PARTLY NON-LOCAL POTENTIALS IN THE MODEL TRITON

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by

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IF YOU FIND MISTAKES IN THIS PUBLICATION, PLEASE CONSIDER THAT THEY ARE THERE FOR A PURPOSE. WE PUBLISH SOMETHING FOR EVERYONE, AND SOME PEOPLE ARE ALWAYS LOOKING FOR MISTAKES !!!

ABSTRACT

Properties of the model triton have been calculated for two classes of phase equivalent potentials. The partly non-local (PNL) potentials are an attempt to simulate what we know about the nucleon-nucleon interaction, combining long-range locality with short-range non-locality. Rank-two separable potentials are also used for comparison, being constructed not only phase equivalent to the total interaction, but with the attractive part phase equivalent to the local part of the partly non-local potentials. Triton binding energies, wavefunctions, form factors and asymptotic normalization constants have been calculated for all these potentials. Comparisons are made between the two classes of potentials and other types of phase equivalent potentials. Deuteron, triton and nuclear matter properties are discussed.

Both classes of potentials give increasing triton binding energy, E_T, as the attractive part of the potential is weakened, with a sharp rise near the limiting condition, when the attractive part of the interaction produces the same binding energy as the complete interaction. In all cases, however, each PNL potential gives 1 to 2 MeV less binding energy than its purely separable counterpart. Two-body properties such as the zero-energy wound integral and deuteron wavefunction greatly influence the triton binding energy.

The PNL potentials give much more realistic form factors than do the separable potentials although this is due mainly to their lower values of E_T. However, none of the potentials reproduce the experimental form factor of ^{3}He near and beyond the diffraction minimum Q^2_{min}. There is a strong resemblance of deuteron and triton form factors. The effects of the neutron and proton form factors is discussed. The new observable, the triton asymptotic normalization constant, C_T, is not sensitive to changes in the PNL potentials except in the limiting case. It does vary, for the separable potentials, increasing with increasing E_T, as the attractive part of the interaction weakens.

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CHAPTER I

INTRODUCTION

This is a study of some of the properties of the model triton using partly non-local potentials. The model triton is a simplified version of the ground state of the neutron-neutron-proton system, the ^{3}H nucleus. If we ignore the electromagnetic force, then neutrons and protons can be regarded as different states of the same particle. Thus the proton-proton-neutron system, the ^{3}He nucleus, can be treated in the same way; the Coulomb energy can be added later as a perturbation. The primary motivation for the study of the three nucleon problem is that we still do not know how two nucleons interact. "There are few problems in modern theoretical physics which have attracted more attention than that of determining the fundamental interaction between two nucleons. It is also true that scarcely ever has the world of physics owed so little to so many" (Gol 60).

The most important component of the interaction between two nucleons is the strong force. Although it is known that the strong interaction is due to the exchange of mesons, the concept of using a potential to describe the

interaction between two nucleons is thought to be a useful one, since once we have specified a potential, then we have a dynamical model, namely the Schrödinger equation, or some equivalent, which can be applied to any nuclear problem. However, the experimental two-body data are not sufficient to determine the nuclear force uniquely. The principal data for the two-nucleon system are the deuteron binding energy and the proton-proton and neutron-proton phase shifts. Even if these were known to arbitrary accuracy and at all energies, it would not be possible to determine a unique potential without making further assumptions about its form, such as locality or separability. It is therefore necessary to study many-nucleon systems in order to gain further information about the nuclear force. After nuclear matter, the three nucleon system is the simplest many-body problem, and one on which experiments can be performed. Yet only in the last few years have theoretical and computational advances been made, which make it possible to solve even the three-body bound state problem using realistic potentials. Scattering problems are being attacked with some success. The four-body problem is an order of magnitude more difficult although the Faddeev equations have been generalized and preliminary attempts made at their solution (Tjo 76).

By studying the three nucleon system we hope to differentiate between potentials which fit the experimental

two-nucleon data. Ideally we would like to learn about such things as the tensor force, the hard core, the non-locality, the neutron-neutron force, three-body forces, relativistic effects, mesonic degrees of freedom, and so on. The first and perhaps only success along these lines came in 1935 when Thomas (Tho 35) showed that a zero-range force, which at that time was compatible with experiment, gave infinite binding energy for the triton. After that time the nuclear data and force became more complicated at a faster rate than our ability to solve the triton. It was 1969 (Del+ 69) before a (variational) calculation of the triton binding energy was performed using a realistic potential. Meanwhile two important advances had been made: since 1962 Mitra and his co-workers had solved the triton using separable potentials with the Schrödinger equation in momentum space (Mit 69); in 1961 the great breakthrough had come when Faddeev (Fad 61) had reformulated the three-body Lippmann-Schwinger equation into a form which removed the singularities that had been a plague since day one. However, these Faddeev equations were three coupled integral equations in six continuous variables and still too complicated to treat directly. Various methods of angular momentum decomposition have been attempted but the most successful is that of Ahmadzadeh and Tjon (AT 65) which reduced the problem to two continuous variables and many coupled equations. For realistic nuclear forces, with tensor forces and the like, the

problem is still more complicated and only recently have numerical techniques and computers advanced to a stage where calculations can be made for realistic nuclear forces.

The current situation for the bound state problem (Tjo 76) is that realistic potentials give 1 to 1.7 MeV less binding for ^{3}H than the experimental value of 8.5 MeV. Moreover there is good agreement between various groups who use different methods of solution. There is, however, some disparity for the dip in the charge form factor of ^{3}He between variational methods (based on the Schroedinger equation in coordinate space) and integral equation methods (based on the Faddeev equations in momentum space. In addition, modifications of the potentials (without changing the two-nucleon phase shifts and binding energy) which increase triton binding energies toward better agreement with experiment, tend also to push out to larger momentum transfer the minimum of the form factor, giving poorer agreement with experiment. The popular explanation for these discrepancies with experiment is that they are due to mesonic effects - three-body forces and exchange currents and the like. This is discussed further in Chapter IV.

A much simpler system than the real triton is the model triton. The ground state of the triton is predominantly a space symmetric state in which the average force between the nucleon pairs is one half the central force in the triplet even state of the two-body system, plus one half the central

singlet even force. (Why this is so will be discussed in Chapter II, where the model triton is treated in more detail.) This state, called the principal S state makes up more than 90% of the ground state probability of the triton, even in the presence of strong tensor forces, and is the only state if we consider a central, spin- and isospin-independent, force. By taking a force which is the average of the singlet and triplet forces, the problem simplifies to that of three identical bosons. Since the complication of spin no longer enters, it is possible to calculate (model) triton properties for a large number of potentials for the same amount of effort as would be required to do a realistic calculation. Moreover effects which are important in the model triton should also be of comparable importance in the real triton.

When comparing properties (which depend on the off-shell T-matrix) of various potentials, it is important that these potentials be strictly phase-equivalent (deG 75, p. 57). Two potentials are said to be phase-equivalent if they produce the same two-body binding energy and identical phase shifts at all energies. All of the potentials used in this study are of this form. The standard potential is the average of two potentials fit to the $\ell=0$ singlet and triplet n-p phase shifts (Tab 65, potential 1 of Table 1). It will be referred to as ST. This potential is then used to generate a set of phase shifts. Together with the binding energy of the model deuteron (0.428 MeV) they constitute the experimental

data to which the other potentials used here are fitted.

Two classes of potentials are constructed (Sin 72, SWB 72):

(i) partly non-local potentials, having a local attractive part and a repulsive separable part adjusted so as to ensure phase equivalence. These will be referred to as YBS plus a number.

(ii) rank-two separable potentials, in which the attractive term is made phase equivalent to the local part of the partly non-local potentials, and again a repulsive term adjusted to give overall phase equivalence. These will be identified by YSS plus a number.

There are a number of reasons for choosing these potentials. The nuclear force is predominantly local at large distances and non-local at short range. The simplest form of non-locality is separability. Thus if the separable part of these potentials is of short range, they will be physically appealing. The local part is a Bargmann potential which unfortunately does not have the one-pion-exchange tail that a realistic potential should have. However, this is a model calculation in which the difference between locality and separability is being studied. The form of the potential makes many formulae expressible in analytic form and simplifies the overall calculation. Another reason for using these potentials is that they are fitted to the average of a

singlet and a triplet force - just what the model triton needs. The particular choice of potentials made here allows a direct comparison to be made with previous work on this problem. Fiedeldey has done similar model triton calculations with rank-two separable potentials and for certain limiting potentials obtained abnormally high triton binding energies. Moreover, he used an inversion procedure similar to the one used here and used the same phase shifts as input! Thus a comparison can be made, not only between two classes of rank-two separable potentials, but also between separable and partly local potentials. Fiedeldey's results inspired workers at McMaster to use his potentials in nuclear matter calculations (SSB 70). They found that those potentials which gave very high triton binding energy gave very low binding energy in nuclear matter and that the variation was much more pronounced. The partly non-local potentials were found to give much less variation in nuclear matter binding energy. The missing part in all these calculations is the triton binding energy for the partly non-local potentials and their phase equivalent separable counterparts.

The system of programs which I have developed allows the calculation of the following properties of the model triton:

E_T	the triton binding energy
$\psi_T(pq)$	the wave function in momentum space
$F_{ch}(Q^2)$	the charge form factor
C_T	the asymptotic normalization constant .

These properties may be calculated for any interaction for which the two-body T-matrix can be obtained in the $\ell=0$ state. Thus any potential local, separable, or non-local may be used; in fact one does not need a potential at all, only the T-matrix.

The three-body parameters most sensitive to the details of the nuclear force are the triton binding energy and the doublet scattering length. Since these are strongly correlated and spin is effectively ignored in this model, the scattering length was not calculated. Wave functions, not being observables, are rather uninteresting in themselves but can be used to calculate the form factors and asymptotic normalizations. Form factor calculations are often done with realistic potentials, and there have been calculations with phase equivalent potentials using short ranged unitary transformations (Haf 73, HKT 72b). For separable potentials, form factor calculations are not common and none to my knowledge have been done for phase equivalent potentials. The asymptotic normalization constant is another observable of the three nucleon system and one just recently coming under study. It is related to the probability that the ground state of the triton consists of a deuteron plus a neutron some distance from the deuteron. In ^{3}He it would be the proton separated from the deuteron. Physically, C_T is important in any process where a neutron (proton) is added to a deuteron or taken from ^{3}H (^{3}He): this occurs in stripping (t,d) and transfer (d,t) reactions, n-d scattering, electrodisintegration or photo-

disintegration of ^{3}He, and low energy elastic scattering of p-^{3}He or n-^{3}H. The asymptotic normalization constant can be extracted from the bound state wave function. Perhaps this parameter will be sensitive to the nuclear force. Skipping ahead, the conclusion is maybe: it is not sensitive at all to variation in the partly non-local potentials except in the limiting case; it is sensitive to the variation in the separable potentials. However, these separable potentials do not seem very realistic as evidenced by their binding energies and form factors.

An outline of the remainder of the thesis follows. Chapter II discusses the model triton, the potentials used, and the inversion procedure used to obtain them.

Chapter III deals with the calculation of the binding energy. For the partly non-local potentials the integral equation approach of the Faddeev equations in the momentum representation is used. For the separable potentials the formalism based on the Schrödinger equation in momentum space is used. It is found that the partly non-local potentials give about 1 to 1.5 MeV less binding energy than do their separable counterparts. Both classes of potentials give increasing binding energy as the attractive part of the potential is weakened and approaches the limit where it gives the same deuteron binding energy as the complete potential. This is very similar to the behaviour of Fiedeldey's rank-two separable potentials (Fie 69b, FM 72) although the variation

is much less.

In Chapter IV the extraction of the wave function and calculation of form factors is discussed. For the partly non-local potentials the wave function is obtained from the Faddeev amplitudes obtained during the calculation of the binding energy. For the separable potentials the spectator functions are used to calculate the wave function. Using these, the body form factor $F(Q^2)$ can be calculated. Various nucleon form factors are folded in to obtain the charge form factors. The form factors near zero momentum transfer depend strongly on the triton binding energy: there is an almost linear relation between the triton binding energy and the slope of the form factor near zero momentum transfer. At higher momentum transfer more structure is observed, but there is a very strong correlation between the "deuteron" form factors and the triton form factors.

The asymptotic normalization constant is treated in Chapter V. For the partly non-local potentials the variation in C_T is rather negligible (except for the limiting case) and the value of C_T is considerably less than for the standard potential. For the separable potentials the values of C_T increase as the attractive part weakens. The values are comparable to or greater than that of the standard potential, and greater than those of the partly non-local potentials.

Chapter VI discusses the results in more detail. Comparison of the triton and nuclear matter properties is made

for these potentials and those of Fiedeldey. Various properties of the two-body system are examined and related to the triton properties calculated. There is an almost linear relation between the triton binding energy and the zero-energy wound integral. The difference in the triton binding energy from that of the standard potential is proportional to the distortion of the deuteron wave function.

Chapter VII contains a summary of results and conclusions.

CHAPTER II

THE MODEL TRITON AND POTENTIALS USED

This chapter discusses the model triton and the phase equivalent potentials used in this calculation. The real triton and realistic potentials are discussed to see what assumptions and approximations are made and how the model triton compares with our knowledge of the physical system.

THE MODEL TRITON

There are four possible combinations of three nucleons: nnn (the trineutron or ^{3}n), nnp (the triton or ^{3}H), npp (the helion or ^{3}He), and ppp (the triproton or ^{3}Li). Of these, the trineutron and triproton do not form a bound system. The two bound states ^{3}H and ^{3}He each have total angular momentum $J = \frac{1}{2}$ and positive parity. No bound excited states exist. The binding energy of ^{3}H is 8.482 MeV and that of ^{3}He is 7.718 MeV; the difference, 0.764 MeV, is mainly due to the Coulomb interaction between the two protons in the helium nucleus.

The strong interaction conserves the quantum numbers J, π, and J_z. In addition T_z, the z-component of the isospin, is conserved if charge conservation is assumed, and T^2 if charge independence is assumed. Since the binding energies of ^{3}H and ^{3}He are approximately equal, and since the trineutron

and triproton are not bound, we infer that the bound pair form an isospin doublet; thus $T = \frac{1}{2}$. The total orbital angular momentum L and spin S are not "good" quantum numbers but can be used for classification of states. S must be $\frac{1}{2}$ or $\frac{3}{2}$ and coupled with L to give total angular momentum $J = \frac{1}{2}$, restricting L to 0, 1, or 2. This allows the states $(^{2S+1}L_J)$: $^1S_{1/2}$, $^1P_{1/2}$, $^3P_{1/2}$ and $^3D_{1/2}$. There are actually 16 independent states (DP 69) with $J = T = \frac{1}{2}$ and positive parity since the space, spin, and isospin components may be symmetric, anti-symmetric or of mixed symmetry, coupled together to give an overall anti-symmetric state.

The most important of these states is the L = 0 space-symmetric mixed spin and isospin state, known as the principal S state. For a central, spin and isospin independent force, as will be used in this calculation, it is the only state. Calculations with realistic potentials, even those with strong tensor forces, show that this state makes up almost 90% of the probability of the ground state function of the true triton. An L = 0 state is invariant under all rotations. The parity operator for this system is equivalent to a 180° rotation, thus does not alter the wave function. Therefore the parity is positive.

Applying some simple quantum mechanical arguments, let us investigate this state further. Figure 1 gives a schematic picture. For a ground state the total energy is minimized. this means that the kinetic energy should be small and the

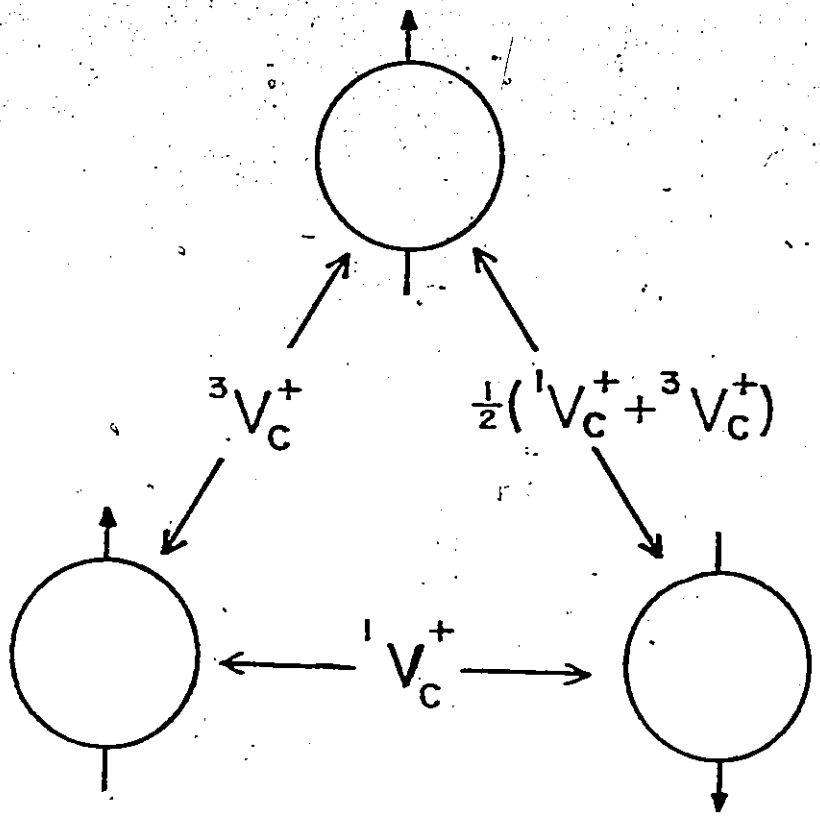


Figure 1. The principal S state of the triton.

potential energy great (and negative). The former constraint implies a space symmetric state with relative orbital angular momentum zero between all pairs so that there are no nodes in the wavefunction and the curvature thus is kept to a minimum. A space symmetric state is also best for potential energy, since the wavefunction is large when the particles are close together taking advantage of the attractive nuclear force. Anti-symmetry with respect to exchange of two particles requires that the two neutrons have opposite spin. Their interaction thus must be in the singlet state. The spin of the proton must therefore be parallel to that of one of the neutrons; the interaction in this case is in the triplet state. The third interaction is between the n-p pair with spins anti-parallel. This can be divided as one half singlet and one half triplet:

$$\alpha(p)\beta(n) = \frac{1}{2}[\alpha(p)\ \beta(n) + \alpha(n)\beta(p)] + \frac{1}{2}[\alpha(p)\beta(n) - \alpha(n)\beta(p)]. \quad (1)$$

The three pair interactions sum to $\frac{3}{2}$ (singlet and triplet), or on average

$$V_{eff} = \frac{1}{2}({}^1V_c^+ + {}^3V_c^+). \quad (2)$$

POTENTIALS IN NUCLEAR PHYSICS

By far the most important force acting between two nucleons is the strong interaction. In the same way that the electromagnetic force can be attributed to the exchange of photons, the nuclear force can be attributed to the exchange

of mesons. The success of the potential concept, within the framework of quantum mechanics, as applied to problems in atomic physics, has led to its application in nuclear physics. The great advantage of a potential is that once it has been determined, we have a dynamical model, the Schrödinger equation, which can be applied to any nuclear problem.

The interaction of the nucleon and meson fields gives rise to the emission of virtual particles, the lightest of which are pions. This violates energy conservation by $\Delta E \sim m_\pi c^2$ if their momentum q is small. The uncertainty principle requires that these pions be reabsorbed in time

$$\Delta t \sim \hbar/\Delta E \sim 5\times10^{-24} \text{ seconds.} \qquad (3)$$

If the pion is absorbed by a second nucleon instead of being reabsorbed, then the second nucleon acquires the extra momentum, q, and the process appears as elastic scattering; see figure 2. If we do not examine times shorter than 10^{-24} seconds, this interaction appears as a force, which can be described non-relativistically as a potential. The second nucleon must be sufficiently close so that the emitted pion can reach it, that is within a distance $c\,\Delta t \sim 1.5$ fm. Thus the nucleon-nucleon (N-N) potential has a range of the order 1 to 2 fm. The electromagnetic interaction by comparison has infinite range since the exchanged photons have zero mass. The N-N potential does however have a tail which decays as

$$\frac{e^{-\mu r}}{r} , \qquad (4)$$

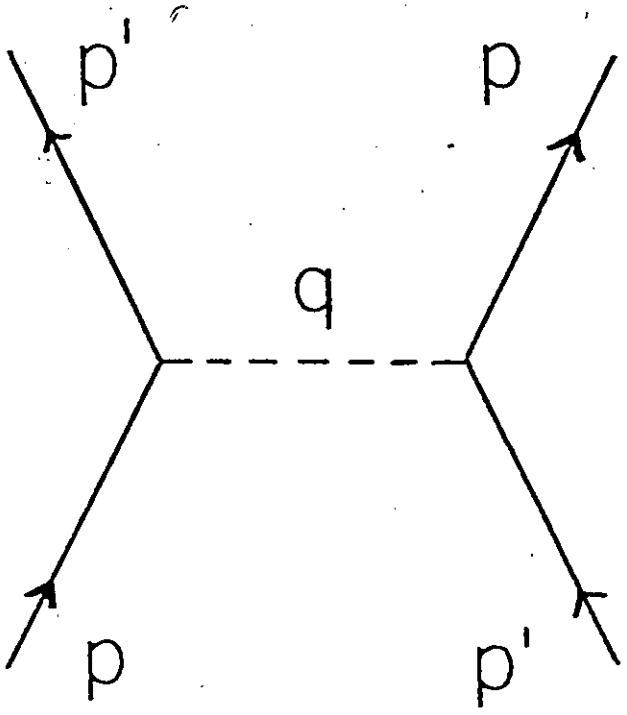


Figure 2. Diagram showing two nucleons interacting via meson exchange.

where $\mu = mc/\hbar$ and m is the mass of the exchanged particle. Beyond 3 fm the N-N interaction is given by the one-pion-exchange potential (OPEP). This can be shown by theoretical arguments, and by phenomenological fits to the experimental data in which the pion mass and coupling constant are treated as free parameters. In addition to single pion exchange, heavier particles or several pions may be emitted. The range is inversely proportional to the mass of the exchanged particles, hence they will be important at shorter distances only. In principle one could use the OPE potential to describe the long range part of the interaction, the two-pion-exchange (TPE) to describe the intermediate region, and so on to more pions and heavier mesons. However, even the TPE is difficult and for practical reasons at some stage this procedure has to be abandoned and some sort of phenomenological description used for short distances. If any potential is to describe this meson soup, it is undoubtedly non-local.

Most reasonable potentials incorporate only the OPEP and fits to the experimental data (HJ 62, Rei 68, TRS 75). The most realistic potentials these days (CDR 72, Cot+ 73) incorporate OPEP to describe the interaction at greater than 3 fm along with 2π and ω-meson exchange which dominate the interaction between 1 and 3 fm. The remainder of the interaction is treated phenomenologically. Since multiple-pion exchanges are so difficult to treat, and the pions are strongly correlated, they can be simulated to some extent by the

one-boson-exchange (OBE) potentials in which the 2π exchange is described in terms of the ρ- and ε-mesons.

Once we have decided on a potential we can use the Schrödinger equation to describe the system. In operator form it is

$$(H_0+V)\,|\psi\rangle = E\,|\psi\rangle \tag{5}$$

where H_0 is the kinetic energy, V the potential and E the energy of the system. For two particles in the coordinate representation it may be written

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi(\underline{r}) + \int d\underline{r}'\ \langle \underline{r}|V|\underline{r}'\rangle\psi(\underline{r}') = E\psi(\underline{r})\ . \tag{6}$$

For a local potential

$$\langle \underline{r}|V|\underline{r}'\rangle = V(\underline{r})\,\delta(\underline{r}-\underline{r}') \tag{7}$$

so that eq. 6 reduces to the familiar form.

