THE LICHTENBAUM CONJECTURE AT THE PRIME 2

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Contents

Abstract

The Analytical Class Number Formula, a classical result of Dirichlet, asserts that two important transcendental invariants associated to a number field F, namely the first non-zero Taylor coefficient of the Dedekind zeta-function, ζ_F, at 0 and the regulator of the group of units of F differ by a rational number. Moreover, this rational number is the quotient of two algebraic invariants of F, namely the ideal class number and the order of the group of the roots of unity in F. The Lichtenbaum Conjecture attempts to exhibit the same type of relation between the first non-zero Taylor coefficient of ζ_F at $1-m$ for $m \geq 2$ and the Borel regulator in K-theory. They differ by the quotient of the orders of the torsion parts of consecutive higher K-groups (the even K-groups appear as generalizing the ideal class group, while the odd ones appear as generalizing the group of units).

The study of this conjecture is done at each prime p using p-adic Chern characters from K-theory to étale cohomology and an interplay between étale cohomology duality results and Iwasawa theory results. Using a different regulator the Lichtenbaum Conjecture has been proved at all odd primes for all abelian number fields by Kolster, Nguyen Quang Do and Fleckinger. We develop similar methods and succeed to obtain a description of the 2-powers appearing in the formula for the case when m is odd. We also note that a motivic context is possible for the formulation of the Lichtenbaum Conjecture.

Chapter 1

Introduction

The **Riemann zeta-function** has been in the attention of mathematicians for a long time. In the beginning it was cleverly manipulated as a formal infinite series by Euler. With the work of Riemann many of its properties were revealed. It is defined by

$$\zeta(s) := \sum_{n\geq 1} \frac{1}{n^s}$$

for all $s \in \mathbb{C}$, $Re\,(s) > 1$. It has a meromorphic continuation to the whole complex plane $\mathbb{C}$ with a unique simple pole at $s = 1$ and residue

$$\lim_{s\to 1}(s-1)\zeta(s) = 1.$$

Moreover, there is a functional equation relating its values at s and $1-s$:

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

(where $\Gamma(s)$ is the Γ-function, a non-zero meromorphic function with simple poles at all negative integers). The values of the zeta-function at even positive integers can be given in terms of Bernoulli numbers (Euler), the coefficients of the power series of $x/(e^x-1)$ around 0. Using the functional equation we obtain the values of zeta-function at odd negative integers in terms of Bernoulli numbers too. The values at odd positive integers are more mysterious. Regarding their rational/irrational nature, few facts are known: for example, we know that $\zeta(3)$ is irrational (Apéry) and that there are infinitely many irrationals in the set $\{\zeta(2n+1) \mid n \geq 1\}$. But one has that:

$$\zeta(1+2n) = (-1)^n \cdot \frac{2^{2n+1}}{(2n)!} \cdot \pi^{2n} \cdot \zeta^*(-2n)$$

for $n \geq 1$, where $\zeta^*(-2n)$ denotes the first non-vanishing coefficient in the Taylor expansion around $-2n$, called the **special value** of the zeta-function at this integer. This is a consequence of the functional equation, and of the fact that the Γ-function has a simple pole at $-n$ which is compensated by the simple zero of the zeta-function at $-2n$.

This points out the need to study the special values of the zeta-function at non-positive integers.

We can see easily that $\zeta(0) = -\frac{1}{2}$. But the algebraic interpretation of such a computation can be grasped only if we work over general algebraic number fields (finite field extensions of $\mathbb{Q}$). Let F be an algebraic number field having r_1 real embeddings and r_2 pairs of complex conjugate embeddings. Let $\mathcal{O}_F$ denote the ring of integers in F (the ring of the roots of monic polynomials with integer coefficients which belong to F), and for any non-trivial ideal I of $\mathcal{O}_F$ define $N(I) := |\mathcal{O}_F/I|$. The **Dedekind zeta-function** of F is then defined by:

$$\zeta_F(s) := \sum_I \frac{1}{N(I)^s},$$

with I running over all non-trivial ideals of $\mathcal{O}_F$. This is again convergent for $Re\,(s) > 1$, can be extended to a meromorphic function on $\mathbb{C}$, and satisfies a functional equation relating $\zeta_F(s)$ and $\zeta_F(1-s)$. It also has a simple pole at $s = 1$ that corresponds, this time, to a zero of order $r_1 + r_2 - 1$ at $s = 0$. The special value $\zeta_F^*(0)$ is given by:

Theorem(Analytic Class Number Formula of Dirichlet) 1.1

$$\zeta_F^*(0) = -\frac{h_F}{w_F} \cdot R_F,$$

where h_F is the class number of F (defined as the order of the class group of F, i.e. the group of fractional ideals of $\mathcal{O}_F$ modulo principal ideals), $w_F := |\mu(F)|$ (the *number of roots of unity of F), and R_F is the Dirichlet regulator (the covolume of the image lattice of the logarithmic embedding of $\mathcal{O}_F^\times/\mu(F)$ into the real vector space $\mathbb{R}^{r_1+r_2-1}$).*

Now the special value $\zeta(0) = -\frac{1}{2}$ tells us that the class group of $\mathbb{Z}$ is trivial, that $\mathbb{Z}$ contains only 2 roots of unity (1 and -1), and that the regulator of $\mathbb{Q}$ is trivial.

We consider from now on the special values at $1-n$ for $n \geq 2$. The order of vanishing d_n of $\zeta_F(s)$ at $s = 1-n$, determined using the functional equation, is:

$$d_n = \begin{cases} r_1 + r_2 & \text{if } n \geq 3 \text{ is odd} \\ r_2 & \text{if } n \geq 2 \text{ is even.} \end{cases}$$

Once the K-groups (see chapter 3) were introduced it became clear that one could try to interpret algebraically the special values of zeta-functions at $1-n$ for $n \geq 2$. Firstly, it is known that

$$K_0(\mathcal{O}_F) \cong \mathbb{Z} \oplus Cl(F), \; K_1(\mathcal{O}_F) \cong \mathcal{O}_F^\times.$$

Hence Dirichlet's result can be rewritten as follows:

$$\zeta_F^*(0) = -\frac{|\mathrm{tors}_{\mathbb{Z}} K_0(\mathcal{O}_F)|}{|\mathrm{tors}_{\mathbb{Z}} K_1(\mathcal{O}_F)|} \cdot R_F.$$

Moreover, by defining higher regulator maps

$$\rho_n(F) : K_{2n-1}(\mathcal{O}_F) \longrightarrow \mathbb{R}^{d_n},$$

and proving that their kernels are finite, and that their images are lattices of rank d_n, Borel obtained that the rank of $K_{2n-1}(\mathcal{O}_F)$ is exactly the order of vanishing of ζ_F at $1-n$. In addition, one obtains

$$\zeta_F^*(1-n) = q_n \cdot R_n(F),$$

where $R_n(F)$ is the covolume of the image lattice of $\rho_n(F)$, called the **Borel regulator** (see Chapter 4), and q_n is a non-zero rational number. Also, Borel proved that

$K_{2n}(\mathcal{O}_F)$ is finite, completing the study of the abstract structure of Quillen's K-groups.

Another indication of a possible generalization of Dirichlet's result comes from the computation of the K-theory of finite fields done by Quillen, which can be formulated in terms of Galois cohomology too:

$$K_{2n}(\mathbb{F}_q) \cong H^2(\mathbb{F}_q, \mathbb{Z}(n)) = 0, \text{and}$$

$$K_{2n-1}(\mathbb{F}_q) \cong H^1(\mathbb{F}_q, \mathbb{Z}(n)) \cong H^0(\mathbb{F}_q, \mathbb{Q}/\mathbb{Z}(n)) = \mathbb{Z}/(q^n - 1)\mathbb{Z}.$$

On the other hand, the zeta-function associated to the finite field $\mathbb{F}_q$ is

$$\zeta_{\mathbb{F}_q}(s) = \frac{1}{1 - q^{-s}}.$$

Thus we obtain for $n \geq 1$ the following:

$$\zeta_{\mathbb{F}_q}(-n) = -\frac{|K_{2n}(\mathbb{F}_q)|}{|K_{2n-1}(\mathbb{F}_q)|} = -\frac{|H^2(\mathbb{F}_q, \mathbb{Z}(n))|}{|H^1(\mathbb{F}_q, \mathbb{Z}(n))|}.$$

Influenced by these results, Lichtenbaum (1971) came up with a conjectural generalization of the class number formula:

Lichtenbaum Conjecture 1.2 *For all $n \geq 2$ and all number fields F we have:*

$$\zeta_F^*(1-n) = \pm 2^t \cdot \frac{|K_{2n-2}(\mathcal{O}_F)|}{|\mathrm{tors}_{\mathbb{Z}} K_{2n-1}(\mathcal{O}_F)|} \cdot R_F$$

for some $t \in \mathbb{Z}$.

A special case was already formulated at that time:

Birch-Tate Conjecture 1.3 *For a totally real field F we have*

$$\zeta_F(-1) = \pm \frac{K_2(\mathcal{O}_F)}{w_2(F)}.$$

Here for any $n \geq 1$ we denote by $w_n(F)$ the largest m such that the diagonal action of $\mathrm{Gal}(\bar{F}/F)$ on $\mu_m^{\otimes n}$ is trivial, i.e.

$$w_n(F) = |H^0(F, \mathbb{Q}/\mathbb{Z}(n))|.$$

The motivation for the Birch-Tate conjecture was Tate's proof of an analogous formula in the function field case. The short account which follows is due to Kolster [44]. Let X be a smooth projective connected curve over a finite field $\mathbb{F}_q$ of characteristic p. Denote by N_r the number of rational points of $\bar{X} := X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ over $\bar{\mathbb{F}}_{q^r}$. We associate to X the zeta-function

$$Z_X(T) := \exp\Big(\sum_{r=1}^{\infty} N_r \cdot \frac{T^r}{r}\Big)$$

viewed as a formal power series in $\mathbb{Q}[[T]]$, and define

$$\zeta_X(s) := Z_X(q^{-s})$$

for all $s \in \mathbb{C}$. Weil proved that

$$Z_X(T) = \frac{P_1(T)}{(1-T)(1-qT)}$$

where $P_1(T) := \prod_{i=1}^{2g}(1-\alpha_i T)$, g being the genus of X and the α_i are the eigenvalues of the Frobenius automorphism acting on the Jacobian variety J of X, all of them having absolute value $q^{1/2}$. Denote by F the function field of X. Let $\mathcal{O}_F$ be the integral closure of $\mathbb{F}_q[T]$ in F. It is known that

$$\zeta_X(s) = \prod_{\mathfrak{p}} \frac{1}{1-N(\mathfrak{p})^{-s}},$$

where $\mathfrak{p}$ runs over all primes in F. To obtain an analogue of the Dedekind zeta-function we also consider the product over only the finite primes:

$$\zeta_{\mathcal{O}_F}(s) := \prod_{\mathfrak{p}\text{ finite}} \frac{1}{1-N(\mathfrak{p})^{-s}}.$$

This differs from the previous one by the product of the Euler factors at the infinite places:

$$\zeta_{\mathcal{O}_F}(s) = \zeta_X(s) \cdot \prod_{v|\infty}(1-N(v)^{-s}).$$

The main tool to be used is étale cohomology, which was invented by Grothendieck and other authors to prove the Weil Conjectures for smooth projective varieties over

$\mathbb{F}_q$. It appears also as a generalization of Galois cohomology, and we will view $H^*_{ét}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))$ for a fixed prime $l \neq p$ as the Galois cohomology of the Galois group of the maximal unramified extension of F. To simplify the presentation we will assume that $\mu_l \subset \mathbb{F}_q$, and let $F_\infty := F(\mu_{l^\infty})$. This way we obtain a $\mathbb{Z}_l$-extension F_∞/F, which means that $\Gamma := \mathrm{Gal}(F_\infty/F)$ is isomorphic to $\mathbb{Z}_l$, and is generated by the Frobenius automorphism γ. Let G_∞ be the Galois group of the maximal abelian unramified pro-l-extension of F_∞. It has a Λ-module structure, where $\Lambda := \mathbb{Z}_l[[T]]$ is the Iwasawa algebra. Using the Hochschild-Serre spectral sequence and Kummer duality we obtain:

$$H^1_{ét}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \cong H^1_{ét}(\bar{X}, \mathbb{Q}_l/\mathbb{Z}_l(n))^\Gamma \cong \mathrm{Hom}(G_\infty, \mathbb{Q}_l/\mathbb{Z}_l(n))^\Gamma \cong J_l(n-1)^\Gamma,$$

where J_l is the l-primary part of the Jacobian of X. But the order of $J_l(n-1)^\Gamma$ is equal to the order of the kernel of $1-q^{n-1}\gamma$ acting on J_l, that is the l-part of $P_1(q^{1-n})$. So:

$$\zeta_X(1-n) \sim_l \frac{|H^1_{ét}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))|}{(1-q^{n-1})(1-q^n)},$$

with $\sim_l$ denoting the equality of the l-adic valuation of the two terms. Now, using the computations

$$|H^0_{ét}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))| = q^n - 1, \ |H^2_{ét}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))| = q^{n-1} - 1,$$

and the isomorphisms

$$H^i_{ét}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \cong H^{i+1}_{ét}(X, \mathbb{Z}_l(n))$$

for all $i \geq 0$, we obtain:

Theorem 1.4 *Let X be a smooth projective connected curve over $\mathbb{F}_q$,* $\mathrm{char}(\mathbb{F}_q) = p$. *Then for all $n \geq 2$, and all primes $l \neq p$ we have:*

$$\zeta_X(1-n) \sim_l \pm \frac{|H^2_{ét}(X, \mathbb{Z}_l(n))|}{|H^1_{ét}(X, \mathbb{Z}_l(n))| \cdot |H^3_{ét}(X, \mathbb{Z}_l(n))|}.$$

Finally, using the exact localization sequence in étale cohomology

$$0 \longrightarrow H^1_{\text{ét}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \longrightarrow H^1_{\text{ét}}(\mathcal{O}_F, \mathbb{Q}_l/\mathbb{Z}_l(n)) \longrightarrow \oplus_{v|\infty} H^0(F_v, \mathbb{Q}_l/\mathbb{Z}_l(n-1))$$
$$\longrightarrow H^2_{\text{ét}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \longrightarrow 0,$$

and the computations

$$H^2_{\text{ét}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) = 0, \ |H^0_{\text{ét}}(F_v, \mathbb{Q}_l/\mathbb{Z}_l(n-1))| = N(v)^{n-1} - 1,$$

we obtain:

Theorem 1.5 *Let F be a global field of characteristic $p > 0$. For all $n \geq 2$, and all primes $l \neq p$, we have:*

$$\zeta_{\mathcal{O}_F}(1-n) \sim_l \pm \frac{|H^2_{\text{ét}}(\mathcal{O}_F, \mathbb{Z}_l(n))|}{w_n(F)}.$$

This represents a cohomological analog of the Lichtenbaum Conjecture.

Let us refocus on the number field situation. Iwasawa's work on $\mathbb{Z}_p$-extensions, the creation of certain $\mathbb{Z}_p[[T]]$-modules (analogous of the previous Jacobians), whose characteristic polynomials are closely related to p-adic L-functions via the Main Conjecture (proved by Wiles for p odd and any totally real field, and for $p = 2$ and $\mathbb{Q}$) led to important progress in the study of the Lichtenbaum Conjecture. Let

$$h_n(F) := \prod_p |H^2_{\text{ét}}(\mathcal{O}_F^S, \mathbb{Z}_p(n))|.$$

The state of the Lichtenbaum conjecture and related results are comprised in the next theorem (refer to the comments that follow for clarifying concepts and notations):

Theorem 1.6 *(1) (Wiles [82]) Let F be a totally real number field, and $n \geq 2$ an even integer. Then:*

$$\zeta_F(1-n) = \pm \frac{h_n(F)}{w_n(F)}$$

up to powers of 2. If F is abelian over $\mathbb{Q}$, then the formula also gives the correct powers of 2.

(2) (Kolster [41]) Let E/F be a CM-extension with E an abelian number field and $n \geq 3$ an odd integer. Let χ be the non-trivial Artin character of $\mathrm{Gal}(E/F)$. *Then:*

$$L(E/F, \chi, 1-n) = \pm \frac{2^{r_1+1}}{Q_n} \cdot \frac{h_n^-}{w_n(E)}.$$

For general CM-extensions the result holds up to powers of 2.

(3) (Kolster, Nguyen Quang Do, Fleckinger [46]) Let F be an abelian number field, and $n \geq 2$. Then:

$$\zeta_F^*(1-n) = \pm \frac{h_n(F)}{w_n(F)} \cdot R_n^{Bei}(F)$$

up to powers of 2.

Statement (1) follows essentially from the work of Wiles in [82]. Note that no regulators are involved, and the zeta-function does not vanish at $1-n$ (n even, and F totally real). The analogous relative situation that does not include regulators is presented in (2). Here χ is the non-trivial Artin character of $\mathrm{Gal}(E/F)$, and $L(E/F, \chi, s)$ is the Artin L-function. Then we have

$$\zeta_E(s) = \zeta_F(s) \cdot L(E/F, \chi, s).$$

For $n \geq 1$ an odd integer the order of vanishing of $\zeta_F(s)$ and $\zeta_E(s)$ at $1-n$ is the same, so that $L(E/F, \chi, 1-n)$ must be a non-zero rational number. For $n = 1$ we have the relative class number formula:

$$L(E/F, \chi, 0) = \frac{2^{r_1}}{Q} \cdot \frac{h^-}{w(E)},$$

where $h^- = h_E/h_F$ is the relative class number, and Q is the Q-index defined as $Q := [\mathcal{O}_E^\times : \mathcal{O}_F^\times \cdot \mu(E)]$, and can be 1 or 2. Statement (2) takes care of odd values $n \geq 3$. Finally, statement (3) treats the abelian case at all negative integers. It uses a different regulator, the so-called Beilinson regulator $R_n^{Bei}(F)$ (see definitions of regulators in Chapter 4). To obtain Lichtenbaum's K-theoretical formulation, one needs to study the p-adic Chern characters between K-theory and étale cohomology (see Chapter 3).

In this thesis we compute the missing power of 2 for abelian number fields F and odd $n \geq 3$. Our main result is comprised in the following theorem (Theorem 10.12):

Theorem 1.7 *For a complex abelian number field F and $n \geq 3$ odd, we have*

$$\zeta_F^*(1-n) = \pm 2^{\mu} \cdot \frac{h_n(F)}{w_n(F)} \cdot R_n^{Bei}(F).$$

Here $\mu := \mu(\bar{U}_{\infty,F^+}/\bar{C}_{\infty,F^+})$, where F^+ is the maximal real subfield of F (refer to Chapter 10 for the presentation of the objects involved).

Note that the sign can be easily determined from the functional equation. Also, it is believed that $\mu = 0$ for $p = 2$ (in the case p odd this is true - refer to Tsuji [78] and Burns and Greither [11]). Here $\bar{U}_{\infty,F^+}$ is the projective limit (with respect to norms) of unit groups $U(F_r^+)$, where F_∞^+/F^+ is the cyclotomic $\mathbb{Z}_p$-extension of F^+ and F_r^+ are the finite levels ($r \geq 1$). The group $\bar{C}_{\infty,F^+}$ is obtained by the same construction for the subgroups of circular units.

Using the comparison of the motivic groups with the K-groups, and the comparison of the motivic regulator with the Beilinson regulator, we show that one can formulate this result in motivic terms (its 2-adic part is discussed in detail, refer to Chapter 4). The motivic formulation provides a clean formula for real abelian fields too (refer to Theorem 10.13).

Theorem 1.8 *For an abelian number field F and $n \geq 3$ odd, we have*

$$\zeta_F^*(1-n) = \pm 2^{\mu} \cdot \frac{h_n(F)}{w_n(F)} \cdot R_n^{\mathcal{M}}(F).$$

Here $\mu := \mu(\bar{U}_{\infty,F^+}/\bar{C}_{\infty,F^+})$, where F^+ is the maximal real subfield of F.

Chapter 2

Galois and Étale cohomology

In this chapter we introduce Galois and continuous Galois cohomology, and their generalizations étale and continuous étale cohomology. The Galois cohomology apparatus would suffice for our future needs, but essential results like Poitou-Tate duality are better formulated in the context of étale cohomology.

Given a topological group G, and a topological G-module M, let us set $\mathcal{D}^i(G,M) := \{f : G^{i+1} \to M | f \text{ continuous}\}$. By introducing on $\mathcal{D}^*(G,M)$ the following differentials

$$(\delta f)(g_0,\ldots,g_{i+1}) := \sum_{s=0}^{i+1}(-1)^s f(g_0,\ldots,\hat{g}_s,\ldots,g_{i+1}),$$

where $\hat{g}_s$ means the omission of g_s, $\mathcal{D}^*(G,M)$ becomes a complex. We let G act on this complex by:

$$(gf)(g_0,\ldots,g_i) := g\cdot f(g^{-1}g_0,\ldots,g^{-1}g_i),$$

where the dot on the right hand side represents the action of G on M. Now the subcomplex of G-invariants $\mathcal{C}^*(G,M) := \mathcal{D}^*(G,M)^G$ is by definition the **standard complex of (continuous) cochains on G with values in M**. This allows us to define $H^i(G,M) := H^i(\mathcal{C}^*(G,M))$, the usual cohomology of the standard complex. If F is a field, F^{sep} is a separable closure of F, $G = G_F := \mathrm{Gal}(F^{sep}/F)$, and M is a continuous G_F-module with discrete topology, then $H^i(G_F,M)$ is just $H^i(F,M)$, the classical **Galois cohomology group**. If M is a finitely generated $\mathbb{Z}_p$-module with its p-adic topology, and M is also a continuous G-module, then we obtain the **continuous (p-adic) cohomology groups** (see Tate [77]). We describe here some properties:

(1) If G is a topological group, and

$$0 \longrightarrow M \xrightarrow{\alpha} N \xrightarrow{\beta} P \longrightarrow 0$$

is an exact sequence of continuous G-modules such that M inherits the subspace topology, and β has a continuous section, then there is a long exact sequence with connecting morphisms δ_*

$$\ldots \longrightarrow H^i(G,M) \xrightarrow{\alpha_*} H^i(G,N) \xrightarrow{\beta_*} H^i(G,P) \xrightarrow{\delta_*} H^{i+1}(G,M) \longrightarrow \ldots$$

(2) If M is a finitely generated $\mathbb{Z}_p$-module with its p-adic topology, and G a topological group acting continuously on M through $\mathbb{Z}_p$-linear automorphisms, then we have an exact sequence

$$0 \to \varprojlim{}^1 H^{i-1}(G, M/p^r M) \to H^i(G,M) \to \varprojlim H^i(G, M/p^r M) \to 0.$$

Here, if $\ldots \longrightarrow G_r \xrightarrow{f_r} G_{r-1} \xrightarrow{f_{r-1}} \ldots \xrightarrow{f_2} G_1$ is an inverse system of abelian groups, and we define $\phi : \prod_{r=1}^{\infty} G_r \to \prod_{r=1}^{\infty} G_r$ by $\phi((g_r)_{r\geq 1}) = (g_r - f_{r+1}(g_{r+1}))_{r\geq 1}$, then we set

$$\varprojlim G_r := \ker \phi, \text{ and}$$

$$\varprojlim{}^1 G_r := \operatorname{coker} \phi,$$

which is the first derived functor of the projective limit. If all G_r are finite or if, for each r, $\operatorname{im}(G_{r+s} \longrightarrow G_r)$ is independent of s for all large enough s (these are the **Mittag-Leffler conditions**), then

$$\varprojlim{}^1 G_r = 0.$$

(3) Let M be a torsion free G-module. We tensor with M the exact sequence $0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{Q}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0$ and we obtain the exact sequence $0 \longrightarrow M \longrightarrow V \longrightarrow$

$W \longrightarrow 0$ where V is a finite dimensional $\mathbb{Q}_p$-vector space, M is an open compact subgroup, and W is a discrete divisible p-primary torsion group. Let G be a compact group. Then the kernel of the connecting morphism $H^{i-1}(G,W) \xrightarrow{\delta_i} H^i(G,M)$ is the maximal divisible subgroup of $H^{i-1}(G,W)$ and its image is the torsion subgroup of $H^i(G,M)$.

(4) There is a continuous analog of the Hochschild-Serre spectral sequence that provides us with a five term exact sequence. Let G be a compact topological group, $H \subset G$ a closed normal subgroup such that $G \longrightarrow G/H$ has a continuous section. For any topological G-module M, we have a five term exact sequence

$$0 \longrightarrow H^1(G/H, M^H) \longrightarrow H^1(G,M) \longrightarrow \bar{M} \longrightarrow H^2(G/H, M^H) \longrightarrow H^2(G,M)$$

where $\bar{M}$ is a certain subgroup of $H^1(H,M)^{G/H}$.

Remarks 2.1 *A refinement of the lower cohomology groups is provided by the* **(modified) Tate cohomology**, *which we will present shortly for the discrete case: G a finite group, M a G-module. The original reason for introducing it was to unify the homology and cohomology theories. (We will not use the homology.) Let*

$$N_G : M \longrightarrow M,\ m \mapsto \sum_{\sigma \in G} \sigma m,$$

be the **norm map**, *and*

$$\mathbb{Z}[G] \longrightarrow \mathbb{Z},\ \sum_{\sigma\in G} n_\sigma \sigma \mapsto \sum_{\sigma \in G} n_\sigma,$$

be the **augmentation map**. *Let $I_G := \ker(\mathbb{Z}[G] \to \mathbb{Z}) =$ the $\mathbb{Z}$-submodule of $\mathbb{Z}[G]$ generated by the elements $\sigma - 1$, $\sigma \in G$, the* **augmentation ideal**. *It is clear that $I_G M \subset \ker N_G$, and $\mathrm{im} N_G \subset M^G$. Consequently, N_G induces the map*

$$N^* : H_0(G,M)(= M/I_G M =: M_G) \longrightarrow H^0(G,M)(= M^G).$$

The unifying definition of Tate is then:

$$\hat{H}^0(G,M) := \mathrm{coker} N^* = M^G/N_G M,\ \hat{H}^{-1}(G,M) := \ker N^* = \ker N_G / I_G M,$$

$$\hat{H}^r(G,M) := H^r(G,M) \text{ for r} \geq 1,\ \hat{H}^{-r}(G,M) := H_{r-1}(G,M) \text{ for r} \geq 2,$$

the left hand side being the homology of G with coefficients in M. If G is cyclic of finite order, generated by $\sigma \in G$, then we obtain periodicity modulo 2: $\hat{H}^r(G,M) \cong \hat{H}^{r+2}(G,M)$ for all $r \in \mathbb{Z}$, and

$$\hat{H}^0(G,M) \cong H^2(G,M) = \ker(\sigma - 1)/\mathrm{im}N_G,$$

$$\hat{H}^{-1}(G,M) \cong H^1(G,M) = \ker N_G/\mathrm{im}(\sigma - 1).$$

The **Herbrand quotient** *is then defined as follows:*

$$h(M) := \frac{|\hat{H}^0(G,M)|}{|\hat{H}^{-1}(G,M)|} = \frac{|H^2(G,M)|}{|H^1(G,M)|},$$

and it is easy to see that it is multiplicative on short exact sequences, and that it is 1 *if M is finite.*

Remarks 2.2 *It is well known from classical Galois cohomology theory that $H^0(F, F^{sep\times}) \cong F^\times$, $H^1(F, F^{sep\times}) = 0$ (Hilbert's 90), and that $H^2(F, F^{sep\times}) \cong \mathrm{Br}(F)$, the Brauer group of F. Also if $m \geq 1$, and m is prime to* $\mathrm{char}(F)$*, then taking the long exact sequence of the* **Kummer sequence** *of Galois modules*

$$0 \longrightarrow \mu_m \longrightarrow F^{sep\times} \xrightarrow{\times m} F^{sep\times} \longrightarrow 0$$

we obtain

$$0 \longrightarrow \mu_m(F) \longrightarrow F^{sep\times} \xrightarrow{\times m} F^{sep\times} \longrightarrow H^1(F,\mu_m) \longrightarrow 0, \text{and}$$

$$0 \longrightarrow H^2(F,\mu_m) \longrightarrow Br(F) \xrightarrow{\times m} Br(F).$$

Remarks 2.3 *The important modules M we are going to deal with are: finite modules, $\mathbb{Q}_p/\mathbb{Z}_p(m)$, and $\mathbb{Z}_p(m)$, where $m \in \mathbb{Z}$. We present shortly the m-th Tate twist. Let F be a field, and $E_\infty := F(\mu_{p^\infty})$ (attach all p-power roots of unity). The Galois* *group* $G_\infty := \mathrm{Gal}(E_\infty/F)$ *acts naturally on* μ_{p^∞} *giving rise to the* **cyclotomic character** $\rho : G_\infty \to \mathbb{Z}_p^*$ *defined by* $\zeta^\sigma = \zeta^{\rho(\sigma)}$ *for all* $\sigma \in G_\infty$ *and all* $\zeta \in \mu_{p^\infty}$. *Now, the module*

$$\mathcal{T} := \lim_{\longleftarrow} \mu_{p^r}$$

is called the **Tate-module**. *As a* $\mathbb{Z}_p$*-module* $\mathcal{T}$ *is isomorphic to* $\mathbb{Z}_p$, *but* G_∞ *acts by* $t^\sigma := \rho(\sigma) \cdot t$ *for* $\sigma \in G_\infty$. *In general, let* M *be a* $\mathbb{Z}_p$*-module with a (continuous)* G_∞*-action, represented by* $m \mapsto m^\sigma$. *Let* $m \in \mathbb{Z}$. *The* **m-th Tate twist** $M(m)$ *of* M *is defined to be the* $\mathbb{Z}_p$*-module* M *with the new* G_∞*-action* $\sigma *_{(m)} x := \rho(\sigma)^m \cdot x^\sigma$. *For example,* $\mu_{p^r} = \mathbb{Z}/p^r\mathbb{Z}(1)$, $\mathcal{T} = \mathbb{Z}_p(1)$, *and* $\mu_{p^\infty} = \mathbb{Q}_p/\mathbb{Z}_p(1)$. *Given two* $\mathbb{Z}_p$*-modules* M *and* N *with* G_∞*-actions, we make* $\mathrm{Hom}_{\mathbb{Z}_p}(M, N)$ *into a* G_∞*-module as follows: for* $f \in \mathrm{Hom}_{\mathbb{Z}_p}(M, N)$ *and* $\sigma \in G_\infty$ *we define* $f^\sigma(x) := (f(m^{\sigma^{-1}}))^\sigma$. *For example, it is not hard to see that*

$$\mathrm{Hom}_{\mathbb{Z}_p}(M(m), \mathbb{Q}_p/\mathbb{Z}_p) \cong \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)(-m) \cong \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p(-m))$$

(as G_∞*-modules). On the tensor product we have a diagonal action, so that we can define* $\mathbb{Z}_p(m)$ *as:*

$$\mathbb{Z}_p(m) = \begin{cases} \mathcal{T}^{\otimes m} & \text{if } m > 0 \\ \mathbb{Z}_p & \text{if } m = 0 \\ \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{T}^{(-m)}, \mathbb{Z}_p) & \text{if } m < 0, \end{cases}$$

Generally, $M(m) \cong M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(m)$. *See below the computations of some cohomology groups with coefficients in these modules.*

We present now some of the functorial properties of the cohomology groups. Let M be a (continuous) G-module, M' a (continuous) G'-module, and $\alpha : G' \to G$, $\beta : M \to M'$ two morphisms. α and β are **compatible** if $\beta(\alpha(g)x) = g\beta(x)$ for all $g \in G'$ and $x \in M$. In this context we obtain a morphism of complexes

$$\mathcal{D}^*(G, M) \longrightarrow \mathcal{D}^*(G', M'),\ f \mapsto \beta \circ f \circ \alpha,$$

as well as homomorphisms

$$H^q(G, M) \longrightarrow H^q(G', M').$$

(1) Let H be a closed subgroup of G, M a (continuous) H-module, and $\mathrm{Ind}_H^G(M) := \{\phi : G \to M \mid \phi \text{ (continuous) map such that } \phi(hg) = h\phi(g) \text{ for all } h \in H\}$, the **induced G-module**. The map $\beta : \mathrm{Ind}_H^G(M) \to M$, $\beta(\phi) = \phi(1)$, and the inclusion $H \hookrightarrow G$ are compatible, so we obtain homomorphisms

$$H^q(G, \mathrm{Ind}_H^G(M)) \longrightarrow H^q(H, M),$$

which are isomorphisms by Shapiro's Lemma.
(2) Let H be a normal closed subgroup of G and M a (continuous) G-module. Then M^H is a G/H-module. The projection $G \to G/H$ and the injection $M^H \hookrightarrow M$ are compatible. We obtain a homomorphism

$$\mathrm{inf}_G^{G/H} : H^q(G/H, M^H) \longrightarrow H^q(G, M),$$

named **inflation.**
(3) Let H be a closed subgroup of G and M a (continuous) G-module. The inclusion $H \hookrightarrow G$ and the identity map $M \xrightarrow{id} M$ are compatible. We obtain then a homomorphism in cohomology

$$\mathrm{res}_H^G : H^q(G, M) \longrightarrow H^q(H, M),$$

named **restriction**.
(4) Let H be an open subgroup of G (closed of finite index), and S be a set of left coset representatives for H in G, $G = \cup_{s \in S} sH$. Let M be a (continuous) G-module. We define

$$\mathrm{Norm}_{G/H}(x) := \sum_{s \in S} sx$$

for each $x \in M^H$. This definition is independent of S, and G fixes $\mathrm{Norm}_{G/H}(x)$. Thus we have a homomorphism

$$\mathrm{Norm}_{G/H} : M^H \longrightarrow M^G.$$

We obtain then a **corestriction** homomorphism

$$\mathrm{cor}_G^H : H^q(H, M) \longrightarrow H^q(G, M),$$

for all q. Namely: for any G-module M, there is a canonical homomorphism of G-modules

$$\phi \mapsto \sum_{s \in S} s\phi(s^{-1}) : \mathrm{Ind}_H^G(M) \longrightarrow M.$$

It induces naturally a map in cohomology which we can compose with the Shapiro isomorphism to obtain cor_G^H:

$$H^q(H, M) \xrightarrow{\cong} H^q(G, \mathrm{Ind}_H^G(M)) \longrightarrow H^q(G, M).$$

Let X be a scheme, and $X_{ét}$ the (small) étale site on X. A presheaf P of abelian groups on $X_{ét}$ is a contravariant functor $P : X_{ét} \longrightarrow Ab$ (Ab is the category of abelian groups), and all presheaves form an abelian category $\mathcal{P}(X_{ét})$ which has enough injectives. The sheaves on $X_{ét}$, i. e. presheaves with sections determined by local data, form a full subcategory denoted $\mathcal{S}(X_{ét})$. Grothendieck proved that the natural functor $\iota : \mathcal{S}(X_{ét}) \longrightarrow \mathcal{P}(X_{ét})$ has a left adjoint $\alpha : \mathcal{P}(X_{ét}) \longrightarrow \mathcal{S}(X_{ét})$. This implies that $\mathcal{S}(X_{ét})$ is an abelian category with generators and has enough injectives. This allows us to define for a sheaf $S \in \mathcal{S}(X_{ét})$ the group $H^i_{ét}(X, S) := R^i\Gamma(X, S)$ called the **i-th étale cohomology group** of $X_{ét}$ with coefficients in S, where $\Gamma(X, \cdot) : \mathcal{S}(X_{ét}) \longrightarrow Ab$ is the section functor and R^i denotes the i-th right derived functor.

Remarks 2.4 *(1) Let X be a scheme, and $\mathbb{G}_{m,X}$ a sheaf on $X_{ét}$ defined by $\mathbb{G}_{m,X}(Y \to X) := \Gamma(Y, \mathcal{O}_Y)^\times$, the group of units of the ring of sections of the structure sheaf $\mathcal{O}_Y$, for all $Y \to X$ in $X_{ét}$. If d is an integer invertible in $\Gamma(X, \mathcal{O}_X)$, i.e. X is a scheme over $\mathbb{Z}[1/d]$, we have the Kummer exact sequence of étale sheaves*

$$0 \longrightarrow \mu_{d,X} \longrightarrow \mathbb{G}_{m,X} \xrightarrow{\times d} \mathbb{G}_{m,X} \longrightarrow 0,$$

where $\mu_{d,X}$ is the kernel of the multiplication by d. Then:

$$H^1_{ét}(X, \mathbb{G}_{m,X}) \cong \mathrm{Pic}(X),$$

where Pic(X) *is the group of isomorphism classes of locally free, rank one $\mathcal{O}_X$-modules (see Hartshorne [31]).*

(2) We will be interested mostly in affine schemes $X = \mathrm{Spec}(\mathcal{O}_F^S)$. Here $\mathcal{O}_F^S$ is the ring of S-integers in the number field F, where S is the union of the set of places of F which are above the rational prime p and the set of the real infinite places of F. In this case we denote $H^*_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_F^S), -)$ by $H^*_{\text{ét}}(\mathcal{O}_F^S, -)$, or even $H^*(\mathcal{O}_F^S, -)$. We have:

$$H^0_{\text{ét}}(\mathcal{O}_F^S, \mathbb{G}_m) \cong \mathcal{O}_F^{S\times} =: U_F^S,$$

the group of units of $\mathcal{O}_F^S$. Also, as seen in (1), we have:

$$H^1_{\text{ét}}(\mathcal{O}_F^S, \mathbb{G}_m) \cong \mathrm{Cl}(\mathcal{O}_F^S),$$

the S-ideal class group of F. Taking the long exact sequence of the Kummer sequence just presented, we obtain the exact sequence:

$$0 \longrightarrow U_F^S/p^\nu \longrightarrow H^1_{\text{ét}}(\mathcal{O}_F^S, \mu_{p^\nu}) \longrightarrow {}_{p^\nu}\mathrm{Cl}(\mathcal{O}_F^S) \longrightarrow 0,$$

where μ_{p^ν} is the group of p^ν-th roots of unity regarded as a sheaf in the étale topology, $\mathbb{Z}/p^\nu\mathbb{Z}(1) := \mu_{p^\nu}$. Passing to the injective limit (the étale cohomology commutes with it), we obtain:

$$0 \longrightarrow U_F^S \otimes \mathbb{Q}_p/\mathbb{Z}_p(1) \longrightarrow H^1_{\text{ét}}(\mathcal{O}_F^S, \mathbb{Q}_p/\mathbb{Z}_p(1)) \longrightarrow {}_p\mathrm{Cl}(\mathcal{O}_F^S) \longrightarrow 0.$$

This is one hint that étale cohomology plays an important role in the study af algebraic number fields. Some of these results hold for any commutative ring R: $H^1_{\text{ét}}(R, \mathbb{G}_m) \cong \mathrm{Pic}(R)$, and $H^2_{\text{ét}}(R, \mathbb{G}_m) \cong \mathrm{Br}(R)$, where the Picard group is defined in terms of rank one projective R-modules, and the Brauer group in terms of Azumaya R-algebras (see Milne [57]). If F is a field then by Hilbert's Theorem 90, $\mathrm{Pic}(F) = H^1_{\text{ét}}(F, \mathbb{G}_m) = 0$. Moreover, using the same Kummer sequence one obtains:

$$H^0_{\text{ét}}(R, \mathbb{Z}/p^\nu\mathbb{Z}(1)) \cong \mu_{p^\nu}(R), \quad H^1_{\text{ét}}(R, \mathbb{Z}/p^\nu\mathbb{Z}(1)) \cong R^\times \rtimes \mathrm{tor}_p\mathrm{Pic}(R),$$

$$H^2_{\text{ét}}(R, \mathbb{Z}/p^\nu\mathbb{Z}(1)) \cong \mathrm{Pic}(R)/p^\nu \rtimes \mathrm{tors}_p\mathrm{Br}(R)$$

(semidirect products).

