INFINITE EXCHANGE SYSTEMS

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SCOPE AND CONTENTS: It is shown that the concept of a finite matroid can be extended to systems of arbitrary cardinality. These systems are included in the class of exchange systems. Properties of particular as well as abstract exchange systems are studied. We show that the collection of all circuits of a graph is a matroid and characterize their dendroids. The dual is defined for a subclass of exchange systems and we show it coincides with the original dual for finite matroids. The dual for the circuit matroid of a tree, as well as other examples, is shown to be a matroid whose dual coincides with the original matroid.

PRELIMINARIES

The notation to be employed is the usual one of set theory. Inclusion ($\supset$) will be taken in the broad sense and strict inclusion will be explicitly stated or written $\supsetneq$. The cardinal power of a set E will be denoted by $|E|$.

DEFINITIONS. A partially ordered set is a pair (E,P) where E is a set and P is a reflexive, anti-symmetric, transitive relation on E. For example, if E is a set of subsets of a set X then $(E,\supset)$ is a partially ordered set. That is, for $A,B \in E$, APB if and only if $A \supset B$.

Let (E,P) be a partially ordered set. A chain $\mathcal{C}$ in E is a subset of E such that if $A,B \in \mathcal{C}$ then APB or BPA.

A graph X is a pair (V(X),E(X)) where V(X) is a set and E(X) is a set of unordered pairs of distinct elements of V(X), that is, E(X) is a symmetric, irreflexive relation on V(X). The unordered pair of elements x,y will be denoted by [x,y]. The elements of V(X) will be called the vertices of X and the elements of E(X) the edges of X.

A subgraph Y of a graph X is a graph whose vertex and edge sets are respectively subsets of the vertex and edge sets of X. An edge e is incident with a vertex x if and only if e = [x,y] for some vertex y. Two edges e = [x,y] and $e' = [x',y']$ are adjacent if and only if exactly two of x,x',y,y' are equal.

Let X and Y be graphs. By $X \cup Y$ and $X \cap Y$ we mean the graphs defined by:

$$V(X \cup Y) = V(X) \cup V(Y)$$

$$E(X \cup Y) = E(X) \cup E(Y), \text{ and}$$

$$V(X \cap Y) = V(X) \cap V(Y)$$

$$E(X \cap Y) = E(X) \cap E(Y) \text{ respectively.}$$

If $x \in V(X)$ then (x) denotes the subgraph of X for which $V((x)) = \{x\}$ and $E((x)) = \emptyset$. If $e = [x,y] \in E(X)$ then (e) denotes the subgraph of X for which $V((e)) = [x,y]$ and $E((e)) = \{e\}$. If Y is a subgraph of X we define $X \backslash Y$ to be the smallest subgraph of X with $E(X \backslash Y) = E(X) - E(Y)$.

Let X and Y be graphs. By a <u>homomorphism</u> φ of X into Y we mean a function $\varphi : V(X) \to V(Y)$ such that $[\varphi(x), \varphi(y)] \in E(Y)$ whenever $[x,y] \in E(X)$. This homomorphism is written $\varphi : X \to Y$. φ induces $\varphi^{\#} : E(X) \to E(Y)$ as follows: for $[x,y] \in E(X)$, $\varphi^{\#}([x,y]) = [\varphi(x), \varphi(y)]$. φ is an <u>isomorphism</u> if and only if φ and $\varphi^{\#}$ are one-one and onto. $\varphi(e)$ will frequently be written for $\varphi^{\#}(e)$.

Let X be a graph. A <u>path</u> P joining x and y is a subgraph of X such that $V(P)$ is the set of elements of a finite sequence $(x_0, x_1, \ldots, x_n)$ with $x_0 = x$ and $x_n = y$, and

$$E(P) = \{[x_i, x_{i+1}] \mid 0 \leqq i \leqq n-1\} .$$

We shall denote the path by $[x_0, \ldots, x_n]$. A graph is <u>connected</u> if any two vertices in X are joined by a path in X, otherwise it is <u>disconnected</u>. A maximal connected subgraph is called a <u>component</u> of X.

Let X be a graph. For $x \in V(X)$ we let

$$V(X;x) = \{ y \mid [x,y] \in E(X) \} \text{, and}$$

$d(X;x) = |V(X;x)|$. $d(X;x)$ or d_x is called the <u>degree</u> of x in X. For $x \in V(X)$

and $d_x \geqq 3$, x is called a branch vertex. If d_x is finite for every vertex of X then X is said to be locally finite.

An Euler graph is a graph with d_x positive and even or infinite for every $x \in V(X)$. A circuit is a connected Euler graph where every vertex has degree 2. If the graph is finite we call it a finite circuit, otherwise it is an infinite circuit. A ray is a connected graph with a vertex x_0 such that

$$d_x = \begin{cases} 2, & x \neq x_0, \\ 1, & x = x_0 . \end{cases}$$

R is denoted by $R = [x_0, \ldots)$. x_0 is called the origin or initial vertex of R. A path $P = [x_0, \ldots, x_n]$, is non-degenerate if it contains a circuit, closed if $x_0 = x_n$, simple if $d_{x_i} \leqq 2$, $0 \leqq i \leqq n$, and simple closed if it is a circuit.

A tree is a connected graph with no finite circuits. A graph is circuit connected if and only if every edge is in a circuit.

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INTRODUCTION

The term matroid was first introduced into the literature by Whitney [12] as a pair $(E, \mathfrak{A})$ where E is a finite set and $\mathfrak{A}$ is a collection of subsets of E with the properties :

(i) if $A \subset B \in \mathfrak{A}$ then $A \in \mathfrak{A}$, and

I

(ii) if $A, B \in \mathfrak{A}$, $|A| = n$, $|B| = n+1$, then there is $b \in B$ such that $A' = A \cup \{b\} \in \mathfrak{A}$, $|A'| = n+1$.

A member of $\mathfrak{A}$ is called an independent set and a maximal such is a base. I(ii) occurs in a paper by Steinitz [11] and such a property is known as a Steinitz exchange relation, see for example Bleicher and Preston [6] (lemma 4).

Whitney also shows that I is equivalent as an axiom system to :

let r be a function on the subsets of E to the non-negative integers such that

(i) $r(\emptyset) = 0$,

II (ii) $r(A \cup \{b\}) = r(A)$ or $r(A) + 1$, and

(iii) if $r(A \cup \{b_1\}) = r(A \cup \{b_2\}) = r(A)$ then $r(A \cup \{b_1, b_2\}) = r(A)$.

$A \in \mathfrak{A}$ if and only if $r(A) = |A|$.

This form was extended to include infinite sets E by Rado [10] by enlarging $\mathfrak{A}$ to contain any set $X \subset E$ for which every finite subset belongs to $\mathfrak{A}$. Such a condition as this is known as one of finitary character. A basic property of these systems is that all bases have the same cardinality.

Edmonds [2] (section 1.5, prop.1) shows that I(i) and the axiom that all maximal members of $\mathfrak{A}$ have the same cardinality is again an equivalent axiom system to I. He further shows in proposition 3 that this is equivalent to the system

III (i) if $A \in \mathfrak{A}$ and $B \subset A$, $B \neq A$ then $B \notin \mathfrak{A}$, and

(ii) if $A, B \in \mathfrak{A}$ with $a \in A-B$ and $b \in A \cap B$ there is $C \in \mathfrak{A}$ with $a \in C \subset A \cup B - \{b\}$.

III is the form used by Tutte [5]. A property of matroids defined by scheme III that has been used to good advantage by Tutte is that of possessing a dendroid, a dendroid D being a subset of E such that $D \cap A \neq \emptyset$ for $A \in \mathfrak{A}$ and is minimal such. Note that one can always ask for this condition on an arbitrary collection of subsets and it would be reasonable to then call such a set a dendroid for that collection.

As a result of the many equivalent forms developed by the authors there is some confusion in terminology and a section (1.4) of Edmonds [2] sorts out some of this terminology. For example, Tutte's dendroids occuring in III are the bases in I.

Another form of generalization has been made by Dlab [7] who uses the following axiom scheme for a set E and system $\mathfrak{A}$ on E :

(i) if $A \subset B \in \mathfrak{A}$ and $A \neq B$ then $A \notin \mathfrak{A}$,

IV (ii) if $A, B \in \mathfrak{A}$ then for $a \in A$ there is $b \in B$ such that $(A-\{a\}) \cup \{b\} \in \mathfrak{A}$, and

(iii) for $X \subset E$ there is $Y \in \mathfrak{A}$, $X \subset Y$ if for every finite subset of X there is a member of $\mathfrak{A}$ containing it.

IV(iii) is also of finite character and IV is shown in [7] p.562 to be equivalent to

Rado's extension of II. This is then equivalent to the system called proper dependence relations in [6] (lemmas 7 and 8).

Among the many interesting properties that systems I and III have is that of possessing a dual. That is, if $\mathfrak{A}$ is a matroid then there is a matroid $\mathfrak{A}^*$ called the dual of $\mathfrak{A}$ for which $(\mathfrak{A}^*)^* = \mathfrak{A}$, and the dendroids for $\mathfrak{A}^*$ are the complements of the dendroids for $\mathfrak{A}$. This property of always having a dual is not shared by graphs as is shown by Whitney [13] (theorem 29). There is a remark in section 1.4 of [2] to the effect that Whitney's dual using I is then Tutte's III using the correspondence between these equivalent systems.

Our purpose in this thesis is to show that the concept of a matroid can be extended to some infinite systems without the usual axiom of finite character and to obtain a dual with the properties given in the last paragraph.

The pair $(E,\mathfrak{A})$ is an exchange system if E is a set and $\mathfrak{A}$ is a non-empty collection of non-empty subsets of E satisfying III(ii). If in addition the system satisfies III(i) and has a dendroid we call it a matroid.

In chapter one some general properties of exchange systems are studied. 1.9 is an example of an exchange system without a dendroid. 1.10 furnishes two examples of exchange systems, one of which has only finite members and the other only infinite members .These were suggested by those of Minty [8] (p. 489 exercises 2.2 and 2.3). 1.15 and 1.16 show that these are matroids and characterize their dendroids while 3.17 and 3.18 show in what manner they are dual one to the other.

For any exchange system with dendroids 1.17 shows the manner in which they satisfy the system IV(i),(ii) which is shown in [12] (section 7) to be equivalent to I. 1.31, 1.47 and 1.48 are concerned with the cardinality that a dendroid can have. One of the properties that one would like a dendroid to have is the local covering property (l.c.p.) introduced in 1.22. In 1.39 an equivalence relation on the set of dendroids for a matroid is defined and 1.45 shows that l.c.p. is compatible with this equivalence. For finite matroids there is just a single equivalence class. 2.61 implies that there are infinite systems with a large number of equivalence classes. 1.46 shows that each class determines a distinct submatroid of the system. 1.71 to 1.75 are concerned with the relation between dendroids for the finite sets in a matroid $\mathfrak{M}$ and dendroids for $\mathfrak{M}$ itself. 1.77 states that under suitable conditions there are matroids $(E,\mathfrak{A})$ and $(E,\mathfrak{B})$ with $\mathfrak{A} \subset \mathfrak{B}$, and 2.4 is a realization of this.

The first part of chapter two develops examples of infinite matroids with considerably more structure than those of chapter one, while the second part shows for any graph X with a circuit that the collection of circuits is a matroid. We call this a circuit matroid to distinguish it from other possible exchange systems derivable from a graph. The essential steps are in 2.20 and 2.25. The third part of chapter two characterizes the dendroids for the circuit matroid of any graph while the final section develops some of the special properties of dendroids for a circuit matroid.

Chapter three is concerned with the dual of a matroid. The definition of a dual used by Tutte [5] and others does not readily lend itself to our systems. Hence we redefine the dual for a matroid $\mathfrak{A}$ and denote it by $\mathfrak{A}^{\#}$. After a few preliminaries we show in 3.14 that for any matroid whose elements are finite sets that $\mathfrak{A}^{\#}$ is a matroid with $(\mathfrak{A}^{\#})^{\#} = \mathfrak{A}$ and that the dendroids for the one are the complements of the dendroids of the other. In particular for finite matroids we obtain $\mathfrak{A}^{\#} = \mathfrak{A}^{*}$. We next show that for the matroids constructed as examples that the dual is again a matroid with the desired properties. In 3.23 we show that a circuit matroid with the l.c.p. has as its dual a matroid again with the same properties and since 2.73 shows that circuit matroid of a tree has the l.c.p. Hence for this infinite system whose elements are infinite we have not only generalized the concept of a matroid but also that of the dual matroid .

The final part of chapter three contains some scattered results on the dual of a circuit matroid. In particular 3.37 describes what the elements of the dual look like in this case.

There is the possibility that the method of contraction used in chapter two for a characterization of a dendroid for circuit matroids can be extended to any graph since this is much the same idea as used by Nash-Williams [9](p. 227 and 230) for the decomposition of a graph into circuits and rays.

CHAPTER I

EXCHANGE SYSTEMS

DEFINITION (1.1) Let E be a fixed set. $\mathfrak{A}$ is a *system* on E if and only if $\mathfrak{A}$ is a non-empty collection of non-empty subsets of E, and is an *inductive system* if the partially ordered set $(\mathfrak{A}, \supset)$ is inductive, i.e., $\bigcap\mathfrak{B} \in \mathfrak{A}$ for any chain $\mathfrak{B} \subset \mathfrak{A}$.

DEFINITION (1.2) A system $\mathfrak{A}$ on E has the *exchange property* if for any $A,B \in \mathfrak{A}$ and $a \in A-B$, $b \in A \cap B$ there exists $C \in \mathfrak{A}$ with $a \in C \subset A \cup B - \{b\}$. A system with the exchange property will be called an *exchange system*.

DEFINITION (1.3) Let $\mathfrak{A}$ be a system on E. $A \in \mathfrak{A}$ is a *minimal set* in $\mathfrak{A}$ if $B \subset A$ and $B \neq A$ implies $B \notin \mathfrak{A}$.

LEMMA (1.4) *Let* $\mathfrak{A}$ *be an exchange system on* E. *If* $A \in \mathfrak{A}$ *is not minimal and* $a \in A$ *then there exist proper subsets* B *and* C *of* A *such that* $a \notin B$, $a \in C$ *and* $B,C \in \mathfrak{A}$.

Proof. A not minimal in $\mathfrak{A}$ means there is a member of $\mathfrak{A}$ that is a proper subset of A. If $C \subset A$, $C \neq A$, and $a \in C$, then there is $b \in A-C$. By the exchange property there is $B \in \mathfrak{A}$ with $b \in B \subset A \cup C - \{a\} = A - \{a\}$. Similarly, if $B \subset A$, $B \neq A$, $a \notin B$ then there is $C \in \mathfrak{A}$ with $a \in C \subset A \cup B - \{b\}$, $a \in C \subset A - \{b\}$, where $b \in A \cap B$.

LEMMA (1.5) *If $\mathfrak{A}$ is an inductive exchange system on* E *then for* a $\in \bigcup\mathfrak{A}$ *there is* A $\in \mathfrak{A}$ *with* a $\in$ A *and* A *minimal in* $\mathfrak{A}$.

Proof. a $\in \bigcup\mathfrak{A}$ implies a $\in$ B $\in \mathfrak{A}$ for some B. Take a maximal chain $\mathfrak{B}$ in $\mathfrak{A}$ such that a $\in$ B for each B $\in \mathfrak{B}$. Then a $\in \bigcap\mathfrak{B}$ = A which is in $\mathfrak{A}$. If A is not minimal then there is C $\in \mathfrak{A}$ with a $\in$ C $\subset$ A, C $\neq$ A by lemma (1.4). But then $\mathfrak{B} \cup \{C\}$ is a chain in $\mathfrak{A}$ properly containing $\mathfrak{B}$ with a $\in$ C, a contradiction.

LEMMA (1.6) *Let $\mathfrak{A}$ be an inductive exchange system on* E. *Then the collection of minimal sets of* $\mathfrak{A}$ *is an exchange system on* E.

Proof. The minimal sets of $\mathfrak{A}$ are the intersections of maximal chains from $\mathfrak{A}$ and belong to $\mathfrak{A}$. Hence they are non-empty and by lemma (1.5) at least one such set exists. This collection is then a system on E.

Now take A and B minimal in $\mathfrak{A}$ with a $\in$ A–B and b $\in$ A$\cap$B. By the exchange property there is a C $\in \mathfrak{A}$ such that a $\in$ C $\subset$ A$\cup$B – $\{b\}$. By the inductiveness of $\mathfrak{A}$ there is a minimal C$'$ $\in \mathfrak{A}$ with a $\in$ C$'$ $\subset$ C, so that a $\in$ C$'$ $\subset$ A$\cup$B – $\{b\}$, as required. Hence the system of minimal sets has the exchange property.

NOTATION (1.7) If $\mathfrak{A}$ is a system on E then the collection of minimal sets of $\mathfrak{A}$ will be denoted by $\mathfrak{A}_{min}$.

REMARK (1.8) Lemma (1.6) clearly holds if $\mathfrak{A}$ is an exchange system on E consisting of finite subsets of E.

REMARK (1.9) The following examples show that there exist exchange systems which have no minimal sets.

(i) Let E be any infinite set and take $\mathfrak{A}$ to be the system $\mathfrak{A} = \{A \subset E \mid E-A \text{ is finite}\}$. If $A,B \in \mathfrak{A}$ and $a \in A-B$ and $b \in A \cap B$ then $A \cup B - \{b\}$ is in $\mathfrak{A}$ for $|E - (A \cup B - \{b\})| \leqq |E-A| + 1 < \aleph_0$. Thus $\mathfrak{A}$ has the exchange property. If $A \in \mathfrak{A}$ and $a \in A$ then $A - \{a\} \in \mathfrak{A}$; hence $\mathfrak{A}$ has no minimal sets.

(ii) Let $\mathfrak{A}$ be the collection of all non-empty open sets of a T_1-space E with no isolated points. Let $A \in \mathfrak{A}$ with a,b in A. Using the T_1 property there is C in $\mathfrak{A}$ with $a \in C$ and $b \notin C$. Hence $a \in A \cap C = C' \subset A - \{b\}$, with C' in $\mathfrak{A}$. Thus no member of $\mathfrak{A}$ is minimal. Now if $A,B \in \mathfrak{A}$ with $a \in A-B$ and $b \in A \cap B$ then this same set C' has $a \in C' \subset A - \{b\} \subset A \cup B - \{b\}$ and $\mathfrak{A}$ has the exchange property.

REMARK (1.10) The following examples show there are exchange systems all of whose members are finite and minimal and exchange systems all of whose members are infinite and minimal. The first is a well known example of an infinite exchange system

(i) Let E be any infinite set and k any positive integer. Consider the system

$$\mathfrak{M}_k = \{A \subset E \mid |A| = k\}.$$

Clearly every set in $\mathfrak{M}_k$ is finite with exactly k elements and hence each member is minimal. If $A,B \in \mathfrak{M}_k$ with $a \in A \cap B$ and $b \in A-B$ then $a \in A \cup B - \{b\}$ which has at least k elements. Thus there is a C contained in $A \cup B - \{b\}$ with $a \in C$ and $|C| = k$. Thus $C \in \mathfrak{M}_k$ and $\mathfrak{M}_k$ has the exchange property.

(ii) Let E be any infinite set and k any positive integer. Consider the system

$$\mathcal{N}_k = \{A \subset E \mid |E-A| = k\} .$$

Clearly $A \in \mathcal{N}_k$ implies A is infinite and if $B \subset A$, $B \neq A$ then $|E-B| > k$, hence the members of $\mathcal{N}_k$ are minimal. Let $A,B \in \mathcal{N}_k$ with $a \in A-B$ and $b \in A \cap B$. Since $E-(A \cup B - \{b\})$ has k or fewer elements there is C in $A \cup B - \{b\}$ with $a \in C$ and $|E-C| = k$, so that $C \in \mathcal{N}_k$. Thus $\mathcal{N}_k$ has the exchange property.

<u>DEFINITION</u> (1.11) Let $\mathfrak{A}$ be a system on E, and

$$\vartheta = \{D \subset E | D \cap A \neq \phi \text{ for each } A \in \mathfrak{A}\}.$$

For $D \in \vartheta$ let F_D be the set of functions $f:D \to \mathfrak{A}$ with $f(x) \cap D = \{x\}$ for each $x \in D$.D is a <u>dendroid</u> if D is a minimal set in ϑ (relative to inclusion). We shall denote the set of all dendroids for $\mathfrak{A}$ by $\vartheta_{\mathfrak{A}}$. It is to be noted that $E \in \vartheta$ and that F_D may be empty. If $f \in F_D$, then f is one-one.

<u>REMARK</u> (1.12) If $\mathfrak{A} \subset \mathfrak{B}$ are systems on E and D is a dendroid for $\mathfrak{A}$ such that $B \cap D \neq \phi$ for each $B \in \mathfrak{B}$, then D is a dendroid for $\mathfrak{B}$.

<u>REMARK</u> (1.13) <u>If</u> $\mathfrak{A}$ <u>is a system on</u> E <u>consisting of finite subsets of</u> E, <u>then dendroids for</u> $\mathfrak{A}$ <u>exist</u>.

Take ϑ as in (1.11), and partially order ϑ by inclusion. Take any chain ϑ_0 in ϑ and let $D = \bigcap_{X \in \vartheta_0} X$. If $D \cap A = \phi$ for some $a \in \mathfrak{A}$ where $A = \{a_1,\ldots,a_n\}$ then for each i there is $X_i \in \vartheta_0$ such that $a_i \notin X_i$, $1 \leqq i \leqq n$. Now $\bigcap_{1 \leqq i \leqq n} X_i \in \vartheta$ and $(\bigcap_{1 \leqq i \leqq n} X_i) \cap A = \phi$, which is a contradiction. Thus ϑ is inductive and minimal sets exist. These minimal sets are the dendroids for $\mathfrak{A}$

<u>LEMMA</u> (1.14) <u>Let</u> $\mathfrak{A}$ <u>be a system on</u> E. D <u>is a dendroid for</u> $\mathfrak{A}$ <u>if and only if</u> $F_D \neq \phi$. <u>Moreover, if</u> $\mathfrak{A}$ <u>has the exchange property and</u> D <u>is a dendroid for</u> $\mathfrak{A}$ <u>then</u> $|F_D| = 1$. <u>When this is the case the single function in</u> F <u>is written as</u> f_D.

<u>Proof</u>. Let D be a dendroid for $\mathfrak{A}$ and take $x \in D$. Then $(D - \{x\}) \cap A_x = \phi$ for some $A_x \in \mathfrak{A}$ so that the function $f: D \to \mathfrak{A}$ with $f(x) = A_x$ for each $x \in D$ belongs to F_D.

Conversely, if $f \in F_D$ then for $x \in D$, $f(x) \cap (D - \{x\}) = \phi$ and $f(x) \in \mathfrak{A}$. Thus D is minimal in E with $D \cap A \neq \phi$ for all $A \in \mathfrak{A}$, that is, D is a dendroid for $\mathfrak{A}$.

Let $\mathfrak{A}$ be an exchange system and D a dendroid for $\mathfrak{A}$. Suppose f and g are in F_D with $f \neq g$. Then for some $x \in D$, $f(x) \neq g(x)$ with $y \in f(x) + g(x)$ and $x \in f(x) \cap g(x)$. Using the exchange property there is $A \in \mathfrak{A}$ with $y \in A \subset f(x) \cap g(x) - \{x\}$, and as a result $D \cap A = \phi$, a contradiction. Hence $f \in F_D$ is unique.

<u>REMARK</u> (1.15) <u>Using (1.13) the systems</u> $\mathfrak{M}_k$ <u>of (1.10) (i) have dendroids. For each</u> $k > 1$ <u>the dendroids of</u> $\mathfrak{M}_k$ <u>are precisely the systems</u> $\mathfrak{N}_{k-1}$.

If $D \in \mathfrak{N}_{k-1}$ then $A \subset E$ with $A \cap D = \phi$ implies $A \subset E - D$ and so $|A| < k$, and hence $A \notin \mathfrak{M}_k$. For each x in D the set $\{x\} \cup (E-D) = A$ has $|A| = k$ and $A \cap D = \{x\}$. Thus $A \in \mathfrak{M}_k$ and $f_D(x) = \{x\} \cup (E-D)$.

Conversely, if D is a dendroid for $\mathfrak{M}_k$ then for $x \in D$, $f_D(x) \cap D = \{x\}$.

Because $E-D \supset f_D(x) - \{x\}$, $|E-D| \geqq k-1$. If $E-D \neq f_D(x) - \{x\}$ then there is $A \subset E - D$ with $|A| = k$. i.e., $A \in \mathfrak{M}_k$ and $A \cap D = \phi$, a contradiction. Hence $D \in \mathfrak{N}_{k-1}$, and $\vartheta_{\mathfrak{M}_k} = \mathfrak{N}_{k-1}$.

<u>REMARK</u> (1.16) <u>The systems $\mathfrak{N}_k$ of (1.10) (ii) have dendroids and</u> $\vartheta_{\mathfrak{N}_k} = \mathfrak{M}_{k+1}$.

Let D be a dendroid for $\mathfrak{N}_k$. For $x \in D$ there is $f_D(x) \in \mathfrak{N}_k$ with $D \subset (E - f_D(x)) \cup \{x\}$. Hence $|D| \leqq k+1$. If $|D| \leqq k$ then there is $A \in \mathfrak{N}_k$ with $D \subset E - A$, and so $A \cap D = \phi$. Thus $|D| = k+1$ and $D \in \mathfrak{M}_{k+1}$.

Take any $D \in \mathfrak{M}_{k+1}$, $(|D| = k+1)$. For $x \in D$ consider $A = (E-D) \cup \{x\}$. $|E-A| = |D - \{x\}| = k$. Hence $A \in \mathfrak{N}_k$ and $D \cap A = \{x\}$. If $B \in \mathfrak{N}_k$ with $D \cap B = \phi$ then $D \subset E-B$ and so $|D| \leqq |E-B| = k$, a contradiction. Hence $D \in \vartheta_{\mathfrak{N}_k}$ and so $\vartheta_{\mathfrak{N}_k} = \mathfrak{M}_{k+1}$.

<u>LEMMA</u> (1.17) <u>Let $\mathfrak{A}$ be an exchange system on E and D a dendroid for $\mathfrak{A}$. For each $a \in D$, $D' = (D - \{a\}) \cup \{a'\}$ is a dendroid for $\mathfrak{A}$ if and only if $a' \in f_D(a)$, and in this case $f_D(a) = f_{D'}(a')$ while for $x \in D - \{a\}$, $f_D(x) = f_{D'}(x)$ if and only if $a' \notin f_D(x)$.</u>

<u>Proof</u>. If $D' \cap A = \phi$ for some $A \in \mathfrak{A}$, then $(D - \{a\}) \cap A = \phi$ so that $A = f_D(a)$ and $a' \notin f_D(a)$. Hence if $a' \in f_D(a)$ then $D' \cap A \neq \phi$ for any $A \in \mathfrak{A}$. Now if $a' \in f_D(x)$ let $x \in A_x \subset f_D(x) \cup f_D(a) - \{a'\}$ where $A_x \in \mathfrak{A}$ by the exchange property. If $a' \notin f_D(x)$ let $A_x = f_D(x)$. Then for $x \neq a$ $f_D(a) \cap D' = \{a\}$ and $A_x \cap D' = \{x\}$. Hence D' is a dendroid with $f_D(a) = f_{D'}(a')$, $f_D(x) = f_{D'}(x)$ if $a' \notin f_D(x)$ and $f_D(x) \neq f_{D'}(x)$ if $a' \in f_D(x)$.

Conversely, let D' be a dendroid. Then $\emptyset \neq D' \cap f_D(a) = ((D - \{a\}) \cup \{a'\}) \cap f_D(a) = \{a'\} \cap f_D(a)$. Hence $a' \in f_D(a)$ and $f_D(a) = f_{D'}(a')$. For $x \in D - \{a\}$ with $a' \notin f_D(x)$, $\emptyset = (D - \{x\}) \cap f_D(x) = (D' - \{a',x\}) \cap f_D(x) = (D' - \{a',x\}) \cap f_D(x) = (D' - \{x\}) \cap f_D(x)$. Hence $f_D(x) = f_{D'}(x)$. If $a' \in f_D(x)$, $x \in D - \{a\}$ then $f_D(x) \cap (D' - \{x\}) = f_D(x) \cap ((D - \{a,x\}) \cup \{a'\}) \neq \emptyset$ and so $f_D(x) \neq f_{D'}(x)$.

REMARK (1.18) *Let* D *be a dendroid for an exchange system* $\mathfrak{A}$ *on* E. *If* $B = E-D$ *and* $b \in B$, $a \notin B$ *then* $B_0 = (B - \{b\}) \cup \{a\}$ *is not the complement of a dendroid for* $\mathfrak{A}$ *if and only if there is* $Y \in \mathfrak{A}$ *with* $Y \subset B_0$. *In this case the set* Y *is unique and* $Y = f_D(a)$. This follows from lemma (1.17) because $E - B_0 = D_0 = ((E-B) - \{a\}) \cup \{b\} = (D - \{a\}) \cup \{b\}$ is a dendroid if and only if $b \in f_D(a)$. Hence if D_0 is not a dendroid $f_D(a) \subset B_0$ and is unique in $\mathfrak{A}$ with this property because of the exchange property.

LEMMA (1.19) *Let* $\mathfrak{A}$ *be an exchange system on* E *and* D *a dendroid for* $\mathfrak{A}$. *Then* $f_D(a) \in \mathfrak{A}_{min}$ *for each* $a \in D$.

Proof. If $f_D(a)$ is not minimal in $\mathfrak{A}$ then by (1.4) there is $A \in \mathfrak{A}$, $a \notin A \subset f_D(a)$, which is a contradiction to D being a dendroid because $D \cap A \subset D \cap (f_D(a) - \{a\}) = \emptyset$.

LEMMA (1.20) Let $\mathfrak{A}$ be an exchange system on E and D a dendroid for $\mathfrak{A}$.

(i) If $A \cap D$ is finite for some $A \in \mathfrak{A}$ then

$$A \subset \cup \{f_D(a) \mid a \in A \cap D\}$$

with equality if $|A \cap D|$ is least, the minimum taken over all dendroids for $\mathfrak{A}$.

(ii) If $A \cap D = \{a_1, a_2\}$ then $A \supset f_D(a_1) + f_D(a_2)$.

Proof. (i) Let $A \cap D = \{a_1, \ldots, a_n\}$, and suppose $y \in A - \bigcup_{i=1}^{n} f_D(a_i)$. Take $A_0 = A$ and inductively define the sets A_k for $1 \leqq k \leqq n$ as follows. If $a_k \notin A_{k-1}$ take $A_k = A_{k-1}$.
If $a_k \in A_{k-1}$ take $A_k \in \mathfrak{A}$ with $y \in A_k \subset (A_{k-1} \cup f_D(a_k)) - \{a_k\}$. By the exchange property this is possible since $y \in A_0$. For each k, $\{a_1, \ldots, a_k\} \cap A_k = \phi$ while $A_k \subset A \cup \bigcup_{i=1}^{k} f_D(a_i)$ by this construction. In particular $A_n \subset A \cup \bigcup_{i=1}^{n} f_D(a_i)$ so that $D \cap A_n \subset \{a_1, \ldots, a_n\}$ while $\{a_1, \ldots, a_n\} \cap A_n = \phi$. Thus $D \cap A_n = \phi$, which is impossible. Thus $A \subset \cup \{f_D(a) \mid a \in A \cap D\}$.

