SMOOTH FINITE CYCLIC GROUP ACTIONS ON POSITIVE DEFINITE FOUR-MANIFOLDS

by

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Abstract

Smooth actions of odd order cyclic groups on closed positive definite simply connected 4-manifolds are considered. For such an action, by studying its associated instanton one Yang-Mills equivariant moduli space, it is proved that the fixed point pattern of the singular set and the isotropy representations are the same as those of an equivariant connected sum of complex projective spaces acted linearly by the same group. Under certain assumptions, questions regarding the number of distinct possible isotropy representations at singular points arising in smooth actions and equivariant connected sums of algebraic actions on 4-dimensional complex projective spaces are answered.

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Notations

$\mathcal{A}$ *The space of connections on some bundle over a four-manifold .*

$\mathcal{G}$ *The gauge group of some bundle over a four-manifold.*

C_n *The cyclic group of order* n.

${}_RM$ *a left* $R-$*module.*

Σ_m *the* m*-symmetric group.*

$Gl(n, R)$ *the general linear group, or the group of automorphisms of* R^n.

(X, G) *the space* X *is a* G*-space.*

$\text{Fix}(G, X)$ *the fixed set of the group* G *in* X.

Introduction

Metric spaces together with their groups of isometries influenced the birth of modern geometry and topology. Today the study of group actions on manifolds occupies a central place in the development of these disciplines. Compact Lie group actions on manifolds have been extensively studied by mathematicians including P. A. Smith, G. D. Montgomery and Zippin, G. E. Bredon, R. S. Palais. From their work and that of others, multiple problems have remained yet to be investigated. In this thesis we analyze actions of odd order cyclic groups on positive definite 4-manifolds in the smooth category. Tools like the Lefschetz Fixed-Point Formula (Theorem 1.4) and the G-Signature Theorem (Corollary 0.50) used for this analysis come from the more general setups of topological and locally linear actions (Definition 0.13) on 4-manifolds but the smooth case allowed us also the employment of the equivariant moduli space developed by Hambleton-Lee in [17]. There are more advantages offered by considering smooth actions: in view of Theorem 1.1, it is enough to analyze the smooth 4-manifold model $\#\mathbb{C}P^2$, where we know that the induced action on integral homology admits a representation by permutations (Proposition 1.11). In the non-smooth case this latter statement is not known yet to be true. Locally linear cyclic group actions on positive definite 4-manifolds like E_8 have been studied by Edmonds in [12] and from there it was conjectured that such an action induces signed permutation representations in homology. One can distinguish two directions for the study of group actions. One is to describe non-isomorphic groups that can act in a given way under certain conditions, and the other is to classify all actions of a given group supported by a given space. For the first direction we point out the results of [17] and [35] where it is shown that the groups that can act locally linearly on $\mathbb{C}P^2$ are those that can act linearly. For the second direction, the results of [10] give some progress towards a classification: the singular sets of locally linear actions of cyclic groups on $\mathbb{C}P^2$ are compared with those of linear actions. Our work continues this development and we extend the results of [16] to the case of an action by permutations in homology.

In Chapter 1 we use the discussion presented in [9] to prove that the fixed point set of the action of an odd order cyclic group on a positive definite 4-manifold X consists of isolated fixed points and 2-spheres. Chapter 2 and Chapter 3 describe properties of the equivariant moduli space that are going to be used in the proof of Theorem 3.7, the main result. We give a detailed description of the fixed point data and isotropy representations of equivariant connected sums of linear actions of odd order cyclic groups on $\mathbb{C}P^2$'s. The cases of trivial and nontrivial induced permutation representations in homology are considered separately and we answer negatively a question posed in [11]: is it true that the number of distinct orbit types arising in an effective locally linear action of an odd order cyclic group on a positive definite 4-manifold cannot be larger then the maximum number of orbit types appearing on $\mathbb{C}P^2$ acted linearly by the same group? Actually Proposition 4.6 shows that the number of orbit types has no bound as long as we allow a connected sum of any large length. As an application of the equivariant moduli space we prove in Chapter 5 the smooth case of a conjecture also stated in [11] (see Chapter 5, Conjecture 1.).

Chapter 0

In this chapter G denotes a compact Lie group and X denotes a manifold. If not otherwise specified, X will be assumed a topological manifold. All the rings considered are commutative integral domains with unit.

Definition 0.1. *The action of the Lie group G on the smooth manifold X is called* **smooth** *if the map of the action, $\mu\colon G \times X \longrightarrow X$, is smooth.*

It is true, but non-trivial to prove, that in order to have a smooth action, it is enough to have $g\colon X \longrightarrow X$ smooth, for any $g \in G$ (see [25]).

Definition 0.2. *Let R be a ring. A group homomorphism $\rho\colon G \longrightarrow Gl(n, R)$ is called an n-dimensional* **R-representation** *of G.*

Thus any representation ρ of a group G on the free R-module R^n is equivalent to an $R[G]$-module structure on R^n by $(\Sigma_g r_g g) \cdot m = \Sigma_g r_g \rho(g) m$.

Definition 0.3. *The representations $r, \rho\colon G \longrightarrow Gl(n, R)$ are* **equivalent** *if there exists $A \in Gl(n, R)$ such that $r(g) = A\rho(g)A^{-1}$, for any $g \in G$.*

From now on, two group representations will be different up to equivalence.