As mentioned earlier, étale cohomology generalizes Galois cohomology. Let F be a field, $X = \mathrm{Spec}(F)$, and fix a separable algebraic closure F^{sep} with Galois group $G_F := \mathrm{Gal}(F^{sep}/F)$. In other words, we choose a geometric point $\bar{x} \to \bar{X} = \mathrm{Spec}(F^{sep})$ and set $G_F = \pi_1(\bar{X}, \bar{x})$ (see Milne [57]). For $P \in \mathcal{P}(X_{\acute{e}t})$, we define the stalk at $\bar{x}$ by

$$P_{\bar{x}} := \varinjlim P(\mathrm{Spec}(K)),$$

with the direct limit running over all fields K, $[K : F] < \infty$. $P_{\bar{x}}$ is a discrete G_F-module via $\sigma^* : \mathrm{Spec}(F^{\sigma}) \longrightarrow \mathrm{Spec}(F)$, $\sigma \in G_F$. On the other hand, given a discrete G_F-module M, we define

$$S_M : X_{\acute{e}t} \longrightarrow Ab, (Y \to X) \mapsto \mathrm{Hom}_{G_F}(H(Y), M)$$

where $H : Fin\acute{E}t/X \longrightarrow G_F\text{-}Sets$ is the functor defined by $H(Y) := \mathrm{Hom}_X(\bar{x}, Y)$, and $Fin\acute{E}t/X$ is the category of étale schemes of finite type over X. Then S_M is in fact a sheaf (Milne [57]), and the correspondence we just defined induces an equivalence of categories $\mathcal{S}(X_{\acute{e}t}) \sim G_F\text{-}Mod$. Since $\Gamma(X, S) = S_{\bar{x}}^{G_F}$, we obtain

$$H^i_{\acute{e}t}(\mathrm{Spec}(F), S) \cong H^i(F, S_{\bar{x}})$$

for $i \geq 0$, and $S \in \mathcal{S}(X_{\acute{e}t})$.

The continuous étale cohomology theory of Jannsen [35] brings better context and tools (for example, spectral sequences). For a scheme X, $\mathcal{S}(X_{\acute{e}t})$ is an abelian category with enough injectives, and hence the same is true for the category of inverse systems $\mathcal{S}(X_{\acute{e}t})^{\mathbb{N}}$. Then for $(S_n, \phi_n) \in \mathcal{S}(X_{\acute{e}t})^{\mathbb{N}}$, and $i \geq 0$, we set $H^i(X, (S_n, \phi_n)) := R^i\Gamma^{\mathbb{N}}(X, (S_n, \phi_n))$, where $\Gamma^{\mathbb{N}}(X, -)$ is induced by $\Gamma(X, -)$. If p is invertible in X, and $S = (S_{p^\nu})$ is an p-adic sheaf (for example, $\mathbb{Z}_p$), then for $i \geq 0$ the **i-th continuous étale cohomology** group of X with coefficients in S is defined by:

$$H^i_{cont}(X, S) := H^i(X, (S_{p^\nu})).$$

In this thesis we use the description of étale cohomology in terms of Galois cohomology. Let F be a number field, p be a rational prime, and let $\mathcal{O}_F^S$ be the ring of S-integers of F, where S is the set of all primes above p and all infinite primes. Let $G_S(F)$ denote the Galois group of the maximal algebraic extension of F that is unramified outside S. Then we have the identification (Milne [57]):

$$H^*_{\text{ét}}(\text{Spec}(\mathcal{O}_F^S), \mu_{p^r}^{\otimes n}) \cong H^*(G_S(F), \mu_{p^r}^{\otimes n})$$

($G_S(F)$ acts diagonally on $\mu_{p^r}^{\otimes n}$). We will drop Spec from the notation when there is no danger of confusion. Because the Mittag-Leffler conditions are satisfied we will have:

$$H^*_{\text{ét}}(\text{Spec}(\mathcal{O}_F^S), \mathbb{Z}_p(n)) = \varprojlim_r H^*_{\text{ét}}(\text{Spec}(\mathcal{O}_F^S), \mu_{p^r}^{\otimes n}) \cong \varprojlim_r H^*(G_S(F), \mu_{p^r}^{\otimes n}).$$

Also, naturally:

$$H^*_{\text{ét}}(\text{Spec}(\mathcal{O}_F^S), \mathbb{Q}_p/\mathbb{Z}_p(n)) = \varinjlim_r H^*_{\text{ét}}(\text{Spec}(\mathcal{O}_F^S), \mu_{p^r}^{\otimes n}) \cong \varinjlim_r H^*(G_S(F), \mu_{p^r}^{\otimes n}).$$

A very important tool in étale cohomology is the **localization sequence**:

Theorem(Soulé [72]) 2.5 *Let R be a Dedekind domain, $F = \text{Quot}(R)$ be its quotient field, p be a prime number invertible in R, and i a positive integer. Then:*

$$H^0_{\text{ét}}(R, \mu_{p^\nu}^{\otimes i}) \cong H^0_{\text{ét}}(F, \mu_{p^\nu}^{\otimes i}),$$

and there is a long exact sequence:

$$0 \to H^1_{\text{ét}}(R, \mu_{p^\nu}^{\otimes i}) \to H^1_{\text{ét}}(F, \mu_{p^\nu}^{\otimes i}) \to \oplus_v H^0_{\text{ét}}(\kappa(F_v), \mu_{p^\nu}^{\otimes (i-1)})$$

$$\ldots \to H^k_{\text{ét}}(R, \mu_{p^\nu}^{\otimes i}) \to H^k_{\text{ét}}(F, \mu_{p^\nu}^{\otimes i}) \to \oplus_v H^{k-1}_{\text{ét}}(\kappa(F_v), \mu_{p^\nu}^{\otimes (i-1)})$$

$$\to H^{k+1}_{\text{ét}}(R, \mu_{p^\nu}^{\otimes i}) \to \ldots$$

(here $\kappa(F_v)$ is the residue field at the place v of F).

Also, if E/F is a Galois extension of number fields, $G = \text{Gal}(E/F)$, for M an étale coefficient sheaf on $\mathcal{O}_F^S$ (for example: $\mathbb{Z}_p(n)$), we have a first quadrant **Hochschild-Serre spectral sequence**

$$E_2^{s,t} = H^s(G, H^t(\mathcal{O}_E^S, M)) \Rightarrow H^{s+t}(\mathcal{O}_F^S, M),$$

and for M a discrete coefficient sheaf (for example: $\mathbb{Q}_p/\mathbb{Z}_p(n)$) such that $cd(G, M) < \infty$ (see Kahn [38]), we have a second quadrant **Tate spectral sequence**:

$$E_2^{-s,t} = H_s(G, H^t(\mathcal{O}_E^S, M)) \Rightarrow H^{-s+t}(\mathcal{O}_F^S, M).$$

These are very useful in the study of descent and codescent for number fields. We mention here only one result to be used in chapter 10 (see Kolster [43]) for the prime $p = 2$.

Proposition 2.6 *Let E/F be a Galois 2-extension, $\Delta := \mathrm{Gal}(E/F)$, and $n \geq 2$ an integer. Then:*

$$H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))^\Delta \cong H^1_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_2(n)).$$

Our first concrete example is the computation of $H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(n))$ for a prime p, and a field F. Let us introduce the number

$$w_n^{(p)}(F) := \max\,\{p^\nu \mid \mathrm{Gal}(F(\mu_{p^\nu})/F) \text{ has exponent dividing } n\}.$$

If there is no maximum, then $w_n^{(p)}(F) := p^\infty$, and $\mathbb{Z}/p^\infty\mathbb{Z} := \mathbb{Q}_p/\mathbb{Z}_p$. Note also that for each n we have $w_{-n}^{(p)}(F) = w_n^{(p)}(F)$.

Proposition 2.7

$$H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(n)) \cong \mathbb{Z}/w_n^{(p)}(F)\mathbb{Z}.$$

.

Proof: $\zeta_{p^\nu}^{\otimes n}$ is invariant under $\sigma \in \mathrm{Gal}(\bar{F}/F)$ when $\sigma^n\zeta_{p^\nu} = \zeta_{p^\nu}$. This means that $\zeta_{p^\nu}^{\otimes n}$ is invariant under all of $\mathrm{Gal}(\bar{F}/F)$ exactly when the group $\mathrm{Gal}(F(\mu_{p^\nu})/F)$ has exponent n.□

Proposition 2.8 *Let $p = 2$, F a field with $\mathrm{char}(F) \neq 2$, let a be the maximal positive integer such that $\mu_{2^a} \subset F(\zeta_4)$, and let n be an integer, and b be the maximal power of 2 dividing n. Then we have:*
(1) If $\zeta_4 \in F$, then $w_n^{(2)}(F) = 2^{a+b}$.
(2) If $\zeta_4 \notin F$ and n is odd, then $w_n^{(2)}(F) = 2$.

(3) If $\zeta_4 \notin F$, F is non-exceptional (i.e. $\zeta_{2^j} - \bar{\zeta}_{2^j} \in F$ for some $j \geq 3$), and n is even, then $w_n^{(2)}(F) = 2^{a+b-1}$.
(4) If $\zeta_4 \notin F$, F is exceptional, and n is even, then $w_n^{(2)}(F) = 2^{a+b}$.

We note that if F is a totally real field, then $w_n^{(2)}(F) = w_n^{(2)}(F(\zeta_4))$ for an even n.

Proposition 2.9 *Let p be an odd prime, F be a field, $d := [F(\mu_{2p}) : F]$, a be the maximal positive integer such that $\mu_{p^a} \subset F(\zeta_{2p})$, but $\mu_{p^{a+1}} \not\subset F(\zeta_{2p})$ and let n be an integer, and b be the maximal power of p dividing n. Then:*

$$w_n^{(p)}(F) = \begin{cases} 1 & if\ n \not\equiv 0 \ mod\ d \\ p^{a+b} & if\ n \ \equiv 0 \ mod\ d. \end{cases}$$

The following facts are known about the finitely generated p-adic étale cohomology groups for rings of integers in number fields F (due to Soulé [72]):
(1) $H^0(\mathcal{O}_F, \mathbb{Z}_p(n)) = 0$ for $n \in \mathbb{Z} \setminus \{0\}$ (see Lemma 9.1).
(2) For $i \geq 3$ and p odd : $H^i(\mathcal{O}_F, \mathbb{Z}_p(n)) = 0$.
(3) For $i \geq 3$:

$$H^i(\mathcal{O}_F, \mathbb{Z}_2(n)) \cong \begin{cases} 0 & \text{for } i+n \text{ odd} \\ (\mathbb{Z}/2\mathbb{Z})^{r_1(F)} & \text{for } i+n \text{ even.} \end{cases}$$

The central tools in our computations are the local and global Poitou-Tate dualities. The main reference for this subject is Neukirch, Schmidt, Wingberg [61].

Let K be a p-local field (a finite field extension of $\mathbb{Q}_p$), $G_K := \mathrm{Gal}(\bar{K}/K)$, where $\bar{K}$ is a fixed algebraic closure of K. Since $cd(G_K) = 2$, there is a dualizing module of G_K, namely, $\mu(\bar{K}) = \mu$. Let M be a finite G_K-module. Then $H^i(K, M)$ is finite for any i. Let M' denote $\mathrm{Hom}(M, \mu)$ or $\mathrm{Hom}(M, \mathbb{G}_m)$. G_K acts on M' according to: $(g\phi)(x) := g(\phi(g^{-1}x))$, $g \in G_K$, $\phi \in M'$, $x \in M$. We have for example: $\mathbb{Z}/l^\nu\mathbb{Z}(n)' \cong \mathbb{Z}/l^\nu\mathbb{Z}(1-n)$, $\mathbb{Q}_l/\mathbb{Z}_l(n)' \cong \mathbb{Z}_l(1-n)$, $\mathbb{Z}_l(n)' \cong \mathbb{Q}_l/\mathbb{Z}_l(1-n)$, where l is a rational prime. The pairing $M \times M' \longrightarrow \mu$ induces a pairing

$$H^i(K, M) \times H^{2-i}(K, M') \longrightarrow H^2(K, M) \cong \mathbb{Q}/\mathbb{Z}.$$

Theorem (Poitou-Tate local duality) 2.10 *Let K be a p-local field, and M a finite coefficient module. Then for all i the pairing defined above*

$$H^i(K, M) \times H^{2-i}(K, M') \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

is a perfect pairing of finite groups, functorial in K and M. In particular, $H^i(K, M) = 0$ for $i \in \mathbb{Z} \setminus \{0, 1, 2\}$.

Corollary 2.11 *Let K be a p-local field. Each $H^i(K, \mathbb{Q}_l/\mathbb{Z}_l(n))$ is a discrete torsion group, each $H^i(K, \mathbb{Z}_l(n))$ is a profinite group, and there is a perfect pairing*

$$H^i(K, \mathbb{Q}_l/\mathbb{Z}_l(n)) \times H^{2-i}(K, \mathbb{Z}_l(1-n)) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

for all i and n. In particular, $H^i(K, \mathbb{Q}_l/\mathbb{Z}_l(n)) = 0$ and $H^i(K, \mathbb{Z}_l(n)) = 0$ for all $i \in \mathbb{Z} \setminus \{0, 1, 2\}$. Also, $H^2(K, \mathbb{Q}_l/\mathbb{Z}_l(1)) \cong \mathbb{Q}_l/\mathbb{Z}_l$ and $H^2(K, \mathbb{Q}_l/\mathbb{Z}_l(n)) = 0$ for all $n \neq 1$.

If K is a local field, then $\hat{H}^i(K, M) \cong H^i(K, M)$ for $i \geq 1$. For $i = 0$ the same is true only if K is non-archimedean. In the archimedean case we have:

$$\hat{H}^0(\mathbb{R}, M) = \hat{H}^0(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), M) = M^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}/(1+J)M, \text{ and } \hat{H}^0(\mathbb{C}, M) = 0.$$

For $p = 2$ the following is known:

Proposition 2.12 *(1) ($\mathbb{Q}_2/\mathbb{Z}_2(n)$-coefficients) For $n \in \mathbb{Z}$ we have*

$$\hat{H}^0(\mathbb{R}, \mathbb{Q}_2/\mathbb{Z}_2(n)) \cong \begin{cases} 0 & \text{for } n \text{ even} \\ \mathbb{Z}/2\mathbb{Z} & \text{for } n \text{ odd, and} \end{cases}$$

$$\hat{H}^0(\mathbb{R}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$

Also, for $i \geq 1$ and $n \in \mathbb{Z}$ we have

$$\hat{H}^i(\mathbb{R}, \mathbb{Q}_2/\mathbb{Z}_2(n)) \cong \begin{cases} 0 & \text{for } i+n \text{ even} \\ \mathbb{Z}/2\mathbb{Z} & \text{for } i+n \text{ odd, and} \end{cases}$$

$$\hat{H}^i(\mathbb{R}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$

If K is a 2-local field, then $H^i(K, \mathbb{Q}_2/\mathbb{Z}_2(n))$ is the Portryagin dual of a finitely generated $\mathbb{Z}_2$-module for all i and n. For $n \in \mathbb{Z} \setminus \{0,1\}$ we have

$$\hat{H}^i(K, \mathbb{Q}_2/\mathbb{Z}_2(n)) \cong \begin{cases} \mathbb{Z}/w_n^{(2)}(K)\mathbb{Z} & \textit{for } i = 0 \\ (\mathbb{Q}_2/\mathbb{Z}_2)^{ef} \oplus \mathbb{Z}/w_{n-1}^{(2)}(K)\mathbb{Z} & \textit{for } i = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

(2) ($\mathbb{Z}_2(n)$-coefficients) For n even we have

$$\hat{H}^0(\mathbb{R}, \mathbb{Z}_2(n)) \cong \mathbb{Z}/2\mathbb{Z}.$$

For n odd we have

$$\hat{H}^0(\mathbb{R}, \mathbb{Z}_2(n)) = 0.$$

For K a 2-local field, and $n \in \mathbb{Z} \setminus \{0\}$ we have

$$\hat{H}^0(K, \mathbb{Z}_2(n)) = 0.$$

Proof: Refer to Rognes, Weibel [66].□

Remark 2.13 *For p odd, and K a p-local field we will be using only the fact*

$$H^2(K, \mathbb{Q}_p/\mathbb{Z}_p(n)) = 0$$

for all $n \in \mathbb{Z} \setminus \{1\}$. The cohomology for archimedean local fields plays no role in this case.

Now, let F be a number field with r_1 real embeddings and r_2 pairs of complex embeddings, $[F : \mathbb{Q}] = r_1 + 2r_2$, and $\mathcal{O}_F^S$ the ring of S-integers in F, where S is the set of all finite primes of F above p and of all infinite primes of F. For each place v of F let F_v denote the v-completion of F, and $\kappa(F_v)$ the residue field at v. By local duality, we have:

$$\hat{H}^i(F_v, M) \cong \hat{H}^{2-i}(F_v, M')$$

for all i, and M, $M = \mathbb{Q}_p/\mathbb{Z}_p(n)$ or $M = \mathbb{Z}_p(n)$. Let

$$\beta^i(\mathcal{O}_F^S, M) : H^i(\mathcal{O}_F^S, M) \longrightarrow \oplus_{v \in S} \hat{H}^i(F_v, M)$$

be the sum of the homomorphisms induced by the completion maps

$$\mathcal{O}_F^S \to F \to F_v$$

for all $v \in S$. Let also

$$\gamma^i(\mathcal{O}_F^S, M) : \oplus_{v\in S}\hat{H}^i(F_v, M) \longrightarrow H^{2-i}(\mathcal{O}_F^S, M')^{\#}$$

be the direct sum of the local Poitou-Tate duality isomorphisms composed with the Pontryagin dual of the map $\beta^{2-i}(\mathcal{O}_F^S, M')$.

Definition 2.14

$$\text{Ш}^i(\mathcal{O}_F^S, M) := \ker \beta^i(\mathcal{O}_F^S, M).$$

When $i = 1$ this is the Tate-Shafarevich group.

Theorem (Poitou-Tate global duality) 2.15 *There is a natural perfect pairing*

$$\text{Ш}^i(\mathcal{O}_F^S, M) \times \text{Ш}^{3-i}(\mathcal{O}_F^S, M') \to \mathbb{Q}/\mathbb{Z}$$

for $i = 1, 2$ and for M finite, $M = \mathbb{Q}_p/\mathbb{Z}_p(n)$ or $M = \mathbb{Z}_p(n)$.

These groups are finite when M is finite, discrete torsion when $M = \mathbb{Q}_p/\mathbb{Z}_p(n)$, and profinite when $M = \mathbb{Z}_p(n)$.

There is also a natural 9-term exact sequence

$$0 \to H^0(\mathcal{O}_F^S, M) \xrightarrow{\beta^0} \bigoplus_{v\in S} \hat{H}^0(F_v, M) \xrightarrow{\gamma^0} H^2(\mathcal{O}_F^S, M')^{\#}$$

$$\to H^1(\mathcal{O}_F^S, M) \xrightarrow{\beta^1} \bigoplus_{v\in S} H^1(F_v, M) \xrightarrow{\gamma^1} H^1(\mathcal{O}_F^S, M')^{\#}$$

$$\to H^2(\mathcal{O}_F^S, M) \xrightarrow{\beta^2} \bigoplus_{v\in S} H^2(F_v, M) \xrightarrow{\gamma^2} H^0(\mathcal{O}_F^S, M')^{\#} \to 0.$$

Hence for each $i = 0, 1, 2$, $H^i(\mathcal{O}_F^S, M)$ is finite when M is finite, discrete torsion when $M = \mathbb{Q}_p/\mathbb{Z}_p(n)$, and profinite when $M = \mathbb{Z}_p(n)$.

Moreover, in certain situations the Tate's 9-term exact sequence breaks up into an exact 6-term sequence and a short exact sequence, as the following theorem proves.

Theorem([7], [66]) 2.16 *Let F be a number field, and $n \geq 2$ (or F a totally real number field, and $n \leq -1$ odd). Then:* $\underline{|||}^2(\mathcal{O}_F^S, \mathbb{Q}_p/\mathbb{Z}_p(n)) = 0$, *so* $\beta^2(\mathcal{O}_F^S, \mathbb{Q}_p/\mathbb{Z}_p(n))$ *is injective, and* $\gamma^1(\mathcal{O}_F^S, \mathbb{Q}_p/\mathbb{Z}_p(n))$ *is surjective.*

For $p = 2$, Rognes and Weibel proved also the following facts about the structure of lower cohomology groups (in [66]).

Proposition 2.17 *Let $p = 2$. The group $H^i(\mathcal{O}_F^S, \mathbb{Q}_2/\mathbb{Z}_2(n))$ is the Pontryagin dual of a finitely generated $\mathbb{Z}_2$-module, and $H^i(\mathcal{O}_F^S, \mathbb{Z}_2(n))$ is a finitely generated $\mathbb{Z}_2$-module for any i and n. For $n \geq 2$ the groups $H^0(\mathcal{O}_F^S, \mathbb{Q}_2/\mathbb{Z}_2(n)) \cong \mathbb{Z}/w_n^{(2)}(F)$ and $H^2(\mathcal{O}_F^S, \mathbb{Z}_2(n))$ are finite, and*

$$H^1(\mathcal{O}_F^S, \mathbb{Z}_2(n)) \cong \mathbb{Z}_2^r \oplus \mathbb{Z}/w_n^{(2)}(F),$$

$$H^1(\mathcal{O}_F^S, \mathbb{Q}_2/\mathbb{Z}_2(n)) \cong (\mathbb{Q}_2/\mathbb{Z}_2)^r \oplus H^2(\mathcal{O}_F^S, \mathbb{Z}_2(n)),$$

where $r = r_2$ if n is even, and $r = r_1 + r_2$ if n is odd.

Chapter 3

Background in Algebraic K-theory. Étale Chern characters.

The "classical" K-theory (refer to [59]) introduced three important functors whose relations with algebraic number theory were fruitful from the beginning.

Let A be an associative ring with unity (we will be interested only in commutative rings), and let $\mathcal{P}(A)$ denote the category of finitely generated projective A-modules. The **Grothendieck group** $K_0(A)$ is defined as the quotient $\mathcal{F}/\mathcal{R}$, where $\mathcal{F}$:= the free abelian group on the isomorphism classes of projective modules in $\mathcal{P}(A)$, and $\mathcal{R}$:= the subgroup generated by the elements $[P+Q]-[P]-[Q]$, for all P, $Q \in \mathcal{P}(A)$). For a local ring A, $K_0(A) \cong \mathbb{Z}$, and for a Dedekind domain A, $K_0(A) \cong \mathbb{Z} \oplus Cl(A)$, where $Cl(A)$ is the ideal class group of A (i.e. the group of isomorphism classes of invertible ideals). Particularly, for the ring of algebraic integers $\mathcal{O}_F$ in a number field F,

$$K_0(\mathcal{O}_F) \cong \mathbb{Z} \oplus Cl(\mathcal{O}_F).$$

Next, $K_1(A)$ is defined as $GL(A)^{ab}$, the abelianization of $GL(A)$, where $GL(A) := \cup_r GL_r(A)$, the embeddings being $GL_r(A) \hookrightarrow GL_{r+1}(A)$,

$$M \mapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}.$$

Of course, one can see it as $H_1(GL(A), \mathbb{Z})$. More important, it turns out that the commutator subgroup of $GL(A)$ is exactly the subgroup of **elementary matrices** $E(A) := \cup_r E_r(A)$, where $E_r(A)$ is generated by the matrices $e^r_{ij}(\lambda)$, the matrices whose only non-trivial off-diagonal entry is λ in the (i, j)th position, $1 \leq i \neq j \leq r$, $\lambda \in A$, and whose diagonal entries are 1. For a commutative ring A, the determinant gives a surjection $GL(A) \to A^\times$ split by $GL_1(A) \hookrightarrow GL(A)$. Let $SL(A)$ be the group of matrices of determinant 1. One can prove that the commutator subgroup of $SL(A)$ is exactly $E(A)$, and that the group $SK_1(A) := SL(A)^{ab} = SL(A)/E(A)$ sits in the following split exact sequence: $0 \to SK_1(A) \to K_1(A) \to A^\times \to 0$. If A is a local ring, then $SK_1(A) = 0$. The same is true for the ring of algebraic integers $\mathcal{O}_F$ in a number field F,

$$SK_1(\mathcal{O}_F) = 0$$

(this is a profound result of Bass, Milnor, and Serre).

In order to define the last "classical" K-group, we need to introduce the **Steinberg group**, $St(A)$. The r-th Steinberg group $St_r(A)$ is defined as the quotient of the free group on symbols $x^r_{ij}(\lambda)$ for $1 \leq i \neq j \leq r$, and for all $\lambda \in A$, modulo the normal subgroup generated by the words:

(1) $x^r_{ij}(\lambda) \cdot x^r_{ij}(\mu) \cdot x^r_{ij}(\lambda + \mu)^{-1}$ for all i, j, and $\lambda, \mu \in A$

(2) $[x^r_{ij}(\lambda), x^r_{kl}(\mu)]$ for $i \neq l, k \neq j$, and all $\lambda, \mu \in A$

(3) $[x^r_{ij}(\lambda), x^r_{jk}(\mu)] \cdot x^r_{ik}(\lambda\mu)^{-1}$ for $i \neq k$, for all $\lambda, \mu \in A$

(the brackets signify the commutator operator). There are natural homomorphisms $St_r(A) \to St_{r+1}(A)$, and natural surjections $\phi_r : St_r(A) \to E_r(A)$, given by $\phi_r(x^r_{ij}(\lambda)) = e^r_{ij}(\lambda)$, because for the elementary generators $e^r_{ij}(\lambda)$ the previous expressions are all equal to unity. Passing to the injective limit we obtain the surjection $\phi : St(A) \to E(A)$, and we define $K_2(A) := \ker \phi$. It turns out that $K_2(A) = H_2(E(A), \mathbb{Z})$ (in fact, even more is true, namely the extension $0 \to K_2(A) \to St(A) \to E(A) \to 0$ is a universal central extension - see Milnor [59]). It is much more difficult to understand this group. Nevertheless, if F is a field there is a presentation of $K_2(F)$ as the free abelian group on the symbols $\{a, b\}$ with $a, b \in F$ subject to the relations

$\{a_1a_2, b\} = \{a_1, b\}\{a_2, b\}$ for all $a_1, a_2, b \in F^\times$

$\{a, b\} = \{b, a\}^{-1}$ for all $a, b \in F^\times$

$\{a, 1-a\} = 1$ for all $a \in F^\times$, $a \neq 1$

(important result due to Matsumoto). There are many homomorphisms from $K_2(F)$ to different groups arising in the theory of fields with valuations (the tame symbol), fields containing roots of unity (the Galois symbol), local fields (the norm-residue symbol). Also, Matsumoto's result led to the definition of the **Milnor K-groups,**

$$K_r^M(F) := F^{\otimes r} / < u_1 \otimes \ldots \otimes u_r | u_i + u_{i+1} = 1 \text{ for some } i > .$$

See below their relation with Galois cohomology groups.

The work of Quillen [62] introduced the higher analogs of classical K-groups, improved the understanding of their functorial properties, and allowed the creation of very interesting conjectures. The topological and categorical background necessary for presenting these theories can be found in Srinivas [76], as well as complete proofs of many results stated below.

Let $BGL(A)$ be the classifying space of $GL(A)$, i.e the connected space - all spaces considered are CW-complexes - with $\pi_1(BGL(A)) \cong GL(A)$, $\pi_r(BGL(A)) = 0$ for all $r \geq 2$, unique up to homotopy equivalence. Applying Quillen's plus construction to the pair $(BGL(A), E(A))$ ($E(A)$ is a perfect normal subgroup of $\pi_1(BGL(A)) \cong GL(A)$), which consists in attaching 2-cells and 3-cells to $BGL(A)$, we obtain a space $BGL(A)^+$ and an inclusion $i : BGL(A) \hookrightarrow BGL(A)^+$ such that

(1) $i_* : \pi_1(BGL(A)) \to \pi_1(BGL(A)^+)$ is the natural quotient map $GL(A) \to GL(A)^{ab} = GL(A)/E(A)$, and

(2) for any local coefficient system L on $BGL(A)^+$,

$$i_* : H_r(BGL(A), i^*L) \to H_r(BGL(A)^+, L)$$

is an isomorphism for all $r \geq 0$.

These properties characterize $BGL(A)^+$ up to homotopy equivalence. Quillen then defined the higher K-groups as follows:

$$K_r(A) := \pi_r(BGL(A)^+), \text{ for } r \geq 1.$$

Of course, we have Hurewicz homomorphisms:

$$K_r(A) = \pi_r(BGL(A)^+) \to H_r(BGL(A)^+, \mathbb{Z}) \cong H_r(BGL(A), \mathbb{Z}), \text{ for } r \geq 1.$$

Let $F(A)$ be the homotopy fiber of $BGL(A) \hookrightarrow BGL(A)^+$. For a continuous map of pointed spaces $f : (X, x_0) \to (Y, y_0)$, the homotopy fiber F is the set of all pairs (x, ω) where $x \in X$ and $\omega : [0, 1] \to Y$ is a path with $\omega(0) = f(x)$ and $\omega(1) = y_0$ - this construction does not depend on the base points. There is a long exact sequence

$$\ldots \to \pi_r(F) \to \pi_r(X) \to \pi_r(Y) \to \pi_{r-1}(F) \to \ldots$$

The study of $F(A)$ provides a proof for the fact that the lower Quillen K-groups are isomorphic with the classical ones, and, moreover, that $K_3(A) \cong H_3(St(A), \mathbb{Z})$.

There is a natural H-space structure on $BGL(A)^+$ (i.e. there is a composition law on it that satisfies the group axioms up to homotopy), induced by the direct sum

$$(M, N) \mapsto \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}.$$

This allowed Loday [54] to define natural products

$$K_r(A) \otimes K_s(A) \to K_{r+s}(A)$$

for all $r, s \geq 1$, with explicit definitions for the case $r = s = 1$. (Moreover, Milnor and Moore proved that the Hurewicz maps are injective up to torsion, and that the $\mathbb{Q}$-space $K_r(A) \otimes_{\mathbb{Z}} \mathbb{Q} \subset H_r(BGL(A), \mathbb{Q}) = H_r(GL(A), \mathbb{Q})$ is identified with the subspace of primitive elements for the comultiplication of the natural Hopf algebra structure on $H_*(GL(A), \mathbb{Q})$.)

Let $f : A \to B$ be a homomorphism of commutative rings. This induces immediately a natural homomorphism of groups $f^* : K_r(A) \to K_r(B)$. If we suppose that

B is a free A-module of rank d, then the isomorphisms $GL_r(B) \cong GL_{rd}(A)$ induce maps $BGL(B)^+ \to BGL(A)^+$, and then the norm (also called trace) homomorphisms $N : K_r(B) \to K_r(A)$ for all $r \geq 1$. For $r = 1$, we obtain the natural norm $B^\times \to A^\times$. The construction extends to $r = 0$ too. The homomorphism f^* is multiplicative, i.e. $f^*(xy) = f^*(x)f^*(y)$ for $x \in K_s(A)$, and $y \in K_s(A)$. We have also the **projection formula**: $N(xf^*(y)) = N(x)y$ for $x \in K_r(B)$, and $y \in K_r(A)$. We can deduce from here that the composition $N \circ f^*$ is exactly the multiplication by d in $K_r(A)$, $r \geq 0$.

The introduction of K-theory with coefficients by Browder [12] has enriched considerably the perspectives. Generally, if X is a connected H-space and q is an integer, $q \geq 2$, we denote by X/q the homotopy fibre of the multiplication by q on X, and we define:

$$\pi_r(X, \mathbb{Z}/q\mathbb{Z}) := \pi_{r-1}(X/q), \text{ for } r \geq 3.$$

As mentioned before, we obtain then a long exact sequence

$$\ldots \to \pi_r(X) \stackrel{\times q}{\to} \pi_r(X) \to \pi_r(X, \mathbb{Z}/q\mathbb{Z}) \to \pi_{r-1}(X) \stackrel{\times q}{\to} \ldots$$

Also, we have natural mod q Hurewicz homomorphisms:

$$h_q : \pi_r(X, \mathbb{Z}/q\mathbb{Z}) \to H_r(X, \mathbb{Z}/q\mathbb{Z}).$$

For $r \geq 2$ we set

$$K_r(A, \mathbb{Z}/q\mathbb{Z}) := \pi_r(BGL(A)^+, \mathbb{Z}/q\mathbb{Z}),$$

and we have short exact sequences

$$0 \to K_r(A)/q \to K_r(A, \mathbb{Z}/q\mathbb{Z}) \to K_{r-1}(A)(q) \to 0.$$

Moreover, we have a product structure:

$$K_r(A, \mathbb{Z}/q\mathbb{Z}) \otimes K_s(A, \mathbb{Z}/q\mathbb{Z}) \to K_{r+s}(A, \mathbb{Z}/q\mathbb{Z}), r, s \geq 2$$

(if q is a power of 2, there are two different product structures, and if q is odd, the product structure is unique).

If p is a prime number, using reduction homomorphisms we then define:

$$K_r(A, \mathbb{Z}_p) := \varprojlim_j K_s(A, \mathbb{Z}/p^j\mathbb{Z}).$$

If A is the ring of integers of a number field, then we have:

$$K_r(A, \mathbb{Z}_p) \cong K_r(A) \otimes \mathbb{Z}_p$$

as $K_r(A)$ is finitely generated in this case (see below the results of Borel, and refer to Soulé [75]).

Let us present shortly the second definition of K-theory given by Quillen. Let $\mathcal{C}$ be a small exact category. Quillen defines a new category $Q\mathcal{C}$ which has the same objects as $\mathcal{C}$, but a morphism $X \to Y$ in $Q\mathcal{C}$ is defined as an isomorphism class of diagrams

$$X \overset{j}{\twoheadleftarrow} Z \overset{i}{\rightarrowtail} Y$$

where i is an admissible monomorphism in $\mathcal{C}$, i.e. there is an exact sequence $0 \to Z \overset{i}{\to} Y \to Y' \to 0$ in $\mathcal{C}$, and j is an admissible epimorphism in $\mathcal{C}$, i.e. there is an exact sequence $0 \to X' \to Z \overset{j}{\to} X \to 0$ in $\mathcal{C}$. Then for all $r \geq 0$ we define:

$$K_r(\mathcal{C}) := \pi_{r+1}(BQ\mathcal{C}, \{0\}).$$

(refer to [76] for the definition of the classifying space of a small category). Now, if A is a commutative ring, and $\mathcal{P}(A)$ is the category of finitely generated projective A-modules, it turns out that $K_r(\mathcal{P}(A)) \cong \pi_r(BGL(A)^+)$ for all $r \geq 1$. Also, this version allows the definition of K-theory of schemes. If X is an arbitrary scheme, let $\mathcal{P}(X)$ be the category of locally free sheaves of finite rank. We define $K_r(X) := K_r(\mathcal{P}(X))$. Of course, if $X = \mathrm{Spec}(A)$ then we have a natural equivalence of categories $\mathcal{P}(X) \approx \mathcal{P}(A)$.

The following facts are known (refer to [76] for proofs):

(Resolution Theorem) Let $\mathcal{M}$ be an exact category, $\mathcal{P} \subset \mathcal{M}$ a full additive subcategory, closed under extensions in $\mathcal{M}$, and assume that

(1) if $0 \to M' \to M \to M'' \to 0$ is exact in $\mathcal{M}$ and $M', M'' \in \mathcal{P}$, then $M' \in \mathcal{P}$

(2) for any object $M \in \mathcal{M}$, there is a finite resolution $0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to M \to 0$, with $P_j \in \mathcal{P}$.
Then $BQ\mathcal{P} \to BQ\mathcal{M}$ is a homotopy equivalence, and hence $K_r(\mathcal{P}) \cong K_r(\mathcal{M})$ for all r.

(Dévissage Theorem) Let $\mathcal{A}$ be an abelian category, $\mathcal{B}$ a full abelian subcategory that is closed under taking subobjects, quotients and finite products in $\mathcal{A}$. Let us assume that each object $M \in \mathcal{A}$ has a finite filtration in $\mathcal{A}$, $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ with $M_i/M_{i-1} \in \mathcal{B}$ for all $i \geq 1$.
Then $BQ\mathcal{B} \to BQ\mathcal{A}$ is a homotopy equivalence, and hence $K_r(\mathcal{B}) \cong K_r(\mathcal{A})$ for all r.

(Localization Theorem) Let $\mathcal{A}$ be an abelian category, $\mathcal{B}$ a full abelian subcategory that is closed under taking subobjects, quotients and extensions in $\mathcal{A}$ (so-called Serre subcategory), and let $\mathcal{C}$ be the quotient abelian category $\mathcal{A}/\mathcal{B}$. Then there is a long exact sequence

$$\ldots \to K_{r+1}(\mathcal{C}) \to K_r(\mathcal{B}) \to K_r(\mathcal{A}) \to K_r(\mathcal{C}) \ldots \to K_0(\mathcal{A}) \to K_0(\mathcal{C}) \to 0.$$

A direct application of the Localization Theorem and Resolution Theorem is the following result:

Theorem 3.1 *Let A be a Dedekind domain with quotient field $Q(A) = F$. There is a long exact sequence*

$$\ldots \to K_{r+1}(F) \to \oplus_{\mathfrak{m}} K_r(A/\mathfrak{m}) \to K_r(A) \to K_r(F) \ldots \to K_0(A) \to K_0(F) \to 0$$

where $\mathfrak{m}$ runs over the maximal ideals of A.

Proof: Let $\mathcal{M}(A)$ be the category of finitely generated A-modules, and $\mathcal{T}(A)$ its full subcategory of torsion A-modules. Then $\mathcal{M}(A)/\mathcal{T}(A)$ is equivalent to the category $\mathcal{P}(F)$ of finite dimensional F-vector spaces. Also, since A is a Dedekind domain, any $M \in \mathcal{M}(A)$ has projective dimension over $A \leq 1$. Therefore, using the Resolution Theorem, we obtain $K_r(\mathcal{M}(A)) \cong K_r(\mathcal{P}(A)) = K_r(A)$. Finally, note that $\{A/\mathfrak{m} \mid \mathfrak{m} \text{ maximal in } A\}$ is a set of representatives for the isomorphism classes of simple objects in $\mathcal{T}(A)$. Because K_r commutes with finite products and filtered direct limits, it suffices to consider only the case when A is a discrete valuation ring. Then $M \mapsto \mathrm{Hom}_{\mathcal{T}(A)}(A/\mathfrak{m}, M)$ is an equivalence of $\mathcal{T}(A)$ with $\mathcal{P}(\mathrm{End}_{\mathcal{T}(A)}(A/\mathfrak{m}))$. This proves that for a Dedekind ring $K_r(\mathcal{T}(A)) \cong \oplus_{\mathfrak{m}} K_r(A/\mathfrak{m})$. Our sequence is provided now directly by the Localization Theorem.□

The complete computation of K-theory of finite fields was done by Quillen in [62]:

Theorem 3.2 *If $\mathbb{F}_q$ is the field with q elements, then*
$K_0(\mathbb{F}_q) \cong \mathbb{Z}$,
$K_{2m}(\mathbb{F}_q) = 0$,
$K_{2m-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^m - 1)\mathbb{Z}$ *for* $m \geq 1$.

Let F be an algebraic number field, and $\mathcal{O}_F$ its ring of integers. In [63] Quillen proved that

$$K_r(\mathcal{O}_F) \text{ is finitely generated for all } r \geq 0.$$

Applying Theorems 3.1 and 3.2 we obtain:

$$K_{2m-1}(\mathcal{O}_F) \cong K_{2m-1}(F) \text{ for all } m \geq 2$$

(Soulé [74]), and the fact that

$$0 \to K_{2m}(\mathcal{O}_F) \to K_{2m}(F) \to \oplus_{\mathfrak{m}} K_{2m-1}(\kappa(F_{\mathfrak{m}})) \to 0$$

is an exact sequence for all $m \geq 1$ ($\kappa(F_{\mathfrak{m}})$ denotes as usual the residue field of F at $\mathfrak{m}$). For $m = 1$ we obtain the kernel of the above mentioned tame symbol λ:

$$0 \to K_2(\mathcal{O}_F) \to K_2(F) \xrightarrow{\lambda} \oplus_{\mathfrak{m}} K_1(\kappa(F_{\mathfrak{m}})) \to 0$$

where the tame symbol λ is given component-wise by :

$$\lambda_{\mathfrak{m}} : K_2(F) \to K_1(\kappa(F_{\mathfrak{m}})) = \kappa(F_{\mathfrak{m}})^{\times},$$

$\lambda_{\mathfrak{m}}(\{u, v\}) = (-1)^{v_{\mathfrak{m}}(u) v_{\mathfrak{m}}(v)} \cdot \frac{u^{v_{\mathfrak{m}}(v)}}{v^{v_{\mathfrak{m}}(u)}} \mod \mathfrak{m}$. This is why $K_2(\mathcal{O}_F)$ is called the *tame kernel.* Its properties have very interesting arithmetic implications.

