SELFISH ROUTING WITH OBLIVIOUS USERS

GAME THEORETICAL ANALYSIS

FOR

SELFISH ROUTING WITH OBLIVIOUS USERS

By

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A Thesis

Submitted to the School of Graduate Studies

in Partial Fulfillment of the Requirements

for the Degree

Master of Science

McMaster University

MASTER OF SCIENCE (2008) McMaster University

COMPUTING AND SOFTWARE Hamilton, Ontario

TITLE: Game Theoretical Analysis for Selfish Routing with Oblivious Users

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NUMBER OF PAGES: ix , 61

Abstract

We extend the original selfish routing setting by introducing users who are oblivious to congestion. Selfish routing captures only the behavior of selfish users, who choose the cheapest route based on the current traffic congestion without caring about the effects of their routing on their fellow users. However, it is more likely that a certain number of network users will be oblivious to congestion. For example, data from low-level QoS services will be routed on predefined routes with no adaptability to network congestion, while data from high-level QoS services will be routed on the fastest paths available. Networks with selfish users may lead to a stable state or Nash equilibrium, where no selfish users can decrease his or her travel time by changing his or her route unilaterally. Traffic equilibria refer to Nash equilibria in networks only with selfish users, and oblivious equilibria refer to Nash equilibria in networks with both oblivious users and selfish users. We study the performance degradation of networks at oblivious equilibrium with respect to the optimal performance.

Our model has a fraction α of oblivious users, who choose predefined shortest paths on the network, and the remaining are selfish users. Considering networks with linear latency functions, first we study parallel links networks with two nodes, and then general topologies. We provide two tight upper bounds of the price of anarchy, which is the ratio of the worst total cost experienced by both oblivious users and selfish users, over the optimal total cost

when all users are centrally coordinated. Our bounds depend on network parameters such as α, the total demand, the latency functions, the total cost of a traffic equilibrium flow, and the total cost of an optimal flow. The dependency of our bounds on network parameters seems to be inevitable considering the fact that the price of anarchy can be arbitrary large depending on network parameters as oblivious users may choose an arbitrarily expensive path.

Acknowledgements

I would like to appreciate all help and support for me to complete this thesis. First of all, I could not have finished my research and thesis without the help of my supervisor, George Karakostas. I thank him for teaching me to fish, instead of getting me a fish. Second, I express my gratitude to my colleagues who supported my research. Especially, Hao Xia gave me valuable cooperation in research. In the end, I am grateful to my mother, sister, and Li Liu, who are wishing me the best luck always.

Hamilton, Ontario, Canada — Taeyon Kim

May, 2008

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Chapter 1

Introduction

1.1 Selfish routing with oblivious users

When a commuter drives a car, just before leaving home or work, he or she probably makes a sensible decision to choose the fastest path to the destination considering the current traffic congestion. To choose a route, he or she considers his or her travel time, but does not care about others' travel time who will be delayed by his or her action. We use the term *selfish users* to refer to non-cooperative network users, who choose the cheapest path connecting one's origin to one's destination based on the current traffic congestion, without considering the effects of their routing on others. Selfish routing captures this behavioral aspect of selfish users in network routing. We use a game theoretical framework to model selfish routing.

Game theory provides a framework to model and analyze situations where each rational player chooses the best strategy, based on the others' strategies, to optimize what he or she can gain. For example, it can be used to analyze equilibrium problems such as producers and consumers in markets, auctions, load balancing, network routing, and so on. Games may lead to a stable state, or Nash equilibrium, where no player can gain anything by changing his or her strategy unilaterally. Specifically, non-cooperative games, where individual players are in competition with each other, can be used to model and analyze selfish routing. The network on which the game is being played is usually called a *traffic network.*

We consider non-atomic selfish routing games, which have infinitely many selfish users, each carrying an infinitesimal amount of flow. In non-atomic cases, we suppose that one user switching routes gives a negligible change to the current traffic congestion since each user carries a very small amount of flow. Selfish routing may lead to a traffic equilibrium, where no selfish user can reduce his or her travel time by changing his or her route unilaterally. We use the term *traffic equilibrium* for flows at Nash equilibrium in traffic networks. Wardop's principle [26] for selfish routing states that

> at equilibrium, for each origin-destination pair the travel costs on all the routes actually used are equal to or less than the travel costs on all unused routes.

One interesting question is how much worse the performance of a network at traffic equilibrium is than the optimal performance on the same network. The performance of a network is measured by the sum of all user travel times, also called the total cost or social cost. It is a well-known fact that the performance of a network at traffic equilibrium may not be optimal. Koutsoupias and Papadimitriou [17] initiated the study of the coordination ratio, also called the *price of anarchy.* In selfish routing, the price of anarchy is the worst ratio of the total cost experienced by selfish users over the optimal performance that would be achieved if all users were coordinated by a central authority. Roughgarden and Tardos [24] showed that, in networks with linear latency functions, the price of anarchy is bounded by 4/3.

In the past, several variations of the original selfish routing setting have been studied. For example, Roughgarden [20] studied Stackelberg scheduling strategies, which have a combination of selfish users and centrally coordinated users. The latter route their flow with the aim of improving the price of anarchy. Karakostas and Viglas [16] studied the combination of selfish users and malicious users. The latter use their traffic in an effort to maximize traffic congestion. Those extended models capture different behavioral aspects of users on the same network. We also extend the original selfish routing setting by introducing oblivious users.

Our extended model is based on the following observation: The original selfish routing is based on the impractical assumption that every user can measure latencies of all possible paths connecting one's origin to one's destination, and that the measurement can be performed instantly. But this assumption may be unrealistic in large-scaled networks such as the Internet. It is more realistic that a certain number of users may consult a routing table based on predefined network parameters such as physical distances or hops between routers. A similar phenomenon is observed when travelers choose a route by observing the shortest path on a map without knowing (or caring) about traffic congestion on routes. We call such users *oblivious to congestion.*

More specifically, our model has a fraction α of oblivious users, and a fraction $1 - \alpha$ of selfish users. We consider networks with linear latency functions. Every edge e has a latency function, $l_e(f_e) = a_e f_e + b_e$, for the traffic delay caused (and experienced) by the edge flow f_e. While oblivious users choose a predefined shortest path on the network without any flow, i.e., the shortest path when we define edge distances as $l_e(0) = b_e$, selfish users minimize their own travel cost on the same network. We use the term *oblivious equilibrium* for flows with a fraction $\alpha > 0$ of oblivious users, and selfish users at Nash equilibrium. First we study parallel links networks with two nodes, and then general topologies. As the result of the study, we provide two upper bounds of the price of anarchy at oblivious equilibrium in both cases, and tight examples for them. Our bounds depend on network parameters such as the fraction α of oblivious users, the total flow, the coefficients of latency functions, the total cost of a Nash flow (a flow at Nash equilibrium), and the total cost of an optimal flow. The dependency of our bounds on the network parameters seems to be unavoidable considering the following fact: the price of anarchy at oblivious equilibrium can be arbitrary large depending on the network parameters as oblivious users can use an arbitrarily expensive path if the path is the shortest when no flows are circulating.

1.2 Previous work

In selfish routing, one of the fundamental questions is to show the existence and uniqueness of traffic equilibria. Schmeidler [25] proved the existence of a Nash equilibrium in non-atomic games for mixed and for pure strategies. In mixed strategies, users have a probability distribution over strategies (or routes), and gain the expected payoff that depends on the individual strategy and on other players' strategies. In pure strategies, users choose a strategy with probability 1. For the equilibrium existence in pure strategies, he additionally assumed the anonymity of users, i.e., individual payoffs depend only on the aggregation of users' strategies without caring about users' identities. Aashtiani and Magnanti [1] studied the existence and uniqueness of traffic equilibria in a generalized network model, which accommodates multiple origin-destination pairs, positive continuous path latency functions, nonnegative continuous demand functions of its shortest travel time, and general link congestion effects on other links. The existence of traffic equilibrium in the model is established using Brouwer's fixed-point theorem. They also showed that travel times are unique under monotonicity conditions of path latency functions and demand functions, and edges carry a uniquely determined amount of flow in networks with strictly monotone latency functions and positive demand functions.

Koutsoupias and Papadimitriou [17] initiated the study of the price of anarchy in selfish routing. Roughgarden and Tardos [24] studied the price of anarchy in the additive model, where path latencies are the sum of the latencies of all edges on the path. They showed that, in networks with linear latency functions, the price of anarchy is bounded by 4/3. In networks with general latency functions, they showed that the total cost experienced by selfish users is no more than the optimal total cost of the same instance having twice as much demand. Roughgarden [21] also showed that the price of anarchy is independent of the network topology. In other words, the simple network topology with two nodes and two parallel links always provides the worst examples of the price of anarchy for an arbitrary class of standard latency functions. A latency function l is standard if $x \cdot l(x)$ is convex on $[0, \infty)$ [21].

There have been efforts to study the price of anarchy in generalized models. Perakis [19] studied the price of anarchy in networks with non-separable, and asymmetric latency functions. Latency functions are non-separable in the sense that edge latencies also depend on the flows on other edges, and they are asymmetric in the sense that different non-cooperative users' strategies (routes) affect their cost differently. She showed that, in networks with linear latency functions, the price of anarchy is bounded by $\frac{4}{4-c^2}$, where c^2 is the degree of asymmetry, and has value 1 for symmetric latency functions. On the other hand, Correa et al. [8] studied the price of anarchy in networks with capacities and latency functions that are continuous, and nondecreasing. They proved that the price of anarchy is at most $(1 - \beta(\mathcal{L}))^{-1}$, where $\mathcal{L}$ is a family of continuous nondecreasing latency functions, and $\beta(\mathcal{L}) = \sup_{l \in \mathcal{L}} \sup_{v \geq 0} \frac{1}{vl(v)} \max_{x \geq 0}\{x\,(l(v) - l(x))\}$. They computed $(1 - \beta(\mathcal{L}))^{-1}$ for several classes of latency functions. For linear latency functions, $(1 - \beta(\mathcal{L}))^{-1}$ is $\frac{4}{3}$.

Selfish behavior of network users can be regulated to correspond to an optimal flow by imposing the so-called optimal taxes on network edges. Here, an optimal flow minimizes the total latency without including taxation in the network. Beckmann et al. [4], and Dafermos and Sparrow [10] showed that an optimal flow f with edge latency functions $l_e(f_e)$ is at Nash equilibrium with marginal cost functions $l_e^*(f_e) = l_e(f_e) + l_e'(f_e)f_e$, where $x \cdot l_e(x)$ is a convex function for each edge e, and l_e' denotes the derivative $\frac{d}{dx}l_e(x)$ of l_e. The marginal cost function has the first term $l_e(f_e)$ describing the per-unit latency experienced by the edge flow f_e, and the second term $l_e'(f_e)f_e$ capturing the increased congestion caused by the flow. It is well known that marginal cost pricing, i.e., imposing $l_e'(f_e)f_e$ on network edges as taxes, regulates selfish users to correspond to an optimal flow. Furthermore, Yang and Huang [27] (see also [14], [12]) studied pricing network edges for multiple classes of selfish users (or called heterogeneous selfish users), who have a different sensitivity to taxes according to their classes. Each class has an associated value-of-time (VOT), which describes how users trade off travel times for tax charges, and vice versa. Because the time-based and money-based equilibrium problems do not necessarily give the same equilibrium solution when VOTs are nonlinear, for both equilibrium problems, they showed the existence of a set of optimal taxes in multi-commodity networks when latency functions are differentiable, convex, and monotonically increasing.