For a separable potential the dependence on $\underline{r}$ and $\underline{r}'$ factors:

$$\langle \underline{r}|V|\underline{r}'\rangle = \lambda f(\underline{r})\,f(\underline{r}') \tag{8}$$

or equivalently in momentum space

$$\langle \underline{p}|V|\underline{p}'\rangle = \lambda g(\underline{p})\,g(\underline{p}') \tag{9}$$

where

$$g(\underline{p}) = (2\pi)^{-3/2}\int f(\underline{r})\,e^{-i\underline{p}\cdot\underline{r}}\,d\underline{r}\ . \tag{10}$$

It is most useful in momentum space where the Schrödinger equation becomes

$$(\frac{\hbar^2}{2\mu})p^2\psi(\underline{p}) + \lambda g(\underline{p}) \int d\underline{p}' g(\underline{p}')\psi(\underline{p}') = E\psi(\underline{p}) \quad . \quad (11)$$

This is an algebraic equation rather than a differential equation, and whose solution can immediately be written down,

$$\psi(\underline{p}) = N \frac{g(p)}{p^2 - (\frac{2\mu}{\hbar^2})E} \qquad (12)$$

where N is a normalization constant.

THE T-MATRIX

An equivalent of the Schrödinger equation and one better suited to scattering problems is the Lippmann-Schwinger (LS) equation

$$T = V - V(H_0 - E - i\varepsilon)^{-1} T \qquad (13)$$

where the $i\varepsilon$ means we are to approach the real axis at E from above, thus ensuring the correct boundary conditions; i.e. outgoing scattered waves. We usually write T not as a function of E but as a function of $z = (2\mu/\hbar^2)E$. Sandwiched between plane wave states eq. 13 becomes

$$\langle \underline{p}|T(z)|\underline{k}\rangle = \langle \underline{p}|V|\underline{k}\rangle - \frac{2\mu}{\hbar^2} \int d\underline{k}' \frac{\langle \underline{p}|V|\underline{k}'\rangle\langle \underline{k}'|T|\underline{k}\rangle}{k'^2 - z - i\varepsilon} \quad . \quad (14)$$

Three cases can be distinguished:

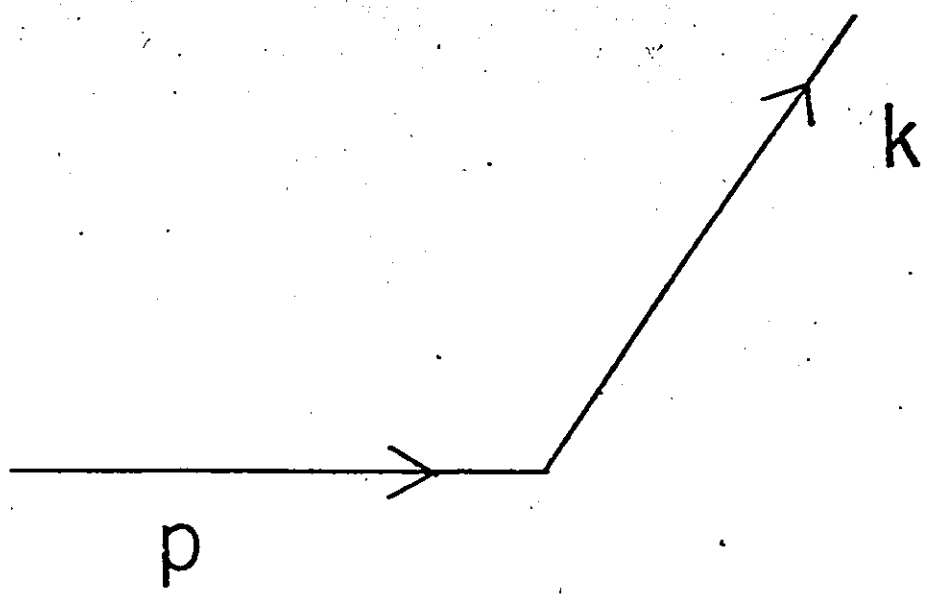


Figure 3. Scattering of two particles. The initial centre-of-mass momentum is $\underline{p}$; the final centre-of-mass momentum is $\underline{k}$. In elastic scattering $|\underline{p}|^2 = |\underline{k}|^2$.

$\langle \underline{p}|T(z)|\underline{k}\rangle$ the fully off-shell T-matrix, (15)
where $|p|^2 \neq |k|^2 \neq z$;

$\langle \underline{p}|T(k^2)|\underline{k}\rangle$ the half-shell T-matrix, (16)
where $|p|^2 \neq |k|^2$; and

$\langle \underline{p}|T(k^2)|\underline{k}\rangle$ the on-shell T-matrix, (17)
where $|p|^2 = |k|^2$.

In quantum mechanics, scattering cross sections are bilinear functions of the scattering matrix, whose matrix elements are complex functions of energy and angle. Assuming rotational invariance, each matrix element may be expressed as a sum of terms, each referring to a particular angular momentum state, and depending on one real parameter, called a phase shift. At a given energy, only a finite number of angular momenta are important, so all the scattering data can be converted into a small number of real parameters. The determination of these is called phase shift analysis.

Only the on-shell T-matrix elements are determined by the two-nucleon scattering data. For a central potential the T-matrix can be expanded as

$$\langle \underline{p}|T(z)|\underline{k}\rangle = \frac{1}{2\pi^2}\frac{\hbar^2}{m}\sum_{\ell}(2\ell+1)t_{\ell}(p,k;z)P_{\ell}(\cos\theta_{pk}) \quad . (18)$$

The on-shell T-matrix elements are then related to the phase shifts by the relation

$$t_{\ell}(k,k;k^2) = -\exp[i\delta_{\ell}(k)]\sin\delta_{\ell}(k)/k \quad . \qquad (19)$$

The half-shell T-matrix enters into calculations of nucleon-nucleon bremsstrahlung, but experiments have not yielded much

information about the T-matrix. This forces us to look at many nucleon systems in order to gain information about the off-shell T-matrix. Many nucleon systems can be described in terms of the off-shell T-matrix provided only pair interactions are assumed. In particular, the three-nucleon system can be described by the Faddeev equations, which, as will be seen in Chapter III, depend on the fully off-shell T-matrix. Potentials which have the same on-shell T-matrix will in general have different off-shell T-matrix elements, and thus give different results in three- and many-body systems. By comparing the results which various potentials predict for these systems with the experimental values, some will be preferable to others and thus tie down the N-N interaction a bit further.

THE PHASE EQUIVALENT POTENTIALS

The information available for the two nucleon system comes mainly from scattering experiments; it is digested somewhat and expressed as phase shifts in various spin and isospin angular momentum states. In addition, the n-p bound state, the deuteron, provides us with additional information: the binding energy E_D, the electric quadrupole moment Q_D, the magnetic dipole moment μ_D, and the electric form factor $F(Q^2)$ from electron scattering. Since all of these quantities depend on the various spin and isospin states and the D state probability of the deuteron, they cannot be used directly in this model calculation.

It is possible to generate reasonable model phase shifts and E_D which will be taken as the experimental data. All the potentials used are phase-equivalent to the standard Tabakin potential V_{ST} (Tab 65, potential 1 of Table 1); that is, they have the same phase shifts and "deuteron" binding energy. This potential is the average of two potentials fitted to the experimental $\ell=0$ n-p singlet and triplet phase shifts respectively. It produces a model deuteron binding energy of 0.428 MeV.

The potentials used are of two forms (Sin 72):

i) partly non-local potentials

$$\langle k|V_{YBS}|p\rangle = \langle k|V_1|p\rangle + n(k)n(p) \ , \qquad (20)$$

ii) rank-two separable potentials

$$\langle k|V_{YSS}|p\rangle = -g(k)g(p) + h(k)h(p) \qquad (21)$$

where $n(k)$, $g(k)$, and $h(k)$ are separable terms and V_1 is an attractive local potential - in our case a two parameter Bargmann eq. (30). The inversion procedure consists of choosing a form for V_1 or g and solving for n or h so that the V_{YBS} and V_{YSS} give exactly the same phase shifts and binding energy as V_{ST}.

The inversion procedure used is that of Y. Singh (Sin 72, SWB 72) who applied the inversion method of Fuda (Fud 70) to non-local potentials of the form

$$V(r,r') = V_1(r,r') + n(r)n(r') \qquad (22)$$

The details are given by Singh and only the recipe for con

structing them is given here. The basic ingredients are:

$\delta(k)$ phase shifts over all momenta $0 \leq k < \infty$

E_D "deuteron" binding energy $(E_D = (\hbar^2/m)k_D^2)$

V_1 arbitrary attractive part of the potential.

These are subject to only two minor constraints

i) $E_B \geq E_D$, that is the attractive part of the potential acting by itself must produce a greater binding energy E_B in the two body system than that of the complete potential $(E_B = (\hbar^2/m)k_B^2)$

ii) $0 \leq \delta_1(k)-\delta(k) \leq \pi$, that is the phase shifts produced by the attractive part of the potential alone $\delta_1(k)$ must be greater than the phase shifts of the complete potential for all momenta; but not too much greater.

The Fredholm determinant of the complete interaction is given by

$$D(k) = (1 + \frac{k_D^2}{k^2})\exp[-\frac{2}{\pi} P \int_0^\infty \frac{\delta(p)pdp}{p^2-k^2}]\exp[-i\delta(k)]. \quad (23)$$

Using potential V_1 we can generate $\delta_1(p)$, E_B, $\phi_B(p)$ and $\phi_p(k)$ where ϕ_B and ϕ_p are the bound state and scattering wavefunctions. From these we can obtain the Fredholm determinant of V_1

$$D_1(k) = (1 + \frac{k_B^2}{k^2})\exp[-\frac{2}{\pi} P \int_0^\infty \frac{\delta_1(p)pdp}{p^2-k^2}]\exp[-i\delta_1(k)]. \quad (24)$$

Then using

$$D_2(k) = D(k)/D_1(k) \quad (25)$$

and

$$D_2(k) = 1 - \int_0^\infty \frac{[\tilde{n}(p)]^2 p^2 dp}{k^2-p^2+i\varepsilon} - \frac{\tilde{n}_B^2}{k^2+k_B^2} \qquad (26)$$

(where $\tilde{D}_2$, $\tilde{n}$ and $\tilde{n}_B$ are defined in terms of n, ϕ_B and ϕ_p) equating the real and imaginary parts gives

$$[\tilde{n}(k)]^2 = \frac{1}{k} \text{Im } [D(k)/D_1(k)] \qquad (27)$$

$$\tilde{n}_B^2 = \frac{\pi}{2}(k_B^2-k_D^2)\exp[-\frac{2}{\pi} P \int_0^\infty \frac{\delta_1(p)pdp}{p^2+k_B^2}] . \qquad (28)$$

This gives $\tilde{n}(k)$ in the basis of the eigenfunctions of V_1. In the usual plane wave basis n(k) is given by

$$n(k) = \cos\delta_1(k)\tilde{n}(k) + \frac{2}{\pi} P \int_0^\infty \frac{\tilde{n}(p)\langle k|V_1|\phi_p\rangle}{p^2-k^2} p^2 dp + \phi_B(k)\tilde{n}_B . \qquad (29)$$

Choosing a two parameter Bargmann potential for V_1 considerably simplifies the procedure. Writing

$$V_1(r) = -(8b^2\beta)\frac{e^{-2br}}{(1+\beta e^{-2br})^2} \qquad (30)$$

where $\beta = (b+a)/(b-a)$, then there are analytic formulae

$$k_B = a \qquad (31)$$

$$D_1(k) = (k-ia)/(k+ib) \quad a,b>0 \qquad (32)$$

$$\tan\delta_1(k) = k(a+b)/(k^2-ab) \qquad (33)$$

as well as for $|\phi_p\rangle$ and $|\phi_B\rangle$

Moreover for the potentials YSS, since $-g(k)g(p)$ is to be phase-equivalent to V_1, it is sufficient to have the Fredholm determinants equal, or

$$\text{Im } D_1(k) = -\frac{k(a+b)}{k^2+b^2} = -kg^2(k) \quad . \tag{34}$$

Hence

$$g(k) = \frac{(a+b)^{1/2}}{(k^2+b^2)^{1/2}} \quad . \tag{35}$$

Using the above procedure again for inversion allows the calculation of h(k).

Six pairs of potentials are used for this study. All have the same range parameter, b = 1.0 fm^{-1} but different strength parameters a for the potentials ranging from a = .1016 fm^{-1} (the limiting case) to a = 0.5 fm^{-1} corresponding to about 10 MeV binding for the attractive part. The parameters a,b of the potentials are listed in Table 1, along with other properties of the potentials. Four of these pairs have been used in nuclear matter calculations (SWP 77) so that a comparison with nuclear matter results can be made. This will be discussed in Chapter VI.

CHAPTER III

THE THREE-BODY EQUATIONS: THE BINDING ENERGY

In this chapter the method of solving the model triton is presented. It is applicable to any interaction for which the two-body T-matrix in the $\ell=0$ partial wave can be defined. Only pair interactions are considered. First the coordinate systems and basis states are defined. Then we turn to the Faddeev equations. A sketch of their angular momentum reduction is presented. The homogeneous equation is solved by successive approximations using a Padé approximant to check for convergence. Triton binding energies are calculated for the partly non-local potentials. The binding energies of the separable potentials are calculated by a simpler method; this is described in Appendix A.

THE BASIS STATES

There are several coordinate systems in use for three-nucleon systems. The one used for this calculation is defined in conjunction with figure 4. Consider three particles having

masses m_1, m_2, m_3 (1-a)

positions x_1, x_2, x_3 (1-b)

and momenta k_1, k_2, k_3. (1-c)

From the position variables one can construct three basic

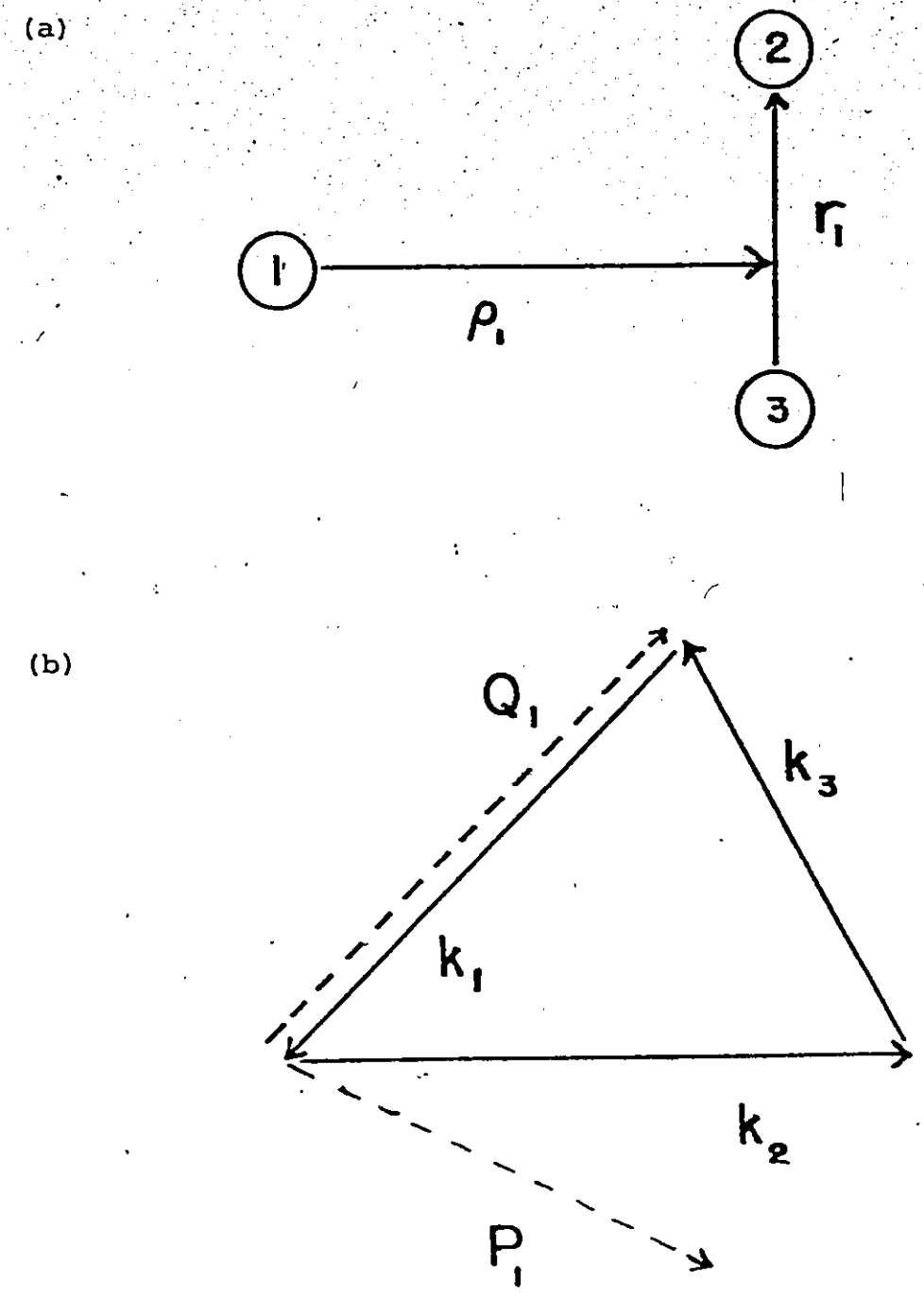


Figure 4. Coordinates for the three-body system; (a) coordinate space representation (b) momentum space representation.

vectors

$$\underline{r}_1 = \underline{x}_2 - \underline{x}_3 \qquad (2\text{-a})$$

$$\underline{\rho}_1 = \frac{m_2\underline{x}_2 + m_3\underline{x}_3}{m_3+m_2} - \underline{x}_1 \qquad (2\text{-b})$$

$$\underline{X} = \frac{m_1\underline{x}_1 + m_2\underline{x}_2 + m_3\underline{x}_3}{m_1+m_2+m_3} \; . \qquad (2\text{-c})$$

$\underline{r}_1$ is the relative separation of particles 2 and 3, $\underline{\rho}_1$ is the distance of the centre-of-mass of the 2-3 pair from particle 1, and $\underline{X}$ is the centre-of-mass of the three particles. The combination $(\underline{r}_2, \underline{\rho}_2, \underline{X})$ and $(\underline{r}_3, \underline{\rho}_3, \underline{X})$ can be defined by cyclicly permuting the indices 1,2 and 3. The conjugate coordinates in momentum space are defined:

$$\underline{P}_1 = \mu_{23} \left(\frac{\underline{k}_2}{m_2} - \frac{\underline{k}_3}{m_3}\right) \qquad (3\text{-a})$$

$$\underline{Q}_1 = \mu_1 \left(\frac{\underline{k}_2+\underline{k}_3}{m_2+m_3} - \frac{\underline{k}_1}{m_1}\right) \qquad (3\text{-b})$$

$$\underline{P} = (\underline{k}_1+\underline{k}_2+\underline{k}_3) \qquad (3\text{-c})$$

where the mass ratios used are

$$\mu_{23} = \frac{m_1(m_2+m_3)}{m_1+m_2+m_3} \qquad (4\text{-a})$$

$$\mu_1 = \frac{m_2 m_3}{m_2+m_3} \qquad (4\text{-b})$$

$$M = m_1+m_2+m_3 \; . \qquad (4\text{-c})$$

For equal masses m, $\mu_{23} = \frac{1}{2}$ m, $\mu_1 = \frac{2}{3}$ m, and M = 3m. These

coordinates are commonly used in triton calculations using separable potentials. For calculations using the Faddeev equations it is common to use a more symmetric set (Lov 64) differing by $\sqrt{2\mu_1}$, $\sqrt{2\mu_{23}}$ and $\sqrt{2M}$ respectively and having units differing by $\hbar m^{1/2}$. It seems less confusing, at least to me, to have the "momenta" in the usual units fm^{-1} and keep $\hbar$ and m explicitly. The symmetric form is rather convenient, so using $\sqrt{2\mu/m}$ instead of $\sqrt{2\mu}$, the variables used are

$$\underline{p}_1 = (\underline{k}_2 - \underline{k}_3)/2 \qquad = \underline{P}_1 \tag{5-a}$$

$$\underline{q}_1 = (\underline{k}_2 + \underline{k}_3 - 2\underline{k}_1)/\sqrt{12} = \sqrt{4/3}\ \underline{Q}_1 \tag{5-b}$$

$$\underline{P}_c = (\underline{k}_1 + \underline{k}_2 + \underline{k}_3)/\sqrt{6} \quad = \underline{P}\ /\sqrt{6}\ . \tag{5-c}$$

The kinetic energy of the system is given by

$$H_0 = \hbar^2 \left[\frac{k_1^2}{2m_1} + \frac{k_2^2}{2m_2} + \frac{k_3^2}{2m_3}\right] \tag{6-a}$$

$$= \hbar^2 \left[\frac{P_1^2}{2\mu_{23}} + \frac{Q_1^2}{2\mu_1} + \frac{P^2}{2M}\right] \tag{6-b}$$

$$= (\hbar^2/m)\left[p_1^2 + q_1^2 + P_c^2\right]\ . \tag{6-c}$$

In the centre-of-mass sytem $\underline{P} = \underline{X} = \underline{O}$, thus only the variables p_1 and q_1 are needed to describe the system. It is convenient to use other coordinate systems $(\underline{p}_2, \underline{q}_2)$ and $(\underline{p}_3, \underline{q}_3)$ which can be defined from equations (5) by permuting the indices 1, 2 and 3. The three systems are related by linear relations (with Jacobian of unity)

$$\underline{p}_i = -\alpha_{ij}\underline{p}_j - \beta_{ij}\underline{q}_j \qquad (7\text{-a})$$

$$\underline{q}_i = \beta_{ij}\underline{p}_j - \alpha_{ij}\underline{q}_j \qquad (7\text{-b})$$

where

$$\alpha_{ij} = \left[\frac{m_i m_j}{(m_i+m_k)(m_j+m_k)}\right]^{1/2} = \alpha_{ji} \qquad (8\text{-a})$$

$$\beta_{ij} = (1-\alpha_{ij}^2)^{1/2} = -\beta_{ji} \qquad (8\text{-b})$$

where i, j, k are cyclic. For equal masses $\alpha_{ij} = \frac{1}{2}$ and $\beta_{ij} = \frac{\sqrt{3}}{2}$.

The three particle state can be represented in four ways as

$$|\underline{k}_1\underline{k}_2\underline{k}_3\rangle = |\underline{p}_1\underline{q}_1\rangle_1 = |\underline{p}_2\underline{q}_2\rangle_2 = |\underline{p}_3\underline{q}_3\rangle_3 \; . \qquad (9)$$

The subscript on the ket refers to the coordinate system and the arguments inside to the value of the momenta. For identical particles the states under interchange are related; for example exchanging particles 2 and 3 means

$$P(1)|\underline{k}_1\underline{k}_2\underline{k}_3\rangle = |\underline{k}_1\underline{k}_3\underline{k}_2\rangle \qquad (10\text{-a})$$

$$= |-\underline{p}_1\underline{q}_1\rangle_1 = |-\underline{p}_3\underline{q}_3\rangle_2 = |-\underline{p}_2\underline{q}_2\rangle_3 \; . \qquad (10\text{-b})$$

Note how equations (7) have been used in writing the last two parts of equation (10-b).