Another application of the localization sequence is that

$$K_{2m-1}(\mathcal{O}_F^S) \otimes \mathbb{Z}_p \cong K_{2m-1}(\mathcal{O}_F) \otimes \mathbb{Z}_p \text{ for all } m \geq 2,$$

where p is a prime number and S is a finite set of primes of F containing all infinite and p-adic primes.

Moreover, Borel [7] proved that $K_{2m}(\mathcal{O}_F)$ is finite for all $m \geq 1$. Borel also computed the rank of K-groups:

$$\mathrm{rank}_{\mathbb{Z}} K_{2m-1}(\mathcal{O}_F) = \begin{cases} r_2(F) & \text{if } m \text{ is even} \\ r_1(F) + r_2(F) & \text{if } m \text{ is odd} \end{cases}$$

for all $m \geq 2$. Dirichlet's Theorem gives us the rank of $K_1(\mathcal{O}_F) \cong \mathcal{O}_F^\times$

$$\mathrm{rank}_{\mathbb{Z}} K_1(\mathcal{O}_F) = r_1(F) + r_2(F) - 1.$$

For finite fields we note that K-theory groups are equal to the Galois cohomology groups:

$$K_{2m}(\mathbb{F}_q) \cong H^2(\mathbb{F}_q, \mathbb{Z}(m)) = 0, \text{and}$$

$$K_{2m-1}(\mathbb{F}_q) \cong H^1(\mathbb{F}_q, \mathbb{Z}(m)) \cong H^0(\mathbb{F}_q, \mathbb{Q}/\mathbb{Z}(m)) = \mathbb{Z}/(q^m - 1)\mathbb{Z}.$$

This is one of the computations that suggested the close relation between K-groups and étale cohomology groups. We will present here Soulé's construction of étale Chern classes and characters. For general constructions of Chern characters from K-theory to various cohomology theories Gillet's article [25], and Schneider's survey article [65] should be consulted.

Let p be a prime number, F an algebraic number field, S a finite set of primes of F containing all infinite and p-adic primes, and $A := \mathcal{O}_F^S$ the ring of S integers in F.

Let P be a finitely generated projective A-module having bounded rank over all residue fields $\kappa(F_v)$, and $\rho : G \to \mathrm{Aut}(P)$ a representation of a discrete group G over P. In [29] Grothendieck defined the Chern classes

$$ch_r(\rho) \in H^{2r}_{\acute{e}t}(A, G; \mathbb{Z}/p^\nu\mathbb{Z})$$

for all $r \geq 0$, $\nu \geq 1$, where on the right side we have the étale equivariant cohomology with trivial G-action on A. Among the properties of these classes we mention their functoriality, i.e. if f is a compatible system of morphisms, then $ch_r(f^*(\rho)) = f^*(ch_r(\rho))$, and their additivity, i.e. if $0 \to P_1 \to P_2 \to P_3 \to 0$ is an exact sequence of $A[G]$-modules as above, and ρ_1, ρ_2, ρ_3 the corresponding representations, then $ch_r(\rho_2) = ch_r(\rho_1) \cup ch_r(\rho_3)$ (we are using here the cup product on the étale equivariant cohomology).

We apply this construction to the canonical representation of the general linear group over A, $id_n : GL_n(A) \to \mathrm{Aut}(A^n)$, for every $n \geq 1$. We obtain classes $ch_r(id_n) \in H^{2r}_{ét}(A, GL_n(A); \mathbb{Z}/p^\nu\mathbb{Z}(r))$, and it turns out that the system $(ch_r(id_n))_n$ is well behaved with respect to the natural inclusions $j_n : GL_n(A) \hookrightarrow GL_{n+1}(A)$, namely $j_n^*(ch_r(id_n)) = ch_r(id_{n+1})$. This allows us to pass to the injective limit and to obtain classes $ch_r(id) \in H^{2r}_{ét}(A, GL(A); \mathbb{Z}/p^\nu\mathbb{Z}(r))$. On the other hand, Künneth formula induces a map

$$H^{2r}_{ét}(A, GL(A); \mathbb{Z}/p^\nu\mathbb{Z}(r)) \to \oplus_{j=0}^{2r-j}\mathrm{Hom}(H_{2r}(GL(A), \mathbb{Z}/p^\nu\mathbb{Z}), H^j_{ét}(A, \mathbb{Z}/p^\nu\mathbb{Z}(r))),$$

which in turn gives the maps

$$ch_{j,r}(id) : H_{2r-j}(GL(A), \mathbb{Z}/p^\nu\mathbb{Z}) \to H^j_{ét}(A, \mathbb{Z}/p^\nu\mathbb{Z}(r))$$

for $0 \leq j \leq 2r$. Composing this map with the mod p^ν Hurewicz map h_{p^ν}, we obtain functorial homomorphisms, called **étale Chern characters**

$$ch_{j,r} : K_{2r-j}(\mathcal{O}_F^S, \mathbb{Z}/p^\nu\mathbb{Z}) \to H^j_{ét}(\mathcal{O}_F^S, \mathbb{Z}/p^\nu\mathbb{Z}(r))$$

for $r \geq 1$, $j = 1, 2$. Passing to the projective limit we obtain the morphisms:

$$ch^{(p)}_{j,r} : K_{2r-j}(\mathcal{O}_F^S, \mathbb{Z}_p) \to H^j_{ét}(\mathcal{O}_F^S, \mathbb{Z}_p(r))$$

for $r \geq 1$, $j = 1, 2$.

Theorem(Soulé [72], Dwyer-Friedlander [20]) 3.3 *The étale Chern characters*

$$ch^{(p)}_{j,m} : K_{2m-j}(\mathcal{O}_F^S, \mathbb{Z}_p) \to H^j_{ét}(\mathcal{O}_F^S, \mathbb{Z}_p(m))$$

are surjective for odd primes p, $m \geq 1$, $j = 1, 2$, $2m - j \geq 1$.

This is an essential step in the study of the **Quillen-Lichtenbaum Conjecture** which predicts that for odd primes p these maps are isomorphisms. For $2m - j \leq 3$ this is true, as a consequence of the work of Merkurjev and Suslin [55], [56] on K_2 and K_3 of fields. For higher K-groups Kahn [38] has shown that these maps are in fact canonically split.

As an important immediate consequence of the previous result and Borel's computations in [7] we obtain new information about the structure of the étale cohomology groups $H^j_{\acute{e}t}(\mathcal{O}^S_F, \mathbb{Z}_p(m))$ for $j = 1, 2$ and $m \geq 2$.

Corollary 3.4 *The group $H^2_{\acute{e}t}(\mathcal{O}^S_F, \mathbb{Z}_p(m))$ is finite and trivial for almost all primes p.*

$$\operatorname{rank}_{\mathbb{Z}} H^1_{\acute{e}t}(\mathcal{O}^S_F, \mathbb{Z}_p(m)) = \begin{cases} r_2(F) & \text{if } m \text{ is even} \\ r_1(F) + r_2(F) & \text{if } m \text{ is odd} \geq 2. \end{cases}$$

Also, $H^2_{\acute{e}t}(\mathcal{O}^S_F, \mathbb{Q}_p/\mathbb{Z}_p(m)) = H^2_{\acute{e}t}(F, \mathbb{Q}_p/\mathbb{Z}_p(m)) = 0$ *for* $m \geq 2$.

A profound impact on Quillen-Lichtenbaum Conjectures has the **Kato Conjecture** which asserts that the **Galois symbols**

$$g_{n,q} : K^M_n(F)/q \to H^n(F, \mathbb{Z}/q\mathbb{Z}(n))$$

for q prime to the characteristic of F are isomorphisms. For $q = 2$ the conjecture is called **Milnor Conjecture**, because it appeared in relation with Milnor's work on quadratic forms. This conjecture is true for $q = 2$ and for global fields by the work of Tate [77] who described K_2 in terms of continuous Galois cohomology, and by the work of Bass and Tate [3] who proved that $K^M_n(F) \cong (\mathbb{Z}/2\mathbb{Z})^{r_1(F)}$. It is true also for $n = 2$ and any field F, by the work of Merkurjev and Suslin [55]. Voevodsky [80] proved the complete Milnor Conjecture for arbitrary fields of characteristic 0, by using motivic (co)homologies. Finally, what is relevant for us is the fact that if for an odd prime p Kato Conjecture holds for all $q = p^\nu$ and all fields of characteristic zero, then the Quillen-Lichtenbaum Conjectures hold for the prime p (refer to [44] and [66]).

For the prime 2 we have the following result obtained separately by Rognes and Weibel in [66], and by Kahn in [39]:

Theorem 3.5 *For $j = 1, 2$ and $2m - j \geq 2$ the 2-adic Chern character*

$$ch^{(2)}_{j,m} : K_{2m-j}(\mathcal{O}_F^S) \otimes \mathbb{Z}_2 \to H^j_{\text{ét}}(\mathcal{O}_F^S, \mathbb{Z}_2(m))$$

is an isomorphism if $2m - j \equiv 0, 1, 2, 7 \bmod 8$,
surjective with kernel $\cong (\mathbb{Z}/2\mathbb{Z})^{r_1(F)}$ if $2m - j \equiv 3 \bmod 8$,
injective with cokernel $\cong (\mathbb{Z}/2\mathbb{Z})^{r_1(F)}$ if $2m - j \equiv 6 \bmod 8$.
In the case $m \equiv 3 \bmod 4$, there is an exact sequence

$$0 \to K_{2m-1}(\mathcal{O}_F^S) \otimes \mathbb{Z}_2 \to H^1_{\text{ét}}(\mathcal{O}_F^S, \mathbb{Z}_2(m)) \to (\mathbb{Z}/2\mathbb{Z})^{r_1(F)}$$
$$\to K_{2m-2}(\mathcal{O}_F^S) \otimes \mathbb{Z}_2 \to H^2_{\text{ét}}(\mathcal{O}_F^S, \mathbb{Z}_2(m)) \to 0.$$

The map $H^1_{\text{ét}}(\mathcal{O}_F^S, \mathbb{Z}_2(m)) \to (\mathbb{Z}/2\mathbb{Z})^{r_1(F)}$ is called the **signature map**. The kernel of this map can be interpreted as a subgroup of a totally positive étale cohomology group (refer to [15]). The order 2^{δ_F} of the cokernel of this map is called the **signature defect.**

Let

$$h_m(F) := \prod_p |H^2_{\text{ét}}(\mathcal{O}_F^S, \mathbb{Z}_p(m))|, \; h_m^{(p)}(F) := |H^2_{\text{ét}}(\mathcal{O}_F^S, \mathbb{Z}_p(m))|, \text{ and}$$

$$w_m(F) := \prod_p |H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(m))|, \; w_m^{(p)}(F) := |H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(m))|.$$

Corollary 3.6 *The 2-powers in the factorization of the K-theory groups are:*

$$|\text{tors}\, K_{2m-1}(F)| \sim_2 \begin{cases} w_m(F) & \text{if } m \equiv 0,1 \bmod 4 \text{ or } F \text{ is totally complex} \\ 2^{r_1(F)} w_m(F) & \text{if } m \equiv 2 \bmod 4 \\ \frac{w_m(F)}{2} \sim_2 1 & \text{if } m \equiv 3 \bmod 4 \text{ and } r_1(F) > 0 \end{cases}$$

and

$$|K_{2m-2}(\mathcal{O}_F^S)| \sim_2 \begin{cases} h_m(F) & \text{if } m \equiv 0,1 \bmod 4 \text{ or } F \text{ is totally complex} \\ 2^{-r_1(F)} h_m(F) & \text{if } m \equiv 2 \bmod 4 \\ 2^{\delta_F} h_m(F) & \text{if } m \equiv 3 \bmod 4 \text{ and } r_1(F) > 0. \end{cases}$$

Corollary 3.7 *The 2-adic Chern character induces an injection:*

$$ch^{(2)}_{1,m} : \tilde{K}_{2m-1}(F) \otimes \mathbb{Z}_2 \hookrightarrow \tilde{H}^1_{\text{ét}}(\mathcal{O}^S_F, \mathbb{Z}_2(m)),$$

where $\tilde{}$ *denotes the torsion-free part of the corresponding finitely generated abelian groups. We also have:* $[\tilde{H}^1_{\text{ét}}(\mathcal{O}^S_F, \mathbb{Z}_2(m)) : \tilde{K}_{2m-1}(F) \otimes \mathbb{Z}_2]$ *is equal to* 1 *if* $m \equiv 0, 1, 2 \bmod 4$ *or* F *is totally complex, and it is equal to* $2^{r_1(F)-1-\delta_F}$, *otherwise.*

Proof: Indeed, from the previous theorem the case $m \equiv 0, 1, 2 \bmod 4$ or F is totally complex is clear. We are using here the following fact: If A is a subgroup of an abelian group B and A and B have equal finite ranks, then $[\tilde{B} : \tilde{A}] = [B : A \cdot \text{tors}\, B]$. Let us suppose now that $r_1(F) > 0$ and $m \equiv 3 \bmod 4$. Then $\text{tors}\, H^1_{\text{ét}}(\mathcal{O}^S_F, \mathbb{Z}_2(m)) \cong \mu_2$, $K_{2m-1}(F) \otimes \mathbb{Z}_2$ is torsion free and it is the kernel of the signature map. This gives us: $[\tilde{H}^1_{\text{ét}}(\mathcal{O}^S_F, \mathbb{Z}_2(m)) : \tilde{K}_{2m-1}(F) \otimes \mathbb{Z}_2] = 2^{r_1(F)-\delta_F-1}$.□

Chapter 4

Higher regulators for algebraic number fields

Let F be an algebraic number field, let $\mathcal{O}_F$ be its ring of integers and denote $X = \mathrm{Spec}(F)$.

We show here the construction (following Neukirch [65]) of a canonical homomorphism, called the **Borel n-th regulator map**,

$$\rho_n : K_{2n-1}(\mathcal{O}_F) \to [\mathbb{R}(n-1)^{X(\mathbb{C})}]^+,$$

which is defined for each $n \geq 1$, where $\mathbb{R}(n-1) = (2\pi i)^{n-1}\mathbb{R}$, $X(\mathbb{C}) = \mathrm{Hom}(F, \mathbb{C})$ (on which complex conjugation, J or $^-$, acts naturally, $+$ meaning as usual the fixed points under J).

We start with the following homomorphism:

$$r_n : K_{2n-1}(\mathbb{C}) \to \mathbb{R}(n-1).$$

For $n = 1$, $K_1(\mathbb{C}) = \mathbb{C}^*$, r_1 is in fact the map given by

$$r_1 : \mathbb{C}^* \to \mathbb{R}, r_1(z) = \log|z|.$$

For $n \geq 2$ the construction of the map r_n is based on three essential facts (refer to Rapoport [65]):

(1) There is a canonical map (Hurewicz map):

$$K_q(\mathbb{C}) = \pi_q(BGL^+(\mathbb{C})) \to H_q(BGL^+(\mathbb{C})) \cong H_q(GL(\mathbb{C}), \mathbb{Z})$$

(2) There is a canonical pairing:

$$H^q(GL(\mathbb{C}), \mathbb{R}(n-1)) \times H_q(GL(\mathbb{C}), \mathbb{Z}) \stackrel{<,>}{\longrightarrow} \mathbb{R}(n-1).$$

(3) There are the "Borel regulator elements" in the continuous cohomology of the topological group $GL(\mathbb{C})$ with coefficients in $\mathbb{R}(n-1)$, invariant under the involution induced by complex conjugation on $GL(\mathbb{C})$ and $\mathbb{R}(n-1)$:

$$b_{2n-1} \in H_c^{2n-1}(GL(\mathbb{C}), \mathbb{R}(n-1)).$$

(These elements allow one to see $H_c^*(GL(\mathbb{C}), \mathbb{R}(n-1))$ as the free exterior algebra generated by the classes $\frac{i}{(2\pi i)^n} b_{2n-1}$ of degree $2n-1$ - see Rapoport [65] for details.) Their images under the canonical maps

$$H_c^{2n-1}(GL(\mathbb{C}), \mathbb{R}(n-1)) \to H^{2n-1}(GL(\mathbb{C}), \mathbb{R}(n-1)),$$

denoted also by b_{2n-1}, are then used to define r_n as the following composite of maps

$$K_{2n-1}(\mathbb{C}) \longrightarrow H_{2n-1}(GL(\mathbb{C}), \mathbb{Z}) \stackrel{<b_{2n-1},>}{\longrightarrow} \mathbb{R}(n-1).$$

The map r_n is called **Borel regulator map**.

Having dealt with the field of complex numbers, we can address the general case of number fields by using embeddings. Namely, each complex embedding $\sigma : F \to \mathbb{C}$ induces, due to functoriality reasons, a map $\sigma_* : K_{2n-1}(F) \to K_{2n-1}(\mathbb{C})$. So we obtain a homomorphism

$$K_{2n-1}(F) \to K_{2n-1}(\mathbb{C})^{X(\mathbb{C})}, a \to (\sigma_*)_{\sigma \in X(\mathbb{C})},$$

functorial in F. Next we consider the following composition of maps:

$$K_{2n-1}(F) \longrightarrow K_{2n-1}(\mathbb{C})^{X(\mathbb{C})} \stackrel{r_n}{\longrightarrow} \mathbb{R}(n-1)^{X(\mathbb{C})}.$$

Because it acts on $X(\mathbb{C})$, on $K_{2n-1}(\mathbb{C})$, and on $\mathbb{R}(n-1)$, complex conjugation acts also on the last two groups in the previous sequence. Moreover, r_n is compatible with these actions.

Definition 4.1 *The* **Borel n-th regulator map**, ρ_n, *is defined as the composite*

$$\rho_n : K_{2n-1}(\mathcal{O}_F) \to K_{2n-1}(F) \to [\mathbb{R}(n-1)^{X(\mathbb{C})}]^+,$$

where the first homomorphism comes from the K-theory functoriality, and the second has just been presented.

The basic properties about these regulator maps are:

Theorem 4.2 *(1)*(**Dirichlet's Unit Theorem**)
For $n = 1$, the map ρ_1 induces an isomorphism

$$(K_1(\mathcal{O}_F) \oplus \mathbb{Z}) \otimes \mathbb{R} \cong [\mathbb{R}^{X(\mathbb{C})}]^+,$$

where $\mathbb{Z}$ is embedded diagonally in $\mathbb{R}^{X(\mathbb{C})}$.
(2) (**Borel**) *For $n \geq 2$ we have:*

(2.1) $\ker \rho_n$ *is finite, equal to* $\mathrm{tors}_{\mathbb{Z}} K_{2n-1}(\mathcal{O}_F)$.

(2.2) $\mathrm{im}\, \rho_n$ *is a lattice in* $\mathbb{R}(n-1)^{X(\mathbb{C})}$.

(2.3) The map ρ_n induces an isomorphism

$$K_{2n-1}(\mathcal{O}_F) \otimes \mathbb{R} \cong [\mathbb{R}(n-1)^{X(\mathbb{C})}]^+.$$

(2.4) The covolume $R_n = R_n(F)$ of the lattice $\mathrm{im}\, \rho_n$, *called the* **Borel regulator**, *satisfies*

$$\zeta_F^*(1-n) = R_n \pmod{\mathbb{Q}^\times}.$$

Next we give the definition of the **Beilinson regulator map**. In order to do this, we need the definitions of absolute cohomology and Deligne cohomology. Following closely Schneider's survey [65], we present them for a general smooth projective variety X over $\mathbb{Q}$.

In the context of Quillen's plus construction one can define Adams operations $\{\psi^k\}_{k\geq 1}$ on the groups $K_i(A)$ on any affine scheme $\mathrm{Spec}(A)$, A a commutative ring. These operations induce the following decomposition:

$$K_i(A) \otimes \mathbb{Q} = \oplus_{n\geq 0} K_i^{(n)}(A)$$

where

$$K_i^{(n)}(A) := \{x \in K_i(A) \otimes \mathbb{Q} \mid \psi^r(x) = r^n x \text{ for all } r \geq 1\}.$$

Quillen [64] and Jouanolou [37] generalized this decomposition to the K-groups of a smooth projective variety X. The **absolute cohomology** of X is then defined by

$$H^i_{\mathcal{A}}(X, \mathbb{Q}(n)) := K^{(n)}_{2n-i}(X).$$

Let us prepare now the necessary context for defining the Deligne cohomology. The de Rham cohomology $H^*_{DR}(X(\mathbb{C}))$ is the cohomology of the complex of sheaves of holomorphic differential forms

$$\Omega^{\cdot} : \mathcal{O}_{X(\mathbb{C})} \to \Omega^1 \to \Omega^2 \to \dots$$

on $X(\mathbb{C})$ and the de Rham filtration $F^{\cdot} H^*_{DR}$ is induced by the naive filtration of this complex:

$$F^p H^i_{DR}(X(\mathbb{C})) = \mathrm{im}(H^i(\Omega^{\cdot}_{\geq p}) \to H^i(\Omega^{\cdot})),$$

where the right hand side is derived from the first arrow of the natural short exact sequence of complexes

$$\Omega^{\cdot}_{\geq p} \longrightarrow \Omega^{\cdot} \longrightarrow \Omega^{\cdot}_{<p}$$

where

$$\begin{array}{rl} \Omega^{\cdot}_{\geq p} : & \qquad\qquad\qquad\qquad 0 \to \quad \Omega^p \to \Omega^{p+1} \to \dots \\ \Omega^{\cdot} : & \mathcal{O}_{X(\mathbb{C})} \to \Omega^1 \to \dots \to \Omega^{p-1} \to \Omega^p \to \Omega^{p+1} \to \dots \\ \Omega^{\cdot}_{<p} : & \mathcal{O}_{X(\mathbb{C})} \to \Omega^1 \to \dots \to \Omega^{p-1} \to 0 \end{array}$$

Recall that we have the Hodge decomposition on the singular cohomology $H^i(X(\mathbb{C}), \mathbb{C})$ as follows:

$$H^i(X(\mathbb{C}), \mathbb{C}) = \bigoplus_{p+q=i, p,q \geq 0} H^{pq}.$$

The complex conjugation on $X(\mathbb{C})$ induces a $\mathbb{C}$-linear involution J_∞ on $H^i(X(\mathbb{C}), \mathbb{C})$ and we have $J_\infty(H^{pq}) = H^{qp}$. Moreover, there is a canonical isomorphism between singular cohomology and de Rham cohomology:

$$H^i(X(\mathbb{C}),\mathbb{C}) \cong H^i_{DR}(X(\mathbb{C})).$$

The de Rham filtration defined previously is related to the Hodge decomposition as follows:

$$F^p H^i_{DR}(X(\mathbb{C})) = \bigoplus_{p'\geq q} H^{p'q}.$$

From the existence of the Hodge decomposition we know that the hypercohomology spectral sequence $H^j(X(\mathbb{C}),\Omega^i) \Rightarrow H^{i+j}_{DR}(X(\mathbb{C}))$ degenerates, so that the map $H^i(\Omega^{\cdot}_{\geq p}) \to H^i(\Omega^{\cdot})$ is injective and we have that $H^i(\Omega^{\cdot}_{<p}) = H^i_{DR}(X(\mathbb{C}))/F^p$.

Now, we define the real **Deligne cohomology** $H^i_{\mathcal{D}}(X_{\mathbb{C}},\mathbb{R}(p))$ of $X_{\mathbb{C}}$ as the cohomology of the following complex

$$\mathbb{R}(p)_{\mathcal{D}} : \mathbb{R}(p) \to \mathcal{O}_{X(\mathbb{C})} \to \Omega^1 \to \ldots \to \Omega^{p-1} \to 0,$$

where the first arrow is the inclusion $\mathbb{R}(p) := (2\pi\sqrt{-1})^p\mathbb{R} \subset \mathbb{C} \subset \mathcal{O}_{X(\mathbb{C})}$. From the following short exact sequence of complexes

$$0 \to \Omega^{\cdot}_{<p}[-1] \to \mathbb{R}(p)_{\mathcal{D}} \to \mathbb{R}(p) \to 0$$

we obtain the long exact cohomology sequence

$$\ldots \to H^i(X(\mathbb{C}),\mathbb{R}(p)) \to H^i_{DR}(X(\mathbb{C}))/F^p \to H^{i+1}_{\mathcal{D}}(X_{\mathbb{C}},\mathbb{R}(p)) \to$$

$$\to H^{i+1}(X(\mathbb{C}),\mathbb{R}(p)) \to \ldots$$

On the singular cohomology we have a real structure, namely

$$H^i(X(\mathbb{C}),\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = H^i(X(\mathbb{C}),\mathbb{C})$$

given by the $\mathbb{R}$-linear involution $\bar{\ }$ on the right hand side which is induced by the complex conjugation on the coefficients. Also, on the analytic de Rham cohomology we have a real structure

$$H^i_{DR}(X_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C} = H^i_{DR}(X(\mathbb{C}))$$

defined by GAGA, the algebraic de Rham cohomology $H^i_{DR}(X_{\mathbb{R}})$ of $X_{\mathbb{R}}$. The DR-conjugation acting on the right hand side is induced by the complex conjugation on $(X(\mathbb{C}), \Omega^\cdot)$. Because (by Deligne [17], Prop. 1.4) $\bar{H}^{pq} = H^{qp}$, the de Rham filtration is defined over $\mathbb{R}$, and under the canonical identification $H^i_{DR}(X(\mathbb{C})) = H^i(X(\mathbb{C}), \mathbb{C})$ the DR-conjugation corresponds to $\bar{J}_\infty$, we are led to define the real Deligne cohomology of $X_{\mathbb{R}}$ as follows

$$H^i_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(p)) := H^i_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(p))^{DR-conj},$$

the subspace of elements invariant under the DR-conjugation.

Note that for $p > \dim X$, as

$$\mathbb{C} = \mathbb{R}(p-1) \oplus \mathbb{R}(p),$$

we obtain the following computation:

$$H^1_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(p)) = H^0(X(\mathbb{C}), \mathbb{C}/\mathbb{R}(p))^+$$

$$= H^0(X(\mathbb{C}), \mathbb{R}(p-1))^+ = [\mathbb{R}(p-1)^{X(\mathbb{C})}]^+,$$

the + meaning fixed elements under complex conjugation.

Note also that the complex $\mathbb{R}(n)_{\mathcal{D}}$ on Spec($\mathbb{C}$) reduces for $n > 0$ to degrees 0 and 1, $\mathbb{R}(n) \to \mathbb{C}$, and via the projection $\pi_{n-1} : \mathbb{C} \to \mathbb{R}(n-1)$ it is isomorphic to $\mathbb{R}(n-1)[-1]$. We obtain the following computation:

$$H^q_{\mathcal{D}}(\operatorname{Spec}\mathbb{C}, \mathbb{R}(n)) = \begin{cases} \mathbb{R}(n-1) & q = 1 \\ 0 & q \neq 1. \end{cases}$$

Let us turn to the main subject. The Beilinson construction is based on the existence of the n-th **Chern class** $c_n \in H^{2n}_{\mathcal{D}}(B.GL_N, \mathbb{Q}(n))$ (see Esnault, Viehweg [5]) which in turn defines an element $c_n \in H^{2n}_{\mathcal{D}}(B.GL_N, \mathbb{R}(n))$. Consider also the evaluation morphism of simplicial schemes

$$e : \operatorname{Spec}\mathbb{C} \times B.GL_N(\mathbb{C}) \longrightarrow B.GL_N$$

(on the left side the simplicial set $B.GL_N(\mathbb{C})$ is viewed as a scheme, namely disjoint union of points). Finally, using Künneth formula (the coefficient system $\mathbb{R}(n)$ is a $\mathbb{R}$-vector space), we obtain:

$$\begin{aligned} e^\star(c_n) &\in H^{2n}_{\mathcal{D}}(\operatorname{Spec}\mathbb{C} \times B.GL_N(\mathbb{C}), \mathbb{R}(n)) \\ &\cong H^1_{\mathcal{D}}(\operatorname{Spec}\mathbb{C}, \mathbb{R}(n)) \otimes H^{2n-1}(B.GL_N(\mathbb{C}), \mathbb{R}) \\ &\overset{\pi_{n-1}}{\cong} H^{2n-1}(B.GL_N(\mathbb{C}), \mathbb{R}(n-1)) = H^{2n-1}(GL_N(\mathbb{C}), \mathbb{R}(n-1)) \end{aligned}$$

The element produced is invariant under both actions of $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$ on the discrete group $GL_N(\mathbb{C})$ and on $\mathbb{R}(n-1)$. The construction is compatible with increasing N which is taken large enough. It is also known that

$$H^{2n-1}(GL_N(\mathbb{C}), \mathbb{R}(n-1)) = \operatorname{Hom}(H_{2n-1}(GL_N(\mathbb{C}), \mathbb{Z}), \mathbb{R}(n-1)).$$

Using again the Hurewicz map we obtain the following composite:

$$\begin{aligned} K_{2n-1}(\mathbb{C}) = \pi_{2n-1}(BGL(\mathbb{C})^+) &\to H_{2n-1}(GL(\mathbb{C}), \mathbb{Z}) \\ &\to \mathbb{R}(n-1) = H^1_{\mathcal{D}}(\operatorname{Spec}\mathbb{C}, \mathbb{R}(n)). \end{aligned} \tag{4.1}$$

One last computation shows that, writing

$$X_{\mathbb{C}} = \operatorname{Spec} F \otimes \mathbb{C} = \prod_{\sigma: F \to \mathbb{C}} \mathbb{C},$$

we obtain:

$$\begin{aligned} H^1_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n)) &= \Big[\bigoplus_{\sigma: F \to \mathbb{C}} H^1_{\mathcal{D}}(\operatorname{Spec}\mathbb{C}, \mathbb{R}(n)) \Big]^+ \\ &= \Big[\bigoplus_{\sigma: F \to \mathbb{C}} \mathbb{R}(n-1) \Big]^+. \end{aligned}$$

Definition 4.3 *The* **Beilinson n-th regulator map***, denoted ρ_n^{Bei}, is defined as the composite of maps*

$$\rho_n^{Bei} : K_{2n-1}(\mathcal{O}_F) \to K_{2n-1}(\mathbb{C})^{X(\mathbb{C})} \to H^1_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n)),$$

the latter map having each component the map (4.1). We then define the **Beilinson regulator***, R_n^{Bei}, as the covolume of the lattice $\rho_n^{Bei}(K_{2n-1}(\mathcal{O}_F))$.*

The relation between the two regulator maps and the corresponding regulators is expressed in the next result (refer to Rapoport's article in [65] and Burgos Gil [14]):

Theorem 4.4 *(1) The Borel regulator map is equal to* 2 *times the Beilinson regulator map.*
(2) The functorial properties of both regulators are the same, and they differ by a power of 2.

Remark 4.5 *(1) We note first that, based on the localization sequence and on the fact that the higher even K-groups of a finite field are finite, we have for $n > 1$:*

$$K_{2n-1}(\mathcal{O}_F) \otimes \mathbb{Q} \cong K_{2n-1}(F) \otimes \mathbb{Q}$$

(in fact, the two groups are isomorphic without the tensoring by $\mathbb{Q}$ as we saw earlier.) We also saw that for $n > 1$ the homomorphism $\rho_n \otimes \mathbb{Q}$ is injective, defining a $\mathbb{Q}$-structure on $H^1_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n))$ and that the covolume of its image is equal (modulo $\mathbb{Q}^\times$) to $\zeta^*_F(1-n)$. Beilinson's conjecture predicts the analogous facts for $H^1_{\mathcal{A}}(X, \mathbb{Q}(n))$.
(2) From the compatibility of the Chern character (equal to the regulator up to some constant in $\mathbb{Q}^\times$) with the Adams operations, the preceding observation implies:

$$K_{2n-1}(F) \otimes \mathbb{Q} = H^1_{\mathcal{A}}(X, \mathbb{Q}(n)).$$

In this final part we introduce a new regulator using motivic cohomology.

Let $H^i_{\mathcal{M}}(F, \mathbb{Z}(n))$ denote Voevodsky's motivic cohomology groups for Spec$F$ (refer to [80] for the complicated machinery behind their definition - we will be using only important properties of them). They are related to the algebraic K-theory groups of F via the following motivic version of the third-quadrant Bloch-Lichtenbaum spectral sequence (refer to [8], [38]):

$$E_2^{s,t} = H^{s-t}_{\mathcal{M}}(F, \mathbb{Z}(-t)) \Rightarrow K_{-s-t}(F),\ s,t \leq 0.$$

We have:

$$\text{rank}\, H^1_{\mathcal{M}}(F, \mathbb{Z}(n)) = \text{rank}\, K_{2n-1}(F),$$

as Kahn [38] and Voevodsky [80] have shown that

$$H^1_{\mathcal{M}}(F, \mathbb{Z}(n)) \otimes \mathbb{Z}_2 \cong H^1_{\acute{e}t}(F, \mathbb{Z}_2(n))$$

and we then use Theorem 3.5. Moreover, for a finite Galois extension of number fields, E/F, such that the Galois group G is a 2-group (in particular, a CM-extension), we have Galois descent:

$$H^1_{\mathcal{M}}(E, \mathbb{Z}(n))^G \cong H^1_{\mathcal{M}}(F, \mathbb{Z}(n)).$$

Indeed, the kernel and the cokernel of the map

$$H^1_{\mathcal{M}}(F, \mathbb{Z}(n)) \longrightarrow H^i_{\mathcal{M}}(E, \mathbb{Z}(n))^G$$

are annihilated by $|G|$. As $|G|$ is a 2-power, we can detect them by tensoring with $\mathbb{Z}_2$. But, using Kahn's result, we obtain the corresponding map in étale cohomology where it is known that the Galois descent holds (refer to chapter 2). Finally, the 2-torsion of $H^1_{\mathcal{M}}(F, \mathbb{Z}(n))$ is the 2-torsion of $H^1_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_2(n))$, whose order is $w_n^{(2)}(F)$ (refer to chapter 2).

Much less it is known about $H^2_{\mathcal{M}}(\mathcal{O}_F^S, \mathbb{Z}(n))$. Nevertheless, we will define it here as follows:

$$H^2_{\mathcal{M}}(\mathcal{O}_F^S, \mathbb{Z}(n)) := \prod_p H^2_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_p(n))$$

and its order will be $h_n(F)$, where

$$h_n(F) := \prod_p |H^2_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_p(n))|.$$

Thus, in our Lichtenbaum formulas, we can think of $h_n(F)$ and $w_n(F)$ as orders of motivic cohomology groups.

Definition 4.6 *We define for $n \geq 2$ the n^{th}* **motivic regulator** *$R_n^{\mathcal{M}}(F)$ of F as the covolume of the lattice $\rho_n^{Bei}(H^1_{\mathcal{M}}(F, \mathbb{Z}(n))$.*

Let E/F be a CM-extension of number fields and let $n \geq 3$ be odd. Then, as we have just seen, $H^1_{\mathcal{M}}(F, \mathbb{Z}(n))$ and $H^1_{\mathcal{M}}(E, \mathbb{Z}(n))$ have the same rank $r_1(F)$, and

$H^1_{\mathcal{M}}(F, \mathbb{Z}(n))$ injects into $H^1_{\mathcal{M}}(E, \mathbb{Z}(n))$. Let us denote by $\tilde{A}$ the torsion free quotient $A/\text{tors}\, A$ of an abelian group A. Kolster defines in [43] the higher Q-indices in terms of étale cohomology groups (2-adically):

$$Q_n := [\tilde{H}^1_{\acute{e}t}(E, \mathbb{Z}_2(n)) : \tilde{H}^1_{\acute{e}t}(F, \mathbb{Z}_2(n))].$$

These Q-indices have the following motivic interpretation:

Proposition 4.7 *Let E/F be a CM-extension of number fields and $n \geq 3$ odd. Then*

$$Q_n = [\tilde{H}^1_{\mathcal{M}}(E, \mathbb{Z}(n)) : \tilde{H}^1_{\mathcal{M}}(F, \mathbb{Z}(n))].$$

Proof: We have the following equality of finite indices:

$$[\tilde{H}^1_{\mathcal{M}}(E, \mathbb{Z}(n)) : \tilde{H}^1_{\mathcal{M}}(F, \mathbb{Z}(n))] = [H^1_{\mathcal{M}}(E, \mathbb{Z}(n)) : H^1_{\mathcal{M}}(F, \mathbb{Z}(n)) \cdot \text{tors}\, H^1_{\mathcal{M}}(E, \mathbb{Z}(n))].$$

Let σ be a generator of $G := \text{Gal}(E/F)$, and let $x \in H^1_{\mathcal{M}}(E, \mathbb{Z}(n))$. Then x/x^σ is torsion, as $H^1_{\mathcal{M}}(E, \mathbb{Z}(n))^G = H^1_{\mathcal{M}}(F, \mathbb{Z}(n))$ has finite index in $H^1_{\mathcal{M}}(E, \mathbb{Z}(n))$. Also, xx^σ belongs to $H^1_{\mathcal{M}}(F, \mathbb{Z}(n))$. We obtain:

$$x^2 = xx^\sigma \cdot (x/x^\sigma) \in H^1_{\mathcal{M}}(F, \mathbb{Z}(n)) \cdot \text{tors}\, H^1_{\mathcal{M}}(E, \mathbb{Z}(n)).$$

Hence the index $[\tilde{H}^1_{\mathcal{M}}(E, \mathbb{Z}(n)) : \tilde{H}^1_{\mathcal{M}}(F, \mathbb{Z}(n))]$ can be computed by tesoring with $\mathbb{Z}_2$, as it is 1 or 2. It follows:

$$[\tilde{H}^1_{\mathcal{M}}(E, \mathbb{Z}(n)) : \tilde{H}^1_{\mathcal{M}}(F, \mathbb{Z}(n))] = [\tilde{H}^1_{\mathcal{M}}(E, \mathbb{Z}(n)) \otimes \mathbb{Z}_2 : \tilde{H}^1_{\mathcal{M}}(F, \mathbb{Z}(n)) \otimes \mathbb{Z}_2]$$

$$= [\tilde{H}^1_{\acute{e}t}(E, \mathbb{Z}_2(n)) : \tilde{H}^1_{\acute{e}t}(F, \mathbb{Z}_2(n))] = Q_n. \square$$

In the same context, we can compare the motivic regulators of F and E.

Proposition 4.8 *Let E/F be a CM-extension of number fields and let $n \geq 3$ be odd. Let $r_1 = [F : \mathbb{Q}]$. Then:*

$$\frac{R_n^{\mathcal{M}}(E)}{R_n^{\mathcal{M}}(F)} = \frac{2^{r_1}}{Q_n}.$$

Also,

$$\frac{R_n^{Bei}(E)}{R_n^{Bei}(F)} = \frac{2^{\delta_F+1}}{Q_n}$$

if $n \equiv 1 \, mod \, 4$, *and*

$$\frac{R_n^{Bei}(E)}{R_n^{Bei}(F)} = \frac{2^{r_1}}{Q_n}$$

if $n \equiv 3 \, mod \, 4$.

