SOME FIBRED KNOTS WITH BI-ORDERABLE KNOT GROUPS

By

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The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a project entitled "**Some Fibred Knots with Bi-orderable Knot Groups**" by **Wangshan Lu** in partial fulfillment of the requirements for the degree of **Master of Science**.

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TO MY FATHER

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Abstract

This project aims to give an overview of knots, orderability of knot groups, and to construct knots for which the knot groups enjoy some nice properties.

To accomplish this, we first present some preliminary results concerning knots and knot groups. We then introduce the Alexander polynomial, and explain the idea of a *special* polynomial originally introduced by Linnell, Rhemtulla and Rolfsen. By investigating the conditions on a special polynomial, we classify all the special Alexander polynomial of fibred knots of degree less than 10. Finally we construct examples of fibred knots which have a special Alexander polynomial.

Chapter 1

Knots and Knot Groups

In this chapter, we give some preliminary results about knots and the orderability of knot groups.

1.1 Definition of Knots

Definition 1.1.1. A *link* L of m components is a subset of S^3, or of $\mathbb{R}^3$, that consists of m disjoint, piecewise linear, simple closed curves. A link of one component is a *knot*.

If not stated as "unoriented" it is often assumed that an orientation is associated with a link given by a specified direction along each component curve.

There is a natural way to draw a link in the plane using a projection $p : \mathbb{R}^3 \to \mathbb{R}^2$. Each line segment of L projects to a line segment in $\mathbb{R}^2$. The projections of two such segments intersect in at most one point which for disjoint segments is not an end point, and that no point belongs to the projections of three segments. The image of L in $\mathbb{R}^2$ together with "over and under" information at the crossings is called a *diagram* of L. The crossing number of a diagram is the number of crossings in the diagram. The crossing number of a link is the minimal number of crossings needed for a diagram of the link.

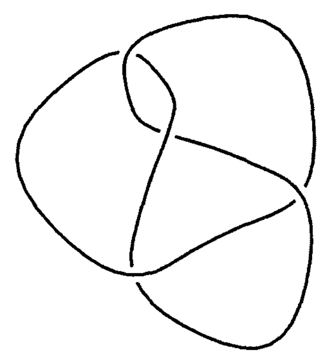


Figure 1.1: A Knot Diagram for Figure-8-Knot

Having defined knots, the natural question is: How do we distinguish knots, that is to say, how do we know whether two knots are the same or different?

It is natural to make the definition below, from a topological point of view.

Definition 1.1.2. Two links L_1 and L_2 in S^3 are *equivalent* if there is an orientation-preserving piecewise linear homeomorphism $h : S^3 \to S^3$ such that $h(L_1) = (L_2)$.

Intuitively, two knots are the same if one knot can be transformed to another without cutting and rejoining. This corresponds to the notion of "isotopic" in topology. To be explicit, it means that besides the conditions in the definition above, we want h to be isotopic to the identity map, that is, there exists $h_t : S^3 \to S^3$ for $t \in [0, 1]$ so that $h_0 = 1$ and $h_1 = h$ and $(x, t) \mapsto (h_t x, t)$ is a piecewise linear homeomorphism of $S^3 \times [0, 1]$ to itself. It can be shown that these two definitions are equivalent using a theorem of piecewise linear topology.

The simplest knot is the so called "unknot" which is defined as the knot bounding an embedded piecewise linear disc in S^3. We can construct the *sum* of two knots by tying one and then the other in the same piece of string. See figure 1.2. For oriented knots, they should be tied according to their orientations. It is easily seen that the sum is well-defined. We denote the sum of two knots K_1 and K_2 by $K_1 + K_2$.

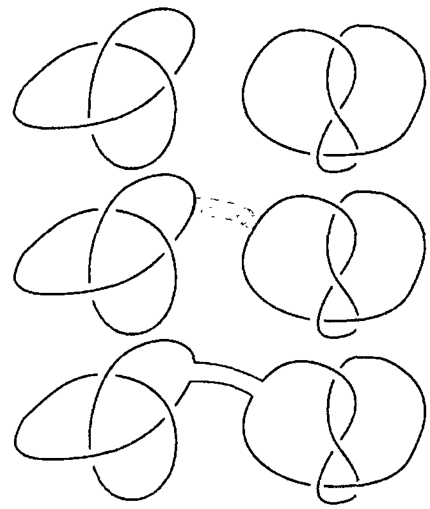


Figure 1.2: Sum of Two Knots

It follows from the definition of the sum that this addition is commutative and associative. It has been proved that no knot other that the unknot has an additive inverse. See [10] for details.

Giving the above two definitions, we can define the building block for a complex knot.