Take D in $\mathcal{D}_{\mathfrak{A}}$ such that $|D \cap A|$ is least. If for some $a \in D \cap A$ there is $y \in f_D(a) - A$ then $D' = (D - \{a\}) \cup \{y\}$ is a dendroid by (1.17) and $|A \cap D'| < |A \cap D|$, a contradiction. Hence for $|A \cap D|$ least

$$A = \cup \{f_D(a) \mid a \in A \cap D\}.$$

(ii) Let $A \cap D = \{a_1, a_2\}$. Then $A \subset f_D(a_1) \cup f_D(a_2)$. Without loss of generality suppose $y \in f_D(a_1) - f_D(a_2)$. $D' = (D - \{a_1\}) \cup \{y\}$ is a dendroid by (1.17) with $f_D(a_1) = f_{D'}(y)$ and $f_{D'}(a_2) = f_D(a_2)$. Now $A \cap D' = A \cap ((D - \{a_1\}) \cup \{y\}) = \{a_2\}$ and so $A = f_{D'}(a_2) = f_D(a_2)$ and $A \cap D = \{a_2\}$, a contradiction. Thus $A \supset f_D(a_1) + f_D(a_2)$.

REMARK (1.21) Let $\mathfrak{A}$ be an exchange system, D a dendroid for $\mathfrak{A}$. Then given any $A \in \mathfrak{A}_{min}$ with $D \cap A$ finite then for each $a' \in A$ there is a $D' \in \vartheta_{\mathfrak{A}}$ such that $f_{D'}(a') = A$.

By (1.20) for $|D \cap A|$ least, $A = \cup\{f_D(a) \mid a \in A \cap D\}$. Because $A \in \mathfrak{A}_{min}$, $A \cap D = \{a\}$ and $A = f_D(a)$. Now $a' \in f_D(a)$. Hence $D' = (D - \{a\}) \cup \{a'\}$ is a dendroid and $A = f_{D'}(a')$ by (1.17).

DEFINITION (1.22) A system $\mathfrak{A}$ on E is weakly locally finite if for every $A \in \mathfrak{A}$ there exists a dendroid D for $\mathfrak{A}$ such that $A \cap D$ is finite. $\mathfrak{A}$ is locally finite if there exists a dendroid D for $\mathfrak{A}$ such that $D \cap A$ is finite for every $A \in \mathfrak{A}$. The term "locally finite" will also be applied to such a dendroid. A dendroid D for $\mathfrak{A}$ will be said to have the local covering property (l.c.p.) if and only if

(1.22.1) $$A \subset \cup \{f_D(x) \mid x \in A \cap D\}$$

for every $A \in \mathfrak{A}$. If every dendroid for $\mathfrak{A}$ has the l.c.p. then $\mathfrak{A}$ itself will be said to have the l.c.p.

DEFINITION (1.23) An exchange system $\mathfrak{A}$ all of whose elements are incomparable and with $\vartheta_{\mathfrak{A}} \neq \phi$ will be called a matroid.

REMARK (1.24) If $\mathfrak{A}$ is a weakly locally finite exchange system then $\mathfrak{A}_{min}$ is a matroid.

REMARK (1.25) *If* $\mathfrak{A}$ *is an exchange system and* $D \in \vartheta_{\mathfrak{A}}$ *is locally finite then* D *has the l.c.p.* This is lemma (1.20)(i).

REMARK (1.26) *If* $\mathfrak{A}$ *is a weakly locally finite exchange system then remark* (1.21) *is a converse to lemma* (1.19).

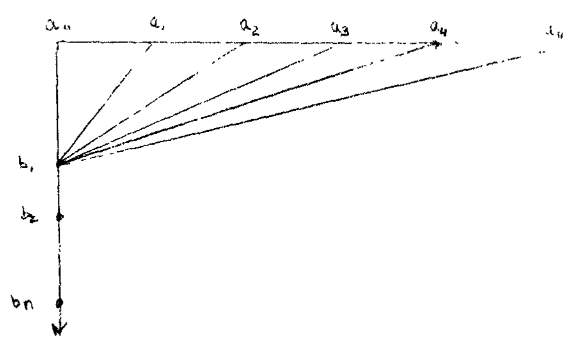
REMARK (1.27) *If a system* $\mathfrak{A}$ *on* E *consists of finite subsets of* E *then every dendroid for* $\mathfrak{A}$ *is locally finite*. In particular this is true for the system $\mathcal{M}_k$ of (1.10). For the system $\mathcal{N}_k$ of (1.10) the dendroids are the members of $\mathcal{M}_{k+1}$, which are finite, and so all the dendroids for $\mathcal{N}_k$ are locally finite.

REMARK (1.28) *Not all exchange systems have the l.c.p.* Consider the graph X shown in figure 1. Let E be the set E(X) of edges in X. Each circuit (finite or infinite) in X has a unique set of edges E(C). Let the system $\mathfrak{A}$ on E be

$$\mathfrak{A}(X) = \{E(C) \mid C \text{ a circuit in } X\}.$$

$\mathfrak{A}(X)$ is an exchange system called the circuit matroid for X.

FIGURE 1



Let the edges be $\alpha_i = [a_i, a_{i+1}]$ and $\gamma_i = [b_1, a_i]$ for $i \geqq 0$, and $\beta_i = [b_i, b_{i+1}]$ for $i > 1$.

Every finite circuit is of the form (a_i, b_1, a_j) $i \neq j$ and every infinite circuit is of the form $(\ldots, b_n, \ldots b_1, a_i, \ldots)$.

A dendroid for the circuit matroid is

$$D = \{\alpha_i \mid i \geqq 0\} \text{ with } f_D(a_i) = E((a_i, b_1, a_{i+1})).$$

The circuit $(\ldots b_n, \ldots, b_1, a_i, \ldots)$ meets every member of D while

$$\bigcup_{i \geqq 0} f_D(\alpha_i) = \{\alpha_i \mid i \geqq 0\} \cup \{\gamma_i \mid i \geqq 0\} \not\supset \{\beta_j\},\ j > 1.$$

Hence D does not have the l.c.p.

<u>LEMMA</u> (1.29) <u>Let</u> D <u>be a dendroid for</u> $\mathfrak{A}$ <u>with the l.c.p.</u> <u>Then</u>

(1.29.1) $$\bigcup \mathfrak{A} = \bigcup \{f_D(x) \mid x \in D\}.$$

<u>Proof</u>. For $A \in \mathfrak{A}$, $A \subset \bigcup \{f_D(a) \mid a \in A \cap D\} \subset \bigcup \{f_D(x) \mid x \in D\}$. Hence $\bigcup \mathfrak{A} \subset \{f_D(x) \mid x \in D\}$ and the equality holds.

<u>COROLLARY</u> (1.30) <u>If</u> $\mathfrak{A}$ <u>is an exchange system and</u> D <u>is locally finite then</u> $$\bigcup \mathfrak{A} = \bigcup \{f_D(x) \mid x \in D\}.$$

<u>Proof</u>. This follows immediately from (1.25).

<u>LEMMA</u> (1.31) <u>Let</u> $\mathfrak{A}$ <u>be an exchange system with some</u> $D \in \vartheta_{\mathfrak{A}}$ <u>finite</u>. <u>Then all members of</u> $\vartheta_{\mathfrak{A}}$ <u>are finite and have the same cardinality</u>.

<u>Proof</u>. Suppose D_0 is a dendroid with $|D_0| > |D|$. Choose a dendroid D_1 with $|D_1| = |D|$ and $|D_1 - D_0|$ least. If $x \in D_1 - D_0$ then because $D_0 \cap f_{D_1}(x) \neq \phi$ there is $y \in D_0$, $y \neq x$ and $y \in f_{D_1}(x)$.

By lemma (1.17) $D_2 = (D_1 - \{x\}) \cup \{y\}$ is a dendroid and $|D_2| = |D_1| = |D|$ while $|D_2 - D_0| < |D_1 - D_0|$ which contradicts the choice of D_1. Hence $D_1 \subset D_0$ and thus $D_1 = D_0$ by the minimality of a dendroid. By symmetry then, if one dendroid is finite they are all finite and they have a common cardinality.

LEMMA (1.32) *Let $\mathfrak{A}$ be an exchange system with dendroid D_0. Then there is an exchange system $\mathfrak{A}_0 \subset \mathfrak{A}_{min}$ for which D_0 is a locally finite dendroid.*

Proof. Take

$$\mathfrak{A}_0 = \{A \in \mathfrak{A}_{min} | A \cap D_0 \text{ is finite}\}.$$

Let A, B $\in \mathfrak{A}_0$ with $a \in A-B$ and $b \in A \cap B$. Then there is $C \in \mathfrak{A}$ with $a \in C \subset A \cup B - \{b\}$. Since $A \cap D_0$ is finite and $B \cap D_0$ is finite so is $C \cap D_0$. By lemma (1.20)(i) there is a dendroid D with $C = \cup \{f_D(x) | x \in D \cap C\}$. Hence $a \in f_D(x) \subset C \subset A \cup B - \{b\}$ for at least one $x \in D \cap C$. $f_D(x)$ is a minimal set in $\mathfrak{A}$ and $f_D(x) \cap D_0$ is finite. Thus $\mathfrak{A}_0$ is an exchange system. $A \in \mathfrak{A}_0$ implies $A \cap D_0$ is finite by the definition of $\mathfrak{A}_0$ and $f_{D_0}(x) \in \mathfrak{A}_0$ for each $x \in D_0$. Thus D_0 is locally finite on $\mathfrak{A}_0$.

LEMMA (1.33) *Let $\mathfrak{A}$ be a system on E and $X \subset E$ such that $X \cap A$ is finite and non-empty for each $A \in \mathfrak{A}$. For $Y \subset X$ there is a Z minimal in X such that $Y \subset Z \subset X$ and $Z \cap A \neq \phi$ for each $A \in \mathfrak{A}$. Moreover, if $Y = \phi$ then Z is a (locally finite) dendroid for $\mathfrak{A}$.*

Proof. Take a chain $\mathcal{C}$ of subsets of X such that for $C \in \mathcal{C}$, $Y \subset C$ and $C \cap A \neq \phi$ for any $A \in \mathfrak{A}$. Suppose $(\cap \mathcal{C}) \cap A = \phi$ for some $A \in \mathfrak{A}$. Take $C_0 \in \mathcal{C}$ and denote the elements of $C_0 \cap A$ by $a_1, \ldots, a_n$. Then there is $C_i \in \mathcal{C}$ with $a_i \notin C_i$, $1 \leqq i \leqq n$. Hence $C = \bigcap_{0 \leqq i \leqq n} C_i \in \mathcal{C}$ and $C \cap A = \phi$, a contradiction. Hence minimal elements Z exist in X containing Y with $Z \cap A \neq \phi$ for each $A \in \mathfrak{A}$. Clearly, if $Y = \phi$ then for each $z \in Z$ there is $A \in \mathfrak{A}$ with $(Z - \{z\}) \cap A = \phi$. Hence Z is a dendroid for $\mathfrak{A}$ and is locally finite by definition (1.22).

THEOREM (1.34) If D_0 and D_1 are locally finite dendroids for an exchange system $\mathfrak{A}$ then $|D_0| = |D_1|$.

Proof. Let

$$\mathfrak{A}_i = \{A \in \mathfrak{A}_{min} | A \cap D_i \text{ is finite}\}, \ i = 0,1.$$

By lemma (1.33) there is $D_i' \subset D_i$ such that D_i' is a locally finite dendroid for $\mathfrak{A}_{1-i}$ whence

$$|D_i| \geqq |D_i'| = |D_{1-i}| \geqq |D_{1-i}'| \text{ for } i = 0,1 \text{ by lemma (1.32)}$$

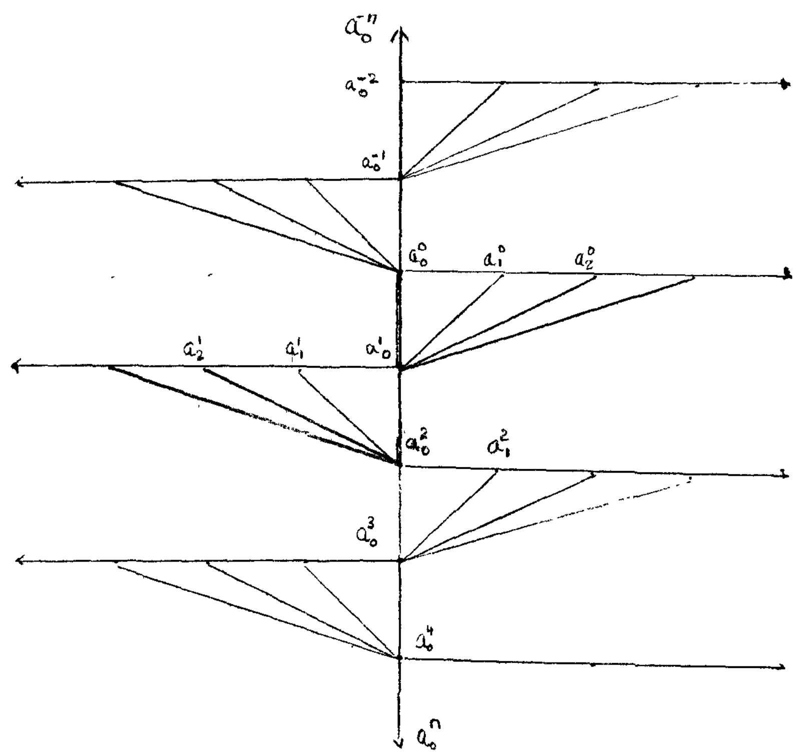
Thus $|D_0| = |D_1|$.

LEMMA (1.35) Let $\mathfrak{A}$ be a weakly locally finite exchange system and D a locally finite dendroid for $\mathfrak{A}_{min}$. Then D is a locally finite dendroid for $\mathfrak{A}$ and thus $\mathfrak{A}$ is locally finite.

Proof. By lemma (1.20), $A \in \mathfrak{A}$ implies $A = \bigcup_{i=1}^{n} A_i$ where $A_i \in \mathfrak{A}_{min}$, for $1 \leqq i \leqq n$. Hence $D \cap A = \bigcup_{i=1}^{n} (D \cap A_i)$, a finite non-empty set. By definition (1.22) D is a locally finite dendroid for $\mathfrak{A}$ and $\mathfrak{A}$ is locally finite.

REMARK (1.36) Lemma (1.29) does not have a converse. By a modification of the graph of figure 1 we obtain a circuit matroid of a graph X and a dendroid D for X such that $\cup \{f_D(x) \mid x \in D\} = E(X)$ and D does not have the l.c.p. More over this graph (shown in figure 2) is such that its circuit matroid is weakly locally finite, and hence weakly locally finite does not imply the l.c.p.

FIGURE 2



D is the set of edges marked in red. It is clear that $E(C) \cap D \neq \phi$ for every circuit C of the graph. For $e \in D$, $f_D(e) = E(C)$, where

C is given by:

$$e = [a_0^0, a_0^1],\ C = (\ldots, a_0^{-n},\ \ldots, a_0^0, a_0^1, \ldots, a_0^n, \ldots)$$

$$e = [a_0^0, a_1^0],\ C = (\ldots, a_0^{-n}, \ldots, a_0^0, a_1^0, a_0^1, \ldots, a_0^n, \ldots)$$

$$e = [a_i^k, a_{i+1}^k],\ C = (a_i^k, a_{i+1}^k, a_0^{k+1}),\ |k| + i \neq 0\ .$$

Since every edge is in a finite circuit, $\cup \{f_D(e) | e \in D\} = E(X)$.

Any circuit C of the form

$$C = (\ldots, a_i^k, a_{i-1}^k, \ldots, a_0^k, a_0^{k+1}, \ldots, a_0^n, \ldots) \text{ with } k > 0$$

has $E(C) \not\subset \bigcup_{e \in E(C) \cap D} f_D(e)$ and hence D does not have the l.c.p.

It is shown at the end of Chapter II that the circuit matroid $\mathfrak{U}(X)$ of a graph X is always weakly locally finite.

The following definition arises naturally from (1.17)

<u>DEFINITION</u> (1.37) Two dendroids D_0, D_1 of an exchange system $\mathfrak{U}$ are <u>adjacent</u> if and only if $D_0 = D_1$ or there is $x \in D_i$ and a $y \in f_{D_i}(x)$ such that $D_{1-i} = (D_i - \{x\}) \cup \{y\}$, i = 0 or 1.

<u>REMARK</u> (1.38) Two distinct dendroids D_0, D_1 of an exchange system $\mathfrak{U}$ are adjacent if and only if $|D_0 + D_1| = 2$. This is clear from the definition, $D_0 + D_1 = \{x,y\}$.

<u>REMARK</u> (1.39) The transitive closure of the relation of adjacency is an equivalence relation on $\vartheta_{\mathfrak{U}}$ for an exchange system $\mathfrak{U}$.

<u>THEOREM</u> (1.40) <u>Let</u> $\mathfrak{U}$ <u>be an exchange system and</u> $D, D' \in \vartheta_{\mathfrak{U}}$. D <u>is equivalent to</u> D′ <u>if and only if</u> D + D′ <u>is finite</u>.

<u>Proof</u>. If D and D′ are equivalent then it is clear from the definition (1.37) and remark (1.38) that D + D′ is finite.

Conversely, if $D + D'$ is non-empty and finite then without loss of generality there is $x \in D' - D$. Since $D \cap f_{D'}(x)$ is finite lemma (1.20) gives

$$x \in f_{D'}(x) \subset \cup \{f_D(y) | y \in f_{D'}(x) \cap D\}.$$

Thus $x \in f_D(y)$ for some $y \in f_{D'}(x) \cap D$ and $y \neq x$ for $x \notin D$ and so $y \notin D'$. Now $D_1 = (D - \{y\}) \cup \{x\}$ is a dendroid adjacent to D for $D + D_1 = \{x,y\}$ and $D_1 + D' = (D + D') - \{x,y\}$, i.e., $|D_1 + D'| = |D + D'| - 2$. By iterating this procedure we can find a dendroid D_n equivalent to D such that $|D_n + D'| \leqq 1$. Since D_n and D' are minimal, $D_n = D'$ and so D and D' are equivalent.

<u>NOTATION</u> (1.41) If D is a dendroid for a system $\mathfrak{A}$ and $X \subset D$ then $\sum_{x \in X} f_D(x)$ is to be the set of all y in exactly one of the $f_D(x)$ with $x \in X$.

<u>LEMMA</u> (1.42) <u>Let</u> $\mathfrak{A}$ <u>be an exchange system and</u> D <u>a dendroid for</u> $\mathfrak{A}$ <u>with the l.c.p. Then</u> $\sum^*_{a \in D \cap A} f_D(a) \subset A$ <u>for any</u> $A \in \mathfrak{A}$.

<u>Proof</u>. Take $A \in \mathfrak{A}$. By definition $A \subset \cup \{f_D(a) | a \in D \cap A\}$. Take any $s \in f_D(a_0)$, $a_0 \in A$ and $s \in \bigcup_{a \in A \cap D} f_D(a) - A$. By the exchange principle there is $B \in \mathfrak{A}$, $s \in B \subset (A \cup f_D(a_0)) - \{a_0\}$. Now $B \subset \bigcup_{b \in B \cap D} f_D(b) = \cup \{f_D(a) | a \in A \cap D - \{a_0\}\}$, and so there is $a_0 \neq a$, $a_0, a \in A \cap D$ with $s \in f_D(a) \cap f_D(a_0)$. Thus it follows

$$\sum^*_{a \in A \cap D} f_D(a) \subset A.$$

COROLLARY (1.43) *If* D *is a dendroid for an exchange system* $\mathfrak{A}$ *and* $D \cap A$ *is finite for some* A *in* $\mathfrak{A}$ *then* $\sum^{*}_{a \in D \cap A} f_D(a) \subset A$.

Proof. By lemma (1.20)(i), $A \subset \bigcup_{a \in D \cap A} f_D(a)$ and hence the B of the above lemma is such that $B \cap D$ is finite. This makes the same argument as above valid and the result follows.

REMARK (1.44) Lemma (1.42) is a straight forward generalization of lemma (1.20)(ii).

THEOREM (1.45) *Let* $\mathfrak{A}$ *be an exchange system and* D_0, D_1 *equivalent dendroids for* $\mathfrak{A}$. *Then* D_0 *has the l.c.p. if and only if* D_1 *has the l.c.p. i.e., l.c.p. is compatible with the equivalence relation defined on* $\vartheta_{\mathfrak{A}}$.

Proof. Using lemma (1.42) it is sufficient to show this for adjacent dendroids. Assume that D has the l.c.p. and let D_0 be adjacent to D. Let A be a fixed member of $\mathfrak{A}$ and let $B = \bigcup \{f_D(x) \mid x \in D \cap A\}$. $A \subset B$ because D has the l.c.p. Take $y \in f_D(x)$, $y \neq x$ and let $D_0 = (D - \{x\}) \cup \{y\}$ be an adjacent dendroid with $f_{D_0}(y) = f_D(x)$ by lemma (1.17).

If $x \in A$ then y can only be in A or B–A whilst if $x \notin A$ then $x \in E-B$ and y can be in A, B–A or E–B. This gives rise to the following six cases to consider.

(i) $x \in E-B$ and $y \in E-B$,

(ii) $x \in E-B$ and $y \in A$,

(iii) $x \in E-B$ and $y \in B-A$ with $f_D(x) \cap A = \phi$,

(iv) $x \in A$ and $y \in A$,

(v) $x \in A$ and $y \in B-A$,

(vi) $x \in E-B$, $y \in B-A$ and $f_D(x) \cap A \neq \phi$.

Case (i). for each $a \in A \cap D$, $f_D(a) \subset B$ whence $y \notin f_D(a)$ and so $f_{D_0}(a) = f_D(a)$ by lemma (1.17), and $D_0 \cap A = D \cap A$. Thus

$$\bigcup_{a \in D_0 \cap A} f_{D_0}(a) = \bigcup_{a \in D \cap A} f_D(a) = B \supset A.$$

Case (ii). $D_0 \cap A = D \cap A \cup \{y\}$. For each $a \in A \cap D$, either $f_D(a) = f_{D_0}(a)$ or $f_{D_0}(a) \supset f_D(a) + f_D(x)$. Hence for all such a, $f_{D_0}(a) \cup f_{D_0}(y) = f_D(a) \cup f_D(x)$. Thus

$$\bigcup_{a \in A \cap D_0} f_{D_0}(a) = (\bigcup_{a \in D \cap A} f_D(a)) \cup f_D(x) \supset B \supset A.$$

Case (iii). $D_0 \cap A = D \cap A$. If $a \in D_0 \cap A$ and $y \in f_D(a)$ then as in (ii) $f_{D_0}(a) \supset f_D(a) + f_D(x) \supset f_D(a) \cap A$ while if $y \notin f_D(a)$ then $f_{D_0}(a) = f_D(a) \supset f_D(a) \cap A$. Thus

$$\bigcup_{a \in A \cap D_0} f_{D_0}(a) \supset A \cap (\bigcup_{a \in A \cap D} f_D(a)) = A.$$

Case (iv). $D_0 \cap A = (D \cap A - \{x\}) \cup \{y\}$. Again for $a \in D \cap A$, $a \neq x$, $f_{D_0}(a) \cup f_{D_0}(y) = f_D(a) \cup f_D(x)$. Hence

$$\bigcup_{a \in D_0 \cap A} f_{D_0}(a) = \bigcup_{a \in D \cap A} f_D(a) = B \supset A.$$

Case (v). $D_0 \cap A = D \cap A - \{x\}$. By lemma (1.43) there is $a_0 \in D \cap A$ with $a_0 \neq x$ such that $y \in f_D(a_0)$. Since we only want to cover A and either $f_{D_0}(a) = f_D(a)$ or $f_{D_0}(a) \supset f_D(a) + f_D(x)$ it is only necessary to show that for every $p \in f_D(x) \cap A$, $p \in f_{D_0}(a)$ for some $a \in A \cap D_0$. Consider $D' = (D - \{x\}) \cup \{p\}$. This is adjacent to D and is the type in case (iv). Hence $\bigcup_{a \in D' \cap A} f_{D'}(a) \supset A$. Also $D_0 = (D' - \{p\}) \cup \{y\}$ is a dendroid adjacent to D' because $y \in f_D(x) = f_{D'}(p)$. Again by lemma (1.43) $y \in f_{D'}(b)$ for some $b \neq p$, $b \in D' \cap A$, and $f_{D_0}(b) \supset f_{D'}(b) + f_{D'}(p)$. Thus $p \in f_{D_0}(b)$ as required (note $x \in f_{D_0}(a_0)$) and so

$$A \subset \bigcup_{a \in A \cap D_0} f_{D_0}(a) .$$

Case (vi). $x \in E-B$, $y \in B-A$ and $f_D(x) \cap A \neq \phi$. Take $t \in f_D(x) \cap A$. Then $t \in f_D(a)$ for some $a \in D \cap A$. Let $D_1 = (D - \{x\}) \cup \{t\}$, which is adjacent to D and falls into case (ii). Thus $A \subset \bigcup_{a \in A \cap D_1} f_{D_1}(a)$ with $y \in f_{D_1}(t) = f_D(x)$. Then $(D_1 - \{t\}) \cup \{y\} = (D - \{x\}) \cup \{y\} = D_0$ is adjacent to D and is then case (v). Thus again $A \subset \bigcup_{a \in A \cap D_0} f_{D_0}(a)$.

We have now shown that $A \subset \bigcup_{a \in A \cap D_0} f_{D_0}(a)$ in all possible situations and since A was arbitrary in $\mathfrak{A}$ the theorem follows.

<u>LEMMA</u> (1.46) <u>Let</u> $\mathfrak{A}$ <u>be an exchange system and</u> $D_0 \in \vartheta_{\mathfrak{A}}$. <u>Let</u> $\vartheta_0 = \{D \in \vartheta_{\mathfrak{A}} | D + D_0 \text{ is finite}\}$, <u>and let</u> $\mathfrak{A}_1 = \{f_D(x) | x \in D \in \vartheta_0\}$. $\mathfrak{A}_1$ <u>is an exchange system and is exactly the system of lemma</u> (1.32), <u>i.e.,</u> $\mathfrak{A}_0 = \{A \in \mathfrak{A}_{min} | A \cap D_0 \text{ is finite}\}$.

<u>Proof</u>. Since $\mathfrak{A}_0$ is an exchange system it is sufficient to show that $\mathfrak{A}_0 = \mathfrak{A}_1$.

Take $f_D(x) \in \mathfrak{A}_1$. By lemma (1.19) $f_D(x) \in \mathfrak{A}_{min}$. Now $f_D(x) \cap D_0 \subset f_D(x) \cap ((D_0 + D) \cup D_0 \cap D) \subset f_D(x) \cap (D_0 + D) \cup \{x\}$ and is finite because $D_0 + D$ is finite. Thus $f_D(x) \in \mathfrak{A}_0$ and so $\mathfrak{A}_1 \subset \mathfrak{A}_0$.

Conversely, if $A \in \mathfrak{A}_{min}$ and $A \cap D_0$ is finite then there is $D \in \vartheta_{\mathfrak{A}}$ such that $D + D_0$ is finite and $A \cap D$ is least. If $|A \cap D| \neq 1$, then there is $a \in A \cap D$ such that $f_D(a)$ and A are incomparable. Take $b \in f_D(a) - A$ and consider the dendroid $D' = (D - \{a\}) \cup \{b\}$. $D' \cap A = D \cap A - \{a\}$ and $D' + D$ is finite, which contradicts $A \cap D$ being least with this property. Hence there is $D \in \vartheta_0$ with $A = f_D(a)$ for some $a \in A$. Thus $\mathfrak{A}_0 \subset \mathfrak{A}_1$ and the lemma follows.

<u>COROLLARY</u> (1.47) <u>If</u> D <u>is any dendroid for</u> $\mathfrak{A}_0$ <u>with the l.c.p. then</u> $|D_0| \leqq |D|$.

<u>Proof.</u> D has the l.c.p. implies each $a \in D_0$ is in some $f_D(x)$. Hence $D_0 \subset \bigcup_{x \in D} (f_D(x)) \cap D_0$. But $f_D(x) \cap D_0$ is finite. Hence $|D_0| \leqq \sum_{x \in D} |f_D(x) \cap D_0| \leqq |D| \aleph_0$.

If D is finite so is D_0 and $|D| = |D_0|$ by lemma (1.31). Thus $|D_0| \leqq |D|$ in both cases.

<u>REMARK</u> (1.48) In order to have $|D_0| \leqq |D|$ it suffices that for each $a \in D_0$ there be an $x \in D$ with $a \in f_D(x)$.

<u>DEFINITION</u> (1.49) Let $\mathfrak{A}$ be a system on E. $\mathfrak{A}$ <u>separates</u> <u>points</u> of E if and only if for any two distinct elements $a,b \in E$ there is $A \in \mathfrak{A}$ with $a \in A$ and $b \notin A$.

REMARK (1.50) Let $\mathfrak{A}$ be a system on E. If $\cup \mathfrak{A} \neq E$ then the system $\mathfrak{L}$ given by

$$\mathfrak{L} = \mathfrak{A} \cup \{\{x\} | x \in E - \cup \mathfrak{A}\}$$

is a system on E such that $\cup \mathfrak{L} = E$. Moreover, $\mathfrak{A}$ is an exchange system if and only if $\mathfrak{L}$ is an exchange system and D is a dendroid for $\mathfrak{A}$ if and only if $D \cup (E - \cup \mathfrak{A})$ is a dendroid for $\mathfrak{L}$. Notice that it is necessary that $\cup \mathfrak{A} = E$ if $\mathfrak{A}$ is to separate points of E.

For this reason, unless otherwise stated, it will be assumed that a system $\mathfrak{A}$ on E has the property $\cup \mathfrak{A} = E$.

LEMMA (1.51) Let $\mathfrak{A}$ be a system on E. For $a \in E$ let $\eta_a = \{A \in \mathfrak{A} | a \in A\}$, and define a relation N on E by aNb if and only if $\eta_a = \eta_b$. N is an equivalence relation on E, and $\bar{a} = \cap \eta_a$, where $\bar{a} = \{b | b \in E, aNb\}$.

Proof. Obvious.

NOTATION (1.52) For $A \subset E$ set $\bar{A} = \{\bar{a} | a \in A\}$, and $\bar{\mathfrak{A}} = \{\bar{A} | A \in \mathfrak{A}\}$.

THEOREM (1.53) Let $\mathfrak{A}$ be a system on E. Then in the notation of (1.52), $\bar{\mathfrak{A}}$ is a point separating system on $\bar{E}$. Further $\mathfrak{A}$ is an exchange system if and only if $\bar{\mathfrak{A}}$ is an exchange system and D is a dendroid for $\mathfrak{A}$ if and only if $\bar{D}$ is a dendroid for $\bar{\mathfrak{A}}$ and D is a representative set for $\bar{D}$.

Proof. $\bar{a} \in \bar{A} \in \bar{\mathfrak{A}}$ if and only if $a \in A \in \mathfrak{A}$, for $\bar{a} \in \{\bar{b} | b \in A\}$ implies $\cap \eta_a = \cap \eta_b$ for some $b \in A$, and so $a \in A$.

Take $\bar{a} \neq \bar{b}$. Since $\eta_a \neq \eta_b$ and $\cup \mathfrak{A} = E$ it follows without loss of generality that there is an $\bar{A} \in \bar{\mathfrak{A}}$ such that $\bar{a} \in \bar{A}$, $\bar{b} \notin \bar{A}$.

If $\mathfrak{A}$ is an exchange system take $\bar{a} \in \bar{A}-\bar{B}$ and $\bar{b} \in \bar{A} \cap \bar{B}$. Then $a \in A-B$ and $b \in A \cap B$. Hence there is $C \in \mathfrak{A}$ with $a \in C$ and $C \subset A \cup B - \{b\}$, and so $\bar{a} \in \bar{C} \subset \bar{A} \cup \bar{B} - \{\bar{b}\}$ and $\bar{C} \in \bar{\mathfrak{A}}$.

Conversely, if $\bar{\mathfrak{A}}$ is an exchange system and $A,B \in \mathfrak{A}$ with $a \in A-B$ and $b \in A \cap B$ then $\bar{a} \in \bar{A}-\bar{B}$, $\bar{b} \in \bar{A} \cap \bar{B}$ and hence there is $\bar{C} \in \bar{\mathfrak{A}}$ with $\bar{a} \in \bar{C} \subset \bar{A} \cup \bar{B} - \{\bar{b}\}$. Thus $a \in C \subset A \cup B - \{b\}$ and $\mathfrak{A}$ is an exchange system.