Neglecting spin, the states can be expressed as

$$|\underline{p}\underline{q}\rangle_i = \sum_{\substack{\ell m \\ LM}} |p\ell m; qLM\rangle_i Y_{\ell m}(\hat{\underline{p}}) Y_{LM}(\hat{\underline{q}}) \; . \qquad (11)$$

The ground state of the triton has total angular momentum L, therefore coupling the states of equation (11) one gets

$$|pqLM\ell l\rangle_i = \sum_{\substack{\ell L \\ mM}} \langle \ell m l M | LM\rangle |p\ell m; qlM\rangle_i \,. \qquad (12)$$

These form a complete set and are normalized as

$${}_i\langle pqLM\ell l | p'q'L'M'\ell'l'\rangle_i = \frac{\delta(p-p')}{p^2}\,\frac{\delta(q-q')}{q^2}\,\delta_{LL'}\,\delta_{MM'}\,\delta_{\ell\ell'}\,\delta_{ll'} \,. \qquad (13)$$

The discrete quantum numbers $LM\ell l$ will often be denoted collectively as α.

THE FADDEEV EQUATIONS

Solution of three-particle systems had long been hindered by the fact that the three-particle Lippmann-Schwinger equation

$$T = V - VG_0T \qquad (14)$$

had a kernel that was not a Schmidt operator. (By a Schmidt operator K, we mean that tr $KK^+ < \infty$.) Consequently no generalized Fredholm theory would work, and the area was full of pitfalls for the unwary. This is because the interaction contains delta functions corresponding to one of the particles going straight through without interacting with either of the others. It was not until 1960 that Faddeev (Fad 61) was able to reformulate the problem in such a way that the singularities were removed. The secret of his success lies in the fact that his equations refer to the two-body T-matrix

(exact solution of the pair interaction) rather than the two-body potential.

The Faddeev equations are three coupled integral equations

$$T^{(1)}(s) = T_1(s) - T_1(s) G_0(s) [T^{(2)}(s) + T^{(3)}(s)]$$

$$T^{(2)}(s) = T_2(s) - T_2(s) G_0(s) [T^{(3)}(s) + T^{(1)}(s)] \qquad (15)$$

$$T^{(3)}(s) = T_3(s) - T_3(s) G_0(s) [T^{(1)}(s) + T^{(2)}(s)]$$

where

$\frac{\hbar^2}{m} s$ is the energy of the three-particle system

T_i is the two-particle T-matrix for particles j and k (ijk cyclic) in the three-particle Hilbert Space.

G_0 is the Green function for three free particles: $G_0(s) = (H_0 - \frac{\hbar^2}{m} s)^{-1}$

$T^{(i)}$ is a scattering matrix describing the sum of all interactions in which particles j and k interact last

$T = T^{(1)} + T^{(2)} + T^{(3)}$ is the three-particle T-matrix.

If we write eqs. (15) as $T^{(i)} = T_i + K^{ij} T^{(j)}$ with a kernel K that is a 3×3 matrix, then it can be shown that some power of K is a Schmidt operator. For s in most parts of the complex plane, K^2 is sufficient; for bound state and three-particle scattering the fifth power may be needed (Fad 65). The key point is that K^2 has elements of the form $T_i(s) G_0(s) T_j(s) G_0(s)$ where $i \neq j$. This absence of diagonal

elements is due to the fact that all the two-particle subsystems have already been taken into account exactly. The usual Fredholm methods of solution can then be used for solution. All previous treatments could only treat one of the subsystems exactly.

Equations (15) are usually solved in the momentum representation. After removing the centre-of-mass coordinates there are still two vectors or six independent variables describing the three particle system. To reduce the dimensionality of the problem some form of angular momentum reduction is needed. Various attempts have been made (ElB+73, has a review), but the one most widely used is that of Ahmadzadeh and Tjon (AT 65) and generalizations of it (HKT 70, MT 70). The six continuous variables are reduced to two continuous variables and sums over many discrete quantum numbers. In the representation defined by equation (12), and where $n = \underline{k}_1'' \underline{k}_2'' \underline{k}_3''$, the first of equations (15) can be written

$$ {}_1\langle pq\alpha|T^{(1)}(s)|n\rangle = {}_1\langle pq\alpha|T_1(s)|n\rangle - \sum_{i=2}^{3} \sum_{\alpha_i} \int p_i'^2 dp_i' \int q_i'^2 dq_i' $$

$$ {}_1\langle pq\alpha|T_i(s)|p_i'q_i'\alpha_i\rangle_i \times \frac{{}_i\langle p_i'q_i'\alpha_i|T^{(i)}(s)|n\rangle}{p_i'^2+q_i'^2-s} . \qquad (16) $$

After a great deal of angular momentum algebra (AT 65) equation (16) simplifies to a form which contains the (fully off-shell) two-particle T-matrix. Isospin can be added without difficulty. Spin, too, may be included easily, provided the force does not mix spins. For identical particles

$$ {}_1\langle pq\alpha|T^{(j)}(s)|n\rangle = {}_j\langle pq\alpha|T^{(1)}(s)|n\rangle \qquad (17) $$

so that the three equations (15) effectively reduce to one. Equation (16) thus can be written (MT 69)

$$ \psi(pq\beta) = \phi(pq\beta) - \frac{8}{\pi q\sqrt{3}} \sum_{\beta'} F(\beta|\beta') \int_0^\infty q'dq' $$

$$ \times \int_{L(qq')}^{U(qq')} \frac{p'dp' t_\beta(p,(p'^2+q'^2-q^2)^{1/2};s-q^2)\psi(p'q'\beta')}{p'^2+q'^2-s} \qquad (18) $$

where β is α plus spin and isospin

$F(\beta|\beta')$ is a spin-isospin coefficient

$\psi(pq\beta) = {}_1\langle pq\alpha|T^{(1)}(s)|n\rangle$

$\phi(pq\beta) = {}_1\langle pq\alpha|T_1(s)|n\rangle$

t_β is the "β-th" partial wave of the two particle T-matrix, t[†]

$L(qq') = |2q-q'|/\sqrt{3}$

$U(qq') = (2q+q')/\sqrt{3}$.

For the triton $S = \frac{1}{2}$ and $T = \frac{1}{2}$, therefore, the spin-isospin between particles 2 and 3 can only assume values (0,1) or (1,0), which gives us a coupled two channel equation. In the model triton we further assume that the singlet and triplet two body interactions are equal in strength. The average interaction, as explained in Chapter II, is then

$$ t_{eff} = \frac{1}{2}({}^3t_1 + {}^1t_1) \equiv t \ . \qquad (19) $$

The further assumption made for this calculation is that only the $\ell=0$ part of the interaction be used. The coupled equations

[†] t is related to T_i by

$$ \langle \underline{pq}|T_i(s)|\underline{p'q'}\rangle = \langle \underline{p}|t(s-q^2)|\underline{p'}\rangle \delta(\underline{q}-\underline{q'}) $$

(18) then reduce to one

$$\psi(pq\beta) = \phi(pq\beta) - \frac{8}{\pi q\sqrt{3}} \int_0^{\infty} q'dq' \int_L^U p'dp' t(p,(p'^2+q'^2-q^2)^{1/2};s-q^2) \frac{\psi(p'q'\beta)}{p'^2+q'^2-s} . \quad (20)$$

The inhomogeneous term ϕ depends on what problem we wish to solve: the bound state, elastic scattering of a nucleon off a deuteron, or three particle scattering. For the bound state the first term is absent and equation (20) is homogeneous.

METHOD OF SOLUTION

Reduction of equation (20) to matrix form is not practical for two reasons. The limits on p' vary for different q', and the resulting size of the matrix which would have to be inverted is large. However, the problem can be solved by successive iterations (MT 69, Cul 74). The problem is, given K, to find the value of s for which equation (20) is solvable. We do this indirectly by introducing a parameter λ and study the solution of

$$\psi(p,q;s) = \lambda \iint K(p,q;p',q';s)\psi(p',q';s)dp'dq' \quad (21)$$

for fixed s. Assuming some initial approximation ψ^0, successive iterations ψ^1, ψ^2, ... can be generated

$$\psi^{n+1}(p,q;s) = \iint K(p,q;p'q';s)\psi^n(p,q;s)d\underline{p}'d\underline{q}' \quad (22)$$

$$n = 0,1,2,...$$

Convergence has been achieved if successive approximations have the same functional dependence on p and q and differ only by a scalar constant λ; that is

$$\frac{\psi^{n+1}(p,q;s)}{\psi^{n}(p,q;s)} = \lambda \qquad \text{for all } p,q. \tag{23}$$

λ of course depends on s; thus s must be adjusted until $\lambda=1$. Then $s=s_T$ where $E_T = (\hbar^2/m)|s_T|$ is the triton binding energy.

ψ^0 is arbitrary except that it must not be orthogonal to the eigenfunction of the ground state. This is ensured by taking any nodeless space-symmetric function. Convergence is enhanced by choosing a form approximately equal to the eigenfunction. The unitary pole solution, if available, should be excellent. What was done in this study was to use the solution for one potential as the starting point for a second potential. Also, since several guesses for s were needed before $\lambda \to 1$, the solution for one value of s was used as the first approximation for the next value of s, and so on until λ converged to 1 and s to s_T.

The technique of Padé approximants was used to advantage here. We write, for some values of p and q,

$$F(\alpha) = \psi^0 + \alpha\psi^1 + \ldots + \alpha^n\psi^n \quad . \tag{24}$$

The Padé technique allows this series to be written uniquely as a ratio of polynomials

$$F_{[N,M]}(\alpha) = \frac{P_N(\alpha)}{Q_M(\alpha)} \tag{25}$$

where $M+N < n$, $Q_M(0) = 1$, and $P_N(0) = \psi^0$. Now, successive terms in our iterations of equation (22) look like $\psi^{n+1} = \lambda\psi^n$. This appears as a term $(1-\lambda\alpha)$ in the denominator; or a root of the polynomial $Q_m(\alpha)$ at

$$\alpha = 1/\lambda \ . \tag{26}$$

Therefore, if we iterate several times, obtaining the Padé approximate for $M \geq 1$, and solving for the root α_r of Q_M near unity, we find that α_r converges more quickly than other convergence tests. For example, the [N,1] Padé approximant corresponds to the ratio test

$$\lambda_r = \frac{\psi^{N+1}(pq\beta)}{\psi^N(pq\beta)} \ . \tag{27}$$

In fact picking a few scattered values of p and q and checking the zeros of the Padé denominator seems better than the integrated ratio test

$$\frac{\iint |\psi^{n+1}(pq\beta)|^2 p^2 q^2 dpdq}{\iint |\psi^n(pq\beta)|^2 p^2 q^2 dpdq} \tag{28}$$

The Padé denominator for M-k picks up the last k terms of the series (eq. 24), so is sensitive to the rate of change of the eigenfunction with respect to successive iterations. Although not exploited here, this information could undoubtedly be used to speed up convergence

The advantage which was exploited was the fact that λ, as determined by the Padé method, converged much faster than the amplitude ψ. Thus an incorrect value of s could be quickly spotted and corrected for, without waiting for ψ to converge. Also, for s near s_T, λ is almost linear with respect to s, so choosing successive values of s is straightforward. Moreover $d\lambda/dE_T \approx 0.05$ for all potentials for which this method was used, so that after picking only one value for s and iterating until λ converged, a reasonable guess for E_T could be made. Refining E_T requires several more choices of s, of course.

NUMERICAL METHODS

To solve equation (20) the integrals must be evaluated by some quadrature formula, implying some set of grid points $\{p_k\}$ and $\{q_\ell\}$. Thus we have

$$\psi^{n+1}(p_k q_\ell \beta) = -\frac{8}{\pi q_\ell \sqrt{3}} \sum_{j=1}^{N} q_j \omega_j \sum_{i=1}^{M} p_i \omega_i t(p_k, \sqrt{p_i^2+q_j^2-q_\ell^2}; s-q_\ell^2)$$

$$\frac{1}{p_i^2+q_j^2-s}\psi(p_i q_j \beta), \quad k=1,2\ldots K \;:\; \ell = 1,2,\ldots L \qquad (20)$$

There are a few things to note:

(i) We are dealing with the bound state, therefore $s<0$ and the T-matrix is needed only for negative energies. It is equivalent to the K-matrix and is real. ψ is therefore also real so no complex arithmetic is needed.

(ii) The T-matrix $t(p_k,\bar{p};s-q_\ell^2)$ is needed only for the set of points $\{p_k\}$ and $\{s-q_\ell^2\}$ at which the Faddeev amplitude ψ is to be calculated. Therefore it need be calculated only once, and interpolated over the second variable $\bar{p}$ as needed. This is important since interpolation over three variables would be prohibitive, and calculating t many times is costly since a matrix inversion must be performed.

(iii) If the set of points $\{q_\ell\}$ is chosen to be the same as the set $\{q_j\}$ required for the integration, there is a further simplification; namely, that $\psi(p_i q_j)$ need only be interpolated over the variable p_i. The interpolation is thus more accurate and can be done more quickly.

Various types of mappings were used to choose the integration points. Since Gauss-Legendre quadrature was used throughout for the integrations, the region $(0,\infty)$ had to be mapped onto $(-1,1)$. The transformations used were

$$p = \frac{x+1}{1-x} \qquad (30a)$$

$$p = c \tan \frac{\pi}{4}(x+1) \qquad (30b)$$

where c is a parameter. Note that when x=0, then p=c, so half the points p_i are less than c and half greater. The weights are $(\frac{dp}{dx})_{x=x_i}$ times the Gauss Legendre weights. c varying from 0.5 to 2 was used to increase the density of points where the functions were varying most rapidly

The T-matrix was calculated from the $\ell=0$ partial wave of the Lippmann-Schwinger equation with normalization of equation (2-19)

$$t(p,k;z) = v(p,k) - \frac{2}{\pi}\int_0^\infty \frac{k'^2 dk' v(p,k') t(k',k;z)}{k'^2 - z} \qquad (31)$$

by the usual matrix inversion technique (HT 70). There is no need even to subtract out the singularity since $z = s - q_\ell^2 < 0$. If the potential is well enough behaved, the same set of grid points can be used for inversion to get t, as are used for the triton calculation. If not, sufficient points may be used that accurate interpolation can be performed to obtain t on the required grid.

One of the chief headaches throughout the calculation, and in particular the wavefunction calculation of the next chapter, is interpolation. There is generally a trade-off between speed and accuracy. Where interpolation was necessary, various fiddle functions were used to flatten out the functions so as to enhance interpolation accuracy. The forms

$$\left(1 + \frac{p^2}{a^2}\right)^m \quad \text{and} \quad \left(1 + \frac{q^2}{b^2}\right)^n \qquad (32)$$

were found quite useful, and simple. The parameters were chosen so as to reasonably smooth the function; m and n are basically determined by the asymptotic regions. The parameters a and b then were chosen to make the regions near the origin as flat as possible. For the potentials used in this

calculation m from 0.5 to 1.0 was used, and n anywhere from 1 to 4. The amplitude $\psi(pq)$ drops off much more rapidly as a function of q than as a function of p.

Interpolation was performed using either p or x as the variable. Near p=0 the functions are quite linear with respect to p^2. For interpolations over two variables a six point Lagrange formula was used in most cases. Where possible cubic spline interpolation was used, particularly for interpolations over one variable. For two variables cubic spline interpolation though quite accurate is costly.

Up to 16 points were used in the quadrature formulae for both the p and q variables. Various adjustments of c, the mesh points, and interpolation procedures were tried until convergence seemed achieved. Convergence proceeded in two stages. To obtain the binding energy the input energy had to be varied until the parameter λ (see eq. (23)) was equal to unity. To obtain the wavefunction further iterations were necessary; the criterion used was that $\psi^{n+1}(pq) = \psi^{n}(pq)$ to 1 part in 1000 at every point on the grid. Four to six iterations were usually enough to ensure a converged value for λ; another four or so were needed to get ψ to converge.

Various checks were made to ensure that the computer code was working. For the standard potential V_{ST} a binding energy of 8.80 MeV was obtained, compared with the code for a rank two separable potential which gives 8.806 MeV. In this case the two-body T-matrix can be calculated analytically

so the interpolation is unnecessary. For a separable potential the equations reduce to integration over a single variable. The Padé method was applied to this simplified problem and compared with a separable calculation. For a Yamaguchi potential the agreement is quite good, 9.295 MeV for the separable code versus 9.343 for the Padé method. Haftel (Haf 73) has done a similar calculation to mine for unitarily transformed phase-equivalent potentials. For his standard potential I obtained a value of 8.12 MeV compared to his 7.65 MeV. However, the parameters he lists for his potential give a deuteron binding energy of 0.383 MeV, not 0.416 as he states. Adjusting one of his parameters only slightly from 1.55 to 1.475 gives a deuteron binding energy of 0.4158 MeV and brings the triton binding energy down to 7.8 MeV. Since his parameters are not given to sufficient accuracy, a meaningful comparison of the two calculations cannot be made.

Overall it is believed that an accuracy of ±0.1 MeV has been achieved. In comparing the potentials of this calculation one might postulate that since they are similar, any numerical errors will be in the same direction and that 0.02 MeV is reasonable. Calculations were carried out until the energy was given to three decimal places. The numbers have been rounded to two places in the table.

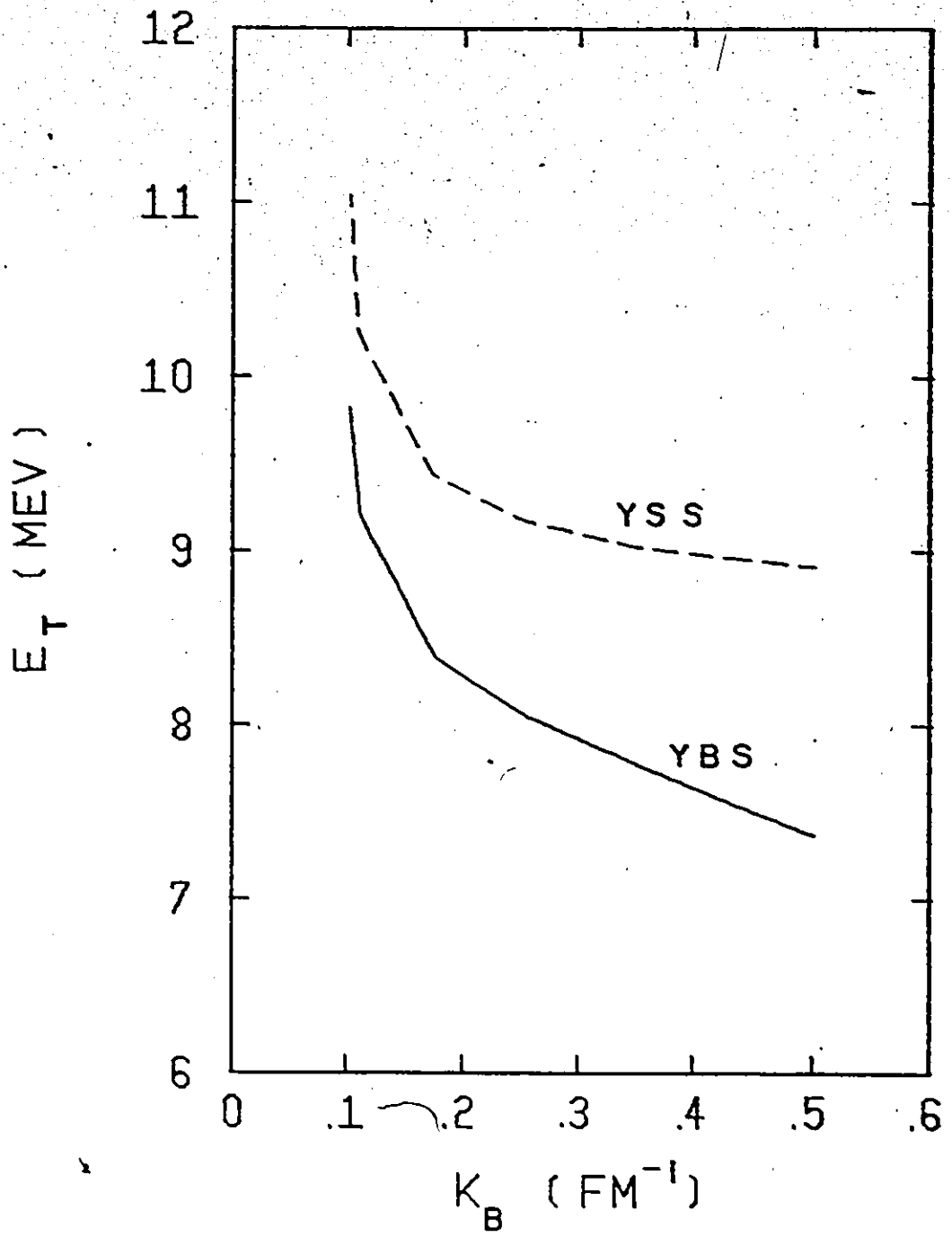


Figure 5. The triton binding energies for the two classes of potentials, as a function of the strength of the attractive part of the interaction.

BINDING ENERGIES

The iteration procedure described above was used for the calculation of the triton binding energies for the partly non-local potentials of Table I. For their equivalent separable potentials the formalism of Tabakin (Tab 65) for the model triton was used. It is more efficient from a calculational point of view and depends on matrix methods. The method is outlined in Appendix A. The calculations were performed using a computer code from H. Fiedeldey modified to work on the CDC-6400 and for form factors of general type g(p) and h(p). A further extension allows the calculation of the spectator functions.

The binding energies for the two classes of potentials are displayed in figure 5. Several observations may be made:

(i) Each separable potential is more strongly bound by 1 to 1.5 MeV compared with its equivalent partly non-local potential.

(ii) The variation in binding energy for the two classes of potentials is about the same, 2 MeV in both cases. The partly non-local potentials vary from 7.37 MeV to 9.82 MeV, while the separable potentials range from 8.91 MeV to 11.04 MeV.

(iii) For both classes of potentials the binding energy increases as the attractive part of the potential is weakened, that is, as the limiting condition $E_B \to E_D$ is approached.

This latter type of behaviour is similar to that found by Fiedeldey (Fie 69b, FM 72) for phase equivalent rank-two separable potentials with $g(k) = \gamma/(k^2+b^2)$ and $h(k)$ determined from the same phase shifts and E_D as used in this calculation. In his case the variation was much greater, ranging from about 8 to 16 MeV. Further discussion of these binding energies is postponed until Chapter VI.

CHAPTER IV

WAVEFUNCTION AND FORM FACTOR

The Low equation (1) may be used to extract the triton wavefunction from the Faddeev amplitudes obtained while calculating the triton binding energy. The wavefunction can then be used to calculate the form factor. The method used here is that of Haftel (Haf 73). For the separable potentials, a quantity equivalent to the Faddeev amplitudes can be extracted from the spectator functions, and used to calculate the wavefunctions. The charge form factor has been calculated for the two types of potentials. In most cases the proton charge distribution has been folded in with the neutron charge form factor taken as zero; however the effect of the neutron form factor has been examined.

THE WAVEFUNCTION

The triton wavefunction ψ_T can be obtained from the residue of the Faddeev amplitude near the triton binding energy. The starting point in the analysis is the Low equation (rather than the usual Lippmann-Schwinger equation)

$$T = V - VGV \qquad (1)$$

where T is the three particle T-matrix

V is the three particle potential, $V = V_{12}+V_{23}+V_{31}$

$G = (H-E)^{-1}$ is the three particle Green function for the complete interaction $H = H_o + V$

E is the energy of the three-body system.

