Portfolio Optimization under Value at Risk, Average Value at Risk and Limited Expected Loss Constraints

PORTFOLIO OPTIMIZATION UNDER VALUE AT RISK, AVERAGE VALUE AT RISK AND LIMITED EXPECTED LOSS CONSTRAINTS

BY

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A THESIS

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS AND STATISTICS

AND THE SCHOOL OF GRADUATE STUDIES

OF McMASTER UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE IN MATHEMATICS

Master of Science in Mathematics (2014)	McMaster University
(Mathematics and Statistics)	Hamilton, Ontario, Canada

TITLE:	Portfolio Optimization under Value at Risk, Average Value at Risk and Limited Expected Loss Constraints
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NUMBER OF PAGES:	xiv, 106

To mum, dad, John, family, friends, and family away from home

Abstract

In this thesis we investigate portfolio optimization under Value at Risk, Average Value at Risk and Limited expected loss constraints in a framework, where stocks follow a geometric Brownian motion. We solve the problem of minimizing Value at Risk and Average Value at Risk, and the problem of finding maximal expected wealth with Value at Risk, Average Value at Risk, Limited expected loss and Variance constraints. Furthermore, in a model where the stocks follow an exponential Ornstein-Uhlenbeck process, we examine portfolio selection under Value at Risk and Average Value at Risk constraints. In both geometric Brownian motion (GBM) and exponential Ornstein-Uhlenbeck (O.U) models, the risk-reward criterion is employed and the optimal strategy is found. Secondly, the Value at Risk, Average Value at Risk and Variance is minimized subject to an expected return constraint. By running numerical experiments we illustrate the effect of Value at Risk, Average Value at Risk, Limited expected loss and Variance on the optimal portfolios. Furthermore, in the exponential O.U model we study the effect of mean-reversion on the optimal strategies. Lastly we compare the leverage in a portfolio where the stocks follow a GBM model to that of a portfolio where the stocks follow the exponential O.U model.

Acknowledgements

I am extremely grateful to my Lord God Almighty, for keeping me and guiding me to begin and complete this thesis. This work would not have been accomplished without close guidance and assistance of my supervisor Dr. Traian Pirvu. Your patience and dedication is priceless. I am grateful for the opportunity to learn from you.

I would like to express my profound gratitude to the professors and post-doctorates in McMaster University who guided me as a student and as a teaching assistant: Dr Westcott, Dr Lovric, Dr Clements, Post-doc Jevtic and Post-doc Harper. A special thanks goes to Dr David Lozinski and Dr Tom Hurd for their advise and support as I pursued my studies in McMaster.

My sincere thanks go to my friends and colleagues : Joycelyne Ewusie, Regina Kampo, Lily Yevugah, Oumar Mbodji, Quentin Shao and Minsuk Kwak. Additionally, I express my thanks to the Department of Mathematics and Statistics for providing the resources I needed to complete this work.

I am ever grateful to my mum and dad, for their encouragement, support and love in difficult moments. Last but not least, I would like to thank John Egyir Aggrey. Your company and patience was more than I ever asked for, thank you.

Notation and abbreviations

VaR Value at Risk

AVaR Average Value at Risk

LEL Limited Expected Loss

CVaR Conditional Value at Risk

GBM Geometric Brownian Motion

OU Ornstein-Uhlenbeck

CaR Capital at Risk

RVaR Relative Value at Risk

Optimization problems in the GBM model

P1 Minimizing VaR within the class of deterministic strategies.

P2 Maximizing expected wealth with VaR constraint within the class of deterministic strategies.

P3 Minimizing AVaR within the class of deterministic strategies.

P4 Maximizing expected wealth with AVaR constraint within the class of deterministic strategies.

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Introduction

1.1 Background

Every investor's intent is to maximize return while minimizing risk. This concept has been the bases of investments and is a guideline to understanding investor behavior. It is one of the building blocks of modern portfolio theory pioneered by the mean-variance efficiency frontier of Markowitz (Markowitz, 1952) and the reward-to-variability ratio of Sharpe (Sharpe, 1962). Portfolio theory uses decision-making tools to solve the problem of managing risky investments (Nawrocki, 1999). The risk management era was established in 1973. In this year the Bretton Woods system of fixed exchange rate collapsed and in the same year the Black-Scholes option pricing formula was born. Ever since then, there has been great amounts of volatility in the market and various tools such as forwards, futures, hedges,options etc have been created to offset this inherent risk (Linsmeier and Pearson, 2000). This has brought about the need for precise measurement of volatility in the system. Hence, various risk measures are proposed.

The most well known risk is the variance. The popularity of the variance is mainly due to its simplicity. However, in terms of effectiveness it is not favored, because it equally penalizes gains and losses. The investor's view of risk is not symmetric but asymmetric about the mean. Variance on the other hand operates on the principle that there is symmetry about the mean (Konno and Yamazaki, 1991). Due to the problems produced by the variance, downside risk measures were introduced to deal with both symmetric and asymmetric distributions. Markowitz exhibited the fact that if the distribution is symmetric then the variance and the downside risk produce the same results (Nawrocki, 1999). In this thesis we use Value at Risk (VaR), Average Value at Risk (AVaR) and Limited Expected Loss (LEL), which are common examples of downside risk measures.

1.2 Value at Risk

The current derivative market is extremely complex and with the introduction of High Frequency Trading (HFT) there exist high stock price volatility (Zhang, 2010). Investors are concerned about the risk in their portfolios and one popular risk measure used is the Value at Risk. Let us start with a formal definition of VaR. The VaR at level $\lambda \in (0,1)$ of a position X is given by

$$VaR_\lambda(X) = inf\{m|P[X+m<0] \leq \lambda\}.$$

In other words, VaR estimates the loss of a portfolio over a given time period with a particular confidence interval. One advantage of VaR is that it is simple to estimate, however "there is a risk in the value at risk itself" (Jorion, 1996). This risk measure tends to give a false sense of security as it fails to shed light on the loss beyond the VaR level. Secondly, VaR is not sub-additive, hence not a coherent risk measure. In order to tackle the flaws of VaR other risk measures such as AVaR have been introduced in the financial world.

1.3 Average Value at Risk

We begin with a formal definition of AVaR. The AVaR at level $\lambda \in (0,1)$ of a position X is given by

$$AVaR_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda VaR_u(X)du.$$

Another common risk measure is Conditional Value at Risk (CVaR) which is formally defined as

$$E[-X| - X \geq VaR_\lambda(X)].$$

When X has a continuous distribution AVaR and CVaR are the same (Follmer and Schied, 2002). An advantage of AVaR is that it captures what the VaR ignores, which is the information on how much one can loose on average if one goes beyond the VaR level (Svetlozar T. Rachev and Fabozzi, 2011). Average Value at Risk is a coherent risk measure, meaning it has the following properties : monotonicity, homogeneity, sub-additivity and translation invariance (Philippe Artzner and Eber, 1999).

1.4 Limited expected loss measure

Let us begin with a formal definition of Limited expected loss measure (LEL). The LEL at level $\lambda \in (0, 1)$ of a position X is given by

$$LEL_{\lambda}(X) = \frac{1}{\lambda} \int_0^{\lambda} VaR_u^Q(X) du,$$

where $VaR_{\lambda}^Q(X)$ is the VaR evaluated under a risk neutral measure. LEL was proposed by Basak and Shapiro (Basak and Shapiro, 2001) and is a variant of AVaR (Yamai and Yoshiba, 2002).

1.5 Related literature

Significant amounts of literature are available in the field of portfolio optimization, current ones related to this thesis are Dmitrasinovic-Vidovic and Ware (2006),Gordana Dmitrasinovic-Vidovic and Li (2011), Susanne Emmer and Korn (2001) and Dmitrasinovic-Vidovic and Ware (2011). In the paper of Dmitrasinovic-Vidovic and Ware (2006), Susanne Emmer and Korn (2001) and Gordana Dmitrasinovic-Vidovic and Li (2011), the stocks follow a GBM, Capital at Risk (CaR) is the risk measured consider. Dmitrasinovic-Vidovic and Ware (2011) assumes that the stocks follows an exponential O.U under CaR constraint.

1.6 Thesis outline and contribution

In this thesis, the risky assets are assumed to follow a GBM in chapter two and an exponential O.U in chapter three. Chapter two is split into two sub-chapters: in one we restrict the model to deterministic strategies and in the next sub-chapter, we restrict to constant strategies. We maximize expected return under VaR, AVaR, LEL and variance constraints. In chapter three, we consider maximizing expected return under VaR and AVaR constraints. The minimal VaR and AVaR portfolios are found in the GBM and the exponential O.U. Furthermore, we minimize VaR, AVaR and variance subject to an expected return constraint. The novelty of this work is the addition of AVaR and LEL constraints. The portfolio optimization problem is approached using the risk-reward criterion. In the GBM model the parameters are time dependent, the constant/deterministic strategies are evaluated and compared. We apply a dimension reduction technique to transform the m-dimensional optimization problem into a 1-dimensional problem which is solved in closed form or numerically. For the exponential O.U model, we restrict to constant strategies. Furthermore, non linear systems of equations are solved to obtain the optimal strategies for maximizing expected wealth under VaR and AVaR constraints. In both the GBM and exponential O.U models, we prove that minimizing VaR, AVaR and variance under an expected return constraint produces the same optimal strategy. Numerical experiments show that AVaR constrains the investor to take less risk as compared to VaR in both models. In the GBM model the most conservative constraint is variance, followed by LEL, AVaR and VaR. However, variance is a symmetric risk measure, LEL is a Q-measure and VaR ignores the losses beyond the confidence threshold α. AVaR on the other hands captures all the attributes that the aforementioned risk measures ignore, hence considered to be the best risk measure. We compare the different classes of optimal strategies and find that time dependent optimal portfolio strategies produce better results than the constant optimal strategies as expected, however the differences are small. The importance of diversification is illustrated in the GBM model. For the exponential O.U, the numerical results show that the mean-reversion rate has a strong effect on optimal strategies. The numerical experiments show that the confidence threshold has a strong effect on optimal strategies in the GBM and the exponential O.U models. Lastly, we compare the optimal strategies in the GBM model to the optimal strategies in the exponential O.U model and the results show that optimal strategies in GBM model introduce more leverage. Numerical examples and figures are illustrated at the end of each chapter (for chapter three) or sub-chapter (for chapter two). Various efficient frontiers are illustrated. Chapter four contains the conclusion and directions for further research. All the mathematical proofs are contained in the appendix (chapter five).

Chapter 2

Geometric Brownian motion framework

In this chapter we assume a market is consisting of a money market and multiple risky assets (stocks). The stocks follow a geometric Brownian motion (GBM). Risk-Reward criterion is utilized with VaR, AVaR, LEL and variance serving as the risk measures. That means we maximize expected utility of the portfolio at some finite horizon subject to VaR, AVaR, LEL and variance constraints. Market parameters are deterministic through out the chapter. The chapter is split into two sub chapters: in one sub chapter we restrict to deterministic trading strategies and in the other to constant trading strategies.

2.1 Deterministic trading strategy

There is notable research investigating continuous time portfolio selection under various risk measures. In the papers of Gordana Dmitrasinovic-Vidovic and Li (2011) and Dmitrasinovic-Vidovic and Ware (2006) the stocks are assumed to follow a GBM model. The risk measure used are VaR, Relative Value at Risk (RVaR) and Capital at Risk (CaR). They maximize expected return subject to these risk measure. In this chapter we follow the approach developed in Dmitrasinovic-Vidovic and Ware (2006) and solve for portfolios which maximize expected return under VaR, AVaR, LEL and variance constraints. The novelty of this chapter is that we introduce AVaR and LEL as a risk measure (the AVaR, VaR, LEL and variance constraints should not exceed a given threshold).

2.1.1 Assumptions

The following assumptions hold in this section:

- Assets are traded continuously over a finite time horizon $[0, T]$ in a frictionless market.
- $m+1$ assets are traded where one of the assets is risk free (the money market) and m are the stocks.
- Stocks follow a GBM model, i.e,

$$dS_i(t) = S_i(t)\Big(b_i(t)dt + \sum_{j=1}^{m} \sigma_{ij}(t)dW^j(t)\Big).$$

  $(\sigma_{ij}(t))_{i,j=1,...m}$ is deterministic and denotes an invertible volatility matrix; $b_i(t)$ is deterministic and denotes the drift of the stock, $b(t) = (b_1(t), ..., b_m(t))'$. $W(t) = (W^j(t))_{j=1,..m}$ is a m-dimensional Brownian motion.

- Number of shares of asset i held in the portfolio is denoted by $N_i(t)$. The fraction of wealth $X(t)$ invested in the risky stock i is denoted by $\pi_i(t) = \frac{N_i(t)S_i(t)}{X(t)}$. $\pi(t) = (\pi_1(t), ..., \pi_m(t))' \in \Re^m$ is deterministic.

- The money market is represented by $S_0(t)$,

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt$$

The interest rate denoted by $r(t)$ is deterministic and $0 < r(t) < b_i(t)$.

- $\alpha \in (0, 0.5)$, denotes the confidence level.

Notations:

The following notations apply throughout this chapter

- The m-dimensional column vectors with each component equal to 1 is denoted by $\mathbf{1}$.
- The Euclidean norm of a matrix or vector is denoted by $||.||$. The $L^2([0,T], \Re^n)$ denotes the set of $\Re^n$-valued, square-integrable functions defined on $[0,T]$, with its natural inner product $\langle f, g\rangle_t = \sum_{i=1}^n \int_0^t f_i(s)g_i(s)ds$. Here $||f||_t = \sqrt{\langle f, f\rangle_t}$ is the corresponding norm.
- π_c denotes the area of a circle with radius 1.

2.1.2 Wealth dynamics

Proposition 2.1.2.1 *The investor invests $\pi(t)$ in the risky stocks and $1 - \pi(t)$ in the money market. The initial wealth $X(0)$ is positive. The wealth at time t, denoted* *by $X(t)$ satisfies the following self-financing equation:*

$$dX(t) = X(t)((r(t) + B(t)'\pi(t))dt + \pi(t)'\sigma(t)dW(t)), \tag{2.1}$$

where $B(t) := b(t) - r(t)\mathbf{1}$. Thus

$$X(t) = X(0)\exp\left(\int_0^t (r(s) + B(s)'\pi(s) - \frac{1}{2}||\sigma(s)'\pi(s)||^2)ds + \int_0^t \pi(s)'\sigma(s)dW(s)\right). \tag{2.2}$$

Proof of Proposition 2.1.2.1: see appendix 5.1.1

2.1.3 Expectation of the wealth

Proposition 2.1.3.1 *The expectation of the wealth $X(t)$ is*

$$E(X(t)) = X(0)R(t)\exp\left(\int_0^t (B(s)'\pi(s))ds\right), \tag{2.3}$$

and the variance is

$$\begin{aligned} Variance[X(t)] &= X(0)^2R(t)^2\Big(\exp\Big(2\int_0^t B(s)'\pi(s)ds \\ &+ \int_0^t ||\sigma(s)'\pi(s)||^2 ds\Big) - \exp\left(2\int_0^t B(s)'\pi(s)ds\right)\Big). \end{aligned} \tag{2.4}$$

Proof of Proposition 2.1.3.1 See appendix 5.1.2.

2.1.4 Value at Risk (VaR)

Let us begin with a formal definition.

Definition 2.1.4.1 *VaR is the maximum amount one can likely lose over a period at a specific confidence level* α*, i.e,*

$$VaR_\alpha(X(t)) = inf\{m|P(Loss(t) \leq m) \geq 1-\alpha\}, \qquad Loss(t) = X(0) - X(t).$$

If for example $\alpha = 5\%$ *then there is a* 95% *probability that the loss will not exceed* $VaR_\alpha(X(t))$*. Equivalently, there is a* 5% *probability that the loss can exceed* $VaR_\alpha X(t)$*.*

The following proposition gives a formula for $VaR_\alpha(X(t))$.

Proposition 2.1.4.1 *For a wealth process* $X(t)$ *and a given confidence level* α*, it follows that*

$$\begin{aligned} VaR_\alpha(X(t)) &= X(0)\Big(1 - R(t)\exp\Big(\int_0^t (B(s)'\pi(s) - \frac{1}{2}||\sigma(s)'\pi(s)||^2)ds \\ &+ (N^{-1}(\alpha))\sqrt{\int_0^t ||\sigma(s)'\pi(s)||^2 ds}\Big)\Big). \end{aligned} \tag{2.5}$$

Proof of Proposition 2.1.4.1: See appendix 5.1.3.

Define the function $VaR(\alpha, \pi, t)$ by

$$\begin{aligned} VaR(\alpha, \pi, t) &= X(0)\Big(1 - R(t)\exp\Big(\int_0^t (B(s)'\pi(s) - \frac{1}{2}||\sigma(s)'\pi(s)||^2)ds \\ &+ (N^{-1}(\alpha))\sqrt{\int_0^t ||\sigma(s)'\pi(s)||^2 ds}\Big)\Big). \end{aligned} \tag{2.6}$$

2.1.5 Average Value at Risk (AVaR)

Let us begin with a formal definition.

Definition 2.1.5.1 *The AVaR is defined by*

$$AVaR_\alpha(X(t)) = \frac{1}{\alpha}\int_0^\alpha VaR_u(X(t))du.$$

Proposition 2.1.5.1 *For a wealth process $X(t)$ and a given confidence level α, it follows that*

$$\begin{aligned} AVaR_\alpha(X(t)) &= X(0)\Big(1 - \frac{R(t)}{\alpha}\Big(\exp\int_0^t B(s)'\pi(s)ds \\ &+ \ln\Big(N\Big(N^{-1}(\alpha) - \sqrt{\int_0^t ||\sigma(s)'\pi(s)||^2 ds}\Big)\Big)\Big). \end{aligned} \tag{2.7}$$

Proof of Proposition 2.1.5.1: See appendix 5.1.4

Define the function $AVaR(\alpha, \pi, t)$ by

$$\begin{aligned} AVaR(\alpha,\pi,t) &= X(0)\Big(1 - \frac{R(t)}{\alpha}\Big(\exp\int_0^t B(s)'\pi(s)ds \\ &+ \ln\Big(N\Big(N^{-1}(\alpha) - \sqrt{\int_0^t ||\sigma(s)'\pi(s)||^2 ds}\Big)\Big)\Big). \end{aligned} \tag{2.8}$$

2.1.6 Limited expected loss measure

Let us begin with a formal definition.

Definition 2.1.6.1 *The LEL is defined by*

$$LEL_\alpha(X(t)) = \frac{1}{\alpha}\int_0^\alpha VaR_u^Q(X(t))du,$$

where $VaR_u^Q(X)$ is the VaR evaluated under a risk neutral measure given the threshold u.

Proposition 2.1.6.1 *For a wealth process $X(t)$ and a given confidence level α, it follows that*

$$LEL_\alpha(X(t)) \;=\; X(0)\Big(1-\frac{R(t)}{\alpha}\Big(N\Big(N^{-1}(\alpha)-\sqrt{\int_0^t ||\sigma(s)'\pi(s)||^2 ds}\Big)\Big)\Big). \quad (2.9)$$

Proof of Proposition 2.1.6.1 See appendix 5.1.5

Define the function $LEL(\alpha,\pi,t)$ by

$$LEL(\alpha,\pi,t) \;=\; X(0)\Big(1-\frac{R(t)}{\alpha}\Big(N\Big(N^{-1}(\alpha)-\sqrt{\int_0^t ||\sigma(s)'\pi(s)||^2 ds}\Big)\Big)\Big) \quad (2.10)$$

2.1.7 Portfolio optimization by minimizing VaR

Let us introduce the class of admissible portfolio denoted by Q. The market price of risk is denoted by

$$\Theta(t)=\sigma(t)^{-1}B(t) \quad (2.11)$$

and

$$R(t)=\exp\Big(\int_0^t r(s)ds\Big). \quad (2.12)$$

In this section the VaR is minimized, i.e

$$(P1) \qquad \min_{\pi\in Q} VaR(\alpha,\pi,T).$$

Theorem 2.1.7.1 *Let*

$$\varepsilon^* = ||\Theta||_T + N^{-1}(\alpha). \quad (2.13)$$

The optimal strategy for $(P1)$ *is*

$$\pi_{\varepsilon^*}(t) = \frac{\max(0, \varepsilon^*)}{||\Theta||_T} (\sigma(t)\sigma(t)')^{-1} B(t), t \in [0, T]. \tag{2.14}$$

The minimal VaR is

$$VaR(\alpha, \pi_{\varepsilon^*}, T) = X(0) \left(1 - R(T) \exp\left(\frac{1}{2}(\max(0, \varepsilon^*))^2\right)\right). \tag{2.15}$$

Proof of the theorem: See appendix 5.1.6.

2.1.8 Portfolio optimization by maximizing expected wealth with VaR constraint

In this section a risk-reward criterion is used. That is, the expected wealth is maximized given that VaR does not exceed a threshold.

$$(P2) \qquad \max_{\pi \in Q} E(X(T)) \text{ subject to}$$

$$VaR(\alpha, \pi, T) \leq C,$$

where C is a given constant. It is further assumed that $0 < C < X(0)$.

Theorem 2.1.8.1 *Recall that* $\Theta(t)$ *is defined by* (2.11)*. Let*

$$\varepsilon_1 = ||\Theta||_T + (N^{-1}(\alpha)) + \sqrt{(||\Theta||_T + N^{-1}\alpha)^2 - 2\ln\left(\frac{1}{R(T)}\left(1 - \frac{C}{X(0)}\right)\right)}. \tag{2.16}$$

The optimal strategy for $(P2)$ *is*

$$\pi_{\varepsilon_1}(t) = \frac{\varepsilon_1}{||\Theta||_T}(\sigma(t)\sigma(t)')^{-1}B(t). \tag{2.17}$$

The maximum expected wealth subject to VaR constraint is

$$\begin{aligned} E[X^*(T)] &= X(0)R(T)\exp\Big(||\Theta||_T\Big(||\Theta||_T + (N^{-1}(\alpha) \\ &+ \sqrt{(||\Theta||_T + (N^{-1}(\alpha)))^2 - 2\ln\left(\frac{1}{R(T)} - \frac{C}{R(T)X(0)}\right)}\Big)\Big), \end{aligned} \tag{2.18}$$

where X^* *is the wealth process corresponding to* π_{ε_1}

Proof of Theorem 2.1.8.1 See appendix 5.1.7

2.1.9 Portfolio optimization by minimizing AVaR

In this section the optimization problem considered is

$$(P3) \qquad \inf_{\pi\in Q} AVaR(\alpha, \pi, T)$$

Theorem 2.1.9.1 *The optimization problem*

$$\sup_{\varepsilon\geq 0}\Big[\varepsilon||\Theta||_T + \ln(N(N^{-1}(\alpha) - \varepsilon))\Big]$$

has a unique solution denoted as ε^{**}. *The optimal strategy for* $(P3)$ *is*

$$\pi_{\varepsilon^{**}}(t) = \frac{\varepsilon^{**}}{||\Theta||_T}(\sigma(t)\sigma(t)')^{-1}B(t), t\in[0,T]. \tag{2.19}$$

The minimal AVaR is

$$AVaR(\alpha, \pi_{\varepsilon^{**}}, T) = X(0)\left(1 - \frac{R(t)}{\alpha}\left(\exp(\varepsilon^{**}||\Theta||_T) + \ln(N(N^{-1}(\alpha) - \varepsilon^{**}))\right)\right). \tag{2.20}$$

Proof of Theorem 2.1.9.1: See appendix 5.1.8

2.1.10 Portfolio optimization by maximizing expected wealth with AVaR constraint

In this section the expected wealth is maximized while constraining the AVaR.

$$(P4) \qquad \max_{\pi\in Q} E(X(T)) \text{ subject to}$$

$$AVaR(\alpha, \pi, T) \leq C^*,$$

where C^* is a given constant. We assume that $0 < C^* < X(0)$.

