ORDERED TOPOLOGICAL VECTOR SPACES AND GROUPS

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BY

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SCOPE AND CONTENTS: We introduce three new classes of ordered topological vector spaces and investigate some of their properties. We show that they are good enough for the validity of the analogues of the Banach-Steinhaus theorem for sets of positive linear mappings, and this helps us obtain some results about positive Schauder bases. We use to advantage one of these classes of spaces to give an affirmative answer to a question raised in ([35]) about the order-bounded sets in the tensor products of ordered locally convex spaces.

We prove some results on the continuity of sublinear mappings of ordered vector spaces equipped with the order bound sc-topology (see the text for the definition).

Finally, we prove a type of closed graph theorem for ordered topological abelian groups.

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In Memory of my Parents

Haji Syed Peer*

and

Sufia Bibi

*He died when the author was away preparing this dissertation at McMaster

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INTRODUCTION

Barrelled and quasibarrelled spaces form important classes of locally convex topological vector spaces. In view of the important role played by these spaces in functional analysis, Husain ([10]) found it desirable to generalize them. He has, thus, introduced and studied in ([10]) two new classes of locally convex spaces which he has labelled as the classes of countably barrelled and countably quasibarrelled spaces; they generalize the classes of barrelled and quasibarrelled spaces respectively. Motivated by the idea of Husain ([10]), we introduce and study, in Chapter I, a class of locally convex vector lattices (more generally, a class of ordered locally convex spaces) which we call a class of countably order-quasibarrelled vector lattices, abbreviated to C.O.Q. vector lattices (respectively, a class of countably order-quasibarrelled vector spaces, abbreviated to C.O.Q. vector spaces). The class of C.O.Q. vector lattices (C.O.Q. vector spaces) generalizes that of order-infrabarrelled Riesz spaces (respectively, order-infrabarrelled spaces) introduced by Wong ([46]) (respectively [47]). We prove a Banach-Steinhaus type theorem for sets of lattice homomorphisms on C.O.Q. vector lattices (respectively, for sets of positive linear maps on C.O.Q. vector spaces). We obtain analogues of two well known theorems in the theory of bases, namely the weak basis theorem and the isomorphism theorem, for order-infrabarrelled spaces with positive Schauder bases - these two results gain nothing in generality if we proceed to consider C.O.Q. vector spaces, because it is known that a topological vector space with a basis is separable ([41], page 144), and we show, in

Chapter I, that a separable C.O.Q. vector space is order-infrabarrelled space, and hence a C.O.Q. vector space with a basis is an order-infra-barrelled space. We also prove that the inductive tensor product of two C.O.Q. vector spaces, under certain conditions, is again a C.O.Q. vector space.

Suppose that (E_1, u_1) and (E_2, u_2) are locally convex spaces ordered by generating cones K_1 and K_2 respectively, and that the closure $\bar{K}_p$ of K_p in $E_1 \otimes_p E_2$ is a cone, where K_p is the projective wedge in the algebraic tensor product $E_1 \otimes E_2$ and $E_1 \otimes_p E_2$ is the algebraic tensor product $E_1 \otimes E_2$ equipped with projective tensor product topology. If S and T are order-bounded subsets of E_1 and E_2 respectively, then the closure $\bar{\Gamma}(S \otimes T)$ of the convex circled hull of $(S \otimes T)$ in $E_1 \otimes_p E_2$ is order-bounded for the order structure determined by $\bar{K}_p$ ([35] , pages 186,187). Peressini and Sherbert ([35], page 187) have asked the following question: When is each order-bounded set in $E_1 \otimes_p E_2$ for $\bar{K}_p$ contained in a set of the form $\bar{\Gamma}(S \otimes T)$ for suitable order-bounded sets S and T in E_1 and E_2 respectively? Equivalently, when does the topology of bi-order-bounded convergence on $\mathcal{B}(E_1, E_2)$ coincide with the topology $O(\mathcal{B}(E_1, E_2), E_1 \otimes E_2)$ on $\mathcal{B}(E_1, E_2)$? In order to give an affirmative answer to this question, we introduce, in Chapter II, a subclass of C.O.Q. vector lattices (more generally, a subclass of C.O.Q. vector spaces) which we call a class of order-(DF)-vector lattices (respectively, a class of order-(DF)-vector spaces). The motivation for this class has, in fact, been derived from the notion of Grothendieck ([7]). We give many examples to justify the existence of such a class in ordered vector spaces.

In Chapter III, we introduce and study, what we call, a class of order-quasiultrabarrelled vector lattices, abbreviated to O.Q.U. vector lattices, (more generally, a class of order-quasiultrabarrelled vector spaces, abbreviated to O.Q.U. vector spaces). This class replaces that of order-infrabarrelled Riesz spaces (respectively, order-infrabarrelled vector spaces) in situations where local convexity is not assumed. We, then, show that the Banach-Steinhaus type theorems and the isomorphism theorem that we have proved in Chapter I can be carried over to this class. The closed graph theorem is one of the most fundamental results of functional analysis. Over the years, there have been many efforts to generalize this theorem for various classes of topological vector spaces and for topological groups. (For an illuminating account of this, see Husain [8] and [9]). In Chapter III, section 4, we present a closed graph theorem for O.Q.U. vector spaces.

In ([40]), Simons has introduced a class of topological vector spaces which he calls a class of upper bound spaces. He has characterized such spaces in terms of, what he calls, k-pseudometrics. Subsequently in [16(a)], Iyahen has observed that the class of upper bound spaces coincides with that of semiconvex spaces which he has studied in [16(a),(b)]. In Chapter IV, we study the class of semiconvex spaces with order structure and call it the class of ordered semiconvex spaces. We, then, extend some of the results, known for ordered locally convex spaces, to ordered semiconvex spaces via k-pseudometrics. We also introduce what we call "the order bound sc-topology" in ordered semiconvex spaces and prove some theorems about the continuity of sublinear maps of ordered vector spaces equipped with the order bound sc-topology.

In ([9], Chapter V), Husain has obtained various versions of the closed graph and the open homomorphism theorems for topological groups; in fact he has proved a very general theorem from which almost all existing open homomorphism and closed graph theorems for topological groups can be deduced. Motivated by his work, we prove, in Chapter V, a closed graph type theorem for ordered topological abelian groups.

Chapter 0 contains some basic material for quick reference.

In each section, statements of definitions, lemmas, propositions and theorems and their corollaries are marked r.n.m, where r is the chapter number, n is the section number within the chapter and m is the statement number within the section.

Throughout this dissertation, we use $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ to denote the set of real numbers, the set of positive real numbers and the set of natural numbers respectively.

CHAPTER 0

PRELIMINARIES

The purpose of this chapter is to make this dissertation more readable by including in it some basic material from topological and ordered topological vector spaces and groups, which will be needed in later chapters.

Throughout this dissertation we consider only the real vector spaces and Hausdorff topological vector spaces and groups.

0.1 Topological vector spaces

Most of the results of this section can be found in Husain ([8]).

0.1.1 Definition: (a) A set E is said to be a topological vector space over $\mathbb{R}$ (abbreviated to t.v.s.) if E as a pointset is a topological space and a vector space over $\mathbb{R}$ such that the mappings:

$$(x,y) \longrightarrow x + y ,$$

and

$$(\lambda, x) \longrightarrow \lambda x$$

are continuous in both variables together, for $x,y \in E$ and $\lambda \in \mathbb{R}$.

We write (E,u) to mean a topological vector space E with topology u.

(b) A t.v.s. (E,u) is said to be Hausdorff if the topology u of E is Hausdorff.

0.1.2 Definition: Let E be a vector space. (a) A nonempty subset A of E is said to be convex if for each $x, y \in A$ and $0 \le \lambda \le 1$, $\lambda x + (1-\lambda)y \in A$. The intersection of convex sets is either empty or convex.

(b) A subset A of E is said to be circled if for each $x \in A$ and $|\lambda| \le 1$, $\lambda \in \mathbb{R}$, $\lambda x \in A$.

(c) A subset A of E is said to absorb another subset B of E if there exists $\alpha > 0$ such that $\lambda B \subset A$ for all $|\lambda| \le \alpha$, $\lambda \neq 0$.

(d) A subset A of E is said to be absorbing if it absorbs each point of E.

0.1.3 THEOREM: In a t.v.s. (E,u) there exists a basis η of u-closed neighbourhoods of 0 such that

(i) each $U \in \eta$ is circled and absorbing, and

(ii) for each $U \in \eta$ there exists a $V \in \eta$ such that $V + V \subset U$.

Conversely, if E is a vector space and η is a filter-base satisfying conditions (i) and (ii), then there exists a unique topology u on E such that (E,u) is a t.v.s. and η is a neighbourhood basis of 0 in E.

0.1.4 Definition: (a) A t.v.s. (E,u) is said to be metrizable if there exists a countable neighboourhood basis of 0 in E.

These neighbourhoods can be chosen to satisfy (i) and (ii) of 0.1.3.

(b) A complete metrizable t.v.s. is called an F-space.

Locally convex spaces

0.1.5 <u>Definition:</u> (a) A t.v.s. is said to be locally convex space, abbreviated to l.c.s., if there exists a neighbourhood basis of 0 in it consisting of convex sets.

(b) A complete metrizable l.c.s. is called a Frèchet space.

There exists a metrizable t.v.s. which is not l.c.s. and there exists an l.c.s. which is not metrizable. (see Husain [8], page 16, for examples).

0.1.6 <u>Definition:</u> (a) Let E be a vector space. A real-valued function p defined on E is said to be a semi-norm if the following conditions are satisfied:

(i) $p(\lambda x) = |\lambda| p(x)$ for all $x \in E$ and $\lambda \in \mathbb{R}$, and

(ii) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in E$.

Clearly $p(0) = 0$ and $p(x) \geq 0$ for all $x \in E$.

(b) A semi-norm p on a vector space E is said to be a norm if $p(x) = 0$ implies $x = 0$.

(c) A norm is denoted by $||.||$ and satisfies (i) and (ii). A norm is always a semi-norm, but not conversely.

0.1.7 <u>PROPOSITION:</u> Let (E,u) be a t.v.s.. There is a one-to-one correspondence between the set of all convex, circled, closed subsets containing 0 as their interior point and the set of all continuous semi-norms p defined on (E,u).

Since there exists a neighbourhood basis of 0 consisting of closed, circled, convex sets in an l.c.s. (E,u), the topology u can also be defined by a subset of all continuous semi-norms on (E,u). The set of all semi-norms on a vector space E defines a topology which is the finest locally convex topology on E.

0.1.8 PROPOSITION: An l.c.s. (E,u) is metrizable if u can be described by a countable set of continuous semi-norms.

0.1.9 Definition: (a) A t.v.s. (E,u) is a normed space if u can be defined by a norm.

(b) A t.v.s. is a Banach space if it is normed and complete in the metric induced by the norm.

Inductive limits and direct sums of l.c. spaces

0.1.10 Definition: (a) Let $\{E_\alpha;\alpha\in I\}$ be a family of l.c. spaces and f_α a linear map of E_α into a vector space E, for each α. Suppose $E = \bigcup_\alpha f_\alpha(E_\alpha)$. Let u be the finest locally convex topology on E such that each f_α is continuous. (E,u) is called the inductive limit of $\{E_\alpha;\alpha\in I\}$.

(b) The algebraic direct sum $E = \bigoplus_\alpha E_\alpha$ of a family $\{E_\alpha;\alpha\in I\}$ of l.c. spaces endowed with the finest locally convex topology u such that the embedding $i_\alpha : E_\alpha \longrightarrow E$ is continuous for each α, is the inductive limit of $\{E_\alpha;\alpha\in I\}$. (E,u) is called the direct sum of $\{E_\alpha;\alpha\in I\}$.

Polars and bipolars

0.1.11 Definition: (a) Let E be an l.c.s.. The set E* of all linear real-valued functions on E is called the algebraic dual of E; the set E' of all continuous linear real-valued functions on E is called the topological dual of E; clearly $E' \subset E^* \subset \mathbb{R}^E$.

(b) A member $x' \in E'$ (respectively $x^* \in E^*$) is called a continuous linear functional (respectively linear functional). The value of $x' \in E'$ at $x \in E$ is denoted by $\langle x, x' \rangle$.

Clearly the mapping: $(x, x') \longrightarrow \langle x, x' \rangle$ of $E \times E'$ into $\mathbb{R}$ is bilinear (i.e., linear in each variable); a linear functional $x' : E \longrightarrow \mathbb{R}$ is continuous if it is bounded on an open neighbourhood of 0.

(c) The coarsest locally convex topology for which the mapping: $x \longrightarrow \langle x, x' \rangle$, for each $x' \in E'$, is continuous is called the weak topology on E and is denoted by ω or $\sigma(E, E')$. In the same way, one defines the weak* topology, denoted by ω^* or $\sigma(E', E)$, on E' as the coarsest one for which the mapping: $x' \longrightarrow \langle x, x' \rangle$, for each $x \in E$, is continuous.

$\sigma(E', E)$ is precisely the topology of simple convergence on E', which in turn, is induced from the product topology defined on $\mathbb{R}^E$.

(d) For each subset A of E, the set of all $x' \in E'$ such that $\langle x, x' \rangle \leq 1$, is called the polar of A and is denoted by A°. $A^{\circ\circ} = \{x \in E; \langle x, x' \rangle \leq 1$ for all $x' \in A^\circ\}$ is called the bipolar.

0.1.12 <u>PROPOSITION</u>: Let A, B and A_α (α runs over an index set) be subsets of an l.c.s. E.

(a) If $A \subset B$, then $A^\circ \supset B^\circ$.

(b) If $\lambda \neq 0$, $(\lambda A)^\circ = \lambda^{-1} A^\circ$.

(c) $(\bigcup_\alpha A_\alpha)^\circ = \bigcap_\alpha A_\alpha^\circ$.

(d) For each set A, A° is a $\sigma(E',E)$-closed convex set containing 0.

(e) If A is circled, so is A°.

(f) $A^{\circ\circ}$ = the convex closure (under $\sigma(E,E')$) of $A \cup \{0\}$.

(g) $A^{\circ\circ\circ} = A^\circ$.

(h) If A is a subspace of E, A° is a $\sigma(E',E)$-closed subspace of E'.

(i) Let A be a subspace of E. $A^{\circ\circ} = \bar{A}$ if A is $\sigma(E,E')$-closed. (The same is true for a subspace A of E').

Barrelled and quasibarrelled spaces

0.1.13 __Definition:__ (a) Let (E,u) be an l.c.s.. A subset B of E which is closed, convex, circled and absorbing is called a barrel.

In an l.c.s. each closed, circled, convex neighbourhood of 0 is a barrel. But the converse is not true in general.

(b) A subset A of (E,u) is called bornivorous if it absorbs all u-bounded subsets of E.

(c) An l.c.s. (E,u) is called a barrelled space (quasibarrelled space) if every barrel (bornivorous barrel) in (E,u) is a neighbourhood of 0.

Every Baire l.c.s. is a barrelled space. In particular Fréchet and Banach spaces are barrelled.

Every barrelled space is quasibarrelled, but not conversely.

0.1.14 __PROPOSITION:__ (a) Inductive limits of barrelled (quasibarrelled) spaces are barrelled (quasibarrelled).

(b) Each quotient space of a barrelled (quasibarrelled) space is also a barrelled (quasibarrelled) space.

(c) Any topological direct sum of barrelled (quasibarrelled) spaces is a barrelled (quasibarrelled) space.

(d) The topological product of barrelled (quasibarrelled) spaces is again barrelled (quasibarrelled).

(e) A closed subspace of a barrelled space need not be barrelled.

(f) Let (E,u) be an l.c.s.; then E with the finest locally convex topology is a barrelled space.

<u>REMARK:</u> A barrelled space is neither necessarily metrizable nor complete.

0.1.15 <u>PROPOSITION:</u> (a) If a quasibarrelled space is sequentially complete, then it is barrelled.

(b) The quasi-completion and the completion of a quasibarrelled space are both barrelled.

0.1.16 <u>PROPOSITION:</u> (a) Let (E,u) and (F,v) be two l.c. spaces and f a linear, continuous and almost open mapping of E into F. If (E,u) is barrelled (quasibarrelled), so is (F,v).

(b) Let (E,u) be a barrelled space and (F,v) any l.c.s.. Then (i) any linear map f of E into F is almost continuous; (ii) any linear map g of F onto E is almost open.

Countably barrelled and countably quasibarrelled spaces

Countably barrelled and countably quasibarrelled spaces are due to T. Husain ([10]). Almost all the results for countably barrelled and countably quasibarrelled spaces, which are given here for the sake of completeness, are due to him.

0.1.17 Definition: Let (E,u) be an l.c.s.. (E,u) is said to be countably barrelled (countably quasibarrelled) if each $\sigma(E',E)$-bounded ($\beta(E',E)$-bounded) subset of E' which is the countable union of equicontinuous subsets of E' is itself equicontinuous.

0.1.18 THEOREM: An l.c.s. (E,u) is countably barrelled (countably quasibarrelled) if each barrel (bornivorous barrel) which is the countable intersection of closed, circled, convex neighbourhoods of 0 is itself a neighbourhood of 0 in E.

0.1.19 COROLLARY: Every barrelled (quasibarrelled) space is countably barrelled (countably quasibarrelled).

0.1.20 THEOREM: (BANACH-STEINHAUS)

Let (E,u) be a countably barrelled space. Let $\{f_n; n = 1,2, \cdots\}$ be a $\sigma(E',E)$-bounded sequence in E'. If $\{f_n; n\in \mathbb{N}\}$ converges to a linear functional f pointwise, then $f\in E'$ and $\{f_n; n\in \mathbb{N}\}$ converges to f uniformly on each precompact subset of (E,u).

The proof of the following proposition is different from that of Husain ([10], Proposition 6), and our proof avoids duality theory.

0.1.21 PROPOSITION: Let (E,u) be a countably barrelled space and (F,v) any l.c.s.. Let $\{H_n; n\in \mathbb{N}\}$ be a sequence of equicontinuous subsets of linear maps of E into F such that $H = \bigcup_{n\geq 1} H_n$ is pointwise bounded. Then H is equicontinuous.

PROOF: Let V be a closed, convex, circled neighbourhood of 0 in F. Let $W_n = \bigcap_{f\in H_n} f^{-1}(V)$; then, clearly W_n is a closed, convex, circled neighbourhood of 0 in E. Now, let

$$W = \bigcap_{n\geq 1} W_n = \bigcap_{f\in H} f^{-1}(V) .$$

Then W is clearly a closed, convex, circled set in E; also it is absorbing, because H is pointwise bounded. Hence W is a barrel which is the countable intersection of closed, circled, convex neighbourhoods W_n of 0 in E and hence is itself a neighbourhood of 0 in E. This shows that H is equicontinuous, because $W = \bigcap_{f\in H} f^{-1}(V)$.

0.1.22 COROLLARY: Let (E,u) be a countably barrelled space and (F,v) any l.c.s.. If $\{f_n; n\in \mathbb{N}\}$ is a pointwise bounded sequence of continuous linear maps of (E,u) into (F,v), then $\{f_n; n\in \mathbb{N}\}$ is equicontinuous.

0.1.23 THEOREM: Let $\{E_\alpha; \alpha\in I\}$ be a family of countably barrelled spaces and $\{f_\alpha; \alpha\in I\}$ a family of linear maps into a vector space E. Let E be endowed with the finest locally convex topology u such that each f_α is continuous. Then (E,u) is countably barrelled.

0.1.24 COROLLARY: Let $\{E_\alpha;\alpha\in I\}$ be a family of countably barrelled spaces and (E,u) its inductive limit. Then (E,u) is also countably barrelled.

0.1.25 COROLLARY: Let (E,u) be a countably barrelled space and M a closed subspace of (E,u). Then the quotient E/M is countably barrelled.

0.1.26 COROLLARY: Let $\{E_\alpha;\alpha\in I\}$ be a family of countably barrelled spaces. Then its direct sum is also countably barrelled.

0.1.27 PROPOSITION: ([15(b)]) The (projective) tensor product of two metrizable countably barrelled spaces is countably barrelled.

Tensor Products

This material is taken from Schaefer ([39]).

Let E,F & G be vector spaces over the same field $\mathbb{R}$; a mapping $f:E\times F \longrightarrow G$ is called bilinear if for each $x\in E$ and each $y\in F$, the partial mappings $f_x:y \longrightarrow f(x,y)$ and $f_y:x \longrightarrow f(x,y)$ are linear. If E, F & G are t.v. spaces, it is not difficult to prove that a bilinear map f is continuous if it is so at (0,0). A bilinear map f is said to be separately continuous if all partial maps f_x and f_y are continuous. If G = $\mathbb{R}$, then a bi-linear map of $E\times F$ into $\mathbb{R}$ is called a bilinear functional on $E\times F$.

Let $B(E,F)$ denote the vector space of all bilinear functionals on $E\times F$. For each pair $(x,y)\in E\times F$, the mapping $f \longrightarrow f(x,y)$ is a linear functional on $B(E,F)$, and hence an element $h_{x,y}$ of the algebraic dual $B(E,F)^*$. It is easily seen that the mapping $g:(x,y) \longrightarrow h_{x,y}$ of $E\times F$ into $B(E,F)^*$ is bilinear. The linear hull of $g(E\times F)$ in $B(E,F)^*$ is denoted by $E\otimes F$ and is called the tensor product of E and F; g is called the canonical bilinear map of $E\times F$ into $E\otimes F$. The element $h_{x,y}$ of $E\otimes F$ is denoted by $x\otimes y$ so that each element of $E\otimes F$ is a finite sum $\Sigma\, \lambda_i(x_i\otimes y_i)$, (the sum over the empty set being 0). We shall write $A\otimes B = g(A\times B)$ for arbitrary subsets $A\subset E$, $B\subset F$; for any subspaces $M\subset E$, $N\subset F$, the symbol $M\otimes N$ shall denote the linear hull of $g(M\times N)$ rather than the set $g(M\times N)$ itself.

It is easy to verify the following rules:

$\lambda(x\otimes y) = (\lambda x)\otimes y = x\otimes(\lambda y)$, $\lambda\in\mathbb{R}$,

$(x_1 + x_2)\otimes y = x_1\otimes y + x_2\otimes y$, and

$x\otimes(y_1 + y_2) = x\otimes y_1 + x\otimes y_2$.

Hence each element $h\in E\otimes F$ is of the form $h = \Sigma x_i\otimes y_i$; obviously, the representation of h is not unique.

0.1.28 <u>Definition</u>: Let E_1 and E_2 be l.c. spaces. Consider the family $\mathbb{P}$ of all locally convex topologies on $E_1\otimes E_2$ for which the canonical bilinear map $g:E_1\times E_2 \longrightarrow E_1\otimes E_2$ is continuous. The upper bound u_p of $\mathbb{P}$ is a locally convex topology, called the Projective (tensor product) topology on $E_1\otimes E_2$. If η_1 and η_2 are neighbourhood bases of 0 in E_1 and E_2 respectively, then the family of convex, circled hulls

$\{\Gamma(U \otimes V); U \in \eta_1, V \in \eta_2\}$,

is a neighbourhood basis of 0 for u_p; thus the projective topology is the finest locally convex topology on $E_1 \otimes E_2$ for which the canonical bilinear map is continuous.

NOTATIONS: (i) $B(E,F:G)$ denotes the vector space of all bilinear mappings of ExF into G.

(ii) If E,F and G are t.v. spaces, then $\mathcal{B}(E,F:G)$ denotes the vector space of continuous bilinear maps.

(iii) $L(E,F)$ denotes the vector space of all linear mappings of E into F, and if E,F are t.v. spaces, then $\mathcal{L}(E,F)$ denotes the vector space of all continuous linear mappings of E into F.

0.1.29 PROPOSITION: (a) Let E_1,E_2 be vector spaces over $\mathbb{R}$ and let g be the canonical bilinear map of $E_1 x E_2$ into $E_1 \otimes E_2$. For any vector space G over $\mathbb{R}$, the mapping $h \longrightarrow h \circ g$ is an isomorphism of $L(E_1 \otimes E_2, G)$ onto $B(E,F:G)$.

(b) The algebraic dual of $E_1 \otimes E_2$ can be identified with $B(E,F)$; under this identification, each vector space of linear functionals on $E_1 \otimes E_2$ is a vector space of bilinear functionals on $E_1 x E_2$, and conversely.

0.1.30 PROPOSITION: (a) Let E_1, E_2 and G be l.c. spaces and let $E_1 \otimes E_2$ be provided with the projective topology. Then the isomorphism $h \longrightarrow h \circ g$ of 0.1.29(a) maps the space of continuous linear mappings $\mathcal{L}(E_1 \otimes E_2, G)$

onto the space of continuous bilinear mappings $\mathcal{B}(E_1,E_2;G)$.

(b) The dual of $E_1 \otimes E_2$ for the projective topology can be identified with the space $\mathcal{B}(E_1,E_2)$ of all continuous bilinear functionals on $E_1 \times E_2$. Under this identification, the equicontinuous subsets of $(E_1 \otimes E_2)'$ are the equicontinuous sets of bilinear functionals on $E_1 \times E_2$.

0.1.31 <u>Definition</u>: Let E_1 and E_2 be l.c. spaces. The inductive (tensor product) topology u_i on $E_1 \otimes E_2$ is defined to be the finest locally convex topology on $E_1 \otimes E_2$ for which the canonical bilinear map is separately continuous.

<u>REMARK</u>: u_i is the inductive limit topology with respect to the family of mappings

$$\{g_x,\ g_y^\circ;\ x \in E_1,\ y \in E_2\} \text{ of } E_2 \text{ (respectively, } E_1) \text{ into } E_1 \otimes E_2 .$$

<u>Bases</u>

The following material on bases is collected from Marti ([25]).

0.1.32 <u>Definition</u>: Let E be a t.v.s. and E^* its algebraic dual. Let $\{x_i\}$ and $\{f_i\}$ be sequences in E and E^* respectively. $\{x_i,f_i\}$ is called a biorthogonal system for E if $f_i(x_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

0.1.33 <u>Definition:</u> A sequence $\{x_i\}$ in a t.v.s. E is a basis for E if for each $x \in E$ there is a unique sequence $\{\alpha_i\}$ in $\mathbb{R}$ such that $x = \lim_n \sum_{i=1}^{n} \alpha_i x_i$ in the topology of E.

Evidently, each expansion coefficient α_i, $f_i(x) = \alpha_i$, defines a linear functional f_i on E. However, the coefficient functionals f_i need not necessarily be continuous.

0.1.34 <u>Definition:</u> A basis for a t.v.s. E with continuous coefficient functionals f_i, defined by $f_i(x) = \alpha_i$, $x \in E$, $i = 1, 2, \cdots$, is called a Schauder basis for E.

0.1.35 <u>THEOREM:</u> In a Banach (or even Fréchet) space E every basis $\{x_i\}$ is a Schauder basis.

Let E be a t.v.s. over $\mathbb{R}$, E' its topological dual and let Λ be an index set of arbitrary cardinality.

0.1.36 <u>Definition:</u> The double family $\{x_\lambda, f_\lambda\}$ is a biorthogonal system if $x_\lambda \in E$ and $f_\lambda \in E^*$ for all $\lambda \in \Lambda$ and if $f_\lambda(x_\mu) = \delta_{\lambda\mu}$, where $\delta_{\lambda\mu}$ is the Kronecker delta. $\{x_\lambda, f_\lambda\}$ is maximal with respect to E if there is no biorthogonal system which contains $\{x_\lambda, f_\lambda\}$ properly.

0.1.37 <u>Definition:</u> A biorthogonal system $\{x_\lambda, f_\lambda\}$ is a generalized basis for E if $f_\lambda(x) = 0$, $\lambda \in \Lambda$, implies $x = 0$ for all $x \in E$. $\{x_\lambda, f_\lambda\}$ is total if the finite linear combinations of $\{x_\lambda\}$ are dense in E (i.e., if $\{x_\lambda\}$ is total in E). If Λ is (not) countable, such a basis is called an (extended) Markushevich basis for E.