Inserting a complete set of states (including the one bound state) gives

$$T_1 = V - \frac{V|\psi_T\rangle\langle\psi_T|V}{(\hbar^2/m)(s_T-s)} - \iint \frac{V|pq\rangle\langle pq|V\,dpdq}{(\hbar^2/m)(p^2+q^2-s)} \quad . \qquad (2)$$

Thus a pole at the bound state is given by the middle term of equation (2). Sandwiching T between the states ${}_1\langle pq\beta|$ and $|\psi_T\rangle$ gives

$${}_1\langle pq\beta|T(s)|\psi_T\rangle \doteq \frac{{}_1\langle pq\beta|V|\psi_T\rangle\langle\psi_T|V|\psi_T\rangle}{(\hbar^2/m)(s-s_T)} \quad . \qquad (3)$$

To simplify this, note that $V|\psi_T\rangle = (\frac{\hbar^2}{m} s_T - H_0)|\psi_T\rangle$ so that

$${}_1\langle pq\beta|V|\psi_T\rangle = (\hbar^2/m)(s_T-p^2-q^2)\,{}_1\langle pq\beta|\psi_T\rangle \quad . \qquad (4)$$

Also $\langle\psi_T|V|\psi_T\rangle \equiv \langle V\rangle$, a number independent of p, q and s. Substituting, the triton wavefunction is obtained

$${}_1\langle pq\beta|\psi_T\rangle = N\,\frac{{}_1\langle pq\beta|T(s)|\psi_T\rangle}{p^2+q^2-s_T} \qquad (5)$$

where s is near s_T but otherwise is arbitrary and N is a normalization constant determined from

$$\int_0^\infty p^2dp \int_0^\infty q^2dq \;|{}_1\langle pq\beta|\psi_T\rangle|^2 = 1 \quad . \qquad (6)$$

Using $T = T^{(1)} + T^{(2)} + T^{(3)}$ the wavefunction can be obtained

$${}_1\langle pq\beta|T(s)|\psi_T\rangle = {}_1\langle pq\beta|T^{(1)}(s)+T^{(2)}(s)+T^{(3)}(s)|\psi_T\rangle. \qquad (7)$$

The first term is just the Faddeev amplitude of chapter 3. Using equation (3-17) and inserting a complete set of states gives

$$ {}_1\langle pq\beta|T^{(j)}(s)|\psi_T\rangle = \sum_{\beta'} \int_0^\infty p'^2 dp' \int_0^\infty q'^2 dq' \; {}_j\langle pq\beta|p'q'\beta'\rangle_1 $$

$$ {}_1\langle p'q'\beta'|T^{(1)}(s)|\psi_T\rangle \quad . \qquad (8) $$

The coupling coefficients ${}_j\langle pq\beta|p'q'\beta'\rangle_1$ can be evaluated using equations (3-7) to (3-13) along with some angular momentum coupling formulae. The final result (Haf 73) is

$$ {}_1\langle pq\beta|\psi_T\rangle = N \frac{\delta_{LO}}{p^2+q^2-s_T} \{\psi(pq\beta)\delta_{\ell o}\delta_{Lo} $$

$$ + \delta_{\ell L}\sqrt{2\ell+1} \int_{-1}^{1} \psi\left[\left(\frac{p^2}{4} + \frac{\sqrt{3}pqx}{2} + \frac{3q^2}{4}\right)^{1/2}, \left(\frac{3p^2}{4} - \frac{\sqrt{3}pqx}{2} + \frac{q^2}{4}\right)^{1/2}, \beta\right] $$

$$ P_\ell(x)dx \qquad \ell, L \text{ even} $$

$$ = 0 \qquad \text{for } \ell \text{ odd} \qquad (9) $$

where we recall β = (ℓLLM, + isospin), and $x = \hat{p}\cdot\hat{q}$. Notice that even though the interaction acts only in relative S waves, the partial wave sum is over all even ℓ. This comes from trying to express p_2 and q_2 (of coordinate system 2) in terms of the variables of coordinate system 1 and the angle between them. For the separable potentials a quantity similar to the residue of the Faddeev amplitudes can be extracted from the spectator functions. After changing coordinate systems

equation (9) also applies. The details are included in Appendix A.

Gauss-Legendre quadrature was used to calculate the integral. Using ψ on the grid $\{p_k\}\{q_\ell\}$, interpolation over two variables is needed. The function was first smoothed using the forms of equation (3-32) and either a six point bivariate Lagrange formula was used or a bivariate cubic spline. The latter is better by far but more time consuming. This probably represents the least accurate part of the calculation.

THE FORM FACTOR

Energetic electron beams can be used to probe the charge distribution of the nucleus. In the first Born approximation (which is equivalent to single-photon exchange and valid for light nuclei), the cross section for electron scattering by a spin-zero non-relativistic nucleus is proportional to the Fourier transform of the nuclear charge density. We can write

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} |F_{ch}(Q^2)|^2 \qquad (10)$$

where $\underline{Q}$ is the momentum transfer, $(d\sigma/d\Omega)_{\text{Mott}}$ is the scattering due to a point nucleus of the same charge, and $F_{ch}(Q^2)$ is the charge form factor defined as

$$F_{ch}(Q^2) = \int e^{i\underline{Q}\cdot\underline{x}}\, \rho_{ch}(\underline{x})\, d\underline{x} \qquad (11)$$

where $\rho_{ch}(x)$ is the charge density of the nucleus (normalized to unity). By measuring $F_{ch}(Q^2)$ we can learn about $\rho(\underline{x})$ from the inverse Fourier transform. To a large extent comparison of electron scattering data with theoretical predictions is used to judge the "goodness" of a wavefunction.

Relativistic effects can be considered. The Mott scattering term of equation (10) is an improvement on the Rutherford scattering formula, since it takes into consideration the spin of the electron and the recoil of the nucleus. The (three-) vectors of equation (10) should really be replaced by four-vectors and the integral performed over space-time. However, if the energy (time component) transferred to the nucleus is small, as it must be for the nucleus to recoil coherently, we can still use the integral over the spatial charge distribution. If the nucleus has spin, magnetic effects become important and the problem gets much worse. A variety of different relativistic corrections have been proposed and there exist almost endless possibilities for confusion. Friar (Fri 73) has carefully examined the relativistic corrections to the scattering of electrons by a nonrelativistic nucleus, particularly ^2H, ^3He and ^4He. These corrections can be tacked on at the end. For the moment the nonrelativistic treatment is followed. The form factor for the (point) distribution of nucleons is given by equation (11) with the the subscripts dropped. For an A particle system the point density is given by

$$\rho(\underline{x}) = (1/A) \sum_{i=1}^{A} \int [d\underline{x}_1 \ldots d\underline{x}_{i-1} d\underline{x}_{i+1} \ldots d\underline{x}_A \psi^*(\underline{x}_1 \ldots \underline{x}_A)\psi(\underline{x}_1 \ldots \underline{x}_A)]_{\underline{x}_i=\underline{x}} \tag{12}$$

where the $\underline{x}$'s are the distances from the centre of mass. To get the charge density the charge distribution of the nucleons must be folded in,

$$\rho_{ch}(\underline{x}) = \int \rho(\underline{x}')\rho_{nucleon}(|\underline{x}-\underline{x}'|)d\underline{x}' \ . \tag{13}$$

For three identical particles equation (12) becomes

$$\rho(\underline{x}) = \iint d\underline{x}_2 d\underline{x}_3 \ \psi^*(\underline{x}_1\underline{x}_2\underline{x}_3)\psi(\underline{x}_1\underline{x}_2\underline{x}_3)|_{\underline{x}_1=\underline{x}} \tag{14}$$

and the form factor is

$$F(Q^2) = \iiint d\underline{x}_1 d\underline{x}_2 d\underline{x}_3 \ |\psi(\underline{x}_1\underline{x}_2\underline{x}_3)|^2 \ e^{i\underline{Q}\cdot\underline{x}_1} \ . \tag{15}$$

Our wavefunctions are defined in momentum space as functions of $\underline{p}$ and $\underline{q}$ (and $\underline{P}_c$). Conjugate to these coordinates are the combinations

$$\underline{u} = \underline{x}_2 - \underline{x}_3$$

$$\underline{v} = \frac{2}{\sqrt{3}} \left(\frac{\underline{x}_2+\underline{x}_3}{2} - \underline{x}_1\right) \tag{16}$$

$$\underline{w} = \sqrt{6} \ (\underline{x}_1+\underline{x}_2+\underline{x}_3) \ .$$

Writing $\psi(\underline{u}\ \underline{v}) \equiv \psi(\underline{x}_1\underline{x}_2\underline{x}_3)$ and using the c.m. constraint $\delta(\underline{x}_1+\underline{x}_2+\underline{x}_3)$, equation (15) becomes

$$F(Q^2) = \iint d\underline{u}d\underline{v} \, |\psi(\underline{u}\underline{v})|^2 \, e^{-i\underline{Q}\cdot\underline{v}/\sqrt{3}} \, . \qquad (17)$$

Transforming to momentum space gives

$$F(Q^2) = \iint d\underline{p}d\underline{q}\psi^*(\underline{p}\,\underline{q})\psi(\underline{p},\underline{q}-\underline{Q}/\sqrt{3}) \, . \qquad (18)$$

Our wavefunction is

$$\psi(\underline{p}\,\underline{q}) = \sum_{\beta} {}_1\langle pq\beta|\psi_T\rangle Y_{\ell m}(\hat{\underline{p}}) Y_{LM}(\hat{\underline{q}}) \, . \qquad (19)$$

Thus (Haf 73)

$$F(Q^2) = \sum_{\beta} \int_0^{\infty} p^2 dp \int_0^{\infty} q^2 dq \, \langle\psi_T|pq\beta\rangle_1$$

$$\int d\hat{q}_1\langle p, \sqrt{q^2 - 2\underline{Q}\cdot\underline{q}/\sqrt{3} + Q^2/3}, \beta|\psi_T\rangle Y^*_{\ell m}(\hat{\underline{q}}) Y_{LM}(\hat{\underline{q}-\underline{Q}/\sqrt{3}}) \qquad (20\text{-a})$$

$$= \frac{1}{2} \sum_{\ell \text{ even}} \int_0^{\infty} p^2 dp \int_0^{\infty} q^2 dq \, \langle\psi_T|pq\beta\rangle_1$$

$$\int_{-1}^{1} dx \, {}_1\langle p, \sqrt{q^2 - 2Qqx/\sqrt{3} + Q^2/3}, \beta|\psi_T\rangle P_\ell(x_0) \qquad (20\text{-b})$$

where $x_0 = (q - Qx/\sqrt{3})/\sqrt{q^2 - 2qQx/\sqrt{3} + Q^2/3}$.

Compensation for relativistic effects can be included by replacing Q by Q_r (Fri 73) where

$$Q_r^2 = Q^2\left[1 - \frac{\hbar^2 Q^2}{2Mc^2}\right] \qquad (21)$$

(M is the triton mass). The effect is mainly to push out the

calculated form factor a bit, (.22 fm^{-2} at $Q^2 = 10.0\ fm^{-2}$), and is twice the usual Lorentz-contraction correction. To obtain the charge form factor for comparison with experiment, the nucleon charge form factors must be folded in. In momentum space this corresponds simply to multiplication by the nucleon charge form factors. For the triton and helion this gives

$$F_{ch}^{^3H}(Q^2) = [f_{ch}^{p}(Q^2) + 2\ f_{ch}^{n}(Q^2)]F(Q_r^2) \quad (22\text{-a})$$

$$2\ F_{ch}^{^3He}(Q^2) = [2\ f_{ch}^{p}(Q^2) + f_{ch}^{n}(Q^2)]F(Q_r^2) \quad (22\text{-b})$$

CALCULATIONS AND RESULTS

The grid on which $\langle pq\beta|\psi_T\rangle$ was defined corresponds to Gauss-Legendre points transformed from (-1,1) to $(0,\infty)$ by equations (3-30). Thus the integration over p in equation (20) can be done "exactly", that is, no interpolation of ψ_T is needed. The function was smoothed as discussed in chapter III and cubic spline interpolation used. Gauss-Legendre points were used for the integration depending on the value of ℓ. Using terms $\ell = 0, 2$ and 4 in the sum seemed sufficient to give about three figures of accuracy. As mentioned before the main uncertainty is in the obtaining of ψ_T accurately.

The analytic "three-pole fit" (Jan+ 66) to the proton form factor was used; the neutron form factor was taken to be zero. This gives identical charge form factors to ^3H and ^3He in this model. The charge form factors are plotted in figure o

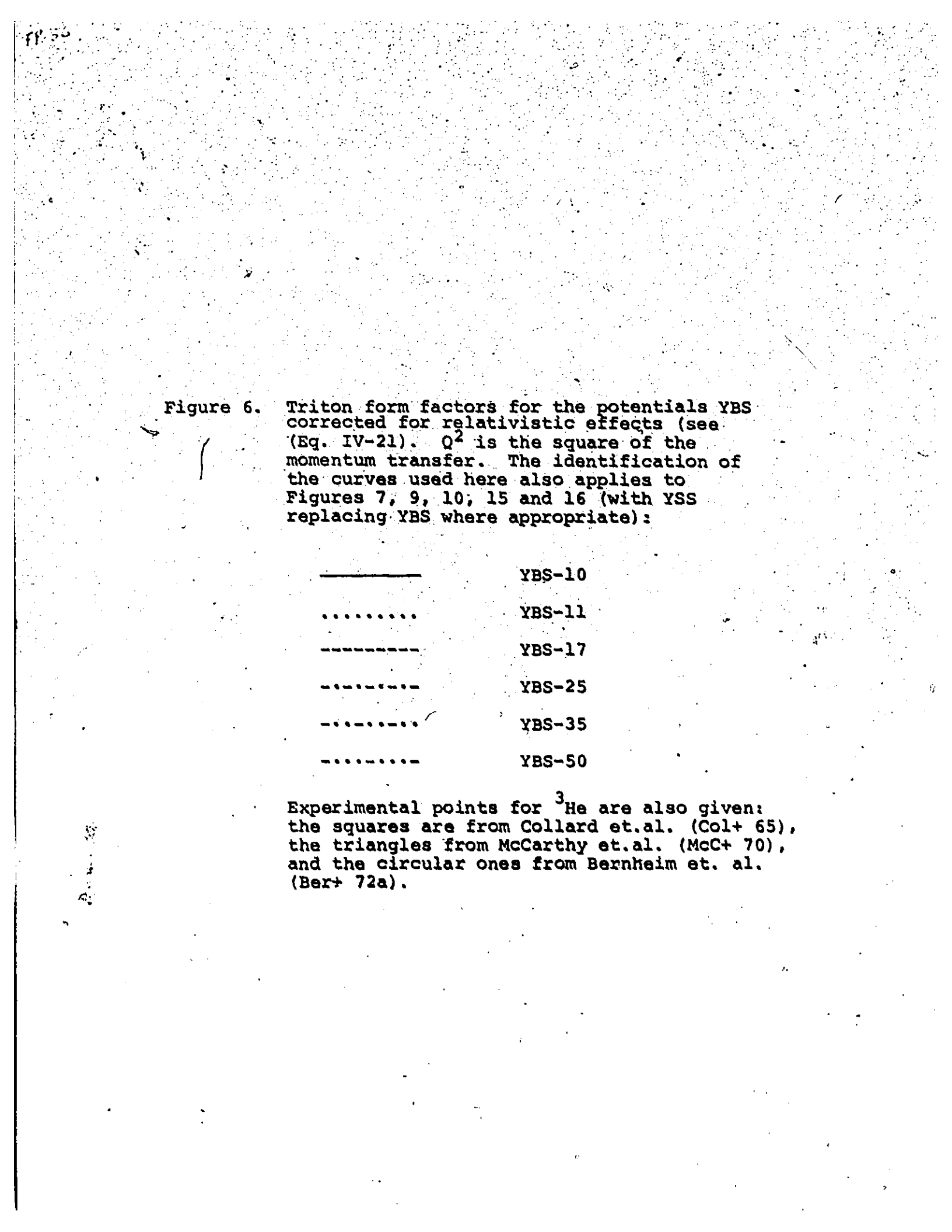
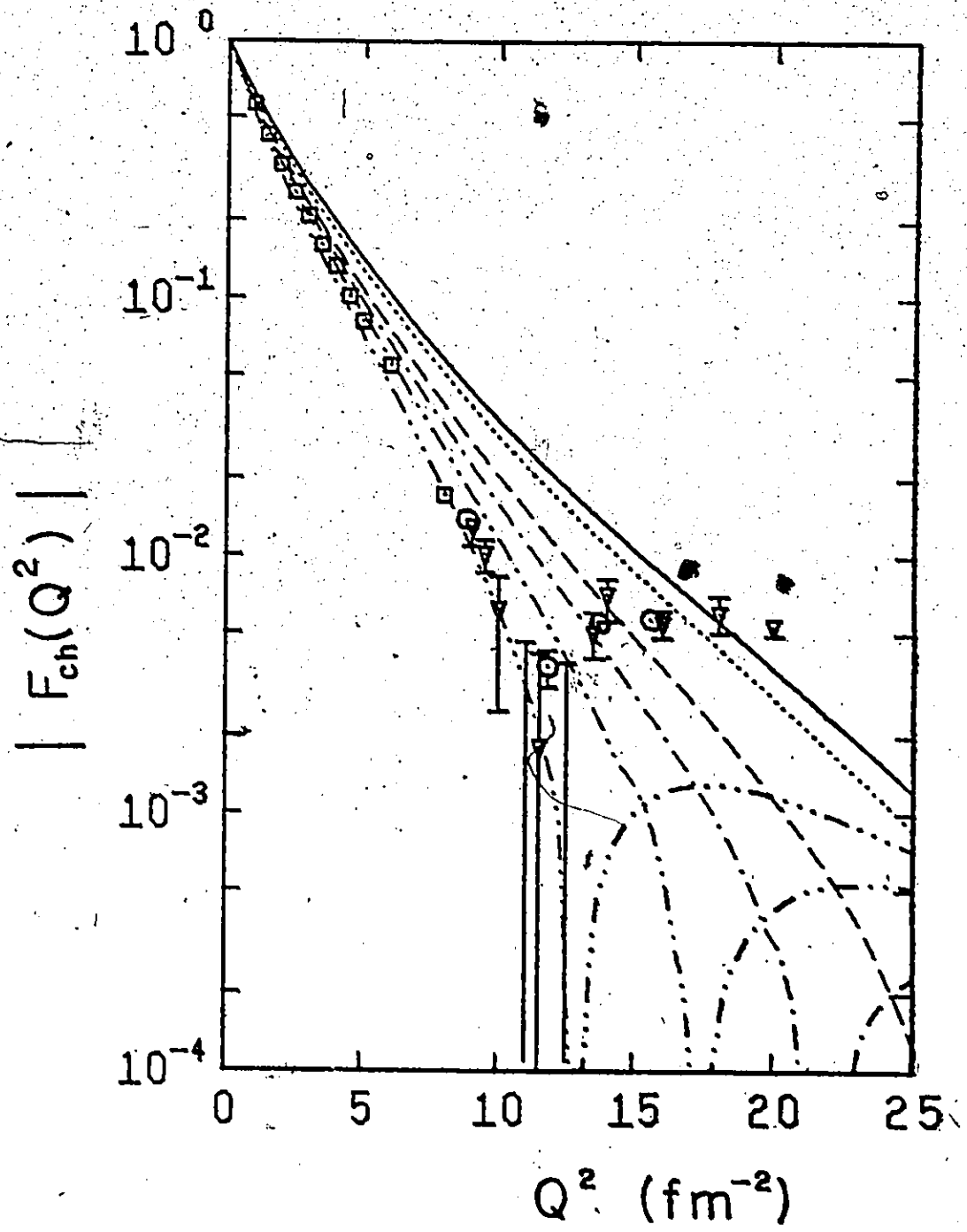


Figure 6. Triton form factors for the potentials YBS corrected for relativistic effects (see (Eq. IV-21). Q^2 is the square of the momentum transfer. The identification of the curves used here also applies to Figures 7, 9, 10, 15 and 16 (with YSS replacing YBS where appropriate):

————	YBS-10
.....	YBS-11
----------	YBS-17
-.-.-.-.-	YBS-25
-.-.-.-.-	YBS-35
-.-.-.-.-	YBS-50

Experimental points for ^{3}He are also given: the squares are from Collard et.al. (Col+ 65), the triangles from McCarthy et.al. (McC+ 70), and the circular ones from Bernheim et. al. (Ber+ 72a).


10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$|F_{ch}(Q^2)|$
0
5
10
15
20
25
Q^2 (fm^{-2})

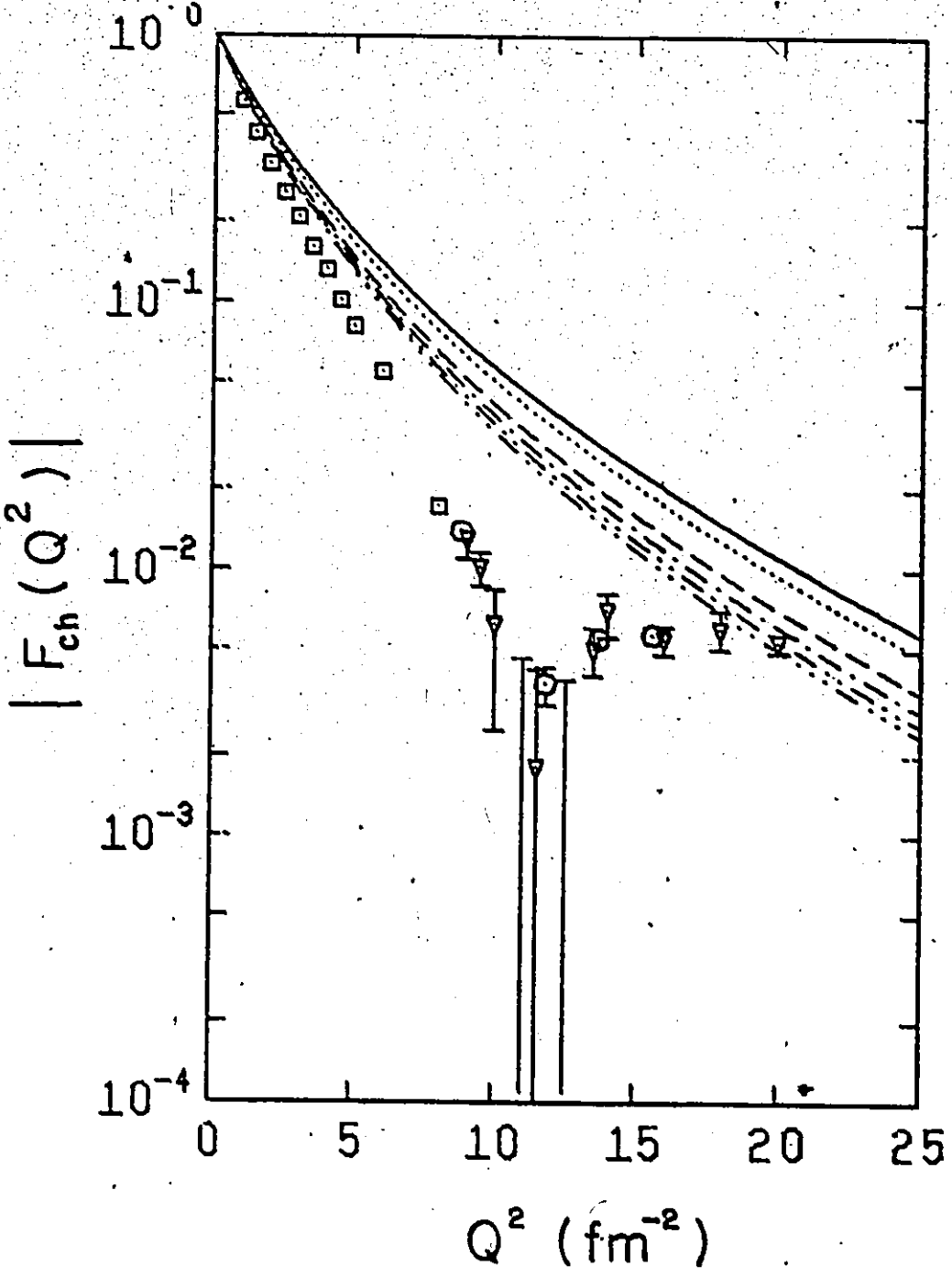


Figure 7. Triton form factors for the potentials YSS. Legend as in Figure 6.

for the partly non-local potentials and in figure 7 for the separable potentials. The relativistic correction is included and experimental points (Col+ 65, McC+ 70, Ber+ 72a) are included for comparison.

