balanced G-design is allowed to contain repeated blocks, one obtains immediately

Theorem 1.15. If there exist two balanced G-designs D_1 and D_2, where D_1 has parameters v, k, λ_1, $\bigcup_{s=1}^{p-1} \{k_s\}$ and D_2 has parameters v, k, λ_2, $\bigcup_{s=1}^{p-1} \{k_s\}$, then there exists a balanced G-design D_3 with paramters v, k, λ_3 and $\bigcup_{s=1}^{p-1} \{k_s\}$, where $\lambda_3 = a_1 \lambda_1 + a_2 \lambda_2$ with a_1 and a_2 being non-negative integers not both equal to zero.

Theorem 1.16. Let G be a bipartite graph. If there exist two balanced G-designs D_1 and D_2 where p = 2 in both cases, D_1 has parameters v_1, k, λ, k_1 and D_2 has parameters v_2, k, λ, k_1, then there exists a balanced G-design D_3 with parameters v_1 v_2, k, λ and k_1.

As this theorem will not be used at all, the proof, which can be obtained by following the outline of the proof of Theorem 2.3 in [12], will be omitted.

Theorem 1.17. If there exists a BIBD (v', k', λ') and a balanced G-design D_1 with paramters $v = k', k, \lambda = 1, \bigcup_{s=1}^{p-1} \{k_s\}$, then there exists a balanced G-design D_2 with parameters $v = v', k, \lambda = \lambda'$ and $\bigcup_{s=1}^{p-1} \{k_s\}$.

Proof. Obviously each block of a BIBD (v', k', λ') yields one balanced G-design with the same parameters as D_1; the collection of all G-blocks obtained in the same way from all the blocks of the BIBD evidently constitute a required design.

It is a routine matter to show that Theorems 1.15 – 1.17 remain valid if one replaces in the statements the phrase "balanced G-design" by "resolvable balanced G-design" or by "strongly balanced G-design". Theorems 1.15 and 1.17 remain valid also if "balanced G-design" is replaced by "cyclic balanced G-design" while an analogue to Theorem 1.16 concerning cyclic balanced G-design is in the following lemma.

Lemma 1.18. Let G be a bipartite graph. If there exist two cyclic balanced G-designs D_1 and D_2, where D_1 has parameters v_1, k, λ, k_1 and D_2 has parameters v_2, k, λ, k_1, then there exists a cyclic balanced G-design D_3 with parameters $v_1 \cdot v_2, k, \lambda, k_1$ if $p = 2$ in both D_1 and D_2, and furthermore v_1 and v_2 are relatively prime.

Table 1

Design Terminology	Graph-theoretical terminology
element	vertex
link (unordered) pair of elements	edge
difference of two elements	edge-length
two elements are linked	two vertices are joined by an edge
complete replication	factor

Figure 1

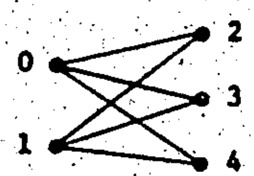

Figure 1(a): B*

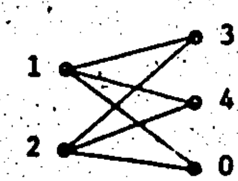

Figure 1(b): C(B)*

B_1^* B_2^*

Figure 1(c)

CHAPTER II: Balanced Bipartite Designs

§2.1. Balanced Bipartite Designs with Arbitrary λ

A block B in this chapter as well as in Chapter III means a set of $k = k_1 + k_2$ elements which are divided into two subsets, B^1 and B^2; B^1 with k_1 elements and B^2 with k_2 elements. Without loss of generality, let $k_1 \leq k_2$. Two elements of B are said to be linked in B if and only if they belong to different subsets; in other words, the elements of a block correspond to the vertices of a complete bipartite graph K_{k_1,k_2}. An alternative to Definition 1.4 can now be given as follows.

Definition 2.1. A balanced bipartite design (briefly BBD) with parameters b, v, r, k, λ, k_1 and k_2 with $k_1 + k_2 = k \leq v$ is an arrangement of v elements into b blocks such that each block contains k elements, each element occurs in exactly r blocks and any two distinct elements are linked in exactly λ blocks.

The two subsets B^1 and B^2 of a block B in a BBD are in fact the similarity classes $\mathcal{C}_1$ and $\mathcal{C}_2$ when $k_1 \neq k_2$. Hence p, the number of similarity classes is two and ℓ_1 (or ℓ_2), the number of elements which is linked to an arbitrary element of $\mathcal{C}_1$ (or $\mathcal{C}_2$) in a block is actually k_2 (or k_1). It was proved in §1.2 that for p = 2, the number of blocks r_s, where s = 1 or 2, in which an element i belongs to the class $\mathcal{C}_s$ does not depend on the particular element i. Hence (1.6) becomes

$$(2.1) \quad \begin{cases} e = k_1 k_2 \\ g = r_1 k_2 + r_2 k_1 \end{cases}$$

where e is the number of links in a block and g is the number of elements which are linked to an arbitrary element.

It was also shown in § 1.2 that the parameters are not independent; in fact, a BBD can be described by just four parameters v, k, λ and k_1. Therefore such a design will be denoted by BBD $(v,k,\lambda;k_1)$. The necessary conditions for the existence of a BBD $(v,k,\lambda;k_1)$, (1.7)-(1.9) become

(2.2) $$b = \frac{\lambda v(v-1)}{2k_1k_2},$$

(2.3) $$r = \frac{\lambda k(v-1)}{2k_1k_2},$$

(2.4) $$r_1 = \frac{\lambda(v-1)}{2k_2},$$

and

(2.5) $$r_2 = \frac{\lambda(v-1)}{2k_1}.$$

Furthermore, substituting (2.2) into (1.2) gives

$$\lambda(v-1) = \frac{2k_1k_2r}{k_1+k_2},$$

which together with (1.4) and (2.1) gives

(2.6) $$(k_2 - k_1)(k_2r_1 - k_1r_2) = 0.$$

Therefore one has either

(a) $k_1 = k_2$, k even and $r_1 = r_2 = \frac{1}{2}r$ or

(b) $k_2r_1 = k_1r_2$ which implies that $2k_1r_2 = g = \lambda(v-1)$, that is, if v is even, then λ must be even as well.

Fisher's inequality, $b \geq v$, does not always hold in a BBD. For example, if $v = 4$, $k_1 = k_2 = 2$ and $\lambda = 2$, then $b = 3$ and a BBD with these parameters exists as shown in Theorem 2.16.

Given integers u,w, denote by (u,w) the greatest common divisor of u and w, and by {u,w} the least common multiple of u and w.

The necessary condition for the existence of a BBD $(v,k,\lambda;k_1)$ is

__Lemma 2.2.__ If there exists a BBD $(v,k,\lambda;k_1)$ with $k_1 < k_2$, then $v \equiv 2k_1k_2h/xy + 1 \pmod{2k_1k_2\,(x,y)/xy}$ where $x = (\lambda,d)$, $y = (\lambda, 2\ell)$ and $d = (k_1, k_2)$, $\ell = \{k_1, k_2\}$ and h is a non-negative integer $\leq x-1$ such that $\lambda h\,(2k_1k_2h + xy) \equiv 0 \pmod{x^2y^2}$.

__Proof.__ Equations (2.4) and (2.5) imply that $v \equiv 1 \pmod{2\ell/y}$ and equation (2.2) implies that $v(v-1) \equiv 0 \pmod{2k_1k_2/(\lambda, 2k_1k_2)}$, that is, $v(v-1) \equiv 0 \pmod{2\ell d/\frac{xy}{(x,y)}}$. Hence if we let $v \equiv 2\ell t/y + 1$ $\pmod{2\ell d(x,y)/xy}$ for some non-negative integer $t \leq d-1$, then equation (2.2) implies that $z = \lambda v(2\ell t/y)/2\ell d$ is an integer and z can be reduced to $z = \lambda vt/dy$.

Equation (2.4) implies that $d \mid \lambda(v-1)$, but $(\lambda,d) = x$ implies that $\frac{d}{x} \nmid \frac{\lambda}{x}$; hence $\frac{d}{x} \mid v-1$. Consequently, z being an integer implies that $\frac{d}{x} \mid t$; that is, $t = hd/x$ for some h which is a non-negative integer $\leq x-1$ such that z is integral. But z is integral if and only if $z^1 = \frac{\lambda h}{xy}\left(\frac{2\ell dh}{xy} + 1\right)$ is integral, and since $2\ell d = 2k_1k_2$, the statement in the lemma is proved.

For the case where $k_1 = k_2 = n$, we have a similar result.

Lemma 2.3. If there exists a BBD $(v,k,\lambda;n)$ where $n = \frac{1}{2}k$, then $v = \frac{n^2h}{x^2} + 1 \pmod{\frac{2n^2}{(\lambda, 2n^2)}}$ where $x = (n,\lambda)$ and h is a non-negative integer $\leq 2x - 1$ such that $\lambda h(n^2h + x^2) \equiv 0 \pmod{2x^4}$.

Proof. The proof is similar to that of Lemma 2.2. Equations (2.3) and (2.2) imply that $v \equiv 1 \pmod{n/x}$ and $v(v-1) \equiv 0 \pmod{2n^2/(\lambda, 2n^2)}$ respectively. If we let $v \equiv nt/x + 1 \pmod{2n^2/(\lambda, 2n^2)}$ for some non-negative integer $t \leq 2n - 1$, then b being integral implies that $z = \lambda vt/2xn$ is an integer.

Equation (2.3) states that $n|\lambda(v-1)$, hence $\frac{n}{x} | v-1$ and consequently z being an integer implies that $\frac{n}{x}|t$; that is, $t = nh/x$ for some h which is a non-negative integer $\leq 2x - 1$ such that z is integral. The proof is complete since the last statement is true if and only if $z^1 = \frac{\lambda h}{2x^2} (\frac{n^2h}{x^2} + 1)$ is integral.

In the construction of a BBD $(v,k,\lambda; k_1)$, one could make use of some BIBD's which are known to exist. For instance, a BIBD (19,9,4) was constructed by employing finite field GF (19) [7] and since a BBD (9,4,1;2) can be constructed [cf. Lemma 2.6], one obtains a BBD (19,4,4;2) from Theorem 1.17. However, the advantages of applying Theorem 1.17 are obviously limited: for example, since there exists a BIBD (25,9,3) [7], one thus obtains a BBD (25,4,3;2); yet by using a direct construction, a BBD (v,4,1;2) (and consequently, by applying Theorem 1.15, also a BBD $(v,4,\lambda;2)$, $\lambda \geq 2$) can be constructed for all $v \equiv 1 \pmod 8$. Therefore, all the results in this chapter will be obtained via direct constructions.

If $B_i = \{B_i^1 ; B_i^2\}$, where $B_i^1 = (a_{i1}^1, \ldots, a_{ik_1}^1)$ and $B_i^2 = (a_{i1}^2, a_{i2}^2, \ldots, a_{ik_2}^2)$, is a block, then δ_{rs} is the edgelength between elements a_{ir}^1 and a_{is}^2 for all r,s. As given in §1.3, $\delta_{rs} = \min(|a_{ir}^1 - a_{is}^2|, v - |a_{ir}^1 - a_{is}^2|)$ and $1 \leq \delta_{rs} \leq [\frac{1}{2}v]$. If B_i is a base block of order m, then in addition to (1.10) which states that $1 \leq m \leq v$, $mk \geq v$ and $m|v$, m must also satisfy

(2.7) let $v = mq$, then $q|k_1$ and $q|k_2$ for $k_1 \neq k_2$ and either $q|k_1$ or $q|k$ for $k_1 = k_2$.

Given a set of parameters, we will attempt to construct a cyclic BBD $(v,k,\lambda; k_1)$ first by trying to exhibit a collection $\mathcal{B}$ of base blocks. In this chapter, a base block is assumed to have order v unless otherwise stated.

<u>Lemma 2.4.</u> Let $\lambda|k_1$ or $\lambda|k_2$. There exists a cyclic BBD $(v,k,\lambda; k_1)$ for all $v \equiv 1 \pmod{2k_1k_2/\lambda}$, $v \geq k_1k_2 + 1$.

<u>Proof</u>. Without loss of generality, let $k_2 = \lambda z$, also let $v = 2k_1zt + 1$ where t is a positive integer, then $b = vt$ and the set of edgelengths is $E = \{1,2,\ldots, k_1zt\}$. For $i = 1,2,\ldots,t$, consider the sets

$$S_i = \bigcup_{q=1}^{k_1z} \{(i-1)k_1z + q\},$$

$$U_i = \bigcup_{q=1}^{z} \{(i-1)k_1z + k_1q\},$$

$$W_i = \bigcup_{q=1}^{z} \{(2t - i+1)k_1z + (1 - q)k_1\},$$

then $U_i \cap W_j = \phi$ for all i,j and $E = \bigcup_{i=1}^{t} S_i$.

A collection $\mathcal{B}$ of t base blocks, $B_i = \{B_i^1 ; B_i^2\}$ for $i = 1,2,\ldots,t$, will be constructed as follows:

Let $$B_1^1 = (\bigcup_{f=0}^{k_1-1} \{f\}) \text{ and } B_1^2 = (\bigcup_{j=1}^{\lambda} M_j)$$

where $$M_j = \begin{cases} U_j & \text{for } j = 1,2,\ldots,t \\ W_{j-t} & \text{for } j = t+1, t+2,\ldots, 2t. \end{cases}$$

M_j's are defined for all j on account that $v \geq k_1k_2 + 1$ implies $2t \geq k_2/z = \lambda$. B_1 is well defined too since $B_1^1 \cap M_j = \phi$ for all j and $M_i \cap M_j = \phi$ for all $i \neq j$. It can be verified that the sets of edge-lengths between the elements of B_1^1 and the elements of U_j and between the elements of B_1^1 and the elements of W_j are the same and are in fact equal to S_j for each $j = 1,2,\ldots,t$.

Let P be a permutation of degree v with 2z t-cycles, which are $(k_1q, k_1(q+z), \ldots, k_1q + (t-1)k_1z)$ and

$((2tz - q+1)k_1, (2tz - z - q+1)k_1,\ldots,(tz + z - q+1)k_1)$ for $q = 1,2,\ldots,z$, and also $(k_1 - 1)\ 2zt + 1$ 1-cycles, which involve the rest of the elements and hence can be omitted. Apply P on B_1 i times repeatedly to generate base block B_{i+1}, $i = 1,2,\ldots, t-1$, we get a set of t base blocks. It is easy to see that, since $P(U_i) = U_{i+1}$ and $P(W_i) = W_{i+1}$ (with subscripts taken modulo t), each element of E occurs exactly λ times in these t base blocks. Hence a cyclic design with the given parameters exists.

As the construction of a cyclic BBD with any set of parameters is similar to the one above, only the collection $\mathcal{B}$ of base blocks will be given in the proofs unless the procedure is different from the one just described.

§2.2 Balanced Bipartite Designs with λ=1 or 2.

The necessary conditions for the existence of a BBD with $\lambda = 1$ or 2, together with the sufficient conditions in some cases, are given in this section.

Lemma 2.5. Let $k_1 \leq k_2$. If there exists a BBD $(v,k,1;\ k_1)$, then $v \equiv 1 \pmod{n^2}$ for $k_1 = k_2 = n \equiv 1 \pmod 2$ and $v \equiv 1 \pmod{2k_1k_2}$ otherwise.

Proof. Let $k_1 < k_2$. Consider Lemma 2.2 in which $x = (\lambda, d)$, $y = (\lambda,\ 2\ell)$ and $0 \leq h \leq x-1$; but $\lambda = 1$ implies that both x and y equal 1 and $h = 0$, hence $v \equiv 1 \pmod{2k_1k_2}$.

Let $k_1 = k_2 = n$, then $x = (n,\lambda) = 1$ and h is either 0 or 1. Lemma 2.3 implies that $v \equiv 1 \pmod{2n^2}$ for $h = 0$ and $v \equiv n^2 + 1 \pmod{2n^2}$ for $h = 1$, but in the later case, the condition $\lambda h(n^2h + x^2) \equiv 0 \pmod{2x^4}$ implies that $n^2 + 1$ must be even, that is, n must be odd. Hence in the case where n is odd, $v \equiv 1 \pmod{n^2}$ and in the case where n is even $v \equiv 1 \pmod{2n^2}$.

Letting λ be 1 in Lemma 2.4, we get the following result.

Lemma 2.6. A cyclic BBD $(v,k,1;k_1)$ exists for all $v \equiv 1 \pmod{2k_1k_2}$ where $k_1 \leq k_2$.

The lemma above was originally proved in [20]. The two preceding lemmas together imply

Theorem 2.7. Let $k_1 < k_2$ or $k_1 = k_2 = n \equiv 0 \pmod 2$. A BBD $(v,k,1;\ k_1)$ exists if and only if $v \equiv 1 \pmod{2k_1k_2}$.

For $k_1 = k_2 = n \equiv 1 \pmod 2$, it has not been determined if the condition $v \equiv n^2 + 1 \pmod{2n^2}$ is also sufficient for the existence of a BBD $(v, 2n, 1; n)$ except in the case where $n = 3$, as shown in Lemmas 2.10 and 2.11; however some results have been obtained for general n.

Theorem 2.8. Let $k_1 = k_2 = n$ and $n \equiv 1 \pmod 2$. If there exists a BBD $(v', 2n, 1; n)$ where $v' = (2x + 1)\, n^2 + 1$ and $x \geq 0$, then there exists a BBD $(v, 2n, 1; n)$ for all admissible $v \geq v'$.

Proof. This is a generalization of the case where $n = 3$, which is given in [6]. Lemma 2.5 states that v being admissible implies that $v \equiv 1$ or $n^2 + 1 \pmod{2n^2}$ for n odd; but a BBD $(v, 2n, 1; n)$ where $v \equiv 1 \pmod{2n^2}$ always exists as stated in Lemma 2.6, hence only the case where $v \equiv n^2 + 1 \pmod{2n^2}$ is considered.

Assume that a BBD exists on v' elements where $v' = (2x + 1)\, n^2 + 1$, we will construct a BBD $(v, 2n, 1; n)$, denoted by D, with v elements where $v = (2z + 1)\, n^2 + 1$, and $z = x + y$, $y \geq 0$.

Take two sets S_1 and S_2, where S_1 contains v' elements, S_2 contains $2yn^2 + 1$ elements and they have one element a in common. The total number of distinct elements is then $v' + 2yn^2 = v$ and we let the set of elements in D be the union of sets S_1, S_2. Since there exists a BBD on either v' or $2yn^2 + 1$ elements, we construct a BBD D_1 on the elements of S_1 and a BBD D_2 on the elements of S_2. Let all these blocks be included in D, then in these $q = \frac{1}{2}[(2x + 1)^2\, n^2 + 2x + 1] + y\,(2yn^2 + 1)$ blocks, every pair of distinct elements in the same set S_1 or S_2 are linked exactly once and element a is linked to every element in D once.

Divide the elements other than a in each set S_i into subsets of n elements each. Take a subset of S_1, say S_{1i}, and a subset of S_2, say S_{2j}, to form a block B_{ij} in the design D, that is $B_{ij} = \{S_{1i}; S_{2j}\}$. The number of blocks formed this way is $2yn^2 (2x + 1)$ and in these blocks each element in S_1 other than element a is linked to each element in S_2 other than a once and once only.

The total number of blocks in D is
$q + 2yn^2 (2x + 1) = \frac{1}{2}[(2z + 1)^2 n^2 + 2z + 1] = b$. Since the other requirements are also satisfied, these b blocks do form a BBD (v, 2n, 1; n).

In view of the theorem above, for $k_1 = k_2 = n$ and n odd, if we can construct a BBD (v, 2n, 1; n) for some $v \equiv n^2 + 1 \pmod{2n^2}$, then we can construct an infinite series of the designs. We do not know of the existence of a BBD for any $v \equiv n^2 + 1 \pmod{2n^2}$, $n \geq 5$, but we know that a cyclic design does not exist for any $v \equiv n^2 + 1 \pmod{2n^2}$ and $n \geq 3$.

<u>Lemma 2.9</u>. Let $k_1 = k_2 = n \equiv 1 \pmod 2$ and $n \geq 3$. A cyclic BBD (v, 2n, 1; n) does not exist for any $v \equiv n^2 + 1 \pmod{2n^2}$.

<u>Proof</u>. Assume that a cyclic BBD (v, 2n, 1; n) exists for $v = 2n^2 t + n^2 + 1$, $t \geq 0$. We have $b = vt + \frac{1}{2}v$ and a collection $\mathcal{B}$ of all the base blocks in the design. The edgelength $\ell = \frac{1}{2}v$ should occur $\frac{1}{2}\lambda = \frac{1}{2}$ times in the base blocks [cf. §1.3]. Let $B = \{B^1 ; B^2\}$ be a base block in $\mathcal{B}$ with order m in which ℓ occurs; obviously $m < v$.

Without loss of generality, let 0 be an element of B^1 and ℓ an element of B^2, then there are two possibilities:

(i) $m \in B^1$, which implies that for $2 \leq i \leq v/m - 1$, im belongs to B^1 and for $0 \leq j \leq v/m - 1$, $\ell + jm$ belong to B^2. So far there are v/m edgelengths with the value ℓ, therefore edgelength ℓ occurs at least once in base block B, a contradiction.

(ii) $m \in B^2$, which implies that im belongs to B^1 for i even, $i \leq v/m - 1$ and jm belongs to B^2 for j odd, $j \leq v/m - 1$. But v/m must be even, otherwise such a base block cannot exist. One must have $v/2m \leq n$, but $v/2m = n$ implies that $n|v$, which is not true, hence $v/2m < n$. We also have $\ell + im \in B^2$ for i even, $i < v/m - 1$ and $\ell + jm \in B^1$ for j odd, $j \leq v/m - 1$.

In B^1, the set S_1 of elements im for all i and the set S_2 of elements $\ell + jm$ for all j are either equal or disjoint. If S_1 and S_2 are disjoint, then there are v/m edgelengths with the value ℓ, again a contradiction; hence S_1 and S_2 are equal and only $v/2m$ edgelengths are of the value ℓ. But $v/2m < n$ implies that there are elements in B^1 which do not belong to $S_1 = S_2$. Consider one of these elements, say a, one could start another chain like the one starting with 0, that is, $a + im \in B^1$ for i even, $i < v/m - 1$ and $a + jm \in B^2$ for j odd, $j \leq v/m - 1$. In this chain, there are again $v/2m$ edgelengths with value ℓ, which bring the total of such edgelengths to v/m, a contradiction.

Hence a cyclic BBD $(v, 2n, 1; n)$ with $v \equiv n^2 + 1 \pmod{2n^2}$, n odd and $n \geq 3$ does not exist.

Now consider the case where $n = 3$. We will determine whether non-cyclic designs exist for $v \equiv 10 \pmod{18}$.

Theorem 2.10. There exists a BBD $(v, 6, 1; 3)$ if and only if $v \equiv 1 \pmod 9$ and $v \geq 10$.

Proof. The necessity was proved in Lemma 2.5; also, Lemma 2.6 implies that a cyclic BBD $(v, 6, 1; 3)$ exists for all $v \equiv 1 \pmod{18}$. Therefore we are concerned only with the case where $v \equiv 10 \pmod{18}$ and $v \geq 10$.

A BBD $(28, 6, 1; 3)$ exists as shown below [6] and Theorem 2.8 implies that a BBD $(v, 6, 1; 3)$ also exists for $v \equiv 10 \pmod{18}$ and $v \geq 28$.

Let the 28 elements in the design be partitioned into 4 orbits T_x, $x = 1,2,3,4$, each with 7 elements, by the permutation

$A = (0_1 1_1 \ldots, 6_1)(0_2 1_2 \ldots 6_2)(0_3 1_3 \ldots 6_3)(0_4 1_4 \ldots 6_4)$.

In an orbit T_x, $x = 1,2,3,4$, consider the pure differences $i-j$ between all the unordered pairs of elements (i_x, j_x) in T_x, where $i \neq j$ and $0 \leq i, j \leq 6$. Let

$$\delta(i_x, j_x) = \begin{cases} |i-j| & \text{if } 1 \leq |i-j| \leq 3 , \\ 7-|i-j| & \text{if } 4 \leq |i-j| \leq 6 . \end{cases}$$

then $\delta(i_x, j_x) \in \{1,2,3\}$ for all the unordered pairs of distinct elements (i_x, j_x) in T_x.

Consider two different orbits T_x and T_y, $1 \leq x < y \leq 4$, and the mixed differences between all ordered pairs of elements (i_x, j_y) with $i_x \in T_x$, $j_y \in T_y$ and $0 \leq i, j \leq 6$.

Let

$$\delta(i_x, j_y) = \begin{cases} j-i & \text{if } j \geq i, \\ 7+j-i & \text{if } j < i, \end{cases}$$

then $\delta(i_x, j_y) \in \{0,1,2,3,4,5,6\}$ for all i,j,x,y with $0 \leq i, j \leq 6$, $1 \leq x < y \leq 4$.

Let E_1 (or E_2) be a set of distinct values of $\delta(i_x, j_x)$ (or $\delta(i_x, j_y)$), in other words, $E_1 = \{\ell_x: 1 \leq \ell \leq 3 \text{ and } 1 \leq x \leq 4\}$,

$E_2 = \{\ell_{xy}: 0 \leq \ell \leq 6 \text{ and } 1 \leq x < y \leq 4\}$.

Then, since $b = 42 = 6 \times 7$, we need a collection $\mathcal{B}$ of 6 base blocks B_i, $i = 1,2,\ldots 6$, each of order 7 and such that each value in either E_1 or E_2 occurs once and once only in the 6 base blocks. We say that ℓ_x occurs in B_u if $\ell_x = \delta(i_x, j_x)$ for some pair of elements i_x, j_x which are linked in B_u; the same holds for ℓ_{xy}.

Let $\mathcal{B}$ consist of the following base blocks:

$B_1 = \{(0_1, 2_2, 1_3); (1_1, 0_2, 0_4)\}$, $B_2 = \{(0_1, 4_2, 6_3); (2_1, 3_3, 3_4)\}$,

$B_3 = \{(0_1, 2_2, 6_3); (4_2, 5_3, 4_4)\}$, $B_4 = \{(0_1, 6_2, 5_4); (3_1, 2_3, 6_4)\}$,

$B_5 = \{(0_1, 6_2, 6_4); (5_2, 6_3, 1_4)\}$, $B_6 = \{(0_1, 3_2, 2_4); (6_2, 1_3, 5_4)\}$.

