TAXATION AND BEHAVIOUR UNDER UNCERTAINTY

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ABSTRACT

of

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This dissertation examines the economic effects of alternative taxation policies on household consumption and investment decisions under uncertainty. A consideration of the alternative approaches to a theory of decision-making under uncertainty (Part 1) reveals that maximisation of expected utility is consistent with rational behaviour in a world of uncertainty. However, an alternative criterion of minimising the probability that the risky outcome will fall below some critical level (the "safety-first" principle) may also appear as rational. Consequently, we consider both types of behavioural models.

In a single-period framework (Part 2), the particular decision process we consider is that of an individual allocating his initial wealth between riskless assets with a secure rate of return and risky assets with a random rate of return. We then investigate the effects of taxation on risk-taking (both social and private).

We observe that a chance-constrained portfolio choice model can be interpreted as a reasonable description of the investor's concern for safety, and that the qualitative results regarding portfolio separation (implying that optimal risky asset ratios are independent of initial wealth) and the effects of taxation (suggesting that a proportional or a lump-sum tax with full loss offsets encourages a movement towards the

riskier assets) are the same both for a normal distribution of asset returns and the alternative assumption of a lognormal securities market.

Part 2 also investigates the effects of a simple progressive tax schedule (a linear tax with a marginal tax rate which applies both above and below an exemption level) on risk-taking. Assuming only that the investor is a risk-avertor and that the risky asset is superior (or alternatively, assuming only decreasing absolute risk-aversion), we show that linearly progressive taxation of investment income (with full loss offsets) encourages the demand for the risky asset, and that exempting risky capital gains (losses) from taxation discourages (total) risk-taking. We further show that a linearly progressive tax on investment income leads to greater risk-taking than a flat rate proportional tax where both of these taxes lead to equal losses of expected utility for the investor, or alternatively, where they yield the same expected revenue.

Part 3 considers models of portfolio choice and consumption allocation in an intertemporal context. Apart from taxation of non-asset income and the case where the rate of return on riskless investment is zero, the kind of a priori restrictions we have placed on single-period preferences are no longer sufficient to determine the effects of taxation. However, given the relative magnitudes of the income elasticities of consumption and of the risky asset demand, assumptions on the risk-aversion measures allow us to determine these results. This analysis, therefore, also indicates the kind of empirical knowledge that is required in order to meaningfully discuss the implications of alternative taxation policies.

We further pursue the framework of intertemporal consumption-portfolio allocation to analyse the long debated issue of the differential incidence of a consumption (expenditure) tax rather than an (investment) income tax. We find that under some reasonably interpretable conditions, the differential incidence of a consumption tax is to encourage risk-taking and discourage saving more effectively than an investment income tax. This result is in conflict with the general consensus in the literature that an income tax discourages saving as compared to a consumption tax. We, therefore, conclude in this context that with the introduction of uncertainty, the implications of fiscal policy are modified in an important way.

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PART 1: INTRODUCTION

"...Our lack of economic knowledge is, in good part, our difficulty in modelling the ignorance of the economic agent" (Arrow [4], p. 1).

CHAPTER I

SOME INTRODUCTORY NOTES ON TAXATION, RISK AND UNCERTAINTY

In this chapter, we indicate the nature of the problem to be studied and explain why the problem is worth discussing. We shall also attempt to justify our basic approach to the treatment of uncertainty.

I.1 The Problem: Its Definition and Importance

In general terms the subject matter of this dissertation can be described as an analysis of the consequences of government budget policy on household behaviour in a world of uncertainty. By government budget policy, we mean the alternative tax-transfer schemes that the government can adopt in order to attain its objectives (e.g., proportional vs. progressive taxes, taxes on consumption rather than taxes on income). In a single period framework, the particular decision process we consider is that of an individual allocating his initial wealth [endowment] between riskless assets with a secure rate of return and risky assets with a random rate of return. In the language of portfolio theory both these types of activities are known as investments, and, in particular, investment in the risky asset is termed risk-taking. We will use the phrase risk-taking increases when, given the amount of initial wealth, the demand for risky assets increases. In an inter-temporal context, we simultaneously allow consumption decisions as well as the investment decisions just mentioned. One of the major differences between the two types of decision processes is that while the investment fund is fixed by initial wealth in the former, the individual can, by adjusting consumption

expenditures, vary the fund to be allocated to portfolio investments in the latter. Within these frameworks of individual decision-making, alternative taxation policies and changes therein may have many effects (e.g., on the rate of interest, on wages and other incomes, on saving and investment, on profits and losses, on consumption habits). But, since we are dealing with models of individual behaviour, we focus on the effects of alternative taxation policies on the variables that are under the control of these economic agents, i.e., on investment, consumption and saving strategies with a particular emphasis on risk-taking. A full account of the economic consequences of taxation policies, however, would have to be sought in a general equilibrium context and is not attempted here. Nevertheless, we shall argue that the present topic, i.e., the effects of alternative taxation policies on individual decision-making under uncertainty, is a significant one.

Throughout our analysis we make the assumption of perfect markets in the competitive sense, i.e., the rate of return on investment is independent of the amount invested. In other words, the investor is "small" compared to the market. For this reason, our analysis is not quite applicable to investment in real capital where the prospect of the return is thought to be very much dependent on the amount of investment. Thus, it is generally held that the main applicability of the present analysis is in explaining Stock Exchange investment.[1] This is clearly the case with models of single-period analysis (Part 2 of this thesis). Fortunately, not all of our discussion need be given such a narrow

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[1] For instance, see Hicks [31] for a similar remark. Also note that we do not consider borrowing by the investor. Dropping this restriction might modify some of our results.

interpretation: as soon as we incorporate such primitive decisions as consumption and saving in an inter-temporal setting, the preceding remark regarding the specific applicability of our analysis loses its strength. Given that the future is necessarily uncertain (there being very few future's markets) and that the individual economic agents are "small" compared to the economy, the analysis of Part 3, intertemporal analysis, would seem to be quite appropriate for the discussion of the broader questions of fiscal policy as it affects most rational decision-making units in the economy.

Lump-sum taxes, even though they affect behaviour do not change any relative prices in the economy and, thus, are thought to be the least distortionary of all taxes.[2] However, most real-world taxes are not of this kind and in many instances lump-sum taxes are not desirable on grounds of economic equity and justice. Typically, therefore, taxes change market allocations. In this dissertation, however, our main concern is not with the welfare consequences of such changes but with the descriptive question of what precise effects will alternative budgetary policies have on different economic variables. Such knowledge can also serve as an input in devising an optimal tax structure under uncertainty.[3] "Therefore", as Myrdal has argued, "the problem of

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[2]In the language of price theory, we say that a lump-sum tax generates an income effect but no substitution effect. Some other taxes, depending on the particular context, may yield a similar result. But, a priori, it may not be obvious which other taxes are, in effect, equivalent to a lump-sum tax.

[3]The problem of optimum taxation involves choosing a tax structure, given the government's revenue requirements, such that social welfare, in some sense, is at a maximum. The fundamental work in this area is that of Mirrlees [47]. However, the discussion has, so far, been in the context of a world of certainty.

incidence has logical priority also from the point of view of welfare or justice" ([52], p. 186).[4]

To conclude this section, let us refer, by way of an illustration, to an important issue of tax reform. One of the major innovations of the Carter Commission [60] was to argue for a comprehensive tax base. The principal rivals for this title are income and consumption [expenditure]. The Commission argued for income, while Nicholas Kaldor [32] argued for an expenditure tax. The principal rationale behind Kaldor's argument was the belief that a tax assessed on expenditure does not discriminate against either saving or risk-taking as opposed to an income tax. On the other hand, Musgrave [51] felt that if all savings were for future consumption, this choice was unimportant. He had earlier demonstrated this to be the case under the conditions of certainty in his celebrated text [50]. Later on in this study (Chapters V and VI) we will see that Musgrave's conclusion, although valid in a world of certainty, no longer holds once we are in the realm of uncertainty. This remark, it is hoped, will indicate the importance of a systematic study in order to meaningfully discuss fiscal policy questions under uncertainty.

I.2 Treatment of Uncertainty and Probability Theory

In the last section, we have used the words "risk" and "uncertainty" without caring to discuss how they were precisely defined. This section attempts to clarify our stand on probability theory and its relation to decision-making under "uncertainty".

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[4]Myrdal uses the term incidence in the sense of the comparative static effects of changes in the alternative taxation policies on different economic variables. This is distinct from the conventional notion of the term. Musgrave ([50], pp. 207-208) defines incidence as the resulting change in the distribution of income.

Most decisions require choosing an act a_i from a given set $A = \{a_i\}$, $i=1,\ldots,n$. If any such act leads to a unique set of consequences $\{u_i\}$, one could label the procedure as choice under certainty. However, when an act a_i leads to one of a set of consequences $U = \{u_{ij}\}$, $j=1,\ldots,m$ depending upon which element of the set $S = \{s_j\}$ of the m-states of nature[5] obtains, we are in the realm of decision-making under uncertainty. We may denote the set $U = \{u_{ij}\}$ as the set of utility payoffs. Essentially, therefore, the state-dependence of the payoff structure of an act characterises choice under uncertainty.[6]

As we have learned from decision-making under certainty, rational behaviour means behaviour according to some ordering of alternative acts in terms of their relative desirability. But, this relative desirability of alternative acts depends, in an uncertain world, upon which state of nature prevails and this the individual agents are not supposed to know (at least completely). Thus we have an immediate problem: the ordering of uncertain consequences. This ordering depends on how one describes these uncertain consequences and to this consideration we now turn.

Although the agent is ignorant of the state of nature that will prevail, he might very well possess a subjective probability distribution over the states of nature. (Behaviour need not be a random phenomenon!) Such subjectivity, as Arrow [3] noted, may very well stem from observations

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[5]Arrow defines a state of nature as a description of the world so complete that, if true and known, the consequences of every action would be known ([3], p. 45).

[6]It may be noted that this definition of uncertainty is distinct from game-theoretic uncertainty. In the latter event, the set of states $S = \{s_j\}$ are not the strategies employed by Nature but rather by other conscious agents. The type of uncertainty we are interested in is better described as "games against nature".

on the external world. In the literature, there exists a considerable debate regarding the meaning (or merit, or both) of alternative representations of the agent's information (however incomplete) concerning the states of nature, and, consequently, regarding the alternative characterisations of behaviour under uncertainty.[7]

Most of this debate was concerned with the appropriateness (or the lack thereof) of probability theory in describing uncertain consequences. For instance, Shackle felt that if the individual does not have the ability to repeat the experiment indefinitely often, then probabilities, being essentially long-run frequency ratios, are irrelevant to his conduct. In general, it is clear that probability if defined as an objective measure of relative frequencies implies that probability statements cannot describe all kinds of ignorance. Both Fisher [23] and Keynes [33], on the other hand, viewed probability as a measure of degree of belief (subjective). Even then, probabilities, according to Keynes [33], are not necessarily measurable. This led to Knight's distinction between risk and uncertainty. He defined risk as measurable uncertainty, while true uncertainty was not quantifiable: "the essence of the situation [uncertainty] ... [is] neither entire ignorance nor complete and perfect information, but partial knowledge" (p. 199). "Knight's uncertainties", commented Arrow, "seem to have surprisingly many of the properties of ordinary probabilities, and it is not clear how much is gained by the distinction" ([3], p. 18).

In sum, the major source of discontent of these authors with

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[7]See, for instance, Keynes [33], Knight [35], Fisher [23] and Shackle [66], where each of the authors had something different to suggest.

the probability theory related to their view that probability statements cannot describe all types of uncertainties. However, Ramsey has argued that this need not be insuperable: "The kind of measurement of belief with which probability is concerned is not this kind but is a measurement of belief qua basis of action" ([57], p. 171). Following the advance made by Ramsey, Von Neumann and Morgenstern [74] later established the following: If a person is able to express preferences between every possible pair of gambles, where the gambles are taken over some basic set of alternatives, then one can introduce utility associations to the basic set of alternatives in such a manner that, if the person is guided solely by the utility expected value, he is acting in accord with his true tastes.[8] This result only requires that the individual is consistent in his tastes. It should be noted that this result justifies expected utility maximisation without having to interpret probabilities as long-run frequency ratios. In other words, probability distribution is relevant even when only one event is to be observed. Furthermore, this procedure of assigning utilities to outcomes can be accomplished without his being consciously aware of making his decisions in this manner.

Later, Savage [64] has generalised this result. Rather than assuming subjective probability as Von Neumann - Morgenstern do, he generates a subjective probability measure as a consequence of his axioms of consistency. These results, therefore, render any distinction between risk and uncertainty -- in the sense of Knight -- quite irrelevant for

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[8]This particular wording of the Von Neumann-Morgenstern result is due to Luce and Raiffa ([41], p. 21).

purposes of a theory of decision-making. We may, therefore, use these two concepts synonymously.

The expected utility theorem, therefore, appears as the fundamental postulate of rational behaviour under uncertainty. Indeed, most of the discussion to follow -- Chapters IV through VII -- are expected utility analyses of the effects of taxation on household behaviour.

There are situations, however, where additional criteria may appeal to the intuition as being rational.[9] One such class of approaches involves, as a criterion of ordering, minimisation of the probability that outcome will fall below some critical level.[10] These may be referred to as Safety-First models. Arrow [3] and Marschak [46] have argued that the above rule would be a special case of the Von Neumann-Morgenstern theory, while Roy [59], who also proposed this rule in the theory of portfolio choice, did not favour such an interpretation. This is our subject-matter in Chapters II and III, where we show that Domar and Musgrave [16] also had a similar idea; and, furthermore, that Naslund's stochastic programming model [53] also has a modified safety-first interpretation.

I.3 Specific Role of this Thesis

It is commonly believed that the true driving force behind modern

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[9] One other approach toward a theory of uncertainty is that of Shackle. For a critical review of his contribution, the reader is referred to Arrow [3], Baumol and Graaf [25] and Egerton [18]. However, his critics usefully point out that his theory is not capable of explaining portfolio diversification which is at variance with everyday experience and, thus, Shackle's theory cannot be a useful alternative in the present context.

[10] These approaches would, in effect, alter the objective function and not the underlying approach to probability theory.

capitalist growth is entrepreneurship -- the willingness to take risk. There has been much concern that taxes may discourage risk-taking. But, in economic theory the conclusions have been quite different. Both Tobin [72] and Domar-Musgrave come to the conclusion that an increase in the proportional tax rate on portfolio income will, with full loss offset, increase the holding of the risky asset in a portfolio of a given size.[11] For no loss offsets, the effect is indeterminate.

Leaving until later chapters the discussion of whether such results are intuitively plausible or not, we note that the theoretical foundation of the above-mentioned analyses is quite unsatisfactory. Consequently, the predictions [the effects of taxation] can be no more reliable than the behavioural models from which they emerge. We review here the approaches of these authors:

(a) Mean-Variance Approach. Recall that Tobin's analysis [72] -- and this led to a decade of such studies -- requires that individuals chose the optimum portfolio allocation considering only the mean and the variance of alternative portfolio returns. The limitations of this framework are well known. However, to illustrate, we mention the following:

(i) First, there is no inherent reason why the first two moments of a probability distribution will provide sufficient information to choice under uncertainty. Consider the following example due to Mossin ([49], p. 26), where

	y	$f_1(y)$	y	$f_2(y)$
A	2	1/4	4	3/4
B	6	3/4	8	1/4

[11] By a portfolio of given size or a fixed portfolio, we mean that as in single-period models, the fund (initial endowment) out of which the investor allocates his portfolio is fixed.

y denotes the outcome with $f_1(y)$ and $f_2(y)$ as alternative probability distributions. In both cases, the means are equal to 5 and the variance 3. But there does not seem to be any convincing reason why one should be indifferent between the two choices;

(ii) One way to justify sole concern for mean-variance is that we have a quadratic utility function,[12] i.e.,

$$U(y) = y-by^2, \; b > 0. \qquad \ldots(I\text{-}1)$$

But such restrictions on preferences are unsatisfactory for at least two reasons:

(a) Marginal utility, $U'(y)$, becomes negative at finite incomes, which is contrary to experience and to non-satiety axioms we usually accept in consumer theory;

(b) Even if one restricts attention to that range of y for which $U'(y)$ is strictly positive, the demand for risky assets decreases with wealth;

(iii) A second way of justifying the mean-variance approach is to assume a two-parameter distribution, e.g., the Normal. This may be appropriate in certain circumstances, but, may not capture many important situations (e.g., the simple example given in the table above cannot be ruled out as "unrealistic").

(b) Domar-Musgrave Approach. In Chapter II of this dissertation we shall argue that a closer examination of the approach taken by Domar and

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[12]This can be seen as follows: Taking expectation of the utility function, (I-1), we have

$$E[U(y)] = E[y-by^2] = \mu-b[\mu^2+\sigma^2],$$

where μ is the mean and σ^2, the variance of y. Thus expected utility is completely determined by the mean and the variance of the underlying probability distribution when there is a quadratic utility function.

Musgrave leads, contrary to their original assertion, to the following result: a change in the proportional tax rate on portfolio income with full loss offset has no effect on risk-taking.

With these reservations on the traditional analysis in mind, we are, therefore, left with no basis for any claim that increased taxation encourages risk-taking or discourages it. This thesis is thus concerned with re-examining this and related questions. We do so in two steps:

(1) First, in keeping with the traditional analysis (Fisher and Keynes [34]), we concentrate on pure portfolio theory, i.e., portfolio allocation out of a given initial endowment. The major questions asked in this context are, whether a proportional tax encourages risk-taking and, whether progression would change this result. In the process, we focus on behavioural models of both the general expected utility of wealth and the "safety-first" types.

(2) Next, we depart from tradition and consider models of portfolio choice and consumption allocation in an intertemporal context -- an area that has been inadequately discussed by fiscal theorists. In these models, as has already been mentioned, households simultaneously determine the optimal size and the optimal composition of the portfolio. We devote a considerable space (Chapters V and VI) to the discussion of simple two-period models which capture this feature of intertemporal preferences. Since, our main interest is to derive the implications of temporal ("delayed" or future) uncertainty on current decisions, this would also seem to be appropriate. However, in Chapter VII, we examine a discrete-time dynamic programming model with an infinite horizon in an attempt to discover if the results of the two-period models still hold in

a more general framework.

The main question we ask in these contexts is whether the results of pure portfolio theory (Part 2 of this thesis) depend in any fundamental way on the assumption of a fixed investable wealth. This would simultaneously tell us whether increased taxation (of different types) favours a substitution of present over future consumption through increased or decreased risk-taking. We must, however, bear in mind that giving up a dollar of current consumption does not necessarily result in a certain increase in future consumption because of the uncertainty of capital risk: "the more one saves, the more one stands to lose" (Sandmo [63], p. 353).[13] The other major question we raise relates to the differential incidence[14] of consumption rather than investment income taxes (Chapter VI). In particular, this provides a formal test of Kaldor's proposition that a tax on consumption does not discriminate against risk-taking and saving, while an income tax does. The last chapter (VIII) attempts to summarise the main findings of our study and their possible interpretations.

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[13] This is strictly true of investment in risky assets. However, the investor saves both in the safe and in the risky forms.

[14] Differential incidence of a change in the taxation policy measures the effect of a policy change such that the net revenue due to changes in the tax parameters is nil. See Musgrave ([50], chapter 10) for a discussion of this and other incidence concepts.

PART 2: SINGLE-PERIOD ANALYSIS

"...Single-period analysis is always a first step".

(Hicks, J.R.: Capital and Growth.)

CHAPTER II

SAFETY-FIRST, RISK-TAKING AND TAXATION

This chapter analyses several models which use the minimisation of the probability that the return from investment will fall below some critical level as a criterion for ordering uncertain prospects. Domar-Musgrave [16], Roy [59] and Marschak [46] have all offered alternative approaches to the problem. Under some conditions, as has also been mentioned in Chapter I, the approaches of Domar-Musgrave and Marschak become special cases of the expected utility theory. Roy, on the other hand, deliberately avoided any expected utility interpretation.[1] Instead, he minimises an upper bound of the probability that investment income will fall below the disaster level. In this chapter, we examine the behavioural implications of such restrictions on preferences. We find that none of these approaches satisfactorily achieves their stated goal, i.e., to consider the minimisation of the probability of loss. It is also shown that, if we adopt Roy's formulation, investors do not hold risky assets if a safe asset is available. Such extreme caution does not seem very reasonable. However, in the next chapter, it will be shown that a <u>chance-constrained programme</u>[2] of the type employed by Naslund [53] can be viewed as a generalisation of Roy's safety-first principle in the sense that it

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[1]As a matter of fact, Roy believed, in the tradition of the frequency school, that probability statements cannot describe all types of uncertainties, which, as we have shown in Chapter I, cannot refute the expected utility hypothesis.

[2]Optimisation problems where the constraints appear in the form of probability statements can be described as a <u>chance-constrained programme</u>.

allows for an interior solution (i.e., diversification in the presence of a safe asset).

In terms of the effects of taxation, it is found in Section II.2 that a rigorous interpretation[3] of the Domar-Musgrave model of investment behaviour yields the result that a change in the proportional tax rate on investment income with full loss offsets has no effect on risk-taking, and that, in the absence of loss offsets, the same tax reduces the demand for risky assets. This is at variance with their original conclusion that a proportional tax with full loss offsets encourages risk-taking, and that the result is indeterminate where losses cannot be offset at all.

II.1 Three Approaches to Safety-First

According to Roy, "... for [a] large number of people some idea of a disaster exists, and the principle of safety-first asserts that it is reasonable and probable in practice that an individual will seek to reduce as far as is possible the chance of such a catastrophe occurring" (p. 432). Marschak states that there is some number d such that if $X \leq d$, where X is some random outcome (e.g., investment returns), the firm is bankrupt and, therefore, he assumes that the possibility of the occurrence of this situation would be minimised. This idea is also embedded in the rationale underlying the Domar-Musgrave model of investment behaviour under uncertainty: "... of all possible questions which the investor may ask, the most important one, it appears to us, is concerned with the probability

[3]By a rigorous interpretation we mean that a theory of rational behaviour means behaviour according to some ordering of the consequences. Thus, for instance, the only interpretation of an indifference map in a certain space is necessarily that of a real-valued representation of preferences in that space (i.e., corresponding to any indifference map, there is an underlying utility function determined in a special unique way).

of actual yield being less than zero, that is, with the probability of loss. This is the essence of risk." (p. 396).[4] While the intuitive basis of these reasonings are alike, their analytical treatments, as will be argued in this section, have been quite different. More importantly, we show that the resulting behaviour in all these models fails to capture the intuition propounded in the statement of their models.

(a) Marschak's Suggestion. Arrow [3] has shown that the rule of minimising the probability that income falls below some critical level (say, zero), if viewed in an expected utility sense, implies that the utility is zero for all incomes below the critical level and unity for all incomes above it. This is clearly seen as follows: Given

$$\left.\begin{array}{l} U(X) = 0 \text{ for } X \leq 0 \\ U(X) = 1 \text{ for } X > 0 \end{array}\right\}, \qquad \ldots(II\text{-}1)$$

and for any probability density function of X, expected utility is given by

$$\left.\begin{array}{l} EU(X) = P(X > 0) \\ \text{or,} \quad EU(X) = (1-\alpha) \end{array}\right\}, \qquad \ldots(II\text{-}2)$$

where $\alpha = \int_{-\infty}^{0} f(X)\,dX$, i.e., the probability of loss, and E is the expectation operator over the probability distribution of X. Clearly, therefore, maximisation of (II-2) is equivalent to minimising the probability of loss. Thus, a whole hearted acceptance of the Domar-Musgrave investment rationale would imply a utility function of the type given by (II-1).

It is interesting to note that in an attempt to generalise the utility function, (II-1), Marschak suggested that

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[4] The underlining, except for the last word, is due to the present author.

$$\left.\begin{array}{l} U(X) = -\nu \text{ for } X \leq 0 \\ U(X) = X \text{ for } X > 0 \end{array}\right\} . \qquad \ldots(II\text{-}3)$$

Expected utility is now given by

$$EU(X) = -\alpha\nu + (1-\alpha)\beta, \qquad \ldots(II\text{-}4)$$

where α is as defined above and

$$\beta = \int_0^\infty Xf(X)dX / \int_0^\infty f(X)dX$$

(i.e., expected profits for $X > 0$).

From (II-4) Marschak suggested that maximising expected utility would imply minimisation of the probability of loss, α , <u>for a given β</u>. In the appendix (Section A.II.1) we indicate that this is not a very profound result. In particular, we shall argue that <u>unless</u> β is held at its optimum value with respect to expected utility, maximisation of expected utility does not correspond with the minimisation of the probability of loss. In general, therefore, the intuitive rationale of a straight-forward maximisation of Marschak's utility function is unclear.

<u>(b) "Yield-Risk" and Expected Utility</u>. As mentioned in the preceding paragraphs, the expected utility function given by equation (II-2) would seem to capture the Domar-Musgrave investment rationale. But, when they come to model uncertainty Domar-Musgrave do not follow this approach. Neither, as it turns out, do they retain the investment rationale they started with. Instead, they argue that in making decisions, financial investors weigh the advantage of a greater return, or <u>yield</u>, against the disadvantage of a probable loss, or <u>risk</u>. In order to give precise meaning to these terms "yield" and "risk", they appeal to the properties of the investor's subjective probability distribution over the anticipated returns. In particular, they assume that the investor focuses his attention

on the mathematical expectation of the percentage yield, y, and in addition, on y's positive or "gain" component, g, and its negative or "risk" component, r. i.e.,

$$r \equiv (-) \int_{-\infty}^{0} Xf(X)dX, \qquad \text{...(II-5)}$$

and

$$g \equiv \int_{0}^{\infty} Xf(X)dX. \qquad \text{...(II-6)}$$

Thus, the yield, y, is defined

$$y \equiv \int_{-\infty}^{\infty} Xf(X)dX = (g-r). \qquad \text{...(II-7)}$$

They did recognise the usefulness of a more precise coefficient of risk, but left that task for later analysts. In effect they assumed that "the investor will consider changes in y and r only" (p. 397). It would be interesting to examine whether the "yield-risk" criterion of ordering is consistent with their earlier argument of minimising the probability of loss as being the essence of risk.[5]

In this connection the following fundamental result is due to Richter: "<u>when the investor maximises expected utility and his portfolio preferences can be described in terms of the first n portfolio income moments, then his utility is an n-th degree polynomial in income</u>" ([58], p. 154).[6] Thus, the yield-risk criterion of ordering investment opportunities implies a utility function of the form (notice that "yield" is the first moment while "risk" is a <u>truncated</u> first moment):

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[5]It may be argued that the yield-risk criterion allows a trade-off (between yield and risk), while the objective of minimising risk apparently does not, and hence we would expect the two processes to be different. However, we note that unless one describes how risk is minimised, any such remark may not be warranted.

[6]The underlining is this author's.

$$\left.\begin{array}{l} U = \gamma + cX \text{ for } X \geq 0 \\ U = \gamma + bX \text{ for } X \leq 0 \end{array}\right\}, \qquad \ldots(II\text{-}8)$$

where b and c are positive constants.

Taking the expectation, we have

$$EU(X) = \gamma + b\int_{-\infty}^{0} Xf(X)dX + c\int_{0}^{\infty} Xf(X)dX,$$

or, $$EU(X) = \gamma + c\ (\text{yield}) + (c-b)(\text{risk}), \qquad \ldots(II\text{-}9)$$

where $(b-c) > 0$. Extending the argument of the appendix (Section A.II.1), one could argue that, given the optimal level of yield, maximisation of (II-9) leads to a minimum of risk. Thus, it is in this rather loose sense that maximisation of the Domar-Musgrave expected utility is related to the objective of minimisation of the probable loss and not the probability of loss.[7] Thus we conclude that the yield-risk rule of ordering is, in general, inconsistent with the objective of minimising the probability of loss.

(c) Safety-First or Safety All? Compared to the approaches of Domar-Musgrave or Marschak, Roy's is more direct. He assumes that the investor knows the mean (μ) and the standard deviation (σ) of the investment returns, X. This allows him to obtain an upper bound of the probability of disaster $P(X \leq d)$, where d denotes the disaster level using the Tchebycheff

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[7]The yield-risk utility function is related to the objective of minimising probable loss in the same sense as Marschak's utility function is related to the objective of minimising the probability of loss. It is easy to check that these two types of utility functions cannot be equivalent: Define $\theta = \int_{-\infty}^{0} Xf(X)dX / \int_{-\infty}^{0} f(X)dX$ as some index of "average loss"(negative) and using the definitions (II-5) through (II-7), we can rewrite (II-4) as

$$EU(X) = (\text{yield}) + (1+\frac{\nu}{\theta})(\text{risk}). \qquad \ldots(II\text{-}4)'$$

Since θ depends on the choice of the strategy, preferences described by (II-4)' and (II-9) cannot be equivalent.

theorem:[8]

Let $g(X)$ be a non-negative function of the random variable X. For every $K > 0$, we then have,

$$P(g(X) \geq K) \leq \frac{Eg(X)}{K} \cdot \qquad \text{...(II-10)}$$

By taking $g(X) = (X-\mu)^2$ and $K = k^2\sigma^2$, we obtain, for every $k > 0$, the Bienayme-Tchebycheff inequality:

$$P(|X-\mu| \geq k\sigma) \leq (\frac{1}{k})^2. \qquad \text{...(II-11)}$$

Furthermore, for $k = (\frac{\mu-d}{\sigma})$, we have

$$P(|X-\mu| > \mu-d) \leq (\frac{\sigma}{\mu-d})^2, \qquad \text{...(II-12)}$$

and, in particular,

$$P(X \leq d) = P(\mu-X \geq \mu-d) \leq (\frac{\sigma}{\mu-d})^2. \qquad \text{...(II-13)}$$

Instead of minimising $P(X \leq d)$, Roy minimises the upper bound of this probability as given by the r.h.s. of (II-13). Notice that the derivation of (II-13) is independent of any specific assumption regarding the nature of the distribution of asset returns.[9]

The investor's problem, therefore, for a two-asset case, is given by:

$$\min_{\{x_1,x_2\}} (\frac{\sigma}{\mu-d})^2 \quad \text{s.t. } M = x_1 + x_2, \qquad \text{...(II-14)}$$

---

[8]See Cramer ([12], pp. 182-183) for a discussion of Tchebycheff's and related inequalities.

[9]It is interesting to note that if we are willing to assume a known distribution, we can obtain the exact deterministic equivalent of the probability $P(X \leq d)$. In particular, if X is normally distributed with μ and σ^2 as its first two moments,

$$P(X \leq d) = 1-P(X \geq d) = 1/2 - F(\theta),$$

where $F(\theta) = \frac{1}{\sqrt{2\pi}} \int_\theta^0 e^{-t^2/2}dt$, $\theta = (\frac{d-\mu}{\sigma})$ and $t = (\frac{X-\mu}{\sigma})$.

Under the normal distribution, therefore, minimising $(\frac{\sigma}{\mu-d})$ is in fact equivalent to minimising $P(X \leq d)$.

where $\sigma^2 \equiv x_1^2\sigma_{11} + 2x_1x_2\sigma_{12} + x_2^2\sigma_{22}$,

$\mu \equiv \mu_1x_1 + \mu_2x_2$,

and $\mu_i (\equiv E(r_i))$ is the expected return on the i-th asset, σ_{ij} are the elements of the variance-covariance matrix, $[\sigma_{ij}]$, and M is initial wealth. As sketched in the appendix (Section A.II.2) the solution to the above problem is given by the following demand functions:

$$x_1 = \frac{M}{D}\{\sigma_{22}(\mu_1 - \frac{d}{M}) - \sigma_{12}(\mu_2 - \frac{d}{M})\}, \qquad \text{...(II-15)}$$

$$x_2 = \frac{M}{D}\{\sigma_{11}(\mu_2 - \frac{d}{M}) - \sigma_{12}(\mu_1 - \frac{d}{M})\}, \qquad \text{...(II-16)}$$

where

$$D \equiv (\sigma_{11} - \sigma_{12})(\mu_2 - \frac{d}{M}) - (\sigma_{12} - \sigma_{22})(\mu_1 - \frac{d}{M}).$$

Assuming the first asset to be riskless, i.e., $\sigma_{11} = \sigma_{12} = 0$, the above demand functions reduce to

$$\left.\begin{aligned} x_1 &= M, \\ x_2 &= 0. \end{aligned}\right\} \qquad \text{...(II-17)}$$

It must be emphasized that this result is not a special case applicable to the two-asset model alone, but is due to the singularity of the $[\sigma_{ij}]$-matrix in the presence of a riskless asset. Thus we have the following conclusion:[10]

<u>The safety-first investors, according to Roy's formulation, do not hold</u>

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[10] After an early version of this work was presented in a seminar, it was discovered that this result has also been pointed out by Pyle and Turnovsky [56]. However, apart from our somewhat different approach, this brief presentation should help eliminate the confusion on this issue. References may be made to Telser ([71], pp. 2-3), Hahn ([27], pp. 283-288), Lintner ([39], pp. 18-19), Naslund (p. 293) and, even after the publication of Pyle and Turnovsky, to Allingham ([2], p. 213) and Levy and Sarnat ([38], pp. 1829-34).

risky assets if a safe asset is available. In other words, the liquidity preference curve is a vertical straight line.[11]

In geometric terms, the analysis of portfolio behaviour for the one safe-one risky asset case implies that the function, $h(\mu,\sigma) = 0$, obtained by minimising σ^2 given the values of μ and M is a straight line of positive slope. This is so because once we specify the expected value of the portfolio, its standard deviation is uniquely determined. As Mossin ([49], p. 46) has observed, this means that no real minimisation of σ^2 is involved, so that the set of efficient portfolios coincides with the set of attainable ones. To see this, we note that from

$$\left.\begin{aligned} \mu &= E(\Sigma_i r_i x_i), \\ M &= \Sigma_i x_i, \end{aligned}\right\} i = 1,2$$

we obtain, letting x_1 be safe,

$$x_2(\mu_2 - r_1) = \mu - r_1 M. \qquad \text{...(II-18)}$$

In conjunction with $\sigma^2 = x_2^2\sigma_{22}$, (II-18) yields

$$\mu = r_1 M + (\mu_2 - r_1)\frac{\sigma}{\sqrt{\sigma_{22}}}, \qquad \text{...(II-19)}$$

which is plotted below (Figure II.1).[12] In order to minimise the probability

---

[11] Notice that our remark on liquidity preference is valid in the sense that traditional portfolio analysis gives "money" the interpretation of a safe asset. In any case, our result that the liquidity preference curve is vertical is contrary to the comments made by Hahn in the discussion of Tobin's paper [73]. Hahn raised a question concerning the relative merits of Tobin's theory and that of Roy. He was of the opinion that both of these theories explained diversification and the downward sloping liquidity preference function.