0.1.38 THEOREM: (a) A generalized basis for E is a maximal biorthogonal system with respect to E.

(b) Every Schauder basis for E is a Markushevich basis for E.

0.1.39 Definition: Let E and F be topological vector spaces. A sequence (x_n) in E is said to be similar to a sequence (y_n) in F if for all sequences $(a_n) \subset \mathbb{R}$,

$$\sum_{n=1}^{\infty} a_n x_n \text{ converges (in E) iff } \sum_{n=1}^{\infty} a_n y_n \text{ converges (in F).}$$

0.2 Topological groups

In this section, we collect those definitions and results from the theory of topological groups which are needed in Chapter V. Most of the results of this section can be found in Husain ([9]).

0.2.1 Definition: A topological space G that is also a group is called a topological group if the mapping $g_1:(x,y) \longrightarrow xy$ of GxG onto G is continuous in both variables together and if the inversion map $g_2:x \longrightarrow x^{-1}$ of G onto G is also continuous.

If the group operation is addition instead of multiplication, xy and x^{-1} should be regarded as $x + y$ and $- x$ respectively. The identity of a multiplicative group will be denoted by e and that of an additive group by 0.

If we put $UV = \{xy;x\in U,y\in V\}$ and $U^{-1} = \{x^{-1};x\in U\}$, where U and V are subsets of a group G (in the additive case, $U + V = \{x + y;x\in U,y\in V\}$, $- U = \{-x;x\in U\}$), then the continuity of the mappings g_1 and g_2 can be expressed as follows:

g_1 is continuous in both x and y if for each neighbourhood W of xy there exist a neighbourhood U of x and a neighbourhood V of y such that

$UV \subset W$. Similarly, g_2 is continuous if for each neighbourhood W of x^{-1}, there exists a neighbourhood U of x such that $U^{-1} \subset W$.

0.2.2 <u>THEOREM:</u> A group G, endowed with a topology u, is a topological group if the mapping: $(x,y) \longrightarrow xy^{-1}$ of $G_u \times G_u$ onto G_u is continuous in both variables together.

0.2.3 <u>THEOREM:</u> Let G_u be a topological group. Then the right and left translations $r_a : x \longrightarrow xa$, $l_a : x \longrightarrow ax$, the inversion mapping: $x \longrightarrow x^{-1}$ and the inner automorphisms: $x \longrightarrow axa^{-1}$ are all homeomorphisms.

0.2.4 <u>COROLLARY:</u> Let F be a closed, P an open, A any subset of a topological group G and $a \in G$. Then aF, Fa, F^{-1} are closed; aP, Pa, P^{-1}, AP and PA are all open.

Thus, it follows that the products PQ and QP of two open sets P and Q in a topological group are open. However, the product of two closed subsets in a topological group need not be closed.

0.2.5 <u>Definition:</u> A subset U of a group G is said to be symmetric if $U = U^{-1}$. In case G is an additive group, U is symmetric if $U = -U$.

0.2.6 <u>PROPOSITION:</u> In a topological group there exists a fundamental system {U} of symmetric neighbourhoods of e.

0.2.7 <u>PROPOSITION:</u> Let {U} be the system of all neighbourhoods of e in a topological group G. Then for any subset A of G, $\bar{A} = \cap AU = \cap UA$.

The above proposition says that for any subset A, $\bar{A} \subset AU$ and $\bar{A} \subset UA$ for each neighbourhood U of e.

0.2.8 <u>COROLLARY</u>: Let U be any neighbourhood of e in a topological group G. Then there is a neighbourhood V of e such that $\bar{V} \subset U$.

0.2.9 <u>THEOREM</u>: In each topological group G, there exists a fundamental system (i.e., a basis) {U} of closed neighbourhoods of the identity such that

(a) each U is symmetric,

(b) for each U in {U}, there exists a V in {U} such that $V^2 \subset U$, and

(c) for each U in {U} and $a \in G$, there exists a V in {U} such that $V \subset a^{-1}Ua$ or $aVa^{-1} \subset U$.

Conversely, given a group G with a filter base {U} satisfying (a) - (c), there exists a unique topology u on G such that G_u is a topological group and {U} forms a fundamental system of neighbourhoods of e.

<u>REMARK</u>: In case the group is additive, the conditions of the theorem can be stated as follows:

(a') each U is symmetric, i.e., $U = -U$,

(b') for each U in {U}, there exists a V in {U} such that $V + V \subset U$, and

(c') for each U in {U}, there exists a V in {U} such that $V \subset -a + U + a$ or $a + V - a \subset U$.

In case G is abelian, one can dispense with (c) and (c').

0.2.10 THEOREM: For a topological group G, the following statements are equivalent:

(a) G is a T_0-space.

(b) G is a T_1-space.

(c) G is a T_2-space (or Hausdorff space).

(d) $\cap U = \{e\}$, where $\{U\}$ is a neighbourhood basis of e.

0.3 Ordered vector spaces

This section is composed of those definitions and results from the theory of ordered vector spaces which are needed in the later chapters. Most of the material of this section can be found in Peressini ([32]), Jameson ([17]), and Luxemburg and Zaanen ([23]).

0.3.1 Definitions: An ordered vector space is a vector space E equipped with a transitive, reflexive and antisymmetric relation $\leq$ satisfying the following conditions:

(O_1) If $x \leq y$, $x,y \in E$, then $x + z \leq y + z$ for all $z \in E$.

(O_2) If $x \leq y$, $x,y \in E$, then $\alpha x \leq \alpha y$ for all $\alpha \in \mathbb{R}^+$.

The positive cone (or simply, cone) K in an ordered vector space E is defined by $K = \{x \in E; x \geq 0\}$, where 0 denotes the zero element in E. The cone K satisfies the following properties:

(C_1) $K + K \subseteq K$,

(C_2) $\lambda K \subseteq K$, $\lambda \in \mathbb{R}^+$, and

(C_3) $K \cap (-K) = \{0\}$.

In particular, it follows from (C_1) and (C_2) that K is a convex set in E. On the other hand, if K is a subset of a vector space E satisfying (C_1), (C_2) and (C_3) then $x \leq y$ if $y - x \in K$ defines an order relation $\leq$ on E with respect to which E is an ordered vector space with positive cone K. Thus, for a given vector space E, there is a canonical one-to-one correspondence between the collection of order relations with properties (O_1) and (O_2) and the collection of all subsets of E with properties (C_1), (C_2) and (C_3).

A subset K of E containing 0 and satisfying (C_1) and (C_2) is called a wedge.

Suppose that E is an ordered vector space. If $x, y \in E$ and $x \leq y$, then the set $[x,y] = \{z \in E; x \leq z \leq y\}$ is called the order interval between x and y. A subset B of E is order bounded if there exists $x, y \in E$ such that $B \subseteq [x,y]$.

Suppose that K is the positive cone in an ordered vector space E. K is said to be a generating cone if E is the linear subspace spanned by K, i.e. $E = K - K$. A subset D of an ordered vector space E is majorized (respectively minorized) if there exists an element $z \in E$ such that $z \geq d$ (respectively $z \leq d$) for all $d \in D$. If every pair $x, y \in D$ is majorized (minorized) in D, then D is directed $(\leq)$ [respectively directed $(\geq)$]. Observe that the cone K is generating iff E is directed $(\leq)$.

An element $e \in E$ is an order-unit if for each $x \in E$ there is a $\lambda > 0$ such that $x \leq \lambda e$. The cone K in E generates E if E contains an order-unit. E is almost Archimedean if for some $y \in K$ and all $\alpha > 0$, $-\alpha y \leq x \leq \alpha y$ implies $x = 0$. E is Archimedean if for some $y \in K$ and all $\alpha > 0$, $\alpha x \leq y$ implies $x \leq 0$. Clearly every Archimedean ordered vector space is almost Archimedean.

If B is a subset of an ordered vector space E and if $x \in E$ has the following properties:

(a) $x \geq b$ for all $b \in B$, and

(b) $z \geq x$ whenever $z \geq b$ for all $b \in B$,

then x is called the supremum of B and we write $x = \sup(B)$. The infimum of B, written $\inf(B)$, is defined by replacing $\geq$ by $\leq$ in (a) and (b). If the $\sup(x,y) = x \vee y$ and the $\inf(x,y) = x \wedge y$ of every pair $x, y \in E$ exist, then E is called a vector lattice. Since $x \vee y = -\{(-x) \wedge (-y)\}$, E is a vector lattice if the supremum (or infimum) of every pair of elements of E exists.

If E is a vector lattice and $x \in E$, we define

$$x^+ = \sup(x,0),\ x^- = (-x)^+,\ |x| = \sup(x,-x).$$

x^+ and x^- are called the positive part and the negative part respectively of x, while $|x|$ is referred to as the absolute value of x.

If E is a vector lattice, the following identities are direct consequences of (O_1) and (O_2):

$$z - (x \vee y) = (z - x) \wedge (z - y) \quad (1)$$

$$\left.\begin{aligned} z + (x \vee y) &= (z + x) \vee (z + y) \\ z + (x \wedge y) &= (z + x) \wedge (z + y) \end{aligned}\right\} \quad (2)$$

$$\alpha(x \vee y) = \alpha x \vee \alpha y,\ \alpha(x \wedge y) = \alpha x \wedge \alpha y,\ \alpha \in \mathbb{R}^+ \quad (3)$$

If we replace z by x + y in (1), we obtain the identity $x + y = x \vee y + x \wedge y$. (4)

In particular, if y = 0 in (4), then

$$x = x^+ - x^-. \quad (5)$$

It follows from (5) that the cone in a vector lattice E is always generating. It is easy to verify that $|x| = x^+ + x^-$. (6)

0.3.2 <u>PROPOSITION:</u> A vector lattice is distributive, i.e.

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) ,$$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ for all } x,y,z \in E.$$

0.3.3 <u>PROPOSITION:</u> If $x,y,z \in E$, E a vector lattice, then,

(a) $|x + y| \leq |x| + |y|$.

(b) $||x| - |y|| \leq |x - y|$.

(c) $|(x \vee z) - (y \vee z)| \leq |x - y|$.

(d) $|x^+ - y^+| \leq |x - y|$, $|x^- - y^-| \leq |x - y|$.

(e) $(x + y)^+ \leq x^+ + y^+$, $(x + y)^- \leq x^- + y^-$.

(f) $(x + y) \wedge z \leq x \wedge z + y \wedge z$, if $x,y,z \in K$, K positive cone in E.

0.3.4 <u>PROPOSITION:</u> Let (E,K) be a vector lattice with the positive cone K. If $x,y,z \in E$ such that $0 \leq z \leq x + y$, then there exist $x_1, y_1 \in E$ such that $0 \leq x_1 \leq x$, $0 \leq y_1 \leq y$, $z = x_1 + y_1$. In otherwords, if $x,y \in E$, then (*) $[0,x] + [0,y] = [0,x + y]$, for all $x,y \in K$. This, in turn, is equivalent to the property that $[x,y] + [u,v] = [x + u, y + v]$ for all $x,y,u,v \in E$.

An ordered vector space satisfying (*) is said to have the decomposition property. Thus, every vector lattice has the decomposition property.

If (E,K) is an ordered vector space with positive cone K and if M is a linear subspace of E, then M is an ordered vector space for the order determined by the cone $K \cap M$. If g is the canonical map of E onto the quotient space E/M, then $g(K)$ is a wedge in E/M; $g(K)$ is not necessarily a cone in E/M ([32], 1.1.7).

If $\{(E_\alpha, C_\alpha); \alpha \in I\}$ is an arbitrary family of ordered vector spaces, then the product space $\prod_\alpha E_\alpha$ is an ordered vector space for the order structure determined by the cone $K = \prod_\alpha C_\alpha$. Similarly, the direct sum $\bigoplus_\alpha E_\alpha$ is an ordered vector space for the cone $C = \bigoplus_\alpha C_\alpha$.

These order structures on subspaces, quotients, products and direct sums of ordered vector spaces will be referred to as canonical order structures. Observe that the decomposition property and the property of being a vector lattice are preserved in the formation of products and direct sums.

A linear map T from an ordered vector space (E_1, K_1) into an ordered vector space (E_2, K_2) is called

(a) positive if $T(K_1) \subseteq K_2$,

(b) strictly positive if $Tx > 0$ whenever $x > 0$,

(c) order-bounded if T maps each order-bounded set in E_1 into an order-bounded set in E_2.

Clearly every strictly positive linear map is positive and every positive linear map is order-bounded.

The collection $K(E_1, E_2)$ of all positive linear mappings of an ordered vector space E_1 into an ordered vector space E_2 is a wedge in the vector space of all linear mappings of E_1 into E_2. Moreover, if the cone K_1 in E_1 is generating, then the wedge $K(E_1, E_2)$ is a cone.

We use $L^b(E_1, E_2)$ to denote the vector space of all order-bounded linear maps of E_1 into E_2. The vector space $L^b(E, \mathbb{R})$ of all order-bounded linear functionals on an ordered vector space E will be denoted by E^b, and the wedge $K(E, \mathbb{R})$ by K^*. The linear hull E^+ of K^* in E^b, i.e., $E^+ = K^* - K^*$ will be referred to as the order dual of E. If E is an ordered vector space with the decomposition property and a generating cone, then $E^+ = E^b$.

Lattice ideals and lattice homomorphisms

0.3.5 Definitions: A subset B of a vector lattice E is solid if $y \in B$ whenever $x \in B$ and $|y| \leq |x|$. A linear subspace M of E is a lattice ideal (abbreviated to l-ideal) if M is a solid subset of E. Every l-ideal M in a vector lattice E is a sublattice of E; i.e., for $x, y \in M$, the $\sup(x,y)$ and $\inf(x,y)$ in E lie in M.

0.3.6 PROPOSITION: If E is a vector lattice and M is an l-ideal in E, then the quotient space $\dot{E} = E/M$ is a vector lattice for the order structure determined by the canonical image $\dot{K}$ in $\dot{E}$ of the cone K in E.

0.3.7 Definition: A linear map from one vector lattice to another is called a lattice homomorphism (abbreviated to l-homomorphism) if it preserves the lattice operations.

Observe that l-homomorphisms are monotonic (i.e., $f(x) \leq f(y)$ whenever $x \leq y$). If an l-homomorphism has an inverse, then this is also an l-homomorphism.

The relation $x \vee y + x \wedge y = x + y$ shows that for a linear map f to be an l-homomorphism it is sufficient that it preserves either $\vee$ or $\wedge$.

0.3.8 PROPOSITION: If E and F are vector lattices and f a linear map from E to F, then the following statements are equivalent:

(i) f is an l-homomorphism.

(ii) $f(x^+) \wedge f(x^-) = 0$ for all $x \in E$.

(iii) if $x \wedge y = 0$, then $f(x) \wedge f(y) = 0$.

(iv) $f(|x|) = |f(x)|$ for all $x \in E$.

0.4 Ordered topological vector spaces

The material of this section can be found in Peressini ([32]), Schaefer ([39]) and Jameson ([17]).

0.4.1 Definition: An ordered vector space which is also a t.v.s. is called an ordered t.v.s..

We use the notation (E,C,u) to indicate that E is an ordered t.v.s. with positive cone C and topology u.

REMARK: Observe that we do not assume the cone C in an ordered t.v.s. to be closed.

0.4.2 Definition: (a) Let (E,C) be an ordered vector space with positive cone C. Let A be a subset of E. Then the full hull [A] of A is defined by

$$[A] = (A + C) \cap (A - C) = \{z \in E; x \leq z \leq y,\ x,y \in A\}.$$

Clearly $A \subset [A]$ and $[[A]] = [A]$.

(b) If $A = [A]$, then A is said to be full.

<u>REMARK</u>: If A is circled (convex), so is $[A]$.

0.4.3 <u>Definition</u>: Let (E,C,u) be an ordered t.v.s.. The cone C is normal for u if there exists a neighbourhood basis η of 0 for u consisting of full sets.

We can assume that each member of η is circled, because the full hull of a circled set is circled. Moreover, if (E,C,u) is ordered l.c.s., then we can assume that the sets in η are convex.

For examples of normal cones, see ([32], page 65).

0.4.4 <u>PROPOSITION</u>: Let (E,C,u) be an ordered t.v.s.. The following statements are equivalent:

(a) C is normal for u.

(b) There exists a neighbourhood basis $\{V\}$ of 0 for u such that $0 \leq x \leq y$, $y \in V$ implies $x \in V$.

(c) For any two nets $\{x_\beta ; \beta \in I\}$ and $\{y_\beta ; \beta \in I\}$ in (E,C,u), if $0 \leq x_\beta \leq y_\beta$ for all $\beta \in I$ and if $\{y_\beta ; \beta \in I\}$ converges to 0 for u, then $\{x_\beta ; \beta \in I\}$ converges to 0 for u.

(d) Given a u-neighbourhood V of 0, there is a u-neighbourhood W of 0 such that $0 \leq x \leq y$, $y \in W$ implies $x \in V$.

A very useful consequence of the normality restriction on positive cone in an ordered t.v.s. is given in the following result.

0.4.5 **PROPOSITION:** Let (E,C,u) be an ordered t.v.s.. If C is normal for u, then every order-bounded subset of E is u-bounded.

0.4.6 **Definition:** An ordered t.v.s. (E,C,u) equipped with a (Hausdorff) locally convex topology u is called an ordered l.c.s..

0.4.7 **PROPOSITION:** If (E,C,u) is an ordered l.c.s., then the following statements are equivalent:

(a) C is normal for u.

(b) There is a family $\{p_\alpha;\alpha\epsilon I\}$ of semi-norms generating the topology u such that $0 \leq x \leq y$ implies $p_\alpha(x) \leq p_\alpha(y)$ for all $\alpha\epsilon I$; equivalently $p_\alpha(z+w) \geq p_\alpha(z)$ for all $z,w\epsilon C$, $\alpha\epsilon I$.

0.4.8 **COROLLARY:** The closure $\bar{C}$ of a normal cone C in an ordered l.c.s. (E,C,u) is a normal cone.

0.4.9 **Definition:** Let (E,C,u) be an ordered t.v.s.. Let $\mathcal{B}$ be the family of all u-bounded subsets of E. The cone C is called a strict $\mathcal{B}$-cone if $\{A\cap C - A\cap C; A\epsilon \mathcal{B}\}$ is fundamental for $\mathcal{B}$.

0.4.10 **PROPOSITION:** (Nachbin-Namioka-Schaefer). If (E_1,C_1,u_1) and (E_2,C_2,u_2) are ordered t.v.spaces and if the cone C_2 in E_2 is normal, then each of the following conditions implies that every positive linear mapping T of (E_1,C_1,u_1) into (E_2,C_2,u_2) is continuous:

(a) The cone C_1 in (E_1,u_1) has a non-empty interior.

(b) (E_1,u_1) is a bornological space ordered by a sequentially complete strict $\mathcal{B}$-cone and (E_2,C_2,u_2) is an l.c.s..

(c) (E_1,C_1,u_1) is a metrizable t.v.s. of second category ordered by a complete generating cone C_1 and (E_2,u_2) is an l.c.s..

0.4.11 <u>COROLLARY</u>: Let (E,C,u) be a t.v.s. ordered by a cone C; then each of the following conditions on (E,C,u) implies that every positive linear functional on (E,C,u) is continuous:

(a) C has a non-empty interior.

(b) (E,u) is a bornological space and C is a sequentially complete strict $\mathcal{B}$-cone.

(c) (E,u) is a metrizable t.v.s. of second category ordered by a complete generating cone C.

<u>Locally convex vector lattices</u>

0.4.12 <u>Definition</u>: An ordered t.v.s. (E,C,u) which is a vector lattice is called a topological vector lattice (abbreviated to t.v.l.) if there exists a neighbourhood basis of 0 for u consisting of solid sets. In addition, if (E,u) is an l.c.s., then (E,C,u) is called a locally convex vector lattice (abbreviated to l.c.v.l.).

A vector lattice (E,C) equipped with a norm $||.||$ is called a normed vector lattice if $|x| \leq |y|$ implies $||x|| \leq ||y||$; if E is complete for this norm topology, then $(E,C,||.||)$ is called a Banach lattice.

Since a solid subset A of a vector lattice (E,C) is circled and the convex hull of A is solid ([32], Proposition 2.4.1), it follows that in an l.c.v.l. (E,C,u) there is a neighbourhood basis of 0 for u consisting of convex, solid sets. Also, if E is a normed vector lattice, the unit ball in E is solid; in particular, E is an l.c.v.l. for the topology generated by the norm.

Let (E,C) be a vector lattice. The mappings: $(x,y) \longrightarrow x \vee y$ and $(x,y) \longrightarrow x \wedge y$ of ExE into E are called the lattice operations.

0.4.13 <u>PROPOSITION:</u> If (E,u) is a t.v.s. which is a vector lattice, then (E,u) is a t.v.l. iff the cone C in (E,u) is normal and the lattice operations are continuous.

0.4.14 <u>PROPOSITION:</u> (a) the closure $\bar{B}$ of a solid set B in a t.v.l. (E,C,u) is solid.

(b) If (E,C,u) is a t.v.l., there is a neighbourhood basis of 0 consisting of closed, solid sets. If M is an l-ideal in (E,C,u), the closure $\bar{M}$ of M is an l-ideal in (E,C,u).

0.4.15 <u>PROPOSITION:</u> If (E,C,u) is a t.v.s. which is a vector lattice and if the lattice operations in (E,C,u) are continuous, then the cone C in (E,u) is a strict B-cone. In addition, since u is a Hausdorff topology, C is closed.

0.4.16 <u>PROPOSITION:</u> If (E,C,u) is a t.v.l. and if M is a vector sublattice of E, then M is a t.v.l. for the topology induced by (E,u). If M is an l-ideal in (E,C,u), then the quotient space E/M is a t.v.l. for the quotient topology. An arbitrary product of t.v. lattices is a t.v.l..

0.4.17 <u>PROPOSITION:</u> Let (E,C,u) be an l.c.s. which is a vector lattice. Then the following statements are equivalent:

(a) (E,C,u) is an l.c.v.l..

(b) For any nets $\{x_\alpha;\alpha\in I\}$ and $\{y_\alpha;\alpha\in I\}$ in E, if $\{y_\alpha;\alpha\in I\}$ converges to 0 in (E,u) and $|x_\alpha| \le |y_\alpha|$ for all $\alpha\in I$, then $\{x_\alpha;\alpha\in I\}$ converges to 0 in (E,u).

(c) There is a family $\{p_\beta;\beta\in J\}$ of semi-norms on E that generates u such that $|x| \le |y|$ implies $p_\beta(x) \le p_\beta(y)$ for all $\beta\in J$.

<u>REMARK:</u> A semi-norm p on a vector lattice (E,C) is called a lattice semi-norm if $|x| \le |y|$ implies that $p(x) \le p(y)$.

0.4.18 <u>PROPOSITION:</u> Let (E,C,u) be an l.c.v.l.. Then the completion $(\tilde{E},\tilde{u})$ of (E,u) is an l.c.v.l. for the order structure determined by the closure $\tilde{C}$ of C in $(\tilde{E},\tilde{u})$.

0.5 <u>Ordered groups and topological groups</u>

In this section, we collect some definitions and results in ordered groups and topological groups from Jameson ([17]); we need them in Chapter V. All abelian groups are written additively.

0.5.1 __Definition:__ An ordered abelian group is an abelian group E equipped with a transitive, reflexive and antisymmetric relation $\leq$ satisfying the following condition:

(G_1) If $x \leq y$, then $x + z \leq y + z$ for all $x,y,z,\in E$.

The positive set K in an ordered abelian group E is defined by $K = \{x\in E; x \geq 0\}$; it has the following properties:

(P_1) $K + K \subseteq K$.

(P_2) $K \cap (-K) = \{0\}$.

On the other hand, if K is a subset of an abelian group E satisfying (P_1) and (P_2), then $x \leq y$ iff $y - x\in K$ defines an order relation $\leq$ on E with respect to which E is an ordered abelian group with positive set K. Such orderings are in one-to-one correspondence with semigroups in E containing 0.

If F is a subgroup of an ordered abelian group (E,K), then the restriction to F of the ordering in E is clearly the ordering of F associated with the semigroup $F\cap K$.

0.5.2 __Definition:__ An ordered abelian group which is also a topological group is called an ordered topological abelian group.

0.5.3 __Definition:__ An ordered topological abelian group (E,K,u) is said to be locally full if there exists a neighbourhood basis of 0 consisting of full sets.

REMARK: For the definition of full sets, see 0.4.2.

0.5.4 THEOREM: Let (E,C,u) be an ordered topological abelian group; let $\mathcal{B}$ be a neighbourhood basis of 0 in E. The following statements are equivalent:

(i) E is locally full.

(ii) Given $U \in \mathcal{B}$, there exists $V \in \mathcal{B}$ such that if $x, y \in C$ and $x + y \in V$, then $x, y \in U$.

REMARK: The relation between U and V in (ii) can be expressed in the order notation as follows: If $0 \leq x \leq y \in V$, then $x \in U$.

The property of C expressed in (ii) is a special case of a relation between pairs of subsets of a topological group which is defined as follows:

0.5.5 Definition: Let A and B be subsets of a topological group E and $\mathcal{B}$ a neighbourhood basis of 0 in E. A is allied to B (written A al B) if, given $U \in \mathcal{B}$, there exists $V \in \mathcal{B}$ such that $a \in A$, $b \in B$ and $a + b \in V$ implies $a, b \in U$. If A al A, we say that A is self-allied.

REMARK: In view of the above definition, the result 0.5.4 says that (E,C,u) is locally full iff C is self-allied.

CHAPTER I

COUNTABLY ORDER-QUASIBARRELLED VECTOR LATTICES AND SPACES

In this chapter, we introduce and study a new class of l.c.v. lattices (more generally, ordered l.c. spaces) which generalizes that of order-infrabarrelled Riesz spaces (respectively, order-infrabarrelled spaces) introduced in ([46]) (respectively, in [47]).

1.1 C.O.Q. vector lattices

In this section, we define and characterize, what we call, C.O.Q. vector lattices and show that they form a proper generalization of order-infrabarrelled Riesz spaces ([46]). We establish their relations with countably barrelled and countably quasibarrelled l.c.v. lattices - the concepts of countably barrelled and countably quasibarrelled l,c. spaces have been introduced by Husain in ([10]). We prove a Banach-Steinhaus type theorem for lattice homomorphisms on C.O.Q. vector lattices.

Let (E,C,u) be an l.c.v.l.. A subset B of E is called order-bornivorous if B absorbs all order-bounded subsets of E. Wong ([46]) calls an l.c.v.l. (E,C,u) an order-infrabarrelled Riesz space if each order-bornivorous barrel in (E,C,u) is a neighbourhood of 0 in E.

1.1.1 **Definition:** Let (E,C,u) be an l.c.v.l.. (E,C,u) is called a countably order-quasibarrelled vector lattice (abbreviated to C.O.Q. vector lattice) if every order-bornivorous barrel which is the countable intersection of closed, circled, convex neighbourhoods of 0 in (E,C,u) is itself a neighbourhood of 0.

Let (E,C) be a vector lattice and B a subset of E. The set K(B) defined by

$$K(B) = \{x \in E; [-|x|, |x|] \subseteq B\}$$

is called the solid kernel of B; it is the largest solid set contained in B. If B is convex, so is K(B) ([46], page 18).

1.1.2 <u>THEOREM</u>: An l.c.v.l. (E,C,u) is a C.O.Q. vector lattice iff each solid barrel which is the countable intersection of closed, convex, solid neighbourhoods of 0 in (E,C,u) is itself a neighbourhood of 0.