One can verify that these base blocks satisfy the requirement above; hence the 42 blocks generated from them by A indeed form a BBD (28, 6, 1; 3) and A is the automorphism of the design.

Lemma 2.9 implies that a cyclic BBD (10, 6, 1; 3) does not exist; assume that a non-cyclic design with the same parameters exist. Without loss of generality, let $B_1 = \{(0, a, b); (1, 2, 3)\}$,

$B_2 = \{(0, c, d); (4, 5, 6)\}$ and $B_3 = \{(0, e, f); (7, 8, 9)\}$, where a and $b \in B_2^2 \cup B_3^2$. If a and b belong to different subsets B_2^2, B_3^2, then

$\{c,d\} \subset B_3^2$ and $\{e,f\} \subset B_2^2$, which together imply that c and e (and other pairs as well) are linked twice, a contradiction.

Now if a and b belong to the same subset, say a = 4 and b = 5, then elements 4 and 5 are linked to six common elements in B_1 and B_2. Hence 4 and 5 must be linked to two other common elements and each other; but r = 3 implies that these links must occur in the same block, which is impossible. Therefore a BBD (10, 6, 1; 3) does not exist, and the proof of the theorem is complete.

In the case where v = 10 and n = 3, we have, however,

Lemma 2.11. A cyclic BBD (10, 6, λ; 3) exists for $\lambda \geq 2$.

Proof. For $\lambda = 2$, b = 10, let B be a base block of order 10, where B = {(0,1,2); (3,6,9)}. For $\lambda = 3$, b = 15, let H be a base block of order 5, where H = {(0,1,3); (5,6,8)} and B_1 be a base block of order 10, where B_1 = {(0,2,5); (3,4,9)}. It is easy to see that $\mathcal{B}_1$ = {B} and $\mathcal{B}_2$ = {H, B_1} form a cyclic BBD (10, 6, λ; 3) under the cyclic automorphism C with λ = 2 and 3 respectively. The proof is complete when Theorem 1.15 is applied to the results.

Now consider the case where $\lambda = 2$; again one gets the necessary conditions by applying Lemmas 2.2 and 2.3 to our case.

Lemma 2.12. Let $k_1 \leq k_2$. If there exists a BBD (v, k, 2; k_1), then $v \equiv 1$ or $3/4\, n^2 + 1 \pmod{n^2}$ for $k_1 = k_2 = n \equiv 2 \pmod 4$ and $v \equiv 1 \pmod{k_1 k_2}$ otherwise.

Proof. We follow the procedure in the proof of Lemma 2.5. Let $k_1 < k_2$. In Lemma 2.2, $y = (\lambda, 2t) = 2$ and $x = (\lambda, d) = 1$ if either k_1 or k_2 is odd, $x = (\lambda, d) = 2$ if neither of them is odd. Let $h = 0$, then $x = 1$ or 2 and $v \equiv 1 \pmod{k_1k_2(x,y)/x} \equiv 1 \pmod{k_1k_2}$. Let $h = 1$, then $x = 2$ and $v \equiv k_1k_2/2 + 1 \pmod{k_1k_2}$ if $2(2k_1k_2 + 4) \equiv 0 \pmod{16}$ which is not true. Hence the necessary condition when $k_1 < k_2$ is $v \equiv 1 \pmod{k_1k_2}$.

Let $k_1 = k_2 = n$. In Lemma 2.3, we have $x = (n, \lambda)$; therefore $x = 1$ if n is odd and $x = 2$ if n is even. Since $0 \leq h \leq 2x - 1$, when $h = 0$, $x = 1$ or 2 and $v \equiv 1 \pmod{n^2}$. Let $h = 1$, then $x = 1$ or 2 and $v \equiv n^2/x^2 + 1 \pmod{n^2}$ if $2(n^2 + x^2) \equiv 0 \pmod{2x^4}$. But the later congruence is true only when $x = 1$, in which case, the former congruence is $v \equiv n^2 + 1 \pmod{n^2}$, which is the same as $v \equiv 1 \pmod{n^2}$, $v \geq n^2 + 1$. Let $h = 2$, then $x = 2$ and $v \equiv n^2/2 + 1 \pmod{n^2}$ if $4(2n^2 + 4) \equiv 0 \pmod{32}$, which is not true. Consider the last case where $h = 3$, $x = 2$, then $v \equiv 3/4n^2 + 1 \pmod{n^2}$ if $6(3n^2 + 4) \equiv 0 \pmod{32}$ which is true only when $n \equiv 2 \pmod 4$. Hence the necessary condition is $v \equiv 1$ or $3/4\, n^2 + 1 \pmod{n^2}$ for $n \equiv 2 \pmod 4$ and $v \equiv 1 \pmod{n^2}$ otherwise.

An analogue to Lemma 2.6 with $\lambda = 1$ being replaced by $\lambda = 2$ is

Lemma 2.13. Let $k_1 \leq k_2$. A cyclic BBD $(v, k, 2; k_1)$ exists for all $v \equiv 1 \pmod{k_1k_2}$.

Proof. Since $\lambda = 2$, Lemma 2.4 implies that the statement is true if either k_1 or k_2 is even. Let both of k_1 and k_2 be odd and $v = k_1k_2 s + 1$ for some $s \geq 1$; but Lemma 2.6 states that $v = k_1k_2 s + 1$ for s even is a sufficient condition for the existence of a cyclic BBD $(v, k, 1; k_1)$ and consequently of a cyclic BBD $(v, k, 2; k_1)$ when Theorem 1.15 is applied. Hence only the case where s is odd will be considered, that is, let $s = 2t + 1$.

We have $v = 2t\, k_1k_2 + k_1k_2 + 1$; then $b = v(2t + 1)$ and we will construct $2t + 1$ base blocks B_i, $i = 1,2,\ldots, 2t + 1$, each of order v. As usual, B_i is denoted by $B_i = \{B_i^1 ; B_i^2\}$ where

$B_i^1 = (a_{i1}^1, a_{i2}^1, \ldots, a_{ik_1}^1)$ and $B_i^2 = (a_{i1}^2, a_{i2}^2, \ldots, a_{ik_2}^2)$ [cf. §2.1].

For $i = 1,2,\ldots,t$, let $B_{t+i} = B_i$ where $a_{ij}^1 = j-1$ for $j = 1,2,\ldots,k_1$ and $a_{iq}^2 = (i-1)\, k_1k_2 + q\, k_1$ for $q = 1,2,\ldots, k_2$. Now let $a_{2t+1,j}^1 = j-1$ for $j = 1,2,\ldots,k_1$ and $a_{2t+1,q}^2 = t\, k_1k_2 + q\, k_1$ for q for $q = 1,2,\ldots, k_2$.

The set of edgelengths is $E = \{1,2,\ldots, t\, k_1k_2 + ½(k_1k_2 + 1)\}$. It is easy to see that each element of the subset E^1 of E where $E^1 = \{1,2,\ldots, t\, k_1k_2\}$, occurs twice in the 2t base blocks B_i, $i = 1,2,\ldots, 2t$ and in B_{2t+1}, each element of the subset $E - E^1$ except $t\, k_1k_2 + ½(k_1k_2 + 1)$ occurs twice and $t\, k_1k_2 + ½(k_1k_2 + 1) = ½v$ occurs once only. Hence the proof is complete.

Lemmas 2.12 and 2.13 together give us

<u>Theorem 2.14</u>. Let $k_1 < k_2$ or $k_1 = k_2 = n \not\equiv 2 \pmod 4$. A BBD $(v, k, 2; k_1)$ exists if and only if $v \equiv 1 \pmod{k_1k_2}$.

An analogue of Theorem 2.8 for the case where $\lambda = 2$ is

<u>Theorem 2.15</u>. Let $k_1 = k_2 = n \equiv 2 \pmod 4$. If there exists a BBD $(v', 2n, 2; n)$ where $v' = (x + 3/4)n^2 + 1$ and $x \geq 0$, then there exists a BBD $(v, 2n, 2; n)$ for all admissible $v \geq v'$.

Proof. Due to Lemma 2.13, we need not consider all admissible $v \geq v'$ but only the v's which have values $v = (z + 3/4) n^2 + 1$ where $z = x + y$ and $y \geq 0$.

As in the proof of Theorem 2.8, take two BBD's $(v_1, 2n, 2; n)$ where $v_1 = v'$ and $v_2 = yn^2 + 1$ such that these two designs, called D_1 and D_2, respectively, have one element a in common.

Divide the elements in D_1 other than a into $xn + 3u + 1$ subsets S_{1i} of n elements each, where $u = \frac{1}{4}(n-2)$, and one subset S of $\frac{1}{2}n$ elements. Divide the elements D_2 other than a into yn subsets, S_{2j} of n elements each. For each $i = 1, 2, \ldots, xn + 3u + 1$, each $j = 1,2,\ldots, yn$, we can form a block B_{ij} by letting $B_{ij} = \{ S_{1i}; S_{2j}\}$. Let $\overline{B_{ij}} = B_{ij}$ for each i, j, hence the number of blocks constructed this way is $q = 2yn (xn + 3u + 1)$, let the collection of these blocks be denoted by $\mathcal{B}_1$.

Take a block from $\mathcal{B}_1$, say B_{11}, divide the elements of S_{11} into two subsets S^1, S^2, both with $\frac{1}{2}n$ elements, then construct three new blocks $B_{111} = \{(S^1 \cup S^2); S_{21}\}$, $B_{112} = \{(S^1 \cup S); S_{21}\}$ and $B_{113} = \{(S^2 \cup S); S_{21}\}$. Repeat this procedure for every block B_{ij} in $\mathcal{B}_1$ in which $B_{ij}^1 = S_{11}$, that is, for every block B_{1j}, $j = 1,2,\ldots, yn$. We obtain a new collection $\mathcal{B}_2$ from $\mathcal{B}_1$ by including into $\mathcal{B}_2$ all blocks B_{ij}, $\overline{B_{ij}}$, with $i > 1$ and $j \geq 1$, and B_{1j1}, B_{1j2}, B_{1j3} instead of B_{1j}, $\overline{B_{1j}}$ for all $j \geq 1$. Hence the number of blocks in $\mathcal{B}_2$ is $q + yn$.

We can construct a BBD $(v, 2n, 2; n)$, denoted by D, with $v = (z + 3/4) n^2 + 1$ as follows: let the elements of D be all the elements of D_1 and D_2, let the blocks of D be all the blocks in D_1, D_2 and $\mathcal{B}_2$.

We see that the total number of elements is $(x + 3/4)n^2 + 1 + yn^2 = (z + 3/4)n^2 + 1 = v$ and the total number of blocks is $[(x + 3/4)n^2 + 1]\ (x + 3/4) + (yn^2 + 1)\ y + 2yn(xn + 3u + 1) + yn$ $= [(x + y + 3/4)n^2 + 1]\ (x + y + 3/4) = b$. Furthermore, every pair of elements g and h are linked twice in blocks in D_i if they both belong to D_i, $i = 1$ or 2 and they are linked twice in blocks of $\mathcal{B}_2$ if they belong to different D_i's and are neither equal to a. The element a of course is linked to each element of D_1 or D_2 twice. Hence the proof is complete.

For $n = k_1 = k_2 \equiv 2 \pmod 4$ and $n \geq 6$, we do not know if a BBD $(v, 2n, 2; n)$ exists for any $v \equiv (x + 3/4)\ n^2 + 1$ and $x \geq 0$, but in the case where $n = 2$, we have

<u>Theorem 2.16</u>. A BBD $(v, 4, 2; 2)$ exists if and only if $v \equiv 0$ or $1 \pmod 4$, $v \geq 4$.

<u>Proof</u>. The necessity follows from Lemma 2.12 and Lemma 2.13 implies that a cyclic BBD $(v, 4, 2; 2)$ exists for all $v \equiv 1 \pmod 4$. Theorem 2.15 implies that a BBD $(v, 4, 2; 2)$ with $v = 4x$ exists whenever a design with $v = 4x'$, $1 \leq x' \leq x$, exists; but a BBD $(4, 4, 2; 2)$ exists as proved by the following 3 base blocks: $\{(0,3); (1,2)\}$, $\{(0,1); (2,3)\}$ and $\{(0,2); (1,3)\}$.

<u>Remark 1</u>. The Theorem was first proved in [12], but the proof here is simpler because of the use of Theorem 2.15.

<u>Remark 2</u>. Although we have been proving the sufficient conditions for the existence of designs with $\lambda = \lambda'$, where $\lambda' = 1$ or 2, in this section, the sufficient conditions hold also for the same designs for $\lambda = q\lambda'$, where $q \geq 1$, in view of Theorem 1.15.

§2.3 Balanced Bipartite Designs with $k_1 = 1$ or 2

The necessary and sufficient condition for the existence of a BBD with $k_1 = 1$, $k_2 \geq 2$, the necessary condition for the existence of a BBD with $k_1 = 2$, $k_2 \geq 3$, together with the sufficient condition in the case where $v \geq 2k - 3$ and v odd, are given in this section.

Let $k_1 = 1$. Only cases where $k_2 \geq 2$ are considered, since a BBD with $k_1 = k_2 = 1$ always exists.

The necessary conditions (2.2) – (2.5) now become:

$$b = \frac{\lambda v(v-1)}{2k_2}, \quad r = \frac{\lambda k(v-1)}{2k_2}, \quad r_1 = \frac{\lambda(v-1)}{2k_2} \quad \text{and} \quad r_2 = \frac{\lambda(v-1)}{2} .$$

If r_1 is an integer, then so are b, r and r_2. Theorem 1.15 implies that it is sufficient to search for a minimum value of λ such that r_1 is integral for some v. Now λ being minimal implies that $\lambda \mid 2k_2$. Consider the value y, where $1 \leq y \leq k_2$ and $y \mid k_2$. Let $k_2 = xy$ and λ be a mulsiple of y, say $\lambda = zy$, then $r_1 = \frac{z(v-1)}{2x}$. We have either $z = 1$ or $z = 2$, since $z \geq 3$ and $(z,x) = s$, then $s = 1$ implies that λ is not minimal and $s > 1$ is equivalent to the case where λ is a multiple of sy.

Let $z = 1$, then $\lambda = y$ and r_1 being integral implies that $v \equiv 1 \pmod{2x}$; similarly $\lambda = 2y$ and $v \equiv 1 \pmod{x}$ for $z = 2$.

In general, if $G(k_2) = \{y_i,\ i = 1,2,\ldots q : \exists\ x_i \text{ with } x_i y_i = k_2\}$, then $\lambda \in \{y_i, 2y_i, i = 1,2,\ldots,q\}$ and $v \equiv 1 \pmod{2x_i}$ for $\lambda = y_i$, $v \equiv 1 \pmod{x_i}$ for $\lambda = 2y_i$.

We have thus proved the necessary part of the following theorem.

Theorem 2.17. A cyclic BBD $(v,k,\lambda;\ 1)$ exists if and only if $v \equiv 1 \pmod{2x_i}$ for $\lambda = py_i$ and $v \equiv 1 \pmod{x_i}$ for $\lambda = 2py_i$, where $p = 1,2,\ldots,\ y_i \in G(k_2)$ and $x_iy_i = k_2$.

Proof. As $b = r_1v$, the sufficiency will be proved for arbitrary x,y, where $xy = k_2$, and a minimum λ by constructing $t = r_1$ base blocks $B_1, B_2,\ldots, B_t$, each of order v. Consider the following two cases:

Case 1: v odd; then either $v = 2xt+1$, $\lambda = y$ or $v = xt + 1$, $\lambda = 2y$ with x being even. Let $w = (v-1)/2t = xy/\lambda$, then w is an integer. In the base block B_1, let $B_1^1 = (0)$ and $B_1^2 = (1,2,\ldots,\ \lambda w)$ where $\lambda w = xy = k_2$. The elements of B_1 are distinct since $k \leq v$ implies $\lambda \leq 2t$. Also, let P be a permutation of degree v with a 1-cycle, (0), and $2w$ t-cycles $(j,w+j,\ldots,\ (t-1)w+j)$ for $j = 1,2,\ldots,w,tw+1,\ tw+2,\ldots,(t+1)w$. Apply P on B_1 $t-1$ times to generate base blocks $B_2,\ B_3,\ldots,\ B_t$. It can easily be checked that each element of $E = \{1,2,\ldots,tw\}$ occurs in these t base blocks exactly λ times. Hence a cyclic BBD with the given parameters exists.

Case 2: v even; then $v = xt+1$ where x and t are both odd and $\lambda = 2y$. Let $m = \frac{1}{2}(x-1)$, $S_i = \{(i-1)m+p,\ p = 1,2,\ldots,m\}$ and $R_i = \{(t+i-1)m+t+p,\ p = 1,2,\ldots,m\}$ for $i = 1,2,\ldots,t$. Then $S_i \cap R_j = \phi$ for $i,j = 1,2,\ldots,t$ and the set of edgelengths in $E = \bigcup_{i=1}^{t} S_i \cup \bigcup_{j=1}^{\frac{1}{2}(t+1)} (tm+j)$.

Consider base block B_i, $1 \leq i \leq t$, let $B_i^1 = (0)$ and

$B_i^2 = \bigcup_{j=1}^{2y} F_{ij} \cup \bigcup_{\ell=1}^{y} \{h_{i\ell}\}$, where for $i = 1$,

$$F_{1j} = \begin{cases} S_j \text{ for } j = 1,2,\ldots,t \\ R_{j-t} \text{ for } j = t+1, t+2, \ldots, 2t \end{cases}$$

$$h_{1\ell} = \begin{cases} tm + \tfrac{1}{2}(t+3) - \ell \text{ for } \ell = 1,2,\ldots, \tfrac{1}{2}(t+1) \\ tm + \ell \text{ for } \ell = \tfrac{1}{2}(t+3), \tfrac{1}{2}(t+5),\ldots, t . \end{cases}$$

The elements of B_1 are distinct since $k \leq v$ implies that $y \leq t$.

Let P be a permutation of degree v with two 1-cycles (0) and $(tm + \tfrac{1}{2}(t+1))$, two $\tfrac{1}{2}(t-1)$-cycles $(tm+1, tm+2, \ldots, tm+\tfrac{1}{2}(t-1))$ and $(tm+t, tm+t-1, \ldots, tm+\tfrac{1}{2}(t+3))$, and also 2m t-cycles $(j, m+j, \ldots, (t-1)m+j)$ where $j = 1,2,\ldots,m, tm+t+1, tm+t+2, \ldots, tm+t+m$.

Let $g_{i\ell}$, where $i > 1$ be defined by:

$$g_{i\ell} = \begin{cases} tm + \tfrac{1}{2}(t+1) & \text{for } i = \ell = 2,3,\ldots,y \\ tm + \ell & \text{for } i = \ell + 1 \text{ and } \ell = 1,2,\ldots, \tfrac{1}{2}(t-1) \\ tm + t & \text{for } i = \ell + 1 \text{ and } \ell = \tfrac{1}{2}(t+1), \tfrac{1}{2}(t+3),\ldots,t-1 \\ \infty & \text{otherwise} \end{cases}$$

For $i = 2,3,\ldots,t$, if we let

$$F_{ij} = P\,(F_{i-1,j}) \quad \text{where } j = 1,2,\ldots,2y$$

and

$$h_{i\ell} = \begin{cases} g_{i\ell} & \text{if } g_{i\ell} \neq \infty \\ P(h_{i-1,\ell}) & \text{if } g_{i\ell} = \infty \end{cases} \quad \text{where } \ell = 1,2,\ldots, y,$$

we get all the base blocks B_i, $i = 2,3,\ldots,t$. It is a routine matter to check that these t blocks form a base of a cyclic BBD.

Now let $k_1 = 2$; only cases where $k_2 \geq 3$ are considered since the necessary conditions for designs with $k_1 = k_2 = 2$ are different.

<u>Lemma 3.18</u>. If there exists a BBD $(v,k,\lambda;\ 2)$, then

for $\lambda = py_i$, $v \equiv 1 \pmod{4x_i}$;

for $\lambda = 2py_i$, $v \equiv 1 \pmod{x_i}$ where x_i is odd, y_i, v are even and

$v \equiv 1 \pmod{2x_i}$ otherwise;

for $\lambda = 4py_i$, $v \equiv 1 \pmod{4x_i}$ where $x_i \equiv 2 \pmod 4$, v even and

$v \equiv 1 \pmod{x_i}$ otherwise;

x_i's are all the factors of k_2, $y_i = k_2/x_i$ and $p = 1,2,\ldots$

<u>Proof</u>. The necessary conditions (2.2) – (2.5) now become

$$b = \frac{\lambda v(v-1)}{4k_2}, \quad r = \frac{\lambda k(v-1)}{4k_2}, \quad r_1 = \frac{\lambda(v-1)}{2k_2}, \text{ and } r_2 = \frac{\lambda(v-1)}{4}.$$

The procedure of the proof is similar to that of Theorem 2.17; we will find a minimum value for λ such that b, r, r_1 and r_2 would be integral for some v. Consider the value y, where $1 \leq y \leq k_2$ and $y|k_2$; let $k_2 = xy$ and $\lambda = zy$, then z has values 1, 2 or 4.

Let $z = 1$, then r_1 being an integer implies that $v \equiv 1 \pmod{2x}$ and b being an integer implies that $v(v-1) \equiv 0 \pmod{4x}$, hence $v \equiv 1 \pmod{4x}$. Let $z = 2$, then we have $v \equiv 1 \pmod x$ and $v(v-1) \equiv 0 \pmod{2x}$, hence $v \equiv 1 \pmod x$ for x odd, y, v even and $v \equiv 1 \pmod{2x}$ otherwise. For $z = 4$, assuming that $x \nmid (v-1)$ but $4x|(v-1)$, then both $k = xy+2$ and v must be even, which in turn implies that $x \equiv 2 \pmod 4$. Therefore $v \equiv 1 \pmod{4x}$ for $x \equiv 2 \pmod 4$ and $v \equiv 1 \pmod x$ otherwise.

Lemma 2.19. There exists a cyclic BBD $(v,k,\lambda; 2)$ for v odd and $v \geq 2k-3$.

Proof. Consider the cases where $v \equiv 1 \pmod{4x}$, $z = 1$ and $v \equiv 1 \pmod{2x}$, $z = 2$ for arbitrary x,y and minimum λ. Let $m = (v-1)/x$ and $t = z(v-1)/4x$, which are both integers, then $b = vt$, and t base blocks $B_1, B_2, \ldots, B_t$, each of order v, will be constructed.

Let $S_i = \{(i-1)x + q,\ q = 1,2,\ldots,x\}$ where $i = 1,2,\ldots,m$, then $S_i \cap S_j = \phi$ for $i \neq j$. Also let $B_1^1 = (0, xt)$ and $B_1^2 = (\bigcup_{j=1}^{y} F_{t+j})$ where for $j = 1,2,\ldots,y$, $F_{t+j} = S_{t+j}$ if $1 \leq j \leq t$ and $F_{t+j} = S_{2t+j}$ if $t+1 \leq j \leq 2t$. Now $v \geq 2k-3$ implies that $m \geq 2y$, but $t = \frac{1}{4}zm$, hence we get $t+y \leq m$. Furthermore, $4t = m$ for $z = 1$ and $y \leq \frac{1}{2}m = t$ for $z = 2$, hence in either case, the y subsets F_j's of B_1^2 are mutually disjoint.

Let P_1 and P_2 be two permutations of degree v, each with x t-cycles and $v-xt$ 1-cycles (which are omitted), i.e. the x t-cycles of P_1 are $(tx+j,\ (t+1)x+j,\ldots,\ (2t-1)x+j)$ for $j = 1,2,\ldots,x$; and those of P_2 are $(3tx+j,\ (3t+1)x+j,\ \ldots,\ (4t-1)x+j)$ for $j = 1,2,\ldots,x$.
For $v = 4xt+1$, $\lambda = y$, the set of edge-lengths is $E = S_1 \cup S_2 \cup \ldots \cup S_{2t}$. Applying the sum of P_1 and P_2 on B_1 $t-1$ times, we obtain all the required base blocks. For $v = 2xt+1$, $\lambda = 2y$, then $E = S_1 \cup S_2 \cup \ldots \cup S_t$ and applying the permutation P_1 on B_1 repeatedly, we get all the t base blocks.

Now consider the last case: $v = xt+1$, $\lambda = 4y$ where $x \equiv 0 \pmod 4$ and t is odd (otherwise $v = 2xp+1$ for $t = 2p$, a cyclic design exists for such v and $\lambda = 2y$ as proved above). Let $x = 2m$ and $S_i = \{(i-1)m+q,\ q = 1,2,\ldots,m\}$ for $i = 1,2,\ldots,2t$, then $S_i \cap S_j = \phi$ for $i \neq j$ and

$E = S_1 \cup S_2 \cup \cdots \cup S_t$. Now let $B_1^1 = (0, mt)$ and $B_1^2 = (\bigcup_{j=1}^{2y} S_{t+j})$ where $2y \le t$ since $2k - 3 \le v$. Let P be a permutation of degree v, with m t-cycles $(tm+j, (t+1)m+j, \ldots, (2t-1)m+j)$ for $j = 1,2,\ldots,m$. Applying P on B_1 repeatedly, we get all the t base blocks.

For the case where v is even or v is odd and $k \le v < 2k - 3$, the existence of corresponding designs will be proved for $k = 5$ and 6 in §2.4.

§2.4. Balanced Bipartite Designs with $3 \le k \le 6$.

Obviously, for given fixed v, k and k_1, there is always a minimal value λ_{min} of λ such that the necessary conditions (2.2) – (2.5) are satisfied. In this section, we show through a sequence of lemmas that if $k = 3,4,5$ or 6, a BBD with parameters v, k, k_1 and λ_{min} always exists, with the following three exceptions: (i) $v = k = 5$, $k_1 = 2$, $\lambda_{min} = 3$, (ii) $v = k = 6$, $k_1 = 3$, $\lambda_{min} = 3$ and (iii) $v = 10$, $k = 6$, $k_1 = 3$, $\lambda_{min} = 1$. We show that a BBD does not exist for all $\lambda \equiv 3 \pmod 6$ in case (i) or (ii) and for $\lambda = 1$ in case (iii); [cf. Theorem 2.10] and that, on the other hand, a BBD $(5,5,\lambda;\ 2)$ or a BBD $(6,6,\lambda;\ 3)$ exists for all $\lambda \equiv 0 \pmod 6$ and a BBD $(10,6,\lambda;\ 3)$ exists for all $\lambda \ge 2$ [cf. Lemma 2.11].

In view of Theorem 1.15, the existence of a BBD with λ_{min} implies that a BBD $(v, k, \lambda;\ k_1)$ with $3 \le k \le 6$ always exists except in the three cases mentioned above, whenever the necessary conditions (2.2) – (2.5) are satisfied.

