BOUNDED PSEUDO-COMPLETE LOCALLY CONVEX *-ALGEBRAS

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SCOPE AND CONTENTS: We introduce a new class of locally convex algebras called BP*-algebras. They form a considerably larger class than that of MQ*-algebras which were introduced and studied by T. Husain and Rigelhoff. We discuss continuity of positive linear functionals on these algebras. We characterize extreme points of normalized positive functionals and obtain several results on representations of these algebras.

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INTRODUCTION

In this dissertation we shall be concerned with a certain class of topological algebras, called BP*-algebras. We discuss herein, among other properties, the continuity of positive functionals and the identification of the extreme points of the set of normalized positive functionals on these algebras. We also investigate representations of BP*-algebras by operators on a Hilbert space and conclude with the realization of these algebras as algebras of functions.

The theory of Banach spaces has been extended to a theory of general topological vector spaces with very fruitful results, particularly, in the special case of Locally convex spaces. It was therefore natural to define some analogous extensions of the theory of Banach algebras. The first major work in this field was that of E.A. Michael [19]. He defined LMC-algebras to be an algebra with a Locally convex topology given by a family of semi-norms $\{p_\lambda\}$ satisfying the condition

$$p_\lambda(xy) \leq p_\lambda(x) \cdot p_\lambda(y).$$

The theory of LMC-algebras achieves much of it's success by maintaining some of the properties which are known for Banach algebras. For example, they have jointly continuous multiplication. However, the notion of LMC-algebras does exclude those topological algebras in which multiplication is continuous only in each variable separately such as Example 2.4 and $L^{\omega}[0,1]$, [4], eventhough the latter has jointly continuous multiplication.

In order to generalize LMC-algebras, G.R. Allan and others began studying general Locally convex algebras. The concept of pseudo-complete Locally convex algebra is due to G.R. Allan [1]. In a later paper [2], Allan and co-authors introduce a class of pseudo-Banach algebras based on [1] and the notion of bound structure in an algebra.

The original motivation for introducing BP*-algebras comes from [1] and [2] and a paper [17] of T. Husain and Rigelhoff on "Representations of MQ*-algebras". We extend the class of MQ*-algebras to a new class of Locally convex algebras called BP*-algebras.

Chapter 1 contains those definitions and results from topological vector spaces, pseudo-complete and pseudo-Banach algebras, LMC and Banach algebras which will be recalled in subsequent chapters. We have included proofs of some of the most fundamental results.

In Chapter II we provide several examples to show that a BP*-algebra need not be an MQ*-algebra. We discuss some of the permanence properties of these algebras. We show that A_e (the algebra A with unit e adjoined) is bounded pseudo-complete if and only if A is bounded pseudo-complete.

Chapter III concerns itself primarily with continuity of positive functionals on BP*-algebras. We start with a fundamental

Lemma that every positive functional on a BP*-algebra is admissible, thus generalizing the corresponding result for MQ*-algebras. By means of an example we show that not every positive functional on a BP*-algebra is continuous. However, we have established continuity of positive functionals on BP*-algebra in its inductive limit topology. We have shown that every positive functional on a barrelled BP*-algebra is continuous and obtain some interesting corollaries.

In Chapter IV, we prove an extremely important result concerning the extreme points of the set P(A) of normalized positive functionals. P(A) is shown to be weakly compact convex set, thus extending the result for MQ*-algebras. It is then shown that indecomposable positive functionals are exactly the extreme points of P(A), thus generalizing the corresponding result for Banach *-algebras and LMC*-algebras.

Finally in Chapter V, we investigate representations of BP*-algebras in B(H). One of the main results provides a necessary and sufficient condition for a cyclic representation to be irreducible. We end up with a theorem which states that the Gelfand representation is a continuous *-isomorphism of a semi-simple, symmetric BP*-algebra A onto a dense self-adjoint separating subalgebra $\hat{A}$ of C(M(A), τ_0), where τ_0 is the topology of uniform convergence on equi-continuous subsets of M(A). We use this theorem to prove normability of the inductive limit topology on a semi-simple, symmetric BP*-algebra.

CHAPTER 1

PRELIMINARIES

1. Topological Vector Spaces

Definition 1.1. A linear space E over K (real or complex field) endowed with a Hausdorff topology u is called a TVS (Topological Vector Space) if the mappings

(1) $(x,y) \to x + y$ of $E \times E$ into E and

(2) $(\lambda,x) \to \lambda x$ of $K \times E$ into E

are continuous.

We shall denote a TVS by E_u.

Definition 1.2. A subset A of a linear space is called convex if for all x, y ε A, $\lambda x + (1-\lambda) y \in A$, where $o \leq \lambda \leq 1$. A is called balanced (circled) if for each x ε A, $\lambda x \in A$ whenever $|\lambda| \leq 1$. A balanced convex set is called absolutely convex. This is equivalent to saying that for all x and y in A, $\lambda x + \mu y$ belongs to A whenever $|\lambda| + |\mu| \leq 1$. A is said to absorb a subset B if there exists $\lambda > o$ such that $B \subset \lambda A$. A is called absorbing if A absorbs each point in E. That is for each x ε E, there exists $\lambda > o$ such that $x \in \lambda A$.

Theorem 1.3. In a TVS E_u there exists a fundamental system $\mathcal{U}$ of u-closed neighbourhoods of the origin such that

(1) Each U in $\mathcal{U}$ is circled and absorbing.

(2) For each U in $\mathcal{U}$, there exists a V in $\mathcal{U}$ such that $V + V \subset U$.

(3) $\bigcap U = \{0\}$, where U is in $\mathcal{U}$.

Conversely if E is a real or complex vector space and $\mathcal{U}$ is a filter-base satisfying conditions (1) to (3), then there exists a topology u on E such that E_u is a TVS and is a fundamental system of neighbourhoods of the origin.

Proof: See Theorem 1, Page 13 [15].

Definition 1.4. A TVS is called a locally convex space if there exists a fundamental system of convex neighbourhoods of the origin.

Clearly each normed space is a locally convex space, where the norm $||\cdot||$ defines a fundamental system $\{\lambda U\}$ of convex neighbourhoods of 0, where $\lambda > o$ and $U = \{x \in E: \ ||x|| \leq 1\}$.

Theorem 1.5. A locally convex space E is metrizable if and only if it is separated (Hausdorff) and there is a countable base of neighbourhoods of the origin.

Proof: See Theorem 4, Page 16 [24].

Definition 1.6. A complete metrizable locally convex space is called a Fréchet space.

Proposition 1.7. For each $\alpha \in \Delta$, let E_α be a locally convex space and f_α a linear mapping of E_α into a vector space

E, so that $\cup f_\alpha(E_\alpha)$ spans E. Then there is a finest convex topology u on E under which all the mappings f_α are continuous. A base of 0-neighbourhoods for this topology is formed by the set $\mathcal{U}$ of all absolutely convex subsets U of E, such that, for each α, $f_\alpha^{-1}(U)$ is a neighbourhood of 0 in E_α.

<u>Proof</u>: See Proposition 4, Page 79 [24].

<u>Definition 1.8</u>. A locally convex space E with this topology u is called the inductive limit of the convex spaces E_α by the mappings f_α.

<u>Remark 1.9</u>. If the locally convex spaces E_α are vector subspaces of E, whose union spans E and the linear mappings f_α are all restrictions to E_α of the identity mapping of E, then the <u>inductive limit topology</u> is the finest locally convex topology on E which induces on each E_α a topology coarser than the given topology and an absolutely convex set U is a 0-neighbourhood in E if and only if $U \cap E_\alpha$ is a 0-neighbourhood in E_α for each α.

<u>Proposition 1.10</u>. Let f be a linear mapping of an inductive limit E_u of E_α $(\alpha \in \Delta)$ into a locally convex space F. Then f is continuous if and only if the composition mapping $f \circ f_\alpha$ is continuous for each α.

<u>Proof</u>: See Proposition 3, Page 18 [15].

<u>Definition 1.11</u>. Let E be a locally convex space. A subset B of E is called a <u>barrel</u> if it is closed, absolutely convex and absorbing. E is called a <u>barrelled space</u> if every

barrel in E is a neighbourhood of the origin. A subset B of E is said to be <u>bounded</u> if for each neighbourhood V of the origin, there is a $\lambda > o$ such that $B \subset \lambda V$. A subset M of E is said to be <u>bornivorous</u> if M absorbs every bounded subset of E. E is said to be <u>bornological</u> if each convex set which is bornivorous is a neighbourhood of the origin. A locally convex space E is called <u>quasi-barrelled</u> if each barrel of E which is bornivorous is a neighbourhood of the origin. A TVS E is said to be <u>quasi-complete</u> if every closed bounded set of E_u is complete. It is called <u>sequentially complete</u> if each Cauchy sequence converges in it. A locally convex space which is barrelled and in which each bounded set is relatively compact is called a <u>Montel</u> space.

<u>Theorem</u>. Suppose a second finer topology v is given on a TVS E_u. If v has a base of neighbourhoods of 0 consisting of u-closed sets, then every subset of E which is complete (respectively sequentially complete) with respect to u, is also complete (respectively sequentially complete) with respect to v.

<u>Proof</u>: See (4)(b) Page 210 [18] or Proposition 8 [6].

<u>Proposition 1.12</u>. Every complete locally convex space is quasi-complete and every quasi-complete space is sequentially complete.

<u>Proof</u>: See 7.4 Page 480 [12].

<u>Proposition 1.13</u>. Each barrelled or bonological space is quasi-barrelled, but not conversely.

Proof: See Proposition 9 Page 22 [15].

Proposition 1.14. A sequentially complete quasi-barrelled space is barrelled. In particular every quasi-complete quasi-barrelled space is barrelled.

Proof: See (1) Page 368 [18].

Proposition 1.15. Every sequentially complete bornological space is barrelled.

Proof: See (2), 28, Page 379 [18]. The proof also follows by combining Proposition 1.13 and Proposition 1.14 above.

Proposition 1.16. A metrizable quasi-complete locally convex space is a Frechet space.

Proof: See 4(b) Page 22 [15].

Proposition 1.17. Every Baire space (which cannot be written as a countable union of nowhere dense sets) is a barrelled space. In particular every Frechet space is barrelled.

Proof: See 6, Page 19 [15].

Proposition 1.18. An inductive limit of barrelled space is barrelled.

Proof: See Proposition 6, Page 81 [24].

Proposition 1.19. A closed subspace F of a complete uniform space E is complete. Every complete subspace M of a Hausdorff uniform space is closed.

Proof: See Proposition 7, Page 16 [6].

Definition 1.20. Let E and F be two TVS. The set of all linear and continuous mappings of E into F forms a vector space and is denoted by L(E,F). Let $\mathfrak{S}$ be the class of subsets of E. One can define a topology on L(E,F) of uniform convergence over sets M in $\mathfrak{S}$ as follows: Let V be a neighbourhood of the origin in F and M in $\mathfrak{S}$. Let T(M,V) denote the set of all linear and continuous mappings f in L(E,F) such that $f(M) \subset V$. The collection T(M,V), where M runs over $\mathfrak{S}$ and V over neighbourhoods of the origin in F, forms a sub-basis for a topology called the $\mathfrak{S}$-topology. If $\mathfrak{S}$ consists of all finite subsets of E, then $\mathfrak{S}$-topology is called the topology of simple convergence. A subset H of L(E,F) is called simply bounded if H is bounded in the simple convergence topology of L(E,F). A subset H of L(E,F) is said to be equi-continuous if, for each neighbourhood V of the origin in F, $\bigcap_{f \in H} f^{-1}(V)$ is a neighbourhood of the origin in E.

Theorem 1.21. Let E be a barrelled space and F any locally convex space. Then each simply bounded subset H of L(E,F) is equi-continuous.

Proof: See Theorem 5, Page 26 [15].

Definition 1.22. Let E be a locally convex space. The set E' = L(E,R) of all linear and continuous mappings of E into the reals R is called the topological dual or simply dual of E.

Clearly $E' \subset E^* \subset R^E$, where E^* is the algebraic dual. The value of the Linear functional $x' \in E'$ at $x \in E$ is denoted by $<x,x'>$. The coarsest locally convex topology on E for which the mapping: $x \to <x,x'>$ for each $x' \in E'$, is continuous, is called the <u>weak topology</u> $\sigma(E,E')$ on E. In the same way, the weak*-topology $\sigma(E',E)$ on E' is the coarsest topology for which the mapping: $x' \to <x,x'>$ for each $x \in E$, is continuous. $\sigma(E',E)$ is precisely the topology of simple convergence on E', which, in turn, is induced from the product topology on R^E. For each subset A of E, the set of all $x' \in E'$ such that $|<x,x'>| \leq 1$, for all $x \in A$, is called the <u>polar</u> of A, and is denoted by A^0. The <u>bipolar</u> A^{00} is the set of all $x \in E$ such that $|<x,x'>| \leq 1$ for all $x' \in A^0$.

<u>Proposition 1.23</u>. The polar A^0 of any subset A is absolutely convex and $\sigma(E',E)$ - closed.

<u>Proof</u>: See Proposition 9, Page 34 [24].

<u>Proposition 1.24</u>. A subset M' in E', the dual of a locally convex space E_u, is equi-continuous if and only if there exists a u-neighbourhood U of the origin in E_u such that $M' \subset U^0$.

<u>Proof</u>: See Proposition 14, Page 28 [15].

<u>Theorem 1.25</u>. If E is a locally convex space with dual E' and U is a neighbourhood of the origin, then U^0 is $\sigma(E',E)$ - compact.

<u>Proof</u>: See Theorem 6, Page 61 [24].

<u>Corollary 1.26</u>. If A' is equi-continuous, then $\overline{A'}$ is $\sigma(E',E)$ - compact.

Proof: Follows from Theorem 1.25 above.

Definition 1.27. Let u and v be two locally convex topologies on a vector space E. u and v are said to be equivalent ($u \sim v$) if E_u and E_v have the same topological dual i.e. $E'_u = E'_v$. In the sense of Bourbaki, u is said to be compatible with the duality between E_v and E'_v if $E'_u = E'_v$.

Theorem 1.28. Let E_u be a locally convex space with dual E'. Then the following statements are equivalent,

(a) E_u is barrelled.

(b) Each $\sigma(E',E)$ - bounded subset of E' is equi-continuous.

Proof: See Theorem 7, Page 30 [15].

Corollary 1.29. If E_u is a barrelled space, then in E' the following sets are the same (a) equi-continuous (b) relatively compact (c) $\sigma(E',E)$ - bounded.

Proof: Follows from Theorem 1.28 above and the Corollary 8, Page 30 [15].

Definition 1.30. The $\mathfrak{S}$-topology on a locally convex space E, where $\mathfrak{S}$ consists of all absolutely convex $\sigma(E',E)$ - compact subsets of the dual E' of E, is called the Mackey topology and is denoted by $\tau(E,E')$. A locally convex space E_u is said to be a Mackey space if $u = \tau(E,E')$.

Proposition 1.31. A quasi-barrelled space E_u is a Mackey space..

Proof: See Proposition 16, Page 31 [15].

Corollary 1.32. Every bornological space (in particular a metrizable locally convex space) is a Mackey space.

Proof: Recall Proposition 1.13 and Proposition 1.31 above.

Proposition 1.33. Let E be a Fréchet space and F a barrelled space. Let f be a linear mapping of E onto F so that the graph of f is closed. Then f is open.

Proof: See Corollary 5, Page 41 [15].

Proposition 1.34. If there is a continuous open mapping of a Fréchet space onto a Hausdorff locally convex space F, then F is a Fréchet space.

Proof: See Proposition 13, Page 119 [24].

Definition 1.35. In a vector space, a closed (or open) line segment [a,b] (or (a,b)) is the set of points of the form $\lambda a + (1-\lambda)b$ with $0 \leq \lambda \leq 1$ (or $0 < \lambda < 1$). A point x of a convex set C is called an extreme point of C if no proper open segment through x can be chosen which lies in C. In other words if $x = \lambda x_1 + (1-\lambda)x_2$, $0 < \lambda < 1$, $x_1, x_2 \in C$, then $x = x_1 = x_2$.

Theorem 1.36. [Krein-Milman]. Every non-empty convex compact subset of a Hausdorff locally convex space, has extreme points and is the closed convex hull of its extreme points [Convex hull of an arbitrary set M is the intersection of all the convex subsets of E containing it].

Proof: See Theorem 1, Page 138 [24].

2. Pseudo-Complete locally Convex Algebras.

Definition 1.37. A locally convex algebra is an associative linear algebra A over the complex field C, equipped with a topology T such that

(1) (A;T) is a Hausdorff locally convex TVS.

(2) For any element x_o of A, the maps $x \to x_o x$ and $x \to x x_o$ of A into itself are continuous.

Clearly a locally m-convex algebra (see definition 1.69) and, in particular, each normed algebra is a locally convex-algebra, where an algebra E with a norm $||\cdot||$ is called a normed algebra if

$$||xy|| \leq ||x|| \; ||y|| \text{ for all } x,y \in E .$$

Definition 1.38. Let A be a locally convex algebra. An element x of A is bounded if, for some non-zero complex number λ, the set $\{(\lambda x)^n : n = 1,2,3,\ldots\}$ is a bounded subset of A. The set of all bounded elements of A will be denoted by A_o.