Definition 0.4. *If H is a subgroup of G acting on X, then*

$$X^H = \{x \mid gx = x, \forall g \in H\}$$

denotes the **fixed point set of** *H.*

If $G(x) = \{gx \mid g \in G\}$ denotes the **orbit through** $x \in X$, and $G_x = \{g \mid gx = x\}$ denotes the **isotropy group of** x, then it is easy to see that $G_x \sim G_y$, i.e. $G_y = gG_xg^{-1}$, for any $y \in G(x)$, for some $g \in G$. This fact asserts that the isotropy groups at any two points of one orbit are conjugate. The set of cosets G/G_x is into one-to-one correspondence with the orbit point set $G(x)$.

Definition 0.5. *We denote by* $X_{(H)} = \{x \mid G_x \sim H\}$ *the subset of* X *consisting of all points of isotropy* H.

Definition 0.6. *We say that an orbit* $G(x)$ *has* **orbit type** G/H *if* $H \sim G_x$.

Thus $X_{(H)}$ is the union of all orbits of type G/H. The points of X whose isotropy groups are exactly H are denoted $X_H = X_{(H)} \bigcap X^H$ and let's notice that $X_{(G)} = X_G = X^G$. Also, in the case when G is abelian, $X_{(H)} = X_H \subset X^H$.

Definition 0.7. *The action of* G *on* X *is* **effective** *if there is no element in* G*, different from the identity element* e*, that fixes every point of* X.

Definition 0.8. *An effective action of* G *on* X *is* **semifree** *if the only isotropy types are* G *and* $\{e\}$.

Definition 0.9. *An effective action of* G *on* X *is* **pseudofree** *if for any isotropy type* H, X^H *is a discrete set.*

Definition 0.10. *If the group* G *acts on* X*, a* **slice** *at* $x \in X$ *is a* G_x*-invariant subset* S *in* X *such that*

1. $x \in S$

2. *S is closed in $G(S)$*

3. *$G(S)$ is a neighbourhood of $G(x)$*

4. $G_x(S) = S$

5. $gS \bigcap S \neq \emptyset \Longrightarrow g \in G_x$

Theorem 0.11. *The connected subset S of X is a slice at $x \in X$ if and only if the map*

$$G \times_{G_x} S \longrightarrow X,$$

defined by $[g,s] \mapsto g(s)$, is a G-equivariant embedding onto an open neighbourhood of the orbit $G(x)$.

Proof. See [7], pp. 82-83. ■

More explicitly, if S is a slice and if there exists $f\colon U \longrightarrow G$, a local cross-section of $\pi\colon G \longrightarrow G/G_x$, then $F(u,s)\colon U \times S \longrightarrow X$, $F(u,s) = f(u)s$ is a diffeomorphism onto an open subset of X which extends the natural bijection $G/G_x \xrightarrow{F_x} G \cdot x$.

Definition 0.12. *A slice S in $x \in X$ is called* **linear** *if the representation of the action of G_x on S is equivalent to an orthogonal representation of G_x on an Euclidean space.*

Definition 0.13. *The action of G on X is* **locally smooth**, *or* **locally linear**, *if in any point x of X, there exists a linear slice, S.*

Proposition 0.14. *A smooth action is a locally smooth action.*

Proof. See [7], VI. Cor 2.4. ∎

Theorem 0.15. *If G acts on X and H is a subgroup of G, then $M_{(H)}$ is a manifold, locally closed in X. Moreover, if X is a topological (smooth) manifold, then $M_{(H)}$ is topological (smooth) manifold, and the closure $\overline{M_{(H)}}$ contains only orbits of types less or equal to H. The manifold $M_{(H)}$ is the total space of a G/H-bundle over $M_{(H)}/G$ with structure group $N(H)/H$, where $N(H)$ is the normalizer of H in G.*

Proof. Like in [7], IV. Thm 3.3. ∎

Corollary 0.16. *If in* Theorem 0.15 *G is commutative, then G/H acts freely on M_H and M_H is a G/H-principal bundle over M_H/G.*

Remark 0.17. *Let (X, G) be a G-space and let's assume that for any $x \in X$ there is a slice S through x. Then $(G \times_{G_x} S)_{(G_x)} \simeq G/G_x \times S^{G_x}$.*

Proposition 0.18. [7], IV. Lemma 5.1 *If G acts (locally) smoothly on X, and H is a subgroup of G, then $N(H)$ acts (locally) smoothly on M^H.*

If the action of G on X is smooth, then G inherits a representation on TX_x by

$$gv = dg_x(v) \in TX_{g(x)}, \forall v \in TX_{g(x)},$$

where dg_x is the derivative at x of the action $g\colon X \longrightarrow X$.

Remark 0.19. *If G is finite cyclic and $G = \langle g \rangle$, then $X^G = X^g$.*

Indeed, $X^g = \{x \mid gx = x\}$ and, because $x \in X^g \Longrightarrow g^2x = gx = x \Longrightarrow x \in X^{g^2}$, we have $X^g \subset X^{g^2} \subset \cdots \subset X^{g^{m-1}} \subset X$, where m is the order of G. But, if $x \in X^{g^{m-1}}$, then $g^{m-1}x = x \Longrightarrow g^m x = gx \Longrightarrow x = gx \Longrightarrow x \in X^g$. Thus, $X^g = X^{g^2} = \cdots = X^{g^{m-1}}$ and $X^G = X^g$.