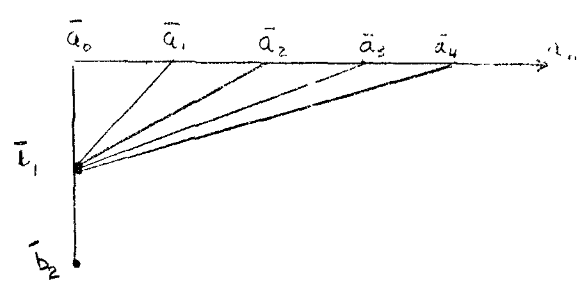
Let D be a dendroid for $\mathfrak{A}$ and $a,b \in D$ with $\bar{a} = \bar{b}$. Then $b \in A$ for every $A \in \mathfrak{A}$ with $a \in A$. Hence $b \in f(a) \cap D$ for $f \in F_D$ and so $b = a$. Thus D is a representative set for $\bar{D}$. Now for $a \in D$ there is $A \in \mathfrak{A}$ with $a \in A$, $b \notin A$ for $b \in D - \{a\}$, viz. $A = f(a)$. Hence $\bar{a} \in \bar{A} \in \bar{\mathfrak{A}}$, $\bar{b} \notin \bar{A}$ for $\bar{b} \in \bar{D} - \{\bar{a}\}$. Clearly, if $\bar{D} \cap \bar{A} = \phi$ for some $\bar{A} \in \bar{\mathfrak{A}}$ then $D \cap A = \phi$. Hence $\bar{D}$ is a dendroid for $\bar{\mathfrak{A}}$.

Conversely, let B be a dendroid for $\bar{\mathfrak{A}}$ and let D be a representative set for B. Then $\bar{D} = B$. Let $\bar{b} \in B$, $B \cap \bar{A} = \{\bar{b}\}$. If $x \in D$ with $x \in A$ then $\bar{x} \in \bar{A}$ and $\bar{x} \in \bar{B}$. Hence $\bar{x} = \bar{b}$ so that $D \cap A = \{x\}$. Clearly if $x \in D$ then $\bar{x} = \bar{b} \in B$ so that for $x \in D$ there is $A \in \mathfrak{A}$ with $A \cap D = \{x\}$. If $D \cap A = \phi$ for some $A \in \mathfrak{A}$ then for all $a \in A$, $a \notin D$ so that for all $\bar{a} \in \bar{A}$, $\bar{a} \notin \bar{D} = B$. Then $B \cap \bar{A} = \phi$, a contradiction and so D is a dendroid for $\mathfrak{A}$.

REMARK (1.54) In view of theorem (1.53) it will be assumed that a system on E is point separating unless otherwise stated.

REMARK (1.55) The circuit matroid of the graph in remark (1.28) is not point separating while the circuit matroid for the graph in remark (1.36) is point separating. Note that the quotient system for the circuit matroid of remark (1.28) is not the circuit matroid of the resulting graph shown in figure 3.

FIGURE 3



DEFINITION (1.56) Let $\mathfrak{A}$ be a system on E. A set $S \subset E$ is a separator of $\mathfrak{A}$ if and only if for each $A \in \mathfrak{A}$ with $A \cap S \neq \phi$, $A \subset S$. Let $\mathcal{S}(\mathfrak{A})$ be the set of separators of $\mathfrak{A}$.

LEMMA (1.57) $\mathcal{S}(\mathfrak{A})$ is a complete atomic Boolean algebra of sets.

Proof. (i) ϕ and E belong to $\mathcal{S}(\mathfrak{A})$ as the smallest and largest elements respectively.

(ii) If $S \in \mathcal{S}(\mathfrak{A})$ then $(E-S) \in \mathcal{S}(\mathfrak{A})$ for if $A \cap S \neq \phi$ then $A \subset S$, hence $A \cap (E-S) \neq \phi$ implies $A \subset E-S$.

(iii) Let $\mathcal{C}$ be a chain in $\mathcal{S}(\mathfrak{A})$. If $A \cap \cap\mathcal{C} \neq \phi$ then $A \cap S \neq \phi$ for each $S \in \mathcal{C}$ and $A \subset S \in \mathcal{C}$. Thus $A \subset \cap\mathcal{C}$ and $\cap\mathcal{C} \in \mathcal{S}(\mathfrak{A})$.

(iv) Using (ii) and (iii), $E - \cup \mathcal{C} = \bigcap_{S \in \mathcal{C}} (E-S) \in \mathcal{S}(\mathfrak{A})$. Hence $\cup \mathcal{C} \in \mathcal{S}(\mathfrak{A})$.

(v) For each $a \in E$ let $\mathcal{S}_a$ be the collection of all separators containing a and let $S_a = \cap \mathcal{S}_a$. $E \in \mathcal{S}_a$ and so $a \in S_a \in \mathcal{S}(\mathfrak{A})$. If $T \in \mathcal{S}(\mathfrak{A})$, $a \notin T \subset S_a$ then $a \in (E-T) \cap S_a = S_a$. Thus $T = (E-S_a) \cap S_a = \phi$ and S_a is an atom of $\mathcal{S}(\mathfrak{A})$.

Each $S \in \mathcal{S}(\mathfrak{A})$ is the union of the atoms it contains; i.e.

$$S = \cup \{S_a | a \in S\}.$$

The atoms of $\mathcal{S}(\mathfrak{A})$ have been called <u>elementary separators</u> by Tutte [5].

<u>DEFINITION</u> (1.58) Let $\mathfrak{A}$ be a system on E and $S \subset E$. An S-<u>chain</u> is a finite sequence $(A_1,\ldots,A_n)$ such that

(i) $A_i \in \mathfrak{A}$, $A_i \subset S$, $i = 1,\ldots,n$ and

(ii) $A_i \cap A_{i+1} \neq \phi$, $i = 1,\ldots,n-1$ if $n > 1$.

<u>DEFINITION</u> (1.59) Let $\mathfrak{A}$ be a system on E. $S \subset E$ is <u>connected</u> if and only if given any $a,b \in S$ there is an S-chain whose first term contains a and whose last term contains b.

A <u>component</u> of $\mathfrak{A}$ is a subsystem $\mathfrak{A}' \subset \mathfrak{A}$ such that $\cup \mathfrak{A}'$ is a maximal connected subset of E.

<u>LEMMA</u> (1.60) <u>Let $\mathfrak{A}$ be a system on E and I an index set such that $(S_\alpha)_{\alpha \in I}$ is a family of connected sets with $x \in \bigcap_{\alpha \in I} S_\alpha$. Then $\cup \{S_\alpha | \alpha \in I\}$ is connected.</u>

Proof. Let $a,b \in \bigcup_{\alpha\in I} S_\alpha$ with $a \in S_\alpha$, $b \in S_\beta$. Since $x \in S_\alpha \cap S_\beta$ there are S_α-chains and S_β-chains $(A_1,\dots,A_n)$ and $(B_1,\dots,B_m)$ in S_α and S_β respectively with $a \in A_1$, $x \in A_n$, $x \in B_1$, $b \in B_m$. Hence $(A_1,\dots,A_n,B_1,\dots,B_m)$ is a $\bigcup_{\alpha\in I} S_\alpha$-chain with $a \in A_1$, $b \in B_m$. Hence $\bigcup_{\alpha\in I} S_\alpha$ is connected.

COROLLARY (1.61) By Zorn's lemma maximal connected sets exist and every connected set is contained in a maximal connected set.

COROLLARY (1.62) Components of $\mathfrak{A}$ exist and any two components are disjoint.

THEOREM (1.63) The atoms of $\mathcal{S}(\mathfrak{A})$ are exactly the maximal connected subsets of E.

Proof. Take S an atom in $\mathcal{S}(\mathfrak{A})$. If $S = \{a\}$ then clearly S is connected. If a, b are two points in S then define recursively the following countable collection of sets R_k. There is $A \in \mathfrak{A}$ with $a \in A \subset S$. Set $R_1 = A$. For $k \geqq 2$ let

$$R_k = \{x \mid \text{there exists } B \in \mathfrak{A} \text{ with } x \in B,\ B \cap R_{k-1} \neq \emptyset\}.$$

For each k, $R_k \subset R_{k+1}$ for if $x \in R_k$ there is $B \in \mathfrak{A}$ with $B \cap R_k \neq \emptyset$ and $x \in B$. Hence $x \in R_{k+1}$, and in fact $B \subset R_{k+1}$. $A = R_1 \subset S$. Suppose there is $R_k \not\subset S$. Then there is a least positive integer k with $x \in R_k - S$. Thus there is $B \in \mathfrak{A}$ with $x \in B$, $B \cap R_{k-1} \neq \emptyset$, that is, $B \cap S \neq \emptyset$ and so $B \subset S$, a contradiction. Hence $R_k \subset S$ for all k and $\bigcup_1^\infty R_k \subset S$. Let $B \in \mathfrak{A}$ with $B \cap (\bigcup_1^\infty R_k) \neq \emptyset$. Then $B \cap R_k \neq \emptyset$ for some k and so $B \subset R_{k+1}$. Hence $\bigcup_1^\infty R_k$ is a separator of $\mathfrak{A}$, and since S is minimal, $\bigcup_1^\infty R_k = S$.

It is clear that $A = R_1$ is connected because (A) is an R-chain. Let a,b be in S with $a \in R_1$ and $b \in R_k$, $k > 1$. Choose k the least integer possible. $b \in B \in \mathfrak{A}$, $B \cap R_{k-1} \neq \phi$. There is a least k_1 such that $B \cap R_{k_1} \neq \phi$. If $k_1 \neq 1$, let $b_1 \in B \cap R_{k_1}$, where $b \neq b_1$ by the choice of k. Then there is $B_1 \in \mathfrak{A}$ such that $b_1 \in B_1$, $B \cap R_{k_2} \neq \phi$ with k_2 the least integer possible and $k_2 < k_1$. In a finite number of steps one obtains an S-chain $(B, B_1, \ldots, B_{k_n})$ such that $b \in B$ and $a \in B_{k_n}$. Hence S is connected. If $S = \{a\}$, then $a \in A \in \mathfrak{A}$ implies $\{a\} = A$ and S is maximal connected. If $|S| > 1$ and $S \subset S'$ connected then for $b \in S'$ and $a \in S$ there is an S'-chain $(A_1, \ldots, A_n)$ with $a \in A_1$ and $b \in A_n$, with $A_1 \subset S$. Since $A_2 \cap A_1 \neq \phi$, $A_2 \subset S$ and in the same way $A_n \subset S$. Thus $S' \subset S$ and S is maximal connected in E.

Conversely, suppose S is a maximal connected set in E. Let $S' \in \mathcal{S}(\mathfrak{A})$ with $S' \subset S$. If $S' \neq \phi$ take $a \in S'$ and $b \in S$. Then there is an S-chain from a to b. By a repetition of the argument immediately above this S-chain must be an S'-chain. Hence $S' = S$ and so S is an atom of $\mathcal{S}(\mathfrak{A})$.

LEMMA (1.64) Let $\mathfrak{A}$ be a system on E. $\mathfrak{A}$ is an exchange system if and only if each component of $\mathfrak{A}$ is an exchange system and $\mathfrak{A}$ has a dendroid if and only if each component of $\mathfrak{A}$ has a dendroid.

Proof. Let I be an index set for the collection of components of $\mathfrak{A}$. Each member of $\mathfrak{A}$ belongs to some component and hence $\mathfrak{A} = \bigcup_{\alpha \in I} \mathfrak{A}_\alpha$, a disjoint union (by Corollary (1.62)).

Now suppose that $\mathfrak{A}$ is an exchange system and $A,B \in \mathfrak{A}_\alpha$ with $a \in A-B$ and $b \in A \cap B$. Then $A,B \in \mathfrak{A}$ implies there is $C \in \mathfrak{A}$ with $a \in C \subset A \cup B - \{b\}$. By lemmas (1.60) and (1.63) $C \in \mathfrak{A}_\alpha$ and so each component of $\mathfrak{A}$ has the exchange property. If D is a dendroid for $\mathfrak{A}$ let $D_\alpha = D \cap \bigcup \mathfrak{A}_\alpha$. Each $a \in D$ is in a unique D_α by corollary (1.62) and hence $a \in f_D(a)$ implies $f_D(a) \in \mathfrak{A}_\alpha$. If $D_\alpha \cap A = \phi$ for some $A \in \mathfrak{A}_\alpha$ then $D \cap A = \phi$ with $A \in \mathfrak{A}$, a contradiction. Thus each D_α is a dendroid for $\mathfrak{A}_\alpha$. Note that $D_\alpha \neq \phi$, for there is $A \in \mathfrak{A}_\alpha$ and $D \cap A \neq \phi$.

Conversely, suppose each $\mathfrak{A}_\alpha$ has the exchange property. If $A,B \in \mathfrak{A}$ with $a \in A-B$ and $b \in A \cap B$ then there is $C \in \mathfrak{A}$ with $a \in C \subset A \cup B - \{b\}$, because A and B belong to the same component of $\mathfrak{A}$ by lemma (1.60) and corollary (1.62).

Let D_α be a dendroid for $\mathfrak{A}_\alpha$ for each $\alpha \in I$ and let $D = \bigcup_{\alpha\in I} D_\alpha$. If $A \cap D = \phi$ then $A \cap D_\alpha = \phi$ for the unique α with $A \in \mathfrak{A}_\alpha$. For each $a_\alpha \in D_\alpha$ there is $A_\alpha \in \mathfrak{A}_\alpha$ with $D_\alpha \cap A_\alpha = \{a_\alpha\}$. Since $A \in \mathfrak{A}$ and the D_α's are pairwise disjoint, $A \cap D = \{a_\alpha\}$ and D is a dendroid for $\mathfrak{A}$.

<u>DEFINITION</u> (1.65) A system $\mathfrak{A}$ on E will be a <u>connected system</u> on E if and only if $\mathfrak{A}$ has a single component.

<u>REMARK</u> (1.66) Given any complete Boolean algebra $\mathfrak{B}$ of subsets of E one can define an equivalence relation R on E by aRb if and only if $\mathfrak{B}_a = \mathfrak{B}_b$, where $\mathfrak{B}_x = \cap \{B \in \mathfrak{B} | x \in B\}$. Then $\mathfrak{B}$ is the algebra of all R-saturated subsets of E (i.e., subsets which are unions of

equivalence classes modulo R). It follows from theorem (1.63) that in the case of the algebra $\mathcal{S}(\mathfrak{A})$ the R-saturated sets are exactly those which are unions of $\mathfrak{A}$-components of E.

<u>DEFINITION</u> (1.67) Let $\mathfrak{A}$ be a system on E and S a subset of E. If there is $A \in \mathfrak{A}$ with $A \cap S \neq \phi$ then $\mathfrak{A}|S = \{A \cap S | A \in \mathfrak{A},\ A \cap S \neq \phi\}$ is the <u>retraction</u> of $\mathfrak{A}$ to S. If there is $A \in \mathfrak{A}$ with $A \subset S$ then $\mathfrak{A}_S = \{A \in \mathfrak{A} | A \subset S\}$ is the <u>restriction</u> of $\mathfrak{A}$ to S.

<u>REMARK</u> (1.68) Let $\mathfrak{A}$ be a system on E and $S \subset E$. Then $\mathfrak{A}_S \subset \mathfrak{A}|S$.

<u>REMARK</u> (1.69) <u>If $\mathfrak{A}$ is an exchange system and $\mathfrak{A}|S \neq \phi \neq \mathfrak{A}_S$ then $\mathfrak{A}|S$ and $\mathfrak{A}_S$ are exchange systems</u>. Take $A,B \in \mathfrak{A}|S$ with $a \in A-B$ and $b \in A \cap B$. Then $A = A' \cap S$ and $B = B' \cap S$ with $A', B' \in \mathfrak{A}$, $a \in A'-B'$ and $b \in A' \cap B'$. Hence there is $C' \in \mathfrak{A}$ with $a \in C' \subset A' \cup B' - \{b\}$ and so $a \in C = C' \cap S \subset A \cup B - \{b\}$ with $C \in \mathfrak{A}|S$.

If $A,B \in \mathfrak{A}_S$ then $A,B \in \mathfrak{A}$ and hence $a \in C \in \mathfrak{A}$ with $C \subset A \cup B - \{b\} \subset S$. Thus $C \in \mathfrak{A}_S$, and $\mathfrak{A}|S$ and $\mathfrak{A}_S$ have the exchange property.

<u>REMARK</u> (1.70) If $\mathfrak{A}$ is a system on E with a dendroid D then it does not follow that $\mathfrak{A}|S$ has a dendroid, for $\mathfrak{A}|S$ may have no minimal elements. Moreover, even though $D \cap S$ has the intersection property for $\mathfrak{A}_S$ this need not be a dendroid because for some $a \in D \cap S$ there is no guarantee that $f_D(a) \subset S$.

<u>LEMMA</u> (1.71) <u>Let</u> $\mathfrak{A}$ <u>be a system on</u> E <u>and let</u> $\mathfrak{F}$ <u>be the collection of finite subsets of</u> E <u>in</u> $\mathfrak{A}$. <u>If there is</u> $A \in \mathfrak{A}$ <u>such that</u> $A \notin \mathfrak{F}$ <u>and</u> $B \subset A$ <u>implies</u> $B \notin \mathfrak{F}$ <u>then there is a subset</u> M <u>of</u> E <u>such that</u> $\mathfrak{A}_M \cap \mathfrak{F} = \emptyset$.

<u>Proof</u>. Let $\mathcal{M} = \{X \subset E \mid \mathfrak{A}_X \cap \mathfrak{F} = \emptyset\}$. $A \in \mathcal{M} \neq \emptyset$. Let $\mathcal{C}$ be a chain in $(\mathcal{M},\subset)$. If $F \in \mathfrak{F}$ with $F \subset \bigcup \mathcal{C}$ then there is a finite subchain $\mathcal{C}'$ of $\mathcal{C}$ with $F \subset \bigcup \mathcal{C}' \in \mathcal{C}$, a contradiction. Thus maximal sets exist in $(\mathcal{M},\subset)$.

<u>COROLLARY</u> (1.72) <u>If</u> $\mathfrak{A}$ <u>is an exchange system and</u> $\mathfrak{F} \neq \emptyset \neq \mathfrak{A}_M$ <u>then both</u> $\mathfrak{F}$ <u>and</u> $\mathfrak{A}_M$ <u>are exchange systems</u>.

<u>Proof</u>. This follows directly from remark (1.69).

PROPOSITION (1.73) <u>Let</u> $\mathfrak{A}$ <u>be a system on</u> E <u>and</u> $\mathfrak{F}$ <u>the collection of finite subsets of</u> E <u>in</u> $\mathfrak{A}$. <u>If</u> $\mathfrak{F} \neq \emptyset$ <u>then in the notation of lemma</u> (1.71) <u>for every maximal</u> $M \in \mathcal{M}$, E–M <u>is a dendroid for</u> $\mathfrak{F}$ <u>and every dendroid for</u> $\mathfrak{F}$ <u>is of this form</u>.

<u>Proof</u>. Let M be maximal in $\mathcal{M}$. Since $(E-M) \cap F \neq \emptyset$ implies $F \in \mathfrak{A}_M$ for each $F \in \mathfrak{F}$ it follows from the definition of $\mathcal{M}$ that $(E-M) \cap F = \emptyset$. By lemma (1.33) $\mathfrak{F}$ has a dendroid $D \subset E - M$. Now if $D \subsetneq E-M$ then by the maximality of M there is $F \in \mathfrak{F}$ with $F \subset M \cup ((E-M)-D)$. But then $D \cap F \subset D \cap M \cup D \cap ((E-M)-D) = \emptyset$ and hence $D = E-M$.

Conversely, let D be a dendroid for $\mathfrak{F}$. Then $F \not\subset E-D$ for any $F \in \mathfrak{F}$. Take Y maximal in $\mathcal{M}$ such that $Y \supset E-D$. Then by the above, $E-Y \subset D$ is a dendroid for $\mathfrak{F}$ and hence $E-Y = D$, i.e., $Y = E-D$.

THEOREM (1.74) Let D be a dendroid for a system $\mathfrak{A}$ on E. Then $D = D_0 \cup D_1$ (disjoint) where D_0 is a dendroid for $\mathfrak{F}$ and D_1 is a dendroid for $\mathfrak{A}_M$ for some suitable $M \in \mathcal{M}$, in fact $M = E - D_0$.

Proof. Since $D \cap F \neq \emptyset$ for any $F \in \mathfrak{F}$, D contains a dendroid D_0 for $\mathfrak{F}$ by lemma (1.33). By the proof of proposition (1.73) this determines the maximal set $M = E-D_0$ in $\mathcal{M}$. Let $D_1 = D-D_0$. For $x \in D_1$, $f_D(x) \cap D_0 = \emptyset$ and so $f_D(x) \subset M$, and if $A \cap D_1 = \emptyset$ then $A \cap D_0 \neq \emptyset$ so that $A \notin \mathfrak{A}_M$. Hence D_1 is a dendroid for $\mathfrak{A}_M$.

PROPOSITION (1.75) Let $\mathfrak{B} \subset \mathfrak{A}$ be a maximal sub-exchange system such that $\cup \mathfrak{B}$ contains no member of $\mathfrak{F}$. If D_1 is a dendroid for $\mathfrak{B}$ then D_1 can be extended to a dendroid for $\mathfrak{A}$.

Proof. By lemma (1.71) there is a maximal M in $\mathcal{M}$ such that $\mathfrak{A}_M \cap \mathfrak{F} = \emptyset$ and $M \supset \cup \mathfrak{B}$. By proposition (1.73) $D_0 = E-M$ is a dendroid for $\mathfrak{F}$. Let $D = D_0 \cup D_1$ and take $x \in D_1$. $f_{D_1}(x) \cap D = f_{D_1}(x) \cap D_1 = \{x\}$ since $f_{D_1}(x) \subset M$. If $A \cap D_0 = \emptyset$ then $A \subset M$ and so $A \in \mathfrak{B}$ and $A \cap D_1 \neq \emptyset$. Hence it only remains to show that for $x \in D-D_1$ there is $f_D(x) \in \mathfrak{A}$ with $f_D(x) \cap D = \{x\}$. $f_{D_0}(x)$ is finite and so $f_{D_0}(x) \cap D_0 = \{x\}$ and $f_{D_0}(x) \cap D_1$ is finite. Take $A \in \mathfrak{A}$ such that $A \cap D_0 = \{x\}$ and $|A \cap D_1|$ is least. Suppose $a \in D_1 \cap A$. Then there is $C \in \mathfrak{A}$ with $x \in C \subset A \cup f_{D_1}(a) - \{a\}$ by the exchange property. Since $f_{D_1}(a) \cap D_0 = \emptyset$, $C \cap D_0 = \{x\}$ and $|C \cap D_1| < |A \cap D_1|$. Hence $A \cap D_1 = \emptyset$ and therefore $A = f_D(x)$. Thus it follows that D is a dendroid for $\mathfrak{A}$.

REMARK (1.76) A circuit matroid $\mathfrak{A}$ has the property that for $A, B \in \mathfrak{A}$, $A + B$ is the union of disjoint elements from $\mathfrak{A}$. Hence if $D \in \vartheta_{\mathfrak{A}}$ and $D \cap A$ is finite then $A = \sum_{a \in D \cap A} f_D(a)$. This is seen as follows. $\sum_{a \in A \cap D} f_D(a) \subset A$ by corollary (1.43). Now $A + (\sum_{a \in D \cap A} f_D(a))$ is a union from $\mathfrak{A}$ while $D \cap (A + \sum_{a \in D \cap A} f_D(a)) = \phi$ and so $A = \sum_{a \in A \cap D} f_D(a)$. This means in general that in the notation of proposition (1.75), $f_{D_0}(a) \neq f_D(a)$ for $f_D(a)$ must be infinite if $f_{D_0}(a) \cap D_1 \neq \phi$.

THEOREM (1.77) Let $\mathfrak{A}$ be a matroid on E with the l.c.p. and such that $A \cup D \neq E$ for any $A \in \mathfrak{A}$ and $D \in \vartheta_{\mathfrak{A}}$. Then there is a matroid $\mathfrak{L}$ properly containing $\mathfrak{A}$. Moreover, the dendroids for $\mathfrak{L}$ are precisely the sets $D \cup \{a\}$ where $D \in \vartheta_{\mathfrak{A}}$ and $a \in E-D$.

Proof. Let $\mathfrak{L} = \{E-D \mid D \in \vartheta_{\mathfrak{A}}\} \cup \mathfrak{A}$.

Dendroids are incomparable and so are their complements. The elements of $\mathfrak{A}$ are incomparable. Take $A \in \mathfrak{A}$ and $D \in \vartheta_{\mathfrak{A}}$. Then $A \not\subset E-D$ and if $A \supset E-D$ then $A \cup D = E$, contrary to hypothesis. Thus the elements of $\mathfrak{L}$ are incomparable.

To show that $\mathfrak{L}$ is an exchange system take $a \in A-B$ and $b \in A \cap B$. If $B = E-D$ for some $D \in \vartheta_{\mathfrak{A}}$ then $(B - \{b\}) \cup \{a\} \neq E-D'$ for any $D' \in \vartheta_{\mathfrak{A}}$ if and only if $f_D(a) \subset (B - \{b\}) \cup \{a\} \subset A \cup B - \{b\}$, by remark (1.18). Since $\mathfrak{A}$ is an exchange system we need only consider the remaining case: B is not the complement of a dendroid for $\mathfrak{A}$, but $A = E-D$ for some $D \in \vartheta_{\mathfrak{A}}$. Then $B-A = B \cap D$, and if $E - ((A - \{b\}) \cup \{x\}) \notin \vartheta_{\mathfrak{A}}$ for any $x \in D \cap B$, then by remark (1.18) and the l.c.p. of $\mathfrak{A}$, $B \subset \bigcup_{x \in D \cap B} f_D(x) \subset (A - \{b\}) \cup B \cap D$. Thus $b \in B \cap (E-D) \subset (A - \{b\}) \cap (E-D) \subset A - \{b\}$, a contradiction. Thus there is some $x \in B \cap D$ such that $(A - \{b\}) \cup \{x\}$ is the complement of a dendroid for $\mathfrak{A}$ and so there is $C \in \mathfrak{L}$ with $a \in C \subset A \cup B - \{b\}$. This proves that $\mathfrak{L}$ is an exchange system.

Let $D_0 \in \vartheta_{\mathfrak{A}}$ and $a \in E - D_0$. Set $D = D_0 \cup \{a\}$ and take $B \in \mathfrak{L}$. If $B \cap D = \phi$ then $B \notin \mathfrak{A}$ and so $B = E-D'$ for some $D' \in \vartheta_{\mathfrak{A}}$. But then $D' \supset D = D_0 \cup \{a\}$, a contradiction on the minimality of D'. Now consider the function $f: D \to \mathfrak{L}$ given by

$f(a) = (E-D) \cup \{a\}$,

$f(x) = (E-D) \cup \{x\} = (E-((D_0 - \{x\}) \cup \{a\})) \cup \{x\} \in \mathfrak{L}$ if $a \in f_{D_0}(x)$,

$f(x) = f_{D_0}(x)$ if $a \notin f_{D_0}(x)$.

In each case $f(y) \cap D = \{y\}$ for $y \in D$ and so $f \in F_D$ and D is a dendroid for $\mathfrak{L}$ by lemma (1.14).

Conversely, let D be a dendroid for $\mathfrak{L}$, and consider $D_1 = \{x \in D | f_D(x) = E-D' \text{ for some } D' \in \vartheta_{\mathfrak{A}}\}$. Clearly $D_1 \neq \phi$, for then D is a dendroid for $\mathfrak{A}$ which always has a proper extension to one for $\mathfrak{L}$ by the first part. Take $a \in D_1$, $f_D(a) = E-D'$, $D' \in \vartheta_{\mathfrak{A}}$. Then $E-D' \subset (E-D) \cup \{a\}$. That is, $D' \supset D \cap (E - \{a\}) = D - \{a\}$. By the first part, $D' \cup \{a\} \in \vartheta_{\mathfrak{L}}$ and so $D' \cup \{a\} = D$.

<u>LEMMA</u> (1.78) <u>Let</u> $\mathfrak{A}$ <u>be a system on</u> E <u>and</u> $D \subset E$. <u>If</u> B <u>is minimal in</u> E <u>containing</u> D <u>such that</u> $B \cap A \neq \emptyset$ <u>for</u> $A \in \mathfrak{A}$ <u>then for</u> $a \in B-D$ <u>there is</u> $A \in \mathfrak{A}$ <u>such that</u> $A \cap B = \{a\}$. <u>If</u> $B' \subset B$ <u>and</u> $B' \cap A \neq \emptyset$ <u>for</u> $A \in \mathfrak{A}$ <u>then</u> $B' - D = B - D$.

<u>Proof</u>. If $B-D \neq \emptyset$ take $a \in B-D$. If $A \cap (B - \{a\}) \neq \emptyset$ for any $A \in \mathfrak{A}$ then $B - \{a\} \supset D$ and $B - \{a\}$ has non-empty intersection with all members of $\mathfrak{A}$, a contradiction on the minimality of B. For each such a let f(a) be such a set. Take $b \in B - B'$. Then if $b \notin D$ we have $f(b) \cap (B - \{b\}) = \emptyset$ and thus $f(b) \cap B' = \emptyset$, a contradiction. Thus $B' - D = B - D$.

CHAPTER 2

EXAMPLES OF EXCHANGE SYSTEMS

NOTATION (2.1) Let E be an arbitrary infinite set and $\{X_i\}_{i=2}^{\infty}$ a partition of E with each X_i infinite. Let $E_i = \bigcup_{j=2}^{i} X_j$ for $i \geqq 2$ and let $\mathcal{M}_i$ be the exchange system on E_i given in example (1.10). Let $\mathcal{M}_2 = \mathfrak{A}_2$ and for $k > 2$ let

$$\mathfrak{A}_k = \{A \in \mathcal{M}_k \mid A \not\supset B \in \mathcal{M}_s \text{ for any } s < k\}$$

and define the system $\mathfrak{A}_\infty$ on E by $\mathfrak{A}_\infty = \bigcup_{k=2}^{\infty} \mathfrak{A}_k$.

If $A \subset X_k$ for some k, and $|A| = k$, then $A \in \mathfrak{A}_k$ and hence $A \in \mathfrak{A}_\infty$. In addition the sets $\mathfrak{A}_k$ are clearly disjoint.

LEMMA (2.2) *Let* $A \subset E$ *and* $s \geqq 2$ *Then* $|A_s| \geqq s$ *if and only if there is* $B \in \mathfrak{A}_\infty$ *with* $B \subset A$ *and* $B \in \mathfrak{A}_k$ *for some* $k \leqq s$.

Proof. If $|A_s| \geqq s$ for some $s \geqq 2$, there is a least integer k with $|A_k| \geqq k$. Take $B \subset A_k$ such that $|B| = k$. Then $B \in \mathcal{M}_k$. Also, if $C \subset B$, then by the minimality of k, $|C_t| < t$ for $t < k$. Hence $C_t \notin \mathcal{M}_t$ for $t < k$ and thus $B \in \mathfrak{A}_k$ as required.

Conversely, if $|A_s| < s$ for all $s \geqq 2$ then $B \subset A$ and $B \in \mathcal{M}_s$ implies $s = |B| \leqq |A_s| < s$, a contradiction.

LEMMA (2.3) $\mathfrak{A}_\infty$ *is a point separating, connected system on* E *with* $\bigcup \mathfrak{A}_\infty = E$. *The elements of* $\mathfrak{A}_\infty$ *are finite and incomparable*.

<u>Proof</u>. Each $A \in \mathfrak{A}_\infty$ is in a unique $\mathfrak{A}_k$ which is contained in $\mathcal{M}_k$ and thus is finite.

Let $A \in \mathfrak{A}_\infty$ and $B \subsetneq A$. Then $A \in \mathfrak{A}_k$ for a unique k and $k' = |B| < |A| = k$. By definition $A \in \mathfrak{A}_k$ implies $B \notin \mathcal{M}_{k'}$, and so $B \notin \mathfrak{A}_\infty$. Thus the elements of $\mathfrak{A}_\infty$ are incomparable.