We divide the construction of BBD's with $3 \le k \le 6$ into three categories, depending on whether $k_1 = 1$, 2 or 3.

For $k_1 = 1$, $3 \le k \le 6$, the necessary and sufficient condition for the existence of a BBD is stated in Theorem 2.17.

For $k_1 = 2$, $5 \le k \le 6$, the necessary condition for the existence of a BBD is stated in Lemma 2.18 and the sufficient condition for the existence of a BBD with v odd and $v \ge 2k - 3$ is given in Lemma 2.19. Also, Theorems 2.7 and 2.14 provide the necessary and sufficient conditions for the existence of a BBD with $\lambda = 1$ and $\lambda = 2$ respectively. Therefore, we need to construct only BBD's $(v, k, \lambda; 2)$ where $k = 5$ or 6, $\lambda \ge 3$ and v is even, or v is odd and $k \le v < 2k - 3$.

Let $k = 5$, we will consider the first exception:

<u>Lemma 2.20</u>. A BBD $(5, 5, \lambda; 2)$ does not exist for $\lambda \equiv 3 \pmod 6$.

<u>Proof</u>. Assume that there is a BBD with $v = 5 = k$, $k_1 = 2$, $\lambda \equiv 3 \pmod 6$, and $b = r = 5\lambda/3$. Consider two arbitrary and distinct elements x and y, and let U_i, $i = 1,2,3,4$ be the following four sets of blocks:

$$U_1 = \{B:\ x,y \in B^1\}, \quad U_2 = \{B:\ x,y \in B^2\}$$

$$U_3 = \{B:\ x \in B^1, y \in B^2\}, \quad U_4 = \{B:\ x \in B^2, y \in B^1\}.$$

Denoting $|U_i| = u_i$, we have obviously $b = r = u_1 + u_2 + u_3 + u_4$. On the other hand, $r_1 = u_1 + u_3 = u_1 + u_4$, which implies that $u_3 = u_4$. But then $\lambda = u_3 + u_4 = 2u_3$ which is impossible since λ is odd.

<u>Corollary 2.21</u>. A BBD $(5,5,\lambda; 2)$ exists if and only if $\lambda \equiv 0 \pmod 6$.

<u>Proof</u>. The necessity follows from the relations (2.2) - (2.5) and from Lemma 2.20. On the other hand, a cyclic BBD $(5,5,6; 2)$ exists as

proved by the following two base blocks:

$$B_1 = \{(0,1)\ ;\ (2,3,4)\} \text{ and } B_2 = \{(0,2);\ (1,3,4)\}.$$

Since v odd and $k \le v < 2k - 3$ where $k = 5$ implies that v can take value 5 only, we need to consider only the case where v is even.

In Lemmas 2.22 -2.30 which follow, we omit the proof of necessity since it always follows from conditions (2.2) - (2.5).

Lemma 2.22. A cyclic BBD (v, 5, 4; 2) exists if and only if $v \equiv 1 \pmod 3$.

Proof. Let $v = 3t + 1$, $t > 1$ and v be even. A collection $\mathcal{B}$ of t base blocks is given by

$$B_i = B_{i+1} = \{(0,3i+1);\ (\tfrac{1}{2}(3i-1),\ \tfrac{1}{2}(3i+1),\ \tfrac{1}{2}(3i+3))\}$$

$$\text{for } i = 1,3,5,\ldots,\ t-4;$$

$$B_{t-2} = \{(0,3t-5);\ (\tfrac{1}{2}(3t-7),\ \tfrac{1}{2}(3t-5),\ \tfrac{1}{2}(3t-3))\},$$

$$B_{t-1} = B_t = \{(0,3t-2);\ (\tfrac{1}{2}(3t-7),\ \tfrac{1}{2}(3t-5),\ \tfrac{1}{2}(3t-3))\}.$$

Lemma 2.23. A cyclic BBD (v,5,12; 2) exists for any $v \ge 5$.

Proof. Because of Lemma 2.22, we have to consider only $v \equiv 0$ or $2 \pmod 6$.

Let $v = 6t$, $t \ge 1$; a collection of $6t - 1$ base blocks $\mathcal{B}$ is given by

$$B_1 = B_2 = B_3 = \{(0,4);\ (1,2,3)\},\ B_4 = B_5 = \{(0,3);\ (1,2,v-1)\}$$

$$B_{i+x} = \{(0,i+1);\ (\tfrac{1}{2}i-1, \tfrac{1}{2}i,\ \tfrac{1}{2}i+1)\} \text{ where } x = 0,1,$$

$$B_{i+y} = \{(0,\ i+4);\ (\tfrac{1}{2}i+1,\ \tfrac{1}{2}i+2,\ \tfrac{1}{2}i+3)\} \text{ where } y = 2,3,4,$$

$$B_{i+5} = \{(0,\ i+4);\ (\tfrac{1}{2}i,\ \tfrac{1}{2}i+2,\ \tfrac{1}{2}i+4)\}$$

for $i \equiv 0 \pmod 6$, $6 \le i \le 6t-6$.

Let $v = 6t+2$, $t \geq 1$, a collection of $6t+1$ base blocks $\mathcal{B}$ is given by

$B_1 = B_2 = \{(0,5);\ (1,2,3)\}$, $B_3 = B_4 = \{(0,5);\ (1,2,4)\}$,

$B_5 = B_6 = B_7 = \{(0,4);\ (1,2,3)\}$,

and for $i \equiv 2 \pmod 6$, $8 \leq i \leq 6t-4$,

$B_{i+x} = \{(0,i+2);\ (\tfrac{1}{2}i,\ \tfrac{1}{2}i+1,\ \tfrac{1}{2}i+2)\}$ where $x = 0,1,2$

$B_{i+y} = \{(0,i+4);\ (\tfrac{1}{2}i+1,\ \tfrac{1}{2}i+2,\ \tfrac{1}{2}i+3)\}$ where $x = 3,4,5$.

We have dealt with the case where $k_1 = 2$ and $k = 5$, now consider the case where $k_1 = 2$ and $k = 6$. But v odd and $k \leq v < 2k-3$ imply that $v = 7$, we have

<u>Lemma 2.24</u>. A cyclic BBD (7,6,8; 2) exists.

<u>Proof</u>. We have $b = 21$. Let three base blocks be:

$B_1 = \{(0,1);\ (2,3,5,6)\}$, $B_2 = \{(0,2);\ (3,4,5,6)\}$

and $B_3 = \{(0,3);\ (1,2,4,6)\}$; they generate the required design.

<u>Lemma 2.25</u>. A cyclic BBD (v,6,8; 2) exists for all $v \geq 6$.

<u>Proof</u>. The case where v odd is proved in Lemmas 2.19 and 2.24. Let $v = 2t$, $t \geq 3$, then $b = v(t-1) + t$. We will construct a base block H of order $\tfrac{1}{2}v$ and $t-1$ base blocks B_i, $1 \leq i \leq t-1$, each of order v.

Let $H = \{(0,t);\ (1,\ t-1,\ t+1,\ 2t-1)\}$, then the elements 1 and $t-1$ of the set of edgelengths $E = \{1,2,\ldots,t\}$ occur twice each in H. Define the remaining base blocks as follows:

$B_1 = \{(0,1);\ (2,3,\ 2t-2,\ 2t-1)\}$,

$B_2 = \{(0,2);\ (1,\ t,\ t+1,\ t+2)\}$ and for $i = 3,4,\ldots,\ t-1$

$B_i = \{(0,1);\ (i-1,\ i+1,\ 2t-i,\ 2t-i+2)\}$.

It is easy to check that each element of E except 1, t-1 and t occurs eight times in these t-1 base blocks, whereas elements 1 and t-1 each occurs only six times, and t four times.

Let $k_1 = k_2 = 2$. The necessary and sufficient conditions for the existence of a BBD $(v,4,\lambda;\ 2)$ for $\lambda = 1$ and 2 are given in Theorems 2.7 and 2.16 respectively. For the case where $\lambda > 2$, we have

Lemma 2.26. A BBD $(v,\ 4,\ 4;\ 2)$ exists for all $v \geq 4$.

Proof. In view of Theorem 2.16, we have to consider only the case $v \equiv 2$ or 3 (mod 4).

Let $v = 4t + 3$, $t \geq 1$, then $b = v(2t+1)$; a collection $\mathcal{B}$ of $2t+1$ base blocks is given by

$B_1 = B_2 = \{(0,4);\ (1,2)\}$, $B_3 = \{(0,4);\ (1,3)\}$,

$B_i = B_{i+1} = \{(0,\ 2i+1);\ (i,\ i+1)\}$ for $i = 4,6,\ldots,\ 2t$.

Let $v = 4t+2$, $t \geq 1$, then $b = (4t+1)(2t+1)$. Let A be a permutation of degree v on the v elements, that is,

$A = (0_1\ 1_1 \ldots (2t)_1)(0_2\ 1_2 \ldots (2t)_2)$, and $\mathcal{B}$ be a collection of $4t+1$ base blocks B_i each of order $2t+1$, where

$B_1 = B_2 = B_3 = \{(0_1,1_2);\ (1_1,2_2)\}$, $B_4 = \{(0_1,2_2);\ (1_1,1_2)\}$,

$B_5 = \{(0_1,1_1);\ (1_2,2_2)\}$ and, in addition, for $t \geq 2$

$B_i = B_{i+1} = B_{i+2} = B_{i+3} = \{(0_1, 1_2); ((\frac{1}{2}(\frac{1}{2}i+1))_1, (\frac{1}{2}(\frac{1}{2}i+3))_2)\}$,

for $i \equiv 2 \pmod 4$, $6 \le i \le 4t - 2$.

One can verify, as in the proof of Theorem 2.10, that the b blocks obtained by applying A on the $4t + 1$ base blocks B_i repeatedly form the required design.

Let us consider the last category: $k_1 = 3$. But $3 \le k \le 6$ implies that $k_1 = k_2 = 3$, $k = 6$, in which case r_1 and r_2 are redundant and (2.2), (2.3) become $b = \frac{\lambda v(v-1)}{18}$, $r = \frac{\lambda(v-1)}{3}$. Consequently we have the following necessary conditions:

(2.8) $v \equiv 1 \pmod 9$ for $\lambda \equiv 1$ or $2 \pmod 3$,

(2.9) $v \equiv 0,1 \pmod 3$ for $\lambda \equiv 3$ or $6 \pmod 9$,

(2.10) $v \ge 6$ for $\lambda \equiv 0 \pmod 9$.

The necessary and sufficient condition for the existence of a BBD $(v,6,1;\ 3)$ is $v \equiv 1 \pmod 9)$, $v > 10$ as proved in Theorem 2.10 and a BBD $(10,6,\lambda;\ 3)$ exists only for $\lambda > 1$ as proved in Lemma 2.11.

<u>Lemma 2.27</u>. A cyclic BBD $(v,6,3;\ 3)$ exists for $v \equiv 1 \pmod 3$.

<u>Proof</u>. Let $v = 3t+1$, then $v \ge k$ implies that $t \ge 2$. Assuming that v is odd, let $t = 2m$, we get $v = 6m+1$ and $b = mv$. Consider the following m base blocks:

$B_i = \{(0,1,2);\ (3i,3i+3,v-3i+2)\}$, $B_{i+1} = \{(0,1,2);(3i,3i+3,v-3i-1)\}$

where i is odd, $1 \le i \le p$, $p = m-1$ if m is even and $p = m-4$ if m is odd, $m \ge 5$. In the later case, the remaining 3 base blocks are

$B_{m-2} = B_{m-1} = B_m = \{(0,1,2);\ (3m-6,3m-3,3m)\}$. For $m = 1$, let the base block be $\{(0,1,3);\ (2,4,5)\}$ and for $m = 3$, let the 3 base blocks be the same as the 3 remaining base blocks in the general case above.

If v is even, let $s = \frac{1}{2}(t-1)$, then $s \geq 1$ and $v = 6s+4$, $b = sv+v/2$. The case $s = 1$ was proved in Lemma 2.11, hence let $s \geq 2$. Let H be the base block of order v/2, $H = \{(0,2,3s+3);\ (3s+2,\ 3s+4,1)\}$ and the s base blocks, each of order v be

$$B_i = \{(0,1,2);\ (3i,3i+1,\ v-3i+1)\} \text{ for } i = 1,2,\ldots,s-2,$$

$$B_{s-1} = \{(0,1,2);\ (3s-3,3s-2,3s+1)\} \text{ and}$$

$$B_s = \{(0,2,3);\ (3s-1,3s+1,v-3s+2)\}.$$

We will now deal with the second exception.

<u>Lemma 2.28</u>. A BBD $(6,6,\lambda;\ 3)$ exists if and only if $\lambda \equiv 0 \pmod 6$.

<u>Proof</u>. (2.9) and (2.10) state that a BBD $(6,6,\lambda;\ 3)$ exists only if $\lambda \equiv 0 \pmod 3$. Let $\lambda = 6s+3$, then $b = 10s+5 = r$. Assume that a design with these parameters exists, then consider three arbitrary elements x, y and z. Without loss of generality, let U_i, $i = 1,2,3,4$ be the following four sets of blocks:

$$U_1 = \{B:\ x,z \in B^1,\ y \in B^2\},\quad U_2 = \{B:\ x \in B^1,\ y,z \in B^2\}$$

$$U_3 = \{B:\ x,y,z \in B^1\},\quad U_4 = \{B:\ x,y \in B^1,\ z \in B^2\}.$$

Denoting $|U_i| = u_i$, we have obviously $u_1 + u_2 = \lambda$ and $u_1 + u_2 + u_3 + u_4 = b$. Since both x and y must be linked to z in λ blocks, we get $u_1 + u_4 = u_2 + u_4 = \lambda$, which implies that $u_1 = u_2 = \lambda/2$, a contradiction. Hence a BBD $(6,6,\lambda;\ 3)$ does not exist for $\lambda \equiv 3 \pmod 6$; however, a cyclic design with these parameters exists for $\lambda \equiv 0 \pmod 6$. For example, let $\lambda = 6$, then $b = r = 10$. Let F be a base block of order 1, where $F = \{(0,2,4);\ (1,3,5)\}$, H be a base block of order 3, where

$H = \{(0,1,2);\ (3,4,5)\}$, and B be a base block of order 6, where $B = \{(0,1,3);\ (2,4,5)\}$. Hence we have a cyclic BBD (6,6,6;3).

Lemma 2.29. A cyclic BBD (v,6,3;3) exists for $v \equiv 0 \pmod 3$, $v \geq 9$.

Proof. Let $v = 3t$, $t \geq 3$; assuming first that v is odd, then t is odd and $b = \frac{1}{2}v(t-1)+t$. We will construct a base block H of order t, and $\frac{1}{2}(t-1)$ base blocks B_i, $i = 1,2,\ldots,\frac{1}{2}(t-1)$, each of order v. Let $H = \{(0,t,2t);\ (\frac{1}{2}(t-1),\ \frac{1}{2}(3t-1),\ \frac{1}{2}(5t-1))\}$ and $\frac{1}{2}(t-3) = 3z+x$, for $z \geq 0$ and $x = 0$, 1 or 2. We have the following three cases for $z = 1$:

(a) $x = 0$, then $t = 9$ and $B_1 = \{(0,1,2);\ (3,12,17)\}$,
$B_2 = \{(0,1,2);\ (3,14,26)\}$, $B_3 = \{(0,1,2);\ (6,8,23)\}$ and
$B_4 = \{(0,1,2);\ (9,11,20)\}$;

(b) $x = 1$, then $t = 11$ and $B_1 = \{(0,1,2);\ (3,13,22)\}$,
$B_2 = \{(0,1,2);\ (3,15,32)\}$, $B_3 = \{(0,1,2);\ (6,9,16)\}$,
$B_4 = \{(0,1,2);\ (10,12,25)\}$, $B_5 = \{(0,2,3);\ (7,17,29)\}$;

(c) $x = 2$, then $t = 13$ and $B_1 = \{(0,1,2);\ (3,16,25)\}$,
$B_2 = \{(0,1,2);\ (3,18,38)\}$, $B_3 = \{(0,1,2);\ (13,15,28)\}$
$B_4 = B_5 = \{(0,1,2);\ (6,10,19)\}$ and $B_6 = \{(0,2,3);\ (7,10,30)\}$,

For $z \geq 2$, let $B_j = \{(0,1,2);\ (3j,3j+3,v-3j+2)\}$ and $B_{j+1} = \{(0,1,2);\ (3j,3j+3,v-3j-1)\}$ where j is odd, $1 \leq j \leq z-1$ if z is even and $1 \leq j \leq z-4$ if z is odd, in which case, we have three more base blocks, $B_{z-2} = B_{z-1} = B_z = \{(0,1,2);\ (3z-6,3z-3,3z)\}$. The rest of the base blocks are as below,

(a') $x = 0$, then $t = 6z+3$ and for $\ell = 1,2,\ldots,2z+1$,
$B_{z+\ell} = \{(0,1,2);\ (3z+3\ell,3z+2+3\ell,\ v-3z+2-3\ell)\}$;

(b') $x = 1$, then $t = 6z+5$ and $B_{z+1} = \{(0,1,2);\ (3z+3,3z+6,9z+7)\}$,

$B_{z+2} = \{(0,2,3);\ (3z+4,9z+8,v-3z-1)\}$, and for $\ell = 1,2,\ldots,2z$,

$B_{z+2+\ell} = \{(0,1,2);\ (3z+4+3\ell,3z+6+3\ell,v-3z-2-3\ell)\}$;

(c') $x = 2$, then $t = 6z+7$ and $B_{z+1} = \{(0,2,3);\ (3z+4,3z+7,v-3z-6)\}$,

$B_{z+2} = B_{z+3} = \{(0,1,2);\ (3z+3,3z+7,9z+10)\}$ and for $\ell = 1,2,\ldots,2z$,

$B_{z+3+\ell} = \{(0,1,2);\ (3z+7+3\ell,3z+9+3\ell,v-3z-5-3\ell)\}$.

For $z = 0$, the $\frac{1}{2}(t-1) = x+1$ base blocks are given in (a'), (b') and (c').

Now let v be even, then t is even as well and $t \geq 4$, $b = v(\frac{1}{2}t-1) + v/2 + v/3$. Let F be a base block of order $v/3$, where $F = \{(0,t,2t);\ (1,t+1,2t+1)\}$, H be a base block of order $v/2$, where $H = \{(0,\frac{1}{2}(3t)+2,\frac{1}{2}(3t)+3);\ (\frac{1}{2}(3t),2,3)\}$ and the $\frac{1}{2}t-1$ base blocks, each of order v be:

$B_i = \{(0,1,2);\ (3i,3i+3,v-3i+2)\}$ for $i = 1,2,\ldots,u$,

$B_{\frac{1}{2}t-j} = \{(0,1,2);\ (\frac{3t}{2}-3j+1, \frac{3t}{2}-3j+2, v-\frac{3t}{2}+3j)\}$ for $j = 1,2,\ldots,w$;

where u and w vary according to the value of t, i.e.

(a) $t \equiv 0 \pmod 6$, then $u = t/3 - 2$, $w = t/6 - 1$, and the remaining two base blocks are $B_{\frac{t}{3}-1} = \{(0,1,2);\ (t-3,t,t+2)\}$, $B_{t/3} = \{(0,2,3);\ (t-2,t,v-t+1)\}$,

(b) $t \equiv 2 \pmod 6$, then $u = \frac{t-8}{3}$, $w = \frac{t-8}{6}$, and the remaining three base blocks are $B_{\frac{t-5}{3}} = \{(0,1,2);\ (t-5,t-3,t-2)\}$, $B_{\frac{t-2}{3}} = \{(0,1,2);\ (t,t+3,v-t-1)\}$, $B_{\frac{t+1}{3}} = \{(0,2,3);\ (t-4,t,t+2)\}$;

(c) $t \equiv 4 \pmod 6$, then $u = \frac{t-4}{3}$, $w = \frac{t-4}{6}$ and the remaining base block is $B_{\frac{t-1}{3}} = \{(0,1,3);\ (t,t+1,v-t+3)\}$.

Lemma 2.30. A cyclic BBD $(v,6,9;3)$ exists for $v \equiv 2 \pmod 3$, $v \geq 8$.

Proof. Assume that v is odd and let $s = \frac{1}{2}(v-1)$, then $s \equiv 2 \pmod 3$, $s \geq 5$ and $b = vs$. Let the s base blocks, each of order v be:
$B_1 = B_2 = \{(0,1,2);\ (3,4,6)\}$, $B_3 = \{(0,1,3);\ (2,4,5)\}$, $B_4 = \{(0,1,3);\ (2,4,6)\}$, $B_5 = \{(0,2,3);\ (4,5,6)\}$ and for $i \equiv 0 \pmod 3$, $6 \leq i \leq s-2$,
$B_i = B_{i+1} = \{(0,1,2);\ (i+1,i+3,v-i-1)\}$, $B_{i+2} = \{(0,1,2);\ (i+1,i+2,v-i+1)\}$.

Consider the case where v is even. Let $v = 6t+2$, where $t \geq 1$, then $b = 3tv + v/2$. Let the base block of order $v/2$ be $H = \{(0,3,6);\ (3t+1,3t+4,3t+7)\} \bmod v$ and the $3t$ base blocks, each of order v be as follows:

(a) for $t = 1$, $B_1 = \{(0,1,2);\ (3,5,7)\}$, $B_2 = \{(0,1,3);\ (2,5,6)\}$ and $B_3 = \{(0,1,3);\ (2,6,7)\}$;

(b) for $t \geq 2$, $B_i = B_{i+1} = B_{i+2} = \{(0,1,2);\ (i+2,i+5,v-i)\}$, $B_{i+3} = B_{i+4} = B_{i+5} = \{(0,1,2);\ (i+2,i+5,v-i-3)\}$, where $i \equiv 1 \pmod 6$, $1 \leq i \leq p$ and p has values according to whether t is even or odd. If t is even, then $p = 3t - 11$ and the remaining 6 base blocks are:
$B_{3t-5} = B_{3t-4} = B_{3t-3} = \{(0,1,2);\ (3t-3,3t+1,v-3t+5)\}$,
$B_{3t-2} = B_{3t-1} = \{(0,1,2);\ (3t-3,3t,v-3t+2)\}$ and
$B_{3t} = \{(0,1,2);\ (3t-2,3t,v-3t+2)\}$; if t is odd, then $p = 3t-14$ and the remaining 9 base blocks are:
$B_{3t-8} = B_{3t-7} = B_{3t-6} = B_{3t-5} = \{(0,1,2);\ (3t-6,3t-3,v-3t+8)\}$,
$B_{3t-4} = B_{3t-3} = B_{3t-2} = \{(0,1,2);\ (3t-3,3t,3t+1)\}$,
$B_{3t-1} = \{(0,1,2);\ (3t-6,3t-3,3t)\}$ and $B_{3t} = \{(0,1,2);\ (3t-2,3t,v-3t+2)\}$.

The results of §2.4 imply the following theorems:

<u>Theorem 2.31.</u> A necessary and sufficient condition for the existence of a BBD $(v,3,\lambda;\,1)$ is

$v \equiv 1 \pmod 4$ for $\lambda \equiv 1 \pmod 2$

$v \equiv 1 \pmod 2$ for $\lambda \equiv 2 \pmod 4$

$v \geq 3$ for $\lambda \equiv 0 \pmod 4$

<u>Theorem 2.32.</u> A necessary and sufficient condition for the existence of a BBD $(v,4,\lambda;\,1)$ is

$v \equiv 1 \pmod 6$ for $\lambda \equiv 1$ or $5 \pmod 6$

$v \equiv 1 \pmod 3$ for $\lambda \equiv 2$ or $4 \pmod 6$

$v \equiv 1 \pmod 2$ for $\lambda \equiv 3 \pmod 6$

$v \geq 4$ for $\lambda \equiv 0 \pmod 6$

<u>Theorem 2.33.</u> A necessary and sufficient condition for the existence of a BBD $(v,4,\lambda;\,2)$ is

$v \equiv 1 \pmod 8$ for $\lambda \equiv 1 \pmod 2$

$v \equiv 0$ or $1 \pmod 4$ for $\lambda \equiv 2 \pmod 4$

$v \geq 4$ for $\lambda \equiv 0 \pmod 4$

<u>Theorem 2.34.</u> A necessary and sufficient condition for the existence of a BBD $(v,5,\lambda;\,1)$ is

$v \equiv 1 \pmod 8$ for $\lambda \equiv 1 \pmod 2$

$v \equiv 1 \pmod 4$ for $\lambda \equiv 2 \pmod 4$

$v \equiv 1 \pmod 2$ for $\lambda \equiv 4 \pmod 8$

$v \geq 5$ for $\lambda \equiv 0 \pmod 8$.

<u>Theorem 2.35.</u> A necessary and sufficient condition for the existence of a BBD $(v,5,\lambda;2)$ is

$v \equiv 1 \pmod{12}$ for $\lambda \equiv 1$ or $5 \pmod{6}$

$v \equiv 1 \pmod{6}$ for $\lambda \equiv 2$ or $10 \pmod{12}$

$v \equiv 1 \pmod{4}$, $v \neq 5$ for $\lambda \equiv 3 \pmod{6}$

$v \equiv 1 \pmod{3}$ for $\lambda \equiv 4$ or $8 \pmod{12}$

$v \equiv 1 \pmod{2}$ for $\lambda \equiv 6 \pmod{12}$

$v \geq 5$ for $\lambda \equiv 0 \pmod{12}$

<u>Theorem 2.36.</u> A necessary and sufficient condition for the existence of a BBD $(v,6,\lambda;1)$ is

$v \equiv 1 \pmod{10}$ for $\lambda \equiv 1$ or 3 or 7 or $9 \pmod{10}$

$v \equiv 1 \pmod{5}$ for $\lambda \equiv 2$ or 4 or 6 or $8 \pmod{10}$

$v \equiv 1 \pmod{2}$ for $\lambda \equiv 5 \pmod{10}$

$v \geq 6$ for $\lambda \equiv 0 \pmod{10}$

<u>Theorem 2.37.</u> A necessary and sufficient condition for the existence of a BBD $(v,6,\lambda;2)$ is

$v \equiv 1 \pmod{16}$ for $\lambda \equiv 1 \pmod{2}$

$v \equiv 1 \pmod{8}$ for $\lambda \equiv 2$ or $6 \pmod{8}$

$v \equiv 1 \pmod{4}$ for $\lambda \equiv 4 \pmod{8}$

$v \geq 6$ for $\lambda \equiv 0 \pmod{8}$

Theorem 2.38. A necessary and sufficient condition for the existence of a BBD $(v,6,\lambda;\ 3)$ is

$v \equiv 1 \pmod 9$, $v \neq 10$ for $\lambda = 1$

$v \equiv 1 \pmod 9$ for $\lambda \equiv 1$ or $2 \pmod 3$, $\lambda \neq 1$

$v \equiv 0$ or $1 \pmod 3$, $v \neq 6$ for $\lambda \equiv 3$ or $15 \pmod{18}$

$v \equiv 0$ or $1 \pmod 3$ for $\lambda \equiv 6$ or $12 \pmod{18}$

$v > 6$ for $\lambda \equiv 9 \pmod{18}$

$v \geq 6$ for $\lambda \equiv 0 \pmod{18}$

Chapter III: Resolvable Balanced Bipartite Designs

§3.1. Introduction

A block B in this chapter is also denoted by $B = \{ B^1; B^2 \}$ where $B^1 = (a_1^{\,1}, a_2^{\,1}, \ldots, a_{k_1}^{\,1})$; $B^2 = (a_1^{\,2}, a_2^{\,2}, \ldots, a_{k_2}^{\,2})$, and two elements $a_i^s, a_j^t \in B$ are linked in B if and only if $s \neq t$. We also assume that $k_1 \leq k_2$. Definition 1.8 for a resolvable balanced G-design can be restated as follows:

Definition 3.1. A resolvable balanced bipartite design with parameters b, v, r, k, λ, k_1 and k_2 (briefly RBBD $(v,k,\lambda; k_1)$ is a BBD $(v,k,\lambda; k_1)$, the b blocks of which can be partitioned into r complete replications F_i each with b/r blocks such that for any F_i, each element occurs in exactly one block of F_i.