Clearly every element of a normed algebra is bounded because, for each $x \neq 0$, $\lambda = 1/_{||x||}$ satisfies the definition. Hence for a normed algebra A, $A = A_o$.

Notation. Let $\mathcal{B}_1$ denote the collection of all subsets B of A such that

(a) B is absolutely convex and $B^2 \subset B$.

~~(b)~~ B is bounded and closed.

For each $B \in \mathcal{B}_1$,

$A(B) = \{\lambda x: \lambda \in C, x \in B\}$ is, by (a) and (b), a subalgebra generated by B and

$$||x||_B = \inf \{\lambda > o: x \in \lambda B\} \ (x \in A(B))$$

is a norm on A(B) which makes A(B) a normed algebra, with closed unit ball B. It will always be assumed that A(B) carries the topology induced by this norm.

Proposition 1.39. The topology induced on A(B) from A, is coarser than the norm topology on A(B).

Proof: Let V be an absolutely convex neighbourhood of the origin for the locally convex topology induced from A. Since B is bounded, there exists $\lambda > o$ such that $\lambda B \subset V$. λB is clearly contained in A(B). Hence $\lambda B \subset V \subset A(B)$. B being a closed unit ball in A(B), λB is a neighbourhood of the origin in the norm topology on A(B). This proves that the norm topology on A(B) is finer than the induced topology on A(B).

Q.E.D.

Definition 1.40. A sub-collection $\mathcal{B}_2$ of $\mathcal{B}_1$ is said to be basic (in $\mathcal{B}_1$) if for every B_1 in $\mathcal{B}_1$, there is some B_2 in $\mathcal{B}_2$ such that $B_1 \subset B_2$.

For example, the class of positive multiples of the unit ball in a normed algebra is basic in the class of all bounded subsets.

Proposition 1.41. Let A be a locally convex algebra and let $\mathcal{B}_2$ be any basic sub-collection of $\mathcal{B}_1$. Then

$$A_o = \cup\{A(B) : B \in \mathcal{B}_2\} .$$

Proof: Let D denote the right hand side. Let $x \in D$. Then $x \in A(B)$ for some B in $\mathcal{B}_2$. Then, if $\lambda > ||x||_B$, the set $\{(\lambda^{-1}x)^n: n = 1,2,3,\ldots\}$ is bounded in A(B). The norm topology on A(B) being finer than the induced one, and a set bounded in a finer topology remains bounded in the coarser topology, it follows the set $\{(\lambda^{-1}x)^n: n \geq 1\}$ is bounded in A. Consequently $x \in A_o$. That is $D \subset A_o$.

Conversely, if $x \in A_o$, there is some complex $\lambda (\neq o)$ such that the set

$$S = \{(\lambda x)^n: n \geq 1\}$$

is bounded in A. Clearly $S^2 \subset S$. It is easy to see that the closed absolutely convex hull of S belongs to $\mathcal{B}_1$ [Recall Lemmas 1.3 and 1.4 [19] and [24]]. Thus there is some B in $\mathcal{B}_2$ such that $S \subset B$. Hence $\lambda x = z \in B$. Therefore $x \in A(B)$. That is $A_o \subset D$ and hence $A_o = D$.

Q.E.D.

Definition 1.42. The locally convex algebra A is called pseudo-complete if and only if each of the normed algebras A(B) ($B \in \mathcal{B}_1$) is a Banach algebra.

Proposition 1.43. If A is sequentially complete, then A is pseudo-complete.

Proof: We have to show that A(B) is complete or sequentially complete for each B $\varepsilon \mathcal{B}_1$. Consider a Cauchy sequence $\{z_n\}$ in A(B) in the norm topology. $\{z_n\}$ being norm-bounded, there exists a $\lambda > o$ such that

$$\{\lambda z_n\} \subset B \subset A$$

where B is a closed unit ball in A(B). Since each B $\varepsilon \mathcal{B}_1$ is closed in the topology of A and by proposition 1.39, the norm topology on A(B) is finer than the induced one on A(B), $\{\lambda z_n\}$ is also a Cauchy sequence in the induced topology and sequential completeness of A implies that $\lambda z_n \to b \varepsilon B$ in the weaker topology. By the Theorem on Page 7, $\lambda z_n \to b$ in the finer topology on A(B). Since $\lambda^{-1} b \varepsilon$ A(B), we have shown that A(B) is a sequentially complete normed algebra and thus is a Banach algebra. This proves A is pseudo-complete.

Q.E.D.

Proposition 1.44. If $\mathcal{B}_1$ contains a basic sub-collection $\mathcal{B}_2$ such that A(B) is a Banach algebra for every B $\varepsilon \mathcal{B}_2$, then A is pseudo-complete.

Proof: Let B $\varepsilon \mathcal{B}_1$. We can choose B_2 in $\mathcal{B}_2$ such that $B \subset B_2$. Clearly $A(B) \subset A(B_2)$. Since B is closed in A, it is closed in $A(B_2)$; since the topology of A(B) is stronger than its topology as a subspace of $A(B_2)$, it follows by the theorem quoted in the above proposition that A(B) is complete for every B $\varepsilon \mathcal{B}_1$. Hence A is pseudo-complete.

Q.E.D.

We give here some definitions before we give a counter example for the converse of proposition 1.43. Let E be a TVS. We recall that a sequence $\{x_n\}$ ($x_n \varepsilon E$) is Cauchy in the sense of Mackey if there is some sequence of positive numbers $\{\varepsilon_n\}$, tending to zero, and a bounded subset B of E, such that for all n, $x_n - x_m \varepsilon \varepsilon_n B$ whenever $m > n$. $\{x_n\}$ converges in the sense of Mackey to x, if there is some sequence of positive numbers $\{\varepsilon_n\}$ tending to zero, and a bounded subset B of E, such that $x_n - x \varepsilon \varepsilon_n B$ for all n. Finally E is Mackey complete if every Cauchy sequence, in the sense of Mackey, is convergent in the sense of Mackey. Any sequence which is Cauchy or convergent in the sense of Mackey is respectively Cauchy or convergent in the topology of E. Also if E is sequentially complete, it is Mackey complete. We now give an example of a locally convex algebra which is pseudo-complete but not Mackey complete.

Example. Let A be the algebra of all complex polynomials; let A be given the topology T of uniform convergence on the compact subsets of the positive real line R^+. A_o consists just of the constant functions and that $\mathcal{B}_1$ has a greatest member B_o, namely the set of all constant functions not exceeding unity in absolute value. Thus, by proposition 1.44, A is pseudo-complete.

Now let the sequence $\{p_n\}$ in A be defined by

$$p_n(x) = \sum_{r=o}^{n} (-1)^{r+1} x^{2r+1}/(2r+1)! \quad .$$

Then $p_n(x)$ converges to sinx uniformly on compact subsets of R^+. Let

$$B = \{p: p \in A. \ |p(x)| \leq e^x \ (x \geq o)\}.$$

Then B is T-bounded and $e^{-x} \ |p_n(x)-\sin x|$ tends to zero uniformly on R^+; therefore $\{p_n\}$ is Cauchy in the sense of Mackey but does not converge to an element of A. Thus A is not Mackey-complete. Consequently A is not sequentially complete.

Q.E.D.

Proposition 1.45. (a) A closed subalgebra of a pseudo-complete algebra is itself pseudo-complete.

(b) If A has no identity, then A_e (the algebra A with a unit e adjoined) is pseudo-complete if and only if A is pseudo-complete.

(c) If the locally convex algebra A has an identity e, then for the collection $\mathcal{B}_1$ as defined before,

$$\{B \in \mathcal{B}_1 : e \in B\}.$$

is a basic sub-collection of $\mathcal{B}_1$.

Proof: See Proposition 2.8 [1].

Notation. With the notation as in (c) above, if A is a locally convex algebra with identity e, we shall write

$$\mathcal{B} = \{B \in \mathcal{B}_1 : e \in B\} .$$

If A has no identity, we shall write $\mathcal{B}$ for $\mathcal{B}_1$.

Theorem 1.46. If A is commutative and pseudo-complete, then $\mathcal{B}$ is outer-directed by inclusion (i.e. if B, C $\varepsilon \mathcal{B}$, then there is some D such that D $\varepsilon \mathcal{B}$, and $B \cup C \subset D$).

Proof: We recall from Theorem 1.21 that each simply bounded subset H of L(E,F) is equi-continuous where E is a barrelled space and F is any locally convex space. We first assume that A has an identity e. Let B, C $\varepsilon \mathcal{B}$; we show that BC is bounded in A.

For each b ε B, let L_b be the linear map from A(C) into A defined by $L_b(c) = bc$, where c runs over A(C). Let $B' = \{L_b : b \varepsilon B\}$. Multiplication in A being separately continuous, we see that each L_b is continuous and that B' is bounded in the topology of simple convergence on A(C), i.e. B' is a simply bounded subset of L(A(C), A). Since A(C) is a Banach algebra which is a Banach space and hence a barrelled space, it follows by the above quoted theorem that B' is an equi-continuous subset of L(A(C), A). Hence BC is bounded in A.

Since A is commutative, $(BC)^2 = B^2C^2 \subset BC$, so that the closed absolutely convex hull of BC belongs to $\mathcal{B}_1$. Hence, since $e \varepsilon B \cap C$, there is some D in $\mathcal{B}$ such that

$$B \cup C \subseteq BC \subseteq D.$$

This proves the theorem when A has an identity. If A has no identity, we form A_e and use Proposition 1.45 above.

Q.E.D.

Corollary 1.47. If A is commutative and pseudo-complete, then A_o is a subalgebra of A.

Proof: Let x, y ε A_o. It is easy to see that $\lambda x \in A_o$ for any $\lambda \in C$. There exist non-zero complex numbers μ and λ such that

$$B = \{(\lambda x)^n : n \geq 1\} , \quad C = \{(\mu y)^n : n \geq 1\}$$

are both bounded in A. Since BC is bounded by the Theorem 1.46, clearly $xy \in A_o$. We know by Proposition 1.41, that

$$A_o = \cup \{A(B) : B \in \mathcal{B}\} .$$

Then $x \in A(B)$ for some B in $\mathcal{B}$, and $y \in A(C)$ for some C in $\mathcal{B}$. Since $\mathcal{B}$ is outer-directed by the inclusion relation, one can show that $x + y \in A(D)$ for some D in $\mathcal{B}$. Hence $x + y \in A_o$. We have therefore established that A_o is a subalgebra of A.

Q.E.D.

Definition 1.48. If A is a locally convex algebra and $x \in A$, we define the radius of boundedness, $\beta(x)$ of x by

$$\beta(x) = \inf \{\lambda > o : \{(\lambda^{-1}x)^n\}_{n \geq 1} \text{ is bounded}\} .$$

Proposition 1.49. If A is any locally convex algebra, then for any $x \in A$, $\beta(x) < \infty$ if and only if $x \in A_o$.

Proof: See Proposition 2.14 [1].

Proposition 1.50. If A is locally convex and if $x \in A_o$, then

$$\beta(x) = \inf \{||x||_B : B \in \mathcal{B}, x \in A(B)\} .$$

Proof: See Corollary 2.17 [1].

Definition 1.51. Let A have an identity e and let $x \in A$. Then the spectrum of x denoted by $\sigma_A(x)$ or $\sigma(x)$, is a subset of the extended complex plane C* defined as follows:

(a) for $\lambda \neq \infty$, $\lambda \in \sigma(x)$ if and only if $\lambda e - x$ has no inverse belonging to A_o.

(b) $\infty \in \sigma(x)$ if and only if $x \notin A_o$.

The spectral radius $r_A(x)$ of the element x in A will be defined by

$$r(x) = \sup \{|\lambda|: \lambda \in \sigma(x)\}$$

Proposition 1.52. Let A be a locally convex algebra and let $x \in A$. Then $\sigma(x) \neq \emptyset$ and if A is pseudo-complete, then $\sigma(x)$ is closed.

Proof: See Corollary 3.9 [1].

Theorem 1.53. Let A be a locally convex algebra and let $x \in A$. Then $\beta(x) \leq r(x)$ and if A is pseudo-complete, then $\beta(x) = r(x)$.

Proof: See Theorem 3.12 [1].

Proposition 1.54. Let A be pseudo-complete and let $x \in A_o$. Then

(1) $\sigma_A(x) = \bigcap \{\sigma_{A(B)}(x): B \in \mathcal{B},\ x \in A(B)\}$.

(2) $r_A(x) = \inf \{r_{A(B)}(x): B \in \mathcal{B},\ x \in A(B)\}$.

Proof: See Proposition 3.13 [1].

Theorem 1.55. Let A be a locally convex algebra with continuous inversion and let $x \in A$. Then

$$Sp(x) = \{\lambda \in C: \lambda e - x \text{ has no inverse in } A\}.$$
$$\subset \sigma(x) \subset \overline{Sp(x)}$$

and if A is pseudo-complete, then $\sigma(x) = \overline{Sp(x)}$.

Proof: See Theorem 4.1 [1].

Proposition 1.56. If A is pseudo-complete and has continuous inversion, then $x \in A_o$ if and only if Sp(x) is bounded.

Proof: See Corollary 4.2 [1].

3. Projective Limits

Definition 1.57. Let $\{X_\alpha: \alpha \in \Delta\}$ be an indexed family of topological spaces X_α, the index set Δ being directed by the relation $\leq$. Suppose that whenever $\alpha \leq \beta$, there is given a continuous map $\pi_{\alpha\beta}$ from X_β into X_α such that

(1) $\pi_{\alpha\alpha}$ is the identity map on X_α (all α) and

(2) if $\alpha \leq \beta \leq \gamma$, then $\pi_{\alpha\beta} \circ \pi_{\beta\gamma} = \pi_{\alpha\gamma}$.

We write for the product space

$$P = \Pi\{X_\alpha: \alpha \in \Delta\}$$

and define the projective limit $\Lambda\{X_\alpha: \alpha \in \Delta\}$ of the spaces X_α, with respect to the maps $\pi_{\alpha\beta}$, to be the subspace Q of P where

$$Q = \Lambda\{X_\alpha : \alpha \in \Delta\}$$
$$= \{(x_\alpha)_{\alpha\in\Delta} \in P : \pi_{\alpha\beta}(x_\beta) = x_\alpha \ (\alpha \leq \beta)\} .$$

Proposition 1.58. If each X_α is a non-empty compact Hausdorff space, then Q is also a non-empty compact Hausdorff space.

Proof: See Proposition 6.2 [1] or Chapter 1, §9, No. 6, Proposition 8 [5].

Definition 1.59. Let A be commutative and pseudo-complete. Then the carrier space of A_o, denoted by M_o, is the set of all non-identically zero multiplicative linear functionals on A_o, in the weak *-topology $\sigma(M_o, A_o)$.

Proposition 1.60. The space M_o is a non-empty compact Hausdorff space.

Proof: See Corollary 6.5 [1].

Proposition 1.61. Let A be commutative and pseudo-complete, and let $x \in A_o$. Then x is invertible in A_o if and only if $\phi(x) \neq 0$ for every $\phi \in M_o$.

Proof: See Lemma 6.6 [1].

Theorem 1.62. Let A be commutative and pseudo-complete and let $x \in A_o$. Then

$$\sigma(x) = \{\phi(x) : \phi \in M_o\} .$$

Proof: See Theorem 6.7 [1].

4. Pseudo-Banach Algebras

Definition 1.63. Let A be a commutative algebra with identity e. A bound structure for A is a non empty collection $\mathcal{B}$ of subsets of A such that

(1) B is absolutely convex, $B^2 \subset B$, $e \in B$ for each B in $\mathcal{B}$

(2) given B_1, B_2 in $\mathcal{B}$, there exists B_3 in $\mathcal{B}$ and $\lambda > o$ such that $B_1 \cup B_2 \subset \lambda B_3$

$(A, \mathcal{B})$ is called a bound algebra.

For $B \in \mathcal{B}$, let $A(B) = \{\lambda b: \lambda \in C, b \in B\}$. In view of (1), A(B) is the subalgebra of A generated by B. The Minkowski functional of B defines a sub-multiplicative semi-norm $||\cdot||_B$ on A(B). If each $||\cdot||_B$ is a norm and if A(B) is a Banach algebra with respect to $||\cdot||_B$, then $(A, \mathcal{B})$ is called a complete bound algebra.

From (2), $A_o = \cup \{A(B) : B \in \mathcal{B}\}$ is a subalgebra of A. If A is complete, and if $A = A_o$, then A is a pseudo-Banach algebra.

Theorem 1.64. (Gelfand-Mazur Theorem). If the pseudo-Banach algebra is a field, then A is isomorphic to the complex field .

Proof: See Corollary 1.7 [2].

Proposition 1.65. The (Jacobson) radical of a pseudo-Banach algebra A, is equal to the ideal

$$R(A) = \{a \in A: x(a) = o, \ x \in X_A\}.$$

where X_A is the character space (the set of all non-zero multiplicative linear functionals on A).