Theorem 2.1.10.1 *The equation*

$$\varepsilon||\Theta||_T + \ln(N(N^{-1}(\alpha) - \varepsilon)) - \ln\left(\frac{\alpha(X(0) - C^*)}{R(T)X(0)}\right) = 0,$$

has a positive root denoted by γ_2. *The optimal strategy for* $(P4)$ *is*

$$\pi_{\gamma_2}(t) = \frac{\gamma_2}{||\Theta||_T}(\sigma(t)\sigma(t)')^{-1}B(t). \tag{2.21}$$

The maximum expected wealth with the AVaR constraint is

$$E[\hat{X}(T)] = X(0)R(T)\exp(\gamma_2||\Theta||_T), \tag{2.22}$$

where $\hat{X}$ is the wealth process corresponding to π_{γ_2}.

Proof of Theorem 2.1.10.1: See appendix 5.1.9

2.1.11 Portfolio optimization by maximizing expected return subject to LEL constraint

In this section the $E[X(T)]$ is maximized subject to LEL constraint, i.e

$$(P5) \qquad \max_{\pi \in Q} E[X(T)] \text{ subject to}$$

$$LEL(\alpha, \pi, T) \leq C^{**}$$

where C^{**} is a constant. We assume that $0 < C^{**} < X(0)$.

Theorem 2.1.11.1 *The optimal strategy for $(P5)$ is*

$$\pi_{\breve{\varepsilon}}(t) = \frac{\breve{\varepsilon}}{||\Theta||_T}(\sigma(t)\sigma(t)')^{-1}B(t), t \in [0, T], \tag{2.23}$$

where $\breve{\varepsilon} = N^{-1}(\alpha) - N^{-1}\left(\frac{\alpha(X(0)-C^{**})}{X(0)R(T)}\right)$. The maximal expected wealth subject to LEL constraint is

$$E[\check{X}(T)] = X(0)R(T)\exp(\breve{\varepsilon}||\Theta||_T),$$

where $\check{X}$ denotes the wealth corresponding to the optimal strategy $\pi_{\breve{\varepsilon}}$.

Proof of Theorem 2.1.11.1: See appendix 5.1.10

2.1.12 Portfolio optimization by maximizing expected return subject to variance constraint

In this section the $E[X(T)]$ is maximized subject to variance constraint, i.e

$$(P6) \qquad \max_{\pi \in Q} E[X(T)] \text{ subject to}$$

$$variance(X(T)) \leq C^a$$

where C^a is a constant. We assume that $0 < C^a < X(0)$.

Theorem 2.1.12.1 *The equation*

$$\exp(2\varepsilon||\Theta||_T + \varepsilon^2) - \exp(2\varepsilon||\Theta||_T) = \frac{C^a}{X(0)^2 R(T)^2}$$

has a unique positive solution denoted as $\breve{\varepsilon}^*$. The optimal strategy for $(P6)$ is

$$\pi_{\breve{\varepsilon}^*}(t) = \frac{\breve{\varepsilon}^*}{||\Theta||_T}(\sigma(t)\sigma(t)')^{-1}B(t), t \in [0, T]. \tag{2.24}$$

The maximal expected wealth subject to variance constraint is

$$E[\check{X}^*(T)] = X(0)R(T)\exp(\breve{\varepsilon}^*||\Theta||_T),$$

where $\check{X}^*$ denotes the wealth corresponding to the optimal strategy $\pi_{\breve{\varepsilon}^*}$.

Proof of Theorem 2.1.12.1: See appendix 5.1.11

2.1.13 Portfolio optimization by minimizing Variance, VaR and AVaR subject to expected return constraint

In this section a risk-reward criterion is used.

$$\begin{aligned}
&(P7) \qquad \min_{\pi \in Q} VaR(\alpha, \pi, T) \text{ subject to } E[X(T)] = M, \\
&(P8) \qquad \min_{\pi \in Q} AVaR(\alpha, \pi, T) \text{ subject to } E[X(T)] = M, \\
&(P9) \qquad \min_{\pi \in Q} Variance[X(T)] \text{ subject to } E[X(T)] = M,
\end{aligned}$$

where M is a constant. It is further assumed that $M > X(0)\exp(rT)$.

Theorem 2.1.13.1 *Let*

$$\zeta = \ln\left(\frac{M}{X(0)R(T)}\right). \tag{2.25}$$

The optimal strategy for (P7), (P8) and (P9) is

$$\pi_*(t) = \frac{\zeta}{||\Theta||_T^2}(\sigma(t)\sigma(t)')^{-1}B(t). \tag{2.26}$$

Proof of Theorem 2.1.13.1: See appendix 5.1.12

2.1.14 Numerical analysis

To keep the exposition simple, the interest rate and the volatility matrix are assumed constant. Three stocks are considered. The drift of the stock i is denoted by $b_i(t)$, where $b_i(t) = u_i + \beta_i \cos(\varphi_i t)$. Here φ_i denotes the economic cycle, u_i denotes the average rate of return and β_i denotes the deviation around u_i. The variance covariance matrix is denoted by $\Gamma(t)dt$, and

$$\Gamma(t) = \sigma(t)\sigma(t)' = v(t)\rho(t)v(t),$$

where the diagonal matrix $v(t)$ denotes the standard deviation and $\rho(t)$ denotes the correlation matrix. We take $v(t) = diag[0.2, 0.25, 0.3]$,

$$\rho(t) = \begin{bmatrix} 1.00 & -0.60 & -0.80 \\ -0.60 & 1.00 & 0.50 \\ -0.80 & 0.50 & 1.00 \end{bmatrix}.$$

The time horizon considered is 8 years and time granularity for the parameters below are yearly. We have $u_1 = 0.08$, $u_2 = 0.1$, $u_3 = 0.12$ and $r = 0.05$. Let $\varphi_1 = \varphi_2 = \varphi_3 = 0.75$, $\beta_1 = 0.75\beta, \beta_2 = 0.5\beta, \beta_3 = 0.25\beta$ and $\beta = 0.015$. Take $C = C^* = 0.7 \cdot X(0)$.

Optimal Strategy with Risk Constraints

In this section, the optimal strategy for the maximal expected wealth with VaR, AVaR, LEL and variance constraints (i.e optimal for problems (P2), (P4), (P5) and (P6)) is considered and illustrated in figure 2.1 to 2.4. The highest portfolio weight is in stock 1 as it is the least volatile. The optimal strategy for stock 1 in P2 is higher than that of P4, followed by P5 and P6. Variance is the most conservative risk measure, followed by LEL and AVaR. This is due to the fact that variance is a symmetric risk measure and penalizes both loss and gains. This makes the portfolio seem more risky and hence calculates a lower optimal strategy. LEL on the other hand is a downside risk measure and hence shows the risk associated with losses only, however it operated under the assumption that the investor is risk neutral. VaR is a another downside risk measure but it ignores the losses beyond the α threshold. AVaR deals with the disadvantages associated with variance, LEL and VaR , hence is considered to be the best risk measure. Furthermore, stock 3 is more volatile than stock 2, however, the portfolio weight in stock 3 is higher in stock 2 due to diversification effect introduced by the correlation matrix.

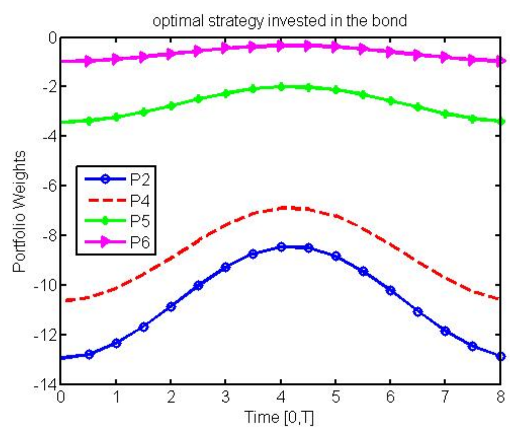

Figure 2.1: Optimal portfolio in money market

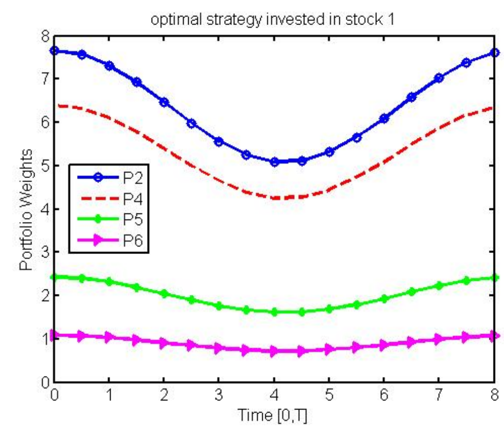

Figure 2.2: Optimal portfolio in stock 1

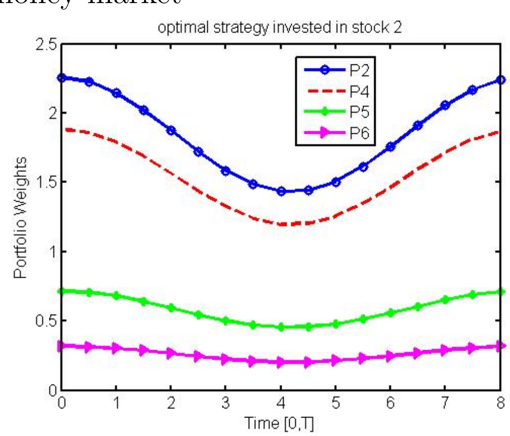

Figure 2.3: Optimal portfolio in stock 2

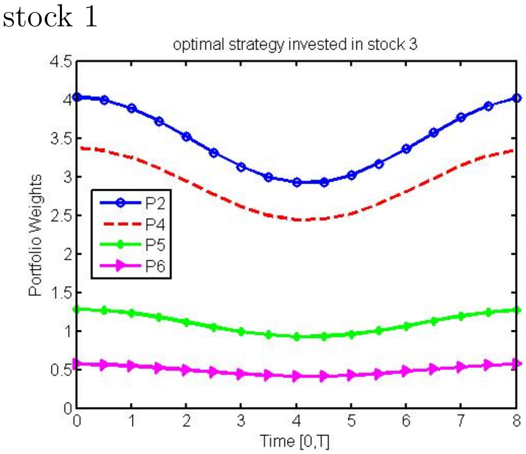

Figure 2.4: Optimal portfolio in stock 3

Expected Return with Risk Constraints

As illustrated in fig 2.5 the maximal expected return occurs in the order P2 followed by P4, P5 and P6. This is intuitive since variance is the most conservative risk measure compared to LEL, AVaR and VaR. The expected return is plotted where $T \in [0, 8]$.

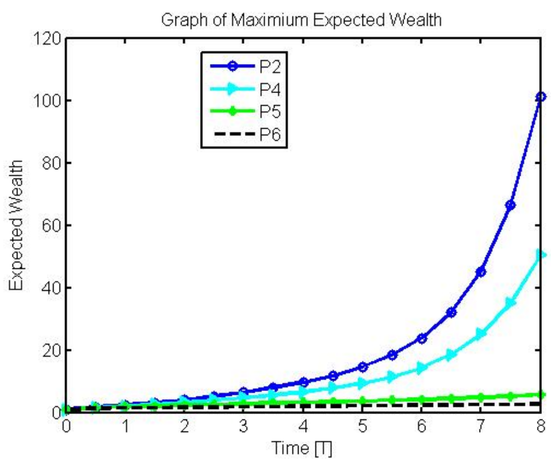

Figure 2.5: Expected return for P2, P4, P5 and P6

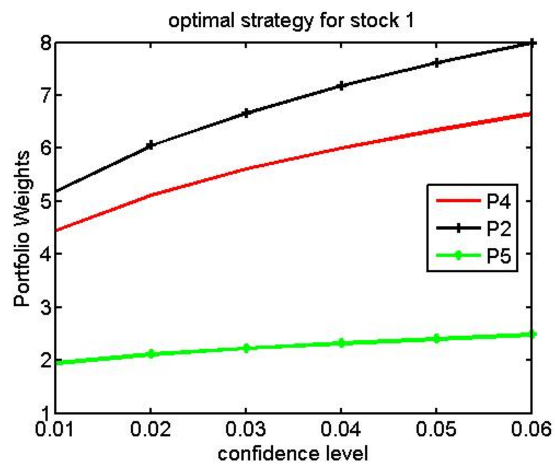

Figure 2.6: Optimal strategy with various confidence intervals

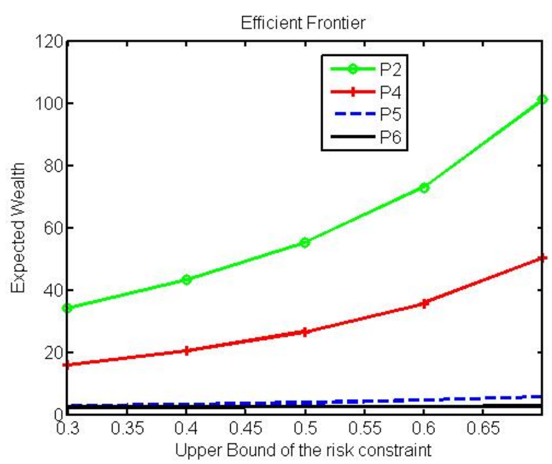

Figure 2.7: Efficient frontier

Optimal Strategy with different confidence interval

In fig 2.6 the plot shows the optimal strategy in stock 1 for P2, P4 and P5 for the confidence level, $\alpha \in [0.01, 0.06]$. As α increases the portfolio weight increases for P2, P4 and P5.

Efficient Frontier

In figure 2.7, we take $C = C^* = [0.3, 0.7]$. The plot shows that as C increases, the portfolio weights also increase for P2, P4, P5 and P6. The reason being that the higher the risk threshold, the higher the risk appetite of the investor. Hence, the amount invested in the stock increases.

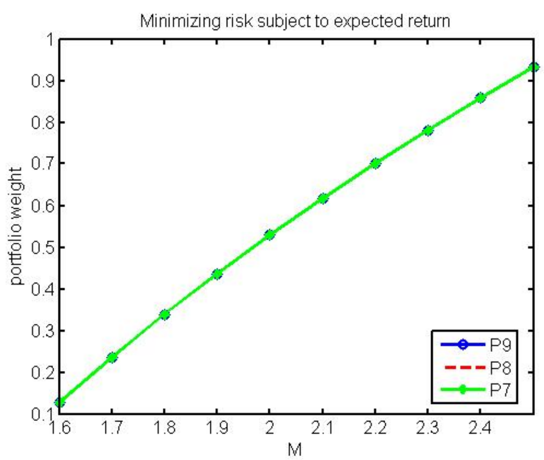

Figure 2.8: Optimal strategy invested in the stock P7, P8 and P9

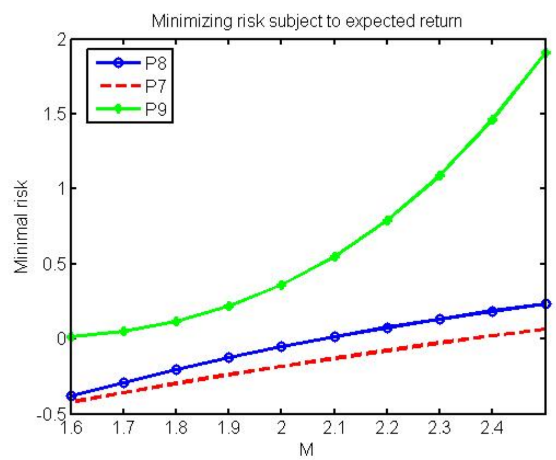

Figure 2.9: Minimized risk for P7, P8 and P9

Minimal VaR, AVaR and Variance with an expected return constraint

In this section P7, P8 and P9 are considered, with $M = [1.1 \cdot X(0), 1.3 \cdot X(0)]$. The minimal risk is illustrated in fig 2.9. The minimal variance is significantly higher than that of the minimal VaR or minimal AVaR. Furthermore, the minimal VaR is lower than minimal AVaR.

2.2 Constant trading strategies

Portfolio optimization in a Black-Scholes setting within the class of constant strategies is studied in the paper of Susanne Emmer and Korn (2001). Capital-at-Risk (CaR) and variance are the risk measures considered in that paper. In this section we restrict to constant portfolio strategies as well.

2.2.1 Assumptions

The same assumptions as in the previous chapter apply, in addition

- $\pi(t)$ is assumed to be constant.

  Since $||\sigma(t)'\pi||^2 = \pi'\sigma(t)\sigma(t)'\pi$, it follows that

$$\begin{aligned} \int_0^T ||\sigma(t)'\pi||^2 dt &= \int_0^T \pi'\sigma(t)\sigma(t)'\pi dt \\ &= \pi'\Sigma\pi > 0. \end{aligned}$$

  Here $\Sigma = (\Sigma_{ij})_{(i,j)=1,\ldots,m}$ is given by

$$\Sigma_{ij} = \int_0^T (\sigma(t)\sigma(t)')_{ij} dt \tag{2.27}$$

- Assume that Σ_{ij} is invertible.

2.2.2 Wealth dynamics

Corollary 2.2.2.1 *The investor invests π in the risky stock and $1-\pi$ in the money market. The initial wealth $X(0)$ is positive. The wealth follows a self-financing strategy, thus*

$$dX(t) = X(t)(r(t)dt + B(t)'\pi dt + \pi'\sigma(t)dW(t)). \tag{2.28}$$

Therefore

$$X(T) = X(0)\exp\left(\int_0^T (r(t) + B(t)'\pi)\,dt - \frac{1}{2}\pi'\Sigma\pi + \int_0^T \pi'\sigma(t)dW(t)\right). \tag{2.29}$$

2.2.3 Expectation of the wealth

Corollary 2.2.3.1 *The expectation of the portfolio at some $T > 0$ is*

$$E[X(T)] = X(0)R(T)\exp\left(\int_0^T B(t)'\pi dt\right), \tag{2.30}$$

and the variance is

$$\begin{aligned} var[X(T)] &= X(0)^2R(T)^2\Big(\exp\left(2\int_0^T B(t)'\pi dt + \pi'\Sigma\pi\right) \\ &\quad - \exp\left(2\int_0^T B(t)'\pi dt\right) \end{aligned} \tag{2.31}$$

2.2.4 Value at Risk (VaR)

Corollary 2.2.4.1 *For a wealth process $(X(t))_{t\in[0,T]}$, a given α confidence threshold and time T,*

$$VaR_\alpha(X(T)) = X(0)\left(1 - \exp\left(\int_0^T (r(t) + B(t)'\pi)dt - \frac{1}{2}\pi'\Sigma\pi + N^{-1}(\alpha)\sqrt{\pi'\Sigma\pi}\right)\right). \tag{2.32}$$

Define the function $VaR(\alpha, \pi, T)$ by

$$VaR(\alpha, \pi, T) := X(0)\left(1 - \exp\left(\int_0^T (r(t) + B(t)'\pi)dt - \frac{1}{2}\pi'\Sigma\pi + N^{-1}(\alpha)\sqrt{\pi'\Sigma\pi}\right)\right). \tag{2.33}$$

2.2.5 Average Value at Risk (AVaR)

Corollary 2.2.5.1 *For a wealth process $(X(t))_{t\in[0,T]}$, a given α confidence threshold and time T,*

$$AVaR_\alpha(X(T)) = X(0)\left(1 - \frac{R(T)}{\alpha}\left(\exp\left(\int_0^T B(t)'\pi dt + \ln(N(N^{-1}(\alpha) - \sqrt{\pi'\Sigma\pi}))\right)\right)\right). \tag{2.34}$$

Define the function $AVaR(\alpha, \pi, T)$ by

$$AVaR(\alpha, \pi, T) := X(0)\left(1 - \frac{R(T)}{\alpha}\left(\exp\left(\int_0^T B(t)'\pi dt + \ln(N(N^{-1}(\alpha) - \sqrt{\pi'\Sigma\pi}))\right)\right)\right). \tag{2.35}$$

2.2.6 Limited Expected Loss

Corollary 2.2.6.1 *For a wealth process* $(X(t))_{t\in[0,T]}$ *and a given confidence threshold* α *at time* T*,*

$$LEL_{\alpha}(X(T)) = X(0)\left(1 - \frac{R(T)}{\alpha}(N(N^{-1}(\alpha) - \sqrt{\pi'\Sigma\pi}))\right). \tag{2.36}$$

Define the function $LEL(\alpha, \pi, T)$ by

$$LEL(\alpha, \pi, T) := X(0)\left(1 - \frac{R(T)}{\alpha}(N(N^{-1}(\alpha) - \sqrt{\pi'\Sigma\pi}))\right). \tag{2.37}$$

2.2.7 Portfolio optimization by minimizing VaR

In this section the minimum VaR of the portfolio is calculated. The optimization problem considered is

$$(P10) \qquad \min_{\pi\in Q} VaR(\alpha, \pi, T).$$

The portfolio optimization problems are projected onto the family of surfaces,

$$Q_{\varepsilon} = \{\pi : \pi'\Sigma\pi = \varepsilon^2\}.$$

It follows that

$$Q = \bigcup_{\varepsilon\geq 0} Q_{\varepsilon}.$$

Recall that Σ is defined by (2.27).