[12] Notice that we have plotted (II-19) such that $d < r_1 M$. The fact that this is legitimate can easily be checked from Roy's derivation of the inequality (II-13). In particular, for that inequality to hold, Roy needed $k = (\mu - d)/\sigma > 0$, which at $(x_1 = M, x_2 = 0)$ requires that $r_1 M > d$. Alternatively, this condition can also be derived from the Kuhn-Tucker conditions for a corner optimum of Roy's problem. However, for $r_1 M < d$, the investor's problem cannot obviously be discussed in the above framework. It may still be pointed out that the more he saves in the form of the safe asset, the smaller is his probability of avoiding disaster.

of disaster, Roy requires that the investor maximises the steepness of the line drawn from (0,d) such that it is tangent to (II-19). It can be seen from Figure II.1 that such a procedure leads to the corner solution given by the equations (II-17). In the next chapter, we shall attempt a re-interpretation of a chance-constrained portfolio choice model as a modified safety-first problem such that it allows diversification in the presence of a safe asset.

II.2 The Effects of Taxation[13]

In their diagrammatic exposition, Domar and Musgrave draw an indifference map on the assumptions that the marginal utility of income (yield) falls with rising income and that the marginal disutility of risk rises with increasing risk. But, as we have shown in the last section, the appropriate indifference curves generated by the expected utility function (II-9) are parallel straight lines of slope $(c/b-c)$, a positive constant. From the nature of such indifference curves, Richter concluded:

> ...such indifference curves are incapable of generating the movement towards higher risk-taking suggested by Domar and Musgrave. Under neither proportional income tax nor lump-sum tax shifts will there by any change in the portfolio ([58], p. 157).

However, as we shall presently see, Richter's conclusion is only valid in the full loss-offset case. To see this, we superimpose linear indifference curves while retaining the optimum asset curves[14] as drawn by Domar-Musgrave.

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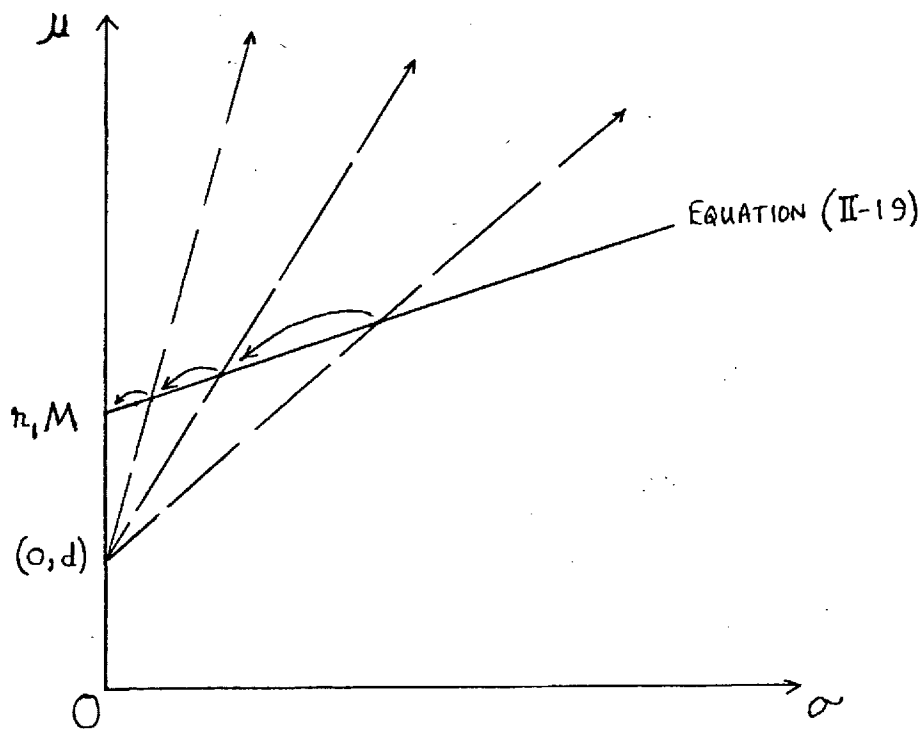
[13] Despite our reservations regarding these approaches to the investor's problem, we nevertheless analyse the effects of taxation in the Domar-Musgrave model when interpreted in the expected utility sense. This is largely because their paper has been the seminal work in the area of taxation and risk-taking.

[14] These curves give the locus of the maximum yield subject to a given level of risk for both "cash" and "non-cash" assets from which the investor may choose. The derivation is detailed in Domar-Musgrave (pp. 397-402).

Figure II.1: Safety-First and Portfolio Equilibrium



Case 1: Full Loss Offset. When a proportional tax on investment income is imposed the values of risk and yield are both reduced at the same rate. This is so because the government shares equally in both losses and gains. Therefore, each point on the pre-tax optimum asset curve OGB in Figure II.2 moves along a ray from the origin. Suppose that the pre-tax equilibrium is given by the point M on OGB. With the imposition of a tax at the rate $t(0 < t < 1)$ the investor, while retaining the same portfolio, finds himself at M_t (on the new optimum asset curve OG_tB_t) such that the ratio MM_t/OM equals the tax rate. Notice that the slope of OG_tB_t at M_t is equal to the slope of OGB at M since risk and yield are, again, in the same ratio as before the imposition of the tax. This together with linear indifference curves requires that M_t is also the post-tax equilibrium after the individual has adjusted his risk-taking behaviour. The optimum asset composition, therefore, remains unchanged. Thus, while total risk[15] (and yield) remains unchanged at M, private risk-taking is, however, reduced (so is the private yield) to or_t (oy_t).

Intuitively, the fact that the investor fails to compensate for the tax (increased tax rate) can be explained in the following terms:

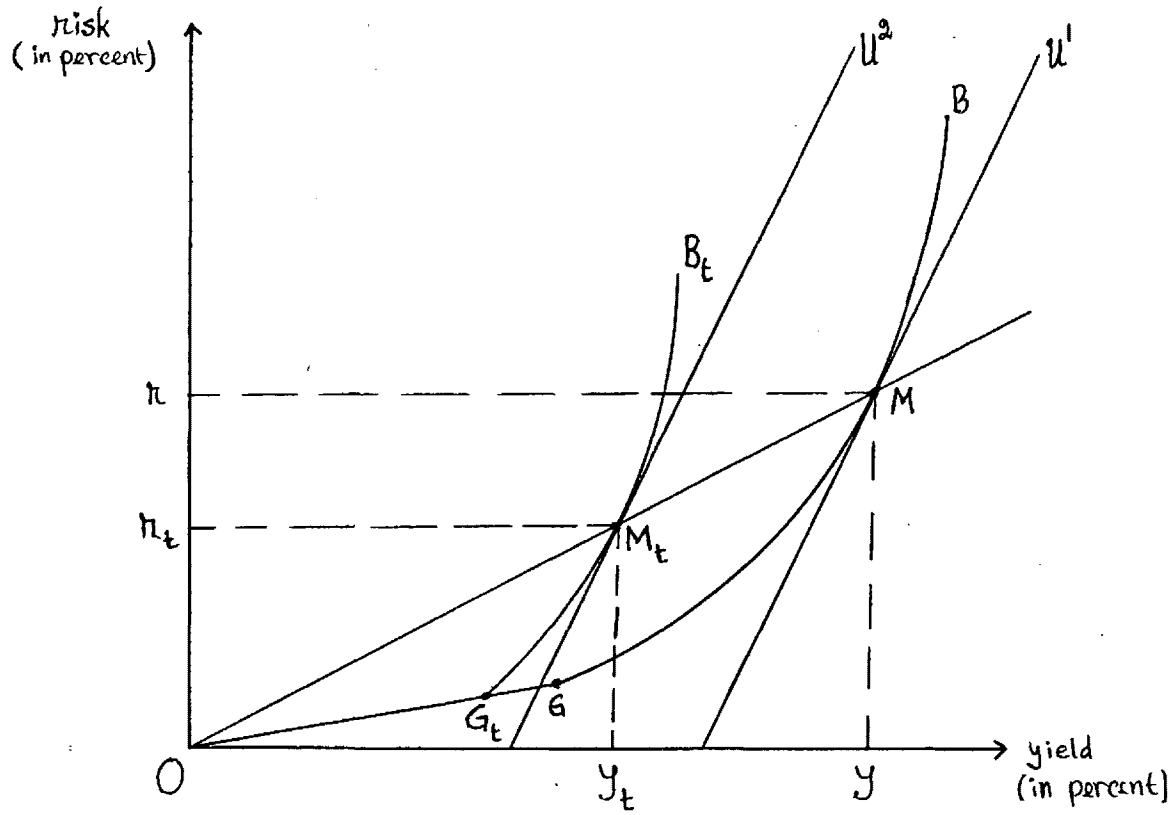
(a) Proportional taxation reduces risk and yield by the same rate and, thus, is unable to render riskier assets any more attractive than before the tax;

(b) Although the investor's yield is reduced, he is unwilling to offset this by additional risk-taking due to the constancy of the marginal

---

[15] We use the terms private and total (social) risks in the sense defined by Domar-Musgrave: private risk (yield) is that which is borne by the investor, and the total risk (yield) denotes the total risk-taking (yield) in the economy and, thus, includes the part borne by the government as well.

Figure II.2: Income Tax with Full Loss Offsets



utility of income. Hence we have shown the following:

An investor who maximises expected utility and is only concerned with "risk" and "yield" of alternative asset combinations do not change his portfolio as a result of a (change in) proportional income tax with full loss offsets.

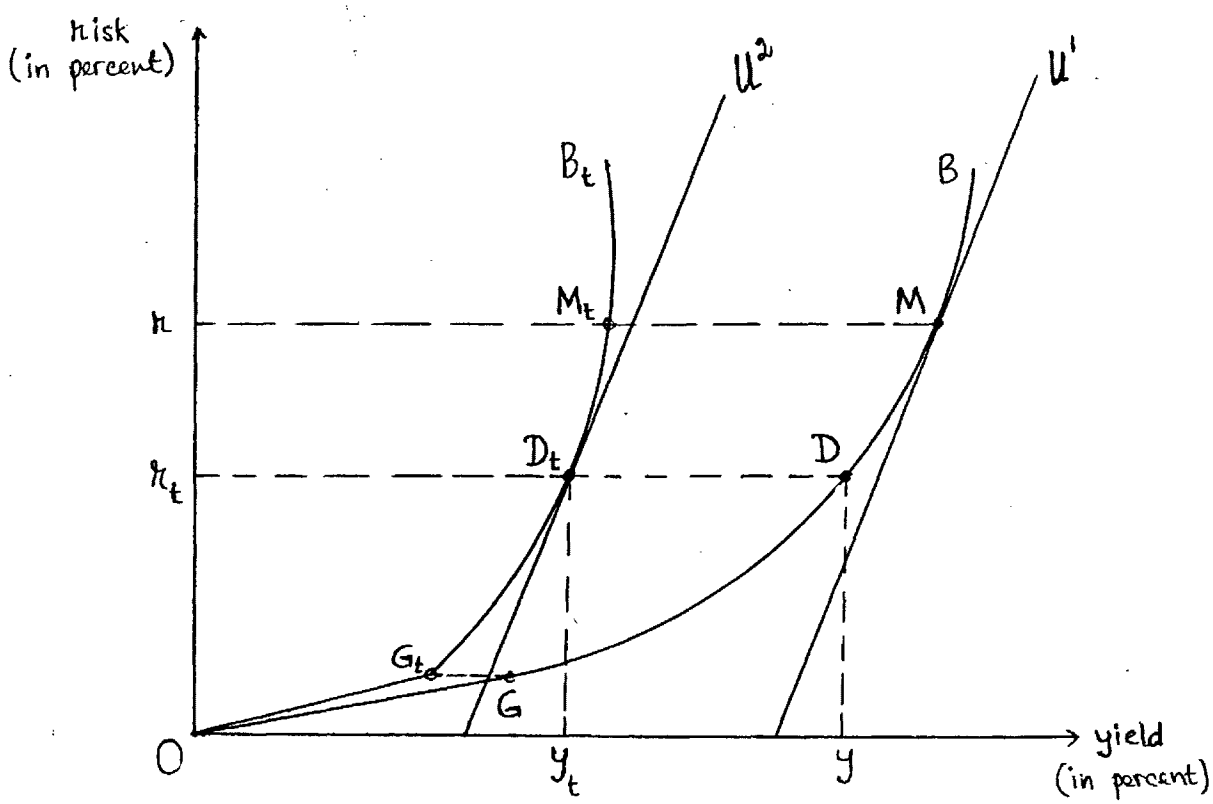
In the appendix (Section A.II.3), we provide an algebraic proof of this result for the case of normally distributed asset returns.

Case 2: No Loss Offset. By imposing a tax without loss offsets, the government shares in the investor's gains, while leaving his losses unchanged. Equation (II-7) would now read

$$y_t = (1-t)g-r,$$

where y_t denotes after-tax yield. Domar and Musgrave define the percentage reduction in yield as the degree of tax sensitiveness (this equals $(y-y_t)/y = (1+r/y)t$). With a tax without loss offsets, each point on the (pre-tax) optimum asset curve OGB (Figure II.3) suffers a reduction in yield in accordance with its degree of tax sensitiveness and moves horizontally to the left. This horizontal movement is porportional to (r/y). OG_tB_t is then the new optimum asset curve. Further, as the ratio (r/y) rises with the level of risk, the upper part of OG_tB_t bends to the left. With the imposition of the tax, the investor -- whose pre-tax equilibrium is at M -- now finds himself at M_t. Since the slope of the indifference curves remains constant, M_t, or for that matter any point above it cannot be the new portfolio equilibrium. This is so because the horizontal shifts in the optimum asset curves rises with the level of risk: for instance, the shift from M to M_t is greater than from D to D_t. The post-tax equilibrium, therefore, is at some point D_t which

Figure II.3: Income Tax without Loss Offsets



is below M_t. At D_t both private risk and yield is reduced compared with M_t. Moreover, total risk (and yield) corresponding to the new equilibrium is given by D which is again below M. Or_t is the new level of risk (both total and private), while Oy is the new total yield and Oy_t is the private yield. Again, the intuitive reasoning behind this result would run as follows: Taxation makes the riskier assets less attractive thereby discouraging risk-taking, while the constancy of marginal utility of income does not allow the investor to offset the reduction in yield.[16] We, therefore, obtain the following result:

<u>An "yield-risk" investor reduces the amount of private risk-taking due to a (an increase in) proportional tax on portfolio income when losses cannot be offset which leads to lower total risk-taking (and a lower total yield as well)</u>.

In view of the preceding discussion, it is of some interest to note that Lepper [37] uses a utility function similar to the ones discussed above except that it is quadratic. She also defines <u>semi-variance</u>, $\int_{-\infty}^{d} X^2 f(X)dX$, as the coefficient of risk and the utility function is written in the form

$$EU(X) = a\int_{-\infty}^{d} X^2 f(X)dX + (a+1)\mu, \qquad \text{...(II-20)}$$

where

$$U(X) = aX^2 + (a+1)X \text{ for } X < d,$$

$$U(X) = ad^2 + (a+1)X \text{ for } X \geq d,$$

$-1 < a < 0$ and d is a positive constant. Although (II-20) generates

---

[16] Recall that the ambiguity of the effect of taxation in the no loss offset case in the original Domar-Musgrave treatment was due to their assumed slope of the indifference curves.

linear indifference curves in the (mean, semi-variance)-space, Lepper's results are somewhat different from those obtained above. In particular, under a proportional tax on income with full loss offset, the marginal rate of return per unit of risk (i.e., the slope of the opportunity locus) is larger at points corresponding to the pre-tax efficient portfolios. This causes a substitution in favour of the riskier assets which is missing in the "yield-risk" model. This difference is due to the fact that Lepper uses the semi-variance as the measure of risk and taxation reduces this by the square of the tax rate. However, under a proportional tax without loss offset, the mean return (yield) associated with any amount of risk is reduced by slightly more than in the case of full loss offsets. But the "risk" associated with the portfolio is unaffected by the tax. The marginal rate of return per unit of risk is, therefore, reduced by the tax. This leads to a reduction in the demand for riskier assets -- the same conclusion as in the "Case-2" of the yield-risk model.

II.3 Conclusion

In this chapter we have reviewed three early approaches towards a theory of risk-taking behaviour. These theories were particularly concerned with the objective of minimisation of the probability of loss. It has been discovered that none of them succeeded in describing the behaviour that would have been consistent with their stated goals. In terms of the effects of taxation, we have closely examined the Domar-Musgrave model and found, contrary to the original conclusion by these authors, that a proportional tax on investment income does not affect risk-taking when losses can be fully offset, but this tax reduces risk-taking when losses cannot be offset at all. We have to recognize that these results are

obtained from a model whose intuitive basis, as has been shown, is quite unclear and they certainly have to be tested in much wider contexts (i.e., more realistic models of behaviour) before any firm conclusions can be drawn.

Appendix to Chapter II

A.II.1 On a Comment Due to Marschak.[17] In the context of the utility function

$$EU(X) = -\alpha\nu + (1-\alpha)\beta,$$

where

$$\alpha = \int_{-\infty}^{0} f(X)dX, \quad \beta = \int_{0}^{\infty} Xf(X)dX / \int_{0}^{\infty} f(X)dX$$

and $0 \leq \alpha \leq 1$, Marschak commented that, for a given β, maximisation of $EU(X)$ would minimise α. In this section we show that although the above statement is correct, maximisation of expected utility subject to a given β is not equivalent to maximising expected utility unless β is fixed at its optimum value with respect to expected utility. By the same token, minimising the probability of loss (α) subject to a given β is not equivalent to minimising the probability of loss. Further, the connection between maximisation of expected utility and minimisation of probability of loss breaks down whenever both α and β are decision variables.

This is seen as follows. Constant expected utility contours are given by

$$C = -\alpha\nu + (1-\alpha)\beta,$$

or

$$\beta = \frac{\alpha\nu + C}{1-\alpha}, \qquad \text{...(A-II-1)}$$

where both $d\beta/d\alpha$ and $d^2\beta/d\alpha^2$ are positive. We can, therefore, draw the constant expected utility contours in the (α,β)-space as in Figure A.II.1.

Typically, however, α and β would be related so that we are constrained to a subset of values (say, given by an implicit function $g(\alpha,\beta) = 0$ of which the north-west boundary, AB, is efficient). This

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[17] The argument of this section of the appendix is due to D.W. Butterfield.

Figure A.II.1: Constant Expected Utility Contours

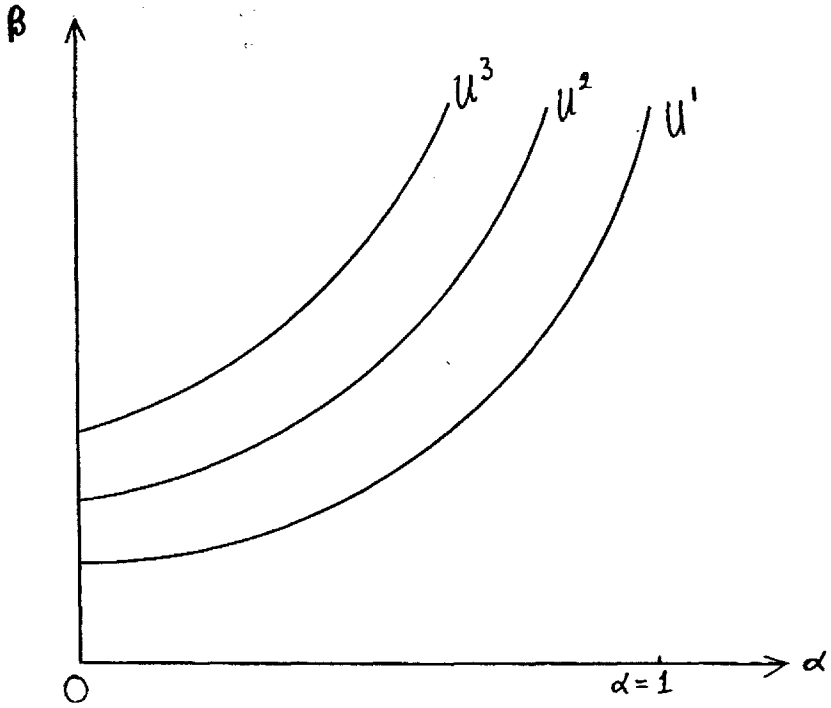
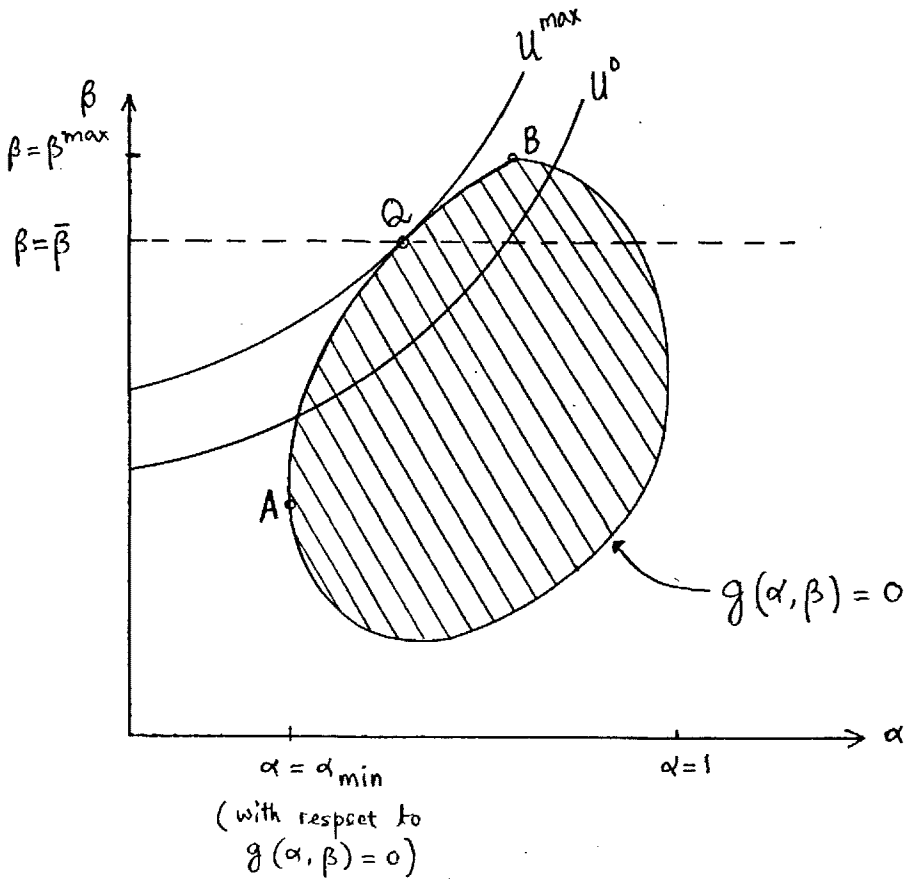


Figure A.II.2: Maximisation of Expected Utility



feasible set, however, would be different for different probability distributions. In Figure A.II.2, we draw the feasible set such that it is bounded above. Clearly, in the presence of such a constraint set, expected utility is maximum at Q which corresponds, when α is at a minimum to the level of β given by $\bar{\beta}$. Minimisation of α subject to a given β, therefore does not lead to a utility maximum unless β equals $\bar{\beta}$.

A.II.2 Outline Derivation of the Equations (II-15) and (II-16) The problem given by equation (II-14) in the text can be stated as the following Lagrangean:

$$\underset{\{x_1,x_2\}}{\text{Min}} \quad L = \frac{x_1^2\sigma_{11} + 2x_1x_2\sigma_{12} + x_2^2\sigma_{22}}{(x_1\mu_1 + x_2\mu_2 - d)^2} + \lambda(M - x_1 - x_2).$$

The first-order conditions for a minimum are:

$$\frac{\partial L}{\partial x_1} = \frac{1}{\Delta}\{2(\mu-d)^2(x_1\sigma_{11} + x_2\sigma_{12}) - 2(\sigma^2)(\mu-d)\mu_1\} - \lambda = 0; \qquad \text{... (A-II-2)}$$

$$\frac{\partial L}{\partial x_2} = \frac{1}{\Delta}\{2(\mu-d)^2(x_1\sigma_{12} + x_2\sigma_{22}) - 2(\sigma^2)(\mu-d)\mu_2\} - \lambda = 0; \qquad \text{... (A-II-3)}$$

and,

$$\frac{\partial L}{\partial \lambda} = M - x_1 - x_2 = 0; \qquad \text{... (A-II-4)}$$

where,

$$\mu \equiv \mu_1x_1 + \mu_2x_2,$$

$$\sigma^2 \equiv x_1^2\sigma_{11} + 2x_1x_2\sigma_{12} + x_2^2\sigma_{22},$$

and

$$\Delta \equiv (\mu-d)^4.$$

From (A-II-2) and (A-II-3), we have

$$(\mu-d)\ \{\ (x_1\sigma_{11} + x_2\sigma_{12}) - (x_1\sigma_{12} + x_2\sigma_{22})\} = (\mu_1 - \mu_2)\ \sigma^2. \qquad \ldots(\text{A-II-5})$$

Equation (A-II-5) together with (A-II-4) yields

$$x_1 = \frac{M}{D}\ \{\ \sigma_{22}(\mu_1 - d/M) - \sigma_{12}(\mu_2 - d/M)\} \qquad \ldots(\text{A-II-6})$$

and

$$x_2 = \frac{M}{D}\ \{\ \sigma_{11}(\mu_2 - d/M) - \sigma_{12}(\mu_1 - d/M)\}\ , \qquad \ldots(\text{A-II-7})$$

where $D \equiv (\sigma_{11} - \sigma_{12})(\mu_2 - d/M) - (\sigma_{12} - \sigma_{22})(\mu_1 - d/M)$. Equation (A-II-6) and (A-II-7) are numbered (II-15) and (II-16) respectively in the text.

A.II.3 <u>Effects of an Income Tax with Full Loss Offset: An Algebraic Proof</u>

Richter has suggested the use of the following utility function as a description of the behaviour of the Domar-Musgrave investor:

$$EU(X) = a + b \int_{-\infty}^{0} Xf(X)dX + c\int_{0}^{\infty} Xf(X)dX. \qquad \ldots(\text{A-II-8})$$

But

$$\int_0^\infty Xf(X)dX = g = 1/(\sigma\sqrt{2\pi}) \int_0^\infty X \exp\ [-\tfrac{1}{2}(\tfrac{X-\mu}{\sigma})^2]dX$$

assuming $X \sim N(\mu,\sigma^2)$.

Or, $$g = \sigma/\sqrt{2\pi}\int_{-\mu/\sigma}^{\infty} t \exp\ (\ -t^2/2)dt + \mu/\sqrt{2\pi}\int_{-\mu/\sigma}^{\infty}\exp(-t^2/2)dt$$

$$= \sigma/\sqrt{2\pi}[-\exp(-t^2/2)]_{-\mu/\sigma}^{\infty} + \mu(\tfrac{1}{2} + F(\mu/\sigma))$$

$$= \sigma/\sqrt{2\pi}\ \exp(-\tfrac{1}{2}(\mu^2/\sigma^2)) + \mu(\ \tfrac{1}{2} + F(\mu/\sigma)). \qquad \ldots(\text{A-II-9})$$

Definitionally,

$$r = -\int_{-\infty}^{0} Xf(X)dX = g - \int_{-\infty}^{\infty} Xf(X)dX = g - \mu.$$

Therefore,

$$r = \sigma/\sqrt{2\pi}\ \exp\ (-\tfrac{1}{2}(\mu^2/\sigma^2)) + \mu(\tfrac{1}{2} + F(\mu/\sigma)) - \mu,$$

or,

$$r = \sigma/\sqrt{2\pi}\ \exp(-\tfrac{1}{2}(\mu^2/\sigma^2)) + \mu(F(\mu/\sigma) - \tfrac{1}{2}). \qquad \text{...(A-II-10)}$$

Also, we can rewrite (A-II-8) as

$$EU(X) = a + c\mu - (c-b) \int_{-\infty}^{0} Xf(X)dX. \qquad \text{...(A-II-11)}$$

From (A-II-10) and (A-II-11) we obtain

$$EU(X) = a + c\mu + (c-b)[(\sigma/\sqrt{2\pi})\exp(-\tfrac{1}{2}(\mu/\sigma)^2) + \mu(F(\mu/\sigma)-\tfrac{1}{2})],$$

$$= a + (c-b)(\sigma/\sqrt{2\pi})\exp(-\tfrac{1}{2}[\mu^2/\sigma^2]) + \mu[(c-b)F(\mu/\sigma)+(b+c)/2].$$

Writing $(c-b) = p$, $(b+c)/2 = q$ and ignoring a, we have

$$EU(X) = p\sigma/\sqrt{2\pi}\exp(-\tfrac{1}{2}\mu^2/\sigma^2) + \mu(pF(\mu/\sigma) + q). \qquad \text{...(A-II-12)}$$

Taking the special case of 2-assets only (one safe-one risky), we can write the investor's problem as (in the presence of a proportional tax t):

$$\text{Max } L = p[(1-t)x_2(\sqrt{\sigma_{22}}/\sqrt{2\pi})\exp(-\tfrac{1}{2}(r_1x_1 + \mu_2x_2)^2/(x_2\sqrt{\sigma_{22}})^2)]$$

$$+ (1-t)(r_1x_1 + \mu_2x_2)\ \{pF[(r_1x_1 + \mu_2x_2)/x_2\sqrt{\sigma_{22}}] + q\}$$

$$- \lambda\ (x_1 + x_2 - M), \qquad \text{...(A-II-13)}$$

where r_1 is the return on the safe asset, μ_2 is that on the risky asset, x_i are the amounts invested in them, M is total wealth. Differentiating and setting $\partial L/\partial x_1 = 0$, we obtain

$$q/p + F(z) = \lambda/p(1-t)r_1, \qquad \text{...(A-II-14)}$$

where $z = (r_1x_1 + \mu_2x_2)/x_2\sqrt{\sigma_{22}}$. Similarly for $\partial L/\partial x_2 = 0$, we have

$$(\sqrt{\sigma_{22}}/r_1)(1/\sqrt{2\pi})\exp(-\tfrac{1}{2}z^2) + \mu_2/r_1F(z) + (q/p)(\mu_2/r_1)$$

$$= \lambda/p(1-t)r_1. \qquad \text{...(A-II-15)}$$

Combining (A-II-14) and (A-II-15), we have

$$(q/p)(r_1 - \mu_2) = (\sqrt{\sigma_{22}}/\sqrt{2\pi})\exp(-\tfrac{1}{2}z^2) - (r_1 - \mu_2)F(z) \text{..(A-II-16)}$$

Differentiating (A-II-16) w.r.t. 't'

$$-(\sqrt{\sigma_{22}}/\sqrt{2\pi})\exp(-\tfrac{1}{2}z^2)(z)[(r_1/x_2\sqrt{\sigma_{22}})dx_1/dt - (r_1x_1/x_2^2\sqrt{\sigma_{22}})dx_2/dt]$$

$$- (r_1 - \mu_2)F'(z)[(r_1/x_2\sqrt{\sigma_{22}})dx_1/dt - (r_1x_1/x_2^2\sqrt{\sigma_{22}})\ dx_2/dt] = 0.$$

Using $dx_2/dt = -dx_1/dt$, in view of the budget constraint, and simplifying we can rewrite the above as

$$dx_1/dt\ \{-(r_1/x_2)(z)(1/\sqrt{2\pi})\exp(-\tfrac{1}{2}z^2) - ((r_1x_1)/x_2^2)(z)$$

$$(1/\sqrt{2\pi})\exp(-\tfrac{1}{2}z^2) - (r_1 - \mu_2)(r_1/x_2\sqrt{\sigma_{22}})F'(z) - (r_1 - \mu_2)$$

$$((r_1x_1)/x_2^2\sqrt{\sigma_{22}})F'(z)\} = 0,$$

or,

$$dx_1/dt\{-(r_1M/x_2)[(z/x_2)(1/\sqrt{2\pi})\exp(-\tfrac{1}{2}z^2) + ((r_1 - \mu_2)/x_2\sqrt{\sigma_{22}})$$

$$F'(z)]\} = 0. \qquad \text{...(A-II-17)}$$

For (A-II-17) to hold, either $dx_1/dt = 0$, or $(r_1M/x_2) = 0$, or the terms inside the square brackets add up to zero. First, we notice that $(r_1M/x_2) \neq 0$, unless $x_2 = \infty$. Secondly, for the terms in square brackets to sum to zero, we require, in view of the relation $F'(z) = (1/\sqrt{2\pi})$ $\exp(-\tfrac{1}{2}z^2)$,

$$(r_1x_1 + \mu_2x_2)/(x_2^2\sqrt{\sigma_{22}}) = (\mu_2 - r_1)/(x_2\sqrt{\sigma_{22}}),$$

or,

$$(r_1)(x_1/x_2) + \mu_2 = \mu_2 - r_1,$$

or, $\quad x_1 + x_2 = 0$, which is nonsence.