5. The Space L^ω

The algebra $A = L^\omega[0.1]$ (cf: Arens [4]) consists of all complex-valued measurable functions f on [0,1] such that $f \in L^p[0,1]$ (p = 1,2,...), functions equal almost everywhere not being distinguished. A is topologized by the whole collection of L^p-norms $\{||\cdot||_p: p = 1,2,3,...\}$. Then A is a complete metrizable locally convex *-algebra (the involution being complex conjugation) with jointly continuous multiplication.

Theorem 1.66. $L^\infty \subset L^\omega \subset L^p$ but $L^\infty \neq L^\omega \neq L^p$ for $p \geq 1$. The identity mappings $L^\infty \to L^\omega \to L^p$ are continuous, but their inverses are not. L^∞ is dense in L^ω and L^ω is dense in each $L^p (p \geq 1)$.

Proof: See theorem 1. [4].

Theorem 1.67. L^ω is a convex metric commutative ring with the property that if U is a convex open set in L^ω containing O and if $UU \subset U$, then U coincides with the whole space L^ω.

Proof: See Theorem 2 [4].

6. Locally Multiplicatively Convex Topological Algebras

Definition 1.68. A subset U of an algebra A is called idempotent if $UU \subset U$: it is called m-convex (multiplicatively convex) if it is convex and idempotent.

Definition 1.69. A topological algebra is called Locally m-convex if there exists a basis for the neighbourhoods of the origin consisting of sets which are m-convex and symmetric (balanced).

Clearly every Banach algebra is locally m-convex. L^{ω} is not locally m-convex.

Definition 1.70. An *-algebra is an algebra with a mapping which assigns to every $x \in A$ an $x^* \in A$ such that for $x, y \in A$

(a) $(x^*)^* = x$ (b) $(xy)^* = y^*x^*$ (c) $(x+y)^* = x^* + y^*$ and (d) $(\lambda x)^* = \bar{\lambda}x^*$ for any scalar λ.

Definition 1.71. A symmetric *-algebra is a *-algebra with identity and the property that $(e + x^*x)^{-1}$ exists for every $x \in A$.

Proposition 1.72. Every locally m-convex algebra has a continuous inversion on the set of regular elements.

Proof: See Proposition 2.8 [19].

Proposition 1.73. If f is a multiplicative linear functional on a symmetric *-algebra, then

$$f(x^*) = \overline{f(x)}$$

Proof: See Lemma 6.4(b) [19].

Proposition 1.74. If A is a symmetric *-algebra of complex valued functions on a space T, then A is closed under conjugation (i.e. if $x \in A$, then x^* defined by $x^*(t) = \overline{x(t)}$ for $t \in T$, is in A).

Proof: See Corollary 6.5 [19].

Proposition 1.75. Let A be a symmetric *-algebra, and B a commutative, semi-simple and complete locally m-convex symmetric *-algebra. Then any homomorphism h from A into B is a *-homomorphism (i.e. $h(x^*) = [h(x)]^*$ for all $x \in A$).

Proof: See Lemma 6.6 [19].

Proposition 1.76. Let T be a topological space, $t \in T$ and A a separating subalgebra of $C_t(T)$ which is closed under conjugation. Then A is dense in $C_t(T)$ with the compact open topology.

Proof: See Proposition 6.8 [19].

Notation $S(A) = \{x \in A: r_A(x) \leq 1\}$.

Proposition 1.77. The following conditions on a topological algebra are equivalent,

(1) A is a Q-algebra (the set of invertible elements is open).

(2) S(A) is a neighbourhood of 0.

Proof: See Lemma E.2 [19].

Proposition 1.78. Every element of a Q-algebra has a compact spectrum.

<u>Proof</u>: See Lemma E.3 [19].

<u>Remark</u>: Every Banach algebra is a Q-algebra. The algebra $C(\Omega_o)$ is not a Q-algebra [19].

<u>7. Positive Functionals</u>

<u>Definition 1.79</u>. A linear functional f on a *-algebra A is said to be <u>positive functional</u> if $f(x^*x) \geq o$ for each $x \in A$. A positive functional f is said to be <u>extendable</u> if $f(x^*) = \overline{f(x)}$ and if there exists an $a > o$ such that for all $x \in A$, $|f(x)|^2 \leq af(x^*x)$. A positive functional f is said to be <u>admissible</u> if

$$\text{Sup}\ \{\frac{f(y^*x^*xy)}{f(y^*y)} : y \in A,\ f(y^*y) \neq o\} < \infty .$$

<u>Theorem 1.80</u>. If f is a positive functional on a *-algebra. Then for all x, $y \in A$,

(a) $f(y^*x) = \overline{f(x^*y)}$.

(b) $|f(x^*y)|^2 \leq f(x^*x)\cdot f(y^*y)$

If A has a unit e, then it follows that

(c) $|f(x)|^2 \leq f(e)\cdot f(x^*x)$.

<u>Proof</u>: See Theorem (21.16) [14].

<u>Theorem 1.81</u>. If A_e is the algebra A with unit adjoined and f is a positive functional on A, then a necessary and sufficient condition for the existence of a positive functional

$\tilde{f}$ on A_e such that $\tilde{f}|_A = f$ is that f be extendable.
Then

$$|f(x)|^2 \leq \tilde{f}(e)\, f(x^*x) .$$

Proof: See (C.3) Page 470 [14].

Theorem 1.82. Every positive functional on a Banach *-algebra with a unit is bounded or continuous.

Proof: See Corollary 21.20 [14].

Definition 1.83. Let f and g be positive functionals on a *-algebra. g is said to be dominated by f if there exists a positive number λ such that λf - g is a positive functional on A. A positive functional f is said to be indecomposable if every positive functional g dominated by f is a multiple of f.

Theorem 1.84. If P(A) denotes the set of all positive functionals on a Banach *-algebra with identity e such that f(e) = 1, then P(A) is compact in the weak *-topology.

Proof: See 4, Page 266 [21].

Theorem 1.85. A positive functional f in P(A) where A is a Banach *-algebra, is indecomposable if and only if f is an extreme point of the set P(A).

Proof: See I, Page 266 [21].

Theorem 1.86. Positive multiplicative members of P(A) are exactly the extreme points of P(A).

Proof: See Remark Page 272 [21]. Use also Theorem 1.85 above.

8. Extentions of Positive Functionals

Definition 1.87. A set K in the real linear space X is called a cone if

(1) $x \in K$ and $\alpha \geq o$ imply that $\alpha x \in K$.

(2) $x, y \in K$ implies that $x + y \in K$.

(3) $x \in K$, $x \neq o$ imply that $-x \notin K$.

The following theorem due to M. Krein plays an important role on extensions of positive functionals.

Theorem 1.88. Suppose K is a cone in the real locally convex space X. Assume K contains interior points and let Y be a subspace in X which contains at least one interior point of K. Then every positive linear functional f on Y can be extended to a positive linear functional F on X.

Krein's theorem can be employed to prove the following theorem on extensions of positive functionals.

Theorem 1.89. Suppose A_1 is a closed *-subalgebra of a symmetric Banach *-algebra A with identity. Then every positive functional f_o on A_1 can be extended to a positive functional on A.

Proof: See III, Page 304 [21].

9. Representations

Definition 1.90. Let A be a *-algebra and H a Hilbert space. A homomorphism $T: x \to T_x$ of A into the algebra of bounded operators on H is called a *-representation of A provided $T_{x^*} = T_x^*$, where T_x^* is the adjoint operator of T_x. A subspace H_1 of H is said to be invariant under the representation T if $T_x(H_1) \subset H_1$ for all $x \in A$. A representation T of an algebra A by operators on H is said to be cyclic if there is a vector $\zeta \in H$ such that the linear subspace $\{T_x\zeta: x \in A\}$ is dense in H and ζ is called a cyclic vector. Let f be a positive functional on a *-algebra A and let T be a *-representation of A on a Hilbert space H. Then f is said to be representable by T if there is a cyclic vector $\zeta \in H$ such that $f(x) = \langle T_x\zeta, \zeta\rangle$. A representation T of an algebra A is irreducible if {0} and H are the only closed subspaces of H that are invariant under all T_x.

Theorem 1.91. Let f be an admissible, hermitian ($f(x^*) = \overline{f(x)}$) positive functional on a *-algebra A. Then there exists a *-representation T of A on a Hilbert space H with the following properties:

(1) There is a linear mapping $x \to \zeta_x$ of A onto dense subspace of H such that

$$f(y^*xy) = \langle T_x\zeta_y, \zeta_y\rangle \text{ for all } x, y \in A.$$

(2) If A has a unit e, then the representation T has

a cyclic vector ζ and

$$f(x) = \langle T_x\zeta,\zeta\rangle \ .$$

The *-representation T is called an associated *-representation of f.

<u>Proof</u>: See Theorem 4.54, Page 215 [23].

<u>Theorem 1.92</u>. Every *-representation T of a Banach *-algebra A by bounded operators B(H) on a Hilbert space H is a continuous mapping of A into B(H), supplied with the norm topology.

<u>Proof</u>: See Theorem 21.22, Page 320 [14].

<u>Proposition 1.93</u>. (a) Every positive functional which is representable is also extendable and admissible.

(b) If a positive functional is representable, then it is representable by its associated representation.

<u>Proof</u>: See Corollary 4.5.9, Page 217 [23].

<u>Theorem 1.94</u>. A cyclic representation $x \to T_x$ of a *-algebra A with identity in B(H) is irreducible if and only if the positive functional f defined by

$$f(x) = \langle T_x\zeta,\zeta\rangle$$

is indecomposable for each cyclic vector ζ in H.

<u>Proof</u>: See Theorem Page 265 [21].

<u>Generalized Weierstrass Theorem</u>. Let X be an arbitrary

space and C(X;c) denote the set of all continuous real-valued functions on X with the compact open topology c. If D is a separating subset of C(X;c), containing a non-zero constant, then A(D), the algebra generated by D, is dense in C(X,c).

<u>Proof</u>: See Theorem 3.3, Page 282 [11].

CHAPTER II

BP*-ALGEBRAS

As a generalization of MQ*-algebras [17], we introduce a class of locally convex-algebras, designated as BP*-algebras. We provide several examples of these algebras which ensure that BP*-algebras form a considerably larger class than that of MQ*-algebras. We discuss some of the permanence properties. We modify an example of V. Ptak to show that, unlike MQ*-algebras, the continuous, open, homomorphic images of BP*-algebras are not necessarily BP*-algebras.

Definition 2.1. Let A be a commutative pseudo-complete locally convex-algebra (Definition 1.42) with a continuous involution. A is called a BP*-algebra if $A = A_o$ i.e. every element of A is bounded (recall that A_o is a subalgebra of A by Corollary 1.47).

Remark Denote by $\mathcal{B}^* = \{B \in \mathcal{B}_1 : B = B^*\}$. A locally convex *-algebra A will be called *-pseudo-complete if A(B) is complete for every $B \in \mathcal{B}^*$. Since $\mathcal{B}^* \subset \mathcal{B}_1$, clearly if A is pseudo-complete, then A is *-pseudo-complete. Consequently every BP*-algebra is *-pseudo-complete. Hence in a BP*-algebra every A(B) will be a *-subalgebra of A for each $B \in \mathcal{B}^*$. In all our discussions involving positive functionals, it will be assumed that $\mathcal{B}^*$ is a basic subcollection of $\mathcal{B}_1$, so that in view of Proposition 1.41, A = $\cup\{A(B): B \in \mathcal{B}^*\}$ where each A(B) is a Banach *-algebra. The induced involution is continuous in the norm topology on A(B) because we recall from the Remark 1.39 that the topology induced on A(B) is coarser than the norm topology on A(B).

Definition 2.2. A commutative sequentially complete locally m-convex-Q-algebra with a continuous involution is called an MQ*-algebra [17].

The following proposition and examples justify the existence of this new class of locally convex-algebra.

Proposition 2.3. Every MQ*-algebra A is a BP*-algebra but not conversely.

Proof: It is clear that every locally m-convex algebra is a locally convex algebra with jointly continuous multiplication. Since by Proposition 1.43, every sequentially complete locally convex algebra is a pseudo-complete locally convex algebra, it follows that A is a pseudo-complete locally convex *-algebra. One knows from Proposition 1.72 that every locally m-convex algebra has a continuous inversion and every element of a Q-algebra has a compact spectrum. Consequently Proposition 1.56 yields that every element of A is bounded. Hence A is a BP*-algebra.

The converse will follow from the following examples which show that not every BP*-algebra is an MQ*-algebra.

<u>Example 2.3</u>. (Example of a BP*-algebra which is complete locally m-convex *-algebra but not a Q-algebra). Let Ω_o be the space of ordinals < Ω (the first ordinal with uncountably many predecessors) with the order topology. In view of example 3.7 [19], $C(\Omega_o)$, the algebra of continuous, complex-valued functions, with the compact open topology, is a commutative complete locally m-convex-algebra with identity and continuous involution. Since sequential completeness implies pseudo-completeness (Proposition 1.43), $C(\Omega_o)$ is a pseudo-complete locally convex *-algebra. One knows from [13] that every complex-valued continuous function on Ω_o is bounded. In fact every $f \in C(\Omega_o)$ is bounded in the sense of Definition 1.38; for let $f \neq o$ (the case $f = o$ is trivial).

Choose

$$\alpha = 1/\sup_{x\in\Omega_o}|f(x)| > o .$$

Let N(K,1) be a neighbourhood of zero in the compact open topology, where K is a compact subset of Ω_o. We wish to show that the set

$$\{(\alpha f)^n : \ n \geq 1\} .$$

is bounded in $C(\Omega_o)$. It is sufficient to show that

$$[\alpha|f(x)|]^n \leq 1 , \quad n \geq 1, \quad \text{and } x \in K.$$

Clearly

$$\alpha = 1/\sup_{x\in\Omega_o}|f(x)| \leq 1/\sup_{x\in K}|f(x)|,$$

consequently

$$[\alpha f(x)]^n \leq [\alpha \sup_{x \in K} |f(x)|]^n \leq 1 , \quad n = 1,2,3,...$$

and this proves that every $f \in C(\Omega_o)$ is bounded in the sense of Definition 1.38.

Furthermore by the remark on Page 59 [19], $C(\Omega_o)$ is not a Q-algebra. We have therefore shown that A is a BP*-algebra but not an MQ*-algebra.

<u>Example 2.4</u>. (Example of a BP*-algebra which is a complete locally convex algebra but not locally m-convex and hence not a Q-algebra.) Let $C_b(R)$ denote the algebra of bounded continuous complex-valued functions on the real numbers R(pointwise operations and complex conjugation as *-operation). Denote the set of strictly positive real-valued continuous functions on R which vanish at infinity by $C_o^+(R)$. The family of semi-norms $\{P_\phi: \phi \in C_o^+(R)\}$ determine a locally convex topology β on $C_b(R)$ where

$$P_\phi(f) = \sup\{|f(x)\phi(x)|: \ x \in R\}, \ f \in C_b(R).$$

The space $(C_b(R),\beta)$ is a locally convex *-algebra (multiplication is continuous in each factor separately) since

$$P_\phi(fg) \leq M(f)\cdot P_\phi(g), \ g \in C_b(R).$$

where M(f) is the supremum of $|f|$. Completeness follows from Theorem 3.6 [25]. That each P_ϕ fails to be submultiplicative follows from Example 3 [9]. Since a topological algebra E is locally m-convex if and only if the topology is generated by a family of sub-multiplicative semi-norms, it follows $(C_b(R),\beta)$ is not locally m-convex and hence not an MQ*-algebra. Since completeness or sequential completeness implies pseudo-completeness, $(C_b(R),\beta)$ is a pseudo-complete locally convex *-algebra. It remains to show that every element is bounded in the β-topology. Let $f \in C_b(R)$, and $\phi \in C_o^+(R)$. Let $||f||_b$ and $||\phi||_\infty$ be the sup-norms in $C_b(R)$ and $C_o^+(R)$ respectively. Choose $\lambda = ||f||_b > o$ if $f \neq o$: (The case when $f = o$ is trivial). Clearly

$$P_\phi(f^n/||f||_b^n) = \sup|\phi f^n/||f||_b^n|$$

$$\leq ||\phi||_\infty .$$

If $N(P_\phi,1)$ denotes the unit ball in the β-topology, then the set

$$\{(f/||f||_b)^n, \quad n \geq 1\} \subset ||\phi||_\infty \quad N(P_\phi,1)$$

and this shows that every element of $(C_b(R),\beta)$ is bounded in the sense of Definition 1.38.

We have, therefore, finally shown that $(C_b(R),\beta)$ is a BP*-algebra which is not an MQ*-algebra.

<u>Example 2.5</u>. (Example of a BP*-algebra which is neither complete, nor locally m-convex, and hence not a Q-algebra.) Consider the algebra C[0,1] of all continuous complex-valued functions on the closed unit interval [0,1]. A norm p is defined

on this algebra by

$$p(f) = \sup \{|f(x)\phi(x)|: \ x \in [0,1]\},$$

where

$$\phi(x) = \begin{cases} x & 0 \le x \le 1/2 \\ 1 - x & 1/2 \le x \le 1 \end{cases}$$

By second paragraph on Page 21 [9], (C[0,1], p) is a normed linear space which is neither complete nor locally m-convex and hence not a Q-algebra. It is indeed a locally convex algebra since the multiplication is continuous in each factor separately. As in Example 2.4, one can show every element of (C[0,1], p) is bounded in the sense of Definition 1.38. The following arguments will show that this algebra is, in fact, pseudo-complete.