Corollary 2.2.7.1 *Set* $\Upsilon = \int_0^T B(t)dt$ *and* $\Omega = \sqrt{\Upsilon'\Sigma^{-1}\Upsilon}$*. The optimal strategy for*

$(P10)$ is

$$\pi_{\varepsilon^*} = \frac{\max(0, \varepsilon^*)}{\Omega} \Sigma^{-1} \Upsilon \tag{2.38}$$

where

$$\varepsilon^* = \Omega + N^{-1}(\alpha). \tag{2.39}$$

The minimal VaR is given by

$$VaR(\alpha, \pi_{\varepsilon^*}, T) = X(0) \left(1 - R(T) \exp\left(\frac{1}{2} \max(0, \varepsilon^*)^2\right)\right). \tag{2.40}$$

2.2.8 Portfolio optimization by maximizing expected wealth with VaR constraint

This is a form of Markowitz mean-variance optimization, the difference being that the variance is replaced by VaR. The optimization problem is

$$(P11) \qquad \max_{\pi \in Q} E[X(T)] \text{ subject to } VaR(\alpha, \pi, T) \leq C,$$

where C is a constant. It is assumed that $0 < C < X(0)$.

Corollary 2.2.8.1 *Set $\Upsilon = \int_0^T B(t)dt$ and $\Omega = \sqrt{\Upsilon' \Sigma^{-1} \Upsilon}$. Let*

$$\varepsilon_1 = \Omega + N^1(\alpha) + \sqrt{(\Omega + N^{-1}(\alpha))^2 - 2 \ln\left(\frac{1}{R(T)}\left(1 - \frac{C}{X(0)}\right)\right)}. \tag{2.41}$$

The optimal strategy for $(P11)$ is

$$\pi_{\varepsilon_1} = \frac{\varepsilon_1}{\Omega} \Sigma^{-1} \Upsilon. \tag{2.42}$$

The maximum expected wealth subject to the VaR constraint is

$$E[X^{\pi_{\varepsilon_1}}(T)] = X(0)R(T)\exp\Big(\Omega\Big(\Omega + N^1(\alpha) + \sqrt{(\Omega + N^{-1}(\alpha))^2 - 2\ln\Big(\frac{1}{R(T)}\Big(1 - \frac{C}{X(0)}\Big)\Big)}\Big)\Big), \quad (2.43)$$

where $X^{\pi_{\varepsilon_1}}$ is the wealth corresponding to π_{ε_1}.

2.2.9 Portfolio optimization by minimizing AVaR

In the section the optimization problem considered is

$$(P12) \qquad \inf_{\pi\in Q} AVaR(\alpha,\pi,T).$$

Corollary 2.2.9.1 *The optimization problem*

$$\sup_{\varepsilon\geq 0}\Big[\varepsilon\Omega + \ln(N(N^{-1}(\alpha) - \varepsilon))\Big]$$

has at finite solution denoted by $\hat{\varepsilon}$. The optimal strategy for $(P12)$ is

$$\pi_{\hat{\varepsilon}} = \frac{\hat{\varepsilon}}{\Omega}\Sigma^{-1}\Upsilon. \quad (2.44)$$

The minimal AVaR is

$$AVaR(\alpha,\pi_{\hat{\varepsilon}},T) = X(0)\Bigg(1 - \frac{R(T)}{\alpha}\Big(\exp\Big(\hat{\varepsilon}\Omega + \ln(N(N^{-1}(\alpha) - \hat{\varepsilon})\Big)\Big)\Bigg). \quad (2.45)$$

2.2.10 Portfolio optimization by maximizing expected wealth with AVaR constraint

In this section the AVaR replaces the variance in a Markowtiz Mean-Variance optimization problem, so the optimization problem considered is

$$(P13) \qquad \max_{\pi \in Q} E[X(T)] \text{ subject to } AVaR(\alpha, \pi, T) \leq C^*,$$

where C^* is a constant. Assume that $0 < C^* < X(0)$.

Corollary 2.2.10.1 *The equation*

$$\varepsilon\Omega + \ln(N(N^{-1}(\alpha) - \varepsilon)) - \ln\left(\frac{\alpha(X(0) - C^*)}{R(T)X(0)}\right) = 0, \tag{2.46}$$

has one positive root denoted by $\bar{\varepsilon}$. *The optimal strategy for* $(P13)$ *is given by*

$$\pi_{\bar{\varepsilon}} = \frac{\bar{\varepsilon}}{\Omega}\Sigma^{-1}\Upsilon. \tag{2.47}$$

The maximum expected wealth with the AVaR constraint is

$$E[X^{\pi_{\bar{\varepsilon}}}(T)] = X(0)R(T)\left(\exp\left(\bar{\varepsilon}\Omega\right)\right), \tag{2.48}$$

where $X^{\pi_{\bar{\varepsilon}}}$ *is the wealth corresponding to* $\pi_{\bar{\varepsilon}}$.

2.2.11 Portfolio optimization by maximizing expected wealth subject to LEL constraint

In this section the expected wealth is maximized subject to LEL constraint, i.e

$$(P14) \qquad \max_{\pi \in Q} E[X(T)] \text{ subject to}$$

$$LEL(\alpha, \pi, T) \leq C^{**}$$

where C^{**} is a constant. We assume that $0 < C^{**} < X(0)$.

Corollary 2.2.11.1 *The optimal strategy for* $(P14)$ *is*

$$\pi_{\varepsilon^a}(t) = \frac{\varepsilon^a}{\Omega}\Sigma^{-1}\Upsilon, t \in [0, T], \tag{2.49}$$

where $\varepsilon^a = N^{-1}(\alpha) - N^{-1}\left(\frac{\alpha(X(0)-C^{**})}{X(0)R(T)}\right)$*. The maximal expected wealth subject to LEL constraint is*

$$E[X^a(T)] = X(0)R(T)\exp(\varepsilon^a\Omega),$$

where X^a *denotes the wealth corresponding to the optimal strategy* π_{ε^a}*.*

2.2.12 Portfolio optimization by maximizing expected return subject to variance constraint

In this section the $E[X(T)]$ is maximized subject to variance constraint, i.e

$$(P15) \qquad \max_{\pi \in Q} E[X(T)] \text{ subject to}$$

$$variance(X(T)) \leq C^b$$

where C^b is a constant. We assume that $0 < C^b < X(0)$.

Corollary 2.2.12.1 *The equation*

$$\exp(2\varepsilon\Omega + \varepsilon^2) - \exp(2\varepsilon\Omega) = \frac{C^b}{X(0)^2 R(T)^2}$$

has a unique positive solution denoted as $\check{\varepsilon}^{**}$. The optimal strategy for $(P15)$ is

$$\pi_{\check{\varepsilon}^{*}}(t) = \frac{\check{\varepsilon}^{**}}{\Omega}\Sigma^{-1}\Upsilon, t \in [0, T]. \tag{2.50}$$

The maximal expected wealth subject to variance constraint is

$$E[\check{X}^{**}(T)] = X(0)R(T)\exp(\check{\varepsilon}^{**}\Omega),$$

where $\check{X}^{**}$ denotes the wealth corresponding to the optimal strategy $\pi_{\check{\varepsilon}^{**}}$.

2.2.13 Portfolio optimization by minimizing Variance, VaR and AVaR subject to expected return constraint

In this section a risk-reward criterion is used.

$$(P16) \qquad \min_{\pi \in Q} VaR(\alpha, \pi, T) \text{ subject to } E[X(T)] = M,$$
$$(P17) \qquad \min_{\pi \in Q} AVaR(\alpha, \pi, T) \text{ subject to } E[X(T)] = M,$$
$$(P18) \qquad \min_{\pi \in Q} Variance[X(T)] \text{ subject to } E[X(T)] = M,$$

where M is a constant. It is further assumed that $M > X(0)\exp(rT)$.

Theorem 2.2.13.1 *Let*

$$\zeta = \ln\left(\frac{M}{X(0)R(T)}\right). \tag{2.51}$$

The optimal strategy for (P16), (P17) and (P18) is

$$\pi_*(t) = \frac{\zeta}{\Omega}\Sigma^{-1}\Upsilon. \tag{2.52}$$

2.2.14 Numerical analysis

The previous data is used in this section as well. In figures 2.10 to 2.13 the plot illustrates the optimal strategy for P11,P13, P14 and P15. The portfolio weights invested in stocks in P11 are higher than P13, followed by P14 and P15. From figure 2.14 we see that P2 and P4 yields a higher expected return than P11 and P13 respectively. From figure 2.15 we see that P5 has a higher expected return than P15. However the difference between P6 and P15 is negligible. Although deterministic optimal strategies perform better than constants strategies, from the plots we see that the improvement is marginal.

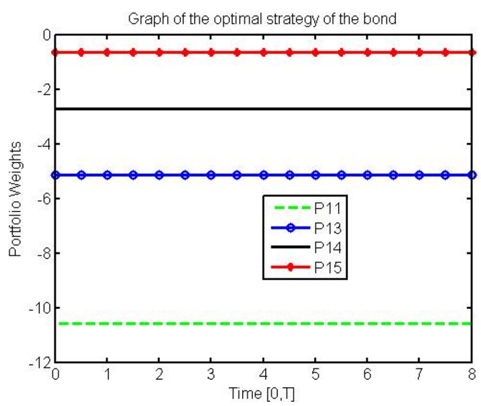

Figure 2.10: Constant optimal portfolio in money market

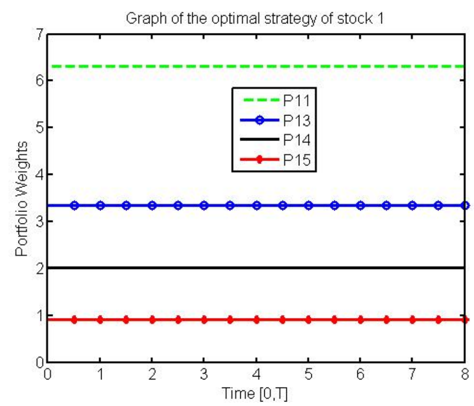

Figure 2.11: Constant optimal portfolio in stock 1

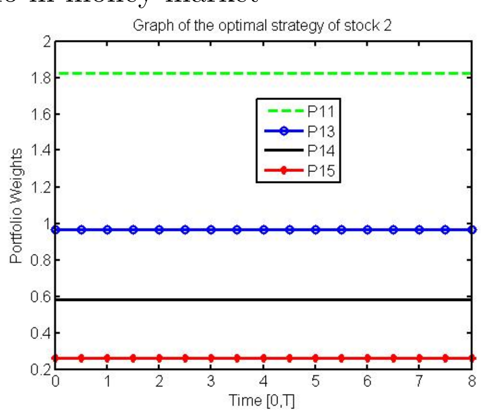

Figure 2.12: Constant optimal portfolio in stock 2

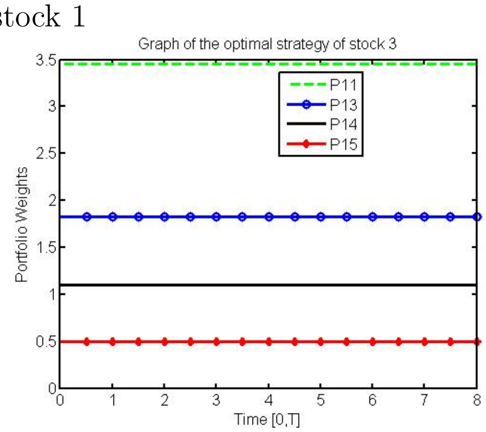

Figure 2.13: Constant optimal portfolio in stock 3

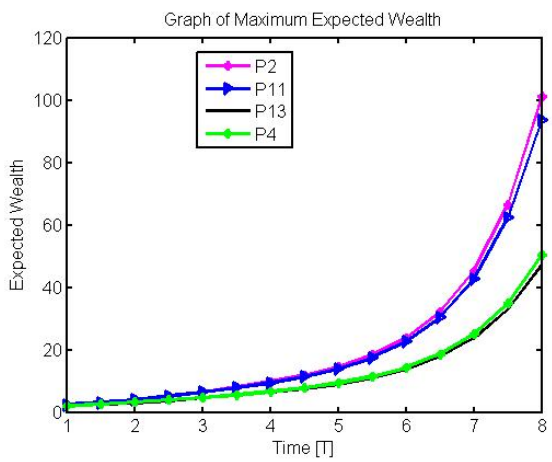

Figure 2.14: Expected return for P2,P4, P11 and P13

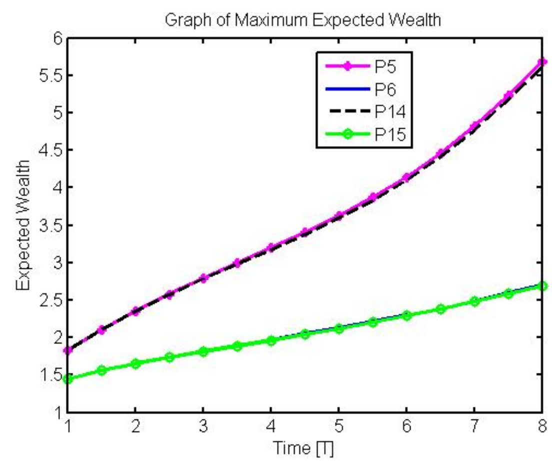

Figure 2.15: Expected return for P5, P6, P14 and P15

Chapter 3

Exponential Ornstein Uhlenbeck process

Over the last past decades the mean-reverting model for stocks and commodity dynamics have received great attention (d' Aspremont, 2008). Commodities, futures and exchange rates are intuitively assumed to be mean-reverting and there is significant research to support this assumption, [see (Anderson, 2007), (Schwartz, 1997), (Lutz, 2010), (Gil-Alana, 2000) and (Ali Lari-Lavassani and Ware, 2001)]. In the work of Dmitrasinovic-Vidovic and Ware (2011), the risky assets are assumed to follow an exponential Ornstein Uhlenbeck process. They used Capital-at-Risk (CaR) while performing portfolio optimization and they restrict to the class of constant strategies. Short selling of the stocks and the money market is permitted and unconstrained. In this chapter the same assumptions are made. The novelty of our work is that VaR and AVaR risk measures are used in portfolio optimization.

3.1 Assumptions

The following assumptions pertain to this model

- The securities are perfectly divisible.
- Negative positions are possible.
- Re-balancing of the portfolio occurs at no transaction cost.
- $m + 1$ assets are traded continuously over a finite horizon $[0, T]$.
- m assets follow the exponential O.U model.

$$\frac{dS_i(t)}{S_i(t)} = k_i(A_i - \ln(S_i(t)))dt + \sum_j \sigma_{ij} dW_j(t), \qquad S_i(0) > 0, i = 1, ..., m. \quad (3.1)$$

  The mean-reversion rate, $k_i > 0$ and $A_i \in R$. Here $W(t) = (W_1(t), W_2(t)...W_m(t))'$. denotes a m-dimensional Brownian motion.

- The matrix $\sigma = (\sigma_{ij})$ where $i = 1, ...m$ and $j = 1, ...m.$ is invertible and denotes the volatility matrix.
- One of the assets is risk less and it is denoted as $S_0(t)$,

$$S_0(t) = \exp(rt). \quad (3.2)$$

- Let $N_i(t)$ denote the number of shares of stock i held in the portfolio at t, and $X(t)$ the wealth at time t. Then $\pi_i = \frac{N_i(t)S_i(t)}{X(t)}$ is the proportion of wealth invested in stock i .

- $\pi = (\pi_1, \pi_2, ..., \pi_m)'$ and is assumed constant.

Furthermore, (3.1) can be written as

$$dS_i(t) = S_i(t)(\psi_i - \theta_i \ln(S_i(t)))dt + \sigma_i dW(t)), \qquad i = 1, ..., m. \tag{3.3}$$

Here

$$\psi_i = k_i A_i, \qquad \theta_i = k_i \text{ and } \sigma_i = (\sigma_{i1}, ...\sigma_{im})'.$$

3.2 Wealth Equation

Proposition 3.2.1 *The self-financing equation for the wealth is*

$$dX(t) = X(t)\left(rdt + \sum_{i=1}^{m} \pi_i[(\psi_i - r - \theta_i \ln(S_i(t)))dt + \sigma_i dW(t)]\right). \tag{3.4}$$

Moreover

$$\begin{aligned} X(t) &= X(0)\exp\Big(\Big(r + \pi'(\psi - r\boldsymbol{1}) - \frac{||\pi'\sigma||^2}{2}\Big)t + \pi'\Xi(t,\theta) \\ &+ \sum_{i}^{m} \pi_i\sigma_i \exp(-\theta_i t)\int_0^t \exp(\theta_i s)dW(s)\Big), \end{aligned} \tag{3.5}$$

where

$$\omega_i = \psi_i - \frac{||\sigma_i||^2}{2}, \qquad Z_i(t) = \ln(S_i(t)), \qquad K(t,\theta) = \int_0^t \exp(-\theta s)ds$$

and

$$\Xi_i(t,\theta_i) = K(t,\theta_i)(\omega_i - Z_i(0)\theta_i) - \omega_i t$$

Proof of Proposition 3.2 See appendix 5.2.1

3.3 Expectation of the wealth

Proposition 3.3.1 *The expectation of the wealth is*

$$E(X(t)) = X(0)\exp\left(rt + \pi' g(t) - \frac{t||\pi'\sigma||^2}{2} + \frac{1}{2}\pi' G(t,\theta)\pi\right) \tag{3.6}$$

and the variance is

$$Var(X(t)) = X(0)^2 \exp\left(2rt + 2\pi' g(t) - t||\pi'\sigma||^2 + \pi' G(t,\theta)\pi\right)\left(\exp(\pi' G(t,\theta)\pi) - 1\right). \tag{3.7}$$

Here

$$g(t) = (\psi - r)t + \Xi(t,\theta) \tag{3.8}$$

and

$$G_{ij}(t,\theta) = K(t, \theta_i + \theta_j)\sigma_i \sigma_j'. \tag{3.9}$$

Proof of Proposition 3.3.1 See appendix 5.2.2

3.4 Value at Risk

Proposition 3.4.1 *The VaR for the gain/losses, $X(t) - X(0)$ is*

$$VaR_\alpha(X(t)) = X(0)\left(1 - \exp\left(rt + \pi g(t) - \frac{t}{2}||\pi'\sigma||^2 + N^{-1}(\alpha)\sqrt{\pi' G(t,\theta)\pi}\right)\right). \tag{3.10}$$

Proof of Proposition 3.4.1 See appendix 5.2.3

3.4.1 Properties of the Value at Risk

Define the function $VaR(\alpha, \pi, t)$ by

$$VaR(\alpha, \pi, t) := X(0)\left(1 - \exp\left(rt + f(\pi, t)\right)\right), \tag{3.11}$$

where

$$f(\pi, t) := \pi' g(t) - \frac{t}{2}||\pi'\sigma||^2 + N^{-1}(\alpha)\sqrt{\pi' G(t, \theta)\pi}. \tag{3.12}$$

Lemma 3.4.1.1 *The function $\pi \to f(\pi, t)$ is concave.*

Proof of Lemma 3.4.1.1 See appendix 5.2.4

3.5 Average Value at Risk

Proposition 3.5.1 *The AVaR for the gain/losses, $X(t) - X(0)$ is*

$$\begin{aligned} AVaR_\alpha(X(t)) &= X(0)\Big(1 - \frac{1}{\alpha}\exp\Big(rt + \pi' g(t) - \frac{t}{2}||\pi'\sigma||^2 \\ &+ \frac{\pi' G(t, \theta)\pi}{2} + \ln(N(N^{-1}(\alpha) - \sqrt{\pi' G(t, \theta)\pi}))\Big)\Big). \end{aligned} \tag{3.13}$$

Proof of Proposition 3.5.1 See appendix 5.2.5

Standing Assumption: Let

$$\bar{G}(t, \theta) := G(t, 0) - G(t, \theta)$$

where $G(t, 0) := t\sigma_i\sigma_j'$. Assume that $\bar{G}(t, \theta)$ is positive definite.

Remark: The assumption is met if $\sigma_i = \sigma_j$.

3.5.1 Properties of the Average Value at Risk

Define the function $AVaR(\alpha, \pi, t)$ as

$$AVaR(\alpha, \pi, t) \;\; := \;\; X(0)\Big(1 - \frac{1}{\alpha}\exp\Big(rt + f_1(\pi, t))\Big)\Big), \tag{3.14}$$

where

$$f_1(\pi, t) \;\; := \;\; \pi' g(t) - \frac{t}{2}||\pi'\sigma||^2 + \frac{\pi' G(t,\theta)\pi}{2} + \ln(N(N^{-1}(\alpha) - \sqrt{\pi' G(t,\theta)\pi}).$$

Lemma 3.5.1.1 *The function $\pi \to f_1(\pi, t)$ is concave.*

Proof of Lemma 3.4.1.1 See appendix 5.2.6

3.6 Portfolio optimization by minimizing VaR

The optimization problem considered is

$$(P19) \qquad \min_{\pi} VaR(\alpha, \pi, T)$$

Theorem 3.6.1 *If $g(T)'G(T,\theta)^{-1}g(T) > (N^{-1}(\alpha))^2$, then the equation*

$$1 = g(T)'\left(aT\sigma\sigma' - N^{-1}(\alpha)G(T,\theta)\right)^{-1} G(T,\theta)\left(aT\sigma\sigma' - N^{-1}(\alpha)G(T,\theta)\right)^{-1} g(T) \tag{3.15}$$

has a unique positive solution denoted by a^*. *Then the optimal strategy for* $(P19)$ *is*

$$\pi^* = \left(T\sigma\sigma' - \frac{N^{-1}(\alpha)G(T,\theta)}{a^*}\right)^{-1} g(T). \tag{3.16}$$

Else if $g(T)'G(T,\theta)^{-1}g(T) \leq (N^{-1}(\alpha))^2$ *then* (3.15) *has no positive solution and the optimal strategy is* $\pi^* = 0$.

Proof of Theorem 3.6.1 See appendix 5.2.7

3.7 Portfolio optimization by maximizing expected wealth subject to VaR constraint

The optimization problem considered is

$$(P20) \qquad \max_{\pi} E[X(T)] \text{ subject to}$$

$$VaR(\alpha, \pi, T) \leq C,$$

where C is a constant. It is futher assumed that $0 < C < X(0)$.