Proposition 0.20. *If G is a finite group acting smoothly on the manifold X, then $(TX)^g = TX^g, \forall g \in G$.*

Proof. Let v be a vector in T_xX^g. Then there exists a path $\gamma \subset X^g$ such that $\gamma(0) = x$ and $\frac{d}{dt}|_{t=0}(\gamma(t)) = v$. Of course, $x \in TX_x$ and $\gamma \subset X$.
$gv = dg_x(v) = \frac{d}{dt}|_{t=0}(g\gamma(t)) = \frac{d}{dt}|_{t=0}(\gamma(t)) = v$.
Therefore, $v \in (T_xX)^g$, and, indeed $T_xX^g \subset (T_xX)^g$.
For the other way around, because G is compact, we obtain a G-invariant metric, $\langle , \rangle$, on X by averaging the action on an arbitrary metric. Let's pick $v \in (T_xX)^g$ Then $gv = v$. Let also $\gamma(t) = exp_x v(t)$ be the geodesic in the direction of v. The tangent vector in 0 to the curve $g\gamma$ is $\frac{d}{dt}|_{t=0} g\gamma(t) = dg_x(v) = gv = v$. Since $x \in X^g$, we have also $g\gamma(0) = gx = x = \gamma(0)$. Then, locally around 0, γ and $g\gamma$ coincide and, because the metric is invariant, γ is a local geodesic in X^g. Thus $v \in T_xX^g$. ∎

0.1 Elementary representation theory

We present now basic results concerning the finite dimensional K- representations of finite groups, with $K = \mathbb{R}$ or $K = \mathbb{C}$. Let V be a finite dimensional K-vector space and G be a finite group. A representation of G on V is a group homomorphism $\rho \colon G \longrightarrow Gl(V)$. We often say V is a representation of G.

Definition 0.21. *If W is a G-invariant subspace of V, the corestriction of ρ to W, $\rho^W: G \longrightarrow Gl(W), g \mapsto \rho(g)|_W$, is a* **subrepresentation** *of V.*

Because V is a finite dimensional vector space, and therefore any G-invariant subspace is a direct summand, we have the theorem

Theorem 0.22. *Every representation is a direct sum of irreducible representations.*

and a result concerning the number of non isomorphic irreducible representations of G on complex vector spaces:

Lemma 0.23 (Schur). *If $\rho: G \longrightarrow Gl(V)$ and $\theta: G \longrightarrow Gl(W)$ are two irreducible complex representations, and $T: V \longrightarrow W$ is a G-equivariant linear transformation, then:*

1. *If V is not isomorphic to W, then $T = 0$.*
2. *If $V = W$ and ρ is equivalent to θ, then $T = cI$, where $c \in \mathbb{C}$ and I is the identity.*

Proof. This is an immediate consequence of the fact that $\operatorname{Ker} T$ and $\operatorname{Im} T$ are subrepresentations. The space V needs to be complex only for the second assertion. ∎

We denote by $\chi_V(g)$ the trace of $\rho(g)$ when $g \in G$.

Definition 0.24. *The map $\chi_V: G \longrightarrow \mathbb{C}$ is called the* **character** *of the representation V.*

If $\rho: G \longrightarrow Gl(V)$ and $\theta: G \longrightarrow Gl(W)$ are representations , we list the basic properties of characters (for proofs, see [29]):

1. $\rho + \theta \colon G \longrightarrow Gl(V \oplus W)$ is a representation and its character is

$$\chi_{V \oplus W} = \chi_V + \chi_W \tag{1}$$

2. $\rho \centerdot \theta \colon G \longrightarrow Gl(V \otimes W)$ is a representation and its character is

$$\chi_{V \otimes W} = \chi_V \chi_W \tag{2}$$

Always $\chi(1) = \dim V$ and $\chi(hgh^-1) = \chi(g)$, where 1 is the identity and $h, g \in G$. This means that the characters of G are **class functions** on G. Because of these properties and Lemma 0.23 we have a more precise statement for Theorem 0.22:

Proposition 0.25. *If $G\colon \longrightarrow Gl(V)$ is a representation with character χ, and all the irreducible representations of G are $C_1, \ldots, C_n$, with characters $\chi_1, \ldots, \chi_n$, respectively, then $\chi = \sum m_k \chi_k$ and V is isomorphic to $\bigoplus_{k=1}^{n} m_k C_k$, where $m_k \geq 0$ is called the* **multiplicity** *of C_k in V and $m_k C_k$ denotes the direct sum of C_k by itself m_k times.*

Definition 0.26. *The ring $R(G) = \bigoplus_{k=1}^{n} \mathbb{Z}\chi_k$ is the* **representation ring** *of the group G.*

When the representations are real, the representation ring is denoted by $RO(G)$. On $R(G)$ one can define a scalar product, $\langle\, , \rangle$ by

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{m} \sum_{g \in G} \chi_1(g) \overline{\chi_2(g)}$$

Then the following results (see [29]) are straightforward:

Proposition 0.27.

i. When χ is the character of an irreducible representation we have $\langle \chi, \chi \rangle = 1$.

ii. When χ_1 and χ_2 are the characters of two different representations of G, we have $\langle \chi_1, \chi_2 \rangle = 0$.

iii. The multiplicities m_k from Proposition 0.25 *are independent of the decomposition of V.*

iv. Two representations with the same character are equivalent.