The most noteworthy feature of these results is that the separable potentials give larger form factors than the partly nonlocal potentials and have no minimum in the form factor, whereas the partly nonlocal potentials either have a minimum or appear to be going to have one just beyond the range of the graph. The other feature to note is that the form factors are quite dependent on the triton binding energy. Near $Q^2 = 0$ the slope of the form factor is almost linearly related to the binding energy of the triton. A larger binding energy corresponds to less slope, and using the relation

$$F(Q^2) = 1 - \frac{1}{6} \langle r_{rms} \rangle^2 Q^2 + \ldots . \qquad (23)$$

it can be seen that this means a smaller root-mean-square radius. To obtain a more quantitative feeling for this, an approximate r.m.s. radius is calculated using the form factor at $Q^2 = 0.5$ fm^{-2}. (The limit $Q^2 \to 0$ did not work too well for some numerical reasons.) The plots of $F(Q^2)$ on a semi-logarithmic graph are almost linear as a function of Q^2, so

$$F(Q^2) = e^{-\frac{a^2 Q^2}{6}} \qquad (24)$$

was assumed. Expanding in a Taylor series gives a form like eq. (23), so a can be associated with the r.m.s. radius. It

turns out that the values of a are almost linear with respect to E_T. They are listed in Table II, and range from 1.64 to 1.99 fm. The approximate linearity holds true for both classes of potentials, although the limiting cases tend to depart somewhat from the line. Using the approximation of eq. (24) the values of a for the ^{3}H and ^{3}He experimental data are 1.689 and 1.845 fm compared with the actual root-mean-square radii of 1.70 and 1.88 fm respectively. None of the potentials achieve a minimum at the experimental value of 11.6 fm^{-2}, although Potential YBS-50 comes close at 13.1 fm^{-2}, and none come near the secondary maximum. For potential YBS-17 which has the binding energy (8.4 MeV) nearest the binding energy of ^{3}He, the form factor minimum is at 27 fm^{-2}. The separable potential YSS-50 with binding of 8.91 MeV shows no indication of giving a minimum. The minima in the form factors as plotted are due to the form factor changing sign. The nucleon form factors cannot change the position of the minimum (see eq. (22)) since both the neutron and proton form factors are positive and, although the neutron form factor is increasing, the proton form factor is decaying at a faster rate. The nucleons do affect the value of the form factor and in particular the height of the secondary maximum.

A great deal too much fuss† seems to be made about the height of the secondary maximum and the fact that calculated form factors never seem to reach the experimental values.

†See for example many of the references quoted on pp 90-91 of (KT 74).

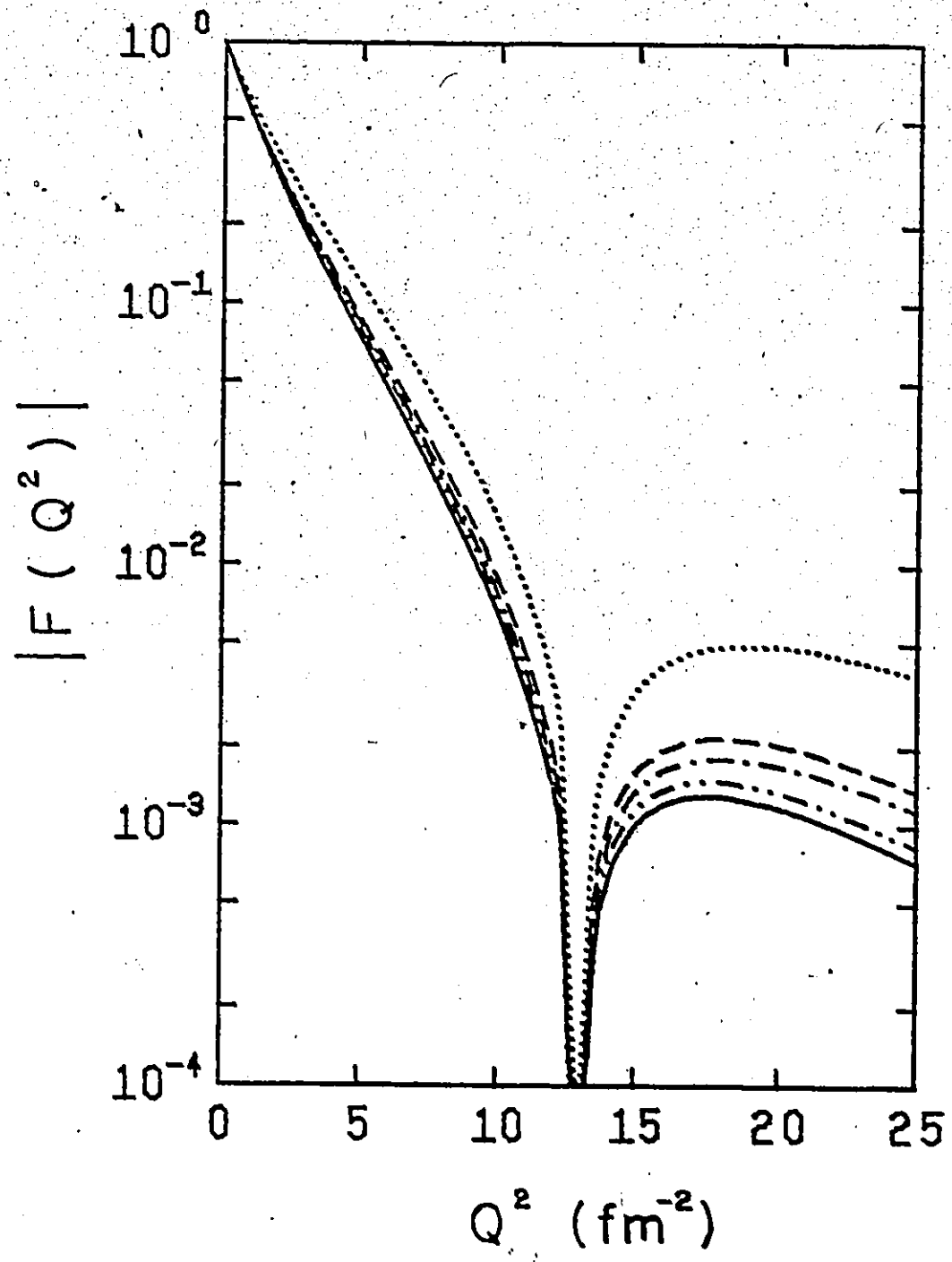


Figure 8. The effect of the nucleon form factors on the triton form factor for potential YBS-50. The dotted line is the body form factor, the solid line the form factor using f^{p}_{ch} only (Jan+ 66). The dashed, dash-dot and dash-dot-dot lines show the effect of the neutron form factor using the maximal (Ber+ 72b) standard (Jan+ 66) and minimal (Ber+ 72b) fits to f^{n}_{ch} respectively.

This is not at all surprising since most potentials give a minimum at too large a value of Q^2 compared with experiment. It has been shown that the form factor must lie below some envelope[†] (Dre+ 74). If the secondary maximum lies at larger Q, then it is necessarily constrained to have a smaller value. Unless the experimental minimum is fitted by the calculated form factor, the height of the secondary maximum does not seem to be a greatly useful quantity by which to judge the potential.

For comparison the deuteron form factors for the potentials are shown in figures 9 and 10. There is an extremely strong correlation between the deuteron form factors and the triton form factors, particularly in the position of the minima.

The role of the neutron charge form factor is briefly examined here. The neutron charge form factor is important in heavy nuclei (Ber+ 72b) and calculations have been done (BS 75) using realistic ^3H and ^3He wavefunctions. The neutron form factor is rather poorly defined experimentally but minimal and maximal fits (Ber+ 72b) can be used as well as the standard fit (Jan+ 66). The difference between the minimal and maximal values draws in the calculated minimum of ^3H by 0.4 fm^{-2} and moves out the minimum of ^3He by 0.2 fm^{-2}. The secondary maxima of ^3H and ^3He can be increased by factors of 1.5 and 1.1 respectively (BS 75). In our case the nucleon form factors have no effect on the position of the minima, but

[†]For example for ^{208}Pb, the envelope Cq^{-4} was used with C determined from the position of the last maximum.

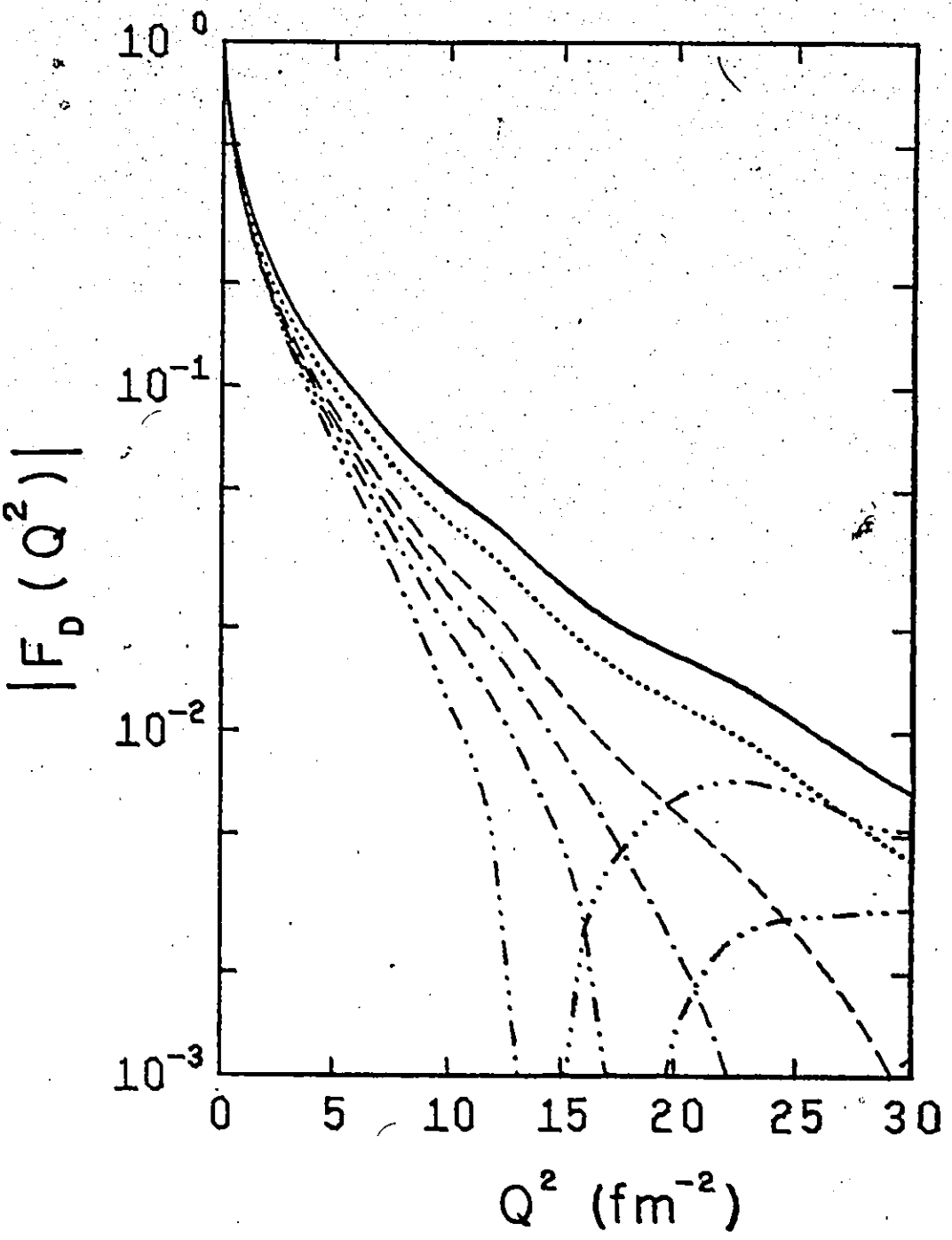


Figure 9. The deuteron body form factor for the partly non-local potentials. Curves are labelled as in Figure 6.

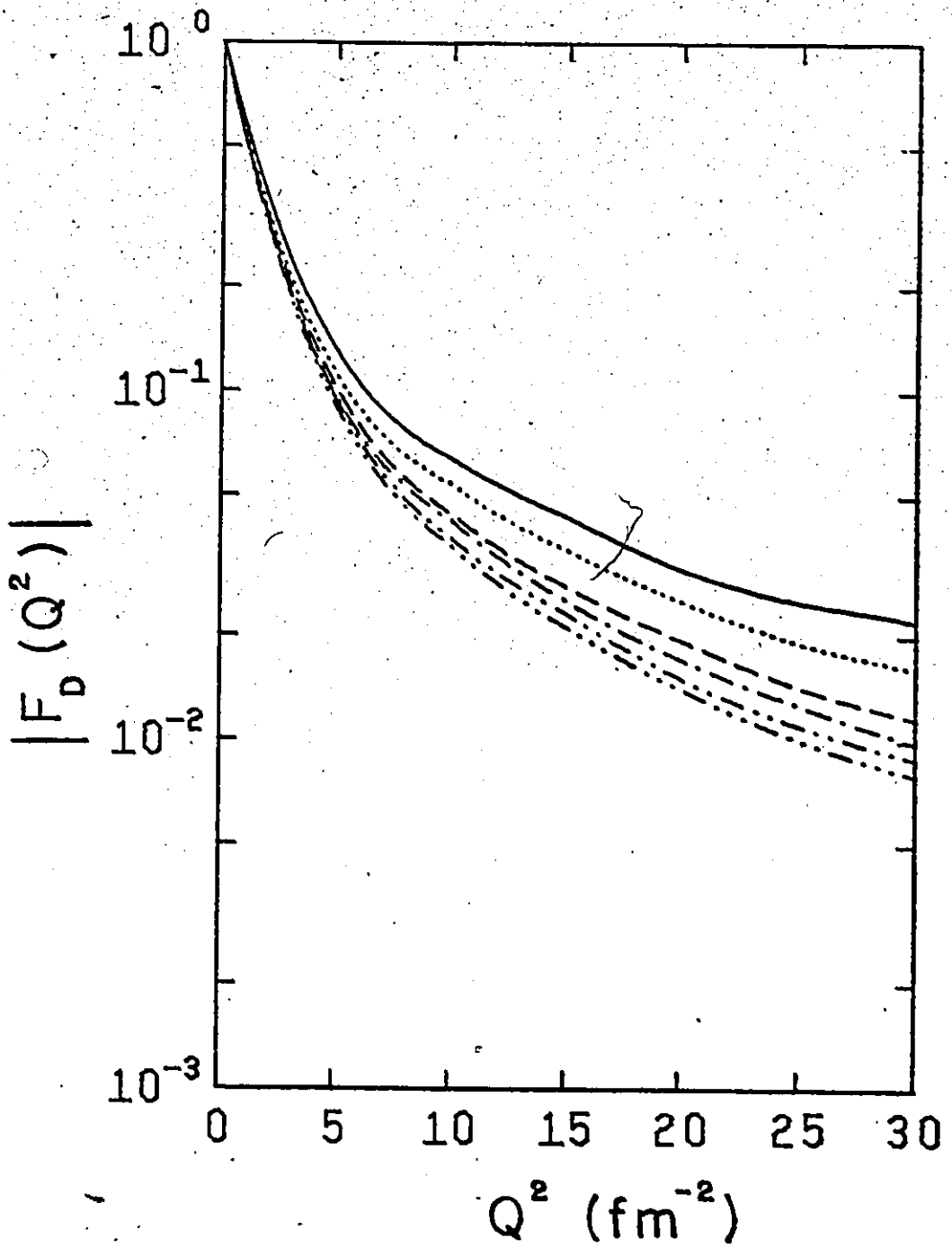


Figure 10. The deuteron body form factor for the separable potentials. Curves are labelled as in Figure 6.

have considerable effect on the value of the form factors. The role of the nucleon form factors is depicted in figure 8, where potential YBS-50 is used since it corresponds most closely to the experimental data. The upper curve is that of the body form factor and the lower curve the charge form factor using the proton charge form factor only, as in figures 6 and 7. The neutron form factor increases the form factor somewhat, particularly in the region of the secondary maxima. The effects of the maximal, minimal and standard fits to the neutron form factor are depicted for ^3H where the effect is greatest. For ^3He the role of the neutron form factor is reduced by a factor of 4. At $Q^2 = 25$ fm^{-2} the form factor using the maximal neutron form factor is almost double the result using only the proton form factor.

The problem of the ^3He charge form factor

One of the problems of the three nucleon system is the inability to simultaneously fit the binding energy and the ^3He charge form factor near and beyond the diffraction minimum. Realistic potentials typically give triton binding energies 1 to 1.7 MeV below the experimental value of 8.5 MeV (KT 74). Not only do these potentials not fit E_T but they give a charge form factor minimum at momentum transfer squared, Q^2_{min}, which is too large (12-18 fm^{-2} compared with 11.6 fm^{-2} experimentally) and a secondary maximum too small by a factor anywhere between 3 and 8 (KT 74). Moreover, phase shift equivalent potentials which increase the value of E_T toward

better agreement with experiment also increase Q^2_{min} to values even further away from the experimental value. This is expected as larger E_T causes smaller r.m.s. radius, and therefore shifts the diffraction minimum out in Q-space. Tensor forces, higher partial waves and relativistic corrections do not help this E_T vs. Q^2_{min} dilemma very much (BKT 74). (The further complication of the large discrepancy between Q^2_{min} as obtained from variational calculations and as obtained via Faddeev methods appears now much reduced with the improvement of the momentum space Faddeev results (BKT 75). For the Reid soft core potential (Rei 68) for example, the most elaborate variational method gives $E_T = 7.2 \pm 0.2$ MeV and $Q^2_{min} = 13.3$ fm^{-2} (SS 74), coordinate space Faddeev calculations give $E_T = 7.0$ MeV and $Q^2_{min} = 14.0$ fm^{-2} (LG 73), while the momentum space Faddeev results yield $E_T = 6.98$ MeV and $Q^2_{min} = 13.9$ fm^{-2} (BKT 75). The remarkable agreement among the three methods for E_T and Q^2_{min} as well as the height of the secondary maximum and the probabilities of the various states is an indication of the reliability of the results.)

This indicates a role for other effects such as three-body forces, charge dependent forces and meson exchange currents. Brayshaw (Bra 73) from a comparison of the form factor data on ^{3}H and ^{3}He concludes that there is evidence for a strong attractive three-body force. Theoretical estimates indicate that as much as 2.3 MeV in the correct direction could be obtained (KT 74). Haftel (Haf 75, Haf 76)

has shown that considerable improvement can be obtained using three-body unitary transformations that leave E_T constant. This is another way to invoke three-body forces. Meson exchange corrections to charge form factor results by Kloet and Tjon show improved fits to Q^2_{min} and the secondary maximum (KT 76).

CHAPTER V

THE ASYMPTOTIC NORMALIZATION CONSTANT

It has been proposed (Lim 73) that the triton asymptotic normalization constant, C_T, be treated as an observable on a par with the charge radius, form factor, and binding energy. The stimulus came from an earlier extraction of C_T from a modified phase-shift analysis of elastic p - ^{3}He scattering in which a marked improvement in the weighted variance followed from using peripheral values for the phase-shifts in the higher partial waves (BH 72). As an object of theoretical calculations, C_T^2 is a measure of the probability that the neutron (proton) and deuteron are a considerable distance apart in the ^{3}H (^{3}He) nucleus. The quantity can be extracted from various nuclear scattering and transfer reactions as well as the electrodisintegration and photodisintegration of ^{3}He. It is possible to extract the triton asymptotic normalization constant from the triton momentum space wavefunctions. This is done for both classes of our potentials to see if it is a quantity sensitive to the details of the nuclear force.

To begin with, let us look at the deuteron, where the dimensionless asymptotic normalization constant C_D is defined by

$$\phi_D^s(\underline{r}) \xrightarrow[r\to\infty]{} C_D \sqrt{2k_D}\, \frac{e^{-k_D r}}{r}\, Y_{oo}(\hat{\underline{r}}) \qquad (1)$$

where the deuteron binding energy is $E_D = (\hbar^2/m)k_D^2$. In the two nucleon problem the asymptotic normalization is of little interest, being given by (Wil 63)

$$C_D^2 = \frac{1}{1-k_D r_t} \qquad (2)$$

where r_t is the effective range for the triplet n-p interaction (evaluated at the deuteron binding energy). Since all potentials whether or not they contain tensor force, hard core, non-locality, or whatever must closely reproduce the deuteron binding energy (2.225 MeV) and the effective range (1.76 fm), C_D^2 is fixed at 1.69. There appears to be no simple relation like this for the triton.

The triton asymptotic normalization constant (to be called C rather than C_T from now on) is defined by

$$\psi(\underline{r}_1\underline{\rho}_1) \xrightarrow[r\to\infty]{} C\sqrt{2\alpha}\, \frac{e^{-\alpha\rho_1}}{\rho_1}\, Y_{oo}(\hat{\underline{\rho}}_1)\, \phi_D^s(r_1)\, Y_{oo}(\hat{\underline{r}}_1)\, \frac{\chi_1\eta_2}{\sqrt{2}} \qquad (3)$$

where: $\underline{r}_1$ and $\underline{\rho}_1$ are the coordinate distances as defined in equation (3-2),

$\alpha = \sqrt{(4/3)(m/\hbar^2)(E_T - E_D)}$,

$E_B = E_T - E_D$ is the separation energy of the neutron in ^3H,

ϕ_D^s is the s-state deuteron radial wavefunction, normalized to $\int |\phi_D^s(r_1)|^2 r_1^2 dr_1 = 1$,

and $\chi_1\eta_2/\sqrt{2}$ is the spin-isospin function corresponding to

corresponding to n-d($^3S_1$) in the triton. As with the deuteron, the parameters are chosen so that the wave function is given everywhere by the RHS of equation (3), then C = 1. C greater than unity implies a hole in the wavefunction of size $\ln|C|/\alpha$ (BH 72).

Any measurement which samples mainly the asymptotic region n-d (p-d) of ^{3}H (^{3}He) should enable one to extract C. In the inner region where the wavefunction is not completely described by the asymptotic form, it may be necessary to introduce some model as an approximation, which hopefully does not much affect the extraction of C. There are a number of experiments from which C has been extracted. A sketch is presented here but for a more complete listing with references see the review article by Kim and Tubis (KT 74).

Various nuclear transfer reactions, in particular stripping and pickup reactions of the form (t,d) and (d,t), have been used to extract values of C(^{3}H). A schematic picture of the process is given in figure 11 to give an illustration of equation (3). The value of C is important in the analysis of these experiments using some model, such as the distorted-wave-Born approximation or the peripheral model.

The ^{2}S channel for elastic n-d scattering shows unique, anomalous features around threshold. Forward and partial-wave dispersion relations can be used to estimate C^2.