<u>PROOF</u>: Assume that (E,C,u) is a C.O.Q. vector lattice. Let $B = \bigcap_{n \geq 1} V_n$ be a solid barrel such that each V_n is a closed, convex, solid neighbourhood of 0 in (E,C,u). Since B is solid and absorbing, it follows that B is an order-bornivorous set in (E,C,u). Since a solid set is always circled, it follows that B is an order-bornivorous barrel which is the countable intersection of closed, convex, circled neighbourhoods of 0. Hence B is a neighbourhood of 0 in (E,C,u), because (E,C,u) is a C.O.Q. vector lattice, by assumption.

Conversely, assume that each solid barrel which is the countable intersection of closed, convex, solid neighbourhoods of 0 in (E,C,u) is itself a neighbourhood of 0. Let $B = \bigcap_{n \geq 1} V_n$ be an order-bornivorous barrel such that each V_n is a closed, convex, circled neighbourhood of 0 in (E,C,u). Let K(B) be the solid kernel of B. Then

$$K(B) = \{x \in E; [-|x|, |x|] \subseteq B\}$$

$$= \{x \in E; [-|x|, |x|] \subseteq V_n \text{ for all } n \geq 1\}$$

$$= \bigcap_{n \geq 1} \{x \in E; [-|x|, |x|] \subseteq V_n\}$$

$$= \bigcap_{n \geq 1} K(V_n) .$$

Thus, $K(B) = \bigcap_{n \geq 1} K(V_n)$. Since $K(V_n)$ is the largest solid set contained in V_n, and since (E,C,u) is a locally convex vector lattice, there exists a convex, solid neighbourhood U_n of 0 in (E,C,u) such that

$$U_n \subseteq K(V_n) \subset V_n$$

and hence $K(V_n)$ is a neighbourhood of 0 in (E,C,u). Since V_n is closed, we have

$$K(V_n) \subseteq \overline{K(V_n)} \subset V_n .$$

But $\overline{K(V_n)}$ is a solid set, by 0.4.14(a), and $K(V_n)$ is the largest solid set contained in V_n. Hence,

$$\overline{K(V_n)} = K(V_n)$$

and this shows that $K(V_n)$ is closed. Also $K(V_n)$ is convex, because V_n is convex. Similarly $K(B)$ is closed and convex. Since B is order-bornivorous, it follows that $K(B)$ is order-bornivorous ([45], Lemma 3.1), and this implies that $K(B)$ is absorbing. Thus, $K(B) = \bigcap_{n \geq 1} K(V_n)$ is a solid barrel which is the countable intersection of closed, convex, solid neighbourhoods of 0 in (E,C,u). Hence $K(B)$ is a neighbourhood of 0 in (E,C,u), by assumption. But $K(B) \subset B$ and hence B is a neighbourhood of 0 in (E,C,u). This completes the proof.

Let (E,C,u) be an l.c.v.l. with the topological dual E'. For any $x \in C$, define

$$p_x(f) = |f|(x) = \sup(\langle y,f\rangle; y \in [-x,x]) \text{ for all } f \in E'.$$

Then the family $\{p_x; x \in C\}$ defines a locally solid topology, which we denote by $\sigma_s(E',E)$, on E' such that $(E', \sigma_s(E',E))'$ is the l-ideal in E'^b generated by E([46], page 19). The topology $\sigma_s(E',E)$ is called the locally solid topology associated with $\sigma(E',E)$. Similarly, we have the topology $\sigma_s(E,E')$ on E. It can easily be verified that

$$\sigma(E',E) \subset \sigma_s(E',E) \subset \beta(E',E).$$

1.1.3 THEOREM: An l.c.v.l. (E,C,u) is a C.O.Q. vector lattice iff every $\sigma_s(E',E)$-bounded subset of E' which is the countable union of equicontinuous subsets of E' is itself equicontinuous.

PROOF: Assume that (E,C,u) is a C.O.Q. vector lattice. Let $\{H_n; n \in \mathbb{N}\}$ be a countable family of equicontinuous subsets of E' such that $H = \bigcup_{n\geq 1} H_n$ is $\sigma_s(E',E)$-bounded. Clearly $H^\circ = \bigcap_{n\geq 1} H_n^\circ$. Since H is $\sigma_s(E',E)$-bounded, it follows that H° is order-bornivorous. Hence H° is an order-bornivorous barrel in E and each H_n° is a closed, circled, convex neighbourhood of 0 because each H_n is equicontinuous. Hence H° is a neighbourhood of 0 in (E,C,u) because (E,C,u) is a C.O.Q. vector lattice by assumption. This shows that $H^{\circ\circ}$ is equicontinuous. But $H \subseteq H^{\circ\circ}$ and hence H is equicontinuous.

Conversely, assume that every $\sigma_s(E',E)$-bounded subset of E' which is the countable union of equicontinuous subsets of E' is itself equicontinuous. Let $B = \bigcap_{n\geq 1} V_n$ be an order-bornivorous barrel such that each V_n is closed, circled, convex neighbourhood of 0 in (E,C,u). Clearly, $\bigcup_{n\geq 1} V_n^\circ \subset B^\circ$. But B° is $\sigma_s(E',E)$-bounded because B is order-bornivorous, and hence $\bigcup_{n\geq 1} V_n^\circ$ is $\sigma_s(E',E)$-bounded. Since each V_n° is an equicontinuous subset of E', it follows, by assumption, that $\bigcup_{n\geq 1} V_n^\circ$ is equicontinuous. Hence

$$(\bigcup_{n\geq 1} V_n^\circ)^\circ = \bigcap_{n\geq 1} V_n^{\circ\circ} = \bigcap_{n\geq 1} V_n = B$$

is a neighbourhood of 0 in (E,C,u), and the proof is complete.

1.1.4 <u>PROPOSITION</u>: Let (E,C,u) be an l.c.v.l.. (a) If (E,C,u) is order-infrabarrelled, then (E,C,u) is a C.O.Q. vector lattice.

(b) If (E,C,u) is countably barrelled, then it is a C.O.Q. vector lattice.

(c) If (E,C,u) is a C.O.Q. vector lattice, then it is countably quasi-barrelled.

<u>PROOF:</u> (a) The proof is obvious from the definitions concerned.

(b) Let $B = \bigcap_{n\geq 1} V_n$ be an order-bornivorous barrel such that each V_n is closed, circled, convex neighbourhood of 0 in (E,C,u). Clearly B is a barrel which is the countable intersection of closed, circled, convex neighbourhoods of 0 in E and hence, by hypothesis, is a neighbourhood of 0. This shows that (E,C,u) is a C.O.Q. vector lattice.

(c) Let $B = \bigcap_{n\geq 1} V_n$ be a bornivorous barrel which is the countable intersection of closed, convex, circled neighbourhoods V_n of 0 in E. Since the positive cone in an l.c.v.l. is normal, every order-bounded set in E is u-bounded. Hence, a fortiori, B is an order-bornivorous barrel and since by hypothesis, (E,C,u) is a C.O.Q. vector lattice, it follows that B is a neighbourhood of 0 in (E,C,u). This shows that (E,C,u) is countably quasi-barrelled.

1.1.5 COROLLARY: Every barrelled l.c.v.l. (in particular, every Fréchet and Banach lattice) is a C.O.Q. vector lattice.

We now give examples to show that a C.O.Q. vector lattice need not be order-infrabarrelled, need not be countably barrelled, and that a countably quasibarrelled l.c.v.l. need not be a C.O.Q. vector lattice. But first we prove a proposition which will be used to give an example of a C.O.Q. vector lattice which is not order-infrabarrelled, although the proposition is interesting in its own right. We need the following definitions and terminology to prove that proposition:

Let X be a completely regular Hausdorff space, and C(X) the vector space of all continuous real-valued functions on X. Let C(X) be ordered by the positive cone

$$K = \{f\in C(X); f(x) \geq 0 \text{ for all } x\in X\} .$$

Endowed with the compact-open topology u_c, $(C(X),K,u_c)$ is an l.c.v.l..

A subset B of X is said to be C(X)-pseudocompact if each function in C(X) is bounded on B. The support of $F\in C(X)'$, which we write as supp(F), is the smallest closed subset A of X such that F(f) = 0 for every $f\in C(X)$ vanishing on A. It has already been observed in ([28]) that supp(F) always exists and is compact. If $B\subseteq C(X)'$, then the support of B, written supp(B), is defined to be the closure of $\bigcup(\text{supp}(f); f\in B)$.

1.1.6 PROPOSITION: Let X be a completely regular Hausdorff space. Then the following statements are equivalent:

(a) $(C(X),K,u_c)$ is countably barrelled.

(b) $(C(X),K,u_c)$ is a C.O.Q. vector lattice.

(c) Every $C(X)$-pseudocompact subset of X which is the closure of a countable union of compact sets is actually compact.

PROOF: That (a) implies (b) follows from 1.1.4(b). The equivalence of (a) and (c) has been established in ([28], Theorem 7.2). We now show that (b) implies (c) which completes the proof. Let B be a $C(X)$-pseudocompact subset of X such that $\beta = \text{cl}(\bigcup_{n\geq 1} K_n)$, where each K_n is a compact subset of X. Observe that X is homeomorphic to $\hat{X}\subset(C(X)',\sigma(C(X)',C(X)))$ under the map $x \longrightarrow \hat{x}$, defined by

$$\hat{x}(f) = f(x) \text{ for all } f\in C(X) .$$

Define $\hat{A}_n = \{\hat{x}\in C(X)'; (x)= \text{supp}(\hat{x})\subseteq K_n \text{ \& } |\hat{x}(f)| \leq 1 \text{ for every } f\in C(X) \text{ with } p_{K_n}(x) \leq 1\}$. Then by Lemma 3.1 in([28]), it follows that $\hat{A}_n$ is equicontinuous. It now follows from

$$|\hat{x}|(|f|) = \sup\{|\hat{x}(g)|; |g| \leq |f| \text{ , } g\in C(X)\}$$

that $\hat{A} = \bigcup_{n\geq 1} \hat{A}_n$ is a $\sigma_s(C(X)',C(X))$-bounded subset of $C(X)'$, since $B = \text{cl}(\bigcup_{n\geq 1} K_n)$ is $C(X)$-pseudocompact. Hence $\hat{A}$ is equicontinuous by assumption. But then by Lemma 3.1 in ([28]), it follows that supp$(\hat{A})$ is a compact subset of X. This shows that B is compact, because

$$\mathrm{supp}(\hat{A}) = \mathrm{cl}(\bigcup_{n\geq 1} \mathrm{supp}(\hat{A}_n)) = \mathrm{cl}(\bigcup_{n\geq 1} K_n) = B .$$

This completes the proof.

1.1.7 <u>EXAMPLES</u>: (i) Let $X = [0,\Omega)$ be the space of ordinals less than the first uncountable ordinal. Then $(C(X),K,u_c)$ is a C.O.Q. vector lattice by 1.1.6 because the closure of a countable union of compact subsets of $X = [0,\Omega)$ is compact ([28], page 336). But, since X is pseudocompact, it follows from ([46], Proposition 2.7) that $(C(X),K,u_c)$ is not order-infrabarrelled.

(ii) Let $(E,C;u)$ be any metrizable l.c.v.l.. Then $(E',C',\beta(E',E))$ is an l.c.v.l. ([39], Theorem 5.7.4). Also $(E',\beta(E',E))$ is a complete (DF)-space ([21], page 396). But then, it follows from Husain ([10], Propositions 1 and 4) that $(E',\beta(E',E))$ is countably barrelled (see also [14], Example (i)). Thus $(E',C',\beta(E',E))$ is a countably barrelled l.c.v.l. and hence, in view of 1.1.4(b), it is a C.O.Q. vector lattice. But $(E',C',\beta(E',E))$ need not be an order-infrabarrelled Riesz space, because it need not be quasibarrelled.

<u>REMARK</u>: The l.c.spaces considered by G. Köthe ([21], §31) (also see Grothendieck [7], pages 71 and 88) to show that the strong dual of a metrizable l.c.s. need not be quasibarrelled, are in fact l.c.v. lattices and hence the above example makes sense in our situation.

(iii) Consider $m = \ell^\infty$, the Banach lattice of all bounded real sequences with the supremum-norm $||\cdot||$ and the order structure determined by the positive cone

$$K = \{x = (x_i)_{i=1}^{\infty} \in m; x_i \geq 0 \text{ for all } i\}.$$

Consider the vector subspace E of m defined as follows:

$$E = \{x + \lambda e; x \in E_0, \lambda \in R\},$$

where E_0 is the subspace of m consisting of all finite sequences, and e is the sequence having one in every coordinate. Let $K_E = K \cap E$. Then $(E, K_E, ||\cdot||)$ is an order-infrabarrelled Riesz space ([30], page 114). Hence by 1.1.4(a), it is a C.O.Q. vector lattice. We show, however, that E is not countably barrelled. For each n, let f_n be the linear functional on E defined by the equation

$$f_n(x + \lambda e) = nx_n.$$

It is well-defined, and each f_n is continuous. Let $H = \bigcup_{n \geq 1}\{f_n\}$. Then H is pointwise bounded and each singleton $\{f_n\}$ is equicontinuous. But H is not equicontinuous ([43], page 1). This shows that $(E, K_E, ||\cdot||)$ is not countably barrelled (Husain [10], Definition 1).

(iv) Consider the Banach lattice C(I) of all continuous real-valued functions on I = [0,1] for the norm

$$||f|| = \sup\{|f(t)|; t \in I\}, \quad f \in C(I)$$

and for the order structure determined by the positive cone

$$K = \{f \in C(I); f(t) \geq 0 \text{ for all } t \in I\}.$$

Let E be the vector subspace consisting of all elements $f \in C(I)$ which vanish in a neighbourhood (depending on f) of t = 0. Then $(E, K_E, ||.||)$, $K_E = E \cap K$, is a normed vector lattice and hence countably quasibarrelled. We, however, show that it is not a C.O.Q. vector lattice. For each n = 1,2,..., define

$$V_n = \{f \in E; |f(\tfrac{1}{n})| \leq \tfrac{1}{n}\} .$$

Clearly, each V_n is a closed, convex, solid neighbourhood of 0; it is solid because, if $|g| \leq |f|$ and $f \in V_n$, then $|g(\frac{1}{n})| = |g|(\frac{1}{n}) \leq |f|(\frac{1}{n}) = |f(\frac{1}{n})| \leq \frac{1}{n}$ and hence $g \in V_n$. Now, let

$$V = \bigcap_{n \geq 1} V_n = \{f \in E; |f(\tfrac{1}{n})| \leq \tfrac{1}{n} \text{ for all } n \geq 1\} .$$

Then, V is a solid barrel ([46], page 20). But it is known from ([39,], page 70, Exercise 14) that V is not a neighbourhood of 0 in E. Hence, in view of 1.1.2, $(E, K_E, ||.||)$ is not a C.O.Q. vector lattice.

(x) Consider the normed vector lattice $(\phi, K, ||.||)$ of all real sequences with only finitely many non-zero coordinates, where $K = \{x = \{x_n\} \in \phi; x_n \geq 0$ for all $n\}$ is the positive cone and $||.||$ is the supremum norm. Since $(\phi, K, ||.||)$ is a normed vector lattice, it is quasibarrelled and hence countably quasi-barrelled. We show, however, that $(\phi, K, ||.||)$ is not a C.O.Q. vector lattice. For each n = 1,2,···, let $V_n = \{x = \{x_n\} \in \phi; |x_n| \leq \frac{1}{n}\}$. Then, each V_n is a closed, convex, solid neighbourhood of 0 in ϕ. Now, let $B = \bigcap_{n \geq 1} V_n$. Then, B is a barrel in ϕ ([44], page 224, Example 1). Clearly B is a solid subset of ϕ. Thus, B is a solid barrel in ϕ; but it is not a neighbourhood of 0 ([44], page 224, Example 1). Hence, it follows from 1.1.2, that $(\phi, K, ||.||)$ is not a C.O.Q. vector lattice.

We now show that, under certain conditions, a C.O.Q. vector lattice is countably barrelled and that a countably quasibarrelled l.c.v.l. is a C.O.Q.

vector lattice.

1.1.8 PROPOSITION: Let (E,C,u) be a C.O.Q. vector lattice with the positive cone C and topological dual E'. If C' is a strict $\mathcal{B}$-cone in $(E',\sigma(E',E))$, then (E,C,u) is a countably barrelled l.c.v.l..

PROOF: Let $B = \bigcap_{n\geq 1} V_n$ be a barrel in (E,C,u) such that each V_n is a closed, convex, circled neighbourhood of 0. Since B is a barrel and hence absorbing, $B°$ is a $\sigma(E',E)$-bounded subset of E'. Since C' is a strict $\mathcal{B}$-cone in $(E',\sigma(E',E))$ it follows from ([45], Proposition 3.5) that $B°$ is $\sigma_s(E',E)$-bounded. But then, B is order-bornivorous. Hence, B is a neighbourhood of 0 in E, because (E,C,u) is a C.O.Q. vector lattice. This shows that (E,C,u) is countably barrelled.

NOTATIONS: Let (E,C,u) be an l.c.v.l. with its topological dual E'. We write $E_{|\sigma|}^{*}$ for $(E',\sigma_s(E',E))'$ and E'' for $\beta(E',\beta(E',E))'$.

1.1.9 PROPOSITION: Let (E,C,u) be a C.O.Q. vector lattice satisfying the condition

(A): The topology $\beta(E,E')$ on E is the relative topology induced by $\beta(E_{|\sigma|}^{*},E')$, and the $\sigma(E',E)$-closure of each $\sigma_s(E',E)$-bounded set in E' is $\sigma_s(E',E)$-bounded. Then (E,C,u) is a countably barrelled l.c.v.l..

PROOF: Let $(H_n;n \geq 1)$ be a sequence of equicontinuous subsets of E' such that $H = \bigcup_{n\geq 1} H_n$ is $\sigma(E',E)$-bounded. The condition (A) implies that each $\sigma(E',\tilde{E})$-bounded subset of E' is $\sigma_s(E',E)$-bounded ([46], Proposition 5.5). Hence, H is $\sigma_s(E',E)$-bounded. But then H is equicontinuous, because (E,C,u) is a C.O.Q. vector lattice. Thus, (E,C,u) is a countably barrelled l.c.v.l..

Let (E,C,u) be an l.c.v.l.. The completion $(\tilde{E},\tilde{u})$ of (E,u) is an l.c.v.l. for the order structure determined by the closure $\tilde{C}$ of C in $(\tilde{E},\tilde{u})$ by 0.4.18. We have shown in 1.1.4(c) that every C.O.Q. vector lattice is countably quasibarrelled and Husain ([10]) has shown that the completion of a countably quasibarrelled space is countably barrelled. Combining these results we have:

1.1.10 <u>PROPOSITION</u>: If (E,C,u) is a C.O.Q. vector lattice, then the completion $(\tilde{E},\tilde{u})$ of (E,u) is a countably barrelled l.c.v.l. for the order structure determined by the closure $\tilde{C}$ of C in $(\tilde{E},\tilde{C})$.

1.1.11 <u>COROLLARY</u>: If (E,C,u) is an order-infrabarrelled Riesz space, then $(\tilde{E},\tilde{C},\tilde{u})$ is barrelled l.c.v.l..

Since a quasicomplete (in particular, complete) countably quasi-barrelled space is countably barrelled ([10], Proposition 4 and Corollary 4), we have the following:

1.1.12 <u>PROPOSITION</u>: A quasi-complete (and in particular, complete) C.O.Q. vector lattice is a countably barrelled l.c.v.l..

1.1.13 <u>COROLLARY</u>: A quasicomplete (and in particular, complete) order-infrabarrelled Riesz space is a barrelled l.c.v.l..

A topological vector lattice (E,C,u) is called locally order complete if there exists a neighbourhood basis of 0 for u consisting of solid and order complete sets. An ordered t.v.s. (F,K,v) is called boundedly order complete if every v-bounded directed (≤) subset has a supremum ([32], page 139).

1.1.14 **COROLLARY:** If (E,C,u) is a C.O.Q. vector lattice which is locally order complete and boundedly order complete, then (E,C,u) is a countably barrelled l.c.v.l..

PROOF: In view of ([32], Proposition 4.1.5), (E,C,u) is topologically complete. Also, since the topology u is Hausdorff, the positive cone C is closed ([32], Proposition 2.4.13). Hence by 1.1.12 it follows that (E,C,u) is countably barrelled.

1.1.15 **COROLLARY:** A locally order complete and boundedly order complete order-infrabarrelled Riesz space is a barrelled l.c.v.l..

Let $(x_\alpha;\alpha\in D) = (x_\alpha)$ be a subset of a vector lattice (E,C), directed upwards (written as $x_\alpha\uparrow$). If $x = \sup x_\alpha$ exists in E, then we write $x_\alpha\uparrow x$. A vector subspace M of (E,C) is called lattice ideal (abbreviated to l-lattice) if M is solid in E. An l-ideal M is called a normal subspace of E if it follows from $x_\alpha\uparrow x$ in E with x_α in M for all α that $x\in M$.

1.1.16 **PROPOSITION:** Let (E,C,u) be a countably quasibarrelled l.c.v.l..

Suppose that the topological dual E' of E is a normal subspace of E^b. Then, (E,C,u) is a C.O.Q. vector lattice.

PROOF: Let $\{H_n; n = 1,2,\cdots\}$ be a countable family of equicontinuous subsets of E' such that $H = \bigcup_{n\geq 1} H_n$ is $\sigma_s(E',E)$-bounded. Since E' is a normal subspace of E^b, it follows from ([46], Proposition 3.2) that H is $u_b(E')$-bounded, where $u_b(E')$ is order bound topology on E'. But $\beta(E',E) \subset u_b(E')$([46], page 25). Hence H is $\beta(E',E)$-bounded. But (E,C,u) is a countably quasibarrelled l.c.v.l. and hence H is equicontinuous. This shows that (E,C,u) is a C.O.Q. vector lattice.

1.1.17 COROLLARY: Let (E,C,u) be as in 1.1.16. If E' is $\sigma_s(E',E)$-complete, then (E,C,u) is a C.O.Q. vector lattice.

PROOF: E' is $\sigma_s(E',E)$-complete iff E' is a normal subspace of E^b ([30], Lemma 4). Hence, in view of 1.1.16, (E,C,u) is a C.O.Q. vector lattice.

1.1.18 PROPOSITION: Let (E,C,u) be a countably quasibarrelled l.c.v.l. satisfying the condition (B): The topology $\beta(E^*_{|\sigma|},E')$ on $E^*_{|\sigma|}$ is the relative topology induced by $\beta(E^*,E')$ and the $\sigma_s(E',E)$-closure of each $\beta(E',E)$-bounded set in E' is $\beta(E',E)$-bounded. Then (E,C,u) is a C.O.Q. vector lattice.

PROOF: Let $\{H_n; n \geq 1\}$ be a sequence of equicontinuous subsets of

E' such that $H = \bigcup_{n\geq 1} H_n$ is $\sigma_s(E',E)$-bounded. The condition (B) implies that each $\sigma_s(E',E)$-bounded set in E' is $\beta(E',E)$-bounded ([46], page 25). But then, H is equicontinuous, because (E,C,u) is countably quasibarrelled. This shows that (E,C,u) is a C.O.Q. vector lattice.

It is interesting to find conditions under which a C.O.Q. vector lattice is order-infrabarrelled. The following proposition is useful in this direction.

1.1.19 PROPOSITION: Let (E,C,u) be a C.O.Q. vector lattice. Then every $\sigma_s(E',E)$-bounded subset of E' is $\beta(E',E)$-bounded.

PROOF: Let B be a $\sigma_s(E',E)$-bounded subset of E'. Then, we claim that

$$\sup\{|\langle x,y\rangle|; x\in A, y\in B\} < \infty,$$

where A is a weakly bounded subset of E, and this proves that B is $\beta(E',E)$-bounded. Suppose

$$\sup\{|\langle x,y\rangle|; x\in A, y\in B\} = \infty.$$

Then there exists a sequence $\{y_n; n = 1,2,\cdots\}$ in B such that $\sup\{|\langle x,y_n\rangle|; x\in A\} > n$ for each n. Now, since $\{y_n; n = 1,2,\cdots\}$ is $\sigma_s(E',E)$-bounded and (E,C,u) is a C.O.Q. vector lattice, it follows from 1.1.3 that $\{y_n; n = 1,2,\cdots\}$ is equicontinuous, and hence strongly bounded, which contradicts our assumption that $\sup_{x\in A, n}|\langle x,y_n\rangle| = \infty$. This completes the proof.

1.1.20 COROLLARY: Let (E,C,u) be a C.O.Q. vector lattice. Then (E,C,u) is an order-infrabarrelled Riesz space iff it is a quasibarrelled l.c.v.l.. In particular, a metrizable (or even bornological) C.O.Q. vector lattice is order-infrabarrelled.

PROOF: Assume that (E,C,u) is an order-infrabarrelled Riesz space. Then it is definitely a quasibarrelled l.c.v.l. ([46], Corollary 2.3). Now, conversely, assume that (E,C,u) is quasibarrelled. Let H be a $\sigma_s(E',E)$-bounded subset of E'. Then H is $\beta(E',E)$-bounded by 1.1.19. But then H is equicontinuous, because (E,C,u) is quasibarrelled. Thus every $\sigma_s(E',E)$-bounded subset of E' is equicontinuous, and this implies that (E,C,u) is an order-infrabarrelled Riesz space ([46], Proposition 2.2).

We recall that a linear map f of a vector lattice into a vector lattice is called lattice-homomorphism (abbreviated to l-homomorphism) if it preserves the lattice operations.

We now prove a Banach-Steinhaus type theorem for l-homomorphisms on C.O.Q. vector lattices.

1.1.21 THEOREM: Let (E,C,u) be a C.O.Q. vector lattice and (F,K,v) any l.c.v.l.. Let $\{H_n; n = 1,2,\cdots\}$ be a countable family of equicontinuous sets of l-homomorphisms of E into F such that $H = \bigcup_{n\geq 1} H_n$ is pointwise bounded. Then H is equicontinuous.

PROOF: Let V be a closed, convex, solid neighbourhood of 0 in F. Let

$$U_n = \bigcap_{f\in H_n} f^{-1}(V) .$$

Clearly, U_n is a closed, convex neighbourhood of 0 in E. Since $f \in H_n$ is an l-homomorphism and V is solid, it follows that $f^{-1}(V)$ is solid and hence U_n is solid. Let

$$U = \bigcap_{n \geq 1} U_n = \bigcap_{n \geq 1} \bigcap_{f \in H_n} f^{-1}(V) = \bigcap_{f \in H} f^{-1}(V).$$

Then U is a closed, convex and solid subset of E. Since H is simply bounded, it follows that U is absorbing. Thus, U is a solid barrel which is the countable intersection of closed, convex, solid neighbourhoods of 0 in the C.O.Q. vector lattice (E,C,u), and hence U is a neighbourhood of 0 in E in view of 1.1.2. But then H is equicontinuous, because $U = \bigcap_{f \in H} f^{-1}(V)$.

1.1.22 <u>COROLLARY:</u> Let (E,C,u) be an order-infrabarrelled Riesz space and (F,K,v) any l.c.v.l.. Let H be any set of continuous l-homomorphisms of E into F such that H is pointwise bounded. Then H is equicontinuous.

1.1.23 <u>COROLLARY:</u> Let (E,C,u) and (F,K,v) be as in 1.1.21. Let $\{f_n; n = 1,2,\cdots\}$ be a pointwise bounded sequence of continuous l-homomorphisms of E into F. Then $\{f_n; n = 1,2,\cdots\}$ is equicontinuous.

1.1.24 <u>COROLLARY:</u> Let (E,C,u) and (F,K,v) be as in 1.1.22. Let $\{f_n; n = 1,2,\cdots\}$ be a pointwise bounded sequence of continuous l-homomorphisms of E into F. Then $\{f_n; n = 1,2,\cdots\}$ is equicontinuous.