Let N(p,1) denote the unit ball in (C[0,1], p). That is

$$N(p,1) = \left[f: \ |f(x)| \le \begin{cases} \frac{1}{x}, & 0 < x \le \frac{1}{2} \\ \frac{1}{1-x}, & \frac{1}{2} \le x < 1 \end{cases} \right]$$

If $\mathcal{B}_1$ denote the collection of all subsets of this algebra which are absolutely convex, idempotent, closed and bounded, then we claim that

$$B_o = \{f: |f(x)\phi(x)| \le |\phi(x)|\} = \{f: |f(x)| \le 1, \ x \in (0,1)\}$$

is the greatest member of $\mathcal{B}_1$.

It is easy to check that B_o is absolutely convex, idempotent and p-bounded. To see closedness of B_o, we see that for $x \in (0,1)$, α_x defined by

$$\alpha_x(f) = f(x)$$

is p-continuous and

$$B_o = \bigcap_{x \in (0,1)} \alpha_x^{-1}[-1,1]$$

is closed.

We claim that B_o is a greatest member of $\mathcal{B}_1$. That is $B \subset B_o$ for all $B \in \mathcal{B}_1$. If not, there exists an $f \in B$ which is not in B_o. Then for some $x \in (0,1)$, $f(x) > 1$. Since B_o is idempotent, $B_o^n \subset B_o$ for all n. Consequently $f^n \in B_o$ for all n. Boundedness of B_o implies that there exists $\lambda > o$, such that

$$B_o \subset \lambda N(p,1) = N(p,\lambda).$$

Hence

$$|f^n(x)\phi(x)| \leq \lambda \quad \text{for all } n$$

but we can choose n_o such that $|f^{n_o}(x)\phi(x)| > \lambda$ since $f(x) > 1$ for some $x \in (0,1)$. This contradiction yields that B_o is the greatest member of $\mathcal{B}_1$. Clearly $\{B_o\}$ is a basic sub-collection of $\mathcal{B}_1$. We know from Proposition 1.44 that if $\mathcal{B}_1$ contains a basic sub-collection $\mathcal{B}_2$ such that A(B) is a Banach algebra for every B in $\mathcal{B}_2$, then A is pseudo-complete. Consequently

(C[0,1],p) is a pseudo-complete locally convex *-algebra.

Finally we conclude that (C[0,1],p) is a BP*-algebra which is not an MQ*-algebra.

Q.E.D.

Example 2.6. (Example of a BP*-algebra which is neither sequentially complete, nor locally m-convex, nor a Q-algebra) Let $A = L^{\infty}[0,1]$, the algebra of essentially bounded measurable functions with the L^{ω}-topology, generated by the whole collection (countable) of L^p-norms $\{||\cdot||_p : p = 1,2,3,...\}$. A is a metrizable locally convex *-algebra. In view of Theorem 1.66, A cannot be sequentially complete because A is dense in L^{ω}. By Theorem 1.67, L^{ω} is not m-convex. Consequently L^{∞} being a subalgebra of L^{ω}, cannot be a complete locally m-convex algebra.

Furthermore we claim A is not a Q-algebra. Let G denote the set of all invertible elements in L^{∞}. We shall show that G is not open. Suppose G is open, then $N(1) \subset G$, where 1 is the constant function equal to 1 and is an invertible element in L^{∞} and N(1) is a basic neighbourhood of 1, in the L^{ω}-topology.

For $p_i \geq 1$ $(i = 1,2,3,...n)$ and some $\varepsilon > 0$,

$$V = \bigcap_{i=1}^{n} \{f \in L^{\infty}: ||f-1||_{p_i} < \varepsilon\} \subset N(1) \subset G.$$

Define

$$g(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Since $g \in L^{\infty}$ and $||g-1||_{p_i} = o < \varepsilon$, $i = 1,\ldots,n$, this implies that $g \in V \subset N(1) \subset G$, but we know by definition of g that g is not invertible and this contradiction establishes our claim that G is not open.

Moreover L^{∞}, by 8.4 [10], is a pseudo-complete locally convex *-algebra. By (4, Page 96 [3]), L^{∞} is the bounded part of L^{ω}. Hence it follows easily that every element of L^{∞} is, in fact, bounded in the relativized L^{ω}-topology. (See the proof of Proposition 2.7, below)

We have therefore shown that A is a BP*-algebra but not a Q-algebra and hence not an MQ*-algebra.

Q.E.D.

1. Permanence Properties

Proposition 2.7. A closed subalgebra A_1 of a BP*-algebra A is also a BP*-algebra.

Proof: Since by Proposition 1.45(a), closed subalgebra of a pseudo-complete algebra is pseudo-complete, A_1 is a pseudo-complete locally convex *-algebra with induced continuous involution. We wish to show that every element of A_1 is bounded in the relative topology.

Let $x \in A_1$. Since $A = A_o$ (set of all bounded elements of A) and A_1 is a subalgebra of A, x is a bounded element of A, which means there exists a non-zero complex number λ, such that the set

$$\{(\lambda x)^n : n = 1,2,3,\ldots\}$$

is a bounded subset of A. Clearly this set is also bounded in A_1, in the relative topology. In other words x is a bounded element of A_1. Consequently A_1 is a BP*-algebra.

Q.E.D.

Proposition 2.8. Let A be a BP*-algebra, and I a closed two sided *-ideal in A. Then A/I is a locally convex *-algebra every element of which is bounded. A/I is not necessarily a BP*-algebra.

Proof: As is well-known (Proposition 2.4(e) [19]), A/I is a locally convex *-algebra with involution $(x + I)^* = x^* + I$. Since the canonical map is a continuous homomorphism of A onto A/I, it is easy to see that every element of A/I is bounded (recall that a continuous image of a bounded set is bounded).

The following example will show that A/I need not be pseudo-complete, where I is a closed ideal in a pseudo-complete locally convex algebra A. We shall need the following theorems due to Morris [20] and Ptak [22].

Let X be a Hausdorff space. If F is a closed subset of X, let

$$I_F = \{f \in C(X): f \text{ vanishes on } A\}.$$

Then I_F is a closed ideal in the algebra C(X) of continuous functions with compact open topology.

Theorem [Morris and Wulbert] The closed ideals in C(X) are in one-to-one correspondence with the closed subsets of X via

$F \to I_F$ if and only if X is completely regular.

<u>Theorem</u> [Ptak] Let F be a closed subset of a completely regular space X. Then on $C_F(X) = C(X)/I_F \subset C(F)$, the quotient-topology from C(X) and the relative compact open topology from C(F) coincide on $C_F(X)$.

V. Ptak has shown that there exists a completely regular space X such that

(1) C(X) is complete

(2) X is not normal.

Let A = C(X) where X is completely regular, non-normal space as given above by Ptak. Therefore A, by (1), is a pseudo-complete locally convex algebra since we know that completeness implies pseudo-completeness. As a consequence of (2), there exists a closed set F (by Tietze extension theorem [11]) such that not every f in C*(F) can be lifted to a function in C*(X), where C*(X) denote the bounded continuous functions on X. Then in view of the above consequence and Morris- Wulbert theor

$$C^*(X)/I_F = C^*_F(X) \text{ is a proper subset of } C^*(F).$$

The following simple argument shows that $C^*_F(X)$ is uniformly dense in C*(F).

Let $N(\varepsilon) = \{f \in C^*(F): |f| \leq \varepsilon \text{ on } F\}$ be a neighbourhood in the sup-topology. There exists $f \in C^*(X)$ such that $f(x) \neq 0$ for $x \notin F$ and $f(F) = 0$. Clearly $g = f + \varepsilon/2 \in N(\varepsilon) \cap C^*_F($ This shows that $C^*_F(X)$ is uniformly dense in C*(F). Since C*(F) is complete in the sup-topology and $C^*_F(X)$ is properly dense in

$C^*(F)$, $C^*_F(X)$ cannot be complete.

Now let B be a subset of $A' = C(X)/I_F = A/I_F$ such that

$$B = \{f \in C^*_F(X): |f(F)| \leq 1\}.$$

Ptak's theorem says that the topology of $C^*_F(X)$ is, in fact, the relative topology of $C^*(F)$. B is, then, closed, bounded, absolutely convex and idempotent. The span of B,

$$A'(B) = C^*(X)/I_F = C^*_F(X) \text{ with sup norm.}$$

In view of the above arguments, $A'(B)$ is not complete. Consequently A/I_F is not pseudo-complete.

Q.E.D.

Remark In view of the above example, one easily concludes that a continuous, open, homomorphic image of a BP*-algebra need not be a BP*-algebra. However these results do hold in case of MQ*-algebras (Proposition 7, 8 [17]).

2. Adjunction of a unit e (The algebra A with unit adjoined)

The non-unitary case can be reduced to the unitary one by a standard operation called the adjunction of a unit. We endow the set $A_e = C \times A$ with operations

$$(\lambda, x) + (\mu, y) = (\lambda+\mu,\ x+y)$$

$$\alpha(\lambda, x) = (\alpha\lambda,\ \alpha x)$$

$$(\lambda, x)(\mu, y) = (\lambda\mu,\ xy + \lambda y + \mu x)$$

which transform A_e into an algebra with a unit $e = (0,1)$. We identify A with the subalgebra of A_e composed of the elements (x,o), and we write $\lambda e + x$ instead of (λ,x). We topologize A_e as the product of A with the complex field C. If A is a locally convex algebra, so is A_e. By proposition 1.45 (ii), A_e is pseudo-complete if and only if A is pseudo-complete.

We have now the following proposition.

Proposition 2.9. A_e is bounded pseudo-complete if and only if A is bounded pseudo-complete.

Proof: Suppose A_e is bounded pseudo-complete. Then by Proposition 1.45 (b), A is a pseudo-complete locally convex-algebra. A is identified with a subalgebra of A_e consisting of elements $\lambda e + x$ with $\lambda = o$. Since every element of A_e is bounded, it follows that every element of A is bounded (See proof of Proposition 2.7). Hence A is bounded pseudo-complete.

Conversely assume A is bounded pseudo-complete. Then, for any $x \varepsilon A$, there exists an $\alpha_1 > o$ such that the set

$$\{(\alpha_1 x)^n : n = 1,2,3,\ldots\}$$

is bounded in A. We observe, first of all, that if $\alpha_2 < \alpha_1$, then the set

$$\{(\alpha_2 x)^n : n = 1,2,3,\ldots\}$$

is also bounded. For, if V is an absolutely convex neighbourhood of the origin in A, there exists $\lambda > o$ such that the set

$$\{(\alpha_1 x)^n\}_{n \geq 1} \subset \lambda V \quad \text{and}$$

$$\alpha_2^n x^n = \frac{\alpha_2^n}{\alpha_1^n} (\alpha_1^n x^n) \subset \frac{\alpha_2^n}{\alpha_1^n} (\lambda V) \subset \lambda V,$$ since $\alpha_2/\alpha_1 < 1$ and V is balanced. This shows that the set

$$\{(\alpha_2 x)^n : n = 1,2,3,...\}$$

is bounded in A.

Moreover since every element of the normed algebra C is bounded, the set

$$\{(\frac{1}{|\lambda|} \lambda)^n : n = 1,2,3,...\}$$

is bounded in C.

Let $\alpha = \min (\frac{1}{|\lambda|}, \alpha_1)$. Then by the above observations the sets

$$\{(\alpha\lambda)^n : n = 1,2,...\} \text{ and } \{(\alpha x)^n : n = 1,2,3,...\}$$

are bounded sets in C and A respectively. Therefore for every absolutely convex neighbourhood U of the origin in C and an absolutely convex neighbourhood V of the origin in A, there exist λ_1 and λ_2 both positive reals such that

$$\{\alpha^n (o, x^n) : n = 1,2,3,...\} \subset \lambda_1 U \times \lambda_2 V .$$

Choose $\lambda_1 < \lambda_2$, then $\lambda_1 U \subset \lambda_2 V$. Consequently

$$\{\alpha^n(0,x)^n : n = 1,2,\ldots\} \subset \lambda_2 (U\times V)$$

which implies that, in the product topology, the set

$$\{\alpha^n(o,x)^n : n = 1,2,3,\ldots\}$$

is bounded in A_e. (Recall that multiplication in A_e shows that $(o,x)^n = (o,x^n)$). Hence $(o,x) \in A_e$, is a bounded element of A_e. Similarly one can show that $(\lambda,o) \in A_e$, is also a bounded element of A_e. We know, by Corollary 1.4, that bounded elements of a commutative pseudo-complete, locally convex algebra form a subalgebra. Since A_e is commutative pseudo-complete, we conclude that

$$(\lambda,x) = (o,x) + (\lambda,o)$$

is a bounded element of A_e. Hence A_e is a bounded pseudo-complete algebra.

Q.E.D.

CHAPTER III

CONTINUITY OF POSITIVE FUNCTIONALS

One of the important properties of an MQ*-algebra is that all it's positive functionals are continuous. The same is not true for BP*-algebras. We give an example to show that not every positive functional on a BP*-algebra is continuous. We discuss continuity of positive functionals on a BP*-algebra in it's inductive limit topology and under certain conditions on these algebras.

The following lemma is from [17].

<u>Lemma</u> Let A be a locally m-convex algebra which is sequentially complete. If $x \in A$ is such that $\lim_{n\to\infty} (p_i(x^n))^{1/n} < 1$ for each i, then there is a $y \in A$ such that $2y - y^2 = x$. If, in addition, A has a continuous involution and $x = x^*$, then there is a $y \in A$ with $2y - y^2 = x$ and $y = y^*$.

<u>Proof</u>: Let $y_n = -\sum_{k=1}^{n} \binom{1/2}{k}(-x)^k$. Since $|\binom{1/2}{k}| \leq 1$ and $\lim_{n\to\infty} [p_i(x^n)]^{1/n} < 1$, the series $\sum_{i=1}^{\infty} \binom{1/2}{k}(-1)^k p_i(x^k)^{1/k}$ is absolutely convergent and it follows that $\{y_n\}$ is a Cauchy sequence. Since A is sequentially complete, there is $y \in A$ such that $y_n \to y$.

For $|t| \leq 1$, the series $f(t) = -\sum_{k=1}^{\infty} \binom{1/2}{k}(-t)^k$ has the property:

$$2f(t) - [f(t)]^2 = t$$

and it follows that $2y - y^2 = x$ and this proves the first statement. If $x = x^*$, then clearly $y_n = y_n^*$ and the continuity of the involution implies that $y = y^*$.

Corollary. Let A be a locally m-convex algebra. If A is either complete or a sequentially complete Q-algebra, then for $x \in A$, $r(x) < 1$ implies there is $y \in A$ with $2y - y^2 = x$. If, in addition, A is a *-algebra and x is hermitian, then y can be chosen to be hermitian.

Proof: This follows from the above lemma since the spectral radius $r(x) = \sup_i \lim_{n\to\infty} [p_i(x^n)]^{1/n}$.

Q.E.D.

1. Admissibility of a Positive Functional on a BP*-algebra.

The following proposition is important.

Proposition 3.1. Every positive functional on a BP*-algebra is admissible.

Proof: Let $x \in A$. Then $x^*x \in A = A_o = \cup\{A(B): B \in \mathcal{B}^*\}$ which implies that $x^*x \in A(B)$ for some B. A(B) being a Banach *-algebra with continuous involution, $r_{A(B)}(x^*x)$ is finite. Given $\varepsilon > o$, then

$$r_{A(B)}\,(x^*x/[r_{A(B)}(x^*x) + \varepsilon]) < 1 .$$

Hence by the above lemma (recall that every Banach *-algebra is an MQ*-algebra), there is a $y \in A(B)$ such that y is hermitian and

$$2y - y^2 = x^*x/[r_{A(B)}(x^*x) + \varepsilon] .$$

For $z \in A$, put $v = z - yz$, then

$$v^*v = z^*z - z^* (x^*x/[r_{A(B)}(x^*x) + \varepsilon])z .$$

Since f is a positive functional, $f(v^*v) \geq o$ and hence from the last equation

$$f(z^*x^*xz) \leq f(z^*z)(r_{A(B)}(x^*x) + \varepsilon) .$$

ε being arbitrary, we easily see that

$$\sup \{\frac{f(z^*x^*xz)}{f(z^*z)} : z \in A, f(z^*z) \neq o\}$$

$$\leq \inf_B \{r_{A(B)}(x^*x): x^*x \in A(B)\}$$

$$= r_A(x^*x),$$ by Proposition 1.54 or Page 413 [1].

We know from Proposition 1.49 that $x \in A_o$ if and only if $r(x)$ is finite. Consequently $r_A(x^*x)$ is finite and this proves that f is admissible.

Q.E.D.

<u>Corollary 3.2</u>. Let A be a BP*-algebra. Then for a positive functional f on A,

$$f(z^*x^*xz) \leq f(z^*z)r(x^*x).$$

Proof: Immediate from the last inequality in the proof of the above proposition.

Corollary 3.3. Let A be a BP*-algebra. Then a positive functional f on A is representable if and only if it is extendable.