Definition 1.1.3. A knot K is a *prime* knot if it is not the unknot, and $K = K_1 + K_2$ implies that K_1 or K_2 is the unknot.

Schubert showed in [9] that every knot can be uniquely decomposed (up to the order in which the decomposition is performed) as a knot sum of prime knots.

So from now on, we will concentrate on prime knots, since many properties of the composite knot could be derived from the properties of each summand knot in a relatively easy way.

Now, we give the definition of meridian and longitude.

Definition 1.1.4. Let K be an oriented knot in S^3 with solid torus neighborhood N. A *meridian* μ of K is a non-separating simple closed curve in ∂N that bounds a

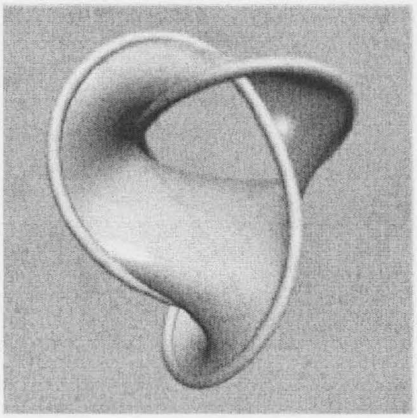


Figure 1.3: Seifert Surface of Figure-8 Knot drawn using Jack van Wijk's SeifertView

disc in N. A *longitude* λ of K is a simple closed curve in ∂N that is homologous to K in N and null-homologous in the exterior of K.

1.2 Seifert Surfaces

It is very interesting that every link in S^3 can be regarded as the boundary of a surface embedded in S^3. The surface will be used to construct the Alexander polynomial of a given knot.

Definition 1.2.1. A *Seifert surface* for an oriented link L in S^3 is a connected compact oriented surface contained in S^3 that has L as its oriented boundary.

Remark 1.2.1. While the condition "connected" and "compact" is reasonable for us to get a nice surface, "oriented" is a crucial condition for us to construct the infinite cyclic cover. This eliminates the possibility of a Mobius band being a Seifert surface of some link.

Theorem 1.2.1. *Any oriented link in S^3 has a Seifert surface.*

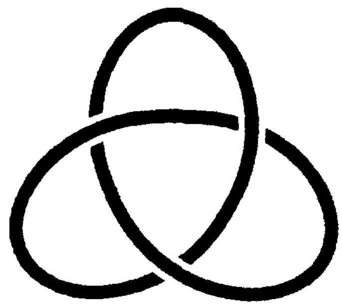


Figure 1.4: Trefoil Knot

1.3 Fibred Knots

Definition 1.3.1. A *fibred* knot or link L is one for which $S^3 - L$ fibres over S^1, with fibres open surfaces, each of whose closures has L as its boundary.

Many knots and links are fibred, for example, the unknot (See Fig 1.4.), trefoil knot, and figure-eight knot are fibred knots. The sum of two fibred knots is fibred.(See pp. 323-326 of [12].)

The commutator subgroup of the group of a fibred knot is finitely generated. In deed, the finite generation of the commutator subgroup characterizes fibred knots as follows.

Theorem 1.3.1. *(Stallings)The complement of a knot fibres locally trivially over S^1 with Seifert surfaces as fibres if the commutator subgroup G of the knot group is finitely generated.(See Theorem 5.1 in [11].)*

The Alexander polynomial has very good properties when the given knot is fibred, which we will illustrate in the coming chapter.

1.4 Knot Groups

Intuitively, we can see that different knots are likely to shape different complement spaces, that is, the complement of the knot. In [8], Gordon and J. Luecke proved that knots are determined by their complements.

Theorem 1.4.1. *If K_1 and K_2 are unoriented knots in S^3 and there is an orientation preserving homeomorphism between their complements, then K_1 and K_2 are equivalent (as unoriented knots).*

This is not necessarily true for links with more than one component.

Looking at the topology of the complement thus is a good way to study knots. We give the definition of a knot group.

Definition 1.4.1. *The group of a link L* in S^3 is defined to be $\pi_1(S^3 - L)$, the fundamental group of the complement of L.

The fundamental group of the complement determines the topology of complement for prime knots.

Theorem 1.4.2. *If K_1 and K_2 are prime knots in S^3 and $\pi_1(S^3-K_1)$ to $\pi_1(S^3-K_2)$ are isomorphic groups, then $S^3 - K_1$ and $S^3 - K_2$ are homeomorphic. (See Theorem 11.9 in [10].)*

The theorem above is not true for composite knots. The square knot, which is a knot sum of a trefoil knot and its mirror image, and the granny knot, which is a knot sum of two trefoil knots having the same orientation, have the same knot group but different knot complement.(See p. 62 in [12].) See Fig 1.5.