Our special interest resides in the complex or real representations of $G = \langle g \rangle = C_m$ which, according to the general results above, are described by

Proposition 0.28.

i. When $m = 2k$ there are irreducible representations $V_0, V_1, \ldots, V_k$ with characters $\chi_0, \chi_1, \ldots, \chi_k$, respectively, where:

1. V_0 is the trivial degree-one representation with $\chi_0 = 1$.

2. V_k is the degree-one representation $gv = -v$ with $\chi_k = -1$.

3. For $j \neq 0, k$, the representations V_j are the degree-two representations given in the canonical basis of $\mathbb{R}^2$ by the matrices

$$\rho_j(g) \begin{pmatrix} \cos(t_j) & -sin(t_j) \\ \sin(t_j) & \cos(t_j) \end{pmatrix}$$

with $\chi_j = 2\cos(t_j)$.

ii. When $m = 2k-1$ we have the same kinds of irreducible representations as in (i) *except V_k which cannot occur.*

Definition 0.29. *A* G-**homotopy** *between the* G*-spaces* X *and* Y *is a* G*-equivariant map* $F\colon [0,1]\times X \longrightarrow Y$*, where* G *acts trivially on* $[0,1]$ *and diagonally on* $[0,1]\times X$*. The* G*-equivariant maps* $f_0,\ f_1\colon X \longrightarrow Y$ *are* G **-homotopic** *if there exists a* G*-homotopy* $F\colon [0,1]\times X \longrightarrow Y$ *such that* $F(0,x) = f_0(x)$ *and* $F(1,x) = f_1(x)$*. A* G*-equivariant map* $f\colon X \longrightarrow Y$ *is a* G-**homotopy equivalence** *if there exists a* G*-equivariant map* $h\colon Y \longrightarrow X$ *such that* fh *is* G*-homotopic to the identity map* 1_Y *and* hf *is* G*-homotopic to the identity map* 1_X*. The map* $f\colon X \longrightarrow Y$ *is a* **weak** G-**homotopy equivalence** *if there exists a* G*-equivariant map* $h\colon Y \longrightarrow X$ *such that* fh *and* hf *are only homotopic to* 1_Y *and* 1_X*, respectively.*

Let's notice also that the condition that the maps $f_t\colon X \longrightarrow Y$ be G-equivariant, for all $t\in[0,1]$, $f_t(x) = F(t,x)$, provides an equivalent definition for a G-homotopy F.

0.2 Γ-equivariant G-principal and Γ-equivariant vector bundles.

In this section G denotes the group of a bundle over X while the compact Lie group Γ acts on the Hausdorff space X. For more detail we refer to [33], pp. 67.

Definition 0.30. *For* Γ *and* G *groups and for a group homomorphism* $\phi\colon \Gamma \to Aut(G)$ *we define the* **semidirect product** *of* Γ *and* G *relative to* ϕ *to be the group* $\Gamma \times_\phi G$ *having the same group set as* $\Gamma \times G$ *and with the* *group multiplication*

$$(\gamma, g)(\gamma', g') = (\gamma\gamma', \phi(\gamma)(g)g')$$

where γ, $\gamma' \in \Gamma$ *and* g, $g' \in G$.

Notice that for a trivial homomorphism ϕ the semidirect product is the product $\Gamma \times G$ with the usual group multiplication. Let $E \xrightarrow{p} X$ be a G-principal bundle and let $\Gamma \xrightarrow{\phi} Aut(G)$ be a topological action of the Lie group Γ on the Lie group G.

Definition 0.31. *A* (Γ, ϕ, G)*-bundle is a* G*-principal bundle* $E \xrightarrow{p} X$ *such that*

1. E *and* X *are* Γ*-spaces and* p *is* Γ*-equivariant.*
2. $\gamma(eg) = (\gamma e)(\phi(\gamma)(g))$, $\forall \gamma \in \Gamma$, $\forall e \in E$, $\forall g \in G$.

Remark 0.32. *If* ϕ *is trivial then there is a lift of the action of* Γ *on* X *to an action on the principal bundle* $E \rightarrow X$*, i.e. there exists a map* $\Gamma \rightarrow Aut(E)$*, where* $Aut(E)$ *is the infinite dimensional group of bundle automorphisms of* E.

Definition 0.33. *A* G*-principal bundle* $E \xrightarrow{p} X$ *is a* **Γ-equivariant principal bundle** *if it is a* (Γ, ϕ, G)*-bundle as in* Definition 0.31 *with* ϕ *trivial.*

Remark 0.34. *For* Γ*-equivariant principal* G*-bundles we have the identity:*

$$(\gamma e)g = \gamma(eg), \ \forall \gamma \in \Gamma, \forall g \in G. \tag{3}$$

Definition 0.35. *Let x be a Γ-fixed point on X and let $E \xrightarrow{p} X$ be a Γ-equivariant principal bundle with bundle group G acting freely to the right on E. Then the* **isotropy representation** *at x in E is the group homomorphism $\alpha_x \colon \Gamma \to G$ satisfying $\gamma e = e\alpha_x(\gamma)$, $\forall \gamma \in \Gamma$. Note that α_x is independent of the choice of $e \in p^{-1}(x)$ up to G-conjugacy.*