To show that $\mathfrak{A}_\infty$ is connected let x_i, $i = 1,2$, be two distinct points in E. Then without loss of generality $x_i \in X_{k_i}$ with $k_1 \leqq k_2$. Consider any set $A \subset E$ such that $A \cap X_j = \phi$ for $j > k_2$, $|A \cap X_{k_2}| = 2$ and $|A \cap X_j| = 1$ for $2 \leqq j < k_2$. Then by lemma (2.2) $A \in \mathfrak{A}_\infty$. From all these possible A's choose one such that $x_i \in A \cap X_{k_i}$, $i = 1,2$. Then $\{x_1,x_2\} \subset A \in \mathfrak{A}_\infty$ and so $\mathfrak{A}_\infty$ is connected. Since the x_i are arbitrary, $\bigcup \mathfrak{A}_\infty = E$.

To show that $\mathfrak{A}_\infty$ is point separating take $A_i \subset X_i$ such that $x_i \in A_i$, $|A_i| = k_i$. Then $A_i \in \mathfrak{A}_\infty$ and if $k_1 \neq k_2$, $A_1 \cap A_2 = \phi$. If $k_1 = k_2$ then because X_{k_1} is infinite one can still choose $A_1 \cap A_2 = \phi$. Thus $\mathfrak{A}_\infty$ is point separating.

<u>THEOREM</u> (2.4) $\mathfrak{A}_\infty$ <u>is a matroid</u>.

<u>Proof</u>. Using lemma (2.3) it is sufficient to show that $\mathfrak{A}_\infty$ has the exchange property. Let $A,B \in \mathfrak{A}_\infty$ with $a \in A-B$, $b \in A \cap B$ then there are unique integers n, k and j such that $B \in \mathfrak{A}_n$, $a \in X_k$ and $b \in X_j$. We consider two cases: (i) $n \geqq k$ and (ii) $n < k$.

Case (i). $(B - \{b\}) \cup \{a\} \subset E_n$ with $|(B - \{b\}) \cup \{a\}| = |B| = n$. By lemma (2.2) and the minimality of B there is $Z \in \mathfrak{A}_\infty$ with $a \in Z \subset (B - \{b\}) \cup \{a\} \subset A \cup B - \{b\}$.

Case (ii). Take $x \in (B-A) \cap X_\nu$ with ν as large as possible. Let $t = \min\{m \mid |(A_m - \{b\}) \cup \{x\}| = m\}$. (Note $|(A - \{b\}) \cup \{x\}| = |A|$).

Again by lemma (2.2) and the minimality of A there is $Z \in \mathfrak{A}_\infty$ with $x \in Z \subset (A - \{b\}) \cup \{x\}$, $|Z| = s$. Then $Z = (Z - \{x\}) \cup \{x\} = (A_s - \{b\}) \cup \{x\}$.

(α) If $s \geqq j$ then $b \in A_s$ and $s = |Z| = |A_s - \{b\}| + 1$, so that $|A_s| = s$. This gives $s = |A|$ and $a \in Z = (A - \{b\}) \cup \{x\}$. (By lemma (2.2)).

(β) If $s < j$ then $s = |Z| = |A_s| + 1$ so that $|A_s - \{b\}) \cup \{x\}| = s$. That is, $s = t$ and $|A_s| = t-1$.

Now $B - B_s \subseteq A_n - A_s$ by the choice of x. Hence $A_s = A_n - (A_n - A_s) \subseteq A_n - (B - B_s)$, and hence $s - 1 = |A_s| \leqq |A_n| - |B - B_s| = (|A_n| - |B|) + |B_s| < |B_s| < s$, a contradiction, and $n < k$ implies $s \geqq j$. Thus in all cases there is $Z \in \mathfrak{A}_\infty$ with $a \in Z \subset A \cup B - \{b\}$; this then proves that $\mathfrak{A}_\infty$ has the exchange property.

Since $\mathfrak{A}_\infty$ consists of only finite subsets of E, $\mathfrak{A}_\infty$ has a dendroid by remark (1.13). Hence $\mathfrak{A}_\infty$ is a matroid.

<u>LEMMA</u> (2.5) <u>For each</u> $A \in \mathfrak{A}_\infty$ <u>there is</u> $D \in \vartheta_{\mathfrak{A}_\infty}$ <u>with</u> $A \subset D$.

<u>Proof</u>. Take $A \in \mathfrak{A}_\infty$, that is, $A \in \mathfrak{A}_k$, k unique and $|A| = k$. Select $a_j \in X_j$ for $j > k + 1$ and $\{a_2,\ldots,a_{k+1}\}$ a k element set in X_{k+1}. Let

$$D = E_k \cup (X_{k+1} - \{a_2,\ldots,a_{k+1}\} \cup \bigcup_{j=k+2}^{\infty} (X_j - a_j) \quad .$$

$$= E - \{a_j \mid j \geqq 2\} \ .$$

Clearly $A \subset D$.

$B \cap D = \phi$, a contradiction. Thus the first of the properties follows.

Now take $x \in D$ where $x \in X_p$ for a unique p and set $q = \max.(p,k+1)$; take $B = \{x\} \cup \{q_j \mid 2 \leq j \leq q\}$. Then $|B \cap E_q| = q$ and $B \in \mathcal{M}_q$. If $C \subseteq B$ then $|C \cap E_s| < s$ for all s. Hence $B \in \mathfrak{A}_\infty$ and $B \cap D = \{x\}$. This $B = f_D(x)$ and D is a dendroid.

THEOREM (2.6) The dendroids for $\mathfrak{A}_\infty$ are precisely the subsets D of E satisfying

(i) $|E_s - D| < s$ for all $s \geq 2$, and

(ii) for each $s \geq 2$ there is $t \geq s$ with $|E_t - D| = t-1$.

Proof. Take any $D \subset E$ with the properties (i) and (ii) and take $A \subset E$ such that $A \cap D = \phi$. Now $A \subset E - D$ so $A_s \subset E_s - D$ for each $s \geq 2$ and hence $|A_s| \leq |E_s - D| < s$. Thus $A \notin \mathfrak{A}_\infty$ by lemma (2.2).

To show that D is a dendroid take $x \in D$ with $x \in X_s$ for some s. Take the first $t \geq s$ such that $|E_t - D| = t - 1$ and let $A = \{x\} \cup (E_t-D)$. Then $|A| = t$ and $A \subset E_t$ so that $A \in \mathcal{M}_t$. For any $r < t$

$$A_r = \begin{cases} (E_r-D) \cup \{x\}, & \text{for } s \leq r. \\ (E_r-D) & , \text{for } r < s. \end{cases}$$

Thus
$$|A_r| = \begin{cases} |E_r-D| + 1 \leq (r-2)+1 & \text{if } s \leq r. \\ |E_r-D| < r & \text{if } r < s. \end{cases}$$

In either case $|A_r| < r$ so that $A \in \mathfrak{A}_\infty$ with $A \cap D = \{x\}$ and hence D is a dendroid.

Conversely, let D be a dendroid for $\mathfrak{A}_\infty$ and suppose $|E_s-D| \geq s$ for some $s \geq 2$. Then by lemma (2.2) there is $B \in \mathfrak{A}_\infty$, $B \subset E_s-D$ and then

Let $B \in \mathfrak{U}_\infty$. If $B \cap D = \phi$ then $|B \cap X_j| \leqq 1$ for $j > k + 2$. Hence $B \in \mathfrak{U}_t$ for some $t \leqq k + 1$. But $B \cap E_k = \phi$, thus $B \in \mathfrak{U}_{k+1}$. However, $|B \cap E_{k+1}| \leqq k$, which is a contradiction.

To show the second part take any $s \geqq 2$ and $x \in X_s \cap D$. Take $y \in f_D(x)$ with $y \in X_t$ and t as large as possible. Then $t \geqq s$, and $f_D(x) - \{x\} \subset E_t - D$ so that $t-1 \leqq |E_t - D| < t$. Thus for $s \geqq 2$ there is $t \geqq s$ such that $|E_t - D| = t-1$.

<u>COROLLARY</u> (2.7) <u>For</u> $D \in \vartheta_{\mathfrak{U}_\infty}$ <u>and</u> $x \in D$, $f_D(x) = (E_t-D) \cup \{x\}$ <u>where</u> $x \in X_s$ <u>and</u> t <u>is the first integer such that</u> $t \geqq s$ <u>and</u> $|E_t-D| = t-1$.

<u>Proof</u>. The proof of theorem (2.6) shows that $|f_D(x)| = t$, and $|E_t-D| = t-1$. Hence $f_D(x) = (E_t-D) \cup \{x\}$. t is the first integer equal to or greater than s with property (ii) of theorem (2.6) otherwise there is an integer p with $s \leqq p < t$ and $|(E_p-D) \cup \{x\}| = p$ and hence an $A \in \mathfrak{U}_\infty$, $A \subsetneq f_D(x)$, a contradiction.

<u>THEOREM</u> (2.8) <u>In the notation of theorem</u> (1.77) <u>the matroid</u> $\mathfrak{L}$ <u>containing</u> $\mathfrak{U}_\infty$ <u>has the l.c.p</u>.

<u>Proof</u>. If $A \in \mathfrak{U}$, $D \in \vartheta_{\mathfrak{U}_\infty}$, $A \cup D = E$, then $E-D = A-D$ any $|E_t-D| < |A|$ for all t, which contradicts property (ii) of $\vartheta_{\mathfrak{U}_\infty}$. Thus $\mathfrak{U}_\infty$ satisfies the conditions of theorem (1.77). Let D be a dendroid for $\mathfrak{L}$. Then $D = D_0 \cup \{a\}$ for some $D_0 \in \vartheta_{\mathfrak{U}_\infty}$ and $a \in E-D_0$. For each $x \in D_0$ the proof of theorem (1.77) shows that $f_D(x) \supset f_{D_0}(x) - \{a\}$, while if $a \in f_{D_0}(x)$ then $f_D(x) = (E-D) \cup \{x\}$. Thus if $x \in B \cap D$ with $x = a$ or $a \in f_{D_0}(x)$ then $\bigcup_{y \in B \cap D} f_D(y) \supset (E-D) \cup B \cap D \supset B$. If no such x exists and $B \in \mathfrak{U}$ then

$\bigcup_{y \in B \cap D} f_D(y) \supseteq \bigcup_{y \in B \cap D_0} f_{D_0}(y) \supset B$ since $\mathfrak{A}_\infty$ has the l.c.p. (consisting of only finite sets). If $B \notin \mathfrak{A}_\infty$ then $B = E - D'$ for some $D' \in \vartheta_{\mathfrak{A}_\infty}$. Now $(E-D') \cap E_p$ is finite for all p and we need only consider the case where $B \cap D$ is infinite. Thus there is $y \in B \cap D \cap X_p$ for some $p > t$ where $a \in X_t$. Now $f_{D_0}(y) = (E_s - D) \cup \{y\}$, $s \geqq p > t$. Hence $a \in f_{D_0}(y)$ for every $B \in \mathcal{L} - \mathfrak{A}_\infty$, and some $y \in B \cap D$.

REMARK (2.9) If $\mathcal{B}$ is any system of incomparable sets containing $\mathfrak{A}_\infty$ then the finite sets in $\mathcal{B}$ are the sets of $\mathfrak{A}_\infty$. If $B \in \mathcal{B}$ is finite then there is a least integer k with $B \subset E_k$. If $|B_s| < s$ for all $s \leqq k$ then take $Y \subset X_k$ with $Y \cap B = \phi$, $|Y| = k - |B|$. Then $B \subset B \cup Y \subset E_k$ with $|B \cup Y| = k$. By lemma (2.2) there is $A \in \mathfrak{A}_\infty$, with $A \subset B \cup Y$. If $A \notin \mathfrak{A}_k$ then $A \subset A \cap B \cup A \cap Y = A \cap B$, a contradiction. But then $A = B \cup Y \supsetneq B$, a contradiction. Thus $|B_s| \geqq s$ for some $s \leqq k$, and then $B \supset A \in \mathfrak{A}_\infty$ by lemma (2.2), and thus $B = A \in \mathfrak{A}_\infty$.

REMARK (2.10) $\mathfrak{A}_\infty$ is not the circuit matroid for any graph X. For any integer $k > 2$ take $A \subset X_k$ with $|A| = k$. Then $A \in \mathfrak{A}_\infty$. Take $B = (A - \{a\}) \cup \{b\}$ with $a \in A$ and $b \in X_{k-1}$. Then $B \in \mathfrak{A}_\infty$, and $A + B = \{a,b\}$ which is not the union of members of $\mathfrak{A}_\infty$. Thus $\mathfrak{A}_\infty \neq \mathfrak{A}(X)$ for any graph X.

<u>THEOREM</u> (2.11) <u>Let</u> $\mathfrak{m}$ <u>be an arbitrary infinite cardinal</u>, $1 \leqq n \leqq \mathfrak{m}$, <u>and</u> k <u>an integer with</u> $k \geqq -n+1$ <u>if</u> n <u>is finite. There is an exchange system</u> $\mathfrak{A}_k$ <u>of</u> $\mathfrak{m}$ <u>incomparable elements which separates points by disjoint members, is connected and has dendroids. If</u> n <u>is infinite then the elements of</u> $\mathfrak{A}_k$ <u>each have cardinality</u> n.

<u>Proof</u>. Let E be a set of cardinality $\mathfrak{m}$ and let σ be the first ordinal of cardinal n. Let $\{X_\alpha \mid 0 \leqq \alpha < \sigma\}$ be a partition of E into n disjoint sets each of cardinality $\mathfrak{m}$. Define $\mathfrak{A}_k$ as the collection of all subsets $A \subset E$ which satisfy

(1) $A \cap X_\alpha$ is finite for each α,

(2) $A \cap X_\alpha$ is a one element set for almost all α, and

(3) $\Sigma(|A \cap X_\alpha| - 1) = k$, where the sum extends over all α with $|A \cap X_\alpha| \neq 1$.

Let A_α stand for $A \cap X_\alpha$ for each α, $0 \leqq \alpha < \sigma$. If $A \in \mathfrak{A}_k$ and $B \subset A$ then $|B_\alpha| \leqq |A_\alpha|$. Thus $\Sigma(|B_\alpha| - 1) \leqq \Sigma(|A_\alpha| - 1) = k$, with equality if and only if $B = A$. Hence $\mathfrak{A}_k$ consists of incomparable elements. It is clear that if n is infinite that $\mathfrak{A}_k$ is non-empty and when n is finite the restriction $k \geqq -n+1$ ensures the same (the worst case being $k = -n+1$ which is the discrete one, $\mathfrak{A}_k$ consisting of all one element sets).

For $A \in \mathfrak{A}_k$, $A_\alpha \neq \phi$ for some α. Let $|A_\alpha| = p$, finite. Then $\mathfrak{M}_\alpha = \{B \subset X_\alpha \mid |B| = p\}$ is such that for each $B \in \mathfrak{M}_\alpha$ the set $A' = (A - A_\alpha) \cup B$ is a member of $\mathfrak{A}_k$. Thus $|\mathfrak{A}| \geqq \mathfrak{m}$. For $A \in \mathfrak{A}_k$ let $I = \{\alpha \mid |A_\alpha| \neq 1\}$. It is clear for a finite given set of ordinals I each less than σ that there are at most $\mathfrak{m}^{|I|} = \mathfrak{m}$ sets $A \in \mathfrak{A}_k$ with $I_A = I$.

Hence $|\mathfrak{A}_k| \leqq mn = m$, and therefore $|\mathfrak{A}_k| = m$.

If n is infinite then the fact that $A \cap X_\alpha$ is finite for each α implies $|A| = n$ for each $A \in \mathfrak{A}_k$.

Let a_0, a_1 be distinct points of E. Take any positive integer $r > |k|$ and choose $R^{(i)} \subset E - \{a_0, a_1\}$, $i = 0,1$, in such a way that $R^{(0)} \cap R^{(1)} = \phi$, $|R^{(i)} \cap X_\alpha| \leqq 1$ for each α, and $|\{\alpha | R^{(i)} \cap X_\alpha = \phi\}| = r$, $i = 0,1$. Let α_i be the unique ordinal for which $a_i \in X_{\alpha_i}$, $i = 0,1$, and select $S^{(i)} \subset X_{\alpha_i} - (R^{(0)} \cup R^{(1)} \cup \{a_0, a_1\})$ such that $|S^{(i)}| = r + k$ and $S^{(0)} \cap S^{(1)} = \phi$. Then $A^{(i)} = R^{(i)} \cup S^{(i)} \cup \{a_i\}$ are two distinct sets belonging to $\mathfrak{A}_k$:

$$\Sigma(|(S^{(i)} \cup R^{(i)} \cup \{a_i\}) \cap X_\alpha| - 1)$$
$$= |S^{(i)} \cap X_{\alpha_i}| - 1 + |a_i| + \Sigma(|R^{(i)} \cap X_\alpha| - 1)$$
$$= (r + k) - r = k.$$

For $A^{(0)}$ and $A^{(1)}$ take $c \in A^{(0)} - \{a_0\}$. Then the set $(A^{(0)} - \{c\}) \cup \{a_1\}$ clearly has properties 1), 2), and 3) and so is in $\mathfrak{A}_k$ and $\mathfrak{A}_k$ is connected with $\bigcup\mathfrak{A}_k = E$.

To obtain a dendroid for $\mathfrak{A}_k$ take $B \subset E$ such that $|B_0| = k$ and $|B_\alpha| = 1$ for $0 < \alpha < \sigma$ if k is non-negative, and take $B_i = \phi$ for $0 \leqq i \leqq |k|$ and $|B_\alpha| = 1$ for $|k| < \alpha < \sigma$ if k is negative. In either case it is clear that $\Sigma(|B_\alpha| - 1) = k - 1$. Let $D = E - B$. Take $A \in \mathfrak{A}_k$. If $A \cap D = \phi$ then $A \subset B$ and $\Sigma(|A_\alpha| - 1) \leqq \Sigma(|B_\alpha| - 1) = k - 1$, a contradiction. For $x \in D$ consider $A = \{x\} \cup B$. Then A_α is finite for each α, and is a one element set for almost all α and $\Sigma(|A_\alpha| - 1) = \Sigma(|B_\alpha| - 1) + |\{x\}| = k$ and $A \in \mathfrak{A}_k$ with $A \cap D = \{x\}$. Thus D is a dendroid for $\mathfrak{A}_k$ and for $x \in D$, $f_D(x) = \{x\} \cup \complement D$.

COROLLARY (2.12) The dendroids for $\mathfrak{U}_k$ are precisely the complements of the sets $Y \subset E$ with $Y \cap X_\alpha$ finite for almost all α, and $\Sigma(|Y \cap X_\alpha| - 1) = k - 1$. Further, for any dendroid D and $x \in D$ $f_D(x) = \{x\} \cup \complement D$.

Proof. Let Y be any set of the form in the statement of the corollary. If $A \cap \complement Y = \phi$ then $A \subset Y$ and $A_\alpha \subset Y_\alpha$ for each α. Thus $\Sigma(|A \cap X_\alpha| - 1) \leqq k - 1$ and $A \notin \mathfrak{U}_k$. Take $x \in \complement Y$. Then $\Sigma(|(Y \cup \{x\}_\alpha| - 1) = k$. Hence $Y \cup \{x\} \in \mathfrak{U}_k$ and $(Y \cup \{x\}) \cap \complement Y = \{x\}$. Thus $\complement Y$ is a dendroid and $f_{\complement Y}(x) = \complement Y \cup \{x\}$.

Conversely, let D be a dendroid and $x \in D$. Then $f_D(x) - \{x\}$ has the property $\Sigma(|(f_D(x) - \{x\})_\alpha| - 1) = k - 1$. Hence by the above paragraph, $x \cup \complement f_D(x)$ is a dendroid (containing D). Thus $D = \{x\} \cup \complement f_D(x)$, $\complement D = f_D(x) - \{x\}$, and $f_D(x) = \complement D \cup \{x\}$, for $D \in \vartheta_{\mathfrak{U}_k}$.

COROLLARY (2.13) $\mathfrak{U}_k$ has the l.c.p.

Proof. For $A \in \mathfrak{U}_k$, $A \subset (E - D) \cup (A \cap D) = \bigcup_{a \in D \cap A} f_D(a)$.

REMARK (2.14) For $A \in \mathfrak{U}_k$ and D a dendroid for $\mathfrak{U}_k$, $\sum_{a \in A \cap D} f_D(a) = A \cap D$.

LEMMA (2.15) Let X be a graph and $\mathfrak{U}(X)$ the collection of all circuits in X. If C_1 and C_2 are in $\mathfrak{U}(X)$ then $C_1 + C_2$ is the union of edge disjoint members of $\mathfrak{U}(X)$.

Proof. Take $C_1, C_2 \in \mathfrak{A}(X)$. C_1 and C_2 are locally finite Euler graphs. Hence by [4], p. 835 so is $C_1 + C_2$. In view of this our lemma is a consequence of Veblen's theorem [1, Kapitel II, §5, Satz 11].

COROLLARY (2.16) *Let* C *be a graph and* $\mathfrak{A}(X) \neq \phi$. *Then* $\mathfrak{A}(X)$ *is an exchange system*.

Proof. Take C_1 and C_2 in $\mathfrak{A}(X)$ with $e_1 \in C_1 - C_2$ and $e_2 \in C_1 \cap C_2$. Then there is a circuit $C \subset C_1 + C_2$ with $e_1 \in C$. Hence $e_1 \in C \subset C_1 \cup C_2 - \{e_2\}$ and $\mathfrak{A}(X)$ is an exchange system.

DEFINITION (2.17) If X is a graph and $\mathfrak{A}(X) \neq \phi$ then $\mathfrak{A}(X)$ is called the *circuit matroid* of the graph X. In the following we shall only occasionally distinguish between a circuit and its edge set.

LEMMA (2.18) *Let* X *be a graph and* $\mathfrak{F}$ *the collection of finite circuits in* $\mathfrak{A}(X)$. *If* $\mathfrak{F} \neq \phi$ *then* $\mathfrak{F}$ *is an exchange system with* $\vartheta_{\mathfrak{F}} \neq \phi$.

Proof. Let F_1 and F_2 be in $\mathfrak{F}$, hence in $\mathfrak{A}(X)$. Then $F_1 + F_2$ is a disjoint union of circuits by lemma (2.15), each of which is finite. Thus as in the corollary above $\mathfrak{F}$ is an exchange system. By remark (1.13) $\vartheta_{\mathfrak{F}} \neq \phi$.

REMARK (2.19) When X is finite then $\mathfrak{A}(X)$ is a finite collection of finite circuits. If $\mathfrak{A}(X) \neq \phi$ then $\vartheta_{\mathfrak{A}(X)}$ has been characterized by Tutte [5], as follows. For each component X_i of X let T_i be a spanning tree. Then $\bigcup_i E(X_i \backslash T_i)$ is a dendroid for $\mathfrak{A}(X)$ and all dendroids are of this form.

<u>LEMMA</u> (2.20) <u>Let</u> X <u>be a connected graph with</u> $\mathfrak{A}(X) \neq \phi$. <u>If</u> $\mathfrak{F} \neq \phi$ <u>then the dendroids for</u> $\mathfrak{F}$ <u>are precisely the sets</u> $E(X \backslash T)$ <u>where</u> T <u>is a spanning tree of</u> X.

<u>Proof</u>. By lemma (1.73) the dendroids for $\mathfrak{F}$ are the sets $E(X) - M$ where M is maximal such that $\mathfrak{A}(X)_M \cap \mathfrak{F} = \phi$.

Clearly T is maximal in X with no finite circuits if and only if T is a spanning tree. Hence $M = E(T)$ and $D = E(X \backslash T)$ is a dendroid for $\mathfrak{F}$ and all dendroids are of this form.

<u>LEMMA</u> (2.21) <u>Let</u> X <u>be a graph and</u> T <u>a</u> (<u>spanning</u>) <u>tree of</u> X. <u>If</u> $\mathfrak{A}(T) \neq \phi$ <u>then</u> $\bigcup\mathfrak{A}(T) = T_0$ <u>is a tree with</u> $\mathfrak{A}(T_0) = \mathfrak{A}(T)$, <u>and</u> T_0 <u>is circuit connected</u>.

<u>Proof</u>. If C is a circuit in T then C is a circuit in T_0 and vice-versa. Hence it is sufficient to show that T_0 is circuit connected.

Let x_i $(i = 1,2)$ be two distinct vertices in T_0. Then there is a path P joining x_1 and x_2 in T. Let C_i be circuits in T_0 with $x_i \in V(C_i)$, $i = 1,2$. If x_1 and x_2 are in either C_1 or C_2 there is nothing to prove. Otherwise, if $C_1 \cap C_2 = \phi$ then take one of the rays R_i in C_i with initial vertex x_i, $i = 1,2$. Then $R_1 + P + R_2$

is a circuit in T, hence in T_0, containing x_1 and x_2.

If $C_1 \cap C_2 \neq \phi$ take $x \in V(C_1) \cap V(C_2)$ such that $\rho(x_1, x_2)$ (= distance) is least, and R_i the ray in C_i with initial vertex x and containing x_i. Then $R_1 + R_2$ is a circuit in T (in T_0) containing x_1 and x_2. Hence T_0 is a tree.

Let $e_1, e_2 \in E(T_0)$ with $e_i \in C_i$, $i = 0,1$, circuits in T_0. Let P be the path joining the vertices x_i not containing the edges e_i, $i = 0,1$, where x_i is a vertex of e_i. Now if $e_i \notin E(C_{1-i})$, $i = 0$ or 1 consider the circuit $R_1 + R_2 + P$ where R_i is the ray in C_i containing e_i, edge disjoint with P and whose terminal vertex coincides with that of P at e_i. $R_1 + R_2 + P$ is in $\mathfrak{A}(T_0)$ and thus T_0 is circuit connected.

<u>COROLLARY</u> (2.22) <u>Let $\mathfrak{A}(X)$ be the circuit matroid of a connected graph X and D a dendroid for $\mathfrak{A}(X)$. Then $D = D_0 \cup E(X\setminus T)$ for a suitable spanning tree T of X and dendroid D_0 of $\mathfrak{A}(T_0)$.</u>

<u>Proof</u>. By lemma (1.74) $D = D_0 \cup D_1$ where D_1 is a dendroid for $\mathfrak{F}$ and D_0 a dendroid for $\mathfrak{A}_{E(X)-D_1}$. By lemma (2.20) $D_1 = E(X\setminus T)$ for a suitable spanning tree T of X and $\mathfrak{A}_{E(X)-E(X\setminus T)} = \mathfrak{A}_{E(T)} = \mathfrak{A}(T) = \mathfrak{A}(T_0)$. Hence D_0 must be a dendroid for $\mathfrak{A}(T_0)$.

<u>REMARK</u> (2.23) Although no further use is made of it we add here a simple characterization of trees for which $T = T_0$.

LEMMA (2.24) A tree is a union of circuits if and only if it has no end vertices of degree one.

Proof. Let T be a tree which is the union of circuits and x a vertex in T. Then $x \in e \in E(T)$ and e belongs to a circuit in T. Hence $d(T;x) \geqq 2$ and so T has no vertices of degree one.

Conversely, if $d(T;x) \geqq 2$ for each $x \in V(T)$ take a maximal path P in T containing x. If P is not a circuit it has an initial vertex x_0 which must be of degree one by maximality, a contradiction. Hence T is the union of circuits.

THEOREM (2.25) Let T be a tree with $\cup\mathfrak{U}(T) = T$. Then $\vartheta_{\mathfrak{U}(T)} \neq \phi$.

Proof. Let S be a connected subgraph of T such that S contains no ray. Fix x_0 on the boundary of $S(\mathcal{B}(S)$, see for example, [3], p. 346) and set $N = (V(T) - V(S)) \cup \{x_0\}$. For $x \in V(T)$ let $E_x = \{[x,y] \in E(T) | \rho(x_0,y) > \rho(x_0, x)\}$. Note that $E_{x_0} = E(T;x_0)$. Let ε be a choice function for the family $(E_x)_{x \in N}$ such that $\varepsilon(x_0) \notin E(S)$. This is possible since $x_0 \in \mathcal{B}(S)$.

We claim that

(1) $$D = \bigcup_{x \in \mathcal{B}(S)-\{x_0\}} Ex \quad \cup \quad \bigcup_{x \in N} (E_x - \{\varepsilon(x)\})$$

is a dendroid for $\mathfrak{U}(T)$.

Let C be a circuit in T.

Case (i) $V(C) \subset N$. Let x be the closest vertex to x_0. Then $|E_x \cap E(C)| = 2$ and so $|D \cap E(C)| \geqq 1$.

Case (ii) $V(C) \not\subset N$. Then $V(C) \not\subset V(S)$ and there is $y \in \mathfrak{B}(S)$, $y \in V(C)$. If $y = x_0$ then $|E_{x_0} \cap E(C)| = 2$ and so $|D \cap E(C)| \geqq 1$. If $y \neq x_0$ then $|E_y \cap E(C)| \neq 0$ and so $D \cap E(C) \neq \phi$.

Thus D intersects all circuits in T.

For $x \in \mathfrak{B}(S) - \{x_0\}$, $E_x \cap E(S) = \phi$. Take $e \in D$, $e = [x,y]$. Then at least one of x,y is in N.

Given $x \in N$ define a sequence $y_0, y_1, \dots$ by $y_0 = x$ and y_{i+1} by $\varepsilon(y_i) = [y_i, y_{i+1}]$ for $i \geqq 0$. Then $R_x = [y_0, y_1, \dots]$ is a ray in T and $E(R_x) \cap D = \phi$.

Now if x and y are in N then $[x,y] \neq \varepsilon(x')$ for any x'. Hence $R_x \cap R_y = \phi$ and $C = R_x \cup (e) \cup R_y$ is a circuit in T with $E(C) \cap D = \{e\}$.

If $x \in N$ and $y \notin N$ then $y \in \mathfrak{B}(S) - \{x_0\}$. Take P to be the path joining x and x_0. $P \subset S \cup (e)$ and again $C = R_x \cup P \cup R_{x_0}$ is a circuit in T with $D \cap E(C) = \{e\}$. Hence D is a dendroid for $\mathfrak{A}(T)$.

<u>COROLLARY</u> (2.26) <u>Let</u> $\mathfrak{A}(X)$ <u>be the circuit matroid for a connnected graph</u> X. <u>Then</u> $\vartheta_{\mathfrak{A}(X)} \neq \phi$, if $\mathfrak{N}(X) \neq \emptyset$.

<u>Proof</u>. Let T be a spanning tree of X. Then $E(X\backslash T)$ is a dendroid for $\mathfrak{F}$ if $\mathfrak{F} \neq \phi$. By the above, we have a dendroid D for $\mathfrak{A}(T)$ if $\mathfrak{A}(T) \neq \phi$. Hence by lemma (1.75) $D \cup E(X\backslash T)$ is a dendroid for $\mathfrak{A}(X)$. Note that if $\mathfrak{F} = \phi$ then $\mathfrak{A}(T) = \mathfrak{A}(X)$ while if $\mathfrak{A}(T) = \phi$, $E(X\backslash T)$ meets all circuits.

THEOREM (2.27) *Let* X *be a graph with* $\mathfrak{A}(X) \neq \phi$. *Then* $\vartheta_{\mathfrak{A}(X)} \neq \phi$.

Proof. By corollary (2.26) each component Y of X has a dendroid D_Y for $\mathfrak{A}(Y)$. Let $D = \cup\{D_Y \mid Y \text{ a component of } X\}$. If C is a circuit in X then C belongs to a unique component Y of X. Hence $D \cap E(C) \supset D_Y \cap E(C) \neq \phi$.

For $y \in D_Y$, $f_{D_Y}(y)$ is a circuit in Y and hence in X. If Y' is a component of X, $Y' \neq Y$ then $E(Y') \cap E(Y) = \phi$. Hence $f_{D_Y}(y) \cap D = \{y\}$ and D is a dendroid for the circuits of X, i.e., for $\mathfrak{A}(X)$.