1.2 <u>Permanence properties</u>

In this section, we investigate various properties of C.O.Q. vector lattices. We give an example to show that an l-ideal of a C.O.Q. vector lattice need not be a C.O.Q. vector lattice, and then we give a set of conditions under which it can be a C.O.Q. vector lattice.

1.2.1 THEOREM: The locally convex direct sum of a family of C.O.Q. vector lattices is a C.O.Q. vector lattice.

PROOF: Let $\{(E_\alpha, C_\alpha, u_\alpha); \alpha \in I\}$ be a family of C.O.Q. vector lattices. Let $E = \bigoplus_{\alpha \in I} E_\alpha$ (the algebraic direct sum of $\{E_\alpha; \alpha \in I\}$); $u = \bigoplus_\alpha u_\alpha$ (the locally convex sum topology on E); $C = \bigoplus_\alpha C_\alpha$ and let $i_\alpha : E_\alpha \longrightarrow E$ be the injection map for each $\alpha \in I$. Then (E,C,u) is an l.c.v.l. ([45], Corollary 3.20). To show that (E,C,u) is a C.O.Q. vector lattice, let $B = \bigcap_{n \geq 1} V_n$ be a solid barrel in E such that each V_n is a closed, convex, solid neighbourhood of 0 in E. Clearly $i_\alpha^{-1}(B) = \bigcap_{n \geq 1} i_\alpha^{-1}(V_n)$. Since each i_α is a continuous l-homomorphism of E_α into E, it follows that $i_\alpha^{-1}(V_n)$ is a closed, convex, solid neighbourhood of 0 in E_α. Clearly, $i_\alpha^{-1}(B)$ is a solid barrel in E_α. Hence $i_\alpha^{-1}(B)$ is a neighbourhood of 0 in E_α which implies that B is a neighbourhood of 0 in E. This shows that (E,C,u) is a C.O.Q. vector lattice.

1.2.2 PROPOSITION: Let (E,C,u) be a C.O.Q. vector lattice and (F,K,v) any l.c.v.l.. Let f be a positive, linear, continuous and almost open map of (E,C,u) into (F,K,v). Then (F,K,v) is a C.O.Q. vector lattice.

PROOF: Let $B = \bigcap_{n \geq 1} V_n$ be an order-bornivorous barrel such that each V_n is a closed, convex, circled neighbourhood of 0 in (F,K,v). Clearly,

$f^{-1}(B) = \bigcap_{n \geq 1} f^{-1}(V_n)$. Then $f^{-1}(B)$ is a barrel, because f is continuous and linear. That $f^{-1}(B)$ is order-bornivorous follows from the fact that f is positive and linear. Since f is continuous and each V_n is a neighbourhood of 0 in F, it follows that $f^{-1}(V_n)$ is a closed, convex, circled neighbourhood of 0 in (E,C,u). Hence $f^{-1}(B)$ is a neighbourhood of 0 in (E,C,u), because (E,C,u) is a C.O.Q. vector lattice. Since f is almost open, $\overline{f(f^{-1}(B))}$ is a neighbourhood of 0 in (F,K,v). Observe that

$$\overline{f(f^{-1}(B))} \subset \bar{B} = B,$$

because B is closed. Hence B is a neighbourhood of 0 in (F,K,v). This shows that (F,K,v) is a C.O.Q. vector lattice.

1.2.3 COROLLARY: Let (E,C,u) and (F,K,v) be as in 1.2.2. Let f be a positive, linear, continuous and open map of E into F. Then (F,K,v) is a C.O.Q. vector lattice.

1.2.4 COROLLARY: Let (E,C,u) be a C.O.Q. vector lattice and M a closed l-ideal in E. Then E/M is a C.O.Q. vector lattice.

The following example shows that an l-ideal of a C.O.Q. vector lattice need not be a C.O.Q. vector lattice.

1.2.5 EXAMPLE: Let C(I) be the Banach lattice of all continuous real-valued functions on I = [0,1], with the supremum norm $||.||$ and the order structure determined by the positive cone

$$K = \{f \in C(I); f(t) \geq 0 \text{ for all } t \in I\}.$$

$(C(I),K,||\cdot||)$ is a C.O.Q. vector lattice, because it is a countably barrelled l.c.v.l.. Now consider the vector subspace E consisting of all elements $f \in C(I)$ which vanish in a neighbourhood (depending on f) of $t = 0$. Then $(E,K_E,||\cdot||)$, $E \cap K = K_E$, is an l-ideal in $C(I)$; but it has been shown in 1.1.7(iv) that it is not a C.O.Q. vector lattice.

In the following proposition, we show that, under certain conditions, any l-ideal of a C.O.Q. vector lattice is of the same type. But first we recall a definition from ([38], page 92): The <u>codimension</u> of a vector subspace M of a vector space E is the (algebraic) dimension of the quotient vector space E/M.

1.2.6 <u>PROPOSITION</u>: Let (E,C,u) be a C.O.Q. vector lattice such that C' is a strict $\mathcal{B}$-cone in $(E',\sigma(E',E))$. If M is an l-ideal of countable codimension in E, then M is a C.O.Q. vector lattice under the induced topology.

<u>PROOF</u>: By 1.1.8, (E,C,u) is a countably barrelled l.c.v.l.. Clearly M is an l.c.v.l. in the relative topology. Furthermore, M is countably barrelled in view of ([13], Theorem 3.1). Hence M is a C.O.Q. vector lattice by 1.1.4(b). This completes the proof.

1.3 <u>C.O.Q. vector spaces and Banach-Steinhaus type theorem</u>

In this section, we consider the underlying space to be ordered l.c.s. instead of l.c.v.l., and thus we have C.O.Q. vector spaces. We prove a powerful theorem called a Banach-Steinhaus type theorem for C.O.Q. vector spaces, which will be used, in the next section, to obtain some results about positive bases in order-infrabarrelled spaces. We show that these results gain nothing in generality if we consider C.O.Q. vector spaces.

We first recall a definition from ([47], page 53): An ordered l.c. space (E,C,u) is called an order-infrabarrelled space if each order-bornivorous barrel is a neighbourhood of 0 in (E,C,u).

1.3.1 <u>Definition:</u> An ordered l.c. space (E,C,u) is called a C.O.Q. vector space if every order-bornivorous barrel which is the countable intersection of closed, circled, convex neighbourhoods of 0 in (E,C,u) is itself a neighbourhood of 0.

<u>REMARK:</u> Clearly, every order-infrabarrelled space is a C.O.Q. vector space. The following theorem shows when the converse is true.

1.3.2 <u>THEOREM:</u> Let (E,C,u) be a separable C.O.Q. vector space. Then (E,C,u) is an order-infrabarrelled space.

<u>PROOF:</u> Let B be an order-bornivorous barrel in (E,C,u). Then, the set $E\setminus B$ is open in E, and hence $E\setminus B$ is separable. Let $D = \{x_n; n \geq 1\}$ be a countable dense subset of $E\setminus B$. Since $x_n \notin B$, for each n, there exists $x_n' \in E'$ such that $\langle x_n, x_n' \rangle > 1$ and $x_n' \in B^\circ$. Let $D' = \{x_n'; n \geq 1\} = \bigcup_{n\geq 1}\{x_n'\}$. Then, $B \subset D'^\circ = (\bigcup_{n\geq 1}\{x_n'\})^\circ = \bigcap_{n\geq 1}\{x_n'\}^\circ$, because $D' \subset B^\circ$ and so $B = B^{\circ\circ} \subset D'^\circ$. Since B is order-bornivorous and $B \subset D'^\circ$, it follows that D'° is order-bornivorous. Also $\{x_n'\}^\circ$ is a closed, convex, circled neighbourhood of 0 in E, because $\{x_n'\}$ is equicontinuous. Thus D'° is an order-bornivorous barrel which is the countable

intersection of closed, convex, circled neighbourhoods of 0 in the C.O.Q. vector space E, and hence D'° is a neighbourhood of 0 in E, and $\mathrm{int}(D'^{\circ}) \neq \emptyset$. As $D \subset E \setminus \mathrm{int}(D'^{\circ})$ ([5], page 260), we have $E \setminus B \subset \bar{D} \subset E \setminus \mathrm{int}(D'^{\circ})$ and hence $\mathrm{int}(D'^{\circ}) \subset B$. This implies that B is a neighbourhood of 0 in (E,C,u). Hence (E,C,u) is an order-infrabarrelled space.

1.3.3 <u>COROLLARY</u>: Let (E,C,u) be a separable C.O.Q. vector lattice. Then (E,C,u) is an order-infrabarrelled Riesz space.

Let (E,C,u) be an ordered l.c.s. with generating cone C, and $E' \subseteq E^b$. Then there exists a locally convex topology $O(E',E)$ on E' such that $\{[-x, x]^{\circ};\ x \in C\}$ is a neighbourhood basis of 0 for $O(E',E)$ which is called the topology on E' of uniform convergence on the order-bounded subsets of E ([33], page 203).

1.3.4. <u>PROPOSITION</u>: Let (E,C,u) be an ordered l.c.s. with generating cone C, and $E' \subseteq E^b$. Then the following statements are equivalent:

(a) (E,C,u) is a C.O.Q. vector space.

(b) Each $O(E',E)$-bounded subset of E' which is the countable union of equicontinuous subsets of E' is itself equicontinuous.

<u>PROOF</u>: (b) $\Longrightarrow$ (a): Let $B = \bigcap_{n \geq 1} V_n$ be an order-bornivorous barrel in E such that each V_n is a circled, closed, convex neighbourhood of 0 in E. Then clearly $\bigcup_{n \geq 1} V_n^{\circ} \subset B^{\circ}$. Since B is order-bornivorous, B° is $O(E',E)$-bounded and hence, $\bigcup_{n \geq 1} V_n^{\circ}$ is $O(E',E)$-bounded. Also each V_n° is equicontinuous. Hence, by (b), it follows that $\bigcup_{n \geq 1} V_n^{\circ}$ is equicontinuous. But then $(\bigcup_{n \geq 1} V_n^{\circ})^{\circ} = \bigcap_{n \geq 1} V_n^{\circ\circ} = \bigcap_{n \geq 1} V_n = B$ is a neighbourhood of 0. This implies that (E,C,u) is a

C.O.Q. vector space.

(a) $\Longrightarrow$ (b): Let $(H_n; n = 1,2,\cdots)$ be a sequence of equicontinuous subsets of E' such that $H = \bigcup_{n\geq 1} H_n$ is $\sigma(E',E)$-bounded. Then $H^\circ = \bigcap_{n\geq 1} H_n^\circ$ is an order-bornivorous barrel such that each H_n° is a closed, convex, circled neighbourhood of 0 in E. Hence H° is a neighbourhood of 0 in E, because, by (a), (E,C,u) is a C.O.Q. vector space. Hence $H^{\circ\circ}$ and so H is equicontinuous, because $H \subseteq H^{\circ\circ}$, and this completes the proof.

We now prove a Banach-Steinhaus type theorem for C.O.Q. vector spaces.

1.3.5 THEOREM: Let (E,C,u) be a C.O.Q. vector space and (F,K,v) any ordered l.c.s. with normal cone K. Let $(H_n; n = 1,2\cdots)$ be a countable family of equicontinuous sets of positive linear maps of E into F such that $H = \bigcup_{n\geq 1} H_n$ is pointwise bounded. Then H is equicontinuous.

PROOF: Let V be a closed, convex, circled, full neighbourhood of 0 in F. Then, clearly $U_n = \bigcap_{f\in H_n} f^{-1}(V)$ is a circled, closed, convex neighbourhood of 0 in E. Let $U = \bigcap_{n\geq 1} U_n = \bigcap_{n\geq 1} \bigcap_{f\in H_n} f^{-1}(V) = \bigcap_{f\in H} f^{-1}(V)$. Then U is closed, convex and circled; it is also absorbing because H is pointwise bounded. Now we wish to show that U is order-bornivorous. Let $[x,y]$ be any order interval in E. Since H is pointwise bounded, there exists $\lambda > 0$ such that $f(x), f(y) \in \lambda V$ for all $f \in H$. But then, $[f(x), f(y)] \subset \lambda V$ for all $f \in H$, because V is full. Since f is positive and linear, it follows that

$$f([x,y]) \subset [f(x), f(y)] \subset \lambda H \text{ for all } f \in H.$$

This shows that $[x,y] \subset \lambda \bigcap_{f\in H} f^{-1}(V) = \lambda U$ for some $\lambda > 0$, and hence $U = \bigcap_{f\in H} f^{-1}(V)$

is order-bornivorous. Thus $U = \bigcap_{n\geq 1} U_n$ is an order-bornivorous barrel which is the countable intersection of closed, circled, convex neighbourhoods of 0 in E. Since (E,C,u) is a C.O.Q. vector space, it follows that U is a neighbourhood of 0 in E. But $U = \bigcap_{f\in H} f^{-1}(V)$ and hence H is equicontinuous.

1.3.6 <u>COROLLARY</u>: Let (E,C,u) and (F,K,v) be as in 1.3.5. Let $\{f_n; n = 1,2,\cdots\}$ be a sequence of continuous, positive, linear maps of E into F such that $\{f_n; n = 1,2,\cdots\}$ is pointwise bounded. Then $\{f_n; n = 1,2,\cdots\}$ is equicontinuous.

<u>PROOF</u>: Taking $H_n = \{f_n\}$ in 1.3.5, we obtain the result.

1.3.7 <u>COROLLARY</u>: Let (E,C,u) and (F,K,v) be as in 1.3.5, and suppose that the positive cone K in F is closed. If $\{f_n; n = 1,2,\cdots\}$ is a sequence of continuous, positive, linear maps of E into F such that $\{f_n; n \geq 1\}$ converges pointwise to a map f from E into F, then f is continuous, positive and linear.

<u>PROOF</u>: Clearly f is linear. Since K is closed in F, it follows that f is positive. Since $\{f_n; n \geq 1\}$ is pointwise bounded, it is equicontinuous by 1.3.6. But then, in view of ([3], Chapter III, §3, Proposition 4), f is continuous.

<u>REMARK</u>: Since an order-infrabarrelled space is a C.O.Q. vector space, the results 1.3.5, 1.3.6 and 1.3.7 are obviously true for order-infrabarrelled spaces.

1.4 C.O.Q. vector spaces and bases

Let (E,C,u) be an ordered l.c. space. An u-(Schauder) basis $\{x_n,f_n\}$ is called a positive u-(Schauder) basis if $\{x_n\}\subset C$ and all f_n's are positive ([26]). A t.v.s. which possesses a basis is separable ([41], page 144). Hence a C.O.Q. vector space (E,C,u) with a positive basis is an order-infrabarrelled space, in view of 1.3.2. We, therefore, obtain analogues of the weak basis theorem ([25], page 128) and the isomorphism theorem ([25], page 123) for positive Schauder bases in order-infrabarrelled spaces only.

1.4.1 THEOREM: Let (E,C,u) be an order-infrabarrelled space with normal cone C. Then every positive $\sigma(E,E')$-Schauder basis in E is an u-Schauder basis.

PROOF: Let $\{x_i,f_i\}$ be a positive $\sigma(E,E')$-Schauder basis in E. Then $\{x_i,f_i\}$ is a weak Markuschevich basis ([25], Theorems 9.1.10 and 9.5.4). Thus, $\{x_i\}$ is a total set in E. This implies that we can choose, for each $x\in E$, a sequence $(y_n)\subset E$ such that $y_n \longrightarrow x$ and $y_n\in sp\{x_i; 1 \le i \le n\}$. Consider the sequence $\{T_n; n \ge 1\}$ of continuous, positive linear maps $T_n: E \longrightarrow E$ defined by

$$T_n(x) = \sum_{i=1}^{n} f_i(x)x_i, \text{ for } x\in E,\ n = 1,2,\cdots$$

Observe that $\lim_n T_n(x) = x$ in the weak toplogy of E; hence $\{T_n; n \ge 1\}$ is pointwise bounded. But then, by 1.3.6 and the Remark following 1.3.7, $\{T_n; n \ge 1\}$ is equicontinuous. Hence,

$$x = x + \lim_n T_n(x - y_n)$$

$$= x + \lim_n \sum_{i=1}^{n} f_i(x - y_n)x_i$$

$$= \lim_n y_n + \lim_n\left(\sum_{i=1}^{n} f_i(x)x_i - y_n\right)$$

$$= \lim_n \sum_{i=1}^{n} f_i(x)x_i \, .$$

and so $\{x_i, f_i\}$ is a Schauder basis for u.

1.4.2 <u>COROLLARY</u>: Let (E,C,u) be an order-infrabarrelled space with the normal cone C such that C has the non-empty interior. Then every positive $\sigma(E,E')$-basis in E is a u-Schauder basis.

<u>PROOF</u>: Let $\{x_n, f_n\}$ be a positive $\sigma(E,E')$-basis in E. Since the positive cone C has the non-empty interior, it follows that each f_n is in E' ([32], 2.2.17). Hence, $\{x_n, f_n\}$ is a positive $\sigma(E,E')$-Schauder basis in E. The result now follows from 1.4.1.

Let (E,C,u) be an ordered l.c.s.. Then the topology $O(E,E')$ on E of uniform convergence on order-bounded subsets of E' is consistent with the dual system $\langle E,E'\rangle$ if the positive cone C is closed and generating and if E' is a full subspace of E^* ([32], 3.2.4). We now obtain the analogue of 1.4.1 for $O(E,E')$.

1.4.3 <u>THEOREM</u>: Let (E,C,u) be an order-infrabarrelled space such that E' is a full subspace of E^* and C is closed, normal and generating. Then every positive $O(E,E')$-Schauder basis in E is a u-Schauder basis.

PROOF: Let $\{x_n,f_n\}$ be a positive $O(E,E')$-Schauder basis in E. Then for each $x \in E$,

$$x = \lim_p \sum_{n=1}^{p} f_n(x)x_n = \sum_{n=1}^{\infty} f_n(x)x_n$$

in $O(E,E')$-topology. Since convergence for $O(E,E')$ implies convergence for $\sigma(E,E')$, it follows that each $x \in E$ has the weak series expansion $\sum_{n=1}^{\infty} f_n(x)x_n$. In order to show that $\{x_n,f_n\}$ is actually a $\sigma(E,E')$-Schauder basis, we have to show that the unique sequence $\{f_n\}$ corresponding to x in the $O(E,E')$-expansion of x is also unique for the $\sigma(E,E')$ series expansion of x. If $\sum_{n=1}^{\infty} g_n x_n = 0$, convergence being in the weak topology, we have

$$f_m\left(\sum_{n=1}^{\infty} g_n x_n\right) = \sum_{n=1}^{\infty} g_n f_m(x_n) = 0, \text{ for } m = 1,2,\dots$$

Since $\{x_n,f_n\}$ is an $O(E,E')$-basis in E, we have $f_m(x_n) = \delta_{nm}$. Hence $g_m = 0$ for each m. This implies that the sequence $\{f_n\}$ is unique for the weak series expansion of x. Thus $\{x_n,f_n\}$ is a positive $\sigma(E,E')$-Schauder basis for E. But then, in view of 1.4.1, $\{x_n,f_n\}$ is a u-Schauder basis in E.

1.4.4 COROLLARY: Let (E,C,u) be an order-infrabarrelled space such that E' is a full subspace of E^* and C is closed, normal, generating and has non-empty interior. Then every positive $O(E,E')$-basis in E is a u-Schauder basis.

PROOF: Since C has non-empty interior, every positive linear functional on E is continuous. Hence every positive $O(E,E')$-basis for E is a positive $O(E,E')$-Schauder basis. The result now follows from 1.4.3.

If (E,C,u) is an l.c.v.l., then the topology $O(E,E')$ on E of uniform convergence on order-bounded subsets of E' is always consistent with the dual system $\langle E,E'\rangle$, and in this case $O(E,E') = \sigma_s(E,E')$.

The following corollaries are immediate consequences of 1.4.3 and 1.4.4.

1.4.5 <u>COROLLARY</u>: Let (E,C,u) be an order-infrabarrelled Riesz space. Then, every positive $\sigma_s(E,E')$-Schauder basis in E is an u-Schauder basis.

1.4.6 <u>COROLLARY</u>: Let (E,C,u) be an order-infrabarrelled Riesz space such that C has non-empty interior. Then, every positive $\sigma_s(E,E')$-basis in E is an u-Schauder basis.

Let (E,C,u) be an l.c.v.l.. A $\sigma_s(E,E')$-basis $\{x_i,f_i\}$ for E is called lattice theoretically absolutely convergent if for each $x \in E$ the sequence $\{\sum_{i=1}^{n} |f_i(x)x_i|\}$ is majorized in E, where $\sum_{i=1}^{\infty} f_i(x)x_i$ is the basis expansion of x. Since E' is an l-ideal in E^b, it follows that $\{x_i,f_i\}$ is a lattice theoretically absolutely convergent basis in E iff it is a $\sigma_s(E,E')$-basis such that $\sum_{i=1}^{\infty} p_f(f_i(x)x_i) < \infty$ for each $x \in E, f \in C'$, where $\{p_f; f \in C'\}$ is the family of semi-norms generating the topology $\sigma_s(E,E')$ ([27]).

1.4.7 <u>PROPOSITION</u>: Let (E,C,u) be an order-infrabarrelled Riesz space. If $\{x_i,f_i\}$ is lattice theoretically absolutely convergent positive $\sigma_s(E,E')$-Schauder basis in E, then $\{x_i,f_i\}$ is an unconditional u-Schauder basis in E.

<u>PROOF</u>: Let Ω denote the class of all finite subsets of $\mathbb{N}$. Consider the continuous, linear, positive maps $T_\sigma : E \longrightarrow E$ defined by

$$T_\sigma(x) = \sum_{i\in\sigma} f_i(x)x_i, \text{ for each } \sigma\in\Omega, x\in E.$$

For $f\in E'$ and $\sigma\in\Omega$, we have

$$|f(T_\sigma(x))| \le \sum_{i\in\sigma} |f(f_i(x)x_i)| \le \sum_{i\in\sigma} |f||f_i(x)x_i| \le \sum_{i=1}^{\infty} p_{|f|}(f_i(x)x_i) < \infty .$$

Thus, the set $\{T_\sigma;\sigma\in\Omega\}$ is pointwise bounded and hence, by 1.3.5 and the Remark following 1.3.7, it is equicontinuous. Let U be a neighbourhood of 0 in E. Then we can choose a circled neighbourhood V of 0 in E such that $V + V \subset U$. Since $\{T_\sigma;\sigma\in\Omega\}$ is equicontinuous, there exists a neighbourhood W of 0 in E such that

$$T_\sigma(W) \subset V \text{ for all } \sigma\in\Omega.$$

In view of 1.4.5, $\{x_i, f_i\}$ is a u-Schauder basis in E; hence there exists an integer $n > 0$ such that $x - T_{\sigma_n}(x) \in V \cap W$, where $\sigma_n = \{1, 2, \cdots, n\}$. Then,

$$T_\sigma(x - T_{\sigma_n}(x)) \in V \text{ for all } \sigma\in\Omega .$$

Hence, for all $\sigma\in\Omega$ such that $\sigma_n \subset \sigma$, we have,

$$x - T_\sigma(x) = x - T_{\sigma_n}(x) - T_\sigma(x - T_{\sigma_n}(x)) \in V + V \subset U.$$

Thus,

$$\lim_{\sigma\in\Omega} \sum_{i\in\sigma} f_i(x)x_i = x, \text{ for each } x\in E$$

and this shows that $\{x_i, f_i\}$ is an unconditional u-Schauder basis in E.

1.4.8 <u>COROLLARY</u>: Let (E,C,u) be an order-infrabarrelled Riesz space such that C has non-empty interior. If (x_i,f_i) is a lattice theoretically absolutely convergent positive $\sigma_s(E,E')$-basis in E, then it is an unconditional u-Schauder basis in E.

We now obtain an analogue of the isomorphism theorem for order-infrabarrelled spaces with similar positive Schauder bases.

1.4.9 <u>THEOREM</u>: Let (E,C,u) and (F,K,v) be order-infrabarrelled spaces such that the cones C and K are both closed and normal in E and F, respectively. Let (x_n) and (y_n) be positive Schauder bases in E and F, respectively. Then (x_n) is similar to (y_n) iff there exists a positive topological isomorphism $T:E \longrightarrow F$ such that $Tx_n = y_n$ for all $n \in \mathbb{N}$.

<u>PROOF</u>: Assume that there is a positive isomorphism $T:E \longrightarrow F$ such that $Tx_n = y_n$ for all $n \in \mathbb{N}$. Then, clearly $\sum_{n=1}^{\infty} a_n x_n$ converges iff $T(\sum_{n=1}^{\infty} a_n x_n) = \sum_{n=1}^{\infty} a_n Tx_n = \sum_{n=1}^{\infty} a_n y_n$ converges. Hence, we get similarity. Conversely, assume that (x_n) and (y_n) are similar positive Schauder bases in E and F respectively. Let $(f_n; n \in \mathbb{N}) \subset K'$ be such that (x_n,f_n) is a biorthogonal system. Then, for $x \in E$, $x = \sum_{n=1}^{\infty} f_n(x)x_n$. Consider the sequence $\{T_m; m \geq 1\}$ of continuous, positive linear maps $T_m:E \longrightarrow F$ defined by $T_m(x) = \sum_{n=1}^{m} f_n(x)x_n$, $m = 1,2,\cdots$. Define T by $T(x) = \sum_{n=1}^{\infty} f_n(x)y_n$. T is well-defined, because, by similarity of the bases, $\sum_{n=1}^{\infty} f_n(x)y_n$ is convergent. T is one-to-one: $Tx = 0$ implies $\sum_{n=1}^{\infty} f_n(x)y_n = 0$, and since (y_n) is a Schauder basis, $f_n(x) = 0$ for all $n \in \mathbb{N}$, and hence $x = 0$, because (f_n) is separating. This shows that T is one-to-one.

T is onto: For $y \in F$, $y = \sum_{n=1}^{\infty} b_n y_n$, and by similarity, $\sum_{n=1}^{\infty} b_n x_n = \sum_{n=1}^{\infty} f_n(x) x_n$ converges in (E,u) to some $x \in E$; hence $Tx = y$ and this shows that T is onto.

Observe that $T_m(x) \longrightarrow T(x)$ for $x \in E$, by definition. Hence $\{T_m; m \geq 1\}$ is a sequence of continuous, positive, linear maps of E into F, converging pointwise to T, and hence T is continuous, positive and linear, by 1.3.7 and the Remark following 1.3.7. By symmetry the same is true of T^{-1} and hence T is the required positive isomorphism of E onto F.

1.5 Tensor products of C.O.Q. vector spaces

In this section, we show that the inductive tensor product of two C.O.Q. vector spaces, under certain conditions, is again a C.O.Q. vector space.

Let E_1 and E_2 be vector spaces ordered by positive cones K_1 and K_2 respectively.

1.5.1 Definition: ([35], page 182). A wedge K in the tensor product $E_1 \otimes E_2$ is said to be compatible with E_1 and E_2 if $x \otimes y \in K$ whenever $x \in K_1$ and $y \in K_2$. The smallest compatible wedge in $E_1 \otimes E_2$ is called the projective wedge K_p defined by

$$K_p = \{\sum_{n=1}^{m} x_n \otimes y_n ; x_n \in K_1, y_n \in K_2\}, \text{ m finite} .$$

1.5.2 PROPOSITION: ([35], 2.3). If K_1 and K_2 are generating in E_1 and E_2 respectively, and if K is a compatible wedge in $E_1 \otimes E_2$, then K generates $E_1 \otimes E_2$.