As an extended study of pricing network edges, Cole et al. [7] studied how much taxes can decrease the total disutility, which includes the total latency and the total taxation in the network. They showed that, in networks with linear latency functions, the marginal cost pricing does not decrease the total disutility of a Nash flow. They also showed that levying taxes can not improve the total disutility of a Nash flow more than edge removals. Then they showed that there is an n-node network with nonlinear latency functions in which imposing taxes, or edge removals, can improve the total disutility of a Nash flow by a $\lfloor \frac{n}{2} \rfloor$ factor. In addition, Karakostas and Kolliopoulos [15] studied how much taxation has an influence on the coordination ratio in multi-commodity networks with heterogeneous selfish users. They studied the coordination ratio of the total disutility over the social cost including the total taxation, and showed that the ratio when taxation is used is better than the ratio when no taxation is used for several classes of latency functions. In addition, they bounded the coordination ratio of the total disutility, when the marginal costs are imposed as taxes, over the optimal total cost in networks with homogeneous users for certain classes of latency functions.

There have been efforts to study network models with finitely many users, each carrying a non-infinitesimal amount of flow. Fotakis et al. [13] studied the existence of a Nash equilibrium in atomic unsplittable selfish routing games. In atomic unsplittable cases, finitely many users carry a non-infinitesimal amount of flow, and route the flow on a path without splitting. They showed that there exist single-commodity networks with linear latency functions, for which a Nash equilibrium in pure strategies can not exist. However, they showed the existence of a Nash equilibrium for multi-commodity networks with linear latency functions. Orda et al. [18] studied the uniqueness of a Nash equilibrium for atomic splittable selfish routing games. In atomic splittable cases, networks have a finite number of selfish users, who are allowed to carry a non-infinitesimal amount of flow and route the flow fractionally over many paths. They proved the uniqueness of a Nash equilibrium in two-node multiple-links networks with latency functions that are continuous, convex, and differentiable.

For the price of anarchy in atomic splittable cases, Roughgarden [23] showed that it is no larger than that in non-atomic cases. Awerbuch et al. [2] studied the price of anarchy in atomic unsplittable selfish routing games for networks with linear latency functions. They showed that in both mixed and pure strategies the price of anarchy is at most $\frac{3+\sqrt{5}}{2}$ for weighted demand cases, where users may carry a different amount of flow. They also showed that in pure strategies the price of anarchy is bounded by $\frac{5}{2}$ for unweighted demand cases, where users carry unit flow. Christodoulou and Koutsoupias [6] studied the price of anarchy in atomic selfish routing games with finitely many users, who carry unit flow. They showed that in single-commodity networks with linear latency functions the price of anarchy is at most $\frac{5N-2}{2N+1}$, where N is the number of users, which is asymptotically the same as in [2].

So far, the price of anarchy is the ratio of the average latency experienced by selfish users over that by optimal flows. However, there have been studies to measure the degradation of the network performance due to anarchy with respect to the maximum latency that users may experience, instead of the average latency. The price of anarchy relative to the maximum latency is the ratio of the maximum latency experienced by selfish users over that of flows which minimize the maximum latency. Roughgarden [22] studied this new version of the price of anarchy in non-atomic selfish routing games. He showed that the price of anarchy relative to the maximum latency is $n-1$ for single-commodity networks with n vertices, and with latency functions that are arbitrary continuous, and nondecreasing. Christodoulou and Koutsoupias [6] also studied the same price of anarchy in atomic selfish routing games with finitely many users, who carry unit flow. They showed that the price of anarchy relative to the maximum latency is at most $\frac{5}{2}$ for single-commodity networks with linear latency functions.

On the other hand, Farzad et al. [11] studied the flow-free selfish routing model, in which a user traveling on an edge only causes delays to following users who use the edge afterward. They showed that, in multi-commodity networks with splittable non-atomic users and polynomial latency functions of degree d, the price of anarchy is bounded by $(d+1)^{d+1}$. They also showed that, in networks with unsplittable atomic users and linear latency functions, the price of anarchy is at most $3+\sqrt{2}$.

1.3 Our contribution

Our contribution is to introduce a new class of oblivious users to the selfish routing setting, and to provide tight upper bounds of the price of anarchy at oblivious equilibrium. The selfish routing setting has the assumption that all network users route their flow selfishly, and that they can measure all path latencies in every moment. However, it is more likely to have a different class of users who are oblivious to congestion because they are not willing (or able) to measure the path latencies as often as needed. For example, data from low-level QoS services in telecommunications networks may be routed on predefined paths with no adaptability to congestion, while data from high-level QoS services are routed on the fastest paths available to them, given the network congestion at the time.

We study the degradation of the network performance caused by both selfish users and oblivious users. We consider networks with linear latency functions, and provide two tight upper bounds of the price of anarchy at oblivious equilibrium. The bound for parallel links networks with two nodes is:

$$\frac{C(\tilde{f})}{C(f^{opt})} \leq (1-\alpha)\frac{C(f^*)}{C(f^{opt})} + \alpha \max\{1, \alpha r\}, \text{ if } \tilde{f^o_{e_s}} = \alpha d \geq f^*_{e_s},$$

and

$$\frac{C(\tilde{f})}{C(f^{opt})} \leq \frac{4}{3}, \text{ otherwise.}$$

In our bound, we have the fraction α of oblivious users, and demand d. $C(\tilde{f})$ is the total cost of an oblivious equilibrium flow $\tilde{f}$, $C(f^{opt})$ is the total cost of an optimal flow f^{opt}, and $C(f^*)$ is the total cost of a traffic equilibrium flow f^*. A flow $\tilde{f^o}$ denotes the proportion of the flow $\tilde{f}$ for oblivious users. We define $r = \sum_{e \in \mathcal{E}^{opt}} a_{e_s}/a_e$, where $\mathcal{E}^{opt}$ is the set of the edges that an optimal flow f^{opt} uses, and e_s denotes the predefined shortest edge. The a_e and b_e are the coefficients of linear latency functions $l_e(x) = a_e x + b_e$. The bound for general topologies is:

$$\frac{C(\tilde{f})}{C(f^{opt})} \leq \frac{4\left(1-\alpha+\alpha D \sum_{i\in K}\sum_{e\in P_i^s}(a_e D + b_e)/C(f^{opt})\right)}{3+\alpha}.$$

General topology networks have multiple commodities, and K is the index set for commodities. P_i^s is the predefined shortest path for commodity i, and D is the total demand of all commodities. They are tight for every fraction $\alpha \in [0,1]$ of oblivious users. Furthermore, our bounds generalize the previous result of the price of anarchy without oblivious users, which Roughgarden and Tardos [24] studied in networks with linear latency functions. Note that for $\alpha = 0$, our bounds are exactly the 4/3 bound of [24].

1.4 Organization

Chapter 2 provides the background for selfish routing. In Chapter 3, we extend the selfish routing setting to include the concept of oblivious users. We study the price of anarchy at oblivious equilibrium, and show tight bounds for parallel link networks and general topologies. Chapter 4 has conclusions and open problems.

Chapter 2

Selfish Routing

2.1 The Model

We consider a directed network $G = (V, E)$ with vertex set V, and edge set $E \subseteq V \times V$. An edge is represented by an ordered pair of two vertices. We have k origin-destination vertex pairs $(s_1, t_1), ..., (s_k, t_k)$, and define $K = \{1, ..., k\}$. We use the term *commodity* i to refer to an origin-destination pair (s_i, t_i). Let $\mathcal{P}_i$ denote a set of all simple paths for commodity i. A simple path is a path with no repeated vertices on the path. We define the set of all paths $\mathcal{P} = \bigcup_{i \in K} \mathcal{P}_i$. A flow is a function $f : \mathcal{P} \to \{0\} \cup \mathcal{R}^+$, mapping each path $p \in \mathcal{P}$ to a nonnegative amount of traffic on it. We will use the notation f_p instead of $f(p)$ for simplicity. We also denote a flow vector $(f_p)_{p \in \mathcal{P}}$ by f. Each origin-destination pair (s_i, t_i) has an associated flow demand of rate d_i. A flow f is feasible if $\sum_{p \in \mathcal{P}_i} f_p = d_i$ for all $i \in K$.

Each edge $e \in E$ is given a latency function $l_e(f_e)$ which gives the traffic delay caused (and experienced) by edge flow f_e, where $f_e = \sum_{p \in \mathcal{P}: e \in p} f_p$. We assume that latency functions l_e are nonnegative, differentiable, nondecreasing, and convex. We consider the additive model, where path latencies are the sum of the latencies of all edges on the path, i.e., $l_p(f) = \sum_{e \in p} l_e(f_e)$. We define the total cost $C(f)$ of a flow f as the total latency experienced by f, i.e., $C(f) = \sum_{e \in E} l_e(f_e) f_e$. The total cost $C(f)$ can also be defined as $\sum_{p \in \mathcal{P}} l_p(f) f_p$.

Finally, we consider non-atomic games, which have infinitely many users (or agents), each carrying an infinitesimal amount of flow. In non-atomic cases, we suppose that a route switching of a single user has no influence on the path latencies, but the aggregate route switching of users can have an influence on them. We call the triple (G, d, l) an instance of selfish routing, where $d = (d_i)_{i \in K}$ and $l = (l_e)_{e \in E}$.

2.2 Flows at traffic equilibrium

We study selfish flows at traffic equilibrium. First we consider that each user carries a measurable (non-infinitesimal) amount of flow, and then an infinitesimal amount of flow will be considered. A feasible flow f for instance (G, d, l) is at traffic equilibrium if each agent of the flow routes its traffic on the minimum-latency path available to it. To choose the minimum-latency path, each agent of the flow measures the path latencies with respect to the remaining flow. If agents include the latency caused by itself to the measurement, agents might always find a path with smaller latency to reroute itself. Hence, the network might not lead to an equilibrium state. Dafermos and Sparrow [10] (see also [24]) provided the following definition for Nash flows (or flows at traffic equilibrium).

Definition 2.2.1. [10; 24] A flow f feasible for instance (G, d, l) is at traffic equilibrium if for all $i \in K$, $p_1, p_2 \in \mathcal{P}_i$, and $\delta \in (0, f_{p_1}]$, we have $l_{p_1}(f) \leq l_{p_2}(\tilde{f})$, where

$$\tilde{f}_p = \begin{cases} f_p - \delta & \text{if} \quad p = p_1 \\ f_p + \delta & \text{if} \quad p = p_2 \\ f_p & \text{if} \quad p \notin \{p_1, p_2\}. \end{cases}$$

Definition 2.2.1 states that a flow f is at traffic equilibrium if agents for commodity i, each carrying an amount δ of flow, can not find a path with smaller latency in P_i. When δ tends to 0, i.e., each agent carries an infinitesimal amount of flow, Definition 2.2.1 corresponds to the next lemma.

Lemma 2.2.2. [24] *A flow f feasible for instance (G, d, l) is at traffic equilibrium if and only if for every $i \in K$ and $p_1, p_2 \in \mathcal{P}_i$ with $f_{p_1} > 0, l_{p_1}(f) \leq l_{p_2}(f)$.*

Lemma 2.2.2 is equivalent to Wardop's principle [26]. Since any agent for commodity i can not find a path that has smaller latency in P_i, the paths used by the agents have the same latency. Let $L_i(f)$ denote the same (or common) latency of traffic equilibrium flow f for commodity i. Then $l_p(f_p) = L_i(f)$, $\forall p \in \mathcal{P} : f_p > 0$, and $l_p(0) \geq L_i(f)$, $\forall p \in \mathcal{P} : f_p = 0$, for $\forall i \in K$. By using the common latencies, the total cost can be defined as $C(f) =$

$\sum_{i \in K} L_i(f) d_i$.