REMARK (2.28) It is clear that if D is a dendroid for $\mathfrak{A}(X)$ and Y is a component of X then $D \cap E(Y)$ is a dendroid for $\mathfrak{A}(Y)$ when $\mathfrak{A}(Y) \neq \phi$. Thus one needs only consider connected graphs in obtaining a characterization of the dendroids of a graph X.

A CHARACTERIZATION OF $\vartheta_{\mathfrak{A}(X)}$ FOR A GRAPH X

REMARK (2.29) If X is connected then the dendroids for $\mathfrak{A}(X)$ are given by $E(X\setminus T) \cup D$ where T is a spanning tree of X and D is a dendroid for $T_0 = \cup\mathfrak{A}(T)$, by corollary (2.22). Further, a dendroid for X is the union of dendroids on each component of X by remark (2.28). Thus we need only characterize the dendroids of circuit matroids (infinite) of trees.

DEFINITION (2.30) Let X be a tree. The pair (φ, Y) is X-admissible if and only if:

(i) Y is a graph,

(ii) $\varphi : V(X) \to V(Y)$ is an onto function,

(iii) for each $y \in V(Y)$ the graph $\varphi^{-1}(y)$ in X is connected and contains no ray (here $\varphi^{-1}(y)$ denotes the subgraph of X whose vertex set is $\varphi^{-1}(y)$),

(iv) $[x,y] \in E(X)$ and $\varphi(x) \neq \varphi(y)$ implies $[\varphi(x), \varphi(y)] \in E(Y)$,

(v) $[p,q] \in E(Y)$ implies there is $[x,y] \in E(X)$ with $\varphi(x) = p$, $\varphi(y) = q$.

REMARK (2.31) If φ is a graph isomorphism from X to Y then (φ, Y) is X-admissible.

LEMMA (2.32) *Let* (φ, Y) *be X-admissible and* P *a path in* X. *Then* $\varphi(P)$ *is a path in* Y.

Proof. Let $P = (x_0, \dots, x_n)$ be a path in X. We use induction on n. If $n = 1$ the statement follows directly from condition (2.30)(iii). Let $P' = (x_0,\dots,x_{n-1})$. By the induction hypothesis $\varphi(P')$ is a path. We distinguish two cases.

Case (i). $\varphi(x_n) = \varphi(x_{n-1})$. In this case $\varphi(P) = \varphi(P')$ and hence $\varphi(P)$ is a path.

Case (ii) $\varphi(x_n) \neq \varphi(x_{n-1})$. Suppose $\varphi(x_n) = \varphi(x_i) = y$ for some i, $0 \leqq i \leqq n-2$. Then there is a path Q in $\varphi^{-1}(y)$ joining x_i and x_n. Since X is a tree, Q is a segment of P containing x_i and x_n, and hence $\varphi(x_j) = y$ for $i \leqq j \leqq n$. In particular $\varphi(x_{n-1}) = \varphi(x_n)$, a contradiction. It follows that $\varphi(x_n) \neq \varphi(x_i)$ $0 \leqq i \leqq n-1$ and hence that $\varphi(P)$ is a path.

LEMMA (2.33) Let (φ,Y) be X-admissible and $[p,q] \in E(Y)$. Then there is a unique edge $[x,y] \in E(X)$ with $[\varphi(x), \varphi(y)] = [p,q]$.

Proof. By (2.30)(v) there is $[x_0, x'_0] \in E(X)$ with $[\varphi(x_0), \varphi(x'_0)] = [p,q]$. By (2.30)(iii) $\varphi^{-1}(p)$ and $\varphi^{-1}(q)$ are connected subgraphs of X. Suppose $[x_1,x'_1] \in E(X)$ with $p = \varphi(x_1)$ and $q = \varphi(x'_1)$. There are paths P_i joining x_i and x'_i for $i = 0,1$ (with P_i degenerate if $x_i = x'_i$).
$[x_1,x_0] \cup P_0 \cup [x'_0,x'_1] \cup P_1$ is a closed path in X which is nontrivial if $x_i \neq x'_i$ for i equal 0 or 1. Since X is a tree this implies that the edge $[x,y] \in E(X)$ for which $[\varphi(x), \varphi(y)] = [p,q]$ is unique.

<u>LEMMA</u> (2.34) <u>If</u> (φ,Y) <u>is</u> X-<u>admissible and</u> S <u>is connected in</u> X <u>then</u> $\varphi(S)$ <u>is connected in</u> Y.

<u>Proof</u>. Take p,q in $V(\varphi(S))$. Then there is x,y in S such that $\varphi(x) = p$ and $\varphi(y) = q$. Since S is connected there is a path P in S from x to y. Hence by lemma (2.32), $\varphi(P)$ is a path in $\varphi(S)$ from $\varphi(x) = p$ to $\varphi(y) = q$.

<u>LEMMA</u> (2.35) <u>Let</u> (φ,Y) <u>be</u> X-<u>admissible</u>. <u>Every simple path</u> Q <u>in</u> Y <u>is the image of a path</u> P <u>in</u> X <u>and</u> Y <u>is a tree</u>. <u>Further, if</u> Q <u>is non-degenerate then</u> P <u>can be chosen such that if</u> P' <u>is a simple path in</u> X <u>with</u> $\varphi(P') = Q$ <u>then</u> P <u>is a subpath of</u> P'.

<u>Proof</u>. If Q is a vertex then by (2.30)(ii) there is a vertex $x \in V(X)$ with $\varphi(x) = Q$.

Suppose $Q = (q_0,\ldots,q_n)$. By (2.33) there is $[x_i,z_i]$ unique in $E(X)$ with $[\varphi(x_i), \varphi(z_i)] = [q_i,q_{i+1}]$ for $0 \leqq i < n$. Now $\varphi(z_i) = \varphi(x_{i+1})$ for $0 \leqq i \leqq n-2$, and by (2.30)(iii) there is a simple path P_i joining z_i and x_{i+1}. Since X is a tree each P_i is unique. Thus

$$P = [x_0,z_0] \cup P_0 \cup [x_1,z_1] \cup \ldots \cup P_{n-2} \cup [x_{n-1},z_{n-1}]$$

is a simple path in X with $\varphi(P) = Q$.

Let P' be a path in X such that $\varphi(P') = Q$. For $0 \leqq i < n$ lemma (2.33) implies that $[x_i,z_i]$ is an edge in P', and every path P'_i joining z_i and x_{i+1} contains a unique simple path joining z_i and x_{i+1}, since X is a tree. Hence P_i is a subpath of P'_i

for $0 \leqq i < n-1$. Thus P' contains P, and if P' is simple, P is a subpath of P'.

If C is a simple closed path in Y then $C = C_0 \cup C_1$ where C_0 and C_1 are edge disjoint paths from say q_0 to q_1. Now $C_i = \varphi(P_i)$, and there is x_i, x'_i in P_i such that $\varphi(x_i) = q_0$, and $\varphi(x'_i) = q_1$ for $i = 0,1$. By (2.30)(iii) there are paths Q and Q' in X joining x_0, x_1 and x'_0, x'_1 respectively. Then

$$Q \cup P_0 \cup Q' \cup P_1$$

is a closed non-trivial path in X, a contradiction. Hence Y is a tree.

<u>COROLLARY</u> (2.36) If $S \subset Y$ is connected then $\varphi^{-1}(S)$ is connected.

<u>Proof</u>. Obvious from condition (2.30)(iii).

<u>LEMMA</u> (2.37) <u>Let</u> (φ, Y) <u>be</u> X-<u>admissible</u>. <u>If</u> $C \in \mathfrak{A}(X)$ <u>then</u> $\varphi(C) \in \mathfrak{A}(Y)$ <u>and for</u> $C' \in \mathfrak{A}(Y)$ <u>there is</u> $C \in \mathfrak{A}(X)$, <u>unique</u>, <u>with</u> $\varphi(C) = C'$.

<u>Proof</u>. Let $C = (\ldots x_{-k}, \ldots, x_0, \ldots, x_k, \ldots)$ and for $k \geqq 0$ let $P_k = (x_{-k}, \ldots, x_k)$. By lemma (2.32) $\varphi(C) = \bigcup_{k \geqq 0} \varphi(P_k)$, i.e., $\varphi(C)$ is a union of an increasing sequence of paths. Hence $\varphi(C)$ is one of the following: a path, a ray, or a circuit. Suppose that $\varphi(C)$ is not a circuit. Because Y is a tree $\varphi(C)$ contains a vertex y_0 of degree 1. If $\varphi(C) = y_0$ then $\varphi^{-1}(y_0)$ contains C, a contradiction to (2.30)(iii). Let y_1 be the unique vertex of $\varphi(C)$ adjacent to y_0. Without loss

of generality we may assume $\varphi(x_0) = y_0$ and $\varphi(x_1) = y_1$. By lemma (2.33) this implies $\varphi(x_k) = y_0$ for all $k \leqq 0$ and so $\varphi^{-1}(y_0)$ contains the ray $[x_0, x_{-1}, \ldots]$, a contradiction to (2.30)(iii). It follows that $\varphi(C)$ is an infinite circuit.

Let $C' = (\ldots, y_{-k}, \ldots, y_0, \ldots, y_k, \ldots)$ be an infinite circuit in Y and $P'_k = (y_{-k}, \ldots, y_0, \ldots, y_k)$. Then $C' = \bigcup_{k \geqq 0} P'_k$. By lemma (2.35) P'_k is the image of a unique simple path P_k in X, and lemma (2.35) implies P_k is a proper subpath of P_{k+1} for $k \geqq 0$ such that both ends of P_{k+1} are different than those of P_k. Hence $C = \bigcup_{k \geqq 0} P_k$ is a circuit for C contains countably many distinct edges in both directions from any given vertex.

To show the uniqueness of this C let $i = 0,1$ and suppose C_i is a circuit in X with $\varphi(C_i) = C'$ and for $k \geqq 0$ $P_i^{(k)}$ is the simple subpath of C_i such that $\varphi(P_i^{(k)}) = P'_k$, in the notation of the last paragraph. Then $\bigcup_{k \geqq 0} P_i^{(k)} = C_i$ as in the first paragraph of this lemma. Lemma (2.35) shows that $P_i^{(k)}$ is a subpath of $P_{1-i}^{(k)}$ for $k \geqq 0$ and $i = 0,1$. Hence $P_i^{(k)} = P_{1-i}^{(k)}$ for $i = 0,1$ and $C_0 = C_1$. Thus for $C' \in \mathfrak{A}(Y)$ there is a unique $C \in \mathfrak{A}(X)$ with $\varphi(C) = C'$.

COROLLARY (2.38) *If* (φ, Y) *is* X-*admissible then* φ *induces a one-one onto correspondence* φ_c *between* $\mathfrak{A}(X)$ *and* $\mathfrak{A}(Y)$ *which is compatible with the exchange property.*

Proof. Define $\varphi_c : \mathfrak{A}(X) \to Y$ be means of φ as follows. For each $A \in \mathfrak{A}(X)$ there is a unique circuit C_A in X with $E(C_A) = A$ and vice-versa. The same holds for $\mathfrak{U}(Y)$. Define $\varphi_c(A) = B \in \mathfrak{U}(Y)$ where $B = E(\varphi(C_A))$. The uniqueness statement of the lemma above shows that φ_c is one-one.

For $B \in \mathfrak{U}(Y)$ this lemma also shows there is a unique circuit C_A in X with $\varphi(C_A) = C_B$. Hence $\varphi_c(A) = B$, and φ_c is both one-one and onto.

Using the fact that $\mathfrak{U}(Y)$ has exchange, if $e \in B_1 - B_2$, $e' \in B_1 \cap B_2$, then $e \in B_3 \subset B_1 \cup B_2 - \{e'\}$. Hence $\varphi^{-1}(e) \in \varphi_c^{-1}(B_3)$ $\varphi_c^{-1}(B_3) \subset \varphi_c^{-1}(B_1) \cup \varphi_c^{-1}(B_2) - \varphi^{-1}(e')$, i.e., if $\varphi_c(A_i) = B_i, i = 1,2,3$ then $\varphi^{-1}(e) \in A_1 - A_2$, $\varphi^{-1}(e') \in A_1 \cap A_2$ and $\varphi^{-1}(e) \in A_3 \subset A_1 \cup A_2 - \{\varphi^{-1}(e')\}$.

<u>REMARK</u> (2.39) If (φ, Y) is X-admissible it follows from lemma (2.33) that there exists a one-one function $\Psi : E(Y) \to E(X)$ given by $\Psi([p,q]) = [x,y]$, where $\varphi(x) = p$, $\varphi(y) = q$. By an abuse of notation we shall denote Ψ by φ^{-1}. Thus if $Q \subset E(Y)$, then $\varphi^{-1}(Q)$ denotes the set $\{\varphi^{-1}(e) \mid e \in Q\}$. If $e = [x,y]$ is in $\varphi^{-1}(E(Y))$ we shall denote by $\varphi(e)$ the edge $[\varphi(x), \varphi(y)] \in E(Y)$. If $D \subset \varphi^{-1}(E(Y))$ we set $\varphi(D) = \{\varphi(e) \mid e \in D\}$. However for C a circuit in X we still have $\varphi(E(C)) = E(\varphi(C))$ as in the corollary above.

<u>NOTATION</u> (2.40) Let X be a graph and r an equivalence relation on $V(X)$. For $x \in (V(X)$ the subgraph X_x of X is defined by

$$V(X_x) = \{y \in V(X) \mid yrx\} \text{ and}$$
$$E(X_x) = \{[x,y] \in E(X) \mid \{x,y\} \subset V(X_x)\}.$$

Denote the equivalence class of x by $\bar{x}$ and let X/r be the quotient graph of X defined by

$$V(X/r) = \{\bar{x} \mid x \in V(X)\} \text{ and}$$

$$E(X/r) = \{[\bar{x},\bar{y}] \mid \bar{x} \neq \bar{y} \text{ and there is } x' \in \bar{x},\ y' \in \bar{y} \text{ with } [x', y'] \in E(X)\}$$

Let φ_r be the natural mapping from X to X/r given by $\varphi_r(x) = \bar{x}$.

For S a subgraph of X define the relation s on $V(X)$ by xsy if and only if $x = y$, or $\{x,y\}$ is in a component of S. s will be called the equivalence _induced_ by S.

LEMMA (2.41) _Let_ X _be a tree and_ r _an equivalence relation on_ $V(X)$. _Then_ $(\varphi_r, X/r)$ _is_ X-_admissible if and only if each_ X_x _is connected and contains no ray. Similarly, if_ S _is a subgraph of_ X _the equivalence_ s _induced by_ S _on_ $V(X)$ _makes_ $(\varphi_s, X/s)$ X-_admissible if and only if each component of_ S _is connected and contains no ray_.

Proof. Clearly properties (i),(ii),(iv) and (v) of definition (2.30) are satisfied for $(\varphi_r, X/r)$. Since $\varphi_r^{-1}(\bar{x}) = X_x$, $(\varphi_r, X/r)$ is X-admissible if and only if X_x is connected and contains no ray.

For the equivalence relation s induced by S, $X_x = \varphi_s^{-1}(\bar{x})$, which is the component of S containing any x' with $x'sx$ if $\bar{x} \in \varphi_s(S)$ and is just x otherwise, where $\{x\} = \bar{x}$. Hence $(\varphi_s, X/s)$ is X-admissible if and only if each component of S is connected and contains no ray.

COROLLARY (2.42) *There is a one-one correspondence between the subgraphs S of X with no isolated vertices such that each component of S is connected and contains no ray and the equivalence relations r on $V(X)$ such that $(\varphi_r, X/r)$ is X-admissible.*

Proof. Obvious.

DEFINITION (2.43) An equivalence relation on $V(X)$ such that $(\varphi_r, X/r)$ is X-admissible will be called an *admissible equivalence* on X and X/r will be called a *contraction* of X.
In the same way if a subgraph S of X induces an admissible equivalence s on X then S will be called X-admissible and X/s will be written X/S.

LEMMA (2.44) *Let (φ, Y) be X-admissible. There is an admissible equivalence r on X such that $\varphi = \tau \circ \varphi_r$ where $\tau : X/r \to Y$ is a graph isomorphism.*

Proof. Define r by xry if and only if $\varphi(x) = \varphi(y)$. Then $X_x = \varphi^{-1}(\varphi(x))$ is connected and contains no ray by definition and so r is an admissible equivalence on X by lemma (2.41).

Define $\tau : X/r \to Y$ by $\tau(\bar{x}) = \varphi(x)$. Clearly τ is well defined and onto $V(Y)$. For $e' \in E(Y)$, $e' = \varphi(e)$ where $e \in [x,y]$ and $\varphi(x) \neq \varphi(y)$. Hence $\bar{e} = [\bar{x},\bar{y}] \in E(X/r)$ and $\tau(\bar{e}) = [\tau(\bar{x}),\tau(\bar{y})]$, i.e. $\tau(\bar{e}) = e'$. For $\bar{e} \in E(X/r)$, $\bar{e} = [\bar{x},\bar{y}]$, $e = [x,y] \in E(X)$, unique. $\varphi(e) = [\varphi(x),\varphi(y)] = [\tau(\bar{x}),\tau(\bar{y})] = \tau(\bar{e})$. Hence τ is a graph isomorphism.

<u>REMARK</u> (2.45) Using remark (2.31), corollary (2.38) and lemma (2.44), it is clear that any relation concerning dendroids of $\mathfrak{U}(X)$ and dendroids of $\mathfrak{U}(Y)$ where (φ, Y) is X-admissible is essentially given by a relation concerning dendroids of $\mathfrak{U}(X)$ and $\mathfrak{U}(X/S)$ respectively where X/S is some contraction of X.

<u>LEMMA</u> (2.46) <u>Let</u> X/S <u>be a contraction of</u> X. <u>Then</u> $\{\varphi^{-1}(A) \mid A \in \vartheta_{\mathfrak{U}(X/S)}\} \subset \vartheta_{\mathfrak{U}(X)}$. <u>If</u> $D \in \vartheta_{\mathfrak{U}(X)}$ <u>then</u> $\varphi(D) \in \vartheta_{\mathfrak{U}(X/S)}$ <u>if and only if</u> $D \subset \varphi^{-1}(E(X/S))$.

<u>Proof</u>. We shall not distinguish between circuits and their edge sets.

Let A be a dendroid for $\mathfrak{U}(X/S)$ and let $\varphi^{-1}(A) = D$. For $\bar{e} \in A$, $\varphi^{-1}(f_A(\bar{e})) \in \mathfrak{U}(X)$ and $\varphi^{-1}(f_A(\bar{e})) \cap D = \varphi^{-1}(f_A(\bar{e}) \cap A)$, a single element of $E(X)$. The same lemma (2.37) shows $\varphi(C)$ is a circuit in X/S for each circuit in X. Hence, if $C \cap D \neq \phi$, then $\varphi(C) \cap A = \varphi(C \cap D) = \phi$, a contradiction. Thus $\varphi^{-1}(A)$ is a dendroid for $\mathfrak{U}(X)$.

For the second part, if $e \in D \in \vartheta_{\mathfrak{U}(X)}$ with $\varphi(e) \notin E(X/S)$ then $\varphi(e) = \varphi(f_D(e) \cap D) = \varphi(f_D(e) \cap \varphi(D) \notin E(X/S)$, while $\varphi(f_D(e))$ is a circuit in X/S. Thus $D \subset \varphi^{-1}(E(X/S))$ is necessary.

Suppose that $D \subset \varphi^{-1}(E(X/S))$. For any circuit in X, $C \cap D \neq \phi$ and so $\varphi(C \cap D) = \varphi(C) \cap \varphi(D) = C' \cap \varphi(D) \neq \phi$ for any circuit C' in X/S by lemma (2.37). Since $f_D(e) \cap D = \{e\}$, $\varphi(f_D(e) \cap \varphi(D) = \{\varphi(e)\}$ and $\varphi(f_D(e)) = f_{\varphi(D)}(\varphi(e))$. This proves the lemma.

REMARK (2.47) Let X be a tree and let X be rooted at x_0. In the notation of theorem (2.25) $|E(X;x)-E_x| = 1$ for all $x \neq x_0$ and $E_{x_0} = E(X;x_0)$.

LEMMA (2.48) Let X be a circuit connected tree and D a dendroid for $\mathfrak{U}(X)$. Then for some $x_0 \in V(X)$, $E(X;x_0)-D = [x_0,y]$.

Proof. It is clear that $E(X;x) - D \neq \phi$ for all vertices x. Suppose $|E(X;x)-D| > 1$ for all $x \in V(X)$. Choose x_0 arbitrary in $V(X)$. Let $[x_0,x_1]$ and $[x_{-1},x_0]$ be distinct edges in $E(X;x_0) - D$. This forms a path $P_1 = (x_{-1},x_0,x_1)$. Suppose that P_k has been defined with $E(P_k) \cap D = \phi$. Let $[x_k,x_{k+1}]$ be an edge in $E(X;x_k)$ not before used and not in D. $x_{k+1} \neq x_j$ for $0 \leqq j \leqq k$ since X is a tree. Do the same at vertex x_{-k}. Then $P_{k+1} = (x_{-k-1},\ldots,x_{k+1})$ is a path with $P_{k+1} \cap D = \phi$. $C = \bigcup_{k \geqq 0} P_k$ is a circuit in X and $C \cap D = \phi$, a contradiction. Thus the lemma follows.

DEFINITION (2.49) If X is a tree and D a dendroid for $\mathfrak{U}(X)$ then any vertex x with $|E(X;x)- D| = 1$ is said to be D-saturated.

NOTATION (2.50) Let X be a tree with dendroid D for $\mathfrak{U}(X)$. Fix a D-saturated vertex x_0 and an edge $e_0 \in E_{x_0} \cap D$. Given any edge e of X let x_e, y_e be the two ends of e and let the notation be so chosen that $\rho(x_0,x_e) < \rho(x_0,y_e)$.

For $e \in D$, $f_D(e) = E(C)$ where C is a circuit which may be written $C = R_e^+ \cup (e) \cup R_e^-$ where R_e^- and R_e^+ are the rays from

x_e and y_e respectively in C not containing $e = [x_e, y_e]$. Let

$A_D = V(R^-_{e_0}) \cup \bigcup_{e \in D} V(R^+_e)$

$E_D = D \cup E(R^-_{e_0}) \cup \bigcup_{e \in D} E(R^+_e)$.

For each $x \in V(S)$ let S_x be the component of S containing x, where S is any subgraph of X.

<u>LEMMA</u> (2.51) <u>In the notation of</u> (2.50), <u>for</u> e_0, e_1, <u>and</u> e_2 <u>distinct in</u> D <u>we have</u> $V(R^+_{e_1}) \cap V(R^+_{e_2}) = \phi = V(R^+_{e_1}) \cap V(R^-_{e_0})$.

<u>Proof</u>. Suppose $x \in V(R^+_{e_1}) \cap V(R^+_{e_2})$. Then there are distinct paths

$P_1 = (x_0, \ldots, x_{e_1}, y_{e_1}, \ldots, x)$ and $P_2 = (x_0, \ldots, x_{e_2}, y_{e_2}, \ldots, x)$.

Suppose $x \in V(R^+_{e_1}) \cap V(R^-_{e_0})$. Then there are distinct paths

$P_1 = (x_0, \ldots, x_{e_1}, y_{e_2}, \ldots, x)$ and $P_2 = (x_0, \ldots, x)$, $E(P_2) \cap D = \phi$.

In either case this gives a contradiction, X being a tree.

<u>COROLLARY</u> (2.52) <u>For each</u> $x \in A_D$ <u>there is a unique</u> $e \in D$ <u>with</u> $x \in V(R^+_e)$ <u>or</u> $x \in V(R^-_{e_0})$.

<u>Proof</u>. This is immediately clear from the above.

<u>LEMMA</u> (2.53) <u>In the notation of</u> (2.50) $V(S_{x_1}) \cap V(S_{x_2}) = \phi$ <u>for</u> x_1, x_2 <u>distinct in</u> $V(S_D) \cap A_D$ <u>and</u> $\bigcup_{x \in V(S_D) \cap A_D} S_x = S$.

Proof. Let $x_i \in V(S_D) \cap A_D$, $i = 1,2$ and $V(S_{x_1}) \cap V(S_{x_2}) \neq \phi$. Then $S_{x_1} = S_{x_2}$ and there is a path P in S_D joining x_1 and x_2. Let e_i be the unique edge in D associated with x_i. Let $P_i = (x_0,\ldots,x_{e_i},y_{e_i},\ldots,x)$. Then $P_1 \cup P \cup P_2$ is a non-degenerate closed path if $e_1 \neq e_2$. Hence $e_1 = e_2 = e$. Now there is a path Q in $R_e^- \cup (e) \cup R_e^+$ distinct from P joining x_1 and x_2 and again $P \cup Q$ is a non-degenerate closed path unless $P = (x_1) = Q$. Thus $V(S_{x_1}) \cap V(S_{x_2}) \neq \phi$ implies $x_1 = x_2$.

For the second part let $y \in V(S_D)$. Then there is a path in X joining x_0 and y. Because $|E_{x_0} - D| = 1$ there is either an $e \in D \cap E(P)$, or an $x \in V(R_{e_0}^-) \cap V(P)$ closest to y. In either case there is an $x \in A_D \cap V(P)$ closest to y and $y \in V(S_x)$. Hence

$$\bigcup_{x \in V(S) \cap A_D} S_x = S.$$

REMARK (2.54) S_D is the union of all paths P with initial vertex x in A_D and $E(P) \cap E_D = \phi$.

LEMMA (2.55) In the notation of (2.50), there is no ray in S_x for each $x \in V(S_D) \cap A_D$.

Proof. If R'' is a ray in S_x then there is a ray R' in S_x with initial vertex x.

If $x \in V(R_e^+)$ set $P = (y_e,\ldots,x)$, a subpath of R_e^+ and if $x \in V(R_{e_0}^-)$ set $P = (x_0,\ldots,x)$, a subpath of $R_{e_0}^-$. In either case

$R = P \cup R'$ is a ray from some $e \in D$ with $E(R) \cap D = \phi$. By the uniqueness of $R_e^- \cup (e) \cup R_e^+$ for each $e \in D$, R' is a ray with $E(R') \subset E_D$, a contradiction to R' being in S_D.

PROPOSITION (2.56) Let X be a tree with dendroid D for $\mathfrak{U}(X)$ rooted at a D-saturated vertex x_0. Then S_D is an X-admissible subgraph and if s is the induced equivalence relation then $\bar{D}$ is a dendroid for the contraction X/S and $\bar{D}$ has the form

(2.56.1) $$\bar{D} = \bigcup_{\bar{x} \in V(X/S)} (E\bar{x} - \bar{n}(\bar{x}))$$

for a suitable choice function $\bar{n}$ for the family $(E\bar{x})$ $\bar{x} \in V(X/S)$. X/S is to be rooted at $\bar{x}_0 = \{x_0\}$, and x_0 is a $\bar{D}$-saturated vertex of X/S.

Proof. Lemmas (2.53) and (2.55) show that no component of S_D contains a ray. By lemma (2.41) S_D is then X-admissible. Now $E(S_D) = E(X) - E_D$ and so $\varphi^{-1}(E(X/S)) = E_D$, i.e., $D \subset \varphi^{-1}(E(X/S))$. Thus by lemma (2.46) $\varphi(D) = \bar{D}$ is a dendroid for $\mathfrak{U}(X/S)$. Since $|E_{x_0} - D| = 1$ and $E_{x_0} \subset E_D$, $\bar{x}_0 = \{x\}$ and $|E_{\bar{x}_0} - \bar{D}| = 1$, i.e., $\bar{x}_0$ is a $\bar{D}$-saturated vertex and X/S can be rooted at $\bar{x}_0$.

We now show that $|E_{\bar{x}} - D| = 1$ for each $\bar{x} \in V(X/S)$. As is indicated in corollary (2.52) the set A_D is a representative set for $V(X/S)$. Take $\bar{x} \in V(X/S)$. $[\bar{x},\bar{y}] \in E_{\bar{x}}$ if and only if $\rho(\bar{x}_0,\bar{x}) < \rho(\bar{x}_0,\bar{y})$ and there is $[x,y] \in E(X) - E(S_D)$, with $\varphi(x') = \bar{x}$ $\varphi(y') = \bar{y}$ and $\bar{x} \neq \bar{y}$. If $\rho(x_0,y') < \rho(x_0,x')$ then $P = (x_0,\ldots,y')$ is a subpath of $P' = (x_0,\ldots,y',x')$ and $\varphi(P)$ is a subpath of

$\varphi(P')$ so that $\rho(\bar{x}_0,\bar{y}) < \rho(\bar{x}_0,\bar{x})$, a contradiction. Thus we have $[\bar{x},\bar{y}] \in E_{\bar{x}}$ if and only if this edge is the image of an edge in $E_{x'} - E(S_D)$ for some $x' \in V(S_X)$, i.e., $E_{\bar{x}} = \overline{\bigcup_{x' \in V(S_X)} E_{x'}}$.
Let $X_1 \in A_D$ with $\varphi(x_1) = \varphi(x') = \bar{x}$. $\bar{e} \in E_{\bar{x}} - \bar{D}$ if and only if $\bar{e} = \varphi(e')$, e' unique and $e' \in E_{x'} - D$, $e' \notin E(S_D)$, and $x' \in V(S_{x_1})$. Thus $e' = [x'_e, y'_e] \in E(R^-_{e_0}) \cup \bigcup_{e \in D} E(R^+_e)$. Lemma (2.53) implies that $V(S_x) \cap A_D = \{x_1\}$, and thus that $x'_e = x_1$. The lemma further implies there is a unique y in A_D with $[x_1,y] \in E_D - D$. Thus there is a unique $e \in \bigcup_{x' \in V(S_x)} (Ex' - D) \cap E_D$ with $\varphi(e) = \bar{e}$. That is, $E_{\bar{x}} - \bar{D} = \overline{\bigcup_{x' \in V(S_x)} (E_{x'} - D) \cap E_D} = \bar{e}$, as desired.

Thus we may define a function $\bar{n} : V(X/S) \to (E_{\bar{x}})\bar{x} \in V(X/S)$ by taking $\bar{n}(\bar{x})$ to be the unique edge in $E_{\bar{x}} - \bar{D}$. Then

$$\bar{D} = \bigcup_{\bar{x} \in V(X/S)} (E_{\bar{x}} - \bar{n}(\bar{x})).$$

<u>COROLLARY</u> (2.57) <u>Let</u> X <u>be an infinite tree</u>. <u>Then</u> D <u>is a dendroid for</u> $\mathfrak{A}(X)$ <u>if and only if there is a contraction</u> X/S <u>of</u> X <u>and a dendroid</u> Q <u>of</u> $\mathfrak{A}(X/S)$ <u>of the form</u> (2.56.1) <u>such that</u> $D = \varphi^{-1}(Q)$.

<u>Proof</u>. The proposition shows that every $D \in \vartheta_{\mathfrak{A}(X)}$ gives rise to such an S and Q.

Conversely, for such an S and Q, $\varphi^{-1}(Q)$ is a dendroid for $\mathfrak{A}(X)$ by lemma (2.51).

Lemma (2.58) *Let X be a tree and S an X-admissible subgraph of X. Then there is $x \in V(S)$ such that $|E(X;x) \cap E(S)| \leq 1$.*

Proof. Suppose not. Take $x_0 \in V(S)$, and $[x_0,x_1] \in E(S)$. Set $P_1 = (x_0,x_1)$. Suppose that $P_n = (x_0,\ldots,x_n)$ has been obtained with P_i a proper subpath of P_j for $i < j \leq n$, and $E(P_n) \cap E(S) = \phi$. Because $|E(X;x_n) \cap E(S)| > 1$ there is x_{n+1}, $[x_n,x_{n+1}] \in E(X;x_n) \cap E(S)$ $x_{n+1} \neq x_{n-1}$. Again $x_{n+1} \neq x_j$ for $0 \leq j < n$ because X is a tree. Thus $R = \bigcup_{n \geq 0} P_n$ is a ray in S, a contradiction.