Proof: It is enough to prove that:

$$\rho_{n,E}^{\mathcal{M}}(H_{\mathcal{M}}^1(F, \mathbb{Z}(n))) = 2^{r_1} \cdot R_n^{\mathcal{M}}(F).$$

Once we have this, using the previous result we obtain:

$$\frac{R_n^{\mathcal{M}}(E)}{R_n^{\mathcal{M}}(F)} = \frac{\rho_{n,E}^{\mathcal{M}}(H_{\mathcal{M}}^1(E, \mathbb{Z}(n)))}{R_n^{\mathcal{M}}(F)} = \frac{\rho_{n,E}^{\mathcal{M}}(H_{\mathcal{M}}^1(F, \mathbb{Z}(n)))/Q_n}{R_n^{\mathcal{M}}(F)} = \frac{2^{r_1}}{Q_n}.$$

The functorial properties of the regulator maps imply the following commutative diagram:

$$\begin{array}{ccc} K_{2n-1}(E) \otimes \mathbb{Q} & \stackrel{\rho_{n,E}^{\mathcal{M}}}{\longrightarrow} & [\mathbb{R}(n-1)^{\mathrm{Hom}(E,\mathbb{C})}]^+ \\ \downarrow & & \downarrow \\ K_{2n-1}(F) \otimes \mathbb{Q} & \stackrel{\rho_{n,F}^{\mathcal{M}}}{\longrightarrow} & [\mathbb{R}(n-1)^{\mathrm{Hom}(F,\mathbb{C})}]^+ \end{array}$$

The first vertical arrow is the trace in K-theory, $Tr_{E/F}$, and the second vertical arrow is defined as follows: for each pair $(\sigma, \bar{\sigma})$ of conjugate complex embeddings of E in $\mathbb{C}$ the corresponding component $(\mathbb{R}(n-1), \mathbb{R}(n-1))^+$ is mapped to the component $\mathbb{R}(n-1)$ corresponding to $\sigma|_F$ via $(x, x) \mapsto 2x$. It follows that the covolume of $\rho_{n,E}^{\mathcal{M}}(H_{\mathcal{M}}^1(F, \mathbb{Z}(n)))$ is equal to the covolume of $\rho_{n,F}^{\mathcal{M}}(2H_{\mathcal{M}}^1(F, \mathbb{Z}(n)))$, that is $2^{r_1} \cdot R_n^{\mathcal{M}}(F)$. For the Beilinson regulators one considers the analogous index in K-theory and uses theorem 3.7.□

We would like to compare the Beilinson regulator and the motivic regulator. We have seen that 2-adically we can compare the two types of cohomology groups and the K-groups. Unfortunately, a direct map between the integral K-groups and the integral motivic groups it is not known, unless one assumes that the Beilinson-Soulé Conjecture, $H^i_{\mathcal{M}}(F, \mathbb{Z}(n)) = 0$ for all $i \leq 0$ and all $n \geq 1$, is valid. If this is true, then there is an edge morphism in the Bloch-Lichtenbaum spectral sequence

$$K_{2n-1}(F) \longrightarrow H^1_{\mathcal{M}}(F, \mathbb{Z}(n)).$$

Assuming that the Beilinson-Soulé Conjecture holds for F, then for all $n \geq 2$:

$$R_n^{Bei}(F) = R_n^{\mathcal{M}}(F)$$

if $n \equiv 0, 1, 2 \bmod 4$ or F is totally complex, and

$$R_n^{Bei}(F) = 2^{r_1 - 1 - \delta_F} \cdot R_n^{\mathcal{M}}(F),$$

otherwise (δ_F is the signature defect introduced in chapter 3) .

The existence of this motivic regulator and the conjectural relation between motivic cohomology groups and K-groups suggest a motivic formulation of the Lichtenbaum Conjecture.

Chapter 5

Class field theory for abelian pro-p extensions of number fields

In order to deal with abelian pro-p extensions of number fields Jaulent (refer to [36]) introduced the **p-adification** of classical class field theory. For any number field F we denote by F^{ab} the maximal abelian extension of F and by $F^{(p)}$ the maximal abelian pro-p extension of F.

We start with the local case. Let F be an l-adic local field (i.e. a finite extension of $\mathbb{Q}_l$). There is a map, the local Artin map,

$$F^\times \longrightarrow \mathrm{Gal}(F^{ab}/F)$$

which is injective, but not surjective. Let $\hat{\mathbb{Z}} = \prod_q \mathbb{Z}_q$ denote the completion of $\mathbb{Z}$ with respect to the subgroup topology. The pro-finite completion of $F^\times$ is

$$\varprojlim_H F^\times/H \cong \hat{\mathbb{Z}} \times U_F,$$

where H ranges over a cofinal sequence of open, subgroups of finite index, and U_F denotes as usual the unit group of F. The local Artin map extends to a topological isomorphism

$$\hat{\mathbb{Z}} \times U_F \cong \mathrm{Gal}(F^{ab}/F).$$

Now, let $\hat{F} := F^\times \otimes \mathbb{Z}_p$ denote the p-adic completion of F. Restricting the previous isomorphism to p-primary parts we obtain the following topological isomorphism

$$\hat{F} \cong \mathrm{Gal}(F^{(p)}/F).$$

Denote also $\hat{U}_F := U_F \otimes \mathbb{Z}_p$. This is equal to $U_{F,1}$, the group of principal units, if $l = p$, and is equal to $\mu(F)(p)$, if $l \neq p$. Under this isomorphism $\hat{U}_F$ is mapped onto the inertia group T_F. In particular, we note that the inertia group T_F is finite when $l \neq p$.

Moreover, the closed subgroups of $\hat{F}$ are precisely the norm groups $N_{L/F}(\hat{L})$ of abelian subextensions L/F of $F^{(p)}/F$ (here, $N_{L/F}(\hat{L})$ denotes the intersection of all norm groups $N_{E/F}(\hat{E})$ with E/F running through the finite subextensions of L/F). We obtain the following canonical isomorphisms

$$\hat{F}/N_{L/F}(\hat{L}) \cong \mathrm{Gal}(L/F), \text{ and}$$

$$\hat{U}_F/N_{L/F}(\hat{U}_L) \cong T(L/F),$$

where $T(L/F)$ is the inertia subgroup of $\mathrm{Gal}(L/F)$.

Because $F^\times$ is a finitely generated abelian group if F is an l-adic local field, we have

$$\hat{F} \cong \varprojlim_n F^\times/(F^\times)^{p^n}.$$

This is why we are defining

$$\hat{F} := \varprojlim_n F^\times/(F^\times)^{p^n}, \text{ and}$$

$$\hat{U}_F := \hat{F}$$

for $F = \mathbb{R}$ or $F = \mathbb{C}$. These groups are trivial except in the situation $p = 2$ and $F = \mathbb{R}$ when they are equal to $\mathbb{Z}/2\mathbb{Z}$.

We continue with the global case. Now, let F be a finite extension of $\mathbb{Q}$ and v a finite or infinite prime of F. Let F_v and U_{F_v} denote the completion of F at v and the local units of F_v, respectively. If v is infinite, we put $U_{F_v} := F_v^\times$. Let J_F denote the idéle group of F and C_F the idéle class group of F. The global Artin map provides a topological epimorphism

$$J_F \longrightarrow \mathrm{Gal}(F^{ab}/F),$$

factoring through C_F. The kernel of the induced map $C_F \to \mathrm{Gal}(F^{ab}/F)$ - called the subgroup of universal norms of C_F - is the connected component of the identity of C_F, which in turn is the maximal divisible subgroup of C_F (see Artin-Tate [2], Chap 14, Prop. 10). The p-adification $\hat{J}_F$ of J_F is defined as the restricted topological product of all $\hat{F}_v$ with respect to the local units $\hat{U}_{F_v}$. Because the kernel of the natural map $J_F \to \hat{J}_F$ vanishes under the Artin map (it is the maximal p-divisible subgroup of J_F), we obtain a continuous epimorphism

$$\hat{J}_F \longrightarrow \mathrm{Gal}(F^{(p)}/F).$$

Put

$$\hat{F} := F^\times \otimes \mathbb{Z}_p \hookrightarrow \varprojlim_n F^\times/(F^\times)^{p^n}.$$

It is known that

$$F^\times/(F^\times)^{p^n} \longrightarrow \prod_v F_v^\times/(F_v^\times)^{p^n}$$

has kernel of exponent 2, so that it vanishes in the projective limit. The natural map

$$\hat{F} \longrightarrow \hat{J}_F$$

is injective, and continuous, and $\hat{F}$ is closed in $\hat{J}_F$. This allows us to define

$$\hat{C}_F := \hat{J}_F/\hat{F},$$

the p-adification of the idéle class group C_F. Since the passage from J_F to $\hat{J}_F$ eliminates the p-divisible part of J_F, we obtain a topological isomorphism

$$\hat{C}_F \cong \mathrm{Gal}(F^{(p)}/F).$$

Moreover, the class field theory of abelian pro-p extensions of F can be described as follows: There is a one-to-one correspondence between closed subgroups of $\hat{C}_F$ and abelian pro-p extensions of F. If L/F is an abelian pro-p extension of F, then there are canonical isomorphisms

$$\hat{J}_F/N_{L/F}(\hat{J}_L)\hat{F} \cong \hat{C}_F/N_{L/F}(\hat{C}_L) \cong \mathrm{Gal}(L/F),$$

$$\hat{U}_{F_v}/N_{L/F}(\hat{J}_L)\hat{F} \cap \hat{U}_{F_v} \cong T_v(L/F), \text{ and}$$

$$\hat{F}_v/N_{L/F}(\hat{J}_L)\hat{F} \cap \hat{F}_v \cong D_v(L/F),$$

where $T_v(L/F)$ is the inertia subgroup and $D_v(L/F)$ is the decomposition subgroup of $\mathrm{Gal}(L/F)$ at the prime v.

We are ready now to study Galois groups of important abelian pro-p extensions of a number field F.

Let S be a finite set of primes of F (it could contain both finite and infinite primes). Let S_p be the set of finite primes of F above p, and S_∞ the set of infinite primes of F.

Let $M_S(F)$ denote the maximal abelian S-ramified pro-p extension of F (that is, the maximal abelian pro-p extension, which is unramified outside primes in S). Let $H_S(F)$ be the Hilbert S-class field of F, that is the maximal abelian unramified p-extension of F in which all primes in S decompose (so-called S-decomposed). We have then:

Proposition 5.1 *There are canonical four term exact sequences*

$$\hat{U}_F \longrightarrow \prod_{v\in S} \hat{U}_{F_v} \longrightarrow \mathrm{Gal}(M_S(F)/F) \longrightarrow \mathrm{Gal}(H(F)/F) \longrightarrow 0, \textit{ and}$$

$$\hat{U}_F^S \longrightarrow \prod_{v\in S} \hat{F}_v \longrightarrow \mathrm{Gal}(M_S(F)/F) \longrightarrow \mathrm{Gal}(H_S(F)/F) \longrightarrow 0.$$

Proof: Class field theory provides us with the following canonical isomorphisms

$$\hat{J}_F/\prod_{v\notin S} \hat{U}_{F_v} \cdot \hat{F} \cong \mathrm{Gal}(M_S(F)/F),$$

$$\hat{J}_F/\prod_{v} \hat{U}_{F_v} \cdot \hat{F} \cong \mathrm{Gal}(H(F)/F), \text{ and}$$

$$\hat{J}_F / \prod_{v\in S} \hat{F}_v \cdot \prod_{v\notin S} \hat{U}_{F_v} \cdot \hat{F} \cong \mathrm{Gal}(H_S(F)/F).$$

By taking quotients we obtain

$$\mathrm{Gal}(M_S(F)/H(F) \cong \prod_{v} \hat{U}_{F_v} \cdot \hat{F} / \prod_{v\notin S} \hat{U}_{F_v} \cdot \hat{F} \cong \prod_{v\in S} \hat{U}_{F_v} / (\prod_{v\in S} \hat{U}_{F_v}) \cap (\prod_{v\notin S} \hat{U}_{F_v} \cdot \hat{F})$$

$$\cong \mathrm{coker}(\hat{U}_F \to \prod_{v\in S} \hat{U}_{F_v})$$

and

$$\mathrm{Gal}(M_S(F)/H_S(F) \cong \prod_{v\in S} \hat{F}_v / (\prod_{v\in S} \hat{F}_v) \cap (\prod_{v\notin S} \hat{U}_{F_v} \cdot \hat{F}) \cong \mathrm{coker}(\hat{U}_F^S \to \prod_{v\in S} \hat{F}_v). \square$$

Definition 5.2 *The kernel* $\ker(\hat{U}_F \to \prod_{v\in S_p} \hat{U}_{F_v})$ *is called the* **Leopoldt kernel**. *It is a free* $\mathbb{Z}_p$*-module.*

Leopoldt conjectured that this kernel is trivial. Brumer [13] proved this assertion in the following abelian cases:

Theorem(Leopoldt's Conjecture) 5.3 *If* F *is abelian over* $\mathbb{Q}$ *or* $\mathbb{Q}(\sqrt{-d})$, $d > 0$, *then the Leopoldt kernel is trivial.*

Remark 5.4 *A simple diagram chase on the sequences of Proposition 5.1 shows that* $\ker(\hat{U}_F \to \prod_{v\in S} \hat{U}_{F_v}) \cong \ker(\hat{U}_F^S \to \prod_{v\in S} \hat{F}_v)$. *This gives us an alternative description of the Leopoldt kernel, namely,* $\ker(\hat{U}_F^{S_p} \to \prod_{v\in S_p} \hat{F}_v)$. *So, they are both trivial if* F *is abelian. Also, in the abelian case, we have that* $\ker(\hat{U}_F \to \prod_{v\in S_2\cup S_\infty} \hat{U}_{F_v})$ *is trivial.*

Remark 5.5 *The group* $\mathrm{Gal}(M_S(F)/F)$ *is a finitely generated* $\mathbb{Z}_p$*-module, because the groups of local units are finitely generated and the Hilbert class field is a finite extension of* F. *Moreover, from Proposition 5.1 and the finiteness of Hilbert fields, we obtain*

$$\mathrm{rk}_{\mathbb{Z}_p}\mathrm{Gal}(M_S(F)/F) = (r_1(F) + 2r_2(F)) - (r_1(F) + r_2(F) - 1 - \delta_F) = 1 + r_2(F) - \delta_F.$$

This is also the number of independent $\mathbb{Z}_p$*-extensions of* F *(see next chapter).*

Chapter 6

Iwasawa theory of number fields. Galois groups as Λ-modules.

Definition 6.1 *Let p be a prime number, and let F be a number field or a completion of a number field at one of its places. An extension F_∞/F is called a* **$\mathbb{Z}_p$-extension** *if* $\mathrm{Gal}(F_\infty/F) \cong \mathbb{Z}_p$.

Using the bijective correspondence between the closed subgroups of $\mathbb{Z}_p$, namely 0, $p^r\mathbb{Z}_p$ for all $r \geq 0$, and finite normal subextensions of F_∞/F, we obtain a unique tower

$$F_0 := F \subset F_1 \subset F_2 \subset \ldots \subset F_\infty = \bigcup_{r \geq 0} F_r$$

such that $[F_r : F] = p^r$ and $\mathrm{Gal}(F_r/F) \cong \mathbb{Z}/p^r\mathbb{Z}$. The most important example of a $\mathbb{Z}_p$-extension of F is the **cyclotomic $\mathbb{Z}_p$-extension** of F. It is defined as the unique $\mathbb{Z}_p$-extension of F inside $E_\infty := \bigcup_{r \geq 0} E_r$, where $E_r := F(\zeta_{p^{r+\delta}})$, corresponding to the subgroup of the Galois group $\mathrm{Gal}(E_\infty/F) \cong \mathbb{Z}_p \times \Delta$ ($\Delta \cong \mathrm{Gal}(E_0/F)$ is finite) that is isomorphic to $\mathbb{Z}_p$. Here $\delta = 0$ if p is odd, and $\delta = 1$ if $p = 2$.

We summarize now the ramification properties of $\mathbb{Z}_p$-extensions of number fields. Let F be a number field. Let S_p be the set of primes of F above p. For a $\mathbb{Z}_p$-extension of F, F_∞/F, there is at least one prime which ramifies in F_∞/F. Also, F_∞/F is **p-ramified**, i.e. it is unramified at all primes not belonging to S_p, and there is an $e \geq 0$

such that each ramified prime in F_∞/F_e is totally ramified. Moreover, if F_∞/F is the cyclotomic $\mathbb{Z}_p$-extension, then all primes in S_p are ramified in F_∞/F. We put $\mathbb{R}$ as the only $\mathbb{Z}_p$-extension of $\mathbb{R}$.

For a number field or a completion of it, F, and a $\mathbb{Z}_p$-extension of F, F_∞/F, consider an abelian pro-p-extension of F_∞, say K_∞, such that K_∞/F is still Galois and denote $G := \mathrm{Gal}(K_\infty/F)$. The study of the Galois group $X := \mathrm{Gal}(K_\infty/F_\infty)$ is of great importance (especially when some ramification restrictions apply to K_∞). It acquires immediately a $\mathrm{Gal}(F_\infty/F)$-module structure, because of the exact sequence of $\mathbb{Z}_p$-modules $0 \to X \to G \to \Gamma \to 0$, where from now on $\Gamma \cong \mathrm{Gal}(F_\infty/F)$ denotes a multiplicative version of $\mathbb{Z}_p$, with a fixed topological generator γ. Namely, for $x \in X$ we define the action of γ as follows: $x^\gamma := \tilde{\gamma}x\tilde{\gamma}^{-1}$, where $\tilde{\gamma}$ is a lift of γ to G. The action is well defined because X is abelian.

We introduce now important facts from the theory of compact Γ-modules or, more accurately, compact Λ-modules, where Λ is defined below.

Definition 6.2 *The* **Iwasawa algebra** *of* Γ*, denoted* $\mathbb{Z}_p[[\Gamma]]$*, is defined as follows:*

$$\mathbb{Z}_p[[\Gamma]] := \varprojlim \mathbb{Z}_p[\Gamma/\Gamma^{p^r}].$$

Let $\omega_r = (1+T)^{p^r} - 1$, $r \geq 1$, and $\omega_0 = T$. Note that $\omega_{r+1} = (\omega_r + 1)^p - 1$. It is easily proved that the map $\gamma \bmod \Gamma^{p^r} \mapsto 1 + T \bmod (\omega_r)$ provides an isomorphism $\mathbb{Z}_p[\Gamma/\Gamma^{p^r}] \cong \mathbb{Z}_p[T]/(\omega_r)$ compatible with the projective limits. Thus we obtain the following description of the Iwasawa-algebra

$$\mathbb{Z}_p[[\Gamma]] \cong \varprojlim \mathbb{Z}_p[T]/(\omega_r).$$

On the other hand, we have the ring of formal power series $\mathbf{\Lambda} := \mathbb{Z}_p[[T]]$, which is a noetherian local Krull domain, with Krull dimension 2. The lattice of prime ideals consists (0), (p), (p,T) and all prime ideals $(P(T))$ with $P(T)$ irreducible and **distinguished** (i.e. a monic polynomial having all other coefficients divisible by p). Moreover, for this ring one has an Euclidean Algorithm (given $f = \sum_{i=0}^{\infty} a_i T^i \in \Lambda$ such that $p \mid a_i$, for $0 \leq i \leq n-1$, $p \nmid a_n$, and $g \in \Lambda$, there is a unique presentation

$g = q \cdot f + r$, where $q \in \Lambda$, and $r \in \mathbb{Z}_p[T]$, $\deg r \leq n-1$), and a Weierstrass Preparation Theorem (given $f = \sum_{i=0}^{\infty} a_i T^i \in \Lambda$ such that $p \mid a_i$, for $0 \leq i \leq n-1$, $p \nmid a_n$, it has a unique presentation $f(T) = P(T) \cdot U(T)$ where $P(T)$ is a distinguished polynomial of degree n, and $U(T) \in \Lambda^\times$, i.e. it is an invertible power series; a general f can by presented uniquely as $f(T) = p^\mu \cdot P(T) \cdot U(T)$, where $\mu \geq 0$).

Using the Euclidean Algorithm we obtain a well-defined morphism of rings $\Lambda \to \mathbb{Z}_p[T]/(\omega_r)$, which is surjective because Λ is complete with respect to the (p,T)-adic topology, and it is injective because of the Krull Intersection theorem: $\cap_{r \geq 1}(p,T)^r = 0$. We are ready to conclude that

$$\mathbb{Z}_p[[\Gamma]] \cong \Lambda.$$

The structure of compact Λ-modules is known up to pseudo-isomorphisms. Generally, let R be a noetherian Krull domain and M a compact Λ-module. M is called **pseudo-null** if $M_{\mathfrak{p}} = (0)$ for all $\mathfrak{p}$ prime with height$(\mathfrak{p}) \leq 1$ (let us denote by $\mathrm{Spec}^1(R)$ the set of these primes). In the case $R = \Lambda$ this amounts to M being finite. A morphism of R-modules $f : M \to N$ is called **pseudo-isomorphism** if $\ker f$ and $\mathrm{coker} f$ are pseudo-null. The notation is $M \sim N$. For $R = \Lambda$ the kernel and cokernel should be finite. The relation " $\sim$ " is an equivalence relation on R-torsion modules.

Theorem(the structure of Λ-modules) 6.3 *Let M be a finitely generated Λ-module. Then there are integers $d \geq 0$, $m \geq 0$ and $n_i \geq 1$, $1 \leq i \leq m$, such that*

$$M \sim \Lambda^d \oplus \bigoplus_{i=1}^{m} \Lambda/\mathfrak{p}_i^{n_i}$$

where $\mathfrak{p}_i \in \mathrm{Spec}^1(\Lambda)$. The integers d, m, n_i and the prime ideals $\mathfrak{p}_i$ are unique up to a permutation of indices.

This theorem allows us to associate to M the following invariants:

Definition 6.4 *Let M be a finitely generated Λ-module. Suppose that*

$$M \sim \Lambda^d \oplus \bigoplus_{i=1}^{m} \Lambda/\mathfrak{p}_i^{n_i}.$$

(1) Λ *-modules of the form*

$$E(d;\mathfrak{p}_1^{n_1},\cdots,\mathfrak{p}_m^{n_m}) := \Lambda^d \oplus \bigoplus_{i=1}^{m} \Lambda/\mathfrak{p}_i^{n_i}$$

are called **elementary** Λ **-modules.**

(2) The **divisor** *of* M *is defined as*

$$\mathrm{div}(M) = \mathrm{div}(E(d;\mathfrak{p}_1^{n_1},\cdots,\mathfrak{p}_m^{n_m})) := \sum_{i=1}^{m} n_i\mathfrak{p}_i.$$

(3) The product of ideals $\prod_{i=1}^{m}\mathfrak{p}_i^{n_i}$ *is called the* **characteristic ideal** *of* M *and any generator of it, say* $G(T)$*, is called a* **characteristic power series** *of* M*. Moreover, using the Weierstrass Preparation Theorem, we have uniquely* $G(T) = p^{\mu}\cdot F(T)\cdot U(T)$*, where* $F(T)$ *is a distinguished polynomial and* $U(T)\in\Lambda^{\times}$*. The polynomial* $p^{\mu}\cdot F(T)$ *is called the* **characteristic polynomial** *of* M *and it is denoted by* **char***(M). The exponent* μ *is called the* μ**-invariant** *of* M*, and* $\lambda = \deg F(T)$ *is called the* λ-**invariant** *of* M.

The following results are proved by Iwasawa in [34].

Lemma 6.5 *Let* M *be a finitely generated* Λ*-torsion module. Then the following assertions are equivalent:*

(a) M^{Γ} *is finite.*

(b) M_{Γ} *is finite.*

(c) $\mathrm{char}(M)(0)\neq 0$.

If any of these is true, then we have:

$$\frac{|M^{\Gamma}|}{|M_{\Gamma}|} = |\mathrm{char}(M)(0)|_p = p^{-v_p((\mathrm{char}(M)(0))}.$$

If we replace $\mathbb{Z}_p$ by the integral closure of $\mathbb{Z}_p$ in a finite extension of $\mathbb{Q}_p$, the previous result remains true as do most of the results in Iwasawa theory.

Regarding Λ-rank and Λ-freeness of a Λ-module, we have:

Lemma 6.6 *Let M be a finitely generated Λ-module. We have:*
(1) $\mathrm{rk}_\Lambda M = \mathrm{rk}_{\mathbb{Z}_p} M_\Gamma - \mathrm{rk}_{\mathbb{Z}_p} M^\Gamma$.
(2) If $M^\Gamma = 0$ and M_Γ is $\mathbb{Z}_p$-torsion free, then M is a free Λ-module and $\mathrm{rk}_\Lambda M = \mathrm{rk}_{\mathbb{Z}_p} M_\Gamma$.

Lemma 6.7 *Let $0 \to M \to N \to P \to 0$ be an exact sequence of finitely generated Λ-modules. We have:*
(1) If N is a free Λ-module and P is $\mathbb{Z}_p$-torsion free, then M is a free Λ-module.
(2) If P is a free Λ-module and M is $\mathbb{Z}_p$-torsion free, then N is a free Λ-module.

Another important characteristic of finitely generated Λ-torsion modules is the asymptotic behavior of the orders of the groups of coinvariants $M_{\Gamma^{p^n}}$ which can be thought as the quotients $M/\omega_n M$. Let us denote $\nu_{n,m} := \omega_m/\omega_n$ for all $m \geq n \geq 0$, and $\xi_n := \nu_{n,n+1}$ for $n \geq 1$, $\xi_0 := \omega_0 = T$.

Theorem(Iwasawa [34]) 6.8 *Let M be a finitely generated Λ-torsion module with Iwasawa invariants μ and λ.*

(a) There exists n_0, such that $M/\nu_{n_0,n} M$ is finite for all $n \geq n_0$.

(b) If n_0 is chosen as in (a), then for some constant ν depending only on M we have

$$|M/\nu_{n_0,n} M| = p^{\mu p^n + \lambda n + \nu} \text{ for all } n \gg 0.$$

Corollary 6.9 *(a) Let E be a non-trivial elementary Λ-torsion module, and let us assume that the $\mathbb{Z}_p$-torsion in $E/\omega_n E$ is bounded independent of n. Then $E \cong \oplus_{i=1}^t \Lambda/(\xi_{n_i})$. In particular, $E/\omega_n E$ is $\mathbb{Z}_p$-torsion free for large n.*

(b) Let M be a finitely generated Λ-module such that it is fixed by $\Gamma^{p^{n_0}}$ for some $n_0 \geq 0$. Then $M \sim \oplus_{i=1}^t \Lambda/(\xi_{n_i})$.

The next result is crucial for our computations in chapter 10. It is a generalization of Lemma 5.3 in [46].

Lemma 6.10 *Let F_∞/F be an arbitrary $\mathbb{Z}_p$-extension with Galois group Γ, and let*

$$0 \to A \to M \xrightarrow{f} N \to B \to 0$$

be an exact sequence of Λ-modules, such that $\mathrm{char}(M)(T) = h(T)\cdot \mathrm{char}(N)(T)$, *where* $h(T) \in \mathbb{Z}[T]$, $h(0) \neq 0$. *Let $f_1 : M^\Gamma \to N^\Gamma$ and $f_2 : M_\Gamma \to N_\Gamma$ be the induced maps in cohomology.*

If any of the four groups A^Γ, A_Γ, B^Γ, B_Γ is finite, the same is valid for the kernels and cokernels of f_1 and f_2 and we have

$$\frac{|\ker(f_1)|}{|\mathrm{coker}(f_1)|} = p^{-v_p(h(0))} \cdot \frac{|\ker(f_2)|}{|\mathrm{coker}(f_2)|}.$$

Proof: We have immediately that:

$$\mathrm{char}(A)(T) = \mathrm{char}(B)(T) \cdot h(T).$$

It follows that the finiteness of the groups A^Γ, A_Γ, B^Γ, B_Γ are equivalent statements, and, if they are finite, their Herbrant quotients are related as follows:

$$\frac{|A^\Gamma|}{|A_\Gamma|} = \frac{|B^\Gamma|}{|B_\Gamma|} \cdot p^{-v_p(h(0))}.$$

Taking kernels and cokernels along the following natural diagram:

$$\begin{array}{ccccccccccccc}
0 & \to & (\frac{M}{A})^\Gamma & \to & N^\Gamma & \to & B^\Gamma & \to & (\frac{M}{A})_\Gamma & \to & N_\Gamma & \to B_\Gamma & \to 0 \\
 & & \uparrow \alpha & & \uparrow f_1 & & & & \uparrow \beta & & \uparrow f_2 & & \\
 & & M^\Gamma & = & M^\Gamma & & & & M_\Gamma & = & M_\Gamma & &
\end{array}$$

we obtain:

$$\frac{|\ker(\alpha)|}{|\mathrm{coker}(\alpha)|} \cdot \frac{1}{B^\Gamma} \cdot \frac{|\ker(f_2)|}{|\mathrm{coker}(f_2)|} = \frac{|\ker(f_1)|}{|\mathrm{coker}(f_1)|} \cdot \frac{|\ker(\beta)|}{|\mathrm{coker}(\beta)|} \cdot \frac{1}{B_\Gamma}.$$

But $\ker(\alpha) = A^\Gamma$, the short sequence

$$0 \to \mathrm{coker}(\alpha) \to A_\Gamma \to \ker(\beta) \to 0$$

is exact, and $\mathrm{coker}(\beta) = 0$. The result is now immediate.□

We go back now to our Galois groups. Two of these Galois groups are defined as follows:

Definition 6.11 *Let F be a number field and let S be a finite set of primes of F. We denote by $L^S_{\infty,F}$* **the maximal abelian, unramified and S-decomposed** *(i.e. all primes lying above primes in S are totally decomposed)* **pro-p-extension** *of F_∞, and we set $X^S_{\infty,F} := \mathrm{Gal}(L^S_{\infty,F}/F_\infty)$ (the extension $L^S_{\infty,F}/F$ is Galois because of the maximality). Because the infinite primes are automatically decomposed in unramified extensions, they play no role in the previous definition. If $S = \emptyset$, then the index S is dropped from the notation. We use also the notation $X'_{\infty,F}$ when $S = S_p$, the set of all primes of F above p. Next, if $S = S_p \cup S_\infty$, i.e. the union of the set of all primes above p and the set of all infinite primes, we denote by $M_{\infty,F}$* **the maximal abelian S-ramified pro-p-extension** *of F_∞, and we set $\mathcal{X}_{\infty,F} := \mathrm{Gal}(M_{\infty,F}/F_\infty)$. When $S = S_p$, we denote the corresponding objects $M^f_{\infty,F}$ and $\mathcal{X}^f_{\infty,F}$ (this is relevant only in the case $p = 2$). $\mathcal{X}_{\infty,F}$ is called the* **standard Iwasawa module.**

Let us focus on $X^S_{\infty,F}$, where S is a finite set of primes. Let $H^S_{F_r}$ be the maximal unramified S-decomposed p-extension of F_r (so-called Hilbert S-class field of F_r). Let $A^S_{F_r}$ denote the p-Sylow subgroup of the S-ideal class group of F_r. By class field theory we have $\mathrm{Gal}(H^S_{F_r}/F_r) \cong A^S_{F_r}$, and these isomorphisms are compatible with restriction maps and norms. From here we can obtain another description of $X^S_{\infty,F}$, namely

$$X^S_{\infty,F} \cong \varprojlim A^S_{F_r},$$

and we see that $X^S_{\infty,F}$ is a finitely generated Λ-torsion module. Moreover, we have

$$A^S_{F_r} \cong \mathrm{Gal}(H^S_{F_r}/F_r) \cong X^S_{\infty,F}/\nu_{n_0,r} Y^S_{\infty,F}$$

for all $r \geq n_0$, where n_0 is chosen such that all ramified primes in F_∞/F_{n_0} are totally ramified (it exists for any $\mathbb{Z}_p$-extension), and $Y^S_{\infty,F} := \mathrm{Gal}(L^S_{\infty,F}/H^S_{F_{n_0}}F_\infty)$. The first isomorphism is defined by the Artin symbol $[\mathfrak{A}] \mapsto \left(\frac{H^S_{F_r}/F_r}{\mathfrak{A}}\right)$, and the inverse of the second one is given by restriction. It follows that the isomorphism $A^S_{F_r} \cong X^S_{\infty,F}/\nu_{n_0,r}Y^S_{\infty,F}$ is given by $[\mathfrak{A}] \mapsto x \bmod \nu_{n_0,r}Y^S_{\infty,F}$ where $x_{|H^S_{F_r}} = \left(\frac{H^S_{F_r}/F_r}{\mathfrak{A}}\right)$. The isomorphisms are compatible with norms and projections, and with inclusions and multiplications by $\nu_{n,r}$. Observing that $Y^S_{\infty,F} \sim X^S_{\infty,F}$, because their quotient is $A^S_{F_{n_0}}$, hence finite, and applying Theorem 6.8 we obtain:

Theorem 6.12 *Let $\mu_S = \mu(X^S_{\infty,F})$ and $\lambda_S = \lambda(X^S_{\infty,F})$ be the Iwasawa invariants of $X^S_{\infty,F}$. Then there is a constant ν_S, such that for $r \gg 0$,*

$$|A^S_{F_r}| = p^{\mu_S \cdot p^r + \lambda_S \cdot n + \nu_S}.$$

This shows among other things that in order to have $\mu_S = 0$ it is necessarily and sufficient to have $\mathrm{rank}_{\mathbb{F}_p} A^S_{F_r}$ bounded independent of r. It is known much more in the abelian case:

Theorem(Ferrero-Washington) 6.13 *Let F be an abelian number field, and let S be a finite set of primes. Then $\mu_S = 0$.*

Moreover, Greenberg conjectured that for a real abelian field $X^S_{\infty,F}$ is even finite, i.e. $\lambda_S = 0$.

An approximation of Λ-structure of $\mathrm{Gal}(L_\infty(F)/L'_\infty(F))$ is known in the case of the cyclotomic $\mathbb{Z}_p$-extension (more generally for a $\mathbb{Z}_p$-extension in which no p-adic prime is infinitely decomposed).

Proposition 6.14

$$\mathrm{Gal}(L_\infty(F)/L'_\infty(F)) \sim \oplus_{i=1}^t \Lambda/(\xi_{n_i})$$

with $\sum_{i=1}^t \deg \xi_{n_i} \leq s_p$, where s_p, the number of primes above p is assumed finite.

Proof:(Iwasawa [34], Th.9) Using directly Corollary 6.9 or using the class field theory sequences provided in the next chapter (see the proof of Proposition 8.1) one can obtain the short exact sequence:

$$0 \to \mathrm{coker}(\bar{U}_{\infty,F} \to \bar{U}'_{\infty,F}) \to \mathrm{coker}(\oplus_{v\in S_p}\bar{U}_{\infty,F_v} \to \oplus_{v\in S_p}\mathcal{F}_{\infty,F_v})$$

$$\to \mathrm{Gal}(L_\infty(F)/L'_\infty(F)) \to 0$$

and study the two cokernels.□

Corollary 6.15 *Let m be an integer, $m \geq 2$. Then $X_{\infty,F}(m)^\Gamma$ is finite if and only if $X'_{\infty,F}(m)^\Gamma$ is finite.*

Proof: We have the following short exact sequence

$$0 \to \mathrm{Gal}(L_\infty(F)/L'_\infty(F)) \to X_{\infty,F} \xrightarrow{\phi} X'_{\infty,F} \to 0.$$

Then, m-twisting and using the previous Proposition, we obtain that $(\ker\phi)(m)^\Gamma$ and $(\ker\phi)(m)_\Gamma$ are finite. The invariant-coinvariant long exact sequence associated to the short exact sequence above (or the multiplicative behavior of the characteristic polynomial with respect to exact sequences together with Lemma 6.5) finishes the proof.□

Remark 6.16 *The Corollary 6.15 is still true for F_r and Γ^{p^r}-invariants, where $r \geq 0$. Note that $X_{\infty,F} = X_{\infty,F_r}$. The same for the $'$-version. Therefore we can state: $X_{\infty,F}(m)^{\Gamma^{p^r}}$ is finite if and only if $X'_{\infty,F}(m)^{\Gamma^{p^r}}$ is finite.*

We prove here that $\mathrm{Gal}(L_\infty(E)/L'_\infty(E))$ is finite, where $E = F(\mu_{2p})$.

Theorem 6.17 *Let $m \geq 2$. Then $X_{\infty,E}(m)^{\Gamma^{p^r}}$ and $X'_{\infty,E}(m)^{\Gamma^{p^r}}$ are finite for all $r \geq 0$.*

Proof: It is known that $X'_{\infty,E_r}(m)_{\Gamma^{p^r}}$ can be seen as a subgroup of $H^2(\mathcal{O}'_{E_r}, \mathbb{Z}_p(m-1))$ (see chapter 8). But this group is finite, because the group $K_{2(m-1)-2}(\mathcal{O}'_{E_r}) \otimes \mathbb{Z}_p$ is finite, and the Chern character that links them has finite kernel and co-kernel (see chapter 3). It follows that $X'_{\infty,E_r}(m)_{\Gamma^{p^r}}$ is finite, and finally, using the previous remarks, $X_{\infty,E}(m)^{\Gamma^{p^r}}$ will be finite.□

Theorem 6.18 *With the previous notations, we have $X_{\infty,E} \sim X'_{\infty,E}$.*

Proof: Let $m \geq 2$. Then Theorem 6.17 implies that $\mathrm{char}((\ker\phi)(m))$ is relatively prime to $\omega_r(T)$ for all $r \geq 0$ (here γ acts on $X_{\infty,E}(m)$ as multiplication by $1+T$, which is multiplication by $\kappa(\gamma)^{-m} \cdot (1+T)$ on the original $X_{\infty,E}$, and $X_{\infty,E}(m)^{\Gamma^{p^r}} = \ker(X_{\infty,E}(m) \xrightarrow{\times\omega_r} (X_{\infty,E}(m)))$. It follows that $\mathrm{char}((\ker\phi)(m))$ is relatively prime to $\xi_r(T)$ for all $r \geq 0$. Using Proposition 6.14 to which we apply first the m-twist, we conclude that $(\ker\phi)(m)$ is finite. The underlying abelian group $\ker\phi$ is then finite.□

We move our attention to the other important Galois group $\mathcal{X}_{\infty,F}$, the standard Iwasawa module. It is known that $\text{rank}_{\mathbb{Z}_p}\mathcal{X}_{\infty,F}/\omega_0\mathcal{X}_{\infty,F} = r_2(F) + \delta(F)$ (see chapter 5). Using similar formulas for coinvariants at each finite level, and passing to the projective limit, we obtain that $\text{rank}_\Lambda \mathcal{X}_{\infty,F} \geq r_2(F)$, with equality if and only if $\delta(F_n)$ are bounded independent of n.

Definition 6.19 *The difference* $\delta_\infty(F) := \text{rank}_\Lambda \mathcal{X}_{\infty,F} - r_2(F)$ *is called the* **Weak Leopoldt defect** *and the* $\mathbb{Z}_p$*-extension* F_∞/F *is said to satisfy the* **weak Leopoldt Conjecture** *if* $\delta_\infty = 0$.

The Weak Leopoldt Conjecture is implied be the Leopoldt Conjecture.

The Weak Leopoldt Conjecture has a cohomological characterization, which leads further to significant progress in the study of the Λ-module structure of $\mathcal{X}_{\infty,F}$. Let $\Omega_S(F)$ denote the maximal algebraic S-ramified pro-p-extension of F, $G_S(F) := \text{Gal}(\Omega_S(F)/F)$, and $G_S(F_\infty) := \text{Gal}(\Omega_S(F)/F_\infty)$.

Theorem 6.20 *(a) There is an exact sequence*

$$0 \to (H^2(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p)^\Gamma)^\# \to D_F \to \mathcal{X}^\Gamma_{\infty,F} \to 0.$$

(b) The Weak Leopoldt Conjecture holds if and only if

$$H^2(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p) = 0.$$

Proof: Consider the following Hochschild-Serre spectral sequence:

$$E_2^{r,s} = H^r(\Gamma, H^s(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p) \Rightarrow E^{r+s} = H^{r+s}(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p).$$

Because the cohomological p-dimension of Γ is 1, $E_2^{r,s} = 0$ for all $r \geq 2$, $s \geq 0$ and we obtain the following exact sequences:

$$1 \to E_2^{1,q-1} \to E^q \to E_2^{0,q} \to 1$$

for all $r \geq 1$. Taking $q = 2$ we obtain:

$$0 \to H^1(\Gamma, H^1(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p)) \to H^2(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p)$$

$$\to (H^2(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p)^\Gamma) \to 0.$$

Now, since $G_S(F_\infty)$ acts trivially on $\mathbb{Q}_p/\mathbb{Z}_p$, and since $\mathcal{X}_{\infty,F}$ is actually the abelianization of $G_S(F_\infty)$ we have

$$H^1(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p) \cong \mathrm{Hom}_{\mathbb{Z}_p}(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p) \cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{X}_{\infty,F}, \mathbb{Q}_p/\mathbb{Z}_p)$$

which is exactly the Pontrjagin-dual of $\mathcal{X}_{\infty,F}$, denoted $\mathcal{X}^{\#}_{\infty,F}$, and on which Γ acts by $f^\gamma(x) = f(x^{\gamma^{-1}})$ for all $x \in \mathcal{X}_{\infty,F}$. It follows easily that

$$H^1(\Gamma, H^1(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p)) \cong (\mathcal{X}^{\#}_{\infty,F})_\Gamma \cong (\mathcal{X}^{\Gamma}_{\infty,F})^{\#}.$$

The middle term $H^2(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p)$ is isomorphic with the Leopoldt kernel D_F (see Kolster [45] for a proof). Thus the assertion (a) is proved.