If the isomorphism between those two groups preserves the peripheral structure of the knots, we could conclude those knot are actually equivalent.

Theorem 1.4.3. *If there exists an isomorphism from $\pi_1(S^3 - K_1)$ to $\pi_1(S^3 - K_2)$ which sends $[\lambda_1]$ to $[\lambda_2]$ and $[\mu_1]$ to $[\mu_2]$, where λ_i is a longitude for K_i and μ_i is a meridian for K_i, then K_1 and K_2 are equivalent knots. (See Theorem 11.8 in [10].)*

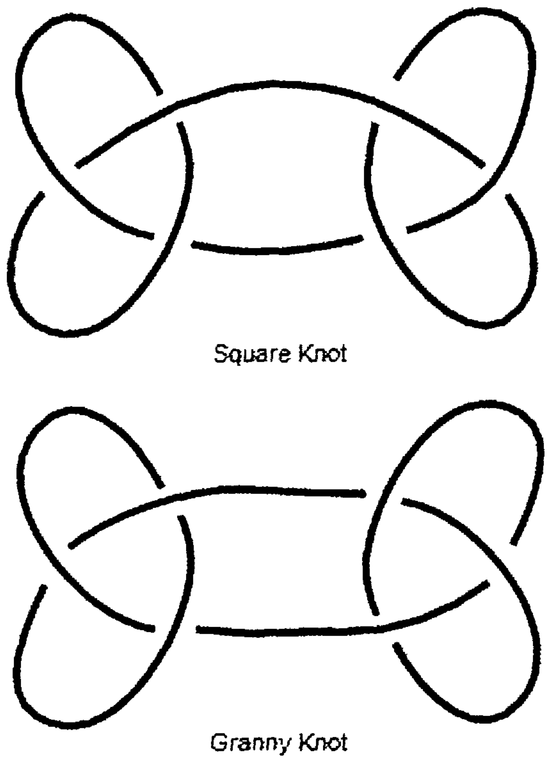


Figure 1.5: Square knot and Granny knot

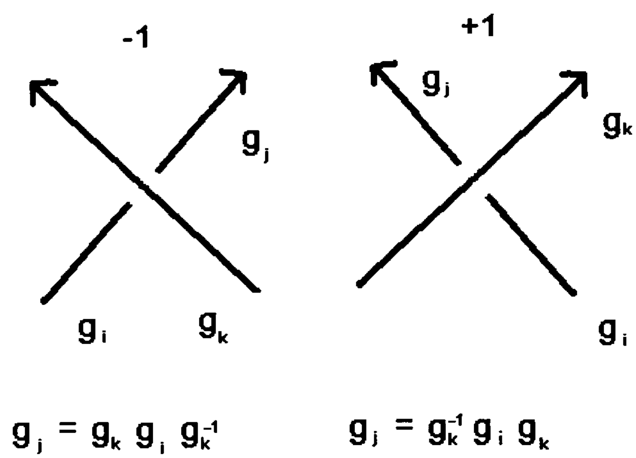


Figure 1.6: Two possible relations

There is a simple way to determine a group presentation of a given knot group (simple in principle, the actual process is tedious). It is called the *Wirtinger presentation*, $G =< g_1, g_2, ..., g_m; r_1, r_2, ..., r_n >$ where g_i's are generators, and r_j's are relations.

We shall select a diagram of the link L: Select an orientation of L for convenience. Corresponding to the ith segment of the diagram with the usual breaks at underpasses take a group generator g_i. For each crossing take a relation as indicated in Figure 1.6. Suppose at the crossing c the over-pass arc is labeled g_k and the underpass is labeled g_i as it approaches c and g_j as it leaves c. Then $r_c = g_k g_i g_k^{-1} g_j^{-1}$ if the sign of the crossing is negative and $r_c = g_k^{-1} g_i g_k g_j^{-1}$ if the sign is positive. See figure 1.6

The proof can be found in [10].

1.5 Group Orderability

An ordered group is a group with a compatible order on it, while an orderable group is a group which can be equipped with such an order.

Definition 1.5.1. A group G is said to be *left-ordered* (respectively, *right-ordered*) if there is a strict total ordering $<$ of its elements such that for all $f, g, h \in G$, $f < g$ implies $hf < hg$ (respectively, $fh < gh$). If a group has an ordering such that it is both left-ordered and right-ordered, then we say that this group is *bi-ordered.*

Note that a group is left-orderable if and only if it is right-orderable.

An orderable group has various good algebraic properties.