From two-body photodisintegration and electrodisintegration data of ^{3}He values can be assigned to C. Separable interactions are usually used to analyse the photodisintegration

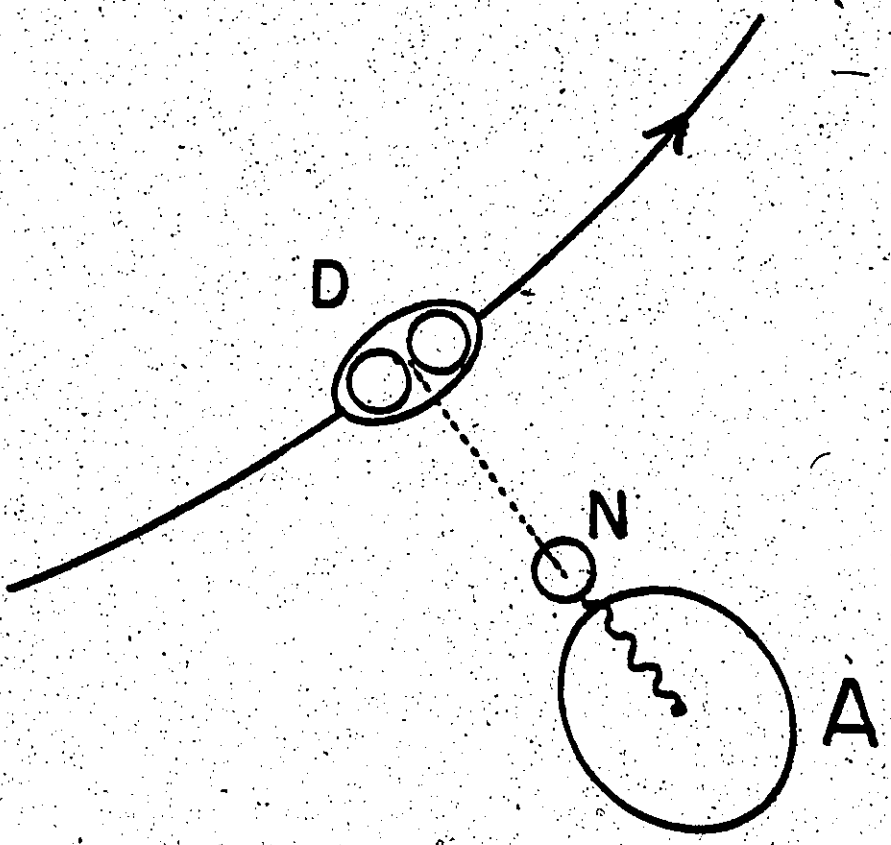


Figure 11. A transfer or stripping reaction in which the triton asymptotic normalization constant will be important: A is a nucleus, N a nucleon and D a deuteron.

data; they consistently give values of C greater than those found using other methods. This may be related to our results in which the separable potentials give the larger values of C.

One of the most interesting methods of extracting C is from the low energy elastic scattering of p - ^{3}He and n - ^{3}H (BH 72). The idea is the same as that used in nucleon-nucleon scattering (Czi+ 59) in which the higher partial waves with larger impact parameters are assumed to be purely peripheral and the corresponding phase shifts computed explicitly. The lower partial waves are arbitrary and used as the variables in a least squares fit to the data. C is adjusted to give the best χ^2 fit to the data.

From an analysis of a number of these experiments Lim (Lim 73) obtains $C^2(^3He) = 3.4$ and $C^2(^3He) = 3.0$, about a ten percent difference. Another analysis (GGP 73) from a weighted average of various experiments gives $C^2(^3H) = 3.0 \pm 0.6$. The analysis of p - ^{3}He and n - ^{3}H scattering data suggests that $C^2(^3H) \approx C^2(^3He)$ if Coulomb effects are properly taken into account in extracting C.

There are also a number of theoretical calculations of C. From the triton wavefunction for the Reid soft core potential a value of $C^2 = 2.86$ has been obtained (KT 72). Short range phase equivalent modifications of the same potential give only small variations (KT 74). Other potentials give quite large variations: the Darwich-Green potential (DG 67) gives $C^2(^3H) = 3.8$ and the Malfliet-Tjon potential (MT 69) gives 4.5 (OB 73); for various separable interactions the

values of $C^2(^3H)$ range from 2.58 to 3.84, and two values of $C^2(^3He)$ are 2.79 and 3.26 (GL 75). For the potentials studied in this thesis $C^2(^3H) \approx 3.3$ for all the partly non-local potentials (except the limiting case), but varies from 3.43 to 4.5 for the rank-two separable counterparts.

There are a number of methods to obtain C from the triton wavefunction. The first calculation was that of Kim and Tubis (KT 72) who extrapolated the triton wavefunction in momentum space $\psi(\underline{p},\underline{q})$ to the (unphysical) value $q^2 = -\alpha^2$ where there is a pole. A second method (GL 75), valid for separable potentials, is an integral relation involving the form factor g(p) and the spectator function G(Q). The third method (LG 76) is a generalization of the second, and involves an integral over the deuteron wavefunction, the potential and a quantity similar to the Faddeev amplitude of chapter III. The formula resembles integrals employed in obtaining the triton wavefunction and should be particularly useful for calculating the asymptotic normalization of wavefunctions obtained from local potentials.

The first method suffers from the drawback that extrapolations tend to be unstable and perhaps not as accurate (BIO 75). It has the advantage that very little computation is involved (once the triton wavefunction has been calculated). The second method is not general enough for our potentials. The third method is quite recent, too recent to be incorporated into this work. It is in principle exact but the integrals

involved are prohibitively difficult for the general potentials used in this calculation, and involve evaluating the potential and wavefunction at complex momenta. An improved extrapolation procedure, similar to that of Kim and Tubis was therefore selected for the calculation of C.

In momentum space equation (3) is written as

$$\langle \underline{P}_1 \underline{Q}_1 | \psi^{asy} \rangle = C \sqrt{\frac{4\alpha}{\pi}} \frac{1}{Q_1^2+\alpha^2} Y_{oo}(\hat{\underline{Q}}_1) \phi_D^s(P_1) Y_{oo}(\hat{\underline{P}}_1) \frac{\chi_1 \eta_2}{\sqrt{2}} . \quad (4)$$

Comparing with the relevant portion of the triton wavefunction

$$\langle \underline{P}_1 \underline{Q}_1 | \psi_T \rangle = \psi_T(P_1 Q_1) Y_{oo}(\hat{\underline{P}}_1) Y_{oo}(\hat{\underline{Q}}_1) \frac{\chi_1 \eta_2}{\sqrt{2}} \quad (5)$$

and noting that ψ_T has a pole at $Q_1^2 = -\alpha^2$, gives

$$C = \lim_{Q_1^2 \to -\alpha^2} \left[\frac{\psi_T(P_1 Q_1)(Q_1^2+\alpha^2)}{\sqrt{\frac{4\alpha}{\pi}} \phi_D^s(P_1)} \right] . \quad (6)$$

The quantity in square brackets will be referred to as $f(P_1, Q_1)$. The value of C can be obtained by picking a value for P_1, plotting $f(P_1, Q_1)$ as a function of Q_1^2 for small Q_1, and finally extrapolating to $Q_1^2 = -\alpha^2$.

Kim and Tubis used the polynomial form

$$f(P,Q) = C + \sum_{n=1}^{N} a_n(P) (Q^2+\alpha^2)^n \quad (7)$$

to obtain C. Note that C is independent of P_1, but in practice

will show some variation due to the extrapolation procedure and inaccuracies in the wavefunction. For the Kim and Tubis calculation using wavefunctions calculated using the Reid potential, this method gives an extremely stable value of C for P_1 ranging from 0.36 fm^{-1} to 8.33 fm^{-1}. In our case the method is not nearly as accurate because:

i) We have to extrapolate about twice as far.
$E_B = E_T - E_D$ is typically 9 MeV compared with
$E_B = 6.70-2.23 = 4.47$ MeV for the Reid potential.

ii) The small "deuteron" binding energy (.43 MeV or 0.01 fm^{-2}) causes the Green function singularity $(P_1^2 + \frac{3}{4} Q_1^2 - s_T)^{-1}$ of ψ_T to be very close to the pole at $Q_1^2 = -\alpha^2 = -\frac{3}{4} S_B$, if P_1 is small. Thus the function which we extrapolate has to be modified.

This "kinematical curvature" can be removed by writing

$$\psi_T(P_1,Q_1) = \tilde{\psi}_T(P_1,Q_1)/(P_1^2 + \tfrac{3}{4} Q_1^2 - s_T)$$

and

$$\psi_D^s(P_1) = g_D(P_1)/(P_1^2 - s_D) \quad . \tag{8}$$

Then,

$$f(P_1,Q_1) = \left[\frac{\tilde{\psi}_T(P_1,Q_1)\,(Q_1^2 + \alpha^2)}{\sqrt{\frac{4\alpha}{\pi}}\, g_D(P_1)}\right] \frac{P_1^2 - s_D}{P_1^2 + \frac{3}{4} Q_1^2 - s_T} \quad . \tag{9}$$

The last factor can be written as

$$\left[1 + \frac{\frac{3}{4} Q_1^2 - s_B}{P_1^2 - s_D}\right]^{-1} \quad . \tag{10}$$

Notice that it is equal to 1 at $Q_1^2 = \frac{4}{3} s_B = -\alpha^2$, but varies rapidly for small P_1, since s_D is so small (- 0.01 fm^{-2}). For large P_1, it is almost constant and equal to unity.

There is another wrinkle in the treatment. The coordinates in terms of which ψ_T is determined are p_1 and q_1, not P_1 and Q_1. This means that $q_1 = \sqrt{3/4}\, Q_1$, and involves a normalization change of ψ_T by $(3/4)^{-3/4}$. In these coordinates equation (9) becomes

$$C = \lim_{q^2+s_B} \left[\frac{1}{(12)^{1/4}\sqrt{\frac{\alpha}{\pi}}} \frac{\langle pq\beta_0 | \tilde{\psi}_T \rangle (q^2-s_B)}{g_D(p)}\right] \frac{p^2-s_D}{p^2+q^2-s_T} . \qquad (11)$$

The quantity in square brackets, $\tilde{f}(p,q)$, can be written in the form of equation (7) and extrapolated to obtain C. A least squares fit to equation (7) was done using q^2 of up to 0.5 fm^{-2}. A wider range of p could be used than for f(P,Q) of equation (9). The values of C^2 are listed in Table II for both classes of potentials. The values given are an average for various values of p , since the extrapolation procedure is not stable as for the Kim and Tubis calculation. For the partly non-local potentials the value of C^2 is almost constant at 3.3. The behaviour of C^2 for the separable potentials is quite different. The values of C^2 increase as the attractive part of the potential is weakened, going from $C^2 = 3.43$ for potential YSS-50 to $C^2 = 4.5$ for the limiting case YSS-10. It is also noteworthy that for all the separable potentials C is

larger than it is for any of the partly non-local potentials, except the limiting case YBS-10.

We expect the asymptotic normalization to be a peripheral quantity of the three nucleon system and like the quartet scattering length, ^4a, rather insensitive to the detailed trinucleon dynamics. It may be sensitive to the tail of the interaction only. Ours is a model calculation so the results may not be directly comparable with the experimental value of C_T or with other extractions from realistic trinucleon wave functions. Separable potentials do not have a realistic form in the asymptotic region; the large amplitude of the wave functions and form factors for large momenta is probably responsible for the variation in C for the potentials YSS.

The only other calculation of C with phase equivalent potentials, generated by unitary transformations, finds very small variation in C (KT 74). This indicates that C is not sensitive to potential changes at very short distances. However, in the study of E_T and ^2a, McGurk and Fiedeldey (MF 75b) claim that the unitary transformations being of range $\leq$ 1 fm are too restrictive, and that variations in P_D and the N-N potential between 1 and 2 fm are allowed consistent with experimental deuteron properties. Thus it may be that C is sensitive to this region. Our calculations and those of Orlov and Belyaev (OB 73) indicate large variations in C for

potentials without an OPE tail. It may be that one could use the triton asymptotic normalization constant to distinguish between potentials differing at intermediate distances.

CHAPTER VI

DISCUSSION

This chapter is mainly concerned with the relations between the triton properties and the characteristics of the two nucleon interaction. The relation between the triton binding energy and the binding energy per particle of nuclear matter is also discussed. The results of the calculations described in previous chapters are discussed and compared with those of other model calculations and with more realistic calculations. Particular attention is paid to results obtained with other phase equivalent potentials. The most relevant works are those of Fiedeldey and McGurk (FM 72), Haftel (Haf 73) and Singh (SWB 72). Fiedeldey works almost exclusively with rank-two separable potentials which have an attractive form factor

$$g(k) = \gamma/(k^2 + \beta^2) \qquad (1)$$

and a repulsive term $h(k)$ being calculated so as to maintain phase equivalence with the shifts $\delta(k)$ of the same standard Tabakin potential as used in this calculation. Singh has done nuclear matter calculations using some of the same potentials as used in my triton calculations. The partly non-local potentials have a local attractive Bargmann part

plus a repulsive separable part chosen to maintain phase equivalence. The separable potentials have an attractive term phase equivalent to the Bargmann and of the form

$$g(k) = \sqrt{a+b}/\sqrt{b^2+k^2} \quad . \qquad (2)$$

The repulsive term h(k) is again chosen for phase equivalence. Haftel has calculated triton properties and nuclear matter binding energy and saturation density for phase equivalent potentials obtained by unitary transformations of a two term Yukawa potential acting in the relative ℓ=0 state.

A number of review articles contain information particularly relevant to this discussion. A well referenced summary of recent work on the three body problem is contained in the review by Kim and Tubis (KT 74). The two and three body problems are discussed by Levinger (Lev 74). An excellent review of off-shell behaviour by Srivastava and Sprung (SS 75a) was consulted frequently, not only in this chapter but throughout the thesis. For more information on nuclear matter the reader is referred to the review by Sprung (Spr 72). A short survey of our knowledge about the deuteron is contained in another review by Sprung (Spr 75).

Triton binding energies and I_0

At very low energies the nucleon nucleon K-matrix can be expressed as an off-shell effective range formula (SS 70)

$$[K_0(p,q;k^2)]^{-1} = \frac{1}{a_0} - \frac{1}{2} r_0 k^2 - \frac{I_0}{a_0^2} (p^2+q^2-2k^2) \qquad (3)$$

where a_0 and r_0 are the usual on-shell scattering length and effective range, respectively, and the off-shell parameter, I_0, is the zero-energy wound integral,

$$I_0 = \int_0^\infty [(r-a_0)-u_0(o,r)]rdr. \qquad (4)$$

u_0 is the wave function at zero-energy, normalized to $u_0(o,r) \xrightarrow[\infty]{} r-a_0$.

It was suggested by Sprung (Spr 70) that a relation should exist between E_T and I_0, based on the belief that the triton is principally a low-energy system. For separable potentials in the model triton Fiedeldey (FM 72) found an almost linear relation between E_T and I_0, provided the attractive part g(k) remains strong enough. There exists also a strong correlation between nucleon-deuteron reaction quantities and I_0 (Bru+ 74). Fuda (Fud 75) has shown that even in the coupled $^3S_1$-$^3D_1$ state the N-N K-matrix can be characterized by a single parameter, I_{ot}. For realistic phase-equivalent N-N interactions McGurk and Fiedeldy (MF 75a) have found that fixing the other off-shell parameters and allowing I_{os} or I_{ot} to vary gives surprisingly linear relations between E_T (or ^2a) and I_{os} or I_{ot}; in fact, for fixed triplet interaction E_T and ^2a are almost completely determined by I_{os}. Charge symmetry restricts the allowed variation of I_{os} (SS 75b, Sau 74), and the deuteron D-state probability restricts that of I_{ot} (MF 75a).

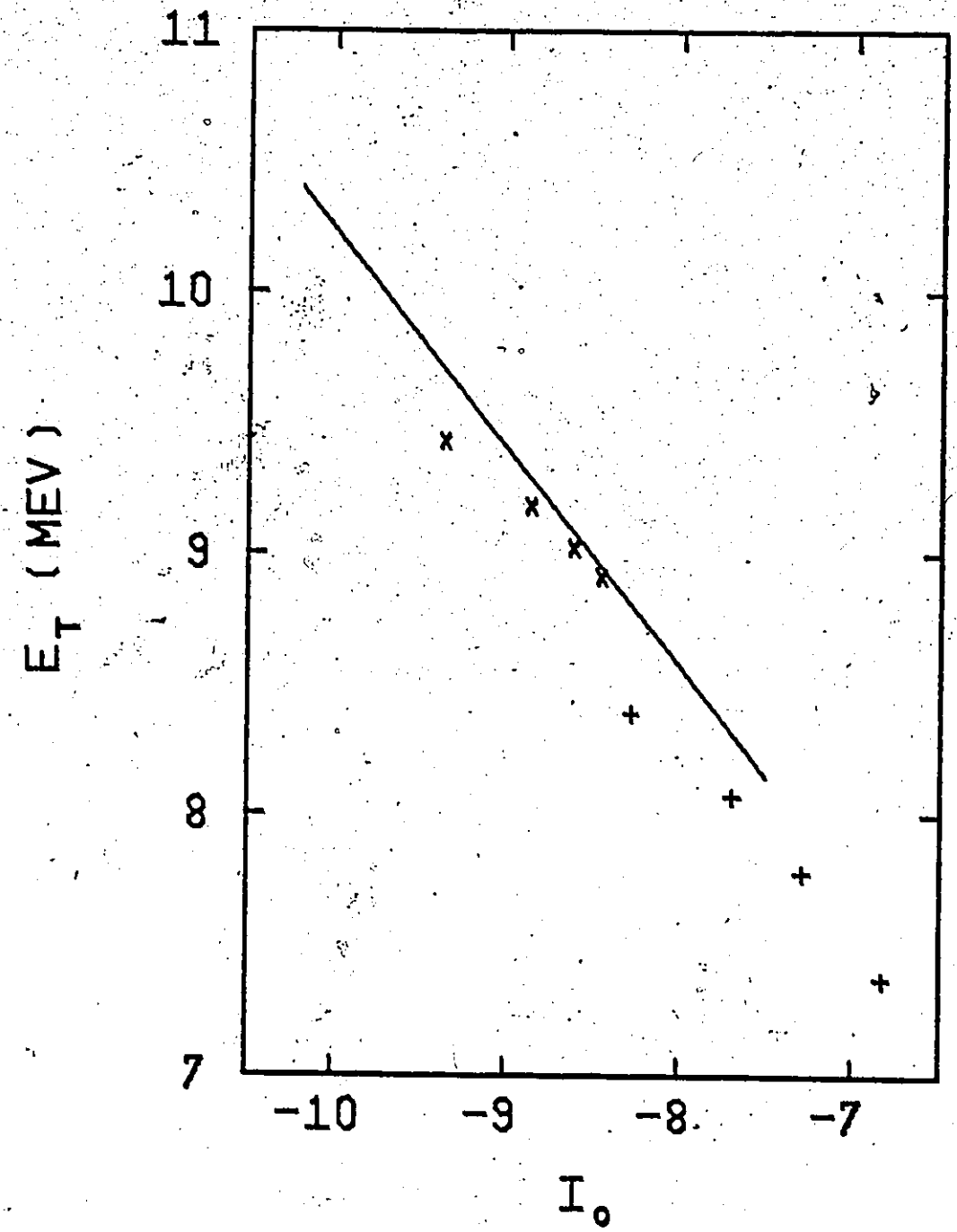


Figure 12. Relation of the triton binding energy to the zero-energy wound integral. + refers to potentials YBS and x to potentials YSS. The line is from Fiedeldey and McGurk's separable potential results (FM 72).

For the model triton calculations using the partly non-local and separable potentials of this thesis, the same trend holds as for Fiedeldey's model calculations; that is E_T increases with increasing $|I_o|$. However, the relation is not as linear as for Fiedeldey's rank-two separable potentials and we get two curves; see Fig. 12. Fiedeldey's line is included for comparison although his points, too, show a slight curvature though not as pronounced. The values for the limiting cases are not included in Fig. 12 although they are listed in Table I. The linear relation in these cases no longer holds at all and the points fall much below the line. I_o and E_T do, however, change in the same direction. The same behaviour was true of Fiedeldey's limiting potentials.

The separable potentials YSS, for given I_o, yield slightly more binding energy than do the partly non-local potentials. The feature of separable potentials giving more binding energy than local potentials is quite common in triton calculations.

Comparison of the binding energies of the triton and nuclear matter

Soon after developing his procedure for generating phase-equivalent rank-two separable potentials (Fie 69a), Fiedeldey calculated triton binding energies using these potentials (Fie 69b, FM 72). He found that some of the limiting potentials gave very large binding energies in the triton, up to 16.2 MeV. When these potentials were applied

to nuclear matter (SSB 70), it was found that the range of variation of the binding energy per particle, E_{NM}, was even greater. This was not surprising since nuclear matter is a more dense system than the triton. What was surprising, however, was that the variation in binding energy was opposite in direction to that for the triton; that is, potentials which increased the binding energy of the triton decreased the binding energy of nuclear matter even more drastically.

A number of calculations using phase equivalent potentials generated by short-ranged unitary transformations of local potentials have shown that usually, as one would expect, the variation of E_T and E_{NM} is in the same direction. For the Reid soft core potential (Rei 68) phase-equivalent modifications which kept the deuteron electric form factor roughly the same, gave variations ΔE_T and ΔE_{NM} which were of the same sign for all cases where both E_T and E_{NM} were calculated (HT 71, HKT 72b). The triton was much less sensitive to variations than was nuclear matter. Phase equivalent modifications of a two term Yukawa potential in the S state were studied by Haftel (Haf 73) in both the triton and nuclear matter. For potentials whose deuteron wave functions were almost the same, the variations of E_T and E_{NM} were in the same direction, again with much smaller variations in E_T than in E_{NM} (1 MeV compared to 26 MeV). The variation in triton binding energy was comparable to that of nuclear matter

at "density" k_F = 0.6 to 1.0 fm^{-1}, much below the empirical value of k_F = 1.36 fm^{-1}. Much larger variations of E_T occurred when the deuteron wave function was allowed to vary more. Then ΔE_T and ΔE_{NM} were usually, but not always, of the same sign.

When Singh (Sin 72, SWB 72) developed his method of constructing classes of partly non-local potentials and equivalent separable ones, he calculated nuclear matter properties and found (see Fig. 13):

(i) the partly non-local potentials gave more binding in nuclear matter than the equivalent purely separable ones,

(ii) no drastic drop in E_{NM} is found as the limiting condition $E_B \approx E_D$ is approached for the partly non-local potentials,

iii) the purely separable potentials behave much like those of Fiedeldey as the limiting condition is reached.

One of the incentives for the investigations of this thesis was to make the comparison complete by calculating triton binding energies for Singh's potentials. The same features that characterize Fiedeldey's potentials are apparent for both the separable and partly non-local potentials, namely that as the limiting condition is approached, the triton binding energy increases sharply although the variation is not as extreme as for Fiedeldey's potentials.