Proof: Assume f is representable. Then by Proposition 1.93(a), it follows that f is extendable.

Conversely let f be extendable. Let A_e be the algebra A with unit adjoined. We know by Proposition 2.10 that A_e is a BP*-algebra with involution given by $(x,\alpha)^* = (x^*,\bar{\alpha})$. Since f is extendable, there is a positive functional $\tilde{f}$ on A_e (recall Theorem 1.81) such that

$$\tilde{f}|A = f .$$

By Proposition 3.1, $\tilde{f}$ is admissible. Hence by Theorem 1.91, there exists an *-representation T on a Hilbert space H such that T has a cyclic vector η such that

$$\tilde{f}(x) = \langle T_x\eta,\eta\rangle .$$

Proceeding as in Theorem 1 [17], let $N = \{\eta \in H: T_x\eta = o$ for all $x \in A\}$ and write $\zeta = \zeta_1 + \zeta_2$ where $\zeta_1 \in N$ and $\zeta_2 \in N^{\perp}$ (the orthogonal complement of N). It follows that $f(x) = \langle T_x\zeta_2,\zeta_2\rangle$. Let H_o be the closure of $\{T_x\zeta_2: x \in A\}$, then H_o is cyclic with cyclic vector ζ_2. Consequently f is representable.

Q.E.D.

Corollary 3.4. Let f be a positive functional on a BP*-algebra A with identity e. Then

(1) $|f(x)| \leq r(x)f(e)$ for each $x \in A_h$ (the set of all hermitian elements of A).

(2) $|f(x)|^2 \leq r(x^*x)f(e)^2$.

Proof: Let $x \in A_h$. Then as in the proof of Proposition 3.1,

$$r_{A(B)}(x/[r_{A(B)}(x) + \varepsilon]) < 1,$$

and there exists y in A(B) such that y is hermitian and

$$2y - y^2 = x/[r_{A(B)}(x) + \varepsilon].$$

Let $z = e - y$, then $z = z^*$ and $z^*z = z^2 = e - x/[r_{A(B)}(x) + \varepsilon]$. f being a positive functional, $f(z^*z) \geq o$, and ε being arbitrary, we obtain as in the proof of Proposition 3.1, that

$$f(x) \leq r_A(x)f(e) .$$

Since $r_A(-x) = r(x)$, we apply the above to $-x$ and obtain

$$-f(x) \leq f(-x) \leq r_A(x)f(e) .$$

Consequently

$$|f(x)| \leq r(x)f(e) .$$

Furthermore since x^*x is hermitian, we know from (c) of Theorem 1.80 [Cauchy-Schwartz inequality] that $|f(x)|^2 \leq f(x^*x)f(e)$ for any positive functional f. x^*x being hermitian, we obtain from (1) that

$$|f(x)|^2 \leq f(x^*x)f(e) \leq r(x^*x)(f(e))^2$$

which proves (2).

Q.E.D.

2. Continuity of a Positive Functional

The following example shows that not every positive functional on a BP*-algebra is continuous.

Example As shown in example 2.6 of Chapter II, $A = L^{\infty}[0,1]$, is a BP*-algebra. As in [10] we define a positive functional F on A by

$$F(f) = \int_{o}^{1} \frac{f(t)dt}{t(\log t-1)^{2}} \quad (f \varepsilon A) .$$

For p = 1,2,3,... , $N \geq 1$, we define

$$f_{p,N}(t) = \min \{(4t)^{-1/2p}, N\}.$$

Then $||f_{p,N}||_p \leq [\int_{o}^{1} (4t)^{-1/2}dt]^{1/p} = 1$. (*)

But $F(f_{p,N}) \geq \int_{o}^{\eta} \frac{Ndt}{t(\log t-1)^{2}}$, where $\eta = \frac{N^{-2p}}{4}$.

$$= N/(2p\log N + \log 4 + 1) \quad (**)$$

which tends to infinity as $N \to \infty$, for each $p \geq 1$. Now if F were continuous, there would exist a positive integer p and a constant k > o such that

$$|F(f)| \leq k \; ||f||_{p}, \quad \text{for all } f \varepsilon A$$

but in view of (*) and (**), this gives a contradiction on taking $f = f_{p,N}$ and letting $N \to \infty$. Hence the positive functional F defined on the BP*-algebra $L^{\infty}[0,1]$ cannot be continuous.

Before we prove some results on the continuity of a positive functional, we shall require the following proposition.

<u>Proposition</u> Let A be a topological *-algebra and E a topological vector space. Then a linear mapping $f: A \to E$ is continuous if and only if $f|A_h$ is continuous.

<u>Proof</u>: In general continuity implies continuity of restriction in the relative topology for if U is open in E, then

$$(f|A_h)^{-1}(U) = A_h \cap f^{-1}(U)$$

is open in A_h.

Conversely suppose $f|A_h = \phi$ is continuous. Then by definition of restriction $\phi(x) = f(x)$ for $x \in A_h$. Consider a net $\{z_\alpha\}$ in A such that $z_\alpha \to z$ in A. We wish to show that $f(z_\alpha) \to f(z)$. One knows that if x is an element of a *-algebra, then x can be written uniquely in the form $x = h_1 + ih_2$, where h_1 and h_2 belong to A_h. Hence let $z_\alpha = x_\alpha + iy_\alpha$ and $z = x + iy$, x, y, x_α, y_α are in A_h and $x_\alpha \to x$ and $y_\alpha \to y$. Linearity and continuity of ϕ imply that

$$f(z_\alpha) = f(x_\alpha) + if(y_\alpha) = \phi(x_\alpha) + i\phi(y_\alpha) \to \phi(x) + i\phi(y)$$

which is equal to $f(x) + if(y) = f(z)$. Hence continuity of f is established.

Q.E.D.

Theorem 3.5. Let A be a BP*-algebra with identity e. If A is a Q-algebra, then every positive functional f on A is continuous.

Proof: By (1) of Corollary 3.4, if f(e) = o, f is identically zero and hence automatically continuous. The positivity of f implies that f(e) > o. Consider the set

$$W = \{x \in A: \quad r(x) \leq \varepsilon/f(e)\}$$

where ε is a given positive number. A being a Q-algebra, W is a neighbourhood of the origin. In view of Corollary 3.4 $W \cap A_h$ is a neighbourhood of o in A_h and

$$f(W \cap A_h) \leq \varepsilon$$

so that $f|A_h$ is continuous. Hence f is continuous by the above proposition.

Q.E.D.

Corollary 3.6. Let A be as in the above theorem without ident If f is an extendable positive functional on A, then f is continuous.

Proof: Since A is BP*, it follows by Proposition 2.9, A_e is a BP*-algebra. The Q-property is also preserved in A_e. Extendability of f ensures the existance of a positive functional $\tilde{f}$ on A (Theorem 1.81) such that $\tilde{f}|A = f$. The above theorem implies $\tilde{f}$ is continuous. Consequently f is continuous on A.

Q.E.D.

The following observations will be useful in some of the subsequent results.

Remark 3.7. It may well happen that there are no non-trivial multiplicative linear functionals on a BP*-algebra. Consider for example the algebra $L^{\infty}[0,1]$ in Example 2.6, where it was shown that it is a BP*-algebra. One knows from Page 104 [3], that there are no non-trivial multiplicative linear functionals on the algebra $L^{\omega}[0,1]$. For, if f is a non-trivial multiplicative linear functional on $A = L^{\omega}[0,1]$, then f is also multiplicative restricted to the algebra C, say, of all continuous complex-valued functions on [0,1]. There is thus some point t_o in [0,1] such that $f(x) = x(t_o)$ for every x in C (f cannot annihilate C since C contains the identity of A). But there is certainly some continuous $\mathbb{C}$*-valued function z in $L^{\omega}[0,1]$ such that $z(t_o) = \infty$, $z(t) \geq o$ for all t. But then there is some x_n in $C(n=1,2,\ldots)$ such that $x_n(t_o) = 1$ and $nx_n(t) \leq z(t)$ for all t. Thus $f(z) \geq n$ $(n=1,2,\ldots)$ which is a contradiction.

Since by Theorem 1.66, $L^{\infty}[0,1]$ is dense in $L^{\omega}[0,1]$, there are no non-trivial multiplicative linear functionals on $L^{\infty}[0,1]$ because, otherwise, by continuity of multiplication one can extend it to a non-trivial multiplicative linear functional on $L^{\omega}[0,1]$ which is impossible by virtue of the above arguments.

Remark 3.8. We know by Definition 12.1 [19] that a topological algebra is called <u>functionally continuous</u> if every multiplicative linear functional on A is continuous.

A BP*-algebra or, more generally a pseudo-complete locally convex-algebra may not be functionally continuous.

As in Example 2.3, $C(\Omega_o)$ is a pseudo-complete locally convex-algebra. Ω_o satisfies the assumptions of the following proposition of E.A. Michael (Page 52 [19]) and hence $C(\Omega_o)$ is not functionally continuous.

<u>Proposition</u> Let T be a countably compact, non-compact, completely regular space, which is either locally compact or first countable. Let A = C(T) with the compact open topology. Then

(a) A is complete, commutative, locally m-convex algebra

(b) A is not functionally continuous

(c) Every multiplicative linear functional on A is bounded (i.e. sends bounded sets into bounded sets).

<u>Proof</u>: See Proposition 12.2 [19].

The purpose of the above discussions is to show that M(A), the set of all non-zero continuous multiplicative linear functionals on a BP*-algebra A may be empty. This forces us to impose some conditions on the algebra A, so that M(A) is non-empty. Since there is a one-to-one correspondence between non-zero continuous multiplicative linear functionals and closed maximal ideals, <u>semi-simplicity</u> will sometimes be assumed on the algebra A. However, in the following results on continuity of positive functionals, we assume M(A) is non-empty.

<u>Proposition 3.9</u>. On every symmetric, BP*-algebra A, there exists a family $\{p_f : f \in M(A)\}$ of sub-multiplicative semi-norms satisfying

$$p_f(x^*x) \leq [p_f(x)]^2 \quad .$$

Proof: By Proposition 1.73, $f(x^*) = \overline{f(x)}$ for a multiplicative linear functional f on a symmetric *-algebra A. Consequently every multiplicative linear functional on a symmetric *-algebra is a positive functional. Since every member of M(A) is admissible, we have by Proposition 3.1.

$$p_f(x) = \sup \{[\frac{f(z^*x^*xz)}{f(z^*z)}]^{1/2} : z \in A,\ f(z^*z) \neq o\}.$$

$$< \infty \ .$$

Clearly $p_f(x) \geq o$ and $p_f(\lambda x) = |\lambda|\ p_f(x)$.
For sub-additivity we observe that

$$f[z^*(x^* + y^*)(x+y)z]$$

$$= f(z^*x^*xz) + f(z^*y^*yz) + f(z^*x^*yz) + f(z^*y^*xz)$$

but by Cauchy-Schwartz inequality,

$$|f(z^*x^*yz)|^2 \leq f(z^*x^*xz)\ f(z^*y^*yz)$$

and

$$|f(z^*y^*xz)|^2 \leq f(z^*y^*yz)\ f(z^*x^*xz) \quad .$$

Consequently

$$f[z^*(x+y)^*(x+y)z] \leq [f(z^*x^*xz)^{1/2} + f(z^*y^*yz)^{1/2}]^2 \ .$$

Hence $p_f(x+y) \leq p_f(x) + p_f(y)$.
Since f is multiplicative, we shall show that

(1) $p_f(xy) \leq p_f(x) \cdot p_f(y)$

and

(2) $p_f(x^*x) \leq [p_f(x)]^2$

To show (1), we see that

$$p_f(xy) = \sup \left[\left\{\frac{f(z^*(xy)^*(xy)z)}{f(z^*z)}\right\}^{1/2} : z \in A,\ f(z^*z) \neq o\right]$$

$$= \sup \left[\left\{\frac{f(z^*y^*(x^*x)yz)}{f(z^*z)}\right\}^{1/2} : z \in A,\ f(z^*z) \neq o\right]$$

Set $yz = u$, then $f(u^*u) = f(z^*y^*yz) = f(z^*z) \cdot f(y^*y)$. We then obtain

$$p_f(xy) = \sup \left\{\frac{f(u^*x^*xu)}{f(u^*u)} \cdot \frac{f(z^*y^*yz)}{f(z^*z)}\right\}^{1/2}$$

where $f(u^*u) \neq o$, $f(z^*z) \neq o$. As z runs over A, u runs over $yA \subset A$. Consequently,

$$p_f(xy) \leq p_f(x) \cdot p_f(y).$$

Now to see (2), we observe that $p_f(x) = p_f(x^*)$, hence by (1), we see that

$$p_f(x^*x) \leq p_f(x^*) \cdot p_f(x)$$

$$\leq [p_f(x)]^2 .$$

Q.E.D.

<u>Corollary 3.10</u>. $p_f(x) \leq r(x)$ for all $f \in M(A)$.

<u>Proof</u>: By Corollary 3.2,

$$p_f(x) \leq [r(x^*x)]^{1/2} = r(x)$$

Corollary 3.11. Let A be a BP*-algebra which is also a Q-algebra. For $f \in M(A)$, let

$$B' = \{x \in A: p_f(x) \leq 1\}.$$

Then B' is absolutely convex, idempotent and closed in the topology generated by the family $\{p_f: f \in M(A)\}$ and B' is a neighbourhood of the origin in A. Hence the topology generated by this family of sub-multiplicative semi-norms is a locally m-convex topology coarser than the initial topology of A.

Proof: The first part is easy to verify. For the second part we know by Proposition 1.77 that

$$S(A) = \{x \in A: r(x) \leq 1\}$$

is a neighbourhood of the origin. By Corollary 3.10, $B' \supset S(A)$. Hence B' is a neighbourhood of the origin and hence the topology generated by $\{p_f: f \in M(A)\}$ is coarser than the initial topology. In view of the fact that a topological algebra is locally m-convex if and only if the topology is generated by a family of sub-multiplicative semi-norms, we finally conclude that the family $\{p_f: f \in M(A)\}$ generates a locally m-convex topology.

Q.E.D.

We prove some more results on the continuity of a positive functional on a BP*-algebra. We shall assume as before M(A) is non-empty.

We need the following lemma which is analogous to Theorem 1.62.

Lemma 3.12. Let A be a bounded pseudo-complete locally convex-algebra. Then the spectrum

$$\sigma(x) = \{\phi(x): \phi \in M(A)\}$$

where M(A) denotes the set of all non-zero continuous multiplicative linear functionals on A.

Proof: For $x \in A = A_o$, $\infty \notin \sigma(x)$. For finite λ, $\lambda \in \sigma(x)$ if and only if $\lambda e - x$ has no inverse in A i.e. if and only if $\phi(\lambda e - x) = o$ for some $\phi \in M(A)$. (Recall M(A) is assumed non-empty) i.e. if and only if $\phi(x) = \lambda$ for some $\phi \in M(A)$ and this proves the lemma.

Q.E.D.

Theorem 3.13. Every positive functional f on a barrelled BP*-algebra A with identity e is continuous.

Proof: We assume $f(e) > o$ because, for $f(e) = o$, the result is trivial.

Let $\varepsilon > o$ be given. Consider,

$$W = \{x \in A: r(x) \leq \varepsilon/f(e) \leq 1\}$$

By the above lemma, the spectral radius

$$r(x) = \sup \{|\phi(x)|: \phi \in M(A)\} \qquad (*)$$

Then

$$W = \bigcap_{\phi \in M(A)} \{x \in A: |\phi(x)| \leq 1\}.$$

$$= [M(A)]^o, \text{ polar of } M(A).$$

Clearly W is weakly closed in A and hence closed in A. W is an absolutely convex subset of A (Proposition 1.23). Since $A = A_o$ and by Proposition 1.49, $x \in A_o$ if and only if $r(x) < \infty$, it follows by (*) that $\sup_{\phi \in M(A)} |\phi(x)|$ is finite. Consequently M(A) is a weakly bounded subset of A' and hence its polar $[M(A)]^o = W$ is an absorbent subset of A. Using the barrelled property of A, it follows that W is a neighbourhood of the origin in A. As in theorem 3.5, one can show that f is continuous.

Q.E.D.

<u>Corollary 3.14</u>. Every positive functional f on a sequentially complete quasi-barrelled BP*-algebra A with identity is continuous.

<u>Proof</u>: Since by Proposition 1.14, a sequentially complete quasi-barrelled space is barrelled, it follows f is continuous.

<u>Corollary 3.15</u>. Every positive functional f on a sequentially complete bornological BP*-algebra A with identity is continous.

<u>Proof</u>: Continuity of f follows by Corollary 3.14, because every sequentially complete bornological space is barrelled (Proposition 1.15).

<u>Corollary 3.16</u>. Every positive functional f on a Fréchet BP*-algebra A with identity is continuous.

<u>Proof</u>: Use the fact that every Frechet space is barrelled (Proposition 1.17). Continuity of f is then immediate from the theorem.

Corollary 3.17. Every positive functional f on a Montel BP*-algebra A with identity is continuous.

Proof: Since by Definition (1.11), a Montel space is barrelled, the continuity of f follows by the theorem.

Q.E.D.