If G is left-orderable, then G is torsion-free.

If G is bi-orderable, then:

1. G has no generalized torsion (product of conjugates of a nontrivial element being trivial),

2. G has unique roots: $g^n = h^n$ implies $g = h$,

3. If $[g^n, h] = 1$ in G then $[g, h] = 1$, where $[x, y] = xyx^{-1}y^{-1}$

There are many results on orderable groups. See [1] for an exposition.

Many groups of interest in topology are orderable.

Theorem 1.5.1. *If M is any connected surface other than the projective plane or Klein bottle, then $\pi_1(M)$ is bi-orderable. (See Theorem 1.4 in [4].)*

Specifically, although bi-orderability is not common in knot groups, we have following proposition.

Proposition 1.5.2. *Classical link groups are right-orderable. (See Proposition 3.1 in [3].)*

The group of the trefoil knot is isomorphic to B_3, the braid group on 3 strands, which has presentation of $< x, y \mid x^2 = y^3 >$. (See pp. 52 and 61 in [12].)

And it has been proved that when $n > 2$, B_n is left-orderable but not bi-orderable. (See [1].)

Chapter 2

The Alexander Polynomial

The tools of algebra are extremely useful in the study of topology, and their application to knots is no exception. There are a lot of algebraic invariants associated with a given knot. Here we introduce the Alexander polynomial, which is the first polynomial of a sequence of related polynomials. Although it is not a complete invariant of knots, it is quite efficient at distinguishing two knots.

2.1 The Definition of the Alexander Polynomial

In this section, we construct the Alexander polynomial using the Seifert surface of a given knot.

First, we review some algebra and establish some notation.

Suppose that M is a module over a commutative ring R. It will be assumed that R has an identity, i.e., a $1 \in R$ such that $1x = x$ for all $x \in R$. A module is *free* if any element in it can be uniquely expressed as a linear sum of elements in *basis*; the module of n-tuples of elements of R is the canonical example of a free R-module. A *finite presentation* for M is an exact sequence $F \xrightarrow{\alpha} E \xrightarrow{\phi} M \to 0$ where E and F are free R-modules with finite bases. If α is represented by the matrix A with respect to bases $e_1, e_2, \ldots, e_m$ and $f_1, f_2, \ldots, f_n$ of E and F, then the matrix A, of m rows and n columns, is a *presentation matrix* for M.

Definition 2.1.1. Suppose M is a module over a commutative ring R, having an $m \times n$ presentation matrix A. The r-th elementary ideal ϵ_r of M is the ideal of R generated by all the $(m - r + 1) \times (m - r + 1)$ minors of A.

Proposition 2.1.1. *Suppose that F is a connected, compact, orientable surface with non-empty boundary, piecewise linearly contained in S^3. Then the homology groups $H_1(S^3 - F; \mathbb{Z})$ and $H_1(F; \mathbb{Z})$ are isomorphic, and there is a unique non-singular bilinear form β: $H_1(S^3 - F; \mathbb{Z}) \times H_1(F; \mathbb{Z}) \to \mathbb{Z}$ with the property that $\beta([c], [d]) = lk(c, d)$ for any oriented simple closed curves c and d in $S^3 - F$ respectively.*

Next, we construct the Alexander polynomial of a given link.

Now suppose that F is a Seifert surface for an oriented link L in S^3, so that $\partial F = L$. Let N be a regular neighborhood of L, a disjoint union of solid tori that "fatten" the components of L. Let X be the closure of $S^3 - N$. Then $F \cap X$ is F with a (collar) neighborhood of ∂F removed. Thus $F \cap X$ is just a copy of F and , just to simplify notation, it will be regarded as actually being F. This F has a regular neighborhood $F \times [-1, 1]$ in X, with F identified with $F \times 0$ and the notation chosen so that the meridian of every component of L enters the neighborhood at $F \times -1$ and leaves it at $F \times 1$. Let $i^{\pm}$ be the two embedded images $F \to S^3 - F$ defined by $i^{\pm}(x) = x \times \{\pm 1\}$ and, if c is an oriented simple closed curve in F, let $c^{\pm} = i^{\pm} c$.