2.3 Convex programming

2.3.1 Optimal flows

We study the properties of optimal flows, which minimize the total cost. We formulate the optimization problem of finding an optimal flow in convex programming. Convex programs have the following form.

$$
\begin{aligned}
\text{(CP)} \quad & \text{minimize} \quad && f^{cp}(x) && \\
& \text{subject to} \quad && g_i(x) \leq 0 && i = 1, 2, ..., m \\
& && h_j(x) = 0 && j = 1, 2, ..., r \\
& && x \in X, &&
\end{aligned}
$$

where X is a non-empty convex set in $\mathcal{R}^n$. We want to find the point $x \in X$ that minimizes the convex and real-valued objective function $f^{cp} : \mathcal{R}^n \to \mathcal{R}$ on X. A function f^{cp} is convex if for any $x, y \in X$ and any $\lambda \in [0, 1]$,

$$f^{cp}(\lambda x + (1 - \lambda) y) \leq \lambda f^{cp}(x) + (1 - \lambda) f^{cp}(y).$$

The relations $g_i(x) \leq 0$ are the inequality constraints, where $g_i(x)$ are convex functions on X. The relations $h_j(x) = 0$ are equality constraints, where $h_j(x)$ are affine functions on X. Affine functions are vector-valued functions of the form $h_j(x) = Ax + b$, where A is a matrix and b is a column vector.

Constrained programs can be transformed to unconstrained programs by the method of Lagrange multipliers. We denote Lagrange multipliers of (CP) by $\lambda = (\lambda_1, \lambda_2)$, where $\lambda_1 \in \mathcal{R}^m$ and $\lambda_2 \in \mathcal{R}^r$. We define the Lagrangian function of (CP) as:

$$\mathcal{F}(x, \lambda) = f^{cp}(x) + {\lambda_1}^T g(x) + {\lambda_2}^T h(x).$$

Furthermore, define $\Omega = \{x \in X \mid g(x) \leq 0, h(x) = 0\}$, and $\widehat{\mathcal{F}}(\lambda) = \inf_x \mathcal{F}(x, \lambda)$.

We outline some important theorems of convex programming, which will be used to study properties of optimal flows and Nash flows. We state the theorems without proof [3]. In the next theorem, given a set S, *int* of S is the set of all interior points of S.

Theorem 2.3.1. [3] *Let f and g be convex and let h be affine. Suppose that there is an $\hat{x}$ such that $g(\hat{x}) < 0$, $h(\hat{x}) = 0$, and $0 \in int\{h(x) \mid x \in X\}$. Then*

$$\inf \{f^{cp}(x) \mid x \in \Omega\} = sup_{\lambda_1 \geq 0} \widehat{\mathcal{F}}(\lambda).$$

In what follows, we use the gradient $\nabla f^{cp}(x) = (\frac{\partial f^{cp}}{\partial x_1}, ..., \frac{\partial f^{cp}}{\partial x_n})$, provided f^{cp} is differentiable at x. The next theorem states the equivalence between convex programs and variational inequality problems.

Theorem 2.3.2. [5] *Let $\hat{x}$ be a solution to the optimization problem (CP), where f^{cp} is continuously differentiable and Ω is a nonempty closed convex set* *in $\mathcal{R}^n$. Then $\hat{x}$ is a solution of the (variational inequality) problem*

$$\nabla f^{cp}(\hat{x})^T(x - \hat{x}) \geq 0 \qquad \forall x \in \Omega. \tag{2.1}$$

And, if f^{cp} is a convex function and $\hat{x}$ is a solution to (2.1), then $\hat{x}$ is a solution to the optimization problem (CP).

Due to Theorem 2.3.1, the problem (CP) is equivalent to finding a saddle point $(\hat{x}, \hat{\lambda})$ of the function $\mathcal{F}$, i.e.,

$$\mathcal{F}(\hat{x}, \lambda) \leq \mathcal{F}(\hat{x}, \hat{\lambda}) \leq \mathcal{F}(x, \hat{\lambda}) \qquad \forall x, \forall \lambda.$$

The conditions for the saddle point can be described by the following Karush-Kuhn-Tucker (KKT) conditions. For a vector f^{cp} of functions f_i^{cp}, we define the gradient $\nabla f^{cp}(x) = (\nabla f_1^{cp}(x), ..., \nabla f_m^{cp}(x))$, provided f_i^{cp} are differentiable at x.

Theorem 2.3.3. [5] *Assume that the optimal value of (CP) is finite and that there is an $\hat{x}$ such that $g(\hat{x}) < 0$, $h(\hat{x}) = 0$, and $0 \in int\{h(x) \mid x \in X\}$. In order that a given vector $\hat{x}$ is an optimal solution to (CP), it is necessary and sufficient that a vector $\hat{\lambda}$ exists such that $(\hat{x}, \hat{\lambda})$ is a saddle-point of the Lagrangean function $\mathcal{F}$ of (CP). Equivalently, a vector $\hat{x}$ is an optimal solution if and only if there is a vector $\hat{\lambda}$ of Lagrange multipliers which, together with $\hat{x}$, satisfies the KKT conditions for (CP):*

$$\nabla f^{cp}(\hat{x}) + \nabla g(\hat{x})\hat{\lambda}_1 + \nabla h(\hat{x})\hat{\lambda}_2 = 0 \tag{2.2}$$

$$g(\hat{x}) \leq 0, \quad \hat{\lambda}_1 \geq 0, \quad \hat{\lambda}_1^T g(\hat{x}) = 0 \tag{2.3}$$

$$h(\hat{x}) = 0 \tag{2.4}$$

So far, we have summarized theorems of convex programming. From now, we use convex programming and its theorems to study properties of optimal flows. The problem of finding an optimal flow can be formulated as the following convex program.

$$\begin{aligned}
\text{(NLP1)} \quad \text{minimize} \quad & C(f) = \sum_{e \in E} l_e(f_e) f_e & & \\
\text{subject to} \quad & \sum_{p \in P_i} f_p = d_i & \forall i \in K & \quad (2.5) \\
& f_e = \sum_{p \in \mathcal{P}: e \in p} f_p & \forall e \in E & \quad (2.6) \\
& f_p \geq 0 & \forall p \in \mathcal{P}. & \quad (2.7)
\end{aligned}$$

Constraints (2.5) enforce demand conservation for each commodity i. Constraints (2.6) state how to get edge flows from path flows. Constraints (2.7) are the non-negativity constraints for path flows. The objective function $C(f) = \sum_{e \in E} l_e(f_e) f_e$ is convex since we have assumed that each edge latency function l_e is convex and nondecreasing. Let H denote a set of feasible flows, i.e.,

$$H = \{f \mid f_p \geq 0, \ \forall p \in \mathcal{P}, \text{ and } \sum_{p \in \mathcal{P}_i} f_p = d_i, \ \forall i \in K\}.$$

The set H is convex since for any $f_1, f_2 \in H$ and any $\lambda \in [0,1]$,

$$\lambda f_1 + (1-\lambda) f_2 \in H.$$

In what follows, let c'_e denote the derivative $\frac{d}{dx}c_e(x)$ of c_e, where $c_e(f_e) = l_e(f_e)f_e$. Here, $c'_e(f_e) = l_e(f_e) + l'_e(f_e)f_e$, and l'_e denotes the derivative $\frac{d}{dx}l_e(x)$ of l_e. We also define $c_p(f) = \sum_{e\in p} c_e(f_e)$, and $c'_p(f) = \sum_{e\in p} c'_e(f_e)$. The next lemma states the conditions of optimal flows, which are similar with that of Nash flows. Each agent of optimal flows can not find a path with smaller latency if we define path latency functions as $c'_p(f)$ instead of $l_p(f)$. We present the proof of this well known lemma for completeness.

Lemma 2.3.4. *A flow f is an optimal solution to (NLP1) if and only if for every $i \in K$ and $p_1, p_2 \in P_i$ with $f_{p_1} > 0$, $c'_{p_1}(f) \le c'_{p_2}(f)$.*

Proof. Let us define $g_p(f) = -f_p$ and $h_i(f) = d_i - \sum_{p\in\mathcal{P}_i} f_p$ from the constraints (2.7) and (2.5), respectively. We associate Lagrange multipliers λ_1 with $g = (g_p)_{p\in\mathcal{P}}$, and λ_2 with $h = (h_i)_{i\in K}$ to formulate the following Lagrangean function:

$$\mathcal{F}(f,\lambda) = \sum_{p\in P} c_p(f) + \sum_{p\in\mathcal{P}} \lambda_{1,p} g_p(f) + \sum_{i\in K} \lambda_{2,i} h_i(f).$$

From the KKT conditions, Theorem 2.3.3, a flow f is an optimal solution to (NLP1) if and only if there exists a vector (λ_1, λ_2) that, together with f, satisfies the following conditions:

$$c'_p(f) - \lambda_{1,p} - \lambda_{2,i} = 0, \quad \forall p \in \mathcal{P}_i, \quad \forall i \in K \tag{2.8}$$

$$-f_p \leq 0, \quad \forall p \in \mathcal{P} \tag{2.9}$$

$$c'_p(f) - \lambda_{2,i} \geq 0, \quad \forall p \in \mathcal{P}_i, \quad \forall i \in K \tag{2.10}$$

$$\left(c'_p(f) - \lambda_{2,i}\right) f_p = 0, \quad \forall p \in \mathcal{P}_i, \quad \forall i \in K \tag{2.11}$$

$$d_i - \sum_{p \in \mathcal{P}_i} f_p = 0, \quad \forall i \in K \tag{2.12}$$

Hence, a flow f feasible for (NLP1) that satisfies the conditions below is optimal. Note that a feasible flow f satisfies the conditions (2.9) and (2.12), and there exists $\lambda_{1,p} \geq 0$ satisfying the condition (2.8). The lemma follows from the following conditions, which have been drawn from the conditions (2.10) and (2.11).

$$c'_p(f) = \lambda_{2,i}, \quad \forall p \in \mathcal{P}_i : f_p > 0, \ \forall i \in K$$

$$c'_p(f) \geq \lambda_{2,i}, \quad \forall p \in \mathcal{P}_i : f_p = 0, \ \forall i \in K.$$

□

Lemma 2.2.2 states the conditions of Nash flows, and Lemma 2.3.4 states the conditions of optimal flows. From the similarity between the conditions of Nash flows and optimal flows, the next proposition shows the relationship between the two flows.

Proposition 2.3.5. *An optimal flow f for instance (G, d, l) is at traffic equi-*

librium for instance (G, d, l^*), where marginal cost functions l_e^* are defined as $l_e^*(f_e) = c_e'(f_e) = l_e(f_e) + l_e'(f_e)f_e$.

2.3.2 Nash flows

In networks with linear latency functions, problems of finding a traffic equilibrium for an instance (G, d, l) can be formulated as the following convex program [4]:

$$
\begin{aligned}
\text{(NLP2)} \quad \text{minimize} \quad & \sum_{e \in E} \int_0^{f_e} l_e(t)dt & \\
\text{subject to} \quad & d_i = \sum_{p \in P_i} f_p & \forall i \in K \\
& f_e = \sum_{p \in \mathcal{P}: e \in p} f_p & \forall e \in E \\
& f_p \geq 0 & \forall p \in \mathcal{P}.
\end{aligned}
$$

Under the assumption that latency functions l_e are convex and nondecreasing, the objective function of (NLP2) is convex. We show that an optimal solution to (NLP2) is a Nash flow for instance (G, d, l) in the next proposition. We present the proof of this well known fact for completeness reasons.

Proposition 2.3.6. *Let a flow f be an optimal solution to (NLP2). Then a flow f is at traffic equilibrium for instance (G, d, l).*

Proof. From Lemma 2.3.4, a flow f is an optimal solution to (NLP2) if and only if for every $i \in K$ and $p_1, p_2 \in \mathcal{P}_i$ with $f_{p_1} > 0$, $l_{p_1}(f_{p_1}) \leq l_{p_2}(f_{p_2})$. Then the proposition follows from Lemma 2.2.2. □

We can also formulate traffic equilibrium problems as variational inequality problems. Let $\langle x, y \rangle$ denote the inner product of two vectors x and y. Let H be a set of feasible flows for (NLP2), i.e., $H = \{f \mid f_p \geq 0,\ \forall p \in \mathcal{P}, \text{ and } \sum_{p \in \mathcal{P}_i} f_p = d_i,\ \forall i \in K\}$. The next proposition follows from Theorem 2.3.2.