1.5.3 PROPOSITION: ([35], 2.4). If E_1 and E_2 are vector spaces ordered by the cones K_1 and K_2 respectively, then each of the following conditions implies that the projective wedge K_p is a cone in $E_1 \otimes E_2$.

(1) The cone K_n^* of all positive linear functionals on E_n is total in the algebraic dual E_n^*, for $n = 1$ or $n = 2$.

(2) There is a strictly positive linear functional on E_n, for $n = 1$ or $n = 2$.

Let E_1 and E_2 be l.c. spaces ordered by generating cones K_1 and K_2 respectively, and suppose K_p is the projective cone in $E_1 \otimes E_2$. Then if S and T are order-bounded subsets of E_1 and E_2 respectively, then the convex circled hull $\Gamma(S \otimes T)$ of $S \otimes T$ is order-bounded for the order structure determined by K_p([35], page 186). The partial mappings g_x and g_y, $x \in E_1$, $y \in E_2$ associated with the canonical bilinear mapping g of $E_1 x E_2$ into $E_1 \otimes E_2$ are order-bounded ([35], page 188).

1.5.4 THEOREM: Let (E_1, K_1, u_1) and (E_2, K_2, u_2) be C.O.Q. vector spaces, where K_n is a generating cone in E_n, $n = 1,2$. Let K_p be the projective cone in $E_1 \otimes E_2$. Then $(E_1 \otimes E_2, K_p, u_i)$, where u_i is the inductive (tensor product) topology, is a C.O.Q. vector space.

PROOF: Let $B = \bigcap_{n \geq 1} V_n$ be an order-bornivorous barrel in $E_1 \otimes E_2$ such that each V_n is a closed, convex, circled neighbourhood of 0 in $E_1 \otimes E_2$. Let g_x and g_y $(x \in E_1, y \in E_2)$ be the partial mappings associated with the canonical bilinear mapping $g: E_1 x E_2 \longrightarrow E_1 \otimes E_2$. Then g_x and g_y $(x \in E_1, y \in E_2)$ are order-bounded maps. Clearly, $g_x^{-1}(B) = \bigcap_{n \geq 1} g_x^{-1}(V_n)$. Since g_x is continuous, $g_x^{-1}(V_n)$, for each n, is a closed, convex, circled neighbourhood of 0 in E_2.

Since B is order-bornivorous and g_x is order-bounded, it follows that $g_x^{-1}(B)$ is order-bornivorous. Clearly $g_x^{-1}(B)$ is closed, circled, and convex. Thus, $g_x^{-1}(B)$ is an order-bornivorous barrel which is the countable intersection of closed, convex, circled neighbourhoods of 0 in E_2. Hence $g_x^{-1}(B)$, $x \in E_1$, is a neighbourhood of 0 in E_2. Similarly $g_y^{-1}(B)$, $y \in E_2$, is a neighbourhood of 0 in E_1. This shows that B is a u_i-neighbourhood of 0 in $E_1 \otimes E_2$. Hence $E_1 \otimes E_2$ is a C.O.Q. vector space.

1.5.5 COROLLARY: Let (E_1, K_1, u_1) and (E_2, K_2, u_2) be order-infrabarrelled spaces, where K_n is generating in E_n, $n = 1,2$. Suppose that K_p is the projective cone in $E_1 \otimes E_2$. Then $(E_1 \otimes E_2, K_p, u_i)$, where u_i is the inductive (tensor product) topology, is an order-infrabarrelled space.

CHAPTER II

ORDER-(DF)-VECTOR LATTICES AND SPACES

In order to give an affirmative answer to a question posed in ([35]) about order-bounded sets in the tensor products of ordered l.c. spaces, we introduce and study, in this chapter, a subclass of C.O.Q. vector lattices (spaces) which we call the class of order-(DF)-vector lattices (spaces). We give many examples to justify the existence of such a class in ordered t.v. spaces. The motivation for this class has been derived from the notion of (DF)-spaces due to Grothendieck ([7]).

2.1 Order-(DF)-vector lattices

2.1.1 **Definition:** An ordered vector space (E,C) is said to possess a countable fundamental system of order-bounded sets, say $(B_n; n\in \mathbb{N})$, if every order-bounded set B in E is contained in some B_n.

2.1.2 **Definition:** An l.c.v.l. (E,C,u) is called an order-(DF)-vector lattice if (a) it has a countable fundamental system of order-bounded sets, and (b) every order-bornivorous barrel which is the countable intersection of closed, circled, convex neighbourhoods of 0 in E is itself a neighbourhood of 0.

REMARK: Clearly every order -(DF)-vector lattice is a C.O.Q. vector lattice.

2.1.3 THEOREM: Let (E,C,u) be an l.c.v.l.. Then (E,C,u) is an order-(DF)-vector lattice iff (i) it has a countable fundamental system of order-bounded sets, and (ii) every solid barrel which is the countable intersection of closed, convex, solid neighbourhoods of 0 in E is itself a neighbourhood of 0.

PROOF: All that we need to show is the equivalence of condition (ii) and condition (b) of 2.1.2. But this equivalence has already been established in 1.1.2.

2.1.4 THEOREM: Let (E,C,u) be an l.c.v.l.. Then (E,C,u) is an order-(DF)-vector lattice iff (i) it has a countable fundamental system of order-bounded sets, and (ii) every $\sigma_s(E',E)$-bounded subset of E' which is the countable union of equicontinuous subsets of E' is itself equicontinuous.

PROOF: Clearly, it is enough if we establish the equivalence of condition (ii) and condition (b) of 2.1.2. But this follows from 1.1.3.

2.1.5 COROLLARY: Every C.O.Q. vector lattice (and hence, a fortiori every order-infrabarrelled Riesz space) with a countable fundamental system of order-bounded sets is an order-(DF)-vector lattice.

PROOF: It is obvious, in view of the definitions concerned.

2.1.6 COROLLARY: Every C.O.Q. vector lattice (in particular, every countably barrelled l.c.v.l. and every order-infrabarrelled Riesz space) with an order unit is an order-(DF)-vector lattice.

PROOF: Let e be an order-unit in a C.O.Q. vector lattice (E,C,u). Then the class $\{n[-e,e];n = 1,2,\cdots\}$ is a fundamental system of order-bounded sets in E; clearly it is countable. The result now follows from 2.1.5.

Let (E,C) be an ordered vector space. A subset H of the positive cone C is said to exhaust C if for each $x \in C$, there exist $h \in H$ and $\lambda > 0$ such that $x \leq \lambda h$ ([32], page 121). We call H an exhausting subset of C.

2.1.7 COROLLARY: Every C.O.Q. vector lattice (in particular, every countably barrelled l.c.v.l. and every order-infrabarrelled Riesz space) with a countable exhausting subset of the positive cone is an order-(DF)-vector lattice.

PROOF: Let (E,C,u) be a C.O.Q. vector lattice with a countable exhausting subset H of C. Since C is generating, $\{[-t,t];t \in C\}$ is a fundamental system of order-bounded sets in E. Suppose $H = \{h^{(n)};n = 1,2,\cdots\}$. Since H is an exhausting subset of C, for each $t \in C$, there exist $h^{(n)} \in H$ and an integer $m > 0$ such that $t \leq mh^{(n)}$. This implies that $[-t,t] \subset m[-h^{(n)},h^{(n)}]$. Hence the class $\{m[-h^{(n)},h^{(n)}];(m,n) \in \mathbb{N} \times \mathbb{N}\}$ is a fundamental system of order-bounded sets in E; clearly it is countable. The result now follows from 2.1.5.

2.1.8 EXAMPLES: (a) Let ϕ be the vector space of all real sequences with only finitely many non-zero components. Then (ϕ,K), where K is the positive cone defined by $K = \{x = \{x_i\} \in \phi; x_i \geq 0$ for all $i\}$, is a vector lattice. It has a countable exhausting subset of K([32], page 122). Hence, as shown in 2.1.7, (ϕ, K) has a countable fundamental system of order-bounded sets.

REMARK: (1) ϕ does not contain order-units ([32], page 10). (2) $(\phi, K, ||.||)$ is not an order-(DF)-vector lattice, by 1.1.7(v),

(b) Consider the vector space $\kappa(X)$ of all real-valued, continuous functions with compact support in a σ-compact, locally compact Hausdorff space X. Then $(\kappa(X), C)$, where $C = \{f \in \kappa(X); f(t) \geq 0$ for all $t \in X\}$ is the positive cone, is a vector lattice. It has a countable exhausting subset of C ([32], page 122) and hence has a countable fundamental system of order-bounded sets.

REMARK: $\kappa(X)$ does not contain order-units ([32], page 122).

(c) Consider the vector space $m = \ell^\infty$ of all bounded real sequences; it is a Banach lattice with supremum norm $||.||$ and order structure determined by the cone $K = \{x = \{x_i\} \in m; x_i \geq 0$ for all $i\}$. Thus it is a C.O.Q. vector lattice. The element $e = \{e_n\}$ defined by $e_n = 1$ for all $n \in \mathbb{N}$ is an order-unit in m. Hence, in view of 2.1.6, $(m, K, ||.||)$ is an order-(DF)-vector lattice.

Similarly, the Banach lattice c of all convergent real sequences, with supremum norm $||.||$ and the usual pointwise ordering, is an order-(DF)-vector lattice.

(d) Consider C(I), the vector space of all continuous real-valued functions on I = [0,1]; it is a Banach lattice with the supremum norm ||.|| and the order structure determined by the cone K = {f∈C(I);f(x) ≥ 0 for all x∈I}. Hence (C(I),K,||.||) is a C.O.Q. vector lattice. The function identically equal to one on I is an order-unit in C(I), and hence, in view of 2.1.6,(C(I),K,||.||) is an order-(DF)-vector lattice.

The next example shows that an order-(DF)-vector lattice need not be an order-infrabarrelled Riesz space.

(e) Let X = [0,Ω) be the space of ordinals less than the first uncountable ordinal, C(X) the vector space of all continuous real-valued functions on X, u_c the compact-open topology on C(X), and K the positive cone in C(X) defined as follows:

$$K = \{f\in C(X); f(t) \geq 0 \text{ for all } t\in X\}.$$

Then, by 1.1.7(i), (C(X),K,u_c) is a C.O.Q. vector lattice which is not an order-infrabarrelled Riesz space. The function identically equal to one on X is an order-unit in C(X); hence, in view of 2.1.6, (C(X),K,u_c) is an order-(DF)-vector lattice. Thus, (C(X),K,u_c) is an order-(DF)-vector lattice which is not an order-infrabarrelled Riesz space.

(f) Any order-unit-normed vector lattice is an order-(DF)-vector lattice, because it is an order-infrabarrelled Riesz space ([30], page 113) and hence a C.O.Q. vector lattice, and it has an order-unit.

We now give an example of an order-(DF)-vector lattice which is not a countably barrelled l.c.v.l..

(g) Consider the Banach lattice $m = \ell^\infty$ of all bounded real sequences with the supremum-norm $||.||$ and the order structure determined by the cone

$$K = \{x = \{x_i\} \in m; x_i \geq 0 \text{ for all } i\} .$$

Consider $E = \{x + \lambda e; x \in E_0, \lambda \in \mathbb{R}\}$ with the supremum norm and the order inherited from m, where E_0 is the subspace of m consisting of all finite sequences and e is the sequence having one in each coordinate. Then $(E, K_E, ||.||)$, where $K_E = E \cap K$, is an order-unit-normed vector lattice ([30], page 113), and hence, in view of the preceding example (f), it is an order-(DF)-vector lattice. But we have shown in 1.1.7(iii) that it is not countably barrelled.

Finally, we give an example which shows that an order-infrabarrelled Riesz space and hence a C.O.Q. vector lattice need not be an order-(DF)-vector lattice. This example also shows that a countably barrelled l.c.v.l. need not be an order-(DF)-vector lattice.

(h) Consider $\ell^1 = \{x = \{x_i\}; x_i \in \mathbb{R}, \sum_{i=1}^{\infty} |x_i| < \infty\}$ with the usual norm $||.||$ and the order structure determined by the cone

$$K = \{x = \{x_i\} \in \ell^1; x_i \geq 0 \text{ for all } i\} .$$

$(\ell^1, K, ||.||)$ is a Banach lattice and hence an order-infrabarrelled Riesz space as well as a countably barrelled l.c.v.l.. But then, in view of 1.1.4(b), it is a C.O.Q. vector lattice. We show, however, that it is not an order-(DF)-vector lattice. To show this, we have to establish that ℓ^1 does not have a countable fundamental system of order-bounded sets. Suppose it has a countable fundamental system of order-bounded sets. Then $\sigma_s(\ell^\infty,\ell^1)$ is metrizable. Since $(\ell^\infty,\sigma_s(\ell^\infty,\ell^1))$ is complete ([32], pages 150 and 151), it follows that $(\ell^\infty,\sigma_s(\ell^\infty,\ell^1))$ is complete, metrizable and hence $\sigma_s(\ell^\infty,\ell^1)$ coincides with the supremum norm topology of ℓ^∞, because they are comparable ([36], page 120, Corollary 1). Thus $\sigma_s(\ell^\infty,\ell^1)$ is normable. Clearly, the positive cone K in ℓ^1 is generating and $\sigma(\ell^1,\ell^\infty)$-closed. This implies that ℓ^1 has an order-unit ([32], page 128). But it is known that ℓ^1 does not have order-units ([32], page 12), and thus we have a contradiction.

▼

REMARK: Since an order-(DF)-vector lattice is a C.O.Q. vector lattice, the Banach-Steinhaus type theorem 1.1.21 is valid for order-(DF)-vector lattices. Furthermore, the results 1.1.8, 1.1.9, 1.1.12, 1.1.19 and 1.1.20 are also valid for order-(DF)-vector lattices.

2.1.9 **PROPOSITION:** Let (E,C,u) be a countably quasibarrelled l.c.v.l. with a countable fundamental system of order-bounded sets. If one of the following conditions is satisfied, then (E,C,u) is an order-(DF)-vector lattice:

(i) The topological dual E' of E is a normal subspace of E^b.

(ii) The topology $\beta(E''_{|\sigma|},E')$ on $E''_{|\sigma|}$ is the relative topology induced by $\beta(E'',E')$ and the $\sigma_s(E',E)$-closure of each $\beta(E',E)$-bounded set in E' is $\beta(E',E)$-bounded.

PROOF: If (i) is satisfied, then the result follows from 1.1.16. If (ii) is satisfied, then the result follows from 1.1.18.

2.1.10 <u>COROLLARY</u>: Let (E,C,u) be a countably quasibarrelled l.c.v.l. with a countable exhausting subset of C (in particular, with an order unit). If one of the conditions (i) and (ii) of 2.1.9 is satisfied, then (E,C,u) is an order-(DF)-vector lattice.

2.1.11 <u>COROLLARY</u>: Let (E,C,u) be a countably quasibarrelled l.c.v.l. with a countable fundamental system of order-bounded sets (in particular, with a countable exhausting subset of C or with an order-unit). If E' is $\sigma_s(E',E)$-complete, then (E,C,u) is an order-(DF)-vector lattice.

<u>PROOF</u>: E' is $\sigma_s(E',E)$-complete iff E' is a normal subspace of E^b ([30], Lemma 4). The result now follows from 2.1.9.

An l.c.s. (E,u) is said to be a (DF)-space if (a) it has a countable fundamental system of u-bounded sets, and (b) every strongly bounded subset M of E' which is the countable union of equicontinuous subsets of E' is itself equicontinuous ⟺ every bornivorous barrel which is the countable intersection of closed, convex, circled neighbourhoods of 0 is itself a neighbourhood of 0 in E ([7]. See also [2?], §29(3)).

2.1.12 <u>PROPOSITION</u>: If (E,C,u) is an order-(DF)-vector lattice, then (i) $E'_{\sigma} = (E', \sigma_s(E',E))$ is a metrizable l.c.v.l., and (ii) the strong dual of E'_{σ}, E'_{b} is an l.c.v.l. which is also a complete (DF)-space.

<u>PROOF</u>: (i) Let $\{B_n; n = 1,2,\cdots\}$ be a countable fundamental system of order-bounded sets in E. Then $\{B_n^{\circ}; n = 1,2,\cdots\}$ form a neighbourhood basis

of 0 for $\sigma_s(E',E)$. This implies that $E|_{\sigma|}$ is metrizable.

(ii) The strong dual of $E|_{\sigma|}$ is an l.c.v.l. ([39], 5.7.4). Since $E|_{\sigma|}$ is metrizable, its strong dual is a complete (DF)-space ([21], §29(3)).

2.1.13. <u>COROLLARY</u>: The strong dual of $E|_{\sigma|}$ is a countably barrelled l.c.v.l..

<u>PROOF</u>: Husain ([10], Propositions 1 and 4) has shown that every (DF)-space is countably quasibarrelled and that a complete countably quasi-barrelled space is countably barrelled. Thus a complete (DF)-space is countably barrelled, and the result now follows from 2.1.12(ii).

2.2 Permanence properties

2.2.1 <u>LEMMA</u>: Let (E,C) be a vector lattice and M an l-ideal in E. Then every order-bounded subset of the vector lattice E/M is the canonical image of an order-bounded subset of E.

<u>PROOF</u>: Let $g:E \longrightarrow E/M$ be the canonical map of E onto E/M. Then, g is an l-homomorphism and hence positive. Let $[t,s]$ be an order interval in E/M. Then, there exist $x,y \in E$ such that $t = g(x)$, $s = g(y)$. We show that $g([x,y]) = [g(x),g(y)]$. Since g is linear and positive, we have

$$g([x,y]) \subseteq [g(x),g(y)].$$

Hence, it is enough if we show that

$$[g(x),g(y)] \subseteq g([x,y]) \ .$$

Let $r \in [g(x),g(y)]$. Then, there exists $z \in E$ such that $r = g(z)$, and $g(x) \leq g(z) \leq g(y)$. Let $z + m \in g(z)$, $m \in M$. Then, there exist $m_1, m_2 \in M$ such that $x + m_1 \leq z + m \leq y + m_2$ ([32], page 37). Hence, $z + m \in [x + m_1, y + m_2] = [x,y] + [m_1,m_2]$, because E, being a vector lattice, has the decomposition property, by 0.3.4. Since M is an l-ideal in E, it is an order ideal ([32], page 43), and hence $[m_1,m_2] \subset M$. This implies that $z + m \in [x,y] + M$. But m is an arbitrary element of M and hence,

$$g(z) \in g([x,y]); \text{ i.e. } r \in g([x,y]).$$

Thus, $g([x,y]) = [g(x),g(y)]$ and hence every order-bounded subset of E/M is the canonical image of an order-bounded subset of E.

2.2.2 PROPOSITION: Let (E,C,u) be an order-(DF)-vector lattice and M a closed l-ideal in E. Then E/M is an order-(DF)-vector lattice.

PROOF: In view of 1.2.4, E/M is a C.O.Q. vector lattice. If $\{B_n; n = 1,2\cdots\}$ is a countable fundamental system of order-bounded sets in E, then it follows from 2.2.1 that $\{g(B_n); n = 1,2,\cdots\}$, where g is the canonical map of E onto E/M, is a countable fundamental system of order-bounded sets in E/M. Hence, it follows from 2.1.5 that E/M is an order-(DF)-vector lattice.

The following example shows that an l-ideal of an order-(DF)-vector lattice need not be of the same type.

2.2.3 EXAMPLE: Consider C(I), the vector space of all continuous real-valued functions on $I = [0,1]$; it has been shown in 2.1.8(d) that $(C(I),K,||.||)$, equipped with the supremum norm $||.||$ and ordered by the cone

$$K = \{f \in C(I); f(t) \geq 0 \text{ for all } t \in I\},$$

is an order-(DF)-vector lattice. Now consider the vector subspace E of $C(I)$ consisting of all elements $f \in C(I)$ which vanish in a neighbourhood (depending on f) of $t = 0$. Then (E,K_E), where $K_E = E \cap K$, is an l-ideal in $C(I)$. It has been shown in 1.1.7(iv) that $(E,K_E,||.||)$ is not a C.O.Q. vector lattice; this implies that it is not an order-(DF)-vector lattice.

2.2.4 PROPOSITION: Let $(E,\mathcal{C},u)$ be an order-(DF)-vector lattice such that C' is a strict $\mathcal{B}$-cone in $(E',\sigma(E',E))$. If M is an l-ideal of countable codimension in E, then M is an order-(DF)-vector lattice under the induced topology.

PROOF: Since an order-(DF)-vector lattice is a C.O.Q. vector lattice, it follows from 1.2.6 that M is a C.O.Q. vector lattice. Let $\{B_n; n = 1,2,\cdots\}$ be a countable fundamental system of order-bounded sets in E; then

$$\{M \cap B_n; n = 1,2,\cdots\}$$

is a countable fundamental system of order-bounded sets in M. Hence M is an order-(DF)-vector lattice.

2.2.5 THEOREM: The locally convex direct sum of a sequence of order-(DF)-vector lattices is an order-(DF)-vector lattice.

PROOF: Let $\{(E_n, C_n, u_n); n = 1,2,3,\cdots\}$ be a countable family of order-(DF)-vector lattices. Let $E = \underset{n}{\oplus} E_n$ (the algebraic direct sum of $\{E_n; n = 1,2,\cdots\}$); $C = \underset{n}{\oplus} C_n$; $u = \underset{n}{\oplus} u_n$ (the locally convex direct sum topology on E); and let $i_n : E_n \longrightarrow E$ be the injection map for each $n \in \mathbb{N}$. Then (E,C,u) is a locally convex vector lattice ([45], Corollary 3.20). In view of 1.2.1, it follows that (E,C,u) is a C.O.Q. vector lattice. Let $\{B_i^{(n)}; i = 1,2,\cdots\}$ be a countable fundamental system of order-bounded sets in (E_n, C_n). Then the family $\left(\overset{k}{\underset{n=1}{\oplus}} B_{i_n}^{(n)}\right)_{k \geq 1}$ is a countable fundamental system of order-bounded sets in E. Thus, (E,C,u) is a C.O.Q. vector lattice with a countable fundamental system of order-bounded sets. Hence, in view of 2.1.5, (E,C,u) is an order-(DF)-vector lattice.

2.3 Order-(DF)-vector spaces and tensor products

In this section, we explain the question posed in ([35]) and then we give an affirmative answer to that question for order-(DF)-vector spaces.

2.3.1 Definition: An ordered l.c.s. (E,C,u) is called an order-(DF)-vector space if (a) it possesses a countable fundamental system of order bounded sets, and (b) every order-bornivorous barrel which is the countable intersection of closed, circled, convex neighbourhoods of 0 is itself a neighbourhood of 0 in E.

<u>REMARK:</u> Clearly every order-(DF)-vector space is a C.O.Q. vector space.

2.3.2 <u>PROPOSITION:</u> A separable order-(DF)-vector space is an order-infrabarrelled space.

<u>PROOF:</u> Since an order-(DF)-vector space is a C.O.Q. vector space, the result follows from 1.3.2.

2.3.3 <u>PROPOSITION:</u> Let (E,C,u) be an ordered l.c.s. with generating cone C, and $E' \subseteq E^b$. Then the following statements are equivalent:

(a) (E,C,u) is an order-(DF)-vector space.

(b) E has a countable fundamental system of order-bounded sets and each $\sigma(E',E)$-bounded subset of E' which is the countable union of equicontinuous subsets of E' is itself equicontinuous.

<u>PROOF:</u> This follows from 1.3.4.

Let E_1 and E_2 be vector spaces ordered by cones K_1 and K_2, respectively. We use $B(E_1,E_2)$ to denote the vector space of all bilinear functionals on $E_1 \times E_2$, $B^b(E_1,E_2)$ to denote the vector space of all order-bounded bilinear functionals on $E_1 \times E_2$, and $\mathcal{B}(E_1,E_2)$ to denote the vector space of all continuous bilinear functionals on $E_1 \times E_2$.

Suppose that (E_1,u_1) and (E_2,u_2) are locally convex spaces ordered by generating cones K_1 and K_2, respectively, and that the closure $\bar{K}_p$ of K_p in $E_1 \otimes_p E_2$ is a cone. (The latter condition is satisfied if K_1' is total in E_1' and K_2 is closed, by [35], 2.8). If S and T are order-bounded subsets of E_1 and E_2, respectively, then the closure $\bar{\Gamma}(S \otimes T)$ of the convex circled hull of $(S \otimes T)$ in $E_1 \otimes_p E_2$ is order-bounded for the order structure determined by $\bar{K}_p$ ([35], pages 186 and 187). Peressini and Sherbert ([35], page 187) have asked the following question:

When is each order-bounded set in $E_1 \otimes E_2$ for $\bar{K}_p$ contained in a set of the form $\bar{\Gamma}(S \otimes T)$ for suitable order-bounded sets S and T in E_1 and E_2, respectively? Equivalently, when does the topology of bi-order-bounded convergence on $\mathcal{B}(E_1,E_2)$ coincide with the topology $O(\mathcal{B}(E_1,E_2),\ E_1 \otimes E_2)$ on $\mathcal{B}(E_1,E_2)$? Here we give an affirmative answer to this question when E_1 and E_2 are both order-(DF)-vector spaces. (See also [35], 2.12).

2.3.4 THEOREM: Let (E_1,K_1,u_1) and (E_2,K_2,u_2) be as above, and suppose $\mathcal{B}(E_1,E_2) \subseteq \mathcal{B}^b(E_1,E_2)$. If (E_1,K_1,u_1) and (E_2,K_2,u_2) are order-(DF)-vector spaces, then the topology of bi-order-bounded convergence on $\mathcal{B}(E_1,E_2)$ coincides with $O(\mathcal{B}(E_1,E_2),\ E_1 \otimes E_2)$.

PROOF: Clearly $E_1 \times E_2$ is an order-(DF)-vector space with a generating cone. Since $E_1 \times E_2$ has a countable fundamental system of order-bounded sets, it follows that the topology u of bi-order-bounded convergence is metrizable. Since this topology is coarser than $O(\mathcal{B}(E_1,E_2),\ E_1 \otimes E_2)$, it suffices to show that the identity map of $(\mathcal{B}(E_1,E_2),u)$ onto $(\mathcal{B}(E_1,E_2),\ O(\mathcal{B}(E_1,E_2),\ E_1 \otimes E_2))$ is continuous; for this, in turn, it suffices to show that every u-null sequence (f_n) in $\mathcal{B}(E_1,E_2)$ is $O(\mathcal{B}(E_1,E_2),\ E_1 \otimes E_2)$-bounded. Consider

$$V = \bigcap_{n \geq 1} f_n^{-1}([-\varepsilon,\varepsilon]), \quad \varepsilon > 0.$$

Since $\{f_n; n = 1,2,\cdots\}$ is u-bounded in $\mathcal{B}(E_1,E_2)$, it follows that V is order-bornivorous. Clearly V is closed, circled, convex, so that V is an order-bornivorous barrel which is the countable intersection of closed, circled, convex neighbourhoods of 0 in $E_1 \times E_2$ and hence V itself is a neighbourhood of 0 in $E_1 \times E_2$, because $E_1 \times E_2$ is an order-(DF)-vector space. This shows that $\{f_n; n = 1,2,\cdots\}$ is equicontinuous on $E_1 \times E_2$ and hence equicontinuous in the dual $\mathcal{B}(E_1,E_2)$ of $E_1 \otimes E_2$ ([39]; Corollary 3.6.2). Hence $\{f_n; n = 1,2,\cdots\}$ is $\sigma(\mathcal{B}(E_1,E_2), E_1 \otimes E_2)$-bounded. This completes the proof.

CHAPTER III

ORDER-QUASIULTRABARRELLED VECTOR LATTICES AND SPACES

In this chapter, we introduce and study a class of t.v. lattices (more generally, a class of ordered t.v. spaces) which replaces that of order-infrabarrelled Riesz spaces (respectively, order-infrabarrelled spaces) in situations where local convexity is not assumed.