Definition 0.36. *A* **(Γ, ϕ, G)-bundle map** *is a Γ-equivariant G-bundle map. Two (Γ, ϕ, G)-bundles E and E' are* **equivalent** *if there exists a Γ-equivariant G-bundle automorphism between E and E'.*

Definition 0.37. *A real or complex vector bundle $E \xrightarrow{p} X$ is a* **Γ-equivariant vector bundle** *if (E, Γ) and (X, Γ) are Γ-spaces and p is Γ-equivariant such that for any $\gamma \in \Gamma$, $\gamma \colon E \to E$ is a bundle map, i.e. $p(\gamma(e)) = p(e)$ and the restrictions to the fibers $\gamma \colon p^{-1}(x) \to p^{-1}(\gamma x)$ are linear maps.*

Remark 0.38. *Let $E \xrightarrow{p} X$ be a Γ-vector bundle and let x be in X. Then, from* 0.37*, there is a representation of the isotropy group Γ_x on the fiber $p^{-1}(x)$.*

It is easy to verify that the usual operations with bundles, like the Whitney sum, the tensor product, the pull-back via Γ-maps, have natural analogues in the category of Γ-vector bundles. The trivial Γ-vector bundle has an additional structure. Thus $X \times F \to X$ is a trivial Γ bundle once we fix a representation of Γ on the fiber F (see Remark 0.38). Therefore we have as many Γ-trivial vector bundles with fiber F as Γ-representations on F.

Definition 0.39. *Two Γ-equivariant vector bundles E and E' are* **equivalent** *if there exists a Γ-equivariant vector bundle automorphism between E and E'.*

If Γ acts smoothly on the smooth manifold X, we denote by $Vect_\Gamma(X)$ the set of equivalent classes of Γ-vector bundles over X. With the Whitney sum operation, $Vect_\Gamma(X)$ becomes a commutative semigroup with the unit equal to the zero-dimensional trivial Γ-bundle over X. Also (see for example [2]) we have the theorem:

Theorem 0.40. *If $E \xrightarrow{p} X$ is a Γ-equivariant (vector or principal) bundle and $F\colon [0,1] \times Y \longrightarrow X$ is a Γ-homotopy, then the pull-backs $f_0^*(E)$ and $f_1^*(E)$ are the same in $Vect_\Gamma(Y)$, where $f_0(y) = F(0,y)$ and $f_1(y) = F(1,y)$.*

For non-equivariant G-principal bundles over a CW-complex X we have the classification theorem:

Theorem 0.41. *The map*

$$[X, BG] \xrightarrow{\phi} \{[G \hookrightarrow P \to X]\}$$

defined by $\phi([f]) = [f^*(EG)]$ is a bijection.
Here $[X, BG]$ is the space of homotopy classes $[f]$ of maps $f\colon X \to BG$, $EG \to BG$ is the universal G-principal bundle constructed by Milnor, and $f^*(EG)$ is the pull-back of EG by f.

For Γ-equivariant G-principal bundles over a CW Γ-complex X T. tom Dieck constructed $E(\Gamma, G) \to B(\Gamma, G)$ a universal Γ-bundle (see [33], pp. 57-60), where $E(\Gamma, G)$ is the naturally Γ-equivariant Milnor join EG. By Theorem 0.40, one obtains:

Theorem 0.42. *The map*

$$Vect_\Gamma X \xrightarrow{\phi} [X, B(\Gamma, G)]^\Gamma$$

defined by $\phi([f]) = [f^*(E(\Gamma, G))]$ *is a bijection. Here* $[X, B(\Gamma, G)]^\Gamma$ *is the space of* Γ*-homotopy classes* $[f]$ *of* Γ*-maps* $f\colon X \to B(\Gamma, G)$.

Let G be a compact Lie group and let X be a smooth G-manifold endowed with a G-invariant metric. Let $Y \subset X$ be a G-invariant closed smooth submanifold. Then the orthogonal complement $TY^\perp$ in TX is a G-vector bundle isomorphic to the quotient of G-bundles $((TX)|_Y)/TY$.

Definition 0.43. *The* **normal bundle** *of* Y *in* X *is the denoted by* $\nu(Y, X)$ *and is defined to be the equivalence class of the* G*-equivariant bundle* $TY^\perp$.

Definition 0.44. *An* **open invariant tubular neighbourhood** *of* Y *in* X *is a smooth* G*-vector bundle* E *on* Y *together with an equivariant diffeomorphism* $\phi\colon E \to X$ *onto a neighbourhood of* Y *in* X *such that the restriction of* ϕ *to the* 0*-section of* Y *of* E *is the inclusion of* Y *in* X*. If the bundle* E *is endowed with an inner product on the fibers, then a* **closed invariant tubular neighbourhood** *of* Y *is given by the restriction of* ϕ *on the disk bundle* $D(E)$.

Theorem 0.45. ([7], VI., Thm. 2.2.) Y *has an invariant tubular neighbourhood in* X.

When S is a commutative semigroup we define $F(S)$ to be the free abelian group generated by the elements of S. The addition in $F(S)$ is denoted by $\oplus$. Then $E(S) = s_1 \oplus s_2 \ominus (s_1 + s_2)$ is a subgroup of $F(S)$, where "$\ominus$" is the inverse operation in $F(S)$ and "$+$" is the addition in S.