It follows that a necessary condition for the existence of a RBBD is

$$v \equiv 0 \pmod{k} \tag{3.1}$$

The number of links in the blocks which constitute a complete replication must divide the total number of links in a design; therefore $\lambda \binom{v}{2} / \frac{v}{k} \cdot k_1 k_2$ must be an integer. Hence another necessary condition is

$$\lambda k v \equiv \lambda k \pmod{2 k_1 k_2},$$

that is,

$$v \equiv 1 \pmod{\frac{2 k_1 k_2}{(2 k_1 k_2, \lambda k)}}, \tag{3.2}$$

where $(2 k_1 k_2, \lambda k)$ represents the greatest common divisor of $2 k_1 k_2$ and λk.

In the case where $k_1 = k_2 = n$, equations (3.1) and (3.2) become

(3.3) $$v \equiv 0 \pmod{2n}$$

(3.4) $$v \equiv 1 \pmod{\frac{n}{(n,\lambda)}}$$

If we let $v = 2n\,t$ for some $t \geq 1$, then (3.4) implies that $n \mid (n,\lambda)(2n\,t-1)$. But $n \nmid (2n\,t-1)$ unless $n = 1$, therefore $n \mid (n,\lambda)$ for $n > 1$, that is, $n \mid \lambda$. We have just proved

Lemma 3.2. If there exists a RBBD $(v,2n,\lambda;\ n)$ then $v \equiv 0 \pmod{2n}$ and $\lambda \equiv 0 \pmod{n}$.

Analogues of Theorems 1.15 - 1.17 for RBBD's are

Theorem 3.3. If there exist two RBBD's $D_1(v,k,\lambda_1;k_1)$ and $D_2(v,k,\lambda_2;k_1)$, then there exists a RBBD $D_3(v,k,\lambda_3;k_1)$ where $\lambda_3 = a_1\lambda_1 + a_2\lambda_2$, a_1 and a_2 being non-negative integers not both equal to zero.

Theorem 3.4. If there exist two RBBD's $D_1(v_1,k,\lambda;k_1)$ and $D_2(v_2,k,\lambda;k_1)$, then there exists a RBBD $D_3(v_1 \cdot v_2, k, \lambda; k_1)$.

Theorem 3.5. If there exist a RBIBD $D_1(v,k^1,\lambda)$ and a RBBD $D_2(k^1,k,1;k_1)$, then there exists a RBBD $D_3(v,k,\lambda;k_1)$.

Corollary. In Theorems 3.3 and 3.5, D_3 is cyclic if both D_1 and D_2 are; and in Theorem 3.4, D_3 is cyclic if both D_1 and D_2 are cyclic and v_1, v_2 are relatively prime.

The proofs of the statements above can be obtained by following the line of proofs of Theorems 1.15 - 1.17. We also have an analogue of Definition 1.6:

<u>Definition 3.6.</u> Two RBBD's D_1 and D_2 are <u>isomorphic</u> if there is a one-to-one mapping A of elements D_1 onto elements of D_2, with induced mappings A^1 of blocks of D_1 onto blocks of D_2 and A^{11} of complete replications of D_1 onto complete replications of D_2, such that, if x is an element, B a block and F a complete replication of D_1, then $x \varepsilon B$ if and only if $A(x) \varepsilon A^1(B)$ and $B \varepsilon F$ if and only if $A^1(B) \varepsilon A^{11}(F)$. If $D_1 = D_2$, then the mapping A is called an <u>automorphism</u> of D_1.

Again we will denote the induced mappings A^1 and A^{11} by A, assuming that no confusion will thus be caused.

A RBBD is said to be <u>cyclic</u> if it has an automorphism C consisting of a single cycle of length v and C is called a <u>cyclic automorphism</u> (cf. Definition 1.13). Again, let the elements of a design be $0,1,\ldots,v-1$ and $C = (01 \ldots (v-1))$, unless otherwise stated.

Elements and blocks of a RBBD D are partitioned by an automorphism A of D into orbits of elements and blocks respectively (cf. Lemma 1.9). We can define orbits of complete replications similarly. Two complete replications F_i and F_j in D are said to be in the same orbit if $A^q(F_i) = F_j$ for $q \geq 1$. Since being in the same orbit is an equivalence relation, the complete replications are partitioned into orbits, each of which can be represented by any complete replication in it and this will be called a <u>base complete replication</u>. The <u>order</u> of a base complete replication in an orbit is the cardinality of the orbit.

§3.2. Resolvable Balanced Bipartite Designs with $k_1 = 1$, $k > 3$

The necessary conditions, together with the sufficient conditions in some cases, for the existence of a RBBD with $k_1 = 1$ and $k \geq 3$ are given in this section.

The congruence relations (3.1) and (3.2), which together with equations (2.2) - (2.5) are the necessary conditions for the existence of a RBBD $(v,k,\lambda;\ k_1)$, are now reduced to

(3.5) $$v \equiv 0 \pmod{k_2 + 1},$$

(3.6) $$v \equiv 1 \left(\bmod \frac{2k_2}{(2k_2\ ,\ \lambda k)}\right).$$

Since $v = k^2$ satisfies both congruences which are linear with relatively prime modulo, (3.5) and (3.6) are equivalent to

(3.7) $$v \equiv k^2 \left(\bmod \frac{2k_2\ k}{(2k_2\ ,\ \lambda k)}\right).$$

Letting $\lambda = 1$, the congruence relation is reduced to

(3.8) $$v \equiv k^2 \pmod{\{2k_2\ ,\ k\}},$$

where $\{2k_2, k\}$ represents the least common multiple of $2k_2$ and k.

Consider the value of $(2k_2, \lambda k)$ in the denominator of (3.7), let $(k_2,\lambda) = x$ and $k_2 = \bar{k}_2 x, \lambda = \bar{\lambda}x$, then there are two possible cases:

(i) $(2k_2,\lambda k) = x$, that is, k is odd and $2\nmid\bar{\lambda}$,

then $v \equiv k^2 \pmod{2\bar{k}_2\ k}$ which is equivalent to

$$v \equiv \begin{cases} k \pmod{2\bar{k}_2\, k} & \text{if } x \text{ is even} \\ \bar{k}_2 k + k \pmod{2\bar{k}_2\, k} & \text{if } x > 1 \text{ is odd} \\ k^2 \pmod{2k_2\, k} & \text{if } x = 1 . \end{cases}$$

Note that when x is even, λ is even also with $\bar{\lambda}$ being odd and when x is odd, λ is odd too.

(ii) $(2k_2, \lambda k) = 2x$, that is, k is even or $\bar{\lambda}$ is even, then

$v \equiv k^2 \pmod{\bar{k}_2\, k}$ which is equivalent to $v \equiv k \pmod{\bar{k}_2\, k}$.

Equations (2.2) - (2.5) state that

$$b = \frac{\lambda v(v-1)}{2k_2}, \quad r = \frac{\lambda k(v-1)}{2k_2}, \quad r_1 = \frac{\lambda(v-1)}{2k_2} \quad \text{and } r_2 = \frac{\lambda(v-1)}{2} ;$$

hence r_1 being an integer implies that b, r and r_2 are integral also.

We will test the admissibility of each v in cases (i) and (ii) by finding out whether r_1 is integral for the particular v. Let $v = 2\bar{k}_2\, k\, t + k$, $t \geq 0$, then $r_1 = \bar{\lambda}(kt + x/2)$, which is integral since x is even. Let $v = 2\bar{k}_2\, k\, t + \bar{k}_2\, k + k$, $t \geq 0$, then $r_1 = \bar{\lambda}(k\, t + \tfrac{1}{2}(k+x))$ which is integral since both x and k are odd. Let $v = 2k_2\, k\, t + k^2$, $t \geq 0$, then $r_1 = \lambda(kt + \tfrac{1}{2}(k+1))$. Lastly let $v = \bar{k}_2\, k\, t + k$, $t \geq 0$, then $r_1 = \tfrac{1}{2}\bar{\lambda}(kt + x)$, which is integral only if $\bar{\lambda}$ is even (since $\bar{\lambda}$ being odd implies that k is even and consequently x is odd).

We could summarize the results in

Lemma 3.7. The necessary condition for the existence of a RBBD $(v,k,\lambda;1)$ is

(I) $v \equiv k^2 \pmod{2k_2k}$ if k is odd, $x = 1$ and $\lambda = 1,2,3,\ldots$

(II) $v \equiv k \pmod{2\bar{k}_2k}$ if k is odd, x is even and $\lambda = qx$, $q = 1,3,5,\ldots$

(III) $v \equiv \bar{k}_2k + k \pmod{2\bar{k}_2k}$ if k is odd and x is odd > 1,

$\lambda = qx$, $q = 1,3,5,\ldots$

(IV) $v \equiv k \pmod{\bar{k}_2k}$ if $\lambda = 2xq$, $q = 1,2,3,\ldots$

where $(k_2, \lambda) = x$, $k_2 = \bar{k}_2x$ and $\lambda = \bar{\lambda}x$.

We will prove that some of the necessary conditions are also sufficient for the existence of a RBBD $(v,k,\lambda;1)$; Theorem 3.3 implies that we need to construct designs with minimum λ only.

Lemma 3.8. Let k be odd. A RBBD $(v,k,\lambda;1)$ exists for $v = k^{2q}$ where q and λ are positive integers.

Proof. $v = k^{2q}$ satisfies the necessary condition since $k^{2q} = 2k_2k\,[k(k^{2q-2} - 1)/2k_2] + k^2$ where $(k^{2q-2} - 1)/2k_2$ is an integer. In view of Theorems 3.3 and 3.4, the proof is complete if a RBBD $(k^2, k, 1; 1)$, say D, can be constructed.

Let the k^2 elements in D be partitioned into k orbits, T_x, of k elements each by the permutation A, where

$A = (0_1\, 1_1 \ldots (k-1)_1)(0_2\, 1_2 \ldots (k-1)_2) \ldots (0_i\, 1_i \ldots (k-1)_i) \ldots (0_k\, 1_k \ldots (k-1)_k)$.

The number of complete replications in D is $r = \frac{1}{2}k(k+1)$ and the number of blocks in each complete replication is k.

We will follow the procedure in the proof of Theorem 2.10.

In an orbit T_x, $1 \le x \le k$, consider the pure differences i-j between all the unordered pairs of elements (i_x, j_x) in T_x, where $i \neq j$ and $0 \le i, j \le k-1$. Let

$$\delta(i_x, j_x) = \begin{cases} |i - j| & \text{if } 1 \le |i - j| \le \tfrac{1}{2}(k-1) \\ k - |i - j| & \text{if } \tfrac{1}{2}(k+1) \le |i - j| \le k-1, \end{cases}$$

then $1 \le \delta(i_x, j_x) \le \tfrac{1}{2}(k-1)$.

Consider two distinct orbits T_x, T_y where $1 \le x < y \le k$ and the mixed differences between all the ordered pairs of elements (i_x, j_y) where $i_x \in T_x$, $j_y \in T_y$ and $0 \le i, j \le k-1$. Let

$$\delta(i_x, j_y) = \begin{cases} j - i & \text{if } j \ge i \\ k + j - i & \text{if } j < i \end{cases},$$

then $0 \le \delta(i_x, j_y) \le k-1$.

Let E_1 (or E_2) be a set of distinct values of $\delta(i_x, j_x)$ (or $\delta(i_x, j_y)$), in other words,

$$E_1 = \{\ell_x : 1 \le \ell \le \tfrac{1}{2}(k-1) \ , \ 1 \le x \le k\}$$

$$E_2 = \{\ell_{xy} : 0 \le \ell \le k-1 \ , \ 1 \le x < y \le k\}$$

Since there are k(k-1) links in a complete replication, we will construct $\tfrac{1}{2}(k+1)$ base complete replications, F_i, such that each of the $\tfrac{1}{2}(k-1)k$ values of E_1 or each of the $\binom{k}{2}k$ values of E_2 occurs once and once only in a block of these base complete replications. A value, ℓ_x say, is said to occur in a block B if $\ell_x = \delta(i_x, j_x)$ for a pair of distinct elements i_x, j_x which are linked in B; similarly, ℓ_{xy} occurs in B if $\ell_{xy} = \delta(i_x, j_y)$

for ordered pair of elements (i_x, j_y) which are linked in B.

For $1 \le i \le \frac{1}{2}(k-1)$, let $F_i = \bigcup_{p=1}^{k} \{B_{ip}\}$ where $B_{ip} = \{(0_p); (\bigcup_{j=1}^{k-1} \{j_{p+i}\})\}$. Then in these $\frac{1}{2}(k-1)k$ blocks, each element of $E_3 \subset E_2$, where $E_3 = \{\ell_{xy} : 1 \le \ell \le k-1,\ 1 \le x < y \le k\}$, occurs once and once only.

Let $F_{\frac{1}{2}(k+1)} = \bigcup_{p=1}^{k} \{B_p\}$ where

$$B_p = \{((p-1)_p); (\bigcup_{j=1}^{\frac{1}{2}(k-1)} \{(p-1+j)_p \cup (p-1)_{p+j}\})\} .$$

Then, in these k blocks, each element of $E_4 \subset E_1 \cup E_2$ where

$E_4 = E_1 \cup \{\ell_{xy} : \ell = 0,\ 1 \le x < y \le k\}$, occurs once and once only.

It is easy to verify that, by applying the permutation A on the elements of these $\frac{1}{2}(k+1)$ base complete replications repeatedly, one obtains $\frac{1}{2}k(k+1)$ complete replications which form a RBBD $(k^2, k, 1; 1)$ as required. The permutation A is in fact an automorphism of the constructed design.

As the construction of a RBBD $(v,k,\lambda;\ k_1)$ is similar to the one above, from now on, only the automorphism A and a set of base complete replications will be given in the proofs, unless the procedures are different.

Similar to Lemma 3.8 we also have

__Lemma 3.9.__ A RBBD $(v,k,\lambda;\ 1)$ exists for λ even and $v = k^q$ where q is a positive integer.

__Proof.__ Since k^q can be expressed as $k^q = \bar{k}_2 k[k(k^{q-2} - 1)/\bar{k}_2 + x] + k$ where $(k^{q-2} - 1)/\bar{k}_2$ is an integer for $q \ge 2$; hence $v = k^q$ satisfies the necessary condition (II) or (IV) of Lemma 3.7 for $q = 1$ or $q \ge 2$.

A RBBD (k,k,2;1) will be constructed, since the existence of such a design, together with Theorems 3.3 and 3.4, would prove the statement of the lemma.

For $b = v = k = r$, a complete replication would consist of just one block. Consider a base block $B_1 = \{(0); (\bigcup_{j=1}^{k-1} \{j\})\}$ and a cyclic automorphism $C = (0\ 1\ 2\ \ldots\ k-1)$; it is easy to see that k blocks can be obtained by applying C on B_1 repeatedly, hence a cyclic RBBD (k,k,2;1) exists.

We will prove that the necessary condition (II) of Lemma 3.7 is also sufficient in

Lemma 3.10. Let k be odd. A RBBD $(v,k,\lambda;1)$ exists for $\lambda = qx$ where q is a positive integer and $v \equiv k(\text{mod } 2\ \bar{k}_2\ k)$ where $\bar{k}_2 = k_2/x$, $x = (k_2, \lambda)$ and x is even.

Proof. Again due to Theorem 3.3, we need to consider only the case where $\lambda = x$. Let $v = 2\bar{k}_2\ k\ t + k$ where $t \geq 1$, since the case where $t = 0$ has been proved in Lemma 3.9. We have $r = k(kt + \frac{1}{2}x)$ and the number of blocks in each complete replication is $z = 2\bar{k}_2\ t + 1$.

Let A be a permutation of degree $v = kz$, that is, $A = (0_1\ 1_1\ \ldots (k-1)_1)\ (0_2\ 1_2\ \ldots\ (k-1)_2)\ \ldots\ (0_z\ 1_z\ \ldots\ (k-1)_z)$. As in the proof of Lemma 3.9, a RBBD (kz, k, x; 1) exists if $kt + \frac{1}{2}x = t + \frac{1}{2}xz$ base complete replications with respect to A can be constructed such that each value of $E_1 = \{\ell_f : 1 \leq \ell \leq \frac{1}{2}(k-1), 1 \leq f \leq z\}$ or of $E_2 = \{\ell_{fg} : 0 \leq \ell \leq k-1, 1 \leq f < g \leq z\}$ occurs x times in the blocks of these base complete replications.

For $i = 1,2,\ldots, t$, let $F_i = \bigcup_{j=1}^{z} \{B_{ij}\}$ be base complete replications, where $B_{ij}^1 = (0_j)$ and

$$B_{ij}^2 = \left(\bigcup_{w=1}^{\frac{1}{2}x} \bigcup_{q=1}^{\bar{k}_2} \{ (2\bar{k}_2 w - 2\bar{k}_2 + q)_{i\bar{k}_2 - \bar{k}_2 + j + q} \cup (2\bar{k}_2 w - \bar{k}_2 + q)_{t\bar{k}_2 + i\bar{k}_2 - \bar{k}_2 + j + q} \} \right)$$

for each $j = 1,2,\ldots,z$ and all the subscripts are taken modulo z. For all ordered pairs (f,g) where $1 \le f < g \le z$, define p to be $p \equiv g-f \pmod{\bar{k}_2}$ and $p > 0$; let

$$E_3 = \bigcup_{m=0}^{\frac{1}{2}x-1} \left\{ \bigcup_{\substack{1 \le f < g \le z \\ 1 \le g-f \le t\bar{k}_2}} \{(2m\bar{k}_2 + p)_{fg}\} \bigcup_{\substack{1 \le f < g \le z \\ t\bar{k}_2 + 1 \le g-f \le 2t\bar{k}_2}} \{(2m\bar{k}_2 + \bar{k}_2 + p)_{fg}\} \right\},$$

then $E_3 \subset E_2$, and each value of E_3 occurs twice in the blocks F_i, $1 \le i \le t$.

We now consider the construction of the other $\frac{1}{2}xz$ base complete replications $F_{t+mz+i} = \bigcup_{j=1}^{z} \{B_{t+mz+i,j}\}$ where $i = 1,2,\ldots,z$ and $m = 0,1,\ldots, \frac{1}{2}x-1$.

Let $B_{t+mz+i,i} = \{(0_i); (\bigcup_{q=1}^{k_2} \{q_i\})\}$ for all i,m, then each value of E_1 occurs $2.\frac{1}{2}x$ times in these $\frac{1}{2}xz$ blocks.

For $j = 1,2,\ldots,\frac{1}{2}(z-1)$; let $f = \min\{i-j, i+j\}$ and $g = \max\{i-j, i+j\}$, where elements $i-j$, $i+j$ are taken modulo z and are greater than zero. Also, let

$$Q_{f,g,m} = \{1, 2, \ldots, k_2\} - \{y^*\} \quad \text{where}$$

$$y^* = \begin{cases} 2m\bar{k}_2 + p & \text{if } 1 \le g - f \le t\bar{k}_2 \\ 2m\bar{k}_2 + \bar{k}_2 + p & \text{if } t\bar{k}_2 + 1 \le g - f \le 2t\bar{k}_2 \end{cases}$$

and p with values given above. Therefore, for $i = 1,2,\ldots,z$, $m = 0,1,\ldots,\frac{1}{2}x-1$ and $j = 1,2,\ldots, \frac{1}{2}(z-1)$, if we let

$$B_{t+mz+i,f} = \{(0_f); (\{0_g\} \bigcup_{q \in Q_{f,g,m}} \{q_g\})\},$$

$$B_{t+mz+i,g} = \{(y^*_g); (\bigcup_{q=1}^{k_2} \{q_f\})\},$$

where y^* is dependent on f, g and m and is the element in $\{1,2,\ldots,k_2\}$ which does not belong to $Q_{f,g,m}$ then in these $\frac{1}{2}xz(z-1)$ blocks each value of E_3 occurs x-2 times and each value of $E_2 - E_3$ occurs x times. Thus the proof is complete.

The necessary condition (IV) in Lemma 3.7 is also sufficient, as shown in

<u>Lemma 3.11</u>. A RBBD $(v,k,\lambda;1)$ exists for $v \equiv k(\mathrm{mod}\ \bar{k}_2 k)$ and $\lambda = 2qx$ where $\bar{k}_2 = k_2/x$, $x = (k_2,\lambda)$ and q is a positive integer.

<u>Proof</u>. In view of Theorem 3.3, we need to consider only the case where $\lambda = 2x$. Let $v = \bar{k}_2 k t + k = kz$, where t is an integer ≥ 1 and $z = \bar{k}_2 t + 1$; the case where $t = 0$ has been proved in Lemma 3.9. The total number of complete replications is $r = k(kt + x)$ and the number of blocks in each complete replication is z.

The procedure is similar to that in the proof of Lemma 3.10.

Let A be a permutation of degree v, $A = (0_1\ 1_1 \ldots (k-1)_1)(0_2\ 1_2 \ldots (k-2)_2) \ldots$ $\ldots (0_z\ 1_z \ldots (k-1)_z)$. A RBBD (kz, k, 2x; 1) exists if $kt + x = zx + t$ complete replications, which are base relative to A, can be constructed such that each value of

$E_1 = \{\ell_f : 1 \le \ell \le [\tfrac{1}{2}k],\ 1 \le f \le z\}$ or of

$E_2 = \{\ell_{fg} : 0 \le \ell \le k-1,\ 1 \le f < g \le z\}$ occurs 2x times in the blocks of these base complete replications.

For $i = 1,2,\ldots,t$, let $F_i = \bigcup_{j=1}^{z} \{B_{ij}\}$ be base complete replications, where $B^1_{ij} = (0_j)$ and $B^2_{ij} = (\bigcup_{w=1}^{x} \bigcup_{q=1}^{\bar{k}_2} \{(w\bar{k}_2 - \bar{k}_2 + q)_j, (w\bar{k}_2 - \bar{k}_2 + j + q)_{[illegible]}\})$, all the subscripts are taken modulo z. For all ordered pairs (f,g), where $1 \le f < g \le z$, define $p \equiv g - f \pmod{\bar{k}_2}$ and $p > 0$. Let

$$E_3 = \bigcup_{1 \le f < g \le z} \bigcup_{m=0}^{x-1} \{(m\bar{k}_2 + p)_{fg}\}$$ then $E_3 \subset E_2$, and each value of E_3

occurs twice in the blocks of F_i, $1 \le i \le t$.

The construction of the other zx base complete replications F_{t+mz+i}, $0 \le m \le x-1$, $1 \le i \le z$, will be divided into two cases:

(i) z is odd; then t is even (since t being odd implies that z is even, irrespective of whether k is even or odd). For $i = 1,2,\ldots,z$, $m = 0,1,\ldots,x-1$ let $B^1_{t+mz+i,i} = (0_i)$ and $B^2_{t+mz+i,i} = (\bigcup_{q=1}^{k_2} \{q_i\})$, then each value of E_1 occurs 2x times in these xz blocks except that when k is even, then k/2 occurs x times.

For each $j = 1,2,\ldots,\tfrac{1}{2}(z-1)$, let $f = \min\{i-j,\ i+j\}$ and $g = \max\{i-j,\ i+j\}$, where elements $i-j$, $i+j$ are taken modulo z and are greater than zero.

Also let $Q_{f,g,m} = \{1,2,\ldots,k_2\} - \{m\bar{k}_2 + p\}$ where $0 \le m \le x-1$ and p as defined above.

$$B_{t+mz+i,f} = \{(0_f);\ (\{0_g\} \bigcup_{q \in Q_{f,g,m}} \{q_g\}),$$

$$B_{t+mz+i,g} = \{((m\bar{k}_2 + p)_g);\ (\bigcup_{q=1}^{k_2} \{q_f\}))\}.$$

It is easy to check that in these blocks each value of $E_2 - E_3$ occurs $2x$ times and each value of E_3 occurs $2x - 2$ times.

(ii) z is even; then let $B_{t+mz+1,j} = \{(0_j);\ (\bigcup_{q=1}^{k_2} \{q_j\})\}$ for i, $j = 1,2,\ldots,z$ and $m = 0,1,\ldots,x-1$, and $F_{t+mz+i} = \bigcup_{j=1}^{z} \{B_{t+mz+i,j}\}$.

The other base complete replications are similar to the ones in case (i). For each $j = 1,2,\ldots,\frac{1}{2}(z-2)$ let $f = \min\{i-j, i+j\}$ and $g = \max\{i-j, i+j\}$ where $i-j$, $i+j$ satisfy the same conditions as they did in case (i); again $Q_{f,g,m} = \{1,2,\ldots,k_2\} - \{m\bar{k}_2 + p\}$ as in case (i). Now for $i = 1,2,\ldots,z-1$, let

$$B_{t+mz+1+i,u} = \{(0_u);\ (\{0_w\} \bigcup_{q \in Q_{f,g,m}} \{q_w\})\},$$

$$B_{t+mz+1+i,w} = \{((m\bar{k}_2+p)_w;\ (\bigcup_{q=1}^{k_2} \{q_u\})\ \},$$

where we have either $u=\min\{i,y\}, w=\max\{i,y\}$ for $y \equiv i+\frac{1}{2}z \pmod{z}$, $y>0$ or $u=f, w=g$ for all $(z-2)/2$ ordered pairs (f,g). It is easy to check that these zx base complete replications satisfy the required conditions and hence the proof is complete.