NOTATION (2.59) Let X be a tree and S an X-admissible subgraph of X. Let X be rooted at x_0. For $x \in V(X)$ set

$$E^x = \bigcup_{z \in V(S_x)} \left(E_z - E(S)\right).$$

Remark (2.60) *If in the notation of (2.50)* $\varphi(x) = \varphi(y)$ then $S_x = S_y$ by lemma (2.53). Hence if $\bar{x} = \bar{y}$ then $E^x = E^y$.

THEOREM (2.61) *Let X be a tree, D is a dendroid for $\mathfrak{U}(X)$ if and only if there is an X-admissible subgraph S of X, a root x_0 of X and a choice function $n : X_S \to (E^x)_{x \in X_S}$ such that*

$$D = \bigcup_{x \in X_S} (E^x - n(x)).$$

X_S is any representative set for the equivalence classes of s induced by S.

Proof. Choose x_0 such that $|E(S;x_0) \cap E(S)| \leq 1$ and root X there. As in proposition (2.56)

$$\overline{E^x} = \overline{\bigcup_{z \in V(S_x)} (E_z - E(S))} = E_{\bar{x}}, \text{ where } \varphi(x) = \bar{x}.$$

Hence $E^x = \varphi^{-1}(E_{\bar{x}})$ for $\varphi(x) = \bar{x}$.

Now suppose that D is a dendroid for $\mathfrak{A}(X)$. Choose S and x_0 as in proposition (2.56).

$$\bar{D} = \overline{\bigcup_{\bar{x} \in V(X/S)} (E_{\bar{x}} - \bar{n}(\bar{x}))} = \bigcup_{\bar{x} \in \varphi(X_S)} (E_{\bar{x}} - \bar{n}(\bar{x}))$$

$$D = \varphi^{-1}(\bar{D}) = \bigcup_{x \in \varphi(X_S)} \varphi^{-1}(E_{\bar{x}} - \bar{n}(\bar{x})) = \bigcup_{x \in \varphi(X_S)} (E^x - \varphi^{-1}(\bar{n}(\bar{x})))$$

$$= \bigcup_{x \in X_S} (E^x - \varphi^{-1}(\bar{n}(\varphi(x)))$$

Write $n = \varphi^{-1} \circ \bar{n} \circ \varphi$. Then

$$D = \bigcup_{x \in X_S} (E^x - n(x)).$$

Conversely, suppose X is rooted at x_0, and S is an X-admissible subgraph of X with representative set X_S such that

$D = \bigcup_{x \in X_S} (E^x - n(x))$, $n : X_S \to \bigcup_{x \in X_S} E^x$, choice function. Then

$$\bar{D} = \overline{\bigcup_{x \in X_S} (E^x - n(x))} = \bigcup_{x \in X_S} \overline{(E^x - n(x))} = \bigcup_{x \in X_S} \overline{E^x} - \overline{n(x)}$$

$= \bigcup_{\bar{x} \in V(X/S)} (E_{\bar{x}} - n'(\bar{x}))$, where $n'(\bar{x}) = \varphi(n(x))$, a choice function on $V(X/S)$. Now apply theorem (2.28) with the distinguished set S being $\bar{x}_0$. This gives D a dendroid for $\mathfrak{A}(X/S)$. Hence by the corollary (2.57), D is a dendroid for $\mathfrak{A}(X)$

Some Properties of Dendroids of Circuit Matroids

LEMMA (2.62) *Let X be an infinite graph and $B \subset E(X)$ such that $B \cap A \neq \emptyset$ for any $A \in \mathfrak{A}(X)$. Then B contains a dendroid for $\mathfrak{A}(X)$. Moreover, if $D_0 \subset B$ is a dendroid for the finite circuits in $\mathfrak{A}(X)$ then there is a dendroid D for $\mathfrak{A}(X)$ with $D_0 \subset D \subset B$.*

Proof. Because $B \cap F \neq \emptyset$ for the finite circuits in $\mathfrak{A}(X)$, B contains a dendroid D_0 for these circuits by lemma (1.33). Now $E(X)-D_0 = E(X \setminus T)$ for a spanning tree of X by lemma (2.20). If $\mathfrak{A}(T) = \emptyset$ then D_0 meets all circuits in $\mathfrak{A}(X)$ and the theorem is proved. Suppose that $\mathfrak{A}(T) \neq \emptyset$, so that $B-D_0$ meets all circuits in T. Then $|E(T;x_0)-(B-D_0)| \leqq 1$ for some $x_0 \in V(T)$. The proof of this is exactly the same as in lemma(2.48). Root T at x_0 and let S' be the minimal subgraph of T whose edge set is $E(T)-(B-D_0)$. If K is a component of S' and R_1, R_2 are rays in K then $R_1 +^{+} R_2$ is finite for otherwise there is a circuit in K not meeting $B-D_0$. Let Λ be an index set for the components of S' which contain an infinite ray and for $\lambda \in \Lambda$ let R_λ be a maximal ray in K_λ. Let S be the minimal subgraph in T whose edge set is $E(S') - \bigcup_{\lambda \in \Lambda} E(R_\lambda)$. Suppose that K is a component of S and R is a ray in K. K is contained in a component of S', and hence $K \subset K_\lambda$ for some $\lambda \in \Lambda$. But then there is a path P joining R and R_λ, so that $R \cup P \cup R_\lambda$ contains a circuit in K_λ, a contradiction. Hence no component of S contains a ray and S is T-admissible. Let $\varphi : T \to T/S$ and put $\varphi(x) = \bar{x}$ for $x \in V(T)$. Now $(E(T)-E(S)) - (B-D_0) = \bigcup_{\lambda \in \Lambda} E(R_\lambda)$, a disjoint union of sets. Hence for

$x \in V(T)$, $| E_{\bar{x}} - \overline{(B-D_0)} | = | \bigcup_{\lambda \in \Lambda} \overline{E(R_\lambda)} \cap E_{\bar{x}} | \leq 1$. Let

$Q = \{ \bar{x} \in V(T/S) \mid E_{\bar{x}} - \overline{(B-D_0)} = \emptyset \}$, and define the choice function

$n : V(T/S) \to \bigcup_{\bar{x} \in V(T/S)} E_{\bar{x}}$ by $n(\bar{x}) = E_{\bar{x}} - \overline{(B-D_0)}$ if $\bar{x} \notin Q$, and arbitrary

if $\bar{x} \in Q$. By corollary (2.57), $H = \bigcup_{x \in V(T/S)} (E_{\bar{x}} - n(\bar{x}))$ is a dendroid for

$\mathfrak{A}(T/S)$, and by the choice of $n(\bar{x})$, $\varphi^{-1}(E_{\bar{x}} - n(\bar{x})) \subset B-D_0$ for each $x \in V(T)$.

Hence $\varphi^{-1}(H) = B' \subset B-D_0$ is a dendroid for $\mathfrak{A}(T)$ by corollary (2.57). Thus

$D_0 \subset D_0 \cup B' = D \subset B$, and D is a dendroid for $\mathfrak{A}(X)$ by corollary (2.22).

<u>LEMMA</u> (2.63) <u>Let</u> X <u>be an infinite connected graph with</u> $Z \subset B \subset E(X)$ <u>and</u> $B \cap A \neq \emptyset$ <u>for</u> $A \in \mathfrak{A}(X)$. <u>Then there is</u> D <u>minimal in</u> B <u>with</u> $Z \subset D$, $D \cap A \neq \emptyset$ <u>for</u> $A \in \mathfrak{A}(X)$. <u>If</u> $e \in D-Z$ <u>there is</u> $A \in \mathfrak{A}(X)$ <u>such that</u> $A \cap D = \{e\}$. <u>If</u> $Z = \emptyset$ <u>then</u> D <u>is a dendroid for</u> $\mathfrak{A}(X)$.

<u>Proof</u>. Let $\mathfrak{F} = \{ A \in \mathfrak{A}(X) \mid A \text{ is finite} \}$. By lemma (1.33) there is D_0 minimal in B such that $Z \subset D_0$ and $D_0 \cap A \neq \emptyset$ for $A \in \mathfrak{F}$. $D_0 \supset D_0'$, a dendroid for $\mathfrak{F}$ and by lemma (1.78), $D_0' - Z = D_0 - Z$.

Now $E(X) - D_0' = E(T)$, T a spanning tree of X by lemma (2.20). Let S be the minimal subgraph of X whose edge set is $E(T) - Z$. S is the disjoint union of trees since T contains no finite circuits. If $\mathfrak{A}(S) = \emptyset$ then every circuit in T meets Z and the theorem is proved. If $\mathfrak{A}(S) \neq \emptyset$ then $\mathfrak{A}(T) \neq \emptyset$. As shown in lemma (2.62), $(B-D_0') \cap A \neq \emptyset$ for $A \in \mathfrak{A}(T)$. $B-D_0' = (B-D_0) \cup (Z-D_0)$. If $A \cap Z = \emptyset$ for some $A \in \mathfrak{A}(T)$ then $A \cap (B-D_0) \neq \emptyset$.

Hence $(B-D_0) \cap A \neq \emptyset$ for $A \in \mathfrak{A}(S)$. By lemma (2.62) there is a dendroid D_1 for $\mathfrak{A}(S)$ contained in $B-D_0$. Let $D = D_0 \cup D_1$. Take $A \in \mathfrak{A}(X)$. If $A \cap D_0 = \emptyset$ then $A \in \mathfrak{A}(S)$ and so $A \cap D_1 \neq \emptyset$, and if $A \cap D_1 = \emptyset$ then A is finite or $A \in \mathfrak{A}(T) - \mathfrak{A}(S)$ and so $A \cap D_0 \neq \emptyset$. Hence $D \cap A \neq \emptyset$ for any $A \in \mathfrak{A}(X)$.

For $e \in D_1$ there is $A \in \mathfrak{A}(S)$, $A \cap D_1 = \{e\}$. Since D_0 and $E(S)$ are disjoint, $A \cap D = A \cap (D_0 \cup D_1) = A \cap D_1 = \{e\}$. For $e \in D_0 - Z$ there is $A \in \mathfrak{F}$ with $A \cap D_0 = \{e\}$. Hence if $A \cap D = \{e\} \cup \{e_1,\ldots,e_n\}$ there is $A' \in \mathfrak{A}(X)$ with $A' \cap D = \{e\}$ and $A' \subset f_{D_0}(e) \cup \bigcup_{1 \leq i \leq n} f_{D_1}(e_i)$ by lemma (1.20), and clearly $A' \cap Z = \emptyset$.

Thus D is minimal in B such that $Z \subset D$ and $D \cap A \neq \emptyset$ for $A \in \mathfrak{A}(X)$. We have also shown that for $e \in D - Z$ there is $A \in \mathfrak{A}(X)$ with $A \cap D = \{e\}$. It is clear that if $Z = \emptyset$ then D is a dendroid for $\mathfrak{A}(X)$.

<u>REMARK</u> (2.64) Lemma (2.63) is the generalization of lemma (1.33) to circuit matroids.

<u>PROPOSITION</u> (2.65) <u>Let</u> S <u>be</u> <u>a</u> <u>graph</u> <u>and</u> $w \in E(S)$ <u>such</u> <u>that</u>

(i) $w \in E(C)$, C <u>an</u> <u>infinite</u> <u>circuit</u>,

(ii) $S\backslash(w) = T_0 \cup T_1$,

(iii) $V(T_0) \cap V(T_1) = \emptyset$,

(iv) T_i <u>is</u> <u>an</u> <u>infinite</u> <u>graph</u> <u>and</u> <u>is</u> <u>circuit</u> <u>connected</u>, $i = 0,1$, <u>and</u> <u>some</u>

(v) $D_i \in \vartheta_{\mathfrak{A}(T_i)}$ <u>meets</u> <u>every</u> <u>ray</u> <u>in</u> T_i, $i = 0,1$.

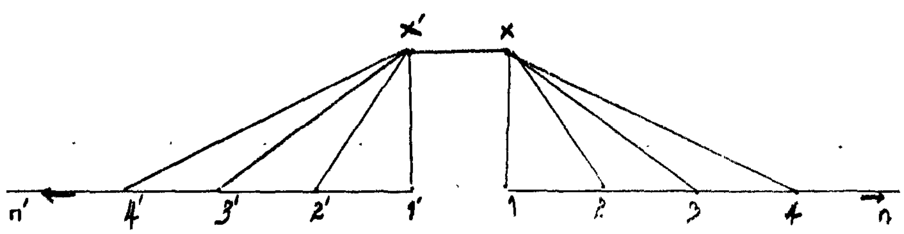
<u>Then</u> $\mathfrak{A}(S)$ <u>does</u> <u>not</u> <u>have</u> <u>the</u> <u>l.c.p.</u>

Proof. If $A \cap (D_0 \cup D_1) = \emptyset$ for some $A \in \mathfrak{A}(S)$ then $A \not\subset E(T_i)$ for $i = 0$ or 1. Hence $w \in A$ and A is infinite, $A = A_0 \cup (w) \cup A_1$ with the subgraph corresponding to A_i being a ray in T_i, $i = 0,1$. But then $A \cap D_i \neq \emptyset$, a contradiction. Thus $D_0 \cup D_1 = D$ is a dendroid for $\mathfrak{A}(S)$, and for $e \in D$, we have $f_D(e) = f_{D_i}(e)$, $i = 0$ or 1. Now $w \in E(C) \not\subset \bigcup_{e \in D \cap E(C)} f_D(e)$. Hence $\mathfrak{A}(S)$ does not have the l.c.p.

REMARK (2.64) Since D meets every ray in T_i above, no infinite circuit has as an edge set an $f_D(e)$, $e \in D$.

EXAMPLE (2.67) The following is a simple example of a graph of the type in proposition (2.65)

FIGURE 4



w is taken as $[x,x']$ and $C = (\ldots,n',\ldots,1',x',x,1,\ldots,n,\ldots)$. The dendroid is the set of red edges. . By adjoining a,b to V(S), and $[x',b]$, $[b,a]$, $[a,x]$, $[x',a]$, $[x,b]$ to E(S) and $[a,b]$, $[a,x']$, $[b,x]$ to D we make $\mathfrak{A}(S)$ into a point separating matroid which does not have the l.c.p.

<u>LEMMA</u> (2.68) <u>Let</u> X <u>be an infinite graph containing a subgraph</u> S <u>of the type in proposition</u> (2.65). <u>Let</u> D_1 <u>be the dendroid for</u> $\mathfrak{A}(S)$ <u>given there. Then</u> D_1 <u>has an extension</u> D, <u>a dendroid for</u> $\mathfrak{A}(X)$.

<u>Proof</u>. $E(X) \cap A \neq \emptyset$ for $A \in \mathfrak{A}(X)$ and $E(X) \supset D$. Hence by lemma (2.63) there is D minimal in $E(X)$ containing D_1 such that $D \cap A \neq \emptyset$ for $A \in \mathfrak{A}(X)$, and for $e \in D - D_1$ there is $A \in \mathfrak{A}(X)$ with $A \cap D = \{e\}$.

Now suppose there is $e_0 \in D_1$ such that $A \cap (D - \{e_0\} \neq \emptyset$ for any $A \in \mathfrak{A}(X)$. Then $(f_{D_1}(e_0) -\{e_0\}) \cap D_1 = \{e_1,\dots,e_n\}$ for some integer n by remark (2.66). Let $e_i = [x_i,y_i]$ for $0 \leqq i \leqq n$, and be indexed such that $f_{D_1}(e_0) -\{e_0\}$ is the edge set of the path $(x_0,\dots,x_1,y_1,\dots,x_n,y_n,\dots,y_0)$.

If $f_D(e_i)$ is infinite let $R_{x_i} \cup (e_i) \cup R_{y_i}$ be the circuit with that edge set. Otherwise let it be $P_i \cup (e_i)$, $1 \leqq i \leqq n$.

Suppose that some $f_D(e_i)$ is infinite. Let j and k be the least and greatest integer respectively in 1 to n such that this is so. Then $(x_0,\dots,x_1) \cup P_1 \cup \dots \cup P_{j-1} \cup (y_{j-1},\dots,x_j) \cup R_{x_j}$ taken with $(y_0,\dots,y_n) \cup P_n \cup \dots \cup P_{k+1} \cup (x_{k+1},\dots,y_k) \cup R_{y_k} \cup (e)$ contains an infinite circuit missing $D -\{e_0\}$, a contradiction. Hence $f_D(e_i)$ is finite for $1 \leqq i \leqq n$. Now consider the path P' joining x_0 and y_0 and not containing e which is given by $P' = (x_0,\dots,x_1) \cup P_1 \cup (y_1,\dots,x_2) \cup \dots \cup P_n \cup (y_1,\dots,y_0)$. This contains a simple path P with the same end points. Hence

$P \cup (e_0) \in \mathfrak{A}(X)$ and $(E(P) \cup \{e_0\}) \cap (D - \{e_0\}) = \emptyset$, a contradiction. Thus D is a dendroid for $\mathfrak{A}(X)$.

<u>THEOREM</u> (2.69) <u>Let</u> X <u>be a graph which contains a subgraph of the type in proposition</u> (2.65). <u>Then</u> $\mathfrak{A}(X)$ <u>does not have the l.c.p.</u>

<u>Proof</u>. Let this subgraph be S, D_1 it's dendroid, w it's distinguished edge and C the related circuit, all as in proposition (2.65). Let D be an extenshion of D_1 as in lemma (2.68). Since $f_{D_1}(e') = f_D(e')$ for $e' \in D_1$, we have

$$w \in E(C) \not\subset \bigcup_{e \in E(C) \cap D} f_D(e) \subset E(S) - \{w\}.$$ Thus $\mathfrak{A}(X)$ does not have the l.c.p.

<u>LEMMA</u> (2.70) <u>Let</u> X <u>be an infinite graph with dendroid</u> D_0 <u>for</u> $\mathfrak{A}(X)$. <u>Let</u> D' <u>be a dendroid for</u> $\mathfrak{A}_0(X) = \{A \in \mathfrak{A}(X) \mid |A \cap D_0|$ is finite $\}$. D' <u>can be extended to a dendroid</u> D <u>for</u> $\mathfrak{A}(X)$.

<u>Proof</u>. If D' is a dendroid for $\mathfrak{A}(X)$ there is nothing to prove. Suppose that D' is not a dendroid for $\mathfrak{A}(X)$. D' meets all finite members of $\mathfrak{A}(X)$ since these are in $\mathfrak{A}_0(X)$. Now $\mathfrak{A}' = \{A \in \mathfrak{A}(X) \mid A \cap D' = \emptyset\} \neq \emptyset$, and hence is an exchange system by corollary (1.72), whose circuits form a disjoint union of trees. Thus $\mathfrak{A}'$ has a dendroid D_1 by theorem (2.25). Take $D = D_1 \cup D'$.

Let $A \in \mathfrak{A}(X)$ and $e \in D_1$ with $A \cap D_1 = \{e\}$. Since D_1 is a dendroid for $\mathfrak{A}'$ we may take $A \in \mathfrak{A}'$. Then $A \cap D' = \emptyset$ and so $A \cap D = \{e\}$.
Let $e \in D'$ and $A \in \mathfrak{A}_0(X)$, A infinite with $A \cap D' = \{e\}$. If $A \cap D_1 \neq \emptyset$ then

we construct a new circuit as follows. Let $e = [x,y]$ and let $A = E(R_x) \cup (e) \cup E(R_y)$ in the notation of (2.50). If $E(R_x) \cap D_1 \neq \emptyset$ take $w = [p,q] \in E(R_x) \cap D_1$ such that $\rho(p,x) < \rho(q,x)$ and $\rho(p,x)$ is least for all such w. Again $f_{D'}(w) = E(R_p) \cup \{w\} \cup E(R_q)$ and $E(R_p) \cap D' = \emptyset$. Take $R'_x = (x,\dots,p) \cup R_p$. If $E(R_x) \cap D_1 = \emptyset$ take $R'_x = R_x$. In either case $E(R'_x) \cap D_1 = \emptyset$. Do the same for R_y. Then $C = R'_x \cup (e) \cup R'_y$ is a circuit such that $E(C) \cap D = \{e\}$. Finally, if $e \in D'$ and $f_{D'}(e)$ is finite then since $f_{D'}(e) \cap D_1$ is finite and $D_1 \cap \bigcup_{w \in D'} f_{D'}(w) = \emptyset$, there is $A \in \mathfrak{A}(X)$ with $A \cap D = \{e\}$, by a finite number of applications of the exchange property. It only remains to show that $D \cap A \neq \emptyset$ for any $A \in \mathfrak{A}(X)$.

If $A \in \mathfrak{A}(X)$ with A finite then $A \cap D' \neq \emptyset$. If $A \cap D' = \emptyset$, then $A \in \mathfrak{A}'$ and hence $A \cap D_1 \neq \emptyset$, while if $A \cap D_1 = \emptyset$ then $A \notin \mathfrak{A}'$ and so $A \cap D' \neq \emptyset$.

Thus D is a dendroid for $\mathfrak{A}(X)$ and is an extension of D'.

<u>PROPOSITION</u> (2.71) <u>Let</u> X <u>be a graph and</u> $\mathfrak{A}(X) \neq \emptyset$. <u>Then</u> $\mathfrak{A}(X)$ <u>is weakly locally finite</u>. <u>In fact, for each</u> $A \in \mathfrak{A}(X)$ <u>and</u> $e \in A$ <u>there is</u> $D \in \vartheta_{\mathfrak{A}(X)}$ <u>with</u> $A = f_D(e)$.

<u>Proof</u>. Take $e \in A \in \mathfrak{A}(X)$. If A is finite remark (1.21) shows that $A = f_D(e)$ for some $D \in \vartheta_{\mathfrak{A}(X)}$.

If A is infinite, A is the edge set of a tree. Let T be a spanning tree of X containing the set of edges A. Set $T_0 = \cup\, \mathfrak{A}(T)$. Let $e = [e_x, e_y]$. Root T_0 at e_x and let D_1 be the dendroid for $\mathfrak{A}(T_0) = \mathfrak{A}(T)$ given by

$$D = \bigcup_{x \in V(T_0)} (E_x - n(x))$$

where $n : V(T_0) \to \bigcup_{x \in V(T_0)} E_x$, a choice function, where

(i) $n(x) = [x,y] \in A \cap E_x$ for $x \neq e_x$, $x \in V(C_A)$,

(ii) $n(e_x) = e$, and

(iii) $n(x)$ is arbitrarily chosen in E_x otherwise.

Clearly $A = f_{D_1}(e)$. Now $D = D_1 \cup E(X\backslash T)$ is a dendroid for $\mathfrak{A}(X)$ and $A \cap E(X\backslash T) = \emptyset$. Hence $A = f_D(e)$ and the proposition is proved.

PROPOSITION (2.72) Let T be a tree, $\mathfrak{A}(T) \neq \emptyset$. Then a necessary and sufficient condition that all dendroids for $\mathfrak{A}(T)$ be locally finite is that no ray of T contain infinitely many branch points.

Proof. Necessity. Suppose there is a ray $R = [x_0, x_1, \ldots) \subset T$ and an infinite sequence $i_1 < i_2 < \ldots$ such that $d(x_{i_j}; T) \geq 3$ for $j \geq 1$. Since T is a tree, there is a circuit C in T such that $x_i \in V(C)$ for $i \geq N$, some integer. Let $T_0 = \cup\, \mathfrak{A}(T)$. Root T_0 at x_{i_1} and consider

$$D = \bigcup_{x \in V(T_0)} (E_x - n(x))$$

where $n : V(T_0) \rightarrow \bigcup_{x \in V(T_0)} E_x$, a choice function such that $n(x_{i_j}) \notin E_{x_{i_j}} \cap E(R)$ for $j \geq 1$, which is possible since x_{i_j} is a branch vertex. Otherwise $n(x)$ may be arbitrarily chosen.

By corollary (2.57) D is a dendroid for $\mathfrak{A}(T_0)$, and hence for $\mathfrak{A}(T)$. Clearly $D \cap E(C)$ is infinite since $[x_k, x_{k+1}] \in D \cap E(C)$ for $k \geq i_j \geq N$.

Sufficiency. Let D be a dendroid for $\mathfrak{A}(T)$ and $C = R_{-1} \cup R_1$ be a circuit in T, R_{-1} and R_1 edge disjoint rays. Put $R = [x_0^i, x_1^i, \ldots)$, $i = -1, 1$. By hypothesis $d(x_k^i; T) = 2$ for $k \geq N$, an integer, and $i = -1, 1$. Now if $e_j = [x_j^i, x_{j+1}^i] \in D$ for $i = -1$ or 1 and $j \geq N$ then $\{ [x_k^i, x_{k+1}^i] \mid k \geq N \} \in f_D(e)$. Hence $|D \cap E(C)| \leq 2(N+1)$, and D is locally finite.

THEOREM (2.73) <u>Let</u> X <u>be</u> <u>a</u> <u>tree</u>, $\mathfrak{A}(X) \neq \emptyset$. <u>Then</u> $\mathfrak{A}(X)$ <u>has</u> <u>the</u> <u>l.c.p.</u>

<u>Proof</u>. Take $A \in \mathfrak{A}(X)$ and D a dendroid for $\mathfrak{A}(X)$. Suppose $e = [e_x, e_y]$, $e \in A - \bigcup \{ f_D(w) \mid w \in D \cap A \}$. Let C_A be the circuit whose edge set is A. Root X at e_x. Put $C = R_x \cup (e) \cup R_y$ in the notation of (2.50). If $E(R_y) \cap D = \emptyset$ take $w \in D \cap A$ with $\rho(w_x, e_x)$ least. Then $R = R_y \cup (e_y, e_x, \ldots, w_x)$ is a ray missing D. Thus $R = R_w^-$ and $e \in f_D(w)$, a contradiction. A similar argument holds if $E(R_x) \cap D = \emptyset$.

Take $w,z \in A \cap D$ such that they are on opposite ends of e and such that $\rho(e_x,w_x)$ and $\rho(e_x,z_x)$ are least. Then $R_w^- + (z_x,\ldots,e_x,e_y,\ldots,w_x) + R_z^-$ is a circuit with no edge in D, a contradiction. Since these are all the possible cases then no such e exists in A and $\mathfrak{A}(X)$ has the l.c.p.

CHAPTER 3

THE DUAL OF A MATROID.

NOTATION (3.1) Let $\mathfrak{A}$ be a matroid on E (definition (1.23)). For $x \in E - D$, and $D \in \vartheta_{\mathfrak{A}}$ put

(i) $f_D^*(x) = \{x\} \cup \{ y \in D \mid x \in f_D(y) \}$,

(ii) $\mathfrak{R}(\mathfrak{A}) = \{ X \subset E \mid |A \cap X| \neq 1 \text{ for any } A \in \mathfrak{A} \}$, and

(iii) $\mathfrak{A}^{\#} = \{ f_D^*(x) \mid x \in E\text{–}D,\ D \in \vartheta_{\mathfrak{A}} \}$.

REMARK (3.2) For fixed $D \in \vartheta_{\mathfrak{A}}$ the sets $f_D^*(x)$ and $f_D^*(y)$ are distinct for $x \neq y$. However, it can occur that $f_D^*(x) = f_{D'}^*(y)$ provided D and D′ are different dendroids for $\mathfrak{A}$.

In terms of the notation just introduced the results of Tutte [5] for matroids $\mathfrak{A}$ on a finite set E can be stated as

(i) $\mathfrak{R}(\mathfrak{A})$ is an exchange system,

(ii) $\mathfrak{A}^{\#} \subset \mathfrak{R}(\mathfrak{A})_{min}$ is an exchange system,

(iii) $\mathfrak{R}(\mathfrak{R}(\mathfrak{A})_{min})_{min} = \mathfrak{A}$, and

(iv) $\vartheta_{\mathfrak{A}^{\#}} = \{ E - D \mid D \in \vartheta_{\mathfrak{A}} \}$.

In fact, one can show that $\mathfrak{A}^{\#} = \mathfrak{R}(\mathfrak{A})_{min}$ in this case. The present chapter is concerned with extensions of Tutte's results to the case where E is infinite.

REMARK (3.3) <u>If the exchange system</u> $\mathfrak{A}$ <u>on</u> E <u>has</u> $\{ \{x\} \mid x \in E \} \cap \mathfrak{A} = \emptyset$ <u>then clearly</u> $E \in \mathfrak{R}(\mathfrak{A})$. <u>In fact, if</u> $D \in \vartheta_{\mathfrak{A}}$ <u>and</u> $|D|$ <u>is infinite then there is</u> $X \in \mathfrak{R}(\mathfrak{A})$ <u>with</u> $|X| = |D|$. This is seen as follows.

Take $D \in \vartheta_{\mathfrak{A}}$ with $|D|$ infinite and well order D by the ordinals less than s, where s is the first ordinal with $|s| = |D|$. Then

$$D = \{ a_p \mid 0 \leqq p < s \} .$$

Define recursively the sets X_p, $0 \leqq p < s$ as follows. Take x_0 in $f_D(a_0)$, $x_0 \notin D$ and set $X_0 = \{x_0\}$. This is possible since $f_D(a_0) \neq \{a_0\}$.

Suppose X_i defined for $0 \leqq i < j$. If $(\bigcup_{i<j} X_i) \cap f_D(a_j) = \emptyset$, select $x_j \in f_D(a_j)$, $x_j \notin D$ and set $X_j = (\bigcup_{i<j} X_i) \cup \{x_j\}$. If $(\bigcup_{i<j} X_i) \cap f_D(a_j) \neq \emptyset$ set $X_j = \bigcup_{i<j} X_i$.

Let $X_s = \bigcup_{p<s} X_p$ and put $X = D \cup X_s$. Clearly $|X_s| \leqq |s| = |D|$ so that $|X| = |D|$.

If $A \in \mathfrak{A}$ then $A \cap D \neq \emptyset$ and thus if $|A \cap (X_s \cup D)| = 1$ then $A \cap X_s = \emptyset$ and $A = f_D(a_j)$ for some j, $0 \leqq j < s$. Now $A \cap X_s = \emptyset$ implies $(\bigcup_{i<j} X_i) \cap f_D(a_j) = \emptyset$ so by definition there is $x_j \in X_j \subset X_s$, $x_j \in f_D(a_j)$, $x_j \notin D$ and hence $A \cap X_s \neq \emptyset$, a contradiction. Thus $X \in \mathfrak{R}(\mathfrak{A})$.

LEMMA (3.4) Let $\mathfrak{A}$ be an exchange system with a dendroid D. Then if $y \in f_D^*(x) - \{x\}$, there is $D_1 \sim D$ such that $f_{D_1}^*(y) = f_D^*(x)$.

Proof. By definition $x \in f_D(y)$. By lemma (1.17) $D_1 = (D - \{y\}) \cup \{x\}$ is a dendroid for $\mathfrak{A}$, $D_1 \sim D$, and if $p \in D - \{y\}$ with $x \notin f_D(p)$ then $f_{D_1}(p) = f_D(p)$ and hence $y \notin f_{D_1}(p)$. Thus $f_{D_1}^*(y) \subset f_D^*(x)$. Since $\sim$ is symmetric we have $f_D^*(x) \subset f_{D_1}^*(y)$, and so $f_D^*(x) = f_{D_1}^*(y)$.

LEMMA (3.5) Let $\mathfrak{A}$ be an exchange system with dendroid D. Then a necessary and sufficient condition that $|A \cap f_D^*(x)| \neq 1$ for any $A \in \mathfrak{A}$ and $x \in E - D$ is that D have the l.c.p.

Proof. Sufficiency. Let D have the l.c.p. and suppose for some $A \in \mathfrak{A}$ and $x \in E - D$ that $A \cap f_D^*(x) = \{y\}$. By lemma (3.4) there is $D_1 \sim D$ with $f_{D_1}^*(y) = f_D^*(x)$ and by theorem (1.45) D_1 has the l.c.p. Now $y \in A$, $A \subset \cup \{ f_{D_1}(a) \mid a \in A \cap D_1 \}$ and thus there is $a \in A \cap D_1$ with $y \in f_{D_1}(a)$, ie., $\{a,y\} \subset A \cap f_{D_1}^*(y) = A \cap f_D^*(x)$, a contradiction. This proves the sufficiency.