3. Inductive Limit Topology on BP*-algebras

Notation We shall denote the members of $\mathcal{B}^*$ by $\{B_\alpha : \alpha \in \Delta\}$ and we shall write A_α for $A(B_\alpha)$.

Proposition 3.18. Let A be a BP*-algebra. Then A also carries the inductive limit topology which is finer than the original topology.

Proof: Since each A_α is a subalgebra of A and the inclusion maps f_α are the restrictions to A_α of the identity mappings of the algebra A, and $\cup f_\alpha(A_\alpha)$ spans A, it follows by Proposition 1.7, that A carries the inductive limit topology. In view of Remark 1.9 an absolutely convex set U is a neighbourhood of 0 in A in the inductive limit topology on A if and only if $U \cap A_\alpha$ is a neighbourhood of 0 in E_α for each α.

Let U be a neighbourhood of the origin in the original topology T. We know that the norm topology on A_α is finer than the induced topology on each A_α. Consequently $U \cap A_\alpha$ is a norm-neighbourhood of 0 in A_α. By what we have shown in the above paragraph, U is a neighbourhood of 0 in A in the inductive limit topology i. This proves that T is coarser than i.

Q.E.D.

The following example will show that the inductive limit topology is not, in general, equal to the original topology on a BP*-algebra.

Example 3.19. Let $A = C(\Omega_o)$ which was shown to be a BP*-algebra in Example 2.3 in the compact open topology k. We shall show that the inductive limit topology i on A is, in fact, the sup-topology s on A, which is different from k.

The inductive limit topology on A is the largest topology on A making each inclusion map i_B

$$A(B) \xrightarrow{i_B} A$$

continuous. Let $\mathcal{B}^*$ be the corresponding collection for A. We claim that $\mathcal{B}^*$ has the greatest member B_o

$$B_o = \{f \in A : |f(x)| \leq 1 \text{ for all } x \in \Omega_o\}$$

p_B and p_{B_o} will denote respectively the norms on A(B) and $A(B_o)$. B_o is clearly absolutely convex, idempotent, closed and bounded. We wish to show that $B \subset B_o$ for all $B \in \mathcal{B}^*$. If not, there exist $f \in B$ such that $f \notin B_o$ which means there exists $x_o \in \Omega_o$ such that $|f(x_o)| > 1$. Since $B^2 \subset B$, $f^n \in B$ for all n. Now $N(\{x_o\}, 1)$ is a neighbourhood of O in A. Since B is bounded, there exists $\lambda > o$ such that

$$B \subset \lambda N(\{x_o\}, 1) = N(\{x_o\}, \lambda) .$$

Choose a positive integer n_o such that $|f(x_o)|^{n_o} > \lambda$.

Now

$$f^{n_o} \in B \subset N(\{x_o\}, \lambda) \Rightarrow |f^{n_o}(x_o)| \leq \lambda.$$

which is a contradiction. Hence $B \subset B_o$ for all $B \in \mathcal{B}^*$.
We also observe that

$$A = A(B_o) \text{ because } f/\sup_{x\in\Omega_o} |f(x)| \in B_o ,$$

Moreover $p_{B_o} \leq p_B$ and the inclusion map

$$(A(B),p_B) \longrightarrow (A(B_o),p_{B_o})$$

is continuous. We have therefore shown that $C(\Omega_o)$, with the sup-topology, is the inductive limit of $\{(A(B),p_B): B \in \mathcal{B}^*\}$. Hence on $C(\Omega_o)$, $i = s \neq k$.

Q.E.D.

Proposition 1.10 which says that "a linear mapping f of an inductive limit of a family $\{E_\alpha^*\}$ of locally convex spaces is continuous if and only if $f \circ f_\alpha$ is continuous" will be instrumental in proving the following results on the continuity of multiplicative and positive linear functionals on BP*-algebras and continuity of *-representations of these algebras by operators on a Hilbert space, in the inductive limit topology.

<u>Theorem 3.20</u>. Every multiplicative linear functional on a bounded pseudo-complete locally convex algebra A, endowed with the inductive limit topology, is continuous.

Proof: Let f be a multiplicative linear functional on A. f_α will be the inclusion map of A_α into A and C is the complex field.

We have:

$$A_\alpha \xrightarrow{f_\alpha} A \xrightarrow{f} C .$$

Then clearly $f \circ f_\alpha$ is a multiplicative linear functional on the Banach algebra A_α. Since every multiplicative linear functional on a Banach algebra is continuous, it follows that, in particular, $f \circ f_\alpha$ is continuous. Hence appealing to Proposition 1.10, f is continuous on A.

Q.E.D.

Remark 3.21. In view of Proposition 1.60, the space M(A), consisting of non-zero multiplicative linear functionals on A, is weakly compact, and the space of continuous members of it is clearly relatively compact. However, if we restrict to the inductive limit topology on A, then each member of M(A) is continuous by Theorem 3.20. Consequently the space of continuous multiplicative (non-zero) linear functionals on A with identity, also denoted by M(A), is compact.

Theorem 3.22. Every positive functional on a BP*-algebra A is continuous in it's inductive limit topology.

Proof: Let f be a positive functional on A. Since each f_α is a *-homomorphism, it is easy to see that $f \circ f_\alpha$ is a positive functional on the Banach *-algebra A_α. Since every positive functional on a Banach *-algebra is continuous, $f \circ f_\alpha$ is continuous on A_α and hence by Proposition 1.10, f is continuous.

Q.E.D.

<u>Theorem 3.23</u>. Let A be a BP*-algebra. Then every *-representation of A by operators on a Hilbert space H, is continuous in the inductive limit topology.

<u>Proof</u>: Let T be a *-representation of A.

Diagramatically:

$$A_\alpha \xrightarrow{f_\alpha} A \xrightarrow{T} B(H)$$

where $T: x \to T_x$ and B(H) denotes the Banach algebra of bounded operators on H supplied with the norm topology and conjugation as involution. Each f_α being *-homomorphism, we have

$$(T \circ f_\alpha)_{x^*} = T[f_\alpha(x^*)] = T[f_\alpha(x)]^*$$

$$= T_{z^*} \text{ where } z = f_\alpha(x)$$

$$= (T_z)^* = [(T \circ f_\alpha)_x]^*$$

which shows $T \circ f_\alpha$ is a *-representation of A_α on H. Since any *-representation of a Banach *-algebra on a Hilbert space is continuous, it follows that each $T \circ f_\alpha$ is continuous. Consequently as above, T is continuous in the inductive limit topology.

Q.E.D.

CHAPTER IV

NORMALIZED POSITIVE FUNCTIONALS ON BP*-ALGEBRAS

Throughout in this chapter, $P(A)$ will denote the set of all those positive functionals on a BP*-algebra A which take values 1 at the identity of A. A positive functional satisfying this condition is said to be normalized. $P(A)$ is shown to be a weakly compact convex set. We characterize the extreme points of $P(A)$. They are identified as positive continuous multiplicative linear functionals on A. In fact the indecomposable positive functionals are exactly the extreme points of $P(A)$.

We recall that the projective limit of an indexed family of topological spaces $\{X_\alpha : \alpha \in \Delta\}$ with respect to the connecting maps $\pi_{\alpha\beta}$ is the subspace Q of the product space $P = \Pi\{X_\alpha : \alpha \in \Delta\}$, such that

$$Q = \{(x_\alpha) \in P: \ \pi_{\alpha\beta}(x_\beta) = x_\alpha(\alpha \leq \beta)\}.$$

and Q is non-empty compact Hausdorff provided each X_α is non-empty compact Hausdorff. We denote the members of $\mathcal{B}$ by $\{B_\alpha : \alpha \in \Delta\}$ where Δ is the index set. For $\alpha, \beta \in \Delta$, we write $\alpha \leq \beta$ to mean $B_\alpha \subseteq B_\beta$. By the Theorem 1.46, $\mathcal{B}$ is outer-directed by inclusion and hence Δ is directed by the partial ordering.

We write A_α for $A(B_\alpha)$ and $P_\alpha = P(A_\alpha)$ will denote the set of all positive functionals on A_α with values 1 at the identity, endowed with the relative weak *-topology $\sigma(P_\alpha, A_\alpha)$. It is also clear that $B_\alpha \subset B_\beta$ implies $A_\alpha \subset A_\beta$ and each A_α is a Banach *-algebra. $P^*(A)$ is the set of all positive functionals on A, and $P^*_\alpha = P^*(A_\alpha)$ will denote the set of all positive functionals on A_α.

The main theorem is as follows:

<u>Theorem 4.1</u>. $P^*(A)$ is homeomorphic to the projective limit of $\{P^*_\alpha : \rho_{\alpha\beta}\}$, where $\rho_{\alpha\beta}$ is defined below.

<u>Proof</u>: $i_{\alpha\beta}: A_\alpha \to A_\beta$ will denote the inclusion maps. Then $(A_\alpha, i_{\alpha\beta})$ will be the direct system of sets, because clearly

(1) $i_{\alpha\alpha}$ is the identity map on A_α for all α, and

(2) $i_{\beta\gamma} \circ i_{\alpha\beta} = i_{\alpha\gamma}$, $\alpha \leq \beta \leq \gamma$.

If ϕ_β is a positive functional on A_β as in the diagram:

$$A_\alpha \xrightarrow{\ i_{\alpha\beta}\ } A_\beta \xrightarrow{\ \phi_\beta\ } C,$$

then $\phi_\beta \circ i_{\alpha\beta}$ is a positive functional on A_α. That is, $\phi_\beta \circ i_{\alpha\beta} \in P^*_\alpha$ when $\phi_\beta \in P^*_\beta$.

We therefore define:

$$\rho_{\alpha\beta}: \ P^*_\beta \longrightarrow P^*_\alpha$$

by

$$\rho_{\alpha\beta}(\phi_\beta) = \phi_\beta \circ i_{\alpha\beta}$$ which is the restriction of ϕ_β to A_α. $\rho_{\alpha\beta}$ is clearly continuous in the weak *-topology. It is not difficult to see that $(P^*_\alpha, \rho_{\alpha\beta})$ is an inverse system of sets because:

(1) $\rho_{\alpha\alpha}$ is an identity on P_α^* for all α and

(2) for $\alpha \leq \beta \leq \gamma$, $\rho_{\alpha\beta} \circ \rho_{\beta\gamma} = \rho_{\alpha\gamma}$ since

$$\rho_{\alpha\beta} \circ \rho_{\beta\gamma}(\phi_\gamma) = \rho_{\alpha\beta}[\rho_{\beta\gamma}(\phi_\gamma)] = \rho_{\alpha\beta}[\phi_\gamma \circ i_{\beta\gamma}]$$

$$= (\phi_\gamma \circ i_{\beta\gamma}) \circ i_{\alpha\beta} = \phi_\gamma \circ (i_{\beta\gamma} \circ i_{\alpha\beta})$$

$$= \phi_\gamma \circ i_{\alpha\gamma} = \rho_{\alpha\gamma}\ (\phi_\gamma)\ .$$

Then

$$Q = \{(\phi_\gamma)\ \varepsilon\ \Pi P_\alpha^*:\ \ \rho_{\alpha\beta}(\phi_\beta) = \phi_\alpha \text{ for } \alpha \leq \beta\},$$

is the projective limit of the spaces $\{P_\alpha^*: \sigma(P_\alpha, A_\alpha)\}$ with respect to the connecting maps $\rho_{\alpha\beta}$.

We observe that if $\phi\ \varepsilon\ P^*(A)$ then as in the diagrame

$$A_\alpha \xrightarrow{\ i_\alpha\ } A \xrightarrow{\ \phi\ } C$$

$\phi \circ i_\alpha\ \varepsilon\ P_\alpha^*$. We then define

$$\rho_\alpha : P^*(A) \longrightarrow P_\alpha^*$$

by

$$\rho_\alpha(\phi) = \phi \circ i_\alpha .$$

It is simple to see that ρ_α is continuous in the weak *-topology. For, let

$$N_1 = N_1\ (\phi \circ i_\alpha, a, \varepsilon)$$

be a weak-neighbourhood of $\rho_\alpha(\phi)$ in P_α^*, where $a\ \varepsilon\ A_\alpha$ and $\phi\ \varepsilon\ P^*(A)$.

In other words

$$N_1 = \{\psi \in P_\alpha^*: |\psi(a) - \phi \circ i_\alpha(a)| < \varepsilon\}.$$

We claim that the weak-neighbourhood $N_2 = N_2(\phi, i_\alpha(a), \varepsilon)$ in $P^*(A)$ is such that

$$\rho_\alpha(N_2) \subset N_1, \quad \text{where}$$

$$N_2 = \{\psi \in P^*(A): |\psi(i_\alpha(a)) - \phi(i_\alpha(a))| < \varepsilon\}.$$

Let $\phi \in \rho_\alpha(N_2)$. Then $\phi = \rho_\alpha(\psi) = \psi \circ i_\alpha$, where $\psi \in N_2$, which implies that

$$|\psi \circ i_\alpha(a) - \phi \circ i_\alpha(a)| < \varepsilon.$$

That is

$$| \quad \phi\,(a) - \phi \circ i_\alpha(a)| < \varepsilon.$$

Consequently $\phi \in N_1$. Hence ρ_α is continuous.

We now define

$$\rho: P^*(A) \longrightarrow \Pi P_\alpha^*$$

by

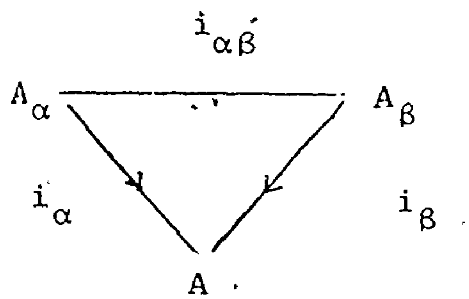
$$\rho(\phi) = (\phi \circ i_\alpha)_\alpha.$$

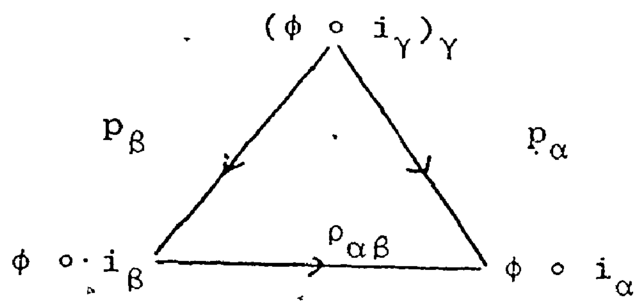
It is shown below that $\rho(\phi)$, in fact, belongs to the projective limit Q; for if $\alpha < \beta$, then (see the diagrams below. p_α's are the projection maps).

$$\rho_{\alpha\beta}(\phi \circ i_\beta) = (\phi \circ i_\beta) \circ i_{\alpha\beta}$$

$$= \phi \circ (i_\beta \circ i_{\alpha\beta})$$

$$= \phi \circ i_\alpha .$$





It remains to show that ρ is a homeomorphism.

(1) ρ is one-to-one.

Suppose $\rho(\phi) = \rho(\psi)$.

Then $(\phi \circ i_\gamma)_\gamma = (\psi \circ i_\gamma)_\gamma$.

which implies that $\phi \circ i_\gamma = \psi \circ i_\gamma$ for all γ.

Hence $\phi = \psi$ on A and this shows ρ is one-to-one.

(2) ρ is onto.

Let $(\phi_\gamma) \varepsilon Q$. Since $A = \cup A_\alpha$, for x in A, x is in A_α for some α.

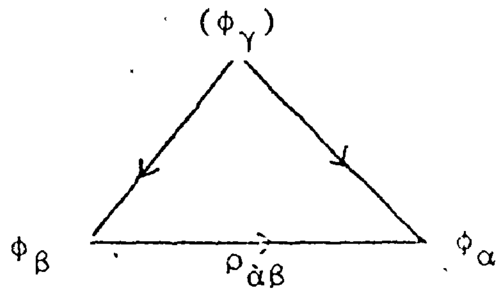
Define $\phi(x) = \phi_\alpha(x)$ for all α.

ϕ is well-defined; for if $x \varepsilon A_\beta$ and

$$\phi(x) = \phi_\beta(x).$$

Then since $(\phi_\gamma)_\gamma \varepsilon Q$, we have

$$\rho_{\alpha\beta}(\phi_\beta) = \phi_\beta \circ i_{\alpha\beta} = \phi_\alpha .$$



Hence $\phi_\beta(x) = \phi_\alpha(x)$. That is ϕ is well defined and $\phi \in P^*(A)$.
Furthermore

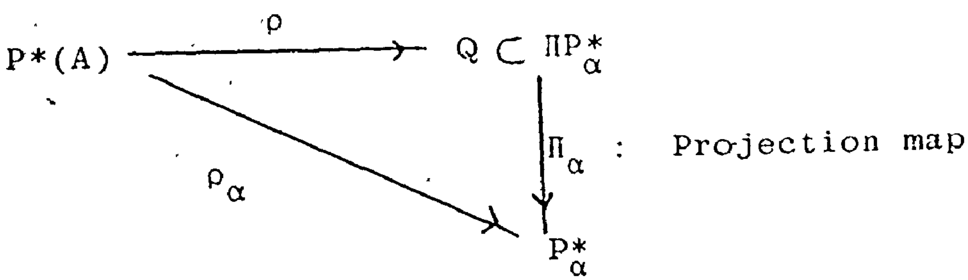
$$\rho(\phi) = (\phi \circ i_\gamma)_\gamma = (\phi_\gamma \circ i_\gamma)_\gamma = (\phi_\gamma)_\gamma .$$

This proves that ϕ is onto.