Definition 0.46. *The* **Grothendieck group** *of* S *is* $K_0(S) = F(S)/E(S)$.

We can also define $K_0(S)$ through the universal property (see [2]): there is a semigroup homomorphism $f : S \longrightarrow K_0(S)$ such that, for any abelian group A and any semigroup homomorphism $\psi : S \longrightarrow A$, there is a unique group homomorphism $\phi : K_0(S) \longrightarrow A$ with the property that $\phi f = \psi$. If S denotes the commutative ring of the (complex or real) representations of the finite group G, with the addition of 1 and the multiplication of 2, then the map $S \ni \rho \mapsto \chi_\rho \in R(G)$ can be extended to a group homomorphism

$$K_0(S) \longrightarrow R(G) \tag{4}$$

This homomorphism turns out to be a ring isomorphism. Let G be a compact Lie group and let X be a compact oriented G-manifold of even dimension $2k$. We assume that G acts on X preserving the orientation, i.e. for any $g \in G$, if $[X] \in H_{2k}(X;\mathbb{Z})$ denotes the fundamental class of X, then $g_{2k}[X] = [X]$, where $g_*\colon H_*(X;\mathbb{Z}) \to H_*(X;\mathbb{Z})$ is the induced action on homology.

Definition 0.47. *The $\mathbb{R}$-bilinear form $B\colon H^k(X;\mathbb{R}) \times H^k(X;\mathbb{R}) \to \mathbb{R}$ given by $B(x,y) = (x \smile y) \frown [X]$ is the intersection form on X.*

Remark 0.48. *From $x \smile y = (-1)^k(y \smile x)$ we have that B is symmetric for k even and skew-symmetric for k odd. Then considering $(\ ,\ )$ a inner product on $H_k(X;\mathbb{R})$ and by averaging the action of the compact group G on $(,)$ we obtain a G-invariant inner product $\langle\ ,\ \rangle$. Then we define $A\colon H^k(X;\mathbb{R}) \to H^k(X;\mathbb{R})$ by $\langle x, Ay\rangle = B(x,y)$.*

If k is even then A is symmetric and it has real eigenvalues and we obtain a G-invariant splitting of $H^k(X;\mathbb{R}) = H^+(A) \oplus H^-(A)$ in positive and negative spaces according to positive, respectively negative, eigenvalues of A. If

χ denotes the total character of the representation of G on $H^k(X;\mathbb{R})$, then χ^+ and χ^- denote the characters of the restricted representations to $H^+(A)$ and $H^-(A)$. If k is odd, then $J = A\sqrt{AA^*}$, with A^* the adjoint of A relative to $\langle,\rangle$, defines a complex structure on $H^k(X;\mathbb{R})$ by $(u+iv)y = uy + J(vy)$ for any $y \in H^k(X;\mathbb{R})$. Since A is G-equivariant, J is G-equivariant and $H^k(X;\mathbb{R})$ becomes a complex G-vector space. Let χ denote the character of the complex representation of G on $H^k(X;\mathbb{R})$.

Definition 0.49. [3] *If k is even, the G-**signature** of X is the virtual character*

$$Sign(G,X) = \chi^+ - \chi^- \in RO(G).$$

*If k is odd, the G-**signature** of X is the virtual character*

$$Sign(G,X) = \chi - \chi^* \in R(G),$$

where $\chi^*(g) = \overline{\chi(g)} \in \mathbb{C}$, for all $g \in G$.

For k even, $Sign(G,X)$ is a generalization of the non-equivariant signature associated with the intersection form on a $4k$-dimensional manifold. In [4], Thm. 6.12., the G-signature theorem gives a description of $Sign(g,X) = Sign(G,X)(g)$ in terms of characteristic classes of orthogonal and unitary groups and of characteristic classes of the tangent and normal bundles to the fixed set X^g. Of great importance in the study of group actions on 4-manifolds is the following corollary (see [4], Prop. 6.18):

Corollary 0.50. *If g is an odd order automorphism of the compact oriented 4-manifold X and we denote by x_j and F_k the isolated fixed points, respectively the fixed surfaces that make up X^g (see* Lemma 1.18*), then*

$$Sign(g,X) = \sum_j -\cot\frac{a_j}{2}\cot\frac{b_j}{2} + \sum_k \csc^2\frac{\theta_k}{2}[F_k]\cdot[F_k]\ ,$$

where $g \equiv \begin{pmatrix} t^{a_j} & \\ & t^{b_j} \end{pmatrix}$ *is a representation of* g *on* TX_{x_j} *with* $ord(g) = m$, $t^m = 1$, a_j, b_j *relatively prime to* m, *and* $[F_k] \in H_2(X; \mathbb{Z})$ *homology classes of* F_k.

Chapter 1

Permutation Representations

For what follows, X is a closed, smooth, compact, oriented, simply connected 4-manifold. The group actions considered are effective and smooth. The finite group G acts on X inducing an action in homology, i.e. we have $\rho\colon G \longrightarrow Aut_{\mathbb{Z}}(H_*(X))$ an integral representation. Unless we specify the coefficients, $H_*(X)$ will denote the integral homology of X. The main result of this chapter is Theorem 1.14 and its proof will become evident after our discussion on the properties of ρ in the case when G is cyclic of odd order, and X has positive definite intersection form. Throughout, C_m denotes a cyclic group of order m.

