SOME RESULTS ON THE SUBALGEBRA INTERSECTION PROPERTY

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ARISING FROM POLIN ALGEBRAS

By

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Abstract

To help answer some questions regarding the complexity classifications of constraint satisfaction problems, Valeriote developed the notion of intersection properties of powers of algebras. Valeriote showed some connections between these intersection properties and the tame congruence theory types of the algebras that satisfy or fail them. This paper uses Polin algebras to refute a conjecture of Valeriote and to motivate further investigation into the connection between intersection properties, tame congruence theory and congruence distributivity.

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I would like to thank my parents Gordon & Laura Williams for their support during my entire university career. Thanks to my wife Alison Williams for her support and understanding. I would like to dedicate my Masters degree work to the memory of my grandfather Archie Williams, who passed during my study.

Contents

1 Introduction

The overarching motivation of this thesis is the classification of all tractable subclasses of the Constraint Satisfaction Problem(CSP). Albeit that this paper is concerned with specific results, primarily using Polin algebras as (counter) examples to draw out connections between tame congruence theory, intersection properties and congruence distributivity, CSPs provide the context for the work.

Much work has been done to find specific restrictions on the form of the constraints of a CSP that will ensure tractability. Especially using constraint relations of finite algebras. For examples of this see [Bulatov et al., 2005, Idziak et al., 2007, Jeavons et al., 1998, Kiss and Valeriote, 2007]. This paper specifically investigates some of the work Valeriote set forth in [Valeriote, 2006]. Valeriote gives certain subalgebra intersection properties and develops their connection with congruence distributivity, tame congruence theory and the CSP. Our main result is a counterexample to Valeriote's Conjecture 2 in [Valeriote, 2006], that if the variety generated by a finite idempotent algebra **A** omits types **1**, **2** and **5** then **A** satisfies the strong k-intersection property for some $k > 0$. Our counterexample involves taking an idempotent reduct of a Polin algebra, and giving a standard pair of subalgebras that fail the strong k-intersection property. We also show that no non-Boolean and non-trivial subvariety of the Polin variety satisfies the strong k-intersection property.

As a standard reference for universal algebra the reader is advised to see [Burris and Sankappanavar, 1981] and [McKenzie et al., 1987] and for constraint satisfaction problems and their connection with finite algebras see [Bulatov et al., 2005].

2 Constraint Satisfaction Problems

Definition 2.1 (Constraint Satisfaction Problem). An instance of the Constraint Satisfaction Problem is a triple (V,A,C) with

- V a set of variables
- A a nonempty finite domain
- C a set of constraints $\{C_1, \ldots, C_q\}$ where each constraint $C_i = (\vec{s_i}, R_i)$ consists of:
  - $\vec{s_i}$, an m_i-tuple of variables from V, called the scope of C_i

- R_i a subset of A^{m_i}, called the constraint relation of C_i

We say that an instance of the constraint satisfaction problem has a solution if there is a mapping $f : V \to A$ such that for each $i \leq q$, $f(\vec{s_i}) \in R_i$. The size of an instance of the CSP is the length of a string containing all scopes and all tuples from all constraint relations of the instance.

Definition 2.2 ($CSP(\Gamma)$)**.** For a domain A and a set Γ of finitary relations over A, $CSP(\Gamma)$ is the collection of all instances of the constraint satisfaction problem with constraint relations coming from Γ.

In general the class of CSPs is **NP**-complete, however, by restricting the constraint relations we can find tractable subclasses of CSPs.

Definition 2.3 ((Globally) Tractable)**.** We call a constraint language Γ globally tractable if there is a polynomial time algorithm that uniformly solves all instances from $CSP(\Gamma)$. We call a constraint language Γ tractable if each finite subset Γ' of Γ is globally tractable.

The natural next step and a problem of current interest is to attempt to classify all of the (globally) tractable constraint languages. In an effort to solve this problem, translations of algebras into CSPs and vice versa have been developed. In this manner we can talk about the complexity of an algebra. This translation allows us to use techniques from universal algebra to classify CSPs and find tractable subclasses based on certain properties of algebras.

Definition 2.4 (Algebra)**.** An algebra $\mathbf{A}$ is a pair $\mathbf{A} = \langle A, F \rangle$ where A is a non-empty set and F is a set of finitary operations on A.

A function $f \in F$ on a set A is idempotent if for all $x \in A$, $f(x, x, \ldots, x) = x$. An algebra $\mathbf{A}$ is idempotent if all of its basic operations $f \in F$ are idempotent.

Definition 2.5. A **variety** is a collection of algebras of the same signature that is closed under taking homomorphic images, subalgebras and direct products. The variety generated by an algebra $\mathbf{A}$, is the closure of $\mathbf{A}$ under taking homomorphic images of subalgebras of cartesian powers of $\mathbf{A}$ and is denoted $V(\mathbf{A})$ or $HSP(\mathbf{A})$. A **subvariety** is a subset of a variety closed under taking homomorphic images, subalgebras and direct products.

For the remainder of the thesis we will concern ourselves only with finite algebras. This is because, for our purposes, we are concerned only with CSPs over a finite domain.