Theorem 3.7.1 *The optimal portfolio strategy for* $(P20)$ *is given by*

$$\tilde{\pi} = \left(T\sigma\sigma'(\lambda^* - 1) + G(T,\theta) - \lambda^* \frac{N^{-1}(\alpha)G(T,\theta)}{a^*}\right)^{-1} g(T)(\lambda^*), \tag{3.17}$$

where (a^*, λ^*) *is a solution of the following system of equations:*

$$||\sqrt{G(T,\theta)}(aT\sigma\sigma'(\lambda - 1) + aG(T,\theta) - \lambda N^{-1}(\alpha)G(T,\theta))^{-1}g(T)(\lambda)||^2 - 1 = 0 \tag{3.18}$$

and

$$\frac{T}{2}||\sigma\left(T\sigma\sigma'(\lambda-1)+G(T,\theta)-\lambda\frac{N^{-1}(\alpha)G(T,\theta)}{a}\right)^{-1}g(T)(\lambda)||^2-N^{-1}(\alpha)a-rT-$$

$$g(T)'(\lambda)\left(T\sigma\sigma'(\lambda-1)+G(T,\theta)-\lambda\frac{N^{-1}(\alpha)G(T,\theta)}{a}\right)^{-1}g(T)+\ln\left(1-\frac{C}{X(0)}\right)=0. \tag{3.19}$$

Proof of Theorem 3.7.1 See appendix 5.2.8

3.8 Portfolio optimization by minimizing AVaR

In this section the optimization problem considered is

$$(P21) \qquad \inf_{\pi} AVaR(\alpha,\pi,T)$$

Theorem 3.8.1 *The optimal strategy for* $(P21)$ *is given by*

$$\breve{\pi}=\Big(T\sigma\sigma'+G(T,\theta)\Big(\frac{\exp(-\frac{(N^{-1}(\alpha)-\breve{a})^2}{2})}{\breve{a}(\sqrt{2\pi_c})N(N^{-1}(\alpha)-\breve{a})}-1\Big)\Big)^{-1}g(T), \tag{3.20}$$

where $\breve{a}$ *is a positive solution of*

$$h_1(a)=1. \tag{3.21}$$

Here

$$h_1(a) \quad := \quad ||\sqrt{G(T,\theta)}(aT\sigma\sigma'+G(T,\theta)(w-a))^{-1}g(T)||^2,$$

$$w := \frac{\exp(-\frac{(N^{-1}(\alpha)-a)^2}{2})}{(\sqrt{2\pi_c})N(N^{-1}(\alpha)-a)}.$$

If $h_1(a) = 1$ *has no positive solutions then* $\breve{\pi} = 0$.

Remark: All the positive solutions of (3.21) are found numerically. Then we pick the one which optimizes $(P21)$.

Proof of Theorem 3.8.1 See appendix 5.2.9

3.9 Portfolio optimization by maximizing expected wealth subject to AVaR constraint

The optimization problem considered is

$$(P22) \qquad \sup_{\pi} E[X(T)] \text{ subject to } AVaR(\alpha, \pi, T) \leq C^*,$$

where C^* is a constant. Assume that $0 < C^* < X(0)$.

Theorem 3.9.1 *The optimal portfolio strategy for* $(P22)$ *is given by*

$$\breve{\pi}^* = \left(T\sigma\sigma'(\breve{\lambda}^* - 1) + G(T,\theta)\Big((1 - \breve{\lambda}^* + \frac{\breve{\lambda}^* \exp(-\frac{(N^{-1}(\alpha)-\breve{a}^*)^2}{2})}{\breve{a}^*(\sqrt{2\pi_c})N(N^{-1}(\alpha)-\breve{a}^*)}\Big)\right)^{-1} g(T)(\breve{\lambda}^*), \tag{3.22}$$

where $(\breve{a}^*, \breve{\lambda}^*)$ *is a solution of the following system of equations:*

$$||\sqrt{G(T,\theta)}(aT\sigma\sigma'(\lambda - 1) + G(T,\theta)(a(1-\lambda) + \lambda w))^{-1} g(T)(\lambda)||^2 - 1 = 0 \tag{3.23}$$

and

$$\frac{T}{2}||\sigma\left(T\sigma\sigma'(\lambda-1)+G(T,\theta)\Big(1-\lambda+\frac{\lambda w}{a}\Big)\right)^{-1}g(T)(\lambda)||^2-\frac{a^2}{2}-rT$$
$$-g(T)'(\lambda)\left(T\sigma\sigma'(\lambda-1)+G(T,\theta)\Big(1-\lambda+\frac{\lambda w}{a}\Big)\right)^{-1}g(T)$$
$$-\ln(N(N^{-1}(\alpha)-a))+\ln\left(\alpha\left(1-\frac{C^*}{X(0)}\right)\right)=0. \tag{3.24}$$

Here

$$w := \frac{\exp(-\frac{(N^{-1}(\alpha)-a)^2}{2})}{(\sqrt{2\pi_c})N(N^{-1}(\alpha)-a)}.$$

Proof of Theorem 3.9.1 See appendix 5.2.10

3.10 Portfolio optimization by minimizing VaR, AVaR and Variance subject to expected wealth constraint

The optimization problems under consideration here are

$$(P23) \qquad \min_{\pi} VaR(\alpha,\pi,T) \text{ subject to } E[X(T)]=M,$$
$$(P24) \qquad \min_{\pi} AVaR(\alpha,\pi,T) \text{ subject to } E[X(T)]=M,$$
$$(P25) \qquad \min_{\pi} Variance[X(T)] \text{ subject to } E[X(T)]=M,$$

where M is a constant. It is further assumed that $M > X(0)\exp(rT)$.

Theorem 3.10.1 *The optimal portfolio strategy for $(P23)$, $(P24)$ and $(P25)$ is given* *by*

$$\tilde{\pi}^a = \left(T\sigma\sigma'(\lambda^*) + G(T,\theta)(2-\lambda^*)\right)^{-1} g(T)(\lambda^*), \tag{3.25}$$

where λ^* *is a solution of the nonlinear equation:*

$$\begin{aligned} &||rT + g(T)'(\lambda)\left(T\sigma\sigma'(\lambda) + G(T,\theta)(2-\lambda)\right)^{-1} g(T) + \frac{1}{2}||\sqrt{G(T,\theta)}(T\sigma\sigma'(\lambda) \\ &+G(T,\theta)(2-\lambda))^{-1} g(T)(\lambda)||^2 - \frac{T}{2}||\sigma\left(T\sigma\sigma'(\lambda) + G(T,\theta)(2-\lambda)\right)^{-1} g(T)(\lambda)||^2 \\ &-\ln\left(\frac{M}{X(0)}\right) = 0. \end{aligned} \tag{3.26}$$

Proof of Theorem 3.10.1 See appendix 5.2.11

3.11 Numerical analysis

A one-dimensional model (i.e $m = 1$) is analyzed. The parameters used are as follows: $T = 1.5, k_1 = 0.5, L_1 = \ln(7), \gamma = 1, X(0) = 100, \alpha = 0.05, S(0) = 5, \sigma = 0.2$ and $C = C^* = 0.5X(0)$.

Optimal portfolio strategy

The optimal strategy for the mean-reverting model in $P20$ and $P22$ are plotted in fig 3.1. AVaR is more consecutive than VaR, hence, the optimal strategy in the stock is higher for P20 than P22.

Efficient Frontier

In this section $C = C^* \in [0.15X(0), 0.5X(0)]$. In fig 3.3, $P20$ has a higher efficient frontier than $P22$ as intuitively expected (since AVaR is more conservative).

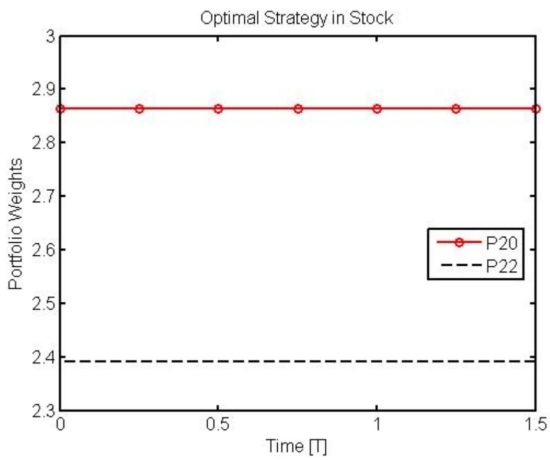

Figure 3.1: Optimal strategy for the stock in a mean-reverting model

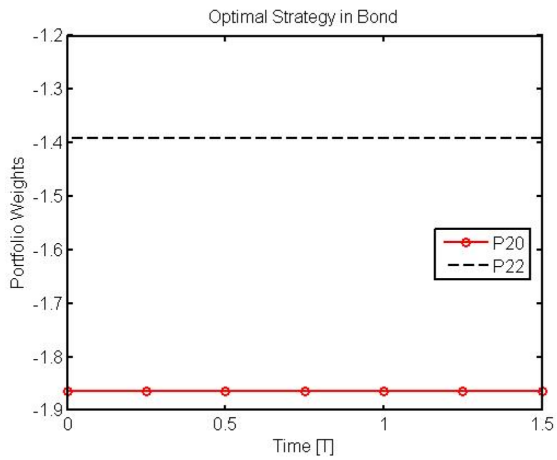

Figure 3.2: Optimal strategy for the money market in a mean-reverting model

As $C = C^*$ increases the investor is less risk averse, this leads to a higher expected return. This concept is reflected in fig 3.3.

The mean-reverting rate impact

We study the effect of the mean-reverting rate on optimal strategies. Let $k_1 \in [0.3, 0.7]$. The optimal strategy in the stock increases with respect to k_1 as illustrated in fig 3.4 . This is due to the fact that as the speed of mean-reversion increases there is a high chance of obtaining profit by holding or shorting more stock depending on the data. The AVaR constraint ($P22$) is proceeding with more caution than the VaR constraint ($P20$).

The impact of confidence level (α)

We study the effect of α on optimal strategies. In this section $\alpha \in [0.01, 0.09]$. As α increases, the investor is allowed to take more risk and this leads to holding more

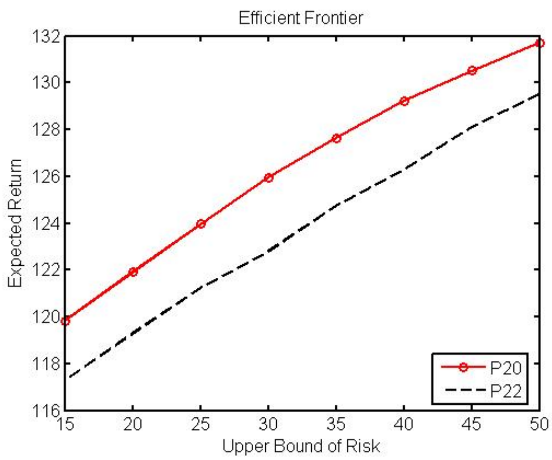

Figure 3.3: Efficient frontier in a mean-reverting model

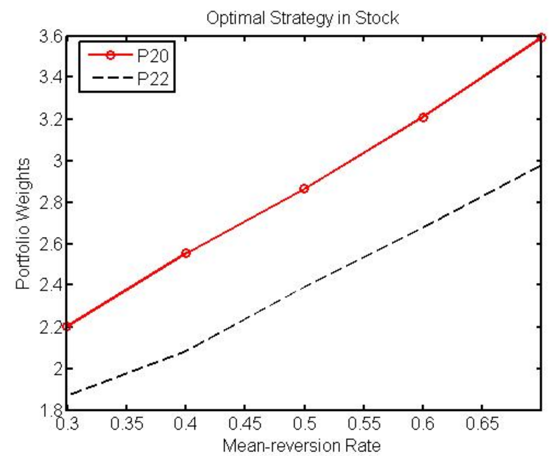

Figure 3.4: Optimal strategy as a function of mean-reverting rate

stock. This is illustrated in fig 3.5 for $P20$ and $P22$.

Minimal VaR, AVaR and Variance subject to an expected return constraint

In this section P23, P24 and P25 are considered, with $M = [1.1 \cdot X(0), 1.3 \cdot X(0)]$. The minimal risk is illustrated in fig 3.6.

Comparing the outcomes of GBM and exponential O.U models

In this section $k_i = 0.27$. Figure 3.7 show that the optimal strategy in the GBM introduces more leverage into the portfolio compared to the optimal strategy in the exponential O.U.This is explained by the fact that in the GBM model, the stock have a higher market price of risk so it is more attractive. This property account for the results in figure 3.7 and figure 3.8.

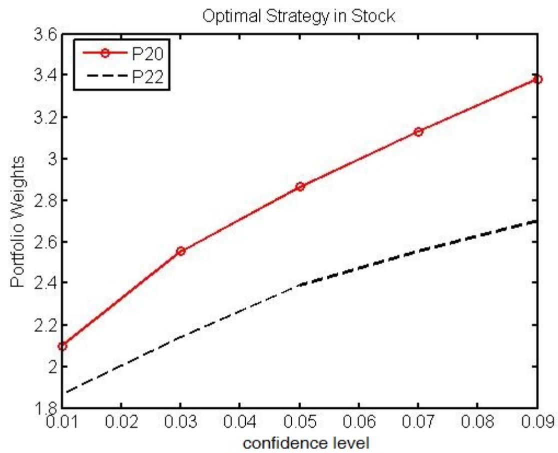

Figure 3.5: Optimal portfolio strategy as a function of α

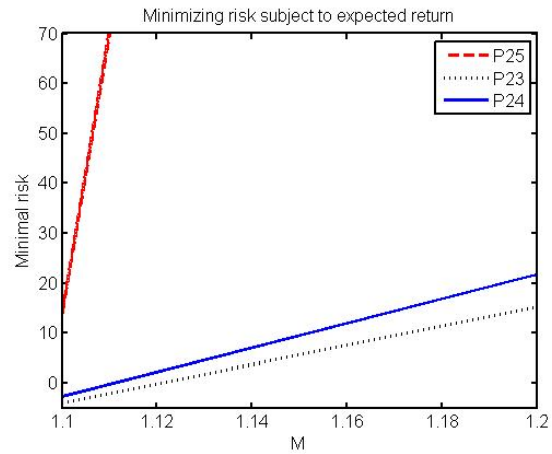

Figure 3.6: Minimized risk for P23, P24 and P25

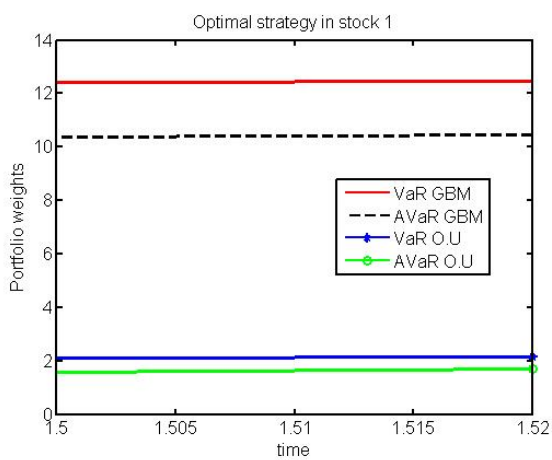

Figure 3.7: Comparing optimal strategy in GBM to exponential O.U in stock 1

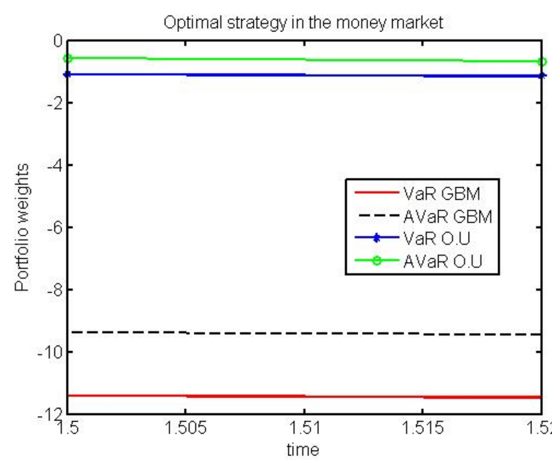

Figure 3.8: Comparing optimal strategy in GBM to exponential O.U in the money market

Chapter 4

Conclusion and further research

In this thesis, the novelty is the addition of AVaR and LEL constraints. The minimal VaR and AVaR portfolios are found for GBM and exponential O.U models. Additionally VaR, AVaR and variance is minimized with an expected return constraint. In the GBM model we maximize expected wealth with VaR, AVaR, LEL and variance constraints. However, in the exponential O.U model we maximize expected wealth with VaR and AVaR constraints only. Using numerical experiments we compare deterministic to constant portfolio strategies. Furthermore, from the plots we notice that variance is the most conservative risk measure, followed by LEL, AVaR and VaR. In the exponential O.U model, the mean-reversion rate is shown to have a strong effect on portfolio selection. Lastly, we examine leverage in the portfolio. The plots show that the optimal strategy in the GBM model introduces more leverage. For further research it would be interesting to introduce correlation constraints in the GBM and exponential O.U models. Another research direction is to consider deterministic strategies in the O.U model.

Chapter 5

Appendix

5.1 Appendix A

Appendix contains all the proofs for the model in which the stocks follow a geometric Brownian motion and the portfolio strategies are deterministic.

5.1.1 Proof of Proposition 2.1.2.1

$$\begin{aligned} d(X(t)) &= \sum_{i=0}^{m} N_i(t)dS_i(t) = N_0(t)dS_0(t) + \sum_{i=1}^{m} N_i(t)dS_i(t) \\ &= (1-\pi(t))X(t)r(t)dt + X(t)\pi(t)(b(t)dt + \sigma(t)dW(t)) \\ &= X(t)((r(t) + B(t)'\pi(t))dt + \pi(t)'\sigma(t)dW(t). \end{aligned}$$

Here $W(t) = (W_1(t), W_2(t)...W_m(t))'$ is a m-dimensional Brownian motion and $B(t) = b(t) - r(t)\mathbf{1}$. Solving the stochastic equation above gives

$$X(t) = X(0)\exp\left(\int_0^t \left(r(s) + B(s)'\pi(s) - \frac{||\sigma(s)'\pi(s)||^2}{2}\right) ds + \int_0^t \pi(s)'\sigma(s)dW(s)\right).$$

$\diamondsuit$

5.1.2 Proof of Proposition 2.1.3.1

$$\begin{aligned}
E[X(t)] &= E\Big[X(0)\exp\left(\int_0^t r(s) + B(s)'\pi(s) - \frac{1}{2}||\sigma(s)'\pi(s)||^2 ds + \int_0^t \pi(s)'\sigma(s)dW(s)\right)\Big] \\
&= X(0)\exp\left(\int_0^t (r(s) + B(s)'\pi(s) - \frac{1}{2}||\sigma(s)'\pi(s)||^2 ds\right) \\
&\quad E\Big[\exp\left(\int_0^t \pi(s)'\sigma(s)dW(s)\right)\Big]. \\
&= X(0)\exp\left(\int_0^t (r(s) + B(s)'\pi(s) - \frac{1}{2}||\sigma(s)'\pi(s)||^2 ds\right) \\
&\quad E\Big[\exp\left(\frac{1}{2}\int_0^t ||\sigma(s)'\pi(s)||^2 ds\right)\Big] \\
&= X(0)\exp\left(\int_0^t (r(s) + B(s)'\pi(s))ds\right) \\
&= X(0)R(t)\exp\left(\int_0^t (B(s)'\pi(s))ds\right). \qquad (5.1)
\end{aligned}$$

The next step is to find the variance of $X(t)$.

$$\begin{aligned}
E[X(t)^2] &= X(0)^2R(t)^2\exp\left(2\int_0^t B(s)'\pi(s)ds - \int_0^t ||\sigma(s)'\pi(s)||^2ds + 2\int_0^t \pi(s)'\sigma(s)dW(s)\right) \\
&= X(0)^2R(t)^2\exp\Big(2\int_0^t B(s)'\pi(s)ds - \int_0^t ||\sigma(s)'\pi(s)||^2ds\Big) \\
&\quad E\Big[\exp\Big(2\int_0^t \pi(s)'\sigma(s)dW(s)\Big)\Big] \\
&= X(0)^2R(t)^2\exp\left(2\int_0^t B(s)'\pi(s)ds + \int_0^t ||\sigma(s)'\pi(s)||^2ds\right).
\end{aligned}$$

$$\begin{aligned}
Variance[X(t)] &= E[X(t)^2] - (E[X(t)])^2 \qquad (5.2)\\
&= X(0)^2R(t)^2\exp\left(2\int_0^t B(s)'\pi(s)ds + \int_0^t ||\sigma(s)'\pi(s)||^2ds\right) \\
&\quad - \left(X(0)R(t)\exp\left(\int_0^t B(s)'\pi(s)ds\right)\right)^2 \\
&= X(0)^2R(t)^2\Big(\exp\left(2\int_0^t B(s)'\pi(s)ds + \int_0^t ||\sigma(s)'\pi(s)||^2ds\right) \\
&\quad - \exp\left(2\int_0^t B(s)'\pi(s)ds\right)\Big).
\end{aligned}$$

Hence the claim.