Theorem 1.1 (Donaldson). ([8], Thm.1.3.1.) *The only positive definite intersection form ($b_+ = n, b_- = 0$) that can be realized by a smooth, compact, simply-connected 4-manifold is the diagonal form , $\lambda = n\langle 1\rangle$.*

Theorem 1.2 (Wall). *Two simply connected smooth manifolds that have equivalent intersection forms are h-cobordant, i.e. they are deformation retracts of the manifold which they bound. In particular, two such manifolds are homotopy equivalent.*

Therefore, according to 1.1, our manifold X, if positive definite, has the same intersection form as $\overset{n}{\underset{1}{\#}}\mathbb{CP}^2$ and, by 1.2 , X is a homotopy-$\overset{n}{\underset{1}{\#}}\mathbb{CP}^2$. We denote by e_i, $i = 1, \ldots, n$, the fundamental class of $\mathbb{CP}^1$ embedded in the standard way in the i-th component of the connected sum $\overset{n}{\underset{1}{\#}}\mathbb{CP}^2$, and the corresponding homology class in X. Then $H_2(X)$ is a free $\mathbb{Z}$-module with basis $\{e_1, \ldots, e_n\}$. Since X is simply connected, we have $H_1(X) = 0$, and, by Poincaré duality, $H_3(X) = 0$ and $H_0(X) \simeq H_4(X) \simeq \mathbb{Z}$.

Suppose now that X is a compact smooth manifold on which the compact Lie group G acts smoothly. By works of Mostow, Palais and Wassermann (see [34]) we know that the associated homology chain complex $C_*(X)$ of X (which is a CW-complex) inherits a simplicial G-action, i.e. G transforms simplices (cells) into simplices. Relevant to this assertion are the following two results from [34]:

Proposition 1.3. *There exists a G-equivariant Morse function on X.*

Proposition 1.4. *The manifold X is G-equivariantly homotopic to the G-complex $(V_1 \times_{H_1} G) \cup_{f_2} (V_2 \times_{H_2} G) \cup \cdots \cup_{f_n} (V_n \times_{H_n} G)$, where the V_i's are radius one H_i-slices, and the f_i's are H_i-equivariant attaching maps for all H_i the subgroups of G.*

Hence $C_*(X)$ is a chain complex of finitely generated $\mathbb{Z}[G]$-modules and, consequently, the same is true about $H_*(X)$.

1.1 Representations of the induced action on homology

Let M and N be $R[G]$-modules and let $\lambda\colon M \times M \longrightarrow N$ be an $R[G]$-symmetric bilinear map. Therefore $\lambda \in Hom_R(M \times M, N)$ is G-equivariant, where $M \times M$ is the product $R[G]$-module given by the product action. λ is equivariant iff λ is fixed by the induced action of G on $Hom_R(M \times M, N)$,

$$(g\lambda)(x,y) = g\lambda(g^{-1}x, g^{-1}y), \forall g \in G, \forall x, y \in M, \text{ i.e.}$$

$$\lambda(gx, gy) = g\lambda(x,y), \forall g \in G, \forall x, y \in M.$$

λ is G-invariant iff

$$\lambda(gx, gy) = \lambda(x,y), \forall g \in G, \forall x, y \in M.$$

In particular, the $\mathbb{Z}$-bilinear intersection pairing $\lambda\colon H^2(X) \times H^2(X) \to H^4(X) \simeq \mathbb{Z}$ given by the cup product $\lambda(x,y) = x \smile y$ is symmetrical and, because $gx \smile gy = g(x \smile y)$, for any $g \in G$ and any $x, y \in H^2(X)$, λ is G-equivariant (i.e. the Poincaré duality isomorphism is G-equivariant).

Proposition 1.5. *Let $e_1, \ldots, e_n$ be the elements of a basis in the free module R^n, and let Σ_n be the symmetric group of order n. Then, the map $P\colon \Sigma_n \longrightarrow Hom_R(R^n, R^n)$, $P(s)e_i = e_{s(i)}$, defines a representation of Σ_n whose kernel is trivial.*

Proof. For any $s, t \in \Sigma_n$ and $i = 1, \ldots, n$ one obtains $P(st)e_i = e_{st(i)} = e_{s(t(i))} = P(s)P(t)e_i$, i.e. $P(st) = P(s)P(t)$. Also, $P(s) = 1 \Leftrightarrow s = 1$, and $P(s^{-1}) = P(s)^{-1}$. Thus Σ_n maps injectively into $Gl(n, R)$. ∎

Definition 1.6. *If G is any group and $f\colon G \longrightarrow \Sigma_n$ is a group-homomorphism, then the map $P \circ f\colon G \longrightarrow Gl(n,R)$ is a degree n R-***representation by permutations** *of the group G.*

Any group is isomorphic to a subgroup of its own automorphisms and $G = \{g_1, \dots, g_m\}$, is isomorphic to a subgroup of $\mathcal{P}(G)$, the permutation group of G. Thus we have the monomorphism $G \ni g \mapsto \gamma(g) \in \mathcal{P}(G)$ where $\gamma(g)(g_i) = gg_i = g_j$.