(3) ρ is continuous.

We observe that

$$\Pi_\alpha \circ \rho = \rho_\alpha \quad \text{for all } \alpha .$$



Since ρ_α is continuous in the weak *-topology (shown before) on $P^*(A)$, it follows ρ is continuous in the weak *-topology on $P^*(A)$.

(4) ρ is open.

Let $\phi \in P^*(A)$. A basic neighbourhood U of ϕ is

$$U = N(\phi, \varepsilon, a_1, a_2, \ldots, a_n)$$

$$= \{\psi \in P^*(A): |\phi(a_i) - \psi(a_i)| < \varepsilon\},\ a_1, a_2, \ldots a_n \in A.$$

Choose β such that $a_1, a_2, \ldots, a_n$, are in A_β. Then

$$U = \{\psi \in P^*(A): |\rho_\beta(\phi)(a_i) - \rho_\beta(\psi)(a_i)| < \varepsilon\}. \qquad \text{(i)}$$

Let $\rho_\beta(\phi) = \phi \circ i_\beta \in P^*_\beta$. Then a basic neighbourhood of $\rho_\beta(\phi) = \phi \circ i_\beta$ in P^*_β is

$$V = N_\beta(\phi \circ i_\beta, \varepsilon, a_1, a_2, \ldots, a_n)$$

$$= \{\psi_\beta \in P^*_\beta: |\rho_\beta(\phi)(a_i) - \psi_\beta(a_i)| < \varepsilon\}.$$

We know that $\Pi_\beta^{-1}(V)$ is a neighbourhood of $\rho(\phi)$ in Q where

$$\Pi_\beta^{-1}(V) = \{(\phi_\gamma)_\gamma \in Q: \phi_\beta \in V\}.$$

We claim that

$$Q \cap \Pi_\beta^{-1}(V) \subset \rho(U).$$

Let $(\psi_\gamma)_\gamma$ belong to $Q \cap \Pi_\beta^{-1}(V)$.
Then $(\psi_\gamma)_\gamma \in \Pi_\beta^{-1}(V)$ implies $\psi_\beta \in V$ and this means that

$$|\phi \circ i_\beta(a_i) - \psi_\beta(a_i)| < \varepsilon . \qquad \text{(ii)}$$

Since ρ is onto Q and $(\psi_\gamma)_\gamma$ is in Q,

$$(\psi_\gamma)_\gamma = \rho(\psi), \quad \text{for some } \psi \in P^*(A)$$
$$= (\psi \circ i_\gamma)_\gamma.$$

Hence $\psi_\beta = \psi \circ i_\beta = \rho_\beta(\psi) \in P^*_\beta$.

Therefore by (ii) $|\rho_\beta(\phi)(a_i) - \rho_\beta(\psi)(a_i)| < \varepsilon$.
In view of (i), it therefore follows that $\psi \in U$. Consequently $(\psi_\gamma)_\gamma = \rho(\psi) \in \rho(U)$, and this proves that ρ is open in Q, or ρ^{-1} is continuous. We have finally established that ρ is a homeomorphism.

Q.E.D.

We specialize the above theorem to the following important corollaries.

<u>Corollary 4.2</u>. P(A) is homeomorphic to the projective limit of $(P_\alpha; \rho_{\alpha\beta})$, where P(A) and P_α consist respectively of those members of P*(A) and P^*_α which are normalized, with their relativized weak *-topologies.

<u>Proof</u>: It follows immediately from the proof of the Theorem 4.1. Another way of looking at the proof is as follows: We know from (6) Page 18 [18] that a continuous one-to-one mapping of a compact space onto a Hausdorff space is a homeomorphism. The proof follows once we have shown that ρ^{-1} is one-to-one continuous map of the compact space Q (See Corollary 4.3 below) onto the Hausdorff space P(A), because then ρ^{-1} is open i.e. ρ is continuous and ρ is then a homeomorphism. (Note that we do not require the continuity of ρ_α).

Q.E.D.

<u>Corollary 4.3</u>. P(A) is a non-empty compact Hausdorff space.

<u>Proof</u>: By Theorem 1.84, each P_α is weakly compact and convex. Consequently the projective limit Q of compact Hausdorff

space, is necessarily compact Hausdorff. Since ρ is a homeomorphism, it follows that P(A) is a compact Hausdorff space.

Q.E.D.

P(A) is <u>convex</u> because if f, g ε P(A), then f(e) = 1 = g(e) and hence $\lambda f + (1-\lambda) g \varepsilon P(A)$. Taking together the above theorem and the corollaries, we have the following theorem which will play an important role in subsequent developments.

<u>Theorem 4.5</u>. P(A) is a weakly compact convex set.

<u>Remark 4.6</u>. P(A) has extreme points by Krein-Milman theorem.

1. Characterization of Extreme Points of P(A)

We now proceed to give a characterization of extreme points which do exist by virtue of the Remark 4.6.

<u>Proposition 4.7</u>. If for any f ε P(A), $f|A_\alpha$ $(=f \circ i_\alpha)$ is an extreme point of $P(A_\alpha)(=P_\alpha)$ for all α, then f is an extreme point of P(A).

<u>Proof</u>: If f ε P(A) and assume

$$f = \lambda f_1 + (1-\lambda)f_2, \text{ where } f_1, f_2 \in P(A).$$

Then it is easy to see that

$$f|A_\alpha = \lambda f_1|A_\alpha + (1-\lambda)f_2|A_\alpha \text{ for all } \alpha, \text{ where}$$

$f_1|A_\alpha$ and $f_2|A_\alpha$ belong to P_α for all α. Since $f|A_\alpha$ is an extreme point of P_α, it follows that

$$f|A_\alpha = f_1|A_\alpha = f_2|A_\alpha \text{ for all } \alpha.$$

Choose an arbitrary point $x \varepsilon A = UA_\alpha$, then $x \varepsilon A_\alpha$ for some α. Consequently $f = f_1 = f_2$ on A. This shows that f is an extreme point of P(A).

Q.E.D.

Proposition 4.8. The positive multiplicative linear functionals on A are extreme points of P(A).

Proof: Let f be a positive multiplicative linear functional on A. Then clearly $f|A_\alpha$ is a positive multiplicative linear functional on A_α for all α. Since by Theorem 1.86 positive multiplicative linear functionals on A_α are exactly the extreme points of P_α for each α, hence by the Proposition 4.7, f is an extreme point of P(A).

Q.E.D.

Definition $f \varepsilon P^*(A)$ is said to be indecomposable if every functional $f_1 \varepsilon P^*(A)$ dominated by f (i.e. $\lambda f - f_1$ is a positive functional for some positive λ) is a multiple of f.

We have the following proposition.

Proposition 4.9. If $f|A_\alpha$ is indecomposable for all α, then f is indecomposable.

Proof: Assume f' is a positive functional dominated by f. Then for some positive number λ, $\lambda f - f'$ is a positive functional on A. Consequently $\lambda f|A_\alpha - f'|A_\alpha$ is a positive functional on A_α for all α. Since $f|A_\alpha$ is indecomposable, it follows that

$$f'|A_\alpha = \mu f|A_\alpha \quad \text{for all } \alpha, \text{ where}$$

μ is a scalar.

Hence f' is a multiple of f, proving, thereby, f' is indecomposable.

Q.E.D.

The following remark may be useful later.

Remark If for any linear functional f on A, $f|A_\alpha$ is a positive functional on A_α for all α, then f is a positive functional on A.

Proof: We know by definition of the BP*-algebra A, that $A = \cup A_\alpha$. Let $x \in A$, then $x \in A_{\alpha_o}$ for some α_o. Then

$$f(x^*x) = f|A_{\alpha_o}(x^*x) \geq 0.$$

Q.E.D.

2. K-BP*-Algebras

Definition A BP*-algebra will be called a K-BP*-algebra if every positive functional on A_α can be extended to a continuous positive functional on A. [This property holds for closed *-subalgebras of symmetric Banach *-algebras. See Theorem 1.89].

For such algebras we establish a sort of converse for Proposition 4.7.

<u>Theorem 4.10</u>. Let A be a K-BP*-algebra with the inductive limit topology, and suppose each A_α is dense in A. Then f is an extreme point of P(A) if and only if $f \circ i_\alpha = f|A_\alpha$ is an extreme point of P_α for all α.

<u>Proof</u>: One part of the theorem is already proved in Proposition 4.7. Assume f is an extreme point of P(A). Consider the equation

$$f|A_\alpha = \lambda f_1 + (1-\lambda)f_2, \ o < \lambda < 1, \ f_1, \ f_2 \ \varepsilon \ P(A_\alpha).$$

Extension property says that there exist continuous positive functionals f' and f" corresponding to f_1 and f_2 respectively such that

$$f'|A_\alpha = f_1 \text{ and } f''|A_\alpha = f_2 \ .$$

Consequently

$$f|A_\alpha = \lambda f'|A_\alpha + (1-\lambda)f''|A_\alpha = [\lambda f' + (1-\lambda f'')] \circ i_\alpha .$$

We know that f ε P(A) is always continuous in the inductive limit topology. Moreover we recall that two continuous functions equal on a dense subspace coincide on the whole. We finally obtain

$$f = \lambda f' + (1-\lambda)f'' \quad f', \ f'' \ \varepsilon \ P(A).$$

Since f is an extreme point of P(A), f = f' = f" and this implies that

$$f|A_\alpha = f_1 = f_2 .$$

This proves that $f|A_\alpha$ is an extreme point of $P(A_\alpha)$.

Q.E.D.

The following theorem identifies the extreme points of $P(A)$. It will be recalled in the next chapter.

<u>Theorem 4.11</u>. Let A be a BP^*-algebra. Then $f \varepsilon P(A)$ is indecomposable if and only if f is an extreme point of the set $P(A)$.

<u>Proof</u>: Assume f is indecomposable. Consider the equation

$$f = \lambda f_1 + (1-\lambda)f_2, \ o < \lambda < 1, \ f_1, \ f_2 \ \varepsilon \ P(A).$$

Then $f - \lambda f_1 = (1-\lambda)f_2$ is a positive functional and f_1 is dominated by f. The assumption implies that $f_1 = \mu f$. Since $f(e) = 1 = f_1(e)$, it follows that $\mu = 1$. Hence $f = f_1$. We can similarly show that $f = f_2$. Hence f is an extreme point of $P(A)$.

Conversely let $f \varepsilon P(A)$ be an extreme point of $P(A)$. Let $f_1 \varepsilon P(A)$ be dominated by f. In other words $f' = f - \lambda f_1$ is a positive functional on A. We wish to show that f_1 is a multiple of f. Now let $f'(e) = o$. Then $f - \lambda f_1 = o$ since $f'(xe) = f'(x) f(e) = o$. That is $f = \lambda f_1$ but $f(e) = 1 = f_1(e)$, which forces $\lambda = 1$. Hence $f_1 = f$ which shows that f is indecomposable under the condition $f'(e) = o$. If $f'(e) > o$, then let $f_2 = f'/\mu$ when $\mu = f'(e)$ and therefore $f_2(e) = 1$. From the equation

$f' = f - \lambda f_1$ we get $\mu f_2 = f - \lambda f_1$. That is, $f = \lambda f_1 + \mu f_2$ with $\lambda, \mu > o$ and $\lambda + \mu = 1$, because $1 = f(e) = f_1(e) = f_2(e)$. Since f is an extreme of P(A), $f = f_1 = f_2$. Thus every functional in P(A) dominated by f coincides with f. Hence f is indecomposable

Q.E.D.

Corollary 4.12. Let A be a K-BP*-algebra with the inductive limit topology. Let each A_α be dense in A. If f is an indecomposable element of P(A), then $f|A_\alpha$ is indecomposable in $P(A_\alpha)$ for any α.

Proof: Let f be indecomposable, then by the above theorem, f is an extreme point of P(A). In view of Theorem 4.10 $f|A_\alpha$ is an extreme point of $P(A_\alpha)$ for any α. By the corresponding theorem for Banach *-algebra (Theorem 1.85), it follows that $f|A_\alpha$ is an indecomposable element of $P(A_\alpha)$. Q.E.D.

The following theorem is a sort of converse for the Proposition 4.8 for K-BP*-algebras.

Theorem 4.13. Let A be a K-BP*-algebra with the inductive limit topology and let each A_α be dense in A. Then the positive multiplicative linear functionals on A are exactly the extreme points of P(A).

Proof: We have already shown in Proposition 4.8 that if f is a positive multiplicative linear functional on A, then f is an extreme point of P(A).

Now assume f is an extreme point of P(A). Then $f|A_\alpha$ is an extreme point of $P(A_\alpha)$ for any α by Theorem 4.10. Theorem 1.86 for Banach *-algebras implies that $f|A_\alpha$ is a multiplicative member of $P(A_\alpha)$ for any α. Consequently f is a positive multiplicative member of P(A).

Q.E.D.

<u>Corollary 4.14</u>. Let A and A_α satisfy the conditions of the above theorem. If A is symmetric, then the kernel of the Gelfand map G: $x \to \hat{x}$ defined by $\hat{x}(\phi) = \phi(x)$ for $\phi \in M(A)$, is the set

$$R = \{x \in A: \quad f(x^*x) = o \ \forall f \in P(A)\}.$$

<u>Proof</u>: Let K_G denote the kernel of the Gelfand map. That is

$$K_G = \{x \in A: \quad G(x) = o\}.$$

We have to show that $R = K_G$.

Let $x \in R$, then $f(x^*x) = o \ \forall f \in P(A)$. Symmetry implies that every multiplicative linear functional on A is a positive functional, hence, in particular, we have

$$\phi(x^*x) = o \ \forall \phi \in M(A)$$

and this yields $\phi(x) = o \ \forall \phi \in M(A)$. Thus $G(x)[\phi] = \phi(x) = o$ for all $\phi \in M(A)$ imply $G(x) = o$. Hence $x \in K_G$.

On the other hand, let $x \in K_G$. Then $\phi(x) = o$ for all $\phi \in M(A)$ and therefore $\phi(x^*x) = o \ \forall \phi \in M(A)$. We may regard x^*x as a continuous linear functional on the conjugate space A^* in the weak *-topology. Each ϕ being multiplicative and positive linear functional on A, is exactly the extreme point of P(A) by

Theorem 4.13. Therefore x^*x vanishes identically on the extreme points of $P(A)$. That is

$$\phi(x^*x) = o \quad \forall \phi \in Ext(P(A))$$

but, by Krein Milman theorem, $P(A)$ is the closed convex hull of it's extreme points. Now x^*x being a continuous linear functional on A^*, we claim that x^*x vanishes identically on $P(A)$; for let $D = Ext(P(A))$. Continuity says that

$x^*x(\overline{convD}) \subset x^*x(\overline{convD})$ where convD denotes the convex hull of D. Since x^*x is a linear functional and $x^*xD = o$, it follows $x^*x(convD) = o$. Consequently $x^*x(\overline{convD}) = x^*x(P(A)) = o$. In other words

$$\phi(x^*x) = o \quad \forall \phi \in P(A).$$

Thus $x \in R$. Hence

$$R = \{x \in A: f(x^*x) = o \ \forall f \in P(A)\}.$$

Q.E.D.

CHAPTER V

REPRESENTATIONS OF BP*-ALGEBRAS

This chapter deals with an investigation of representations of BP*-algebras in the algebra of bounded operators on a Hilbert space H. Necessary and sufficient conditions are obtained for a cyclic representation to be irreducible. We give various characterizations of the *-radical of these algebras. We end up with a theorem which represents a symmetric, semi-simple BP*-algebra as a certain subalgebra of a function algebra.

The following proposition is well-known.

Proposition 5.1. If f is a positive functional on a *-algebra A, then

$$N_f = \{x \in A: f(x^*x) = o\}$$

is a linear subspace of A and the quotient space $K_f = A/N_f$ can be made into a pre-Hilbert space by defining the inner product

$$(x + N_f, y + N_f) = f(y^*x), \quad (x, y \in A);$$

A representation (π_f, K_f) of A is then defined by

$$[\pi_f(x)](y + N_f) = xy + N_f \quad (x, y \in A).$$

Proof: f being a positive functional on A, $x \in N_f$

implies $f(\overline{\lambda}x^*\lambda x) = |\lambda|^2 f(x^*x) = o$, which shows that $\lambda x \in N_f$. Furthermore if $x, y \in N_f$, then

$$f[(x+y)^*(x+y)] = f(x^*x) + f(y^*y) + f(x^*y) + f(y^*x)$$

$$\leq f(x^*x) + f(y^*y) + f(x^*x)^{1/2} f(y^*y)^{1/2} + f(y^*y)^{1/2} f(x^*x)^{1/2}$$ by Cauchy -Schwartz inequality

$$= [f(x^*x)^{1/2} + f(y^*y)^{1/2}]^2 = o.$$

Hence $x + y \in N_f$. Consequently N_f is a linear subspace of A. Using the fact that $f(x^*) = \overline{f(x)}$ and linearity of f, one can verify that $(x + N_f, y + N_f)$ is an inner product.