Definition 2.6 ($\Gamma_{\mathbf{A}}$)**.** For $\mathbf{A}$, a finite algebra, define $\Gamma_{\mathbf{A}}$ to be the collection of all subuniverses of finite cartesian powers of $\mathbf{A}$.

This translation from algebras to CSPs allows us to talk about the tractability of an algebra. CSP($\Gamma_{\mathbf{A}}$) is the collection of all instances of the Constraint Satisfaction Problem where $\mathbf{A}$ is the domain and the constraint relations are members of $\Gamma_{\mathbf{A}}$. We call an algebra $\mathbf{A}$ (globally) tractable if the constraint language $\Gamma_{\mathbf{A}}$ is (globally) tractable. This leads to the problem of classifying all (globally) tractable algebras. Although we do not confine ourselves to idempotent algebras in this paper, the reader should note that by the work of Bulatov, Jeavons and Krohkin in [Bulatov et al., 2005] it is enough to consider only idempotent algebras.

3 Intersection Properties & Types

Definition 3.1 (k-minimal)**.** For $k > 0$, an instance (V,A,C) of the CSP is k-minimal if:

- every k-element subset of V is within the scope of some constraint $C_i \subseteq C$
- for every set $I \subseteq V$ with $|I| \leq k$ and for every pair of constraints $C_i = (s_i, R_i)$ and $C_j = (s_j, R_j)$ whose scopes contain I, the projections of the constraint relations R_i, R_j onto I are the same.

Definition 3.2 (width k)**.** An algebra $\mathbf{A}$ has relational width k if all k-minimal instances of $CSP(\Gamma_{\mathbf{A}})$, with non-empty constraint relations, have a solution.

If an algebra $\mathbf{A}$ has relational width k for some $k > 0$, then $\mathbf{A}$ is globally tractable [Bulatov and Jeavons, 2001]. Further narrowing our scope another problem of note is to determine all algebras with finite relational width k for some $k > 0$. The reader should note that there are a few different notions of width that have been developed in the literature of CSPs (see [Larose and Zádori, 2007]), we will only use the relational width as defined above. A notion partially arising from, and definitely related to width, are intersection properties.

Definition 3.3 (k-equal)**.** Let $n > 0$ and A_i be sets for $1 \leq i \leq n$. For $k > 0$ and $B, C \subseteq \prod_{1 \leq i \leq n} A_i$ we say that B and C are k-equal, denoted $B =_k C$, if for every subset I of $\{1, 2, \ldots, n\}$ of size k, the projection of B and C onto the coordinates I are equal.

Definition 3.4. Let $\mathbf{A}$ be an algebra and $k > 0$.

1. For $n > 0$ and $\mathbf{B}$ a subalgebra of $\mathbf{A}^n$, we denote the set of all subuniverses C of $\mathbf{A}^n$ with $C =_k B$ by $[\mathbf{B}]_k$.

2. $\mathbf{A}$ has the **k-intersection property** if for every $n > 0$ and subalgebra $\mathbf{B}$ of $\mathbf{A}^n$, $\bigcap [\mathbf{B}]_k \neq 0$.

3. $\mathbf{A}$ has the **strong k-intersection property** if for every $n > 0$ and subalgebra $\mathbf{B}$ of $\mathbf{A}^n$, $\bigcap [\mathbf{B}]_k =_k B$.

These properties were developed by Valeriote in [Valeriote, 2006] to further the classification of algebras within the context of CSPs as seen by the following theorem.

Lemma 3.5 ([Valeriote, 2006]). *Let $\mathbf{A}$ be a finite algebra. If $\mathbf{A}$ has relational width k for some $k > 0$ then $\mathbf{A}$ satisfies the k-intersection property.*

Another method observed in the attempt to classify all tractable finite algebras is to use the local invariants of algebras as developed in tame congruence theory. The standard and principal reference for tame congruence theory is Hobby and McKenzie's work [Hobby and McKenzie, 1988]. Another reference source is [Clasen and Valeriote, 2002]. In this thesis we will introduce only the most basic details to serve our purpose, although a good understanding is necessary to verify the work referenced here.

Locally, every algebra, on the level of minimal sets or neighbourhoods, has one of the following five types:

1. unary

2. affine / vector-space

3. 2 element Boolean

4. 2 element lattice

5. 2 element semi-lattice

Every algebra $\mathbf{A}$ has a type set, denoted $typ(\mathbf{A})$, that consists of all i such that type i is witnessed locally on some minimal set of $\mathbf{A}$. Conversely we say a finite algebra $\mathbf{A}$ omits type i if locally that type of behaviour does not occur in $\mathbf{A}$. A variety omits type i if every finite member of it does.

Below we cite three type omitting theorems that closely draw together types, width and the intersection properties.