Necessity. Suppose that D does not have the l.c.p. Then there is $A \in \mathfrak{A}$ with $y \in A - \cup \{ f_D(a) \mid a \in A \cap D \}$. Hence $A \cap f_D^*(y) = \{y\}$.

LEMMA (3.6) *Let $\mathfrak{A}$ be an exchange system with $\vartheta_{\mathfrak{A}} \neq \emptyset$ and $A \in \mathfrak{A}$. Then a necessary and sufficient condition that $|A \cap X| \neq 1$ for any $X \in \mathfrak{A}^{\#}$ is that $A \subset \bigcup \{ f_D(a) \mid a \in A \cap D \}$ for every $D \in \vartheta_{\mathfrak{A}}$.*

Proof. Sufficiency. Take $X \in \mathfrak{A}^{\#}$ with $A \cap X \neq \emptyset$. For $x \in A \cap X$ we can take $X = f_D^*(x)$ for some $D \in \vartheta_{\mathfrak{A}}$ by lemma (3.4). Since $x \in A \subset \bigcup\{f_D(a) \mid a \in A \cap D\}$, $x \in f_D(a)$, $a \in A \cap D$. Hence $a \in f_D^*(x)$ and $|A \cap X| \geqq 2$.

Necessity. $A \not\subset \bigcup \{ f_D(a) \mid a \in A \cap D \}$ for some $D \in \vartheta_{\mathfrak{A}}$ means D does not have the l.c.p. by definition and thus by the second part of lemma (3.5) there is $X \in \mathfrak{A}^{\#}$ with $|X \cap A| = 1$.

REMARK (3.7) *Using lemmas (3.5) and (3.6), $\mathfrak{A}^{\#} \subset \mathfrak{R}(\mathfrak{A})$ if and only if $\mathfrak{A}$ has the l.c.p.*

LEMMA (3.8) *Let $\mathfrak{A}$ be an exchange system with the l.c.p. Then $\mathfrak{A}^{\#}$ consists of incomparable elements which are minimal in $\mathfrak{R}(\mathfrak{A})$.*

Proof. By remark (3.7) $\mathfrak{A}^{\#} \subset \mathfrak{R}(\mathfrak{A})$. Suppose $Y \in \mathfrak{A}^{\#}$, $X \in \mathfrak{R}(\mathfrak{A})$, $y \in Y - X$, and $Y \supset X$. Then $Y = f_D^*(y)$ for some $D \in \vartheta_{\mathfrak{A}}$ by lemma (3.4). Let $x \in X$ and $a \in f_D(x) \cap X$. Then $a \in Y = f_D^*(y)$ and since X is properly contained in Y, $a \neq y$. Hence $a \in D$ and therefore $a \in f_D(x) \cap D$, ie., $a = x$. Then $f_D(x) \cap X = \{x\}$, a contradiction to $X \in \mathfrak{R}(\mathfrak{A})$. Thus the elements of $\mathfrak{A}^{\#}$ are minimal in $\mathfrak{R}(\mathfrak{A})$ and $\mathfrak{A}^{\#}$ consists of incomparable elements.

LEMMA (3.9) *Let $\mathfrak{A}$ be an exchange system with the l.c.p. Then* $E - D$ *is a dendroid for* $\mathfrak{R}(\mathfrak{A})$ *with the l.c.p. for each* $D \in \vartheta_{\mathfrak{A}}$. *In fact,* $E - D$ *is a dendroid for* $\mathfrak{A}^{\#}$.

Proof. If $(E - D) \cap X = \emptyset$ for some $D \in \vartheta_{\mathfrak{A}}$ and $X \in \mathfrak{R}(\mathfrak{A})$ then $D \supset X$ and for $x \in X$, $|f_D(x) \cap X| = 1$, contrary to definition. Now $\mathfrak{A}^{\#} \subset \mathfrak{R}(\mathfrak{A})$ by lemma (3.8) and for $x \in E - D$, $f_D^*(x) \cap (E - D) = \{x\}$ and so $E - D$ is a dendroid for $\mathfrak{R}(\mathfrak{A})$ and since $f_D^*(x) \in \mathfrak{A}^{\#}$, $E - D$ is a dendroid for $\mathfrak{A}^{\#}$.

Take $X \in \mathfrak{R}(\mathfrak{A})$ and $Y = \cup \{ f_D^*(x) \mid x \in (E - D) \cap X \}$. If $y \in (E - D) \cap X$ then $y \in f_D^*(y) \subset Y$. If $y \in D \cap X$ then $|f_D(y) \cap X| \geqq 2$ by remark (3.7) and there is $x \in f_D^*(y) \cap X$, $x \neq y$, ie., $x \in (E - D) \cap X$, $y \in f_D^*(x) \subset Y$ and hence $X = Y$. Thus $E - D$ has the l.c.p.

LEMMA (3.10) *Let $\mathfrak{A}$ be an exchange system with* $E - D \in \vartheta_{\mathfrak{A}^{\#}}$, $D \in \vartheta_{\mathfrak{A}}$ *and* $B \sim E - D$. *Then* $E - B \in \vartheta_{\mathfrak{A}}$ *and* $E - B \sim D$.

Proof. It is sufficient to take B adjacent to $E - D$. Then $B = ((E - D) - \{y\}) \cup \{x\}$. $E - B = (D - \{x\}) \cup \{y\}$. If $A \cap (E - B) = \emptyset$ for some $A \in \mathfrak{A}$ then $A = f_D(x)$ and $y \notin f_D(x)$. Hence $x \notin f_D^*(y)$. Then $B \cap f_D^*(y) = \emptyset$, a contradiction. Clearly $f_D(x) \cap (E - B) = \{y\}$. If $z \in E - B$, $z \neq y$ such that $((E - B) - \{z\}) \cap f_D(z) \neq \emptyset$ then $y \in f_D(z)$. By exchange in $\mathfrak{A}$ there is $A_1 \in \mathfrak{A}$ with $z \in A_1 \subset (f_D(x) \cup f_D(z)) - \{y\}$. Then $A_1 \cap ((E - B) - \{z\}) = \emptyset$. Hence $E - B \in \vartheta_{\mathfrak{A}}$ and $E - B \sim D$

<u>LEMMA</u> (3.11) <u>Let $\mathfrak{A}$ and $\mathfrak{A}^{\#}$ be exchange systems such that $\mathfrak{A}_{min}$ is weakly locally finite</u>. If $\vartheta_{\mathfrak{A}^{\#}} = \{E-D \mid D \in \vartheta_{\mathfrak{A}}\} \neq \phi$, <u>then</u>

$$(\mathfrak{A}^{\#})^{\#} = \mathfrak{A}_{min} \quad .$$

<u>Proof</u>. Let B be a dendroid for $\mathfrak{A}^{\#}$, D a dendroid for $\mathfrak{A}$, and $B = E-D$. Now $B \cap f^{*}_{E-B}(z) = B \cap f^{*}_{D}(z) = \{z\}$ and since $\mathfrak{A}^{\#}$ is an exchange system $f_B(z)$ is unique by lemma (1.14), i.e. $f_B(z) = f^{*}_{E-B}(z)$.

The following relation holds between elements of $\mathfrak{A}_{min}$ and $(\mathfrak{A}^{\#})^{\#}$.

$$\begin{aligned} f^{*}_{B}(y) &= \{y\} \cup \{z \in B \mid y \in f_B(z)\} = \{y\} \cup \{z \in E-D \mid y \in f^{*}_{E-B}(z)\} \\ &= \{y\} \cup \{z \in E-D \mid y \in f^{*}_{D}(z)\} = \{y\} \cup \{z \in E-D \mid z \in f_D(y)\} \\ &= f_D(y). \end{aligned}$$

Hence $(\mathfrak{A}^{\#})^{\#} = \mathfrak{A}_{min}$.

<u>LEMMA</u> (3.12) <u>Let $\mathfrak{A}$ be an exchange system with the l.c.p. If $\mathfrak{A}^{\#}$ is an exchange system and</u> $B \in \vartheta_{\mathfrak{A}^{\#}}$ <u>then</u> $(E-B) \cap A \neq \phi$ <u>for any</u> $A \in \mathfrak{A}$.

<u>Proof</u>. Suppose $(E-B) \cap A = \phi$. Then $A \subset B$. Take $a \in A$. There is $X \in \mathfrak{A}^{\#}$ with $X \cap B = \{a\}$. By definition $X = f^{*}_{D}(a)$ for some $D \in \vartheta_{\mathfrak{A}}$, using lemma (3.4). Then

$$a \in A \cap f^{*}_{D}(a) \subset B \cap f^{*}_{D}(a) = \{a\}, \text{ i.e., } A \cap f^{*}_{D}(a) = \{a\},$$

a contradiction to lemma (3.5). Hence $(E-B) \cap A \neq \phi$ for any $A \in \mathfrak{A}$.

COROLLARY (3.13) Let $\mathfrak{A}$ and $\mathfrak{A}^\#$ be exchange systems on E such that $\mathfrak{A}$ has the l.c.p. and, in addition, for any $Z \subset E$ with $Z \cap A \neq \phi$ there is $D \subset Z$, a dendroid for $\mathfrak{A}$. Then

$$\vartheta_{\mathfrak{A}^\#} = \{E-D \mid D \in \vartheta_{\mathfrak{A}}\}.$$

Proof. By lemma (3.12) any $B \in \vartheta_{\mathfrak{A}^\#}$ is such that $(E-B) \cap A \neq \phi$ for $A \in \mathfrak{A}$. Hence there is $D \subset E-B$, $D \in \vartheta_{\mathfrak{A}}$. By lemma (3.9), $E-D \in \vartheta_{\mathfrak{A}^\#}$, and $E-D \supset B$. Thus $E-D = B$, and

$$\vartheta_{\mathfrak{A}^\#} = \{E-D \mid D \in \vartheta_{\mathfrak{A}}\}.$$

THEOREM (3.14) Let $\mathfrak{A}$ be an exchange system on E with A finite for each $A \in \mathfrak{A}$. Then $\mathfrak{R}(\mathfrak{A})$ is an inductive exchange system with $\mathfrak{R}(\mathfrak{A})_{min} = \mathfrak{A}^\#$, $\mathfrak{A}^\#$ is an exchange system,

$$\vartheta_{\mathfrak{R}(\mathfrak{A})} = \{E-D \mid D \in \vartheta_{\mathfrak{A}}\} = \vartheta_{\mathfrak{A}^\#}, \text{ and } (\mathfrak{A}^\#)^\# = \mathfrak{A}_{min}.$$

Proof. Clearly $\mathfrak{A}$ has the l.c.p. from remarks (1.27) and (1.25). By lemmas (3.8) and (3.9) $\mathfrak{A}^\# \subset (\mathfrak{R}(\mathfrak{A}))_{min}$ and $E-D \in \vartheta_{\mathfrak{R}(\mathfrak{A})}$ for each $D \in \vartheta_{\mathfrak{A}}$. $\vartheta_{\mathfrak{R}(\mathfrak{A})} \subset \vartheta_{\mathfrak{A}^\#}$ by lemma (3.9) if $B \in \vartheta_{\mathfrak{R}(\mathfrak{A})}$ implies $B = E-D$ for some $D \in \vartheta_{\mathfrak{A}}$ and we have equality if this is also true for $B \in \vartheta_{\mathfrak{A}^\#}$. Take B a dendroid for $\mathfrak{R}(\mathfrak{A})$ or $\mathfrak{A}^\#$. If $(E-B) \cap A = \phi$ for some $A \in \mathfrak{A}$ then $A \subset B$. There is $f \in F_B$ such that for $a \in A \subset B$, $f(a) \cap B = \{a\}$. Hence $f(a) \cap A = \{a\}$, a contradiction on $f(a) \in \mathfrak{R}(\mathfrak{A}) \supset \mathfrak{A}^\#$. Now by lemma (1.33) there is $D \subset E-B$, $D \in \vartheta_{\mathfrak{A}}$. Hence $E-D \in \vartheta_{\mathfrak{R}(\mathfrak{A})}$, $E-D = B$. Thus

$$\vartheta_{\mathfrak{R}(\mathfrak{A})} = \{E-D \mid D \in \vartheta_{\mathfrak{A}}\} = \vartheta_{\mathfrak{A}^\#}.$$

Let $X \in \mathfrak{R}(\mathfrak{A})_{min}$ and put

$$\vartheta_X = \{B \subset E | X \subset B,\ B \cap A \neq \phi \text{ for any } A \in \mathfrak{A}\}.$$

$E \in \vartheta_X$. By lemma (1.33) ϑ_X has minimal elements. Let B be such a minimal element and let $D \in \vartheta_{\mathfrak{A}}$ with $D \subset B$. Then $D \cup X \supset B$ and so $D \cup X = B$. If $X \subset D$ then for $x \in X$, $f_D(x) \cap X = \{x\}$, a contradiction on the definition of $\mathfrak{R}(\mathfrak{A})$. For $y \in X-D$, $y \in f_D(a)$ for some $a \in D$ by lemma (1.29). Then $D' = (D - \{a\}) \cup \{y\} \in \vartheta_{\mathfrak{A}}$ and $D' \cup X = B-\{a\} \in \vartheta_X$ and so $a \notin D-X$ by the minimality of B. Hence $y \in X-D$ and $y \in f_D(a)$ implies $a \in X$. Hence $f^*_D(y) \subset X$ and so $f^*_D(y) = X$ by the minimality of X and thus $(\mathfrak{R}(\mathfrak{A}))_{min} = \mathfrak{A}^{\#}$.

Let $\mathfrak{L}$ be a chain in $\mathfrak{R}(\mathfrak{A})$. If $(\cap\, \mathfrak{L}) \cap A = \{a\}$ for some $A \in \mathfrak{A}$; $A = \{a, a_1, \ldots, a_n\}$ then for $1 \leqq i \leqq n$ there is $B_i \in \mathfrak{L}$, $a_i \notin B_i$. Hence $B = B_1 \cap \ldots \cap B_n \in \mathfrak{L}$ and $B \cap A = \{a\}$, a contradiction. Thus $\mathfrak{R}(\mathfrak{A})$ is inductive. By lemma (1.6), $\mathfrak{A}^{\#}$ is an exchange system if $\mathfrak{R}(\mathfrak{A})$ is since for $x \in X \in \mathfrak{R}(\mathfrak{A})$ there is $Y \in \mathfrak{A}^{\#}$ with $x \in Y$.

Suppose that $\mathfrak{R}(\mathfrak{A})$ is not an exchange system. Then for some $X, Y \in \mathfrak{R}(\mathfrak{A})$ with $x \in X-Y$ and $y \in X \cap Y$ there is no $Z \in \mathfrak{R}(\mathfrak{A})$ with $x \in Z \subset X \cup Y - \{y\}$.

Index $X \cap Y$ by the initial ordinal s of cardinal $|X \cap Y|$ so that $X \cap Y = \{x_p | 0 \leqq p < s\}$ with $x_0 = y$.

The following three sequences are to be constructed.

(i) $\{N(p)\}_{0 \leqq p < s}$ such that $N(p) < N(q)$ for $0 \leqq p < q < s$,

(ii) $\{Z_p\}_{0 \leqq p<s}$, $Z_p = X \cap Y - (\{x_0\} \cup \{x_{n(q)} | 0 \leqq q < p\})$, and

(iii) $\{T_p\}_{0 \leqq p<s}$, $T_p \in \mathfrak{A}$, $T_p \cap Z_p = x_{N(p)}$, $T_p \cap (X \cup Y) = \{x_0, x_{N(p)}\}$.

To start the sequences take $Z_0 = X \cup Y - \{x_0\}$. By hypothesis there is $A \in \mathfrak{A}$ with $A \cap Z_0 = \{c\}$. Without loss of generality $c \in X$. Then $|A \cap X| \geqq 2$ and thus $A \cap X = \{x_0, c\}$. If $c \notin Y$ then $A \cap Y = \{x_0\}$, a contradiction. Hence $c = x_t$ for some t, $0 \leqq t < s$. Take $A \in \mathfrak{A}$ with $A \cap Z_0 = \{x_t\}$ and t as small as possible. Set T_0 to be this A and set $N(0) = t$. Then $N(0)$ is defined, $Z_0 = X \cup Y - \{x_0\}$, T_0 is defined and $T_0 \in \mathfrak{A}$, $T_0 \cap Z_0 = x_{N(0)}$, $T_0 \cap (X \cup Y) = \{x_0, x_{N(0)}\}$.

Suppose the three sequences with the stated properties have been obtained for $0 \leqq p < q$, $q < s$. Take $Z_q = X \cup Y - (\{x_0\} \cup \{x_{N(p)} | 0 \leqq p<q\})$. Since $Z_q \subset X \cup Y - \{x_0\}$ there is $A \in \mathfrak{A}$ with $A \cap Z_q = \{c\}$, by hypothesis. Without loss of generality take $c \in X$. Then $|A \cap X| \geqq 2$, finite, and $A \cap (X - \{x_0\}) \subset \{x_{N(p)} | 0 \leqq p < q\}$. Choose an $A \in \mathfrak{A}$ with $A \cap Z_q = \{c\}$ and $|A \cap (X - \{x_0\})|$ least. Suppose $x_{N(p)} \in A$ for some p, $0 \leqq p < q$. Then $c \notin T_p$, $x_{N(p)} \in A \cap T_p$. Hence by exchange there is $A' \in \mathfrak{A}$ with $c \in A' \subset A \cup T_p - \{x_{N(p)}\}$. Then

$$c \in A' \cap Z_q \subset (A \cap Z_q \cup T_p \cap Z_q) - \{x_{N(p)}\} \subset (\{c\} \cup T_p \cap Z_p) - \{x_{N(p)}\} = \{c\},$$

while

$$A' \cap (X - \{x_0\}) \subset (A \cap (X - \{x_0\}) \cup T_p \cap (X - \{x_0\})) - \{x_{N(p)}\} = A \cap (X - \{x_0\}) - x_{N(p)},$$

a contradiction on the choice of A. Thus there is $A \in \mathfrak{A}$ with $A \cap Z_q = \{c\}$, and $A \cap X = \{x_0, c\}$. Then $c \in Y$ for otherwise $A \cap Y = \{x_0\}$.

Thus $c = x_t$ for some t, $0 \leqq t < s$. Now choose $A \in \mathfrak{A}$ such that $A \cap Z_q = \{x_t\}$, $A \cap (X \cup Y) = \{x_0, x_t\}$ and t is the least index possible. Set T_q to be this A and set $N(q)$ to be this t. Since $x_{N(q)} \in Z_p$ by the choice of T_p, $0 \leqq p < q$ we have $N(q) > N(p)$ for $0 \leqq p < q$. Thus $N(q)$ is defined with $N(p) < N(q)$ for $0 \leqq p < q$, Z_q is defined, $Z_q = X \cup Y - (\{x_0\} \cup \{x_{N(p)} | 0 \leqq p < q\})$ and T_q is defined with $T_q \in \mathfrak{A}$, $T_q \cap Z_q = \{x_{N(q)}\}$ and $T_q \cap (X \cup Y) = \{x_0, x_{N(q)}\}$.

Thus if the sequences are defined for $0 \leqq p < q < s$ they are defined for $0 \leqq p \leqq q$ and hence for all p, $0 \leqq p < s$.

Now take $Z = X \cup Y - (\{x_0\} \cup \{x_{N(p)} | 0 \leqq p < s\})$. Then $x \in Z \subset X \cup Y - \{x_0\}$ and by hypothesis there is $A \in \mathfrak{A}$ with $A \cap Z = \{c\}$. Again without loss of generality $c \in X$ and so $|A \cap X| \geqq 2$, finite, and $A \cap X \subset \{x_0\} \cup \{x_{N(p)} | 0 \leqq p < s\}$. Choose such an $A \in \mathfrak{A}$ with $|A \cap (X - \{x_0\})|$ least. If $x_{N(p)} \in A$ then $c \in A - T_p$, $x_{N(p)} \in A \cap T_p$ and by the exchange property in $\mathfrak{A}$ there is $A' \in \mathfrak{A}$ with $c \in A'$, $A' \subset A \cup T_p - \{x_{N(p)}\}$. Again $c \in A' \cap Z \subset (A \cup T_p) \cap Z = \{c\}$ and $A' \cap (X - \{x_0\}) \subsetneqq A \cap (X - \{x_0\})$. Hence for such an A with $|A \cap (X - \{x_0\})|$ least, $A \cap X = \{x_0, c\}$. Then, once more, $x_0 \in A \cap Y$ and $Y - X \subset Z$ so that $c \in Y$. From all such $A \in \mathfrak{A}$ choose one with $A \cap Z = \{x_t\}$ and $A \cap (X \cup Y) = \{x_0, x_t\}$. Since $Z \subset Z_p$ then by the choice of T_p and $x_{N(p)}$ we have $t > N(p)$ for $0 \leqq p < s$. But N is strictly increasing and hence cofinal with s. Thus such a t cannot exist and a final contradiction is reached. This means that the hypothesis is untenable and thus $\mathfrak{R}(\mathfrak{A})$ has the exchange property.

We have now shown that $\mathfrak{A}$ satisfies the conditions of lemma (3.11) and thus $(\mathfrak{A}^{\#})^{\#} = \mathfrak{A}_{min}$.

<u>REMARK</u> (3.15) That $\mathfrak{R}(\mathfrak{A})$ has the exchange property in theorem (3.14) depends essentially on the fact that $|A \cap X|$ is finite for $A \in \mathfrak{A}$ and $X \in \mathfrak{R}(\mathfrak{A})$. This naturally leads to the following statement.

<u>THEOREM</u> (3.16) <u>Let</u> $\mathfrak{A}$ <u>be an exchange system on</u> E <u>with</u> D <u>locally finite for each</u> $D \in \vartheta_{\mathfrak{A}}$. <u>Then</u> $\mathfrak{A}^{\#}$ <u>is an exchange system, and</u> $\mathfrak{A}^{\#} \subset \mathfrak{R}(\mathfrak{A})_{min}$.

<u>Proof</u>. $\mathfrak{A}$ has the l.c.p. by remark (1.25) and definition (1.22). By lemma (3.8), $\mathfrak{A}^{\#} \subset \mathfrak{R}(\mathfrak{A})_{min}$, and consists of incomparable elements.

Take $X, Y \in \mathfrak{A}^{\#}$ with $x \in X+Y$ and $y \in X \cap Y$. By lemma (3.4) we can set $X = f^*_D(y)$ and $Y = f^*_{D'}(y)$ for some $D, D' \in \vartheta_{\mathfrak{A}}$. Because $A \cap D'$ is finite for any $A \in \mathfrak{A}$ and $D' \in \vartheta_{\mathfrak{A}}$, $A \cap (X \cup Y)$ is also finite for any $A \in \mathfrak{A}$. Thus the argument used in theorem (3.14) to show there is Z with $x \in Z \subset X \cup Y - \{y\}$ and $|Z \cap A| \neq 1$ for any $A \in \mathfrak{A}$ is applicable here to obtain the same result.

Let $\mathcal{C}$ be any chain such that for $Z \in \mathcal{C}$; $x \in Z \subset X \cup Y - \{y\}$ and $|Z \cap A| \neq 1$ for any $A \in \mathfrak{A}$. If $A \cap (\cap \mathcal{C}) = \{a\}$ for some $A \in \mathfrak{A}$ then for some $Z \in \mathcal{C}$, $A \cap Z = \{a, a_1, \ldots, a_n\}$ with $A \cap Z \subset A \cap (X \cup Y)$. For $1 \leqq i \leqq n$ there is $Z_i \in \mathcal{C}$ with $a_i \notin Z_i$. Hence $Z' = Z \cap Z_1 \cap \ldots \cap Z_n \in \mathcal{C}$ and $Z' \cap A = \{a\}$, a contradiction. Thus there is $Z \subset E$ with $x \in Z$, $Z \subset X \cup Y - \{y\}$, and minimal such that $|A \cap Z| \neq 1$. We will now show that any such minimal set is a member of $\mathfrak{A}^{\#}$. Let Z be any such minimal element.

Because $D \cup D'$ has finite intersection with each member of $\mathfrak{A}$ there is P minimal containing $X \cup Y - \{x,y\}$ and contained in $D \cup D'$ with $P \cap A \neq \phi$ for any $A \in \mathfrak{A}$ by lemma (1.33) and there is Q minimal in P with $(Z - \{x\}) \subset Q$, $Q \cap A \neq \phi$ for any $A \in \mathfrak{A}$ and finally $R \subset Q$, R a dendroid for $\mathfrak{A}$. By lemma (1.78) $Q \supset P - (X \cup Y)$ and $R \supset Q - (Z - \{x\})$. Now if $x \in Q$ then there is $A \in \mathfrak{A}$, $x \in A$ and $\phi = A \cap (Q - \{x\}) \supset A \cap (Z - \{x\})$ so that $A \cap Z = \{x\}$, a contradiction. Thus $x \notin Q$ and so $x \notin R$. By lemma (1.78), $f_R(a) \cap (Z - \{x\}) = \phi$ for $a \in R - (Z - \{x\})$. Hence by remark (3.7) $f_R(a) \cap Z \neq \phi$ for $a \subset Z$. Thus $f^*_R(x) \subset Z$ and by the minimality of Z, $f^*_R(x) = Z$.

<u>THE DUAL FOR</u> : $\mathcal{M}_k$, $\mathcal{N}_k$, $\mathfrak{A}_\infty$, $\mathfrak{L} \supset \mathfrak{A}_\infty$, $\mathfrak{A}_k$, <u>and</u> $\mathfrak{A}(X)$.

<u>THEOREM</u> (3.17) <u>For</u> $k > 1$ <u>the system</u> $\mathcal{M}_k$ <u>has the properties</u>

(i) $\mathcal{M}_k^\#$ <u>is an exchange system</u>,

(ii) $\mathcal{M}_k^\# \subset \mathfrak{R}(\mathcal{M}_k)_{min.}$,

(iii) $\vartheta_{\mathcal{M}_k^\#} = \{E-D \mid D \in \vartheta_{\mathcal{M}_k}\}$,

(iv) $\mathcal{M}_k$ <u>has the l.c.p.</u>,

(v) $(\mathcal{M}_k^\#)^\# = \mathcal{M}_k$,

(vi) $\vartheta_{\mathcal{M}_k} = \mathcal{N}_{k-1}$, <u>and</u>

(vii) $\mathcal{M}_k^\# = \mathcal{N}_{k-2}$.

<u>Proof</u>. The restriction $k > 1$ is necessary for if $k = 0$, $\mathcal{M}_0 = \{\{\phi\}\}$ which is not an exchange system and if $k = 1$, $\mathcal{M}_1 = \{\{x\} \mid x \in E\}$ and the only dendroid is E. Hence $\mathcal{M}_1^\# = \{\phi\}$ and is not an exchange system.

Since $\mathcal{M}_k$ is a collection of finite sets which is an exchange system, $\mathcal{M}_k$ satisfies the conditions of theorem (3.14). Hence properties (i) through (v) follow from this theorem. Property (vi) is remark (1.15). Using (vi), for $D \in \mathcal{N}_{k-1}$, $x \in E-D$, we have $\{x\} \cup D = \{x\} \cup \{y \in D | x \in \{y\} \cup (E-D)\} = \{x\} \cup \{y \in D | x \in f_D(y)\} = f_D^*(x)$. Hence $\mathcal{M}_k^{\#} = \mathcal{N}_{k-2}$.

<u>THEOREM</u> (3.18) <u>The system</u> $\mathcal{N}_k$ <u>has the properties</u>

(i) $\mathcal{N}_k^{\#}$ <u>is an exchange system</u>,

(ii) $\mathcal{N}_k^{\#} \subset \mathcal{R}(\mathcal{N}_k)_{min}$,

(iii) $\vartheta_{\mathcal{N}_k^{\#}} = \{E-D | D \in \vartheta_{\mathcal{N}_k}\}$,

(iv) $\mathcal{N}_k$ <u>has the l.c.p.</u>,

(v) $(\mathcal{N}_k^{\#})^{\#} = \mathcal{N}_k$,

(vi) $\vartheta_{\mathcal{N}_k} = \mathcal{M}_{k+1}$, and

(vii) $\mathcal{N}_k^{\#} = \mathcal{M}_{k+2}$.

<u>Proof</u>. From theorem (3.17), $\mathcal{N}_k = (\mathcal{M}_{k+2})^{\#}$, thus $\mathcal{N}_k^{\#} = (\mathcal{M}_{k+2}^{\#})^{\#} = \mathcal{M}_{k+2}$ and $(\mathcal{N}_k^{\#})^{\#} = (\mathcal{M}_{k+2})^{\#} = \mathcal{N}_k$.

From remark (1.16), $\vartheta_{\mathcal{N}_k} = \mathcal{M}_{k+1}$ and hence $\mathcal{N}_k$ has the l.c.p.

Again, from theorem (3.17), $\vartheta_{\mathcal{N}_k^{\#}} = \vartheta_{\mathcal{M}_{k+2}} = \mathcal{N}_{k+1}$, hence $\vartheta_{\mathcal{N}_k^{\#}} = \{E-D | D \in \vartheta_{\mathcal{N}_k}\}$.

The properties (i) through (vi) have now been established.

<u>REMARK</u> (3.19) $\vartheta_{\mathcal{M}_k^{\#}} = \mathcal{M}_{k-1}$ <u>and</u> $\vartheta_{\mathcal{N}_k^{\#}} = \mathcal{M}_{k+1}$. This is clear from the statements in theorems (3.17) and (3.18).

<u>THEOREM</u> (3.20) $\mathfrak{A}_\infty$ <u>has the properties</u>

(i) $\mathfrak{A}_\infty$ <u>is an exchange system</u>,

(ii) $\mathfrak{A}_\infty^\#$ <u>is an exchange system</u>,

(iii) $\mathfrak{A}_\infty$ <u>has the l.c.p.</u>,

(vi) $\mathfrak{A}_\infty^\# \subset \Re(\mathfrak{A}_\infty)_{\min}$,

(v) $\vartheta_{\mathfrak{A}_\infty^\#} = \{E-D \mid D \in \vartheta_{\mathfrak{A}_\infty}\}$, <u>and</u>

(vi) $(\mathfrak{A}_\infty^\#)^\# = \mathfrak{A}_\infty$.

<u>Proof</u>. Property (i) is theorem (2.4). Since $\mathfrak{A}_\infty$ is a collection of finite sets the conditions of theorem (3.14) are satisfied and properties (ii) through (vi) hold.

<u>THEOREM</u> (3.21) Let $\mathcal{L} = \{E-D \mid D \in \vartheta_{\mathfrak{A}_\infty}\} \cup \mathfrak{N}_\infty$, <u>the system of theorem</u> (2.8). <u>Then</u> $\mathcal{L}$ <u>has the following properties</u>.

(i) $\mathcal{L}$ <u>is an exchange system</u>,

(ii) $\mathcal{L}$ <u>has the l.c.p.</u>,

(iii) $\mathcal{L}^\# \subset \Re(\mathcal{L})_{\min}$,

(iv) $\mathcal{L}^\#$ <u>is an exchange system</u>,

(v) $\vartheta_{\mathcal{L}^\#} = \{E-D \mid D \in \vartheta_{\mathcal{L}}\}$, <u>and</u>

(vi) $(\mathcal{L}^\#)^\# = \mathcal{L}$.

<u>Proof</u>. (i) is theorem (1.77) while (ii) is theorem (2.8). Hence (iii) follows from lemma (3.9). We now show (iv).

We use the notation of (2.1) and let D be a dendroid for $\mathfrak{A}_\infty$. Define $r: E \to \{n \mid n \geqq 2\}$ by $r(y) = \min\{t \mid y \in X_t\}$, and set $D(y) = |f_D(y)|$ for $y \in D$. For $q \in E-D$, $q \in f_D(y)$ if and only if $r(q) \leqq D(y)$ as shown in corollary (2.7).