All the RBBD's $(v,k,\lambda;\ 1)$ which have been constructed are listed below.

$$
(3.9)\begin{cases}
\text{(I)} & k \text{ odd}, \ \lambda \geq 1 \text{ and } v = k^{2q}, \ q \geq 1, \\
\text{(II)} & k \text{ odd}, \ \lambda = px, \ p \geq 1 \text{ and } v \equiv k \pmod{2k_2k/x} \\
 & \qquad \text{where } x \text{ is even and } x \mid k_2, \\
\text{(III)} & k \text{ odd}, \ \lambda = px, \ p \geq 1 \text{ and } v = k^{2q}, \ q \geq 1 \\
 & \qquad \text{where } x \text{ is odd and } x \mid k_2, \ x > 1, \\
\text{(IV)} & \lambda = p2x, \ p \geq 1 \text{ and } v \equiv k \pmod{k_2k/x} \text{ where } x \mid k_2.
\end{cases}
$$

Note that $v = k^{2q}$ satisfies the necessary condition of (III) in Lemma 3.7 since

$$v = k^{2q} = \frac{k_2k}{x}\,(xk(k^{2q-2} - 1)/k_2 + x) + k,$$

where $xk\,(k^{2q-2} - 1)/k_2 + x = xk(k^{2q-3} + k^{2q-4} + \ldots + k+1) + x$ which is always odd.

Therefore we obtain the following

<u>Theorem 3.12</u>. A necessary and sufficient condition for the existence of a RBBD $(v,\ k,\ \lambda;\ 1)$ where k is even is

$$v \equiv k \pmod{k_2k/x} \quad \text{and} \quad \lambda \equiv 0 \pmod{2x}$$

where $x \mid k_2$.

§3.3. Resolvable Balanced Bipartite Designs with Small k

Let $k_1 = 1$, $k = 3$; Lemma 3.7 states that the necessary condition for the existence of a RBBD $(v,3,\lambda;\ 1)$ is

(I) $v \equiv 9 \pmod{12}$ for $\lambda \geq 1$

(II) $v \equiv 3 \pmod 6$ for $\lambda \equiv 0 \pmod 2$

(III) $v \equiv 0 \pmod 3$ for $\lambda \equiv 0 \pmod 4$

It was proved in [25] that condition (I) is also sufficient. Together with the results in (3.9), we have

Theorem 3.13. A necessary and sufficient condition for the existence of a RBBD $(v,3,\lambda;\ 1)$ is

$v \equiv 9 \pmod{12}$ for $\lambda \equiv 1 \pmod 2$

$v \equiv 3 \pmod 6$ for $\lambda \equiv 2 \pmod 4$

$v \equiv 0 \pmod 3$ for $\lambda \equiv 0 \pmod 4$.

We now consider the case where $k_1 = 2$ and $k \geq 5$. Equations (3.1) and (3.2) imply that for $k_2 = k-2$, we get

$$(3.10)\qquad \begin{cases} v \equiv 0 \pmod{k_2 + 2} \\ v \equiv 1 \left(\bmod \dfrac{4k_2}{(4k_2,\ \lambda(k_2+2))}\right) \end{cases}$$

If $\lambda = 1$, then these equations become

$$(3.11)\qquad \begin{cases} v \equiv 0 \pmod{k_2 + 2} \\ v \equiv 1 \left(\bmod \dfrac{4k_2}{(4k_2,\ k_2+2)}\right) \end{cases}$$

and we have

<u>Lemma 3.14</u>. If there exists a RBBD $(v,k,1;\ 2)$, then

$v \equiv \frac{1}{2}k_2(\frac{1}{2}k_2 + 2) + 1 \pmod{\frac{1}{2}k_2(k_2 + 2)}$ for $k \equiv 0 \pmod 8$,

$v \equiv 4k_2 x + 1 \pmod{4k_2 k}$ where x is an integer $0 \le x \le k_2 + 1$ such that $4k_2 x + 1 \equiv 0 \pmod k$ for k odd.

<u>Proof</u>. If k is even, then v is even as well by (3.11). Let $k_2 = 4u$, then $v \equiv 1 \pmod{8u}$ which implies that v is odd, a contradiction. Let $k_2 = 4u + 2$, then the greatest common divisor in (3.11) becomes $4(4u + 2,\ u + 1)$, where $(4u + 2,\ u + 1) = 1$ if u is even and in that case, $v \equiv 1 \pmod{4u + 2}$, that is v is odd which is a contradiction. Now let u be odd, then $v \equiv 1 \pmod{2u + 1}$ and $v \equiv 0 \pmod{4u + 4}$. Let $v \equiv (2u + 1)x + 1 \pmod{4(u + 1)(2u + 1)}$ where $0 \le x \le 4u + 3$ such that $(2u + 1)x + 1 \equiv 0 \pmod{4u + 4}$. Consider $s = ((2u + 1)x + 1)/4(u + 1)$ where u is odd; s is an integer implies that x must be odd. Hence let $x = 2q + 1$ where $0 \le q \le 2u + 1$, then

$$s = \frac{(2u + 1)(2q + 1) + 1}{4(u + 1)} = \frac{(2u + 1)\,q + u + 1}{2(u + 1)}.$$

We see that q must be even; furthermore, $(u + 1) \nmid (2u + 1)$ implies $(u + 1) \mid q$, that is $q = u + 1$ and $x = 2u + 3$. Hence the necessary condition when $k \equiv 0 \pmod 8$ is $v \equiv (2u + 1)(2u + 3) + 1 \pmod{4(u + 1)(2u + 1)}$ which is $v \equiv \frac{1}{2}k_2(\frac{1}{2}k_2 + 2) + 1 \pmod{\frac{1}{2}k_2(k_2 + 2)}$.

If k is odd, then $v \equiv 1 \pmod{4k_2}$. Let $v = 4k_2x + 1 \pmod{4k_2(k_2 + 2)}$ where $0 \le x \le k_2 + 1$ such that $4k_2x + 1 \equiv 0 \pmod{k_2 + 2}$. Hence the statement is proved.

Consider the simplest case with $k_1 = 2$ and $\lambda = 1$, that is, $k_2 = 3$. Lemma 3.14 states that $v \equiv 12x + 1 \pmod{60}$ where $0 \le x \le 4$ such that $12x + 1 \equiv 0 \pmod 5$. Hence $x = 2$ and the necessary condition for the existence of a RBBD $(v,5,1;\ 2)$ is $v \equiv 25 \pmod{60}$. We have.

<u>Lemma 3.15</u>. There exists a RBBD $(25,5,\lambda;\ 2)$ for $\lambda \ge 1$.

<u>Proof</u>. Let $\lambda = 1$, then $r = 10$. Let A be a permutation where

$A = (0_1\ 1_1 \ldots 4_1)(0_2\ 1_2 \ldots 4_2) \ldots\ldots (0_5\ 1_5 \ldots 4_5)$.

Two complete replications $F_i = \bigcup_{j=1}^{5} \{B_{ij}\}$, $i = 1,2$ are constructed below.

Let $B_{1j} = \{(0_j,\ 2_{j+4});\ (1_{j+1},\ 3_{j+1},\ 4_{j+1})\}$ and

$B_{2j} = \{((j-1)_j,\ (j+1)_j);\ ((j+2)_j,\ (j+1)_{j+1},\ (j+1)_{j+3})\}$ for $j = 1,2,\ldots 5$

and all the integers are taken modulo 5 and the value j_0 should be replaced by j_5.

It is easy to check that these two complete replications are actually base complete replications of the design with respect to A.

We now consider the case where $k_1 = k_2 = n \ge 1$, that is $n = \frac{1}{2}k$ and k is even. Lemma 3.2 states that the necessary condition for the existence of a RBBD $(v,2n,\lambda;\ n)$ is $v \equiv 0 \pmod{2n}$ with $\lambda \equiv 0 \pmod n$.

Take the simplest case, that is, $n = 1$. It is well-known that in this case, the necessary condition is also sufficient, hence we have (cf. [10])

<u>Theorem 3.16</u>. A RBBD $(v,2,\lambda;\ 1)$ exists for all $v \equiv 0 \pmod 2$ and $\lambda \ge 1$.

Proof. Let $\lambda = 1$ and $v = 2t$, and the elements be $\{\infty\} \cup \{0,1,\ldots,2t-2\}$. Let the $r = 2t-1$ complete replications be denoted by $F_i = \bigcup_{j=1}^{t} \{B_{ij}\}$, $1 \le i \le 2t-1$, where $B_{ij} = \{(i-j-1); (i+j-1)\}$ for $j = 1,2,\ldots,t-1$ and $B_{it} = \{(\infty); (i-1)\}$, each element is taken modulo $2t-1$. Again the proof is complete when Theorem 3.3 is used.

For $n \ge 1$, we get the following

Theorem 3.17. Let $k_1 = k_2 = n \ge 1$. If there exists a RBBD $(v,2n,\lambda; n)$ where $v = 2nt$ and $\lambda = ns$, $s \ge 1$, then there exists a RBBD $(v,2n,\lambda; n)$ where $v = 2nqt$, $q \ge 1$ and $\lambda = pns$, $p \ge 1$.

Proof. Let $k_1 = k_2 = n \ge 1$, $\lambda = n$, then $r = v-1$. We will prove that if a RBBD $(2nt, 2n, n; n)$ exists, then a RBBD $(v,2n,n;n)$, where $v = 2nqt$, $q \ge 1$, exists by induction; the rest of the statement follows easily.

It is obvious that the existence of a design with $2nt$ elements implies the existence of a design with $2nqt$ elements when $q = 1$; assume that it is also true for all $q \le q^1$, we now prove that it is true for $q = q^1 + 1$ as well.

(i) If $q = q^1 + 1$ is even, say $q = 2x$, take two copies of a RBBD $(2nxt, 2n, n; n)$ which exists, denote them by D_1 and D_2, we will construct a RBBD $(4nxt,2n,n;n)$, say D. Let a complete replication in D consist of a complete replication in D_1 and one in D_2. Hence, since each complete replication of D_1 or D_2 can appear in only one complete replication of D, there are $2nxt - 1$ complete replications in D constructed this way. Furthermore, each pair of elements belonging to the same D_i, $i = 1$ or 2 are linked n times.

Partition the elements of D_i, $i = 1$ or 2 into $2xt$ subsets of n elements each, represent the subsets by S_{ij}, $j = 1,2,\ldots, 2xt$. Let $2nxt$ complete replications in D be denoted by $F_\ell = \bigcup_{m=1}^{2xt} \{B_{\ell,m}\}$, $\ell = 1,2,\ldots,2nxt$, where for $i=1,2,\ldots,2xt$, $B_{i,m} = \{(S_{1,m}); (S_{2,i+m-1})\}$, and the subscripts are taken modulo $2xt$. Let $F_{2xtz+i} = F_i$ for $i = 1,2,\ldots,2xt$ and $z = 1,2,\ldots, n-1$. Hence there are $2nxt$ complete replications, in whose blocks, any two elements, one from D_1 and the other from D_2, are linked n times. The total number of complete replications is $4nxt - 1 = r$.

(ii) If $q = q^1 + 1$ is odd, say $q = 2x + 1$, the construction of a RBBD $(2nt(2x+1), 2n,n;n)$, D is similar to that in case (i). Take $2x + 1$ copies D_ℓ, $\ell = 1,2,\ldots,2x + 1$, of a RBBD $(2nt,2n,n;n)$ which exists. Again, the $2nt - 1$ complete replications in each D_ℓ together form $2nt - 1$ complete replications in D, in whose blocks pairs of elements from the same D_ℓ's are linked p times.

Partition the elements of D_i, $1 \leq \ell \leq 2x + 1$ into $2t$ subsets of n elements each and denote them by $S_{\ell,j}$, $j = 1,2,\ldots,2t$.

Consider a set of $2(2x+1)$ elements $M = \{m_\ell : m=1,2, \ell = 1,2,\ldots, 2x + 1\}$. Theorem 3.16 implies that a RBBD $(4x+2,2,1;1)$ exists with $4x + 1$ complete replications which are denoted by $F'_p = \bigcup_{\ell=1}^{2x+1} \{B_{p,\ell}\}$, $p = 1,2,\ldots,4x+1$. Without loss of generality, let $B_{4x+1,\ell} = \{(1_\ell); (2_\ell)\}$, $\ell = 1,2,\ldots,2x+1$. By associating $\bigcup_{j=(m-1)t+1}^{mt} S_{\ell,j}$ with each element $m_\ell \in M$, we can construct t complete replications $F_{p,i}$, $i = 1,2,\ldots,t$ in D which are associated with each F'_p, $1 \leq p \leq 4x$ as follows:

For $\ell = 1,2,\ldots,2x+1$, let $B_{p,\ell} \in F'_p$ be denoted by $B_{p,\ell} = \{(m_s); (m'_{s'})\}$ where m_s, $m'_{s'} \in M$ and $m_s \neq m'_{s'}$. Then for $i = 1,2,\ldots,t$, let $F_{p,i} = \bigcup_{\ell=1}^{2x+1} \bigcup_{j=1}^{t} \{B_{p,\ell,i,j}\}$ where

$B_{p,\ell,i,j} = \{(S_{s,(m-1)t+j}); (S_{s',(m'-1)t+j+i-1})\}$ and $(m-1)t+j$ is equal to positive integer modulo mt and $(m'-1)t+j+i-1$ is equal to positive integer modulo m't, except when m'=2 and $(m'-1)t+j-i-1>2t$, then it is $j+i-1$.

In these 4xt complete replications $F_{p,i}$, $p = 1,2,\ldots,4x$, $i = 1,2,\ldots,t$, an element of D_y and an element of D_z where $y \neq z$ are linked once and once only. Therefore, if each of these complete replications are repeated n-1 more times, then $\lambda = n$ is satisfied and the total number of complete replications constructed in D is $(2nt - 1) + n4xt = r$, which is the number D should have. Hence the proof is complete.

We are interested to find out whether a RBBD (2n,2n,n;n) exists for all $n > 1$, since, in view of Theorem 3.17, its existence implies that the necessary condition is also sufficient for the existence of a RBBD (2nt,2n,nq;n) where t and q are positive integers. Furthermore, in a RBBD (2n,2n,n;n), a complete replication consists of just one block, hence a RBBD (2n,2n,n;n) exists if a BBD (2n,2n,n;n) exists. Hence Theorem 2.16 implies

Theorem 3.18. A necessary and sufficient condition for the existence of a RBBD $(v,4,\lambda;2)$ is $v \equiv 0 \pmod 4$ and $\lambda \equiv 0 \pmod 2$.

Consider the case where $n \equiv 1 \pmod 2$, we could generalize part of the proof in Lemma 2.28 to prove

Lemma 3.19. Let $k_1 = k_2 = n \equiv 1 \pmod 2$. A RBBD $(2n,2n,\lambda;\ n)$ does not exist for $\lambda \equiv n \pmod{2n}$.

Proof. Let $\lambda = n(2s+1)$, hence λ is odd and $b = r = (2s+1)(2n-1)$. Assume that a resolvable design with these parameters exists. Consider three arbitrary elements x, y and z. Without loss of generality, let U_i, $i = 1,2,3,4$ be the following four sets of blocks:

$$U_1 = \{B:\ x,z \in B^1,\ y \in B^2\}\ ,\quad U_2 = \{B:\ x \in B^1,\ y,z \in B^2\}$$

$$U_3 = \{B:\ x,y,z \in B^1\}\ ,\quad U_4 = \{B:\ x,y \in B^1,\ z \in B^2\}.$$

Denoting $|U_i| = u_i$ we have obviously $u_1 + u_2 = \lambda$ and $u_1 + u_2 + u_3 + u_4 = b$. Since both x and y must be linked to z in λ blocks, we get $u_1 + u_4 = u_2 + u_4 = \lambda$ which implies that $u_1 = u_2 = \lambda/2$, a contradiction. Hence such a resolvable design does not exist.

However, in the case where $n = 3$, we recall Lemma 2.28 which states that a BBD $(6,6,\lambda;3)$ exists if and only if $\lambda \equiv 0 \pmod 6$, therefore, the second statement of the following theorem is proved when Theorem 3.17 is applied.

Theorem 3.20. A necessary and sufficient condition for the existence of a RBBD $(v,6,\lambda;3)$ is

$$v \equiv 0 \pmod 6,\ v \neq 6 \text{ for } \lambda \equiv 0 \pmod 3$$

$$v \equiv 0 \pmod 6 \quad \text{for } \lambda \equiv 0 \pmod 6.$$

Proof. Let $\lambda = 3$ and $v = 12$, then $r = 11$ and there are two blocks in each complete replications. Let the twelve elements be denoted by $\{\infty\} \cup \{0,1,\ldots,10\}$ and let $A = (\infty)(01\ldots10)$ be a permutation. The complete replication $F = B_1 \cup B_2$, where

$B_1 = \{(\infty,4,7);\ (3,6,9)\}$, $B_2 = \{(0,1,2);\ (5,8,10)\}$, is actually a base complete replication relative to A. Hence a RBBD (12,6,3;3) exists and consequently a RBBD $(v,6,\lambda;3)$ exists for all $v \equiv 0 \pmod{12}$ and $\lambda \equiv 0 \pmod 3$.

Let $v = 12t + 6$, $t \geq 1$, $\lambda = 3$, then $r = 12t + 5$. Let the 12t+6 elements be $\{\infty\} \cup \{0, 1, \ldots, (12t+4)\}$ and let $A = (\infty)(01\ldots(12t+4))$ be a permutation. Let $F = \bigcup_{i=1}^{2t+1} B_i$ be a base complete replication with

$B_i = \{(0,1,2);\ (3i,\ 3i+3,\ v-3i+1)\}$, $B_{i+1} = \{(0,1,2);\ (3i,\ 3i+3,\ v-3i-2)\}$

for i odd, $1 \leq i \leq 2t-3$ and $B_{2t-1} = \{(0,1,2);\ (6t-3,\ 6t,\ 6t+3)\}$,

$B_{2t} = \{(0,1,3);\ (6t-2,\ 6t+9,\ 6t+10)\}$, $B_{2t+1} = \{(\infty,1,2);\ (6t+1,\ 6t+2,\ 6t+3)\}$.

Hence a RBBD $(v,6,\lambda;3)$ exists for $v \equiv 6 \pmod{12}$, $v > 6$, and $\lambda \equiv 0 \pmod 3$, hence the first statement.

When $n = 4$, we get the following theorem easily by constructing a RBBD (8,8,4;4) and then applying Theorem 3.17.

<u>Theorem 3.21</u>. A necessary and sufficient condition for the existence of a RBBD $(v,8,\lambda;4)$ is

$$v \equiv 0 \pmod 8 \text{ and } \lambda \equiv 0 \pmod 4.$$

<u>Proof</u>. Let $v = 8$ and $\lambda = 4$ and let $A = (\infty)(012\ldots6)$ be a permutation on the eight elements, then $r = 7$. Let $F = \{B\}$ be a base complete replication relative to A, where $B = \{(\infty,0,1,3);\ (2,4,5,6)\}$, then we get a RBBD (8,8,4;4) when A is applied to B six times.

CHAPTER IV: Balanced Tripartite Designs

§4.1. Elementary Relations.

A block B in this chapter is denoted by $B = \{B^1;B^2;B^3\}$, that is, B is a set of k elements which are divided into three subsets B^1, B^2 and B^3, with respectively k_1, k_2 and k_3 elements and $k_1 + k_2 + k_3 = k$. Without loss of generality, let $1 \le k_1 \le k_2 \le k_3$. Two elements of B are said to be linked in B if and only if they belong to different subsets. Hence the elements of a block correspond to the vertices of a complete tripartite graph K_{k_1,k_2,k_3}.

An analogue of Definition 1.5 is as below.

Definition 4.1. A balanced tripartite design (briefly BTD) with parameters b,v,r,k,λ,k_1,k_2 and k_3, with $k_1 + k_2 + k_3 = k$, is an arrangement of v elements into b blocks such that each block contains k distinct elements, each element occurs in exactly r blocks and any two distinct elements are linked in exactly λ blocks.

In the case where $k_1 \neq k_2 \neq k_3$, the three subsets of a block in a BTD are in fact the similarity classes $\mathcal{C}_j$, $j = 1,2,3$. In a block B, the number of elements which are linked to an arbitrary element of class $\mathcal{C}_j$ is obviously $k_p + k_q$, where $\{p,q,j\} = \{1,2,3\}$. Let r_{jx} be the number of blocks in which the element x belongs to the class $\mathcal{C}_j$, $j = 1,2,3$, then we get

$$(4.1)\quad \begin{cases} r = r_x = r_{1x} + r_{2x} + r_{3x} \\ e = k_1k_2 + k_2k_3 + k_3k_1 \\ g = r_{1x}(k_2+k_3) + r_{2x}(k_3+k_1) + r_{3x}(k_1+k_2) \end{cases}$$

where e is the number of links in a block and g is the number of elements which are linked to the element x.

The equalities (1.2) - (1.4) now become

$$(4.2)\qquad bk = vr$$

$$(4.3)\qquad \lambda\binom{v}{2} = b(k_1k_2 + k_2k_3 + k_3k_1)$$

$$(4.4)\qquad \lambda(v-1) = r_{1x}(k_2+k_3) + r_{2x}(k_3+k_1) + r_{3x}(k_1+k_2)$$

Therefore we get, from (4.3) and (4.2), that

$$(4.5)\qquad b = \frac{\lambda v(v-1)}{2(k_1k_2 + k_2k_3 + k_3k_1)} = \frac{\lambda v(v-1)}{2e} \quad \text{and}$$

$$(4.6)\qquad r = \frac{\lambda k(v-1)}{2(k_1k_2 + k_2k_3 + k_3k_1)} = \frac{\lambda k(v-1)}{2e}$$

Also, from (4.4), we obtain

$$(4.7)\qquad r_{1x} = \frac{\lambda(v-1) - r(k_1+k_2) - r_{2x}(k_3-k_2)}{k_3 - k_1}.$$

We get two conclusions from these equalities. First of all, the parameters are not independent and therefore a BTD can be denoted by BTD $(v,k,\lambda;\ k_1,k_2)$, where only five parameters are needed to describe a design; secondly, r_{jx}, $j = 1,2,3$, is not independent of the particular element x, therefore a BTD is not necessarily a strongly balanced tripartite design (SBTD). Recall Theorem 1.14 which implies that a cyclic BTD is a SBTD.

In the case where $k_1 = k_2 < k_3$, it follows that $r_{1x} + r_{2x}$ is independent of x and hence

$$(4.8) \qquad r_3 = \frac{\lambda k_3(v-1)}{2(k_2^2+2k_2k_3)}$$

and the BTD is again a SBTD. Similarly, a BTD with $k_1 < k_2 = k_3$ is also a SBTD and r_1 has the same expression as r_3 in (4.8) with k_3 replaced by k_1. In the case where $k_1 = k_2 = k_3$, the design is obviously a SBTD.

In the following sections, the construction of some BTD's with specified parameters are given. First of all, however, we will consider a special case where $\lambda = 1$. Now (4.5) and (4.6) are reduced to $b = \frac{v(v-1)}{2e}$, $r = \frac{k(v-1)}{2e}$. Obviously, if $v = 2e + 1$, then both b, r are integers. Furthermore, we have

<u>Lemma 4.2.</u> There exists a cyclic BTD $(v,k,1;k_1,k_2)$ with $v = 2e + 1$, $k_1 = 1$ or 2, where $e = k_1 k_2 + k_2 k_3 + k_3 k_1$.

<u>Proof.</u> For $v = 2e + 1$, $\lambda = 1$, we get $b = v$, $r = k$, hence a base block $B = (B^1; B^2; B^3)$ of order v will be constructed.

(i) Let $k_1 = 1$, $B^1 = (0)$, $B^2 = (1,2,\ldots,k_2)$ and $B^3 = (2k_2+1, 3k_2+2, \ldots, (k_3+1)k_2 + k_3)$. It is easy to check that each value of the set of edgelengths $E = \{1,2,\ldots,e\}$ occurs once in the base block.

(ii) Let $k_1 = 2$, $B^1 = (0,1)$, $B^2 = (2,4,6,\ldots, 2k_2)$ and $B^3 = (\gamma_1,\gamma_2,\ldots,\gamma_{k_3})$, where for $1 \le j \le k_3$, j odd, $\gamma_j = v - (j+1)k_2 - 2j + 1$ and $\gamma_{j+1} = (j+3)k_2 + 2j + 2$. It is a routine matter to check that each element of E occurs once in this base block. Hence the proof is complete.

§4.2. Balanced Tripartite Designs with $k_1 = k_2$

Let $n = k_1 = k_2$ and $k_3 > n$; then e, the number of links in a block with two subsets of n elements each and one subset of k_3 elements, is $e = n^2 + 2n\, k_3$. The necessary conditions for the existence of a BTD $(v, 2n+k_3, \lambda; n,n)$ are that b, r and r_3 have integral values, where by (4.5), $b = \lambda v(v-1)/2e$, $r = \lambda k(v-1)/2e$ by (4.6) and $r_3 = \lambda k_3(v-1)/2e$ by (4.8). Therefore $r - r_3 = \lambda(v-1)/(n+2k_3)$ must be integral too.

Consider the case $\lambda = 1$; $r - r_3$ being integer implies that $v \equiv 1 \pmod{n + 2k_3}$. Let $v \equiv (n+2k_3)t + 1 \pmod{2n(n+2k_3)}$ where $t = 0,1,\ldots,2n-1$. Hence r being an integer implies that $kt(n+2k_3)/2e = kt/2n = (2n+k_3)t/2n$ is an integer as well. Therefore $k_3t \equiv 0 \pmod{2n}$. In addition, b being an integer implies that $((n+2k_3)t+1)t/2n = ((nt+1)t+2k_3t^2)/2n$, is an integer too. Since $2k_3t^2 \equiv 0 \pmod{2n}$, we have $(nt+1)t \equiv 0 \pmod{2n}$ which implies that $n \mid t$, that is, $t = 0$ or n. Let $t = n$, then n must be odd and k_3 must be even. Hence we have proved that

Lemma 4.3. Let $k_1 = k_2 = n < k_3$. If there exists a BTD $(v,k,1;n,n)$, then

$$v \equiv 1 \pmod{n(n + 2k_3)} \text{ for } n \text{ odd, } k_3 \text{ even,}$$
$$v \equiv 1 \pmod{2n(n + 2k_3)} \text{ otherwise.}$$

Therefore, in the simplest case, $n = 1$, the necessary condition for the existence of a BTD $(v,k,1;1,1)$ is

$$(4.9)\quad \begin{cases} v \equiv 1 \pmod{1 + 2k_3} & \text{for } k_3 \text{ even} \\ v \equiv 1 \pmod{2(1 + 2k_3)} & \text{otherwise} \end{cases}$$

Before we give the construction of some infinite series of BTD's $(v,k,1;1,1)$, we need several definitions (cf [18]).