Theorem 3.6 ([Larose and Zádori, 2007]). *Let* $\mathbf{A}$ *be a finite idempotent algebra. If* $\mathbf{A}$ *has relational width* k *for some* $k > 0$ *then* $V(\mathbf{A})$ *omits types 1 & 2.*

Theorem 3.7 ([Valeriote, 2006]). *Let* $\mathbf{A}$ *be a finite idempotent algebra. If* $\mathbf{A}$ *has the k-intersection property for some* $k > 0$ *then* $V(\mathbf{A})$ *omits types 1 & 2.*

Theorem 3.8 ([Valeriote, 2006]). *Let* $\mathbf{A}$ *be a finite idempotent algebra. If* $\mathbf{A}$ *has the strong k-intersection property for some* $k > 0$ *then* $V(\mathbf{A})$ *omits types 1,2 & 5.*

Definition 3.9. A congruence of an algebra $\mathbf{A}$ is an equivalence relation that is closed under the operations of $\mathbf{A}$. An algebra is congruence distributive if its congruence lattice satisfies the distributive law. A variety is congruence distributive if all of its members are.

Theorem 3.10 ([Hobby and McKenzie, 1988]). *A variety* V *is congruence-distributive iff* $typ(V) \cap \{1, 2, 5\} = 0$ *and all minimal sets* U *in all finite* $\mathbf{A} \in V$ *have* $|U| = 2$.

Given the above theorems, the further work and conjectures of Larose-Zadori and Bulatov, and his own research, Valeriote made the following conjectures in the conclusion of [Valeriote, 2006].

Conjecture 3.11 ([Valeriote, 2006]). If $\mathbf{A}$ is a finite idempotent algebra such that $V(\mathbf{A})$ omits types 1 & 2 then $\mathbf{A}$ satisfies the k-intersection property for some $k > 0$.

Conjecture 3.12 ([Valeriote, 2006]). If $\mathbf{A}$ is a finite idempotent algebra such that $V(\mathbf{A})$ omits types 1, 2 & 5 then $\mathbf{A}$ satisfies the strong k-intersection property for some $k > 0$.

These conjectures, particularly Conjecture 3.12, were the stepping stones for the results that follow.

4 Polin Algebras

In an effort to either give support for, or rule out, these and other conjectures, we look at a particular kind of algebra that sits on the borderline of some of the aforementioned conditions.

Definition 4.1 (Polin Algebra). A Polin algebra $\mathbb{P}$ consists of an "external" Boolean algebra $\mathbb{B}$ with

1. each $a \in \mathbb{B}$ replaced by another Boolean algebra, $\mathbf{S}(a)$
2. every order relation $a \geq b$ replaced by a homomorphism $\xi^a_b : \mathbf{S}(a) \rightarrow \mathbf{S}(b)$
3. every homomorphism being compatible with the order relation. That is, if $a \geq b \geq c$ then $\xi^b_c \circ \xi^a_b = \xi^a_c$ and $\xi^a_a = id_{\mathbf{S}(a)}$.

$\mathbb{P}$ becomes an algebra of type $(2, 1, 1)$ via:

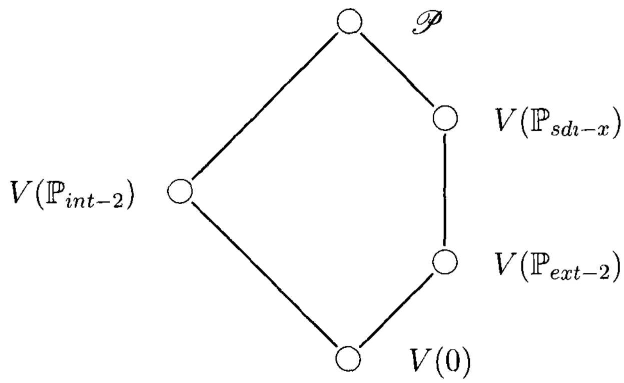
- $(a, s) \cdot (b, t) = (a \cdot b, \xi^a_{ab}(s) \cdot \xi^b_{ab}(t))$ (meet)
- $(a, s)' = (a, s')$ (internal complement)
- $(a, s)^+ = (a', 1)$ (external complement)

Polin's variety $\mathscr{P}$ (the variety of all Polin algebras), is not congruence distributive however, it admits only type 3 minimal sets. These are the primary reasons why, in this paper, we inspect Polin algebras. These two characteristics cause Polin algebras to lie in the middle ground of some of the theorems in Section 3.

A standard result of universal algebra is that a variety is governed by its subdirectly irreducible members [Burris and Sankappanavar, 1981]. Day and Freese give us in [Day and Freese, 1980] that the subdirectly irreducible Polin algebras are:

1. The 2-element external Boolean algebra with 1-element internal Boolean algebras. ($\mathbb{B} = \mathbf{2}$, $\mathbf{S}(1) = \mathbf{S}(0) = \mathbf{1}$) We will denote this algebra as $\mathbb{P}_{ext-2}$.
2. The 1-element external Boolean algebra with 2-element internal Boolean algebra. ($\mathbb{B} = \mathbf{1}$, $\mathbf{S}(0) = \mathbf{2}$) We will denote this algebra as $\mathbb{P}_{int-2}$.
3. An arbitrary external Boolean algebra with the maximal or top element of the Boolean algebra having a 2-element internal Boolean algebra and all other internal Boolean algebras being 1-element. ($\mathbb{B}$ is any Boolean, $\mathbf{S}(1) = \mathbf{2}$ and $\mathbf{S}(a) = \mathbf{1}$ for all $a < 1$) We will denote any of these algebras as $\mathbb{P}_{sdi-x}$; where $x = 2^n + 1$ for some $n \geq 1$, is the size of the algebra.