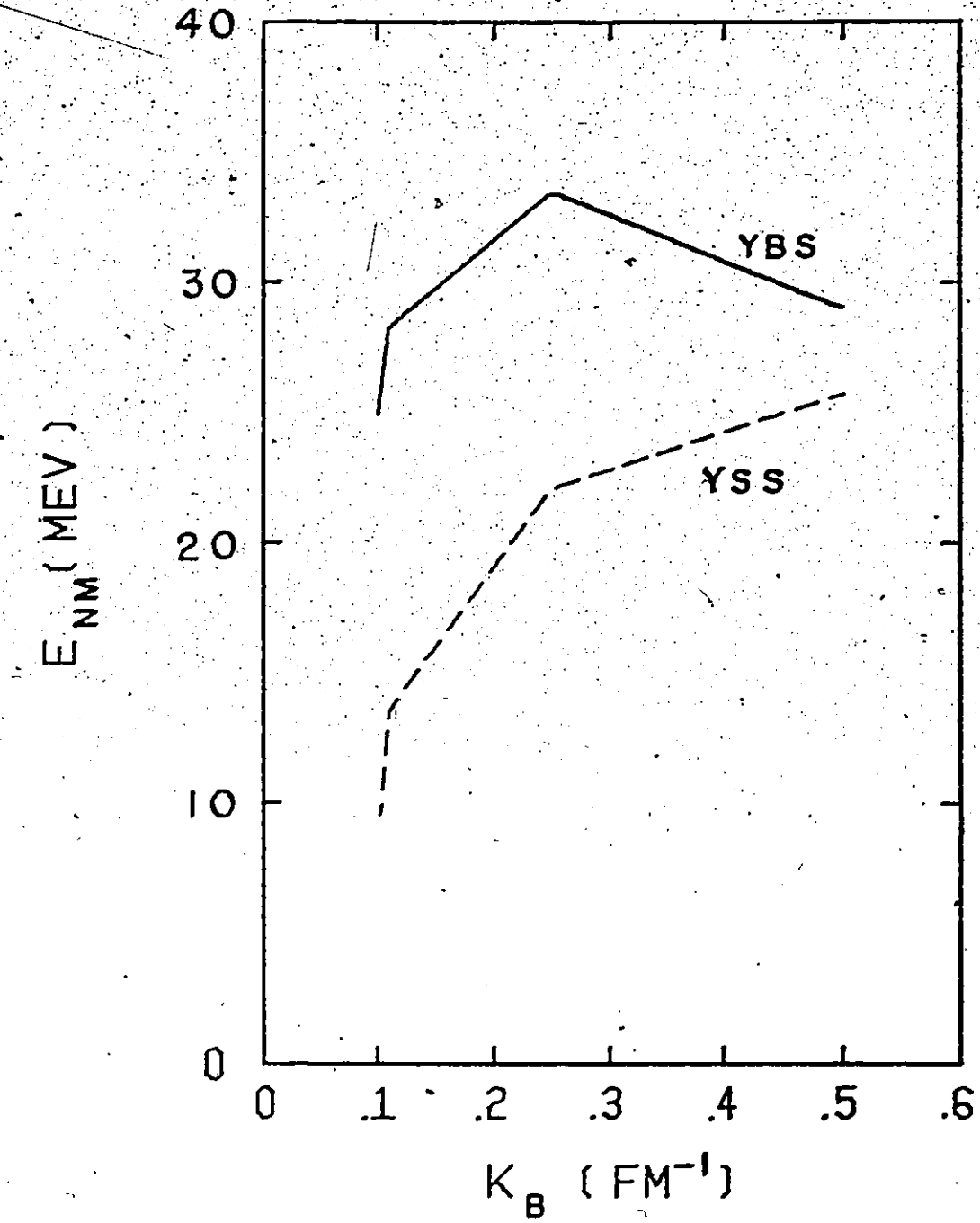


Figure 13. Nuclear matter binding energy per particle for the two classes of potentials, as a function of the strength of the attractive part of the interaction.

Whereas the deuteron wave function and zero-energy wound integral are important features in governing the triton binding energy, the deuteron wave function and wound integral of Bruckner theory are features important for the binding energy per particle in nuclear matter. The wound integral κ is a measure of short range correlations in nuclear matter and is generally linked to the saturation property of the system. It is an integral over the square of the defect wave function and is similar to I_o except that it cannot be negative. For potentials with approximately the same electric form factor, E_{NM} at fixed density is found to be nearly linear in κ, with increasing κ giving decreasing binding energy. For Singh's partly non-local potentials κ decreases as criticality is approached. For the separable potentials κ increases as the limiting condition is approached and E_{NM} decreases sharply. The sharp decrease in E_{NM} for critical potentials is also true for Fiedeldey's potentials (SSB 70). It is peculiar to separable potentials and is associated with the behaviour of their two-body bound state wave function. In the limiting case the repulsive part of the potential $h(k)$ must be orthogonal to the bound state wave function $\psi_B(k)$. The weak repulsion must be suppressed at the deuteron pole ($S_D \gtrsim -0.01$ fm^{-2} for our potentials) but it can be more important further off-shell ($S \gtrsim -3.0$ fm^{-2} in nuclear matter). In view of the large values of I_o produced by the limiting potentials and the proposed constraints (SS 75b) they should

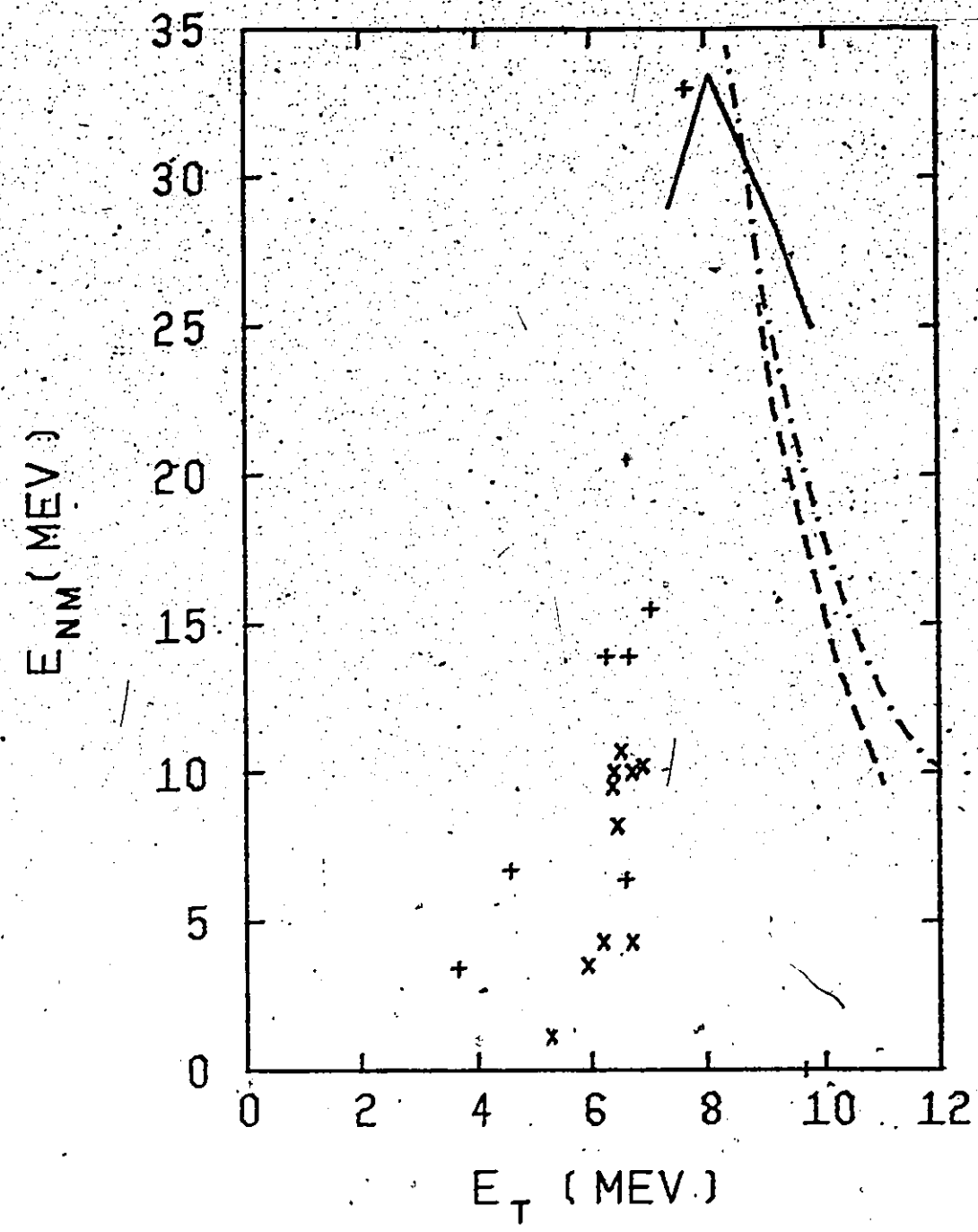


Figure 14. Relation of nuclear matter and triton binding energies for several classes of phase equivalent potentials. The solid line is for potentials YBS, the dashed line for potentials YSS, and the dash-dot line for separable potentials (SSB 70). The other points marked are for unitary transformations of local potentials: + (Haf 73) and x (HKT 72b).

be considered unrealistic anyway. Fig. 14 depicts some values of E_T and E_{NM} obtained from several classes of phase equivalent potentials.

A similar type of variation in binding energy with respect to κ holds in finite nuclei as in nuclear matter, using phase equivalent potentials generated from short-ranged unitary transformations (SS 75a). Thus, for this type of transformation it appears that a variation which produces more binding in the triton will usually produce more binding in finite nuclei and in nuclear matter. We have seen from the calculations of E_T and E_{NM} that this is not the case for rank-two separable potentials or the partly non-local potentials. Finite nucleus calculations have not been done for these potentials.

The deuteron wavefunction and the triton

A number of studies have shown the importance of the deuteron wave function in determining the binding energy of the triton. It can be seen from a comparison of E_T with Fig. 15 and Fig. 16 that a strong correlation exists between the deuteron wave functions and the triton binding energy.

It is easy to see why the deuteron wave function should be so important in the triton, by looking at Fig. 17, where the regions of space where $t(p,k;s)$ is important are indicated. The k dimension is not pictured, being symmetric with p and omitted for clarity. The phase shifts determine

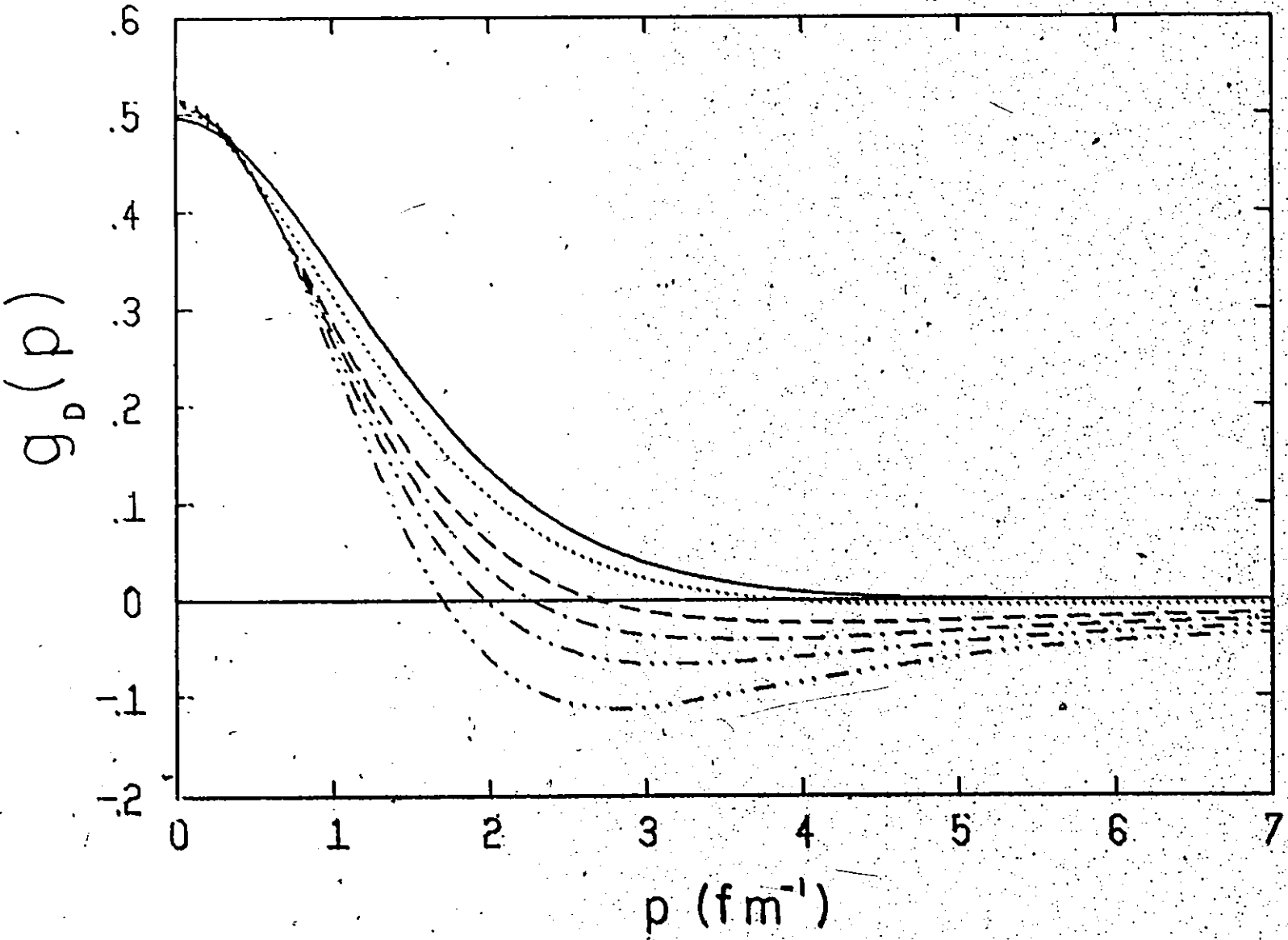


Figure 15. Deuteron momentum space wavefunctions for the partly non-local potentials; $g_D(p) = (p^2+k_D^2)\psi_D(p)$. Curves are labelled as in Figure 6.

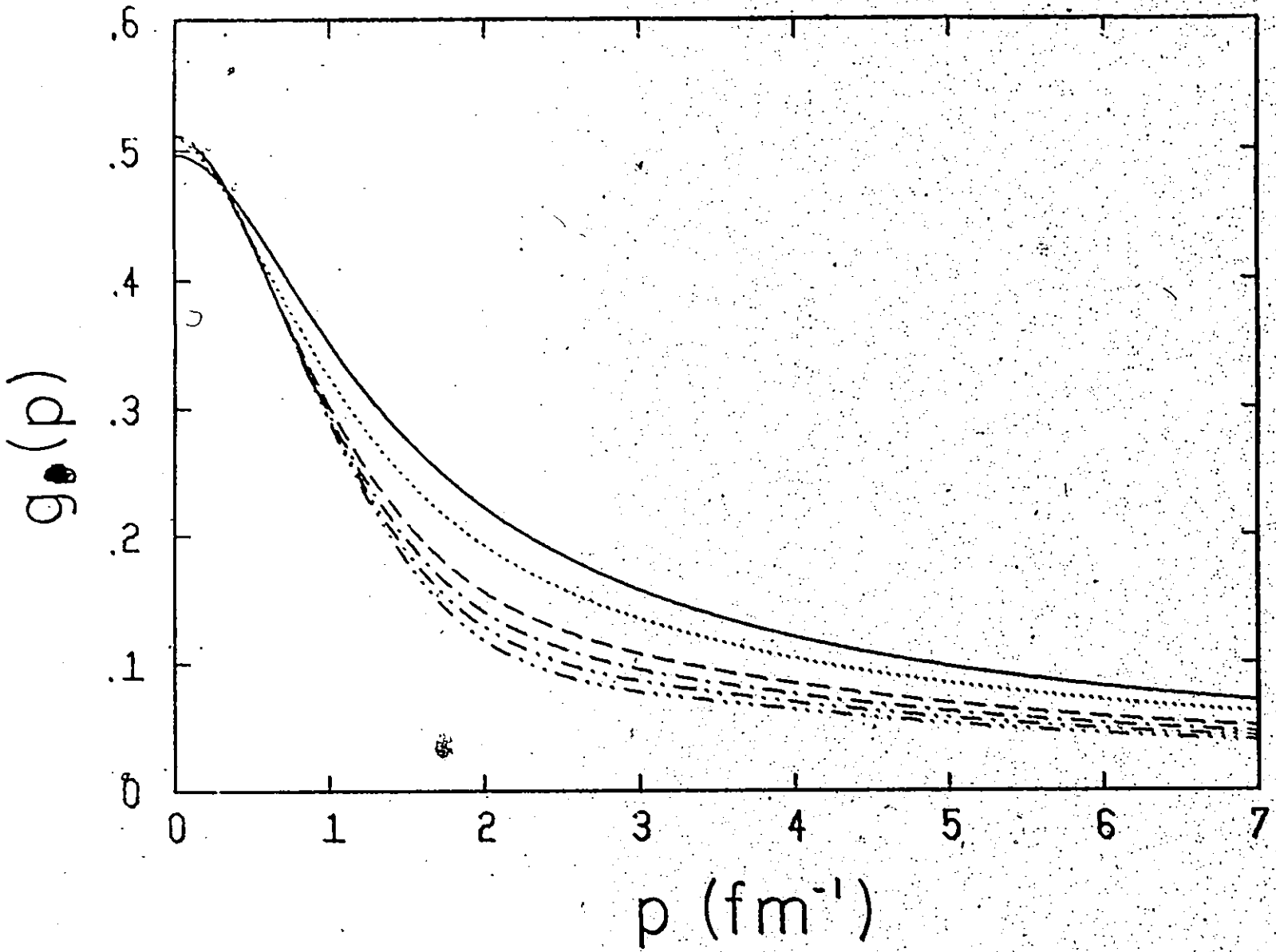


Figure 16. Deuteron momentum space wavefunctions for the separable potentials. Curves are labelled as in Figure 6.

the T-matrix on the (on-shell) line $p^2 = k^2 = s$. At $s = s_D$ the two-body T-matrix has a pole due to the deuteron bound state; at this point it is separable and is completely determined by the deuteron wavefunction

$$\psi_D(p) = g_D(p)/(p^2 - s_D) \quad . \tag{5}$$

Near $s = s_D$ it can be expressed as

$$t(p,k;s) = - \frac{\lambda_D g_D(p) g_D(k)}{1 - \lambda_D \frac{2}{\pi} \int_0^\infty \frac{g_D^2(p) q^2 dq}{q^2 - s}} \tag{6}$$

where λ_D is chosen so as to make the denominator zero at $s = s_D$. In the unitary pole approximation (UPA),

$$V_{UPA}(p,k) = - \lambda_D g_D(p) g_D(k) \tag{7}$$

and $T_{UPA}(p,k;s)$ is given everywhere by equation 6.

From the Lippmann-Schwinger equation, Eq. II-14, it can be seen that

$$T(p,k;s) \xrightarrow[-\infty]{s} V(p,k) \tag{8}$$

for any interaction.

From equation III-18 we can see that the triton depends explicitly on the two-body T-matrix only for values of $s \leq s_T - q^2$. Moreover, we expect the region $s \leq s_T$ to be the most important. In this region the T-matrix is still

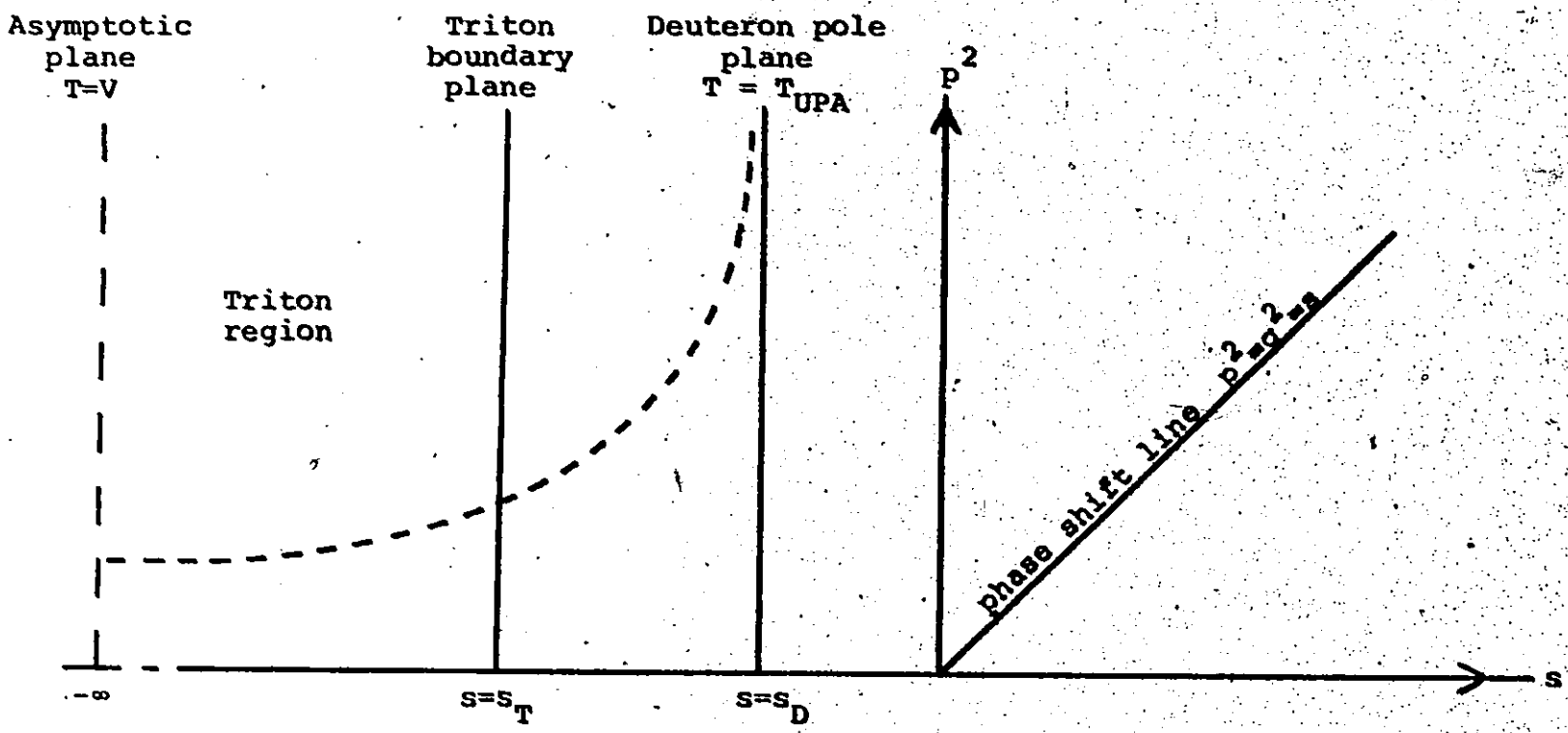


Figure 17. The two-body T-matrix and the triton. p and s space for the T-matrix $t(p,q;s)$; the q-dimension has been omitted for clarity, thus the vertical lines at s_D and s_T and $-\infty$ really represent planes. The dashed line symbolizes the s dependence of the T-matrix, in particular the pole at $s=s_D$.

strongly influenced by the deuteron pole. The dashed line in Fig. 17 symbolizes this effect.

It is thus easy to see how the deuteron wave function and binding energy can influence the triton binding energy and wave function. For example a shift in the position of the deuteron pole to a more negative value will push the dashed line a corresponding distance in the same direction and tend to make the interaction stronger for all $s<s_D$ and all p and k. This is in qualitative accord with the results of Fiedeldey and McGurk (MF 75b), in which the triton binding energy changes about twice as much as the deuteron binding energy for interactions which give slightly more deuteron binding energy but which are otherwise similar. Similarly, it accounts for the fact that some of the one-boson-exchange potentials give E_T nearer the experimental value but give too much deuteron binding energy.

The shape of the deuteron wave function will still be felt in the region $s \lesssim s_T$. Its presence will be particularly noticable in the first variable of the Faddeev amplitude $\psi(p,q)$ since $t(p,\bar{p};s-q^2)$ appears in the kernel of equation II-18. In the UPA the p dependence of ψ is given exactly by $g_D(p)$. Referring to Figs. 15 and 16 again we see that wave functions that are more compressed in momentum space yield less binding in the triton, at least within each class of potentials. This is very similar to the results of Haftel (Haf 73) who finds that the changes in E_T for his uni-

tarily transformed potentials are related to changes in $\psi_D(k)$ in the region 1 $fm^{-1} \lesssim k \lesssim 2$ fm^{-1}, although this does not necessarily mean that the peak of sensitivity of E_T to ψ_D is in this region. Modifications of the T-matrix which maintain off-shell unitarity indicate that there is a strong sensitivity when both momenta p,k are less than 2 fm^{-1}; weaker, potential-dependent sensitivity when either argument is between 2 and 5 fm^{-1}; and essentially no dependence when either argument exceeds 5 fm^{-1} (LMS 73).