We remark that it is possible to introduce a bigger class which replaces that of C.O.Q. vector lattices (spaces) in non-locally convex situations, but we have restricted ourselves to above mentioned class only.

3.1 O.Q.U. vector lattices

In this section, we define and characterize, what we call, O.Q.U. vector lattices, and establish their relations with ultrabarrelled and quasi-ultrabarrelled t.v. lattices. We also give some examples to clarify certain situations.

3.1.1 <u>Definition</u>: ([15(a)], Definition 3.1). (a) A closed, circled subset B of a t.v.s. (E,u) is said to be an ultrabarrel if there exists a sequence $\{B_n; n \in \mathbb{N}\}$, called a defining sequence for B, of circled and absorbing subsets of E such that $B_1 + B_1 \subseteq B$ and $B_{n+1} + B_{n+1} \subseteq B_n$ for all n.

(b) A closed, circled subset B of a t.v.s. (E,u) is said to be bornivorous ultrabarrel if there exists a sequence $\{B_n; n \in \mathbb{N}\}$ of circled bornivorous subsets of E such that $B_1 + B_1 \subseteq B$ and $B_{n+1} + B_{n+1} \subseteq B_n$ for all $n \geq 1$.

Clearly, if B is a circled, closed, convex, absorbing (closed, circled, convex, bornivorous) subset of a t.v.s., then B is an ultrabarrel (a bornivorous ultrabarrel) with $(2^{-n}B)$ as a defining sequence. However, an ultrabarrel (a bornivorous ultrabarrel) need not be convex and need not have a defining sequence of convex sets; for a counter-example see ([15(a)], page 294).

W. Robertson ([37]) calls a t.v.s. (E,u) an ultrabarrelled space if any linear topology on E with a base of u-closed neighbourhoods of 0 is necessarily coarser than u. However, we have the following characterization of ultrabarrelled spaces in terms of ultrabarrels.

3.1.2 <u>THEOREM</u>: ([15(a)], Theorem 3.1). A t.v.s. is an ultrabarrelled space iff every ultrabarrel is a neighbourhood of 0.

3.1.3 <u>Definition</u>: ([15(a)], Definition 5.1). A t.v.s. (E,u) is said to be quasi-ultrabarrelled if every bornivorous ultrabarrel in E is a neighbourhood of 0.

Clearly every ultrabarrelled space is quasiultrabarrelled.

3.1.4 <u>Definition</u>: Let (E,C,u) be an ordered t.v.s.. A closed, circled subset B of E is said to be an order-bornivorous ultrabarrel if there exists a sequence $\{B_n; n = 1,2,\cdots\}$ of circled, order-bornivorous subsets of E such that $B_1 + B_1 \subseteq B$ and $B_{n+1} + B_{n+1} \subseteq B_n$ for all $n \geq 1$.

3.1.5 <u>Definition</u>: A t.v.l. (E,C,u) is called order-quasiultrabarrelled vector lattice (abbreviated to O.Q.U. vector lattice) if every order-bornivorous ultrabarrel in E is a neighbourhood of 0.

3.1.6 PROPOSITION: (a) Every ultrabarrelled t.v.l. is an O.Q.U. vector lattice; (b) every O.Q.U. vector lattice is a quasi-ultrabarrelled t.v.l..

PROOF: (a) Let (E,C,u) be an ultrabarrelled t.v.l., and let B be an order-bornivorous ultrabarrel in E. Then, clearly B is an ultrabarrel in E and hence is a neighbourhood of 0 in E. Thus, (E,C,u) is an O.Q.U. vector lattice.

(b) Let (F,K,v) be an O.Q.U. vector lattice, and let B be a bornivorous ultrabarrel in F. Since the positive cone K in F is normal, every order-bounded set in F is v-bounded; a fortiori, B is an order-bornivorous ultrabarrel in F and hence a neighbourhood of 0. This shows that (F,K,v) is a quasi-ultrabarrelled t.v.l..

Let (E,C) be a vector lattice. A locally solid linear topology u on (E,C) is a linear topology which admits a neighbourhood basis of 0 consisting of solid sets in E.

If u is a locally solid linear topology on (E,C), then we use u°° to denote the locally solid topology derived from u; i.e., the finest locally solid topology coarser than u.

3.1.7 PROPOSITION: If (E,C,u) is an O.Q.U. vector lattice, then (E,C,u°°) is an order-infrabarrelled Riesz space.

PROOF: Let B be an order-bornivorous barrel in (E,C,u°°). Then B is an order-bornivorous ultrabarrel in (E,C,u) and hence a u-neighbourhood of 0. The set B must then be u°°-neighbourhood of 0. Thus (E,C,u°°) is an order-infrabarrelled Riesz space.

3.1.8 <u>COROLLARY</u>: Any locally convex O.Q.U. vector lattice is an order-infrabarrelled Riesz space.

3.1.9 <u>PROPOSITION</u>: Every order-unit-normed vector lattice is an O.Q.U. vector lattice.

<u>PROOF</u>: Let (E,C,u) be an order-unit-normed vector lattice with order-unit e. Then $\{n[-e,e]; n = 1,2,\cdots\}$ is a fundamental system of order-bounded sets in E. Let B be an order-bornivorous ultrabarrel in E. Then there exists $\lambda > 0$ such that $\lambda[-e,e] \subset B$. But the scalar multiples of $[-e,e]$ form a neighbourhood basis of 0 for u ([17], 3.7.1). Hence B is a neighbourhood of 0; this shows that (E,C,u) is an O.Q.U. vector lattice.

3.1.10 <u>EXAMPLES</u>: (a) Let $[a,b]$ be a closed interval on the real line equipped with the Lebesque measure, and let $0 < p < 1$. Then the vector space $L^p([a,b])$ ordered by the cone

$$K = \{f \in L^p([a,b]); f(t) \geq 0 \text{ almost everywhere for all } t \in [a,b]\},$$

and equipped with the complete metrizable topology determined by the neighbourhood basis $(V_n)_{n \geq 1}$ of 0, where

$$V_n = \{f \in L^p; ||f||_p \leq \tfrac{1}{n}\}, ||f||_p = \left(\int_a^b |f|^p\right)^{1/p},$$

is a complete metrizable t.v.l. and hence is an ultrabarrelled t.v.l. ([37], Proposition 12, Corollary). But then in view of 3.1.6(a), it is an O.Q.U. vector lattice.

(b) The vector space

$$\ell^p = \{x = \{x_i\}; x_i \in \mathbb{R}, \sum_{i=1}^{\infty} |x_i|^p < \infty\}, \ 0 < p < 1,$$

ordered by the cone $K = \{x = \{x_i\} \in \ell^p; x_i \geq 0 \text{ for all } i\}$ and equipped with the complete metrizable topology determined by the neighbourhood basis $\{V_n\}_{n \geq 1}$ of 0, where

$$V_n = \{x = \{x_i\} \in \ell^p; ||x||_p \leq \frac{1}{n}\}, \ ||x||_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p},$$

is a complete metrizable t.v.l. and hence, as shown in (a) above, it is an O.Q.U. vector lattice.

The following example shows that a quasiultrabarrelled t.v.l. need not be an O.Q.U. vector lattice.

(c) Consider the Banach lattice C([0,1]) of all continuous real-valued functions on I = [0,1] and the vector subspace E of C(I) as in 1.1.7(iv). As has been shown there, E is an l.c.v.l. which is not a C.O.Q. vector lattice, and hence surely not an order-infrabarrelled Riesz space; but then, it is not an O.Q.U. vector lattice in view of 3.1.8. However, it is a quasiultrabarrelled l.c.v.l., because a normed vector lattice is quasiultrabarrelled ([15(b)], page 610).

We now give an example of an O.Q.U. vector lattice which is not ultrabarrelled.

(d) Consider the Banach lattice m of all bounded real sequences and the vector subspace E of m as in 1.1.7(iii). As has been shown there, E is not countably barrelled and hence surely not barrelled. This implies that E is not ultrabarrelled ([37], page 250). But E is an order-unit-normed vector lattice and hence, in view of 3.1.9, an O.Q.U. vector lattice.

A subset B of a vector space is called semiconvex if $B + B \subset \lambda B$ for some $\lambda > 0$. A t.v.s. E is called almost convex if every bounded subset is contained in some bounded, closed, circled and semiconvex set ([15(a)], page 289).

3.1.11 PROPOSITION: A (Hausdorff) sequentially complete almost convex quasiultrabarrelled t.v.l. is an ultrabarrelled t.v.l..

PROOF: This follows from ([15(a)], Theorem 5.2).

3.1.12 COROLLARY: A (Hausdorff) sequentially complete almost convex O.Q.U. vector lattice is an ultrabarrelled t.v.l..

PROOF: It is obvious in view of 3.1.6(b) and 3.1.11.

3.1.13 Definition: A closed solid subset B of a t.v.l. (E,C,u) is called a solid ultrabarrel if there exists a sequence $(B_n; n \geq 1)$ of solid absorbing subsets of E such that $B_1 + B_1 \subseteq B$ and $B_{n+1} + B_{n+1} \subseteq B_n$ for all $n \geq 1$.

3.1.14 THEOREM: A t.v.l. (E,C,u) is an O.Q.U. vector lattice iff each solid ultrabarrel is a u-neighbourhood of 0.

PROOF: Assume that (E,C,u) is an O.Q.U. vector lattice. Let V be a solid ultrabarrel in (E,C,u) with a defining sequence $(V_n; n = 1,2,\cdots)$. Since V is solid, it follows that V is circled. Similarly, each V_n is circled. Furthermore, since each V_n is solid and absorbing, it follows that each V_n is order-bornivorous. Thus V is a closed and circled set and $(V_n; n = 1,2,\cdots)$ is a sequence of circled order-bornivorous subsets of E such that $V_1 + V_1 \subseteq V$ and $V_{n+1} + V_{n+1} \subseteq V_n$ for all $n \geq 1$. This implies that V is an order-bornivorous ultrabarrel in E and hence V is a neighbourhood of 0 in E.

Conversely, assume that every solid ultrabarrel in E is a neighbourhood of 0. Let V be an order-bornivorous ultrabarrel with a defining sequence (V_n). As has been shown in 1.1.2, K(V) and $K(V_n)$ are closed. Since V_n is order-bornivorous, it follows that $K(V_n)$ is order-bornivorous ([45], Lemma 3.1. The assumption of convexity in this Lemma is unnecessary). We now show that $K(V_1) + K(V_1) \subseteq K(V)$ and $K(V_{n+1}) + K(V_{n+1}) \subseteq K(V_n)$ for all $n \geq 1$. Let $x, y \in K(V_1) = \{z \in E; [-|z|, |z|] \subseteq V_1\}$. Then,

$$[-|x|, |x|] + [-|y|, |y|] \subseteq V_1 + V_1 \subseteq V .$$

Since E has the decomposition property, we get

$$[-(|x| + |y|), (|x| + |y|)] \subseteq V .$$

But then,

$$[-|x + y|, |x + y|] \subseteq V.$$

Hence,

$$x + y \in K(V), \text{ i.e., } K(V_1) + K(V_1) \subseteq K(V) .$$

Similarly, we can show that

$$K(V_{n+1}) + K(V_{n+1}) \subseteq K(V_n) .$$

Thus $K(V)$ is a solid ultrabarrel with defining sequence $\{K(V_n)\}_{n \geq 1}$. Hence $K(V)$ is a neighbourhood of 0 in E. But $K(V) \subset V$ and so V is a neighbourhood of 0 in E. This shows that (E,C,u) is an O.Q.U. vector lattice.

3.1.15 <u>THEOREM</u>: Let (E,C,u) be an O.Q.U. vector lattice, and (F,K,v) a t.v.l.. If H is a pointwise bounded set of continuous l-homomorphisms of E into F, then H is equicontinuous.

<u>PROOF</u>: Let V be a closed, solid neighbourhood of 0 in F, and $B = \bigcap_{f \in H} f^{-1}(V)$. Since each $f \in H$ is an l-homomorphism, $f^{-1}(V)$ is solid and hence B is solid. That B is an ultrabarrel follows from ([15(a)], pages 294 and 296). Hence, B is a neighbourhood of 0 in E, because (E,C,u) is an O.Q.U. vector lattice. But then, since $B = \bigcap_{f \in H} f^{-1}(V)$, it follows that H is equicontinuous.

3.1.16 <u>COROLLARY</u>: Let (E,C,u) and (F,K,v) be as in 3.1.15. Let $\{f_n;$ $n \geq 1\}$ be a pointwise bounded sequence of continuous l-homomorphisms of E into F. Then, $\{f_n; n \geq 1\}$ is equicontinuous.

In 3.1.8, we have shown that a locally convex O.Q.U. vector lattice is an order-infrabarrelled Riesz space. The following example shows that an order-infrabarrelled Riesz space may fail to be an O.Q.U. vector lattice.

3.1.17 EXAMPLE: Consider $\ell^{1/2} = \{x = \{x_i\}; x_i \in R, \sum_{i=1}^{\infty} |x_i|^{1/2} < \infty\}$ with the order structure determined by the cone $K = \{x = \{x_i\} \in \ell^{1/2}; x_i \geq 0$ for all $i\}$, and the topology u determined by the neighbourhood basis $\{V_n\}_{n \geq 1}$ of 0, where

$$V_n = \{x = \{x_i\} \in \ell^{1/2}; ||x||_{1/2} < \frac{1}{n}\}, \quad ||x||_{1/2} = (\sum_{i=1}^{\infty} |x_i|^{1/2})^2 < \infty.$$

$(\ell^{1/2}, K, u)$ is then a complete metrizable t.v.l.. Now, write $||x||_1 = \sum_{i=1}^{\infty} |x_i|$. Then, since $||x||_1 \leq ||x||_{1/2}$, $\ell^{1/2}$ is a subspace of ℓ^1 such that the ℓ^1-norm induced on $\ell^{1/2}$ is coarser than u. Clearly, $(\ell^{1/2}, K, ||\cdot||_1)$ is a normed vector lattice. The space ϕ of all real sequences with only finite number of non-zero components is dense in ℓ^1 under the norm topology, and so since $\phi \subseteq \ell^{1/2}$, $\ell^{1/2}$ is dense in ℓ^1 under the norm topology. Hence the dual of $\ell^{1/2}$ under the norm topology is $m = \ell^{\infty}$ and the norm topology is thus $\tau(\ell^{1/2}, m)$. Also the dual of $\ell^{1/2}$ under u is m, and so $u^{\circ\circ} = \tau(\ell^{1/2}, m)$ ([37], page 256). It is known ([37], page 256) that $(\ell^{1/2}, u^{\circ\circ})$ is barrelled, and hence $(\ell^{1/2}, K, u^{\circ\circ})$ is an order-infrabarrelled Riesz space. We show, however, that it is not an O.Q.U. vector lattice. For $x = \{x_i\} \in \ell^{1/2}$, define $f_n(x) = (x_1, x_2, \cdots, x_n, 0, 0, \cdots)$. Then each f_n is continuous 1-homomorphism of $(\ell^{1/2}, u^{\circ\circ})$ into $(\ell^{1/2}, u)$, and for each $x \in \ell^{1/2}$, $f_n(x) \longrightarrow x$ under u ([37], page 256). But, since the identity map of $(\ell^{1/2}, u^{\circ\circ})$ onto $(\ell^{1/2}, u)$ is not continuous, $\{f_n\}$ is not equicontinuous. Thus the conclusion of 3.1.16 does not hold for $(\ell^{1/2}, u^{\circ\circ})$ and this shows that $(\ell^{1/2}, u^{\circ\circ})$ is not an O.Q.U. vector lattice.

3.2 Permanence properties

In [15(a)], S.O. Iyahen has introduced and studied what he calls *-inductive limits and *-direct sums of topological vector spaces. We use this idea of S.O. Iyahen to show that the *-direct sum of a countable family of O.Q.U. vector lattices is again O.Q.U. vector lattice.

3.2.1 Definition: ([15(a)], page 286). Let E be a vector space, $((E_\alpha, u_\alpha); \alpha \in I)$ a family of topological vector spaces, f_α a linear map of E_α into E and suppose $\bigcup_\alpha f_\alpha(E_\alpha)$ spans E. The set Φ of all linear topologies on E for which each f_α is continuous is not empty, because it contains the trivial topology. The upper bound u of the members of Φ is also in Φ; it is the finest linear topology on E for which all the f_α are continuous. The topology u on E is called the *-inductive limit topology on E induced by $[E_\alpha : f_\alpha]$, and we say that (E,u) is the *-inductive limit of $([E_\alpha : f_\alpha]; \alpha \in I)$.

3.2.2 Definition: ([15(a)], page 288). If for each $\alpha \in I$, E_α is a t.v.s., we shall call the vector space $\sum_{\alpha \in I} E_\alpha$ under the *-inductive limit topology of $([(E_\alpha, u_\alpha) : i_\alpha]; \alpha \in I)$, where the i_α are injection maps, the *-direct sum of $\{E_\alpha\}$.

3.2.3 EXAMPLES: ([15(a)], page 286).

(i) Let E be a t.v.s., F a vector subspace of E and g the canonical map of E onto E/F. Then E/F under its quotient topology is the *-inductive limit of [E:g].

(ii) Let E be a vector space and $\{v_\alpha;\alpha\in I\}$ a family of linear topologies on E. For each $\alpha\in I$, let i_α be the identity map of (E,v_α) into E. If v is the lower bound of the topologies v_α, then (E,v) is the *-inductive limit of $\{[(E,v_\alpha);i_\alpha]\}$.

3.2.4 PROPOSITION: ([15(a)], page 287). Let (E,u) be an *-inductive limit of $\{[(E_\alpha,u_\alpha);f_\alpha]:\alpha\in I\}$. For each $\alpha\in I$, let V_α be a circled neighbourhood of 0 in E_α, and let $U = \bigcup_{\phi} \{\sum_{\alpha\in\phi} f_\alpha(V_\alpha)\}$, where ϕ is a finite subset of I. Then U is a neighbourhood of 0 in (E,u).

If I is countable, then as V_α runs through a base of circled neighbourhoods of 0 in E_α, the above sets form a base of neighbourhoods of 0 for (E,u).

3.2.5 LEMMA: Let $\{(E_n,C_n);n = 1,2,\cdots\}$ be a countable family of ordered vector spaces, u_n a linear topology on E_n for each n such that C_n is normal with respect to u_n. Then $C = \bigoplus_n C_n$ is normal with respect to the *-direct sum topology $u = * - \bigoplus_n u_n$.

PROOF: Let $\eta_n = \{V^{(n)}\}$ be a neighbourhood basis of 0 in E_n such that each $V^{(n)}$ is circled and $0 \le x^{(n)} \le y^{(n)}$, $y^{(n)}\in V^{(n)}$ implies $x^{(n)}\in V^{(n)}$. Let $\{V\}$, $V = \bigcup_J \sum_{n\in J} V^{(n)}$, J a finite subset of $\mathbb{N}$, be a neighbourhood basis of 0 for the *-direct sum topology of $\{E_n;n = 1,2,\cdots\}$, where $V^{(n)}\in\eta_n$. Without loss of generality, we assume that $J = \{1,2,\cdots,m\}$. Let $0 \le x \le y$, $y\in V = \bigcup_m \sum_{n=1}^{m} V^{(n)}$. We show that $x\in V$ and this, in turn, proves that $C = \bigoplus_n C_n$ is normal with respect to *-direct sum topology $u = * - \bigoplus_n u_n$. Now,

$y \in \bigcup_m \sum_{n=1}^{m} V^{(n)}$ implies that $y \in \sum_{n=1}^{m} V^{(n)}$ for some m. Hence $0 \leq x \leq y = \sum_{n=1}^{m} y^{(n)}$, $y^{(n)} \in V^{(n)}$, for some m. This implies that $0 \leq x^{(n)} \leq y^{(n)}$, $n = 1,2,\cdots,m$ and $x = \sum_{n=1}^{m} x^{(n)}$ for some m. Since each C_n is normal with respect to u_n and since $0 \leq x^{(n)} \leq y^{(n)}$, $y^{(n)} \in V^{(n)}$, it follows that $x^{(n)} \in V^{(n)}$, $n = 1,2,\cdots,m$, for some m. Hence $x \in \sum_{n=1}^{m} V^{(n)}$ for some m and this shows that $x \in V = \bigcup_m \sum_{n=1}^{m} V_n$; and the proof is complete.

3.2.6 LEMMA: Let u be a linear topology in a vector lattice (E,C) such that the positive cone C in E is normal with respect to u. Let $\eta = \{U\}$ be a neighbourhood basis of 0 in E for u, consisting of circled and full sets, and let $\eta_s = \{K(U); U \in \eta\}$. Then there exists a unique locally solid linear topology u_s on E such that η_s is a neighbourhood basis of 0 in E for u_s. Furthermore, u_s is finer than u, and u_s is the greatest lower bound of all locally solid linear topologies which are finer than u.

PROOF: Each K(U) is clearly non-empty, solid and absorbing; also for $\lambda \neq 0$,

$$K(\lambda U) = \{x; [-|x|, |x|] \subseteq \lambda U\}$$

$$= \lambda\{x; [-|x|, |x|] \subseteq U\} = \lambda K(U) .$$

Also for $U_1, U_2 \in \eta$,

$$K(U_1 \cap U_2) = \{x; [-|x|, |x|] \subseteq U_1 \cap U_2\}$$

$$= \{x; [-|x|, |x|] \subseteq U_1\} \cap \{x; [-|x|, |x|] \subseteq U_2\}$$

$$= K(U_1) \cap K(U_2);$$

and if

$$U \subseteq V, \ U, V \in \eta$$

then

$$K(U) = \{x; [-|x|, |x|] \subseteq U\}$$
$$\subseteq \{x; [-|x|, |x|] \subseteq V\} = K(V).$$

Hence, there exists a unique locally solid linear topology u_s on E such that η_s is a neighbourhood basis of 0 for u_s. Since $K(U) \subset U$ for each $U \in \eta$, it follows that u_s is finer than u. Now, let u' be any locally solid linear topology on E such that $u' \supset u$. Let W be any u_s-neighbourhood of 0 in E. There exists U such that $K(U) \subseteq W$. Since $u \subset u'$, there exists a solid u'-neighbourhood V of 0 such that $V \subseteq U$, and hence $V \subseteq K(U) \subseteq W$. Thus u' is finer than u_s. This completes the proof.

Although the proofs of 3.2.7 and 3.2.8 below are similar to those of Proposition 3.19 and Corollary 3.20 in ([45]), we shall give them in detail for the sake of completeness. The only difference is that we do not assume convexity and so we have to use 3.2.5 instead of ([39], page 217, 3.2).

Let $\{(E_n, C_n); n \in \mathbb{N}\}$ be a family of vector lattices. Let u_n be a linear topology on E_n, for each n, such that the positive cone C_n is normal with respect to u_n. Let $E = \bigoplus_n E_n$ (the algebraic direct sum of $\{E_n; n \in \mathbb{N}\}$); $u' = * - \bigoplus_n u_n$ (the *-direct sum topology on E); $C = \bigoplus_n C_n$; let $i_n : E_n \longrightarrow E$ be the injection map for each $n \in \mathbb{N}$; and let $p_n : E \longrightarrow E_n$ be the projection map for each n.

3.2.7 PROPOSITION: Let $u_{n,s}$ denote the locally solid linear topology on E_n associated with u_n for each $n \in \mathbb{N}$, and let u'_s be the locally solid linear topology on E, associated with u'. Then u'_s is coarser than $u' = * - \oplus_n u_{n,s}$.

PROOF: In view of 3.2.5, C is normal with respect to $u' = * - \oplus_n u_n$. Let V be any u'_s-neighbourhood of 0 in E. We show that $i_n^{-1}(V)$ is a $u_{n,s}$-neighbourhood of 0 in E_n for each $n \in \mathbb{N}$. Let W be a circled and full u'-neighbourhood of 0 in E such that $K(W) \subseteq V$. For each $n \in \mathbb{N}$, $i_n^{-1}(W)$ is a circled and full u_n-neighbourhood of 0 in E_n and so $K(i_n^{-1}(W))$ is a $u_{n,s}$-neighbourhood of 0 in E_n. We claim that

$$K(i_n^{-1}(W)) = i_n^{-1}(K(W)) .$$

Observe that $K(W) \subset W$ and that $i_n^{-1}(K(W))$ is solid because i_n is 1-homomorphism; hence $i_n^{-1}(K(W)) \subseteq K(i_n^{-1}(W))$, because $K(i_n^{-1}(W))$ is the largest solid set contained in $i_n^{-1}(W)$ and $i_n^{-1}(K(W)) \subset i_n^{-1}(W)$. Conversely, let $x_n \in K(i_n^{-1}(W))$. Then, since $x_n \in K(i_n^{-1}(W)) \subset i_n^{-1}(W)$, we have $i_n(x_n) \in W$. Now let $y \in E$ such that $|y| \leq |i_n(x_n)|$; then $y = i_n(p_n(y))$ and $|p_n(y)| \leq |x_n|$. Since $x_n \in K(i_n^{-1}(W))$, it follows that $p_n(y) \in i_n^{-1}(W)$ because $K(i_n^{-1}(W))$ is solid and $K(i_n^{-1}(W)) \subset i_n^{-1}(W)$. Hence $y = i_n(p_n(y)) \in W$. This shows that $[-|i_n(x_n)|, |i_n(x_n)|] \subseteq W$ and hence that $i_n(x_n) \in K(W)$, i.e., $x_n \in i_n^{-1}(K(W))$. Thus we have established that

$$K(i_n^{-1}(W)) = i_n^{-1}(K(W)) .$$

Since $K(W) \subseteq V$, it follows that

$$K(i_n^{-1}(W)) \subseteq i_n^{-1}(V)$$

for each n, and hence that V is a $* - \oplus_n u_{n,s}$ neighbourhood of 0 in E. Thus u'_s is coarser than $* - \oplus_n u_{n,s}$.

3.2.8 <u>COROLLARY</u>: Let u_n be the locally solid linear topology on E_n for each $n \in \mathbb{N}$. Then the *-direct sum topology $u = * - \bigoplus_n u_n$ on E is a locally solid linear topology and hence (E,C,u) is a topological vector lattice.

<u>PROOF</u>: Let $u_{n,s}$ be the locally solid linear topology on E_n, for each n, associated with u_n. It is then obvious that $u_{n,s} = u_n$ for each n. In view of 3.2.7, u_s' is coarser than $* - \bigoplus_n u_{n,s} = * - \bigoplus_n u_n$. But u_s' is always finer than $* - \bigoplus_n u_n$, and hence it follows that $u_s' = * - \bigoplus_n u_n$. Thus $* - \bigoplus_n u_n$ is a locally solid linear topology on E, and therefore $(E, C, * - \bigoplus_n u_n)$ is a topological vector lattice.

3.2.9 <u>THEOREM</u>: The *-direct sum of a countable family of O.Q.U. vector lattices is an O.Q.U. vector lattice.

<u>PROOF</u>: Let $\{(E_n, C_n, u_n); n \in \mathbb{N}\}$ be a countable family of O.Q.U. vector lattices; $E = \bigoplus_n E_n$ (the algebraic direct sum of $\{E_n; n \in \mathbb{N}\}$); $u = * - \bigoplus_n u_n$ (*-direct sum topology on E), and $C = \bigoplus_n C_n$. By 3.2.8, it follows that (E,C,u) is a t.v.l.. Let $i_n : E_n \longrightarrow E$ be the injection map for each $n \in \mathbb{N}$. We wish to show that (E,C,u) is an O.Q.U. vector lattice. Let V be a solid ultrabarrel in (E,C,u). For each $n \in \mathbb{N}$, i_n is a continuous l-homomorphism, and hence $i_n^{-1}(V)$ is a solid ultrabarrel in (E_n, C_n, u_n) which is an O.Q.U. vector lattice. This implies that $i_n^{-1}(V)$ is a neighbourhood of 0 in E_n. Hence V is a neighbourhood of 0 in (E,C,u) and so (E,C,u) is an O.Q.U. vector lattice.