Proposition 2.3.7. [9] *A flow f^* is an optimal solution to (NLP2) if and only if f^* is a solution to the following variational inequality:*

$$\langle l(f^*), f - f^* \rangle \geq 0, \quad \forall f \in H.$$

Since an optimal point f^* to (NLP2) is at traffic equilibrium for instance (G, d, l), the next corollary follows.

Corollary 2.3.8. *A flow f^* is at traffic equilibrium for instance (G, d, l) if and only if f^* is a solution to the following variational inequality:*

$$\langle l(f^*), f - f^* \rangle \geq 0, \quad \forall f \in H.$$

Additionally, if we define $C^f(x) = \sum_{e \in E} l_e(f_e) x_e$, then the variational inequality can be expressed as follows:

$$C^{f^*}(f^*) \leq C^{f^*}(f), \quad \forall f \in H.$$

2.4 The price of anarchy

The price of anarchy is the worst-possible ratio of the total cost of a Nash flow over the optimal total cost. Following Roughgarden and Tardos [24], and Roughgarden [21], we define the ratio $\rho = \rho(G,d,l) = C(f^*)/C(\hat{f})$, where f^* is a Nash flow and $\hat{f}$ is an optimal flow for instance (G,d,l). Let $\mathcal{L}$ be a class of latency functions that are continuous and non-decreasing. Then the price of anarchy $\mathcal{A}(\mathcal{L})$ is defined as $\mathcal{A}(\mathcal{L}) = \sup_{l \in \mathcal{L}} \rho(G,d,l)$.

Roughgarden and Tardos [24] studied the price of anarchy, and showed that it is bounded by $\frac{4}{3}$ in networks with linear latency functions. Because their proof is rather long, here we present a shorter proof by Correa et al. [8].

Theorem 2.4.1. [24] *Suppose an instance (G,d,l) with linear latency functions. Let a flow f^* be at traffic equilibrium and $\hat{f}$ be an optimal flow for the instance. Then*

$$\frac{C(f^*)}{C(\hat{f})} \leq \frac{4}{3}.$$

Proof. Let f be any feasible flow. We denote linear latency functions as $l_e(x) = a_e x + b_e$. Then

$$\begin{aligned} C(f^*) &= \sum_{e \in E} (a_e f_e^* + b_e) f_e^* \\ &\leq \sum_{e \in E} (a_e f_e^* + b_e) f_e \\ &\leq \sum_{e \in E} (a_e f_e + b_e) f_e + \frac{1}{4} \sum_{e \in E} a_e f^{*2} \end{aligned}$$

$$\leq C(f) + \frac{1}{4}C(f^*).$$

Because f^* is at traffic equilibrium, f^* is a solution to the variational inequality

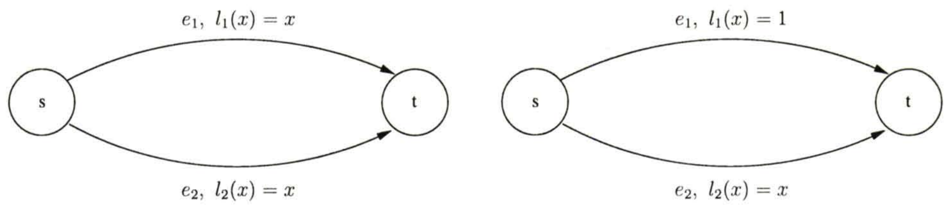
$$\langle l(f^*), f^* \rangle \leq \langle l(f^*), f \rangle, \ \forall f,$$

and the first inequality follows. The second inequality holds since $(f - \frac{1}{2}f^*)^2 \geq 0$. By substituting f with an optimal flow $\hat{f}$, the theorem follows.

□

We illustrate the price of anarchy with the two examples of Figure 2.1. Consider a network with two vertices: origin vertex s and destination vertex t. The network has two parallel edges connecting s to t, and has demand 1. In Figure 2.1(a), the Nash flow $f^* = (f^*_{e_1}, f^*_{e_2})$ is $(\frac{1}{2}, \frac{1}{2})$, and the optimal flow $\hat{f}$ is also $(\frac{1}{2}, \frac{1}{2})$. The ratio ρ of this example is 1, which is not the worst ratio for the class $\mathcal{L}$ of linear latency functions. On the other hand, the example of Figure 2.1(b) provides the worst ratio. In this example, the Nash flow f^* is $(0, 1)$, and $C(f^*) = 1$. The optimal flow $\hat{f}$ is $(\frac{1}{2}, \frac{1}{2})$, and $C(\hat{f}) = \frac{3}{4}$. Hence, the example of Figure 2.1(b) has price of anarchy $\rho = \frac{4}{3}$, which is the worst-possible ratio in networks with linear latency functions [24].

We now present another proof for the same result by using the β-function defined in [8], which will be used to analyze the price of anarchy at oblivious equilibrium. Recall that $C^{f^*}(x) = \sum_{e \in E} l_e(f^*_e)x_e$. The β-function is a powerful tool to separate the cost term $C^f(x)$, which consists of two different



(a) Not worst-possible ratio (b) Worst-possible ratio

Figure 2.1: Two examples for the price of anarchy

flows f and x, into $C(f)$ and $C(x)$.

Definition 2.4.2. [8] Let $\mathcal{L}$ be a class of latency functions that are continuous and nondecreasing. For any function $l \in \mathcal{L}$ and any value $v \geq 0$, we define the β-function as

$$\beta(v, l) := \frac{1}{vl(v)} \max_{x \geq 0}\{x(l(v) - l(x))\},$$

where $0/0$ is 0 by convention. We define

$$\beta(l) := \sup_{v \geq 0} \beta(v, l), \text{ and } \beta(\mathcal{L}) := \sup_{l \in \mathcal{L}} \beta(l).$$

In the next theorem, the price of anarchy at traffic equilibrium is bounded by using the β-function.

Theorem 2.4.3. [8] *Let $\mathcal{L}$ be a family of continuous, nondecreasing latency functions. Consider an instance of the traffic equilibrium problem with latency functions drawn from $\mathcal{L}$. Then the ratio of the total travel time of a Nash flow*

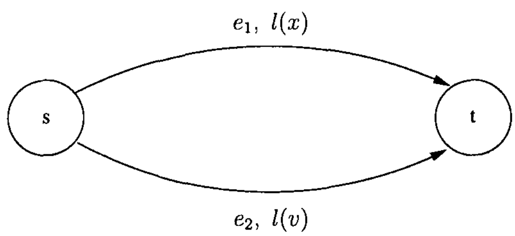


Figure 2.2: Tight example for the bound $(1-\beta(\mathcal{L}))^{-1}$

f^* to that of a system optimum $\hat{f}$ is bounded from above by $(1-\beta(\mathcal{L}))^{-1}$, i.e.,

$$C(f^*) \leq \frac{1}{1-\beta(\mathcal{L})} C(\hat{f}).$$

Proof. Let x be a feasible flow. Due to the definition of β-function,

$$C^{f^*}(x) \leq \sum_{e\in E} \beta(f_e^*, l_e) l_e(f_e^*) f_e^* + \sum_{e\in E} l_e(x_e) \leq \beta(\mathcal{L}) C(f^*) + C(x). \tag{2.13}$$

From Corollary 2.3.8, we have $C(f^*) \leq C^{f^*}(x)$. Then the theorem follows by substituting x with an optimal flow $\hat{f}$. □

Correa et al. [8] showed that the bound $(1-\beta(\mathcal{L}))^{-1}$ given in Theorem 2.4.3 is tight. Consider the network of Figure 2.2, with two vertices s and t, and with two parallel edges e_1 and e_2. The network has (s,t) flow of rate v. The Nash flow f^* is $(v, 0)$, and $C(f^*) = l(v)v$, while the optimal total cost is

$$C(\hat{f}) = \min_{0\leq x\leq v} \{l(x)x + l(v)(v-x)\} = vl(v) - \max_{0\leq x\leq v} \{x\,(l(v) - l(x))\}.$$

As a result, the price of anarchy for this example is

$$\begin{aligned}\rho &= \left(1 - \frac{\max_{0 \leq x \leq v} \{x\,(l(v) - l(x))\}}{vl(v)}\right)^{-1} \\ &= (1 - \beta(v, l))^{-1} \\ &= (1 - \beta(\mathcal{L}))^{-1}. \end{aligned} \tag{2.14}$$

For the equality (2.14), $\mathcal{L}$ contains a latency function l_{sup}, which achieves $\beta(v, l_{sup}) = \beta(\mathcal{L})$ for $v > 0$. Then the bound in Theorem 2.4.3 is tight for $l = l_{sup}$.

Correa et al. [8] give the $\beta(\mathcal{L})$ values for several classes of latency functions. In particular, they show that if $\mathcal{L}$ is a family of continuous, non-decreasing latency functions l satisfying $l(cx) \leq cl(x)$ for all $c \in [0, 1]$, then $\beta(\mathcal{L}) = 1/4$. This implies that the price of anarchy is 4/3 for networks with linear latency functions.

Chapter 3

Oblivious Users

In this chapter, we study the price of anarchy at oblivious equilibrium, whereas we have studied the price of anarchy at traffic equilibrium in the previous chapter. The selfish routing setting in the previous chapter is extended for the routing setting of oblivious equilibrium problems to include the concept of oblivious users. We introduce two bounds of the price of anarchy at oblivious equilibrium for two different topologies: parallel links networks, and general topologies.

3.1 The Model

The oblivious routing setting extends the selfish routing setting. In the oblivious routing setting, a fixed fraction α of the total demand d_i for commodity i consists of oblivious users, and the remaining fraction $(1-\alpha)$ consists of selfish users while in the selfish routing setting the whole demand d_i for commodity i consists of selfish users. Hence, the selfish routing setting is a case of the oblivious routing setting when α is 0. We present the oblivious routing setting based on the selfish routing setting, which has been covered in the previous chapter. In the meantime, we outline the overlapping part of the two settings.

We consider a directed network $G = (V, E)$ with vertex set V, edge set E, and k commodities. Let $K = \{1, ..., k\}$ be an index set for commodities. $\mathcal{P}_i$ denotes a set of all simple paths for commodity i, and $\mathcal{P} = \bigcup_{i \in K} \mathcal{P}_i$. Each origin-destination pair (s_i, t_i) has an associated flow demand of rate d_i. We denote traffic on a path p by f_p. Each edge $e \in E$ is given a latency function $l_e(f_e)$ of the edge flow f_e, where $f_e = \sum_{p \in \mathcal{P}: e \in p} f_p$. We consider the additive model, i.e., $l_p(f) = \sum_{e \in p} l_e(f_e)$, with the assumption that latency functions l_e are nonnegative, differentiable, nondecreasing, and convex. We define the total cost $C(f) = \sum_{e \in E} l_e(f_e) f_e$ or $\sum_{p \in \mathcal{P}} l_p(f) f_p$.

We consider non-atomic games with infinitely many users, each carrying an infinitesimal amount of flow. A fraction $\alpha \in [0, 1]$ of the total demand d_i for commodity i consists of infinitely many oblivious users, who route their flow on the predefined shortest path in the network. The predefined shortest path, which is denoted by P_i^s for commodity i, is the shortest one when path distances are measured by $l_p(0)$. If there are more than one shortest paths, we assume that all oblivious users choose the first one in a lexicographic ordering. The remaining fraction $(1 - \alpha)$ of the total demand d_i for commodity i consists of infinitely many selfish users.

Let f^o denote the proportion of the flow f for oblivious users, and f^* denote the proportion for selfish users. A flow $f = f^o + f^*$ is feasible if $\sum_{p \in \mathcal{P}_i} f_p^o = \alpha d_i$, and $\sum_{p \in \mathcal{P}_i} f_p^* = (1-\alpha)d_i$ for $\forall i \in K$. We call the tuple (G, r, l, α) an instance of oblivious routing, where $d = (d_i)_{i \in K}$ and $l = (l_e)_{e \in E}$. Note that we use the triple (G, d, l) for an instance of selfish routing.