For any $x = 2^n + 1$ and $y = 2^m + 1$ where $m, n \geq 1$, it is not hard to see that $\mathbb{P}_{sdi-x} \in V(\mathbb{P}_{sdi-y})$. Also, it is clear that $\mathbb{P}_{ext-2} \in \mathbb{P}_{sdi-y}$ and that $\mathbb{P}_{int-2} \notin \mathbb{P}_{sdi-y}$. This gives us the subvariety structure of Polin's variety.



In the remainder of the paper we will be dealing with Polin algebras of an order small enough that it is simpler to forgo the formal definition of an element denoted by it's external and internal status (i.e. $(a, s) \in \mathbb{P}$) and merely define each element as a single entity (i.e. $a \in \mathbb{P}$).

Definition 4.2. Let a be an element of some Polin algebra $\mathbb{P}$. We call a^{++} the **support** of a. All elements within a particular internal Boolean algebra of $\mathbb{P}$ will the have the same support. The supports of all elements in $\mathbb{P}$ compose the **representative elements**. The representative elements make up the external Boolean algebra of $\mathbb{P}$ denoted $\mathbb{P}^{++}$.

Lemma 4.3. *For* $\mathbf{A}$, $\mathbf{B} \leq \mathbb{P}^n$, *if* $\mathbf{A} =_k \mathbf{B}$ *for some* $k \geq 2$ *then* $\mathbf{A}^{++} = \mathbf{B}^{++}$

Proof. One can see that if $\mathbf{A} =_k \mathbf{B}$ then $\mathbf{A}^{++} =_k \mathbf{B}^{++}$. However, both $\mathbf{A}^{++}$ and $\mathbf{B}^{++}$ are Boolean algebras, thus we can conclude by the Baker-Pixley theorem [Baker and Pixley, 1975] that $\mathbf{A}^{++} = \mathbf{B}^{++}$. □

Definition 4.4 ($\mathbf{A}_{\langle i,j\rangle}$,$(x, y)_{\langle i,j\rangle}$)**.** For $\mathbf{A} \leq \mathbb{P}^n$ and $1 \leq i, j \leq n$, let $\mathbf{A}_{\langle i,j\rangle}$ denote the projection of $\mathbf{A}$ on to the set of coordinates $\{i, j\}$. Let $(x, y)_{\langle i,j\rangle}$ denote that (x, y) is a member of $\mathbf{A}_{\langle i,j\rangle}$.

5 Results

In an effort to see whether or not Polin algebras satisfied the strong-k intersection property, we began by investigating the smallest Polin algebras. Since both of the two element Polin algebras are Boolean, the first algebras of interest were the three element Polin algebras.

Definition 5.1. Let $\mathbf{P}$ represent the 3-element Polin algebra $\langle\{0, 1, 2\}, ', {}^{+}\wedge\rangle$ where the operations are defined as:

$$\mathbf{P} = \begin{array}{c} \left(\begin{array}{c}\mathbf{2}\\ 1\end{array}\right) \\ \downarrow \\ (\mathbf{0}) \end{array}$$

$'$	
0	0
1	2
2	1

$+$	
0	2
1	0
2	0

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

Note that $\mathbf{P} \equiv \mathbb{P}_{sdi-3}$.

Theorem 5.2. **P** *satisfies the strong 2-intersection property.*

Proof. We want to show that if any two subalgebras of $\mathbf{P}^n$ are 2-equal, then they are 2-equal to their intersection. In symbols: want to show that if $\mathbf{A}, \mathbf{B} \leq \mathbf{P}^n$ such that $\mathbf{A} =_2 \mathbf{B}$ then $\mathbf{A} =_2 \mathbf{A} \cap \mathbf{B} =_2 \mathbf{B}$.

Let $\alpha = \{(0,0),(0,1),(0,2),(1,0),(2,0),(1,1),(2,2)\}$. If $\mathbf{A}, \mathbf{B} \leq \mathbf{P}^n$ and $\mathbf{A} =_2 \mathbf{B}$ then $\mathbf{A}_{\langle i,j\rangle} \cap \alpha = (\mathbf{A} \cap \mathbf{B})_{\langle i,j\rangle} \cap \alpha$ for $1 \leq i,j \leq n$. This follows from Lemma 4.3 and the fact that the set α consists of the 2-fold projections of all representative elements of $\mathbf{P}^n$ and their internal complements. Thus if $\mathbf{A}$ or $\mathbf{B}$ witnesses the projection $(x,y)_{\langle i,j\rangle}$ where $(x,y) \in \alpha$, then $\mathbf{A} \cap \mathbf{B}$ will have an element that witnesses $(x,y)_{\langle i,j\rangle}$. This implies that we need only check instances of projections $(2,1)_{\langle i,j\rangle}$ & $(1,2)_{\langle i,j\rangle}$

Since $(1,2)'_{\langle i,j\rangle} = (2,1)_{\langle i,j\rangle}$ it suffices to check that if the projection $(2,1)_{\langle i,j\rangle}$ is witnessed in $\mathbf{A}$, that it is also witnessed in $\mathbf{A} \cap \mathbf{B}$.