Take $T \in \mathfrak{L}^{\#}$. Take $q \in T$ such that $r(q)$ is least. Then there is a dendroid B of $\mathfrak{L}$ such that $T = f^*_B(q)$ by lemma (3.4). $B = D \cup \{a\}$ where D is a dendroid for $\mathfrak{A}_\infty$ by theorem (1.77). Because $q \in f_B(a) = (E-B) \cup \{a\}$, $r(q) \leqq r(a)$. For $y \in D$, $f_B(y) = f_D(y)$ if and only if $r(a) < D(y)$. There are three possibilities for $D(y)$.

(i) $D(y) < r(q)$. Then $q \notin f_D(y)$, $a \notin f_D(y)$ and so $q \notin f_B(y)$.

(ii) $r(q) \leqq D(y) < r(a)$. Then $q \in f_D(y) = f_B(y)$.

(iii) $r(q) \leqq r(a) \leqq D(y)$. Then $q \in f_D(y)$ and $q \in f_B(y) = (E-D) \cup \{y\}$.

Hence for $y \in D$, $q \in f_D(y)$ if and only if $q \in f_B(y)$. Thus $f^*_B(q) = f^*_D(q) \cup \{a\}$.

Now take any $b \in D$ such that $r(a) \leqq D(b)$. Then $a \in f_D(b)$ and $D' = (D - \{b\}) \cup \{a\}$ is a dendroid for $\mathfrak{A}_\infty$ and $B = D' \cup \{b\}$. Now $r(q) \leqq r(b)$, and thus as in the argument above, $f^*_B(q) = f^*_{D'}(q) \cup \{b\}$.

Let $x_0 \in T_0 - T_1$, $x_1 \in T_0 \cap T_1$, where $T_i \in \mathfrak{L}^{\#}$ for $i = 0,1$. Then $T_i = f^*_{B_i}(q_i) = f^*_{D_i}(q_i) \cup \{a_i\} = Q_i \cup \{a_i\}$, where $B_i = D_i \cup \{a_i\}$, D_i a dendroid for $\mathfrak{A}_\infty$, $r(a_i) > \max\{r(x_0), r(x_1)\}$ as shown in the last paragraph for $i = 0,1$.

By this choice, $x_i \in Q_i$, $x_0 \in Q_0 - Q_1$, $x_1 \in Q_0 \cap Q_1$. $\mathfrak{A}^{\#}_\infty$ has the exchange property as shown in theorem (3.20) and thus there is $Q = f^*_D(q) \subset Q_0 \cup Q_1 - \{x_1\}$ with $x_0 \in Q$, D a dendroid for $\mathfrak{A}_\infty$. Choose D such that $r(q) = \min\{r(y) | y \in Q\}$. Now $r(q) \leqq r(x_0) < r(a_0)$, and since $a_0 \notin Q$, $B = D \cup \{a_0\}$ is a dendroid for $\mathfrak{L}$ and $x_0 \in f^*_B(q) = f^*_D(q) \cup \{a_0\} \subset T_0 \cup T_1 - \{x_1\}$. Thus $\mathfrak{L}^{\#}$ has the exchange property.

Let B be a dendroid for $\mathfrak{L}^{\#}$. Then by lemma (3.12), $(E-B) \cap A \neq \phi$ for $A \in \mathfrak{L}$. Thus $E-B \supset Z$, a dendroid for $\mathfrak{A}_{\infty}$ by lemma (1.33). Z is not a dendroid for $\mathfrak{L}$, and so there is $b \in (E-B)-Z$. Then $D = Z \cup \{b\} \subset E-B$, and is a dendroid for $\mathfrak{L}$ by theorem (1.77). By corollary (3.13),

$$\vartheta_{\mathfrak{L}^{\#}} = \{E-D | D \in \vartheta_{\mathfrak{L}}\}.$$

Thus $\mathfrak{L}$ satisfies the conditions of lemma (3.11) and

$$(\mathfrak{L}^{\#})^{\#} = \mathfrak{L}.$$

This completes the proof of theorem (3.21).

<u>THEOREM</u> (3.22) <u>Let</u> m, n, <u>and</u> k <u>be as in theorem</u> (2.11). <u>If</u> n <u>is finite take</u> $k > -n+1$, <u>and</u> k <u>any integer otherwise. Then</u> $\mathfrak{A}_k$ <u>has the following properties</u>.

(i) $\mathfrak{A}_k$ <u>has the exchange property</u>,

(ii) $\mathfrak{A}_k$ <u>has the l.c.p</u>.

(iii) $\mathfrak{A}_k^{\#} \subset \mathfrak{R}(\mathfrak{A}_k)_{min}$,

(iv) $\vartheta_{\mathfrak{A}_k^{\#}} = \{E-D | D \in \vartheta_{\mathfrak{A}_k}\}$,

(v) $\mathfrak{A}_k^{\#}$ <u>is an exchange system, and</u>

(vi) $(\mathfrak{A}_k^{\#})^{\#} = \mathfrak{A}_k$.

<u>Proof</u>. Property (i) is theorem (2.11) and (ii) is corollary (2.13), while (iii) follows from lemma (3.9).

Take B a dendroid for $\mathfrak{A}_k^{\#}$, and $b \in B$. Then by corollary (2.12), $f_B(b) = \{b\} \cup D = f_D^*(b)$ for some $D \in \vartheta_{\mathfrak{A}_k}$. Hence $B \cap D = \phi$ and thus $B \subset E-D$ so that $B = E-D$, and property (iv) follows.

We now show that $\mathfrak{A}_k^{\#}$ has the exchange property. Let $X,Y \in \mathfrak{A}_k^{\#}$ with $x \in X-Y$ and $y \in X \cap Y$. Then $X = \{x\} \cup D$ and $Y = \{y\} \cup D'$ where D and D' are dendroids for $\mathfrak{A}_k$ and $y \in D$, by corollary (2.12). Because $f_D(y) \cap D' = ((E-D) \cup \{y\}) \cap D' = (E-D) \cap D' \neq \phi$ there is $z \in f_D(y) \cap D'$, $z \neq x$, $z \neq y$. Hence $D_0 = (D - \{y\}) \cup \{z\}$ is a dendroid for $\mathfrak{A}_k$ and $f^*_{D_0}(x) = \{x\} \cup D_0 \subset X \cup Y - \{y\}$, and $\mathfrak{A}_k^{\#}$ has property (v).

We have shown that $\mathfrak{A}_k$ satisfies lemma (3.11), hence $(\mathfrak{A}_k^{\#})^{\#} = \mathfrak{A}_k$, and the proof of the theorem is complete.

THEOREM (3.23) Let X be an infinite graph such that $\mathfrak{A}(X)$ has the l.c.p. Then $\mathfrak{A}(X)$ has the properties:

(i) $\mathfrak{A}(X)$ has the exchange property,

(ii) $\mathfrak{A}(X)^{\#} \subset \mathfrak{R}(\mathfrak{A}(X))_{min}$,

(iii) $\mathfrak{A}(X)^{\#}$ is an exchange system,

(iv) $\vartheta_{\mathfrak{A}(X)^{\#}} = \{E-D \mid D \in \vartheta_{\mathfrak{A}(X)}\}$, and

(v) $(\mathfrak{A}(X)^{\#})^{\#} = \mathfrak{A}(X)$.

Proof. (i) is corollary (2,16). (ii) follows from the l.c.p. condition and lemma (3.9). We now prove that (iii) holds. Let $U,V \in \mathfrak{A}(X)^{\#}$ with $\omega \in U \cap V$, $e \in U-V$. By lemma (3.4) we can take $U = f^*_D(\omega)$, $V = f^*_{D'}(\omega)$, D, D' dendroids for $\mathfrak{A}(X)$. Take $Z = U \cup V - \{\omega\}$. Then $Z \subset D \cup D'$.

If $A \in \mathfrak{A}(X)$ with $A \cap (D \cup D' - \{e\}) = \phi$ then $A = f_D(e) = f_{D'}(e)$ and $\omega \in f_D(e)$. Hence $e \in f^*_{D'}(\omega)$, a contradiction to $e \in U-V$.

By lemma (2.61) there is Q minimal in $D \cup D' - \{e\}$ such that $Q \supset Z$ and $Q \cap A \neq \phi$ for $A \in \mathfrak{A}(X)$. By lemma (2.60) there is R, a dendroid for $\mathfrak{A}(X)$ with $R \subset Q$ and $R-Z = Q-Z$ by lemma (1.78). Because R has the l.c.p. by hypothesis $\bigcup_{e' \in R} f_R(e') = E(X)$.

Take $e' \in R$ such that $e \in f_R(e')$. If $e' \in R-Z$ then $e' \in Q-Z$ and $f_R(e') \cap Z = \phi$ by lemma (1.78). Thus $f_R(e) \cap U = \{e,\omega\}$ by lemma (3.5) and $f_D(e) \cap V = \{\omega\}$, a contradiction to lemma (3.5). Hence $e \in f_R(e')$ implies $e \in Z$ and so $e \in f_R^*(e) \subset Z = U \cup V - \{\omega\}$. Thus $\mathfrak{A}(X)^{\#}$ has the exchange property.

By lemma (3.12), if $B \in \vartheta_{\mathfrak{A}(X)^{\#}}$ then $(E-B) \cap A \neq \phi$ for any $A \in \mathfrak{A}(X)$. By lemma (2.60) there is $D \subset E-B$, $D \in \vartheta_{\mathfrak{A}(X)}$. Hence $E-D \supset B$ and by (ii), $E-D = B$. Thus we have property (v). Now all the conditions for lemma (3.11) are satisfied so that $(\mathfrak{A}(X)^{\#})^{\#} = \mathfrak{A}(X)$, and the proof is complete.

<u>COROLLARY</u> (3.24) <u>Let</u> X <u>be a tree with</u> $\mathfrak{A}(X) \neq \phi$. <u>Then the properties of theorem</u> (3.22) <u>hold for</u> $\mathfrak{A}(X)$.

<u>Proof</u>. By theorem (2.73) $\mathfrak{A}(X)$ has the l.c.p. Hence the conditions of theorem (3.22) hold and the result follows.

Some Further Results on Duals

LEMMA (3.25) *Let $\mathfrak{A}$ be an exchange system. If $X, Y \in \mathfrak{A}^{\#}$ with $y \in X \cap Y$ and $D \in \vartheta_{\mathfrak{A}}$ with the l.c.p. such that $Y = f^*_D(y)$ and $|X - (Y \cup D)|$ is finite then there is $R \in \vartheta_{\mathfrak{A}}$ such that*

(i) $|X - (Y \cup R|$ *is minimal with* $Y = f^*_R(y)$, *and*

(ii) $X \cap \bigcup\{f_R(z) | z \in R - (Y \cup X)\} = \phi$.

Proof. If $\phi = X - (Y \cup D)$ then $X \subset \{y\} \cup D$, and (i) immediately holds. Now $z \in D - (Y \cup X)$ implies $y \notin f_D(z)$. But $f_D(z) \cap D = \{z\}$ and $f_D(z) \cap Y = \phi$ hence $f_D(z) \cap X = \phi$, and (ii) follows.

If $0 \neq |X - (Y \cup D)|$, finite, then there is $R \in \vartheta_{\mathfrak{A}}$ such that (i) holds. We now show that (ii) must hold.

Suppose $a \in X \cap \bigcup\{f_R(z) | z \in R-(Y \cup X)\}$. Then $a \in f_R(z)$, $z \in R-Y$, and $a = z$. Hence $R' = (R - \{z\}) \cup \{a\} \in \vartheta_{\mathfrak{A}}$ and $|X - (Y \cup R')| < |X - (Y \cup R)|$.

Take $q \in R - f^*_R(y)$. Then $f_{R'}(q) \subset f_R(q) \cup f_R(z)$ and thus does not contain y. Hence $f^*_{R'}(y) \subset f^*_R(y)$. But $R' \sim R$ and so R' has the l.c.p. by theorem (1.45). Then lemma (3.5) implies $f^*_{R'}(y) = f^*_R(y)$ which contradicts the choice of R. Thus when (i) holds (ii) must hold, and (i) is always possible.

THEOREM (3.26) *Let $\mathfrak{A}$ be an exchange system and $D_0 \in \vartheta_{\mathfrak{A}}$. Let*

$$\vartheta_0 = \{D \in \vartheta_{\mathfrak{A}} |\ |D+D_0| \text{ is finite}\}. \text{ Then}$$

$\mathfrak{A}^{\#}_0 = \{f^*_D(x) | x \in E-D,\ D \in \vartheta_0\}$ *is an exchange system.*

<u>Proof</u>. Lemmas (1.32) and (1.46) show that

$\mathfrak{A}_0 = \{A \in \mathfrak{A}_{min} \mid |A \cap D_0| \text{ is finite}\}$

$= \{A \in \mathfrak{A}_{min} \mid |A \cap D| \text{ is finite}, D \in \vartheta_0\}$ is an exchange system.

Take $X, Y \in \mathfrak{A}_0^{\#}$ with $x \in X-Y$ and $y \in X \cap Y$. By lemma (3.4) $Y = f_D^*(y)$ for a dendroid D, $D \sim D_0$. Then $|X - (Y \cup D)|$ is finite, and D has the l.c.p. on $\mathfrak{A}_0$ because $A \cap D$ is finite for $A \in \mathfrak{A}_0$. Take R such that properties (i) and (ii) of lemma (3.25) are satisfied. Then, as indicated in the proof of that lemma, R is a dendroid for $\mathfrak{A}_0$ and $R \sim D_0$, and $X-R \neq \phi$ by lemma (3.5).

Take $z \in X-R$. Then $z \in f_R(b)$ only if $b \in X \cup Y$, and $Q \in \vartheta_0$, $Q = (R-\{b\}) \cup \{z\}$. $|X-(Y \cup Q)| < |X-(Y \cup R)|$ and $p \in f_Q^*(b)$ implies $b \in f_Q(p)$. Also, $f_Q(p) = f_R(p)$ if $p \in R - (X \cup Y)$. Hence $z \in f_Q^*(b) \subset Y \cup X - \{y\}$.

Now if $x \in X-R$, take $z = x$. If $x \in X \cap R$ then $|f_R(x) \cap X| \geqq 2$, $y \notin f_R(x)$ and so there is $z \in X-R$ with $z \in f_R(x)$. Take $b = x$. In either case, $x \in f_Q^*(b) \subset X \cup Y - \{y\}$. We must now show that this is in $\mathfrak{A}^{\#}$, and we do this by showing that Q is a dendroid for $\mathfrak{A}$. $Q \sim D_0$, and if $A \in \mathfrak{A}_{min}$ with $Q \cap A = \phi$ then write $Q = (D_0 - T) \cup S$, $T + S$ finite. Then $A \cap D_0 \subset A \cap T$, and thus $A \in \mathfrak{A}_0$, a contradiction. Hence $Q \in \vartheta_{\mathfrak{A}}$ and $f_Q^*(b) \in \mathfrak{A}^{\#}$. Thus $\mathfrak{A}_0^{\#}$ is an exchange system.

<u>LEMMA</u> (3.27) <u>Let</u> $\mathfrak{A}$ <u>be an exchange system on</u> E <u>with</u> D_0, D_1 <u>dendroids for</u> $\mathfrak{A}$ <u>with the l.c.p. If</u> $X_i \in \mathfrak{A}^{\#}$, $f^*_{D_i}(y) = X_i$, $i = 0,1$, $x \in X_0 - X_1$ <u>and</u> $|D_0 + D_1 - (X_0 \cup X_1)|$ <u>is finite then there is</u> $R_i \sim D_i$ <u>with</u> $R_0 + R_1 \subset X_0 \cup X_1$, $f^*_{R_i}(y) = f^*_{D_i}(y)$, $i = 0,1$ <u>and there is</u> $z \in \mathfrak{A}^{\#}$ <u>with</u> $x \in Z \subset X_0 \cup X_1 - \{y\}$.

<u>Proof</u>. Because $|D_0 + D_1 - (X_0 \cup X_1)|$ is finite we can take $R_i \sim D_i$ with $X_1 = f^*_{R_i}(y)$, $i = 0,1$ and $|R_0 + R_1 - (X_0 \cup X_1)|$ minimal. Suppose $p \in R_0 - (X_0 \cup X_1)$ with $q \in f_{R_0}(p) \cap (X_1 - X_0)$. Then for the dendroid $D = (R_0 - \{p\}) \cup \{q\}$ we have $f_D(p') \subset f_{R_0}(p) \cup f_{R_0}(p')$ by lemma (1.42). Hence if $p' \in R_0 - (X_0 \cup X_1)$ then $y \notin f_D(p')$ and so $f^*_D(y) \subset f^*_{R_0}(y)$. By lemma (3.8) this implies $f^*_D(y) = f^*_{R_0}(y)$ and we have a contradiction since $|D + R_1 - (X_0 \cup X_1)| = |R_0 + R_1 - (X_0 \cup X_1)| - 2$.

Now suppose $p \in R_0 - (X_0 \cup X_1)$, $p \notin R_1$. Because R_1 has the l.c.p., $f_{R_0}(p) \subset \bigcup_{q \in f_{R_0}(p) \cap R_1} f_{R_1}(q)$ by theorem (1.44), and

$q \in R_1 - (X_0 \cup X_1)$. Again let $D = (R_0 - \{p\}) \cup \{q\}$, $f_D(p') \subset f_{R_0}(p) \cup f_{R_0}(p')$. If $p' \in R_0 - (X_0 \cup X_1)$ then $y \notin f_D(p')$. Thus $f^*_D(y) = f^*_{R_0}(y)$ and again $|D + R_1 - (X_0 \cup X_1)| = |D + R_1 - (X_0 \cup X_1)| - 2$, a contradiction. By symmetry we have $R_0 + R_1 \subset X_0 \cup X_1$.

By lemma (3.5) there is $z \in f_{R_0}(x) \cap (X_0 \cup X_1)$. Let $D = (R_0 - \{x\}) \cup \{z\}$. We have shown that $z \notin f_{R_0}(p)$ for $p \in R_0 - (X_0 \cup X_1)$, and so $x \in f_D(p) = f_{R_0}(p)$. Thus $x \in f^*_D(x) \subset X_0 \cup \{z\} - \{y\} \subset X_0 \cup X_1 - \{y\}$. Let $Z = f^*_D(x)$ and the lemma is proved.

NOTATION (3.28) Let X be a tree and D a dendroid for $\mathfrak{A}(X)$. If $e \in E(X) - D$ and $w \in f_D^*(e) - \{e\}$ then let $R(D,e;w)$ be the unique ray in C whose initial vertex belongs to (e), does not contain w and does not contain e, where $E(C) = f_D(w)$.

LEMMA (3.29) Let X be a circuit connected tree, D a dendroid for $\mathfrak{A}(X)$, $e \in E(X) - D$, and w, z two distinct edges in $f_D^*(e) - \{e\}$. Then $R(D,e;w) = R(D,e;z)$.

Proof. If $R(D,e;w) \neq R(D,e;z)$ then we put

$$C = \begin{cases} R(D,e;w) + R(D,e;z) & \text{if the rays are not vertex disjoint} \\ (e) \cup R(D,e;w) \cup R(D,e;z), & \text{if the rays are vertex disjoint.} \end{cases}$$

In either case, since X is a tree, C is an infinite circuit, and by definition of $f_D(s)$ for $s \in D$, we have $E(C) \cap D = \emptyset$, a contradiction. Hence $R(D,e;w) = R(D,e;z)$.

REMARK (3.30) For X a circuit connected tree and D a dendroid for $\mathfrak{A}(X)$ with $e \in E(X) - D$, lemma (3.29) shows $R(D,e;w)$ is independent of w for $w \in f_D^*(e) - \{e\}$. $R(D,e;w) \cup (e)$ is then written $S(D,e)$. The initial vertex of e in $S(D,e)$ will be taken as e_x, $e = [e_x,e_y]$. Lemma (3.29) further implies that for $w = [w_x,w_y] \in f_D^*(e) - \{e\}$, $\rho(e_x,w_x) < \rho(e_y,w_x)$. Hence $f_D^*(e) - \{e\}$ is contained in exactly one of the two components of the minimal subgraph of X whose edge set is $E(X) - \{e\}$.

LEMMA (3.31) Let X be a circuit connected tree and D a dendroid for $\mathfrak{A}(X)$. If $e \in E(X)-D$ then

$$R = \{f_D(w) \mid w \in f_D^*(e) - \{e\}\}$$

is a circuit if $|f_D^*(e)| = 2$ and a ray otherwise.

Proof. If $|f_D^*(e)| = 2$, $f_D^*(e) = \{e,w\}$ and $R = f_D(w)$, a circuit.

If $|f_D(e)| \neq 2$ then every edge $e \in E(X)$ is in a circuit and since $\mathfrak{A}(X)$ has the l.c.p. by theorem (2.73), $|f_D^*(e)| > 2$. The intersection of circuits in a tree is a circuit, ray or path and by lemma (3.29) contains the ray $S(D,e)$ but no members of D. Hence, R is a ray.

LEMMA (3.32) Let X be a circuit connected tree and D a dendroid for $\mathfrak{A}(X)$. For $e \in E(X)-D$ let $K(D,e)$ be the subgraph of X consisting of all paths P from e_x to $w \in f_D(e) - \{e\}$ not containing e or w. $K(D,e)$ is X-admissible. If R is a ray from e_x not containing e then $E(R) \cap f_D^*(e) \neq \phi$ and $w \in E(R) \cap D$ is the nearest one to e if and only if $w \in f_D^*(e) - \{e\}$.

Proof. It is clear that $K(D,e)$ is connected and that $D \cap E(K(D,e)) = \phi$. Let $R = [x_0,\ldots)$ be a ray in $K(D,e)$. Then $C = ((e_x,\ldots,x_0)+R) \cup S(D,e)$ is a circuit and $E(C) \cap D = \phi$, a contradiction. Thus $K(D,e)$ contains no ray and is thus X-admissible.

Let R be any ray from e_x not containing e. Then $R \cup S(D,e)$ is a circuit and $D \cap (E(R \cup S(D,e))) = D \cap E(R) \neq \phi$. Let w be the member of $D \cap E(R)$ closest to e. Then $S(D,e) \cup (e_x,\ldots,w_x)$ is a ray from x, edge disjoint from D, and is thus in $f_D(w)$ and $w \in f_D^*(e)$.

Let $w \in f^*_D(e) - \{e\}$ with $w \in E(R)$ for some ray R from e_x not containing e. Let P be the path joining w_x and e_x. Then $P \cup S(D,e)$ is a ray from w_x edge disjoint from D. Hence w is the nearest edge to e in $E(R) \cap D$.

THEOREM (3.33) *Let X be a circuit connected tree with dendroid D for $\mathfrak{U}(X)$ and $e \in E(X)-D$. Root X at e_x and let $\varphi: X \to X/K(D,e)$ with n a choice function for the family $(E_{\bar{x}})_{\bar{x} \in V(X/K(D,e))}$ with $n(\bar{e}_x) = \bar{e}$, where $\bar{x} = \varphi(x)$. Then $D_1 = \varphi^{-1}\left(\bigcup_{\bar{x} \in V(X/K(D,e))} (E_{\bar{x}} - n(\bar{x}))\right)$ is a dendroid for $\mathfrak{U}(X)$ with $f^*_{D_1}(e) = f^*_D(e)$.*

Proof. Since $K(D,e)$ is X-admissible, D_1 is a dendroid for $\mathfrak{U}(X)$ by corollary (2.57). $e \notin D_1$ and $\bar{w} \in (E_{\bar{e}_x} - n(\bar{e}_x))$ if and only if $w \in \bigcup_{x \in V(K(D,e))} E_x \cap D = f^*_D(e) - \{e\}$. Hence $f^*_D(e) = f^*_{D_1}(e)$.

REMARK (3.34) If X is a circuit connected tree and every vertex is a branch vertex then theorem (3.33) shows that for any $f^*_D(e)$ there is $f^*_{D_1}(e) = f^*_D(e)$ with D_1 not equivalent to D since there are infinitely many vertices where the choice function n of that theorem has two or more permissible values. Hence, using lemma (3.4) there is $w \neq e$ and D not equivalent to D_1 with $f^*_D(e) = f^*_{D_1}(w)$. This is a much stronger statement than in lemma (3.4).

LEMMA (3.35) *Let X be a tree and S, S′ two X-admissible subgraphs of X with $V(S) \cap V(S') = \phi$. Then $S \cup S'$ is X-admissible.*

Proof. It is clear that a component of $S \cup S'$ is a component of exactly one of S or S'. Since each such component contains no ray, $S \cup S'$ is X-admissible.

LEMMA (3.36) *If X is a circuit connected tree, D a dendroid for $\mathfrak{U}(X)$, $e = [e_x, e_y] \in E(X)-D$ and S_1 is X-admissible with $V(S_1) \cap V(K(D,e)) = \phi$, then $S = S_1 \cup K(D,e)$ is X-admissible. Root X at e_x. For any dendroid for $\mathfrak{U}(X/S)$ of the form (2.56.1) with $n(\bar{e}_x) = \bar{e}$ and $\varphi: X \to X/S$,*

$$\varphi^{-1}(E_{\bar{e}_x}) = f_D^*(e).$$

Proof. $E_{\bar{e}_x} = \overline{f_D^*(e)}$ since $e \notin S$ and for P a path in $K(D,e)$, $\varphi(P) = \bar{e}_x$. Since S_1 is vertex disjoint from $K(D,e)$, $\varphi(w) \in E(X/S)$ for $w \in f_D^*(e)$. Hence the result follows.

THEOREM (3.37) *Let X be a circuit connected graph and T a spanning tree of X with D_1 a dendroid for $\mathfrak{U}(T)$, $e \in E(T)-D_1$, $D_0 = E(X/T)$, and $D = D_0 \cup D_1$. Then*

(i) D *is a dendroid for* $\mathfrak{U}(X)$,

(ii) $f_D^*(e) \cap D_1 = f_{D_1}^*(e)$,

(iii) $w \in D_0 \cap f_D^*(e)$ *if and only if exactly one vertex of* w *is in* $V(K(D_1,e))$,

(iv) *if D_1' is a dendroid for $\mathfrak{U}(T)$ of the form in theorem (3.33) then $D' = D_0 \cup D_1'$ is a dendroid for $\mathfrak{U}(X)$ with $f_{D'}^*(e) = f_D^*(e)$,*

(v) *if $w \in D_0 \cap f_{D'}^*(e)$ and $f_{D'}(w) = f_{D_0}(w)$ then w is an edge with one vertex in $V(K(D_1',e))$ and the other in $V(S(D_1',e))$, and*

(vi) $R = \cap \{f_D(w) | e \in f_D(w), |f_D(w)| \text{ infinite}\}$ <u>is a ray containing</u> $S(D_1',e)$ <u>if there are two or more members being intersected and an infinite circuit otherwise</u>.

<u>Proof</u>. (i) is remark (2.29).

(ii) follows from the proof of theorem (1.74) which implies $f_D(w) = f_{D_1}(w)$ for $w \in D_1$.

(iii) Let $x \in V(T)-V(K(D_1,e))$, and R the ray in T with initial vertex e_x and x a vertex in R. Let $P = (e_x,\ldots,x)$ be the path in R joining x and e_x. If $e \in E(P)$ and $E(P) \cap D = \phi$ take $R_x = S(D_1,e)+P$. If $e \notin E(P)$ then $E(P) \cap D \neq \phi$ by definition of $K(D_1,e)$. Thus either $e \in E(P)$ and $E(P) \cap D_1 = \phi$, or $E(P) \cap D_1 \neq \phi$. In the latter case take $z \in E(P) \cap D$, $z = [z_x,z_y]$ such that $\rho(z_x,x)$ is least. Put $R_x = R_z^+ \cup (z_x,\ldots,x)$.

In all cases R_x is a ray, $e \notin E(R_x)$, $D \cap E(R_x) = \phi$, and $E(R_x) \cap E(K(D_1,e)) = \phi$. This leaves two possibilities for $w = [x,y] \in D_0 \cap f_D^*(e)$.

(i) $y \in V(K(D_1,e))$. Take $R_y = S(D_1,e) \cup (e_x,\ldots,y)$, a ray with $e \in E(R_y)$.

(ii) $y \notin V(K(D_1,e))$. Construct R_y as R_x was. $e \notin R_y$, $D \cap E(R_y) = \phi$.

If (ii) holds then $R = R_x \cup R_y \cup [x,y]$ is circuit connected and thus there is a circuit C in R, hence in X, with $w \in E(C)$ and by construction $e \notin E(C)$, $E(C) \cap D = \{w\}$, a contradiction. If (i) holds and $V(R_x) \cap V(R_y) = \phi$, then $C = R_x \cup R_y \cup [x,y]$ is an infinite circuit with $e \in E(C)$, $E(C) \cap D = \{w\}$, and $w \in f_D(e)$. If $\phi \neq V(R_x) \cap V(R_y)$

then there is $z \in V(R_x)\ V(R_y) \cap V(S(D_1,e))$ with $\rho(z,e_x)$ least. Then $C = (z,\ldots e_x,e_y,\ldots y) \cup [x,y] \cup (x,\ldots,z)$ is a finite circuit with $e \in E(C)$, $E(C) \cap D = \{w\}$, and $w \in f_D(e)$.

The only remaining possibility is that both x and y are in $K(D_1,e)$. But then $C = (x,\ldots,e_x) \cup (e_x,\ldots,y) \cup [x,y]$ is a circuit with $E(C) \cap D = \{w\}$, and $e \notin E(C) = f_D(w)$. Thus $w \in D_0 \cap f^*_D(e)$ if and only if exactly one vertex of w belongs to $V(K(D_1,e))$.

(iv) Because $K(D_1,e) = K(D_1',e)$ and $f^*_{D_1}(e) = f^*_{D_1'}(e)$, (iii) immediately shows that $f^*_D(e) = f^*_{D'}(e)$.

(v) Because $E_{x'} - D = [x',y'] \in E(S(D_1',e)$ for $x' \in V(S(D_1,e)) - \{e_x\}$ by the construction of D', the proof of (iii) shows that $f_{D'}(w)$ is finite for $w \in D_0 \cap f_D(e)$ exactly when w has a vertex, say x, $x \in V(S(D_1',e))$.

(vi) If there is only one $w \in f_D(e) - \{e\}$ such that $f_D(w)$ is infinite then R is clearly an infinite circuit. The proof of (iii), shows that every infinite $f_D(w)$, $w \in f^*_D(e)$ contains the set $E(S(D_1,e))$. Then exactly as in the proof of lemma (3.31), R is a ray containing $S(D_1,e)$.

THEOREM (3.38) Let T be a tree, $\cup\,\mathfrak{A}(T) = T$. Then the following statements are equivalent:

(i) T is locally finite,

(ii) $f^*_D(e)$ is finite for every $D \in \vartheta_{\mathfrak{A}(T)}$ and every $e \in E(T)-D$,

(iii) $f^*_D(e)$ is finite for some $D \in \vartheta_{\mathfrak{A}(T)}$ and every $e \in E(T)-D$.

<u>Proof</u>. (i) ⇒ (ii). Take any $D \in \vartheta_{\mathfrak{A}(T)}$ and any $e \in E(T)-D$. If $|f^*_D(e)| = 2$ there is nothing to prove and we may thus assume that $|f^*_D(e)| > 2$. $K(D,e)$ is a locally finite tree and contains no ray. It follows from Konigs theorem [1] that $K(D,e)$ is finite. Let be the boundary of $K(D,e)$. For $x \in V$, $E(T;x)$ is finite, and $f^*_D(e) \subset \bigcup_{x \in V} E(T;x)$. Thus, since $\mathfrak{B}$ is finite, $f^*_D(e)$ is.

(ii) ⇒ (iii). This is clear.

(iii) ⇒ (i). Take any $x \in V(T)$ and let C be a circuit with $x \in V(C)$. Since T has the l.c.p. by theorem (2.73), we can take C such that $E(C) \cap D = \{e\}$, $e = [x,y]$. Now $d(x;T) \leqq |\mathfrak{B}(K(D,e))| + 1 \leqq |f^*_D(e)| + 1$. Hence $d(x;T)$ is finite and T is locally finite.

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