3.2 Flows at oblivious equilibrium

A flow f feasible for instance (G, d, l, α) is at oblivious equilibrium if each selfish user routes one's flow on the minimum-latency path available after oblivious users have routed their flow on the predefined shortest path. Selfish users measure path latencies with respect to the remaining flow. The following lemma describes this idea of a flow at oblivious equilibrium.

Lemma 3.2.1. *Suppose a flow $f = f^o + f^*$ feasible for instance (G, d, l, α), where f^o is the oblivious flow that routes demand αd_i on the predefined shortest path P_i^s for commodity i, and f^* is a selfish flow satisfying demand $(1-\alpha)d$. Then the flow f is at oblivious equilibrium if and only if for every $i \in K$ and $p_1, p_2 \in \mathcal{P}_i$ with $f_{p_1}^* > 0, l_{p_1}(f^o + f^*) \leq l_{p_2}(f^o + f^*)$.*

Suppose a flow $f = f^o + f^*$ at oblivious equilibrium for instance (G, d, l, α). Then a flow f^* is at traffic equilibrium for instance $(G, (1-\alpha)r, \tilde{l})$, where $\tilde{l}_e(x) = l_e(f_e^o + x)$. The problem of assigning oblivious users on the predefined shortest path can be solved efficiently in polynomial time by applying shortest path algorithms. Hence, oblivious equilibrium problems can be reduced to user equilibrium problems in terms of the modified latency functions $\tilde{l}$ and demands $(1-\alpha)d$.

We formulate oblivious equilibrium problems as the following convex program after finding an oblivious flow f^o. Let $h_e(x) = \int_0^x l_e(f_e^o + t)dt$. An optimal solution f^* to (NLP3) is a selfish flow for instance (G, d, l, α), and we have a flow f at oblivious equilibrium by combining two flows f^o and f^*.

$$
\begin{aligned}
\text{(NLP3)} \quad \text{minimize} \quad & \sum_{e \in E} h_e(f_e) & \\
\text{subject to} \quad & (1-\alpha)d_i = \sum_{p \in P_i} f_p & \forall i \in K \\
& f_e = \sum_{p \in \mathcal{P}: e \in p} f_p & \forall e \in E \\
& f_p \geq 0 & \forall p \in \mathcal{P}.
\end{aligned}
$$

Suppose a flow $f = f^o + f^*$ is at oblivious equilibrium. If we define path latency functions as $l_p^o(f) = \sum_{e \in p} l_e(0)$, then no oblivious users can find a path with smaller latency. In other words, an oblivious flow f^o of the flow f is at traffic equilibrium for instance $(G, \alpha d, l^o)$, where $l_e^o(x) = l_e(0)$. Hence, a flow f^o is a solution to the following variational inequality (3.1).

$$\langle l^o,\ x - f^o \rangle \geq 0,\ \forall x \in \{f \mid f \text{ is a flow satisfying demand } \alpha d\}. \tag{3.1}$$

A selfish flow f^* of the flow f is at traffic equilibrium for instance $(G, (1-\alpha)d, l^s)$, where $l_e^s(x) = l_e(f_e^o + x)$. Hence, a flow f^* is a solution to the following variational inequality (3.2).

$$\langle l^s,\ x - f^* \rangle \geq 0,\ \forall x \in \{f \mid f \text{ is a flow satisfying demand } (1-\alpha)d\}. \quad (3.2)$$

3.3 Networks of parallel links

We restrict our previous oblivious routing setting to parallel links networks, which have only two nodes: origin node s and destination node t. Parallel links networks have multiple links connecting s to t, and have a single commodity. We denote the total demand of the commodity by d. We assume that oblivious users route their flow on the shortest edge when edge distances are measured by $l_e(0)$. We denote the shortest edge by e_s. If there are more than one shortest edges, we assume that all oblivious users choose the first one in a lexicographic ordering. Furthermore, we assume that each edge latency is linear and strictly increasing, i.e., $l_e(x) = a_e x + b_e$ with $a_e > 0$. Hence, in our case $l_e(0) = b_e$, $\forall e \in E$.

3.3.1 The price of anarchy

We bound the price of anarchy at oblivious equilibrium by separating the total latency $C(\tilde{f})$ into two terms $C(\tilde{f}^o)$ and $C(\tilde{f}^*)$: one is a proportion of the total latency caused by oblivious users, and the other is by selfish users. Suppose a Nash flow f^* exists for instance (G, d, l) and an oblivious equilibrium flow $\tilde{f} = \tilde{f}^o + \tilde{f}^*$ for instance (G, d, l, α). In parallel links networks, the shortest edge e_s is attractive to selfish users as long as the flow on the edge e_s is less than $f^*_{e_s}$. If oblivious users fill the edge e_s more than $f^*_{e_s}$, no selfish users will use the edge e_s.

Proposition 3.3.1. *Let $\tilde{f} = \tilde{f}^o + \tilde{f}^*$ feasible for instance (G, d, l, α) be at oblivious equilibrium, where $\tilde{f}^o$ is an oblivious flow and $\tilde{f}^*$ is a selfish flow. Let f^* feasible for instance (G, d, l) be at traffic equilibrium. If $\tilde{f}^o_{e_s} = \alpha d < f^*_{e_s}$, then $\tilde{f} = f^*$. Otherwise, $\tilde{f}^o_e \tilde{f}^*_e = 0$, $\forall e \in E$.*

Proof. In case $\tilde{f}^o_{e_s} = \alpha d < f^*_{e_s}$, the edge e_s is still attractive to users of a selfish flow $\tilde{f}^*$, some of which will use the edge e_s. As α tends to 0, the total amount of the edge flow, $\tilde{f}_{e_s} = \tilde{f}^o_{e_s} + \tilde{f}^*_{e_s}$, will not be changed, so that a flow $\tilde{f}$ is also at traffic equilibrium for instance (G, d, l). Since Aashtiani and Magnanti [1] showed that edge flows are unique in networks with strictly monotone latency functions, we have $\tilde{f} = f^*$. In case $\tilde{f}^o_{e_s} = \alpha d \geq f^*_{e_s}$, the edge e_s is no longer attractive to users of a selfish flow $\tilde{f}^*$. No selfish users choose the edge e_s. On the other hand, oblivious users only choose the edge e_s. Hence, we have $\tilde{f}^o_e \tilde{f}^*_e = 0$, $\forall e \in E$. □

If $\tilde{f}^o_{e_s} = \alpha d < f^*_{e_s}$, a flow $\tilde{f}$ at oblivious equilibrium has the same flow pattern with a flow f^* at traffic equilibrium. In this case, the price of anarchy at oblivious equilibrium is the same with the price of anarchy at traffic equilibrium 4/3, which has been studied by Roughgarden and Tardos [24]. Hence, for the price of anarchy at oblivious equilibrium, we study the other case $\tilde{f}^o_{e_s} = \alpha d \geq f^*_{e_s}$, where vectors $\tilde{f}^o$ and $\tilde{f}^*$ are orthogonal, i.e., ${\tilde{f}^o}^T \tilde{f}^* = 0$. Since two flows $\tilde{f}^o$ and $\tilde{f}^*$ do not use the same edge, we can separate the total cost $C(\tilde{f})$ into two terms $C(\tilde{f}^o)$ and $C(\tilde{f}^*)$. First we study the portion of the total latency caused by selfish flow $\tilde{f}^*$, i.e., $C(\tilde{f}^*)$, then the portion caused by oblivious flow $\tilde{f}^o$, i.e., $C(\tilde{f}^o)$.

Suppose two flows f^d and $f^{d+\delta}$. A flow f^d is at traffic equilibrium with the total demand d, and a flow $f^{d+\delta}$ is at traffic equilibrium with the total demand $d+\delta$, where $\delta \geq 0$. For every edge e, an edge flow $f_e^{d+\delta}$ is larger than or equal to an edge flow f_e^d. Intuitively, a flow $f^{d+\delta}$ can be constructed from a flow f^d by adding an additional demand δ, while users of f^d keep their routes.

Proposition 3.3.2. *Let f^d and $f^{d+\delta}$ be flows at traffic equilibrium for instances (G, d, l) and $(G, d+\delta, l)$, respectively, where $\delta \geq 0$. Then we have $f_e^d \leq f_e^{d+\delta}$, $\forall e \in E$.*

Proof. Suppose, for the sake of contradiction, that there exists an edge e such that $f_e^d > f_e^{d+\delta}$. Let $L_1(f^d)$ be the common latency for (G, d, l), and $L_2(f^{d+\delta})$ for $(G, d+\delta, l)$. Since $f_e^d > f_e^{d+\delta} \geq 0$, and latency functions are strictly increasing, Wardrop's principle[26] implies that $L_1(f^d) > L_2(f^{d+\delta})$, which contradicts the fact that $d < d+\delta$. □

Suppose a flow f^* at traffic equilibrium in a network. If an edge is removed from the network together with the edge flow from f^*, the remaining flow is still at traffic equilibrium in the modified network, which does not have the edge. No users of the remaining flow can find a path with smaller latency in the modified network.

Proposition 3.3.3. *Let a flow f^* be at traffic equilibrium for instance (G, d, l).*

Let us construct a new instance $(G', d - f^*_{e'}, l)$, *where* $G' = (V, E \setminus \{e'\})$, *by removing an edge* e' *from* G *together with the edge flow* $f^*_{e'}$. *Then flow* $(f^*_e)_{e \in E \setminus \{e'\}}$ *is at traffic equilibrium for the new instance* $(G', d - f^*_{e'}, l)$.

Proof. Let $L(f^*)$ denote the common latency by a traffic equilibrium flow f^* in instance (G, d, l). Due to Wardrop's principle, a flow f^* at traffic equilibrium for instance (G, d, l) satisfies the following necessary and sufficient conditions:

$$l_e(f^*_e) = L(f^*), \quad \forall e \in E : f^*_e > 0$$
$$l_e(0) \geq L(f^*), \quad \forall e \in E : f^*_e = 0.$$

Let us define $f^{*\prime} = (f^*_e)_{e \in E \setminus \{e'\}}$. By removing an edge e' from G together with the edge flow $f^*_{e'}$, the common latency by the flow $f^{*\prime}$ in the new instance $(G', d - f^*_{e'}, l)$ is not changed from $L(f^*)$. Since the flow $f^{*\prime}$ is feasible for the new instance, and satisfies the necessary and sufficient conditions of traffic equilibrium for every $e \in E \setminus \{e'\}$, the proposition follows. □

Suppose that two flows f^* and $\tilde{f}$ carry the same total demand d: f^* is a traffic equilibrium flow, and $\tilde{f}$ is an oblivious equilibrium flow. The flow f^* carries the total demand d, and the flow $\tilde{f}^*$ carries a fraction $(1 - \alpha)$ of the total demand d. Then, on every edge e, the flow f^* carries the traffic of selfish users more than or equal to the flow $\tilde{f}^*$.

Proposition 3.3.4. *Let* f^* *be a flow at traffic equilibrium for instance* (G, d, l), *and* $\tilde{f} = \tilde{f}^o + \tilde{f}^*$ *be a flow at oblivious equilibrium for instance* (G, d, l, α). *Then we have* $f^*_e \geq \tilde{f}^*_e$, $\forall e \in E$.

Proof. In case $\tilde{f}^o_{e_s} = \alpha d < f^*_{e_s}$, from Proposition 3.3.1 we have that $f^* = \tilde{f}$, which implies that $f^*_e = \tilde{f}^*_e$, $\forall e \in E \setminus \{e_s\}$ and $f^*_{e_s} \geq \tilde{f}^*_{e_s}$.