3.2.10 <u>PROPOSITION</u>: Let (E,C,u) be an O.Q.U. vector lattice and (F,K,v) any t.v.l.. If f is an l-homomorphism of E into F, then f is almost continuous.

<u>PROOF</u>: Let V be a closed solid neighbourhood of 0 in F. Then, $f^{-1}(V)$ is solid, because f is an l-homomorphism. Since the closure of a solid set is solid, by 0.4.14(a), it follows that $\overline{f^{-1}(V)}$ is solid. Also, $\overline{f^{-1}(V)}$ is an ultrabarrel ([15(a)], page 294). Hence, $\overline{f^{-1}(V)}$, being a solid ultrabarrel in an O.Q.U. vector lattice (E,C,u), is a neighbourhood of 0 in E, and this shows that f is almost continuous.

3.2.11 <u>PROPOSITION</u>: Let (E,C,u) be an O.Q.U. vector lattice and (F,K,v) any t.v.l.. If g is an l-homomorphism of F onto E, then g is almost open.

<u>PROOF</u>: Let V be a closed solid neighbourhood of 0 in F. Since g is an l-homomorphism, g(V) and hence $\overline{g(V)}$ is solid. Also, $\overline{g(V)}$ is an ultrabarrel ([15(a)], page 294). Thus, $\overline{g(V)}$, being a solid ultrabarrel in an O.Q.U. vector lattice (E,C,u), is a neighbourhood of 0 in E. This shows that g is almost open.

3.2.12 <u>PROPOSITION</u>: Let (E,C,u) be an O.Q.U. vector lattice and (F,K,v) any t.v.l.. If f is a positive, linear, continuous and almost open map of E into F, then (F,K,v) is O.Q.U. vector lattice.

<u>PROOF</u>: Let V be an order-bornivorous ultrabarrel in F. Let B be any order-bounded set in E. Then, there exists $x \in C$ such that $B \subseteq [-x,x]$;

in other words, the family $\{[-x,x];x\in C\}$ is a fundamental system of order-bounded sets in E. Since f is positive and linear, it follows that $f([-x,x])\subset[-f(x),f(x)],x\in C$. Since the cone K in F is normal, there exists a $\lambda > 0$ such that

$$f([-x,x])\subset[-f(x),f(x)]\subset \lambda V$$

and this implies that $[-x,x]\subset\lambda f^{-1}(V),x\in C$ and some $\lambda > 0$. Thus, $f^{-1}(V)$ is an order-bornivorous set in E. Furthermore, since f is continuous, $f^{-1}(V)$ is an ultrabarrel in E ([15(a)], page 294). Hence, $f^{-1}(V)$, being an order-bornivorous ultrabarrel in an O.Q.U. vector lattice E, is a neighbourhood of 0 in E. Since f is almost open, $\overline{f(f^{-1}(V))}$ is a neighbourhood of 0 in F. But $\overline{f(f^{-1}(V))}\subset\overline{V} = V$, because V is closed. This shows that V is a neighbourhood of 0 in (F,K,v), and hence (F,K,v) is an O.Q.U. vector lattice.

3.2.13 <u>COROLLARY</u>: Let (E,C,u) be an O.Q.U. vector lattice and M a closed l-ideal in E. Then E/M is an O.Q.U. vector lattice.

The following example shows that an l-ideal of an O.Q.U. vector lattice need not be an O.Q.U. vector lattice.

3.2.14 <u>EXAMPLE</u>: Consider the Banach lattice C(I) of all continuous real-valued functions on I = [0,1], and the vector subspace E of C(I) as in 1.1.7 (iv). Since C(I) is a complete metrizable t.v.l., it is an ultrabarrelled t.v.l. ([37], Proposition 12, Corollary), and hence an O.Q.U. vector lattice in view of 3.1.6(a). But we have shown in 3.1.10(c) that E, which is an l-ideal in C(I) ([46], page 29), is not an O.Q.U. vector lattice.

3.3 O.Q.U. vector spaces and bases

In this section, we show that the Banach-Steinhaus type theorem and the isomorphism theorem that we have proved for order-infrabarrelled spaces in Chapter 1, can be carried over to O.Q.U. vector spaces.

3.3.1 Definition: An ordered t.v.s. (E,C,u) is called an O.Q.U. vector space if every order-bornivorous ultrabarrel is a neighbourhood of 0 in E.

REMARK: The class of O.Q.U. vector spaces replaces that of order-infrabarrelled spaces in situations where local convexity is not assumed.

3.3.2 THEOREM: Let (E,C,u) be an O.Q.U. vector space and (F,K,v) any ordered t.v.s. with normal cone K. Let H be a pointwise bounded set of continuous positive linear maps of E into F. Then H is equicontinuous.

PROOF: Let V be a closed, circled, full neighbourhood of 0 in F. Then

$$W = \bigcap_{f \in H} f^{-1}(V)$$

is an ultrabarrel ([15(a)], pages 294 and 296). We now wish to show that W is order-bornivorous. Let $[x,y]$ be any order interval in E. Since H is pointwise bounded, there exists $\lambda > 0$ such that $f(x), f(y) \in \lambda V$ for all $f \in H$. But then $[f(x),f(y)] \subset \lambda V$ for all $f \in H$, because V is full. Since f is positive and linear, it follows that $f([x,y]) \subset [f(x),f(y)] \subset \lambda V$ for all $f \in H$ and some $\lambda > 0$. This implies that

$$[x,y] \subset \lambda \bigcap_{f \in H} f^{-1}(V)$$

for some $\lambda > 0$. Hence,

$$W = \bigcap_{f \in H} f^{-1}(V)$$

is a neighbourhood of 0 in (E,C,u) because W is an order-bornivorous ultra-barrel in (E,C,u) which is an O.Q.U. vector space. This shows that H is equicontinuous.

3.3.3 <u>COROLLARY</u>: Let (E,C,u) and (F,K,v) be as in 3.3.2. Let $\{f_n; n \in \mathbb{N}\}$ be a pointwise bounded sequence of continuous positive linear maps of E into F. Then $\{f_n; n \in \mathbb{N}\}$ is equicontinuous.

3.3.4 <u>COROLLARY</u>: Let (E,C,u) and (F,K,v) be as in 3.3.2 and suppose that the positive cone K in F is closed. If $\{f_n; n \in \mathbb{N}\}$ is a sequence of continuous positive linear maps of E into F such that it converges pointwise to a map $f: E \longrightarrow F$, then f is continuous, linear and positive.

<u>PROOF</u>: Clearly f is linear. Since the cone K in F is closed, and since each f_n is positive, it follows that f is positive. Now, since $\{f_n; n \in \mathbb{N}\}$ is pointwise bounded, it is equicontinuous by 3.3.3. But then, by ([3], Chapter III, §3, Proposition 4), it follows that f is continuous.

3.3.5 <u>THEOREM</u>: Let (E,C,u) and (F,K,v) be O.Q.U. vector spaces such that the cones C and K are both closed and normal in E and F respectively. Suppose that $\{x_n\}$ and $\{y_n\}$ are positive Schauder bases in E and F respectively.

Then (x_n) is similar to (y_n) iff there exists a positive isomorphism $T:E \longrightarrow F$ such that $Tx_n = y_n$ for all $n \in \mathbb{N}$.

PROOF: The proof is similar to that of 1.4.9; the only difference is that we have to use 3.3.4 instead of 1.3.7.

3.3.6 PROPOSITION: Let (x_i, f_i) be a positive Schauder basis in an O.Q.U. vector space (E, C, u). Let $(T_n; n \in \mathbb{N})$ be a sequence of continuous, positive, linear maps $T_n: E \longrightarrow E$ defined by

$$T_n(x) = \sum_{i=1}^{n} f_i(x) x_i, \text{ for } x \in E,\ n = 1, 2, \cdots.$$

Then the sequence $(T_n; n = 1, 2, \cdots)$ is equicontinuous and converges uniformly to the identity mapping, on precompact sets.

PROOF: Clearly $T_n(x) \longrightarrow x$ as $n \longrightarrow \infty$ and hence the sequence $(T_n; n = 1, 2, \cdots)$ is pointwise bounded. So, in view of 3.3.3, $(T_n; n \in \mathbb{N})$ is equicontinuous. Since $T_n(x) \longrightarrow x$ as $n \longrightarrow \infty$ for all $x \in E$, it follows that the sequence $(T_n; n = 1, 2, \cdots)$ converges uniformly to the identity mapping, on precompact sets.

3.4 A closed graph theorem

The closed graph theorem is one of the most fundamental results of functional analysis. Over the years, there have been many efforts to generalize this theorem for various classes of topological vector spaces and for topological groups. (For an illuminating account of this, see Husain [8] and [9]). In this section, we present a closed graph theorem for O.Q.U. spaces.

3.4.1 **Definition:** Let (E,C,u) be an ordered t.v.s. and (F,v) any t.v.s.. A mapping $f:E \longrightarrow F$ is semibounded if f maps order-bounded subsets of E into v-bounded subsets of F.

3.4.2 **PROPOSITION:** Let (E,C,u) be an O.Q.U. space and (F,v) any t.v.s.. If $f:E \longrightarrow F$ is a linear semibounded mapping, then it is almost continuous.

PROOF: Let V be a circled neighbourhood of 0 in F and [x,y] any order interval in E. Then there exists $\lambda > 0$ such that $f([x,y]) \subset \lambda V$, because f is semibounded; hence $[x,y] \subset \lambda f^{-1}(V)$, for some $\lambda > 0$. This implies that $f^{-1}(V)$ is order-bornivorous in E. Then, surely, $\overline{f^{-1}(V)}$ is order-bornivorous in E. Furthermore, $\overline{f^{-1}(V)}$ is an ultrabarrel in E ([15(a)], page 294). Thus $\overline{f^{-1}(V)}$, being an order-bornivorous ultrabarrel in an O.Q.U. space (E,C,u), is a neighbourhood of 0 in E which proves that f is almost continuous.

3.4.3 **COROLLARY:** Let (E,C,u) be an O.Q.U. space and (F,K,v) any ordered t.v.s. with normal cone K. If $f:E \longrightarrow F$ is a positive linear mapping, then it is almost continuous.

PROOF: Since f is positive and linear, it is order-bounded, i.e., it sends order-bounded subsets of E into order-bounded subsets of F. But every order-bounded subset of F is v-bounded because the cone K in F is normal. Hence f is semibounded, and the result now follows from 3.4.2.

3.4.4 <u>COROLLARY</u>: Let (E,C,u) be an O.Q.U. vector lattice and (F,K,v) any t.v.l.. If f:E ⟶ F is a lattice homomorphism, then it is almost continuous.

<u>PROOF</u>: Since a lattice homomorphism is always positive, linear, and the cone in a t.v.l. is always normal, the result follows from 3.4.3.

We need the following theorem from Husain ([8]).

3.4.5 <u>THEOREM</u>: ([8], Chapter III, Theorem 4). Let E be a (Hausdorff) t.v.s. and F a complete metrizable t.v.s.. Let f:E ⟶ F be a linear map, the graph of which is closed in ExF. If f is almost continuous, then f is continuous.

3.4.6 <u>THEOREM</u>: Let (E,C,u) be a (Hausdorff) O.Q.U. space and (F,v) any complete metrizable t.v.s.. If f:E ⟶ F is a linear semibounded mapping and has closed graph, then f is continuous.

<u>PROOF</u>: In view of 3.4.2, f is almost continuous. The result now follows from 3.4.5.

3.4.7 <u>COROLLARY</u>: Let (E,C,u) be a (Hausdorff) O.Q.U. space and (F,K,v) a complete metrizable ordered t.v.s. with the normal cone K. If f:E ⟶ F is a positive linear mapping with closed graph, then it is continuous.

<u>PROOF</u>: f is almost continuous, in view of 3.4.3. Hence 3.4.5 applies.

3.4.8 **COROLLARY:** Let (E,C,u) be a (Hausdorff) O.Q.U. vector lattice and (F,K,v) a complete metrizable t.v.l. . If $f:E \longrightarrow F$ is a lattice homomorphism and has closed graph, then f is continuous.

PROOF: This follows immediately from 3.4.4 and 3.4.5. ∎

CHAPTER IV

ORDERED SEMICONVEX SPACES

In ([40]), Simons has introduced the concept of upper bound space and has characterized such spaces in terms of, what he calls, k-pseudometrics. This concept is the same as that of semiconvex space introduced by Iyahen ([16(a)]). The class of semiconvex spaces generalizes that of locally convex spaces. In this chapter, we study, what we call, ordered semiconvex spaces, and extend some of the results, known for ordered l.c. spaces, to ordered semiconvex spaces.

4.1 k-pseudometrics

In this section, we recall some definitions and results from ([40]) and ([16(a),(b)]) for use in the next two sections.

4.1.1 Definition: ([40]). Let E be a vector space. A map $q: E \longrightarrow \mathbb{R}^+$ is called a quasi-semi-norm if

(i) there exists $x \in E$ such that $q(x) \neq 0$,

(ii) $q(\lambda x) = |\lambda| q(x)$, for all $\lambda \in \mathbb{R}$, $x \in E$, and

(iii) there is a number $b \geq 1$ for which $q(x + y) \leq b(q(x) + q(y))$ for all $x, y \in E$.

The number b for which (iii) is satisfied is referred to as multiplier of q.

REMARKS: (a) If $q(x) = 0$ implies $x = 0$, then q is called a quasinorm.

(b) If $b = 1$, then q is a semi-norm.

(c) If $b = 1$ and $q(x) = 0$ implies $x = 0$, then q is a norm.

4.1.2 <u>Definition</u>: ([40]). Let E be a vector space. A map $p:E \longrightarrow \mathbb{R}^+$ is called a k-pseudometric $(0 < k \leq 1)$ if

(i) there is $x \in E$ such that $p(x) \neq 0$,

(ii) $p(\lambda x) = |\lambda|^k p(x)$ for all $x \in E$ and $\lambda \in \mathbb{R}$, and

(iii) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in E$.

<u>REMARKS</u>: (a_1) If $p(x) = 0$ implies $x = 0$, then p is called a k-norm.

(b_1) If $k = 1$, then p is a semi-norm.

(c_1) If $k = 1$ and $p(x) = 0$ implies $x = 0$, then p is a norm.

4.1.3 <u>THEOREM</u>: ([40], Theorem 2). If q is a quasi-semi-norm on a vector space E with multiplier b and $k < \log_{2b} 2$, then there exists a k-pseudometric on E equivalent to q.

4.1.4 <u>Definition</u>: ([16(a)]). A t.v.s. (E,u) is called a semiconvex space if there exists a neighbourhood basis of 0 consisting of semiconvex sets in E.

4.1.5 <u>THEOREM</u>: ([16(b)], [40]). A t.v.s. (E,u) is a semiconvex space iff there exists a family $\{p_\alpha; \alpha \in I\}$ of (u-continuous) k_α-pseudometrics $(0 < k_\alpha \leq 1)$ on E such that the family of sets $V_\alpha = \{x \in E; p_\alpha(x) < 1\}, \alpha \in I$, forms a neighbourhood basis of 0 in E.

4.2 Ordered semiconvex spaces

4.2.1 **Definition:** A (Hausdorff) semiconvex space E with an order determined by a positive cone C in E is called an ordered semiconvex space.

4.2.2 **EXAMPLES:** (i) Any ordered l.c.s. is an ordered semiconvex space.

(ii) Consider $L^p([a,b])$, $0 < p < 1$, $a,b \in \mathbb{R}$, with the order structure determined by the cone $K = \{f \in L^p([a,b]); f(t) \geq 0$ almost everywhere on $[a,b]\}$ and with the topology given by the neighbourhood basis $\{V_n\}_{n \geq 1}$ of 0, where

$$V_n = \{f \in L^p([a,b]); (\int_a^b |f|^p)^{1/p} \leq \frac{1}{n}\}, n \in \mathbb{N}.$$

Clearly V_n is circled, but not convex. However we show that V_n is semiconvex. Let $f,g \in V_n$. Then,

$$(\int_a^b |f|^p)^{1/p} \leq \frac{1}{n}, (\int_a^b |g|^p)^{1/p} \leq \frac{1}{n}.$$

But then,

$$(\int_a^b |f+g|^p)^{1/p} \leq 2^{\frac{1-p}{p}} [(\int_a^b |f|^p)^{1/p} + (\int_a^b |g|^p)^{1/p}]$$

([21], page 157). This implies that $V_n + V_n \subset \lambda V_n$ for some $\lambda > 0$. Thus $L^p([a,b])$, $0 < p < 1$, is an ordered semiconvex space which is not an ordered l.c.s..

(iii) Consider $\ell^p = \{x = \{x_i\}; x_i \in \mathbb{R}, \sum_{i=1}^{\infty} |x_i|^p < \infty\}$, $0 < p < 1$, with the order structure determined by the positive cone $K = \{x = (x_i) \in \ell^p; x_i \geq 0$ for all $i\}$, and with the topology given by the neighbourhood basis $(V_n)_{n \geq 1}$ of 0, where $V_n = \{x = (x_i) \in \ell^p; (\sum_{i=1}^{\infty} |x|^p)^{1/p} \leq \frac{1}{n}\}, n \in \mathbb{N}$. Clearly, V_n is circled but not convex. As in (ii) above, we can show that V_n is semiconvex. Thus, ℓ^p, $0 < p < 1$, is an ordered semiconvex space which is not ordered l.c.s..

Let (E,C) be an ordered vector space. If A is a semiconvex subset of E, so is its full hull $[A]$. Hence it follows that, in an ordered semiconvex space (E,C,u), if the positive cone C is normal, there exists a neighbourhood basis of 0 for u consisting of semiconvex, full sets.

4.2.3 THEOREM: Let (E,C,u) be an ordered semiconvex space. Then the following statements are equivalent:

(a) C is normal for u.

(b) There exists a family $\{p_\alpha; \alpha \in I\}$ of k_α-pseudometrics generating the topology u such that $0 \leq x \leq y$ implies $p_\alpha(x) \leq p_\alpha(y)$ for all $\alpha \in I$. Equivalently, $p_\alpha(z + w) \geq p_\alpha(z)$ for all $z, w \in C$, $\alpha \in I$.

PROOF: (a) $\Longrightarrow$ (b). In view of 0.4.4, there exists a basis $\eta = \{U_\alpha; \alpha \in I\}$ of circled, semiconvex neighbourhoods U_α of 0 for u such that $0 \leq x \leq y$, $y \in U_\alpha$, implies $x \in U_\alpha$, for each $\alpha \in I$. Define

$$q_\alpha(x) = \inf\{\mu > 0; x \in \mu U_\alpha\}, \text{ for each } x \in E.$$

It is well defined, because U_α is absorbing. Clearly $q_\alpha(\lambda x) = |\lambda| q_\alpha(x), \lambda \in \mathbb{R}$. Since $U_\alpha, \alpha \in I$, is semiconvex, $U_\alpha + U_\alpha \subset \nu_\alpha U_\alpha$ for some $\nu_\alpha > 0$. Let $\lambda_0 = q_\alpha(x)$,

$\mu_0 = q_\alpha(y)$. Then if $\lambda > \lambda_0$, & $\mu > \mu_0$, we have $\frac{x}{\lambda}$, $\frac{y}{\mu} \epsilon U_\alpha$, and since U_α is circled, it follows that

$$\frac{\lambda}{\lambda + \mu} \frac{x}{\lambda} \epsilon U_\alpha, \frac{\mu}{\lambda + \mu} \frac{y}{\mu} \epsilon U_\alpha .$$

Since $U_\alpha + U_\alpha \subseteq \nu_\alpha U_\alpha$ for some $\nu_\alpha > 0$, it follows that

$$\frac{\lambda}{\lambda + \mu} \frac{x}{\lambda} + \frac{\mu}{\lambda + \mu} \frac{y}{\mu} \epsilon \nu_\alpha U_\alpha \text{ , for some } \nu_\alpha > 0.$$

This gives

$$x + y \epsilon \nu_\alpha(\lambda + \mu)U_\alpha, \text{ for some } \nu_\alpha > 0.$$

and hence

$$q_\alpha(x + y) \leq \nu_\alpha(q_\alpha(x) + q_\alpha(y)) .$$

Thus q_α is a quasi-semi-norm. If $V_\alpha = \{x \epsilon E ; q_\alpha(x) \leq 1\}$, then we have

$$U_\alpha \subseteq V_\alpha \subseteq (1 + \epsilon)U_\alpha, \epsilon > 0 .$$

and so the q_α-topology coincides with u. Now, define $p_\alpha(x) = \inf\{\Sigma q_\alpha^{k_\alpha}(x_i);$ $\{x_i\}$ ranges over all finite subsets of E such that $\Sigma x_i = x\}$, where $k_\alpha < \log_{2\nu_\alpha} 2$. Then p_α is a k_α-pseudometric on E equivalent to q_α, by 4.1.3. Hence the p_α-topology coincides with q_α-topology and so with u. Finally, we show that $0 \leq x \leq y$ implies $p_\alpha(x) \leq p_\alpha(y)$ for each $\alpha \epsilon I$. We know that $0 \leq x_i \leq y_i$, for each i, and $y_i \epsilon U_\alpha$ implies $x_i \epsilon U_\alpha$. So, if $\lambda > 0$,

$$0 \leq \frac{x_i}{\lambda} \leq \frac{y_i}{\lambda} \text{ and } y_i \in \lambda U_\alpha, \text{ implies } x_i \in \lambda U_\alpha.$$

This shows that $q_\alpha(x_i) \leq q_\alpha(y_i)$ for each i. Hence $\inf\{\Sigma q_\alpha^{k_\alpha}(x_i); \{x_i\}$ is taken over all finite subsets of E such that $\Sigma x_i = x\} \leq \inf\{\Sigma q_\alpha^{k_\alpha}(y_i); \{y_i\}$ is taken over all finite subsets of E such that $\Sigma y_i = y\}$, whenever $0 \leq x \leq y$. This implies that $p_\alpha(x) \leq p_\alpha(y)$, for each $\alpha \in I$, whenever $0 \leq x \leq y$.

(b) $\Longrightarrow$ (a): In view of 4.1.5, the sets $V_\alpha = \{x \in E; p_\alpha(x) < 1\}$, $\alpha \in I$, form a neighbourhood basis of 0 for u. Now, if $y \in V_\alpha$, then $p_\alpha(y) < 1$. Since $p_\alpha(x) \leq p_\alpha(y)$ whenever $0 \leq x \leq y$, it follows that $p_\alpha(x) < 1$. This shows that $x \in V_\alpha$. Thus, $0 \leq x \leq y$ and $y \in V_\alpha$ implies $x \in V_\alpha$. Hence in view of 0.4.4, C is normal for u.

<u>REMARK:</u> It is more convenient to have a condition in terms of k-pseudometrics than in terms of quasi-semi-norms, because quasi-semi-norms can be discontinuous in the topology they generate, unlike k-pseudometrics ([40], page 172).

4.2.4 <u>COROLLARY:</u> If the positive cone C in an ordered semiconvex space (E,u) is normal, then so is its closure $\bar{C}$.

<u>PROOF:</u> Since C is the positive cone in (E,u), we have the following conditions:

(i) $C + C \subseteq C$,

(ii) $\lambda C \subseteq C$, $\lambda > 0$,

(iii) $C \cap (-C) = \{0\}$.

Now, since addition and scalar multiplication are continuous in (E,u), it follows from (i) and (ii) that $\bar{C} + \bar{C} \subseteq \bar{C}$ and $\lambda\bar{C} \subseteq \bar{C}$. Now we wish to show that $\bar{C} \cap (-\bar{C}) = \{0\}$. Let $x \in \bar{C} \cap (-\bar{C})$. Then, $x \in \bar{C}$ and $-x \in \bar{C}$. Hence there exist nets $\{x_n; n \in D\}$, $\{y_n; n \in D\}$ in C such that $x_n \longrightarrow x$ and $y_n \longrightarrow -x$. Since C is normal, there exists a family $\{p_\alpha; \alpha \in I\}$ of u-continuous k_α-pseudometrics generating the topology u such that $p_\alpha(x_n + y_n) \geq p_\alpha(x_n)$ for all $\alpha \in I$. Since each p_α is u-continuous, it follows that, for all $\alpha \in I$, $0 = p_\alpha(x - x) \geq p_\alpha(x) \geq 0$. Since u is Hausdorff, we have $x = 0$. Hence $\bar{C} \cap (-\bar{C}) = \{0\}$. Thus, $\bar{C}$ is a cone. Finally, we have to show that $\bar{C}$ is normal for u. Let $x, y \in \bar{C}$. Then there exist nets $\{z_n; n \in D\}$, $\{t_n; n \in D\}$ in C such that $z_n \longrightarrow x$, $t_n \longrightarrow y$. Since C is normal, $p_\alpha(z_n + t_n) \geq p_\alpha(z_n)$ for all $\alpha \in I$. But the continuity of p_α gives $p_\alpha(x + y) \geq p_\alpha(x)$ for all $\alpha \in I$. Hence, in view of 4.2.3, $\bar{C}$ is normal for u.

Let (E,C) and (F,K) be ordered vector spaces. Baker ([1]) calls a map $f: E \longrightarrow F$ sublinear if (a) $f(x + y) \leq f(x) + f(y)$ for all $x, y \in E$, and (b) $f(\lambda x) = \lambda f(x)$, for $\lambda > 0$, $x \in E$.

4.2.5 LEMMA: Let (E,C,u) and (F,K,v) be ordered semiconvex spaces and $f: E \longrightarrow F$ a sublinear map. Suppose that the cone K in F is normal. If f is continuous at 0, then it is continuous on E.

PROOF: Let $x \leq y \leq z$, $x, y, z \in F$. Then $0 \leq y - x \leq z - x$. Since F has normal cone K, there exists a family $\{p_\alpha; \alpha \in I\}$ of k_α-pseudometrics defining the topology u of E such that $p_\alpha(y - x) \leq p_\alpha(z - x)$ for all $\alpha \in I$.

Then

$$p_\alpha(y) \le p_\alpha(x) + p_\alpha(y - x) \le p_\alpha(x) + p_\alpha(z - x) \le 2p_\alpha(x) + p_\alpha(z).$$

This implies that

$$(*) \qquad p_\alpha(y) \le 3 \max(p_\alpha(x), p_\alpha(z)), \text{ for all } \alpha \in I.$$

Now, let $t, s \in E$. Then

$$f(t + s) \le f(t) + f(s)$$

and so,

$$-f(s) \le f(t) - f(t + s) .$$

Also,

$$f(t) \le f(t + s) + f(-s)$$

and so,

$$f(t) - f(t + s) \le f(-s) .$$

Thus, we have

$$-f(s) \le f(t) - f(t + s) \le f(-s) .$$

But then, in view of (*), we have

$$p_\alpha(f(t) - f(t + s)) \le 3 \max(p_\alpha(f(s)), p_\alpha(f(-s))) .$$

Since p_α and f are continuous at 0, there exists a circled neighbourhood V of 0 in E such that

$$p_\alpha(f(r)) < \varepsilon/3 \text{ for all } r \in V .$$

Hence,

$$p_\alpha(f(t) - f(t + s)) < \varepsilon \text{ for all } s \in V .$$

This shows that f is continuous at t. But $t\epsilon E$ is arbitrary, and hence f is continuous on E.

A k-pseudometric p on an ordered vector space E is said to be monotonic if $p(x) \leq p(y)$ whenever $0 \leq x \leq y$.

4.2.6 **PROPOSITION:** Let (E,C,u) be an ordered semiconvex space on which every monotonic k-pseudometric is continuous. Let (F,K,v) be an ordered semiconvex space with normal cone K. If $f:E \longrightarrow F$ is a sublinear map such that (a) $f(E)\subseteq K$, (b) $f(-x) = f(x)$ for all $x\epsilon E$, and (c) $f(x) \leq f(y)$ whenever $0 \leq x \leq y$, then f is continuous.