$\diamondsuit$

5.1.3 Proof of Proposition 2.1.4.1

The VaR equation can be further written as

$$\begin{aligned} 1-\alpha &= P\Big(-\Big(X(0)\exp\Big(\int_0^t (r(s)+B(s)'\pi(s)-\frac{1}{2}||\sigma(s)'\pi(s)||^2)ds \\ &+ \left(\sqrt{\int_0^t ||\sigma(s)'\pi(s)||^2 ds}\right)\xi - X(0)\Big)\Big) \le VaR\Big), \end{aligned}$$

where $\xi \sim N(0,1)$. Let

$$\Gamma := \int_0^t r(s)+B(s)'\pi(s)-\frac{1}{2}||\sigma(s)'\pi(s)||^2 ds$$

and

$$\Psi := \sqrt{\int_0^t ||\sigma(s)'\pi(s)||^2 ds}.$$

Then

$$1-\alpha = P\left(\exp(\Gamma+\Psi\xi) \ge 1-\frac{VaR}{X(0)}\right) = P\left(\xi \ge \frac{\ln(1-\frac{VaR}{X(0)})-\Gamma}{\Psi}\right).$$

So

$$\alpha = P\left(\xi \le \frac{\ln(1-\frac{VaR}{X(0)})-\Gamma}{\Psi}\right) = N\left(\frac{\ln(1-\frac{VaR}{X(0)})-\Gamma}{\Psi}\right).$$

Solving for VaR in the equation above gives

$$VaR_\alpha(X(t)) = X(0)(1-\exp(\Gamma+N^{-1}(\alpha)\Psi)).$$

Hence the equation for VaR

$$\begin{aligned} VaR_\alpha(X(t)) &= X(0)\Big(1 - \exp\Big(\int_0^t (r(s) + B(s)'\pi(s) - \frac{1}{2}||\sigma(s)'\pi(s)||^2)ds \\ &+ N^{-1}(\alpha)\sqrt{\int_0^t ||\sigma(s)'\pi(s)||^2 ds}\Big)\Big). \end{aligned}$$

$\diamondsuit$

5.1.4 Proof of Proposition 2.1.5.1

From the definition of AVaR we have

$$AVaR_\alpha(X(t)) = \frac{1}{\alpha}\int_0^\alpha VaR_u(X(t))du.$$

$$\begin{aligned} AVaR_\alpha(X(t)) &= \frac{1}{\alpha}\int_0^\alpha X(0)\Big(1 - R(t)\exp\Big(\int_0^t (B(s)'\pi(s) - \frac{1}{2}||\sigma(s)'\pi(s)||^2)ds \\ &+ (N^{-1}(u))\sqrt{\int_0^t ||\sigma(s)'\pi(s)||^2 ds}\Big)\Big)du. \end{aligned}$$

Let $\int_0^t ||\sigma(s)'\pi(s)||^2 ds := \kappa^2$. It follows that

$$\begin{aligned} AVaR_\alpha(X(t)) &= X(0) - \frac{R(t)X(0)}{\alpha}\int_0^\alpha \exp\left(\int_0^t \left(B'(s)\pi(s)ds - \frac{1}{2}\kappa^2 + (N^{-1}(u))\kappa\right) du\right) \\ &= X(0) - \frac{R(t)X(0)}{\alpha}\exp\left(\int_0^t B(s)'\pi(s)ds - \frac{1}{2}\kappa^2\right)\left(\int_0^\alpha \exp\left((N^{-1}(u))\kappa\right) du\right). \end{aligned}$$

Let $N^{-1}(u) = y$ and $u = N(y)$. Therefore, $du = \frac{1}{\sqrt{2\pi_c}} \exp \frac{-y^2}{2} dy$. Hence

$$\begin{aligned} AVaR_\alpha(X(t)) &= X(0) - \frac{R(t)X(0)}{\alpha}\Big(\exp\Big(\int_0^t B(s)'\pi(s)ds - \frac{1}{2}\kappa^2\Big)\Big) \\ &\quad \Big(\frac{1}{\sqrt{2\pi_c}}\int_{-\infty}^{N^{-1}(\alpha)} \exp\Big(y\kappa - \frac{y^2}{2}\Big)dy\Big) \\ &= X(0) - \frac{R(t)X(0)}{\alpha}\Big(\exp\Big(\int_0^t B(s)'\pi(s)ds\Big)\Big) \\ &\quad \Big(\frac{1}{\sqrt{2\pi_c}}\int_{-\infty}^{N^{-1}(\alpha)} \exp\Big(\frac{-(y-\kappa)^2}{2}\Big)dy\Big). \end{aligned}$$

Let us perform the change of variable $z = y - \varepsilon$, hence

$$\begin{aligned} AVaR_\alpha(X(t)) &= X(0) - \frac{R(t)X(0)}{\alpha\sqrt{2\pi_c}}\Big(\exp\int_0^t B(s)'\pi(s)ds\Big)\Big(\int_{-\infty}^{N^{-1}(\alpha)-\kappa} \exp\frac{-z^2}{2}dz\Big) \\ &= X(0)\left(1 - \frac{R(t)}{\alpha}\left(\exp\int_0^t B(s)'\pi(s)ds\right)\left(N(N^{-1}(\alpha)-\kappa)\right)\right). \end{aligned}$$

Finally,

$$\begin{aligned} AVaR_\alpha(X(t)) &= X(0)\Big(1 - \frac{R(t)}{\alpha}\Big(\exp\int_0^t B(s)'\pi(s)ds \\ &\quad + \ln\Big(N\Big(N^{-1}(\alpha) - \sqrt{\int_0^t ||\sigma(s)'\pi(s)||^2 ds}\Big)\Big)\Big)\Big). \end{aligned}$$

$\diamondsuit$

5.1.5 Proof of Proposition 2.1.6.1

Under a risk neutral probability measure Q, the stock dynamics is

$$dS(t) = r(t)S(t)dt + \sigma(t)d\widetilde{W}(t), \tag{5.3}$$

where $\widetilde{W}(t)$ denotes the Brownian motion under a Q-measure. Using the same approach as in the proof of section 2.1.4.1 and 2.1.5.1 we prove that

$$\begin{aligned} VaR^{Q}_{\alpha}(X(t)) &= X(0)\Big(1 - R(t)\exp\Big(-\frac{1}{2}||\sigma(s)'\pi(s)||^2)ds \\ &+ (N^{-1}(\alpha))\sqrt{\int_0^t ||\sigma(s)'\pi(s)||^2 ds}\Big)\Big). \end{aligned} \tag{5.4}$$

Furthermore,

$$LEL_{\alpha}(X(t)) = X(0)\Big(1 - \frac{R(t)}{\alpha}\Big(N\Big(N^{-1}(\alpha) - \sqrt{\int_0^t ||\sigma(s)'\pi(s)||^2 ds}\Big)\Big)\Big). \tag{5.5}$$

5.1.6 Proof of Theorem 2.1.7.1

The portfolio optimization problems are projected onto the family of surfaces,

$$Q_{\varepsilon} = \{\pi : \int_0^T ||\sigma(t)'\pi(t)||^2 dt = \varepsilon^2\}.$$

It follows that

$$Q = \bigcup_{\varepsilon \geq 0} Q_{\varepsilon}.$$

The optimization problem can be solved as

$$\begin{aligned} &\min_{\varepsilon \geq 0} \min_{\pi \in Q_{\varepsilon}} \Big[X(0)\Big(1 - \exp\Big(\int_0^T (r(t) + B(t)'\pi(t) - \frac{1}{2}||\sigma(t)'\pi(t)||^2)dt \\ &+(N^{-1}(\alpha))\sqrt{\int_0^T ||\sigma(t)'\pi(t)||^2 dt}\Big)\Big)\Big] \end{aligned}$$

$$\text{subject to } \int_0^T ||\sigma(t)'\pi(t)||^2 dt = \varepsilon^2 \tag{5.6}$$

In a first step we optimize for a fixed ε, i.e,

$$(P_\varepsilon) \qquad \max_{\pi \in Q_\varepsilon} \int_0^T B(t)'\pi(t)dt \text{ subject to } \int_0^T ||\sigma(t)'\pi(t)||^2 dt = \varepsilon^2.$$

Let the market price of risk be denoted as $\Theta(t) = \sigma(t)^{-1}B(t)$ and $||\Theta||_T = \sqrt{\int_0^T ||\sigma(t)^{-1}B(t)||^2 dt}$. We claim that the optimal strategy for (P_ε) is

$$\hat{\pi}_\varepsilon(t) = -\frac{1}{2\hat{\lambda}}(\sigma(t)\sigma(t)')^{-1}B(t) \qquad t \in [0, T], \tag{5.7}$$

with

$$\hat{\lambda} = -\frac{||\Theta||_T}{2\varepsilon}. \tag{5.8}$$

To prove the claim we first check that $\hat{\pi}_\varepsilon(t)$ satisfies the constraint, i.e,

$$\int_0^T \Big|\Big|\sigma(t)\Big(\frac{\varepsilon}{||\Theta||_T}(\sigma(t)\sigma(t)')^{-1}B(t)\Big)\Big|\Big|^2 = \varepsilon^2 \tag{5.9}$$

Thus, the constraint is satisfied. Let $\pi(t)$ be an adapted strategy which satisfies (5.6). Then the optimality of $\hat{\pi}_\varepsilon(t)$ is equivalent to:

$$\int_0^T B'(t)\hat{\pi}(t)dt + \hat{\lambda}\Big(\int_0^T ||\sigma(t)\hat{\pi}(t)||^2 dt - \varepsilon^2\Big) \geq \int_0^T B(t)'\pi(t)dt + \hat{\lambda}\Big(\int_0^T ||\sigma(t)'\pi(t)||^2 dt - \varepsilon^2\Big). \tag{5.10}$$

Let

$$\begin{aligned} L(\pi,\hat{\lambda}) &= \int_0^T B(t)'\pi(t)dt + \hat{\lambda}\left(\int_0^T ||\sigma(t)'\pi(t)||^2 dt - \varepsilon^2\right) \\ &= \hat{\lambda}\int_0^T \Big(||\sigma(s)'\pi(s)||^2 + \frac{1}{\hat{\lambda}}B(t)'\pi(t)\Big)dt - \hat{\lambda}\varepsilon^2 \\ &= \hat{\lambda}\int_0^T \Big(\pi(t)'(\sigma(t)\sigma(t)')\pi(t) + \frac{1}{\hat{\lambda}}B(t)'\pi(t)\Big)dt - \hat{\lambda}\varepsilon^2 \end{aligned} \tag{5.11}$$

By completing the squares

$$\begin{aligned} L(\pi,\hat{\lambda}) &= \hat{\lambda}\int_0^T \Big(\pi(t) + \frac{1}{2\hat{\lambda}}(\sigma(t)\sigma(t)')^{-1}B(t)\Big)'(\sigma(t)\sigma(t)')\Big(\pi(t) + \frac{1}{2\hat{\lambda}}(\sigma(t)\sigma(t)')^{-1}B(t)\Big)dt \\ &- \int_0^T \frac{1}{4\hat{\lambda}}B(t)'(\sigma(t)\sigma(t)')^{-1}B(t)dt - \hat{\lambda}\varepsilon^2 \\ &= \hat{\lambda}\int_0^T \Big|\Big|\sigma(t)\Big(\pi(t) + \frac{1}{2\hat{\lambda}}(\sigma(t)\sigma(t)')^{-1}B(t)\Big)\Big|\Big|^2 dt - \int_0^T \frac{1}{4\hat{\lambda}}B(t)'(\sigma(t)\sigma(t)')^{-1}B(t)dt \\ &- \hat{\lambda}\varepsilon^2 \end{aligned} \tag{5.12}$$

Let

$$U(\pi(t)) := \Big|\Big|\sigma(t)\Big(\pi(t) + \frac{1}{2\hat{\lambda}}(\sigma(t)\sigma(t)')^{-1}B(t)\Big)\Big|\Big|^2$$

Solving $U(\pi(t)) = 0$, we obtain

$$\pi(t) = -\frac{1}{2\hat{\lambda}}(\sigma(t)'\sigma(t))^{-1}B(t). \tag{5.13}$$

Hence $L(\pi,\hat{\lambda}) \leq L(\hat{\pi}_\varepsilon,\hat{\lambda})$ which proves (5.10). Furthermore,

$$\int_0^T B'(t)\hat{\pi}_\varepsilon(t)dt = \varepsilon||\Theta||_T \tag{5.14}$$

Next, we optimize over $\varepsilon \geq 0$,i.e,

$$\max_{\varepsilon \geq 0} \left[\varepsilon(||\Theta||_T + (N^{-1}(\alpha))) - \frac{1}{2}\varepsilon^2 \right]$$

Let

$$f(\varepsilon) := \varepsilon(||\Theta||_T + (N^{-1}(\alpha))) - \frac{1}{2}\varepsilon^2,$$

where $f(\varepsilon)$ is a concave function.

By the concavity of f, the optimizer is given by

$$f'(\varepsilon^*) = -\varepsilon^* + ||\Theta||_T + (N^{-1}(\alpha)) = 0. \tag{5.15}$$

Solving (5.15), one obtains

$$\varepsilon^* = ||\Theta||_T + (N^{-1}(\alpha)).$$

The optimal strategy is

$$\pi_{\varepsilon^*}(t) = \frac{\max(0, \varepsilon^*)}{||\Theta||_T}(\sigma(t)\sigma(t)')^{-1}B(t).$$

The minimal VaR is

$$VaR_\alpha = X(0)\left(1 - R(T)\exp\left(\frac{1}{2}(\max(0, \varepsilon^*))^2\right)\right).$$

◇

5.1.7 Proof of Theorem 2.1.8.1

Since

$$Q = \bigcup_{\varepsilon \geq 0} Q_\varepsilon,$$

the optimization problem becomes

$$\max_{\varepsilon \geq 0} \max_{\pi \in Q_\varepsilon} \Big[X(0)R(T) \exp\Big(\int_0^T B(t)'\pi(t)dt \Big) \Big] \text{ subject to} \tag{5.16}$$

$$X(0)\Big(1 - \exp\Big(\int_0^T (r(t) + B(t)'\pi(t) - \frac{1}{2}||\sigma(t)'\pi(t)||^2)dt$$

$$+(N^{-1}(\alpha))\sqrt{\int_0^T ||\sigma(t)'\pi(t)||^2 dt}\Big)\Big) \leq C \tag{5.17}$$

$$\int_0^T ||\sigma(t)'\pi(t)||^2 dt = \varepsilon^2. \tag{5.18}$$

The strategy π is admissible if

$$\int_0^T B(t)'\pi(t)dt \geq \ln\left(\frac{X(0) - C}{R(T)X(0)} \right) + \frac{1}{2}\varepsilon^2 - N^{-1}(\alpha)\varepsilon. \tag{5.19}$$

In a first step we optimize over $\pi(t)$ given a fixed ε.Let

$$(P_\varepsilon) \qquad \max_{\pi \in Q_\varepsilon} \Big[\int_0^T B(t)'\pi(t)dt \Big] \text{ subject to}$$

$$\int_0^T ||\sigma(t)'\pi(t)||^2 dt = \varepsilon^2.$$

Let the market risk be denoted as $\Theta(t) = \sigma(t)^{-1}B(t)$ and $||\Theta||_T = \sqrt{\int_0^T ||\sigma(t)^{-1}B(t)||^2 dt}$. We claim that the optimal strategy for (P_ε) is

$$\hat{\pi}_\varepsilon(t) = -\frac{1}{2\hat{\lambda}}(\sigma(t)\sigma(t)')^{-1}B(t) \qquad t \in [0, T], \tag{5.20}$$

with

$$\hat{\lambda} = -\frac{||\Theta||_T}{2\varepsilon}. \tag{5.21}$$

We can argue as in (5.9),(5.10),(5.11),(5.12) and (5.13) for the proof of the claim. If there is an admissible $\pi(t)$, i.e it $\int_0^T ||\sigma(t)'\pi(t)||^2 dt = \varepsilon^2$ and (5.19), then we claim that $\hat{\pi}_\varepsilon(t)$ must also be admissible. Since $\hat{\pi}_\varepsilon(t)$ is optimal for P_ε, it follows that

$$\int_0^T B(t)'\hat{\pi}_\varepsilon(t)dt \geq \int_0^T B(t)'\pi(t)dt.$$

Thus from (5.19),

$$\int_0^T B(t)'\hat{\pi}_\varepsilon(t)dt \geq \ln\left(\frac{X(0) - C}{R(T)X(0)}\right) + \frac{1}{2}\varepsilon^2 - N^{-1}(\alpha)\varepsilon, \tag{5.22}$$

whence $\hat{\pi}_\varepsilon(t)$ is admissible. Substitute $\hat{\pi}_\varepsilon(t)$ into the (5.16) to (5.17) and optimize over $\varepsilon \geq 0$, i.e,

$$\max_{\varepsilon \geq 0} \left[\varepsilon||\Theta||_T\right] \text{ subject to } g(\varepsilon) \geq 0,$$

where

$$g(\varepsilon) := -\frac{1}{2}\varepsilon^2 + \varepsilon(||\Theta||_T + (N^{-1}(\alpha))) - \zeta.$$

Here

$$\zeta := \ln\left(\frac{1}{R(T)}\left(1 - \frac{C}{X(0)}\right)\right).$$

$g(\varepsilon) = 0$ is a quadratic equation and has two root

$$\varepsilon_{1,2} = \frac{-(||\Theta||_T - N^{-1}(\alpha)) \pm \sqrt{(||\Theta||_T + N^{-1}(\alpha))^2 - 4(\frac{-1}{2})(-\zeta)}}{2(-1/2)}.$$

So, the optimum is attained at

$$\varepsilon_1 = ||\Theta||_T + (N^{-1}(\alpha)) + \sqrt{(||\Theta||_T + N^{-1}(\alpha))^2 - 2\zeta}.$$

Thus, the optimal strategy is

$$\pi_{\varepsilon_1}(t) = \frac{||\Theta||_T + (N^{-1}(\alpha)) + \sqrt{(||\Theta||_T + N^{-1}(\alpha))^2 - 2\zeta}}{||\Theta||_T}(\sigma(t)\sigma(t)')^{-1}B(t).$$

Therefore, the maximum expected wealth with the Value at Risk constraint is

$$E(X^{\pi_{\varepsilon_1}}(T)) = X(0)R(T)\exp\Big((||\Theta||_T+(N^{-1}(\alpha))+\sqrt{(||\Theta||_T + N^{-1}(\alpha))^2 - 2\zeta})||\Theta||_T\Big).$$

$\diamondsuit$

5.1.8 Proof of the theorem 2.1.9.1

Since

$$Q = \bigcup_{\varepsilon \geq 0} Q_\varepsilon,$$

the optimization problem is written as

$$\inf_{\pi \in Q} X(0)\left(1 - \frac{R(T)}{\alpha}\left(\exp\Big(\int_0^T B'(t)\pi(t)dt + \ln\Big(N\Big(N^{-1}(\alpha) - \sqrt{\int_0^T ||\sigma(t)'\pi(t)||^2 dt}\Big)\Big)\right)\right)$$

subject to

$$\int_0^T ||\sigma(t)'\pi(t)||^2 dt = \varepsilon^2. \tag{5.23}$$

In a first step we optimize over $\pi(t)$ given a fixed ε, i.e,

$$(P_\varepsilon) \qquad \sup_{\pi \in Q_\varepsilon} \Big[\int_0^T B'(t)\pi(t)dt\Big] \text{ subject to}$$

$$\int_0^T ||\sigma(t)'\pi(t)||^2 dt = \varepsilon^2.$$

We claim that the optimal strategy for (P_ε) is

$$\hat{\pi}_\varepsilon(t) = -\frac{1}{2\hat{\lambda}}(\sigma(t)\sigma(t)')^{-1}B(t) \qquad t \in [0,T], \tag{5.24}$$

with

$$\hat{\lambda} = -\frac{||\Theta||_T}{2\varepsilon} \tag{5.25}$$

See (5.9) to (5.13) for proof of the claim. Next, we optimize over $\varepsilon \geq 0$, i.e,

$$(P_{\varepsilon_1}) \qquad \sup_{\varepsilon \geq 0} f(\varepsilon)$$

where

$$f(\varepsilon) := \varepsilon||\Theta||_T + \ln(N(N^{-1}(\alpha) - \varepsilon)).$$

We claim that $f(\varepsilon)$ is concave.

Proof of claim : From the assumptions stated $\alpha \in (0, 0.5)$, it follows that $N^{-1}(\alpha) \leq$

0. Recall that

$$N(w) = \frac{1}{\sqrt{2\pi_c}} \int_0^w \exp\left(-\frac{t^2}{2}\right) dt \text{ and } N'(w) = \frac{1}{\sqrt{2\pi_c}} \exp\left(-\frac{w^2}{2}\right).$$

Let

$$f(w) := \ln(N(w)).$$

We claim that $f(\cdot)$ is increasing. Moreover if $w < 0$ it is also concave. Since

$$f'(w) = \frac{\exp(\frac{-w^2}{2})}{\sqrt{2\pi_c}N(w)} \geq 0, \tag{5.26}$$

it follows that $f(\cdot)$ is increasing . Moreover,

$$f''(w) = \frac{\exp(-\frac{w^2}{2})(wN(w) - N'(w))}{\sqrt{2\pi_c}(N(w))^2}. \tag{5.27}$$

Since

$$\frac{z}{z^2+1} N'(\pm z) \leq N(-z) \leq \frac{1}{z} N'(\pm z) \text{ where } z > 0 \text{ (see (McKean, 1969))},$$

it follows that $f''(w) \leq 0$. Hence $w \to \ln N(w)$ is concave if $w < 0$.

Let us state the following preparatory lemma.

Lemma 5.1.8.1 *If $g : R^m \to R$ is a concave function and $f : R \to R$ is a concave nondecreasing function, then $f \circ g$ is concave.*

Proof of Lemma: By concavity of g

$$g(\lambda z + (1-\lambda)y) \geq \lambda g(z) + (1-\lambda) g(y)$$

and since f is nondecreasing then,

$$f(g(\lambda z + (1-\lambda)y)) \geq f(\lambda g(z) + (1-\lambda)g(y)) \geq \lambda f(g(z)) + (1-\lambda)f(g(y)).$$

Where the second inequality follows from the concavity of f. Using the Lemma 5.1.8.1 above we conclude that $f(\varepsilon)$ is concave.

$\diamondsuit$

Since $f(\varepsilon)$ is concave it follows that, (P_{ε_1}) has a unique solution denoted by ε^{**}. Thus the optimal strategy is

$$\pi_{\varepsilon^{**}}(t) = \frac{\varepsilon^{**}}{||\Theta||_T}(\sigma(t)\sigma(t)')^{-1}B(t).$$

The minimal average value at risk is

$$AVaR(\alpha, \pi, T) = X(0)\left(1 - \frac{R(T)}{\alpha}\left(\exp(\varepsilon^{**}||\Theta||_T + \ln(N(N^{-1}(\alpha) - \varepsilon^{**}))\right)\right).$$

$\diamondsuit$

5.1.9 Proof of Theorem 2.1.10.1

Since

$$Q = \bigcup_{\varepsilon \geq 0} Q_\varepsilon,$$

the optimization problem is tackled as

$$\max_{\varepsilon\geq 0}\max_{\pi\in Q_\varepsilon}\Big[X(0)R(T)\exp\Big(\int_0^T B(t)'\pi(t)dt\Big)\Big] \text{ subject to} \tag{5.28}$$

$$X(0)\Big(1-\frac{R(T)}{\alpha}\Big(\exp\int_0^T B'(t)\pi(t)dt+\ln\Big(N\Big(N^{-1}(\alpha) -\sqrt{\int_0^T ||\sigma(t)'\pi(t)||^2dt}\Big)\Big)\Big)\Big)\leq C^* \tag{5.29}$$

$$\int_0^T ||\sigma(t)'\pi(t)||^2dt=\varepsilon^2 \tag{5.30}$$

In a first step we optimize over $\pi(t)$ given a fixed ε. In this step (5.29) is redundant, i.e,

$$(P_\varepsilon) \qquad \max_{\pi\in Q_\varepsilon}\Big[\int_0^T B(t)'\pi(t)dt\Big]\text{subject to}$$

$$\int_0^T ||\sigma(t)'\pi(t)||^2dt=\varepsilon^2.$$

We claim that the optimal strategy for (P_ε) is

$$\hat{\pi}_\varepsilon(t)=-\frac{1}{2\hat{\lambda}}(\sigma(t)\sigma(t)')^{-1}B(t) \qquad t\in[0,T], \tag{5.31}$$

with

$$\hat{\lambda}=-\frac{||\Theta||_T}{2\varepsilon} \tag{5.32}$$

See (5.9) to (5.13) for proof of the claim.