In case $\tilde{f}^o_{e_s} = \alpha d \geq f^*_{e_s}$, we have $f^*_{e_s} \geq \tilde{f}^*_{e_s} = 0$ since the edge e_s is no longer attractive to selfish users of $\tilde{f}^*$, and $f^*_{e_s}$ is nonnegative. Let us define $E' = E \setminus \{e_s\}$. To compare edge flows on the remaining edges, i.e., $\forall e \in E'$, let us construct a new network $G' = (V, E')$ by removing the edge e_s from E together with the edge flow f_{e_s}. From Proposition 3.3.3, the flows $(f^*_e)_{e\in E'}$ and $(\tilde{f}^*_e)_{e\in E'}$ are at traffic equilibrium for instances $(G', d - f^*_{e_s}, l)$ and $(G', d - \alpha d, l)$, respectively. Since $d - f^*_{e_s} \geq d - \alpha d$, we have $f^*_e \geq \tilde{f}^*_e$, $\forall e \in E'$ from Proposition 3.3.2 . □

In the next lemma, we bound the cost term $C(\tilde{f}^*)$ with respect to $C(f^*)$. The flow $\tilde{f}^*$ carries a fraction $(1 - \alpha)$ of the total demand d, while the flow f^* carries the total demand d. Then the cost term $C(\tilde{f}^*)$ is less than or equal to $(1 - \alpha)C(f^*)$.

Lemma 3.3.5. *Let f^* be a flow at traffic equilibrium for instance (G, d, l), and $\tilde{f} = \tilde{f}^o + \tilde{f}^*$ be a flow at oblivious equilibrium for instance (G, d, l, α). Then we have $C(\tilde{f}^*) \leq (1 - \alpha)C(f^*)$.*

Proof.

$$\begin{aligned} C(f^*) - C(\tilde{f}^*) &= \sum_{e\in E} \Big(l_e(f^*_e) f^*_e - l_e(\tilde{f}^*_e)\tilde{f}^*_e \Big) \\ &\geq \sum_{e\in E} (f^*_e - \tilde{f}^*_e) l_e(f^*_e) \end{aligned}$$

$$\geq \sum_{e \in E} (f_e^* - \tilde{f}_e^*) L(f^*)$$
$$= \alpha d L(f^*) = \alpha C(f^*).$$

We have $l_e(f_e^*) \geq l_e(\tilde{f}_e^*)$, $\forall e \in E$ since latency functions l_e are increasing, and $f_e^* \geq \tilde{f}_e^*$ from Proposition 3.3.4. Hence, the first inequality holds. Let $L(f^*)$ be the common latency by a traffic equilibrium flow f^* in instance (G, d, l). Wardop's principle implies that $l_e(f_e^*) \geq L(f^*)$, $\forall e \in E$. Since $(f_e^* - \tilde{f}_e^*) \geq 0$ and $l_e(f_e^*) \geq L(f^*) \geq 0$, $\forall e \in E$, the second inequality follows. □

So far, we have bounded the cost term $C(\tilde{f}^*)$ with respect to $C(f^*)$. Now, we study properties of an optimal flow f^{opt} to bound the cost term $C(\tilde{f}^o)$ with respect to $C(f^{opt})$.

An optimal flow always uses the shortest edge e_s, in other words $f_{e_s}^{opt} > 0$. To see this, suppose that no users use the edge e_s. Then the total cost will be decreased by moving a certain amount of flow to the edge e_s since it has the smallest latency when no users are using it.

Proposition 3.3.6. *Let f^{opt} denote the optimal flow that minimizes the total cost. Then we have $f_{e_s}^{opt} > 0$ where the edge e_s is the shortest edge.*

Proof. From Proposition 2.3.5, an optimal flow f^{opt} feasible for instance (G, d, l) is at traffic equilibrium for instance (G, d, l^*), where $l_e^*(x) = l_e(x) + l_e'(x)x$ and $l_e'(x)$ is the derivative $\frac{d}{dx} l_e(x)$ of l_e. Here, $l_e(x) = a_e x + b_e$, and $l_e^*(x) = 2a_e x + b_e$.

Let us choose an edge e with $f_e^{opt} > 0$. Then

$$\begin{aligned} l_e^*(f_e^{opt}) &= 2a_e f_e^{opt} + b_e \\ &> b_e \\ &\geq b_{e_s} = l_{e_s}^*(0). \end{aligned}$$

The first inequality holds due to the assumption that $a_e > 0$. The second inequality holds since b_{e_s} is the smallest among all b_e. Since $l_e^*(f_e^{opt}) > l_{e_s}^*(0)$ and f^{opt} is at traffic equilibrium for instance (G, d, l^*), the proposition follows.

□

In the next lemma, we derive properties of an optimal flow f^{opt} by using the fact that an optimal flow f^{opt} for instance (G, d, l) is at traffic equilibrium for instance (G, d, l^*), and b_{e_s} is the smallest among all b_e.

Proposition 3.3.7. *Let f^{opt} be an optimal flow for instance (G, d, l). Then we have $l_{e_s}(f_{e_s}^{opt}) \leq l_e(f_e^{opt})$, $\forall e \in E$. In addition, we have $a_{e_s} f_{e_s}^{opt} \geq a_e f_e^{opt}$, $\forall e \in \mathcal{E}^{opt}$, where $\mathcal{E}^{opt} = \{e \in E \mid f_e^{opt} > 0\}$.*

Proof. First we show that $l_{e_s}(f_e^{opt}) \leq l_e(f_e^{opt})$, $\forall e \in E$.

$$\begin{aligned} l_e(f_e^{opt}) &= \frac{1}{2}\left(l_e^*(f_e^{opt}) + b_e\right) \\ &\geq \frac{1}{2}\left(l_{e_s}^*(f_{e_s}^{opt}) + b_e\right) \\ &\geq \frac{1}{2}\left(l_{e_s}^*(f_{e_s}^{opt}) + b_{e_s}\right) \\ &= l_{e_s}(f_{e_s}^{opt}). \end{aligned}$$

The first inequality follows from Proposition 3.3.6. The second inequality holds since b_{e_s} is the smallest among all b_e.

We show that $a_{e_s} f_{e_s}^{opt} \geq a_e f_e^{opt}$, $\forall e \in \mathcal{E}^{opt}$. Since f^{opt} is at traffic equilibrium for instance (G, d, l^*), every edge $e \in \mathcal{E}^{opt}$ has the same latency in terms of $l_e^*(x)$. Then we have the following:

$$
\begin{aligned}
& l_e^*(f_e^{opt}) - l_{e_s}^*(f_{e_s}^{opt}) = 0 \\
& \Rightarrow 2\left(a_e f_e^{opt} - a_{e_s} f_{e_s}^{opt}\right) + (b_e - b_{e_s}) = 0 \\
& \Rightarrow \left(a_e f_e^{opt} - a_{e_s} f_{e_s}^{opt}\right) \leq 0.
\end{aligned}
$$

□

In the next lemma, we bound the term $\frac{C(\tilde{f}^o)}{C(f^{opt})}$ by using a lower bound of $C(f^{opt})$ and a upper bound of $C(\tilde{f}^o)$.

Lemma 3.3.8. *Let a flow $\tilde{f} = \tilde{f}^o + \tilde{f}^*$ be at oblivious equilibrium for instance (G, d, l, α). Let f^{opt} be an optimal flow for instance (G, d, l). Then we have $C(\tilde{f}^o) \leq \alpha \max\{1, \alpha r\} C(f^{opt})$, where $r = \sum_{e \in \mathcal{E}^{opt}} (a_{e_s}/a_e)$.*

Proof.

$$
\begin{aligned}
C(f^{opt}) &= \sum_{e \in E} l_e(f_e^{opt}) f_e^{opt} \\
&\geq l_{e_s}(f_{e_s}^{opt}) \sum_{e \in E} f_e^{opt} \\
&= \left(a_{e_s} f_{e_s}^{opt} + b_{e_s}\right) d
\end{aligned}
$$

$$\geq \left(\frac{a_{e_s}}{r} d + b_{e_s}\right) d.$$

From the first part of Proposition 3.3.7, we have $l_{e_s}(f^{opt}_{e_s}) \leq l_e(f^{opt}_e)$, $\forall e \in \mathcal{E}^{opt}$. Hence, the first inequality holds. From the second part of Proposition 3.3.7, we have $a_{e_s} f^{opt}_{e_s} \geq a_e f^{opt}_e$, $\forall e \in \mathcal{E}^{opt}$. We get $r f^{opt}_{e_s} \geq d$ by summing over all $e \in \mathcal{E}^{opt}$. Then

$$\frac{C(\tilde{f}^o)}{C(f^{opt})} \leq \frac{(a_{e_s} \alpha d + b_{e_s})\,\alpha d}{\left(\frac{a_{e_s}}{r} d + b_{e_s}\right) d} \leq \alpha \max\{1, \alpha r\}.$$

For the second inequality, consider two cases $\alpha r \leq 1$ and $\alpha r \geq 1$. In case $\alpha r \leq 1$, the term $(a_{e_s} \alpha d + b_{e_s}) \,/\, \left(\frac{a_{e_s}}{r} d + b_{e_s}\right) \leq 1$. In case $\alpha r \geq 1$, the term $(a_{e_s} \alpha d + b_{e_s}) \,/\, \left(\frac{a_{e_s}}{r} d + b_{e_s}\right) \leq \alpha r$. □

The next theorem is our main result for parallel links networks. We bound the price of anarchy of an oblivious equilibrium flow $\tilde{f}$ by combining the two bounds for $\frac{C(\tilde{f}^o)}{C(f^{opt})}$ and $\frac{C(\tilde{f}^*)}{C(f^{opt})}$.

Theorem 3.3.9. *Let $\tilde{f} = \tilde{f}^o + \tilde{f}^*$ be a flow at oblivious equilibrium for instance (G, d, l, α). Let f^* and f^{opt} be a Nash flow and an optimal flow, respectively, for instance (G, d, l). The price of anarchy $\rho = \frac{C(\tilde{f})}{C(f^{opt})}$ at oblivious equilibrium is:*

$$\rho \leq (1-\alpha)\frac{C(f^*)}{C(f^{opt})} + \alpha \max\{1, \alpha r\}, \text{ if } \tilde{f}^o_{e_s} = \alpha d \geq f^*_{e_s},$$

and

$$\rho \leq \frac{4}{3}, \text{ otherwise.}$$

Proof. In case $\tilde{f}^o_{e_s} = \alpha d < f^*_{e_s}$, we have $\tilde{f} = f^*$ from Proposition 3.3.1. Hence, the price of anarchy at oblivious equilibrium is equal to the price of anarchy at traffic equilibrium, which is $\frac{4}{3}$ [24], and the second part of the theorem follows.

In case $\tilde{f}^o_{e_s} = \alpha d \geq f^*_{e_s}$, we have $\tilde{f}^*_e \tilde{f}^o_e = 0$, $\forall e \in E$ from Proposition 3.3.1. Hence, $C(\tilde{f})$ can be separated into $C(\tilde{f}^*)$ and $C(\tilde{f}^o)$. Then

$$\begin{aligned}
C(\tilde{f}) &= \sum_{e\in E} l_e(\tilde{f}^*_e + \tilde{f}^o_e)(\tilde{f}^*_e + \tilde{f}^o_e) \\
&= \sum_{e\in E} l_e(\tilde{f}^*_e)(\tilde{f}^*_e) + \sum_{e\in E} l_e(\tilde{f}^o_e)(\tilde{f}^o_e) \\
&= C(\tilde{f}^*) + C(\tilde{f}^o) \\
&\leq (1-\alpha)C(\tilde{f}) + C(\tilde{f}^o) \\
&\leq (1-\alpha)C(\tilde{f}) + \alpha \max\{1, \alpha r\} C(f^{opt}).
\end{aligned}$$

The first inequality comes from Lemma 3.3.5, and the second inequality comes from Lemma 3.3.8. □

The bound of the price of anarchy at oblivious equilibrium in Theorem 3.3.9 depends on $C(f^{opt})$ and $C(f^*)$, which can be calculated by convex program (NLP1) and (NLP2), respectively. We consider $C(f^*)$ and $C(f^{opt})$ as network parameters since they can be calculated from the given basic network parameters. However, we also show another bound that does not depend on these two cost functions. We derive the new bound by applying the worst ratio of $C(f^*)$ over $C(f^{opt})$, i.e., 4/3. Then the next corollary follows.