Hence we conclude that $dx_1/dt = 0$.

CHAPTER III

TAXATION AND RISK-TAKING IN A CHANCE-CONSTRAINED PROGRAMMING MODEL OF PORTFOLIO SELECTION

In the last chapter, we indicated the difficulty, so well described by Arrow [4],[1] of modelling the ignorance of an economic agent. We were particularly concerned with the behaviour of a decision maker whose primary concern was with safety. We will pursue the same theme here in the context of an alternative framework. Specifically we will show, for an investor who is concerned with safety, that a chance-constrained programming formulation of the portfolio problem avoids the non-diversification difficulty faced by Roy's approach to the problem. This interpretation of a chance-constrained programme as a "modified" safety-first rule would also provide a refutation of the criticisms made by Borch [8] and Allingham [2] regarding the relevance of this technique for economic analysis.

Naslund, in his application of chance-constrained programming, made the assumption of normally distributed asset returns (X). This led Allingham to argue that "... while chance-constrained programming imposes normality on X it gains no generality over the mean-variance approach, and also appears to provide no stronger results" ([2], p. 215). Furthermore, the assumption of normality implies a non-zero probability that asset returns will be negative without bound and, therefore, is unsatisfactory in a typical portfolio context since individuals cannot lose

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[1]This has been quoted at the beginning of this dissertation.

more than they have invested (i.e., limited liability). More pessimistically, Sengupta conjectured that "decision rules under [chance-constrained programming] may be sensitive to departures from normality, unless some prior analysis shows the contrary" ([65], p. 288). As a partial reply to these reservations, we formulate a chance-constrained portfolio choice model for a lognormal securities market. Following Lintner ([40], p. IV-1), we adopt the following operational definition of a lognormal securities market: "Investors' decisions on the composition of their investment portfolios can be adequately described as selections made on the basis of the logarithmic means and logarithmic variances of the distributions of alternative portfolio rates of return."[2] Also note that this definition of a lognormal securities market does not require the individual asset returns to be always positive; only their sum has to be positive. However, this is not to ignore the intuitive difficulties associated with the assumption of a lognormal distribution (e.g., the zero probability of a depression in the stock market), or, for that matter, any specific assumption regarding the probability distribution of outcomes. The merit of our exercise, perhaps, is in examining how sensitive the earlier results are with respect to alternative assumptions regarding probability distributions.

In this context we demonstrate the diversification of asset holdings and the separation property of portfolio models as well as the result that a proportional tax on investment income stimulates risk-taking.

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[2]Of course, it is true that the weighted sums of lognormal distributions are not strictly lognormal. But Lintner's simulations of portfolio returns [40] indicate that many investors may quite reasonably regard the lognormal approximation to portfolio returns as very acceptable for practical purposes.

These results have originally been shown by Tobin [72] in the context of the mean-variance models[3] and Naslund in the context of a normally distributed chance-constrained problem.

III.1 Safety-First and Chance-Constraints: An Integration

A "chance-constrained programme"(CCP), in its standard form, is written as

$$\begin{aligned} &\max f(cX) \\ &\text{s.t. } P(AX \leq d) \geq \alpha,\ 0 \leq \alpha_i \leq 1, \quad \} \qquad \ldots\text{(III-1)} \\ &\qquad H(A,X,d,c) = 0 \end{aligned}$$

where f is a concave function and A,d, c are, in general, random variables. The implicit function H defines the set of decision rules that are considered in this problem (i.e., we need not maximise over all possible X's). The vector α specifies the probability limits to which constraint violations are admitted. Charnes, Cooper and Thompson [11] have shown that we require $\alpha_i \geq \frac{1}{2}$ for a meaningful problem. For $\alpha_i = 1$ and for only one chance-constraint, the above programme, using the expected value ("E-type") formulation, reduces to

$$\left.\begin{aligned} &\max_{\{x_i\}} E(\sum_i r_i x_i), \\ &\text{s.t. } -\sum_i r_i x_i \leq d, \\ &\qquad \sum_i x_i \leq M, \\ &\qquad x_i \geq 0, \end{aligned}\right\} \qquad \ldots\text{(III-2)}$$

where, as in the previous chapter, r_i denotes the random return on the i-th investment, x_i is the amount of i-th investment, d is the known level

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[3]The "mean-variance" approach to portfolio selection has been discussed in Chapter I.

of "disaster" and M is initial wealth. This may be interpreted as a reasonable description of a safety-first rule and is a standard "linear programming under uncertainty" problem treated by Dantzig ([13], [14]) and others. The difficulties of obtaining a solution are discussed by Madansky [42], and even our assumption $\alpha = 1$ (dropping the subscript as we only have one constraint), i.e., maximum safety, does not greatly help us. If we drop our assumption that $\alpha = 1$ but let α be close to unity, it may be argued that we still retain our concern for safety.

Naslund utilises these results to formulate

$$\left.\begin{array}{l} \max\limits_{\{x_i\}} E(X), \\ \text{s.t. } P(X \geq d) \geq \alpha, \\ \qquad \Sigma_i x_i \leq M, \\ \qquad\quad x_i \geq 0. \end{array}\right\} \quad i=1,\ldots,n. \qquad \ldots(\text{III-3})$$

Assuming $X \sim N(\mu,\sigma^2)$, the chance-constraint reduces to $\frac{1}{2} + F(\frac{\mu-d}{\sigma}) \geq \alpha$, or

$$(\frac{\mu-d}{\sigma}) \geq C, \qquad \ldots(\text{III-4})$$

where $F(\frac{\mu-d}{\sigma}) = \frac{1}{\sqrt{2\pi}} \int_\theta^0 e^{-(\frac{1}{2})t^2} dt$, $t = (\frac{X-\mu}{\sigma})$, $\theta = (\frac{d-\mu}{\sigma})$ and the constant $C = F^{-1}(\alpha-\frac{1}{2})$. As shown in the last chapter, the objective function of the above problem for the one safe-one risky asset case is given by (equation (II-19) of chapter II):

$$\mu \equiv E(X) = r_1 M + (\mu_2 - r_1)(\frac{\sigma}{\sqrt{\sigma_{22}}}), \qquad \ldots(\text{III-5})$$

where the notation is the same as that of chapter II.

We can therefore re-state (III-3) as:

$$\left.\begin{array}{l}\max \mu = r_1 M + (\mu_2 - r_1) \dfrac{\sigma}{\sqrt{\sigma_{22}}}, \\ \text{s.t. } \mu \geq d + \sigma C.\end{array}\right\} \qquad \ldots(\text{III-6})$$

This is represented in Figure III.1. Notice that, in Roy's model, the investor minimizes $P(X \leq d)$ by making the tangent of positive slope drawn from $D(0,d)$ as steep as possible (Figure II.1 of Chapter II). Here the investor is constrained by (III-4) thereby allowing for an interior solution, say, at point E.[4] This interpretation of the CCP, i.e., as a modified safety-first principle, it is hoped, will help establish such programming approaches as "reasonable" hypotheses to follow in an uncertain state of the world.

III.2 Chance-Constrained Programming in a Lognormal Securities Market

III.2.1 Deterministic Equivalent Under the Lognormal Distribution

Before we can analytically solve a chance-constrained programme, as has been shown in the solution to the problem (III-3), we have to transform the chance-constraint into its deterministic equivalent by assuming known distributional properties. In this sub-section we obtain the deterministic equivalent which would be appropriate for a lognormal securities market.

Consider a random variable $X(0 < X \leq \infty)$ such that $Y = \log X$ is normally distributed with mean, m and variance, v^2. The probability density function of X is given by

$$f(X) = \left(\frac{1}{vX\sqrt{2\pi}}\right) \exp\left\{ - \frac{(\log X - m)^2}{2v^2} \right\} . \qquad \ldots(\text{III-7})$$

Restricting our attention to the class of chance constraints of the form

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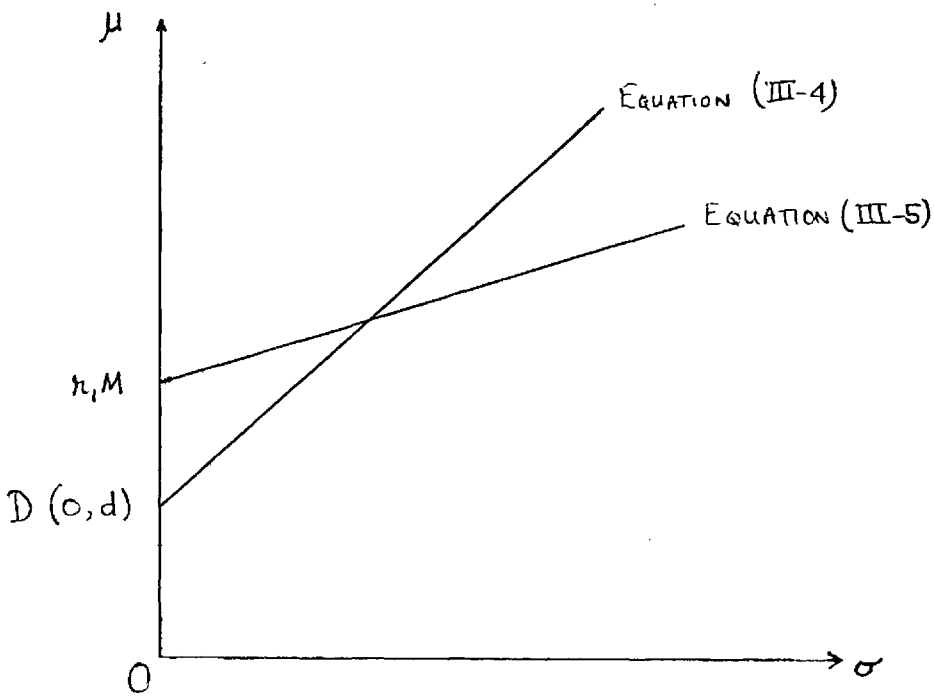
[4]Clearly we require $F^{-1}(\alpha-\frac{1}{2}) > \left(\frac{\mu_2 - r_1}{\sqrt{\sigma_{22}}}\right)$ for an interior solution.

Figure III.1: Safety-First: An Interior Solution



$$P(X \geq d) \geq \alpha, \qquad \ldots(III\text{-}8)$$

and using (III-7), the constraint can be written as:

$$\int_d^\infty \left(\frac{1}{vX\sqrt{2\pi}}\right) \exp\left\{-\frac{(\log X - m)^2}{2v^2}\right\} dX \geq \alpha. \qquad \ldots(III\text{-}9)$$

Setting $\left(\frac{\log X - m}{v}\right) = t$, we can use the symmetry properties of $t \sim N(0,1)$ to simplify (III-9) to

$$\int_{-\tau}^{o} \left(\frac{1}{\sqrt{2\pi}}\right) \exp\left(-\tfrac{1}{2}t^2\right)dt + \int_o^\infty \left(\frac{1}{\sqrt{2\pi}}\right) \exp\left(-\tfrac{1}{2}t^2\right)dt \geq \alpha, \qquad \ldots(III\text{-}10)$$

where $\tau = \left(\frac{m - \log d}{v}\right)$ is clearly positive as we require $(m > \log d)$ for the chance-constraint to be meaningful. In view of $\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\right)\exp\left(-\tfrac{1}{2}t^2\right)dt = 1$ and $-F(Z) = F(-Z)$, we can further simplify (III-10) to

$$F\left(\frac{m - \log d}{v}\right) + \tfrac{1}{2} \geq \alpha, \qquad \ldots(III\text{-}11)$$

where $F(\tau) = \int_{-\tau}^{o} \left(\frac{1}{\sqrt{2\pi}}\right) \exp\left(-\tfrac{1}{2}t^2\right)dt$. Taking the inverse, (III-11) yields

$$m - vF^{-1}(\alpha^*) \geq \log d, \qquad \ldots(III\text{-}12)$$

where we write $F^{-1}(\alpha^*)$ for $F^{-1}(\alpha - \tfrac{1}{2})$. Inequality (III-12) is our desired deterministic equivalent of the chance constraint.

Furthermore, in order to use (III-12) to obtain an explicit solution of a chance-constrained programme, we have to derive the expressions for m and v in terms of the moments of X: the one-to-one mapping between the pair (m,v^2) and the corresponding moments (μ,σ^2) of X is given by

$$\mu = \exp\left(m + \tfrac{1}{2}v^2\right), \qquad \ldots(III\text{-}13)$$

$$\sigma^2 = \mu^2 \left(\exp v^2 - 1\right). \qquad \ldots(III\text{-}14)$$

Given (III-13), (III-14) and simplifying, the parameters (m,v^2) of log X can be expressed as

$$m = 2 \log \mu - \tfrac{1}{2} \log s^2, \qquad \text{...(III-15)}$$

$$v^2 = \log s^2 - 2 \log \mu, \qquad \text{...(III-16)}$$

where μ and s^2 are the first two moments (about the origin) of the random variable X.

III.2.2 The Model With One Safe and Many Risky Assets

The problem is

$$\left.\begin{array}{l} \text{Max } E(\sum_{i=1}^{n} r_i x_i), \\ \text{s.t. } P(\sum_{i=1}^{n} r_i x_i \geq d) \geq \alpha, \\ \qquad \sum_i x_i \leq M, \\ \qquad\quad x_i \geq 0, \text{ all } i, \end{array}\right\} \qquad \text{...(III-17)}$$

where we shall take the first asset to be safe, and let r_i denote the random returns (i=2, ..., n), x_i the amount of wealth held in the form of i-th asset. Using (III-12), the deterministic equivalent of the above programme is:

$$\left.\begin{array}{l} \text{Max } r_1 x_1 + \Sigma_{i=2}^{n} \mu_i x_i, \\ \text{s.t.} \left\{\begin{array}{l} m - vF^{-1}(\alpha^*) \geq \log d, \\ \Sigma_{i=1}^{n} x_i \leq M, \\ x_i \geq o. \end{array}\right. \end{array}\right\} \qquad \text{...(III-18)}$$

Writing the constraints explicitly, we can set up the following Lagrangean[5]

$$L = r_1x_1 + \sum_{i=2}^{n} \mu_i x_i + \lambda_1 \Big[2\{\log (r_1x_1 + \sum_{i=2}^{n} \mu_i x_i)\}$$

$$- \tfrac{1}{2} \{ \log (\sum_{i=2}^{n} x_i^2 s_i^2 + \sum_{\substack{i,j=2 \\ i\neq j}}^{n} x_i x_j s_{ij})\}$$

$$- F^{-1}(\alpha^*) \{ \log (\sum_{i=2}^{n} x_i^2 s_i^2 + \sum_{\substack{i,j=2 \\ i\neq j}}^{n} x_i x_j s_{ij}) - 2 \{ \log(r_1x_1$$

$$+ \sum_{i=2}^{n} \mu_i x_i)\}\}^{\frac{1}{2}} - \log d \Big] - \lambda_2 (\sum_{i=1}^{n} x_i - M), \quad \ldots(\text{III-19})$$

where λ_i are the appropriate dual variables. Using the Kuhn-Tucker conditions, we obtain (in addition to the constraints):

$$\frac{\partial L}{\partial x_1} = r_1 + \lambda_1 \Big[\Big(\frac{2r_1}{r_1x_1+\Sigma_{i=2}^{n}\mu_i x_i}\Big) + F^{-1}(\alpha^*) \Big\{\frac{r_1}{r_1x_1+\Sigma_{i=2}^{n}\mu_i x_i}\Big\}$$

$$\{(\log (\Sigma_{i=2}^{n} x_i^2 s_i^2 + \Sigma_{\substack{i,j=2 \\ i\neq j}}^{n} x_i x_j s_{ij}) - 2\{\log (r_1x_1 +$$

$$\Sigma_{i=2}^{n}\mu_i x_i)\}\}^{-\frac{1}{2}}\Big] - \lambda_2 \leq 0 \qquad \ldots(\text{III-20})$$

and $x_1 \geq 0$;

and

$$\frac{\partial L}{\partial x_h} = \mu_h + \lambda_1 \Big[\Big(\frac{2\mu_h}{r_1x_1+\Sigma_{i=2}^{n}\mu_i x_i}\Big) - \frac{x_h s_h^2 + \Sigma_{\substack{j=2 \\ j\neq h}}^{n} x_j s_{hj}}{\Sigma_{i=2}^{n} x_i^2 s_i^2 + \Sigma_{\substack{i,j=2 \\ i\neq j}}^{n} x_i x_j s_{ij}}$$

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[5]Notice that the terms s_i^2 and s_{ij} are the components of the second-moment of the asset returns, i.e., $s_i^2 \equiv (\sigma_i^2 + \mu_i^2)$ and $s_{ij} \equiv (\sigma_{ij} + \mu_i\mu_j)$ where $[\sigma_{ij}]$ is the true variance-covariance matrix and μ_i, μ_j are the expected asset returns. Often, however, we shall refer to $[s_{ij}]$ as the "variance-covariance" matrix for brevity.

$$-F^{-1}(\alpha^*)\{(\log\ (\Sigma_{i=2}^{n}\ x_i^2 s_i^2 + \Sigma_{\substack{i,j=2\\ i\neq j}}^{n}\ x_i x_j s_{ij})$$

$$-2\ [\log(r_1 x_1 + \Sigma_{i=2}^{n}\ \mu_i x_i)])^{-\frac{1}{2}}\}$$

$$\{(\frac{x_h s_h^2 + \Sigma_{\substack{j=2\\ j\neq j}}^{n}\ x_j s_{hj}}{\Sigma_{i=2}^{n} x_i^2 s_i^2 + \Sigma_{\substack{i,j=2\\ i\neq j}}^{n}\ x_i x_j s_{ij}}) - (\frac{\mu_h}{r_1 x_1 + \Sigma_{i=2}^{n}\mu_i x_i})\}] - \lambda_2 \leq 0$$

and $x_h \geq 0$ and $h = 2, \ldots n.$ ...(III-21)

Considering only the assets that are held, the above relations hold with strict equalities. Simplifying the notation, they can be rewritten as follows:

$$r_1 + \lambda_1 \left[\frac{2r_1}{\mu} + F^{-1}(\alpha^*)\ \frac{r_1}{\mu}\ \{\ (\log\ s^2 - 2\ \log\ \mu)^{-\frac{1}{2}}\}\right] = \lambda_2; \quad \ldots(\text{III-22})$$

$$\mu_h + \lambda_1 \left[\frac{2\mu_h}{\mu} - \frac{x_h s_h^2 + \Sigma_{\substack{j=2\\ j\neq h}}^{n}\ x_j s_{hj}}{s^2} - F^{-1}(\alpha^*)\{(\log\ s^2 - 2\log\mu)^{-\frac{1}{2}}\}\right.$$

$$\left.\left\{\frac{x_h s_h^2 + \Sigma_{\substack{j=2\\ j\neq h}}^{n}\ x_j s_{hj}}{s^2} - \frac{\mu_2}{\mu}\right\}\right] = \lambda_2; \quad \ldots(\text{III-23})$$

where

$$\mu \equiv r_1 x_1 + \Sigma_{i=2}^{n}\mu_i x_i, \quad \ldots(\text{III-24})$$

$$s^2 \equiv \Sigma_{i=2}^{n} x_i^2 s_i^2 + \Sigma_{\substack{i,j=2\\ i\neq j}}^{n} x_i x_j s_{ij}, \quad \ldots(\text{III-25})$$

and $h = 2, \ldots, n.$

Portfolio Separation. From the equation (III-22) we obtain

$$r_1 \{ 1 + \frac{2\lambda_1}{\mu} + \lambda_1 (\frac{F^{-1}(\alpha^*)}{\mu})[(\log s^2 - 2 \log \mu)^{-\frac{1}{2}}]\} = \lambda_2. \qquad \ldots(\text{III-26})$$

Substituting (III-26) into (III-23) and letting h=k, we obtain

$$\mu_k - r_1 = \frac{\lambda\left(\frac{x_k s_k^2 + \Sigma_{\substack{j=2 \\ j\neq k}}^n x_j s_{kj}}{s^2}\right)\{1 + F^{-1}(\alpha^*)[\log s^2 - 2 \log \mu)^{-\frac{1}{2}}]\}}{\{1 + \lambda_1(\frac{2}{\mu} + \frac{F^{-1}(\alpha^*)}{\mu}[(\log s^2 - 2 \log \mu)^{-\frac{1}{2}}])\}}. \qquad \ldots(\text{III-27})$$

Similarly, from (III-26) and (III-23) and letting h=ℓ, we also obtain a relation similar to (III-27).

Dividing (III-27) by the latter expression (h=ℓ), we obtain

$$\frac{\mu_k - r_1}{\mu_\ell - r_1} = \frac{s_k^2 + \Sigma_{\substack{j=2 \\ j\neq k}}^n \frac{x_j}{x_k} s_{kj}}{\frac{x_\ell}{x_k} s_\ell^2 + \Sigma_{\substack{j=2 \\ j\neq \ell}}^n \frac{x_j}{x_k} s_{\ell j}}. \qquad \ldots(\text{III-28})$$

Equation (III-28) indicates that, from the Kuhn-Tucker conditions, we would obtain at most (n-2) equations of this form, in terms of ratios (x_j/x_k), j = 2, ..., n and j≠k (i.e., (n-2) ratios). These ratios, if they exist, are determined by the structure of expected asset returns, μ_i, and the elements of the "variance-covariance" matrix $[s_{ij}]$, but are independent of $F^{-1}(\alpha)$ and the investor's disposable wealth. Thus, we have shown the following result: In a portfolio of one safe and many risky assets, if the investor holds the safe asset, then he holds the risky assets in certain fixed proportions, which are independent of the

size of his disposable wealth and $F^{-1}(\alpha^*)$ [his attitude towards risk]. This is the well-known separation property of portfolio models as discussed by Cass-Stiglitz [10] and Tobin [72], among others. This result says that portfolio allocation decision can be viewed as a two-stage process: first the investor decides on what relative proportions to invest in the different risky assets, and then decides on what proportion of his total portfolio to invest in the safe asset and what proportion in the "composite" risky asset [mutual fund]. The significance of this result has been discussed by Stiglitz [69].

A Proportional Tax on Investment Incomes. From the separation theorem, it follows that for analytical purposes we can aggregate all the risky assets into one and, in effect, consider the two-asset case consisting of the safe asset and this composite risky asset. This allows us to re-state the investor's problem, in the presence of a proportional tax on portfolio income, as

$$\max_x (1-t)\ [rx + (M-x)\mu],$$

$$\text{s.t. } 2\{\log (1-t) + \log [rx + (M-x)\mu] - \tfrac{1}{2}\{(1-t)^2 + \log (M-x)^2 s^2\}$$

$$-F^{-1}(\alpha^*)[\log (1-t)^2 + \log (M-x)^2 s^2 - 2\{\log(1-t) +$$

$$\log [rx + (M-x)\mu]\}]^{\frac{1}{2}} \geq \log d,$$

where $0 < t < 1$ is the proportional tax rate, r is the return on the safe asset and μ is the expected return per dollar of risky investment, and x is the amount (units) of the safe asset held while M is total (initial) wealth. For notational convenience, let us define

$$p \equiv [rx + (M-x)\mu],$$

$$z(p,s,t) \equiv [\log(1-t)^2 + \log\ (M-x)^2 s^2 - 2\{\log(1-t) + \log[rx + (M-x)\mu]\}].$$

This allows us to set up the following Lagrangean:

$$L = (1-t)p + \lambda[2\{\log(1-t) + \log p\} - \tfrac{1}{2}\{\log(1-t)^2 + \log(M-x)^2 s^2\}$$

$$-F^{-1}(\alpha^*)\{z(p,s,t)\}^{\frac{1}{2}} - \log d]. \qquad \ldots(\text{III-29})$$

Maximisation of (III-29) yields the following necessary condition:

$$dL/dx = (1-t)(r-\mu) + \lambda[\{(r-\mu)/p\}\{2 + F^{-1}(\alpha^*)[z(p,s,t)^{-\frac{1}{2}}]\}$$

$$+ \{1/(M-x)\}\{1 + F^{-1}(\alpha^*)[z(p,s,t)^{-\frac{1}{2}}]\}] = 0$$

$$\text{and } x > 0. \qquad \ldots(\text{III-30})$$

Now, differentiating the constraint in (III-29) w.r.t. 't', we obtain

$$dx/dt\{\{(r-\mu)/p\}\{2 + F^{-1}\ (\alpha^*)[z(p,s,t)^{-\frac{1}{2}}]\} + \{1/(M-x)\}$$

$$\{1 + F^{-1}(\alpha^*)\ [z(p,s,t)^{-\frac{1}{2}}]\}\} = 1/(1-t). \qquad \ldots(\text{III-31})$$

Combining (III-30) and (III-31), we have

$$dx/dt = -\ [\frac{\lambda}{(1-t)^2(r-\mu)}]. \qquad \ldots(\text{III-32})$$

Noticing the form of (III-29), it follows that $\lambda < 0$. From (III-30), it can be seen that for an interior solution, we require $(r-\mu) < 0$. Therefore, dx/dt is clearly negative (i.e., increased risk-taking). Hence, we have the following: <u>In portfolios with one safe and one or many risky assets, an increase in the proportional tax rate on investment returns (under full loss offset) increases the demand for risky assets.</u> It should be pointed out that this result is in support of the "classical" result on taxation and risk-taking, which was originally suggested by

Domar and Musgrave[6] and has later been confirmed, in their own contexts, by Tobin [72] and Naslund [53]. The intuitive reasoning behind this result is as follows: By allowing loss-offset privileges, the government now shares part of the risk.

III.2.3 The Model When All Assets Are Risky

For the case of all risky assets, we can proceed in the manner of Section III.2.2. However, equation (III-23) would now be describing (along with the constraints) all the relevant first-order conditions where we let h=1,...,n and, thus, terms of the form $\Sigma_{\substack{j=2\\ j\neq h}}^{n} x_j s_{hj}$ now read $\Sigma_{\substack{j=1\\ j\neq h}}^{n} x_j s_{hj}$.

It can be easily checked that this general model does not permit portfolio "aggregation" [separation] of the type exhibited by the model with one safe asset. Proceeding in the manner exactly analogous to the derivation of (III-28), we obtain

$$\frac{\mu_k-\mu_1}{\mu_\ell-\mu_1} = \frac{s_k^{\,2} + \Sigma_{\substack{j=1\\ j\neq k}}^{n} \frac{x_j}{x_k} s_{kj} - \frac{x_1}{x_k} s_1^{\,2} - \Sigma_{j=2}^{n} \frac{x_j}{x_k} s_{1j}}{\frac{x_\ell}{x_k} s_\ell^{\,2} + \Sigma_{\substack{j=1\\ j\neq \ell}}^{n} \frac{x_j}{x_k} s_{\ell j} - \frac{x_1}{x_k} s_1^{\,2} - \Sigma_{j=2}^{n} \frac{x_j}{x_k} s_{1j}} \quad \text{...(III-33)}$$

As before, these equations are in terms of the ratios x_j/x_k,(j=1,...n; j≠k). However, we have only (n-2) equations in (n-1) ratios. Hence the optimum portfolio allocation cannot be independent of disposable wealth. This type of asymmetry, i.e., aggregation in the presence of one safe asset and its absence in the presence of all risky assets has been

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[6]However, in Chapter II, we have shown that a rigorous interpretation of their model cannot generate such results.

emphasised by Cass and Stiglitz [10].

The Effects of Taxation. To see the effects of taxation in this general case of all risky assets, we consider the special case of two risky assets in the presence of a proportional tax, $0 \leq t \leq 1$, and a lump-sum tax K on investment income. Setting K = 0 allows propositions about proportional taxes. The deterministic equivalent of the investor's problem, using the definition given by (III-25), is:

$$\left.\begin{aligned} &\max_{\{x_1,x_2\}} \quad \mu_1 x_1(1-t) + \mu_2 x_2(1-t) - K \\ &\text{s.t.} \quad x_1 + x_2 \leq M, \\ &\qquad 2\{\log[\mu_1 x_1(1-t) + \mu_2 x_2(1-t) - K]\} - \tfrac{1}{2}\{\log(1-t)^2 \\ &\qquad + \log s^2\} - F^{-1}(\alpha^*)[\log(1-t)^2 + \log s^2 - \\ &\qquad 2\{\log[\mu_1 x_1(1-t) + \mu_2 x_2(1-t) - K]\}]^{\frac{1}{2}} \geq \log d, \\ &\qquad x_1, x_2 \geq 0. \end{aligned}\right\} \quad \ldots\text{(III-34)}$$

As a solution to (III-34), we obtain the following necessary conditions for an optimum (in addition to the wealth and the deterministic constraints):

$$\mu_1(1-t) - \lambda_1 + \lambda_2 \left[\frac{2\mu_1(1-t)}{q} - \frac{x_1 s_1^{\,2} + x_2 s_{12}}{s^2} - F^{-1}(\alpha^*)\{(\log(1-t)^2 + \log s^2 - 2\log q)^{-\frac{1}{2}}\} \left\{\frac{x_1 s_1^{\,2} + x_2 s_{12}}{s^2} - \frac{\mu_1(1-t)}{q}\right\}\right] \leq 0$$

$$\text{and } x_1 \geq 0, \qquad \ldots\text{(III-35)}$$

$$\mu_2(1-t) - \lambda_1 + \lambda_2 \left[\frac{2\mu_2(1-t)}{q} - \frac{x_2 s_2^{\,2} + x_1 s_{12}}{s^2} - F^{-1}(\alpha^*)\{(\log(1-t)^2 + \log s^2 - 2\log q)^{-\frac{1}{2}}\} \left\{ \frac{x_2 s_2^{\,2} + x_1 s_{12}}{s^2} - \frac{\mu_2(1-t)}{q} \right\} \right] \leq 0$$

$$\text{and } x_2 \geq 0, \qquad \ldots\text{(III-36)}$$

where $q \equiv \{\mu_1 x_1(1-t) + \mu_2 x_2(1-t) - K\}$. Differentiating the deterministic constraint of (III-34) w.r.t. 't', we obtain:

$$\frac{dx_1}{dt} \left\{ \frac{2\mu_1(1-t)}{q} - \frac{x_1 s_1^{\,2} + x_2 s_{12}}{s^2} - F^{-1}(\alpha^*)\{ z(q,s,t)^{-\frac{1}{2}}\} \left\{ \frac{x_1 s_1^{\,2} + x_2 s_{12}}{s^2} - \frac{\mu_1(1-t)}{q} \right\} \right\} + \frac{dx_2}{dt} \left\{ \frac{2\mu_2(1-t)}{q} - \frac{x_2 s_2^{\,2} + x_1 s_{12}}{s^2} - F^{-1}(\alpha^*)\{[z(q,s,t)]^{-\frac{1}{2}}\} \left\{ \frac{x_2 s_2^{\,2} + x_1 s_{12}}{s^2} - \frac{\mu_2(1-t)}{q} \right\} \right\} = \frac{2(q+K)}{q(1-t)} - \frac{1}{(1-t)} + F^{-1}(\alpha^*)\{[z(q,s,t)]^{-\frac{1}{2}}\} \left[\frac{q+K}{(1-t)q} - \frac{1}{(1-t)} \right], \qquad \ldots\text{(III-37)}$$

where $z(q,s,t) \equiv \{\log(1-t)^2 + \log s^2 - 2\log q\}$.