Let

$$g_2(\varepsilon):=\varepsilon||\Theta||_T+\ln(N(N^{-1}(\alpha)-\varepsilon))-\ln\left(\frac{\alpha\sqrt{2\pi^*}(X(0)-C^*)}{R(T)X(0)}\right). \tag{5.33}$$

Arguing as in the proof of 5.1.7, substitute $\hat{\pi}_\varepsilon(t)$ into (5.28) and (5.29) and optimize over $\varepsilon \geq 0$, i.e,

$$\max_{\varepsilon \geq 0} \varepsilon ||\Theta||_T \text{ subject to } g_2(\varepsilon) \geq 0.$$

Next, we need the following preparatory Lemma.

Lemma 5.1.9.1 $\{\varepsilon : g_2(\varepsilon) \geq 0, \varepsilon \geq 0\} = [0, \gamma_2]$ *where* γ_2 *is the unique positive root of* $g_2(\varepsilon) = 0$.

Proof of Lemma :Recall that,

$$g_2(x) := x||\Theta||_T + \ln(N(N^{-1}(\alpha) - x)) - \ln\left(\alpha\left(\frac{X(0) - C^*}{R(T)X(0)}\right)\right).$$

Since $C^* < X(0)$, then

$$g_2(0) = \ln(\alpha) - \ln\left(\alpha\left(\frac{X(0) - C^*}{R(T)X(0)}\right)\right) > 0. \tag{5.34}$$

Furthermore,

$$\lim_{x \to \infty} g_2(x) = -\infty \tag{5.35}$$

To proof (5.35), we claim that

$$\lim_{x \to \infty} \frac{\ln(N(-x))}{x} = -\infty. \tag{5.36}$$

Indeed by L'Hopital's rule

$$\lim_{x \to \infty} \frac{\ln(N(-x))}{x} = \lim_{x \to \infty} \frac{-\exp(-\frac{x^2}{2})}{N(-x)} = \lim_{x \to \infty} \frac{x\exp(-\frac{x^2}{2})}{-\exp(-\frac{x^2}{2})} = -x = -\infty.$$

By (5.36), it follows that,

$$\lim_{x\to\infty} g(x) = x\left(1+\frac{\ln(N(-x)}{x}\right) = -x^2 = -\infty.$$

Thus (5.35) is true.

The function $g_2(x)$ is concave when $x \in \Big(N^{-1}(\alpha), \infty\Big)$ (see the proof of section 5.1.8). This yields the results. Thus, the optimal strategy is

$$\pi_{\gamma_2}(t) = \frac{\gamma_2}{||\Theta||_T}(\sigma(t)\sigma(t)')^{-1}B(t).$$

Therefore, the maximum expected wealth with the Average Value at Risk constraint is

$$E[X^{\pi_{\gamma_2}}(T)] = X(0)R(T)\exp(\gamma_2||\Theta||_T).$$

$\diamondsuit$

5.1.10 Proof of the theorem 2.1.11.1

In this section we strict to the family of surfaces such that

$$Q = \bigcup_{\varepsilon\geq 0} Q_\varepsilon.$$

The optimization problem considered is

$$\max_{\pi\in Q_\varepsilon}\max_{\varepsilon\geq 0} X(0)R(T)\exp\left(\int_0^T B(t)'\pi(t)dt\right)$$

$$\text{subject to } X(0)\Big(1 - \frac{R(T)}{\alpha}\Big(N\Big(N^{-1}(\alpha) - \sqrt{\int_0^T ||\sigma(t)'\pi(t)||^2 dt}\Big)\Big)\Big) \leq C^{**}$$

$$\int_0^T ||\sigma(t)'\pi(t)||^2 dt = \varepsilon^2.$$

This can be further written as

$$\max_{\pi \in Q_\varepsilon} \max_{\varepsilon \geq 0} \int_0^T B(t)'\pi(t) dt \text{ subject to}$$

$$\sqrt{\int_0^T ||\sigma(t)'\pi(t)||^2 dt} \leq N^{-1}(\alpha) - N^{-1}\left(\frac{\alpha(X(0) - C^{**})}{X(0)R(T)}\right)$$

$$\int_0^T ||\sigma(t)'\pi(t)||^2 dt = \varepsilon^2.$$

In a first step we optimize over $\pi(t)$ given a fixed ε, i.e

$$(P_\varepsilon) \qquad \max_{\pi \in Q_\varepsilon} \Big[\int_0^T B(t)'\pi(t) dt\Big] \text{ subject to}$$

$$\int_0^T ||\sigma(t)'\pi(t)||^2 dt = \varepsilon^2,$$

Using the same approach as in section 5.1.6 the optimal strategy for (P_ε) is

$$\hat{\pi}_\varepsilon(t) = \frac{\varepsilon}{||\Theta||_T}(\sigma(t)\sigma(t)')^{-1}B(t), \tag{5.37}$$

Next, we optimize over $\varepsilon \geq 0$, i.e,

$$\max_{\varepsilon \geq 0}[\varepsilon||\Theta||_T] \text{ subject to}$$

$$\varepsilon \leq N^{-1}(\alpha) - N^{-1}\left(\frac{\alpha(X(0) - C^{**})}{X(0)R(T)}\right)$$

The optimal strategy is attained at

$$\pi_{\check{\varepsilon}}(t) = \frac{\check{\varepsilon}}{||\Theta||_T}(\sigma(t)\sigma(t)')^{-1}B(t), t \in [0,T], \tag{5.38}$$

where

$$\check{\varepsilon} = N^{-1}(\alpha) - N^{-1}\left(\frac{\alpha(X(0) - C^{**})}{X(0)R(T)}\right). \tag{5.39}$$

The maximal expected wealth subject to LEL constraint is

$$E[\check{X}(T)] = X(0)R(T)\exp(\check{\varepsilon}||\Theta||_T).$$

$\diamondsuit$

5.1.11 Proof of Theorem 2.1.12.1

In this section we strict to the family of surfaces such that

$$Q = \bigcup_{\varepsilon \geq 0} Q_\varepsilon.$$

The optimization problem considered is

$$\max_{\pi \in Q_\varepsilon} \max_{\varepsilon \geq 0} X(0)R(T)\exp\left(\int_0^T B(t)'\pi(t)dt\right) \text{ subject to}$$

$$X(0)^2R(t)^2\Big(\exp\left(2\int_0^T B(t)'\pi(t)dt + \int_0^T ||\sigma(t)'\pi(t)||^2dt\right) - \exp\left(2\int_0^T B(t)'\pi(t)dt\right)\Big) \leq C^a$$

$$\int_0^T ||\sigma(t)'\pi(t)||^2 dt = \varepsilon^2.$$

This can be further written as

$$\max_{\pi \in Q_\varepsilon} \max_{\varepsilon \geq 0} \int_0^T B(t)'\pi(t)dt \text{ subject to}$$

$$\exp\left(2\int_0^T B(t)'\pi(t)dt + \int_0^T ||\sigma(t)'\pi(t)||^2 dt\right) - \exp\left(2\int_0^T B(t)'\pi(t)dt\right) \leq \frac{C^a}{X(0)^2 R(T)^2}$$

In a first step we optimize over $\pi(t)$ given a fixed ε, i.e

$$(P_\varepsilon) \qquad \max_{\pi \in Q_\varepsilon} \left[\int_0^T B(t)'\pi(t)dt\right] \text{ subject to}$$

$$\int_0^T ||\sigma(t)'\pi(t)||^2 dt = \varepsilon^2,$$

Using the same approach as in section 5.1.6 the optimal strategy for (P_ε) is

$$\hat{\pi}_\varepsilon(t) = \frac{\varepsilon}{||\Theta||_T}(\sigma(t)\sigma(t)')^{-1}B(t), \tag{5.40}$$

Next, we optimize over $\varepsilon \geq 0$, i.e,

$$\max_{\varepsilon \geq 0}[\varepsilon ||\Theta||_T]$$

$$\text{subject to } f(\varepsilon) \leq 0,$$

where

$$f(\varepsilon) := \exp(2\varepsilon||\Theta||_T + \varepsilon^2) - \exp(2\varepsilon||\Theta||_T) - \frac{C^a}{X(0)^2 R(T)^2}$$

The unique positive solution to the equation $f(\varepsilon) = 0$ is denoted by $\check{\varepsilon}^*$. The optimal strategy is

$$\pi_{\check{\varepsilon}^*}(t) = \frac{\check{\varepsilon}^*}{||\Theta||_T}(\sigma(t)\sigma(t)')^{-1}B(t), t \in [0, T]. \tag{5.41}$$

The maximal expected wealth subject to variance constraint is

$$E[\check{X}^*(T)] = X(0)R(T)\exp(\check{\varepsilon}^*||\Theta||_T).$$

$\diamondsuit$

5.1.12 Proof of Theorem 2.1.13.1

Since

$$Q = \bigcup_{\varepsilon \geq 0} Q_\varepsilon,$$

the optimization problem (P7) becomes

$$\min_{\pi \in Q} \Big[X(0)\Big(1 - R(T)\exp\Big(\int_0^T (B(t)'\pi(t) - \frac{1}{2}||\sigma(t)'\pi(t)||^2)dt$$

$$+(N^{-1}(\alpha))\sqrt{\int_0^T ||\sigma(t)'\pi(t)||^2 dt}\Big)\Big)] \text{ subject to} \tag{5.42}$$

$$X(0)R(T)\exp\left(\int_0^T B(t)'\pi(t)dt\right) = M. \tag{5.43}$$

The optimization problem is reduced to

$$(P_\pi) \qquad \min_{\pi \in Q}\Big[\int_0^T ||\sigma(t)'\pi(t)||^2 dt\Big] \text{ subject to } \int_0^T B(t)'\pi(t)dt = \zeta.$$

We claim that the optimal strategy for (P_π) is

$$\hat{\pi}(t) = \frac{\zeta}{||\Theta||_T^2}(\sigma(t)\sigma(t)')^{-1}B(t) \qquad t \in [0, T], \tag{5.44}$$

where $\zeta = \ln\left(\frac{M}{X(0)R(T)}\right)$. To prove this claim, let us introduce the Lagrangian

$$\begin{aligned} L(\pi, \hat{\lambda}) &= \int_0^T ||\sigma(t)'\pi(t)||^2 dt + \hat{\lambda}\left(\int_0^T B(t)'\pi(t)dt - \zeta\right) \\ &= \hat{\lambda}\Big[\int_0^T \Big(\frac{1}{\hat{\lambda}}||\sigma(s)'\pi(s)||^2 + B(t)'\pi(t)\Big)dt - \zeta\Big] \\ &= \hat{\lambda}\Big[\int_0^T \Big(\frac{1}{\hat{\lambda}}\pi(t)'(\sigma(t)\sigma(t)')\pi(t) + B(t)'\pi(t)\Big)dt\Big] - \hat{\lambda}\zeta \end{aligned} \tag{5.45}$$

By completing the squares

$$\begin{aligned} L(\pi, \hat{\lambda}) &= \hat{\lambda}\Big[\int_0^T \Big(\pi(t) + \frac{\hat{\lambda}}{2}(\sigma(t)\sigma(t)')^{-1}B(t)\Big)'\frac{1}{\hat{\lambda}}(\sigma(t)\sigma(t)')\Big(\pi(t) + \frac{\hat{\lambda}}{2}(\sigma(t)\sigma(t)')^{-1}B(t)\Big)dt \\ &- \int_0^T \frac{1}{4}B(t)'\hat{\lambda}(\sigma(t)\sigma(t)')^{-1}B(t)dt\Big] - \hat{\lambda}\zeta \\ &= \int_0^T \Big|\Big|\sigma(t)\Big(\pi(t) + \frac{\hat{\lambda}}{2}(\sigma(t)\sigma(t)')^{-1}B(t)\Big)\Big|\Big|^2 dt - \int_0^T \frac{\hat{\lambda}}{4}B(t)'\hat{\lambda}(\sigma(t)\sigma(t)')^{-1}B(t)dt \\ &- \hat{\lambda}\zeta \end{aligned} \tag{5.46}$$

Let

$$U(\pi(t)) := \Big|\Big|\sigma(t)\Big(\pi(t) + \frac{\hat{\lambda}}{2}(\sigma(t)\sigma(t)')^{-1}B(t)\Big)\Big|\Big|^2$$

To minimize the optimization problem we solve the equation $U(\pi(t)) = 0$ and obtain

$$\pi(t) = -\frac{\hat{\lambda}}{2}(\sigma(t)'\sigma(t))^{-1}B(t). \tag{5.47}$$

Substituting (5.47) into the constraint we obtain

$$\int_0^T B(t)' \left(-\frac{\hat{\lambda}}{2} (\sigma(t)'\sigma(t))^{-1} B(t) \right) dt = \zeta. \tag{5.48}$$

Solving (5.48) we obtain

$$\hat{\lambda} = \frac{-2\zeta}{||\Theta||_T^2}, \tag{5.49}$$

where $||\Theta||_T = \sqrt{\int_0^T ||\sigma(t)^{-1}B(t)||^2 dt}$. It follows that the optimal strategy is

$$\pi_*(t) = \frac{\zeta}{||\Theta||_T^2} (\sigma(t)\sigma(t)')^{-1} B(t). \tag{5.50}$$

The next optimization problem considered is (P8).

The optimization problem is reduced to

$$\min_{\pi \in Q} \left[X(0) \left(1 - \frac{R(T)}{\alpha} \left(\exp\Big(\int_0^T B'(t)\pi(t) dt + \ln \Big(N\Big(N^{-1}(\alpha) - \sqrt{\int_0^T ||\sigma(t)'\pi(t)||^2 dt} \Big)\Big) \right) \right) \right]$$

subject to (5.51)

$$X(0)R(T) \exp\left(\int_0^T B(t)'\pi(t) dt \right) = M. \tag{5.52}$$

The problem is solved as

$$\min_{\pi \in Q} \left[\int_0^T ||\sigma(t)'\pi(t)||^2 dt \right] \text{ subject to } \int_0^T B(t)'\pi(t) dt = \zeta.$$

The next optimization problem considered is (P9) which is approached as

$$\begin{aligned} &\min_{\pi\in Q}\Big[X(0)^2R(t)^2\Big(\exp\Big(2\int_0^t B(s)'\pi(s)ds+\int_0^t ||\sigma(s)'\pi(s)||^2ds\Big)\\ -\ &\exp\Big(2\int_0^t B(s)'\pi(s)ds\Big)\Big)\Big] \text{ subject to}\\ &X(0)R(T)\exp\Big(\int_0^T B(t)'\pi(t)dt\Big)=M. \end{aligned} \tag{5.53}$$

The problem is reduced to

$$\min_{\pi\in Q}\Big[\int_0^T ||\sigma(t)'\pi(t)||^2dt\Big] \text{ subject to } \int_0^T B(t)'\pi(t)dt=\zeta.$$

◊

5.2 Appendix B

Appendix B contains the proofs for the mean-reverting model.

5.2.1 Proof of Proposition 3.2

$$\begin{aligned} dX(t) &= \sum_{i=0}^m N_i(t)dS_i(t)\\ &= N_0S_0(t)rdt+\sum_{i=1}^m N_i(t)S_i(t)[(\psi_i-\theta_i\ln(S_i(t)))dt+\sigma_i dW(t)]\\ &= (1-\sum_{i=1}^m \pi_i)X(t)rdt+\sum_{i=1}^m \pi_iX(t)[(\psi_i-\theta_i\ln(S_i(t)))dt+\sigma_i dW(t)]. \end{aligned}$$

The self-financing strategy is then modeled as

$$dX(t) = X(t)\left(rdt + \sum_{i=1}^{m} \pi_i[(\psi_i - r - \theta_i \ln(S_i(t)))dt + \sigma_i dW(t)]\right). \tag{5.54}$$

Let

$$Z_i(t) = \ln(S_i(t)).$$

Using Itô's lemma

$$dZ_i(t) = \frac{S_i(t)(\psi_i - \theta_i \ln(S_i(t)))dt + \sigma_i dW(t))}{S_i(t)} - \frac{S_i(t)^2\sigma^2 dt}{2S_i(t)^2}.$$

So

$$dZ_i(t) = \left(\psi_i - \frac{||\sigma||^2}{2} - \theta_i Z_i(t)dt\right) + \sigma_i dW(t). \tag{5.55}$$

In a second step, let

$$B_i(t) = \exp(\theta_i t)Z_i(t),$$

$$dB_i(t) = \theta_i \exp(\theta_i t)Z_i(t)dt + \exp(\theta_i t)dZ_i(t).$$

By (5.55)

$$\begin{aligned} dB_i(t) &= \theta_i \exp(\theta_i t)Z_i(t)dt + \exp(\theta_i t)\Big[\left(\psi_i - \frac{||\sigma||^2}{2} - \theta_i Z_i(t)dt\right) + \sigma_i dW(t)\Big], \\ &= \exp(\theta_i t)\Big[\left(\psi_i - \frac{||\sigma||^2}{2}\right)dt + \sigma dW(t)\Big]. \end{aligned} \tag{5.56}$$

Let $\omega_i = \psi_i - \frac{||\sigma_i||^2}{2}$. Hence

$$B_i(t) = B_i(0) + \omega_i \int_0^t \exp(\theta_i t)dt + \sigma_i \int_0^t \exp(\theta_i t)dW(t). \tag{5.57}$$

Since $B_i(t) = \exp(\theta_i t)Z_i(t)$ then $Z_i(t) = \exp(-\theta_i t)B_i(t)$ and $Z_i(0) = B_i(0)$. It follows that

$$Z_i(t) = \exp(-\theta_i t)B_i(0) + \omega_i \int_0^t \exp(\theta_i s - \theta_i t)ds + \exp(-\theta_i t)\sigma_i \int_0^t \exp(\theta_i s)dW(s).$$

Let

$$K(t,\theta) := \int_0^t \exp(-\theta s)ds = \exp(-\theta t)\int_0^t \exp(\theta s)ds = \begin{cases} \frac{1-\exp(-\theta t)}{\theta} & \text{if } \theta \neq 0 \\ t & \text{if } \theta = 0 \end{cases}$$

Thus

$$Z_i(t) = \exp(-\theta_i t)Z_i(0) + \omega_i K(t,\theta_i) + \exp(-\theta_i t)\sigma_i \int_0^t \exp(\theta_i s)dW(s), \tag{5.58}$$

and

$$E(Z_i(t)) = \exp(-\theta_i t)Z_i(0) + \omega_i K(t,\theta_i). \tag{5.59}$$

The covariance is calculated as follows

$$\begin{aligned}
Cov(Z_i(t), Z_j(t)) &= E[(Z_i(t) - E(Z_i(t)))(Z_j(t) - E(Z_j(t)))] \\
&= E\Big[(\exp(-\theta_i t)Z_i(0) + \omega_i K(t,\theta_i) + \exp(-\theta_i t)\sigma_i \int_0^t \exp(\theta_i s)dW(s) \\
&\quad - \exp(-\theta_i t)Z_i(0) + \omega_i K(t,\theta_i))(\exp(-\theta_j t)Z_i(0) + \omega_j K(t,\theta_j) \\
&\quad + \exp(-\theta_j t)\sigma_j \int_0^t \exp(\theta_j s)dW(s) - \exp(-\theta_j t)Z_i(0) + \omega_j K(t,\theta_j))\Big] \\
&= E\Big[\left(\exp(-\theta_i t)\sigma_i \int_0^t \exp(\theta_i s)dW(s)\right) \left(\exp(-\theta_j t)\sigma_j \int_0^t \exp(\theta_j s)dW(s)\right) \Big] \\
&= \sum_k \sigma_{ik}\sigma_{jk} \int_0^t \exp(\theta_i s + \theta_j s - (\theta_i t + \theta_j t))ds \\
&= K(t, \theta_i + \theta_j)\sigma_i \sigma_j' \qquad (5.60)
\end{aligned}$$

Let us define

$$G_{ij}(t,\theta) := Cov(Z_i(t), Z_j(t)).$$

In a next step we calculate the wealth process.