Corollary 3.3.10. *Let $\tilde{f} = \tilde{f}^o + \tilde{f}^*$ be a flow at oblivious equilibrium for instance (G, d, l, α). Let f^* and f^{opt} be a Nash flow and an optimal flow,*

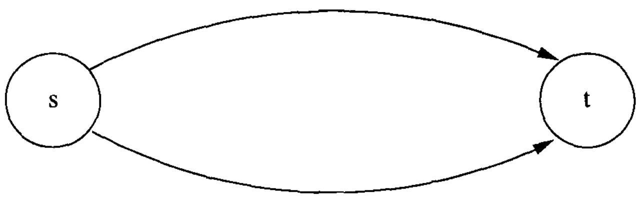


Figure 3.1: A tight example for parallel links networks, with demand 1

respectively, for instance (G, d, l). The price of anarchy $\rho = \frac{C(\tilde{f})}{C(f^{opt})}$ at oblivious equilibrium is:

$$\rho \leq (1-\alpha)\frac{4}{3} + \max\{\alpha, \alpha^2 r\}, \text{ if } \tilde{f}^o_{e_s} = \alpha d \geq f^*_{e_s},$$

and

$$\rho \leq \frac{4}{3}, \text{ otherwise.}$$

3.3.2 A tight example for parallel links networks

We provide a tight example for the bound of the price of anarchy at oblivious equilibrium in Theorem 3.3.9. In case $\tilde{f}^o_{e_s} = \alpha d \leq f^*_{e_s}$, we have $\tilde{f} = f^*$, and the price of anarchy at oblivious equilibrium is bounded by $\frac{4}{3}$. Since $\tilde{f} = f^*$, the tight example 2.1(b) for the price of anarchy at traffic equilibrium shows the tightness of our bound.

In case $\tilde{f}^o_{e_s} = \alpha d \geq f^*_{e_s}$, we consider a parallel links network with demand 1, and with two edges that have latency functions $l(x) = \left(\frac{1-\alpha}{\alpha}\right) x$ and

$l(x) = x + \epsilon$. Figure 3.1 shows this example. Here, α can be an arbitrary value between $(0, 1]$, and non-negative ϵ tends to 0. Since the upper edge has the smallest latency when no flows are circulating, oblivious users route their flow on the upper edge , i.e., $\tilde{f^o} = (\alpha, 0)$, where the first element is the upper edge flow. After the routing of oblivious users, selfish users route $\alpha\epsilon$ amount of flow on the upper edge, and $(1 - \alpha - \alpha\epsilon)$ amount of flow on the lower edge, i.e., $\tilde{f^*} = (\alpha\epsilon, 1 - \alpha - \alpha\epsilon)$. As ϵ tends to 0, we get $\tilde{f} = (\alpha, 1 - \alpha)$. Also, $f^{opt} = (\alpha, 1 - \alpha)$. Hence, $\frac{C(\tilde{f})}{C(f^{opt})} = 1$ and our bound is also 1. To make this example reachable, we set ϵ to 0 under the assumption that oblivious users choose the upper edge.

3.4 General topologies

We study the price of anarchy at oblivious equilibrium in general topologies, the setting of which has been specified in the section 3.1. In addition, let us define $D = \sum_{i \in K} d_i$, where $K = \{1, ..., k\}$ are the commodities. We assume that edge latency functions are linear and increasing, i.e., $l_e(f_e) = a_e f_e + b_e$ with $a_e \geq 0$.

3.4.1 The price of anarchy

In general topologies, we separate the total cost $C(\tilde{f})$ into two terms: $C^{\tilde{f}}(\tilde{f^*})$ and $C^{\tilde{f}}(\tilde{f^o})$. Recall that $C^{\tilde{f}}(\tilde{f^*}) = \sum_{e \in E} l_e(\tilde{f}_e)\tilde{f}^*_e$. Whereas, in parallel links networks, we have separated the total cost into two terms $C^{\tilde{f^*}}(\tilde{f^*})$ and $C^{\tilde{f^o}}(\tilde{f^o})$

due to Proposition 3.3.1, this does not hold any longer in general topologies. The path of oblivious users can share edges with the paths of selfish users. Routing of selfish users might have an influence on $C^{\tilde{f}}(\tilde{f}^o)$, and routing of oblivious users on $C^{\tilde{f}}(\tilde{f}^*)$.

We bound the cost term $C^{\tilde{f}}(\tilde{f}^*)$ with respect to two cost terms $C(\tilde{f})$ and $C(f^{opt})$. First the term $C^{\tilde{f}}(\tilde{f}^*)$ can be bounded by $(1-\alpha)C^{\tilde{f}}(f^{opt})$ as an intermediate step by using the method of variational inequalities. Then the term $C^{\tilde{f}}(f^{opt})$ is separated into $C(\tilde{f})$ and $C(f^{opt})$ by using the β-function.

Lemma 3.4.1. *Let $\tilde{f} = \tilde{f}^o + \tilde{f}^*$ be a flow at oblivious equilibrium for instance (G, d, l, α), and f^{opt} be an optimal flow for instance (G, d, l). Let $\mathcal{L}$ be a class of linear latency functions. Then*

$$C^{\tilde{f}}(\tilde{f}^*) \leq (1-\alpha)\left(\beta(\mathcal{L})C(\tilde{f}) + C(f^{opt})\right).$$

Proof. Since flow $\tilde{f}^*$ is at traffic equilibrium for instance $(G, (1-\alpha)d, \tilde{l})$, where $\tilde{l}_e(x) = l_e(\tilde{f}_e^o + x)$, $\tilde{f}^*$ is a solution to the following variational inequality:

$$\langle \tilde{l}(\tilde{f}^*), \tilde{f}^* \rangle \leq \langle \tilde{l}(\tilde{f}^*), f \rangle, \quad \forall f \in H. \tag{3.3}$$

H is the set of feasible flows for instance $(G, (1-\alpha)d, \tilde{l})$, and $(1-\alpha)f^{opt}$ is in H. If we substitute f with $(1-\alpha)f^{opt}$ in (3.3), we have

$$\begin{aligned} C^{\tilde{f}}(\tilde{f}^*) &= \langle \tilde{l}(\tilde{f}^*), \tilde{f}^* \rangle \\ &\leq (1-\alpha)\langle \tilde{l}(\tilde{f}^*), f^{opt} \rangle \end{aligned}$$

$$= (1-\alpha)\langle l(\tilde{f}), f^{opt}\rangle$$
$$= (1-\alpha)C^{\tilde{f}}(f^{opt}). \tag{3.4}$$

We use the concept of β-function to bound the term $C^{\tilde{f}}(f^{opt})$.

$$C^{\tilde{f}}(f^{opt}) \leq \sum_e \beta(\tilde{f}_e, l_e) l_e(\tilde{f}_e)\tilde{f}_e + \sum_e l_e(f_e^{opt}) f_e^{opt}$$
$$\leq \beta(\mathcal{L})C(\tilde{f}) + C(f^{opt}). \tag{3.5}$$

The first and the second inequalities follow from the definition of β-function: $\beta(\mathcal{L}) = \sup_{l\in\mathcal{L}} \sup_{v\geq 0} \frac{1}{vl(v)} \max_{x\geq 0}\{x\,(l(v) - l(x))\}$. The combination of (3.4) and (3.5) proves the lemma. □

We bound the cost term $C^{\tilde{f}}(\tilde{f}^o)$ in the next lemma. This bound will depend only on the coefficients a_e and b_e, α, and D. Since the term is not bounded with respect to $C(f^{opt})$, the bound of $\frac{C^{\tilde{f}}(\tilde{f}^o)}{C(f^{opt})}$ will include the factor $1/C(f^{opt})$. We assume that the factor $1/C(f^{opt})$ in the bound is a network parameter, since it can be calculated by using the convex program (NLP1).

Lemma 3.4.2. *Let $\tilde{f} = \tilde{f}^o + \tilde{f}^*$ be a flow at oblivious equilibrium for instance (G, d, l, α). Then*

$$C^{\tilde{f}}(\tilde{f}^o) \leq \alpha D \sum_{i\in K} \sum_{e\in P_i^s} (a_e D + b_e).$$

Proof.

$$C^{\tilde{f}}(\tilde{f}^o) = \sum_{e\in E} (a_e \tilde{f}_e + b_e)\tilde{f}_e^o$$

$$\leq \alpha D \sum_{i \in K} \sum_{e \in P_i^s} (a_e \tilde{f}_e + b_e)$$

$$\leq \alpha D \sum_{i \in K} \sum_{e \in P_i^s} (a_e D + b_e).$$

□

We show another bound for the cost term $C^{\tilde{f}}(\tilde{f}^o)$. This time, we bound the term $C^{\tilde{f}}(\tilde{f}^o)$ with respect to $C(f^{opt})$.

Lemma 3.4.3. *Let $\tilde{f} = \tilde{f}^o + \tilde{f}^*$ be a flow at oblivious equilibrium for instance (G, d, l, α), and f^{opt} be an optimal flow for instance (G, d, l). Then*

$$C^{\tilde{f}}(\tilde{f}^o) \leq \frac{n \alpha D \gamma_a}{f^{opt}_{min}} C(f^{opt}),$$

where $n = |V|$, $\gamma_a = \frac{\max_e a_e}{\min_e a_e}$, and $f^{opt}_{min} = \min_{e: f^{opt}_e > 0} f^{opt}_e$.

Proof. Since a flow $\tilde{f}^o$ is at traffic equilibrium for instance $(G, \alpha d, \tilde{l})$, where $\tilde{l}_e(x) = b_e$, $\tilde{f}^o$ is a solution to the following variational inequality:

$$\langle \tilde{l}(\tilde{f}^o), \tilde{f}^o \rangle \leq \langle \tilde{l}(\tilde{f}^o), f \rangle, \quad \forall f \in H.$$

H is the set of feasible flows for instance $(G, \alpha d, \tilde{l})$, and αf^{opt} is in H. If we substitute f with αf^{opt}, then we get the following:

$$\sum_{e \in E} b_e \tilde{f}^o_e \leq \alpha \sum_{e \in E} b_e f^{opt}_e. \tag{3.6}$$

To get a upper bound of $C^{\tilde{f}}(\tilde{f^o})$:

$$\begin{aligned} C^{\tilde{f}}(\tilde{f^o}) &= \sum_{e \in E} \left(a_e \tilde{f}_e \tilde{f^o_e} + b_e \tilde{f^o_e} \right) \\ &\stackrel{(3.6)}{\leq} \sum_{e \in E} a_e \tilde{f}_e \tilde{f^o_e} + \alpha \sum_{e \in E} b_e f_e^{opt}. \end{aligned} \tag{3.7}$$

To get a upper bound of the first term:

$$\begin{aligned} \sum_{e \in E} a_e \tilde{f}_e \tilde{f^o_e} &\leq a_{max} D \sum_{e \in E} \tilde{f^o_e} \\ &\leq a_{max} D \sum_{i \in K} n \tilde{f^o_{p_i^s}} \\ &= a_{max} D \sum_{i \in K} n \alpha d_i \\ &\leq n \alpha D \gamma_a (a_{min} D) \\ &\leq n \alpha D \gamma_a \sum_{e \in E} a_e f_e^{opt} \\ &\leq \frac{n \alpha D \gamma_a}{f_{min}^{opt}} \sum_{e \in E} a_e f_e^{opt2}. \end{aligned} \tag{3.8}$$

Since $\frac{n\alpha D\gamma_a}{f_{min}^{opt}} \geq \alpha$, the combination of (3.7) and (3.8) proves the lemma. □

The next theorem is our main result for general topologies. We bound the price of anarchy of an oblivious equilibrium flow $\tilde{f}$ by combining the two bounds for $\frac{C^{\tilde{f}}(\tilde{f^o})}{C(f^{opt})}$ and $\frac{C^{\tilde{f}}(\tilde{f^*})}{C(f^{opt})}$.