The action of Γ on a non-trivial discrete Γ-module has necessarily fixed points. It follows that: $H^2(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p) = 0 \Leftrightarrow H^2(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p)^\Gamma = 0$.
But $(H^2(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p)^\Gamma)^{\#}$ is $\mathbb{Z}_p$-torsion free being a submodule of D_F, and

$$\mathrm{rank}_{\mathbb{Z}_p}(H^2(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p)^\Gamma)^{\#} = \mathrm{rank}_{\mathbb{Z}_p} D_F - \mathrm{rank}_{\mathbb{Z}_p} \mathcal{X}^{\Gamma}_{\infty,F}$$

$$= \delta_F - (\mathrm{rank}_{\mathbb{Z}_p}(\mathcal{X}_{\infty,F})_\Gamma - \mathrm{rank}_\Lambda \mathcal{X}_{\infty,F}) = \delta_F - (r_2(F) + \delta_F - \mathrm{rank}_\Lambda \mathcal{X}_{\infty,F}) = \delta_\infty.$$

This finishes the proof of (b).□

Proposition 6.21 *The weak Leopoldt Conjecture holds for the cyclotomic $\mathbb{Z}_p$-extension.*

Proof: If F contains a primitive $2p$-th root of unity, then $F_\infty = F(\mu_{p^\infty})$. By Serre [70], the cohomological p-dimension of $G_S(F_\infty)$ is 1; it follows immediately that $H^2(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p) = 0$. Thus all we need is a kind of "going-down" property for the Weak Leopoldt Conjecture. Let E/F be a finite extension of number fields. Let F_∞/F be a $\mathbb{Z}_p$-extension of F. Then $E_\infty := EF_\infty/E$ is a $\mathbb{Z}_p$-extension of E. If E_∞/E satisfies the Weak Leopoldt Conjecture, then $\delta(E_r)$ are bounded independent of r. But it is clear that $\delta(F_r) \leq \delta(E_r)$. We can conclude: $\delta_\infty(E) = 0 \Rightarrow \delta_\infty(F) = 0$.□

We are ready to study the structure of $\mathcal{X}_{\infty,F}$ in the case of a $\mathbb{Z}_p$-extension that satisfies the Weak Leopoldt Conjecture.

Theorem 6.22 *If a $\mathbb{Z}_p$-extension F_∞/F satisfies the Weak Leopoldt Conjecture, then $\mathcal{X}_{\infty,F}$ contains no non-trivial finite Λ-submodule. Moreover, there is an injection $\mathcal{X}_{\infty,F} \hookrightarrow \Lambda^{r_2(F)} \oplus \mathrm{tors}_\Lambda \mathcal{X}_{\infty,F}$ with finite cokernel.*

Proof: We have $H^2(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p) = 0$. We obtain immediately that $H^2(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p)^{\Gamma^{p^r}} = 0$ for all r. Writing the exact sequence from Theorem 6.20 (a) for F_r and Γ^{p^r} we obtain $D_{F_r} = \mathcal{X}_{\infty,F}^{\Gamma^{p^r}}$ for all r (note that $\mathcal{X}_{\infty,F} = \mathcal{X}_{\infty,F_r}$). Let N be a finite Λ-submodule of $\mathcal{X}_{\infty,F}$. It follows that N is fixed under Γ^{p^s} for some s. This means that N is included in $\mathcal{X}_{\infty,F}^{\Gamma^{p^s}}$. But this module is $\mathbb{Z}_p$-torsion free being isomorphic to D_{F_s}. We conclude that $N = 0$.□

As mentioned before, when $p = 2$, $\mathcal{X}_{\infty,F}$ and $\mathcal{X}^f_{\infty,F}$ differ. In fact, there is a surjection (Galois restriction) $\mathcal{X}_{\infty,F} \twoheadrightarrow \mathcal{X}^f_{\infty,F}$ whose kernel we are going to study below.

Theorem 6.23 *The module $\ker(\mathcal{X}_{\infty,F} \twoheadrightarrow \mathcal{X}^f_{\infty,F})$ is a finitely generated Λ-torsion module with trivial λ-invariant. Its μ-invariant is less or equal than $r_2(F)$, with equality if F_∞/F satisfies the Weak Leopoldt Conjecture.*

Proof: From Class Field Theory (see Chapter 5) we have the following commutative diagram:

$$0 \longrightarrow D_F \longrightarrow \hat{U}_F^S \longrightarrow \prod_{v\in S} \hat{F}_v \longrightarrow \mathrm{Gal}(M_S(F)/H_S(F)) \longrightarrow 0$$

$$0 \longrightarrow D_F' \longrightarrow \hat{U}_F^S \longrightarrow \prod_{v\in S_2} \hat{F}_v \longrightarrow \mathrm{Gal}(M_{S_2}(F))/H_S(F)) \longrightarrow 0$$

Here, $S = S_2 \cup S_\infty$, $H_S(F) = H_{S_2}(F)$, and D_F' is the notation for the kernel in the second row. Because $\hat{F}_v$ is $\mathbb{Z}/2\mathbb{Z}$ for a real infinite prime and null for a complex infinite prime, we obtain an exact sequence

$$0 \longrightarrow D_F \longrightarrow D_F' \longrightarrow (\mathbb{Z}/2\mathbb{Z})^{r_1(F)} \longrightarrow \mathrm{Gal}(M_S(F)/M_{S_2}(F)) \longrightarrow 0.$$

Similarly, for all s we have

$$0 \longrightarrow D_{F_s} \longrightarrow D_{F_s}' \longrightarrow (\mathbb{Z}/2\mathbb{Z})^{2^s r_1(F)} \longrightarrow \mathrm{Gal}(M_S(F_s)/M_{S_2}(F_s)) \longrightarrow 0$$

(note that the real infinite primes do not ramify in F_∞/F, so that we obtain $r_1(F_s) = 2^s r_1(F)$). For a real infinite prime v of F one can see that the projective systems $\{\prod_{v_s|v} \mathbb{Z}/2\mathbb{Z}\}$ and $\{\mathbb{Z}/2\mathbb{Z}[T]/T^{2^s}\}$ are isomorphic, so that we obtain:

$$\varprojlim \prod_{v_s|v} \mathbb{Z}/2\mathbb{Z} \cong \varprojlim \mathbb{Z}/2\mathbb{Z}[T]/T^{2^s} \cong \Lambda/2\Lambda.$$

This provides us with a surjection

$$(\Lambda/2\Lambda)^{r_1(F)} \twoheadrightarrow \ker(\mathcal{X}_{\infty,F} \twoheadrightarrow \mathcal{X}^f_{\infty,F}).$$

The assertions follow immediately. If the Weak Leopoldt Conjecture holds, then $\mathrm{coker}(D_{F_s} \to D'_{F_s})$ are finite elementary 2-groups of order bounded independent of s. Their projective limit lives in $\Lambda/2\Lambda$. But $\Lambda/2\Lambda$ contains no non-trivial finite Λ-submodule, hence the previous surjection is an isomorphism. □

For a local field the structure of the two modules is completely understood. Let v be a finite prime of F. Let $F_{v,\infty}/F_v$ be a $\mathbb{Z}_p$-extension of F_v, M_{∞,F_v} the maximal abelian pro-p-extension of $F_{v,\infty}$ and $\mathcal{X}_{\infty,F_v} := \mathrm{Gal}(M_{\infty,F_v}/F_{v,\infty})$. For $r \geq 0$ let $F^{(p)}_{v,r}$ denote the maximal abelian pro-p-extension of $F_{v,r}$. By Class Field Theory, we have that $\mathrm{Gal}(F^{(p)}_v/F_v) \cong \mu(F_v)(p) \oplus \mathbb{Z}_p$ if $v \nmid p$, and $\mathrm{Gal}(F^{(p)}_v/F_v) \cong \mu(F_v)(p) \oplus \mathbb{Z}_p^{[F:\mathbb{Q}_p]+1}$ if $v \mid p$ (as usual, for an abelian group A, $A(p)$ denotes the p-primary part). This tells us that F_v has exactly one $\mathbb{Z}_p$-extension if $v \nmid p$, and it has $[F : \mathbb{Q}_p] + 1$ if $v \mid p$. The maximal abelian unramified pro-p-extension F_v, denoted $F_{v,ur}$, is one of the $\mathbb{Z}_p$-extensions of F_v.

Let L_{∞,F_v} be the maximal abelian unramified pro-p-extension of $F_{v,\infty}$, and we set $X_{\infty,F_v} := \mathrm{Gal}(L_{\infty,F_v}/F_{v,\infty})$. We note that $L_{\infty,F_v} = F_{v,\infty}F_{v,ur}$. Therefore we obtain:

Proposition 6.24 *X_{∞,F_v} is trivial if $F_{v,\infty} = F_{v,ur}$ and it is isomorphic to $\mathbb{Z}_p$ otherwise.*

Now, $\mathcal{X}_{\infty,F_v}$ is a finitely generated Λ-module, because of Nakayama Lemma. Indeed, $\mathcal{X}_{\infty,F_v}/\mathfrak{m}\mathcal{X}_{\infty,F_v} \cong \mathrm{Gal}(F^{(p)}_v/F_\infty)$ is a finitely dimensional $\mathbb{F}_p$-vector space as seen previously (here, as usual, $\mathfrak{m} := (p,T)$). Denote by Ω the maximal pro-p-extension of

F_v and let $G_v := \mathrm{Gal}(\Omega/F_v)$. Using the same technique as in the global case, we see that $\mathcal{X}_{\infty,F_v}^{\Gamma}$ is a quotient of $H^2(G_v, \mathbb{Q}_p/\mathbb{Z}_p)^{\#}$. But, by the local Poitou-Tate duality theorem (see chapter 2) we obtain

$$H^2(G_v, \mathbb{Q}_p/\mathbb{Z}_p)^{\#} \cong H^0(G_v, \mathbb{Z}_p(1)) \cong \varprojlim H^0(G_v, \mu_{p^r}) \cong \varprojlim \mu(F_v)(p) = 0.$$

Proposition 6.25 *(a) When $v \nmid p$, $\mathcal{X}_{\infty,F_v}$ is trivial if $\mu_p \not\subset F_v$, and it is isomorphic to $\mathcal{T}$ otherwise.*
(b) When $v \mid p$, $\mathcal{X}_{\infty,F_v} \cong \Lambda^{[F:\mathbb{Q}_p]} \oplus \mathcal{T}$ if $\mu(F_{v,\infty})(p) = \mu_{p^\infty}$, and it is described by the exact sequence

$$0 \longrightarrow \mathcal{X}_{\infty,F_v} \longrightarrow \Lambda^{[F:\mathbb{Q}_p]} \longrightarrow \mu(F_{v,\infty})(p) \longrightarrow 0$$

if $\mu(F_{v,\infty})(p)$ is finite.

Proof: The discussion just before the proposition shows that

$$\mathcal{X}_{\infty,F_v}^{\Gamma} = 0.$$

It follows that $\mathcal{X}_{\infty,F_v}^{\Gamma^{p^r}} = 0$ for all r, so that $\mathcal{X}_{\infty,F_v}$ contains no non-trivial finite Λ-submodule and multiplication by ω_r is injective for all $r \geq 0$. Using these facts and knowing the structure of the co-invariants, namely $\mathcal{X}_{\infty,F_v}/\omega_r\mathcal{X}_{\infty,F_v} \cong \mathrm{Gal}(F_{v,r}^{(p)}/F_{v,\infty}) \cong \mu(F_{v,r})(p) \oplus \mathbb{Z}_p^{p^r[F:\mathbb{Q}_p]}$ if $v \mid p$, and it is isomorphic to $\mu(F_{v,r})(p)$ otherwise, we can conclude that $\mathrm{rank}_\Lambda \mathcal{X}_{\infty,F_v}$ is $[F:\mathbb{Q}_p]$ if $v \mid p$, and it is 0 if $v \nmid p$. Let us denote

$$\mathcal{T} := \varprojlim \mu(F_{v,r})(p).$$

If $\mu(F_{v,\infty})(p)$ is finite, then $\mathcal{T}$ is null, since $\mathcal{X}_{\infty,F_v}$ contains no non-trivial finite Λ-submodule. If $\mu(F_{v,\infty})(p)$ is infinite, then $\mathcal{T}$ is a direct summand of $\mathcal{X}_{\infty,F_v}$. So, in the case $v \nmid p$, $\mathcal{X}_{\infty,F_v}$ is Λ-torsion, hence null if $\mu(F_{v,\infty})(p)$ is finite, i.e. $\mu_p \not\subset F_v$, and it is isomorphic to $\mathcal{T}$ otherwise. In the case $v \mid p$, $\mathcal{X}_{\infty,F_v}$ is Λ-torsion free, hence the sequence above is exact if $\mu(F_{v,\infty})(p)$ is finite. If $\mu(F_{v,\infty})(p)$ is infinite, then the torsion is exactly $\mathcal{T}$ and this is direct summand. This finishes the proof. $\square$

Chapter 7

Semi-local units modulo cyclotomic units

Let us introduce new important Λ-modules, obtained by taking the projective limit of the corresponding objects at finite levels in a $\mathbb{Z}_p$-extension of a number field F with respect to norms:

$$\bar{U}_{\infty,F} := \varprojlim_r \hat{U}_{F_r},\ \bar{U}^S_{\infty,F} := \varprojlim_r \hat{U}^S_{F_r},$$

$$\bar{U}_{\infty,F_v} := \varprojlim_r \hat{U}_{F_{r,v}},\ \bar{U}_{p,\infty,F} := \oplus_{v\in S}\bar{U}_{\infty,F_v},$$

$$\mathcal{F}_{\infty,F_v} := \varprojlim_r \hat{F}_{r,v}\ for\ v \in S.$$

Here $S := S_p \cup S_\infty$, where S_p is the set of the primes of F above p and S_∞ is the set of the archimedean places We use also the $'$ notation for S_p-numbers, S_p-units and the corresponding Λ-modules. There are diagonal maps:

$$\bar{U}_{\infty,F} \longrightarrow \oplus_{v\in S}\bar{U}_{\infty,F_v},\ \text{and}\ \bar{U}^S_{\infty,F} \longrightarrow \oplus_{v\in S}\mathcal{F}_{\infty,F_v}.$$

Let us define also

$$\bar{C}_{\infty,F} := \varprojlim_r C(F_r) \otimes \mathbb{Z}_p,$$

which we can see embedded either in $\bar{U}_{p,\infty,F}$ or in $\bar{U}^f_{p,\infty,F} := \oplus_{v\in S_p}\bar{U}_{\infty,F_v}$. Here, $C(F)$ denotes the group of circular units. The next lemma comprises all necessary facts used for introducing circular numbers and for recognizing norm-compatible projective sequences which appear later in the presentation.

Lemma 7.1 *Let $\mathbb{Q}(\zeta_d)$ denote the d-th cyclotomic field, where ζ_d is a primitive d-th root of unity, $d \geq 1$. Let $\mu_d = \mu(\mathbb{Q}(\zeta_d)) =< \zeta_d >=$ the group of all d-th roots of unity and $\mu_d^{prim} =$ the set of all primitive d-th roots of unity. We have:*
(1) *If $\zeta_d \in \mu_d^{prim}$ then*

$$N_{\mathbb{Q}(\zeta_d)/\mathbb{Q}}(1-\zeta_d) = \begin{cases} p & if\ d = p^u, p\ prime \\ 1 & if\ d\ is\ not\ a\ prime\ power \end{cases}$$

(2) *Suppose $d = p \cdot d'$, where p is a rational prime.*
(2.1) *If $p \nmid d'$, $\omega \in \mu_d \setminus \{1\}$, $\zeta_d \in \mu_p^{prim}$ then*

$$N_{\mathbb{Q}(\zeta_d)/\mathbb{Q}(\zeta_{d'})}(1-\omega) = (1-\omega)^{p-1},\ and$$

$$N_{\mathbb{Q}(\zeta_d)/\mathbb{Q}(\zeta_{d'})}(1-\omega\cdot\zeta_d) = (1-\omega^p)(1-\omega)^{-1}.$$

(2.2) *If $p \mid d'$, $\omega \in \mu_d \setminus \{1\}$ then*

$$N_{\mathbb{Q}(\zeta_d)/\mathbb{Q}(\zeta_{d'})}(1-\omega) = \begin{cases} (1-\omega)^p & if \omega \in \mu_{d'} \\ 1-\omega^p & if \omega \notin \mu_{d'}. \end{cases}$$

Definition 7.2 *Let F be an abelian number field. We define the* **the group of circular numbers** *of F as the subgroup of $F^\times$ generated by the set of elements $\{-1, N_{\mathbb{Q}(\mu_m)/F\cap\mathbb{Q}(\mu_m)}(1-\zeta_m^a) \mid m \in \mathbb{Z}, m \geq 2, a \in \mathbb{Z}, m \nmid a\}$ (μ_m denotes the group of m-th roots of unity as usual, and ζ_m denotes a primitive m-th root of unity), and we denote it by $D(F)$. The* **group of circular units** *is defined as $C(F) := D(F) \cap U(F)$.*

Note that $D(F)$ and $C(F)$ are Gal$(F/\mathbb{Q})$-modules. Also:
(1) $\mathbb{Q}^\times \subset D(F)$, and $\mu(F) \subset C(F)$.
We see that -1 is in $D(F)$ by definition, and any rational prime p is in $D(F)$ too because it can be written as $p = N_{F\cap\mathbb{Q}(\mu_p)/\mathbb{Q}}(N_{\mathbb{Q}(\mu_p)/F\cap\mathbb{Q}(\mu_p)}(1-\zeta_p))$. If t is the order of $\mu(F)$, then $\mathbb{Q}(\mu_t) \subset F$, and $\zeta_t = (-1)\cdot(1-\zeta_t)^{1-J} \in D(F) \cap U(F)$. Thus $\mu(F) \subset C(F)$.

(2) $D(F)^- := \{\alpha \in D(F) \mid \alpha^{1+J} = 1\} = \mu(F)$.

For any root of unity $\omega \neq 1$ we have $(1-\omega)^{1-J} = -\omega$. Thus for any $\alpha \in D(F)$ we have $\alpha^{1-J} \in \mu(F)$. Hence if $\alpha \in D(F)^-$, then $\alpha^2 = \alpha^{1+J} \cdot \alpha^{1-J} = \alpha^{1-J} \in \mu(F)$. It follows $\alpha \in \mu(F)$. The other inclusion is obvious.

(3) $C(F) = \{\alpha \in D(F) \mid N_{F/\mathbb{Q}}(\alpha) = \pm 1\}$.

For any root of unity ω and any integer s relatively prime to the order of ω we have $(1-\omega)/(1-\omega^s)$ is a unit. Therefore for any $\alpha \in D(F)$ and any $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ we have $\alpha^{1-\sigma} \in U(F)$. It follows that

$$N_{F/\mathbb{Q}}(\alpha) = \alpha^{\sum_\sigma \sigma} \equiv \alpha^{|\mathrm{Gal}(F/\mathbb{Q})|} \bmod C(F),$$

with σ running through all elements in $\mathrm{Gal}(F/\mathbb{Q})$. Thus, if $N_{F/\mathbb{Q}}(\alpha)$ is a unit, then α is a unit. The other inclusion is clear.

If F is a full cyclotomic field, then $C(F)$ is the usual **group of cyclotomic units** (see Washington [81]).

Proposition 7.3 *For a number field F and any $\mathbb{Z}_p$-extension F_∞/F we have exact sequences*

$$0 \to \bar{U}_{\infty,F}/\bar{C}_{\infty,F} \to (\oplus_{v\in S}\bar{U}_{\infty,F_v})/\bar{C}_{\infty,F} \to \mathcal{X}^f_{\infty,F} \to X_{\infty,F} \to 0, \text{ and}$$

$$0 \to \bar{U}^S_{\infty,F}/\bar{C}_{\infty,F} \to (\oplus_{v\in S}\mathcal{F}_{\infty,F_v})/\bar{C}_{\infty,F} \to \mathcal{X}^f_{\infty,F} \to X^S_{\infty,F} \to 0$$

when $S = S_p$, and similar exact sequences when $S = S_p \cup S_\infty$ with $\mathcal{X}^f_{\infty,F}$ replaced by $\mathcal{X}_{\infty,F}$.

Proof: Because the Weak Leopoldt Conjecture is true for the cyclotomic $\mathbb{Z}_p$-extension, the first map in the first sequence of Proposition 5.1 is injective. Taking the projective limit (all groups are finitely generated), we obtain the following exact sequence:

$$0 \to \bar{U}_{\infty,F} \to \oplus_{v\in S}\bar{U}_{\infty,F_v} \to \mathcal{X}^S_{\infty,F} \to X_{\infty,F} \to 0.$$

On the other hand we have a tautological exact sequence:

$$0 \to \bar{U}_{\infty,F}/\bar{C}_{\infty,F} \to \oplus_{v\in S}\bar{U}_{\infty,F_v}/\bar{C}_{\infty,F} \to (\oplus_{v\in S}\bar{U}_{\infty,F_v})/\bar{U}_{\infty,F} \to 0.$$

The first exact sequence in our proposition is now immediate.

The second exact sequence follows in the same manner using the second exact sequence of Proposition 5.1 and Remark 5.4.□

Remark 7.4 *This Proposition shows that $\bar{U}_{\infty,F}/\bar{C}_{\infty,F} \sim X_{\infty,F}$ if and only if $(\oplus_{v\in S}\bar{U}_{\infty,F_v})/\bar{C}_{\infty,F} \sim \mathcal{X}_{\infty,F}$ (and similar equivalences for the other cases). We note that $\bar{C}_{\infty,F}$ can be replaced by $\bar{C}^{\Delta}_{\infty,F}$ if F/F^+ is a CM-extension* ($\Delta := \mathrm{Gal}(F/F^+)$) *and the result still holds.*

In order to relate the characteristic polynomial of the above Λ-modules to the power series giving the p-adic L-functions, we need to work characterwise. A short account of this technique follows (refer to [28]).

Definition 7.5 *Let G be a finite group, M a $\mathbb{Z}_p[G]$-module, and χ a character of G. We define the ring $\mathbb{Z}_p[\chi] := \mathbb{Z}_p[\chi(g)|g \in G]$. This ring is $\mathbb{Z}_p[G]$-module, with the action being $g \cdot x = \chi(g)x$ where $g \in G$ and $x \in M$. The χ-***part*** of M, denoted M_χ, is defined as*

$$M_\chi := M \otimes_{\mathbb{Z}_p[G]} \mathbb{Z}_p[\chi] = \{x \in M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\chi] | g \cdot x = \chi(g)x \text{ for all } g \in G\}.$$

The following facts are well known:

(1) The functor $M \longmapsto M_\chi$ is right exact.

(2) In the case $p \nmid |G|$ the previous functor is even exact, since the ring $\mathbb{Z}_p[G]$ is a direct product of complete discrete valuation rings, and the canonical surjection $M \longrightarrow \prod M_\chi$ is an isomorphism (the product is taken over all irreducible $\mathbb{Q}_p$-characters χ of G).

(3) For any $\Lambda[G]$-module M which is finitely generated over $\mathbb{Z}_p$ (as Λ-module it is automatically Λ-torsion) we have:

$$\mathrm{char}(M) = \prod_\chi \mathrm{char}(M_\chi)$$

(product over all irreducible $\mathbb{Q}_p$-characters of G).

Now, we are preparing the necessary context for stating the **Main Conjecture of Iwasawa Theory** for all primes p.

Let F be a totally real number field, and $E = F(\zeta_{2p})$. Let $\Delta := \mathrm{Gal}(E/F)$ (for $p = 2$ this is $\{\mathrm{id},\ J\}$, with J induced by the complex conjugation). Let F_∞ and E_∞ the corresponding cyclotomic $\mathbb{Z}_p$-extensions. Denote $\Gamma = \mathrm{Gal}(E_\infty/E) \cong \mathrm{Gal}(F_\infty/F)$. Also, $\mathrm{Gal}(E_\infty/F) \cong \Gamma \times \Delta$. This group is pro-cyclic for p odd. For $p = 2$, it is procyclic if and only if $\zeta_{2^k} - \bar{\zeta}_{2^k} \in F$ for some $k \geq 3$ or $\zeta_4 \in F$.

Since E_∞ contains all p-power roots of unity, the Galois group $\mathrm{Gal}(E_\infty/F)$ acts on μ_{p^∞} and we obtain the **cyclotomic character** $\rho : \mathrm{Gal}(E_\infty/F) \longrightarrow \mathbb{Z}_p^\times$ defined by $\sigma \cdot \zeta := \zeta^{\rho(\sigma)}$ for all $\sigma \in \mathrm{Gal}(E_\infty/F)$ and $\zeta \in \mu_{p^\infty}$. The restriction of ρ to Δ is the **Teichmuller character** ω, and the restriction to Γ is denoted by κ.

Let χ be a real one-dimensional p-adic Artin character of F, meaning that the field extension of F associated to χ, denoted usually by F_χ, is real and abelian over F. Artin associated to χ a L-function (for an introduction to Artin L-functions see Koch [40], and for Dirichlet L-functions see Washington [81]). Moreover, there is a p-adic meromorphic (even analytic if $\chi \neq 1$) function $L_p(s, \chi)$ which interpolates the values of L-functions:

$$L_p(1-m, \chi) = L(1-m, \chi\omega^{m-1}) \prod_{\mathfrak{p}|p} (1 - \chi\omega^{-m}(\mathfrak{p}) N(\mathfrak{p})^{m-1})$$

for $m \geq 1$. The construction is due to Kubota and Leopoldt for $F = \mathbb{Q}$ (see Washington [81], page 57), and to Barsky, Cassou-Nogues, and Deligne and Ribet [19] for any totally real number field. For $m \geq 2$ the previous equality gives

$$L_p(1-m, \chi) \sim_p L(1-m, \chi\omega^{-m})$$

(where $\sim_p$ means that they have the same p-adic valuation). If $F_\chi \cap F_\infty = F$, then χ is said to be of type S, and if $F_\chi \subset F_\infty$, it is said to be of type W. $L_p(s, \chi)$ is essentially given by a unique power series $G_\chi(T) \in \mathbb{Z}_p[\chi][[T]]$:

$$L_p(1-s, \chi) = G_\chi(\kappa(\gamma)^s - 1)/H_\chi(\kappa(\gamma)^s - 1),$$

where $H_\chi(T) = \kappa(\gamma)(1+T) - 1$ if χ is of type W, and $H_\chi(T) = 1$ otherwise. Let $G_\chi(T) = \pi^{\mu(G_\chi)} G^*_\chi(T) U(T)$ be the unique decomposition of this power series, where π is a uniformizing parameter of $\mathbb{Z}_p[\chi]$, $G^*_\chi(T)$ is a distinguished polynomial and $U(T)$ is a unit power series. We know that $X_{\infty,F}$ is a noetherian torsion Λ-module.

Main Conjecture 7.6 *Let F be a totally real number field, p a prime and χ a real one-dimensional p-adic Artin character of F of type S. Then:*

$$\mathrm{char}((X_{\infty,F_\chi})_\chi)^*(T) = G^*_\chi(T).$$

It was proved for odd primes and all totally real number fields and for $p=2$ and all abelian real number fields by Wiles in [82] and, using Euler systems, by Rubin in [67] (for odd primes) and by Greither in [28] (for all primes). Instead of F_χ one could take any totally real finite extension of F containing F_χ, say K, such that $K \cap F_\infty = F$. The polynomial $\mathrm{char}((X_{\infty,F_\chi})_\chi)^*(T)$ is independent of the field K. In particular, one could take $K = E^+$, the maximal real subfield of E. Moreover, if p is odd and the character is of order prime to p, then $\mu(X_{\infty,F_\chi}) = \mu(G_\chi)$. In general, this relation between μ-invariants is conjectural (see Greenberg [27]): for p odd both should be 0 - suggested by the abelian case in which it is proved that $\mu(G_\chi) = \mu(X_{\infty,F}) = 0$, and for $p=2$ both are non-trivial and should be equal (if $\chi = 1$ then they are equal as it was proved by Federer [22] and by Greenberg [27]). Let us state the Main Conjecture as a theorem only in the case we will be using:

Theorem 7.7 *Let F be a real abelian field, p a prime and χ a character of F of type S. Then*

$$\mathrm{char}((X_{\infty,F})_\chi)^*(T) = G^*_\chi(T)$$

and the μ-invariants are trivial.

Theorem(Greither [28]) 7.8 *Let p be a prime, F a real abelian number field, $E = F(\zeta_4)$ and χ an even non-trivial character of E. Then the Λ_χ-modules $(\bar{U}_{\infty,E}/\bar{C}_{\infty,E})_\chi$ and $(X_{\infty,E})_\chi$ have the same characteristic polynomials.*

This is essentially a corollary of the Main Conjecture and of the important fact that $\mathrm{char}(\bar{U}_{p,\infty,F}/\bar{C}_{\infty,F})(T)$ divides $G_\chi(T)$ (proved using Coleman operator theory by Gillard [24], Greither [28], and Tsuji [78]). For the trivial character the two characteristic polynomials are 1.

Corollary 7.9 *Let $p = 2$ and let F be a real abelian number field as above. Then*

$$\mathrm{char}(\bar{U}_{\infty,F}/\bar{C}_{\infty,F})(T) = 2^{\mu} \cdot \mathrm{char}(X_{\infty,F})(T).$$

and

$$\mathrm{char}(\bar{U}_{2,\infty,F}/\bar{C}_{\infty,F})(T) = 2^{\mu} \cdot \mathrm{char}(\mathcal{X}_{\infty,F})(T),$$

where $\mu := \mu(\bar{U}_{\infty,F}/\bar{C}_{\infty,F})$.

This consequence is obtained by using property 3 of the χ-parts (see above) and the fact that $\mu(X_{\infty,F}) = 0$ for all abelian fields F.

For the case $p = 2$ and full cyclotomic fields we have also results of Kuz'min [51]. Of course these results are valid for all primes p.

Theorem 7.10 *Let $F = \mathbb{Q}(\zeta_4, \zeta_d)$ with $d \geq 3$ odd, and $\Delta := \mathrm{Gal}(F/F^+) \cong \mathrm{Gal}(F_\infty/F_\infty^+)$, where F_∞/F and F_∞^+/F^+ are the cyclotomic $\mathbb{Z}_2$-extensions. Then:*

$$\tilde{U}_{\infty,F}/\tilde{C}_{\infty,F} \sim X_{\infty,F^+},$$

i.e. these Λ-modules have the same characteristic polynomials.

(refer to Kuz'min [51] Th. 3.1, page 714.) Note that we have used the following pseudo-isomorphisms:

$$X_{\infty,F} \sim X_{\infty,F}^{\Delta}, \text{ and } X_{\infty,F} \sim X'_{\infty,F}$$

Here, the modules $\tilde{U}_{\infty,F}$ and $\tilde{C}_{\infty,F}$ are the Λ-torsion-free part of $\bar{U}_{\infty,F}$ and $\bar{C}_{\infty,F}$ (obtained by working modulo torsion at each finite level). Note also, that $\tilde{C}_{\infty,F}^{\Delta} \cong \tilde{C}_{\infty,F^+}$ (cyclotomic units are invariant under J modulo the subgroup of roots of unity). The same is true for $\tilde{U}_{\infty,F}$.

In Kuz'min's setting a few interesting facts can be proved. Unfortunately, they do not seem to be true in general.

Corollary 7.11 *Let $F = \mathbb{Q}(\zeta_4, \zeta_d)$ with $d \geq 3$ odd, and $\Delta := \mathrm{Gal}(F/F^+)$. Then:*

$$\bar{U}_{\infty,F^+}/\bar{C}^{\Delta}_{\infty,F} \sim \tilde{U}_{\infty,F}/\tilde{C}_{\infty,F} \sim X_{\infty,F^+}.$$

Proof: Only the first pseudo-isomorphism needs some explaining, the latter following directly from the previous theorem. At finite levels we have:

$$1 \to \mu(F_r)^{\Delta} \to C(F_r)^{\Delta} \to \tilde{C}(F_r) \to H^1(\Delta, \mu(F_r)) \to H^1(\Delta, C(E_r)) \text{ and}$$

$$1 \to \mu(F_r)^{\Delta} \to U(F_r)^{\Delta} \to \tilde{U}(F_r) \to H^1(\Delta, \mu(F_r)) \to H^1(\Delta, U(F_r))$$

for all $r \geq 1$. Now, $\mu(F_r)^{\Delta} = \{-1, 1\}$; in the norm-compatible projective limit we obtain 0. Also, $H^1(\Delta, \mu(F_r)) = \mu(F_r)/\mu(F_r)^2$, which in the norm-compatible projective limit gives $\mathbb{Z}/2\mathbb{Z}$. Finally, $\mu(F_r)/(1-J)C(F_r) = 0$, because $-\zeta_N = (1-\zeta_N)^{1-J}$, where $N = 2^r d$, d odd, and, for the same reason, $\mu(F_r)/(1-J)U(F_r) = 0$. Therefore we obtain the following short exact sequences for all r:

$$1 \to \mu(F_r)^{\Delta} \to C(F_r)^{\Delta} \to \tilde{C}(F_r) \to \mathbb{Z}/2\mathbb{Z} \to 0$$

$$1 \to \mu(F_r)^{\Delta} \to U(F_r)^{\Delta} \to \tilde{U}(F_r) \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Tensoring with $\mathbb{Z}_2$, and passing to the projective limit with respect to r, we have:

$$0 \to \bar{C}^{\Delta}_{\infty,F} \to \tilde{C}_{\infty,F} \to \mathbb{Z}/2\mathbb{Z} \to 0, \text{ and}$$

$$0 \to \bar{U}^{\Delta}_{\infty,F} \to \tilde{U}_{\infty,F} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

So:

$$\bar{U}^{\Delta}_{\infty,F}/\bar{C}^{\Delta}_{\infty,F} \sim \tilde{U}_{\infty,F}/\tilde{C}_{\infty,F}$$

(they are in fact isomorphic!). Moreover,

$$\bar{U}^{\Delta}_{\infty,F} = \bar{U}_{\infty,F^+}. \square$$

We present now the proof of the Λ-freeness of $\tilde{C}_{\infty,F}$ in Kuz'min's setting (see [51]).

Definition 7.12 *The* **universal distribution** *on the group* $\frac{1}{N}\mathbb{Z}/\mathbb{Z}$ *for some natural number* N *is the abelian group* $\mathcal{L}_N$ *generated by the symbols* $[\frac{a}{N}]$ *for all* $\frac{a}{N} \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}$ *and the relations*

$$\sum_{b,\ tb=a} \left[\frac{b}{N}\right] = \left[\frac{a}{N}\right]$$

for all $\frac{a}{N} \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}$, $t \mid N$

Kubert [49] proved that $\mathcal{L}_N$ is a free abelian group by exhibiting a basis for it (see Kolster, Nguyen Quang Do [48] and Chapter 9 for the weighted version of universal distributions). Also, we have a natural action of $(\mathbb{Z}/N\mathbb{Z})^\times \cong \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ on $\mathcal{L}_N$, and an isomorphism $\mathcal{L}_N \otimes \mathbb{Q} \cong \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^\times]$. If $N \mid M$ then the natural inclusion $\frac{1}{N}\mathbb{Z}/\mathbb{Z} \hookrightarrow \frac{1}{M}\mathbb{Z}/\mathbb{Z}$, and the multiplication by $\frac{N}{M}$ induce natural maps $i_{N,M} : \mathcal{L}_N \longrightarrow \mathcal{L}_M$ and $N_{N,M} : \mathcal{L}_M \longrightarrow \mathcal{L}_N$.

Now, let p be a fixed prime and let d be a natural number relatively prime to p, and $d \not\equiv 2 \bmod 4$. Let $\delta = 0$ if p is odd, and $\delta = 1$ if $p = 2$. We define

$$\mathcal{L}_{p^\infty d} := \varprojlim \mathcal{L}_{p^{\nu+\delta}d} \otimes \mathbb{Z}_p,$$

the projective limit being taken with respect to the maps $N_{p^{\nu+1+\delta}d,p^{\nu+\delta}d}$. We obtain also a natural action of Λ and of $\Delta := \{-id, id\}(= \{1, J\})$. Two important results are proven by Kuz'min [51]:

$$\mathcal{L}_{p^\infty d} \text{ is a } \Lambda\text{-free module of rank } \phi(p^{1+\delta}d),$$

and for any $i \geq 0$

$$\hat{H}^i(\Delta, \mathcal{L}_{2^\infty d}) = 0.$$

Let p be a fixed prime and d be a natural number relatively prime to p, and $d \not\equiv 2 \bmod 4$. Let $(\zeta_{p^{\nu+\delta}})_{\nu\geq 1}$ be a system of primitive $p^{\nu+\delta}$-roots of unity in $\bar{\mathbb{Q}}$ such that $\zeta^p_{p^{\nu+1+\delta}} = \zeta_{p^{\nu+\delta}}$ for all $\nu \geq 1$. Here $\delta = 0$ if p is odd, and $\delta = 1$ if $p = 2$. Let $F := \mathbb{Q}(\mu_d, \zeta_{p^{1+\delta}})(=: F_1)$, and $F_\nu := \mathbb{Q}(\mu_d, \zeta_{p^{\nu+\delta}})$ for all $\nu \geq 1$. Denoting $F_\infty := \cup_{\nu\geq 1} F_\nu$ we obtain the cyclotomic $\mathbb{Z}_p$-extension of F, F_∞/F. Also, working with the maximal real subfields we obtain the $\mathbb{Z}_p$-extension F^+_∞/F^+. Let $\bar{\Gamma} := \mathrm{Gal}(F_\infty/F)$,

$\Gamma := \mathrm{Gal}(F^+_\infty/F^+)$, $\Delta := \mathrm{Gal}(F/F^+)$ and $\Delta' := \mathrm{Gal}(F_\infty/F^+_\infty)$. Now, $D(F_\nu)$ is the subgroup of $F_\nu^\times$ generated by all numbers of the form $1-\omega'$, $\omega' \in \mu(F_\nu)$, and this set of generators can be written as $\{1 - \omega\zeta^i_{p^{\nu+\delta}} \mid \omega \in \mu_d, i = 0, \dots, p^{\nu+\delta}\}$. Consider the projective limits

$$\tilde{D}_{\infty,F} := \varprojlim D(F_\nu)/\mu(F_\nu) \otimes \mathbb{Z}_p,\ \tilde{C}_{\infty,F} := \varprojlim C(F_\nu)/\mu(F_\nu) \otimes \mathbb{Z}_p$$

(of course in our case $\mu(F_\nu) = \mu_{dp^{\nu+\delta}}$.) They are naturally Λ-modules. Their main property is that they are Λ-free modules (see Kuz'min [51]).