However, at K=0, the r.h.s. of (III-37) reduces to $\frac{1}{1-t}$. Therefore, using (III-35) and (III-36), we obtain

$$\frac{dx_1}{dt} \left\{ \frac{\lambda_1 - \mu_1(1-t)}{\lambda_2} \right\} + \frac{dx_2}{dt} \left\{ \frac{\lambda_1 - \mu_2(1-t)}{\lambda_2} \right\} = \frac{1}{(1-t)} . \qquad \ldots\text{(III-38)}$$

Furthermore, in view of the wealth constraint, this can be rewritten as

$$\frac{dx_1}{dt} = \frac{\lambda_2}{(1-t)^2(\mu_2-\mu_1)} . \qquad \ldots\text{(III-39)}$$

First, we note that $\lambda_2 < 0$ due to the Kuhn-Tucker conditions. Therefore, the condition $\mu_1 > \mu_2$(and if we take this to imply $s_1^2 > s_2^2$) causes a movement towards the "high-risk" asset from the "low-risk" asset. We, therefore, obtain the following: <u>In portfolios with two risky assets, an increase in the proportional tax on portfolio returns causes a shift towards the high expected return (high-risk) asset from the asset with lower expected return (low-risk)</u>. This is a simple extension of the result proved in the preceding section. It is interesting to observe that the removal of the assumption of the existence of a safe asset does not basically alter the effect of a proportional tax.

To see the effects of the lump-sum tax in the presence of a proportional tax, we differentiate the deterministic form of the chance-constraint in (III-34) w.r.t. 'K' and obtain, in a manner analogous to the derivation of (III-39):

$$\frac{dx_1}{dK} = \frac{1}{q}\,[2 + F^{-1}(\alpha^*)\{[z(q,s,t)]^{-\frac{1}{2}}\}]\frac{\lambda_2}{(1-t)(\mu_2-\mu_1)}\,. \qquad \ldots(\text{III-40})$$

As $\lambda_2 < 0$, $\frac{dx_1}{dK} \gtrless 0$ as $\mu_2 \gtrless \mu_1$. The magnitude of the derivative is higher, the higher is the proportional rate t. Also note the appearance of the term $F^{-1}(\alpha^*)$, the investor's attitude to risk, in the determination of risk-bearing. We have therefore the following result: <u>In portfolios with two risky assets and no safe asset, an increase in the lump-sum tax causes a shift towards the "high-risk" asset from the "low-risk" asset.</u>

Notice that the direction of dx_1/dK, is the same even when we set t=0.

III.3 Conclusion

To summarise, the main results of this chapter are as follows. We have shown that a chance-constrained programming model of portfolio selection can be viewed as a "modified" safety-first rule (Section III.1). This modification is of economic significance in the sense that this formulation allows diversification in the presence of a safe asset which is absent in Roy's model. Secondly, in the context of a lognormal securities market, we have shown (Section III.2) that basically all the results of Tobin [72] and Naslund also hold in this model. Therefore, we observe that the departure from normality has not destroyed the basic qualitative results of the chance-constrained programming model of portfolio selection.

CHAPTER IV

PROGRESSION AND RISK-TAKING IN AN EXPECTED UTILITY OF WEALTH MODEL

The models examined in the last two chapters, except for that of Domar and Musgrave [16], have been somewhat outside the main stream of the literature on taxation and risk-taking. The more conventional literature has followed two conceptually distinct lines of development. The first has been the emergence of some notion of risk as an investment criterion in addition to the expected value of the outcome (e.g., returns from portfolio investments). The use of a measure of risk has been facilitated by the characteristics (moments) of the probability distribution of anticipated asset returns.[1] The second line of development has been the acceptance of the postulates of the expected utility hypothesis as the appropriate axioms of rational behaviour under uncertainty.[2] Markowitz first observed the relationship, although somewhat loosely, between a theory of portfolio selection based on the expected utility hypothesis and one based on the two-parameter (essentially mean-variance) approach ([45], pp. 209-210). Subsequently, Borch [9] and Feldstein [19] conclusively established that a theory that attempts to be consistent with both these approaches is a very restrictive one (i.e.,

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[1]This has been suggested by, among others, Makower and Marschak [43], Lange [36], Domar and Musgrave [16] and Hicks [31]. However, only Domar and Musgrave discussed how the imposition of a tax affects the risk as well as the yield of an investment.

[2]This has been examined by Marschak [46], Markowitz [45], Tobin [72] and Arrow [3].

necessitating the use of either a quadratic utility function or a normal distribution of random outcomes). In a similar spirit, we have shown in Chapter II that the third approach to the problem -- the safety-first principle -- if they are to be consistent with expected utility maximisation, again, imply a very restrictive set of preferences.

Apart from the safety-first interpretation, Domar and Musgrave's attempt to investigate the effects of taxation was also, in part, a contribution to the first line of development mentioned above. Many subsequent studies of the problem of taxation and risk-taking have been in the context of the mean-variance framework. These include Tobin [72], Richter [58], Penner [54], Bierwag and Grove [7] and Lepper [37]. The seminal results of these attempts coincide with the original findings of Domar-Musgrave: with full loss offsets, a proportional tax would increase total risk-taking; with no or partial loss offsets, the result is indeterminate. The validity of these results, as we have indicated in Chapters I and II, depends on the restrictive frameworks used. Although, in Chapter III, we have been able to unambiguously determine the effects of taxation on risk-taking for the chance-constrained programming models of portfolio selection, these models still lack sufficient generality as they require specific assumptions regarding the probability distribution of the asset returns. This is a doubtful procedure in the absence of convincing empirical support regarding the behaviour of various rates of return on investments.

Recently, however, Stiglitz [68] and Mossin [48] have attempted to clarify the situation somewhat by analysing the effects of proportional taxes on risk-taking using the more general framework of the expected

utility of wealth hypothesis suggested by Arrow [3]. Interestingly enough, these authors also conclude that a proportional (investment) income tax typically stimulates risk-taking where losses can be fully offset. The treatment of capital gains in these models however, is less straightforward.[3] Taxes without full loss offsets do not seem to yield much in terms of unambiguous results in these more general contexts of expected utility maximisation and hence will not be discussed in the remainder of this study.[4]

The authors previously discussed only considered proportional taxes. In this chapter, we examine the effects of progressive taxes on risk-taking and compare the results with those under a proportional tax system.[5] For analytical convenience, we will take the simplest form of

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[3]Stiglitz [68] has shown that exempting the income from the risky asset ["capital gains"] will lead to an increased risk-taking if relative risk-aversion (to be defined later in this chapter) is less than or equal to unity. Ahsan [1], however, pointed out that exempting capital gains over and above the riskless rate of return discourages risk-taking under much more reasonable restrictions [than Stiglitz'] on behaviour.

[4]We may not regard this as very restrictive. Under many tax laws, e.g., in the U.S. and in Canada, loss offsets are liberal. However, this does not apply to all forms of risk-bearing; and, for increasing marginal tax rates, loss offsets would have to be, to some extent, incomplete. Our results could, therefore, be interpreted as providing an upward bias in predicting the "true" effects of risk-taking.

[5]It is somewhat awkward to relate this attempt to the existing literature on progression and risk-taking. Professor Feldstein [20] provided counter-examples to the previous literature by showing that, in the model he considered, the effect of a progressive tax is ambiguous, and that a proportional tax has no effect on risk-taking. In Chapter V we point out the restrictive nature of Feldstein's model. On the other hand, the models with both lump-sum and proportional taxes (e.g., Richter [58], Naslund [53]) can be interpreted, when the lump-sum is negative, as equivalent to the linearly progressive tax considered here. However, our general formulation would allow us to derive the results that would be obtained if we have a quadratic utility function (i.e., Richter re-interpreted).

progressive tax -- a linear tax which has an exemption level, K, and a proportional tax rate, t, which applies both above and below K, i.e., it extends to a negative average tax rate on incomes below K. The precise definition of "progressiveness", as Atkinson-Stiglitz [5] point out, is open to dispute. Some authors apply the term to cases where the average tax rate rises (as in our case); others restrict the term to schedules where the marginal rate rises (which is not true of the tax considered here).[6]

With this brief survey of the literature serving as an introduction, the rest of the chapter proceeds as follows. In Section IV.1 we outline the features of the basic model and its behavioural implications. Section IV.2 discusses progressive income (investment) taxes, while progressive taxes with capital gains exemptions are discussed in Section IV.3. In Section IV.4 we analyse the comparative effects, (insofar as risk-taking is concerned), of proportional and progressive taxes. Section IV.5 summarises the main results of the chapter.

IV.1 The Portfolio Problem

The household is assumed to allocate its portfolio to maximise expected utility. For simplicity, we shall assume that there are only two assets, a safe asset with a secure rate of return and one risky asset with a random rate of return. The notation is as follows:

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[6] It turns out that a portfolio model with a quadratic tax function where the marginal tax rate rises is not analytically tractable. To indicate the nature of the difficulty involved, note that the standard result -- as will be shown in this chapter -- that decreasing absolute risk-aversion guarantees the non-inferiority of the risky asset no longer holds for such a tax schedule.

Wo = initial wealth;

a = amount invested in the risky asset, $0 \leq a \leq Wo$;

$m = (Wo - a)$ = amount held in the riskless asset;

X = random rate of return on the risky asset with probability distribution $F(x) = P\{X \leq x\}$ and $X \geq -1$;[7]

r = rate of return on riskless investment; $(r \geq 0)$;

W = final wealth;

U(W) = investor's utility function, and it is assumed that U(W) is continuous and is at least twice continuously differentiable with positive and diminishing marginal utilities (i.e., $U'(W) > 0$, $U''(W) < 0$); the restriction $U''(W) < 0$ implies risk-aversion for gambles about final wealth.

The individual, therefore, possessed of Wo, invests an amount 'a' in the risky asset and the remainder in the riskless asset, such as to maximise the expecte d utility of his wealth at the end of the period.[8]

Much of the predictive power of our analysis, as will be seen, derives from our ability to determine how the portfolio allocation between the safe and the risky asset changes as wealth changes. This would clearly depend on the individual's attitude towards risk. We

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[7]Notice that the assumption $X \geq -1$, (i.e., we cannot lose more than what we invested), although appropriate for portfolio models, may not be representative of all conceivable forms of risk-bearing (e.g., personally owned businesses). As a matter of fact, only the result on risk-taking where income from the risky asset is exempted from taxation depend on this assumption.

[8]Our measure of risk-taking, as has already been mentioned in Chapter I, is the individual's demand for the risky asset. In the present notation, this is simply the amount 'a' (out of initial wealth) devoted to the risky asset. This corresponds to the Domar-Musgrave concept of total (or social) risk-taking; while Atkinson-Stiglitz [5] has suggested a(1-t) as a measure of private risk-taking, where t is the marginal tax rate.

shall adopt the measures of risk-aversion introduced by Arrow [3]and Pratt[55]. These are: absolute risk-aversion, $A(W) = [-U''(W)/U'(W)]$ and relative risk aversion, $R(W) = [-U''(W)W/U'(W)]$. Pratt has offered the following interpretation: Consider an individual who is indifferent between an amount $E(W)$ where W is a random variable with mean $E(W)$ and an arbitrarily small variance σ^2; and a certain wealth Wo. Being a risk-avertor, he will only accept the random outcome, W, if the mean $E(W)$ is greater than Wo. Define

$$\pi = E(W) - Wo > 0, \qquad \text{...(IV-1)}$$

where we can conceptualise π as an "insurance premium". Then Pratt shows that for sufficiently concentrated distributions

$$A(W) \simeq 2\left(\frac{\pi}{\sigma^2}\right). \qquad \text{...(IV-2)}$$

Roughly speaking, then, absolute risk-aversion, $A(W)$, equals twice the insurance premium per unit of variance for small risks. If we measure bets not in absolute terms but in proportion to W, then a similar interpretation can be obtained for relative risk-aversion.

However, for our purposes we only require the relationship between wealth elasticities and assumptions on the risk-aversion measures. In a model without taxation Arrow [3] has shown the following results:

(a) The wealth elasticity of the risky asset is greater than, equal to or less than <u>zero</u> as absolute risk-aversion is decreasing, constant or increasing;

(b) The wealth elasticity of the risky asset is greater than, equal to or less than <u>unity</u> as relative risk-aversion is decreasing, constant or increasing.

An extension of these results for the case of linearly progressive

taxation is given in the appendix to this chapter. We find that the second result (b) is modified in the presence of a linear progressive tax schedule. Specifically, we now obtain:

(c) The wealth elasticity of the risky asset is less than unity if relative risk-aversion is constant or increasing.

Thus in the presence of a linearly progressive tax schedule decreasing relative risk-aversion is no longer sufficient to guarantee a wealth elasticity of the risky asset greater than unity.[9]

These results can be interpreted as follows: The first (a) states that if individuals become more risk-averse (in an absolute sense) as they become wealthier, then they allocate absolutely more of their investable wealth to the safe asset. Similarly, the second (proposition (c) above) says that if the individual's risk-aversion (in a relative sense) remains the same or increases as he becomes wealthier, then he allocates relatively more of his portfolio to the safe asset. Proceeding from this framework, we now analyse the effects of taxation.

IV.2 Progressive Income Taxation

In the case of a progressive tax on investment income with full loss offsets, after tax wealth is given by

$$W = Wo + (1-t)\{rWo + a(X-r) - K\} + K, \qquad \ldots(IV\text{-}3)$$

where t is the marginal tax rate on investment income and K is the level of exemption on such income. The investor chooses 'a' to maximise

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[9] Arrow [3] has argued that absolute risk-aversion decreases as wealth increases, and it seems reasonable that risky assets are not inferior. However, his other hypothesis that relative risk-aversion is an increasing function of wealth may not be so obvious, and Stiglitz [67] has discussed the difficulties involved therein.

$$E\{U(W)\} = \int\{U(Wo+(1-t)[rWo+a(X-r)-K]+K)\}dF(x), \qquad \ldots(IV\text{-}4)$$

where the integration is over the range of X. If $U'' < 0$, the necessary and sufficient condition for an interior maximum is

$$E\{U'(W)(X-r)\} = 0. \qquad \ldots(IV\text{-}5)$$

This can be interpreted as stating that the expected marginal utility of a unit of investment in each asset is equated at the optimum (i.e., $E\{U'(W)X\} = E\{U'(W)r\}$). In order to discover how increases in the tax rate affect risk-taking, we differentiate (IV-5) with respect to t,

$$\frac{\partial a}{\partial t} = \frac{a}{1-t} - \frac{(rWo-K)E\{U''(W)(X-r)\}}{(1-t)E\{U''(W)(X-r)^2\}}. \qquad \ldots(IV\text{-}6)$$

Implicit differentiation of (IV-5) also gives

$$\frac{\partial a}{\partial Wo} = (-)\frac{[1+(1-t)r]E\{U''(W)(x-r)\}}{(1-t)E\{U''(W)(X-r)^2\}}. \qquad \ldots(IV\text{-}7)$$

From (IV-6) and (IV-7) we, therefore, have

$$\frac{\partial a}{\partial t} = \frac{a}{1-t} - \frac{(rWo-K)}{[1+(1-t)r]}\frac{\partial a}{\partial Wo}. \qquad \ldots(IV\text{-}8)$$

From (IV-8) it is clear that for r=0,

$$\frac{\partial a}{\partial t} = \frac{a}{1-t} + K\frac{\partial a}{\partial Wo}, \qquad \ldots(IV\text{-}9)$$

where the first term on the r.h.s. of (IV-9) has the interpretation of the substitution effect: $[a/(1-t)] = (\partial a/\partial t)/U=\bar{U}$, and the second term is the usual income effect. Given decreasing absolute risk-aversion, (with $\partial a/\partial Wo > 0$), <u>social</u> risk-taking increases (<u>a</u> rises) and, if we follow Atkinson-Stiglitz and regard a(1-t) as an indicator of <u>private</u> risk-taking, this also increases since we can rewrite (IV-9) as

$$(1-t)\frac{\partial a}{\partial t} - a = \left(\frac{K}{1-t}\right)\frac{\partial a}{\partial Wo}, \qquad \ldots(IV\text{-}10)$$

where the l.h.s. measures the change in private risk-taking. It may be

noted that in the case of a flat rate proportional tax on investment income (i.e., K = 0), as Stiglitz [68] and Mossin [48] have shown, private risk-taking remains unchanged. Also note that if we had a quadratic utility function (this implies increasing absolute risk-aversion and hence $\partial a/\partial Wo < 0$) the effect of the tax on social risk-taking would be indeterminate, but private risk-taking would decrease as the risky asset becomes inferior.[10]

For the general case of $r > 0$, it follows from (IV-8) that the hypothesis of decreasing absolute risk-aversion is not sufficient to determine the direction of changes in risk-taking: the magnitude of $(rWo - K)$ is important. However, if we rewrite (IV-8) as

$$\frac{\partial a}{\partial t} = \left(\frac{a}{1-t}\right) \left[1 - \left\{\frac{r(1-t)}{1+r(1-t)} \left(\frac{Wo}{a} \frac{\partial a}{\partial Wo}\right)\right\}\right] + \frac{K}{[1+r(1-t)]} \frac{\partial a}{\partial Wo}; \qquad \text{...(IV-11)}$$

it is possible to determine the effects on (social) risk-taking. Clearly, given decreasing absolute risk-aversion, total risk-taking increases where relative risk-aversion is constant or increasing. For decreasing relative risk-aversion relative risk-taking, as measured by $(t/a)(\partial a/\partial t)$, would increase if the effective marginal tax rate does not exceed 50%. This can be seen by comparing (IV-11) when multiplied by (t/a), and equation (A-IV-3) of the appendix.

From (IV-8) it is also obvious that the effect of the tax on

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[10] Notice that for a flat rate proportional tax on income (i.e., K=0) this conflict in the predictions between the general formulation and the quadratic utility formulation does not appear. So long as r=0, total risk-taking always increases. In other words, for r=0 (and K=0), the effect of the income tax on risk-taking is independent of the investor's attitude towards risk (except for that he is a risk-avertor).

private risk-taking depends on (rWo-K) and is, therefore, ambiguous.[11] Our major conclusion then is the following: An increase in the marginal tax rate, in a system with a linearly progressive income tax with full loss offsets, leads to an increased demand for the risky asset if (a) the wealth elasticity of the risky asset demand is positive, and is less than or equal to unity (or, alternatively, absolute risk-aversion is decreasing and relative risk-aversion is non-decreasing in wealth); or (b) relative risk-aversion is decreasing and the marginal tax rate does not exceed 50%. The detailed results are presented in Tables IV.1 and IV.2. These results demonstrate the possibility that progressive (investment) income taxation may lead to increased total risk-taking rather than a decrease as might have been speculated. An intuitive rationale for this outcome is that by allowing full loss offsets and an exemption on (risky) income, the government shares part of the risk. Thus, although taxation reduces the probability of large gains, it also reduces the probability of large losses. In other words, the size of the bet has been reduced by taxation.

IV.3 Treatment of Capital Gains

In this section we examine the consequences for risk-taking of exemptions on capital gains. We must point out that, there is no unique way of treating this problem. In Case 1, we follow Stiglitz [68] and take the polar case of taxing the return on the safe asset alone. Under Case 2, we provide an alternative method and show how this modification affects the analysis.

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[11] However, for a flat rate proportional tax (i.e., K=0), equation (IV-8) tells us that private risk-taking decreases, remains constant or increases as absolute risk-aversion decreases, remains constant or increases with wealth.

Table IV.1 Effects of an Income Tax on Risk-Taking: r=0

Type of Absolute Risk-Aversion (A(W))	Type of Risk-Taking	
	Social	Private
Decreasing	+	+
Constant	+	0
Increasing	?	-

Table IV.2 Effects of an Income Tax on Risk-Taking: r>0

Type of Absolute Risk-Aversion (A(W)) \ Type of Risk-Taking	Type of Relative Risk-Aversion (R(W))					
	Decreasing		Constant		Increasing	
	Social	Private	Social	Private	Social	Private
Decreasing	+	?	+	?	+	?
Constant	NA*	NA	NA	NA	+	?
Increasing	NA	NA	NA	NA	?	?

* It is impossible to have constant or increasing absolute risk-aversion with non-increasing relative risk-aversion.

Case 1: Final wealth, after a linearly progressive tax is applied to the income from the safe asset alone, is given by

$$W = (Wo-a)(1+r) + a\ (1+X) - t\ \{(Wo-a)r-K\}. \qquad \ldots(IV\text{-}12)$$

Therefore, maximising expected utility where wealth is defined as above gives

$$E\{(U'(W)(X-r(1-t))\} = 0 \qquad \ldots(IV\text{-}13)$$

as the necessary and sufficient condition for an interior maximum (since we have assumed $U''(W) < 0$). To see the effect of a change in t, we obtain, upon implicit differentiation of (IV-13):

$$\frac{\partial a}{\partial t} = \frac{-rE[U'(W)] + \{(Wo-a)r-K\}E\{U''(W)(X-r(1-t))\}}{E\{U''(W)(X-r(1-t))^2\}}. \qquad \ldots(IV\text{-}14)$$

Hence we need to know the sign of

$$rE[U'(W)] - [(Wo-a)r-K]E\{U''(W)(X-r(1-t))\}, \qquad \ldots(IV\text{-}15)$$

which is positive if absolute risk-aversion is increasing or constant.[12] The result is indeterminate if absolute risk-aversion is decreasing.

However, we can re-arrange the terms in (IV-15) as follows:

$$rE[\{\frac{U''(W)W}{U'(W)} + 1\}\ U'(W)] - E\{\ U''(W)[(rWo-K)\ X + r(Wo+K)]\}. \qquad \ldots(IV\text{-}16)$$

For $X > -1$ (i.e., limited liability), the second term is positive, while the first is positive if $R(W) < 1$. Thus we have shown that: Linear progressive taxation of the income from the safe asset alone increases the demand for the risky asset if (a) absolute risk-aversion is increasing or constant, or (b) relative risk-aversion is less than or equal to

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[12] As will be indicated in the appendix to this chapter, $E\{U''(W)(X-r(1-t))\} \gtrless 0$ as absolute risk-aversion is decreasing, constant or increasing with wealth. Also note that unless $[(Wo-a)r-K]$ is positive, the investor does not pay any taxes.

unity.[13] While restrictions on the value of R(W) are not easily interpretable, the result that increasing absolute risk-aversion would encourage (total) risk-taking can be explained as follows: A tax on the income from the riskless investment is, in part, like a reduction in wealth and hence the assumption of inferiority of the risky asset encourages risk-taking. But, this explains only the income effect. There is also a substitution effect which favours a movement away from the safe asset (notice the first term in expression (IV-15)).

Case 2. Here we modify the treatment of capital gains by asserting that the return on risky assets has two components -- a "safe" component and a risky capital gains (or loss) term. One possible way to conceptualise this idea would be to refer to a bond which offers a known rate of return plus a "liquidity premium" in recognition of the uncertainty as to its end-of-period market value. Consequently, we examine the effects of exempting risky capital gains from taxation.[14] For simplicity, we equate the "safe" component of the return on the risky asset to the return on the safe asset.[15] Final wealth, then, is given by

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[13] It must be pointed out that $R(W) \leq 1$ may well turn out to be implausibly low. In particular, Arrow [3] has shown that if U(W) is to remain bounded as W becomes large, then R(W) cannot tend to a limit below unity.

[14] In Case 1 we examined the consequences for risk-taking of taxing the return on the safe asset alone, while in the present case we are examining a tax on the equivalent of the safe rate of return on the entire portfolio.

[15] Notice that this is merely an algebraic simplicity. Normally, we would expect the "safe" component of the risky return (r_1) to be at least as great as the return on the safe asset, r. In such a case, the derivative $\partial a/\partial t$ will be strengthened in the same direction by the difference (r_1-r). Hence, in equating them we do not lose any generality.

$$W = (Wo-a)(1+r) + a(1+r+y) - t\,(rWo-K)$$

or, $$W = (1+r)Wo + ay - t(rWo - K), \qquad \ldots(IV\text{-}17)$$

where (r+y) equals X. For an interior maximum we require

$$E\{U'(W)y\} = 0, \qquad \ldots(IV\text{-}18)$$

where W is defined by (IV-17). Proceeding in a manner exactly analogous to that presented above, we obtain:

$$\frac{\partial a}{\partial t} = (-)\,\frac{(rWo - K)}{[1+r(1-t)]}\,\frac{\partial a}{\partial Wo}\,. \qquad \ldots(IV\text{-}19)$$

Evidently, $\partial a/\partial t \lesseqgtr 0$ as $\partial a/\partial Wo \gtreqless 0$ (given that $(rWo-K) > 0$, otherwise the individual may not be taxed at all). Hence we have shown that: <u>Linear progressive taxation with exemptions of capital gains over and above the riskless rate of return leads to a decrease, no change or an increase in (total) risk-taking as the wealth elasticity of the demand for the risky asset is greater than, equal to or less than zero</u>. As before, exempting capital gains implies taxing riskless income and this generates an income effect in an obvious way. Moreover, by equating the "safe" component of the risky rate of return to the rate of return on riskless investment, we eliminate any possible substitution among the two forms of investments.

IV.4 The Pure Effect of Progression

Although, the possibility that a proportional (investment) income tax may encourage risk-taking under full loss offsets is somewhat well known, it is almost universally held that a progressive tax is detrimental to risk-taking. In this section we examine the effect of progressivity on risk-taking.[16] In comparing any two taxes, we have to choose a basis

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[16] In Section IV.2 we have seen that a linearly progressive tax on investment income encourages private risk-taking if r=0, while under

for the comparison. The most common criterion is that both taxes yield equal revenue to the government. In the present case, however, there is the additional difficulty that the revenue raised by a tax on investment income is a random variable. Despite some unsatisfactory aspects,[17] we employ the equal expected revenue as one possible basis for comparison. We also examine an alternative criterion in which the taxes to be compared lead to equal reductions in expected utility.

Case 1: The Revenue Criterion. For simplicity, we assume that the return on the safe asset is zero. Therefore, expected revenue under the progressive tax is given by

$$E(R) = \int t(aX-K)dF(x),$$

$$= ta\bar{X} - tK, \qquad \ldots(IV\text{-}20)$$

where $\bar{X}$ denotes the expected value of X, the rate of return on the risky investment. Differentiating (IV-20) totally and equating to zero, i.e., maintaining the same revenue, we have

$$dE(R) = (t\bar{X}da + a\bar{X}dt - tdK - Kdt) = 0$$

or,

$$dt/dE(R) = 0 \quad = \frac{tdK - t\bar{X}da}{(a\bar{X}-K)} \quad . \qquad \ldots(IV\text{-}21)$$

We can now evaluate the change in risk-taking due to a compensated expected revenue change in progression as follows. Notice that the equilibrium condition for the investor's problem, equation (IV-5) with r=0 reads

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similar circumstances, a flat rate proportional tax does not. However, this particular result does not provide a clear comparison of the two taxes since we did not employ any basis for the comparison.

[17] For instance, it may not be possible to compute taxes yielding the same revenue in each state of nature. These difficulties have been discussed by Stiglitz [68].

$$E\{U'(W)X\} = 0. \qquad \text{...(IV-22)}$$

This defines optimal risk-taking as a function of the tax parameters:

$$a = a(t,K), \qquad \text{...(IV-23)}$$

which on total differentiation gives.

$$da = \frac{\partial a}{\partial t}\, dt + \frac{\partial a}{\partial K}\, dK. \qquad \text{...(IV-24)}$$

From (IV-21) and (IV-24) it follows

$$da\Big/_{dE(R)=0} = \frac{\partial a}{\partial t}\left[\frac{tdK - t\bar{X}da}{(a\bar{X}-K)}\right] + \frac{\partial a}{\partial K}\, dK. \qquad \text{...(IV-25)}$$

Rearranging we obtain

$$\frac{da}{dK}\Big/_{dE(R)=0} = \frac{\left[\frac{\partial a}{\partial K} + \frac{t}{(a\bar{X}-K)}\,\frac{\partial a}{\partial t}\right]}{\left[1 + \left(\frac{t\bar{X}}{a\bar{X}-K}\right)\frac{\partial a}{\partial t}\right]}. \qquad \text{...(IV-26)}$$

Recall that $\partial a/\partial t$ (as given by equation (IV-9)) and $\partial a/\partial K$ ($=t(\partial a/\partial Wo)$, which can be seen by differentiating (IV-22)) are both positive if absolute risk-aversion is decreasing. Also notice that we require $\bar{X} > 0$ in order to guarantee an interior solution, and unless $(a\bar{X}-K) > 0$ the investor is not taxed. Hence the following result: <u>When the rate of return on riskless investment is zero and the wealth elasticity of the demand for the risky asset is greater than zero, a compensated expected revenue increase in progression encourages further risk-taking.</u>

Given the results of the preceding sections, this result can be explained as follows. Increased progression comes about through both increases in t and in K. Where the wealth elasticity of the demand for the risky asset is greater than zero, increased K, <u>by itself</u>, encourages risk-taking; while an increase in t, <u>ceteris paribus</u> (as seen in Section IV.2), also encourages risk-taking under these conditions. What

is interesting to observe is that such forces explain the investor's behaviour even when both the changes in K and t take place simultaneously such that the investor's expected tax payments do not change.

Case 2: The Expected Utility Criterion. Let us now consider the effect on risk-taking such that the loss in expected utility -- due to proportional and linear progressive taxes -- is constant. In the process, we assume that the safe asset has a positive rate of return. Moreover, we adapt a diagrammatic device developed by Stiglitz [68].

In order that we may use diagrams, we assume that there are only two states of nature: (1) state θ_1, where the risky asset yields more than r, occurs with a probability p_1; (2) state θ_2, where the risky asset yields less than r, occurs with a probability p_2 such that $p_1+p_2 = 1$. The individual's opportunity locus is depicted in Figure IV.1. If he purchases only the safe asset, he obtains the point S on the 45° line, where

$$W(\theta_1) = W(\theta_2) = Wo(1+r). \qquad \text{...(IV-27)}$$

At the other extreme, if he holds only the risky asset, the end of the period wealth is represented by the point T with

$$W(\theta_1) = W_o[1+X(\theta_1)] \qquad \text{...(IV-28a)}$$

in the first state of nature, and

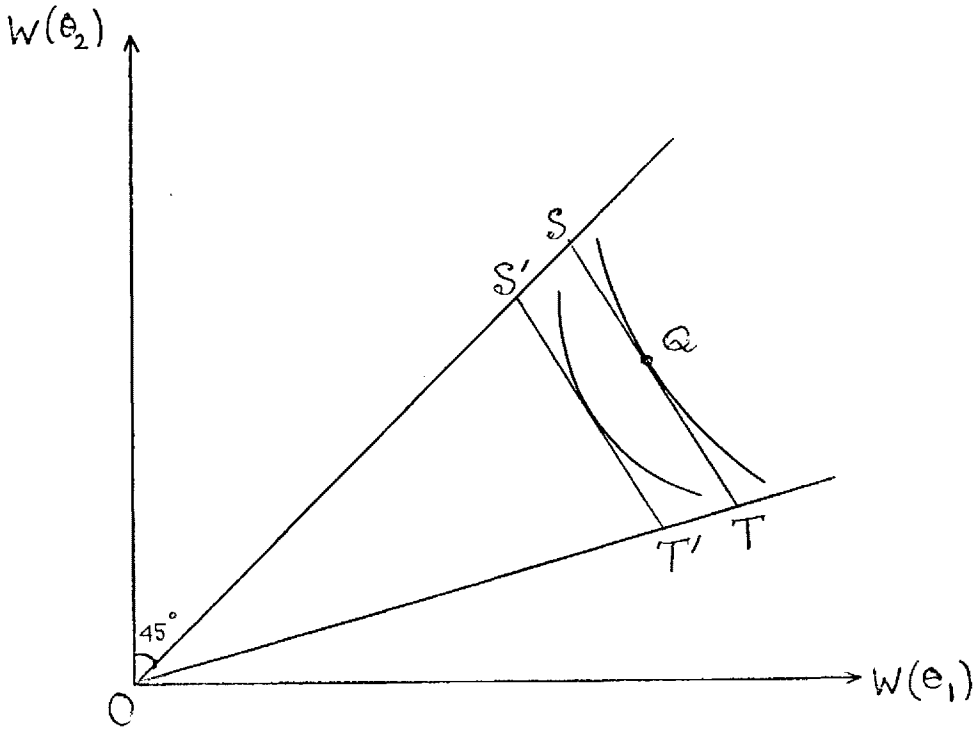
$$W(\theta_2) = W_o[1+X(\theta_2)] \qquad \text{...(IV-28b)}$$

in the other. Obviously, by allocating different proportions between the two assets, the investor can obtain any point along the line ST.