For the remainder of the proof we will assume that we have algebras $\mathbf{A}, \mathbf{B} \leq \mathbf{P}^n$ with $\mathbf{A} =_2 \mathbf{B}$, that both witness a $(2,1)_{\langle i,j\rangle}$ projection. After a possible reordering of coordinates, we can also assume that $i = 1$ and $j = 2$. Our goal is now to show that $\mathbf{A} \cap \mathbf{B}$ witnesses $(2,1)_{\langle 1,2\rangle}$. We will do this by induction on n. The base case where $n = 2$ is clear. Now assume true for all n, prove true for $n+1$.

Case 5.2.1. A $(2,1)_{\langle 1,2\rangle}$ projection comes from an element of the form $a = (2,1,a_3,\ldots,a_{n+1})$ in $\mathbf{A}$ or $\mathbf{B}$ where $a_i = 0$ for some $3 \leq i \leq n+1$.

Without loss of generality assume that $a_{n+1} = 0$ and that $a \in \mathbf{A}$, so that $a = (2,1,a_3,\ldots,a_n,0) \in \mathbf{A}$. By the above Lemma 4.3 $a^{++} \in \mathbf{A} \cap \mathbf{B}$. By our induction hypothesis there exists elements in $\mathbf{A}$ & $\mathbf{B}$ that witness $(2,1)_{\langle 1,2\rangle}$ and differ on at most any one coordinate $3 \leq i \leq n+1$. Call these $f = (2,1,f_3,f_4,\ldots,f_n,x) \in \mathbf{A}$ and $\tilde{f} = (2,1,f_3,f_4,\ldots,f_n,y) \in \mathbf{B}$ that differ only in the $n+1^{th}$-coordinate. $f \wedge a^{++} \in \mathbf{A}$, $\tilde{f} \wedge a^{++} \in \mathbf{B}$ and $f \wedge a^{++} = \tilde{f} \wedge a^{++}$ which implies that $\mathbf{A} \cap \mathbf{B}$ witnesses $(2,1)_{\langle 1,2\rangle}$.

Case 5.2.2. All $(2,1)_{\langle 1,2\rangle}$ projections of both $\mathbf{A}$ & $\mathbf{B}$ come from elements of the form $(2,1,a_3,\ldots,a_{n+1}) \in \mathbf{A}$ and $(2,1,b_3,\ldots,b_{n+1}) \in \mathbf{B}$ where $a_i, b_i \in \{1,2\}$ for all i.

An event of this nature can be broken down into two subcases based on the structure of the external algebra of both **A** & **B**.

Subcase 5.2.2.1. The external Boolean algebra of **A** & **B** is the two element Boolean algebra. That is to say, the support of any element in **A** or **B** is $(2, 2, \ldots, 2)$ or $(0, 0, \ldots, 0)$.

In this case the internal Boolean algebra $\mathbf{A}\backslash\{(0, 0, \ldots, 0)\}$ is 2-equal to the internal Boolean algebra $\mathbf{B}\backslash\{(0, 0, \ldots, 0)\}$. By the Baker-Pixley Theorem [Baker and Pixley, 1975] we can conclude that $\mathbf{A}\backslash\{(0, 0, \ldots, 0)\} = \mathbf{B}\backslash\{(0, 0, \ldots, 0)\}$. Thus any element with a $(2, 1)_{\langle 1,2\rangle}$ projection is a member of both **A** & **B**.

Subcase 5.2.2.2. The external Boolean algebra of **A** & **B** is not the two element Boolean algebra.

A simple, but not completely trivial inspection, reveals that the external Boolean algebra of **A** & **B** will be the four element Boolean algebra, else our analysis becomes an instance of Case 1. The external Boolean algebra will have as representative elements: $(2, 2, \ldots, 2)$, $(0, 0, \ldots, 0)$, $(2, 0, x_3, \ldots, x_{n+1})$, $(0, 2, x_3^+, \ldots, x_{n+1}^+)$.

By our inductive hypothesis we have elements in **A** & **B** that witness $(2, 1)_{\langle 1,2\rangle}$ and only possibly differ in the 3^{rd} coordinate. We will call these $c = (2, 1, x, c_4, \ldots, c_{n+1}) \in \mathbf{A}$ and $\tilde{c} = (2, 1, y, c_4, \ldots, c_{n+1}) \in \mathbf{B}$. $y = x'$ and $y, x \neq 0$ else we are done. Also in our representative elements, clearly either x_3 or x_3^+ equals 2. Assume that $x_3^+ = 2$ and $x_3 = 0$ giving us the element $(0, 2, 2, \ldots, x_{n+1}^+)$; if this was not the case we could just reorder the first two indices to make it so without affecting our $(2, 1)_{\langle 1,2\rangle}$ projection.