Let

$$F(t) = \ln X(t).$$

Using Ito's lemma

$$\begin{aligned}
dF(t) &= \left(r + \sum_{i=1}^{m} \pi_i(\psi_i - r - \theta_i \ln S_i(t))dt + \pi\sigma dW(t)\right) - \frac{||\pi'\sigma||^2}{2}dt, \\
&= \left(r + \pi'(\psi - r\mathbf{1}) - \frac{||\pi'\sigma||^2}{2}\right)dt + \sum_{i=1}^{m} \pi_i(-\theta_i Z_i(t))dt + \pi_i\sigma_i dW(t) \qquad (5.61)
\end{aligned}$$

From (5.55)

$$dZ_i(t) - \left(\psi_i - \frac{||\sigma||^2}{2}\right)dt = -\theta_i Z_i(t)dt + \sigma_i dW(t).$$

Hence

$$\begin{aligned} dF(t) &= \left(r + \pi'(\psi - r\mathbf{1}) - \frac{||\pi'\sigma||^2}{2}\right)dt + \sum_{ij}^{m} \pi_i \left(dZ_i(t) - \left(\psi_i - \frac{||\sigma||^2}{2}\right)dt\right) \\ &= \left(r + \pi'(\psi - r\mathbf{1}) - \frac{||\pi'\sigma||^2}{2} - \pi'\omega\right)dt + \sum_{i}^{m} \pi_i dZ_i(t) \qquad (5.62) \end{aligned}$$

$$F(t) = F(0) + \left(r + \pi'(\psi - r\mathbf{1}) - \frac{||\pi'\sigma||^2}{2} - \pi'\omega\right)t + \pi(Z(t) - Z(0)).$$

Since

$$\begin{aligned} (Z_i(t) - Z_i(0)) &= \exp(-\theta_i t)Z_i(0) + \omega_i K(t,\theta_i) + \exp(-\theta_i t)\sigma_i \int_0^t \exp(\theta_i s)dW(s) - Z_i(0), \\ &= Z_i(0)(\exp(-\theta_i t) - 1) + \omega_i K(t,\theta_i) + \exp(-\theta_i t)\sigma_i \int_0^t \exp(\theta_i s)dW(s) \\ &= -\theta_i t K(t,\theta_i)Z_i(0) + \omega_i K(t,\theta_i) + \exp(-\theta_i t)\sigma_i \int_0^t \exp(\theta_i s)dW(s) \\ &= (\omega_i - Z_i(0)\theta_i)K(t,\theta_i) + \exp(-\theta_i t)\sigma_i \int_0^t \exp(\theta_i s)dW(s). \qquad (5.63) \end{aligned}$$

If $\Xi_i = K(t,\theta_i)(\omega_i - Z_i(0)\theta_i) - \omega_i t$ then,

$$F(t) = F(0) + \left(r + \pi'(\psi - r\mathbf{1}) - \frac{||\pi'\sigma||^2}{2}\right)t + \pi'\Xi(t,\theta) + \sum_{i}^{m} \pi_i \exp(-\theta_i t)\sigma_i \int_0^t \exp(\theta_i s)dW(s). \qquad (5.64)$$

The expectation of the log normal wealth is

$$E[F(t)] = F(0) + \left(r + \pi'(\psi - r\mathbf{1}) - \frac{||\pi'\sigma||^2}{2}\right)t + \pi'\Xi(t,\theta). \qquad (5.65)$$

The variance is

$$\begin{aligned} Var(F(t)) &= E\Big[\left(\sum_i^m \pi_i \exp(-\theta_i t)\sigma_i \int_0^t \exp(\theta_i s) dW(s)\right)\Big] \\ &= \pi'(K(t,\theta_i+\theta_i)\sigma_i\sigma_i')\pi, \\ &= \pi' G(t,\theta)\pi \end{aligned} \tag{5.66}$$

The wealth $X(t)$ can be calculated as

$$X(t) = \exp(F(t)).$$

Thus

$$\begin{aligned} X(t) &= X(0)\exp\Big(\left(r+\pi'(\psi-r\mathbf{1})-\frac{||\pi'\sigma||^2}{2}\right)t+\pi'\Xi(t,\theta) \\ &+ \sum_i^m \pi_i \exp(-\theta_i t)\sigma_i \int_0^t \exp(\theta_i s)dW(s).\Big) \end{aligned} \tag{5.67}$$

Since $F(t)$ is normally distributed the wealth can be represented as $X(t) = \exp(E(F(t)) + \sqrt{Var[F(t)]}\xi$, where ξ is a standard normal variable. Therefore

$$X(t) = \exp\left(F(0) + \left(r+\pi'(\psi-r\mathbf{1})-\frac{||\pi'\sigma||^2}{2}\right)t+\pi'\Xi(t,\theta)+\sqrt{\pi' G(t,\theta)\pi}\xi\right) \tag{5.68}$$

$\diamondsuit$

5.2.2 Proof of Proposition 3.3.1

Let

$$g(t) = (\psi - r)t + \Xi(t,\theta)$$

$$f(\pi,\xi) = \pi' g(t) - \frac{t||\pi'\sigma||^2}{2} + \sqrt{\pi' G(t,\theta)\pi}\xi$$

$$X(t) = \exp(F(0) + (rt + f^*(\pi,\xi)) = X(0)\exp(rt + f^*(\pi,\xi))$$

Since $F(t)$ follows a normal distribution $E(X(t)) = E(\exp(F(t))) = \exp\left(E(F(t)) + \frac{Var(F(t))}{2}\right)$.

Therefore

$$\begin{aligned} E(X(t)) &= \exp\left(F(0) + rt + \pi'(\psi - r)t - \frac{t||\pi'\sigma||^2}{2} + \pi'\Xi(t,\theta) + \frac{1}{2}\pi G(t,\theta)\pi\right) \\ &= X(0)\exp\left(rt + \pi' g(t) - \frac{t||\pi\sigma||^2}{2} + \frac{1}{2}\pi' G(t,\theta)\pi\right) \qquad (5.69) \end{aligned}$$

$$E[X(t)^2] = X(0)^2\exp\left(2rt + 2\pi' g(t) - t||\pi'\sigma||^2 + 2\sum_i^m \pi_i \exp(-\theta_i t)\sigma_i \int_0^t \exp(\theta_i s)dW(s)\right)$$

$$\begin{aligned} Var[X(t)] &= E[X(0)^2\exp\left(2rt + 2\pi' g(t) - t||\pi'\sigma||^2 + 2\sum_i^m \pi_i \exp(-\theta_i t)\sigma_i \int_0^t \exp(\theta_i s)dW(s)\right) \\ &- \left(X(0)^2\exp\left(2rt + 2\pi' g(t) - t||\pi\sigma||^2 + \pi' G(t,\theta)\pi\right)\right)] \\ &= X(0)^2\exp\left(2rt + 2\pi' g(t) - t||\pi'\sigma||^2\right) E\Big[\exp(2\sum_i^m \pi_i \exp(-\theta_i t)\sigma_i \int_0^t \exp(\theta_i s)dW(s)) \\ &- \exp(\pi' G(t,\theta)\pi)\Big] \\ &= X(0)^2\exp\left(2rt + 2\pi' g(t) - t||\pi'\sigma||^2 + \pi' G(t,\theta)\pi\right)\left(\exp(\pi' G(t,\theta)\pi) - 1\right). \qquad (5.70) \end{aligned}$$

◇

5.2.3 Proof of Proposition 3.4.1

$$Loss(t) = -(X(t) - X(0))$$

$$1 - \alpha = P(Loss(t) \leq VaR)$$

Let

$$f^*(\xi, \pi) = \pi' g(t) - \frac{t}{2}||\pi'\sigma||^2 + \sqrt{\pi' G(t,\theta)\pi}\xi$$

where

$$g(t) = (\psi - r)t + \Xi(t, \theta).$$

Then

$$1 - \alpha = P(X(0)(1 - \exp(rt + f^*(\xi, \pi))) \leq VaR)$$

$$1 - \alpha = P\left(\exp(rt + f^*(\xi, \pi)) \geq 1 - \frac{VaR}{X(0)}\right)$$

$$1 - \alpha = P\left(rt + \pi' g(t) - \frac{t}{2}||\pi'\sigma||^2 + \sqrt{\pi' G(t,\theta)\pi}\xi \geq \ln\left(1 - \frac{VaR}{X(0)}\right)\right)$$

$$1 - \alpha = P\left(\sqrt{\pi' G(t,\theta)\pi}\xi \geq \ln\left(1 - \frac{VaR}{X(0)}\right) - rt - \pi' g(t) + \frac{t}{2}||\pi'\sigma||^2\right)$$

$$1 - \alpha = P\left(\xi \geq \frac{\ln(1 - \frac{VaR}{X(0)}) - rt - \pi' g(t) + \frac{t}{2}||\pi'\sigma||^2}{\sqrt{\pi' G(t,\theta)\pi}}\right)$$

$$1 - \alpha = 1 - P\left(\xi \leq \frac{\ln(1 - \frac{VaR}{X(0)}) - rt - \pi' g(t) + \frac{t}{2}||\pi'\sigma||^2}{\sqrt{\pi' G(t,\theta)\pi}}\right)$$

$$\alpha = P\left(\xi \leq \frac{\ln(1 - \frac{VaR}{X(0)}) - rt - \pi' g(t) + \frac{t}{2}||\pi'\sigma||^2}{\sqrt{\pi' G(t,\theta)\pi}}\right)$$

Since ξ is a standard normal with mean 0 and variance 1.

$$\alpha = N\left(\frac{\ln(1-\frac{VaR}{X(0)}) - rt - \pi' g(t) + \frac{t}{2}||\pi'\sigma||^2}{\sqrt{\pi' G(t,\theta)\pi}}\right)$$

$$N^{-1}(\alpha) = \frac{\ln(1-\frac{VaR}{X(0)}) - rt - \pi' g(t) + \frac{t}{2}||\pi'\sigma||^2}{\sqrt{\pi' G(t,\theta)\pi}}$$

$$N^{-1}(\alpha)\sqrt{\pi' G(t,\theta)\pi} = \ln\left(1-\frac{VaR}{X(0)}\right) - rt - \pi' g(t) + \frac{t}{2}||\pi'\sigma||^2$$

$$VaR_\alpha(X(t)) = X(0)\left(1-\exp\left(rt + \pi' g(t) - \frac{t}{2}||\pi'\sigma||^2 + N^{-1}(\alpha)\sqrt{\pi' G(t,\theta)\pi}\right)\right). \tag{5.71}$$

♢

5.2.4 Proof of Lemma 3.4.1.1

Let

$$f(\pi,t) = rt + \pi' g(t) - \frac{t}{2}||\pi'\sigma||^2 + N^{-1}(\alpha)\sqrt{\pi' G(t,\theta)\pi}$$

It follows that $f(\pi,t)$ is equivalent to

$$f(\lambda\pi + (1-\lambda)x, t) > \lambda f(\pi,t) + (1-\lambda) f(x,t), \forall \lambda \in (0,1). \tag{5.72}$$

(5.72) is equivalent to

$$-\frac{t}{2}||(\lambda\pi+(1-\lambda)x)'\sigma||^2 + N^{-1}(\alpha)||(\lambda\pi+(1-\lambda)x)'\sqrt{G(T,\theta)}|| > -\frac{t}{2}(\lambda||\pi'\sigma||^2 + (1-\lambda)||x'\sigma||^2)$$

$$+N^{-1}(\alpha)(\lambda||\pi'\sqrt{G(T,\theta)}|| + (1-\lambda)||x'\sqrt{G(T,\theta)}||).$$

Since, $\alpha \in (0, 0.5)$ and therefore $N^{-1}(\alpha) < 0$ the equation above holds true due to convexity of $||\cdot||$. This means that $f(\pi, t)$ is strictly concave.

$\diamondsuit$

5.2.5 Proof of Proposition 3.5.1

The Average Value of risk is

$$AVaR_\alpha(X(t)) = \frac{1}{\alpha}\int_0^\alpha VaR_u(X(t))du.$$

Therefore from (5.71) the $AVaR$ equation is

$$AVaR_\alpha(X(t)) = \frac{1}{\alpha}\int_0^\alpha X(0)\left(1 - \exp\left(rt + \pi' g(t) - \frac{t}{2}||\pi'\sigma||^2 + N^{-1}(u)\sqrt{\pi' G(t,\theta)\pi}\right)\right) du$$

$$AVaR(\pi, t) = X(0) - \frac{X(0)}{\alpha}\exp\left(rt + \pi' g(t) - \frac{t}{2}||\pi'\sigma||^2\right)\int_0^\alpha \exp(N^{-1}(u)\sqrt{\pi' G(t,\theta)\pi})du$$

Let $N^{-1}(u) = y$, thus $u = N(y)$ and $du = dN(y)$.

$$\begin{aligned} AVaR_\alpha(X(t)) &= X(0) - \frac{X(0)}{\alpha}\exp\left(rt + \pi' g(t) - \frac{t}{2}||\pi'\sigma||^2\right) \\ &\quad \left(\frac{1}{\sqrt{2\pi^*}}\int_{-\infty}^{N^{-1}(\alpha)} \exp\left(\frac{-(y - \sqrt{\pi' G(t,\theta)\pi})^2}{2}\right) dy\right) \end{aligned}$$

Let $z = y - \sqrt{\pi' G(t,\theta)\pi}$, so

$$\begin{aligned} AVaR_\alpha(X(t)) &= X(0) - \frac{X(0)}{\alpha} \exp\left(rt + \pi' g(t) - \frac{t}{2}||\pi'\sigma||^2 + \frac{\pi' G(t,\theta)\pi}{2} \right) \\ &\quad \left(\frac{1}{\sqrt{2\pi_c}} \int_{-\infty}^{N^{-1}(\alpha) - \sqrt{\pi' G(t,\theta)\pi}} \exp\left(\frac{-z^2}{2} \right) dz \right) \\ &= X(0) - \frac{X(0)}{\alpha} \exp\left(rt + \pi' g(t) - \frac{t}{2}||\sigma\pi||^2 + \frac{\pi' G(t,\theta)\pi}{2} \right) \\ &\quad (N(N^{-1}(\alpha) - \sqrt{\pi' G(t,\theta)\pi})). \end{aligned} \tag{5.73}$$

Therefore the AVaR equation is:

$$\begin{aligned} AVaR_\alpha(X(t)) &= X(0)\Big(1 - \frac{1}{\alpha} \exp\Big(rt + \pi' g(t) - \frac{t}{2}||\pi'\sigma||^2 + \frac{\pi' G(t,\theta)\pi}{2} \\ &+ \ln(N(N^{-1}(\alpha) - \sqrt{\pi' G(t,\theta)\pi}))\Big)\Big). \end{aligned} \tag{5.74}$$

$\diamondsuit$

5.2.6 Proof of Lemma 3.5.1.1

$$f_1(\pi, t) := \pi' g(t) - \frac{t}{2}||\pi'\sigma||^2 + \frac{\pi' G(t,\theta)\pi}{2} + \ln(N(N^{-1}(\alpha) - \sqrt{\pi' G(t,\theta)\pi})$$

Let

$$f_3(\pi, t) := rt + \pi' g(t) - \frac{t}{2}||\pi'\sigma||^2 + \frac{\pi' G(t,\theta)\pi}{2},$$

and

$$f_2(\pi, t) := \ln(N(N^{-1}(\alpha) - \sqrt{\pi' G(t,\theta)\pi})).$$

Therefore

$$AVaR(\pi, t) = X(0)\left(1 - \frac{1}{\alpha}\exp\left(f_3(\pi, t) + f_2(\pi, t)\right)\right).$$

$G(T, 0) - G(T, \theta)$ is positive definite, hence $\pi \to \pi' G(t, \theta)\pi - t||\pi'\sigma||^2$ is concave. It follows that $f_3(\pi, t)$ is concave. From the assumptions stated $\alpha \in (0, 0.5)$, hence $N^{-1}(\alpha) < 0$; $\pi \to \sqrt{\pi' G(t, \theta)\pi}$ is convex as illustrated in the preceding sections. This concludes that $\pi \to N^{-1}(\alpha) - \sqrt{\pi' G(t, \theta)\pi}$ is concave. It follows that $f_1(\pi, t)$ is concave.

♢

5.2.7 Proof of Theorem 3.6.1

From (5.71) this optimization problem can be written as

$$\min_{\pi} X(0)\left(1 - \exp\left(rT + \pi' g(T) - \frac{T}{2}||\pi'\sigma||^2 + N^{-1}(\alpha)\sqrt{\pi' G(T, \theta)\pi}\right)\right).$$

The optimization problem is approached as

$$\max_{\pi}\left(rT + \pi' g(T) - \frac{T}{2}||\pi'\sigma||^2 + N^{-1}(\alpha)\sqrt{\pi' G(T, \theta)\pi}\right). \tag{5.75}$$

Due to convexity first order conditions are sufficient for optimality. Therefore,

$$g(T) - T\sigma\sigma'\pi + \frac{N^{-1}(\alpha)G(T, \theta)\pi}{\sqrt{\pi' G(T, \theta)\pi}} = 0.$$

Thus

$$\pi = \left(T\sigma\sigma' - \frac{N^{-1}(\alpha)G(T, \theta)}{\sqrt{\pi' G(T, \theta)\pi}}\right)^{-1} g(T). \tag{5.76}$$

This is an implicit equation for π. We follow the approach of Dmitrasinovic-Vidovic and Ware (2011). Let $a := \sqrt{\pi' G(T,\theta)\pi}$. Therefore

$$\begin{aligned}
a^2 &= g(T)' \left(T\sigma\sigma' - \frac{N^{-1}(\alpha)G(T,\theta)}{a}\right)^{-1} G(t,\theta) \left(T\sigma\sigma' - \frac{N^{-1}(\alpha)G(T,\theta)}{a}\right)^{-1} g(T) \\
&= g(T)' \left(\frac{1}{a}(aT\sigma\sigma' - N^{-1}(\alpha)G(T,\theta))\right)^{-1} G(t,\theta) \left(\frac{1}{a}(aT\sigma\sigma' - N^{-1}(\alpha)G(T,\theta))\right)^{-1} g(T) \\
&= ag(T)' \left(aT\sigma\sigma' - N^{-1}(\alpha)G(T,\theta)\right)^{-1} G(t,\theta) \left(aT\sigma\sigma' - N^{-1}(\alpha)G(T,\theta)\right)^{-1} ag(T) \\
&= a^2 g(T)' \left(aT\sigma\sigma' - N^{-1}(\alpha)G(T,\theta)\right)^{-1} G(t,\theta) \left(aT\sigma\sigma' - N^{-1}(\alpha)G(T,\theta)\right)^{-1} g(T)
\end{aligned}$$

Furthermore,

$$1 = g(T)' \left(aT\sigma\sigma' - N^{-1}(\alpha)G(T,\theta)\right)^{-1} G(t,\theta) \left(aT\sigma\sigma' - N^{-1}(\alpha)G(T,\theta)\right)^{-1} g(T).$$

$$||\sqrt{G(T,\theta)}(aT\sigma\sigma' - N^{-1}(\alpha)G(T,\theta))^{-1} g(T)|| = 1.$$

Let

$$h(a) := ||\sqrt{G(T,\theta)}(aT\sigma\sigma' - N^{-1}(\alpha)G(T,\theta))^{-1} g(T)||^2.$$

Then

$$h(a) := ||(aT(\sqrt{G(T,\theta)}')^{-1}\sigma\sigma'(\sqrt{G(T,\theta)})^{-1} - N^{-1}(\alpha)I)^{-1}(\sqrt{G(T,\theta)}')^{-1} g(T)||^2,$$

where I is an identity matrix. Since $(\sqrt{G(T,\theta)}')^{-1}\sigma\sigma'(\sqrt{G(T,\theta)})^{-1}$ is symmetric positive definite, it has eigenvalues decomposition

$$(\sqrt{G(T,\theta)}')^{-1}\sigma\sigma'(\sqrt{G(T,\theta)})^{-1} = Q'DQ.$$

Here D has diagonal elements which are positive and Q is an orthonormal matrix. Let $z = (\sqrt{G(T,\theta)}')^{-1} g(T)$. Therefore

$$\begin{aligned} h(a) &= ||(aTQ'DQ - N^{-1}(\alpha)I)^{-1}z||^2, \\ &= ||Q'(aTD - N^{-1}(\alpha)I)^{-1}Qz||^2, \\ &= ||z'Q'(aTD - N^{-1}(\alpha)I)^{-1}QQ'(aTD + N^{-1}(\alpha)I)^{-1}Qz \\ &= z'Q'(aTD - N^{-1}(\alpha)I)^{-2}Qz. \end{aligned}$$

Let $\tau = Qz$ and $U = (aTD - N^{-1}(\alpha))^{-1}$ then

$$h(a) = \tau' U^2 \tau$$

and

$$U_i(a) = \frac{1}{aTd_i - N^{-1}(\alpha)},$$

where d_i are the diagonal entries in D. Therefore

$$h(a) = \sum_{i=m}^{m} \frac{\tau_i^2}{(aTd_i - N^{-1}(\alpha))^2}.$$

If $a = 0$ then

$$h(0) = \sum_{i=m}^{m} \frac{\tau_i^2}{(-N^{-1}(\alpha))^2} = \frac{g(T)'G(T,\theta)^{-1}g(T)}{(N^{-1}(\alpha))^2}.$$

If $g(T)'G(T,\theta)^{-1}g(T) > (N^{-1}(\alpha))^2$ then $h(a) = 1$ has a unique positive solution denoted by a^*,(since $h(a)$ is a decreasing function and $h(\infty) = 0$). The optimal strategy π^* is given by

$$\pi^* = \left(T\sigma\sigma' - \frac{N^{-1}(\alpha)G(T,\theta)}{a^*}\right)^{-1} g(T). \tag{5.77}$$

However if $g(T)'G(T,\theta)^{-1}g(T) \leq (N^{-1}(\alpha))^2$ then $h(a) = 1$ has a no positive solution, meaning there are no critical points for $\pi \to \sqrt{\pi' G(t,\theta)\pi}$. Since $\pi \to \sqrt{\pi' G(t,\theta)\pi}$ is not differentiable at 0, 0 is the solution and the optimal strategy is $\pi^* = 0$.

The minimal Value at Risk is

$$VaR(\alpha,\pi,T) := X(0)\left(1 - \exp\left(rT + (\pi^*)'g(T) - \frac{T}{2}||(\pi^*)'\sigma||^2 + N^{-1}(\alpha)\sqrt{(\pi^*)'G(t,\theta)\pi^*}\right)\right). \tag{5.78}$$

◊

5.2.8 Proof of Theorem 3.7.1

From (5.71) and (5.69) the optimization problem is

$$\max_{\pi} X(0)\exp\left(rT + \pi' g(T) - \frac{T||\pi\sigma||^2}{2} + \frac{1}{2}\pi' G(T,\theta)\pi\right) \text{ subject to}$$

$$X(0)\left(1 - \exp\left(rT + \pi' g(T) - \frac{T}{2}||\pi'\sigma||^2 + N^{-1}(\alpha)\sqrt{\pi' G(T,\theta)\pi}\right)\right) \leq C.$$

The optimization problem can be tackled as

$$\max_{\pi}\left(rT + \pi' g(T) - \frac{T||\pi\sigma||^2}{2} + \frac{1}{2}\pi' G(T,\theta)\pi\right)$$

subject to

$$rT + \pi' g(T) - \frac{T}{2}||\pi'\sigma||^2 + N^{-1}(\alpha)\sqrt{\pi' G(T,\theta)\pi} \geq \ln\left(1 - \frac{C}{X(0)}\right).$$

Let us rewrite as

$$\max_{\pi} -[h(\pi,T)] \text{ subject to } -f(\pi,T) \leq -\ln\left(1 - \frac{C}{X(0)}\right),$$

where

$$h(\pi,T) := \frac{T||\pi'\sigma||^2}{2} - \frac{1}{2}\pi' G(T,\theta)\pi$$

and

$$f(\pi,T) := rT + \pi' g(T) - \frac{T}{2}||\pi'\sigma||^2 + N^{-1}(\alpha)\sqrt{\pi' G(T,\theta)\pi}.$$

From (5.72), $f(\pi,T)$ is strictly concave therefore $-f(\pi,T)$ is strictly convex. Next, assume that

$$\bar{G}(T,\theta) = G(T,0) - G(T,\theta) = \begin{cases} \sigma\sigma'(T - \frac{1-\exp(-\theta t)}{\theta}) & \text{if } \theta \neq 0 \\ 0 & \text{if } \theta = 0. \end{cases}$$

$\bar{G}(T,\theta)$ is positive definite if $\theta \neq 0$.

The assumption is met if there is only one stock. The claim follows, if we prove that

$$\bar{G}(T,\theta) = \frac{\theta T + \exp(-\theta T) - 1}{\theta} > 0.$$

Let

$$f(\theta) = \theta T + \exp(-\theta T) - 1.$$

and

$$f'(\theta) = T - T\exp(-\theta T) \geq 0 \text{ if } \theta > 0.$$

Furthermore, if $\theta > 0$, $-h(\pi, T)$ is strictly concave.