Definition 2.1.2. Associated to the Seifert surface F for an oriented link L is the Seifert form $\alpha : H_1(F; \mathbb{Z}) \times H_1(F; \mathbb{Z}) \to \mathbb{Z}$ defined by $\alpha(x, y) = \beta((i^-)_* x, y)$

Now let Y be the space X cut along F. This means that Y is $X - F$ compactified,with two copies, F_- and F_+ of F replacing the removed copy of F (Y is homeomorphic to X less the open neighborhood $F \times (-1, 1)$ of F). Of course, X can be recovered from Y by gluing F_+ and F_- together; thus $X = Y/\phi$, where ϕ is the natural homeomorphism $\phi : F_- \to F \to F_+$. Now take countably many copies of Y and glue them together to form a new space X_∞. More precisely, let $Y_i : i \in \mathbb{Z}$ be spaces homeomorphic to Y, and let $h_i : Y \to Y_i$ be a homeomorphism. Let X_∞ be the space formed from the disjoint union of all the Y_i by identifying $h_i F_-$ with $h_{i+1} F_+$ by means of the homeomorphism $h_{i+1}\phi h_i^{-1}$.

On X_∞ there is a natural self-homeomorphism $t : X_\infty \longrightarrow X_\infty$ defined by $t|Y_i = h_{i+1}h_i^{-1}$. Clearly this is well defined; t is thought of as a translation of X_∞ by "one unit to the right". Hence the infinite cyclic group $< t >$ generated by t acts on X_∞ as a group of homeomorphisms. Thus $< t >$ also acts on $H_1(X_\infty; \mathbb{Z})$.

Theorem 2.1.2. *Let F be a Seifert surface for an oriented link L in S^3 and let A be a matrix, with respect to any basis of $H_1(F; \mathbb{Z})$, for the corresponding Seifert form. Then $tA - A^\tau$ is a matrix that presents the $\mathbb{Z}[t^{-1}, t]$ - module $H_1(X_\infty; \mathbb{Z})$ where A^τ is the transpose of A.*

Finally we reach to the definition of Alexander polynomial.

Definition 2.1.3. The r-th Alexander ideal of an oriented link L is the r-th elementary ideal of the $\mathbb{Z}[t^{-1}, t]$ module $H_1(X_\infty; \mathbb{Z})$. The r-th Alexander polynomial of L is a generator of the smallest principal ideal of $\mathbb{Z}[t^{-1}, t]$ that contains the r-th Alexander ideal. The first Alexander polynomial is called *the Alexander polynomial* and is written $\Delta_L(t)$.

We have a very straightforward way to calculate the Alexander polynomial of a given knot if we know the Wirtinger presentation of the knot. It is called the *free differential calculus* devised by R. H. Fox. Instead of proving it here, we would rather state a step-by-step method of how we use it. The complete proof can be found in [10].

Now given a knot diagram, after labeling each segment(generator), we can immediately get several relations(the number of relations is one less then the number of generators). Each relation is either of the form $g_j = g_k g_i g_k^{-1}$ or $g_j = g_k^{-1} g_i g_k$. Note that we could choose a natural order of the segments such that $j = i+1$. Then we can set a $n \times (n+1)$ matrix with all entries 0, where $(n+1)$ is the number of generators, and n is the number of relations. For the ith relation, let entry (i, k) be $1-t$. Besides, if crossing is negative, let (i,i) be t and (i,j) be -1, otherwise crossing is positive, we let (i,i) be -1 and (i,j) be t. Remember $j = i+1$, so it is quite straightforward.

Now we have a big matrix with mostly zeros. The Alexander polynomial is the greatest common divisor of all n-minors.

While it is simple in principle, it is tedious even with the help of mathematical software when n becomes large. So when we have a nice Seifert surface F for a given knot K, we can use the definition to calculate the Alexander polynomial directly.

The Seifert matrix A is given by $A_{ij} = lk(f_i, f_j^+)$, where f_j^+ is a copy of f_j pushed off F into $S^3 - F$ in the direction defined by the oriented meridian of K. Then it follows that the Alexander polynomial is the determinant of $tA - A^\tau$.

It is more efficient when the knot is presented nicely and systematically.

For the Alexander polynomial regarding a fibred knot, we know the following theorem.

Theorem 2.1.3. *A fibred knot has a monic Alexander polynomial and has genus equal to half the degree of the Alexander polynomial. See [11].*

This is fairly effective at picking out a non-fibred knot! In particular, a knot is not fibred if it does not have 1 as the leading coefficient of its Alexander polynomial.

2.2 Levine's Characterization

In [5], J.Levine gives a characterization of knot polynomials, namely an equivalent condition for a polynomial to be the Alexander polynomial of some knot.

Notice that by the definition of the Alexander polynomial, it is actually a class of polynomials. Within a class every two polynomial differ by a unit multiple, that is, $\pm t^a$.

The necessary and sufficient conditions for a given polynomial $\Delta(t)$ defined over the integers to be the Alexander polynomial of some knot are:

1.$\Delta(1) = \pm 1$

2.$\Delta(t) = t^{2a}\Delta(t^{-1})$ for some integer a

We can always multiply ± 1 to the polynomial to make condition 1 become $\Delta(1) = 1$.