The variation of E_T from that of the standard potential is almost proportional to the distortion of the deuteron wave function for both classes of our potentials. The distortion N_D is defined by

$$N_D = \langle \psi_D - \psi_D^o | \psi_D - \psi_D^o \rangle^{1/2} \qquad (9)$$

where ψ_D^o is the deuteron wavefunction of the standard potential V_{ST} and ψ_D that of the phase equivalent potentials. Potentials which give E_T greater than that of the standard potential give slightly greater values of N_D than those which yield E_T a corresponding amount less than that of V_{ST}. Values of N_D are included in Table I. The magnitude of N_D is not extremely large for any of the potentials, ranging from 0.015 to 0.088. This is less than the cut off value of 0.12 which Fiedeldey found necessary to impose on his to maintain the linear relation between I_o and E_T. In our case the linear relation is not dependent on N_D.

For more realistic interactions including the deuteron

D-state, the wave function is also very important, mainly due to the D-state probability P_D, which is strongly correlated with I_{ot}. E_T decreases monotonically as a function of P_D and tensor force strength (KT 74); linearly if other off-shell constraints are held constant (MF 75a).

A striking correlation exists between the deuteron form factor and the triton binding energy and charge form factor as can be seen by comparing Fig. 6 with Fig. 9 and Fig. 7 with Fig. 10. In particular note the relative position of the minima for the partly non-local potentials. The expression for the deuteron form factor (ℓ=0 component only in our case) is

$$F_D(q^2) = \int \psi_D^*(\underline{k})\psi_D(\underline{k} - \tfrac{1}{2}\underline{q})d\underline{k} \qquad (10\text{-a})$$

$$= \frac{1}{2}\int_0^\infty k^2 dk \psi_D(k) \int_{-1}^{1} \psi_D[(k^2 - kqx + q^2/4)^{1/2}]dx. \qquad (10\text{-b})$$

If ψ_D is always positive, so is F_D, and thus $\log_{10}|F_D|$ will not have a sharp minimum, probably no minimum. None occurs for the separable potentials YSS. Even if ψ_D does become negative, there is no guarantee that $F_D(q^2)$ will, although this is the case for potentials YBS-17 to YBS-50. The expression for the triton form factor is more complicated with various couplings of the p's and q's occurring before the form factor is obtained. If the wave function $\psi_T(p,q)$ were

were separable, which it clearly is not, even for separable potentials (see equation IV-9), then the expression for the form factor, equation IV-20, would bear a strong resemblance to equation 10 of this chapter for the term $\ell=L=0$. The correlation between the minima of the form factors must be more than just a coincidence since a similar situation exists for unitarily transformed potentials (Haf 73). Haftel suggests that changes in E_T are attributable to changes in $F_D(q^2)$ over a range of q^2 centred at $q^2 \approx 12$ fm^{-2}. In view of the strong dependence of F_D and E_T on ψ_D, and that of the triton form factor on E_T it may be simply that E_T and F_D depend on ψ_D in an amazingly correlated way. This has the unfortunate implication that three body bound state calculations do not convey very much information about the nuclear force that is not already available from two-nucleon calculations. This conclusion was shown not so long ago by Brayshaw (Bra 74) for low energy nucleon-deuteron reactions.

CHAPTER VII

CONCLUSION

Triton properties have been calculated for two classes of phase equivalent potentials. The partly non-local potentials are an attempt to simulate what we know about the nucleon-nucleon interaction, that is they combine long range locality with short range non-locality. It would be most realistic to choose a local potential with an OPEP tail, however for simplicity in the inversion procedure, a Bargmann potential has been used. The short range non-locality is simulated by a separable potential, that being the simplest form of non-locality. Rank two separable potentials are also used for comparison, the attractive part chosen to be phase equivalent to the Bargmann part of the partly non-local potentials. All of the potentials have been constructed to be phase equivalent to the standard Tabakin potential (Tab 65) which is an average of a singlet and a triplet potential, each separately fitted to the experimental $\ell=0$ phase shifts. This is a particularly relevant form for the model triton since the effective interaction in the triton principal S state is the average of the singlet and triplet interaction.

Not only are all these potentials phase equivalent

to each other, they are phase equivalent to Fiedeldey's rank two separable potentials which have a Yamaguchi type of attractive form factor. He has calculated the triton binding energy for a large number of potentials of this form. Thus a comparison can be made with a third set of potentials, all being phase equivalent.

When comparing potentials and looking for off-shell behaviour effects it is essential that the potentials be exactly phase shift equivalent. De Groot (DeG 75) has found that even slight on-shell differences of at most four degrees in the phase shifts can cause large off-shell changes in the T-matrix. As far as the triton is concerned, it is not so much that the phase shifts themselves above, say, 50 MeV laboratory energies are important, but the effect which these phase shifts have on the deuteron wave function (Pie 74). For local potentials, ψ_D and the phase shifts are interrelated; that is off-shell effects do not exist but rather high energy on-shell effects. For rank-two separable potentials there is additional freedom which allows a clean separation to be made between the deuteron wave function and the high energy phase shifts.

The triton properties studied in addition to the binding energy are the form factor and asymptotic normalization constant. Form factor calculations with separable potentials are not common, and this to my knowledge is the only one for phase equivalent separable potentials. The "new"

triton observable, the asymptotic normalization constant, is calculated to see whether it is perhaps a sensitive triton property. One would expect it will not be since it is, like the quartet scattering length, a peripheral quantity.

A comparison between the properties of the deuteron, triton, and nuclear matter has been made. This is possible since nuclear matter properties had earlier been obtained for both classes of our potentials as well as those of Fiedeldey.

A summary is presented here of some of the results obtained and conclusions drawn from this study. The main features of the triton are given by the general features of the two nucleon interaction.

(1) E_T is almost linear with respect to I_o for the non-limiting potentials.

(2) Changes in E_T from that of the standard potential are almost linearly related to changes in the deuteron wave function (N_D).

(3) The relation between the deuteron and triton form factors is striking; for some of the partly non-local potentials the minima positions look correlated.

(4) There is a rapid increase in E_T as the attractive part of the potential weakens toward the limiting condition. This seems related to the off-shell parameter I_o, which increases rapidly to unrealistic values.

(5) The rapid decrease in E_{NM} for the limiting separable potentials is a peculiar feature of the separable potentials and related to the fact that the repulsive part of the potential has to become orthogonal to the bound state wave function.

(6) The range of variation of E_T is not as drastic as for Fiedeldey's potentials, although variations in our range parameter b were not investigated.

(7) The partly non-local potentials give less binding energy than do their separable counterparts. Their deuteron wave functions more closely resemble the ψ_D's of local potentials in that they become negative (in momentum space).

(8) The slope of the form factor near $Q^2 = 0$ is a function of E_T. This slope has a lot of effect on the position of the minimum.

(9) The neutron form factor has quite an appreciable effect on the height of the secondary maximum.

(10) The asymptotic normalization constant does not seem to depend on E_T or ψ_D since it is almost constant for all the partly non-local potentials. There is a strong potential dependence in the separable cases; this is possibly related to the incorrect asymptotic behaviour of separable potentials.

In view of the importance of the deuteron wave function several comments can be made. To really see the effect

of separable potentials vs. local potentials it would be interesting to choose the separable potentials not so that g(k) is phase equivalent to the local part of the partly non-local potential, but so that $g(k) \mp g_D(k) = (p^2+k_D^2)\psi_D(k)$ of the corresponding partly non-local potential†. This would be possible provided that the phase shifts $\delta_1(k)$ produced by $g_D(k)$ acting alone do not become less than the input phase shifts $\delta(k)$ of the original potential. Another point is that in realistic calculations the singlet deuteron may play a role similar to that of the deuteron, although being farther away from the region of momentum space important for the triton, it may not be as important relatively. Knowing more about this singlet deuteron would help in further tying down the allowed range in variation of E_T consistent with observed two nucleon data. At present we have virtually no experimental information on the form of this wave function and have considerable freedom in choosing our interaction. Any potential which is used in triton calculations should incorporate as much information as possible about the two-body bound system. This still leaves considerable variation in E_T which must be further restricted before the role of three body forces and meson exchange currents can be fully assessed.

This is not a happy period for potentials. The great optimism of the late sixties that the three nucleon system would yield a great deal of information about the (off-shell)

† In view of the success of the UPA one would expect very little difference in the 3N results with such an interaction.

nucleon-nucleon interactions appears to have been crushed†. Realistic two nucleon potentials that fit the experimental data quite well underbind the triton by 1 to 2 MeV and fail to fit the ^{3}He form factor near and beyond the diffraction minimum. Modifications of potentials, as done in this thesis and in other studies, indicate that any attempts to increase E_T are invariably coupled with a corresponding poorer agreement of the form factor with experiment. Moreover, for potentials obeying the known off-shell constraints, E_T and ^2a lie on the Phillips band independent of the potential model employed (MF 75b). This is a consequence of the fact that E_T and ^2a are only sensitive to the broad detail of the N-N S-matrix in the $^1S_0$ and $^3S_1$-$^3D_1$ states and to certain global features of the residues at the singlet and deuteron poles. Another problem is that of the energy difference (0.764 MeV) in the binding energies of ^{2}H and ^{3}He. The usual method of doing calculations is to invoke charge symmetry ignoring the electromagnetic force and treating ^{3}H and ^{3}He as the same system. Once the binding energy and wave function have been obtained the Coulomb energy can be added as a perturbation. Unfortunately, however, model independent calculations (Fab 72) confirm that there is an uncomfortably large discrepancy of 0.1 MeV between the calculated difference and the experimental difference. This long standing discrepancy is a further indication that all is not well. Thus it appears that nuclear physics cannot be described in terms

† See for example the proceedings of the few nucleon conferences from 1967 to 1976.

of nucleons interacting through a pairwise force, and that the mesonic degrees of freedom will have to be treated explicitly.

Indications are that meson exchange currents and/or three body forces are necessary in order to simultaneously fit E_T and Q^2_{min} (Haf 76, Bra 73). Moreover, the analysis of exchange current contributions to the magnetic moments and electromagnetic form factors is presently hampered by the confused state of theoretical calculations of exchange current operators. This confusion is in part related to a transition in general nuclear phenomenology from formalisms which only contain explicit reference to nucleon coordinates (with the effects of mesonic degrees of freedom expressed via exchange-current and wave function renormalization operators) and formalisms which explicitly contain some mesonic degrees of freedom.

The continuum three body states do not look too promising either. Brayshaw, in his paper "What can we learn from three-body reactions?" (Bra 74), concludes, from his study of neutron-deuteron elastic scattering and deuteron breakup, that we can learn nothing. Low energy reactions provide no new information being essentially determined by the two-nucleon observables and ^2a. If a three-nucleon force is introduced to explain the charge form factor data and/or correct E_T, the trinucleon properties will be essentially determined.

On the other hand, the absence of off-shell effects in scattering seems to be a valuable piece of information. It may make it possible to use three-body reactions as highly accurate probes of hard to measure two-body parameters. Present results for E_T and ^2a do not place worthwhile limits on corrections due to three-body forces, relativistic effects and meson degrees of freedom. Better knowledge of deuteron D-state probability P_D and deuteron polarization P_e in electron deuteron scattering would improve the situation greatly.

Once these two body properties are determined, the accuracy with which trinucleon calculations can now be performed will make it possible to estimate the importance of three-body forces and meson exchange currents. Thus we may be able to make further progress in understanding these, at present, poorly known effects.

APPENDIX A

THE MODEL TRITON WITH TWO TERM SEPARABLE POTENTIALS

The model triton problem reduces to an integral equation in only one continuous variable if we use separable potentials. For a two term separable potential there are two coupled equations. The formalism of Tabakin (Tab 65) based on the three-body Schroedinger equation is used to obtain the triton binding energy and spectator functions for the separable potentials.

In the momentum representation the three-body Schrodinger equation can be written,

$$\langle \underline{k}_1,\underline{k}_2,\underline{k}_3|H_o|\Psi\rangle + \int d\underline{k}_1' d\underline{k}_2' d\underline{k}_3' \langle \underline{k}_1,\underline{k}_2,\underline{k}_3|V|\underline{k}_1',\underline{k}_3',\underline{k}_3'\rangle \Psi(\underline{k}_1',\underline{k}_2',\underline{k}_3') = E\Psi(\underline{k}_1,\underline{k}_2,\underline{k}_3) \quad \text{(A1)}$$

In terms of the coordinates of equation III-3, for pairwise interactions the potential can be written

$$\langle \underline{k}_1,\underline{k}_2,\underline{k}_3|V|\underline{k}_1',\underline{k}_2',\underline{k}_3'\rangle = \delta(\underline{P}-\underline{P}')[\delta(\underline{Q}_1-\underline{Q}_1)\langle \underline{p}_1|V_1|\underline{p}_1'\rangle + \delta(\underline{Q}_2-\underline{Q}_2')\langle \underline{p}_2|V_2|\underline{p}_2'\rangle + \delta(\underline{Q}_3-\underline{Q}_3')\langle \underline{p}_3|V_3|\underline{p}_3'\rangle] \quad \text{(A2)}$$

where V_i is the two-body potential acting between particles j and k (i,j,k) cyclic.

For identical particles $V_1=V_2=V_3 \equiv V$. This may be expanded in the same way as the T-matrix was (see Eq. II-18)

$$\langle \underline{p}|V|\underline{k}\rangle = \frac{1}{2\pi^2}\frac{\hbar^2}{m}\sum_{\ell}(2\ell+1)V_{\ell}(\underline{p},\underline{k})P_{\ell}(\cos\hat{\underline{p}}\cdot\hat{\underline{k}}). \quad \text{(A3)}$$

Using only the $\ell=0$ part of the interaction and taking it to be a rank-two separable potential gives

$$\langle \underline{p}|V|\underline{k}\rangle = \frac{1}{2\pi^2}\frac{\hbar^2}{m}[-g(p)g(k)+h(p)h(k)]. \quad \text{(A4)}$$

Writing Ψ as

$$\Psi(\underline{k}_1\underline{k}_2\underline{k}_3) = \psi(\underline{p}_1,\underline{Q}_1) + \psi(\underline{p}_2,\underline{Q}_2) + \psi(\underline{p}_3,\underline{Q}_3) \quad \text{(A5)}$$

and substituting into (A1), it can be shown that

$$\psi(\underline{p},\underline{Q}) = \frac{g(p)\chi(Q)-h(p)\phi(Q)}{p^2+\frac{3}{4}Q^2-s_T} \quad \text{(A6)}$$

where χ and ϕ are called spectator functions and are functions of the magnitude of $\underline{Q}$ only. They are defined by

$$\chi(Q) = \int d\underline{p}\; g(\underline{p})\Psi(\underline{Q}, -\tfrac{1}{2}\underline{Q}+\underline{p}, -\tfrac{1}{2}\underline{Q}-\underline{p})$$
$$\phi(Q) = \int d\underline{p}\; h(\underline{p})\Psi(\underline{Q}, -\tfrac{1}{2}\underline{Q}+\underline{p}, -\tfrac{1}{2}\underline{Q}-\underline{p}). \quad \text{(A7)}$$

Using equations (A5) and (A6), Ψ can be eliminated from equation (A7) giving

$$[1-\hat{G}(Q)]\chi(Q) + \hat{M}(Q)\phi(Q) = \int_0^{\infty} dQ'Q'^2[G(Q,Q')\chi(Q')-M(Q,Q')\phi(Q')] \quad \text{(A8)}$$

$$\hat{M}(Q)\chi(Q) - [1-\hat{H}(Q)]\phi(Q) = \int_0^{\infty} dQ'Q'^2[-M(Q',Q)\chi(Q') + H(Q,Q')\phi(Q')],$$

where

$$\hat{G}(Q) = \frac{2}{\pi}\int^{\infty} \frac{dQ' Q'^2 g(Q') g(Q')}{Q'^2 + \frac{3}{4}Q^2 - s_T}$$

$$G(Q,Q') = \frac{2}{\pi}\int_{-1}^{1} \frac{dx\ g(u)\ g(v)}{Q^2+Q'^2-QQ'x-s_T} \qquad \text{(A9)}$$

and

$$u^2 = Q'^2 + \frac{1}{4}Q^2 - QQ'x$$
$$v^2 = \frac{1}{4}Q'^2 + Q^2 - QQ'x\ . \qquad \text{(A10)}$$

$\hat{H}$ and H are similar to $\hat{G}$ and G respectively with the g's being replaced by h's. $\hat{M}$ and M are similarly defined, the second g in each case being replaced by h. Since $\hat{G}$, $\hat{H}$, $\hat{M}$, G, H and M are all defined in terms of the given functions g and h, equations (A8) are two coupled integral equations in one variable and can be solved by standard methods.

Method of solution

We choose a quadrature formula with points and weights (p_i, w_i), $i = 1,\ldots,N$. Then we write equations (A8) at the points $Q = p_i$, $i = 1,\ldots N$. This allows eq. (A8) to be written in matrix form as

$$\begin{bmatrix} 1-\hat{\underline{\underline{G}}} & \hat{\underline{\underline{M}}} \\ \hat{\underline{\underline{M}}} & -(1-\hat{\underline{\underline{H}}}) \end{bmatrix} \begin{bmatrix} \underline{\chi} \\ \underline{\phi} \end{bmatrix} = \begin{bmatrix} \underline{\underline{G}} & -\underline{\underline{M}} \\ -\underline{\underline{M}} & \underline{\underline{H}} \end{bmatrix} \begin{bmatrix} \underline{\chi} \\ \underline{\phi} \end{bmatrix} \qquad \text{(A11)}$$

where the quantities $\hat{\underline{G}}$, $\hat{\underline{H}}$, etc. are now matrices or column vectors rather than functions. $\underline{G}$, $\underline{M}$, $\underline{H}$ are N by N matrices

where for example $(\underline{\underline{G}})_{ij} = w_j G(p_i,p_j)$. $\underline{\chi}$ and $\underline{\phi}$ are column vectors of length N, and $\hat{\underline{\underline{G}}}$, $\hat{\underline{\underline{M}}}$ and $\hat{\underline{\underline{H}}}$ are diagonal matrices.

Equation (A11) is a homogeneous set of 2N linear equations. It has a solution when

$$\det \begin{vmatrix} \underline{\underline{G}}-(1-\hat{\underline{\underline{G}}}) & -\underline{\underline{M}}-\hat{\underline{\underline{M}}} \\ -\underline{\underline{M}}-\hat{\underline{\underline{M}}} & \underline{\underline{H}}+(1-\hat{\underline{\underline{H}}}) \end{vmatrix} = \det|E| = 0 \quad . \qquad (A12)$$

All of the matrix elements are functions of s_T. Since we want to solve for s_T, this involves a search for the zero of the determinant. Standard determinant and root finding routines along with quadratures are all that is needed. By mapping the range $(0,\infty)$ into $(-1,1)$ using one of the transformations III-30, Gauss-Legendre quadrature can be used throughout for the integrations.

On obtaining the spectator functions

One might expect that once s_T has been obtained to sufficient accuracy, essentially the same linear equation package as used in evaluating the determinant could be used to obtain $\underline{\chi}$ and $\underline{\phi}$ (to within a multiplicative constant) by solving a 2N-1 equation system. In practice this does not work very well. For example when $\det|E| = \varepsilon$, then the smallest term of the eigenfunction is often positive or negative, depending on the sign of ε. It was found much better to write the set of equations (A11) as

$$\underline{\underline{E}}\underline{\chi} = \lambda\underline{\chi} \qquad (A13)$$

where λ, strictly speaking should be 0 but in practice is some small quantity ε', the magnitude of which depends on how near we are to the solution s_T.

The algorithms that exist for eigenvector routines seem much more stable than those for linear equations. There was no problem in obtaining the eigenvector and there were much smaller changes for successive iterations. In particular, the spectator function always seemed to have the correct signs.

In practice eigenvector routines are much slower than matrix inversion routines. Therefore the determinant method was used until s_T had converged to sufficient accuracy. Then the eigenvector subroutine was called to obtain the spectator functions.

Aside from normalization constants, the quantity $[g(P)\chi(Q)-h(P)\phi(Q)]$ is equivalent to the Faddeev amplitudes $\psi(p,q,\beta)$ obtained in Chapter III, remembering that the co-ordinates (P,Q) and (p,q) are slightly different. Thus the wave function can be obtained by the same techniques as used for the partly non-local potentials.

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Table I. Properties of the Potentials

Potential	$a(fm^{-1})$	$b(fm^{-1})$	E_B(MeV)	I_0	N_D	E_T(MeV)
YBS-10	0.1016	1.0	0.43	-42.01	0.047	9.82
YBS-11	0.11	1.0	0.50	-12.32	0.023	9.20
YBS-17	0.175	1.0	1.27	- 8.28	0.027	8.39
YBS-25	0.25	1.0	2.59	- 7.69	0.042	8.07
YBS-35	0.35	1.0	5.08	- 7.29	0.059	7.78
YBS-50	0.50	1.0	10.37	- 6.83	0.088	7.37
YSS-10	0.1016	1.0	0.43	-46.26	0.082	11.04
YSS-11	0.11	1.0	0.50	-13.38	0.057	10.24
YSS-17	0.175	1.0	1.27	- 9.36	0.034	9.43
YSS-25	0.25	1.0	2.59	- 8.86	0.026	9.10
YSS-35	0.35	1.0	5.08	- 8.61	0.020	9.02
YSS-50	0.50	1.0	10.37	- 8.45	0.015	8.91

Properties of the partly non-local potentials (YBS) and the equivalent separable potentials (YSS). All potentials are phase shift equivalent to the standard Tabakin potential (Tab 65) and have binding energy of 0.428 MeV. a and b are the potential parameters (see equations II-30 and II-35); E_B is the binding energy of the attractive part of the potential acting by itself; I_0 is the zero energy wound integral (see Chapter VI) and N_D is the deuteron overlap integral (see VI-5). E_T is the binding energy of the triton.

Table II. Triton and Nuclear Matter Properties of the Potentials

Potential	E_T (MeV)	$\langle a^2 \rangle^{1/2}$ (fm)	Q^2_{min} (fm^{-2})	C_T^2	E_{NM} (MeV)	k_F^s (fm^{-1})	κ
YBS-10	9.82	1.64	$\sim$36	3.67	24.98	2.1	0.0036
YBS-11	9.20	1.70	$\sim$33	3.26	28.32	2.08	0.0040
YBS-17	8.39	1.76	27.0	3.33			
YBS-25	8.07	1.78	22.5	3.28	33.47	2.05	0.0099
YBS-35	7.78	1.80	17.6	3.26			
YBS-50	7.37	1.81	13.1	3.23	29.0	1.85	0.0296
YSS-10	11.04	1.75	-	4.50	9.56	1.83	0.0165
YSS-11	10.24	1.79	-	4.10	13.60	1.89	0.0129
YSS-17	9.43	1.86	-	3.68			
YSS-25	9.18	1.90	-	3.52	22.13	2.00	0.0071
YSS-35	9.02	1.94	-	3.60			
YSS-50	8.91	1.99	-	3.43	25.73	2.05	0.0050

Triton and nuclear matter properties of the potentials. E_T is the triton binding energy; $\langle a^2 \rangle^{1/2}$ is an approximate r.m.s. radius; Q^2_{min} is the momentum transfer squared at which the form factor has a minimum; C_T^2 is the triton asymptotic normalization constant. The nuclear matter properties (SWB 72) are the binding energy per particle, the saturation ("density") and the wound integral.