Since $\mathbf{A} =_2 \mathbf{B}$, we can guarantee certain other elements in both **A** & **B**. For example there must exist elements that witness $(2, x')_{\langle 1,3\rangle}$ and $(2, x)_{\langle 1,3\rangle}$ that differ on only the 2^{nd} coordinate if any. This gives:

$$\begin{aligned} d &= (2, p, x', d_4, \ldots, d_{n+1}) \in \mathbf{A} & \tilde{d} &= (2, q, x', d_4, \ldots, d_{n+1}) \in \mathbf{B} \\ e &= (2, r, x, e_4, \ldots, e_{n+1}) \in \mathbf{A} & \tilde{e} &= (2, s, x, e_4, \ldots, e_{n+1}) \in \mathbf{B} \end{aligned}$$

Note that in the above elements, the second coordinate (i.e. p, q, r, s) cannot be 0 due to our external Boolean structure (i.e. the fact that the first and third coordinates are non-zero). Now $c \wedge d \wedge e = (2, 1, 1, c_4 \wedge d_4 \wedge e_4, \ldots, c_{n+1} \wedge d_{n+1} \wedge e_{n+1}) = \tilde{c} \wedge \tilde{d} \wedge \tilde{e}$. This gives us that $(2, 1)_{\langle 1,2\rangle}$ is witnessed in $\mathbf{A} \cap \mathbf{B}$

□

By a similar construction to the above, we can see that the other two unique three element Polin algebras also satisfy the strong 2-intersection property. (The proof is very similar with the exception that all projections of interest are settled by a fact similar to Case 5.2.1. That is to say, all projections of interest are 'pulled down' into the lowest possible internal Boolean algebra.) However, when we inspected an idempotent reduct of **P** that maintained omission of types 1,2 & 5, we found an example that failed the strong k-intersection property for all k.

Definition 5.3 ($\mathbf{P}_{id}$). Let $\mathbf{P}_{id}$ represent the reduct of $\mathbf{P}$ $\langle\{0,1,2\},\wedge,f_1,f_2\rangle$. Where $\mathbf{P}_{id}$ is of type $\langle 2,3,3\rangle$ and f_1 & f_2 are defined as follows.

$$f_1(x,y,z) = (x^+ \wedge z)^+ \wedge (z^+ \wedge x)^+ \wedge y$$
$$f_2(x,y,z) = (x' \wedge y' \wedge z')' \wedge (x' \wedge y \wedge z)' \wedge (x \wedge y' \wedge z)' \wedge (x \wedge y \wedge z')'$$

$\mathbf{P}_{id}$ is an idempotent algebra and by Theorems 9.11 and 9.14 of [Hobby and McKenzie, 1988] we have that $typ(\mathbf{P}_{id}) = \{3\}$. Using the UACalc computer software [Freese and Kiss, 2008] we computed the 2 and 3 generated free algebras in the variety generated by **P** and from each of these algebras isolated all of the idempotent terms. Similarly we computed all binary and ternary terms generated by the basic operations of $\mathbf{P}_{id}$. We compared these two sets of terms and determined that $\mathbf{P}_{id}$ witnesses all binary and ternary idempotent terms of **P**. It is open whether $\mathbf{P}_{id}$ is the full idempotent reduct of **P**, however we suspect this may be the case.

Fact 5.4. $f_1(0,2,0) = 2$, $f_1(2,2,2) = f_2(2,2,2) = 2$, otherwise if $x,y,z \in \{0,2\}$ then $f_1(x,y,z) = f_2(x,y,z) = 0$.

Fact 5.5. If $x,y,z \in \{1,2\}$ then $f_1(x,1,z) = 1$, $f_1(x,2,z) = 2$ and $f_2(x,y,z) \in \{1,2\}$.

Theorem 5.6. $\mathbf{P}_{id}$ *fails the strong k-intersection property for any $k > 1$.*

Proof. Fix a $k > 1$. Let **A** be the subset of $\mathbf{P}_{id}^{k+1}$ defined by:

$$\begin{aligned}\mathbf{A} = \quad & \{(2,2,\ldots,2,2)\} \cup \\ & \{(x_1,x_2,\ldots,x_{k+1}) \mid \; x_i = 0 \text{ for some } i < k+1, \\ & \qquad\qquad\qquad\qquad\quad x_j \in \{0,2\} \text{ for } 0 \le j \le k, \\ & \qquad\qquad\qquad\qquad\quad x_{k+1} \in \{1,2\}\}\end{aligned}$$

Note that **A** is closed under $\wedge$. Let $a = (2,2,\ldots,2,2) \in \mathbf{A}$ and $a_x, a_y, a_z \in \mathbf{A}$ such that $a_y \neq a$. By Fact 5.4 and Fact 5.5 one can see that $f_1(a_x,a_y,a_z)$, $f_2(a_x,a_y,a_z) \in \mathbf{A}\backslash a$. If $a_y = a$ and either $a_x \neq a$ or $a_z \neq a$, then again by Fact 5.4 and Fact 5.5 $f_1(a_x,a,a_z), f_2(a_x,a,a_z) \in \mathbf{A}\backslash a$, with the exception of when $a_x = a_z$ over the first k coordinates (e.g. $a_x = a_z = (0,0,2,2,\ldots,2,1/2)$), in which case $f_1(a_x,a,a_z) = a$ and $f_2(a_x,a,a_z) \in \mathbf{A}\backslash a$. Finally, if $a_x = a_y =$