Let H denote the completion of K_f. $L(K_f)$ will denote the vector space of linear transformations on K_f.

The following theorem is important.

<u>Theorem 5.2</u>. Let f be a positive linear functional on a BP*-algebra A. Then

(1) Each T_x is continuous on K_f and hence extendable to an operator in B(H).

(2) If A is a Q-algebra, then the representation $x \to T_x$ of A in B(H) is continuous and has cyclic vector $\zeta_e = e + N_f$ where e is the identity of the algebra.

(3) For each $x \in A$, $f(x) = (T_x\zeta_e, \zeta_e)$.

<u>Proof</u>: By an earlier Corollary 3.2 on the admissibility of positive functionals, we have for the positive functional f

$$f(z^*x^*xz) \leq f(z^*z)r(x^*x).$$

By the definition of the inner product on H

$$(xz + N_f,\ xz + N_f) = f(z^*x^*xz) \qquad \text{and}$$

$$(z + N_f,\ z + N_f) = f(z^*z).$$

Since $(x,x) = ||x||^2$ and $\zeta_z = z + N_f$, by the application of Corollary 3.2, we have

$(xz + N_f,\ xz + N_f) \leq (z + N_f,\ z + N_f)r(x^*x)$ which, in view of the definition of the representation, yields

$$||T_x\zeta_z||^2 \leq ||\zeta_z||^2\ r(x^*x)$$

or

$$||T_x\zeta_z|| \leq ||\zeta_z||\ r(x^*x)^{1/2}.$$

Since every element of A is bounded, $r(x^*x)$ is finite (Proposition 1.49), it follows by the last inequality that T_x is a bounded operator on K_f and has a continuous extension to H which is the completion of K_f. This proves (1).

To prove (2), we observe from

$$||T_x\zeta_z||^2 \leq ||\zeta_z||^2\ r(x^*x)$$

that

$$||T_x||^2 \leq r(x^*x) \qquad \text{(i)}$$

If $W = \{x : r(x) \leq 1\}$, then as in Theorem 3.5 we obtain, where A_h is the hermitian part of A,

$$T(W \cap A_h) \subset V = \{T_x : ||T_x|| \leq 1\} \quad \text{by (i).}$$

The Q-property assertains that W is a neighbourhood of the origin. Consequently $T|A_h$ is continuous. Finally as in Theorem 3.5, T is continuous.

Moreover, since $T_x\zeta_e = x + N_f$ and H is the completion of K_f, it follows that $\{T_x\zeta_e : x \in A\}$ is dense in H and hence ζ_e is the cyclic vector for the representation, thus proving (2).

Furthermore, by the definition of the inner-product,

$$(T_x\zeta_e, \zeta_e) = (x + N_f, e + N_f) = f(x)$$

which establishes (3).

Q.E.D.

1. Irreducibility of Cyclic Representations

The following theorem is well-known for *-algebra with identity (Theorem 1.94) which is employed to prove our next theorem.

Theorem (Page 265 [21]) A cyclic representation $x \to T_x$ of a *-algebra A with identity in $B(H_0)$ is irreducible (No non-trivial subspaces of H are invariant with respect to each T_x) if and only if the positive functional f defined by

$$f(x) = (T_x\zeta, \zeta)$$

is indecomposable for each cyclic vector ζ in H.

Theorem 5.3. Let A be a BP*-algebra with identity. Then a cyclic representation is irreducible if and only if, for each cyclic vector ζ in H, the corresponding positive functional is an extreme point of P(A).

Proof: Assume that the cyclic representation is irreducible. Then by the above theorem the positive functional defined by

$$f(x) = (T_x\zeta, \zeta)$$

is indecomposable for each cyclic vector in H. Consequently by Theorem 4.11, f is an extreme point of P(A).

Conversely assume that the positive functional f corresponding to the cyclic vector ζ is an extreme point of P(A). Then by the same Theorem 4.11, f is indecomposable. Hence the theorem quoted above shows that the cyclic representation $x \to T_x$ is irreducible.

Q.E.D.

2. Complete Family of Representations

Definition A family of representations of a *-algebra A is complete if, for each non-zero vector x_o in A, there exists a representation in the family such that $T_{x_o} \neq o$.

Theorem 5.4. Let A be a BP*-algebra. Then the family of all irreducible *-representations $x \to T_x$ of A in B(H) is complete if and only if the set R(A)

$$R(A) = \{x \in A: f(x^*x) = o \ \ \forall f \in P(A)\} = \{o\}.$$

Proof: Assume the given family is complete. We wish to show that R(A) = {o}. If not, for each $y \in R(A)$, such that $y \neq o$, we have $f(y^*y) = o$ for all $f \in P(A)$. By Theorem 5.2,

$$(T_{y^*y}\zeta_e, \zeta_e) = o$$

T being a *-representation, $(T_y\zeta_e, T_y\zeta_e) = o$ which implies that $||T_y\zeta_e|| = o$. Hence $T_y\zeta_e = o$, ζ_e being a unit cyclic vector. Consequently $T_y = o$ which contradicts that the family is complete. Therefore R(A) = {o}.

Conversely suppose that R(A) = {o} which means that, for any positive functional f in P(A), $f(z^*z) = o$ implies that $z = o$. In other words $z \neq o$ implies that $f(z^*z) \neq o$ for all $f \in P(A)$. In particular for an extreme point f of P(A), defined by

$$f(x) = (T_x\zeta_e, \zeta_e)$$

we have

$$f(z^*z) = (T_{z^*z}\zeta_e, \zeta_e) \neq o$$

and this shows that $||T_z|| \neq o$, for $z \neq o$. The representation $x \to T_x$ being cyclic (ζ_e being a cyclic unit vector), it finally follows by the above theorem that T is irreducible. Consequently the family is complete.

Q.E.D.

Corollary 5.5. Let A be a BP*-algebra. Then the following statements are equivalent

(1) There is an $f \in P(A)$ such that $f(x) \neq o$.

(2) There is $g \in P(A)$ such that $g(x^*x) > o$ where $g(z) = f(x) + if(y)$, $x, y \in A_h$, $z = x + iy \in A$.

(3) There is a *-representation T of A such that $T_x \neq o$.

(4) There is an irreducible *-representation S of A such that $S_x \neq o$.

Proof: $f(e) = 1$ for $f \in P(A)$. By Schwartz inequality (c) Theorem 1.80, we have

$$|f(x)| \leq f(x^*x).$$

Consequently (1) implies (2).

(2) implies (4): As in the proof of the above theorem

$$g(x^*x) = (S_{x^*x}\zeta_e, \zeta_e) \neq o.$$

Consequently $||S_x|| \neq o$. Hence $S_x \neq o$ and S is irreducible.

(4) implies (3): (4) equivalently says that the family of *-representations is complete. Hence, by the above theorem, $R(A) = \{o\}$ which means $x \neq o$ implies $f(x^*x) > o$ and as before there is an *-representation $x \to T_x$ such that $T_x \neq o$.

Finally assume (3) holds. Since $T_x \neq o$, there is unit vector in the Hilbert space (Theorem 5, [17]) such that $||\zeta|| = 1$ and $(T_x\zeta, \zeta) \neq o$.

Define

$$f(x) = (T_x\zeta, \zeta).$$

Then $f(x^*x) = ||T_x\zeta||^2 > o$ so that f is a positive functional. Moreover

$$f(e) = (T_{e^*e}\zeta, \zeta) = ||T_e\zeta||^2 = 1.$$

Consequently $f \in P(A)$ such that $f(x^*x) > o$. Hence (1) is established.

Q.E.D.

3. *-Radical

Definition Let A be a *-algebra. The *-radical R*(A) of A is defined as the intersection of the kernels of all irreducible *-representations of A.

If $R^*(A) = \{o\}$, then A is called *-semi-simple.

Theorem 5.6. Let A be a BP*-algebra. Then

$$R^*(A) = \bigcap\{\text{Ker}T: \ T \text{ is irreducible *-representation of } A\}$$

$$= \{x \in A: \ f(x^*x) = o, \ f \in P(A)\} = R_1$$

$$= \{x \in A: \ f(x^*x) = o, \ f \in \text{ext}(P(A))\} = R_2.$$

Proof: Let $x \in R^*(A)$. Then $T_x = o$ for any irreducible *-representation T of A. Let $f \in P(A)$. Then by an earlier theorem

$$f(x^*x) = (T_{x^*x}\zeta_e, \zeta_e) = (T_x\zeta_e , T_x\zeta_e) = ||T_x\zeta_e||^2$$
$$= o .$$

Hence $x \in R_1$. Consequently $R^*(A) \subset R_1$.

By the above arguments and appealing to Theorem 5.3 it is not difficult to show that $R^*(A) \subset R_2$ and $R_2 \subset R^*(A)$. That is $R_2 = R^*(A)$.

To show that $R_1 \subsetneq R_2$, let $x \in R_1$. Then $f(x^*x) = o$ for all $f \in P(A)$. In particular $f(x^*x) = o$ for all $f \in ext(P(A))$. Hence $R_1 \subset R_2$.

We have therefore shown that

$$R^*(A) \subset R_1 \subset R_2 \text{ and } R_2 = R^*(A),$$

and this establishes the equality of the sets.

Q.E.D.

Corollary 5.7. Let A be a BP*-algebra. Then A is *-semi-simple if and only if the set

$$\{x \in A: \ f(x^*x) = o \ \forall f \in P(A)\} = \{o\}.$$

Proof: If A is *-semi-simple, then by definition $R^*(A) = \{o\}$. Hence by the above theorem

$$\{x \in A: \ f(x^*x) = o \ \forall f \in P(A)\} = \{o\}.$$

Conversely if the given set consists of $\{o\}$ only, it follows by the above theorem that $R^*(A) = \{o\}$. Hence A is *-semi-simple.

Remark. It is well-known that $R^*(A)$ for any *-algebra, is a two sided *-ideal. Unlike MQ*-algebras, if A is a BP*-algebra, then $A/R^*(A)$ is not necessarily a BP*-algebra and this follows by a counter-example constructed in Proposition 2.8.

The following theorem analogous to a result in [8] provides another characterization of the *-radical of a barrelled BP*-algebra.

Theorem 5.8. Let A be a barrelled BP*-algebra with the inductive limit topology, then

(1) $E \subset P(A)$ is equi-continuous if and only if

$$\sup_{f \varepsilon E} f(x^*x) < \infty \quad (x \varepsilon A).$$

(2) If $\mathfrak{F}$ denotes the collection of all equi-continuous subsets of P(A), then for $E \varepsilon \mathfrak{F}$,

$$|x|_E = [\sup \{f(x^*x) : f \varepsilon E\}]^{1/2} \quad (x \varepsilon A)$$

is a semi-norm on A.

(3) $R^*(A) = \{x \varepsilon A : |x|_E = o \text{ for } E \varepsilon \mathfrak{F}\}$.

Proof: (1) Assume E is equi-continuous. Since by the Corollary 1.29, in the dual of a barrelled space, equi-continuous sets and $\sigma(E', E)$ - bounded sets are same, it follows that

$$\sup_{f \varepsilon E} f(x^*x) < \infty.$$

Conversely assume $\sup_{f \varepsilon E} f(x^*x) < \infty \; (x \varepsilon A)$. Then since by (Theorem 3, Page 69 [24]) or Theorem 1.28, any pointwise bounded set of continuous linear mappings of a barrelled space into any locally convex space is equi-continuous, it therefore follows that E is equi-continuous.

(2) Easily follows by Cauchy-Schwartz inequality.

To prove (3), we know from Theorem 5.6, that

$$R^*(A) = \{x \in A: f(x^*x) = o \ \forall f \in P(A)\}.$$

Then it is easy to see that $f(x^*x) = o$ if and only if $|x|_E = o$ for each E in $\mathcal{F}$.

Consequently

$$R^*(A) = \{x \in A: |x|_E = o \text{ for each } E \in \mathcal{F}\}.$$

Q.E.D.

We conclude with the following results.

<u>Lemma 5.9</u>. Let A be a symmetric, semi-simple BP*-algebra. Then the Gelfand representation $x \to \hat{x}$ is a *-isomorphism of A onto a self-adjoint, separating subalgebra $\hat{A}$ of C(M(A)), where $\hat{A}$ is the image of A under $x \to \hat{x}$.

<u>Proof</u>: A being pseudo-complete, we know from Proposition 1.60 that M(A) is a non-empty compact Hausdorff space. Then with every $x \in A$, we associate a complex-valued function $\hat{x}$ on M(A) by

$$\hat{x}(\phi) = \phi(x) \text{ where } \phi \in M(A).$$

Each $\hat{x}$ is a continuous function on M(A) in the weak *-topology. Semi-simplicity of A implies that the radical of A

$$R(A) = \{a \in A: \phi(a) = o \ \forall \phi \in M(A)\} \text{ (Proposition 1.65)}$$
$$= \{o\}.$$

This means that $\hat{x} = o$ implies $x = o$. Hence the Gelfand map is a one-to-one homomorphism. In view of Proposition 1.74, $\hat{A}$ is self-adjoint. Clearly A is separating. Proposition 1.75 guarantees that $x \to \hat{x}$ is a *-homomorphism. Ontoness of $x \to \hat{x}$ is obvious. Hence the Gelfand representation is *-isomorphism of A onto $\hat{A}$, and the lemma is proved.

Q.E.D.

4. Realization of a BP*-Algebra as a Function Algebra

In the following M(A) is the set of non-zero continuous multiplicative linear functionals on A.

Theorem 5.10. Let A be a semi-simple, symmetric BP*-algebra with identity. If τ_0 denotes the topology of uniform convergence on equi-continuous subsets of M(A), then the Gelfand map $x \xrightarrow{G} \hat{x}$ is a continuous *-isomorphism of A onto a dense, self-adjoint, separating subalgebra $\hat{A}$ of $(C(M(A)), \tau_0)$.

Proof: The above lemma ascertains that the Gelfand map G is a *-isomorphism and $\hat{A}$ is a self-adjoint, separating subalgebra.

Appealing to Generalized Weirstrass Theorem [11] or Proposition 1.76, $\hat{A}$ is dense in C(M(A)) in the compact open topology. The compact open topology is finer than τ_0 (because closed equi-continuous subsets are compact and the topology of uniform convergence on members of a larger collection is finer than that on members of a smaller collection and denseness is preserved under coarser topologies). Hence $\hat{A}$ is dense in C(M(A)) with τ

Regarding continuity of G, one knows that for an equi-continuous subset E of M(A), E° is a neighbourhood of zero in A, because by definition of equi-continuity of E,

$$E^\circ = \bigcap_{x' \in E} x'^{-1}(-\infty, 1]$$

is a neighbourhood of zero in A.

Consider a $\mathcal{T}_0$-neighbourhood of zero in C(M(A)) such that

$$N(E, \varepsilon) = \{f \in C(M(A)) : |f(E)| \leq \varepsilon\} ,$$

where E is an equi-continuous subset of M(A).

We claim that εE° is a neighbourhood of zero in A such that

$$G(\varepsilon E^\circ) \subset N(E, \varepsilon).$$

To show this, let $b \in \varepsilon E^\circ$. Then $b = \varepsilon a$, where $a \in E^\circ$ which implies that $|\phi(a)| \leq 1$ for all $\phi \in E$. The definition of the Gelfand map yields $|\hat{a}(\phi)| \leq 1$ for all $\phi \in E$. Then

$$|Gb(\phi)| = |\hat{b}(\phi)| = |\varepsilon a(\phi)| = |\phi(\varepsilon a)|$$

$$= \varepsilon|\phi(a)| = \varepsilon|\hat{a}(\phi)| \leq \varepsilon \quad \text{for all } \phi \in E.$$

Hence $Gb \in N(E, \varepsilon)$, and the continuity of the Gelfand map follows.

Q.E.D.

Corollary 5.11. Let A be as in the above theorem. If A is endowed with the inductive limit topology i, then i is normable

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to the relative topology T on G(A) induced from C(M(A)).

Proof: Since the Gelfand map G is onto G(A), A and G(A) will be identified. A being the inductive limit of barrelled spaces A_α, is barrelled. Since, in the dual of a barrelled space, weakly compact subsets and equi-continuous subsets are same, the τ_0-topology coincides with the compact open topology on C(M(A)). Since every multiplicative linear functional on A is i-continuous, M(A) is compact by Proposition 1.60. Consequently C(M(A)) is a Banach algebra. G(A) being normed, has the Mackey topology. Since $A' = A'_T$, and since the initial topology i and the norm topology T are Mackey topologies, it follows i is identical with T. Consequently i is normable.

Q.E.D.

Corollary 5.12. Let A satisfy the conditions of the above corollary. If A is Frechet, then A is full i.e.

$$A = C(M(A)).$$

Proof: As in the proof of the above corollary, G(A) is barrelled. Since A is Frechet and G is continuous, it follows by Proposition 1.33, that G is open. Hence by Proposition 1.34, G(A) is complete. G(A) being dense in C(M(A)), it finally follows that

$$A = C(M(A)) .$$

Q.E.D.

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