First we study the relation between the two modules. Define

$$\theta(\omega) := \varprojlim\,(1 - \omega^{p^{-\nu-\delta}}\zeta_{p^{\nu+\delta}})$$

for all $\nu \geq 1$. The sequence is compatible with norms, namely:

$$N_{F_{\nu+1}/F_\nu}(1 - \omega^{p^{-\nu-1-\delta}}\zeta_{p^{\nu+1+\delta}}) = 1 - \omega^{p^{-\nu-\delta}}\zeta_{p^{\nu+\delta}}$$

(see Lemma 7.1). Let B_ν be the Galois submodule of $D(F_\nu)/\mu(F_\nu)$ generated by all elements $1 - \omega^{p^{-\nu-\delta}}\zeta_{p^{\nu+\delta}}$ for all $\omega \in \mu_d = \mu_d(F_\nu)$. Thus $D(F_\nu)/\mu(F_\nu) = B_\nu \cdot D(\mathbb{Q}(\mu_d))/\mu_d$. Hence:

$$\tilde{D}_{\infty,F} = \varprojlim B_\nu \otimes \mathbb{Z}_p.$$

We see this way that $\tilde{D}_{\infty,F}$ is generated by the set of elements $\{\theta(\omega) \mid \omega \in \mu_d\}$ as an $\Lambda[\mathrm{Gal}(F/\mathbb{Q})]$-module.

We can note that an element $x \in D(F_\nu)/\mu(F_\nu) \otimes \mathbb{Z}_p$ belongs to the image of the natural projection $\tilde{D}_{\infty,F} \longrightarrow D(F_\nu)/\mu(F_\nu) \otimes \mathbb{Z}_p$ if and only if x belongs to $U^S_{F_\nu}$, where S consists of all primes above p and all infinite primes. Also, it is a well known fact that if ω_m is a primitive m-th root of unity, then $1-\omega_m$ is a unit if and only if m is composite. If $m = p^\nu$, then $1 - \omega_{p^\nu}$ is a prime element in the local field $\mathbb{Q}_p(\omega_{p^\nu})$. We obtain this way the following exact sequence:

$$0 \longrightarrow \tilde{C}_{\infty,F} \longrightarrow \tilde{D}_{\infty,F} \longrightarrow \mathbb{Z}_p \longrightarrow 0,$$

where $\mathbb{Z}_p$ is generated by the element $\theta(1)$.

Theorem 7.13 *Let $F = \mathbb{Q}(\mu_d, \zeta_{p^{1+\delta}})$, where p is a prime, d be a natural number relatively prime to p, and $d \not\equiv 2 \bmod 4$, and $\delta = 0$ if p is odd, and $\delta = 1$ if $p = 2$. Then $\tilde{D}_{\infty,F}$ and $\tilde{C}_{\infty,F}$ are free Λ-modules of rank $[F : \mathbb{Q}]/2$.*

Proof: For each $\nu \geq 1$, define the map

$$\phi_\nu : \mathcal{L}_{p^{\nu+\delta}d} \longrightarrow D(F_\nu)/\mu(F_\nu)$$

as follows:

$$\phi_\nu(0) = 1,\ \phi_\nu\left(\left[\frac{a}{p^{\nu+\delta}d}\right]\right) = 1 - \omega^{p^{-\nu-\delta}a}\zeta^a_{p^{\nu+\delta}},$$

where ω is a fixed primitive d-th root of unity, and $t \in \mathbb{Z}$. This map is an even distribution in the sense of Kubert [49], i.e. satisfies the following relations:

$$\sum_{b,\ tb=a} \phi_\nu\left(\left[\frac{b}{p^{\nu+\delta}d}\right]\right) = \phi_\nu\left(\left[\frac{a}{p^{\nu+\delta}d}\right]\right)$$

for any $a \in \mathbb{Z}$, and $t \mid p^{\nu+\delta}d$, and

$$\phi_\nu\left(\left[\frac{-a}{p^{\nu+\delta}d}\right]\right) = \phi_\nu\left(\left[\frac{a}{p^{\nu+\delta}d}\right]\right).$$

After tensoring with $\mathbb{Z}_p$ we obtain natural epimorphisms compatible with the actions of the groups $\mathrm{Gal}(F_\nu/\mathbb{Q}) \cong (\mathbb{Z}/p^{\nu+\delta}d\mathbb{Z})^\times$ and with norms. Passing to the projective limit with respect to norms, we get an Λ-module epimorphism:

$$\phi : \mathcal{L}_{p^\infty d}/\mathcal{L}^-_{p^\infty d} \longrightarrow \tilde{D}_{\infty,F}$$

(if $\Delta = \{1, J\}$, for a Δ-module A, we define $A^- := \ker(1+J)$ and $A^+ := \ker(1-J)$).

By construction $\tilde{D}_{\infty,F}$ is Λ-torsion-free. Also, because $\mathcal{L}_{p^\infty d}$ is Λ-free, it follows that $\mathcal{L}_{p^\infty d}/\mathcal{L}^-_{p^\infty d}$ and $\mathcal{L}^+_{p^\infty d}$ are Λ-torsion-free, and their Λ-ranks are equal: $\mathrm{rank}_\Lambda\ \mathcal{L}^+_{p^\infty d} = \mathrm{rank}_\Lambda\ \mathcal{L}^-_{p^\infty d} = r_2(F) = [F : \mathbb{Q}]/2$. Moreover, $\mathcal{L}^+_{p^\infty d}$ is in fact Λ-free module because it is a submodule of the Λ-free module $\mathcal{L}_{p^\infty d}$ and $\mathcal{L}_{p^\infty d}/\mathcal{L}^+_{p^\infty d}$ is $\mathbb{Z}_p$-torsion-free (see Lemma 6.7). Now, for any ν there is a natural projection $\bar{D}_{\infty,F} \longrightarrow B_\nu \otimes \mathbb{Z}_p$ (see the previous discussions.) By Sinnott [71], $C(F_\nu)$ has finite index in $U(F_\nu)$. It follows that:

$$\mathrm{rank}_{\mathbb{Z}_p} B_\nu \otimes \mathbb{Z}_p \geq \mathrm{rank}_{\mathbb{Z}_p} U(F_\nu) \otimes \mathbb{Z}_p - \mathrm{rank}_{\mathbb{Z}_p} (D(\mathbb{Q}(\mu_d))/\mu_d) \otimes \mathbb{Z}_p$$

for any $\nu \geq 1$, i.e. $\mathrm{rank}_{\mathbb{Z}_p} B_\nu \otimes \mathbb{Z}_p \geq [F_\nu : \mathbb{Q}]/2 - k$, where k does not depend on ν. This implies that $\tilde{D}_{\infty,F}$ has Λ-rank equal to $[F : \mathbb{Q}]/2$. We conclude that ϕ is an isomorphism.

If p is odd, then $\mathcal{L}_{p^\infty d} = \mathcal{L}^+_{p^\infty d} \oplus \mathcal{L}^-_{p^\infty d}$, so that $\mathcal{L}^+_{p^\infty d} \cong \tilde{D}_{\infty,F}$, hence $\tilde{D}_{\infty,F}$ is Λ-free module.

If $p = 2$, we have

$$\tilde{D}_{\infty,F} \cong \mathcal{L}_{p^\infty d}/\mathcal{L}^-_{p^\infty d} \cong N_\Delta(\mathcal{L}_{p^\infty d}),$$

where N_Δ denotes the norm relative to Δ. But we know that $\hat{H}^0(\Delta, \mathcal{L}_{p^\infty d}) = 0$, i.e. $N_\Delta(\mathcal{L}_{p^\infty d}) = \mathcal{L}^+_{p^\infty d}$. Hence $\tilde{D}_{\infty,F}$ is Λ-free module in this case too.

Finally, the exact sequence

$$0 \longrightarrow \tilde{C}_{\infty,F} \longrightarrow \tilde{D}_{\infty,F} \longrightarrow \mathbb{Z}_p \longrightarrow 0,$$

together with the fact that $\tilde{D}_{\infty,F}$ is Λ-free module imply that $\tilde{C}_{\infty,F}$ is a Λ-free module too (see Lemma 6.7). □

Proposition 7.14 *Let* $F = \mathbb{Q}(\zeta_4, \zeta_d)/F^+$ *with* $d \geq 3$ *odd, and* $\Delta := \mathrm{Gal}(F/F^+)$. *Then:*

$$(\tilde{C}^{\Delta'}_{\infty,F})_\Gamma \cong ((\tilde{C}_{\infty,F})_{\bar{\Gamma}})^\Delta$$

Proof: The isomorphism is obtained by a simple analysis of the five term exact sequences of the two Hochschild-Serre spectral sequences associated to the two pairs of groups, and by using the previous result.□

Chapter 8

Iwasawa modules versus étale cohomology groups

Let p be a prime, K be a number field, and $S := S_p \cup S_\infty$, where S_p is the set of the primes of K above p and S_∞ is the set of the archimedean places of K.

Let K_∞/K be the cyclotomic $\mathbb{Z}_p$-extension of K. As we know, this is defined as the unique $\mathbb{Z}_p$-extension of K inside $K(\mu_{p^\infty}) := \bigcup_{r\geq 0} K(\mu_{p^{r+\delta}})$. Here $\delta = 0$ if p is odd, and $\delta = 1$ if $p = 2$. Let $G_\infty := \mathrm{Gal}(K(\mu_{p^\infty})/K)$, $\Gamma := \mathrm{Gal}(K_\infty/K)$, $\bar{\Gamma} := \mathrm{Gal}(K(\mu_{p^\infty})/K(\mu_{p^{1+\delta}}))$, $\Delta := \mathrm{Gal}(K(\mu_{p^{1+\delta}})/K)$, and $\Delta' := \mathrm{Gal}(K(\mu_{p^\infty})/K_\infty)$. We denote the intermediate fields of K_∞/K by K_r for all $r \geq 0$ ($K_0 = K_1 = K$). Clearly, K_r is the fixed subfield of $K(\mu_{p^{r+\delta}})$ under the Galois group Δ'. Let γ denote a fixed topological generator of Γ, and $\bar{\gamma}$ a fixed topological generator of $\bar{\Gamma}$.

If the prime p is odd then G_∞ is pro-cyclic, as $G_\infty \cong \bar{\Gamma} \times \Delta$ and the order of Δ is relatively prime to p. One works with G_∞ in this case. If $p = 2$, G_∞ is not always pro-cyclic. The group G_∞ is pro-cyclic if and only if either $\zeta_4 \in K$ or $\zeta_{2^k} - \zeta_{2^k}^{-1} \in K$ for some $k \geq 3$. In this case, $G_\infty = \Gamma$ and we work with Γ. A number field K with this property is called a **non-exceptional** number field. If G_∞ is not pro-cyclic, i.e. K does not have the previous property, then K is called an **exceptional** number field. This situation occurs always if $p = 2$ and K is totally real. In the exceptional case we work with Γ, not G_∞ (G_∞ is not pro-cyclic in this case).

It is clear that Γ acts on $\mathbb{Z}_2(n)$ and $\mathbb{Q}_2/\mathbb{Z}_2(n)$ for any integer n if K is a **non-exceptional** field. In the **exceptional** case, Γ acts on $\mathbb{Z}_2(n)$ and $\mathbb{Q}_2/\mathbb{Z}_2(n)$ only if n is an even integer, because in this case Δ' acts trivially on these twisted modules. We are going to state and prove facts only for the case $p = 2$ mentioning in remarks the corresponding formulations and results for odd primes p (the latter case being treated by Schneider in [68]).

We prove first Soulé's and Tate's fundamental results in the previous context for $p = 2$.

Lemma 8.1 *Let $p = 2$ and $n \neq 0$. Assume either K is non-exceptional or n is even. Then we have:*
(1) Let $f(T)$ be the characteristic polynomial of a Γ-torsion module M. Then

$$\mathrm{char}(M(n)) = f(\kappa(\gamma)^{-n}(1+T) - 1).$$

Consequently, for all $n \in \mathbb{Z}$, $\mathrm{char}(\mathbb{Z}_2(n)) = \kappa(\gamma)^{-n}(1+T) - 1$.
(2) (Soulé) $\mathbb{Z}_2(n)^\Gamma = 0$, and $\mathbb{Z}_2(n)_\Gamma$ has finite order $|\mathbb{Z}_2(n)_\Gamma| = p^{v_p(\kappa(\gamma)^{-n}-1)}$.
(3) (Tate) Let M be a discrete Γ-module. Then

$$H^1(\Gamma, M \otimes \mathbb{Q}_2/\mathbb{Z}_2(n)) = 0.$$

Proof: For (1), let $g(T)$ denote the characteristic polynomial of $M(n)$. The generator of Γ acts on $M(n)$ as multiplication by $1+T$, $\gamma *_n x = (1+T)\cdot x$, $x \in M$. Let multiplication by $1+S$ correspond to the action of γ on M. Then $\gamma *_n x = \kappa(\gamma)^n \cdot x^\gamma = \kappa(\gamma)^n \cdot (1+S)\cdot x$. This gives us $S = \kappa(\gamma)^{-n}(1+T) - 1$. Taking $M = \mathbb{Z}_2 \cong \Lambda/(T)$ whose characteristic polynomial is T, we obtain the consequence. To prove (2), let us consider first the case when K is a non-exceptional number field. Then γ cannot act of finite order on all p-power roots of unity, hence we have $\kappa(\gamma)^{-n} \neq 1$. It follows that the characteristic polynomial of $\mathbb{Z}_2(n)$, $\kappa(\gamma)^{-n}(1+T) - 1$, does not vanish at $T = 0$. By Theorem 6.5, we obtain that $\mathbb{Z}_2(n)^\Gamma$ and $\mathbb{Z}_2(n)_\Gamma$ are both finite. But $\mathbb{Z}_2(n)^\Gamma$ cannot be finite unless it is trivial, and the order of $\mathbb{Z}_2(n)_\Gamma$ is given again by Theorem 6.5. Now let K be an exceptional number field, and n an even integer, $n \neq 0$. Because $\mathbb{Z}_2(n)^{\Gamma} = 0$, we obtain $\mathbb{Z}_2(n)^{G_\infty} = 0$. The short exact sequence

$$0 \longrightarrow \Delta' \longrightarrow G_\infty \longrightarrow \Gamma \longrightarrow 0$$

induces in cohomology the following exact sequence:

$$0 \to \mathbb{Z}_2(n)_{\Delta'} \to \mathbb{Z}_2(n)^{G_\infty} \to \mathbb{Z}_2(n)^{\Gamma} \to H^1(\Delta', \mathbb{Z}_2(n)).$$

But $\mathbb{Z}_2(n)^{G_\infty} = 0$, and $H^1(\Delta', \mathbb{Z}_2(n)) = 0$. Therefore $\mathbb{Z}_2(n)^{\Gamma} = 0$.

For (3), since M is a discrete Γ-module, we may assume it is finitely generated. Since tensoring with $\mathbb{Q}_2/\mathbb{Z}_2$ removes the p-torsion part, we only have to consider the case $M = \mathbb{Z}_2$. Using (2) we obtain:

$$H^1(\Gamma, \mathbb{Z}_2 \otimes \mathbb{Q}_2/\mathbb{Z}_2(n)) \cong H^1(\Gamma, \mathbb{Q}_2/\mathbb{Z}_2(n)) \cong \mathbb{Q}_2/\mathbb{Z}_2(n)_\Gamma \cong (\mathbb{Z}_2(-n)^{\Gamma})^{\#} = 0$$

(see the action of γ on $\mathrm{Hom}(\Gamma, M)$ in chapter 2). □

Remark 8.2 *We can make a detailed study of the case when K is an exceptional number field, and n is an even integer, $n \neq 0$, as follows. The short exact sequences*

$$0 \longrightarrow \bar{\Gamma} \longrightarrow G_\infty \longrightarrow \Delta \longrightarrow 0, \text{ and}$$

$$0 \longrightarrow \Delta' \longrightarrow G_\infty \longrightarrow \Gamma \longrightarrow 0$$

induce the following first-quadrant convergent Hochschild-Serre spectral sequences:

$$E_2^{st} = H^s(\Delta, H^t(\bar{\Gamma}, \mathbb{Q}_2/\mathbb{Z}_2(n))) \Rightarrow E^{s+t} = H^{s+t}(G_\infty, \mathbb{Q}_2/\mathbb{Z}_2(n)), \text{ and}$$

$$E_2^{st} = H^s(\Gamma, H^t(\Delta', \mathbb{Q}_2/\mathbb{Z}_2(n))) \Rightarrow E^{s+t} = H^{s+t}(G_\infty, \mathbb{Q}_2/\mathbb{Z}_2(n)).$$

The five-term exact sequence theorem for the first spectral sequence gives the following exact sequence:

$$0 \longrightarrow H^1(\Delta, H^0(\bar{\Gamma}, \mathbb{Q}_2/\mathbb{Z}_2(n))) \longrightarrow H^1(G_\infty, \mathbb{Q}_2/\mathbb{Z}_2(n))$$

$$\longrightarrow H^0(\Delta, H^1(\bar{\Gamma}, \mathbb{Q}_2/\mathbb{Z}_2(n))).$$

But we just proved that $H^1(\bar{\Gamma}, \mathbb{Q}_2/\mathbb{Z}_2(n)) = 0$, *so:*

$$H^1(G_\infty, \mathbb{Q}_2/\mathbb{Z}_2(n)) \cong H^1(\Delta, H^0(\bar{\Gamma}, \mathbb{Q}_2/\mathbb{Z}_2(n))) = H^1(\Delta, \mu_4^{\otimes n}) = \mu_2^{\otimes n} \cong \mathbb{Z}/2\mathbb{Z}.$$

Also, from the second spectral sequence we obtain the following exact sequence:

$$0 \longrightarrow H^1(\Gamma, H^0(\Delta', \mathbb{Q}_2/\mathbb{Z}_2(n))) \longrightarrow H^1(G_\infty, \mathbb{Q}_2/\mathbb{Z}_2(n))$$
$$\longrightarrow H^0(\Gamma, H^1(\Delta', \mathbb{Q}_2/\mathbb{Z}_2(n))) \longrightarrow 0.$$

The cosource of the last arrow is 0 *because* $\mathrm{cd}_p\Gamma = 1$. *We note that* $H^0(\Delta', \mathbb{Q}_2/\mathbb{Z}_2(n)) = \mathbb{Q}_2/\mathbb{Z}_2(n)$, $H^1(\Delta', \mathbb{Q}_2/\mathbb{Z}_2(n)) = \mu_2^{\otimes n} \cong \mathbb{Z}/2\mathbb{Z}$, *and* $H^0(\Gamma, \mu_2^{\otimes n}) = \mu_2^{\otimes n} \cong \mathbb{Z}/2\mathbb{Z}$. *So:*

$$0 \longrightarrow H^1(\Gamma, \mathbb{Q}_2/\mathbb{Z}_2(n)) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

It follows: $H^1(\Gamma, \mathbb{Q}_2/\mathbb{Z}_2(n)) = 0$.

Remark 8.3 *The same result holds for all odd prime p, all abelian number fields, and all non-zero twists, but working with* G_∞ *as defined previously.*

Theorem 8.4 *Let* $p = 2$ *and* $n \neq 0$. *Assume either* K *is non-exceptional or* n *is even. Then we have:*

$$H^1(\mathcal{O}_K^S, \mathbb{Q}_2/\mathbb{Z}_2(-n))^\# \cong \mathcal{X}_{\infty,K}(n)_\Gamma,$$

$$H^1(K_v, \mathbb{Q}_2/\mathbb{Z}_2(-n))^\# \cong \mathcal{X}_{\infty,K_v}(n)_\Gamma,$$

$$H^2(\mathcal{O}_K^S, \mathbb{Q}_2/\mathbb{Z}_2(-n))^\# \cong \mathcal{X}_{\infty,K}(n)^\Gamma,$$

$$H^2(K_v, \mathbb{Q}_2/\mathbb{Z}_2(-n))^\# \cong \mathcal{X}_{\infty,K_v}(n)^\Gamma,$$

where $v \in S_2$.
Also, if $n \in \mathbb{Z}\backslash\{0, 1\}$ *and* K *non-exceptional we have:*

$$\text{Ш}^2(\mathcal{O}_K^S, \mathbb{Z}_2(n)) \cong X'_{\infty,K}(n-1)_\Gamma.$$

Proof: Consider the following Hochschild-Serre spectral sequence:

$$E_2^{s,t} = H^s(\Gamma, H^t(G_S(K_\infty), \mathbb{Q}_2/\mathbb{Z}_2(n)) \Rightarrow E^{s+t} = H^{s+t}(G_S(K), \mathbb{Q}_2/\mathbb{Z}_2(n)).$$

Because the cohomological p-dimension of Γ is 1, $E_2^{s,t} = 0$ for all $s \geq 2$, $t \geq 0$ and we obtain the following exact sequences:

$$1 \to E_2^{1,r-1} \to E^r \to E_2^{0,r} \to 1$$

for all $r \geq 1$. Particularly,

$$1 \to H^1(\Gamma, H^1(G_S(K_\infty), \mathbb{Q}_2/\mathbb{Z}_2(n))) \to H^2(G_S(K), \mathbb{Q}_2/\mathbb{Z}_2(n))$$
$$\to H^2(G_S(K_\infty), \mathbb{Q}_2/\mathbb{Z}_2(n))^\Gamma \to 1,$$

and

$$1 \to H^1(\Gamma, H^0(G_S(K_\infty), \mathbb{Q}_2/\mathbb{Z}_2(n))) \to H^1(G_S(K), \mathbb{Q}_2/\mathbb{Z}_2(n))$$
$$\to H^1(G_S(K_\infty), \mathbb{Q}_2/\mathbb{Z}_2(n))^\Gamma \to 1.$$

The hypotheses on K assure that the Galois group $G_S(K_\infty)$ acts trivially on $\mathbb{Q}_2/\mathbb{Z}_2(n)$.

We obtain $H^2(G_S(K_\infty), \mathbb{Q}_2/\mathbb{Z}_2(n)) = 0$, because the Weak Leopoldt Conjecture holds for the cyclotomic $\mathbb{Z}_2$-extension of K (see Theorem 6.20 and Proposition 6.21). Moreover,

$$H^1(\Gamma, H^1(G_S(K_\infty), \mathbb{Q}_2/\mathbb{Z}_2(n))) \cong \mathrm{Hom}_{\mathbb{Z}_2}(\mathcal{X}_{\infty,K}, \mathbb{Q}_2/\mathbb{Z}_2(n))_\Gamma$$
$$\cong (\mathcal{X}_{\infty,K}(-n)^\#)_\Gamma \cong (\mathcal{X}_{\infty,K}(-n)^\Gamma)^\#.$$

Thus we obtain the second assertion on global fields:

$$H^2(G_S(K), \mathbb{Q}_2/\mathbb{Z}_2(n))^\# \cong \mathcal{X}_{\infty,K}(-n)^\Gamma.$$

Next, we note that:

$$H^1(\Gamma, H^0(G_S(K_\infty), \mathbb{Q}_2/\mathbb{Z}_2(n))) \cong H^1(\Gamma, \mathbb{Q}_2/\mathbb{Z}_2(n)) = 0,$$

by Tate's Lemma ($n \neq 0$). The first global assertion follows:

$$H^1(G_S(K), \mathbb{Q}_2/\mathbb{Z}_2(n))^\# \cong (H^1(G_S(K_\infty), \mathbb{Q}_2/\mathbb{Z}_2(n))^\Gamma)^\#$$

$$\cong (\mathrm{Hom}_{\mathbb{Z}_2}(\mathcal{X}_{\infty,K}, \mathbb{Q}_2/\mathbb{Z}_2(n))^\Gamma)^\# \cong \mathcal{X}_{\infty,K}(-n)_\Gamma.$$

The local versions of the statements can be derived in exactly the same way.

In order to prove the last result, we demonstrate that:

$$\text{Ш}^2(\mathcal{O}_K^S, \mathbb{Q}_2/\mathbb{Z}_2(1-n)) \cong (X'_{\infty,K}(n-1)_\Gamma)^\#,$$

and then take the duals and use Poitou-Tate global duality. In the following diagram the vertical sequences are inflation-restriction exact sequences, while horizontally we have localization maps. All squares are naturally commutative.

$$\begin{array}{ccc}
H^1(\Gamma, \mathbb{Q}_2/\mathbb{Z}_2(1-n)) & \rightarrow & \oplus_{v\in S} H^1(\mathrm{Gal}(K_{\infty,w}/K_v), \mathbb{Q}_2/\mathbb{Z}_2(1-n)) \\
\downarrow & & \downarrow \\
H^1(G_S(K), \mathbb{Q}_2/\mathbb{Z}_2(1-n)) & \rightarrow & \oplus_{v\in S} H^1(K_v, \mathbb{Q}_2/\mathbb{Z}_2(1-n)) \\
\downarrow & & \downarrow \\
H^0(\Gamma, H^1(G_S(K_\infty), \mathbb{Q}_2/\mathbb{Z}_2(1-n))) & \rightarrow & \oplus_{w\in S} H^1(K_{\infty,w}, \mathbb{Q}_2/\mathbb{Z}_2(1-n)) \\
\downarrow & & \\
H^2(\Gamma, \mathbb{Q}_2/\mathbb{Z}_2(1-n)) & &
\end{array}$$

Using the fact that the cohomological p-dimension of Γ is 1, and using again Tate's Lemma ($n \neq 1$), we obtain:

$$H^2(\Gamma, \mathbb{Q}_2/\mathbb{Z}_2(1-n)) = 0, \; H^1(\Gamma, \mathbb{Q}_2/\mathbb{Z}_2(1-n)) = 0,$$

$$\text{and } H^1(\mathrm{Gal}(K_{\infty,w}/K_v), \mathbb{Q}_2/\mathbb{Z}_2(1-n)) = 0.$$

(here w is a fixed place of K_∞ above v).

It follows:

$$\text{Ш}^2(\mathcal{O}_K^S, \mathbb{Q}_2/\mathbb{Z}_p(1-n)) =$$

$$= \ker(H^0(\Gamma, H^1(G_S(K_\infty), \mathbb{Q}_2/\mathbb{Z}_2(1-n))) \rightarrow \oplus_{w\in S} H^1(K_{\infty,w}, \mathbb{Q}_2/\mathbb{Z}_2(1-n)))$$

$$= H^0(\Gamma, \ker(H^1(G_S(K_\infty), \mathbb{Q}_2/\mathbb{Z}_2(1-n)) \to \oplus_{w \in S} H^1(K_{\infty,w}, \mathbb{Q}_2/\mathbb{Z}_2(1-n))))$$
$$= H^0(\Gamma, \ker(H^1(G_S(K_\infty), \mathbb{Q}_2/\mathbb{Z}_2) \to \oplus_{w \in S} H^1(K_{\infty,w}, \mathbb{Q}_2/\mathbb{Z}_2))(1-n))$$
$$= ((X'_{\infty,K})^{\#}(1-n))^{\Gamma} = (X'_{\infty,K}(n-1)_{\Gamma})^{\#}. \square$$

Remark 8.5 *The same result holds if p is odd for all number fields, and all non-zero twists. Again one works with G_∞ in this case.*

Chapter 9

Soulé's and Beilinson's elements in the K-theory of cyclotomic fields

Let p be a prime, $n \geq 1$, $E = \mathbb{Q}(\mu_N)$, $N \geq 3$, $\mathcal{O}_E^S = \mathbb{Z}[\mu_N][\frac{1}{p}]$, and $X = \mathrm{Spec}(E)$ (S is the set of all primes above p and all infinite primes). It is quite difficult to exhibit concrete elements in K-groups and to compute the index of the subgroup generated by them one needs to look at their images via Chern characters. In this chapter we introduce the Soulé's and Beilinson 's elements in odd K-groups. It turns out that they agree in $K_{2n-1}(E) \otimes \mathbb{Z}_p$. Moreover, the Chern character maps them to the modified Soulé-Deligne cyclotomic elements in (continuous) étale cohomology. Finally, the computation of the index of the subgroup generated by these last elements in étale cohomology amounts to computing the index of subgroup generated by the images of circular units in étale cohomology. This leads us to a computation of the index of the Beilinson subgroup in $K_{2n-1}(E) \otimes \mathbb{Z}_p$. The following diagram contains the main groups:

$$\begin{array}{ccccc} \tilde{K}_{2n-1}(E) \otimes \mathbb{Z}_p & \longrightarrow & \tilde{H}^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_p(n)) & = & \tilde{H}^1_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_p(n)) \\ \uparrow & & \uparrow & & \uparrow \\ B_n^{prim}(E) \otimes \mathbb{Z}_p & \longrightarrow & \tilde{D}_n(E) & \longleftarrow & \tilde{C}^S_{\infty,E}(n-1)_\Gamma \end{array}$$

The maps will be defined and studied below.

As we mentioned previously there is a projective system of regular product structures on the $K_*(\mathcal{O}_E^S, \mathbb{Z}_p)$ spectrum, which is unique if p is odd, and there are two product structures if $p = 2$. Soulé constructed in [73] the following maps:

$$\phi_n^p : \mu_N^{prim} \longrightarrow K_{2n-1}(\mathcal{O}_E^S, \mathbb{Z}_p) \cong K_{2n-1}(\mathcal{O}_E^S) \otimes \mathbb{Z}_p\,,$$

given by

$$\phi_n^p(\zeta_N) = (N_{E_r/E}((1-\alpha_r) \cup (\beta_r)^{\cup(n-1)}))_r\,,$$

called the **Soulé's elements in K-theory**. Here ζ_N is a primitive N-th root of unity,

$$(\alpha_r)_r \in \varprojlim \mu_{Np^r}$$

is a projective system such that $\alpha_1^p = \zeta_N$, and

$$(\beta_r)_r \in \varprojlim K_2(\mathcal{O}_{E_r}^S, \mathbb{Z}/p^r)$$

is the projective system of Bott elements such that $\det \circ j_{p^r}(\beta_r) = \alpha_r$ - the maps are defined below (the domain of these maps can be extended to all non-trivial roots of unity). Note that the composition of morphisms

$$K_2(\mathcal{O}_{E_r}^S, \mathbb{Z}/p^r\mathbb{Z}) \xrightarrow{j_{p^r}} K_1(\mathcal{O}_{E_r}^S)(p^r) \xrightarrow{\det} \mu_{p^r}(\mathcal{O}_{E_r}^S)$$

has a natural section defined as the composition of the following maps:

$$\mu_{p^r}(\mathcal{O}_{E_r}^S) \cong \pi_2(BU_E^S, \mathbb{Z}/p^r\mathbb{Z})$$

$$\cong \pi_2(BGL_1(\mathcal{O}_{E_r}^S), \mathbb{Z}/p^r\mathbb{Z}) \longrightarrow \pi_2(BGL(\mathcal{O}_{E_r}^S)^+, \mathbb{Z}/p^r\mathbb{Z}).$$

Using Soulé's formalism of norm(trace) compatible systems of units presented in [73], we obtain the construction. For $p = 2$ one has to use the modification of the Bott elements introduced by Arlettaz, Banaszak and Gajda in [1].

Soulé's elements map to the Soulé-Deligne cyclotomic elements (described below) in étale cohomology via Chern characters.

Now let ω be a fixed N-th root of unity, $\omega \neq 1$. We consider a rational function $f = f_{a,b}(\omega) \in E(t_1, \ldots, t_{n-1})$, where $a = (a_{ij})$, $b = (b_{ij})$ are matrices of integers of size $(n-1)\times 2^{n-2}$ having certain properties described in Neukirch [65], Lemma 3.1, and we set

$$f_{a,b}(\omega)=\prod_j \frac{1-\omega\prod_i t_i^{a_{ij}}}{1-\omega\prod_i t_i^{b_{ij}}} \quad \text{and}$$

$$C_{a,b}=\sum_j\Big(\prod_i a_{ij}^{-1}-\prod_i b_{ij}^{-1}\Big)\neq 0\,.$$

Let $\mathbb{A}^{n-1}_{(f)}$ denote the complement of zeroes and poles of f on $\mathbb{A}^{n-1}_E=\mathrm{Spec}(E[t_1,\ldots,t_{n-1}])$, $T=\{\prod_i t_i(t_i-1)=0\}$, and $T_{(f)}=T\cap\mathbb{A}^{n-1}_{(f)}$.

Following Neukirch [65] we can form the following Loday symbol in the relative absolute cohomology (Deligne [17]):

$$l_{a,b}(\omega)=C_{a,b}^{-1}\cdot\{f_{a,b}(\omega),t_1,\ldots,t_{n-1}\}_{\mathcal{A}}\in H^n_{\mathcal{A}}(\mathbb{A}^{n-1}_{(f)},T_{(f)}.;\mathbb{Q}(n)).$$

By Neukirch [65], Lemma 4.3 and Lemma 4.2, this symbol (in fact its inverse image in $H^n_{\mathcal{A}}(\mathbb{A}^{n-1}_{(f)},T_{(f)}.;\mathbb{Q}(n))$, where the simplicial scheme $T.$ over $\mathbb{A}^{n-1}_E$ is obtained "by resolution of singularities of the divisor with normal crossings" T in $\mathbb{A}^{n-1}_E$; similarly for $T_{(f)}.$) has a unique preimage

$$\tilde{l}_{a,b}(\omega)\in H^1_{\mathcal{A}}(\mathrm{Spec}(E),\mathbb{Q}(n))\cong K_{2n-1}(E)\otimes\mathbb{Q},$$

called **Beilinson's elements in K-theory**. Moreover, under Beilinson regulator map it maps to the n-th polylogarithm. Let $\mathrm{Li}_n(z)$ denote the n-th polylogarithm defined by:

$$\mathrm{Li}_n(z)=\sum_{k=1}^{\infty}\frac{z^k}{k^n},$$

and analytically continued to $\mathbb{C}\setminus[1,\infty)$.

Theorem(Neukirch [65], Th. 1.1; Huber, Wildeshaus [33], Cor. 9.6) 9.1 *For every $n\geq 2$ we have a map of* $\mathrm{Gal}(E/\mathbb{Q})$*-sets*

$$\epsilon_n:\mu_N\longrightarrow H^1_{\mathcal{A}}(\mathrm{Spec}(E),\mathbb{Q}(n))\cong K_{2n-1}(E)\otimes\mathbb{Q}$$

such that, for $\omega\in\mu_N$,

$$\rho_n^{Bei}(\epsilon_n(\omega))=\mathrm{Li}_n(\omega).$$

Proof: Define $\epsilon_n(\omega) = \tilde{l}_{a,b}(\omega)$, and apply Main Lemma 4.1, and Lemma 4.3 from Neukirch [65]. For a different approach see Huber, Wildeshaus [33].□

The two types of elements in K-theory have the same image in p-adic étale cohomology.

Theorem(Huber, Wildeshaus [33]) 9.2 *Let ω denote a fixed N-th root of unity, $N \geq 3$, and n be an integer, $n \geq 2$. The p-adic regulator*

$$\rho_n^p : K_{2n-1}(E) \otimes \mathbb{Q}_p \longrightarrow H^1_{cont}(E, \mathbb{Q}_p(n))$$

takes $\phi_n^p(\omega)$ and $N^{n-1} \cdot n! \cdot \epsilon_n(\omega)$ to the modified Soulé-Deligne (Deligne [18]) cyclotomic elements in (continuous) étale cohomology.

These results lead actually to a proof of the existence of integral Beilinson elements, as in the articles [33] and [32]. Summing up we have:

Theorem 9.3 *Let ω denote a fixed N-th root of unity, $N \geq 3$, and n be an integer, $n \geq 2$. There is an element in $K_{2n-1}(E)$, $B_n(\omega)$, whose image in $K_{2n-1}(E) \otimes \mathbb{Z}_p$ agrees with Soulé's cyclotomic elements. Hence its image in étale cohomology via the Chern character is the modified Soulé-Deligne cyclotomic element $\tilde{c}_n(\zeta_N)$ (described below). Moreover the Beilinson regulator of this element in Deligne cohomology can be computed and is given by polylogarithm functions.*

Let $B_n^{prim}(E)$ denote the subgroup of $\tilde{K}_{2n-1}(E)$ generated by $B_n(\omega)$ and its Galois conjugates for all $\omega \in \mu_N^{prim}$. Here $\tilde{K}_{2n-1}(E)$ denotes the torsion-free part of $K_{2n-1}(E)$. For an arbitrary abelian field F with conductor N, we define $B_n^{prim}(F)$ as the image of $B_n^{prim}(E)$ under the transfer map

$$\mathrm{Tr}_{E/F} : \tilde{K}_{2n-1}(E) \longrightarrow \tilde{K}_{2n-1}(F).$$

Let $\tilde{B}_n^{prim}(F)$ be the image of $B_n^{prim}(F)$ in $\tilde{K}_{2n-1}(F)$.

We describe briefly the elements on the étale cohomology side. Consider the cyclotomic $\mathbb{Z}_p$-extension of $E = \mathbb{Q}(\zeta_N)$, and n an integer, $n \geq 2$. For each $r \geq 0$

define

$$c_n(\zeta_N)_r := \mathrm{cor}_{E(\zeta_{p^r})/E}(\delta(1-\zeta_{Np^r}) \otimes \zeta_{p^r}^{\otimes(n-1)}) \in H^1_{\acute{e}t}(E, \mathbb{Z}/p^r\mathbb{Z}(n)),$$

where δ is the Kummer connecting morphism

$$\delta : E(\zeta_{p^r})^\times \longrightarrow H^1_{\acute{e}t}(E(\zeta_{p^r}, \mathbb{Z}/p^r\mathbb{Z}(1)).$$

By passing to the projective limit with respect to the natural maps between the coefficient sheaves, we define further

$$c_n(\zeta_N) := \varprojlim c_n(\zeta_N)_r \in H^1_{\acute{e}t}(E, \mathbb{Z}_p(n)).$$

We call these the **Soulé-Deligne cyclotomic elements.** These elements have a well understood behavior with respect to corestriction maps. Namely, if l is a prime number, then:

$$\mathrm{cor}_{\mathbb{Q}(\zeta_{Nl})/\mathbb{Q}(\zeta_N)} c_n(\zeta_{Nl}) = \begin{cases} (1 - l^{n-1}\mathrm{Frob}_l^{-1})c_n(\zeta_N) & \text{if } (l, Np) = 1 \\ c_n(\zeta_N) & \text{if } l \mid Np. \end{cases}$$

Here Frob_l is the Frobenius morphism associated to the prime l.

Let $D_n(E)$ be the $\mathbb{Z}_p[\mathrm{Gal}(E/\mathbb{Q})]$-submodule of $H^1_{\acute{e}t}(E, \mathbb{Z}_p(n))$ generated by $c_n(\zeta_N)$. For an arbitrary abelian field F with conductor N, we define $D_n(F)$ as the image of $D_n(E)$ under the corestriction map. Denote by $\tilde{D}_n(F)$ its torsion-free part.

The **modified Soulé-Deligne cyclotomic elements** are defined as follows: $\tilde{c}_n(\zeta_N) := c_n(\zeta_N)$ if $p \mid N$, and $\tilde{c}_n(\zeta_N) := (1 - p^{n-1}\mathrm{Frob}^{-1})c_n(\zeta_N)$ otherwise (refer to Benois, Nguyen Quang Do [10]).

As the Chern character $ch^{(p)}_{1,n} : \tilde{K}_{2n-1}(F) \otimes \mathbb{Z}_p \xrightarrow{\cong} \tilde{H}^1_{\acute{e}t}(\mathcal{O}^S_F, \mathbb{Z}_p(n))$ maps isomorphically $B^{prim}_n(F)$ onto $\tilde{D}_n(F)$, for p odd, we obtain:

$$[\tilde{K}_{2n-1}(F) \otimes \mathbb{Z}_p : B^{prim}_n(F)] = [\tilde{H}^1_{\acute{e}t}(\mathcal{O}^S_F, \mathbb{Z}_p(n)) : \tilde{D}_n(F)].$$

For $p = 2$, using corollary 3.7, we have:

$$[\tilde{K}_{2n-1}(F) \otimes \mathbb{Z}_2 : B^{prim}_n(F) \otimes \mathbb{Z}_2] = 2^{-r_1+1+\delta_F} \cdot [\tilde{H}^1_{\acute{e}t}(\mathcal{O}^S_F, \mathbb{Z}_2(n)) : \tilde{D}_n(F)]$$

if $n \equiv 3 \bmod 4$, and

$$[\tilde{K}_{2n-1}(F) \otimes \mathbb{Z}_2 : B_n^{prim}(F) \otimes \mathbb{Z}_2] = [\tilde{H}^1_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_2(n)) : \tilde{D}_n(F)]$$

if $n \equiv 1 \bmod 4$. The latter index becomes computable once we relate $\tilde{D}_n(F)$ to cyclotomic units.