Definition 4.4. An (A,s)-system, (B,s)-system, (C,s)-system, (D,s)-system, respectively, is a system of s disjoint pairs (p_r, q_r) such that $q_r - p_r = r$ for $r = 1,2,\ldots,s$ and

$$\bigcup_{r=1}^{s} \{p_r, q_s\} = \begin{cases} \{1,2,\ldots,2s\}, & \text{in the case of (A,s)-system,} \\ \{1,2,\ldots,2s-1,2s+1\}, & \text{in the case of (B,s)-system,} \\ \{1,2,\ldots,s,s+2,s+3,\ldots,2s+1\} & \text{in the case of (C,s)-system.} \\ \{1,2,\ldots,s,s+2,s+3,\ldots,2s,2s+2\} & \text{in the case of (D,s)-system.} \end{cases}$$

An (A^+,s)-system, (B^+,s)-system is an (A,s)-system, (B,s)-system respectively, in which $p_s = 1$.

We need the following three lemmas (for their proofs see [18]).

Lemma 4.5. A (C,s)-system exists if and only if there exists an $(A^+,s+1)$-system. A (D,s)-system exists if and only if there exists a $(B^+, s+1)$-system.

Lemma 4.6. An (A^+,s)-system exists if and only if $s \equiv 0$ or 1 (mod 4).

Lemma 4.7. A (B^+,s)-system exists if and only if $s \equiv 2$ or 3 (mod 4), $s \geq 2$.

However, a copy of the system for each s is displayed below.

Let $s = 4m$, $m \geq 1$ an $(A^+, 4m)$-system consists of

(4.10)
- (1) $(r, 4m + 2 - r)$ for $r = 1,2,\ldots,2m$,
- (2) $(2m + 1, 6m)$, (2a) $(4m + 2, 6m + 1)$,
- (3) $(4m + 2 + r, 8m + 1 - r)$ for $r = 1,2,\ldots,m-1$,
- (4) $(7m, 7m + 1)$,
- (5) $(5m + 1 + r, 7m - r)$ for $r = 1,2,\ldots, m-2$.

Notice that if $m = 1$, (2a), (3) and (5) should be omitted; if $m = 2$, (5) should be omitted.

Let $s = 4m + 1$, $m \geq 1$, an $(A^+, 4m + 1)$-system consists of

(4.11)
- (1) $(r, 4m + 3 - r)$ for $r = 1,2,\ldots,m$,
- (2) $(3m + 1, 5m + 2)$,
- (3) $(m + r, 3m + 1 - r)$ for $r = 1,2,\ldots,m$,
- (4) $(3m + 2, 7m + 2)$,
- (5) $(4m + 2 + r, 8m + 2 - r)$ for $r = 1,2,\ldots, m-1$,
- (6) $(6m + 2, 8m + 2)$,
- (7) $(5m + 2 + r, 7m + 2 - r)$ for $r = 1,2,\ldots, m-1$.

Notice that if $m = 1$, (5) and (7) should be omitted, if $m = 0$, the $(A^+,1)$-system is simply $(1,2)$.

Let $s = 4m + 2$, $m \geq 1$, a $(B^+, 4m + 2)$-system consists of

- (I) for $m = 1$, $(4,5),(9,11),(10,13),(2,6),(3,8),(1,7)$;
- (II) for $m \geq 2$,

$$(4.12)\begin{cases}(1) & (r,\ 4m+4-r) \text{ for } r=1,2,\ldots,2m+1,\\ (2) & (2m+2,\ 6m+3),\\ (3) & (6m+2,\ 8m+5),\\ (4) & (5m+1+r,\ 7m+4-r) \text{ for } r=1,2,\ldots,m,\\ (5) & (7m+4,\ 7m+5),\\ (6) & (4m+3+r,\ 8m+4-r) \text{ for } r=1,2,\ldots,m-2.\end{cases}$$

Notice that if $m = 2$, (6) should be omitted, if $m = 0$, a $(B^+,2)$-system does not exist, though a unique $(B,2)$-system exists: $(1,2),(3,5)$.

Let $s = 4m + 3$, $m \geq 0$, a $(B^+, 4m + 3)$-system consists of

$$(4.13)\begin{cases}(I) & \text{for } m=0,\ (2,3),(5,7),(1,4);\\ (II) & \text{for } m\geq 1,\\ & (1)\ (r,\ 4m+5-r) \text{ for } r=1,2,\ldots,2m+1,\\ & (2)\ (2m+2,\ 6m+4),\ (2a)\ (2m+3,\ 6m+3),\\ & (3)\ (7m+4,\ 7m+5),\\ & (4)\ (8m+5,\ 8m+7),\\ & (5)\ (4m+5,\ 6m+5),\\ & (6)\ (4m+5+r,\ 8m+5-r) \text{ for } r=1,2,\ldots,m-1;\\ & (7)\ (5m+4+r,\ 7m+4-r) \text{ for } r=1,2,\ldots,m-2.\end{cases}$$

Notice that if $m = 1$, (5), (6) and (7) should be omitted; if $m = 2$, (7) should be omitted.

The s disjoint pairs of an (A,s)-system or an (A^+,s)-system, or system of any other type, is always denoted by $\{(p_r,q_r),\ r = 1,2,\ldots, s\}$.

We will introduce two more systems:

<u>Definition 4.8.</u> An <u>(A',s)-system</u> is a system of s disjoint pairs (p'_r,q'_r) such that $q'_r - p'_r = r + 1$ for $r = 1,2,\dots,s$ and

$$\bigcup_{r=1}^{s} \{p_r,q_r\} = \{1,2,\dots, 2s\}.$$

This system is equivalent to an (A, s+1)-system which is truncated, that is, an (A, s+1)-system in which either $p_1 = 1$, $q_1 = 2$ or $p_1 = 2s + 1$ $q_1 = 2s + 2$, and in either case, the pair (p_1,q_1) is dropped from the system. As it is essentially the same as a Langford (2,s)-sequence [23], the necessary and sufficient condition for its existence is $s \equiv 0$ or 3 (mod 4).

<u>Definition 4.9.</u> A <u>(F,s)-system</u> is a system of $2s - 1$ subsets of $T = \{1,2,\dots,2s\}$, $\{f_i,g_i\}$, $i = 1,2,\dots, 2s-1$ where

$f_i = s - \frac{1}{2}(i-1)$, $g_i = s + \frac{1}{2}(i + 1)$ for i odd and

$f_i = s + \frac{1}{2}i$, $g_i = s - \frac{1}{2}i$ for i even.

Since $f_{i+1} = g_i$, it is easy to see that $f_1 = s$, $f_{2s} = g_{2s-1} = 2s$ and $\bigcup_{i=2}^{2s-1} \{f_i\} = T - \{s,2s\}$.

We now find out whether the necessary condition (4.9) is also sufficient.

<u>Lemma 4.10.</u> There exists a cyclic BTD (v,k,1; 1,1) where $v \equiv 1 \pmod{2(1 + 2k_3)}$.

<u>Proof.</u> Let $v = 2(1 + 2k_3)t + 1$ where t is an integer ≥ 1. Then $b = vt$ and t base blocks B_i's will be constructed.

For $t \equiv 0$ or $1 \pmod 4$, $t \geq 1$, Lemma 4.6 states that there exists an (A^+,t)-system which is denoted by $\{(p_r,q_r), r = 1,2,\ldots,t\}$. Let a block B_i, $i = 1,2,\ldots,t$ be $B_i = \{(0);\ (i);\ (b_{i1}, b_{i2},\ldots, b_{ik_3})\}$ where for $j = 1,2,\ldots, k_3$, $b_{ij} = v - (p_i + 2jt - t)$.

For $t \equiv 2$ or $3 \pmod 4$, Lemma 4.7 states that a (B^+,t)-system exists and it is also denoted by $\{(p_r,q_r), r = 1,2,\ldots,t\}$. For $i = 1,2,\ldots,t$, let the t blocks be $B_i = \{(0);\ (i),\ (b_{i1}, b_{i2},\ldots, b_{ik_3})\}$ where $b_{ij} = (2j - 1)t + q_i$ and $b_{i\,j+1} = v - (2jt + 3t + 1 - q_i)$ for j odd, $1 \leq j \leq k_3 - 1$, furthermore $b_{ik_3} = (2k_3 - 1)t + q_i$ if k_3 is odd.

It is a routine matter to check that the t blocks constructed are actually t base blocks of the required design. Hence the proof is complete.

In the first necessary condition of (4.9), let $v = et + 1$ where $e = 1 + 2k_3$ and let t be an odd positive integer (otherwise the necessary condition is also sufficient as proved in Lemma 4.10 above). Now k_3 even implies that $e \equiv 1 \pmod 4$ and $e > 1$. We have $b = \frac{1}{2}vt = sv + \frac{1}{2}v$, where $s = \frac{1}{2}(t-1)$.

Lemma 4.11. Let k be even. There exists a cyclic BTD $(v,k,1;1,1)$ for $v \equiv et + 1$, where $e = 1 + 2k_3$ and $t \equiv 1 \pmod 8$.

Proof. Since $s = \frac{1}{2}(t-1)$ and $t \equiv 1 \pmod 8$ imply that $s \equiv 0 \pmod 4$, an (A,s)-system exists which is, as usual, denoted by $\{(p_r,q_r), r = 1,2,\ldots, s\}$. We modify our (A,s)-system to obtain a system of distinct pairs $(\bar{p}_r, \bar{q}_r)$ such that $\bar{q}_r - \bar{p}_r = r$ for $r = 1,2,\ldots, s-1$ and $v - \bar{p}_s - \bar{q}_s = s$; furthermore $\bigcup_{r=1}^{s} \{\bar{p}_r, \bar{q}_r\} = \{1,2,\ldots,s\} \cup \{\tfrac{1}{2}v-s, \tfrac{1}{2}v-s+1, \ldots, \tfrac{1}{2}v-1\}$. For instance, (1) $(\tfrac{1}{2}s - r, \tfrac{1}{2}s + 1 + r)$ for $r = 0,1,\ldots,\tfrac{1}{2}s-1$,

(2) $(\tfrac{1}{2}v - s, \tfrac{1}{2}v - \tfrac{1}{2}s)$,

(3) $(\tfrac{1}{2}v - \tfrac{1}{2}s - r, \tfrac{1}{2}v - \tfrac{1}{2}s + r)$,

for $r = 1,2,\ldots,\tfrac{1}{2}(s-2)$, $r \neq s/4$,

(4) $\bar{p}_s = v/2 - 3s/4$, $\bar{q}_s = v/2 - s/4$.

is a modified (A,s)-system required.

We will construct s base blocks B_i, $i = 1,2,\ldots,s$ each of order v and a base block H of order ½v.

Let $H = \{(0); (v/2); (h_1,h_2,\ldots,h_{k_3})\}$ where $h_j = s + \tfrac{1}{2}(j+1)$, $h_{j+1} = \tfrac{1}{2}v + s + \tfrac{1}{2}(j+1)$ for j odd $1 \leq j < k_3$. Let $B_i = \{(0); (i); (b_{i1}, b_{i2},\ldots,b_{ik_3})\}$, $i = 1,2,\ldots,s$, where

$$b_{ij} = v - (\tfrac{1}{2}k_3 + s(2j-1) + p_i) \text{ for } j = 1,2,\ldots,k_3-1,$$

and

$$b_{ik_3} = \begin{cases} v - (\tfrac{1}{2}k_3 + s(2k_3 - 1) + \bar{p}_i) & \text{for } i \text{ odd}, \\ v - \bar{p}_i & \text{for } i \text{ even}, i \neq s, \\ v - \bar{q}_i & \text{for } i \text{ even}, i = s. \end{cases}$$

Again we can verify that these s+1 base blocks do generate all the blocks of the required design.

We assumed k_3 to be even; in the case where $\tfrac{1}{2}k_3$ is even also, that is $k_3 \equiv 0 \pmod 4$, $k_3 \geq 4$, we can construct another infinite series of designs.

Lemma 4.12. Let $k \equiv 2 \pmod 4$, $k \geq 6$. There exists a cyclic BTD $(v,k,1;1,1)$ for $v = (1+2k_3)t + 1$ where $t \equiv 7 \pmod 8$.

Proof. Since $s = \frac{1}{2}(t-1)$, $s \equiv 3 \pmod 4$ and a (B^+,s)-system exists by Lemma 4.7; a (C,s)-system also exists by Lemma 4.5. As in the proof of Lemma 4.11, we will exhibit one base block H of order $v/2$ and s base blocks B_i, $i = 1,2,\ldots,s$ each of order v.

Let $H = \{(0);\ (v/2);\ (h_1, h_2, \ldots, h_{k_3})\}$ where $h_j = \frac{1}{2}(j+1)(s+1)$ and $h_{j+1} = \frac{1}{2}v + \frac{1}{2}(j+1)(s+1)$, $j = 1,3,\ldots,k_3-1$. For $i = 1,2,\ldots,s$, let $B_i = \{(0);\ (i);\ B_i^3\}$ where $B_i^3 = \{v - p_i - \frac{1}{2}k_3(s+1) - (4j-3)s,\ \frac{1}{2}k_3(s+1) + (4j+1)s + 1 - p_i,\ v - p'_i - (2j-1)(s+1),\ \frac{1}{2}v - p'_i + 2j(s+1);\ j = 1,2,\ldots,k_3/4\}$. The sets $\{(p_r,q_r),\ r = 1,2,\ldots,s\}$ and $\{(p'_r,q'_r),\ r = 1,2,\ldots,s\}$ denote (B^+,s)- and (C,s)-systems respectively. Again we can verify that these are the base blocks for the required design.

Hence we obtain

Theorem 4.13. Let k be odd. The necessary and sufficient condition for the existence of a BTD $(v,k,1;1,1)$ is $v \equiv 1 \pmod{2(1+2k_3)}$.

Let us remark that for k even, we are not able to obtain a necessary and sufficient condition for the existence of a BTD $(v,k,1;1,1)$.

Let us now consider the case for any $\lambda \geq 1$ (where still $k_1 = k_2 = n = 1$). We want to find a minimal λ such that b, r, r_3 (and hence $r - r_3$) are all integers. Now $r - r_3 = \lambda(v-1)/e$ where $e = 1 + 2k_3$, hence $(\lambda,e) \neq 1$ or λ would not be minimal. Consider value y where $1 \leq y \leq e$ and $y|e$; let $e = xy$ and λ be a multiple of y, say $\lambda = yz$, then

$$b = \frac{zv(v-1)}{2x}, \quad r = \frac{zk(v-1)}{2x}, \quad r_3 = \frac{zk_3(v-1)}{2x} \text{ and } r - r_3 = \frac{z(v-1)}{x},$$

where z has value 1 or 2 (cf. proof of Lemma 2.17). Therefore we get

Lemma 4.14. Let $k_1 = k_2 = 1 < k_3$. If there exists a BTD $(v,k,\lambda; 1,1)$, then

$v \equiv 1 \pmod x$ for $\lambda = y$ and both v, k even; when at least one of v, k odd, $v \equiv 1 \pmod{2x}$ for $\lambda = y$ and $v \equiv 1 \pmod x$ for $\lambda = 2y$; where $1 \leq y \leq 1 + 2k_3$ and $xy = 1 + 2k_3$.

Consider the simplest design of the series above, that is, $k_1 = k_2 = 1$, $k_3 = 2$, then the necessary condition can be derived from the lemma above to be

$$(4.14) \quad \begin{cases} \text{(i)} & v \equiv 1 \pmod{10} \text{ for } \lambda = 1, \\ \text{(ii)} & v \equiv 1 \pmod 5 \text{ for } \lambda = 1, \\ \text{(iii)} & v \equiv 1 \pmod 2 \text{ for } \lambda = 5, \\ \text{(iv)} & v \equiv 0 \pmod 2 \text{ for } \lambda = 5. \end{cases}$$

We now show that the necessary conditions in (4.14) are also sufficient. Designs of case (i) have been constructed in Lemma 4.10. Consider case (ii). Let $v = 5t + 1$ where t is odd; Lemma 4.11 implies that a cyclic design exists for $t \equiv 1 \pmod 8$, we have, however

Lemma 4.15. A cyclic BTD $(5t+1,4,1; 1,1)$ exists for all $t \geq 1$.

Proof. We need to construct only designs with $t \equiv 3$ or 5 or 7 (mod 8). Let $s = \frac{1}{2}(t-1)$, then $b = \frac{1}{2}vt = sv + \frac{1}{2}v$ and $v = 10s + 6$. A base block H of order v/2 and s base blocks B_i, $i = 1,2,\ldots,s$, each of order v, will be constructed below.

For $s \equiv 3 \pmod 4$, let $H = \{(0); (5s+3); (i, 5s+4)\}$. Consider an

(A',s)-system (cf. Definition 4.8): $\{(p_r',q_r'),\ r=1,2,\ldots,s\}$, then the blocks B_i, where $B_i = \{(0);\ (i+1);\ (v-(p_i' + s+1),\ v-(p_i'+3s+1))\}$ for $i=1,2,\ldots,s$, with H generate a cyclic BTD with $t \equiv 7 \pmod 8$.

Let $t \equiv 5 \pmod 8$, then $s \equiv 2 \pmod 4$. There exist a (B^+,s)-system and a (D,s)-system by Lemmas 4.7 and 4.5. Let them be denoted by $\{(p_r,q_r),\ r = 1,2,\ldots,s\}$ and $\{(\bar{p}_r,\bar{q}_r),\ r = 1,2,\ldots,s\}$ respectively. Then $H = \{(0);\ (5s + 3);\ (s + 1,\ 6s + 4)\}$ and $B_i = \{(0);\ (i);\ (v-(p_i+s+1),\ 5s+3-\bar{p}_i)\}$, $i = 1,2,\ldots,s$ are the base blocks which generate all the blocks of a cyclic BTD with $t \equiv 5 \pmod 8$.

The only case left is $t \equiv 3 \pmod 8$, that is, $s \equiv 1 \pmod 4$. Consider a modified (A,s)-system $\{(p_r,q_r),\ r = 1,2,\ldots,s\}$ where $\bigcup_{r=1}^{s} \{p_r,q_r\} = \{1,2,\ldots, 2s\}$, $q_r - p_r = r$ for $r = 2,3,\ldots,s$ but $p_1 = (5s-1)/4$, $q_1 = (7s+1)/4$. For instance

(1) $((5s-1)/4,\ (7s+1)/4)$

(2) $(\frac{1}{2}(s+1) - r,\ \frac{1}{2}(s+1) + r)$ for $r = 1,2,\ldots,\frac{1}{2}(s-1)$,

(3) $(\frac{1}{2}(s+1),\ \frac{1}{2}(3s+1))$,

(4) $(\frac{1}{2}(3s-1),\ 2s)$,

(5) $(\frac{1}{2}(3s-1) - r,\ \frac{1}{2}(3s+1) + r)$ for $r = 1,2,\ldots,\frac{1}{2}(s-3)$, $r \neq (s-1)/4$,

is such a system. Then $H = \{(0);\ (5s+3);\ (1,\ 5s+4)\}$ and $B_i = \{(0);\ (i+1);\ (b_{i1},b_{i2})\}$, $i = 1,2,\ldots,s$, where $b_{i1} = v - (p_{i+1} + s + 1)$, $b_{i2} = 5s + 2 - p_{i+1}$ for $i = 1,2,\ldots,s-1$ and $b_{i1} = (13i + 7)/4$, $b_{i2} = (15i + 9)/4$ for $i = s$, are the required base blocks. Hence the proof is complete.

Since $\lambda = 5$ in both cases (iii) and (iv), we combine them together in the following

Lemma 4.16. A cyclic BTD $(v,4,5;\ 1,1)$ exists for all $v \geq 4$.

Proof. First of all, let v be odd, say $v = 2t + 1$, $t \geq 2$, then $b = vt$ and t base blocks B_i, $i = 1,2,\ldots,t$, will be constructed.

Consider the case where $t \equiv 0$ or $2 \pmod 3$. Then $B_i = \{(0); (v-i); (i, v-2i)\}$, $i = 1,2,\ldots,t$ are the required base blocks. Notice that all the elements in each base block are distinct (if they were not, then we would have $v = 3i$; that is, $i = (2t + 1)/3$ which is an integer only when $t \equiv 1 \pmod 3$, a contradiction.) For $t \equiv 1 \pmod 3$, let $t = 3u + 1$, $u \geq 1$ and $w = (2t+1)/3 = 2u + 1$. Also, for $i \in \{1,2,\ldots, 3u + 1\} - \{u, 2u + 1, 3u + 1\}$, let B_i's be the same as those given above, whereas $B_u = \{(0); (u); (3u + 1, 4u + 2)\}$. $B_{2u+1} = \{(0); (2u+1); (1,2u)\}$ and $B_{3u+1} = \{(0); (3u+1); (u, 2u+1)\}$. Therefore a cyclic design exists for all odd v and $\lambda = 5$.

Now let $v = 2t$, $t \geq 2$, then $b = v(t-1) + v/2$. If t is even, say $t = 2s$, then $v = 4s$. Let a base block of order $v/2$ be $H = \{(0); (2s); (s,3s)\}$. Let $\{(f_i,g_i), i = 1,2,\ldots, 2s-1\}$ be a (F,s)-system (cf. Definition 4.9) and for $i = 1,2,\ldots, 2s-1$, let $B_i = \{(0); (v-i); (b_{i1},b_{i2})\}$ where $b_{i1} = f_i$, $b_{i2} = v - (i + f_i)$ for i odd and $b_{i1} = g_i$, $b_{i2} = v - (i + g_i)$ for i even. It is a routine matter to prove that H and B_i, $i = 1,2,\ldots, 2s-1$ are the required base blocks.

If t is odd, say $t = 2s + 1$, then $v = 4s + 2$ and $b = 2sv + v/2$. Let the base blocks of order $v/2 = 2s + 1$ be $H = \{(0); (2s+1); (s,3s+1)\}$ and the 2s base blocks B_i, $i = 1,2,\ldots, 2s$, each of order v be $B_1 = B_{2s} = \{(0); (1); (s+1, 2s+1)\}$, $B_i = \{(0); (i); (x+i, v-y)\}$ for $i = 2,3,\ldots, 2s-1$, where $x = s - i/2$, $y = s - i/2 + 1$ for i even and

$x = y = s - (i-1)/2$ for i odd. Again, it is easy to check that these $2s + 1$ base blocks generate all the blocks of the required design.

The results of (4.14) and Lemmas 4.10, 4.15 and 4.16 can be condensed in

Theorem 4.17. A necessary and sufficient condition for the existence of a BTD $(v,4,\lambda; 1,1)$ is

$$v \equiv 1 \pmod 5 \text{ for } \lambda \not\equiv 0 \pmod 5$$

$$v \geq 4 \qquad \text{for } \lambda \equiv 0 \pmod 5.$$

§4.3. Balanced Tripartite Designs with $k_1 = k_2 = k_3$

Let $k_1 = k_2 = k_3 = n$, then the necessary conditions for the existence of a BTD are that $b = \dfrac{\lambda v(v-1)}{6n^2}$ and $r = \dfrac{\lambda(v-1)}{2n}$ are both integers. Consider the simplest case, $n = 1$; then we are dealing actually with BIBD's with $k = 3$ and the necessary conditions become $b = \lambda v(v-1)/6$ and $r = \lambda(v-1)/2$. The following theorem was proved by Hanani [8] (cf. also [7]);

Theorem 4.18. A necessary and sufficient condition for the existence of a BTD $(v,3,\lambda; 1,1)$ is

$$v \equiv 1 \text{ or } 3 \pmod 6 \text{ for } \lambda \equiv 1 \text{ or } 5 \pmod 6$$

$$v \equiv 0 \text{ or } 1 \pmod 3 \text{ for } \lambda \equiv 2 \text{ or } 4 \pmod 6$$

$$v \equiv 1 \pmod 2 \qquad \text{for } \lambda \equiv 3 \pmod 6$$

$$v \geq 3 \qquad \text{for } \lambda \equiv 0 \pmod 6.$$

Notice that when $\lambda = 1$, a BTD $(v,3,1;1,1)$ is actually a Steiner triple stystem, STS(v), or a BCD $(v,3,1)$.

We now consider a general series for $n \geq 2$ but $\lambda = 1$; we obtain

Lemma 4.19. The necessary condition for the existence of a BTD $(v,k,1;\ n,n)$ where $k = 3n$ is

$$v \equiv 1 \pmod{6n^2} \text{ for } n \equiv 0 \pmod 3$$

$$v \equiv 1 \text{ or } 2n^2 + 1 \pmod{6n^2} \text{ for } n \equiv 1 \text{ or } 2 \pmod 3.$$

Proof. Since r being an integer implies that $v \equiv 1 \pmod{2n}$, let $v \equiv 2nt + 1 \pmod{6n^2}$ where $t = 0,1,\ldots, 3n-1$. Now b being an integer implies that $\frac{(2nt + 1)(2nt)}{6n^2} = \frac{(2nt + 1)t}{3n}$ is an integer. But $n|t$ and $3n \nmid t$ implies that $3|(2nt + 1)$. If $t = n$, then $3|(2n^2 + 1)$ implies that $n^2 \equiv 1 \pmod 3$, that is $n \equiv 1$ or $2 \pmod 3$, if $t = 2n$, then $3|(4n^2 + 1)$ implies that $n^2 \equiv 2 \pmod 3$, which has no solution for n. If $t = 0$, b is integral for all values of n. Hence we get the statement of the lemma.

Consider only the simplest case, that is when $n = 2$. We have, from the Lemma above;

Lemma 4.20. The necessary condition for the existence of a BTD $(v,6,1;\ 2,2)$ is $v \equiv 1$ or $9 \pmod{24}$.