$a_z = a$ then $f_1(a,a,a) = f_2(a,a,a) = a$. Thus $\mathbf{A}$ is a subalgebra of $\mathbf{P}_{id}^{k+1}$. Define $\mathbf{B}$, a similar subset of $\mathbf{P}_{id}^{k+1}$ by:

$$\begin{aligned}\mathbf{B} = \ & \{(2,2,\ldots,2,1)\} \cup \\ & \{(x_1, x_2, \ldots, x_{k+1}) \mid \ x_i = 0 \text{ for some } i < k+1, \\ & \qquad x_j \in \{0,2\} \text{ for } 0 \leq j \leq k, \\ & \qquad x_{k+1} \in \{1,2\}\}\end{aligned}$$

By similar reasons to the above and by Facts 5.4 and 5.5 one can see that $\mathbf{B}$ is also closed under $\wedge$, f_1 & f_2 and is thus also a subalgebra of $\mathbf{P}_{id}^{k+1}$. By construction $\mathbf{A} =_k \mathbf{B}$. However,

$$\begin{aligned}\mathbf{A} \cap \mathbf{B} = \ & \{(x_1, x_2, \ldots, x_{k+1}) \mid \ x_i = 0 \text{ for some } i < k+1, \\ & \qquad x_j \in \{0,2\} \text{ for } 0 \leq j \leq k, \\ & \qquad x_{k+1} \in \{1,2\}\}\end{aligned}$$

$\mathbf{A}$ and $\mathbf{B}$ have k-fold projection $(2,2,\ldots,2)$ in the first k-coordinates, but, $\mathbf{A} \cap \mathbf{B}$ does not witness the projection $(2,2,\ldots,2)$ in the first k-coordinates.

$$\Rightarrow \mathbf{A} \cap \mathbf{B} \neq_k \mathbf{A}$$ □

That $\mathbf{P}_{id}$ fails the strong k-intersection property provides us with a counterexample to Conjecture 3.12.

Theorem 5.7. *$V(\mathbb{P}_{sdi-x})$ fails the strong k-intersection property for all $k > 1$.*

Proof. We prove this by showing that $\mathbb{P}_{sdi-5}$ fails the strong k-intersection property for all k since this algebra is in every non-Boolean subvariety.

$$\begin{tikzcd} & \begin{pmatrix} \mathbf{2} \\ 1 \end{pmatrix} \arrow[dl] \arrow[dr] & \\ (\mathbf{a}) \arrow[dr] & & (\mathbf{b}) \arrow[dl] \\ & (\mathbf{0}) & \end{tikzcd}$$

$\mathbb{P}_{sdi-5} = \langle \{0,1,2,a,b\}, \wedge, ', ^+ \rangle$ where the operations are defined as:

$\wedge$	0	1	2	a	b
0	0	0	0	0	0
1	0	1	1	a	b
2	0	1	2	a	b
a	0	a	a	a	0
b	0	b	b	0	b

$'$	
0	0
1	2
2	1
a	a
b	b

$+$	
0	2
1	0
2	0
a	b
b	a

Fix a $k > 1$. We will look at two specific subalgebras of $\mathbb{P}^{k+1}_{sd\imath-5}$. Call these subalgebras **A** & **B**. **A** and **B** are generated by the following sets of elements.

$$\mathbf{A} = \left\langle \begin{array}{c} (2,1,1,\dots,1,1), \\ (a,1,1,\dots,1,1), \\ (2,a,1,\dots,1,1), \\ (2,1,a,\dots,1,1), \\ \vdots \quad \ddots \quad \vdots \\ (2,1,1,\dots,a,1), \\ (a,1,1,\dots,1,2), \\ (2,a,1,\dots,1,2), \\ (2,1,a,\dots,1,2), \\ \vdots \quad \ddots \quad \vdots \\ (2,1,1,\dots,a,2) \end{array} \right\rangle \qquad \mathbf{B} = \left\langle \begin{array}{c} (2,1,1,\dots,1,2), \\ (a,1,1,\dots,1,1), \\ (2,a,1,\dots,1,1), \\ (2,1,a,\dots,1,1), \\ \vdots \quad \ddots \quad \vdots \\ (2,1,1,\dots,a,1), \\ (a,1,1,\dots,1,2), \\ (2,a,1,\dots,1,2), \\ (2,1,a,\dots,1,2), \\ \vdots \quad \ddots \quad \vdots \\ (2,1,1,\dots,a,2) \end{array} \right\rangle$$

With the exception of the elements $(2,1,1,\dots,1,1)$ and $(2,1,1,\dots,1,2)$ the generating sets of **A** and **B** are identical. Thus it is easy to see that the generating sets of **A** and **B** are k-equal. Since the generating sets of our two algebras are k-equal, we have that $\mathbf{A} =_k \mathbf{B}$. However, we will show that $\mathbf{A} \cap \mathbf{B}$ does not witness the k-fold projection $(2,1,1,\dots,1)_{\langle 1,2,3,\dots,k\rangle}$

The internal algebras that could possibly witness a $(2,1,1,\dots,1)_{\langle 1,2,3,\dots,k\rangle}$ projection are those with support $(2,2,2,\dots,2,x)$ where $x \in \{0,2,a,b\}$. However, since our generating sets only have 1's and 2's in the $k+1^{th}$ coordinate, $x \neq a, b$.