On the same diagram (Figure IV.1) we have drawn indifference curves for different values of expected utility,

$$E\{U(W)\} = p_1U(W(\theta_1)) + p_2U(W(\theta_2)), \qquad \text{...(IV-29)}$$

Figure IV.1 Optimal Portfolio Choice: An Exposition



and clearly the equilibrium is at the point Q, the point of tangency between the indifference curve and the opportunity locus. Also note that

$$a = \frac{QT}{ST},$$

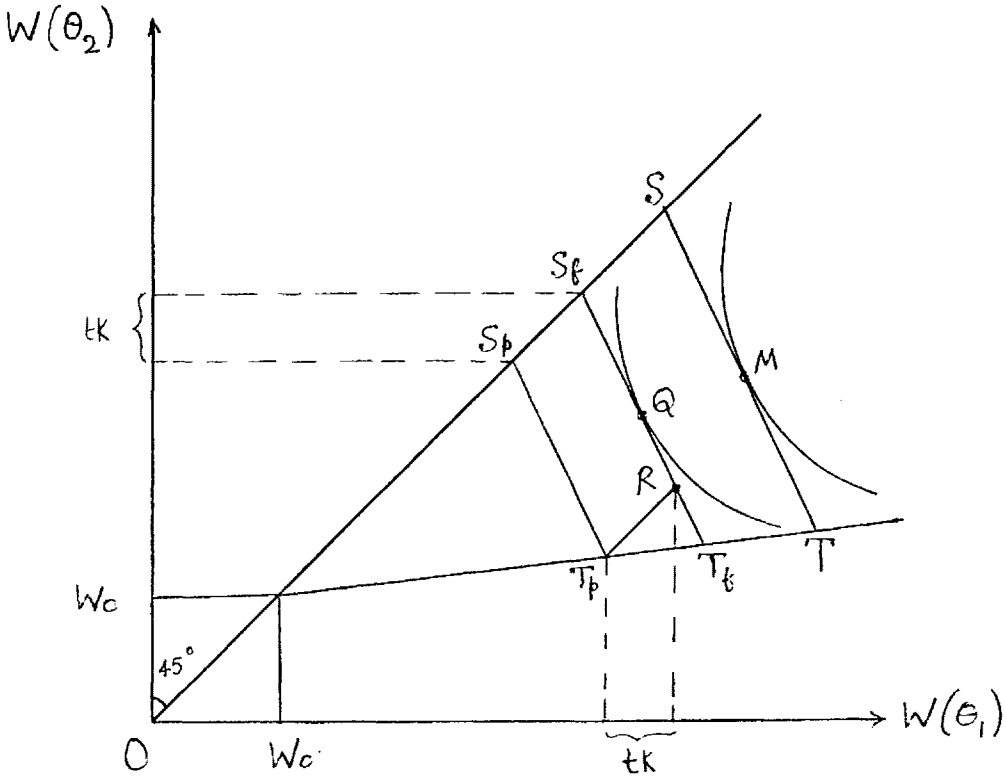
i.e., the proportion of the initial wealth allocated to the risky asset is equal to the ratio of QT to ST.

In Figure IV.2, the opportunity set, in the absence of any tax, is ST and the portfolio equilibrium at M. A flat rate income tax (i.e., K=0) shifts this opportunity set to S_fT_f and the equilibrium to Q. Since the slope of the opportunity set is independent of the parameters of a linear tax schedule, Q also happens to be the optimum position for the equal expected utility progressive tax. But Q represents a portfolio with higher total risk (a) under the progressive tax than the flat tax. This is so because the opportunity set is shorter under the progressive tax being only S_fR: the higher marginal tax rate shifts ST to a position like S_pT_p, while the lump-sum subsidy tK then shifts this bodily to S_fR with R corresponding to a = Wo. Therefore, we conclude: Risk-taking is greater under linear progressive taxation than under strictly proportional income taxation when both types of taxes lead to equal losses of expected utility for the investor. As seen in Figure IV.2, for equal losses of expected utility, the marginal rate of tax is higher in the progressive than in the proportional case, and this alone can account for greater (social) risk-taking in the progressive tax case (see the results of Section IV.2).

IV.5 Conclusion

In this chapter we have presented a general expected utility

Figure IV.2 Proportional vs. Linearly Progressive Taxes



formulation of the portfolio choice problem where the investor faces a simple progressive tax schedule. The analysis shows that a progressive investment income tax would typically encourage (total) risk-taking under very reasonable conditions, while private risk-taking is indeterminate (except where the rate of return on the safe investment is nil, in which case private risk-taking increases, remains constant or decreases as the wealth elasticity of the demand for the risky asset is greater than, equal to or less than zero). Exemptions on capital gains, on the other hand, are most likely to reduce the demand for the risky asset. Comparing a progressive tax and a proportional tax on alternative criteria, we have also shown that the former leads to more risk-taking than the latter.

Appendix to Chapter IV

The relationship between the wealth elasticity of the demand for the risky asset and hypotheses on the risk-aversion functions, as stated in Section IV.1, were originally shown by Arrow [3] in a model without any taxes. Later Stiglitz [68] and Mossin [48] established these results for proportional taxation. In this appendix we extend these relationships for the case of linear progressive taxation on investment income. They can easily be extended for other linear tax schedules.

A.IV.1 Absolute Risk-Aversion and Wealth Elasticities. As shown in the text [equation (IV-7)], the derivative $\partial a/\partial Wo$ has the sign of $E\{U''(W)(X-r)\}$, which can be rewritten as (using the definition of absolute risk-aversion):

$$-E\{A(W)U'(W)(X-r)\}. \qquad \ldots(A\text{-}IV\text{-}1)$$

Now, we show that the expression given by (A-IV-1) has the opposite sign of $\partial A(W)/\partial W$. This is seen as follows. Define $W^* = Wo[1+r(1-t)] + tK$. Then

$$-E\{A(W)U'(W)(X-r)\} = E\{[A(W^*) - A(W)]U'(W)(X-r)\} - A(W^*)E\{U'(W)(X-r)\},$$

and using the first-order condition [equation (IV-5) of the text]

$$-E\{A(W)U'(W)(X-r)\} = E\{[A(W^*) - A(W)]U'(W)(X-r)\}.$$

When $(X-r) > 0$, $W > W^*$ (see equation (IV-3) of the text). Hence if $A(W)$ is increasing with W, $[A(W^*)-A(W)] < 0$, and $\{A(W^*) - A(W)\}(X-r) < 0$. Similarly, when $(X-r) < 0$, $W < W^*$, and if $A(W)$ is increasing with W, $[A(W^*) - A(W)] > 0$, so that $\{A(W^*) - A(W)\}(X-r)$ is still negative. Thus we have shown that

$$-E\{A(W)U'(W)(X-r)\} \gtreqless 0 \text{ as } A(W) \text{ is } \begin{cases} \text{decreasing,} \\ \text{constant,} \\ \text{increasing.} \end{cases}$$

In other words,

$$\frac{\partial a}{\partial Wo} \gtreqless 0 \quad \text{as } A(W) \text{ is } \begin{cases} \text{decreasing,} \\ \text{constant,} \\ \text{increasing.} \end{cases}$$

A.IV.2 Relative Risk-Aversion and Wealth Elasticities. Using the same argument as above, we can show that

$$-E\{R(W)U'(W)(X-r)\} \gtreqless 0 \text{ as } R(W) \text{ is } \begin{cases} \text{decreasing,} \\ \text{constant,} \\ \text{increasing.} \end{cases}$$

Or, $E\{U''(W)(W)(X-r)\} \gtreqless 0$ as $\frac{\partial R(W)}{\partial W} \lesseqgtr 0$.

Using the definition of W, as given by equation (IV-3), we can re-write the above relation as

$$E\{U''(W)(X-r)[Wo+(1-t)(rWo + a(X-r)-K)+K]\} \gtreqless 0 \text{ as}$$

$$\frac{\partial R(W)}{\partial W} \lesseqgtr 0. \qquad \ldots\text{(A-IV-2)}$$

Using the expression for the derivative $\partial a/\partial Wo$ (given by equation (IV-7) of the text), the inequality (A-IV-2) can be rearranged as

$$1 \lesseqgtr \left(\frac{Wo}{a}\frac{\partial a}{\partial Wo}\right) + \frac{tK}{a[1+r(1-t)]}\frac{\partial a}{\partial Wo} \text{ as } \frac{\partial R(W)}{\partial W} \lesseqgtr 0.$$

$$\ldots\text{(A-IV-3)}$$

This result is slightly different from that obtained by Arrow [3]. In particular, even if relative risk-aversion is constant, wealth elasticity of the demand for the risky asset is still less than unity, and decreasing relative risk-aversion no longer guarantees that the investor allocates proportionately more of his portfolio to the risky asset as he becomes wealthier.

PART 3: INTERTEMPORAL ANALYSIS

"...We look upon economic theory as a sequence of conceptual models that seek to express in simplified form different aspects of an always more complicated reality."

(Koopmans, T.C.: Three Essays on the State of Economic Science.)

CHAPTER V

TAXATION IN A TWO-PERIOD TEMPORAL MODEL OF CONSUMPTION AND PORTFOLIO ALLOCATION

In Part 2 of this study (Chapters II, III and IV) we have analysed the effects of taxation on risk-taking in the context of models all of which ignored the consumption-saving decisions. This practice, although analytically convenient, is not a satisfactory description of household decision processes. Certainly the decision to hold one's wealth in various forms is closely related to the objective of attaining future consumption plans that enable the individual to attain maximal expected utility. This view of individual decision-making implies that the discussion of portfolio selection is inseparable from the discussion of optimal consumption decisions under uncertainty. Such an integration of portfolio and consumption decisions allows the individual, by optimally choosing the amount devoted to consumption, to vary the amount of investable wealth. We can consider, therefore, the simultaneous determination of the size and the composition of the optimal portfolio. Recently Sandmo ([61], [62]) and Dreze and Modigliani [17], among others, have attempted such an integration of portfolio choice and consumption allocation over time. In light of this development it is natural to ask whether the effects of taxation on risk-taking [the analysis of Part 2] depend in an essential way on the assumption of a fixed portfolio. Sandmo's own attempt to answer this question was an exploratory first

step.[1] In the present chapter we have used a separable utility function (a special case of Sandmo's general utility function) and have obtained more definitive results. In this sense, this chapter can be viewed as an extension of his results. This analysis will also allow us to examine the effects of taxation on intertemporal consumption decisions under uncertainty. This discussion clarifies the degree to which the predictions of the existing analysis of intertemporal consumption decisions (e.g., Musgrave [50] and Hansen [29]), which was carried out in a world of certainty, carry over to the case of uncertainty. Moreover, as will be shown in the next chapter (VI), the present framework can also be used to discuss the comparative effects of a tax on consumption rather than a tax on investment income.

One other way, in which the present approach differs from those of the preceding chapters, is related to the nature of uncertainty involved. The single-period analysis, as pointed out by Sandmo [62] and Dreze-Modigliani [17], can be interpreted as being concerned with timeless risk prospects, i.e., the uncertainty will be removed before the saving-consumption decision is made, while here we are concerned with temporal risk prospects. In this case the uncertainty about the yield of the risky asset is not going to be removed until the end of the first period,

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[1] Sandmo [62] uses a utility function of the form $U(C_1, C_2)$ and finds that taxation effects are difficult to evaluate when the return on riskless investment is non-zero. For the cases of no return on riskless investment and for taxation of the differential yield (i.e., the risky less the riskless rate of return), the effects on risk-taking are identical to those of the Stiglitz [68] - Mossin [48] single-period analysis.

i.e., until after the saving-consumption decision has been made.[2] Dreze and Modigliani have established that behaviour under temporal uncertainty differs from behaviour under timeless uncertainty and, hence, it is worthwhile to examine whether taxation changes in a world of temporal risk yield the standard results of pure portfolio theory.

Consumption and portfolio allocation decisions are assumed to satisfy maximisation of the individual's expected lifetime utility. The individual is presumed to live for two periods and the (current) consumption-portfolio allocation decision is made in the first period based on the subjective probability distribution of the outcomes. In the second period the household dissaves, consuming all its capital and the exogenous wage incomes. Both proportional and linearly progressive taxes (of the type discussed in Chapter IV) are analysed. It is found that the conditions for stimulatory effects of taxation on risk-taking require both the assumptions on risk-aversion made in the single-period analysis and additional sufficient conditions on the income elasticities of the demand for the risky asset and for consumption. Consumption taxes (both proportional and progressive) are most likely to discourage current consumption, while investment income taxes, depending on the income elasticity of consumption, may either encourage or discourage saving.

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[2]Mossin ([49], pp. 29-30) has offered the following illustration of the distinction between temporal and timeless risks: "Consider the following two situations. In both situations I offer you $1000 if a coin comes up heads, and in both situations the amount will be payable one year from today. However, in one situation the coin is tossed today (a timeless prospect), in the other a year from today (a temporal prospect)".

The rest of this chapter proceeds as follows. In Section V.I we outline the model to be used throughout this and the next chapter, especially noting its assumptions and their behavioural implications. Proportional taxes are analysed in Section V.2, while Section V.3 discusses linearly progressive taxation. Some conclusions are offered in Section V.4. The proofs and some background results are collected in the appendix.

V.1 Consumption and Portfolio Allocation Decisions

The individual makes the portfolio allocation decision in the first period to maximise expected lifetime utility. In the second period he dissaves, consuming all his wage income, capital and the realised investment income. The intertemporal consumption allocation decision can be stated as

$$\left.\begin{array}{l} \max EF(C_1,C_2) = V(C_1) + EU(C_2), \\ \text{s.t. } \left\{ \begin{array}{l} C_1 = Y_1 - a-m, \\ C_2 = Y_2 + a(1+X) + m(1+r), \end{array} \right. \end{array}\right\} \qquad \ldots(V\text{-}1)$$

where both V and U are assumed to be twice continuously differentiable with positive and diminishing marginal utilities, thus guaranteeing risk-aversion and a diminishing marginal rate of substitution between present and future consumption prospects. It must, however, be noted that additive separability is a rather strong assumption (implying that the [expected] marginal utility of consumption in any period is a function of the level of consumption in that period alone) and we shall soon discuss the possible justifications for such an assumption. C_1 and C_2

are respectively current and future consumption,[3] Y_1 and Y_2 are the non-asset incomes received in the first and the second periods,[4] while, as defined in Chapter IV, a and m are the amounts allocated to the risky and the safe assets respectively with X (a random variable) and r as their respective rates of return. The investor's problem, equation (V-1), involves choosing $\{C_1, C_2, a, m\}$. However, by combining the budget constraints we can rewrite C_2 as

$$C_2 = Y_2 + (Y_1 - C_1)(1+r) + a(X-r). \qquad \ldots(V\text{-}2)$$

Now, substituting for C_2 in (V-1) we can eliminate the constraints and restate the investor's problem as one of solving a two-good, two-period problem:

$$\max_{\{C_1, a\}} \; V(C_1) + E\{ U(Y_2 + (Y_1 - C_1)(1+r) + a(X-r))\}. \qquad \ldots(V\text{-}3)$$

The necessary conditions for the existence of an interior maximum are given by

$$V'(C_1) - (1+r)E\{U'(C_2)\} = 0; \qquad \ldots(V\text{-}4)$$

$$E\{U'(C_2)(X-r)\} = 0. \qquad \ldots(V\text{-}5)$$

We may interpret these conditions as follows:

(a) The first says that at an optimum, the marginal rate of time-preference

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[3]The assumption of constant relative prices among any group of commodities allows us to aggregate consumption programmes defined over such commodity groups into a single commodity (denoted by C_1 or C_2). See Hicks ([30], p. 33) for a discussion of this result known as the composite commodity theorem.

[4]To elaborate, we might follow Dreze-Modigliani and define Y_1 to include the net market value of the individual's assets, plus his labour income during the initial period. Similarly, Y_2 will denote the present value of his future labour income, plus additional receipts from sources other than his current assets.

$([V'(C_1)/E\{U'(C_2)\}]-1)$ equals the rate of return on the safe asset, which is analogous to Fisher's famous rule for optimisation over time [23];

(b) The second condition says that the expected marginal utility of a unit of investment in each asset is equalised at an optimum, i.e., $E\{U'(C_2)X\} = E\{U'(C_2)r\}$. This is an extension of the usual interpretation of a portfolio equilibrium as discussed in Chapter IV.

Since we have assumed strict concavity of both $V(C_1)$ and $U(C_2)$, the above conditions are also sufficient for a utility maximum.[5] However, we do state the second-order sufficiency conditions as they allow an interesting economic interpretation. These conditions are:

$$V''(C_1) + (1+r)^2 E\{U''(C_2)\} < 0; \qquad \ldots(V\text{-}6)$$

$$H \equiv \begin{vmatrix} V''(C_1) + (1+r)^2 E\{U''(C_2)\} & -(1+r)E\{U''(C_2)(X-r)\} \\ -(1+r)E\{U''(C_2)(X-r)\} & E\{U''(C_2)(X-r)^2\} \end{vmatrix} > 0. \qquad \ldots(V\text{-}7)$$

From (V-6) and (V-7) it also follows that

$$E\{U''(C_2)(X-r)^2\} < 0. \qquad \ldots(V\text{-}8)$$

Notice that (V-6) and (V-8) are always satisfied given our assumption of diminishing marginal utility in both periods. The interpretation of (V-7), however, requires a consideration of risk-aversion measures in a temporal framework.

In the last chapter we have discussed the measures of risk-

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[5] Strict concavity of V and U makes our additively separable $F(C_1,C_2)$ strictly quasi-concave. See Arrow ([3], Chapter 4) and Stigum[70].

aversion appropriate for single-period analysis. We now investigate whether these measures have to be modified in a temporal context. Dreze and Modigliani have shown that the measure of absolute risk-aversion $A(C_2)$, in a temporal context is $[(-)\frac{U_{22}}{U_2}\Big|_{C_1^*}]$, where $U(C_1,C_2)$ denotes the utility function and C_1^* is the optimal value of current consumption. They also find that a necessary and sufficient condition for this measure to be everywhere independent of C_1 is that the utility function can be written as:

$$U = f(C_1) + g(C_1).\ h(C_2). \qquad \ldots(V\text{-}9)$$

Our separable utility function obviously satisfies this independence property. Thus, our measure of absolute risk-aversion is

$$A(C_2) \equiv (-)\ [U''(C_2)/U'(C_2)], \qquad \ldots(V\text{-}10)$$

and the corresponding measure of relative risk-aversion is

$$R(C_2) \equiv (-)[U''(C_2)C_2/U'(C_2)], \qquad \ldots(V\text{-}11)$$

which are of the indentical form as those of the last chapter. Clearly, a logarithmic utility function,

$$F(C_1,C_2) = \log C_1 + (1-\delta)\log C_2, \qquad \ldots(V\text{-}12)$$

(where δ is a time discount factor) would imply constant relative risk-aversion (of unity) and decreasing absolute risk-aversion.[6] Also notice

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[6]Evidently for constant relative risk-aversion, as stated in the appendix (Section A.V.1), $E\{U''(C_2)(X-r)C_2^2\} = 0$. Using the expression for C_2 [equation (V-2) of the text] the above condition is written as

$$[Y_2 + (Y_1-C_1)(1+r)][U''(C_2)(X-r)] + aE[U''(C_2)(X-r)^2] = 0.$$

The second term is always negative for a risk-avertor, and, hence, for the equality to hold we require $E(U''(C_2)(X-r)] > 0$, which is the condition for absolute risk-aversion to decrease with future consumption (see Section A.V.1 of the appendix).

that the logarithmic utility function (V-12), satisfies the concavity assumptions made earlier in this section. In the subsequent discussion, therefore, we shall use the utility functions in (V-1) and (V-12) as alternative representations of the investor's behaviour.

We are now in a position to discuss the economic interpretation of (V-7) which amounts to the following: In the context of the model described above, for $H > 0$, it is both necessary and sufficient that $0 < \partial C_1/\partial Y_1 < 1$, i.e., $H > 0 \Longleftrightarrow 0 < \partial C_1/\partial Y_1 < 1$, where absolute risk-aversion is decreasing and relative risk-aversion is non-decreasing. In the context of an unrestricted utility function, $U(C_1,C_2)$, Dreze and Modigliani observed that the requirement $0 < \partial C_1/\partial Y_1 < 1$, although representative of actual behaviour, was not guaranteed by the restrictions which they placed on preferences. Clearly, such a requirement, in our model, is an integral part of consistent behaviour.[7] In particular, the existence of the solution to the investor's problem is intimately related to the requirement that $0 < \partial C_1/\partial Y_1 < 1$. One other behavioural

[7]Essentially this is equivalent to assuming non-decreasing relative risk-aversion as one of the fundamental assumptions of our model. This assumption has been debated in the literature (Stiglitz [67]). However, in the present context this need not be a problem: If the assumption of decreasing absolute risk-aversion is accepted in the expected utility of wealth model as being "reasonable" because it guarantees the non-inferiority of the risky asset, we might just as well, accept the hypothesis of non-decreasing relative risk-aversion as "reasonable" in an intertemporal model as it guarantees $0 < \partial C_1/\partial Y_1 < 1$. In any case, all our results will be stated in terms of income elasticities of consumption and asset demand, and since these elasticities are in principle observable, our results are not specific with respect to particular risk-aversion hypotheses. By the same token, the empirical measure of these elasticities, if available, would automatically verify the "reasonableness" of alternative hypotheses on risk-aversion.

implication of our model is stated as follows: In the context of the above model decreasing absolute risk-aversion guarantees the non-inferiority of the risky asset, i.e., $\partial a/\partial Y_1 > 0$. Notice that these results also hold for the alternative assumption of logarithmic preferences. The formal proofs are given in the appendix (Section A.V.1).

Further, in order to interpret some of our results, we shall require the following assumption:

$$(-)\ [V''(C_1)C_1/V'(C_1)] \geq 1, \qquad \ldots(V\text{-}13)$$

which is analogous to our measure of relative risk-aversion given by expression (V-11). These are simply the elasticities of marginal utility of consumption in each period and Arrow ([3], Chapter 3) has shown that they must "hover" around unity for both $V(C_1)$ and $U(C_2)$ to be bounded. The fact that (V-13) does not involve uncertainty (since this is independent of C_2) does not affect the argument in any way (see Arrow [3], pp. 110-111). In particular, it can be shown that if $V(C_1)$ is to remain bounded as C_1 becomes large, then the elasticity of marginal utility of current consumption cannot tend to a limit below unity. Also note that for a logarithmic utility function, which is a close approximation to a bounded utility function, (V-13) holds with strict equality. Moreover, Fellner [22] has estimated these elasticities to be around 1.5 (U.S. data). This may also be taken as a possible justification for making such an assumption. We will refer to (V-13) as the boundedness hypothesis.

To conclude this section, we note that if restrictions of the form $1 > \partial C_1/\partial Y_1 > 0$ and $\partial a/\partial Y_1 > 0$ are taken for granted on grounds of

realism, the additively separable form of the utility function is a very natural case to consider: In such an event these conditions can be interpreted as postulates of consistent behaviour.

V.2 Proportional Taxation

In this section we discuss the effects of proportional taxation of non-asset income, consumption and investment income on consumption and risk-taking.

A Proportional Non-Asset Income Tax. The constraints, after the introduction of a proportional tax on non-asset income (Y_1 and Y_2) at the rate $0 < t < 1$, are

$$\left.\begin{aligned} C_1 &= (1-t)Y_1 - a - m, \\ C_2 &= (1-t)Y_2 + a(1+X) + m(1+r). \end{aligned}\right\} \qquad \ldots(V\text{-}14)$$

On simplification (V-14) yields

$$C_2 = (1-t)Y_2 + [(1-t)Y_1 - C_1](1+r) + a(X-r). \qquad \ldots(V\text{-}15)$$

Now, the problem

$$\max_{\{C_1,a\}} \quad V(C_1) + E\{U(C_2)\} \qquad \ldots(V\text{-}16)$$

where C_2 is defined by (V-15), yields necessary and sufficient conditions that are described by the equations (V-4) through (V-7) of the preceding section. To see the effects of taxation, we differentiate (V-4) and (V-5), where C_2 is given by (V-15), to obtain:

$$\begin{bmatrix} & \\ & H \\ & \end{bmatrix} \begin{pmatrix} \dfrac{\partial C_1}{\partial t} \\ \dfrac{\partial a}{\partial t} \end{pmatrix} = \begin{pmatrix} -(1+r)[Y_2 + Y_1(1+r)]E\{U''(C_2)\} \\ [Y_2 + Y_1(1+r)]\ E\{U''(C_2)(X-r)\} \end{pmatrix}, \qquad \ldots(V\text{-}17)$$

where $|H|$ is given by (V-7). From (V-17) we obtain

$$\frac{\partial C_1}{\partial t} = (-)\left(\frac{1}{1-t}\right) [Y_1 + Y_2(1+r)^{-1}] \frac{\partial C_1}{\partial Y_1} ; \qquad \ldots(V\text{-}18)$$

$$\frac{\partial a}{\partial t} = (-)\left(\frac{1}{1-t}\right) [Y_1 + Y_2(1+r)^{-1}] \frac{\partial a}{\partial Y_1} ; \qquad \ldots(V\text{-}19)$$

where $\partial C_1/\partial Y_1$ and $\partial a/\partial Y_1$ are respectively given by equations (A-V-8) and (A-V-9) of the appendix. We have, therefore, the result that <u>a proportional tax on non-asset income reduces both current consumption and the demand for the risky asset if both are superior goods.</u>[8] This is easily explained: The tax reduces exogenously given non-asset incomes and generates pure income effects. If we assume both present consumption and risk-taking to be superior goods, both are reduced. So far as intertemporal consumption substitution is concerned, this result confirms, (since Y_1 and Y_2 do not involve uncertainty), Musgrave's finding that "a tax on income excluding interest does not affect the choice between present and future consumption" ([50], p. 261).

<u>A Proportional Consumption Tax</u>. In this case, the budget constraints are written as

$$\left.\begin{aligned} C_1 &= (1-t)(Y_1 - a - m); \\ C_2 &= (1-t)(Y_2 + a(1+X) + m(1+r)); \end{aligned}\right\} \qquad \ldots(V\text{-}20)$$

where t now denotes a proportional tax on consumption. The solution to the investor's problem, (V-16), where C_2, in view of (V-20), reads

$$C_2 = [(1-t)\{Y_2 + Y_1(1+r) + a(X-r)\} - C_1(1+r)], \qquad \ldots(V\text{-}21)$$

yields the following necessary and, (given the concavity assumptions),

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[8] Alternatively, the assumptions of non-decreasing relative risk-aversion and decreasing absolute risk-aversion are sufficient to yield this result.

sufficient conditions:

$$V'(C_1) - (1+r)E\{U'(C_2)\} = 0; \qquad \ldots(V\text{-}22)$$

$$(1-t)E\{U'(C_2)(X-r)\} = 0. \qquad \ldots(V\text{-}23)$$

Again, upon implicit differentiation, (V-22) and (V-23) yields:

$$\begin{bmatrix} H \end{bmatrix} \begin{pmatrix} \frac{\partial C_1}{\partial t} \\ \frac{\partial a}{\partial t} \end{pmatrix} = \begin{pmatrix} -(1+r)E\{U''(C_2)\tilde{Y}\} \\ (1-t)E\{U''(C_2)(X-r)\tilde{Y}\} \end{pmatrix} ; \qquad \ldots(V\text{-}24)$$

where $|H|$ gives the appropriate second-order condition for the consumption tax case and

$$\tilde{Y} \equiv [Y_2 + (1+r)Y_1 + a(X-r)]. \qquad \ldots(V\text{-}25)$$

From (V-24) it can be seen that

$$\frac{\partial C_1}{\partial t} = (-)\ (\frac{1}{1-t})\ [Y_1 + Y_2(1+r)^{-1}]\ \frac{\partial C_1}{\partial Y_1}, \qquad \ldots(V\text{-}26)$$

and

$$\frac{\partial a}{\partial t} = (\frac{a}{1-t})\ [1 - \{Y_1 + Y_2(1+r)^{-1}\}\ \frac{1}{a}\ \frac{\partial a}{\partial Y_1}], \qquad \ldots(V\text{-}27)$$

where $\partial C_1/\partial Y_1$ and $\partial a/\partial Y_1$ are respectively given by equations (A-V-10) and (A-V-11) of the appendix. Clearly, if both consumption and the risky asset are superior goods, both current consumption and <u>private</u> risk-taking[9] are reduced by a proportional consumption tax. Total risk-taking, however, would increase, remain constant or decrease as the elasticity of the demand for the risky asset with respect to the present value of non-asset income is smaller than, equal to or greater than unity. However, given the appended discussion on risk-aversion and income

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[9]Recall that in Chapter IV we have defined a(1-t) as the measure of private risk-taking.

elasticities (Sections A.V.2 and A.V.4), we can formally interpret (V-27) as follows:

An increase in the proportional consumption tax

(a) stimulates (total) risk-taking where absolute risk-aversion is decreasing, relative risk-aversion is increasing and the income elasticity of consumption is at least as great as that of the risky asset demand;

(b) has no effect on risk-taking where relative risk-aversion is constant and the income elasticity of consumption equals that of the risky asset demand;[10]

(c) discourages risk-taking where relative risk-aversion is constant (decreasing) and the income elasticity of consumption is smaller (no larger) than that of the risky asset demand.

These results are in conflict with the conventional analysis which does not consider uncertainty. For instance, Musgrave concluded: "A tax on consumption, present and future, is thus equivalent to a tax on income excluding interest. This equivalence holds true provided that there is no saving without subsequent consumption ..." ([50], p. 262). Unlike the case of proportional taxation of non-asset income, a proportional consumption tax generates a substitution effect when we introduce uncertainty via capital risk. The substitution effect encourages risk-taking (see equation (V-27)), i.e.,

$$\left(\frac{a}{1-t}\right) = \left(\frac{\partial a}{\partial t}\right)_{EF(C_1,C_2) = \text{constant}} > 0. \qquad \ldots(V\text{-}28)$$

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[10] This result can be regarded as a precise characterisation of Feldstein's result [20]. Feldstein provided a counter-example to the previous literature by showing that, in the model he considered, a proportional tax has no effect on risk-taking. In the appendix (Section A.V.5) we point out the shortcomings of his demonstration.

This effect arises due to the fact that while the tax reduces current consumption, it also reduces the probability of large gains (and large losses) in future consumption (due to loss offsets), and consequently households allocate relatively more of their portfolio to the risky asset such that future consumption may not turn out to be too small. Thus the substitution effect leaves expected utility at the same level as before the tax. However, the income effect, as usual, tends to reduce risk-taking.

A Proportional Investment Income Tax. The constraints are now modified as follows

$$C_1 = Y_1 - (a + m),$$

$$C_2 = Y_2 + (a + m) + (1-t)(aX+mr).$$

Or simplifying,

$$C_2 = Y_2 + (Y_1 - C_1) + (1-t)\{(Y-C_1)r + a(X-r)\}. \qquad \ldots(V\text{-}29)$$

As a solution to (V-16) we obtain the following first-order conditions:

$$V'(C_1) - (r^*)E\{U'(C_2)\} = 0; \qquad \ldots(V\text{-}30)$$

$$(1-t)\ E\{U'(C_2)(X-r)\} = 0; \qquad \ldots(V\text{-}31)$$

where $r^* \equiv [1+r(1-t)]$ and C_2 is given by (V-29). Proceeding in the manner of the previous cases, we obtain

$$\frac{\partial C_1}{\partial t} = (-)\left(\frac{rE\{U'(C_2)\}}{V''(C_1)} + \left[\frac{(Y_1-C_1)r}{(r^*)} - \frac{rE\{U'(C_2)\}}{V''(C_1)}\right]\frac{\partial C_1}{\partial Y_1}\right), \qquad \ldots(V\text{-}32)$$

or using (V-30)

$$\frac{\partial C_1}{\partial t} = \left(\frac{rC_1}{r^*}\right)\left[1 + \frac{V'(C_1)}{V''(C_1)C_1}\right]\frac{\partial C_1}{\partial Y_1} - \left(\frac{rC_1}{r^*}\right)\left[\frac{Y_1}{C_1}\ \frac{\partial C_1}{\partial Y_1} + \frac{V'(C_1)}{V''(C_1)C_1}\right], \qquad \ldots(V\text{-}33)$$

and

$$\frac{\partial a}{\partial t} = \left(\frac{a}{1-t}\right) \left[1 - \frac{r(1-t)}{(r^*)} \left(\frac{Y_1}{a} \frac{\partial a}{\partial Y_1}\right)\right]$$

$$+ \left(\frac{rC_1}{r^*}\right) \left[1 + \frac{V'(C_1)}{V''(C_1)C_1}\right] \frac{\partial a}{\partial Y_1}, \qquad \ldots(V\text{-}34)$$

where $\partial C_1/\partial Y_1$ and $\partial a/\partial Y_1$ are respectively given by equations (A-V-12) and (A-V-13) of the appendix. First, from equations (V-33) and (V-34) we note that where the return on the riskless investment is zero (which, in an intertemporal model implies a zero marginal rate of time preference at the optimum) the effect of a proportional investment income tax is to leave current consumption unchanged and to increase (total) risk-taking.[11] Private risk-taking in this case remains unchanged. Under logarithmic preferences of the type (V-12), (V-33) reduces to (since $[V''(C_1)C_1/V'(C_1)] = 1$)

$$\frac{\partial C_1}{\partial t} = \left(\frac{rC_1}{r^*}\right) \left[1 - \frac{Y_1}{C_1} \frac{\partial C_1}{\partial Y_1}\right], \qquad \ldots(V\text{-}35)$$

and the second term on the r.h.s. of (V-34) drops out. We therefore obtain the following: <u>For investors whose preferences can be described by a logarithmic utility function, an increase in the proportional tax on investment income will reduce, leave unchanged or increase current consumption as the income elasticity of consumption is greater than, equal to or smaller than unity; this tax will also encourage risk-taking if the income elasticity of the demand for the risky asset is at most as great as unity. Private risk-taking will however decrease</u>. Notice that

[11] This has also been pointed out by Sandmo [62].

although for an income elasticity of asset demand greater than unity risk-taking may decrease, it does not seem very likely given reasonable magnitudes of t and r.[12] It is interesting to note that the effects of an investment income tax on risk-taking for logarithmic preferences are identical to those obtained for a general utility function in a single-period framework.