We will investigate the possibility that the necessary condition in Lemma 4.20 is also sufficient for $v \geq 9$. Consider the case where $v = 9$, we have $b = 3$ and $r = 2$. Let $V = \{0,1,\ldots, 8\}$ and without loss of generality, let $B_1 = \{(0,a_1);\ (1,2);\ (3,4)\}$, $B_2 = \{(0,a_2);\ (5,6);\ (7,8)\}$ and $B_3 = \{(1,a_3);\ (2,a_4);\ (a_5,a_6)\}$ where $a_1 \in \{5,6,7,8\}$, $a_2 \in \{1,2,3,4\}$,

a_3, a_4, a_5, a_6 are distinct and belong to $v - \{0,1,2\}$. Now $r = 2$ implies that $a_2 \notin \{1,2\}$, hence $a_2 \in \{3,4\}$. Consider elements 1 and 2, they are linked with four common elements, namely 0, a_1, 3 and 4 in B_1, so they must be linked to three common elements $\{5,6,7,8\} - \{a_1\}$ and each other in B_3, which is impossible. Hence a BTD (9,6,1;2,2) does not exist. However, we have

Lemma 4.21. A BTD $(9,6,\lambda;\ 2,2)$ exists for $\lambda \geq 2$.

Proof. Let $\lambda = 2$, then $b = 6$, the following six blocks form a BTD (9,6,2; 2,2):

$B_1 = \{(0,6);\ (1,2);\ (3,4)\}$, $B_2 = \{(0,3);\ (5,6);\ (7,8)\}$,
$B_3 = \{(0,4);\ (1,3);\ (5,7)\}$, $B_4 = \{(0,5);\ (2,4);\ (6,8)\}$,
$B_5 = \{(1,7);\ (2,6);\ (5,8)\}$ and $B_6 = \{(1,4);\ (2,8);\ (3,7)\}$.

Let $\lambda = 3$, then $b = v = 9$, the following base block of order 9 generates a cyclic BTD (9,6,3; 2,2): $B = \{(0,1);\ (2,4);\ (3,6)\}$. The proof is complete with the aid of Theorem 1.15.

A cyclic BTD (9,6,2; 2,2) does not exist since this is a special case of the series in the following lemma.

Lemma 4.22. A cyclic BTD $(v,6,\lambda;\ 2,2)$ does not exist for $v \equiv 9 \pmod{24}$ and $\lambda = 1$ or 2.

Proof. Let $v = 24t + 9$, $t \geq 0$, then $\lambda = 1$ implies that $b = vt + v/3$ and $\lambda = 2$ implies that $b = 2vt + 2v/3$. In either case, the existence of a cyclic design implies the existence of at least one base block F of order $v/3^i$, $i = 1,2,\ldots$, where $3^i | v$. Consider the simplest case, $i = 1$; without loss of generality, let $F = \{(0,x);\ (8t+3,\ x+8t+3);\ (16t+6,\ x+16t+6)\}$ be a base block of order $v/3 = 8t + 3$ for some $0 < x \leq 24t + 8$. The value $8t + 3$ occurs twice in the base block, which is a contradiction if $\lambda = 1$, and if $\lambda = 2$, there must exist another base block of order $v/3$ in which $8t + 3$ occurs twice too, a contradiction again. Using the same argument, we obtain the fact that the existence of base blocks of order $v/3^i$, $i \geq 2$, leads to a contradiction.

We do not know if a BTD $(v,6,1;\ 2,2)$ exists for $v = 24t + 9$, $v > 9$, but we have

Lemma 4.23. A BTD $(24t + 9,\ 6,\lambda;\ 2,2)$ exists for all $\lambda \geq 2$ when t is even and for all even λ when t is odd.

Proof. Let $\lambda = 2$ and take $3t + 1$ copies of BTD $(9,6,2;\ 2,2)$ which exists by Lemma 4.21 such that they have one element 0 in common. Denote these designs by D_i, $i = 1,2,\ldots, 3t + 1$. We will construct a BTD $(24t + 9,6,2;\ 2,2)$ for $t \geq 1$, which is denoted by D. Include in D all the blocks in D_i, $i = 1,2,\ldots,3t + 1$, hence the total number of elements in D is indeed $24t + 9$ and the element 0 is linked to every other element λ times.

Partition the elements other than 0 of each D_i into four couples and denote them by $S_{i,j}$, $i = 1,2,\ldots, 3t+1$, $j = 1,2,3,4$. Let $I = \{1,2,\ldots, 3t+1\}$; by Theorem 4.18, we can form a BTD $(3t+1,3,2;\ 1,1)$, D' on the elements of I. Each block of D' is a triple and for each triple, say (x,y,z) of D', we construct 16 blocks for D such that an element of S_{ij} and an element of $S_{i_1 j_1}$, $i, i_1 = x,y,z$, $i \neq i_1$, $j, j_1 = 1,2,3,4$, are linked once and once only. For example, $\{S_{xp};\ S_{ym};\ S_{z\ m+p}\}$, $p,m = 1,2,3,4$ are blocks which satisfy the condition. It is easy to see that two non-zero elements, one in D_i, one in D_j, $i \neq j$, are linked twice and twice only in these newly constructed blocks. The construction of D is complete.

For t even, a BTD (3t + 1, B,3; 1,1) exists by Theorem 4.18. We can construct a BTD (24t + 9, 6,3; 3,3) where t is even by following the argument above with λ equal 3 instead of 2. We complete the proof by applying Theorem 1.15 to the results obtained.

<u>Remark</u>: The idea of the proof of the lemma above, as of Theorem 2.8, comes from [6].

We now consider the second series in Lemma 4.20, which also satisfies the necessary condition for the existence of a BTD (v,6,1; 2,2), namely $v \equiv 1 \pmod{24}$. Since Lemma 4.2 implies that a BTD (25,6,1; 2,2) exists, we let v = 24t + 1, $t \geq 2$. Hence we have b = vt and a design exists if t base blocks B_i, i = 1,2,...,t, each of order v can be constructed. For small t, the construction is shown in

<u>Lemma 4.24</u>. A cyclic BTD (24t + 1, 6,1; 2,2) exists for $1 \leq t \leq 5$.

<u>Proof</u>. Consider cases where $t \geq 2$. Let t = 2, then the two following blocks are base blocks of a cyclic design:

B_1 = ((0,9); (1,7); (23,39)), B_2 = ((0,9); (3,4); (24,40)).

Similarly for t = 3, the three base blocks are B_1 = ((0,13); (6,8); (36,60)),

B_2 = ((0,13); (1,3); (35,59)) and B_3 = ((0,13); (2,9); (33,57)).

For t = 4, the four base blocks are

B_1 = ((0,17); (7,12); (48,80)), B_2 = ((0,17); (1,8); (47,79)),

B_3 = ((0,17); (2,3); (45,77)) and B_4 = ((0,17); (4,11); (44,76)).

Lastly, for $t = 5$, the five base blocks are

$B_1 = \{(0,21);\ (8,16);\ (60,100)\}$, $B_2 = \{(0,21);\ (1,10);\ (59,99)\}$,

$B_3 = \{(0,21);\ (2,6);\ (57,97)\}$, $B_4 = \{(0,21);\ (3,9);\ (56,96)\}$

and $B_5 = \{(0,21);\ (4,14);\ (54,94)\}$.

The case where $t = 1$ has been proved in Lemma 4.2.

We do not know if a BTD exists for each $v = 24t + 1$, $t \geq 6$, but we have

Lemma 4.25. There exists a BTD $(v,6,1;\ 2,2)$ for $v = 24t + 1$ if t has values

(i) $t = q3^n$ where $n = 0,1,2,\ldots$, and $q = 1,2,4,5$ or $q \equiv 1$ or $3 \pmod 6$.

(ii) t is such that a BIBD $(v,k,1)$ exists with $v = 24t + 1$, $k = q3^n$ where q and n are as in (i).

Proof. (i) First of all, let $t = q$. The case where $q = 1,2,4$ or 5 has been proved in Lemma 4.24; so let $q \equiv 1$ or $3 \pmod 6$. Theorem 4.18 states that a BTD $(q,3,1;\ 1,1)$ exists, let such a design be denoted by D' and its elements be $\{1,2,\ldots,q\}$. We will construct a BTD $(24q + 1,6,1;\ 2,2)$, D'', following the same procedure as in the proof of Lemma 4.23.

Lemma 4.24 states that a BTD $(25,6,1;\ 2,2)$ exists, take q copies of this design such that they have one element 0 in common and denote them by D_i, $i = 1,2,\ldots,q$. Let the elements of D'' be all the elements of these q designs and include in D'' all their blocks too. For each

$i = 1,2,\ldots,q$, partition the elements of D_i other than 0 into couples, which are denoted by S_{ij}, $j = 1,2,\ldots,12$. For each block $\{(x);(y);(z)\}$ in D', we can construct 144 blocks for D", for instance, $\{S_{xp}; S_{ym}; S_{z\ p+m}\}$ $p,m = 1,2,\ldots,12$. It can easily be checked that, in these blocks, every pair of elements from different designs D_i, D_j, $i \neq j$, are linked once and once only. Hence all the blocks together constitute the required design D".

Now let $t = q3^n$, where $n = 1,2,\ldots$ and $q = 1,2,4,5$ or $q \equiv 1$ or 3 (mod 6). We will construct a BTD $(24t + 1, 6,1; 2,2)$, D, by a procedure similar to the one used above.

Take 3^n copies of a BTD $(24q + 1, 6,1; 2,2)$ such that they have one element 0 in common and denote them by D_i, $i = 1,2,\ldots,3^n$. Let the elements of D be all the elements in these 3^n designs and include in D all their blocks. Again, for each $i = 1,2,\ldots, 3^n$, partition the elements of D_i other than 0 into 12q couples, which are denoted by S_{ij}, $j = 1,2,\ldots,12q$. We will construct the rest of blocks in n iterations.

Iteration 1. Let $G_{1w} = \bigcup_{i=1}^{3} \bigcup_{j=1}^{12q} S_{3(w-1)+i,j}$, $w = 1,2,\ldots,3^{n-1}$. We construct new blocks $\{S_{3(w-1)+1,p}; S_{3(w-1)+2,m}; S_{3w,\ p+m}\}$, $p,m = 1,2,\ldots,12q$, and relabel the couples in each G_{1w} in any order, that is $G_{1w} = \bigcup_{j=1}^{36q} S_{w,j}$, $w = 1,2,\ldots, 3^{n-1}$.

Iteration u $(1 \leq u \leq n)$. Let $G_{uw} = \bigcup_{i=1}^{3} \bigcup_{j=1}^{3^{u-1}\times 12q} S_{3(w-1)+i,j}$, $w = 1,2,\ldots, 3^{n-u}$. Construct new blocks $\{S_{3(w-1)+1,p}; S_{3(w-1)+2,m}; S_{3w,\ p+m}\}$, $p,m = 1,2,\ldots, 3^{u-1}\times 12q$ and relabel the couples in each G_{uw} in some arbitrary order, that is $G_{uw} = \bigcup_{j=1}^{3^u\times 12q} S_{w,j}$, $w = 1,2,\ldots, 3^{n-u}$.

We stop the process after the n^{th} iteration. Indeed, take any two couples in G_{nw} (w has value 1 only), say $S_{i,j}$ and $S_{\ell,j}$; each element in $S_{i,j}$ is linked with each element of $S_{\ell,j}$ once and once only: they are linked in design D_p, $p = 1,2,\ldots, 3^n$ if they both belong to that design, otherwise they are linked in a block formed during some iteration u, as $S_{i,j}$, $S_{\ell,j}$ must both belong to the same G_{uw} for $u = 1,2,\ldots,n$, $w = 1,2,\ldots, 3^{n-u}$. Hence we have constructed all the blocks of a BTD $(24t + 1, 6, 1; 2,2)$, D.

(ii) If a BIBD $(v,k,1)$ exists for $k = q3^n$, where v,q,n are given, (i), then, since a BTD $(k, 6, 1; 2,2)$ exists by (i), a BTD $(v,6,1; 2,2)$ exists also by Theorem 1.17. This completes the proof.

Chapter V: Balanced Circuit Designs.

§5.1. Introduction and Preliminary Results

A block B in this chapter is a set of k elements, that is, $B = (a_1, a_2, \ldots, a_k)$ such that the elements a_i, a_{i+1} for $i = 1,2,\ldots, k-1$, are linked, the elements a_k, a_1 are linked while any other two elements of B are unlinked. In other words, the elements of a block correspond to the vertices of a circuit. An alternative to the definition given in §1.1 can now be given as follow:

Definition 5.1. A balanced circuit design (briefly BCD) with parameters b,v,r,k and λ is an arrangement of v elements into b blocks such that each block contains k elements, each element occurs in exactly r blocks and any two distinct elements are linked in exactly λ blocks.

The necessary conditions for the existence of a BCD are (1.2) and (1.3); but since e, the number of links in a block is equal to k, we get

(5.1) $$bk = vr$$

(5.2). $$\lambda(v-1) = 2r.$$

Again all the parameters can be expressed through v,k and λ, therefore, we will use for a balanced circuit design the notation BCD (v,k,λ). Since all the elements in a block belong to the same similarity class, a BCD is in fact a SBCD (for proof see §1.2).

In this chapter, we investigate whether the necessary conditions are also sufficient for the existence of a BCD (v,k,λ) for $3 \leq k \leq 6$. In the construction of designs, in addition to the (A,k)-, (B,k)-systems presented by (4.10)-(4.13), we need:

Definition 5.2. An $(\bar{A},k)$-system [a $(\bar{B},k)$-system respectively] is an (A,k)-system [a (B,k)-system] with $p_1 = 1$.

As for (A,k)-, (B,k)-systems, the necessary and sufficient condition for the existence of an $(\bar{A},k)$-system is $k \equiv 0$ or $1 \pmod 4$ and that of a $(\bar{B},k)$-system is $k \equiv 2$ or $3 \pmod 4$ ([5], [18], [23]).

Definition 5.3. A solution to the first Heffter's difference problem for k (briefly, a solution to I.HDP (k)) is a partition of the set {1,2,...,3k} into k triples (x_i, y_i, z_i), $i = 1,2,\ldots,k$ such that for each i, either $x_i + y_i = z_i$ or $x_i + y_i + z_i = 6k + 1$.

It is well-known that a solution to the I.HDP (k) exists for every positive integer k (see for example [18]).

It has been shown (although the results are formulated in a graph theoretical language) that if $k \equiv 0 \pmod 2$, then a BCD (v,k,1) exists whenever $v \equiv 1 \pmod{2k}$. It is obvious that if k is a power of 2, then $v \equiv 1 \pmod{2k}$ is also necessary for the existence of a BCD (v,k,1). Examples of a BCD (33,12,1) in [15] and of a BCD (49,14,1) in [19] show that this condition is not necessary for the existence of a BCD (v,k,1) if k is not a power of 2.

It has been shown in [17] that if $k \equiv 1 \pmod 2$, then a BCD (v,k,1) exists whenever $v \equiv 1 \pmod{2k}$ and the following conjecture is true.

Let the set $S = \{a_1, a_2, \ldots, a_t\} \subset \{1,2,\ldots, 2kt\}$ where k,t are positive integers which satisfy (i) $\sum_{i=1}^{t} a_i \equiv 0 \pmod{2kt+1}$, (ii) $a_i + a_j \not\equiv 0 \pmod{2kt+1}$ for all $i,j = 1,2,\ldots,t$. Then one can find a permutation $a_{i_1}, a_{i_2}, \ldots, a_{i_t}$ of the elements of S such that $\sum_{j=k}^{q} a_{i_j} \not\equiv 0 \pmod{2kt+1}$ for

$k,q = 1,2,\ldots,t$; $q > k$; $q - k < t - 1$.

The conjecture is trivally true for $k = 3,5$ and has been verified also for $k = 7$, but has not been proved in general. An example of a BCD (51,15,1) in [17] shows that the condition $v \equiv 1 \pmod{2k}$ is not necessary for the existence of a BCD $(v,k,1)$ either.

We will prove that if $k = 3,4,5$ or 6, the necessary conditions (5.1), (5.2) for the existence of a BCD (v,k,λ) are also sufficient.

Actually, when $k = 3$ or 4, there is nothing to prove. When $k = 3$, a BCD $(v,3,\lambda)$ is the same as a BIBD $(v,3,\lambda)$ or a BTD $(v,3,\lambda;\ 1,1)$ which exists for all admissible v as shown in Theorem 4.18. When $k = 4$, a BCD $(v,4,\lambda)$ is the same as a BBD $(v,4,\lambda;\ 2)$ which again exists for all admissible v as proved in Theorem 2.33.

Thus we have to deal with the cases $k = 5$ and 6 only. Obviously, for given fixed values of v and k, there is always a minimal value λ_{min} of λ such that the necessary conditions (5.1), (5.2) are satisfied. We show in the next two sections, through a sequence of lemmas, that a BCD (v,k,λ_{min}) always exists for $k = 5$ or 6. In view of Theorem 1.15, this implies that a BCD (v,k,λ) with $k = 5$ or 6 alwyas exists whenever the necessary conditions (5.1) and (5.2) are satisfied.

In what follows in this chapter, the elements of a BCD (v,k,λ) will be represented by residue classes modulo w where $w = v$ or $v-1$. In the later case, additional element ∞ is introduced. A BCD (v,k,λ) will be represented by a collection $\mathcal{B}$ of base blocks with respect to permutation A, where either $A = (012\ldots(v-1))$ or $A = (\infty)(012\ldots(v-2))$. Actually, A is an automorphism of the design and the design is cyclic in the former case.

As mentioned in §1.3, an integer ℓ, $1 \le \ell \le [\frac{1}{2}w]$, is said to occur q times in a base block B_i of order m if qw/m edgelengths in B_i are of the value ℓ. If the element ∞ is present in a base block, then the edgelength between ∞ and any other element linked to it is always considered to be itself ∞. Hence the constructed collection $\mathcal{B}$ of base blocks generates all the blocks of a BCD under A, if every element of the set of edgelengths $E = E_1 \cup E_2$, where $E_1 = \{1,2,\ldots,[\frac{1}{2}w]\}$ and $E_2 = \emptyset$ or $\{\infty\}$ if $w = v$ or $w = v-1$ respectively, occurs exactly λ times in the base blocks, except when w is even, in which case $\frac{1}{2}w$ should occur only $\frac{1}{2}\lambda$ times.

We will denote the set of residue classes modulo w by A_w, which is the permutation $(012\ldots(w-1))$ on the elements of the set, assuming that no misunderstanding will thus be caused. We will also denote the set of all the elements of a BCD (v,k,λ) by V. In the following two sections, we will prove the existence of a BCD (v,k,λ) by constructing a collection $\mathcal{B}$ of base blocks which satisfy the condition in the last paragraph. Notice that from now on, when we say a collection of base blocks, we mean a collection which satisfies the condition. Before we conclude this section, we give

<u>Lemma 5.4.</u> If k is odd and there exists a BIBD (v,k,λ), then there exists a BCD (v,k,λ).

<u>Proof.</u> Each block of a BIBD (v,k,λ) with k odd yields $\frac{1}{2}(k-1)$ blocks of size k of a BCD (cf. Theorem 9.6, [10]); in other words, each block of a BIBD (v,k,λ) yields a BCD $(k,k,1)$.

§5.2 Balanced Circuit Designs with $k = 5$

For $k = 5$, the necessary conditons (5.1) and (5.2) imply that

(5.3) $b = \lambda v(v-1)/10$, $r = \lambda(v-1)/2$.

The necessity parts of the two theorems in this section follow easily from (5.3).

Theorem 5.5 A cyclic BCD $(v,5,1)$ exists if and only if $v \equiv 1$ or $5 \pmod{10}$, $v \geq 5$.

Proof. For the proof of sufficiency see [17].

Lemma 5.6. A BCD $(v,5,2)$ exists for $v \equiv 6 \pmod{10}$, $v \geq 6$.

Proof. Let $v = 10t + 6$ and $V = A_{10t+5} \cup \{\infty\}$. Let A_{10t+5}, $\mathcal{B}$ represent respectively the set of elements, a collection of base blocks, of a cyclic BCD $(10t+5,5,1)$ which exists by the theorem above. It follows that $\mathcal{B}$ can be written as $\mathcal{B} = \mathcal{B}^1 \cup \mathcal{B}^2$, where $\mathcal{B}^1$ contains all the base blocks of order $10t + 5$ and $\mathcal{B}^2$ contains all the base blocks of order $2t + 1$.

Let $B_1 = \{0, 2t+1, 4t+2, 8t+4, \infty\}$ and $F = \{0, 4t+2, 8t+4, 2t+1, 6t+3\}$, then B_1 is a base block of order $10t + 5$ and F a base block of order $2t + 1$. Let $\mathcal{B}$ consist of B_1, F and each base blocks of $\mathcal{B}^1$ twice, then $\mathcal{B}$ generates all the blocks of a BCD $(10t+6,5,2)$.

Before proceeding further, let us remark that "one half" of the Lemmas 5.7, 5.9 and 5.12 below follows from Lemma 5.4 and from the known existence of a BIBD $(v,5,5)$ for $v \equiv 9,13,17 \pmod{20}$ [9], so that these lemmas provide an alternative construction.

Lemma 5.7. If there exists a BCD (10t+6,5,2), then there exists a BCD (10t+7,5,5).

Proof. Let $V = A_{10t+6} \cup \{\infty\}$ and A_{10t+6}, $\mathcal{G}$ be respectively, the set of elements, a collection of base blocks of a BCD (10t+6,5,2). For $t = 0$, let $B_1 = (0,3,1,2,\infty)$, $B_2 = (2,0,1,3,\infty)$ and $H = (1,0,3,4,\infty)$ that is, B_1 and B_2 are base blocks of order 6 and H is a base block of order 3. It is easy to see that $\mathfrak{B}$, consists of blocks of $\mathcal{G}$, B_1, B_2 and H generates all the blocks of a BCD (7,5,5).

For $t \geq 2$, let $\{(p_i,q_i), i = 1,2,\ldots, t-1\}$ be an $(A,t-1)$- or a $(B, t-1)$-system, depending on whether $t \equiv 1, 2 \pmod 4$ or $t \equiv 0,3 \pmod 4$.

For all $t \geq 1$, let H be a base block of order $5t+3$, where $H = (6t+3,\ 0,\ 5t+3,\ t,\ \infty)$ and let B_i, $i = 1,2,\ldots, 3t+2$, be base blocks of order $v = 10t+6$, where $B_i = B_{t+i} = B_{2t+i} = (0,\ 2t+1,\ 6t+3,\ 6t+3+p_i,\ 6t+3-q_i)$ for $i = 1,2,\ldots,t-1$ (that is, these blocks do not exist when $t = 1$), $B_t = B_{2t} = B_{3t} = (0,\ 3t,\ 6t+1,\ 4t+1,\ 7t+3)$; $B_{3t+1} = (0,\ 5t+3,\ t,\ t+x,\ \infty)$ and $B_{3t+2} = (0,x,2x,\ 4t+3+2x,\infty)$,

$$x = \begin{cases} 2t-1 & \text{if } t \equiv 1 \text{ or } 2 \pmod 4 \\ 2t-2 & \text{if } t \equiv 0 \text{ or } 3 \pmod 4. \end{cases}$$

Let $\mathfrak{B}$ consist of blocks of $\mathcal{G}$, B_i, $i = 1,2,\ldots, 3t+2$ and H, then the proof is complete.

Lemma 5.8. If there exists a BCD (10t+7, 5,5) then there exists a BCD (10t+8, 5,10).

Proof. Let $V = A_{10t+7} \cup \{\infty\}$ and let A_{10t+7}, $\mathfrak{D}$ be, respectively, the set of elements, a collection of base blocks of a BCD (10t+7,5,5).

For $t \geq 2$, let $\{(p_i, q_i),\ i = 1,2,\ldots, t-1\}$ be an $(\bar{A}, t-1)$-system or a $(\bar{B}, t-1)$-system depending on whether $t \equiv 1,2 \pmod 4$ or $t \equiv 0,3 \pmod 4$. For $t = 0$, $i = 1,2,\ldots,5$, put $B_i = \{0,1,3,6,\infty\}$, obviously, the base blocks of $\mathcal{L}$, together with B_i, $i = 1,2,\ldots,5$ generate all the blocks of a BCD (8,5,10).

For $t \geq 1$, define $B_{jt+i} = \{0,\ 3t+1-i,\ 6t+3,\ i+1,\ p_{i+1} + i-1\}$ $j = 0,1,\ldots,4$, $i = 1,2,\ldots, t-2$, (again, these blocks do not exist when $t \leq 2$), For $i = 1,2,\ldots,5$, $B_{it-1} = \{0,\ 3t+1,\ 6t+3,\ 4t+1,\ 6t+2\}$ (which are left out when $t \leq 1$), $B_{it} = \{0,\ 2t-1,\ 6t+1,\ 2t,\ 6t+3\}$ and

$$B_{5t+i} = \begin{cases} \{0,\ 2,\ 6,\ 9,\ \infty\} & \text{if } t = 1 \\ \{0,\ 2t,\ 4t-2,\ 6t-5,\ \infty\} & \text{if } t \equiv 1,2 \pmod 4,\ t \geq 2 \\ \{0,\ 2t,\ 4t-2,\ 6t-6,\ \infty\} & \text{if } t \equiv 0,3 \pmod 4,\ t \geq 3. \end{cases}$$

Hence $\mathcal{B} = \mathcal{D} \cup \{B_i,\ i = 1,2,\ldots, 5t+5\}$ generates all the blocks of a BCD $(10t+8,\ 5,\ 10)$.

<u>Lemma 5.9.</u> A cyclic BCD $(v,5,5)$ exists for $v \equiv 9 \pmod{10}$, $v \geq 9$.

<u>Proof.</u> Let $v = 10t+9$, $t \geq 0$ and let $V = A_{10t+9}$. Define $B_{5t+1} = \{0,1,5,4,2\}$, $B_{5t+2} = \{0,1,2,6,3\}$, $B_{5t+3} = \{0,2,5,1,3\}$ and $B_{5t+4} = \{0,1,3,7,4\}$. These four base blocks generate a BCD (9,5,5) which is cyclic. For $t \geq 1$, we have to distinguish two cases:

(1) $t \equiv 0$ or $1 \pmod 4$. Let $\{(p_i, q_i),\ i = 1,2,\ldots,t\}$ be an $(\bar{A},t)$-system. For $j = 0,1,\ldots,4$ put $B_{jt+1} = \{0,\ 3t+2,\ 6t+5,\ 10t+8,\ 6t+4\}$,

$$B_{jt+i} = \begin{cases} \phi & \text{if } t = 1 \\ \{0,\ 3t+3-i,\ 6t+5,\ i,\ i+p_i+2\} & \text{if } t \geq 4,\ i = 2,3,\ldots,t. \end{cases}$$

We can verify that B_i, $i = 1,2,\ldots, 5t+4$, are base blocks which generated a BCD $(10t+9, 5,5)$ which is cyclic.