PROOF: Since the cone K in F is normal, there exists a family $\{q_\alpha;\alpha\epsilon I\}$ of (v-continuous) k_α-pseudometrics defining the topology v of F such that $q_\alpha(t) \leq q_\alpha(s)$ whenever $0 \leq t \leq s$, $t,s\epsilon F$. Define $p_\alpha = q_\alpha \circ f$. Since f is sublinear and $f(E)\subseteq K$, we have

$$0 \leq f(x + y) \leq f(x) + f(y) \text{ for all } x,y\epsilon E .$$

Now,

$$p_\alpha(x + y) = q_\alpha(f(x + y)) \leq q_\alpha(f(x) + f(y)) \leq q_\alpha(f(x)) + q_\alpha(f(y)) .$$

Thus,

$$p_\alpha(x + y) \leq p_\alpha(x) + p_\alpha(y) \text{ for all } x,y\epsilon E .$$

Next, since $f(-x) = f(x)$ for each $x\epsilon E$,

$$p_\alpha(\lambda x) = q_\alpha(f(\lambda x)) = q_\alpha(|\lambda| f(x)) \text{ for all } \lambda\epsilon \mathbb{R} .$$

Hence we have

$$p_\alpha(\lambda x) = |\lambda|^{k_\alpha} q_\alpha(f(x)) \qquad (0 < k_\alpha \leq 1)$$

$$= |\lambda|^{k_\alpha} p_\alpha(x) \qquad \text{for all } \lambda \in \mathbb{R}, x \in E .$$

Thus, we have established that p_α is a k_α-pseudometric on E. In view of the hypothesis (c) & (a),

$$q_\alpha(f(x)) \leq q_\alpha(f(y)) \quad \text{whenever } 0 \leq x \leq y .$$

Hence,

$$p_\alpha(x) \leq p_\alpha(y) \qquad \text{whenever } 0 \leq x \leq y .$$

This implies that p_α is a monotonic k_α-pseudometric on E, for each $\alpha \in I$, and hence is continuous in view of the hypothesis. But then f is continuous at 0, because $p_\alpha = q_\alpha \circ f$. Since the positive cone K in F is normal, it follows from 4.2.5 that f is continuous on E.

<u>REMARK:</u> The result 4.2.6 generalizes Theorem 5 of ([1]).

4.3 <u>The order bound sc-topology</u>

In this section, we introduce, what we call, the order bound sc-topology u_{sc} in ordered vector spaces and prove some interesting theorems about the continuity of sublinear maps on ordered vector spaces equipped with u_{sc}.

4.3.1 <u>Definition:</u> Let (E,C) be an ordered vector space. The order bound sc-topology u_{sc} on E is the finest semiconvex topology u on E for which every order-bounded subset of E is u-bounded.

4.3.2 <u>PROPOSITION:</u> A neighbourhood basis at 0 for the order bound sc-topology on E is the family η of all circled, semiconvex and order-bornivorous subsets of E.

<u>PROOF:</u> It can easily be verified using 0.1.3.

<u>REMARK:</u> Since we can write $\{x\} = [x,x]$, $x \in E$, every member of η is absorbing.

A map from an ordered vector space into an ordered vector space is called order-bounded if it maps order-bounded sets into order-bounded sets.

4.3.3 <u>THEOREM:</u> Let (E,C) be an ordered vector space equipped with the order bound sc-topology u_{sc}, and (F,K,v) any ordered semiconvex space with the normal cone K. Let $f: E \longrightarrow F$ be an order-bounded sublinear map such that

(i) $f(E) \subseteq K$, and

(ii) $f(-x) = f(x)$ for each $x \in E$; then f is continuous.

<u>PROOF:</u> Let $[x,y]$ be an order interval in E. Then there exist $t,s \in F$ such that

$$f([x,y]) \subset [t,s] .$$

because f is order-bounded. Suppose V is a circled, semiconvex, full neighbourhood of 0 in F. Then there exists $\lambda > 0$ such that

$$f([x,y]) \subset [t,s] \subset \lambda V ,$$

because the cone K in F is normal and hence every order-bounded set in F is v-bounded. This implies that

$$[x,y] \subset \lambda f^{-1}(V) \text{ for some } \lambda > 0 ,$$

Thus, $f^{-1}(V)$ is order-bornivorous. Clearly, $f^{-1}(V)$ is circled in view of (ii). To show that $f^{-1}(V)$ is semiconvex, let $x,y \in f^{-1}(V)$. Then,

$$f(x) + f(y) \in V + V \subset \mu V \text{ for some } \mu > 0 ,$$

because V is semiconvex. Since f is sublinear and $f(E) \subseteq K$, it follows that

$$0 \leq f(x + y) \leq f(x) + f(y) \in \mu V \text{ for some } \mu > 0 .$$

Since V is full, $f(x + y) \in \mu V$, and this shows that $f^{-1}(V) + f^{-1}(V) \subset \mu f^{-1}(V)$ for some $\mu > 0$. Thus, $f^{-1}(V)$ is a circled, semiconvex, order-bornivorous set in E which is equipped with the order bound sc-topology u_{sc}. Hence $f^{-1}(V)$ is a neighbourhood of 0 in E. Thus, f is continuous at 0. But then, in view of 4.2.5, f is continuous on E.

4.3.4 <u>COROLLARY</u>: Let (E,C) be an ordered vector space with the generating cone C, and equipped with the order bound sc-topology u_{sc}. Let (F,K,v) be an ordered semiconvex space with normal cone K. Let $f: E \longrightarrow F$ be a sublinear map such that

(i) $f(E)\subseteq K$,

(ii) $f(-x) = f(x)$ for each $x\in E$, and

(iii) $f(x) \leq f(y)$ whenever $0 \leq x \leq y$; then f is continuous.

PROOF: Since the cone C in E is generating, $\{[-x,x];x\in C\}$ is a fundamental system of order-bounded sets in E. We wish to show that f is order-bounded. For this, we show that

$$f([-x,x])\subset[0,f(x)] \text{ , for any } x\in C \text{ .}$$

Let $t\in f([-x,x])$. Then $t = f(y)$ for some y such that $-x \leq y \leq x$, i.e., $0 \leq y \leq x$ or $0 \leq -y \leq x$. In either case, we have $0 \leq f(y) \leq f(x)$, in view of (i), (ii) and (iii). Thus, $t\in[0,f(x)]$. Hence f is order-bounded. But then, it follows from 4.3.3 that f is continuous.

A map f from an ordered vector space (E,C) into an ordered vector space (F,K) is called monotonic increasing if $f(x) \leq f(y)$ whenever $x \leq y$, and $f(0) = 0$ ([1], page 232).

4.3.5 THEOREM: Let (E,C) be an ordered vector space with the generating cone C and equipped with the order bound sc-topology u_{sc}. Let (F,K,v) be an ordered semiconvex space with the normal cone K. If f is a monotonic increasing sublinear map from E into F, then f is continuous.

PROOF: Let V be a circled, semiconvex and full neighbourhood of 0 in F. Let $x,y\in f^{-1}(V)\cap C$; then, since f is monotonic increasing, we have $0 \leq f(x + y) \leq f(x) + f(y)\in V + V\subset \lambda V$ for some $\lambda > 0$, because V is semiconvex. But then, $f(x + y) \in \lambda V$, because V is full. Thus, $x + y \in \lambda(f^{-1}(V) \cap C)$ and this implies

$$f^{-1}(V) \cap C + f^{-1}(V) \cap C \subset \lambda(f^{-1}(V) \cap C) .$$

Hence, $f^{-1}(V) \cap C$ is semiconvex. Define

(*) $$U = f^{-1}(V) \cap C - f^{-1}(V) \cap C .$$

Clearly U is circled and semiconvex. We now wish to show that U is order-bornivorous. But, since the positive cone C in E is generating, it is enough to show that U absorbs $[0,x]$ for any $x \in C$ ([17], page 132. It has been proved for convex sets; but it can easily be extended to semiconvex sets). For this, we first show that

(**) $$f([0,x]) \subset [0,f(x)], x \in C .$$

Let $t \in f([0,x])$; then $t = f(y)$ for some $y \in C$ such that $0 \leq y \leq x$. Since f is monotonic increasing, we have $0 \leq f(y) \leq f(x)$, and hence $t \in [0,f(x)]$. This proves (**). But then, there exists $\mu > 0$ such that $f([0,x]) \subset [0,f(x)] \subset \mu V$, because the positive cone K in F is normal. Hence we have $[0,x] \subset \mu(f^{-1}(V) \cap C)$ for some $\mu > 0$. But, in view of (*), $f^{-1}(V) \cap C \subset U$ and hence $[0,x] \subset \mu U$, for some $\mu > 0$. Thus, U is a circled, semiconvex and order-bornivorous set in (E,u_{sc}) and hence is a neighbourhood of 0 in E. We now claim that

(***) $$U \subseteq f^{-1}(V) .$$

To prove this claim, let $x = x_1 - x_2 \in U$, where $x_1, x_2 \in f^{-1}(V) \cap C$; then $0 \leq f(x_1) \leq f(x_1 - x_2) + f(x_2)$ and from this we have

$$-f(x_2) \leq f(x_1) - f(x_2) \leq f(x_1 - x_2) \leq f(x_1) .$$

because f is monotonic increasing. Since V is circled and $f(x_2) \in V$, it follows that $-f(x_2) \in V$; also $f(x_1) \in V$. But then $f(x_1 - x_2) \in V$, because V is full. This shows that $x = x_1 - x_2 \in f^{-1}(V)$ which proves (***). It is now clear from $U \subseteq f^{-1}(V)$ that $f^{-1}(V)$ is a neighbourhood of 0 in E and hence that f is continuous at 0. In view of 4.2.5, it now follows that f is continuous on E.

4.3.6 <u>COROLLARY</u>: Let (E,C) be an ordered vector space with the generating cone C and equipped with the order bound topology u_b. Let (F,K,v) be an ordered l.c.s. with normal cone K. If f is a monotonic increasing sublinear map from E into F, then f is continuous.

Although the proof of the following proposition is similar to that of ([29], 5.4), we give it in detail for the sake of completeness. We use it to obtain a corollary of 4.3.5.

4.3.7 <u>PROPOSITION</u>: Let (E,C,u) be an ordered t.v.s. which is metrizable and of the second category. Suppose that the positive cone C in E is complete and generating. If u_{sc} is the order bound sc-topology on E, then $u_{sc} \subset u$.

<u>PROOF</u>: Let V be a circled, semiconvex, order-bornivorous subset of E. We claim that V is a u-neighbourhood of 0 in E. Suppose not; then for each u-neighbourhood U of 0 in E and for each integer $n > 0$, we have $U \cap C \not\subset nV$. For, if $U \cap C \subset nV$, then $U \cap C - U \cap C \subset nV - nV \subset \mu nV$

for some $\mu > 0$ because V is circled and semiconvex. Hence, by 5.3 in ([29]), μV and so V is an u-neighbourhood of 0 which contradicts our assumption that V is not an u-neighbourhood of 0. Now, let $\{V_n\}_{n \geq 1}$ be a neighbourhood basis of 0 for (E,u) such that $V_{n+1} + V_{n+1} \subset V_n$ for $n = 1,2,\cdots$. Then we can choose a sequence $\{x_n\}$ in such a way that $x_n \in C \cap V_n \sim nV$. Because the positive cone C in E is u-complete,

$$u\text{-}\lim_{n\to\infty} \sum_{m=1}^{n} x_m \text{ exists in } C.$$

We then let

$$x_0 = u - \lim_n \sum_{m=1}^{n} x_m .$$

Since C is u-closed in E, $x_n \in [0,x_0]$ for each integer $n > 0$. Hence by the assumption, the sequence $\{x_n\}$ is absorbed by V, i.e., $\{x_n\} \subset mV$ for some positive integer m. In particular $x_m \in mV$; but this contradicts our choice of x_m. Hence V must be a u-neighbourhood of 0 in E and this proves that $u_{sc} \subset u$.

4.3.8 <u>COROLLARY</u>: Let (E,C,u) be an ordered t.v.s. which is metrizable and of the second category with complete generating cone C. Let (F,v) be an ordered semiconvex space with normal cone K. If $f: E \longrightarrow F$ is a monotonic increasing sublinear map, then f is continuous on E.

<u>PROOF</u>: By 4.3.7, $u_{sc} \subset u$, where u_{sc} is the order bound sc-topology on E. But then, by 4.3.5, f is continuous on E.

4.3.9 <u>PROPOSITION</u>: Let (E,C) be an ordered vector space equipped with the order bound sc-topology u_{sc} and (F,K,v) any ordered semiconvex space. If f is a positive, linear, open map from E into F, then v is the order bound sc-topology on F.

PROOF: Let V be a circled, semiconvex, order-bornivorous subset of F. Clearly $f^{-1}(V)$ is circled and semiconvex. Since f is positive and linear

$$f([x,y])\subset[f(x),f(y)],$$

for any order interval $[x,y]$ in E. Hence there exists $\lambda > 0$ such that $f([x,y])\subset[f(x),f(y)]\subset\lambda V$, because V is order-bornivorous. Thus, $[x,y]\subset\lambda f^{-1}(V)$ for some $\lambda > 0$, and hence $f^{-1}(V)$ is a circled, semiconvex, order-bornivorous set in (E,u_{sc}) which implies that $f^{-1}(V)$ is a neighbourhood of 0 in E. Since f is open, $f(f^{-1}(V))$ is a neighbourhood of 0 in F. But $f(f^{-1}(V))\subset V$, and hence V is a neighbourhood of 0 in F. This shows that v is the order bound sc-topology on F.

A semiconvex space (E,u) is called a

(i) hyperbornological space if every circled, semiconvex, bornivorous subset of E is a neighbourhood of 0 in E,

(ii) hyperbarrelled space if every closed, circled, semiconvex, absorbing set in E is a neighbourhood of 0 in E.

4.3.10 PROPOSITION: (E,u_{sc}) is a hyperbornological space.

PROOF: Let V be a circled, semiconvex, bornivorous subset of (E,u_{sc}). Since V is bornivorous, it absorbs every u_{sc}-bounded subset of E. But, by the definition of u_{sc}, every order-bounded subset of E is u_{sc}-bounded; a fortiori, V is order-bornivorous. Hence V is a neighbourhood of 0 in (E,u_{sc}), by 4.3.2. This shows that (E,u_{sc}) is a hyperbornological space.

The following lemma generalizes Lemma 6.5.2 in ([6]); we need it in 4.3.12.

4.3.11 LEMMA: Let (E,u) be a t.v.s.; let A be an u-bounded, circled, semiconvex, sequentially complete and sequentially closed subset of E. Suppose that E_A is the linear hull of A. Then there exists a k-pseudometric p_A on E_A for which E_A is complete and the topology on E_A generated by p_A is finer than that induced by u.

PROOF: Define

$$q_A(x) = \inf\{\mu > 0; x \in \mu A\} \text{ , for each } x \in E_A .$$

Then, as has been shown in the proof of 4.2.3, q_A is a quasisemi-norm on E_A with multiplier, say ν. The sets $\lambda A (\lambda > 0)$ form a neighbourhood basis of 0 for the topology on E_A defined by q_A. Since A is u-bounded, the q_A-topology on E_A is finer than that induced by u. To prove the q_A-completeness of E_A, consider a q_A-Cauchy sequence (x_n) in E_A. It follows that

$$\sup_n q_A(x_n) = M < \infty ,$$

so that $x_n \in MA$. Since MA, like A, is sequentially complete in E, and since the topology on E induces a topology on E_A which is coarser than q_A-topology, it follows that there exists $x \in MA$ such that $x_n \longrightarrow x$ in the sense of E. Now we know that

$$q_A(x_m - x_n) = \varepsilon_{mn} \longrightarrow 0 \text{ as } m,n \longrightarrow \infty .$$

So, $x_m - x_n \in \varepsilon_{mn} A$. Letting $m \longrightarrow \infty$, and using the fact that A is sequentially closed in E, we see that $x - x_n \in \varepsilon_n A$, where

$$\varepsilon_n = \limsup_m \varepsilon_{mn} \longrightarrow 0 \text{ as } n \longrightarrow \infty .$$

So, $x \in E_A$ and $q_A(x - x_n) \longrightarrow 0$; i.e., $x_n \longrightarrow x$ in E_A. Now, as in 4.2.3, we define $p_A(x) = \inf(\sum_{i=1}^{n} q_A^k(x_i); \sum_{i=1}^{n} x_i = x$, where n is finite and not fixed), $k < \log_2 2\nu$; then p_A is a k-pseudometric equivalent to q_A.

REMARK: In [16(a)] it has been observed that a semiconvex ultrabarrelled space is hyperbarrelled, and it follows from Theorem 6.2.2 of ([6]) that E_A is semiconvex ultrabarrelled for the p_A-topology. Hence, E_A is a hyperbarrelled space for the topology generated by the k-pseudometric p_A.

4.3.12 THEOREM: Let (E,C;u) be a t.v.s. ordered by a normal cone C, and suppose there is a fundamental system G_0 of order-bounded sets such that each member of G_0 is circled, semiconvex, sequentially complete (and sequentially closed). Then (E,u_{sc}) is a hyperbarrelled space.

PROOF: Let $A \in G_0$; then A is u-bounded, because the cone C in E is normal. Thus A is u-bounded, circled, semiconvex, sequentially complete (and sequentially closed). Hence, if E_A is the linear hull of A, then by 4.3.11 there exists a k-pseudometric p_A on E_A for which E_A is complete. But then, by the above Remark, E_A is a hyperbarrelled space for the p_A-topology. Since every u_{sc}-neighbourhood of 0 in E absorbs A, the p_A-topology on E_A is finer than that induced by u_{sc}. Hence if V is a u_{sc}-closed, circled, semiconvex, absorbing subset of (E,u_{sc}), then $V \cap E_A$ is p_A-closed, circled, semiconvex, absorbing subset of E_A which is a hyper-barrelled space, and hence $V \cap E_A$ is a p_A-neighbourhood of 0 in E_A. Since the sets $\lambda A(\lambda > 0)$ form a basis of p_A-neighbourhoods of 0 in E_A, $\lambda A \subset V \cap E_A$ for some $\lambda > 0$. This implies that $V \cap E_A$ and hence V is order-bornivorous. Thus V is a circled, semiconvex, order-bornivorous set in (E,u_{sc}) and hence is a neighbourhood of 0 in E. This shows that (E,u_{sc}) is a hyperbarrelled space.

CHAPTER V

A CLOSED GRAPH TYPE THEOREM FOR ORDERED TOPOLOGICAL ABELIAN GROUPS

Husain ([9]) has obtained various types of closed graph theorems for topological groups. Motivated by his work, we prove, in this chapter, a kind of closed graph theorem for ordered topological abelian groups.

5.1 Subhomomorphisms and almost continuity

5.1.1 **Definition:** Let (E,C) and (F,K) be ordered abelian groups. A map $f:E \longrightarrow F$ is said to be a subhomomorphism if $f(x+y) \leq f(x) + f(y)$, for all $x,y \in E$.

5.1.2 **LEMMA:** Let (E,C,u) and (F,K,v) be ordered topological abelian groups. Suppose that K is self-allied in F. If a subhomomorphism $f:E \longrightarrow F$ is continuous at 0, then it is continuous on E.

PROOF: Let P be an open set in F such that $y = f(x) \in P, x \in E$. Then $P - y$ is a neighbourhood of 0 in F. There exists a symmetric full neighbourhood V of 0 in F such that $V \subset P - y$. Since f is continuous at 0, there exists a symmetric neighbourhood W of 0 in E such that $f(W) \subset V \subset P - y$. Since $x + W$ is a neighbourhood of x in E, in order to show that f is continuous on E, we need to show that $f(x + W) \subset P$. Let $t \in f(x + W)$; then $t = f(x + w)$, $w \in W$. Now,

$$y = f(x) \leq f(x + w) + f(-w)$$

and from this we have

$$-f(-w) \leq f(x + w) - f(x) \leq f(w) .$$

Since W is symmetric and $w \in W$, it follows that $-w \in W$ and so $f(-w) \in f(W) \subset V$. Since V is symmetric and full, we get

$$f(x + w) - f(x) \in V, \text{ or } f(x + w) \in y + V \subset P$$

for all $w \in W$. Hence $f(x + W) \subset P$, and the proof is complete.

5.1.3 <u>PROPOSITION</u>: Let (E,C,u) and (F,K,v) be ordered topological abelian groups; suppose that K is self-allied in F, and that E is a Baire group. Let $f:E \longrightarrow F$ be a sub-homomorphism such that $f(-x) = f(x), x \in E$. If (a) F is Lindelöf, or (b) F is separable, then f is almost continuous at 0.

<u>PROOF</u>: (a) Suppose F is Lindelöf. Let V be a neighbourhood of 0 in F. Choose a full, symmetric neighbourhood U of 0 such that $U + U \subset V$. Then,

$$f(E) \subset \overline{f(E)} \subset U + f(E)$$

and only a countable subfamily, say $\{U + f(x_n)\}$ covers f(E), because $\overline{f(E)}$, being a closed subspace of a Lindelöf space, is Lindelöf. Since $\bigcup f^{-1}(\hat{U} + f(x_n)) = E$, since the sets of $\{U + f(x_n)\}$ are all homeomorphic nal and since E is a Baire space, it follows that $\overline{f^{-1}(U)}$ has an interior point, say x_0. We now wish to show that

$$f^{-1}(U) + f^{-1}(U) \subset f^{-1}(V) .$$

Let $x, y \in f^{-1}(U)$, so that $f(x), f(y) \in U$. Now $t = f(x) \in f(x + y) + f(-y)$ and so we have $-f(y) \in f(x + y) - f(x) \in f(y) \in U$. Since U is symmetric and full, it follows that

$$f(x + y) \in t + U \subset U + U \subset V .$$

This implies that $x + y \in f^{-1}(V)$ and hence

$$f^{-1}(U) + f^{-1}(U) \subset f^{-1}(V) .$$

Since $f(-x) = f(x)$, it follows that $f^{-1}(U)$ is symmetric and so $\overline{f^{-1}(U)}$ is symmetric, because inverse operation in a topological group is continuous. Hence,

$$0 = x_0 - x_0 \in \overline{f^{-1}(U)} + \overline{f^{-1}(U)} \subset \overline{f^{-1}(U) + f^{-1}(U)} \subset \overline{f^{-1}(V)}$$

and this shows that 0 is an interior point of $\overline{f^{-1}(V)}$. Thus $\overline{f^{-1}(V)}$ is a neighbourhood of 0 in E and hence f is almost continuous at 0.

(b) Suppose F is separable. Let $\{x_n; n = 1, 2, \cdots\}$ be a countable dense subset of F. Let V be a neighbourhood of 0 in F. There exists a full, symmetric neighbourhood U of 0 such that $U + U \subset V$. Clearly, $F = \bigcup_{n \geq 1} (x_n + U)$ which implies that $E = \bigcup_{n \geq 1} f^{-1}(x_n + U)$. Since the sets of $\{x_n + U\}$ are all homeomorphic and since E is a Baire space, it follows that $\overline{f^{-1}(U)}$ has an interior point. It now follows, as in (a), that f is almost continuous at 0.

5.1.4 COROLLARY: 5.1.3 is true for the following pairs E and F of ordered topological abelian groups:

(a) E is a Baire space and F is compact.

(b) E is locally compact or complete metrizable and F is compact.

(c) E and F are both compact.

PROOF: Observe that every compact space is Lindelöf as well as a Baire space and every locally compact or complete metrizable space is a Baire space. Hence 5.1.3 applies.

5.2 Closed graph theorems

Before we prove the two main theorems of this chapter, we recall a theorem from ([1], page 238).

5.2.1 THEOREM: ([1], Theorem 12). Let E and F be topological groups such that F is complete and metrizable. Let G be a closed positive set (subsemigroup) in ExF. If for each neighbourhood V of 0 in F, the closure of the set $G^{-1}[V] = \{x\in E; (x,y)\in G \text{ for some } y\in V\}$ is a neighbourhood of 0 in E, then $G^{-1}[V]$ is itself a neighbourhood of 0 in E.

5.2.2 THEOREM: Let (E,C,u) and (F,K,v) be ordered topological abelian groups with F complete metrizable and K self-allied in F. Let $f:E \longrightarrow F$ be an almost continuous subhomomorphism such that $f(E)\subseteq K$ and $f(-x) = f(x)$ for all $x\in E$. If $G = \{(x,y)\in ExF; f(x) \leq y\}$ is closed, then f is continuous.

PROOF: Let V be a neighbourhood of 0 in F. Then, $\overline{f^{-1}(V)}$ is a neighbourhood of 0 in E, because f is almost continuous. Since $\overline{G^{-1}[V]}$ contains $\overline{f^{-1}(V)}$ ([1], page 239), it follows that $\overline{G^{-1}[V]}$ is a neighbourhood of 0 in E. But then, in view of 5.2.1, $G^{-1}[V]$ is itself a neighbourhood of 0 in E. Since K is self-allied, there exists a neighbourhood U of 0 in F such that $(V - K) \cap K \subseteq U$, in view of 0.5.4, and so $f^{-1}(V - K) \cap f^{-1}(K) \subseteq f^{-1}(U)$. But $f(E) \subseteq K$ and hence $f^{-1}(V - K) \subseteq f^{-1}(U)$. Now, since $G^{-1}[V] = f^{-1}(V - K)$ is a neighbourhood of 0 in E, it follows that $f^{-1}(U)$ is a neighbourhood of 0 in E. This shows that f is continuous at 0. It now follows from 5.1.2 that f is continuous on E.

5.2.3 **THEOREM:** Let (E,C,u) and (F,K,v) be ordered topological abelian groups with E a Baire space, F Lindelöf (or separable) complete metrizable and K self-allied in F. Let $f: E \longrightarrow F$ be a subhomomorphism such that $f(E) \subseteq K$ and $f(-x) = f(x)$ for all $x \in E$. If $G = \{(x,y) \in E \times F; f(x) \leq y\}$ is closed, then f is continuous.

PROOF: In view of 5.1.3, f is almost continuous. But then, by 5.2.2, it follows that f is continuous.

5.2.4 **COROLLARY:** 5.2.3 is true for the following pairs (E,C,u) and (F,K,v) of ordered topological abelian groups:

(a) E is a Baire space and F locally compact space satisfying the second axiom of countability.

(b) E is locally compact or complete metrizable and F as in (a).

(c) E compact and F as in (a).

(d) E a Baire space or locally compact or complete metrizable or compact and F locally compact regular semitopological group satisfying the second axiom of countability.

PROOF: (a) F is complete metrizable by ([9], Chapter IV, §26, Corollary 2), and since it satisfies the second axiom of countability, it is separable. Hence the result follows from 5.2.3.

(b) A locally compact or complete metrizable space is a Baire space and hence the result follows from (a).

(c) A compact space is a Baire space and hence the result follows from (a).

(d) Since F is a topological group by ([9], Chapter II, §17, Corollary 5), the result follows from (a).

REMARK: A topological space E which is also a group is a semi-topological group if the map

$$f:(x,y) \longrightarrow xy$$

of ExE onto E is continuous in each variable separately ([9], page 27).

LIST OF ABBREVIATIONS

t.v.s.	topological vector space
t.v. spaces	topological vector spaces
l.c.s.	locally convex space
l.c. spaces	locally convex spaces
t.v.l.	topological vector lattice
t.v. lattices	topological vector lattices
l.c.v.l.	locally convex vector lattice
l.c.v. lattices	locally convex vector lattices
C.O.Q.	countably order-quasibarrelled
O.Q.U.	order-quasiultrabarrelled

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