In the general case where the utility function is given by (V-1) and $r \neq 0$, the interpretation of (V-33) and (V-34) would run as follows: An increase in the proportional tax on investment income

(a) discourages current consumption where the investor exhibits non-decreasing relative risk-aversion, consumes a constant fraction of income (i.e., $C_1 = kY_1$, where Y_1 is the total income in that period, implying $(Y_1/C_1)\ (\partial C_1/\partial Y_1) = 1$) and has an elasticity of marginal utility of consumption at least as great as unity;

(b) encourages risk-taking where the investor exhibits decreasing absolute risk-aversion, invests a non-increasing proportion of income into the risky asset as income increases and has an elasticity of marginal utility of consumption at least as great as unity. The difficulty in obtaining any straight-forward results in the case of an investment income tax can be intuitively explained as follows. A tax on investment income is effectively a tax on future consumption and hence the substitution effect would work in favour of current consumption and risk-taking (see the first terms in (V-32) and (V-34)). However, these

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[12] For instance, if r = 10% and t = 50% this would require a value of the income elasticity of the risky asset demand greater than 21 for risk-taking to decrease. Also notice that in an intertemporal context, an elasticity greater than unity may not correspond to the hypothesis of decreasing relative risk-aversion. See the appendix discussion in Section A.V.2.

stimulatory effects may or may not be outweighed by the income effects.

One other special case may be of interest. This is the case for an individual, who, before the tax change, is neither a borrower nor a saver, i.e., $C_1 = Y_1$. In such a case, it is seen from (V-32) that an increase in the tax rate stimulates current consumption. In a somewhat different context, Professors Feldstein and Tsiang [21] have interpreted such a result as indicating that if tax decreases are directed towards the low-income groups in which saving is non existent or relatively low, a positive stimulus to saving is more likely.

V.3 Progressive Taxation

In this section we derive results similar to those of the preceding section for the case of linearly progressive taxes. As in the last chapter, the tax schedule is linear with a marginal tax rate which applies both above and below a constant exemption level, K.

A Progressive Non-Asset Income Tax. The budget constraints are now given by

$$C_1 = (1-t)(Y_1-K) + K - (a+m)$$

$$C_2 = (1-t)(Y_2-K) + K + a(1+X) + m(1+r)$$

where t is the marginal rate of tax on non-asset income and K is the level of exemption as defined above. Substituting for m in the expression for C_2, we have

$$C_2 = (1-t)[Y_1(1+r)+Y_2] + Kt(2+r) + a(X-r) - C_1(1+r). \qquad \ldots(V\text{-}36)$$

The necessary and sufficient conditions to the investor's problem given by (V-16) and (V-36) are again described by equations (V-4) and (V-5) where C_2 is now given by (V-36). Proceeding in the manner of the preceding

section we obtain:

$$\frac{\partial C_1}{\partial t} = (-)\ (\frac{1}{1-t})\ \{[Y_1+Y_2(1+r)^{-1}] - K[1+(1+r)^{-1}]\}\frac{\partial C_1}{\partial Y_1}; \qquad \ldots(V\text{-}37)$$

$$\frac{\partial a}{\partial t} = (-)\ (\frac{1}{1-t})\ \{[Y_1 + Y_2(1+r)^{-1}] - K[1 + (1+r)^{-1}]\}\frac{\partial a}{\partial Y_1}; \qquad \ldots(V\text{-}38)$$

where $(\partial C_1/\partial Y_1)$ and $(\partial a/\partial Y_1)$ are respectively given by equations (A-V-8) and (A-V-9) of the appendix. These results can be stated as follows:[13] Where both current consumption and the risky asset are superior goods, a linearly progressive tax on non-asset incomes decreases or increases both current consumption and risk-taking as the present value of non-asset incomes exceeds (in which case the investor is a net tax-payer) or falls short of (in which case the investor is a net receiver) the present value of exemptions on such incomes.

Notice that the stimulatory effects on consumption and risk-taking in the latter case are clearly the implications of a negative (non-asset) income tax, and hence these results accord well with intuition. Also notice that, as in the case of a proportional non-asset income tax, no substitution takes place between present and future consumption.

A Progressive Consumption Tax. Linearly progressive taxation of consumption implies the following budget constraints for the individual:

$$C_1 = (1-t)(Y_1 - a - m) + Kt;$$

$$C_2 = (1-t)[Y_2 + a\ (1+X) + m(1+r)] + Kt;$$

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[13] An alternative interpretation in terms of risk-aversion hypotheses is given by the footnote 8.

where now t is the marginal consumption tax rate and K is the exemption parameter. As before, we can simplify the above constraints and obtain

$$C_2 = [(1-t)\{Y_2 + Y_1(1+r) + a(X-r)\} - C_1(1+r) + Kt(2+r)]. \qquad \ldots(V\text{-}39)$$

Also notice that equations (V-22) and (V-23) in conjunction with (V-39) characterise an optimum for the individual. The effects of a change in t on the investor's saving and risk-taking behaviour are given by:

$$\frac{\partial C_1}{\partial t} = (-)\left(\frac{1}{1-t}\right)\{[Y_1 + Y_2(1+r)^{-1}] - K[1+(1+r)^{-1}]\}\ \frac{\partial C_1}{\partial Y_1}; \qquad \ldots(V\text{-}40)$$

$$\frac{\partial a}{\partial t} = \left(\frac{a}{1-t}\right)\ [1 - \{Y_1+Y_2(1+r)^{-1}\}\ \frac{1}{a}\ \frac{\partial a}{\partial Y_1}\ ] + \frac{2K}{(1-t)}\ \frac{\partial a}{\partial Y_1}\ ; \qquad \ldots(V\text{-}41)$$

where $(\partial C_1/\partial Y_1)$ and $(\partial a/\partial Y_1)$ are respectively given by equations (A-V-10) and (A-V-11) of the appendix (where C_2 is to be read in the manner of (V-39)).

The effect of the consumption tax on consumption is the same as in the case of taxation of non-asset income: for $\partial C_1/\partial Y_1 > 0$,

$$\frac{\partial C_1}{\partial t} \lesseqgtr 0 \text{ as } K \lesseqgtr \left[\frac{Y_1 + Y_2\ (1+r)^{-1}}{1 + (1+r)^{-1}}\right]; \qquad \ldots(V\text{-}42)$$

(i.e., as K $\lesseqgtr$ the equivalent "permanent income stream"). Also note that if there is no investment in the risky asset, the individual is a net receiver if K exceeds his permanent income, and for K equal to his permanent income, he is not affected by the taxation policy. Therefore, the stimulatory effects on current consumption, as argued in the case of progressive taxation of non-asset income, is not surprising. We have,

therefore, the following result: Where consumption is a superior good,[14] linearly progressive taxation of consumption decreases, leaves unchanged or increases current consumption where the equivalent permanent income stream is greater than, equal to or smaller than the exemption level.

In order to interpret the effects on risk-taking we refer to the appendix discussion on relative risk-aversion and income elasticities (Section A.V.4). However, in view of (V-41), we note that unlike the purely proportional tax on consumption, private risk-taking may not be reduced by a linearly progressive tax on consumption. The effects on social risk-taking is stated as follows: An increase in the marginal tax rate in a system with linearly progressive taxation of consumption stimulates risk-taking where (a) absolute risk-aversion is decreasing, relative risk-aversion is non-decreasing and the income elasticity of consumption is at least as great as that of the risky asset demand; or, alternatively, where (b) relative risk-aversion is decreasing, the marginal tax rate is at most 50% and the income elasticity of consumption is no larger than that of the risky asset demand.[15]

A Progressive Investment Income Tax. The appropriate constraints for a progressive tax on investment income are

$$\left.\begin{aligned} C_1 &= (Y_1 - a - m) \\ C_2 &= \{Y_2 + a(1+X) + m(1+r) - t[aX + mr - K] \end{aligned}\right\} ,$$

or,

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[14] See footnote 8.

[15] As a matter of fact, this last statement specifies the conditions for $(t/a)(\partial a/\partial t)$ to be positive.

$$C_2 = [Y_2 + (Y_1 - C_1) + Kt - (1-t)\{(Y_1 - C_1)r + a(X-r)\}], \qquad \ldots(V\text{-}43)$$

where t is the marginal investment income tax rate and K is the exemption level. Again an intertemporal equilibrium is described by the equations (V-30), (V-31) and (V-43). As before, we obtain:

$$\frac{\partial C_1}{\partial t} = \left(\frac{rC_1}{r^*}\right)\left[1 + \frac{V'(C_1)}{V''(C_1)C_1}\right]\frac{\partial C_1}{\partial Y_1} - \left(\frac{rC_1}{r^*}\right)\left[\frac{V'(C_1)}{V''(C_1)C_1} + \frac{Y_1}{C_1}\frac{\partial C_1}{\partial Y_1}\right]$$

$$+ \left(\frac{K}{r^*}\right)\frac{\partial C_1}{\partial Y_1}; \qquad \ldots(V\text{-}44)$$

$$\frac{\partial a}{\partial t} = \left(\frac{a}{1-t}\right)\left[1 - \frac{r(1-t)}{(r^*)}\left(\frac{Y_1}{a}\frac{\partial a}{\partial Y_1}\right)\right] + \left(\frac{rC_1}{r^*}\right)\left[1 + \frac{V'(C_1)}{V''(C_1)C_1}\right]\frac{\partial a}{\partial Y_1}$$

$$+ \left(\frac{K}{r^*}\right)\frac{\partial a}{\partial Y_1}; \qquad \ldots(V\text{-}45)$$

where $(r^*) \equiv [1 + r(1-t)]$, and $\partial C_1/\partial Y_1$, $\partial a/\partial Y_1$ are respectively given by equations (A-V-12) and (A-V-13) of the appendix.

Where the rate of return on the riskless investment is zero, unlike the proportional investment income tax case, current consumption, private (and social) risk-taking are all increased by linearly progressive taxation of investment income. The existence of a lump-sum exemption on investment income generates income effects thus stimulating both current consumption and private risk-taking; social risk-taking, however, is increased when this exemption-generated income effect reinforces the positive substitution effect. The above interpretation requires that both current consumption and the risky asset are superior goods.

In general, however, the effects of taxation are somewhat difficult to interpret. Specifically, even assuming a strictly pro-

portional consumption function (e.g., $C_1 = kY_1$, $k > 0$), equation (V-44) does not allow any straight-forward interpretation. However, for the case of logarithmic preferences the first term in (V-44) and the second term in (V-45) drop out. We are, therefore, led to the following conclusion: For investors whose preferences can be described by a logarithmic utility function, linearly progressive taxation of investment income increases current consumption (or, risk-taking) where the individual allocates a non-increasing proportion of income to consumption (or, to the risky asset) as income increases. Due to the last term in (V-45), our previous remark that an income elasticity of asset demand greater than unity may not decrease risk-taking is further strengthened. Similarly, an income elasticity of consumption in excess of unity is not sufficient to reduce current consumption.

There is one other case where the effect of the tax on risk-taking is unambiguous. This does not require the assumption of logarithmic preferences (see equation (V-45)): Linearly progressive taxation of investment income encourages risk-taking where the investor exhibits decreasing absolute risk-aversion, invests a non-increasing proportion of income in the risky asset as income increases and has an elasticity of marginal utility of consumption at least as great as unity.

V.4 Summary and Conclusion

The main results of this chapter can conveniently be summarised in the following tables.

Table V.1 Effects of Taxation on Current Consumption

Type of tax Base	Type of Taxation: Proportional	Type of Taxation: Progressive
Non-Asset Income	NEGATIVE if $\frac{\partial C_1}{\partial Y_1} > 0$.	POS, ZERO, OR NEG as $K \gtrless \left(\frac{Y_1 + Y_2(1+r)^{-1}}{1 + (1+r)^{-1}}\right)$.
Consumption	NEGATIVE if $\frac{\partial C_1}{\partial Y_1} > 0$.	POS, ZERO, OR NEG as $K \gtrless \left(\frac{Y_1 + Y_2(1+r)^{-1}}{1 + (1+r)^{-1}}\right)$.
Investment Income	(a) ZERO if r=0; or (b) POS, ZERO or NEG as $\frac{Y_1}{C_1} \frac{\partial C_1}{\partial Y_1} \lessgtr 1$ and the utility is logarithmic; or (c) NEGATIVE if (i) $\partial R(C_2)/\partial C_2 \geq 0$, (ii) $(Y_1/C_1)(\partial C_1/\partial Y_1) = 1$ and, (iii) The boundness hypothesis is satisfied. 16	POSITIVE if (a) $r = 0$ and $\partial C_1/\partial Y_1 > 0$; or (b) $(Y_1/C_1)(\partial C_1/\partial Y_1) \leq 1$ and the utility is logarithmic.

[16] Recall that $[(-)\ V''(C_1)C_1/V'(C_1)] \geq 1$ denotes the boundedness hypothesis.

Table V.2 Effects of Taxation on Total (Social) Risk-Taking

Type of tax Base	Type of Taxation: Proportional	Type of Taxation: Progressive
Non-Asset Income	NEGATIVE if $\frac{\partial a}{\partial Y_1} > 0$.	POS, ZERO, OR NEG as $K \gtrless \left(\frac{Y_1+Y_2(1+r)^{-1}}{1+(1+r)^{-1}}\right)$.
Consumption	(a) POSITIVE if (i) $\partial a/\partial Y_1 > 0$, (ii) $\partial R(C_2/\partial C_2 > 0$, and (iii) $(Y_1/a)(\partial a/\partial Y_1) \leq (Y/C_1)(\partial C/\partial Y_1)$; (b) ZERO if (i) $\partial R(C_2)/\partial C_2 = 0$, and (ii) $(Y_1/a)(\partial a/\partial Y_1) = (Y_1/C_1)(\partial C_1/\partial Y_1)$; (c) NEGATIVE if (i) $\frac{\partial R(C_2)}{\partial C_2} < 0$ and $(\frac{Y_1}{a})(\frac{\partial a}{\partial Y_1}) \geq (\frac{Y_1}{C_1})(\frac{\partial C_1}{\partial Y_1})$. or, (ii) $\frac{\partial R(C_2)}{\partial C_2} = 0$ and $\frac{Y_1}{a}\frac{\partial a}{\partial Y_1} > \frac{Y_1}{C_1}\frac{\partial C_1}{\partial Y_1}$.	POSITIVE if (a)(i) $\partial a/\partial Y_1 > 0$, (ii) $\partial R(C_2)/\partial C_2 \geq 0$, and (iii) $(Y_1/a)(\partial a/\partial Y_1) \leq (Y_1/C_1)(\partial C_1/\partial Y_1)$; or, (b)(i) $\partial R(C_2)/\partial C_2 < 0$, (ii) $(Y_1/a)(\partial a/\partial Y_1) \geq (Y_1/C_1)(\partial C_1/\partial Y_1)$, and (iii) $t \leq 1/2$.

continued ...

Table V.2 (continued)...

	POSITIVE	POSITIVE
Investment Income	if (a) $r=0$; or (b) $(Y_1/a)(\partial a/\partial Y_1) \leq 1$ and the utility is logarithmic; or (c) (i) $\partial a/\partial Y_1 > 0$, (ii) $(Y_1/a)(\partial a/\partial Y_1) \leq 1$, and (iii) the Boundedness hypothesis is satisfied.	if (a) $r=0$ and $\partial a/\partial Y_1 > 0$; or (b) $(Y_1/a)(\partial a/\partial Y_1) \leq 1$ and the utility is logarithmic; or (c) (i) $\partial a/\partial Y_1 > 0$, (ii) $(Y_1/a)(\partial a/\partial Y_1) \leq 1$, and (iii) the Boundedness hypothesis is satisfied.

Table V.3 Effects of Taxation on Private Risk-Taking*

Type of tax Base	Type of Tax: Proportional	Type of Tax: Progressive
Consumption	NEGATIVE if $\partial a/\partial Y_1 > 0$.	AMBIGUOUS
Investment Income	(a) ZERO if $r = 0$; (b) NEGATIVE if the utility function is logarithmic.	POSITIVE if $r = 0$ and $\partial a/\partial Y_1 > 0$.

*The present formulation does not yield any prediction of the effects of a non-asset income tax on private risk-taking.

From Table V.1 it is clear that the effects of both proportional and progressive taxation of non-asset income and of consumption are to discourage current consumption for typical individuals (since we would expect the exemption level to be smaller than "permanent income"). These results, however, do not involve any intertemporal substitution of consumption and are pure income effects. On the other hand, the effect of an investment income tax (both proportional and progressive) on consumption is less straight-forward. Specifically, this tax -- essentially on future consumption -- encourages a substitution in favour of current consumption, but the income effects render the total effect somewhat ambiguous. Nevertheless, if we regard an income elasticity to consumption of less than or equal to unity as typical, this tax stimulates current consumption (this, however, requires logarithmic preferences).

Proportional consumption taxation or an investment income tax is most likely to reduce private risk-taking (see Table V.3).

Apart from taxation of non-asset income and where the return on the riskless investment is zero, the results regarding total risk-taking are considerably more ambiguous. Consumption and investment income taxes, although they encourage a substitution in favour of risk-taking, are difficult to interpret. Alternative sets of conditions for these results to be unambiguous are detailed in Table V.2.

The main conclusion, therefore, is that while hypotheses on risk-aversion functions are more or less adequate in determining the effects of taxation on risk-taking in single-period analysis,[17] they are no

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[17] Compare, for instance, the results of Chapter IV(Tables IV-1 and IV-2).

longer adequate in an intertemporal context. In particular, we frequently require the knowledge of the various income elasticities (and their relative magnitudes) in addition to assumptions on risk-aversion. However, these results can still be regarded as meaningful in that, at least in principle, the magnitudes of these income elasticities can be determined.

Appendix to Chapter V[18]

A.V.1 Proof of the Results Stated in Section V.1: In this section we prove the results stated in the text (Section V.1):

(a) $H > 0 \iff 0 < \partial C_1/\partial Y_1 < 1$, where absolute risk-aversion is decreasing and relative risk-aversion is non-decreasing;

(b) $\partial a/\partial Y_1 > 0$ where absolute risk-aversion is decreasing. We will refer to them as propositions (a) and (b) respectively.

Before outlining the proofs, let us state two important results:

Lemma A.V.1 In the context of the model described in Section V.1 of the text, $E\{U''(C_2)(X-r)\} \gtreqless 0$ as absolute risk-aversion decreases, remains constant, or increases as C_2 increases;

Lemma A.V.2 Under the conditions of lemma A.V.1, $E\{U''(C_2)(X-r)C_2\} \lesseqgtr 0$ as relative risk-aversion increases, remains constant, or decreases as C_2 increases.

The proofs of these lemmas, being similar to the one given in the appendix to Chapter IV(Section A.IV.1), are omitted.

In the text we have seen that the necessary and sufficient conditions for an interior optimum of the problem given by (V-3) are described by equations (V-4) and (V-5) of the text. Now, differentiating (V-4) and (V-5) with respect to Y_1, we obtain

$$\begin{bmatrix} H \end{bmatrix} \begin{pmatrix} \dfrac{\partial C_1}{\partial Y_1} \\ \dfrac{\partial a}{\partial Y_1} \end{pmatrix} = \begin{pmatrix} (1+r)^2 E\{U''(C_2)\} \\ -(1+r)E\{U''(C_2)(X-r)\} \end{pmatrix}, \qquad \ldots(A\text{-}V\text{-}1)$$

where $|H|$ is defined by (V-7) of the text.

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[18] I have benefitted from my discussion with Abhijit Sen on some of the appendix results.

Proof of Proposition(b). From (A-V-1), we have

$$\frac{\partial a}{\partial Y_1} = \frac{1}{H} \; [(-)(1+r)V''(C_1)E\{U''(C_2)(X-r)\}]. \qquad \ldots(A\text{-}V\text{-}2)$$

Since $V''(C_1) < 0$, lemma A.V.1 guarantees that $\partial a/\partial Y_1 > 0$. Q.E.D.

Proof of Proposition(a). First we prove the necessity part of this result, i.e., $H > 0 \Rightarrow 0 < \partial C_1/\partial Y_1 < 1$. However, in order to show this we require the following:

Lemma A.V.3 In the conext of the model described in Section V.1 of the text,

$$Z \equiv [E\{U''(C_2(X-r)^2\} + (1+r)E\{U''(C_2)(X-r)\}] < 0,$$

where absolute risk-aversion is decreasing and relative risk-aversion is non-decreasing.

Proof of the Lemma: Multiplying through by $[Y_2(1+r)^{-1} + (a+m)]$, which is positive, we have

$$[Y_2(1+r)^{-1} + (a+m)]\; Z = [Y_2(1+r)^{-1} + m]\; E\{U''(C_2)(X-r)^2\}$$
$$+E\{U''(C_2)(X-r)[a(X-r) + Y_2 + (1+r)(a+m)]\}. \ldots(A\text{-}V\text{-}3)$$

Recalling that $C_2 = [Y_2 + a(1+X) + m(1+r)$, we can rearrange (A-V-3) as follows:

$$[Y_2(1+r)^{-1} + (a+m)]\; Z = [Y_2(1+r)^{-1} + m]\; E[U''(C_2)(X-r)^2]$$
$$+E\{U''(C_2)(X-r)C_2\}. \qquad \ldots(A\text{-}V\text{-}4)$$

Clearly, therefore, given lemma A.V.2, Z is negative. Hence the lemma.

From (A-V-1), we have

$$\frac{\partial C_1}{\partial Y_1} = \{1 - \frac{1}{H} \; [V''(C_1)E\{U''(C_2)(X-r)^2\}]\}. \qquad \ldots(A\text{-}V\text{-}5)$$

If $H > 0$, $\partial C_1/\partial Y_1 < 1$. However, for $\partial C_1/\partial Y_1 > 0$ at the same time, we require

$$H > V''(C_1)E\{U''(C_2)(X-r)^2\}.$$

Noticing the determinantal value of H, this requires

$$[E\{U''(C_2)(X-r)^2\}E\{U''(C_2)\} - (E\{U''(C_2)(X-r)\})^2] > 0.$$

Adding and subtracting $[(1+r)E\{U''(C_2)\}E\{U''(C_2)(X-r)\}]$ to the above expression would imply

$$ZE\{U''(C_2)\} - E\{U''(C_2)(X-r)\}E\{(1+X)U''(C_2)\} > 0.$$

Given lemma A.V.3, the first term is positive. Further, since $X > -1$, the last term is negative by virtue of lemma A.V.1. Therefore, the entire expression is positive. This completes the necessity argument.

Now, we show sufficiency, i.e., $0 < \partial C_1/\partial Y_1 < 1 \Rightarrow H > 0$. Again, since $V''(C_1)E\{U''(C_2)(X-r)^2\} > 0$, equation (A-V-5) implies that $\partial C_1/\partial Y_1 < 1$ obviously requires $H > 0$. Q.E.D.

<u>A.V.2. Relative Risk-Aversion and Income Elasticities.</u> In this section we derive some basic results on the relationship between income elasticities of consumption and of the risky asset demand and hypotheses on relative risk-aversion.

Using the definition of C_2 as given by equation (V-2) of the text, lemma A.V.2 states that

$$[Y_2 + Y_1(1+r)]\ E\{U''(C_2)(X-r)\} + aE\{U''(C_2)(X-r)^2\}$$
$$\lessgtr C_1(1+r)E\{U''(C_2)(X-r)\} \quad \text{as} \quad \frac{\partial R(C_2)}{\partial C_2} \gtrless 0,$$

where $R(C_2)$ denotes relative risk-aversion (see the expression (V-11) of the text). Multiplying through by

$$\left(\frac{1}{a(1+r)E[U''(C_2)(X-r)]}\right)\frac{\partial a}{\partial Y_1},$$

which is positive where absolute risk-aversion, $A(C_2)$, is decreasing, we obtain

$$\frac{1}{a}\frac{\partial a}{\partial Y_1}\{Y_1 + Y_2(1+r)^{-1}\} + \frac{E\{U''(C_2)(X-r)^2\}}{(1+r)E\{U''(C_2)(X-r)\}}\left(\frac{\partial a}{\partial Y_1}\right)$$

$$\lessgtr \left(\frac{C_1}{a}\frac{\partial a}{\partial Y_1}\right) \text{ as } \frac{\partial R(C_2)}{\partial C_2} \gtrless 0 \text{ and } \frac{\partial A(C_2)}{\partial C_2} < 0. \qquad \ldots(\text{A-V-6})$$

In view of equations (A-V-2) and (A-V-5), this reduces to

$$\left(\frac{1}{a}\frac{\partial a}{\partial Y_1}\right)\{Y_1 + Y_2(1+r)^{-1}\} \lessgtr \left[1 + \frac{C_1}{a}\frac{\partial a}{\partial Y_1} - \frac{\partial C_1}{\partial Y_1}\right]. \qquad \ldots(\text{A-V-7})$$

We can state this result as follows:

(a) where absolute risk-aversion is decreasing and relative risk-aversion is increasing

$$\left(\frac{1}{a}\frac{\partial a}{\partial Y_1}\right)\{Y_1 + Y_2\ (1+r)^{-1}\} < 1 \text{ if } \left(\frac{Y_1}{a}\frac{\partial a}{\partial Y_1}\right) \leq \left(\frac{Y_1}{C_1}\frac{\partial C_1}{\partial Y_1}\right);$$

(b) where relative risk-aversion is constant[19]

$$\left(\frac{1}{a}\frac{\partial a}{\partial Y_1}\right)\{Y_1 + Y_2(1+r)^{-1}\} = 1 \text{ iff } \left(\frac{Y_1}{a}\frac{\partial a}{\partial Y_1}\right) = \left(\frac{Y_1}{C_1}\frac{\partial C_1}{\partial Y_1}\right);$$

(c) where relative risk-aversion is decreasing

$$\left(\frac{1}{a}\frac{\partial a}{\partial Y_1}\right)\{Y_1 + Y_1(1+r)^{-1}\} > 1 \text{ if } \left(\frac{Y_1}{a}\frac{\partial a}{\partial Y_1}\right) \geq \left(\frac{Y_1}{C_1}\frac{\partial C_1}{\partial Y_1}\right).$$

Also note that for constant $R(C_2)$, $\left(\frac{1}{a}\frac{\partial a}{\partial Y_1}\right)\{Y_1 + Y_2(1+r)^{-1}\} > 1$ if $\left(\frac{Y_1}{a}\frac{\partial a}{\partial Y_1}\right) > \left(\frac{Y_1}{C_1}\frac{\partial C_1}{\partial Y_1}\right)$.

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[19] As already mentioned in Chapter IV, it is impossible to have non-decreasing absolute risk-aversion where relative risk-aversion is non-increasing.

A.V.3 Taxation and Income Elasticities. In this section we outline the derivation of income elasticities of consumption and the risky asset demand in the context of alternative tax structures.[20]

Non-Asset Income Tax. Differentiating equations (V-4) and (V-5) w.r.t. Y_1 we obtain:

$$\frac{\partial C_1}{\partial Y_1} = (1-t)\{1 - (\frac{1}{H})(V''(C_1)E\{U''(C_2)(X-r)^2\})\}; \qquad \ldots(A\text{-}V\text{-}8)$$

$$\frac{\partial a}{\partial Y_1} = \frac{1}{H}\,[\,(-)\;(1+r)(1-t)V''(C_1)E\{U''(C_2)(X-r)\}]; \qquad \ldots(A\text{-}V\text{-}9)$$

where $|H|$ is given by (V-7) of the text.

Consumption Tax. Similarly from (V-22) and (V-23) we obtain:

$$\frac{\partial C_1}{\partial Y_1} = (1-t)\{1 - (\frac{1}{H})\;(1-t)^2\;E\{U''(C_2)(X-r)^2\}V''(C_1)\}; \qquad \ldots(A\text{-}V\text{-}10)$$

$$\frac{\partial a}{\partial Y_1} = \frac{1}{H}\,[(-)(1-t)^2\;(1+r)V''(C_1)E\{U''(C_2)(X-r)\}]; \qquad \ldots(A\text{-}V\text{-}11)$$

where $|H|$ is implicitly defined by (V-24) of the text.

Investment Income Tax. As in the previous cases, equations (V-30) and (V-31) yield

$$\frac{\partial C_1}{\partial Y_1} = [1 - (\frac{1}{H})(1-t)^2\;V''(C_1)E\{U''(C_2)(X-r)^2\}], \qquad \ldots(A\text{-}V\text{-}12)$$

and

$$\frac{\partial a}{\partial Y_1} = \frac{1}{H}\,[(-)(r^*)(1-t)V''(C_1)E\{U''(C_2)(X-r)\}], \qquad \ldots(A\text{-}V\text{-}13)$$

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[20] It can easily be checked that the income effects $\partial a/\partial Y_1$ and $\partial C_1/\partial Y_1$ are independent of whether a tax is proportional or progressive (in the sense defined here). However, they are dependent on the type of tax; e.g., whether it is a tax on consumption or on investment income.

where $(r^*) \equiv [1+r(1-t)]$ and H is given by the appropriate second-order conditions.

A.V.4 Relative Risk-Aversion, Income Elasticities and Taxation. In Section A.V.2 we derived some results on the relationship between hypotheses on relative risk-aversion and income elasticities of consumption and the risky asset demand. We now ask how these results are affected by the introduction of alternative taxes.

Proportional Consumption Tax. In this case C_2 is given by (V-21) and consequently, the inequality (A-V-6) is modified into

$$\left(\frac{(1-t)}{a}\frac{\partial a}{\partial Y_1}\right)\{Y_1 + Y_2(1+r)^{-1}\} + \frac{(1-t)E\{U''(C_2)(X-r)^2\}}{(1+r)E\{U''(C_2)(X-r)\}}\left(\frac{\partial a}{\partial Y_1}\right)$$

$$\lessgtr \left(\frac{C_1}{a}\frac{\partial a}{\partial Y_1}\right). \qquad \text{...(A-V-14)}$$

Again, in view of equations (A-V-10) and (A-V-11), we obtain

$$\left(\frac{1}{a}\frac{\partial a}{\partial Y_1}\right)\{Y_1 + Y_2(1+r)^{-1}\} \lessgtr \left[1+\left(\frac{1}{1-t}\right)\left\{\frac{C_1}{a}\frac{\partial a}{\partial Y_1} - \frac{\partial C_1}{\partial Y_1}\right\}\right].$$

$$\text{...(A-V-15)}$$

Comparing (A-V-15) to (A-V-7) it is evident that the results of Section A-V-2 are all valid for a proportional consumption tax.

Progressive Consumption Tax. C_2 now is given by equation (V-39) of the text and, proceeding in the above manner, we obtain

$$\left(\frac{1}{a}\frac{\partial a}{\partial Y_1}\right)\{Y_1 + Y_2(1+r)^{-1}\} + \frac{Kt(2+r)}{a(1+r)(1-t)}\left(\frac{\partial a}{\partial Y_1}\right)$$

$$\lessgtr \left[1 + \left(\frac{1}{1-t}\right)\left\{\frac{C_1}{a}\frac{\partial a}{\partial Y_1} - \frac{\partial C_1}{\partial Y_1}\right\}\right] \qquad \text{...(A-V-16)}$$

instead of (A-V-15). In view of the second term on the l.h.s. of the inequality, the preceding results are modified: In the presence of a

linearly progressive tax on consumption,

$$(\frac{1}{a}\frac{\partial a}{\partial Y_1})\{Y_1 + Y_2(1+r)^{-1}\} < 1 \text{ if } (\frac{Y_1}{a}\frac{\partial a}{\partial Y_1}) \leq (\frac{Y_1}{C_1}\frac{\partial C_1}{\partial Y_1})$$

where absolute risk-aversion is decreasing and relative risk-aversion is non-decreasing. Also note that for decreasing relative risk-aversion we can state the following:

$$\frac{Kt(2+r)}{(1+r)(1-t)}(\frac{1}{a}\frac{\partial a}{\partial Y_1}) > [1 - (\frac{1}{a}\frac{\partial a}{\partial Y_1})\{Y_1 + Y_2(1+r)^{-1}\}]$$
$$\text{if } (\frac{Y_1}{a}\frac{\partial a}{\partial Y_1}) \geq (\frac{Y_1}{C_1}\frac{\partial C_1}{\partial Y_1}).$$

Investment and Non-Asset Income Taxes. Proceeding in this manner one can obtain similar results for investment and non-asset income taxes. But this is not pursued here since we do not use such results in order to evaluate any of our results.

A.V.5 Feldstein on Taxation and Risk-Taking. In the context of a two-period model, Feldstein [20] assumes that the second period uncertain income (W) is, in the absence of taxation, entirely consumed in that period. To avoid "the complications of dynamic analysis", he takes this income as the appropriate tax base. He further restricts the utility function such that it implies constant relative risk-aversion and has constant elasticity. We, therefore, have $U(C) = \alpha C^{\beta}$, where α, β are arbitrary constants. In the absence of taxation (C = W), expected utility maximisation requires

$$\alpha \int W^{\beta} f_i(W)dW > \alpha \int W^{\beta} f_j(W)dW \qquad \text{...(A-V-17)}$$

for state-i to be preferred to state-j. After a proportional tax at the rate (1-t), the preceding argument would require that

$$\alpha \int t^{\beta} W^{\beta} f_i(W)dW > \alpha \int t^{\beta} W^{\beta} f_j(W)dW. \qquad \ldots(A\text{-}V\text{-}18)$$

But since $t^{\beta} > 0$ and is non-random, relations (A-V-17) and (A-V-18) are equivalent. Therefore, the preference ordering of the distribtions is unchanged by the introduction of a proportional tax on W. This leads Feldstein to conclude that a proportional tax has no effect on risk-taking.