Let F_∞ be the cyclotomic $\mathbb{Z}_p$-extension of F, and let Γ denote the Galois group of this extension. Using the group of S-cyclotomic units (refer to chapter 7 for details) of the fields at each intermediate level, we construct the following group:

$$\tilde{C}^S_{\infty,F}(n-1)_\Gamma := \varprojlim (\tilde{C}^S(F_r) \otimes \mathbb{Z}/p^r\mathbb{Z}(n-1))_{\Gamma/\Gamma^{p^r}}.$$

For p odd, Soulé constructed a map

$$\tilde{\psi} : \tilde{C}^S_{\infty,F}(n-1)_\Gamma \longrightarrow \tilde{H}^1_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_p(n))$$

(this construction is described in chapter 10). We show in chapter 10 that this map can be constructed also in the case $p = 2$ if F is a complex abelian field or if F is a real abelian and n is odd. This map allows the use of powerful results from Iwasawa theory which in combination with the Poitou-Tate exact sequence lead to the following computation: in the case p odd, one obtains

$$[\tilde{H}^1_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_p(n)) : \text{im}(\tilde{\psi})] = |H^2_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_p(n))|^{\mp}$$

($-$ if n is even, $+$ if n is odd - refer to [46], [10]), and in the case $p = 2$, F is real, and n is odd, we obtain

$$\frac{|\text{coker}(\tilde{\psi})|}{|\ker(\tilde{\psi})|} = 2^{r_1+\mu-1} \cdot |H^2(\mathcal{O}_F^S, \mathbb{Z}_2(n))|$$

(refer to chapter 9).

Finally, it was proven in [10] that:

$$[\text{im}\,(\tilde{\psi}) : \tilde{D}_n(F)] \sim_p \mathcal{E}_n(F) \cdot |\ker(\tilde{\psi})|^{-1}$$

In the case $p = 2$, F is real, and n is odd, there is an undetermined power of 2, which we will call τ. Recent results [48] show that this parameter is trivial (see below).

Putting together all these index computations, in the case $p = 2$, F a real abelian field, and n odd, $n \equiv 3 \bmod 4$, we obtain:

$$2^{r_1 - 1 - \delta_F} \cdot [\tilde{K}_{2n-1}(F) \otimes \mathbb{Z}_2 : B_n^{prim}(F) \otimes \mathbb{Z}_2] = [\tilde{H}^1_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_2(n)) : \tilde{D}_n(F)] =$$

$$|\text{coker}(\tilde{\psi})| \cdot [\text{im}\,(\tilde{\psi}) : \tilde{D}_n(F)] \sim_2$$

$$(2^{r_1 + \mu - 1} \cdot |H^2(\mathcal{O}_F^S, \mathbb{Z}_2(n))| \cdot |\ker(\tilde{\psi})|) \cdot (\mathcal{E}_n(F) \cdot |\ker(\tilde{\psi})|^{-1}) =$$

$$2^{r_1 + \mu - 1} \cdot |H^2(\mathcal{O}_F^S, \mathbb{Z}_2(n))| \cdot \mathcal{E}_n(F).$$

Writing the same computation for the case $n \equiv 1 \bmod 4$, we have:

Theorem 9.4 *For $p = 2$, F real abelian, and $n \geq 3$ odd, we have:*

$$[\tilde{K}_{2n-1}(F, \mathbb{Z}_2) : B_n^{prim}(F) \otimes \mathbb{Z}_2] = 2^{\mu + \delta_F} \cdot |H^2(\mathcal{O}_F^S, \mathbb{Z}_2(n))| \cdot \mathcal{E}_n(F)$$

if $n \equiv 3 \bmod 4$, and

$$[\tilde{K}_{2n-1}(F, \mathbb{Z}_2) : B_n^{prim}(F) \otimes \mathbb{Z}_2] = 2^{r_1 + \mu - 1} \cdot |H^2(\mathcal{O}_F^S, \mathbb{Z}_2(n))| \cdot \mathcal{E}_n(F)$$

if $n \equiv 1 \bmod 4$.

The same computation can be done for an odd p, any abelian field F, and any integer $n \geq 2$ (with no power of 2 in the result).

The index $[\text{im}\,(\tilde{\psi}) : \tilde{D}_n(F)]$ can be computed if one knows the index between certain distribution lattices. We need to introduce a few more things in order to be able to present this index computation.

Let $N \geq 2$, and $n \geq 2$. Let $V_N^{(n-1)}$ be the $\mathbb{Q}$-vector space generated by the symbols $[\omega']$, where ω' is running through all N-th roots of unity ($\omega' \in \mu_N$), and the symbols $[\omega']$ satisfy the **distribution relations of weight** $n - 1$:

$$[\omega'^t] = t^{n-1} \sum_{j=0}^{t-1} [\omega' \zeta_t^j]$$

for all $t \mid N$ (ζ_t denotes a primitive t-th root of unity).

Let ω be a fixed primitive N-th root of unity ($\omega \in \mu_N^{prim}$). The map

$$\phi : \frac{1}{N}\mathbb{Z}/\mathbb{Z} \longrightarrow V_N^{(n-1)}, \ \phi(\frac{r}{N} \bmod \mathbb{Z}) = [\omega^r]$$

is a **distribution of weight** $n-1$ (as in Kubert [49]). We consider here only the case $\dim V_N^{(n-1)} = \phi(N)$, i.e. only **universal distributions of weight** $n-1$. Denote $G = \mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$. In [49] Kubert proved that $< [\omega^\sigma], \sigma \in G >$ form a canonical basis of $V_N^{(n-1)}$.

Definition 9.5 *We let* $\mathcal{L}_N^{(n-1)}(=\mathcal{L}_N)$ *denote the lattice in* $V_N^{(n-1)}$ *spanned by* $[\omega']$, $\omega' \in \mu_N$. $\mathcal{L}_{N,prim}^{(n-1)}(=\mathcal{L}_{N,prim})$ *will denote the lattice spanned by* $[\omega^\sigma]$, $\sigma \in G$. *Moreover, if* F *is an abelian number field of conductor* N, *and* M *is a multiple of* N, *we define the lattice* $\mathcal{L}_F^{(M)}$ *as the sublattice of* $\mathcal{L}_M$ *generated by all traces* $\sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\omega')/\mathbb{Q}(\omega')\cap F)}[\omega'^\sigma]$, *where* $\omega' \in \mu_M$. *This definition is independent of the choice of* M, *because the trace factors through* $\mathbb{Q}(\omega') \cap \mathbb{Q}(\omega_N)$, *and* $\mathcal{L}_N$ *is a sublattice of* $\mathcal{L}_M$. *That is why we will denote this lattice simply* $\mathcal{L}_F$, *and we will take* $M = N$ *as the conductor of* F. *Note that* $\mathcal{L}_F \otimes \mathbb{Q}$ *is a subspace of* V_N, *and* $\dim_{\mathbb{Q}} \mathcal{L}_F \otimes \mathbb{Q} = [F : \mathbb{Q}]$. *Define also* $\mathcal{L}_{F,prim}$ *as previously. They are called* **universal distribution lattice of weight** $n-1$, *and* **primitive universal distribution lattice of weight** $n-1$, *respectively.*

Theorem(Kolster, Nguyen Quang Do [48]) 9.6 *For a full cyclotomic field* $F = \mathbb{Q}(\mu_N)$, $N \in \mathbb{Z}$, $N > 2$, *and* $n \in \mathbb{Z}$, $n \geq 2$, *we have*

$$[\mathcal{L}_F^{(n-1)} : \mathcal{L}_{F,prim}^{(n-1)}] = \prod_{p|N}(p^{f_p(n-1)} - 1)^{r_p},$$

where f_p *is the inertia degree of* p *in* F, *and* r_p *is the number of primes above* p *in* F.

Corollary 9.7 *For a real abelian field field* F *and* $n \geq 2$ *odd, we have:*

$$[\mathcal{L}_F^{(n-1)} : \mathcal{L}_{F,prim}^{(n-1)}] = \prod_{p|N}(p^{f_p(n-1)} - 1)^{r_p}.$$

Recall that for $\mathcal{E}_n(F) := \prod_{\chi,\chi(-1)=(-1)^{n-1}} \prod_{p|N}(1-\chi(p)p^{n-1})$, the **Euler factors**, we have:

$$\mathcal{E}_n(F) = \begin{cases} \prod_{p|N}(p^{f_p^+(n-1)} - 1)^{r_p^+} & \text{if } n \text{ is odd} \\ \prod_{p|N} \frac{(p^{f_p(n-1)}-1)^{r_p}}{(p^{f_p^+(n-1)}-1)^{r_p^+}} & \text{if } n \text{ is even.} \end{cases}$$

We have seen that there is an important map of sets

$$\epsilon_n : \mu_N \setminus \{1\} \longrightarrow K_{2n-1}(\mathbb{Q}(\mu_N)) \otimes \mathbb{Q}$$

such that $(\rho_n^{Bei} \circ \epsilon_n)(\omega') = N^{n-1} \cdot (n-1)! \cdot ((\mathrm{Li}_n(\omega'^{\sigma}))_{\sigma})$, where $\omega' \in \mu_N \setminus \{1\}$. The n-th polylogarithms $\mathrm{Li}_n(\omega')$ satisfy the distribution relations of weight $n-1$. Moreover, one can define

$$\mathrm{Li}_n(1) := \frac{t^{n-1}}{1-t^{n-1}} \cdot \sum_{j=1}^{t-1} \mathrm{Li}_n(\zeta_t^j),$$

where $t \mid N$, $t \neq 1$ (this definition is independent of t). So, ϵ_n can be extended to μ_N. We note that $B_n(\mathbb{Q}(\mu_N))$ (defined previously) is the lattice spanned by all $\epsilon_n(\omega')$ in $K_{2n-1}(\mathbb{Q}(\mu_N)) \otimes \mathbb{Q}$. Under ρ_n^{Bei}, it maps to the full sublattice in $(\mathbb{R}(n-1)^{\mathrm{Hom}(\mathbb{Q}(\mu_N),\mathbb{C})})^+$ spanned by $N^{n-1} \cdot (n-1)! \cdot ((\mathrm{Li}_n(\omega'^{\sigma}))_{\sigma})$, $\omega' \in \mu_N$, $\rho_n^{Bei}(B_n(\mathbb{Q}(\mu_N)))$, which satisfies the distribution relations of weight $n-1$ and the parity relation $\bar{x} = (-1)^{n-1}x$, for all $x \in \rho_n^{Bei}(B_n(\mathbb{Q}(\mu_N)))$. Notice that $B_n^{prim}(\mathbb{Q}(\mu_N))$ (defined previously) equals the image of $\mathcal{L}_{N,prim}$ in $B_n(\mathbb{Q}(\mu_N))$, and consider $\rho_n^{Bei}(B_n^{prim}(\mathbb{Q}(\mu_N)))$. We remind that we have $B_n(\mathbb{Q}(\mu_N)) \hookrightarrow \tilde{K}_{2n-1}(\mathbb{Q}(\mu_N))$. Let now F be an arbitrary abelian number field, with conductor N. There is a canonical map $\tilde{K}_{2n-1}(F) \longrightarrow \tilde{K}_{2n-1}(\mathbb{Q}(\mu_N))$ that is injective, so that one can view $\tilde{K}_{2n-1}(F)$ as a sublattice of $\tilde{K}_{2n-1}(\mathbb{Q}(\mu_N))$. There is also a transfer map $Tr_{\mathbb{Q}(\mu_N)/F} : \tilde{K}_{2n-1}(\mathbb{Q}(\mu_N)) \longrightarrow \tilde{K}_{2n-1}(F)$. We will denote by $B_n(F)$ the sublattice of $\tilde{K}_{2n-1}(F)$ generated by $Tr_{\mathbb{Q}(\omega')/\mathbb{Q}(\omega')\cap F}(\epsilon_n(\omega'))$, for all $\omega' \in \mu_N$. If F is a real abelian field and n is odd then we obtain an isomorphism of lattices with distribution relation of weight $n-1$:

$$\mathcal{L}_F \overset{\epsilon_n(F)}{\cong} B_n(F) \cong \rho_n^{Bei}(B_n(F)).$$

Finally, we note that $B_n^{prim}(F)$ be the image of $\mathcal{L}_{F,prim}$ in $B_n(F)$. Consider also $\rho_n^{Bei}(B_n^{prim}(F))$. Since $\bar{\mathcal{L}}_N \longrightarrow \bar{\mathcal{L}}_F$ is surjective, the same holds for $B_n^{prim}(\mathbb{Q}(\mu_N)) \longrightarrow B_n^{prim}(F)$.

Using theorem 9.7, we obtain:

Theorem(Kolster, Nguyen Quang Do [48], Kolster [43]) 9.8 *For a real abelian field F and $n \geq 2$ odd, we have*

$$[B_n(F) : B_n^{prim}(F)] = \mathcal{E}_n(F).$$

Also, for $p = 2$ we have:

$$[\mathrm{im}(\tilde{\psi}) : \tilde{D}_n(F)] \sim_2 [\mathcal{L}_F^{(n-1)} : \mathcal{L}_{F,prim}^{(n-1)}] \sim_2 \mathcal{E}_n(F)$$

and, consequently,

$$\tau = 0.$$

Chapter 10

The Lichtenbaum Conjecture at the prime 2

Let p be a prime, F be a number field, and $S := S_p \cup S_\infty$, where S_p is the set of the primes of F above p and S_∞ is the set of the archimedean places of F.

Let F_∞/F be the cyclotomic $\mathbb{Z}_p$-extension of F. As we know, this is defined as the unique $\mathbb{Z}_p$-extension of F inside $F(\mu_{p^\infty}) := \bigcup_{r\geq 0} F(\mu_{p^r})$. Let $\Gamma := \mathrm{Gal}(F_\infty/F)$ and $\Delta' := \mathrm{Gal}(F(\mu_{p^\infty})/F_\infty)$. We denote the intermediate fields of F_∞/F by F_r for all $r \geq 0$ ($F_0 = F_1 = F$).

Let $G_\infty := \mathrm{Gal}(F(\mu_{p^\infty})/F)$ and let $\bar{\Gamma} \cong \mathbb{Z}_p$ and Δ its subgroups such that $G_\infty = \bar{\Gamma} \times \Delta$. Let γ denote a fixed topological generator of Γ, and $\bar{\gamma}$ a fixed topological generator of $\bar{\Gamma}$.

If the prime p is odd then G_∞ is pro-cyclic and one works with this group. If $p = 2$, G_∞ is not always pro-cyclic. In the non-exceptional case, $G_\infty \cong \Gamma$. We will work with Γ. In the exceptional case we work with Γ, not G_∞ (G_∞ is not pro-cyclic in this case).

Recall that Γ acts on $\mathbb{Z}_2(n)$ and $\mathbb{Q}_2/\mathbb{Z}_2(n)$ for any integer n if F is a non-exceptional field. In the exceptional case, Γ acts on $\mathbb{Z}_2(n)$ and $\mathbb{Q}_2/\mathbb{Z}_2(n)$ only if n is an even integer, because in this case Δ' acts trivially on these twisted modules.

The case p odd is studied characterwise by Kolster, Nguyen Quang Do, Fleckinger in [46], and [47]. We are approaching here the case $p = 2$, studying essentially the "plus" part (we can not work inthe same characterwise manner as for p odd).

We will be using the following notations (see chapter 6):

$$\mathcal{X}_{2,\infty,F} := \oplus_{v \in S} \mathcal{X}_{\infty,F_v}, \ \mathcal{X}^f_{2,\infty,F} := \oplus_{v \in S_2} \mathcal{X}_{\infty,F_v},$$

$$\bar{U}_{2,\infty,F} := \oplus_{v \in S} \bar{U}_{\infty,F_v}, \ \bar{U}^f_{2,\infty,F} := \oplus_{v \in S_2} \bar{U}_{\infty,F_v}.$$

Lemma 10.1 *Let $p = 2$ and $m \geq 2$. Assume either F is non-exceptional or m is even. Then we have:*

$$\mathrm{tors}_{\mathbb{Z}_2} H^1(\mathcal{O}_F^S, \mathbb{Z}_2(m)) \cong H^0(\mathcal{O}_F^S, \mathbb{Q}_2/\mathbb{Z}_2(m)), \text{ and}$$

$$\tilde{H}^1(\mathcal{O}_F^S, \mathbb{Z}_2(m)) \cong \mathrm{Hom}_\Gamma(\mathcal{X}_{\infty,F}, \mathbb{Z}_2(m)).$$

Proof: For any $\nu \geq 0$, and for any $m \geq 2$, the Hochschild-Serre spectral sequence reads

$$0 \longrightarrow H^1(\Gamma, \mathbb{Z}/2^\nu(m)) \longrightarrow H^1(G_S(F), \mathbb{Z}/2^\nu(m))$$
$$\longrightarrow H^1(\mathrm{Gal}(\Omega_S(F)/F_\infty), \mathbb{Z}/2^\nu(m))^\Gamma \longrightarrow 0,$$

because $\mathrm{cd}_2\Gamma = 1$. Passing to projective limit with respect to ν, we obtain

$$0 \to H^1(\Gamma, \mathbb{Z}_2(m)) \to H^1(G_S(F), \mathbb{Z}_2(m)) \to \mathrm{Hom}_\Gamma(\mathcal{X}_{\infty,F}, \mathbb{Z}_2(m)) \to 0.$$

The latter module in the sequence is evidently $\mathbb{Z}_2$-torsion free, and the first module is finite. The second assertion is then immediate. Moreover, we obtain
$\mathrm{tors}_{\mathbb{Z}_2} H^1(G_S(F), \mathbb{Z}_2(m)) \cong H^1(\Gamma, \mathbb{Z}_2(m)) \cong \mathbb{Z}_2(m)_\Gamma$. Now, it is easy to see that

$$H^1(\Gamma, \mathbb{Z}_2(m)) \cong H^0(\Gamma, \mathbb{Q}_2/\mathbb{Z}_2(m) \cong H^0(G_S(F), \mathbb{Q}_2/\mathbb{Z}_2(m)),$$

using the cohomology with $\mathbb{Q}_2/\mathbb{Z}_2(m)$ coefficients of the sequence

$$0 \to \mathbb{Z}/2^\nu(m) \to \mathbb{Q}_2/\mathbb{Z}_2(m) \xrightarrow{\cdot 2^\nu} \mathbb{Q}_2/\mathbb{Z}_2(m) \to 0,$$

and Tate's Lemma (see Lemma 9.1). Note that $\mathrm{Gal}(\Omega_S(F)/F_\infty)$ acts trivially on $\mathbb{Q}_2/\mathbb{Z}_2(m)$ because m is even.□

Remark 10.2 *If the prime p is odd, the result holds for all $m \geq 2$, and all number fields F (in this case we are working with G_∞). The proof follows the same lines, with the difference that we do not need m to be even, because $\mathrm{Gal}(\Omega_S(F)/F_\infty)$ acts trivially on $\mathbb{Q}_p/\mathbb{Z}_p(m)$ for any m, of course. Also, we have $\mathrm{cd}_p G_\infty = 1$.*

Lemma 10.3 *Let $p = 2$ and $m \geq 2$. Assume either F is non-exceptional or m is even. Then we have the following short exact sequence:*

$$0 \to \bar{C}_{\infty,F}(m)_\Gamma \to \bar{C}^S_{\infty,F}(m)_\Gamma \to H^0(\mathcal{O}^S_F, \mathbb{Q}_2/\mathbb{Z}_2(m)) \to 0.$$

Proof: On finite levels, for any $r \geq 2$, we have an exact sequence

$$0 \to C(F_r) \otimes \mathbb{Z}_2 \to C^S(F_r) \otimes \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0,$$

where the latter group is generated by $N_{F_r(\zeta_4)/F_r}(1-\zeta_{2^r}) \bmod C(F_r) \otimes \mathbb{Z}_2$, and hence a trivial $\mathrm{Gal}(F_r/F)$-module. By m-twisting, passing to the projective limit with respect to r, and using the fact that $\mathbb{Z}_p(m)^\Gamma = 0$ (see Lemma 9.1), we obtain:

$$0 \to \bar{C}_{\infty,F}(m)_\Gamma \to \bar{C}^S_{\infty,F}(m)_\Gamma \to \mathbb{Z}_2(m)_\Gamma \to 0.$$

But we have just seen in the proof of the previous Lemma that

$$H^0(\mathcal{O}^S_F, \mathbb{Q}_2/\mathbb{Z}_2(m)) \cong \mathbb{Z}_2(m)_\Gamma. \square$$

Remark 10.4 *The same result holds for any number field and any $m \geq 2$ in the case the prime p is odd (working with G_∞ coinvariants of modules constructed with respect to the tower $F(\mu_{p^\infty})/F$).*

There is a very important map that helps us relate unit groups and the first cohomology group:

$$\bar{U}^S_{\infty,F}(n-1)_\Gamma \xrightarrow{\phi} H^1(\mathcal{O}^S_F, \mathbb{Z}_2(n)).$$

It has been constructed by Soulé in [72] for the case when p is odd, F a number field, and $m \geq 2$. The same construction works for the case $p = 2$, F a non-exceptional number field, and $m \geq 2$. We show here that it works also in the case $p = 2$, F is a exceptional number field, and n odd, $n \geq 2$.

Let $(\zeta_{2^r}) \in \mathbb{Z}_2(1)$ be a generator of the Tate module (ζ_{2^r} is a primitive 2^r-root of unity, and $\zeta_{2^{r+1}}^2 = \zeta_{2^r}$). Consider $\zeta_{2^r}^{\otimes(n-1)} \in \mu_{2^r}^{\otimes(n-1)}$ (note that $n-1$ is an even integer). We denote by $[\zeta_{2^r}^{\otimes(n-1)}]$ its corresponding image in $H^0(\mathcal{O}_{F_r}^S, \mu_{2^r}^{\otimes(n-1)})$. Also, let $(u_r) \in \bar{U}_{\infty,F}^S$ be a projective system of S-units with respect to norms, that is $N_{F_{r+1}/F_r}(u_{r+1}) = u_r$ for all $r \geq 1$. We denote by $\tilde{u}_r$ the image of u_r in $H^1(\mathcal{O}_{F_r}^S, \mu_{2^r})$ (by Kummer theory, we have $U^S(F_r)/U^S(F_r)^{2^r} \hookrightarrow H^1(\mathcal{O}_{F_r}^S, \mu_{2^r})$). Using the cup product in cohomology, we can define a map ϕ:

$$\phi((u_r \otimes \zeta_{2^r}^{\otimes(n-1)})) = (\mathrm{cor}_{F_r/F}(\tilde{u}_r \cup [\zeta_{2^r}^{\otimes(n-1)}])) \in H^1(\mathcal{O}_F^S, \mathbb{Z}_2(n)),$$

because

$$\varprojlim H^1(\mathcal{O}_F^S, \mu_{2^r}^{\otimes n}) \cong H^1(\mathcal{O}_F^S, \mathbb{Z}_2(n)).$$

As

$$\phi((u_r \otimes \zeta_{2^r}^{\otimes(n-1)})^\gamma) = \phi((u_r \otimes \zeta_{2^r}^{\otimes(n-1)})),$$

we obtain the map we need. We will consider the composition of this map with the map $\bar{C}_{\infty,F}^S(n-1)_\Gamma \to \bar{U}_{\infty,F}^S(n-1)_\Gamma$ and denote it by ψ:

$$\bar{C}_{\infty,F}^S(n-1)_\Gamma \xrightarrow{\psi} H^1(\mathcal{O}_F^S, \mathbb{Z}_2(n)).$$

As $\mathrm{tors}_{\mathbb{Z}_2} \bar{C}_{\infty,F}^S(n-1)_\Gamma = 0 \hookrightarrow \mathrm{tors}_{\mathbb{Z}_2} H^1(\mathcal{O}_F^S, \mathbb{Z}_2(n)) = \mathbb{Z}/2\mathbb{Z}$, we obtain an induced map of torsion-free groups:

$$\tilde{C}_{\infty,F}^S(n-1)_\Gamma \xrightarrow{\tilde{\psi}} \tilde{H}^1(\mathcal{O}_F^S, \mathbb{Z}_2(n)).$$

The restriction of ψ to $\bar{C}_{\infty,F}(n-1)_\Gamma$ (via the injection presented in the previous lemma) will be denoted by ψ too.

Remark 10.5 *A different way of constructing the map ψ over an exceptional field F would be realized by taking Δ-invariants of the corresponding map over $F(\mu_4)$, and use the fact that $H^1(\mathcal{O}_F^S, \mathbb{Z}_2(n)) = H^1(\mathcal{O}_{F(\mu_4)}^S, \mathbb{Z}_2(n))^\Delta$ (refer to Proposition 2.6). The study of the map $\bar{C}_{\infty,F}^{\Delta'}(n-1)_\Gamma \to (\bar{C}_{\infty,F}(n-1)_\Gamma)^\Delta$ becomes then necessary. This has been done in chapter 7 for special cases only.*

Lemma 10.6 *Let $p = 2$ and $n \geq 2$. Assume either F is non-exceptional or n is odd. Then the map ϕ*

$$\bar{U}^S_{\infty,F}(n-1)_\Gamma \xrightarrow{\phi} H^1(\mathcal{O}^S_F, \mathbb{Z}_2(n))$$

is injective.

Proof: Using the fact that the Weak Leopoldt Conjecture holds for cyclotomic $\mathbb{Z}_2$-extensions, we obtain the following exact sequence:

$$0 \to \bar{U}^S_{\infty,F}(n-1) \to \oplus_{v \in S} \mathcal{X}_{\infty,F_v}(n-1) \to \mathcal{X}_{\infty,F}(n-1) \to X'_{\infty,F}(n-1) \to 0.$$

Let D_{n-1} denote the kernel of $\mathcal{X}_{F,\infty}(n-1) \to X'_{\infty,F}(n-1)$. Taking cohomology, we obtain

$$0 \to D^\Gamma_{n-1}/K \to \bar{U}^S_{\infty,F}(n-1)_\Gamma \to \oplus_{v \in S} \mathcal{X}_{\infty,F_v}(n-1)_\Gamma \to (D_{n-1})_\Gamma \to 0,$$

where $K := \oplus_{v \in S} \mathcal{X}_{\infty,F_v}(n-1)^\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^{r_1(F)}$, and a commutative diagram with exact columns

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
0 & \longrightarrow & D^\Gamma_{n-1}/K \\
\downarrow & & \downarrow \\
\mathcal{X}_{\infty,F}(n-1)^\Gamma & \xrightarrow{=} & \mathcal{X}_{\infty,F}(n-1)^\Gamma \\
\downarrow & & \downarrow \\
H^1(\mathcal{O}^S_F, \mathbb{Z}_2(n)) & \longrightarrow & X'_{\infty,F}(n-1)^\Gamma \\
\downarrow & & \downarrow \\
\oplus_{v \in S} \mathcal{X}_{\infty,F_v}(n-1)_\Gamma & \longrightarrow & (D_{n-1})_\Gamma \\
\downarrow & & \downarrow \\
\mathcal{X}_{\infty,F}(n-1)_\Gamma & \xrightarrow{=} & \mathcal{X}_{\infty,F}(n-1)_\Gamma \\
\downarrow & & \downarrow \\
X'_{\infty,F}(n-1)_\Gamma & \xrightarrow{=} & X'_{\infty,F}(n-1)_\Gamma \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

where the first vertical sequence is the Poitou-Tate exact sequence interpreted in terms of (co)invariant groups of Iwasawa modules (see Chapter 9). Note that we are using here the isomorphism

$$X'_{\infty,F}(n-1)_{\Gamma} \cong \underline{|||}^2(\mathcal{O}_F^S, \mathbb{Z}_2(n))$$

(see Theorem 8.4). Diagram chase gives us the result. □

Remark 10.7 *If the prime p is odd, the result holds for any number field F and any integer n, $n \geq 2$, and the proof follows the same lines. One difference would be that the isomorphism*

$$X'_{\infty,F}(n-1)_{G_\infty} \cong \underline{|||}^2(\mathcal{O}_F^S, \mathbb{Z}_2(n))$$

is true with no restriction on the number field F.

Unfortunately, the natural map

$$\bar{C}^S_{\infty,F}(n-1)_{\Gamma} \longrightarrow \bar{U}^S_{\infty,F}(n-1)_{\Gamma}$$

is not always injective. The next lemma depends on the freeness of the Λ-module $\tilde{C}_{\infty,F}$. By Kuz'min (see Theorem 7.13), this property holds for $F = \mathbb{Q}(\mu_d, \zeta_4)$, where d is an odd positive integer (and when p is an odd prime it holds for $F = \mathbb{Q}(\mu_d, \zeta_p)$, where d is relatively prime to p, and $d \not\equiv 2 \bmod 4$).

Lemma 10.8 *Let $F = \mathbb{Q}(\mu_d, \zeta_4)$, where d is an odd positive integer, and n be an odd integer, $n \geq 2$. Then the natural map*

$$\bar{C}^S_{\infty,F}(n-1)_{\Gamma} \longrightarrow \bar{U}^S_{\infty,F}(n-1)_{\Gamma}$$

is injective.

Proof: We will prove first that the map

$$\bar{C}_{\infty,F}(n-1)_{\Gamma} \longrightarrow \bar{U}_{\infty,F}(n-1)_{\Gamma}$$

is injective. This amounts to proving that

$$\tilde{C}_{\infty,F}(n-1)_{\Gamma} \longrightarrow \tilde{U}_{\infty,F}(n-1)_{\Gamma}$$

is injective because

$$\ker(\bar{C}_{\infty,F}(n-1)_{\Gamma} \to \tilde{C}_{\infty,F}(n-1)_{\Gamma}) = \ker(\bar{U}_{\infty,F}(n-1)_{\Gamma} \to \tilde{U}_{\infty,F}(n-1)_{\Gamma})$$

(in fact they are equal to $\mathbb{Z}_2(n)_\Gamma$; $\tilde{C}_{\infty,F}(n-1)^{\Gamma} = 0$ because $\tilde{C}_{\infty,F}(n-1)$ is a free Λ-module, and $\tilde{U}_{\infty,F}(n-1)^{\Gamma} = 0$ because, by construction, $\tilde{U}_{\infty,F}(n-1)$ is a Λ-torsion free module). Now, $(\tilde{C}_{\infty,F})_{\Gamma}$ is a (finitely generated) free $\mathbb{Z}_2$-module, because $\tilde{C}_{\infty,F}$ is a (finitely generated) free Λ-module. Being a $\mathbb{Z}_2$-submodule of it, the kernel of the map, that is $(\tilde{U}_{\infty,F}/\tilde{C}_{\infty,F})(n-1)^{\Gamma}$, is a free $\mathbb{Z}_2$-module.

By Kuz'min [51] (or Greither [28])

$$(\tilde{U}_{\infty,F}/\tilde{C}_{\infty,F})(n-1)^{+} \sim (\tilde{U}^{+}_{\infty,F}/\tilde{C}^{+}_{\infty,F})(n-1) \sim X^{+}_{\infty,F}(n-1).$$

It follows that $((\tilde{U}_{\infty,F}/\tilde{C}_{\infty,F})(n-1)^{\Gamma})^{+}$ is finite since $X^{+}_{\infty,F}(n-1)^{\Gamma}$ is finite. Being at the same time $\mathbb{Z}_2$-torsion free it must be trivial. Also, we can see that $((\tilde{U}_{\infty,F}/\tilde{C}_{\infty,F})(n-1)^{\Gamma})^{-} = 0$. Indeed, applying $1+J$ to the tautological sequence

$$0 \longrightarrow \tilde{C}_{\infty,F}(n-1) \longrightarrow \tilde{U}_{\infty,F}(n-1) \longrightarrow (\tilde{U}_{\infty,F}/\tilde{C}_{\infty,F})(n-1) \longrightarrow 0,$$

and then using the serpent lemma we get that $((\tilde{U}_{\infty,F}/\tilde{C}_{\infty,F})(n-1)^{\Gamma})^{-}$ injects into $(\tilde{C}_{\infty,F}/2\tilde{C}_{\infty,F})(n-1)^{\Gamma} \cong ((\Lambda/2\Lambda)^{r})(n-1)^{\Gamma} = 0$, where r is the Λ-rank of $\tilde{C}_{\infty,F}$.

Now, if A is a $\mathbb{Z}_2[\Delta]$-module, without $\mathbb{Z}_2$-torsion, such that $A^{+} = A^{-} = 0$, then A is null, because $[A : (A^{+} \oplus A^{-})]$ is finite. Applying this idea to $\ker(\bar{C}_{\infty,F}(n-1)_{\Gamma} \longrightarrow \bar{U}_{\infty,F}(n-1)_{\Gamma})$, we obtain that our map is injective.

The next diagram helps us to finish the proof.

$$\begin{array}{ccccc}
0 \to \bar{C}_{\infty,F}(n-1)_{\Gamma} \to & \bar{C}^{S}_{\infty,F}(n-1)_{\Gamma} \to & H^0(\mathcal{O}^S_F, \mathbb{Q}_2/\mathbb{Z}_2(n-1) \to 0 \\
\downarrow & \downarrow & \downarrow \\
\bar{U}_{\infty,F}(n-1)_{\Gamma} \to & \bar{U}^{S}_{\infty,F}(n-1)_{\Gamma} \to & D(n-1)_{\Gamma} \to 0 \\
\downarrow & \downarrow & \downarrow \\
0 \to \bar{U}_{2,\infty,F}(n-1)_{\Gamma} \to & \mathcal{X}_{2,\infty,F}(n-1)_{\Gamma} \to & \oplus_{v\in S} H^0(F_v, \mathbb{Q}_2/\mathbb{Z}_2(n-1)) \to 0
\end{array}$$

In the above diagram only the row sequences are exact, and all squares are commutative. The composition of the last two vertical arrows is canonically injective, so that the first arrow is injective. Because we have just proven that the first arrow of the upper row of vertical arrows is injective, it follows that the middle one,

$$\bar{C}^S_{\infty,F}(n-1)_\Gamma \longrightarrow \bar{U}^S_{\infty,F}(n-1)_\Gamma,$$

is injective.□

Remark 10.9 *For an odd prime p, an abelian field F, and an integer $n \geq 2$ the map $\bar{U}^S_{\infty,F}(n-1)_{G_\infty} \longrightarrow H^1(\mathcal{O}^S_F, \mathbb{Z}_p(n))$ is injective. Its injectivity follows essentially in the same way (see Kolster, Nguyen, Fleckinger [46]), but the approach is character-wise, and there is no restriction on F (see also Tsuji [78], and Belliard, Nguyen [9]).*

The next theorem is one of our main results.

Theorem 10.10 *Let F be an abelian number field and F^+ its maximal real subfield. Let n be an odd integer, $n \geq 3$. Then there is a map*

$$\bar{C}^S_{\infty,F^+}(n-1)_\Gamma \xrightarrow{\tilde{\psi}} H^1(\mathcal{O}^S_{F^+}, \mathbb{Z}_2(n))$$

and

$$\frac{|\mathrm{coker}(\tilde{\psi})|}{|\ker(\tilde{\psi})|} = 2^{r_1(F^+)+\mu-1} \cdot |H^2(\mathcal{O}^S_{F^+}, \mathbb{Z}_2(n))|$$

Proof: The idea of the proof comes from the computation of Bloch and Kato of the Tamagawa numbers of the motives $\mathbb{Z}(n)$ in [5] (see also [46]). Namely, we compare the Poitou-Tate duality exact sequence

$$0 \to \mathcal{X}_{\infty,F^+}(n-1)^\Gamma \to H^1(\mathcal{O}^S_{F^+}, \mathbb{Z}_2(n) \to \oplus_{v\in S}\mathcal{X}_{\infty,F^+_v}(n-1)_\Gamma$$

$$\to \mathcal{X}_{\infty,F^+}(n-1)_\Gamma \to Ш^2(\mathcal{O}^S_{F^+}, \mathbb{Z}_2(n)) \to 0$$

to the natural invariant/coinvariant exact sequence

$$0 \to (\mathbb{Z}/2\mathbb{Z})^{r_1} \to (\bar{U}_{2,\infty,F^+}/\bar{C}_{\infty,F^+})(n-1)^\Gamma \to \bar{C}_{\infty,F^+}(n-1)_\Gamma$$

$$\to \bar{U}_{2,\infty,F^+}(n-1)_\Gamma \to (\bar{U}_{2,\infty,F^+}/\bar{C}_{\infty,F^+})(n-1)_\Gamma \to 0.$$

From the 9-term sequence presented in the Poitou-Tate Global Duality Theorem we obtain the following 6-term exact sequence (see Theorem 2.16):

$$0 \to \oplus_{v\in S}\hat{H}^0(F_v^+, \mathbb{Z}_2(n)) \to H^2(\mathcal{O}_{F^+}^S, \mathbb{Q}_2/\mathbb{Z}_2(1-n))^{\#} \to H^1(\mathcal{O}_{F^+}^S, \mathbb{Z}_2(n)) \to$$

$$\oplus_{v\in S}H^1(F_v^+, \mathbb{Q}_2/\mathbb{Z}_2(1-n))^{\#} \to H^1(\mathcal{O}_{F^+}^S, \mathbb{Q}_2/\mathbb{Z}_2(1-n))^{\#} \to \text{Ш}^2(\mathcal{O}_{F^+}^S, \mathbb{Z}_2(n)) \to 0.$$

Using Theorem 8.4 and Proposition 2.12, we obtain immediately the first sequence to be used. Note that we are using the following identification (based on Proposition 2.12) for the cohomology of $\mathbb{R}$:

$$H^1(\mathbb{R}, \mathbb{Q}_2/\mathbb{Z}_2(1-n))^{\#} = \mathcal{X}_{\infty,\mathbb{R}}(n-1)_{\Gamma_{\mathbb{R}}} = \mathbb{Z}/2\mathbb{Z}.$$

The first group in the tautological sequence is $\bar{C}_{\infty,F^+}(n-1)^{\Gamma} = 0$, and the second $\bar{U}_{2,F^+,\infty}(n-1)^{\Gamma} = (\mathbb{Z}/2\mathbb{Z})^{r_1}$.

Next, we obtain the following commutative diagram:

$$\begin{array}{ccc}
0 & & \\
\downarrow & & \\
(\mathbb{Z}/2\mathbb{Z})^{r_1} & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
(\bar{U}_{2,\infty,F^+}/\bar{C}_{\infty,F^+})(n-1)^{\Gamma} & \xrightarrow{f_1} & \mathcal{X}_{\infty,F^+}(n-1)^{\Gamma} \\
\downarrow & & \downarrow \\
\bar{C}_{\infty,F^+}(n-1)_{\Gamma} & \xrightarrow{\psi} & H^1(\mathcal{O}_{F^+}^S, \mathbb{Z}_2(n)) \\
\downarrow & & \downarrow \\
\bar{U}_{2,\infty,F^+}(n-1)_{\Gamma} & \xrightarrow{h} & \oplus_{v\in S}\mathcal{X}_{\infty,F_v^+}(n-1)_{\Gamma} \\
\downarrow & & \downarrow \\
(\bar{U}_{2,\infty,F^+}/\bar{C}_{\infty,F^+})(n-1)_{\Gamma} & \xrightarrow{f_2} & \mathcal{X}_{\infty,F^+}(n-1)_{\Gamma} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \text{Ш}^2(\mathcal{O}_{F^+}^S, \mathbb{Z}_2(n)) \\
 & & \downarrow \\
 & & 0
\end{array}$$

From class field theory we obtain the following important map of torsion Λ-modules (see chapter 8):

$$\bar{U}_{2,\infty,F^+}/\bar{C}_{\infty,F^+} \xrightarrow{f} \mathcal{X}_{\infty,F^+}.$$

Moreover, using results of Wiles [82], Greither [28], Kuz'min [51], we have:

$$\mathrm{char}(\bar{U}_{2,\infty,F^+}/\bar{C}_{\infty,F^+}) = 2^{\mu} \cdot \mathrm{char}(\mathcal{X}_{\infty,F^+}).$$

(refer to Theorem 7.9). This will allow us to apply Lemma 6.10 to the induced maps in Γ-cohomology,

$$(\bar{U}_{2,\infty,F^+}/\bar{C}_{\infty,F^+})(n-1)^{\Gamma} \xrightarrow{f_1} \mathcal{X}_{\infty,F^+}(n-1)^{\Gamma}$$

$$(\bar{U}_{2,\infty,F^+}/\bar{C}_{\infty,F^+})(n-1)_{\Gamma} \xrightarrow{f_2} \mathcal{X}_{\infty,F^+}(n-1)_{\Gamma}$$

obtaining:

$$\frac{|\ker(f_1)|}{|\mathrm{coker}(f_1)|} = 2^{-\mu} \cdot \frac{|\ker(f_2)|}{|\mathrm{coker}(f_2)|}.$$

The map ψ has been presented a few paragraphs earlier.