An inspection of the internal algebra with support $(2,2,2,\dots,2,2)$ reveals that our generating sets generate no new elements. That is to say, the particular internal Boolean algebra of **A** has only the elements $(2,1,1,\dots,1,1)$, $(2,2,2,\dots,2,2)$ and their respective internal complements. Similarly, **B** has only the elements $(2,1,1,\dots,1,2)$, $(2,2,2,\dots,2,2)$ and their respective internal complements. So no new elements witnessing $(2,1,1,\dots,1)_{\langle 1,2,3,\dots,k\rangle}$ are generated in the internal algebras with support $(2,2,2,\dots,2,2)$.

The next step is to inspect the internal algebra with support $(2,2,2,\dots,2,0)$. However, we will show that our generating sets do not generate any elements with support $(2,2,2,\dots,2,0)$. To see this we inspect the external Boolean algebras of **A** and **B**. The external Boolean algebras of **A** and **B** are generated by the representative elements of our generating sets. One can see that the set of representative elements of the generating sets of **A** and **B** are equivalent.

The representative elements of our generating sets are:

$$\begin{array}{rl} & \{(2,2,2,\ldots,2,2)\} \\ \cup & \{(x_1,x_2,x_3,\ldots,x_k,x_{k+1}) \mid x_i = a \text{ for some } i \neq k+1, x_j = 2 \text{ otherwise}\} \end{array}$$

If we take the closure of this set under the meet operation of our algebra we obtain the set:

$$\{(x_1,x_2,x_3,\ldots,x_k,x_{k+1}) \mid x_i \in \{a,2\}\ \forall\, i \leq k, x_{k+1} = 2\}$$

The closure of this set under the external complement operation is:

$$\begin{array}{rl} & \{(x_1,x_2,x_3,\ldots,x_k,x_{k+1}) \mid x_i \in \{a,2\}\ \forall\, i \leq k, x_{k+1} = 2\} \\ \cup & \{(x_1,x_2,x_3,\ldots,x_k,x_{k+1}) \mid x_i \in \{b,0\}\ \forall\, i \leq k, x_{k+1} = 0\} \end{array}$$

We can see that the above set is closed under the meet operation and thus gives the external Boolean algebra. Note that we are working with representative elements and thus need not check closure under the internal complement operation. We can see that our external Boolean algebra does not contain the element $(2,2,2,\ldots,2,0)$.

Thus we conclude that $\mathbf{A} \cap \mathbf{B}$ does not witness $(2,1,1,\ldots,1)_{\langle 1,2,3,\ldots,k\rangle}$ and $\mathbf{A} \neq_k \mathbf{A} \cap \mathbf{B} \neq_k \mathbf{B}$ □

Corollary 5.8. *The variety generated by any reduct of $\mathbb{P}_{sdi-x}$ for some valid x, fails the strong k-intersection property for all $k > 1$.*

The above corollary is due to the fact that the variety generated by any reduct of $\mathbb{P}_{sdi-x}$ will still contain the pair of subalgebras $\mathbf{A}$ and $\mathbf{B}$, for any $k > 1$, used in Theorem 5.7.

If we strengthen $\mathbf{P}$ by adding an additional unary operation * defined as:

*	
0	1
1	2
2	2

we obtain a new algebra denoted $\mathbf{P}^*$. $\mathbf{P}^*$ still generates a variety that admits only type 3 minimal sets and is not congruence distributive. If we inspect a particular homomorphic image of $(\mathbf{P}^*)^2$ that looks and behaves exactly like $\mathbb{P}_{sdi-5}$ under the $\wedge, '$ and $^+$ operations, but has the additional * operation, we can see that our examples $\mathbf{A}$ and $\mathbf{B}$ from Theorem 5.7 remain unchanged as subalgebras under the additional * operation. This gives us that the variety generated by $\mathbf{P}^*$ (a strengthened version of $\mathbf{P}$ that maintains our desired conditions) fails the strong k-intersection property.

6 Conclusion

Given we now know that omitting types 1, 2 and 5 does not characterize the strong k-intersection property, it raises the question of whether there is a characterization in terms of tame congruence theory. The next logical possibility to inspect appears to be congruence distributivity. We currently have neither direction in this regard however, we do have some work that falls close with 3.10 and the failure of 3.12. Also, E. Kiss and M. Valeriote have shown that if a finite idempotent algebra is in $CD(3)$ then it satisfies the strong 2-intersection property. (An algebra is in $CD(k)$ for some $k > 0$, and by Jónsson generates a congruence distributive variety, if it has a sequence of $k + 1$ Jónsson terms [Valeriote, 2006].) A practical approach may be to look for more varieties that are not congruence distributive but are of type $\{3, 4\}$ and test whether they satisfy strong k-intersection or not.

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