The fact that the above result depends crucially on the tax base considered can be seen most easily by re-interpreting W as final wealth such that W = Wo + Y, where Wo is initial wealth and Y is the uncertain income. Evidently, a proportional wealth (consumption) tax has no effect on risk-taking, while an income tax would, in general, have some effects.

The effect of a consumption tax obtained in an explicit intertemporal context (Section V.2 of the text) brings out further limitations of Feldstein's "counter-example". It is seen, contrary to Feldstein, that the assumption of constant relative risk-aversion is by no means sufficient for a proportional consumption tax to leave risk-taking unchanged; income elasticities are relevant. More specifically, it has been shown that under constant relative risk-aversion, risk-taking would be discouraged where the income elasticity of the risky asset demand exceeds that of consumption.

CHAPTER VI

CONSUMPTION TAXES VERSUS INVESTMENT INCOME TAXES: IMPLICATIONS FOR IMPATIENCE AND RISK-TAKING

The question of the differential effects of a tax on consumption (expenditure) as opposed to a tax on income has been a major controversy in fiscal policy.[1] The existing analysis, by analytically separating the consumption (saving) and investment decisions has failed to come to grips with the problem. This is true both of the applications of the Fisherian theory of saving and of the single-period theories of risk-taking behaviour for analysing the problems of fiscal policy.[2] Both Hansen [29] and Musgrave [50], who develop intertemporal models of income disposal by households under certainty (with no asset choice), tend to agree that a proportional consumption tax does not encourage (nor discourage) a substitution of present over future consumption (consumption in both periods are, however, reduced by the tax if they are normal goods); while an equal revenue proportional income tax encourages a substitution in favour of present consumption. This latter effect is also accompanied by an income effect thus rendering the total effect ambiguous.[3] Nevertheless

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[1] See, for instance, Goode [24] and Kaldor [32].

[2] See Hansen [29] and Musgrave [50] for a discussion of fiscal policy in a Fisherian model of saving behaviour; and part 2 of this study for an analysis of the effects of taxation in models of optimal composition of a given portfolio.

[3] Hansen ([29], p. 147) in fact concluded that the consumption tax will lead to a greater decrease of real consumption than the income tax would. This, he argued, was due to the fact that for equal revenue requirements the consumption tax rate is larger than the income tax rate.

continued ...

this substitution effect (encouraging present consumption) induced by the income tax is often taken to imply that an income tax discriminates against saving.[4] However, in models where opportunities for investing in safe and risky assets are allowed, current saving no longer constitutes future consumption even when we require that the expected present value of life-time consumption equals the expected present value of life-time wealth: the introduction of capital risks makes future consumption a risky prospect. Thus with the introduction of uncertainty the Hansen-Musgrave results become suspect.[5]

Kaldor [32], on the other hand, without formally specifying his model arrives at some interesting conclusions. He argues that a tax on consumption does not discriminate against risk-taking but in fact discriminates in favour of saving; while a tax on investment income (or, a fortiori, a tax on total income) discriminates against both saving and risk-bearing.

Without attempting to settle the issue in its entirety, we plan to investigate the differential incidence of these taxes on household

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This, however, need not be so. In models where all savings are for future consumption (Hansen included),

$$\begin{cases} C_1 = Y_1 - S \\ C_2 = Y_2 + S\,(1+r), \end{cases}$$

where the notation is of usual significance. Clearly a tax on (C_1+C_2) has to be of the same magnitude as a tax on $[Y_1 + Y_2 + Sr]$ to yield equal revenues.

[4]This view is taken, for instance, by Goode and Kaldor.

[5]Indeed in Chapter V, we have shown that in the presence of capital risks, a proportional consumption tax is no longer neutral; the substitution effect encourages risk-taking which may either stimulate or reduce future consumption depending on whether the individual makes a loss or a gain.

consumption and risk-taking decisions in a simple two-period temporal model which we have introduced in the last chapter. For simplicity we compare a consumption tax with a tax on investment income rather than a tax on total income.[6]

It is seen that for low rates of tax (less than 50%) and low expected return on the risky investment (less than 100%), the differential incidence of a consumption tax is to raise both current consumption (i.e., discourage saving) and risk-taking. If these restrictions on tax rates and expected profit rates are satisfied, the differential incidence of a tax on investment income would also, under the additional restriction of zero rate of return on riskless investment, be to encourage current consumption and risk-taking. It is also seen, even more surprisingly, that under the latter set of conditions when both the consumption and the investment income tax rates are equally large, the differential incidence of a consumption tax encourages current consumption and risk-taking more effectively than a tax on investment income. This latter result requires that propensities to consume and to invest in the risky asset are identical under the alternative budgetary policies.

The analysis is performed for the balanced budget incidence of an increase in the consumption tax (or, alternatively, investment income tax) used to finance matching lump-sum transfers to the household. As Diamond [15] has pointed out, this is equivalent to the differential incidence of a consumption tax (or, an investment income tax) increase

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[6]This need not worry us unnecessarily. In terms of our model, wage income plus investment income is total income, and in such contexts, Musgrave has pointed out that "what matters in the case of the income tax is whether or not future interest income will be taxed" ([50], p. 260).

and a lump-sum tax decrease.[7]

VI.1 A Tax on Consumption

As already indicated in Chapter V, the individual makes the portfolio allocation decisions in the first period to maximise expected lifetime utility. In the second period he dissaves, consuming entirely his wage income, the capital and the realised investment income. The intertemporal consumption allocation decision can be stated as[8]

$$\left.\begin{aligned} &\text{maximise } EF(C_1,C_2) = V(C_1) + EU(C_2) \text{ subject to} \\ &\qquad C_1 = (Y_1 - a - m)(1-t) + K_1, \\ &\qquad C_2 = (1-t)[Y_2 + a(1+X) + m(1+r)] + K_2, \end{aligned}\right\} \qquad \ldots(\text{VI-1})$$

where t is the consumption tax rate, K_1 and K_2 are the "matching" lump-sum transfers received in each period, and the rest of the notation is that of Chapter V. C_2 can be rewritten by combining the constraints as

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[7]Since a direct comparison of taxes raising equal expected revenue does not appear to be analytically tractable, it seems worthwhile that an indirect comparison as suggested below would convey some notion of the comparison of two taxes: We combine the analyses for each of the consumption and investment income taxes with lump-sum taxes under matching tax distributions (or equivalently with tax proceeds going back to the same taxpayer). That is, we have not compared the investment income tax with a lump-sum tax in both periods; rather we compare it with a lump-sum tax falling only in the second period, because the investment incomes are realised in this period also. The analogous point is made by Diamond [15] in the context of incidence of an interest income tax in a neoclassical growth model. Incidentally, the above also corresponds to Kaldor's suggestion for comparing different taxes.

[8]Recall that in the last chapter we have assumed that both V and U are twice continuously differentiable with positive and diminishing marginal utilities, thus guaranteeing risk-aversion and the diminishing marginal rate of substitution between present and future consumption prospects.

$$C_2 = (1-t)[Y_1(1+r) + Y_2 + a(X-r)] + (K_1 - C_1)(1+r) + K_2.$$

Now substituting for C_2, we can eliminate the constraint and restate the maximisation as

$$\underset{\{C_1,\ a\}}{\text{maximise}} \quad V(C_1) + E[U\{(1-t)[Y_1(1+r) + Y_2 + a(X-r)] + (K_1-C_1)(1+r) + K_2\}]. \qquad \ldots(\text{VI-2})$$

The necessary and sufficient conditions for the existence of a maximum are given by[9]

$$V'(C_1) - (1+r)E[U'(C_2)] = 0, \qquad \ldots(\text{VI-3})$$

$$(1-t)E[U'(C_2)(X-r)] = 0. \qquad \ldots(\text{VI-4})$$

Differentiating the first-order conditions, (VI-3) and (VI-4), we can obtain:

$$\begin{bmatrix} & & \\ & H & \\ & & \end{bmatrix} \begin{pmatrix} \dfrac{\partial C_1}{\partial t} \\ \dfrac{\partial a}{\partial t} \end{pmatrix} = \begin{pmatrix} -(1+r)E[U''(C_2)\tilde{Y}] \\ (1-t)E[U''(C_2)(X-r)\tilde{Y}] \end{pmatrix} \qquad \ldots(\text{VI-5})$$

and,

$$\begin{bmatrix} & & \\ & H & \\ & & \end{bmatrix} \begin{pmatrix} \dfrac{\partial C_1}{\partial K_1} \\ \dfrac{\partial a}{\partial K_1} \end{pmatrix} = \begin{pmatrix} (1+r)^2 E[U''(C_2)] \\ -(1-t)(1+r)E[U''(C_2)(X-r)] \end{pmatrix}, \qquad \ldots(\text{VI-6})$$

where H is given by the appropriate second-order conditions and $\tilde{Y} \equiv [Y_1(1+r) + Y_2 + a(X-r)]$. Using the expressions for $\partial C_1/\partial Y_1$ and $\partial a/\partial Y_1$ as given by (A-V-10) and (A-V-11) of Chapter V, we can state the above derivatives as follows:

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[9]The interpretation of these conditions are discussed in Chapter V.

$$\frac{\partial C_1}{\partial t} = (-)\left(\frac{Y_1 + Y_2(1+r)^{-1}}{1-t}\right)\frac{\partial C_1}{\partial Y_1}, \qquad \ldots(VI\text{-}7)$$

$$\frac{\partial a}{\partial t} = \left(\frac{a}{1-t}\right)\left[1 - \frac{1}{a}\frac{\partial a}{\partial Y_1}\{Y_1 + Y_2(1+r)^{-1}\}\right], \qquad \ldots(VI\text{-}8)$$

and

$$\frac{\partial C_1}{\partial K_1} = \left(\frac{1}{1-t}\right)\frac{\partial C_1}{\partial Y_1}, \qquad \ldots(VI\text{-}9)$$

$$\frac{\partial a}{\partial K_1} = \left(\frac{1}{1-t}\right)\frac{\partial a}{\partial Y_1}, \qquad \ldots(VI\text{-}10)$$

where the expression $(\frac{a}{1-t})$ has the interpretation of the substitution effect,[10] i.e., $(\frac{1}{1-t}) = \frac{\partial a}{\partial t}\Big/_{EF(C_1,C_2) = \text{constant}}$.

These comparative static results can be easily interpreted. Equations (VI-9) and (VI-10) indicate that lump-sum transfer changes increase consumption and risk-taking by generating simple income effects without influencing the intertemporal consumption substitution possibilities. In view of $(\frac{a}{1-t})$ being the substitution effect in the risk-taking function, equations (VI-7) and (VI-8) can be interpreted in the following manner: a

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[10] It can be seen that both for the consumption and the investment income taxes

$$\frac{\partial a}{\partial t}\Big/_{E(\text{Utility}) = \text{Constant}} = \left(\frac{a}{1-t}\right).$$

For the consumption tax, for instance, for constant expected utility we must have:

let $EF(C_1,C_2) = V(C_1) + E[U(C_2)]$, then $dE(F) = 0$ requires

$$\{V'(C_1) - (1+r)E[U'(C_2)]\}dC_1 + (1-t)E[U'(C_2)(X-r)]\,da$$
$$= aE[U'(C_2)(X-r)]dt - \{(1+r)dK_1 + dK_2\}E[U'(C_2)]$$

This together with equations (VI-8), (VI-10) and (VI-18) yields the result shown.

proportional consumption tax reduces present consumption due to the income effect. A substitution effect arises due to the fact that present consumption is reduced by the tax, but the individual is able to maintain his former level of expected utility by investing more in the risky asset. However, the income effect tends to reduce risk-taking. The total effect on risk-taking, therefore, depends on the magnitude of income elasticities.[11]

If the tax variables are such that the individual, in each period, receives as a lump sum exactly equal to what is expected to be taken from him by the consumption tax, then the tax variables must satisfy[12]

$$\left.\begin{aligned} K_1 &= t(Y_1 - a - m), \\ K_2 &= t[Y_2 + a(1+\bar{X}) + m(1+r)], \end{aligned}\right\} \qquad \ldots(\text{VI-11})$$

where $\bar{X}$ denotes the expected value of X. (VI-11) can also be rewritten as

$$K_2 + K_1(1+r) = t[Y_1(1+r) + Y_2 + a(\bar{X} - r)]. \qquad \ldots(\text{VI-12})$$

To maintain this relationship changes in the tax variables must satisfy

$$\frac{\partial K_2}{\partial t} + (1+r)\frac{\partial K_1}{\partial t} = \bar{Y} + t(\bar{X}-r)\left[\frac{\partial a}{\partial t} + \frac{\partial a}{\partial K_1}\frac{\partial K_1}{\partial t} + \frac{\partial a}{\partial K_2}\frac{\partial K_2}{\partial t}\right], \qquad \ldots(\text{VI-13})$$

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[11] These results have been discussed in detail in Chapter V (Section V.2).

[12] An alternative would be to base the lump-sum transfer on the realised return on the risky asset (X) rather than on the expected return ($\bar{X}$). This would imply a different lump-sum transfer for each individual even if we assumed all individuals to be identical with respect to non-asset income and preferences. Basing the lump-sum transfer on the expected value of X (as here) results in a single lump-sum transfer which is the same for all identical individuals. Also notice that, although the expected tax payments equal the lump-sum transfer received, the individual cannot ignore the taxation policy as he does not realise that this indeed is the case. This is implicit in all balanced-budget studies.

where $\overline{Y} = [Y_1(1+r) + Y_2 + a(\overline{X}-r)]$. But from the first-order conditions we see that

$$\frac{\partial a}{\partial K_1} = (1+r)\frac{\partial a}{\partial K_2}, \qquad \ldots(VI\text{-}14)$$

and consequently (VI-13) can be rewritten as

$$[\frac{\partial K_1}{\partial t} + (1+r)^{-1}\frac{\partial K_2}{\partial t}] = \overline{Y}(1+r)^{-1} + \{t(1+r)^{-1}(\overline{X}-r)\}$$

$$\{\frac{\partial a}{\partial t} + \frac{\partial a}{\partial K_1}[\frac{\partial K_1}{\partial t} + (1+r)^{-1}\frac{\partial K_2}{\partial t}]\},$$

or,

$$[\frac{\partial K_1}{\partial t} + (1+r)^{-1}\frac{\partial K_2}{\partial t}] = \frac{[\overline{Y} + t(\overline{X}-r)\frac{\partial a}{\partial t}]}{[(1+r) - t(\overline{X}-r)\frac{\partial a}{\partial K_1}]}. \qquad \ldots(VI\text{-}15)$$

We can now evaluate the change in risk-taking and in consumption from a consumption tax rate and a lump-sum transfer change which would leave zero expected net revenue in each period. We can call these the "expected revenue compensated changes". Note that the first-order conditions, (VI-3) and (VI-4), implicitly define optimal first period consumption and optimal risk-taking as functions of the tax parameters:

$$a^* = a(t, K_1, K_2), \qquad \ldots(VI\text{-}16)$$

$$C_1^* = C(t, K_1, K_2). \qquad \ldots(VI\text{-}17)$$

From (VI-16) we obtain

$$\frac{\partial a^*}{\partial t} = \frac{\partial a}{\partial t} + \frac{\partial a}{\partial K_1}\frac{\partial K_1}{\partial t} + \frac{\partial a}{\partial K_2}\frac{\partial K_2}{\partial t},$$

or, using (VI-14),

$$\frac{\partial a^*}{\partial t} = \frac{\partial a}{\partial t} + \frac{\partial a}{\partial K_1}[\frac{\partial K_1}{\partial t} + (1+r)^{-1}\frac{\partial K_2}{\partial t}]. \qquad \ldots(VI\text{-}18)$$

This together with (VI-15) yields the subsidy compensated tax derivative

$$\left.\frac{\partial a^*}{\partial t}\right)_{comp} = \frac{\left(\frac{a}{1-t}\right)\left[1 + \left(\frac{\overline{X}-r}{1+r}\right)\frac{\partial a}{\partial Y_1}\right]}{\left[1 - \left(\frac{t}{1-t}\right)\left(\frac{\overline{X}-r}{1+r}\right)\frac{\partial a}{\partial Y_1}\right]} \quad . \qquad \ldots(VI\text{-}19)$$

Since $(\overline{X}-r) > 0$, the numerator is always positive. The sign, therefore, depends on the denominator, i.e.,

$$\left.\frac{\partial a^*}{\partial t}\right)_{comp} \gtreqless 0 \text{ as } 1 \gtreqless \left(\frac{t}{1-t}\right)\left(\frac{\overline{X}-r}{1+r}\right)\frac{\partial a}{\partial Y_1} \quad . \qquad \ldots(VI\text{-}20)$$

Note that for $t = 0.50$ the requirement is

$$1 \gtreqless \left(\frac{\overline{X}-r}{1+r}\right)\frac{\partial a}{\partial Y_1} \quad .$$

Since $\partial a/\partial Y_1 > 0$ and r is usually small (a reasonable upper bound for the real rate of return on a safe asset might be .10), a sufficient condition for (VI-20) to be positive is that $r < \overline{X} \leq 1$ (for $t \leq .50$).

We can therefore conclude that so long as the expected return on the risky asset does not exceed 100% and the consumption tax rate is below 50%, the balanced budget incidence of a consumption tax is to increase risk-taking. If high tax rates (i.e., $t > 0.5$) happen to be combined with high expected profit rates (i.e., $\overline{X} > 1$), risk-taking may actually decrease. The general point, however, is that the result depends on t, $\overline{X}$, r and $\partial a/\partial Y_1$, the first being a government parameter and the rest being determined in the market.

Likewise, we can evaluate the expected revenue compensated change in optimal current consumption. From (VI-17) we have

$$\frac{\partial C_1^*}{\partial t} = \frac{\partial C_1}{\partial t} + \frac{\partial C_1}{\partial K_1}\frac{\partial K_1}{\partial t} + \frac{\partial C_2}{\partial K_2}\frac{\partial K_2}{\partial t} \quad . \qquad \ldots(VI\text{-}21)$$

As before, together with (VI-14) and (VI-15), the above reduces to

$$\frac{\partial C_1^*}{\partial t}\Big/_{comp} = \frac{\partial C_1}{\partial t} + \frac{\partial C_1}{\partial K_1}\left[\frac{\bar{Y} + t(\bar{X}-r)\frac{\partial a}{\partial t}}{(1+r)-t(\bar{X}-r)\frac{\partial a}{\partial K_1}}\right],$$

or, using equations (VI-7) through (VI-10),

$$\frac{\partial C_1^*}{\partial t}\Big/_{comp} = \frac{\left[\frac{a(\bar{X}-r)}{(1+r)(1-t)^2}\frac{\partial C_1}{\partial Y_1}\right]}{\left\{1 - \left(\frac{t}{1-t}\right)\left(\frac{\bar{X}-r}{1+r}\right)\frac{\partial a}{\partial Y_1}\right\}} . \qquad \ldots(VI-22)$$

Notice that (VI-22) admits of similar interpretation as that of (VI-19). As before, the sign of $\frac{\partial C_1^*}{\partial t})_{comp.}$ is the same as that of $\{1 - (\frac{t}{1-t})(\frac{\bar{X}-r}{1+r})\frac{\partial a}{\partial Y_1}\}$, and thus the preceding comments apply. That is so long as the expected return on the risky asset is small (less than 100%) and the consumption tax rate is low (less than 50%), the balanced budget incidence of a consumption tax is to increase current consumption.

However, if we started from a situation of no tax ($t = 0$ and $K_i = 0$), the above expressions are simplified a great deal:

$$\frac{\partial a^*}{\partial t}\Big/^{t=0}_{comp} = a\left\{1+\left(\frac{\bar{X}-r}{1+r}\right)\frac{\partial a}{\partial Y_1}\right\}, \qquad \ldots(VI-19a)$$

$$\frac{\partial C_1^*}{\partial t}\Big/^{t=0}_{comp} = \frac{a(\bar{X}-r)}{(1+r)}\frac{\partial C_1}{\partial Y_1} . \qquad \ldots(VI-22a)$$

Clearly both of these effects are positive given that both current consumption and the risky asset are superior.

VI.2 A Tax on Investment Income

While the model remains the same as that of the preceding section, we now introduce a tax on investment income (rather than a tax on consumption) accompanied by a matching lump-sum transfer. The individual budget constraints can now be written as

$$\left.\begin{aligned} C_1 &= Y_1 - a - m, \\ C_2 &= Y_2 + (a+m) + (1-t)[aX+mr) + K, \end{aligned}\right\} \qquad \ldots(\text{VI-23})$$

where the notation is the same except that t is now the investment income tax rate. On substitution, (VI-23) reduces to

$$C_2 = Y_2 + (Y_1-C_1)\ \{1+(1-t)r\} + (1-t)a(X-r) + K, \qquad \ldots(\text{VI-24})$$

and the maximisation problem reads as

$$\underset{\{C_1,a\}}{\text{maximise}} \quad V(C_1) + E[U\{Y_2 + (Y_1-C_1)[1+(1-t)r] + (1-t)a(X-r)+ K\}]. \qquad \ldots(\text{VI-25})$$

We proceed as before and state the necessary and sufficient conditions for the existence of a maximum:

$$V'(C_1) - (r^*)E[U'(C_2)] = 0\ , \qquad \ldots(\text{VI-26})$$

$$(1-t)E[U'(C_2)(X-r)] = 0, \qquad \ldots(\text{VI-27})$$

where we have defined

$$(r^*) = [1 + r\ (1-t)].$$

We also find that the first-order conditions, (VI-26) and (VI-27), upon implicit differentiation express the derivatives of the consumption and risk-taking functions as follows:

$$\begin{bmatrix} \\ H \\ \\ \end{bmatrix} \begin{pmatrix} \dfrac{\partial C_1}{\partial t} \\ \\ \dfrac{\partial a}{\partial t} \end{pmatrix} = \begin{pmatrix} (-)(r^*)E[U''(C_2)\{(Y_1-C_1)r+a(X-r)\}] \ -rE[U'(C_2)] \\ \\ (1-t)E[U''(C_2)(X-r)\{(Y_1-C_1)r+a(X-r)\}] \end{pmatrix} \qquad \ldots(\text{VI-28})$$

and,

$$\begin{bmatrix} & \\ H & \\ & \end{bmatrix} \begin{pmatrix} \frac{\partial C_1}{\partial K} \\ \\ \frac{\partial a}{\partial K} \end{pmatrix} = \begin{pmatrix} (r^*)E[U''(C_2)] \\ \\ -(1-t)E[U''(C_2)(X-r)] \end{pmatrix} . \qquad \ldots(VI\text{-}29)$$

Furthermore, using the expressions for $\partial C_1/\partial Y_1$ and $\partial a/\partial Y_1$, as given by equations (A-V-12) and (A-V-13) of the last chapter, the above derivatives simplify to:

$$\frac{\partial C_1}{\partial t} = \left(\frac{C_1 r}{r^*}\right) \left\{\frac{V'(C_1)}{V''(C_1)C_1} + 1\right\} \frac{\partial C_1}{\partial Y_1}$$

$$- \left(\frac{C_1 r}{r^*}\right)\left\{\frac{Y_1}{C_1} \frac{\partial C_1}{\partial Y_1} + \frac{V'(C_1)}{V''(C_1)C_1}\right\} , \qquad \ldots(VI\text{-}30)$$

$$\frac{\partial a}{\partial t} = \frac{a}{1-t}\left\{1 - \frac{r(1-t)}{r^*}\left[\frac{Y_1}{a}\frac{\partial a}{\partial Y_1}\right]\right\}$$

$$+ \left(\frac{C_1 r}{r^*}\right)\left(1 + \frac{V'(C_1)}{V''(C_1)C_1}\right) \frac{\partial a}{\partial Y_1} ; \qquad \ldots(VI\text{-}31)$$

and

$$\frac{\partial C_1}{\partial K} = \left(\frac{1}{r^*}\right)\frac{\partial C_1}{\partial Y_1} , \qquad \ldots(VI\text{-}32)$$

$$\frac{\partial a}{\partial K} = \left(\frac{1}{r^*}\right) \frac{\partial a}{\partial Y_1} . \qquad \ldots(VI\text{-}33)$$

Since this is effectively a tax on future consumption, the substitution effect,[13] which equals $(-)\left(\frac{r}{r^*}\right)\left(\frac{V'(C_1)}{V''(C_1)}\right)$ in equation (VI-30), tends to

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[13] As indicated in footnote 10 this can be ascertained by substituting equations (VI-30), (VI-32) and

$$dK/_{\substack{E(\text{utility}) \\ =\text{constant}}} = (Y_1 - C_1)r\,dt + \frac{aE[U'(C_2(X-r)]dt}{E[U'(C_2)]}$$

$$- \frac{(1-t)E[U'(C_2)(X-r)]}{E[U'(C_2)]} dt + \left(r^* - \frac{V'(C_1)}{E[U'(C_2)]}\right)dC_1$$

continued ...

work in favour of current consumption, which may or may not be outweighed by the income effects. Risk-taking may also increase due to the positive substitution effect as in the consumption tax case. Also note that the terms in { } are likely to be positive for any set of conceivable parameter values.[14] For r = 0,

$$\left.\frac{\partial C_1}{\partial t}\right)^{r=0} = 0, \qquad \text{...(VI-30a)}$$

$$\left.\frac{\partial a}{\partial t}\right)^{r=0} = \left(\frac{a}{1-t}\right). \qquad \text{...(VI-32a)}$$

We now analyse the differential incidence of an investment income tax increase and a lump-sum tax decrease on impatience and risk-taking. Since investment income taxes are paid only in the second period, for distributional neutrality, we require that the lump-sum is also paid out in that period such that expected net revenue is nil, i.e.,

$$K = t[a\overline{X} + mr],$$

or

$$K = t[Y_1 - C_1)r + a(\overline{X} - r)]. \qquad \text{...(VI-34)}$$

Again, to maintain (VI-34) changes in the tax variables must satisfy

$$\frac{\partial K}{\partial t} = [(Y_1 - C_1)r + a(\overline{X} - r) + t(\overline{X} - r)\frac{\partial a}{\partial t} + \frac{\partial a}{\partial K}\frac{\partial K}{\partial t}] - tr[\frac{\partial C_1}{\partial t} + \frac{\partial C_1}{\partial K}\frac{\partial k}{\partial t}],$$

---

in the total derivative of the optimal C_1*,

$$dC_1^* = \frac{\partial C_1}{\partial t}\, dt + \frac{\partial C_1}{\partial K}\, dK.$$

[14] Also note that for a logarithmic utility function the factor $(\frac{V'(C_1)}{V''(C_1)C_1} + 1)$ drops out. Once again we refer the reader to Chapter V (Section V.2) for a more satisfactory discussion of these results.

or,

$$\frac{\partial K}{\partial t} = \frac{[(Y_1-C_1)r + a(\bar{X}-r)] + t(\bar{X}-r)\frac{\partial a}{\partial t} - tr\frac{\partial C_1}{\partial t}}{[1 - t(\bar{X}-r)\frac{\partial a}{\partial K} + tr\frac{\partial C_1}{\partial K}]} . \qquad \ldots(\text{VI-35})$$

From the solution to the first-order conditions, (VI-26) and (VI-27), upon implicit differentiation, we obtain

$$\frac{\partial a^*}{\partial t} = \frac{\partial a}{\partial t} + \frac{\partial a}{\partial K}\frac{\partial K}{\partial t} , \qquad \ldots(\text{VI-36})$$

$$\frac{\partial C_1^*}{\partial t} = \frac{\partial C_1}{\partial t} + \frac{\partial C_1}{\partial K}\frac{\partial K}{\partial t} , \qquad \ldots(\text{VI-37})$$

which together with (VI-35) allows us to evaluate the expected revenue compensated changes in consumption and risk-taking. That is,

$$\left.\frac{\partial a^*}{\partial t}\right|_{comp} = \frac{\partial a}{\partial t} + \frac{\partial a}{\partial K}\left[\frac{[(Y_1-C_1)r + a(\bar{X}-r)] + t(\bar{X}-r)\frac{\partial a}{\partial t} - tr\frac{\partial C_1}{\partial t}}{\{1 - t(\bar{X}-r)\frac{\partial a}{\partial K} + tr\frac{\partial C_1}{\partial K}\}}\right],$$

or, using equations (VI-30) through (VI-33),

$$\left.\frac{\partial a^*}{\partial t}\right|_{comp} = \frac{(\frac{a}{1-t})[1 + (\frac{tr}{r^*})\frac{\partial C_1}{\partial Y_1}] + [\frac{a(\bar{X}-r)}{r^*} + (\frac{r}{r^*})(1+\frac{tr}{r^*})\frac{V'(C_1)}{V''(C_1)}]\frac{\partial a}{\partial Y_1}}{[1 - \frac{t(\bar{X}-r)}{r^*}\frac{\partial a}{\partial Y_1} + (\frac{tr}{r^*})(\frac{\partial C_1}{\partial Y_1})]} , \qquad \ldots(\text{VI-38})$$

which is, in general, indeterminate. However, for r=0, this simplifies into

$$\left.\frac{\partial a^*}{\partial t}\right)_{comp}^{r=0} = (\frac{a}{1-t})\left[1 + \frac{\bar{X}\frac{\partial a}{\partial Y_1}}{1 - t\bar{X}\frac{\partial a}{\partial Y_1}}\right] . \qquad \ldots(\text{VI-38a})$$

Clearly, the sign of (VI-38a) is the same as

$$[1 - t\bar{X}\frac{\partial a}{\partial Y_1}]$$

since $0 < \partial a/\partial Y_1 < 1$ and $t < 1$. If $t \leq .5$ is combined with $\overline{X} \leq 2$ risk-taking will increase.

In the preceding manner, from (VI-37) and (VI-35), we obtain

$$\left.\frac{\partial C_1^*}{\partial t}\right)_{comp} = \frac{\partial C_1}{\partial t} + \frac{\partial C_1}{\partial K}\left[\frac{[(Y_1-C_1)r+a(\overline{X}-r)] + t(\overline{X}-r)\frac{\partial a}{\partial t} - tr\frac{\partial C_1}{\partial t}}{(1 - t(\overline{X}-r)\frac{\partial a}{\partial K} + tr\frac{\partial C_1}{\partial K})}\right],$$

or, using equations (VI-30) through (VI-33),

$$\left.\frac{\partial C_1^*}{\partial t}\right|_{comp} = \frac{(\frac{r}{r^*})\frac{V'(C_1)}{V''(C_1)}[\frac{\partial C_1}{\partial Y_1}-1]+\frac{a(\overline{X}-r)}{(1-t)r^*}(\frac{C_1}{Y_1})[\frac{Y_1}{C_1}\frac{\partial C_1}{\partial Y_1} + (\frac{r(1-t)}{r^*})\frac{V'(C_1)}{V''(C_1)C_1}(\frac{Y_1}{a}\frac{\partial a}{\partial Y_1}}{1 - \frac{t(\overline{X}-r)}{r^*}\frac{\partial a}{\partial Y_1} + (\frac{tr}{r^*})(\frac{\partial C_1}{\partial Y_1})} . \qquad \text{...(VI-39)}$$

A set of sufficient conditions for this effect to be positive is

(i) utility is logarithmic,

(ii) $\frac{Y_1}{C_1}\frac{\partial C_1}{\partial Y_1} \geq \frac{Y_1}{a}\frac{\partial a}{\partial Y_1}$, and

(iii) $\overline{X} \leq 1$,

none of which are necessary. For $r = 0$, this again simplifies into

$$\left.\frac{\partial C_1^*}{\partial t}\right|_{\substack{r=0\\ comp}} = \frac{(\frac{a\overline{X}}{1-t})\frac{\partial C_1}{\partial Y_1}}{[1-t\overline{X}\frac{\partial a}{\partial Y_1}]}, \qquad \text{...(VI-39a)}$$

which would be positive if $t \leq 0.5$ and $\overline{X} \leq 2$, neither being necessary. Thus we can conclude that for the case of zero rate of return on riskless investment the balanced-budget incidence of an investment income tax increase and a lump-sum tax decrease is to raise both current consumption and risk-taking if the expected rate of return on the risky asset times the tax rate is less than unity.