The map h appears by taking the invariant/coinvariant sequence of

$$0 \to \bar{U}_{2,\infty,F^+} \to \mathcal{X}_{2,\infty,F^+} \to \oplus_{v\in S_2}\mathbb{Z}_2 \to 0,$$

after twisting, and using $\mathbb{Z}_2(n-1)^{\Gamma} = 0$:

$$0 \to \bar{U}_{2,\infty,F^+}(n-1)_{\Gamma} \xrightarrow{h} \mathcal{X}_{2,\infty,F^+}(n-1)_{\Gamma}$$

$$\to \oplus_{v\in S}\hat{H}^0(F_v^+, \mathbb{Q}_2/\mathbb{Z}_2(n-1)) \to 0.$$

Note that $\hat{H}^0(\mathbb{R}, \mathbb{Q}_2/\mathbb{Z}_2(n-1)) = 0$.

All squares in the diagram are commutative. We will justify here only the commutativity of the $(\psi,\, h)$-square. We see that $H^1(\mathcal{O}_{F^+}^S, \mathbb{Z}_2(n))$ injects in $H^1(\mathcal{O}_F^S, \mathbb{Z}_2(n))$ and the same is true for local 2-adic fields. Moreover,

$$H^1(\mathcal{O}_{F^+}^S, \mathbb{Z}_2(n)) \longrightarrow \oplus_{v\in S_\infty} H^1(F_v^+, \mathbb{Z}_2(n)) \cong (\mathbb{Z}/2\mathbb{Z})^{r_1}$$

factors through $H^1(\mathcal{O}_{F^+}^S, \mathbb{Z}_2(n))/2$ which, via the Bockstein sequence, injects into $H^1(\mathcal{O}_{F^+}^S, \mathbb{Z}/2\mathbb{Z}(n)) \subset F^{+\times}/(F^{+\times})^2$, so that we can view it as the signature map. As the cyclotomic units in F^+ are positive at all real places, the archimedean components of their images via both map compositions are null. The commutativity on the finite components is clear from the definition of the maps.

Now, we take the cardinality of the kernels and the cokernels of the above diagram:

$$\frac{|\ker(f_1)|}{|\mathrm{coker}(f_1)|} \cdot \frac{1}{|\mathrm{coker}(h)|} \cdot \frac{1}{|\text{Ш}^2(\mathcal{O}_{F^+}^S, \mathbb{Z}_2(n))|} = 2^{r_1} \cdot \frac{|\ker(\psi)|}{|\mathrm{coker}(\psi)|} \cdot \frac{|\ker(f_2)|}{|\mathrm{coker}(f_2)|}$$

Using the facts just presented, we obtain:

$$|\mathrm{coker}(\psi)| = 2^{r_1+\mu} \cdot |\text{Ш}^2(\mathcal{O}_{F^+}^S, \mathbb{Z}_2(n))| \cdot \prod_{v \in S} |\hat{H}^0(F_v^+, \mathbb{Q}_2/\mathbb{Z}_2(n-1))| \cdot |\ker(\psi)|$$

The Poitou-Tate global duality exact sequence provides us once more with the following 4-term exact sequence:

$$0 \to \text{Ш}^2(\mathcal{O}_{F^+}^S, \mathbb{Z}_2(n)) \to H^2(\mathcal{O}_{F^+}^S, \mathbb{Z}_2(n)) \to \oplus_{v \in S} H^2(F_v^+, \mathbb{Z}_2(n))$$
$$\to H^0(\mathcal{O}_{F^+}^S, \mathbb{Q}_2/\mathbb{Z}_2(1-n))^{\#} \to 0.$$

The result is now, via local duality and the equality between the cardinality of a finite abelian group and that of its Pontryagin dual:

$$|\mathrm{coker}(\psi)| = 2^{r_1+\mu} \cdot |H^2(\mathcal{O}_{F^+}^S, \mathbb{Z}_2(n))| \cdot |H^0(\mathcal{O}_{F^+}^S, \mathbb{Q}_2/\mathbb{Z}_2(n-1)| \cdot |\ker(\psi)|.$$

Finally, we obtain:

$$|\mathrm{coker}(\psi)| = 2^{r_1+\mu} \cdot |H^2(\mathcal{O}_{F^+}^S, \mathbb{Z}_2(n))| \cdot w_{n-1}(F^+) \cdot |\ker(\psi)|.$$

We then use Lemma 10.3 to pass to cyclotomic S-units. Finally, the passage to the torsion-free part is done by the following basic fact: if A is a subgroup of an abelian group B and A and B have equal finite ranks, then $[\tilde{B} : \tilde{A}] = [B : A \cdot \mathrm{tors}\, B]$.□

For an odd prime p, and F abelian field the injective map used is

$$\bar{C}'_{\infty,F}(n-1)_{G_\infty} \to H^1(\mathcal{O}_F^S, \mathbb{Z}_p(n)).$$

One works with F directly instead of F^+, writing down the same Poitou-Tate sequences, using the computations of cohomology groups from chapter 2 (with no problems at infinite places, and no need for n to be odd). Moreover, one can work characterwise as in Kolster, Nguyen Quang Do, Fleckinger [46], and then reassemble everything.

Theorem ([46] Theorem 5.4) 10.11 *Let F be a complex abelian field, K a totally real subfield, and p an odd prime, $p \nmid [F : K]$. Let $n \in \mathbb{Z}$, $n \geq 2$. For each character χ of $\mathrm{Gal}(F/K)$, such that $\chi(-1) = (-1)^{n-1}$, we have*

$$[H^1(\mathcal{O}_F^S, \mathbb{Z}_p(n))^\chi : \bar{C}'_{\infty,F}(n-1)^\chi_{G_\infty}] = |H^2(\mathcal{O}_F^S, \mathbb{Z}_p(n))^\chi|.$$

Consequently, for any odd n, $n \geq 2$, and any totally real field K we have

$$[H^1(\mathcal{O}_K^S, \mathbb{Z}_p(n)) : \bar{C}'_{\infty,K}(n-1)_{G_\infty}] = |H^2(\mathcal{O}_K^S, \mathbb{Z}_p(n))|,$$

and for any even n, $n \geq 2$, and any complex abelian field F we have

$$[H^1(\mathcal{O}_F^S, \mathbb{Z}_p(n)) : \bar{C}'_{\infty,F}(n-1)_{G_\infty}] = |H^2(\mathcal{O}_F^S, \mathbb{Z}_p(n))^-|.$$

Let

$$h_n(F) := \prod_p |H^2_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_p(n))|,\ h_n^{(p)}(F) := |H^2_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_p(n))|, \text{ and}$$

$$w_n(F) := \prod_p |H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(n))|,\ w_n^{(p)}(F) := |H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(n))|.$$

Note that $H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(n)) \cong \mathrm{tors}_{\mathbb{Z}_p} H^1(\mathcal{O}_E^S, \mathbb{Z}_p(n))$ for all primes p.

We present now our most important result.

Theorem 10.12 *For a complex abelian number field F and $n \geq 3$ odd, we have*

$$\zeta_F^*(1-n) = \pm 2^\mu \cdot \frac{h_n(F)}{w_n(F)} \cdot R_n^{Bei}(F).$$

Here $\mu := \mu(\bar{U}_{\infty,F^+}/\bar{C}_{\infty,F^+})$, where F^+ is the maximal real subfield of F.

Proof: It is known that

$$\zeta_F(s) = \prod_{\chi} L(\chi, s)$$

with χ running through th characters of Gal(F/Q) (or, equivalently, through the primitive Dirichlet characters belonging to F). We can deduce then that:

$$\zeta_F^*(1-n) = \prod_{\chi, \chi(-1)=(-1)^n} L(\chi, 1-n) \cdot \prod_{\chi, \chi(-1)=(-1)^{n-1}} L'(\chi, 1-n).$$

The first part is related with the quotients of lower etale cohomology groups as follows (see [46] and [43]):

For p odd:

$$\prod_{\chi, \chi(-1)=(-1)^n} L(\chi, 1-n) = \begin{cases} \zeta_{F^+}(1-n) \sim_p \frac{h_n^{(p)}(F)}{w_n^{(p)}(F)} & \text{if } n \text{ is even} \\ \frac{\zeta_F^*(1-n)}{\zeta_{F^+}^*(1-n)} \sim_p \frac{h_n^{(p)}(F)/w_n^{(p)}(F)}{h_n^{(p)}(F^+)/w_n^{(p)}(F^+)} & \text{if } n \text{ is odd} \end{cases}$$

Note that $H^j_{ét}(\mathcal{O}_F^S, \mathbb{Z}_p(n)) = H^j_{ét}(\mathcal{O}_{F^+}^S, \mathbb{Z}_p(n)) \oplus H^j_{ét}(\mathcal{O}_F^S, \mathbb{Z}_p(n))^-$ for $j = 1, 2$, and that if n is even, then $H^0(F, \mathbb{Q}_p/\mathbb{Z}_p(n))^-$ is null, and if n is odd, then $H^0(F^+, \mathbb{Q}_p/\mathbb{Z}_p(n))$ is null.

For $p = 2$, $n \geq 3$, and F/F^+ is a CM-extension:

$$\prod_{\chi, \chi(-1)=(-1)^n} L(\chi, 1-n) = \frac{\zeta_F^*(1-n)}{\zeta_{F^+}^*(1-n)} \sim_2 \pm \frac{2^{r_2(F)+1}}{Q_n} \cdot \frac{h_n^{(2)}(F)/w_n^{(2)}(F)}{h_n^{(2)}(F^+)}. \quad (1)$$

Note that $w_n^{(2)}(F^+) = 2$.

Now, we will focus on the second part, the derivative of the L-function. The first essential fact is that it can be expressed in terms of polylogarithms. Recall that $\mathrm{Li}_n(z)$ denotes the n-th polylogarithm and it was defined by:

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n},$$

and analytically continued to $\mathbb{C} \setminus [1, \infty)$. For an abelian number field F, $n \geq 2$, and a primitive Dirichlet character belonging to F, χ, such that $\chi(-1) = (-1)^{n-1}$, we have the following well-known result of Gross:

$$L'(\chi, 1-n) = 2^{-1} \cdot \frac{(n-1)!}{(2\pi i)^{n-1}} \cdot f_\chi^{n-1} \cdot \mathfrak{l}_{n,\chi}, \quad (2)$$

where f_χ is the conductor of χ, and

$$\mathfrak{l}_{n,\chi} := \sum_{\alpha \bmod f_\chi} \chi(\alpha) \cdot \mathrm{Li}_n(e^{2\pi i \alpha / f_\chi})$$

(here $i = \sqrt{-1} = \zeta_4$). The character χ can be also viewed as a character $\tilde{\chi} \bmod N$, where N is the conductor of F, if n is even, and it is the conductor of F^+, if n is odd. In [65] Neukirch proved then that:

$$\prod_{p|N} (1 - \chi(p)p^{n-1}) \cdot f_\chi^{n-1} \cdot \mathfrak{l}_{n,\chi} = N^{n-1} \cdot \mathfrak{l}_{n,\tilde{\chi}}. \quad (3)$$

We compute the covolume of the lattice $\rho_n^{Bei}(B_n^{prim}(F))$ in $(\mathbb{R}(n-1)^{\mathrm{Hom}(F,\mathbb{C})})^+$. Firstly, we do the computation for the field $\mathbb{Q}(\mu_N)$. Let us denote the expression $N^{n-1} \cdot (n-1)! \cdot \mathrm{Li}_n(\omega^\sigma)$, $\omega \in \mu_N^{prim}$, by $\phi(\sigma)$, and let $G := \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$. The real vector space $(\mathbb{R}(n-1)^{\mathrm{Hom}(\mathbb{Q}(\mu_N),\mathbb{C})})^+$ we are working with is also a $\mathbb{R}[G]$-module, and the element

$$\alpha := \sum_{\sigma \in G} \frac{\phi(\sigma)}{(2\pi i)^{n-1}} \sigma \in \mathbb{R}[G].$$

Multiplication by α takes the standard lattice of covolume 1 to the lattice $\rho_n^{Bei}(B_n^{prim}(\mathbb{Q}(\mu_N)))$. So, the covolume we are interested in is given by $\det \alpha$. Viewing it as an element in $\mathbb{C}[G]$, and using characters, this determinant becomes:

$$\det \alpha = \prod_{\chi, \chi(-1)=(-1)^{n-1}} \sum_{\sigma \in G} \frac{\phi(\sigma)\chi(\sigma)}{(2\pi i)^{n-1}}$$

(Washington [81], Lemma 1.2). Next, we have a commutative diagram:

$$\begin{array}{ccc} \tilde{K}_{2n-1}(\mathbb{Q}(\mu_N)) & \longrightarrow & (\mathbb{R}(n-1)^{\mathrm{Hom}(\mathbb{Q}(\mu_N),\mathbb{C})})^+ \\ Tr \downarrow & & \downarrow tr \\ \tilde{K}_{2n-1}(F) & \longrightarrow & (\mathbb{R}(n-1)^{\mathrm{Hom}(F,\mathbb{C})})^+ \end{array}$$

where the trace on the right hand side is defined by

$$tr_{\mathbb{Q}(\mu_N)/F}((a_\sigma)_\sigma) = \Big((\sum_{\sigma, \sigma \bmod H = \bar{\sigma}} a_\sigma) \bar{\sigma} \Big)_\sigma$$

where $H := \mathrm{Gal}(\mathbb{Q}(\mu_N)/F)$, $\bar{\sigma} \in \mathrm{Gal}(F/\mathbb{Q}) \cong G/H$. For $\bar{\sigma} \in G/H$ let $\psi(\bar{\sigma}) := \sum_{\sigma=\bar{\sigma} \bmod H} \phi(\sigma)$, and

$$\beta := \sum_{\bar{\sigma}\in G/H} \frac{\psi(\bar{\sigma})}{(2\pi i)^{n-1}} \bar{\sigma} \in \mathbb{R}[G/H].$$

As previously we obtain:

$$\text{covolume } \rho_n^{Bei}(B_n^{prim}(F)) = \det\beta = \prod_{\chi,\chi(-1)=(-1)^{n-1}} \sum_{\bar{\sigma}\in G/H} \frac{\psi(\bar{\sigma})\chi(\bar{\sigma})}{(2\pi i)^{n-1}}$$

with χ running through the characters of G/H. But:

$$\sum_{\bar{\sigma}\in G/H} \psi(\bar{\sigma})\chi(\bar{\sigma}) = \sum_{\sigma\in G} \phi(\sigma)\tilde{\chi}(\sigma),$$

so that we obtain:

$$\text{covolume } \rho_n^{Bei}(B_n^{prim}(F)) = \prod_{\chi,\chi(-1)=(-1)^{n-1}} \frac{N^{n-1}\cdot(n-1)!}{(2\pi i)^{n-1}} \cdot \mathfrak{l}_{n,\tilde{\chi}}.$$

Using (2) and (3), we obtain:

$$\text{covolume } \rho_n^{Bei}(B_n^{prim}(F)) = 2^{r_2(F)} \cdot \mathcal{E}_n(F) \cdot \prod_{\chi,\chi(-1)=(-1)^{n-1}} L'(\chi, 1-n).$$

This gives immediately the relation between the values of the derivative of L-function at $1-n$ and the n-th Beilinson regulator:

$$\prod_{\chi,\chi(-1)=(-1)^{n-1}} L'(\chi,1-n) = 2^{-r_2(F)} \cdot \mathcal{E}_n(F)^{-1} \cdot [\tilde{K}_{2n-1}(F) : B_n^{prim}(F)] \cdot R_n^{Bei}(F).$$

When F is abelian number field and $n \geq 3$ is odd, the characters are even and they can be considered as characters of F^+. Hence, for $n \geq 3$ odd we can show in the same manner that:

$$\prod_{\chi \text{ even}} L'(\chi,1-n) = 2^{-r_1(F^+)} \cdot \mathcal{E}_n(F^+)^{-1} \cdot [\tilde{K}_{2n-1}(F^+) : B_n^{prim}(F^+)] \cdot R_n^{Bei}(F^+).$$

Finally, as we saw in chapter 9, for p odd, under Chern character, $\tilde{K}_{2n-1}(F)\otimes\mathbb{Z}_p$ maps isomorphically to $\tilde{H}^1_{\acute{e}t}(\mathcal{O}^S_F,\mathbb{Z}_p(n))$, and $B_n(F)\otimes\mathbb{Z}_p$ maps isomorphically to $\tilde{C}^S_{\infty,F}(n-1)_\Gamma$. It follows:

$$[\tilde{K}_{2n-1}(F):B^{prim}_n(F)]\sim_p[\tilde{K}_{2n-1}(F)\otimes\mathbb{Z}_p:B^{prim}_n(F)\otimes\mathbb{Z}_p]$$

$$=\mathcal{E}_n(F)\cdot[\tilde{H}^1_{\acute{e}t}(\mathcal{O}^S_F,\mathbb{Z}_p(n)):\tilde{C}^S_{\infty,F}(n-1)_\Gamma]$$

$$=\begin{cases}|H^2_{\acute{e}t}(\mathcal{O}^S_{F^+},\mathbb{Z}_p(n))| & \text{if } p \text{ is odd and } n \text{ is odd}\\ |H^2_{\acute{e}t}(\mathcal{O}^S_F,\mathbb{Z}_p(n))|/|H^2_{\acute{e}t}(\mathcal{O}^S_{F^+},\mathbb{Z}_p(n))| & \text{if } p \text{ is odd and } n \text{ is even}\end{cases}$$

(see Theorem 10.11). Thus we have the second part (derivative) of the L-function. Multiplying it with the first part presented above, we obtain the result for p odd. For $p=2$, F^+ the maximal real subfield of an abelian number field F, and n odd, we have (refer to theorem 9.4 and theorem 10.10):

$$[\tilde{K}_{2n-1}(F^+)\otimes\mathbb{Z}_2:B^{prim}_n(F^+)\otimes\mathbb{Z}_2]=2^{\mu+\delta_F}\cdot|H^2(\mathcal{O}^S_{F^+},\mathbb{Z}_2(n))|\cdot\mathcal{E}_n(F^+)$$

if $n\equiv 3 \bmod 4$, and

$$[\tilde{K}_{2n-1}(F^+)\otimes\mathbb{Z}_2:B^{prim}_n(F^+)\otimes\mathbb{Z}_2]=2^{r_1+\mu-1}\cdot|H^2(\mathcal{O}^S_{F^+},\mathbb{Z}_2(n))|\cdot\mathcal{E}_n(F^+)$$

if $n\equiv 1 \bmod 4$. Multiplying further with (1) and using theorem 4.8 we obtain the complete result for an abelian number field F and an odd $n\geq 3$. If $n\equiv 3 \bmod 4$ we have:

$$\zeta^*_F(1-n)=\pm\Big(\frac{2^{r_1(F^+)+1}}{Q_n}\cdot\frac{h_n(F)/w_n(F)}{h_n(F^+)}\Big)\cdot$$
$$(2^{\mu+\delta_F}\cdot h_n(F^+)\cdot\mathcal{E}_n(F^+))\cdot(2^{-r_1(F^+)}\cdot\mathcal{E}_n(F^+)^{-1}\cdot R^{Bei}_n(F^+))$$
$$=\pm 2^\mu\cdot\frac{h_n(F)}{w_n(F)}\cdot\Big(\frac{2^{\delta_F+1}}{Q_n}\cdot R^{Bei}_n(F^+)\Big)$$
$$=\pm 2^\mu\cdot\frac{h_n(F)}{w_n(F)}\cdot R^{Bei}_n(F).$$

If $n\equiv 1 \bmod 4$ we have the following computation:

$$\zeta_F^*(1-n) = \pm\Big(\frac{2^{r_1(F^+)+1}}{Q_n} \cdot \frac{h_n(F)/w_n(F)}{h_n(F^+)}\Big)\cdot$$
$$(2^{r_1+\mu-1} \cdot h_n(F^+) \cdot \mathcal{E}_n(F^+)) \cdot (2^{-r_1(F^+)} \cdot \mathcal{E}_n(F^+)^{-1} \cdot R_n^{Bei}(F^+))$$
$$= \pm 2^\mu \cdot \frac{h_n(F)}{w_n(F)} \cdot \Big(\frac{2^{r_1}}{Q_n} \cdot R_n^{Bei}(F^+)\Big)$$
$$= \pm 2^\mu \cdot \frac{h_n(F)}{w_n(F)} \cdot R_n^{Bei}(F).$$

This finishes the proof. □

As we saw in chapter 4, a motivic formulation is possible since we have a motivic regulator and it can be compared with the Beilinson regulator. Also, the result has the same form for both complex and real abelian cases:

Theorem 10.13 *For an abelian number field F and $n \geq 3$ odd, we have*

$$\zeta_F^*(1-n) = \pm 2^\mu \cdot \frac{h_n(F)}{w_n(F)} \cdot R_n^{\mathcal{M}}(F).$$

Here $\mu := \mu(\bar{U}_{\infty,F^+}/\bar{C}_{\infty,F^+})$, where F^+ is the maximal real subfield of F.

Proof: If F is complex, nothing changes (see chapter 4 for the comparison between regulators). If F is real, using the proof of the previous theorem, we obtain the following computation:

$$\zeta_F^*(1-n) = \pm 2^{r_1} \cdot \mathcal{E}_n(F)^{-1} \cdot [\tilde{H}^1(F, \mathbb{Z}(n)) : \mathrm{Im} B_n^{prim}(F)] \cdot R_n^{\mathcal{M}}(F)$$
$$= \pm 2^{r_1} \cdot \mathcal{E}_n(F)^{-1} \cdot 2^{r_1+\mu-1} \cdot h_n(F) \cdot \mathcal{E}_n(F) \cdot R_n^{\mathcal{M}}(F)$$
$$= \pm 2^\mu \cdot \frac{h_n(F)}{w_n(F)} \cdot R_n^{\mathcal{M}}(F).$$

We have denoted by $\mathrm{Im} B_n^{prim}(F)$ the direct product of the images of the Beilinson group under the injective maps $\tilde{K}_{2n-1}(F) \otimes \mathbb{Z}_p \hookrightarrow \tilde{H}^1(F, \mathbb{Z}(n)) \otimes \mathbb{Z}_p$ for all primes p. Also, note that $w_n^{(2)}(F) = 2$ if F is real and n is odd. □

The motivic context and tools seem suitable for future work on the Lichtenbaum Conjecture at the prime 2 for all number fields with no restriction on n.

List of notations

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	Sets of positive integers, integers, rationals, reals, complexes.
F	Number field.
$\bar{F}$	Algebraic closure of a field F.
p, l	Rational primes.
B_n	(Ordinary) Bernoulli number.
$B_{n,\chi}$	Generalized Bernoulli number.
$\zeta(s)$	Riemann ζ-function.
$\zeta_F(s)$	Dedekind ζ-function of a number field F.
$\mathrm{val}_p(x)$	p-adic valuation of x.
$x \sim_p y$	x having the same p-adic valuation as y.
ζ_n	Primitive nth root of unity.
μ_n	Group of nth roots of unity.
$\mu(F)$	Group of all roots of unity in a number field F.
$\mu(F)(p)$	Group of all p-power roots of unity in a number field F.
$\varphi(n)$	Euler φ-function.
d_n	Degree of the field extension $F(\zeta_{p^n})/F$.
M	Largest positive integer M such that $F(\zeta_{p^M}) = F(\zeta_{2p})$.
$\mathbb{Z}_p$	Ring of p-adic integers.
$\mathbb{Q}_p$	Field of p-adic numbers.
$\mathbb{C}_p$	Completion of the algebraic closure of $\mathbb{Q}_p$.
$\mathcal{O}_F$	Ring of integers of a number field F.
$Cl(\mathcal{O}_F)$	Class group of a ring $\mathcal{O}_F$ of integers.
A_F^S	The p-Sylow subgroup of the S-ideal class group of F.
v, w	Prime ideals, places (finite and infinite).
S_p	the set of the primes of a field F above p
S_∞	the set of the archimedean places
$F_{\mathfrak{p}}$	$\mathfrak{p}$-adic completion of a number field F.

$k_{\mathfrak{p}}$	Residue class field of $\mathcal{O}_F$ modulo a prime ideal $\mathfrak{p}$.
e	Ramification index of primes over p in F.
f	Inertia degree of primes over p in F.
g	Degree of the field extension $F/\mathbb{Q}$.
r	Number of primes above p in F.
r_1	Number of real embeddings of $F/\mathbb{Q}$ in $\mathbb{C}$.
r_2	Number of pairs of complex embeddings of $F/\mathbb{Q}$ in $\mathbb{C}$.
$\|T\|$	Cardinality of a set T.
$\|\gamma\|$	Order of an element γ of a group.
$\mathrm{rank}_{p^n}(A)$	p^n-rank of an abelian group A.
$\mathrm{rank}_{\mathbb{Z}}(A)$	$\mathbb{Z}$-rank of an abelian group A.
$\mathrm{tors}(A)$	Torsion subgroup of an abelian group A.
${}_mA$	Maximal subgroup of an abelian group A with exponent dividing m.
$H^n(F, A(k))$	nth cohomology group of $\mathrm{Gal}(\bar{F}/F)$ with k-fold twisted coefficients A.
$H^n_{\acute{e}t}(\mathcal{O}_F, A(k))$	nth étale cohomology group of $\mathrm{spec}\,\mathcal{O}_F$ with k-fold twisted coefficients A.
R^*	Group of units of a ring R.
$K_n(R)$	nth K-group of a ring R.
$R[[T]]$	Ring of formal power series in the indeterminate T over a ring R.
χ	Dirichlet character.
ω	Teichmüller character.
κ	Cyclotomic character.
F_χ	Subfield of $\bar{F}$ fixed by elements of $\ker \chi$.
F_∞	Cyclotomic $\mathbb{Z}_p$-extension of a number field F.
$L(s, \chi)$	L-series attached to a character χ.
$L_p(s, \chi)$	p-adic L-function attached to a character χ.
(a, b)	Greatest common divisor of a and b.
$U(F)$	Group of units of $\mathcal{O}_F$; alternatively, $\mathcal{O}_F^*$.
$C(F)$	Group of cyclotomic units of F.
$\bar{U}_{\infty,F}$	The projective limit of $U(F_r) \otimes \mathbb{Z}_p$ in a $\mathbb{Z}_p$-extension of F (using norms).
$\bar{C}_{\infty,F}$	The projective limit of $C(F_r) \otimes \mathbb{Z}_p$ in a $\mathbb{Z}_p$-extension of F (using norms).

F_∞/F	$\mathbb{Z}_p$-extension of F.
$L^S_{\infty,F}$	The maximal abelian, unramified and S-decomposed pro-p-extension of F_∞.
$M_{\infty,F}$	The maximal abelian S-ramified pro-p-extension of F_∞.
$X^S_{\infty,F}$	The Galois group of the extension $L^S_{\infty,F}/F_\infty$.
$\mathcal{X}_{\infty,F}$	The Galois group of the extension $M_{\infty,F}/F_\infty$.
$\mathrm{Li}_n(s)$	(n-th) Polylogarithm.
$\mathcal{E}_n(F)$	Euler factor of a number field F.

Bibliography

[1] D. Arlettaz, G. Banaszak, W. Gajda: *On 2-adic cyclotomic elements in K-theory and tale cohomology of the ring of integers*, J. Number Theory 82 (2000), 225-255.

[2] M. Artin, J. Tate: *Class Field Theory*, Benjamin New York, Amsterdam 1967.

[3] H. Bass, J. Tate: *The Milnor ring of a global field (with an appendix on euclidean quadratic imaginary fields, by J. Tate)*, Lecture Notes in Math., vol. 342, Springer Verlag, 1973, 349-446.

[4] A. Beilinson: *Higher regulators and values of L-functions*, J. Soviet Math. 30 (1984), 2036-2070.

[5] S. Bloch, K. Kato: "L-functions and Tamagawa numbers of motives" in *The Grothendieck Festschrift*, vol. 1, Progr. Math. 86, Birkhäuser, Boston (1990), 333-400.

[6] A. Borel: *Cohomologie de SL_n et valeurs de fonctions zêta*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 7 (1974), 613-636.

[7] A. Borel: *Stable real cohomology of arithmetic groups*, Ann. Sci. Ecole Norm. Sup. 7 (1977), 613-636.

[8] S. Bloch, S. Lichtenbaum: *A spectral sequence for motivic cohomology*, Univ. of Illinois at Urbana-Champaign Report 1995.

[9] Private communication from J. R. Belliard, T. Nguyen Quang Do: *Formules de classes pour les corps abeliens réels*, preprint 2000.

[10] Private communication from D. Benois, T. Nguyen Quang Do: *La Conjecture de Bloch et Kato pour les motifs* $\mathbb{Q}(m)$ *sur un corps abelien*, preprint 2001.

[11] Private communication from D. Burns, C. Greither: *On the equivariant Tamagawa Number Conjecture for Tate motives*, preprint 2001.

[12] W. Browder: *Algebraic K-Theory with coefficients* $\mathbb{Z}/p$, Geometric Applications of Homotopy Theory I (Proceedings, Evanston, 1977) (M. G. Barratt and M. E. Mahowald, eds.), Lecture Notes in Math., vol. 657, Springer-Verlag, Berlin, Heidelberg, New York, 1978, 40-84.

[13] A. Brumer: *On the units of algebraic number fields*, Mathematika 14 (1967), 121-124.

[14] J. I. Burgos Gil: *The Regulators of Beilinson and Borel*, CRM Monograph Series, Volume 15 (2002).

[15] T. Chinburg, M. Kolster, G. Pappas, V. Snaith: *Galois structure of K-groups of rings of integers*, K-Theory 14 (1998), 319-369.

[16] J. Coates: "p-adic L-functions and Iwasawa's theory" in *Algebraic Number Fields: L-functions and Galois properties*, Durham Symposium 1975, Academic Press, London 1977, 269-353.

[17] P. Deligne: "Le group fondamental de la droite projective moins trois points" in *Galois Groups over* $\mathbb{Q}$, Math. Sci. Res. Inst. Publ. 16, Springer, Berlin (1989), 79-298.

[18] P. Deligne: "Valeurs de fonctions L et périodes d'intégrales" in *Automorphic Forms, Representations and L-functions*, Proc. Symp. Pure Math., vol. 33, part 3, pp. 313-346, American Math. Soc. (1979)

[19] P. Deligne, K. Ribet: *Values of abelian L-functions at negatives integers over totally real fields*, Invent. Math. 59 (1980), 227-286.

[20] W. Dwyer, E. Friedlander: *Algebraic and étale K-theory*, Trans. AMS 292 (1985), 247-290.

[21] L. J. Federer, B. H. Gross: *Regulators and Iwasawa modules*, Invent. Math. 62 (1981), 443-457 (with an appendix by W. Sinnott).

[22] L. J. Federer: "Regulators, Iwasawa modules and the main conjecture for $p = 2$" in *Number theory related to Fermat's last theorem*, ed. by N. Koblitz, Birkhäuser 1982, 289-296.

[23] B. Ferrero, L. C. Washington: *The Iwasawa invariant μ_p vanishes for abelian number fields*, Ann. of Math. 109 (1979), 377-395.

[24] R. Gillard: *Unités cyclotomiques, unités semi-locales et $\mathbb{Z}_l$-extensions II*, Ann. Inst. Fourier 27, Grenoble 29, 4 (1979), 1-15.

[25] H. Gillet: *Riemann-Roch Theorems for Higher Algebraic K-Theory*, Advances in Math. 40 (1981), 203-289.

[26] R. Greenberg: *On the Iwasawa invariants of totally real fields*, Amer. J. Math. 96 (1976), 263-284.

[27] R. Greenberg: *On p-adic L-functions and cyclotomic fields II*, Nagoya Math. J. 67 (1977), 139-158.

[28] C. Greither: *Class groups of abelian fields, and the Main Conjecture*, Ann. Inst. Fourier 42 (1992), 449-499.

[29] A. Grothendieck: *La théorie des classes de Chern*, Bull. Soc. Math. de France 86 (1958), 137-154.

[30] K. Haberland: *Galois Cohomology of Algebraic Number Fields*, VEB Deucher Verlag der Wissenschaften, Berlin 1978.

[31] R. Hartshorne: *Algebraic Geometry*, Graduate Texts in Math. 52, Springer 1977.

[32] A. Huber, G. Kings: *Degeneration of l-adic Eisenstein classes and of the elliptic polylog*, Invent. math. 135 (1999), 545-594.

[33] A. Huber, J. Wildeshaus: *Classical Motivic Polylogarithm according to Beilinson and Deligne*, Doc. Math. J. DMV. 3 (1998) 27-133.

[34] K. Iwasawa: *On $\mathbb{Z}_p$-extensions of algebraic number fields*, Ann. of Math. (2) 98 (1973), 246-326.

[35] U. Jannsen: *Continuous étale cohomology*, Math. Ann. 280 (1988), 207-245.

[36] J.-F. Jaulent: *L'arithmetique des l-extensions*, Thèse, Univ. de Franche-Compté 1986.

[37] J. P. Jouanolou: "Une suite exacte de Mayer-Vietoris en K-théorie algébrique" in *Algebraic K-Theory I*, Lecture Notes in Math. 341, Springer 1973, 293-316.

[38] B. Kahn: *On the Lichtenbaum-Quillen conjecture*, Algebraic K-Theory and Algebraic Topology (P. Goerss, J. F. Jardine, eds.), NATO Proceedings Lake Louise, vol. 407, Kluwer, Dordrecht, 1993, 147-166.

[39] B. Kahn: *The Lichtenbaum-Quillen conjecture at the prime* 2, Univ. of Illinois at Urbana-Champaign Report 1997.

[40] H. Koch: *Algebraic Number Theory*, Encyclopaedia Of Mathematical Sciences 62, Springer 1997.

[41] M. Kolster: *A relation between the 2-primary parts of the Main Conjecture and the Birch-Tate conjecture*, Canad. Math. Bull. 32 (1989), 248-251.

[42] M. Kolster: *Cohomological version of the Lichtenbaum conjecture at* 2, Appendix to [66].

[43] M. Kolster: *Higher relative class number formulae*, submitted to Mathematische Annalen (2001).

[44] M. Kolster: *K-Theory and Arithmetic*, Rapports Centre interuniversitaire en Calcul Mathématique Algébrique, Concordia-Laval-McGill (2000).

[45] M. Kolster: *K-theory and Arithmetic*, Fields Inst. Monogr. Report (1998).

[46] M. Kolster, T. Nguyen Quang Do, V. Fleckinger: *Twisted S-units, p-adic class number formulas, and the Lichtenbaum Conjectures*, Duke Math. J. (1996), 679-717.

[47] M. Kolster, T. Nguyen Quang Do, V. Fleckinger: *Correction to "Twisted S-units, p-adic class number formulas, and the Lichtenbaum Conjectures"*, Duke Math. J. (1997), 641-643.

[48] Private communication from M. Kolster, T. Nguyen Quang Do: *Cyclotomic lattices with distribution relations*, preprint 2002.

[49] D.S. Kubert: *The universal ordinary distribution*, Bull. Soc. Math. France 107 (1979), 179-202.

[50] D.S. Kubert: *The $\mathbb{Z}/2\mathbb{Z}$ cohomology of the universal ordinary distribution*, Bull. Soc. Math. France 107 (1979), 203-224.

[51] L. V. Kuz'min: *On formulas for the class number of the real abelian fields*, Math USSR-Izv. 60, 4 (1996), 695-761

[52] S. Lang: *Cyclotomic Fields I and II*, combined 2nd edition, Graduate Texts in Math. 121, Springer 1990.

[53] S. Lichtenbaum: "Values of zeta functions, étale cohomology, and algebraic K-theory" in *Algebraic K-theory II*, Lecture Notes in Math. 342, Springer, Berlin (1973), 489-501.

[54] J.-L. Loday: *K-théorie algébrique et représentations de groupes*, Ann. Sci. E.N.S. 9 (1976), 309-377.

[55] A. S. Merkurjev, A. A. Suslin: *K-cohomology of Severi-Brauer varieties and the norm residue homomorphism*, Math. USSR 21 (1983), 307-340.

[56] A. S. Merkurjev, A. A. Suslin: *The K_3 of a field*, Math. USSR 36 (1990), 541-565.

[57] J. Milne: *Étale cohomology*, Princeton University Press 1980.

[58] J. Milne: *Arithmetic Duality Theorems*, Perspectives in Mathematics, vol. 1, Academic Press 1986.

[59] J. Milnor: *Introduction to Algebraic K-theory*, Annals of Mathematical Studies, vol. 72, Princeton University Press, New Jersey, 1971.

[60] T. Nguyen Quang Do: *Une étude cohomologique de la partie 2-primaire de $K_2(\mathcal{O})$*, K-Theory 3 (1990), 523-542.

[61] J. Neukirch, A. Schmidt, K. Wingberg: *Cohomology of Number Fields*, A Series of Comprehensive Studies in Mathematics, volume 323, Springer-Verlag Berlin, Heidelberg, 2000.

[62] D. Quillen: *On the cohomology and K-theory for the general linear group over a finite field*, Ann. Math. 96 (1972), 552-586.

[63] D. Quillen: *Finite generation of the groups K_i of rings of algebraic integers*, Springer Lecture Notes, 341 (1973), 179-210.

[64] D. Quillen: *Algebraic K-theory I*, Springer Lecture Notes 341 (1973), 85-147.

[65] M. Rapoport, N. Schappacher, P. Schneider (eds.): *Beilinson's conjectures on special values of L-functions*, Perspectives in Mathematics, vol. 4, Academic Press, Boston, 1988.

[66] J. Rognes, C. Weibel: *Two-primary algebraic K-theory of rings of integers in number fields*, Journal of the AMS 13 (2000), 1-54.

[67] K. Rubin: *The main conjecture*, Appendix to [52].

[68] P. Schneider: *Über gewisse Galoiscohomologiegruppen*, Math. Z. 168 (1979), 181-205.

[69] J.-P. Serre: *Corps Locaux*, Hermann, Paris 1968.

[70] J.-P. Serre: *Cohomologie Galoissienne*, Lecture Notes in Math. 5, Springer 1964.

[71] W. Sinnott: *On the Stickelberger ideal and the circular units of an abelian number field*, Invent. Math. 63 (1980), 181-234.

[72] C. Soulé: *K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale*, Invent. Math. 55 (1979), 251-295.

[73] C. Soulé: *Eléments cyclotomiques en K-théorie*, Astérisque 147/148 (1978), 225-257.

[74] C. Soulé: *Groupes de Chow et K-théorie de variétés sur un corps fini*, Math. Ann. 268 (1984), 317-345.

[75] C. Soulé: *On higher p-qdic regulators*, Algebraic K-Theory, LNM, vol. 854, Springer, Berlin, 1981, 372-401.

[76] V. Srinivas: *Algebraic K-theory*, Birkhäuser 1996.

[77] J. Tate: *Relations between K_2 and Galois cohomology*, Invent. Math. 36 (1976) 257-274.

[78] T. Tsuji: *Semi-local units modulo cyclotomic units*, J. Number Theory 46 (1999), 158-178.

[79] L. Villemot: *Etude du quotient des unités semi-locales par les unités cyclotomiques dans les* $\mathbb{Z}_p$*-extensions des corps de nombres abéliens réels*, Thèse, Orsay 1981.

[80] V. Voevodsky: *The Milnor Conjecture*, Bull. Amer. Math. Soc. 35 (1998), 123-143.

[81] L. Washington: *Introduction to Cyclotomic Fields*, Graduate Texts in Math. 83, Springer-Verlag, New York, 1982.

[82] A. Wiles: *The Iwasawa Conjecture for totally real fields*, Annals of Math. 131 (1990), 493-540.