As we have mentioned in Theorem 2.1.3, the Alexander polynomial $\Delta(t)$ of a fibred knot should satisfy $\Delta(0) = \pm 1$.

Chapter 3

Fibred Knots With Orderable Knot Groups

In [2], it was shown that certain kinds of fibred knots have bi-orderable knot groups.

The criteria for this nice bi-ordering is whether the Alexander polynomial is *special.*

In this chapter we classify all *special* Alexander polynomial of degree less than 10. And we use this result to construct several concrete fibred knots with nice orderings.

There is a rather nice result. It gives a general result on the relation between the orderability of the knot group and the Alexander polynomial.

Theorem 3.0.1. *If K is a fibred knot in S^3, or in any homology 3-sphere, such that all the roots of its Alexander polynomial $\Delta_K(t)$ are real and positive, then its knot group $\pi_1(S^3 \setminus K)$ is bi-orderable. (See Theorem 1.1 in [3].)*

In [2], Linnell, Rhemtulla and Rolfsen obtain a related result which we state in the next section.

3.1 Classification of Special Alexander Polynomials of Degree Less than 10

It has been proved that the restriction on the Alexander polynomial in theorem 3.0.1 can be modified. Next,we give a definition of a *special* polynomial.

Definition 3.1.1. Let $f = \mathbb{Q}[X]$ be a monic polynomial and let $f = f_1 \cdots f_n$ be its factorization into irreducible polynomials (so n is a nonnegative integer and each f_i is irreducible). Then we say that f is a *special* polynomial if each f_i has odd prime power degree, negative constant term, and all roots real. (See Definition 3.1 of [2].)

"Odd prime power" above means $Deg(f_i) = p^d$ where p is odd prime, $d \in \mathbb{Z}$ and $d \geq 1$.

The special polynomial can play a important role of providing the invariance property of the bi-orderings.

Proposition 3.1.1. *Let $f \in \mathbb{Q}[X]$ be a special polynomial, let G be a residually torsion-free nilpotent group, let θ be an automorphism of G, and let $\phi : G/G_2^\tau \to G/G_2^\tau$ be the automorphism induced by θ. Assume that G/G_2^τ has finite rank and that the eigenvalues of θ are roots of f. Then G has a bi-ordering invariant under θ. (See Proposition 3.4 of [2].)*

$G_2^\tau = \{g \in G | g^m \in [G, G]$ for some positive integer $m\}$

G_2^τ is a characteristic subgroup of G, which contains the commutator subgroups of G.

Next, we investigate what the special polynomials look like. Let us briefly review what the Alexander polynomial looks like.

Levine's Characterization of Alexander polynomial is:

1.$\Delta(1) = \pm 1$

2.$\Delta(t) = t^{2a}\Delta(t^{-1})$ for some integer a

We shall classify all the special monic Alexander polynomials of degree less than 10.

By condition 2, we know that the degree of the Alexander polynomial is even given a non-zero constant term.

For degree 2 and degree 4:

Since we want the degree of each factors to be greater than 2, and since 4 is not an odd prime power, there are no special Alexander polynomials of degrees 2 or 4.

For degree 6:

Special polynomial condition:

Since we are not allowing the degree of a factor less than 3, we must have $\Delta(t) = \Delta_1(t)\Delta_2(t)$ where both of $\Delta_1(t)$ and $\Delta_2(t)$ are of degree 3.

Alexander polynomial condition 1:

Since $\Delta(1) = \pm 1$ and the factors are irreducible, both $\Delta_1(1)$ and $\Delta_2(1)$ must be equal to ± 1.

Alexander polynomial condition 2:

This leads to two possibilities:

$\Delta_1(t) = \pm t^{2a}\Delta_2(t^{-1})$

or

$\Delta_1(1)$ and $\Delta_2(1)$ satisfy condition 2 respectively.

If the degree of the factor is odd and satisfies the condition 2, it contradicts condition 1, otherwise we would have non-integer coefficients.

So they must be of the form:

$\Delta_1(t) = t^3 + (\pm 1 - a)t^2 + at - 1$

$\Delta_2(t) = t^3 - at^2 - (\pm 1 - a)t - 1$

where $\Delta_1(t) = \pm t^{2a}\Delta_2(t^{-1})$

since the coefficients of the polynomial are all integers. If the discriminants of these cubic polynomial are non-negative, then all their roots are real. We denote ± 1 above by ϵ below.