(ii) $t \equiv 2$ or $3 \pmod 4$. For $t \geq 3$, let $\{(p_i,q_i), i=1,2,\ldots,t-2\}$ be an $(A,t-2)$-system. Put, for $j = 0,1,\ldots,4$, $B_{jt+i} = \{0, 3t+1+i, 6t+3, i, p_i+i+4\}$, $i = 1,2,\ldots,t-2$, $B_{(j+1)t-1} = \{0, 2t+1, 4t+3, 8t+9, 4t+4\}$ and $B_{(j+1)t} = \{0, 2t+3, 6t+4, 2t+4, 6t+6\}$. These base blocks B_i, $i = 1,2,\ldots, 5t+4$ generate a cyclic BCD $(10t+9, 5,5)$.

Lemma 5.10. A BCD $(v,5,2)$ exists for $v \equiv 0 \pmod{10}$, $v \geq 10$.

Proof. Let $v = 10t$, $t \geq 1$ and let $V = A_{10t-1} \cup \{\infty\}$. For $t \geq 2$, let $\{(p_i,q_i), i = 1,2,\ldots,t-1\}$ be an $(\bar{A},t-1)$-system or a $(\bar{B},t-1)$-system depending on whether $t \equiv 1,2 \pmod 4$ or $t \equiv 0,3 \pmod 4$. Put, for $i = 1,2,\ldots,t-2$,

$$B_i = B_{t+i} = \begin{cases} \phi & \text{if } t = 1 \text{ or } 2 \\ \{0, 3t-1, 6t-1, i+1, p_{i+1}+i-1\} & \text{if } t \equiv 1, 2 \pmod 4,\ t \geq 5 \\ \{0, 3t-1, 6t-1, i+1, p_{i+1}+i\} & \text{if } t \equiv 0,3 \pmod 4,\ t \geq 3 \end{cases}$$

$$B_{t-1} = B_{2t-1} = \begin{cases} \phi & \text{if } t = 1 \\ \{0, 2t-3, 6t-3, 2t-1, 6t\} & \text{if } t \geq 2 \end{cases}$$

$$B_t = \begin{cases} \{0, 4, 1, 5, \infty\} & \text{if } t = 1 \\ \{0, 2t-2, 4t-4, 6t-3, \infty\} & \text{if } t \equiv 1, 2 \pmod 4,\ t \geq 2 \\ \{0, 1, 2, 2t+3, \infty\} & \text{if } t \equiv 0,3 \pmod 4,\ t \geq 3 \end{cases}$$

and finally $B_{2t} = \{0, 2t, 4t, 6t-1, 8t\}$. These $2t$ base blocks are all the base blocks of a BCD $(10t, 5, 2)$.

Lemma 5.11. A BCD (v,5,10) exists for $v \equiv 2 \pmod{10}$, $v \geq 12$.

Proof. Let $v = 10t + 2$, $t \geq 1$ and $V = A_{10t+1} \cup \{\infty\}$. Let A_{10t+1}, $\mathcal{B}$ be respectively, the set of elements, a collection of base blocks of a cyclic BCD (10t+1, 5, 1) which exists by Lemma 5.5. It follows that $\mathcal{B}$ is a union of t base blocks, each of order 10t + 1, $\mathcal{B} = \bigcup_{i=1}^{t} \overline{B}_i$. Let $\overline{B}_j$ be any of these base blocks in which the edgelength (10t+1)/3 does not occur (that is, if $v \not\equiv 1 \pmod 3$, $\overline{B}_j$ may be any of the t base blocks). Let the set of edgelengths which occur in $\overline{B}_j$ be $\{a_1, a_2, a_3, a_4, a_5\}$, then put, for $i = 1,2,\ldots,5$,

$B_{10t+i} = \{0, a_i, 2a_i, 3a_i, \infty\}$. Let $B_{st+i} = \overline{B}_i$, $i = 1,2,\ldots,t$, $i \neq j$, $s = 0,1,\ldots,9$ and $B_{ut+j} = \overline{B}_j$, $u = 0,1,\ldots,6$. These 10t + 2 base blocks generate a BCD (10t+2, 5, 10).

Lemma 5.12. A cyclic BCD (v,5,5) exists for $v \equiv 3 \pmod{10}$, $v \geq 13$.

Proof. Let $v = 10t + 3$, $t \geq 1$ and $V = A_{10t+3}$. We have two cases:

(i) $t \equiv 1$ or $2 \pmod 4$. Let $\{(p_i, q_i), i = 1,2,\ldots, t+1\}$ be a $(\overline{B}, t+1)$-system and for $t \geq 2$, let $\{(p_i^*, q_i^*), i = 1,2,\ldots,t-1\}$ be an (A, t-1)-system. Define

$B_{jt+i} = \{0, 2t+1+i, 6t+3, i+1, p_{i+1} + i-1\}$, $i = 1,2,\ldots, t$, $j = 0,1,2,3$

$$B_{4t+i} = \begin{cases} \phi & \text{if } t = 1 \text{ or } 2 \\ \{0, 3t+1-i, 6t+1, i+2, p_{i+2}^* + i + 2\} & \text{if } t \geq 5,\ i = 1,2,\ldots,t-3 \end{cases}$$

$$B_{5t-2} = \begin{cases} \phi & \text{if } t = 1 \text{ or } 2 \\ \{0, 2t+1, 6t+1, 2, p_2^* + 2\} & \text{if } t \geq 5 \end{cases}$$

$$B_{5t-1} = \begin{cases} \phi \text{ if } t = 1 \\ \{0,5,13,1,2\} \text{ if } t = 2 \\ \{0, 2t-1, 6t+1, 1, p_1^* + 1\} \text{ if } t \geq 5 \end{cases}$$

$$B_{5t} = \begin{cases} \{0, 1, 4, 8, 6\} \text{ if } t = 1 \\ \{0, 3, 7, 16, 10\} \text{ if } t = 2 \\ \{0, 2t, 4t+2, 8t, 4t+1\} \text{ if } t \geq 5. \end{cases}$$

and $B_{5t+1} = \{0, 2t, 4t, 6t, 8t\}$. These 5t+1 base blocks generate a cyclic BCD $(10t + 3, 5,5)$.

(ii) $t \equiv 0$ or 3 (mod 4). Let $\{(p_i,q_i),\ i = 1,2,\ldots, t+1\}$ be an $(\bar{A}, t+1)$-system and let $\{(p_i^*, q_i^*),\ i = 1,2,\ldots,t-1\}$ be a $(B,t-1)$-system. Define

$B_{jt+i} = \{0, 2t+1+i, 7t+3, i+1, p_{i+1}+i-1\}$, $i = 1,2,\ldots, t$, $j = 0,1,2,3$

$B_{4t+i} = \{0, 3t+1-i, 6t+1, i, p_i^*+i+1\}$, $i = 1,2,\ldots,t-1$

$B_{5t} = \{0, 2t+1, 4t+2, 6t+3, 8t+4\}$ and

$B_{5t+1} = \{0, 1, 4t+1, 2t, 6t+1\}$. These 5t+1 base blocks generate a cyclic BCD $(10t+3, 5,5)$.

<u>Lemma 5.13.</u> If there exists a BCD $(10t+3, 5,5)$ then there exists a BCD $(10t+4, 5, 10)$.

<u>Proof.</u> Let $V = A_{10t+3} \cup \{\infty\}$ and let $A_{10t+3}, \mathcal{L}$ be, respectively, the set of elements and a collection of base blocks of a BCD $(10t+3, 5,5)$. Let $\{(p_i,q_i)\ i = 1,2,\ldots, t+1\}$ be an $(\bar{A}, t+1)$-system or a $(\bar{B}, t+1)$-system depending on whether $t \equiv 0,3$ (mod 4) or $t \equiv 1,2$ (mod 4). Define

$B_{jt+i} = \{0, 2t+2+i, 6t+3, i+2, p_{i+2}+i\}$, $i = 1,2,\ldots,t-1$, $j = 0, 1,\ldots,4$,

$B_t = B_{2t} = B_{3t} = \{0, 2t+2, 6t+3, 2, p_2\}$, $B_{4t} = B_{5t} = \{0, 2t+2, 6t+4, 2t+3, \infty\}$,

$$B_{5t+1} = \begin{cases} \{0,\ 2t,\ 4t,\ 6t,\ \infty\} & \text{if } t \equiv 1,2 \pmod 4 \\ \{0,\ 2t+1,\ 4t+2,\ 6t+3,\ \infty\} & \text{if } t \equiv 0,3 \pmod 4 \end{cases} \quad \text{and}$$

$$B_{5t+2} = B_{5t+3} = \begin{cases} \{0,\ p_2-2,\ p_2+q_2-4,\ p_2+q_2+2t-4,\ \infty\}, & \text{if } t \equiv 1,2 \pmod 4 \\ \{0,\ p_2-2,\ p_2+q_2-4,\ p_2+q_2+2t-3,\ \infty\}, & \text{if } t \equiv 0,3 \pmod 4 \end{cases}$$

These $5t+3$ base blocks and all the base blocks in $\mathcal{L}$ together generate a BCD $(10t+4, 5, 10)$.

We are now in a position to formulate the following theorem:

Theorem 5.14. A necessary and sufficient conditon for the existence of a BCD $(v, 5, \lambda)$ is

$v \equiv 1$ or $5 \pmod{10}$ for $\lambda \equiv 1$ or 3 or 7 or $9 \pmod{10}$

$v \equiv 0$ or $1 \pmod 5$ for $\lambda \equiv 2$ or 4 or 6 or $8 \pmod{10}$

$v \equiv 1 \pmod 2$ for $\lambda \equiv 5 \pmod{10}$

$v \geq 5$ for $\lambda \equiv 0 \pmod{10}$.

Proof. The necessity follows from the condition (5.3). The sufficiency follows from Theorem 1.15 and Lemmas 5.5 - 5.13.

§5.3. Balanced Circuit Designs with k = 6

For $k = 6$, the necessary conditions (5.1), (5.2) become

$$(5.4) \qquad b = \lambda v(v-1)/12, \qquad r = \lambda(v-1)/2.$$

Lemma 5.15. A cyclic BCD $(v,6,1)$ exists for $v \equiv 1 \pmod{12}$, $v \geq 13$.

Proof. The existence of a cyclic BCD $(12t+1, 6,1)$ for every $t \geq 1$ follows from [19] Theorem 2.

Lemma 5.16. A BCD $(v,6,6)$ exists for $v \equiv 2 \pmod{12}$, $v \geq 14$.

Proof. Let $v = 12t+2$, $t \geq 1$ and $V = A_{12t+1} \cup \{\infty\}$. Let A_{12t+1}, $\mathcal{L}$ be, respectively, the set of elements, a collection of base blocks of a cyclic BCD (12t+1, 6,1) which exists by the lemma above. As $\mathcal{C}$ is a collection of t base blocks, each of order 12t+1, we can write $\mathcal{C} = \bigcup_{i=1}^{t} \overline{B}_i$, where $\overline{B}_i = \{a_{i1}, a_{i2}, a_{i3}, a_{i4}, a_{i5}, a_{i6}\}$ is a base block. Consider two base blocks $\overline{B}_r$ and $\overline{B}_s$, and let the set of edgelengths occuring in them be $\{b_i, i = 1,2,\ldots,12\}$. (It is obvious that since $\lambda = 1$, $b_j \neq b_k$ for $j \neq k$). In the complete graph K_{12t+1} with the vertex-set A_{12t+1}, form three paths P_1, P_2, P_3 of length four each such that the lengths of edges in P_i are $c_{i1}, c_{i2}, c_{i3}, c_{i4}$ and $\{b_i, i = 1,2,\ldots, 12\} = \bigcup_{i=1}^{3} \{c_{ij}, j = 1,2,3,4\}$; obviously, this is always possible. Define $B_{pt+i} = \overline{B}_i$, $p = 0,1,\ldots,5$, $i = 1,2,\ldots,t$, $i \neq r,s$, $B_{qt+r} = \overline{B}_r$, $B_{qt+s} = \overline{B}_s$, $q = 0,1,\ldots,4$ and $B_{6t+i} = \{0, c_{i1}, c_{i1} + c_{i2}, c_{i1} + c_{i2} + c_{i3}, c_{i1} + c_{i2} + c_{i3} + c_{i4}, \infty\}$, $i = 1,2,3$. These 6t + 1 base blocks generate a BCD (12t + 2, 6,6).

Lemma 5.17. A cyclic BCD (v,6,2) exists for $v \equiv 3 \pmod{12}$, $v \geq 15$.

Proof. Let $v = 12t + 3$, $t \geq 1$ and $V = A_{12t+3}$. Put
$B_i = \{0, 3t+2i, 6t+1+4i, 3i, 1, 3(i+1)\}$, $i = 1,2,\ldots, t$,
$B_{t+i} = \{0, 6t-2i, 6t+1, 6t+2i+3, 1, 4t-2i\}$ $i = 1,2,\ldots, t-1$,
$B_{2t} = \{0, 1, 3, 4t+2, 10t+2, 4t+1\}$ and $F = \{0, 1, 4t+1, 4t+2, 8t+2, 8t+3\}$.
The 2t base blocks, B_i, $i = 1,2,\ldots, 2t$, each of order v and base block F, of order v/3, together generate a cyclic BCD (12t+3, 6,2).

Lemma 5.18. A BCD (v,6,2) exists for $v \equiv 4 \pmod{12}$, $v \geq 16$.

Proof. Let $v = 12t+4$ and $V = A_{12t+3} \cup \{\infty\}$. Let B_i, $i = 1,2,\ldots, t$ and B_{2t} be as in the proof of Lemma 5.17 except that $B_{2t} = \phi$ when $t = 1$.

Define further

$$B_{t+i} = \begin{cases} \phi & \text{if } t = 1 \text{ or } 2 \\ \{0,\ 6t-2i,\ 6t+1,\ 6t+2i+3,\ i,\ 4t-2i\} & \text{if } t \geq 3,\ i = 1,2,\ldots,t-2. \end{cases}$$

$$B'_{2t-1} = \begin{cases} \{0,\ 1,\ 6,\ 12,\ 5,\ \infty\} & \text{if } t = 1 \\ \{0,\ 2t-1,\ 4t+1,\ 8t+3,\ 4t,\ \infty\} & \text{if } t \geq 2 \end{cases}$$

$F_1 = \{0, 1, 4t+1, 4t+2, 8t+2, 8t+3\}$, and $F_2 = \{0, 2t, 4t+1, 6t+1, 8t+2, 10t+2\}$.

The 2t base blocks B_i, $i = 1, 2, \ldots, 2t-2, 2t$, B'_{2t-1}, each of order 12t+3 and 2 base blocks F_1, F_2 each of order 4t+1, together generate a BCD (v,6,2).

Lemma 5.19. A cyclic BCD (v,6,3) exists for $v \equiv 5 \pmod{12}$, $v \geq 17$.

Proof. Let $v = 12t+5$, $V = A_{12t+5}$. Define

$B_i = B_{t+i} = B_{2t+i} = \{0, 3i+7, 1, 4t+6-i, 8t+9+i, 8t+1-2i\}$,

$i = 1,2,\ldots,t-1$, $B_t = B_{2t} = B_{3t} = \{0, 2, 8, 4, 12, 5\}$ and

$B_{3t+1} = \{0, 1, 2, 5, 6, 3\}$, then these 3t+1 base blocks generate a cyclic BCD (12t+5, 6, 3).

Lemma 5.20. A BCD (v,6,2) exists for $v \equiv 6 \pmod{12}$, $v \geq 6$.

Proof. Let $v = 12t+6$, $t \geq 0$ and let $V = A_{12t+5} \cup \{\infty\}$. Define for $i = 1,2,\ldots,t$, $B_i = B_{t+i} = \{0, 3t+1+2i, 6t+3+4i, 6t+3+i, 1, 3i+2\}$ (which do not exist when $t = 0$), and $B_{2t+1} = \{0,1,3,2,4,\infty\}$. These 2t+1 base blocks generate a BCD (12t+6, 6,2).

Lemma 5.21. A cyclic BCD (v,6,2) exists for $v \equiv 1 \pmod 6$, $v \geq 7$.

Proof. Let $v = 6t+1$, $V = A_{6t+1}$ and let $\{(a_i,b_i,c_i),\ i = 1,2,\ldots,t\}$

be a solution to I.HDP (t).

Put

$$B_i = \begin{cases} \{a_i, c_i, b_i, v-a_i, v-c_i, v-b_i\} & \text{if } a_i+b_i = c_i \\ \{a_i, v-c_i, b_i, v-a_i, c_i, v-b_i\} & \text{if } a_i+b_i+c_i = 6t+1, \end{cases}$$

$i = 1,2,\ldots,t$ and these t base blocks generate a BCD (v,6,2) which is cyclic.

<u>Lemma 5.22</u>. If there exists a BCD (12t+7, 6,2) then there exists a BCD (12t+8, 6,6).

<u>Proof</u>. Let $V = A_{12t+7} \cup \{\infty\}$ and let A_{12t+7}, $\mathcal{L}$ be, respectively, the set of elements, a collection of base blocks of a BCD (12t+7, 6,2). If t = 0, let $B_1 = \{0,1,2,3,4,\infty\}$, $B_2 = \{0,2,4,6,1,\infty\}$ and $B_3 = \{0,3,6,2,5,\infty\}$, then B_1, B_2 and B_3, together with base blocks of $\mathcal{L}$, generate a BCD (8,6,6).

If $t \geq 1$, put, for $i = 1,2,\ldots,t$,

$B_i = B_{t+i} = B_{2t+i} = B_{3t+i} = \{0, 3t+2+2i, 6t+5+4i, 6t+4+i, i, 3i+3\}$

$B_{4t+1} = \{0,1,2,3,4,\infty\}$, $B_{4t+2} = \{0,2,4,6,8,\infty\}$ and $B_{4t+3} = \{0,3,6,9,12,\infty\}$.

The base blocks of $\mathcal{L}$ together with base blocks B_i, $i = 1,2,\ldots, 4t+3$, generate a BCD (12t+8, 6,6).

<u>Lemma 5.23</u>. A BCD (v,6,1) exists for $v \equiv 9 \pmod{12}$, $v \geq 9$.

<u>Proof</u>. Let v = 12t+9 and first let t = 0. Let

$V = \{0_1,1_1,2_1,0_2,1_2,2_2,0_3,1_3,2_3\}$ and a permutation of the elements be $A = (0_1\ 1_1\ 2_1)(0_2\ 1_2\ 2_2)(0_3\ 1_3\ 2_3)$. Put $B_1 = \{0_1, 1_1, 0_2, 1_3, 0_3, 1_2\}$, $B_2 = \{1_1, 1_2, 0_2, 0_3, 2_1, 1_3\}$, then B_1 and B_2 are two base blocks under A and they generate a BCD (9,6,1).

Now let $t \geq 1$ and let $V = A_{12t+9}$. Put $B_i = \{0, 2t+3-2i, 4t+4, 6t+8-2i, 10t+9, 8t+7-2i\}$, $i = 1,2,\ldots, t-1$, $B_t = \{0,3,7,6t+8,4,6t+6\}$, $F_1 = \{0,1,4t+3,4t+4,8t+6,8t+7\}$ and $F_2 = \{0,2,4t+3,4t+5,8t+6,8t+8\}$. The t base blocks B_i, $i = 1,2,\ldots,t$, each of order v and the 2 base blocks F_1, F_2 each of order $v/3$, together generate a BCD $(12t+9, 6, 1)$.

<u>Lemma 5.24</u>. A BCD $(v,6,2)$ exists for $v \equiv 10 \pmod{12}$, $v \geq 10$.

<u>Proof</u>. Let $v = 12t+10$, and $V = A_{12t+9} \cup \{\infty\}$. Consider first the case $t = 0$. Let A_9, $\mathcal{C}$ be, respectively, the set of elements, a collection of base blocks of a BCD $(9,6,1)$, which exists by the lemma above. Put $B_1 = \{0,1,3,6,2,\infty\}$, then B_1 and the base blocks in $\mathcal{C}$ generate a BCD $(10,6,2)$.

Now let $t \geq 1$ and let B_i, $i = 1,2,\ldots,t$, F_1, F_2 be the same as in the second paragraph of the proof of Lemma 5.23, and let, in addition, $B_{t+i} = B_i$, $i = 1,2,\ldots,t$, $B_{2t+1} = \{0,1,3, 4t+4, 8t+6, \infty\}$. These $2t+3$ base blocks generate a BCD $(12t+10,6,2)$.

<u>Lemma 5.25</u>. A cyclic BCD $(v,6,6)$ exists for $v \equiv 11 \pmod{12}$, $v \geq 11$.

<u>Proof</u>. Let $v = 12t+11$ and $V = A_{12t+11}$. Define

$$B_{jt+i} = \begin{cases} \phi & \text{if } t = 0 \\ \{0, 3t+4+2i, 6t+9+4i, 6t+6+i, 1, 3i+5\} & \text{if } t \geq 1, \end{cases}$$

$$i = 1,2,\ldots,t, \quad j = 0, 1, \ldots, 5$$

$B_{6t+1} = \{0,1,2,4,8,5\}$, $B_{6t+2} = \{0,1,3,6,9,5\}$, $B_{6t+3} = \{0,1,3,6,10,5\}$, $B_{6t+4} = \{0,1,3,6,5,2\}$ and $B_{6t+5} = \{0,5,1,6,2,4\}$.

These 6t+5 base blocks generate a BCD (12t+11, 6,6).

<u>Lemma 5.26</u>. A BCD (v,6,2) exists for $v \equiv 0 \pmod{12}$, $v \geq 12$.

<u>Proof</u>. Let $v = 12t$, $V = A_{12t-1} \cup \{\infty\}$. Define $B_i = B_{t+i} = \{0, 3t+1+2i, 6t+3+4i, 6t+i, i, 3i+5\}$, $i = 1,2,\ldots,t-1$, $B_t = \{0,1,3,6,10,5\}$ and $B_{2t} = \{0,1,3,6,10,\infty\}$ then these 2t base blocks generate a BCD (12t,6,2).

We can now summarize the results of this section in the following theorem.

<u>Theorem 5.27</u>. A necessary and sufficient condition for the existence of a BCD $(v,6,\lambda)$ is

$$\begin{array}{ll} v \equiv 1 \text{ or } 9 \pmod{12} & \text{for } \lambda \equiv 1 \text{ or } 5 \pmod{6} \\ v \equiv 1 \pmod{4} & \text{for } \lambda \equiv 3 \pmod{6} \\ v \equiv 0 \text{ or } 1 \pmod{3} & \text{for } \lambda \equiv 2 \text{ or } 4 \pmod{6} \\ v \geq 6 & \text{for } \lambda \equiv 0 \pmod{6}. \end{array}$$

<u>Proof</u>. The necessity follows from condition (5.4), the sufficiency follows from Theorem 1.15 and Lemmas 5.15 - 5.26.

BIBLIOGRAPHY

[1] Alekseev, V.E., Skolem method of constructing cyclic Steiner triple systems, Math. Notes 2 (1967), 571-576.

[2] Bose, R.C., On the construction of balanced incomplete block designs, Ann. Eugenics 9 (1939), 353-399.

[3] Bose, R.C., Cameron, J.M., The bridge tournament problem and calibration designs for comparing pairs of objects, J. Res. NBS (Math. and Math. Phys.) 69B (1965), 323-332.

[4] Bose, R.C., Cameron, J.M., Calibration designs based on solutions to the tournament problem, J. Res. NBS (Math. and Math. Phys.) 71B (1967), 149-160.

[5] Davies, R.O., On Langford's problem II, Math. Gaz. 43 (1959), 253-255.

[6] Guy, R.K., Beineke, L.W., The coarseness of the complete graph, Canad. J. Math. 20 (1968), 888-894.

[7] Hall, M. Jr., Combinatorial Theory, Blaisdell, Waltham, Mass. 1967.

[8] Hanani, H., The existence and construction of balanced incomplete block designs, Ann. Math. Stat. 32 (1961), 361-386.

[9] Hanani, H., On balanced incomplete block designs with blocks having five elements, J. Combinat. Theory (A) 12 (1972), 184-201.

[10] Harary, F., Graph Theory, Addison-Wesley, Reading, Mass. 1969.

[11] Hell, P., Rosa, A., Graph decompositions, handcuffed prisoners and balanced P-designs, Discrete Math. 2 (1972), 229-252.

[12] Huang, C., Rosa, A., On the existence of balanced bipartite designs, Utilitas Math. 4 (1973), 55-75.

[13] Huang, C., On the existence of balanced bipartite designs II, accepted for publication in Discrete Math.

[14] Hung, S.H.Y., Mendelsohn, N.S., Handcuffed designs, Notices A.M.S. 20 (1973), 254-255.

[15] Kotzig, A., On the decompositions of complete graphs into 4k-gons (in Russian), Math.-fyz.časopis 15 (1965), 229-233.

[16] Raghavarao, D., Constructions and Combinatorial Problems in Design of Experiments, Wiley, 1971.

[17] Rosa, A., On cyclic decompositons of the complete graph into polygons with odd number of edges, (in Slovak), Čas. pěst. mat. 91 (1966), 53-63.

[18] Rosa, A., A note on cyclic Steiner triple systems, (in Slovak), Mat.-fyz. časopis 16 (1966), 285-290.

[19] Rosa, A., On cyclic decompositions of the complete graph into (4m+2)-gons, Mat.-fyz. časopis 16 (1966), 349-352.

[20] Rosa, A., On certain valuations of the vertices of a graph, Theory of Graphs, Proc. Internat. Sympos. ICC Rome 1966, Dunod, Paris 1967, 349-355.

[21] Rosa, A., Huang, C., Complete classification of solutions to the problem of 9 prisoners, Proc. 25th Summer Meeting of the Canad. Math. Congress, Thunder Bay, June 1971, 553-561.

[22] Rosa, A., Huang, C., Another class of balanced graph designs: balanced circuit designs, submitted to Discrete Math.

[23] Roselle, D.P., Distributions of integers into s-tuples with given differences, Conf. Num. Math., Winnipeg 1971, 31-42.

[24] Skolem, Th., On certain distributions of integers in pairs with given differences, Math. Scand. 5 (1957), 57-68.

[25] Wilson, R.M., Solution of the problem of handcuffed prisoners, unpublished manuscript.

[26] Fisher, R., Incomplete randomized blocks, Ann. Eugen., Lond., 7 (1936), 121-140.