VI.3 The Two Taxes Compared

As we have indicated in the introductory remarks, it is difficult to compare two taxes directly in terms of an analytical model. Such considerations led us to attempt an indirect evaluation of these taxes by comparing each with a lump-sum tax levied on the individual (or equivalently with tax proceeds going back to the same individual paying the tax). This method allowed us to arrive at qualitative results that required information regarding certain government and market determined parameters (i.e., all being observable) and could be reasonably interpreted. One such class of results was obtained for the special case when the return on riskless investment was zero. Let us rewrite these results as follows:

$$\left.\frac{\partial a^*}{\partial t_C}\right|_{\substack{r=0\\ comp}} = \frac{\left(\frac{a}{1-t_C}\right)\left[1 + \bar{X}\frac{\partial a}{\partial Y_1}\right]}{\left[1 - \left(\frac{t_C}{1-t_C}\right)\bar{X}\frac{\partial a}{\partial Y_1}\right]}, \qquad \ldots(\text{VI-19b})$$

$$\left.\frac{\partial C_1^*}{\partial t_C}\right|_{\substack{r=0\\ comp}} = \frac{\left[\frac{a\bar{X}}{(1-t_C)^2}\frac{\partial C_1}{\partial Y_1}\right]}{\left[1 - \left(\frac{t_C}{1-t_C}\right)\bar{X}\frac{\partial a}{\partial Y_1}\right]}; \qquad \ldots(\text{VI-22b})$$

and

$$\left.\frac{\partial a^*}{\partial t_i}\right|_{\substack{r=0\\ comp}} = \left(\frac{a}{1-t_i}\right)\left[1 + \frac{\left(\bar{X}\frac{\partial a}{\partial Y_1}\right)}{\left(1 - t_i\bar{X}\frac{\partial a}{\partial Y_1}\right)}\right], \qquad \ldots(\text{VI-38b})$$

$$\left.\frac{\partial C_1^*}{\partial t_i}\right|_{\substack{r=0\\ comp}} = \frac{\left(\frac{a\bar{X}}{1-t_i}\right)\frac{\partial C_1}{\partial Y_1}}{\left[1 - t_i\bar{X}\frac{\partial a}{\partial Y_1}\right]}, \qquad \ldots(\text{VI-39b})$$

where t_C and t_i denote consumption and investment income tax rates respectively.

If we choose equal tax rates as a basis of comparison, then comparing (VI-19b) and (VI-38b) it is seen that

$$\left.\frac{\partial a^*}{\partial t_C}\right|_{comp}^{r=0} > \left.\frac{\partial a}{\partial t_i}\right|_{comp}^{r=0} . \qquad \text{...(VI-40)}$$

To see this rewrite (VI-38b) as

$$\left.\frac{\partial a^*}{\partial t_i}\right|_{comp}^{r=0} = \frac{\left(\frac{a}{1-t_i}\right)\left[1 + \bar{X}(1-t_i)\frac{\partial a}{\partial Y_1}\right]}{\left[1 - t_i\bar{X}\frac{\partial a}{\partial Y_1}\right]} . \qquad \text{...(VI-38c)}$$

Now, assuming that $\left.\frac{\partial a^*}{\partial t_i}\right|_{comp}^{r=0}$ and $\left.\frac{\partial a^*}{\partial t_C}\right|_{comp}^{r=0}$ are positive, i.e., the denominators are positive, the denominator of (VI-38c) is greater than that of (VI-19b). Furthermore, the numerator of (VI-38c) is smaller than that of (VI-19b). These results require that the propensities to invest in the risky asset is the same whether the individual is paying a consumption tax or an investment income tax.[15] Therefore, we arrive at the interesting conclusion that if the tax rate and the expected rates of profit are such that the balanced budget incidence of a consumption tax and that of a tax on investment income is to encourage risk-taking, then for equally large tax rates, a tax on consumption encourages risk-taking more effectively than a tax on investment income. This assumes identical propensities to invest in the risky asset in the two situations. On

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[15] Comparing equations (A-V-10) through (A-V-13) of last chapter, it can be seen that for r=0 and for the same values of C_1 and C_2, the propensity to consume (and to invest in the risky asset) are the same under the alternative budgetary systems. However, C_1 and C_2 will, in general, not be the same in the two budgetary systems (i.e., consumption tax and investment income tax).

balance, this result may seem to support Kaldor's intuition, but not quite. His main contention was that a tax on consumption would be neutral so far as risk-taking was concerned, while a tax on investment income (or, a fortiori, a tax on income) would discourage risk-taking. On the other hand, the basis of our result is a situation where the balanced budget incidence of both these taxes were to encourage risk-taking.

To see the effects of equal tax rates on consumption, we compare (VI-22b) and (VI-39b) letting $t_C = t_i$. Once again, we start from the situation where both these effects are positive. As before, the denominator of (VI-39b) is greater than that of (VI-22b), while the numerator of (VI-39b) is smaller than that of (VI-22b). Thus under identical propensities to consume and invest in the risky asset,

$$\left.\frac{\partial C_1^*}{\partial t_C}\right|_{\substack{r=0 \\ \text{comp}}} > \left.\frac{\partial C_1^*}{\partial t_i}\right|_{\substack{r=0 \\ \text{comp}}} . \qquad \ldots\text{(VI-41)}$$

Thus we conclude that if the initial rates of tax (equal) and the expected return on the risky asset are such that the balanced budget incidence of a consumption tax (with matching lump-sum transfers) and a tax on investment income (with matching lump-sum transfer) were to stimulate current consumption, a tax on consumption did this more effectively than a tax on investment income. This is contrary to the result under certainty. The general consensus of the literature that a tax on income (or investment income) discourages saving as compared to a tax on expenditure (consumption) is not borne out by this model which incorporates uncertainty via capital risks.

VI.4 Conclusion

We investigated the balanced budget incidence of a tax on consumption (with matching lump-sum transfers) and of a tax on investment income (with a matching lump-sum transfer) in the context of a simple two-period model of consumption and portfolio allocation decisions under uncertainty. The analysis suggests that when the rate of return on the safe investment is zero, when the tax rates (on consumption and on investment income) are low (less than 0.5) and when the expected return on the risky asset is less than 100%, both current consumption and risk-taking are encouraged under both the systems of budget policy. It is also shown that under such circumstances, if the propensities to consume and to invest in the risky asset are identical under the two budget policies and that the two tax rates are equally large, the differential incidence of a consumption tax is to encourage risk-taking and to discourage saving (encourage current consumption) more effectively than that of a tax on investment income. While these results are fairly restrictive, they at least point out that the introduction of uncertainty alters the "standard" results of fiscal policy in an important way.

CHAPTER VII

GENERAL CONSUMPTION TAX IN AN INFINITE HORIZON PORTFOLIO MODEL

In a recent paper Hagen [26] claims to have shown that optimal consumption and risk-taking decisions are not influenced by the imposition of a proportional consumption tax with full loss offset provisions. Although the consumption-saving decision is unaffected by the introduction of a proportional consumption tax in simple intertemporal, say, two-period models under the conditions of certainty, this is not true in similar models under uncertainty. This has been explained in Chapter V. Further, we have shown that a proportional consumption tax influences the intertemporal consumption decision by affecting risk-taking. Thus, given the results of Chapter V, Hagen's result seems surprising.

Hagen uses a dynamic programming model of multiperiod portfolio choice[1] which allows intermediate consumption and investment in two assets: one risky asset with a random rate of return and one riskless asset with a known rate of return. He also assumes an infinite horizon. Except for the time-dimension and, as we shall see, the solution algorithm, the structure of the problem is identical to those discussed in Chapters V and VI. Consequently, therefore, we would expect the predictions obtained from the two-period models to appear as special cases of the results obtainable from a decision process with an

[1]The pioneering work on dynamic programming approaches to multistage decision processes is credited to Richard Bellman [6], while its application to multi-period consumption-portfolio allocation decisions has been advanced by, among others, Hakansson [28].

arbitrary horizon. But, as will be shown, this is not the case. However, Hagen's own explanation for his simple results refers to the simplifying assumptions he made with respect to the investor's preferences, namely, additive one-period preferences (i.e., $U(C_1,C_2,\ldots) = \Sigma_{j=1}^{\infty} \alpha^{j-1} U(C_j)$, $0 < \alpha < 1$) and that the investor's one-period preferences are represented by $U(C) = C^{\beta}$, $0 < \beta < 1$ (i.e., implying constant relative risk-aversion).

The purpose of this chapter is to provide an explanation for the apparently puzzling result obtained by Hagen. This is done in two steps. In Section VII.1 we point out, by means of a counter-example that the restrictions imposed on the investor's preferences, alone, cannot be the explanation for a general consumption tax to leave the optimal consumption and risk-taking decisions unaffected. Then, in Section VII.2, we argue that although the particular solution algorithm used by Hagen, in conjunction with the above-mentioned restrictions on preferences, yields a set of demand functions describing a local maximum to the investor's problem, these demand functions are in their implicit form in the sense of not being completely described by the parameters (and other exogenous variables) of the model, and, consequently, Hagen is incorrect when he uses these demand functions as if they were fully described by the parameters of the model. Section VII.3 concludes this chapter.

VII.1 A Counter-Example

In this section we show that the restrictions imposed by Hagen on the investor preferences, alone, cannot lead to his result. This is considered in a two-period model which is simply a special case of Hagen's

model. We let $U(C_1,C_2)$, where C_j is consumption in period j, represent the investor's preference ordering over various two-period consumption programmes and, following Hagen, we assume that it is of the form:

$$U(C_1,C_2) = C_1^{\beta} + \alpha C_2^{\beta}, \quad 0 < \alpha,\beta < 1, \qquad \text{...(VII-1)}$$

where α expresses the investor's impatience to consume.

The decision problem is, therefore,

$$\max_{\{C_1,a,m\}} \quad EU(C_1,C_2) = C_1^{\beta} + \alpha E[C_2^{\beta}], \qquad \text{...(VII-2)}$$

$$\text{s.t.} \quad \left.\begin{aligned} C_1 &= (1-t)\ [W-a-m] \\ C_2 &= (1-t)\ [a(1+X) + m(1+r)], \end{aligned}\right\} \qquad \text{...(VII-3)}$$

where W is the wealth available for consumption and investment at the beginning of period 1 and the rest of the notation is that of Chapter V. Substituting for m and C_2 in (VII-2) we have

$$\max_{\{C_1,a\}} \quad C_1^{\beta} + \alpha E[\{(1-t)[W(1+r) + a(X-r)] - C_1(1+r)\}^{\beta}]. \qquad \text{...(VII-4)}$$

The first-order conditions for an interior solution are given by:

$$\beta C_1^{\beta-1} - \alpha\beta(1+r)E[C_2^{\beta-1}] = 0, \qquad \text{...(VII-5)}$$

$$(1-t)\alpha\beta E[C_2^{\beta-1}(X-r)] = 0. \qquad \text{...(VII-6)}$$

Differentiating (VII-5) and (VII-6) with respect to the tax rate and initial wealth, and solving, we obtain[2]

$$\frac{\partial C_1}{\partial t} = (-)\ \left(\frac{W}{1-t}\right)\ \frac{\partial C_1}{\partial W}, \qquad \text{...(VII-7)}$$

$$\frac{\partial a}{\partial t} = \left(\frac{a}{1-t}\right)\ \left[1 - \frac{W}{a}\frac{\partial a}{\partial W}\right]. \qquad \text{...(VII-8)}$$

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[2]These results (in their slightly generalised form) are discussed in Chapter V and hence we omit the actual derivation here.

Obviously, $\partial a/\partial t = 0$ when $(W/a)(\partial a/\partial W) = 1$. It is interesting to note that this wealth elasticity of risky asset demand is unity under constant relative risk-aversion utility functions in single-period analysis, but is no longer so even in this simple two-period context. For this wealth elasticity of risk-taking to equal unity we now require that the wealth elasticity of consumption demand is also unity. These results have been discussed in the appendices to Chapters IV and V. This completes our counter-example.

It may further be pointed out that Hagen's argument (p. 287) that $U(C_1, C_2, \ldots)$ and $U(C_1(1-t), C_2(1-t), \ldots)$ represent the same preference ordering is incorrect. To see this let us write down the investor's problem before and after the imposition of the tax:

$$\max_{\{C_1, a\}} \quad C_1^{\beta} + \alpha E[\{[W(1+r) + a(X-r)] - C_1(1+r)\}^{\beta}]; \quad \ldots(\text{VII-9})$$

$$\max_{\{C_1^*, a\}} \quad (C_1^*)^{\beta} + \alpha E[\{(1-t)[W(1+r) + a(X-r)] - C_1^*(1+r)\}^{\beta}]; \quad \ldots(\text{VII-10})$$

where C_1^* denotes after-tax consumption in period 1. Clearly, in terms of the appropriate decision variables (i.e., after-tax consumption), (VII-10) is not a linear transformation of (VII-9) unless the investor chooses to consume all his wealth in the first period.

VII.2 The Infinite Horizon Algorithm[3]

Parallel to the assumptions made in the preceding section we assume that the investor's preference ordering over various infinite horizon consumption programmes takes the following form:

[3]Since the purpose of this section is to point out the inappropriate use (by Hagen) of the infinite horizon algorithm, the exposition follows that of Hagen very closely.

$$U(C_1,C_2,C_3,\ldots) = U(C_1) + \alpha U(C_2,C_3,\ldots), \qquad \ldots(\text{VII-11})$$

where $U(C_2,C_3,\ldots)$ denotes the prospective utility of the consumption program. Applying equation (VII-11), recursively, then

$$U(C_1,C_2,C_3,\ldots) \quad \Sigma_{j=1}^{\infty} \alpha^{j-1} U(C_j). \qquad \ldots(\text{VII-12})$$

In the case of a general consumption tax, as is obvious from equations (VII-3) and (VII-4), the capital on hand at two succeeding decision points is related by the following difference equation:

$$W_{j+1} + [(1-t)\{W_j(1+r) + a_j(X-r)\} - C_j(1+r)], \qquad \ldots(\text{VII-13})$$

where W_{j+1} and C_j are expressed net of the tax and where we assume X is distributed independent of time j. As before, a_j and C_j are the only decision variables facing the investor in every period. We make a further assumption that the investor can borrow any amount as long as he insures that the probability that the terminal value of capital at the end of any period will be non-negative is one, i.e., $Pr\{W_j \geq 0\} = 1$ for all j. This implies that the investor can borrow any amount at the riskless rate, r, insofar as the default risk is non-existent.[4]

Now we introduce the functional $f_j(W_j)$ denoting expected utility obtainable from consumption over all future time, evaluated at decision point j, when capital is W_j and an optimal strategy is followed with respect to consumption and investment in all subsequent periods. Formally then,

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[4]In the context of a dynamic programming formulation, as Hakansson pointed out, this borrowing-lending opportunity is necessary to guarantee an analytic solution to the investor's problem.

$$f_j(W_j) \equiv \max E[U(C_j, C_{j+1}, \ldots)]/W_j. \qquad \ldots(\text{VII-14})$$

From (VII-11) we obtain by the <u>principle of optimality</u>,[5] for all j

$$f_j(W_j) = \max E[U(C_j) + \alpha\{\max E[U(C_{j+1}, C_{j+2}, \ldots)]/W_{j+1}\}]/W_j, \qquad \ldots(\text{VII-15})$$

and from (VII-14) and (VII-15):

$$f_j(W_j) = \max\{U(C_j) + \alpha E[f_{j+1}(W_{j+1})]\}, \text{ all } j. \qquad \ldots(\text{VII-16})$$

Since X is identically distributed in every period j, we drop the time subscript and using (VII-13) and (VII-16), state the investor's decision problem as

$$f(W) = \max_{\substack{0 \le C \le W(1-t) \\ a \ge 0}} \{U(C) + \alpha E[f((1-t)[(W - \frac{C}{1-t})(1+r) + a(X-r)])]\} \qquad \ldots(\text{VII-17})$$

$$\text{s.t.} \qquad \Pr\{(1-t)[a(X-r) + (W - \frac{C}{1-t})(1+r)] \ge 0\} = 1. \qquad \ldots(\text{VII-18})$$

We are now searching for a consumption function $C^*(W)$ and an investment function $a^*(W)$ maximising the functional (VII-17) subject to the constraint (VII-18) where $U(C) = C^\beta$, $0 < \beta < 1$.

Stipulating an arbitrary finite horizon and denoting $C_N(W)$ and $a_N(W)$ as the optimal decisions with respect to consumption and investment in the risky asset when there are N periods left to the horizon and the initial wealth is W, Problem (VII-17)-(VII-18) then is restated as

$$f_N(W) = \max_{\substack{0 \le C_N \le W(1-t) \\ a_N \ge 0}} \{U(C_N) + \alpha E(f_{N-1}((1-t)[a_N(X-r) + (W - \frac{C_N}{1-t})(1+r)]))\} \qquad \ldots(\text{VII-19})$$

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[5] The principle of optimality states that an optimal strategy has the property that whatever the initial state and the initial decision, the remaining decisions must constitute an optimal strategy with regard to the state resulting from the first decision [6].

s.t.

$$\Pr \{(1-t)[a_N(X-r) + (W-\frac{C_N}{1-t})(1+r)] \geq 0 \} = 1, \qquad \ldots(\text{VII-20})$$

where $f_N(W) \equiv 0$ for $N \leq 0$.

For $N = 1$ we have the corner solution $C_1 = W(1-t)$ and $f_1(W) = U[(1-t)W]$ $= (1-t)^\beta W^\beta$ since $f_o(W) = 0$ and $U(C) = C^\beta$.

For $N = 2$ we have

$$f_2(W) = \max_{\substack{0 \leq C_2 \leq W(1-t) \\ a_2 \geq 0}} \{C_2^\beta + \alpha E[(1-t)^\beta [a_2(X-r) + (W-\frac{C_2}{1-t})(1+r)]^\beta]\} \qquad \ldots(\text{VII-21})$$

s.t.

$$\Pr \{(1-t)[a_2(X-r) + (W - \frac{C_2}{1-t})(1+r)] \geq 0 \} = 1. \qquad \ldots(\text{VII-22})$$

Because of the specific form of the utility function, (VII-21) and (VII-22) can be rewritten as

$$f_2(W) = \max_{\substack{0 \leq C_2 \leq W(1-t) \\ a_2 \geq 0}} \{C_2 + \alpha((1-t)[W - \frac{C_2}{1-t}])^\beta . E[(\frac{a_2(X-r)}{(W-\frac{C_2}{1-t})} + 1+r)^\beta]\} \qquad \ldots(\text{VII-23})$$

s.t.

$$\Pr \{ \frac{a_2(X-r)}{(W - \frac{C_2}{1-t})} + (1+r) \geq 0 \} = 1. \qquad \ldots(\text{VII-24})$$

Before we can solve (VII-23) -(VII-24), let us note the following result due to Hakansson. It has been shown that the problem

$$\max_{Z \geq 0} h(Z) = \max_{Z \geq 0} \{E[\{(X-r)Z + (1+r)\}^\beta]\} \qquad \ldots(\text{VII-25})$$

s.t.

$$\Pr \{(X-r)Z + (1+r) \geq 0 \} = 1, \qquad \ldots(\text{VII-26})$$

has a maximum and the maximising value, Z^*, is finite and unique ([28], pp. 593-595). Identifying $a_2/(W-\frac{C_2}{1-t})$ as Z, Hagen argues from the solution to (VII-25)-(VII-26) that the maximum to (VII-23) with respect to a_2 subject to (VII-24) (keeping C_2 constant) is given by

$$f_2(W) = \max_{0 \leq C_2 \leq W(1-t)} \{C_2^{\beta} + \alpha k[(W-\frac{C_2}{1-t})(1-t)]^{\beta}\}, \qquad \ldots(\text{VII-27})$$

where $k=h(Z^*)$ and the maximising value of a_2 is given by the relation

$$\frac{a_2}{(W-\frac{C_2}{1-t})} = Z^*.$$

Hagen then obtains the first-order condition to (VII-27) with respect to C_2 as:

$$\frac{\partial f_2(W)}{\partial C_2} = \beta C_2^{\beta-1} - \alpha\beta k[(1-t)(W-\frac{C_2}{1-t})]^{\beta-1} = 0 \qquad \ldots(\text{VII-28})$$

Solving (VII-28) for C_2^* (optional C_2) we have

$$C_2^* = \frac{[1-(\alpha k)^{1/(1-\beta)}](1-t)^{-1}}{[1-(\alpha k)^{2/(1-\beta)}](1-t)^{-2}} \cdot W, \qquad \ldots(\text{VII-29})$$

and, from the definition of Z

$$a_2^*(W) = (W-\frac{C_2}{1-t})Z^*. \qquad \ldots(\text{VII-30})$$

Notice that for N=2, the above model is identical to the one studied in section VII.1. But when we substitute (VII-29) into (VII-30), the optimal risk-taking function seems to be independent of the tax and consequently the results of the preceding section, equations (VII-7) and (VII-8), are contradicted by this dynamic programming formulation of the

portfolio choice model. This confirms our claim that apparently the infinite horizon model does not yield, as a special case, the results obtained from the simple two-period models.

Returning to the infinite horizon problem, we substitute (VII-29) into (VII-27) to obtain an expression for $f_2(W)$ as a function of W alone. Iteratively then we can solve for $C_N{}^*(W)$, $a_N{}^*(W)$ by letting N=3,..., etc. Proceeding in this manner and defining

$$C^*(W) = \lim_{N\to\infty} C_N{}^*(W),$$

$$a^*(W) = \lim_{N\to\infty} a_N{}^*(W),$$

and assuming convergence, Hagen derives[6]

$$C^*(W) = [1-(\alpha k)^{1/1-\beta}]W(1-t), \qquad \ldots(VII\text{-}31)$$

$$a^*(W) = (\alpha k)^{1/1-\beta} Z^* W. \qquad \ldots(VII\text{-}32)$$

From these results Hagen concludes that the consumption tax has the same effect on net consumption as if initial wealth had been reduced by the consumption tax rate and consequently the optimal risk-taking strategies are unaffected.

But what Hagan fails to realise that although equations (VII-31) and (VII-32) describe a local maximum to the investor's problem (given by equations (VII-17) and (VII-18)), they are still in their <u>implicit</u> form. In particular, $k = h(Z^*)$ is a function of (a,C,t) from the definition of Z^*, and Hagen is incorrect when he treats k as a constant in deriving the first-order condition to the problem:

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[6]Strictly speaking, the convergence of $C_N{}^*(W)$, $a_N{}^*(W)$ together with the uniqueness of $a^*(W)$, $C^*(W)$, which derives from the concavity of U(C) and f(W), guarantees that (VII-31) and (VII-32) are the solutions to the problem described by the equations (VII-17-(VII-18).

$$f_2(W) = \max_{0 \leq C_2 \leq W(1-t)} \{ C_2^{\beta} + \alpha k[(W - \frac{C_2}{1-t})(1-t)]^{\beta} \} \quad ,$$

which (the first-order condition) is

$$C_2^{\beta-1} = \alpha k[(1-t)(W - \frac{C_2}{1-t})]^{\beta-1}$$

Thus we have shown that $C^*(W)$ and $a^*(W)$ as given by equations (VII-31) and (VII-32) are not fully specified in terms of the parameters of the model (in particular the tax rate), and hence cannot be used to analyse the effects of a change in the consumption tax rate (a parameter). This also explains why this infinite horizon algorithm, as used by Hagen, does not yield the results obtained from the simple two-period models as special cases.

VII.3 Conclusion

To conclude this chapter we observe that while continual taxation of the return to saving affects the accumulation of risky capital over time and this may have significant repercussions on optimal consumption and investment strategies under uncertainty, the particular analysis offered by Hagen is misleading. Thus the problem of the effect of a proportional consumption tax on optimal consumption and risk-taking decisions in an infinite horizon model remains unsolved.

PART 4: CONCLUSION

"...[Our] purpose...is not to provide a definite programme for fiscal policy. It is rather to show how problems of economic policy in general, and of fiscal policy in particular, ought to be dealt with in economic analysis"

(Hansen [29], p. xv).

CHAPTER VIII

SUMMARY OF RESULTS AND THEIR IMPLICATIONS

We have been concerned with the economic effects of taxation policy on household saving and investment decisions under uncertainty. A consideration of the alternative approaches to a theory of decision-making under uncertainty reveals that maximisation of expected utility is a fundamental axiom of rational behaviour (Chapter I). We have also observed that the rationale of minimising the probability of loss (i.e., the safety-first principle) may also be regarded as a rational decision-making rule in a world of uncertainty. Both of these approaches have been investigated by earlier writers. Specifically, a utility function with the arguments being the mean and the variance of the investment returns -- a special case of the expected utility approach -- has been very popular for well over a decade (e.g., see Markowitz [44], Tobin [72] and Bierwag and Grove [7]). On the other hand, the seminal work by Domar and Musgrave [16] and two other exploratory papers by Marschak [46] and Roy [59] were related to the rationale of minimising the probability of loss. The limitations of the former approach (i.e., mean-variance) have been reviewed in Chapter I, and in Chapter II we have shown that it has been difficult to construct a model of behaviour that is consistent with the safety-first principle and, in fact, none of the above-mentioned approaches succeeded in describing behaviour that is consistent with the minimisation of the probability of loss.

In Chapter III we have argued that a chance-constrained programming approach to the problem of portfolio choice can be regarded as a reasonable

description of the investor's concern for safety. In this framework it is also possible to unambiguously determine the effects of taxation (proportional and lump-sum) on risk-taking. In particular, we have seen that both a lump-sum tax and a proportional tax on investment income with full loss offsets encourage a movement from the low-risk assets towards the high-risk assets. This result is consistent with the original suggestion of Domar and Musgrave and also with the predictions of the mean-variance models. This result is not surprising since taxation with full loss offsets implies that the government is sharing the investor's risk. This leads to the distinction between total (social) risk (which includes the share of risk-taking of the government) and private risk (which is borne by the investor himself). The chance-constrained formulation does not yield any results on private risk-taking. However, if the utility function is defined in terms of the yield (expected value) and the risk (probable loss)[1] then if losses can be fully offset, a proportional tax on portfolio income leaves total risk-taking unchanged but private risk-taking is reduced; and, in the no-loss offset case, both total and private risk-taking are reduced (Chapter II).

In the context of the chance-constrained programming approach to portfolio selection, we have also observed that the qualitative results (e.g., the effects of taxation and portfolio separation) are the same, both for normally distributed asset returns and for the alternative assumption of a lognormal securities market.

Although these results are fairly straight-forward, the models

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[1]These definitions of "yield" and "risk" have been offered by Domar and Musgrave.

they emerge from are somewhat restrictive. The yield-risk model (as interpreted in Chapter II) implies that the investor is risk-neutral.[2] This is at variance with the common assumption in the theory of risk-taking behaviour that rational decision-makers are risk-averters. The chance-constrained programming models, on the other hand, also lack sufficient generality as they require specific assumptions regarding the probability distribution of the asset returns in order to derive any results. This is unsatisfactory since we do not have conclusive empirical evidence regarding the distribution of rates of return on investments.

The general expected utility of wealth formulation of the portfolio problem suggested by Arrow [3] overcomes the difficulties mentioned above. Mossin [48], Stiglitz [68] and Ahsan [1] have analysed the effects of proportional taxation in this context. These results suggest that a proportional tax on investment income with full loss offsets is most likely to encourage total risk-taking and to discourage (leave unchanged) private risk-taking if the rate of return on the riskless investment, r, is zero (positive). Taxing the equivalent of the riskless rate of return on the entire portfolio (i.e., exempting the risky capital gains (or loss) from taxation) discourages risk-taking if the risky asset is superior.

In Chapter IV we have extended these results for the case of a simple (linearly) progressive tax schedule. Assuming only that the investor is a risk-avertor and that the risky asset is superior, we have shown that linearly progressive taxation of investment income (with full loss offsets)

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[2]This is clear from the property of straight line indifference curves (see Chapter II, Figures II.2 and II.3). Alternatively, the assumption of constancy of the marginal utility of wealth (which implies risk neutrality) is rather restrictive.

encourages (total) risk-taking. The effect on private risk-taking is ambiguous if $r > 0$, but positive if $r = 0$. Likewise, linearly progressive taxation of the equivalent of the riskless return on the entire portfolio is seen to discourage risk-taking by generating a simple income effect. The intuitive reasoning behind these results are as follows. By allowing full loss offsets[3] and an exemption on risky income, income taxation reduces the probability of both large losses and of large gains. Effectively, therefore, the size of the bet has been reduced by taxation of investment income. On the other hand, exempting capital gains implies taxing riskless income and this generates an income effect in an obvious way. Moreover, by taxing the equivalent of the riskless return on the entire portfolio (not just the income from the riskless investment), we eliminate any possible substitution among the two forms of investments.

We have also shown that a linearly progressive tax on investment income leads to greater (total) risk-taking than a flat rate proportional tax where both these taxes lead to equal losses of expected utility for the investor, and also where both these taxes yield the same expected revenue. For instance, if both the tax structures are to lead to equal losses of expected utility, the marginal tax rate has to be higher in the progressive than in the proportional tax case, and this alone accounts for greater risk-taking in the progressive case.

Both the generality of these results (Chapter IV) and the fact that they are reasonably interpretable are surprising, especially in view of the restrictive nature of the results in the preceding literature. However,

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[3]As mentioned in Chapter IV, in these general expected utility models the assumption of incomplete loss offsets does not seem to yield any unambiguous results (e.g., see Stiglitz [68]).

it may still be argued that an ordering over the probability distribution of final wealth is not an adequate description of household decision processes. Evidently, household consumption-saving decisions and the investment (borrowing-lending) decisions are interrelated. This calls for an integration of portfolio and consumption allocation decisions over time. Such an integration allows households, by optimally choosing consumption, to vary the amount of investable wealth. Thus both the size and the composition of the optimal portfolio are simultaneously determined. Hence, it is very important to ask how the predictions of the single-period analysis (obtained under the assumption of a fixed investable wealth) are modified when we make the investment fund a variable. This has been the task of Chapter V.

In this latter context, we have shown that the kind of a priori restrictions that we have placed on the investor's preferences in the single-period analysis (namely, risk-aversion and the non-inferiority of the risky asset) are no longer sufficient (even if we now add the assumption that consumption is also superior) to determine the effects of taxation. Specifically, the effects of taxation of investment income in this two-period model have the same interpretation as those obtained in a single-period framework (Chapter IV) only if we have a logarithmic utility function in the intertemporal context. Notice that the results we obtained in the single-period framework were valid for any concave utility function of wealth. However, there are some special cases where the effects of taxation are unambiguously determined if we assume that the risky asset and consumption are superior (or alternatively, that absolute risk-aversion is decreasing and relative risk-aversion is non-

decreasing). These are as follows:

(a) Taxation of non-asset income (both proportional and the linearly progressive) typically discourages both current consumption and (total) risk-taking by generating simple income effects;

(b) The effect of a consumption tax (both proportional and the linearly progressive), likewise, is to reduce current consumption;

(c) Where the rate of return on riskless investment (r) is zero, a proportional tax on investment income leaves current consumption unchanged and increases it if we have a linearly progressive tax. On the other hand, (total) risk-taking is encouraged by a tax on investment income (both proportional and progressive) if $r = 0$. Further, private risk-taking is discouraged by a proportional consumption tax, and an investment income tax either leaves it unchanged (if the tax is proportional) or encourages it (if the tax is progressive).

Consumption taxes, in general, induce a substitution effect encouraging (total) risk-taking, but this is also accompanied by income effects and, hence, the results are indeterminate (even if $r = 0$). Investment income taxes, on the other hand, generate substitution effects encouraging both risk-taking and current consumption. In these cases we require both assumptions on relative risk-aversion and additional restrictions on the relative magnitudes of the income elasticities of consumption and the risky asset demand. The detailed results have been presented in Tables V.1, V.2 and V.3. However, to give an indication of the general results, let us mention the following: A proportional consumption tax discourages (total) risk-taking where relative risk-aversion is constant and the income elasticity of consumption is no larger

than that of the risky asset demand.

In the last chapter (VII), we have further indicated that if we arbitrarily extend the decision horizon to an infinite one, there is no compelling reason for the above results (obtained in a two-period model) to be drastically modified as, for instance, one might infer from a study by Hagen [26].

The unambiguity of our general results obtained in an intertemporal context, therefore, depends crucially on the relative magnitudes of the income elasticities of consumption and the risky asset demand. Our analysis also indicates the kind of empirical knowledge that is required in order to meaningfully discuss the implications of alternative taxation policies. By the same token, it suggests that if all the necessary measures of elasticities are obtainable, we can also determine which assumptions on the risk-aversion measures are more reasonable than others (either for the society as a whole or for different groups in society).

We further pursue the framework of intertemporal consumption-portfolio decision in Chapter VI, where we investigate the long-debated argument of the comparative effects of a tax on consumption rather than a tax on (investment) income. We find that where the rate of return on the riskless investment is zero, the tax rates are low (i.e., less than 50%) and where the expected rate of return on the risky investment is less than 100%, the balanced budget incidence of a consumption tax and of an investment income tax (with matching lump-sum transfers) is to encourage both current consumption (i.e., discourage saving) and risk-taking. It has also been shown that under these conditions and where the propensities to consume and to invest in the risky asset are identical under the

alternative budgetary policies and the two tax rates are equal, the differential incidence of a consumption tax is to encourage risk-taking and to discourage saving more effectively than a tax on investment income. The general consensus in the literature that a tax on income (investment income) discourages saving as compared to a tax on expenditure (consumption), where the two tax rates are equal, is not borne out by this simple model which incorporates uncertainty via capital risks.

Our important conclusion in this last section, therefore, is that, while the interpretation of the results obtained in this framework require additional knowledge of the investor's preferences (i.e., the propensities to consume and to invest in the risky asset) and other market and government parameters (i.e., the expected rates of returns, the tax rates etc.), which are not implied by the a priori restrictions on preferences alone, it establishes that the introduction of uncertainty affects the implications of fiscal policy in an important way.

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