Case 1: $\epsilon = 1$:

The discriminants of $\Delta_1(t)$ and $\Delta_2(t)$ are

$D_1 = D_2 = a^4 - 10a^3 + 31a^2 - 30a - 23$

After drawing the graph of D_1 and D_1 as a function of a, it is not hard to see that $a < 0$ or $a > 5$ is enough for all the roots real.

Case 2: $\epsilon = -1$:

The discriminants of $\Delta_1(t)$ and $\Delta_2(t)$ are

$D_1 = D_2 = a^4 - 6a^3 + 7a^2 + 6a - 31$

So we have $a < -1$ or $a > 4$ for all roots real.

For degree 8:

Due to the same reason above, we must have two factors of degree 3 and 5. But it contradicts condition 2, which we have discussed above. So there is no degree 8 special Alexander polynomial.

To summarize, we state a proposition below:

Theorem 3.1.2. *In all the Alexander polynomials of degree less than 10, the special ones have one of the following two forms:*

$(t^3 + (1-a)t^2 + at - 1)(t^3 - at^2 - (1-a)t - 1)$ *where* $a < 0$ *or* $a > 5$

or

$(t^3 + (-1-a)t^2 + at - 1)(t^3 - at^2 - (-1-a)t - 1)$ *where* $a < -1$ *or* $a > 4$

and any polynomial having one of these two forms is a special Alexander polynomial.

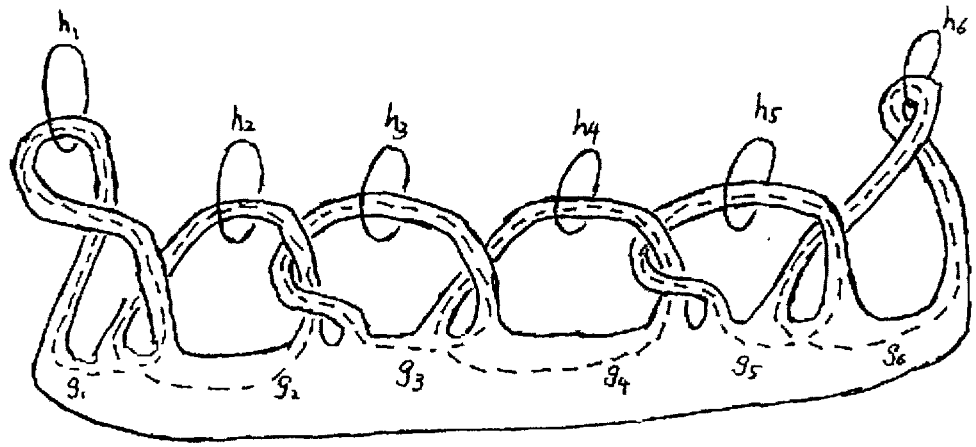
3.2 Construction of Fibred Knots

We describe a construction of fibred knots by Quach in [7] which is originally discovered by Burde in [6].

Let $\Delta(t)$ be an Alexander polynomial such that $\Delta(0) = -1$. Then we can construct a fibred knot with Alexander polynomial $\Delta(t)$. A even stronger version can be found in [7].

Let $\Delta(t) = \lambda^{-2h}(-\Lambda^h + \sum_{k=1}^{h-1} p_k\Lambda^k + 1)$,

$\lambda = \frac{1}{1-t}$, $\Lambda = \lambda(1-\lambda)$, $p_k \in \mathbb{Z}$



$i^+([g_1]) = [h_1]$ $\quad$ $i^+([g_4]) = [h_5]$ $\quad$ up to a non-zero coefficient

$i^+([g_2]) = [h_3]$ $\quad$ $i^+([g_5]) = [h_4]$

$i^+([g_3]) = [h_2]$ $\quad$ $i^+([g_6])$ $[h_6]$

Figure 3.1: A Fibred Knot with Alexander Polynomial: $1-(6+p_2)t+(15+p_1+4p_2)t^2-(21+2p_1+6p_2)t^3+(15+p_1+4p_2)t^4-(6+p_2)t^5+t^6$

We take $h = 3$ for example. See Figure 3.1. The Seifert surface can been easily seen as a surface with $2k$ twisted handles. p_k is the number of the full twists, and the sign of p_k is associated with the orientation as illustrated in the figure.