Theorem 3.4.4. *Let $\tilde{f} = \tilde{f^o} + \tilde{f^*}$ be a flow at oblivious equilibrium for instance (G, d, l, α), and f^{opt} be an optimal flow for instance (G, d, l). Then the price* *of anarchy at oblivious equilibrium is*

$$\frac{C(\tilde{f})}{C(f^{opt})} \leq \frac{4\left(1-\alpha+\alpha D\sum_{i\in K}\sum_{e\in P_i^s}(a_eD+b_e)/C(f^{opt})\right)}{3+\alpha}.$$

Proof. By combining the two terms from Lemmata 3.4.1 and 3.4.2, we have the following:

$$\begin{aligned} C(\tilde{f}) &= C^{\tilde{f}}(\tilde{f}^*) + C^{\tilde{f}}(\tilde{f}^o) \\ &\leq (1-\alpha)C(f^{opt}) + (1-\alpha)\beta(\mathcal{L})C(\tilde{f}) + \alpha D\sum_{i\in K}\sum_{e\in P_{s_i}}(a_eD+b_e). \end{aligned}$$

Hence,

$$\frac{C(\tilde{f})}{C(f^{opt})} \leq \frac{1-\alpha+\alpha D\sum_{i\in K}\sum_{e\in P_{s_i}}(a_eD+b_e)/C(f^{opt})}{1-(1-\alpha)\beta(\mathcal{L})}.$$

Since $\mathcal{L}$ is a set of non-decreasing linear functions, $\beta(\mathcal{L}) = \frac{1}{4}$ [8], and the theorem follows. □

The bound for the price of anarchy at oblivious equilibrium in Theorem 3.4.4 depends on $C(f^{opt})$, which can be calculated by the convex program (NLP1). We show another bound that depends on f^{opt}_{min}, which also can be calculated by convex program (NLP1).

Theorem 3.4.5. *Let $\tilde{f} = \tilde{f}^o + \tilde{f}^*$ be a flow at oblivious equilibrium for instance (G, d, l, α), and f^{opt} be an optimal flow for instance (G, d, l). Then the price* *of anarchy at oblivious equilibrium is*

$$\frac{C(\tilde{f})}{C(f^{opt})} \leq \frac{4\left(1 - \alpha + n\alpha D\gamma_a / f_{min}^{opt}\right)}{3 + \alpha}.$$

Proof. By combining the two terms from Lemmata 3.4.1 and 3.4.3, we have the following:

$$\begin{aligned} C(\tilde{f}) &= C^{\tilde{f}}(\tilde{f}^{*}) + C^{\tilde{f}}(\tilde{f}^{o}) \\ &\leq (1-\alpha)C(f^{opt}) + (1-\alpha)\beta(\mathcal{L})C(\tilde{f}) + \frac{n\alpha D\gamma_a}{f_{min}^{opt}} C(f^{opt}). \end{aligned}$$

Hence,

$$\frac{C(\tilde{f})}{C(f^{opt})} \leq \frac{1 - \alpha + n\alpha D\gamma_a / f_{min}^{opt}}{1 - (1-\alpha)\beta(\mathcal{L})}.$$

Since $\mathcal{L}$ is a set of non-decreasing linear functions, $\beta(\mathcal{L}) = \frac{1}{4}$ [8] , and the theorem follows. □

3.4.2 A tight example for general topologies

We provide a tight example for the bound in Theorem 3.4.4. Suppose the Braess paradox network of Figure 3.2, with a single commodity from vertex s to t and demand 1. Here, we have a fraction α of oblivious users, and α can be an arbitrary value between $[0, 1]$. Oblivious users will follow the path $P_s : s \rightarrow v \rightarrow w \rightarrow t$. So do selfish users. However, an optimal flow will route the first half demand through $P_1 : s \rightarrow v \rightarrow t$, and the other half demand

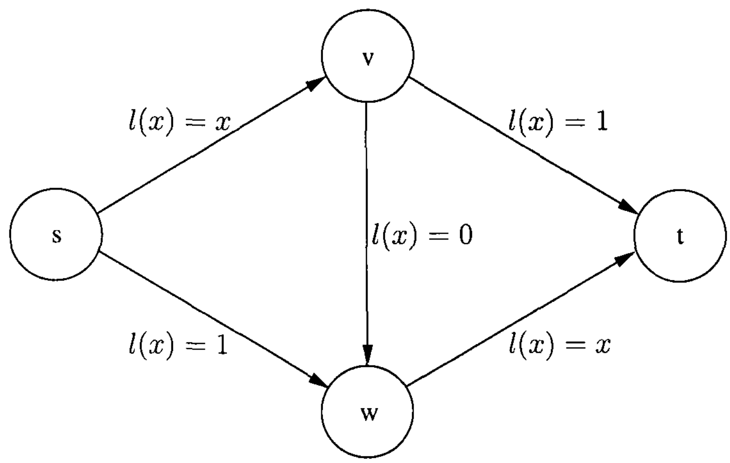


Figure 3.2: A tight example for general topologies, with demand 1

through $P_2 : s \rightarrow w \rightarrow t$. Hence, $C(\tilde{f})$ is 2, $C(f^{opt})$ is 3/2, and the price of anarchy at the oblivious equilibrium is 4/3. Our bound has also the value 4/3.

Chapter 4

Conclusions

We evaluate our bounds of the price of anarchy at oblivious equilibrium, which can be unbounded depending on network parameters. Especially, we discuss how to adjust edge latency functions to minimize the ratio of the total cost caused by both selfish users and oblivious users over the optimal total cost. Because our bounds characterize the network parameters that affect the performance degradation of networks at oblivious equilibrium, our bounds can be utilized to design networks with the aim of minimizing the performance degradation at oblivious equilibrium. However, our bounds do not give a method how to minimize the optimal cost itself. It is an open question how to minimize both the price of anarchy and the optimal total cost in networks with both selfish users and oblivious users.

We evaluate our tight bound for parallel links networks, which is shown in Theorem 3.3.9. Let $\tilde{f}$ be a flow at oblivious equilibrium for instance

(G, d, l, α). Let f^* and f^{opt} be a Nash flow and an optimal flow, respectively, for instance (G, d, l). Our bound is:

$$\frac{C(\tilde{f})}{C(f^{opt})} \leq (1-\alpha)\frac{C(f^*)}{C(f^{opt})} + \alpha \max\{1, \alpha r\}, \text{ if } \tilde{f^o_{e_s}} = \alpha d \geq f^*_{e_s},$$

and

$$\frac{C(\tilde{f})}{C(f^{opt})} \leq \frac{4}{3}, \text{ otherwise.}$$

We discuss the case $\tilde{f^o_{e_s}} = \alpha d \geq f^*_{e_s}$, where the price of anarchy at oblivious equilibrium can be arbitrary large by increasing $r = \sum_{e \in \mathcal{E}^{opt}} a_{e_s}/a_e$. In this case, the other parameters are bounded: $\alpha \in [0, 1]$, and $\frac{C(f^*)}{C(f^{opt})} \in [1, \frac{4}{3}]$. We assume that the fraction α of oblivious users is given.

Our bound has the unbounded term r, while the other term $\frac{C(f^*)}{C(f^{opt})}$ is bounded by $4/3$. The term r can be arbitrary large depending on the coefficients a_e in the situation that oblivious users can suffer arbitrary large congestion by increasing the coefficient a_{e_s}, or that an optimal flow can have very small total cost by decreasing the coefficients $a_e \in \mathcal{E}^{opt}$. Hence, to minimize the term r, the coefficients a_e should be adjusted. Suppose that $a_{e_s} = 0$. In this case, traffic routing becomes so trivial that all users route their flow only on the edge e_s. We would want to avoid this trivial case. In another case, oblivious users and selfish users can be split into two different edges, which is illustrated by Figure 3.1. In this case, we have $r = 1/\alpha$, and our bound is 1. However, if we substitute the term $\frac{C(f^*)}{C(f^{opt})}$ with $4/3$, our bound can not be tight any longer in this example. Because the term $\frac{C(f^*)}{C(f^{opt})}$ can not be set directly from the basic network parameters such as the coefficients a_e and b_e, but be calculated by convex programs, it might be difficult for network designers to determine its value. It is an open problem to remove the the term $\frac{C(f^*)}{C(f^{opt})}$ from the bound without losing the tightness.

We evaluate our tight bound for general topologies, which is shown in Theorem 3.4.4. Let $\tilde{f}$ be a flow at oblivious equilibrium for instance (G, d, l, α). Let f^{opt} be an optimal flow for instance (G, d, l). Our bound is:

$$\frac{C(\tilde{f})}{C(f^{opt})} \leq \frac{4\left(1 - \alpha + \alpha D \sum_{i \in K} \sum_{e \in P_i^s}(a_e D + b_e)/C(f^{opt})\right)}{3 + \alpha}.$$

Suppose that the total demand D is fixed. The parameter α is bounded between 0 and 1. But, the term $\sum_{i \in K} \sum_{e \in P_i^s}(a_e D + b_e)/C(f^{opt})$ can be arbitrarily large by increasing the coefficients a_e in the predefined shortest paths.

The coefficients a_e and b_e, which belong to the edges of the shortest paths, are the network parameters that we can adjust to minimize the ratio, if we suppose that the total demand D is given, and that the fraction α of oblivious users is given. However, as the values of the coefficients a_e and b_e decrease, the term $\sum_{i \in K} \sum_{e \in P_i^s}(a_e D + b_e)/C(f^{opt})$ does not always decrease correspondingly. The term has the denominator $C(f^{opt})$, which will also be decreased as the coefficients a_e and b_e decrease. Therefore, it is not apparent how to adjust the coefficients a_e and b_e before calculating the factor $C(f^{opt})$ with the corresponding coefficients a_e and b_e. Hence, it is an open problem to remove the factor $C(f^{opt})$ from the bound without losing the tightness.

We evaluate another bound for general topologies, which is shown in

Theorem 3.4.5. Let $\tilde{f}$ be a flow at oblivious equilibrium for instance (G, d, l, α). Let f^{opt} be an optimal flow for instance (G, d, l). Our bound is:

$$\frac{C(\tilde{f})}{C(f^{opt})} \leq \frac{4\left(1 - \alpha + n\alpha D\gamma_a / f^{opt}_{min}\right)}{3 + \alpha}.$$

The bound does not depend on the factor $C(f^{opt})$, but it is not tight and depends on the factor $1/f^{opt}_{min}$. Since the bound is not tight, the method of using this bound does not precisely determine the degradation of the network performance. While knowing the drawback of using the non-tight bound, however, we discuss about the conditions of the parameters that minimize the bound. We suppose that parameters α, and n are fixed. The other terms γ_a, and D/f^{opt}_{min} can be arbitrary large by changing the coefficients a_e and b_e.

Let us consider the term γ_a, which is minimized when all coefficients a_e have the same value. This condition of the coefficients a_e implies that each unit of traffic causes the same degree of congestion for every edge. For example, the roads that have the same number of lanes will have the same degree of congestion by the unit traffic. Let us consider the next term D/f^{opt}_{min}. This term captures the smallest fraction of the total demand that an optimal flow splits over paths. For example, if an optimal flow is routed on a single path, the term has the value 1. If an optimal flow is routed on two paths evenly, the term has the value 2. With the increase of the path number that an optimal flow routes on, the value of the term increases , but the ratio does not necessarily increase. Hence, in networks with more paths, the bound might be getting worse with respect to the actual ratio. It is an open problem to remove the term D/f_{min}^{opt} to tighten the bound.

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