$\diamondsuit$

Recall that the optimization problem considered here is

$$\min_{\pi} \left(\frac{T||\pi'\sigma||^2}{2} - \frac{1}{2}\pi' G(T,\theta)\pi \right) \text{ subject to}$$

$$\left(\frac{T}{2}||\pi'\sigma||^2 - N^{-1}(\alpha)\sqrt{\pi' G(T,\theta)\pi} - rT - \pi' g(T) \right) + \ln\left(1 - \frac{C}{X(0)}\right) = 0.$$

Let us introduce the Lagrangian

$$\begin{aligned} L(\pi,\lambda) &= \left(\frac{T||\pi'\sigma||^2}{2} - \frac{1}{2}\pi' G(T,\theta)\pi \right) - \lambda \Big(\frac{T}{2}||\pi'\sigma||^2 \\ &- N^{-1}(\alpha)\sqrt{\pi' G(T,\theta)\pi} - rt - \pi' g(T) + \ln\left(1 - \frac{C}{X(0)}\right) \Big). \end{aligned} \quad (5.79)$$

Furthermore, due to convexity F.O.C are sufficient for optimality. They are

$$\frac{\partial L}{\partial \pi} = T\sigma\sigma'\pi - G(T,\theta)\pi - \lambda \left(T\sigma\sigma'\pi - \frac{N^{-1}(\alpha)G(T,\theta)\pi}{\sqrt{\pi' G(T,\theta)\pi}} - g(T) \right) = 0.$$

Thus,

$$\pi = \left(T\sigma\sigma'(\lambda - 1) + G(T,\theta) - \lambda \frac{N^{-1}(\alpha)G(T,\theta)}{\sqrt{\pi' G(T,\theta)\pi}} \right)^{-1} g(T)\lambda. \quad (5.80)$$

Let $a = \sqrt{\pi' G(T,\theta)\pi}$. Hence,

$$a^2 = g(T)(\lambda)(T\sigma\sigma'(\lambda - 1) + G(T,\theta) - \lambda \frac{N^{-1}(\alpha)G(T,\theta)}{a})^{-1} G(T,\theta)(T\sigma\sigma'(\lambda - 1)$$

$$+G(T,\theta) - \lambda \frac{N^{-1}(\alpha)G(T,\theta)}{a})^{-1} g(T)(\lambda)'.$$

Adapting the same method used in previous sections we obtain

$$1 = g(T)(\lambda)(aT\sigma\sigma'(\lambda-1) + aG(T,\theta) - \lambda N^{-1}(\alpha)G(T,\theta))^{-1}G(T,\theta)(aT\sigma\sigma'(\lambda-1)$$

$$+aG(T,\theta) - \lambda N^{-1}(\alpha)G(T,\theta))^{-1}g(T)(\lambda)'.$$

Let

$$s^*(a,\lambda) := ||\sqrt{G(T,\theta)}(aT\sigma\sigma'(\lambda-1)+aG(T,\theta)-\lambda N^{-1}(\alpha)G(T,\theta))^{-1}g(T)(\lambda)||^2-1 = 0 \tag{5.81}$$

Plugging (5.80) into the VaR constraint one obtains

$$s^{**}(a,\lambda) := \frac{T}{2}||\sigma\left(T\sigma\sigma'(\lambda-1) + G(T,\theta) - \lambda\frac{N^{-1}(\alpha)G(T,\theta)}{a}\right)^{-1} g(T)(\lambda)||^2 - N^{-1}(\alpha)a -$$

$$rT - g(T)'(\lambda)\left(T\sigma\sigma'(\lambda-1) + G(T,\theta) - \lambda\frac{N^{-1}(\alpha)G(T,\theta)}{a}\right)^{-1} g(T) + \ln\left(1 - \frac{C}{X(0)}\right) = 0. \tag{5.82}$$

Equations (5.81) and (5.82) are solved numerically and the optimal solution generated is denoted by a^* and λ^*. It follows that the optimal portfolio strategy is given by

$$\pi^* = \left(T\sigma\sigma'(\lambda^*-1) + G(T,\theta) - \lambda^*\frac{N^{-1}(\alpha)G(T,\theta)}{a^*}\right)^{-1} g(T)(\lambda^*). \tag{5.83}$$

The maximum expected return is

$$E[X^{\pi^*}(T)] = X(0)\exp\left(rT + (\pi^*)'g(T) - \frac{T||(\pi^*)'\sigma||^2}{2} + \frac{(\pi^*)'G(T,\theta)(\pi^*)}{2}\right). \tag{5.84}$$

$\diamondsuit$

5.2.9 Proof of Theorem 3.8.1

From (5.74) the optimization problem can be written as

$$\inf_{\pi} X(0)\left(1-\frac{1}{\alpha}\exp\left(rT+\pi' g(T)-\frac{T}{2}||\pi'\sigma||^2+\frac{\pi' G(T,\theta)\pi}{2}+\ln(N(N^{-1}(\alpha)-\sqrt{\pi' G(T,\theta)\pi}))\right)\right)$$

Let

$$\hat{f}(\pi,T)=rT+\pi' g(T)-\frac{T}{2}||\pi'\sigma||^2+\frac{\pi' G(T,\theta)\pi}{2}+\ln(N(N^{-1}(\alpha)-\sqrt{\pi' G(T,\theta)\pi})).$$

Hence the optimization problem can be tackled as

$$(P1) \qquad \sup_{\pi}\left(\hat{f}(\pi,T)\right).$$

By the convexity of $\hat{f}(\pi,T)$ the optimizer is given by F.O.C.

$$\frac{\partial\hat{f}}{\partial\pi}=g(T)-T\sigma\sigma'\pi+G(T,\theta)\pi-\frac{G(T,\theta)\pi\exp(-\frac{(N^{-1}(\alpha)-\sqrt{\pi' G(T,\theta)\pi})^2}{2})}{\sqrt{\pi' G(T,\theta)\pi}(\sqrt{2\pi_c})(N(N^{-1}(\alpha)-\sqrt{\pi' G(T,\theta)\pi}))}=0. \tag{5.85}$$

Let

$$w:=\frac{\exp(-\frac{(N^{-1}(\alpha)-\sqrt{\pi' G(T,\theta)\pi})^2}{2})}{(\sqrt{2\pi_c})N(N^{-1}(\alpha)-\sqrt{\pi' G(T,\theta)\pi})}. \tag{5.86}$$

Solving for π in (5.85), one obtains

$$\pi=\Big(T\sigma\sigma'+G(T,\theta)\Big(\frac{w}{\sqrt{\pi' G(T,\theta)\pi}}-1\Big)\Big)^{-1}g(T). \tag{5.87}$$

Let $a := \sqrt{\pi' G(T,\theta)\pi}$. Hence,

$$\begin{aligned} a^2 &= g(T)'\Big(T\sigma\sigma' + G(T,\theta)\Big(\frac{w}{a} - 1\Big)\Big)^{-1} G(T,\theta)\Big(T\sigma\sigma' + G(T,\theta)\Big(\frac{w}{a} - 1\Big)\Big)^{-1} g(T) \\ &= g(T)'\Big(aT\sigma\sigma' + G(T,\theta)(w-a)\Big)^{-1} G(T,\theta)\Big(aT\sigma\sigma' + G(T,\theta)(w-a)\Big)^{-1} g(T). \end{aligned}$$

Furthermore,

$$1 = g(T)'\Big(aT\sigma\sigma' + G(T,\theta)(w-a)\Big)^{-1} G(T,\theta)\Big(aT\sigma\sigma' + G(T,\theta)(w-a)\Big)^{-1} g(T).$$

$$||\sqrt{G(T,\theta)}(aT\sigma\sigma' + G(T,\theta)(w-a))^{-1} g(T)|| = 1$$

Let

$$h_1(a) \;:=\; ||\sqrt{G(T,\theta)}(aT\sigma\sigma' + G(T,\theta)(w-a))^{-1} g(T)||^2.$$

One of the solutions of $h_1(a) = 1$ is denoted as $\breve{a}$. Thus, optimal strategy $\breve{\pi}$ is given by

$$\breve{\pi} = \Big(T\sigma\sigma' + G(T,\theta)\Big(\frac{\exp(-\frac{(N^{-1}(\alpha)-\breve{a})^2}{2})}{\breve{a}(\sqrt{2\pi_c})N(N^{-1}(\alpha)-\breve{a})} - 1\Big)\Big)^{-1} g(T). \tag{5.88}$$

If $h_1(a) = 1$ has no positive solutions, meaning there are no critical points. Since $\pi \to \sqrt{\pi' G(t,\theta)\pi}$ is not differentiable at 0, 0 is the solution and the optimal strategy is $\breve{\pi} = \mathbf{0}$, where $\mathbf{0}$ denotes a 0 vector.

The minimal average value at risk is

$$\begin{aligned} AVaR(\alpha,\pi,T) &= X(0)\Big(1 - \frac{1}{\alpha}\exp\Big(rT + (\breve{\pi})' g(t) - \frac{T}{2}||\breve{\pi}'\sigma||^2 + \frac{(\breve{\pi})' G(t,\theta)\breve{\pi}}{2} \\ &\quad + \ln(N(N^{-1}(\alpha) - \sqrt{(\breve{\pi})' G(T,\theta)\breve{\pi}}))\Big)\Big). \end{aligned} \tag{5.89}$$

◊

5.2.10 Proof of Theorem 3.9.1

The optimization problem can be written as

$$\sup_{\pi} X(0)\exp\left(rT + \pi' g(T) - \frac{T||\pi'\sigma||^2}{2} + \frac{1}{2}\pi' G(T,\theta)\pi\right)$$

subject to

$$X(0)\Big(1 - \frac{1}{\alpha}\exp\Big(rT + \pi' g(T) - \frac{T}{2}||\pi'\sigma||^2 + \frac{\pi' G(T,\theta)\pi}{2} \\ + \ln(N(N^{-1}(\alpha) - \sqrt{\pi' G(T,\theta)\pi}))\Big)\Big) \leq C^*.$$

Therefore , the optimization problem can be approached as

$$\sup_{\pi}\left(rT + \pi' g(T) - \frac{T||\pi'\sigma||^2}{2} + \frac{1}{2}\pi' G(T,\theta)\pi\right) \text{ subject to}$$

$$rT+\pi' g(T)-\frac{T||\pi'\sigma||^2}{2}+\frac{1}{2}\pi' G(T,\theta)\pi+\ln\left(N(N^{-1}(\alpha) - \sqrt{\pi' G(T,\theta)\pi})\right) \geq \ln\left(\alpha - \frac{\alpha C^*}{X(0)}\right).$$

The optimization problem is further transformed to

$$\sup_{\pi} -\left(\frac{T||\pi'\sigma||^2}{2} - \frac{1}{2}\pi' G(T,\theta)\pi\right) \text{ subject to}$$

$$-rT-\pi' g(T)+\frac{T||\pi'\sigma||^2}{2}-\frac{1}{2}\pi' G(T,\theta)\pi-\ln\left(N(N^{-1}(\alpha) - \sqrt{\pi' G(T,\theta)\pi})\right) \leq -\ln\left(\alpha - \frac{\alpha C^*}{X(0)}\right).$$

Or

$$(P1) \qquad \inf_{\pi}\left(\frac{T||\pi'\sigma||^2}{2} - \frac{\pi' G(T,\theta)\pi}{2}\right) \text{ subject to}$$

$$\frac{T||\pi'\sigma||^2}{2}-\frac{\pi'G(T,\theta)\pi}{2}-rT-\pi'g(T)-\ln\left(N(N^{-1}(\alpha)-\sqrt{\pi'G(T,\theta)\pi})\right)+\ln\left(\alpha-\frac{\alpha C^*}{X(0)}\right)\leq 0. \tag{5.90}$$

Let us introduce the Lagrangian

$$\begin{aligned} L(\pi,\lambda) &= \frac{T||\pi'\sigma||^2}{2}-\frac{\pi'G(T,\theta)\pi}{2}-\lambda\Big(\frac{T||\pi'\sigma||^2}{2}-\frac{\pi'G(T,\theta)\pi}{2} \\ &\quad -rT-\pi'g(T)-\ln\left(N(N^{-1}(\alpha)-\sqrt{\pi'G(T,\theta)\pi})\right)+\ln\left(\alpha-\frac{\alpha C^*}{X(0)}\right)\Big) \end{aligned}$$

Due to convexity F.O.C are sufficient for optimality. They are

$$\begin{aligned} \frac{\partial L}{\partial \pi} &= T\sigma\sigma'\pi - G(T,\theta)\pi-\lambda\Big(T\sigma\sigma'\pi-G(T,\theta)\pi-g(T) \\ &\quad +\frac{G(T,\theta)\pi\exp(-\frac{(N^{-1}(\alpha)-\sqrt{\pi'G(T,\theta)\pi})^2}{2})}{\sqrt{\pi'G(T,\theta)\pi}(\sqrt{2\pi_c})N(N^{-1}(\alpha)-\sqrt{\pi'G(T,\theta)\pi})}\Big)=0 \end{aligned} \tag{5.91}$$

Let

$$w := \frac{\exp(-\frac{(N^{-1}(\alpha)-\sqrt{\pi'G(T,\theta)\pi})^2}{2})}{(\sqrt{2\pi_c})N(N^{-1}(\alpha)-\sqrt{\pi'G(T,\theta)\pi})}. \tag{5.92}$$

Solving for π in (5.91) one obtains

$$\pi=\left(T\sigma\sigma'(\lambda-1)+G(T,\theta)\Big(1-\lambda+\frac{\lambda w}{\sqrt{\pi'G(T,\theta)\pi}}\Big)\right)^{-1}g(T)(\lambda). \tag{5.93}$$

Set $a=\sqrt{\pi'G(T,\theta)\pi}$. Hence

$$\begin{aligned} a^2 &= g(T)'(\lambda)\left(T\sigma\sigma'(\lambda-1)+G(T,\theta)\Big(1-\lambda+\frac{\lambda w}{a}\Big)\right)^{-1}G(T,\theta) \\ &\quad \left(T\sigma\sigma'(\lambda-1)+G(T,\theta)\Big(1-\lambda+\frac{\lambda w}{a}\Big)\right)^{-1}g(T). \end{aligned}$$

Adapting the same procedure in the previous section we attain

$$\begin{aligned} 1 &= g(T)'(\lambda)\left(aT\sigma\sigma'(\lambda-1)+G(T,\theta)(a(1-\lambda)+\lambda w)\right)^{-1}G(T,\theta) \\ &\quad \left(aT\sigma\sigma'(\lambda-1)+G(T,\theta)(a(1-\lambda)+\lambda w)\right)^{-1}g(T)(\lambda). \end{aligned}$$

Let

$$\breve{s}^*(a,\lambda) := ||\sqrt{G(T,\theta)}(aT\sigma\sigma'(\lambda-1)+G(T,\theta)(a(1-\lambda)+\lambda w))^{-1}g(T)(\lambda)||^2-1. \quad (5.94)$$

Substitute (5.93) into the AVaR constraint then,

$$\begin{aligned} \breve{s}^{**}(a,\lambda) &:= \frac{T}{2}||\sigma\left(T\sigma\sigma'(\lambda-1)+G(T,\theta)\Big(1-\lambda+\frac{\lambda w}{a}\Big)\right)^{-1}g(T)(\lambda)||^2-\frac{a^2}{2}-rT \\ &\quad -g(T)'(\lambda)\left(T\sigma\sigma'(\lambda-1)+G(T,\theta)\Big(1-\lambda+\frac{\lambda w}{a}\Big)\right)^{-1}g(T) \\ &\quad -\ln(N(N^{-1}(\alpha)-a))+\ln\left(\alpha\left(1-\frac{C^*}{X(0)}\right)\right). \end{aligned} \quad (5.95)$$

(5.94) and (5.95) are solved numerically and the solutions generated is denoted by $\breve{a}^*$ and $\breve{\lambda}^*$. It follows that the optimal portfolio strategy is given by

$$\breve{\pi}^* = \left(T\sigma\sigma'(\breve{\lambda}^*-1)+G(T,\theta)\Big(1-\breve{\lambda}^*+\frac{\breve{\lambda}^*\exp(-\frac{(N^{-1}(\alpha)-\breve{a}^*)^2}{2})}{\breve{a}^*(\sqrt{2\pi_c})N(N^{-1}(\alpha)-\breve{a}^*)}\Big)\right)^{-1}g(T)(\breve{\lambda}^*). \quad (5.96)$$

The maximum expected wealth is

$$E[X^{\breve{\pi}^*}(T)] = X(0)\exp\left(rT+(\breve{\pi}^*)'g(T)-\frac{T||(\breve{\pi}^*)'\sigma||^2}{2}+\frac{(\breve{\pi}^*)'G(T,\theta)\breve{\pi}^*}{2}\right) \quad (5.97)$$

♢

5.2.11 Proof of Theorem 3.10.1

The optimization problem (P23) is

$$\min_{\pi} X(0)\left(1-\exp\left(rT+\pi' g(T)-\frac{T}{2}||\pi'\sigma||^2+N^{-1}(\alpha)\sqrt{\pi' G(T,\theta)\pi}\right)\right) \text{ subject to}$$

$$X(0)\exp\left(rT+\pi' g(T)-\frac{T||\pi\sigma||^2}{2}+\frac{1}{2}\pi' G(T,\theta)\pi\right)=M.$$

Since

$$rT+\pi' g(T)-\frac{T||\pi\sigma||^2}{2}=\ln\left(\frac{M}{X(0)}\right)-\frac{1}{2}\pi' G(T,\theta)\pi,$$

it follows that the optimization problem becomes

$$\min_{\pi} X(0)\left(1-\exp\left(\ln\left(\frac{M}{X(0)}\right)-\frac{1}{2}\pi' G(T,\theta)\pi+N^{-1}(\alpha)\sqrt{\pi' G(T,\theta)\pi}\right)\right) \text{ subject to}$$

$$X(0)\exp\left(rT+\pi' g(T)-\frac{T||\pi\sigma||^2}{2}+\frac{1}{2}\pi' G(T,\theta)\pi\right)=M.$$

The optimization problem can be reduced to

$$\min_{\pi}\left(\pi' G(T,\theta)\pi\right)$$

subject to

$$rT+\pi' g(T)-\frac{T||\pi\sigma||^2}{2}+\frac{1}{2}\pi' G(T,\theta)\pi=\ln\left(\frac{M}{X(0)}\right).$$

The Lagrangian is

$$L(\pi,\lambda) = (\pi' G(T,\theta)\pi)-\lambda\Big(rT+\pi' g(T)-\frac{T||\pi\sigma||^2}{2}+\frac{1}{2}\pi' G(T,\theta)\pi-\ln\left(\frac{M}{X(0)}\right)\Big).$$

Furthermore, due to convexity F.O.C are sufficient for optimality. They are

$$\frac{\partial L}{\partial \pi} = 2G(T,\theta)\pi - \lambda\left(G(T,\theta)\pi - T\sigma\sigma'\pi + g(T)\right) = 0.$$

Thus,

$$\pi = \left(T\sigma\sigma'(\lambda) + G(T,\theta)(2-\lambda)\right)^{-1} g(T)\lambda. \tag{5.98}$$

Substituting (5.98) into the constraint we obtain

$$\begin{aligned} s^*(\lambda) \;&:=\; ||rT + g(T)'(\lambda)\left(T\sigma\sigma'(\lambda) + G(T,\theta)(2-\lambda)\right)^{-1} g(T) + \frac{1}{2}||\sqrt{G(T,\theta)}(T\sigma\sigma'(\lambda) \\ &+G(T,\theta)(2-\lambda))^{-1} g(T)(\lambda)||^2 - \frac{T}{2}||\sigma\left(T\sigma\sigma'(\lambda) + G(T,\theta)(2-\lambda)\right)^{-1} g(T)(\lambda)||^2 \\ &- \ln\left(\frac{M}{X(0)}\right) = 0. \end{aligned} \tag{5.99}$$

Equation (5.99) is solved numerically and the optimal solution generated is denoted by λ^*. It follows that the optimal portfolio strategy is given by

$$\tilde{\pi}^a = \left(T\sigma\sigma'\lambda^* - G(T,\theta)(2-\lambda^*)\right)^{-1} g(T)\lambda^*. \tag{5.100}$$

In a next step we tackle the optimization problem (P24)

$$\begin{aligned} \inf_{\pi} \Big[X(0)\Big(1 - \frac{1}{\alpha}\exp\Big(rT + \pi' g(T) - \frac{T}{2}||\pi'\sigma||^2 + \frac{\pi' G(T,\theta)\pi}{2} \\ + \ln(N(N^{-1}(\alpha) - \sqrt{\pi' G(T,\theta)\pi}))\Big)\Big)\Big]. \end{aligned}$$

subject to

$$X(0)\exp\left(rT + \pi' g(T) - \frac{T||\pi'\sigma||^2}{2} + \frac{1}{2}\pi' G(T,\theta)\pi\right) = M$$

Therefore, the optimization problem can be approached as

$$\sup_{\pi}\left(\ln(N(N^{-1}(\alpha)-\sqrt{\pi' G(T,\theta)\pi}))\right) \text{ subject to}$$

$$rT+\pi' g(T)-\frac{T||\pi'\sigma||^2}{2}+\frac{1}{2}\pi' G(T,\theta)\pi=\ln\left(\frac{M}{X(0)}\right).$$

Since the function $x \to \ln\left(N(N^{-1}(\alpha)-x)\right)$ is decreasing it becomes

$$\inf_{\pi}\left(\pi' G(T,\theta)\pi\right)$$

subject to

$$rT+\pi' g(T)-\frac{T||\pi\sigma||^2}{2}+\frac{1}{2}\pi' G(T,\theta)\pi=\ln\left(\frac{M}{X(0)}\right).$$

The optimization problem (P25) can be written as

$$\inf_{\pi}\left[X(0)^2\exp\left(2rT+2\pi' g(T)-T||\pi'\sigma||^2+\pi' G(T,\theta)\pi\right)\left(\exp(\pi' G(T,\theta)\pi)-1\right)\right]$$

subject to

$$X(0)\exp\left(rT+\pi' g(T)-\frac{T||\pi'\sigma||^2}{2}+\frac{1}{2}\pi' G(T,\theta)\pi\right)=M$$

Similarly it can be shown that the optimization problem can be reduced to

$$\inf_{\pi}\left(\pi' G(T,\theta)\pi\right) \text{ subject to}$$

$$rT + \pi' g(T) - \frac{T||\pi'\sigma||^2}{2} + \frac{1}{2}\pi' G(T,\theta)\pi = \ln\left(\frac{M}{X(0)}\right).$$

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