By counting the linking numbers, the Seifert matrix for this knot is

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & p_1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & p_2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

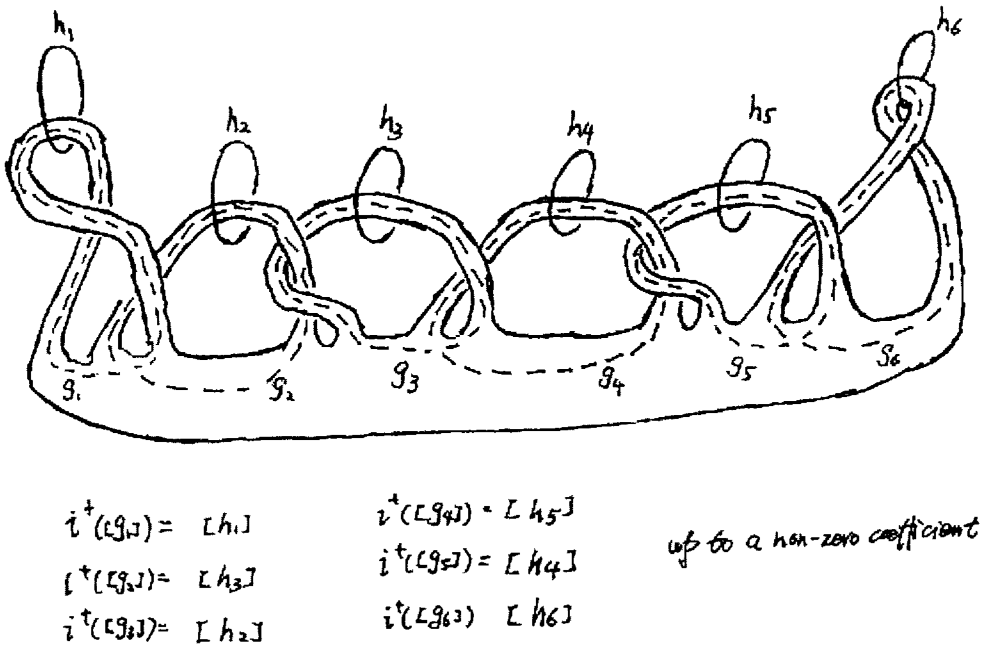


Figure 3.2: i^+ is surjective

So the resulting Alexander polynomial is $1 - (6 + p_2)t + (15 + p_1 + 4p_2)t^2 - (21 + 2p_1 + 6p_2)t^3 + (15 + p_1 + 4p_2)t^4 - (6 + p_2)t^5 + t^6$.

We claim this knot is fibred.

Proof. By [7], we only need to prove i^+ is surjective where $i^+ : \pi_1(F) \to \pi_1(S^3 \backslash F)$ is the induced map from the lifting map of the knot where F is the Seifert surface of the knot. It is sufficient to show that every generator of $\pi_1(S^3 \backslash F)$ has a preimage. While the generators are those loops winding around the handles of the Seifert surface, the generators of $\pi_1(F)$ do map to them. See Figure 3.2.

□

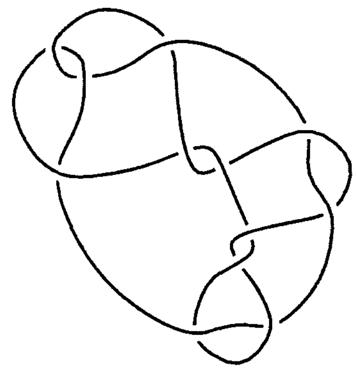


Figure 3.3: Knot 12_{0477a} with A Special Alexander Polynomial

3.3 Examples of Fibred Knots with Special Polynomials

In this section we are going to give a concrete construction of fibred knots which have special polynomials as their Alexander polynomial.

When we are considering all the fibred prime knots of 12 or fewer crossings, there is only one fibred knot which is 12_{0477a} that is, knot of no. 0477a in crossing 12. The Alexander polynomial of this knot is

$1 - 11t + 41t^2 - 63t^3 + 41t^4 - 11t^5 + t^6 = (-1 + 5t - 6t^2 + t^3)(-1 + 6t - 5t^2 + t^3)$.

Figure 3.3 is a diagram for this knot.

Since all the roots of this Alexander polynomial are real and positive, it has been covered previously by Theorem 3.0.1.

For degree 6 case, we let $\epsilon = 1$.

$\Delta(t) = 1+(1-2a)t+(-1+a+a^2)t^2+(-3+2a-2a^2)t^3+(-1+a+a^2)t^4+(1-2a)t^5+t^6$

We have known in last chapter that when $a = -1$ the polynomial is a special Alexander polynomial. And the roots of the Alexander polynomial are not all positive, so it is not covered by Theorem 3.0.1. And this corresponds to $p_1 = 20$ $p_2 = -9$ in

Figure 3.1.

By Theorem 3.1.1 the knot group of this knot is bi-orderable. And by varying the parameter a we get infinitely many fibred knots with special Alexander polynomials and bi-orderable knot groups.

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