Definition 1.7. *The monomorphism $r\colon G \longrightarrow Gl(n,R)$, $g \mapsto P(\gamma(g))$ is the* **regular representation** *of G.*

If G is a cyclic group of order m, and g is a generator, then the matrix associated with the regular representation of G on R^m in the basis $\{e_1, \dots, e_m\}$ is

$$r(g) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & & \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \tag{1.1}$$

where the basis was chosen such that $r(g)e_i = e_{i+1}$, $r(g)e_{m-1} = e_1$. There is also another way to look at the representation matrix 1.1. Let's observe that the table of the group G determines 1.1 and we obtain 1 at the crossing between the line i and the column j whenever $gg_i = g_j$. In general, once we find matrix representations for the generators of a group G, these must also satisfy the relations that define G.

Definition 1.8. *We call a $\mathbb{Z}G$-module M* **indecomposable** *if it cannot be written as a direct sum of two non-trivial $\mathbb{Z}G$-submodules.*

Definition 1.9. *A finitely generated $\mathbb{Z}G$-module M is* **$\mathbb{Z}$-irreducible** *if it has no non-trivial $\mathbb{Z}G$- submodules of lower $\mathbb{Z}$-rank.*

Again, let G be a finite cyclic group of order m, and let g denote a generator. Let ξ be a primitive root of unity, $\xi^m = 1$. A $\mathbb{Z}$-representation of G on the free $\mathbb{Z}$-module M is equivalent to a $\mathbb{Z}[G]$-module structure on M. $\mathbb{Z}[G]$ is not an integral domain ring because $(g-1)(1+g+\cdots+g^{m-1}) = 0$ but we have the non-split exact sequence of $\mathbb{Z}[G]$-modules

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[G] \xrightarrow{g \mapsto \xi} \mathbb{Z}[\xi] \longrightarrow 0 \ ,$$

where the $\mathbb{Z}[G]$-module structure of $\mathbb{Z}[\xi]$ is given by $\mathbb{Z}[G]/(1+g+\cdots+g^{m-1})$, and that of $\mathbb{Z}$ is given by $\mathbb{Z}[G]/(g-1)$. If the G action on M induces a $\mathbb{Z}[G]/(1+g+\cdots+g^{m-1})$-module structure on M then we say that G admits a cyclotomic representation on M.

Definition 1.10. *The representation given by the G-module homomorphism*

$$\mathbb{Z}[G] \ni g \mapsto \xi \in \mathbb{Z}[\xi]$$

is called a **cyclotomic representation.**

As before, taking into account the fact that $\xi^{m-1} = -1-\xi-\cdots-\xi^{m-2}$, in a proper basis of $\mathbb{Z}^{m-1}$, we have the matrix representation:

$$c(\xi) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -1 \\ 1 & 0 & 0 & \dots & 0 & -1 \\ 0 & 1 & 0 & \dots & 0 & -1 \\ 0 & 0 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & & \\ 0 & 0 & 0 & \dots & 1 & -1 \\ & & & & & \end{pmatrix} \tag{1.2}$$

One can see that the representation matrices 1.1 and 1.2 are not equivalent.

Proposition 1.11. *The action of the odd order cyclic group G on X is orientation preserving and G permutes the elements of the standard basis of the $\mathbb{Z}$-free module $H_2(X)$.*

Proof. Let g be a generator of $G = C_m$, and let $\lambda\colon H_2(X) \times H_2(X) \longrightarrow H_4(X) \simeq \mathbb{Z}$ be the intersection form. λ is a symmetrical G-equivariant $\mathbb{Z}$-bilinear map given by the cup product homology operation: $\lambda(x, y) = x \smile y$. The matrix associated with λ in the basis $\{e_i\}_{i=1,\dots,n}$ is given by $\lambda_{ij} = \lambda(e_i, e_j) = \delta_{ij}$. Then g has a $\mathbb{Z}$-representation, still denoted g, $g\colon H_4(X) \longrightarrow H_4(X)$. Since $g \in Aut_{\mathbb{Z}}(H_4(X))$, $ga = \pm a$, for a, the fundamental class, and, because $m = ord(G)$ is odd and $g^m = 1$, we have $ga = a$. Thus G acts trivially on $H_4(X)$, i.e. orientation preserving, and therefore λ is G-invariant.

If g represents a permutation, then $ge_i = e_k$, for any $i = 1, \dots, n$. Then, $\lambda(ge_i, ge_j) = \lambda(e_k, e_l) = \delta_{kl}$. It is clear that k and l are equal if and only if i and j are equal. Thus, $\lambda(ge_i, ge_j) = \delta_{ij}$ for any $g \in G$, therefore λ is G-invariant. Conversely, if, for any $g \in G$ and any i, j, $\lambda(ge_i, ge_j) = \delta_{ij}$, then, writing $ge_i = \sum_k a_i{}^k e_k$, $\lambda(\sum_k a_i{}^k e_k, \sum_l a_j{}^l e_l) = \delta_{ij}$, or $\sum_k (a_i{}^k)^2 = 1$, $a_i^k \in \mathbb{Z}$. Therefore $a_i{}^k = \pm 1$, for one k, and $a_i{}^k = 0$, for all the others.

These facts are described by the exact sequence of multiplicative groups

$$1 \to \{\pm 1\}^n \to Aut(H_*(X), \lambda) \to \Sigma_n \to 1,$$

where $Aut(H_*(X), \lambda)$ represents the group of the automorphisms of $H_*(X)$ that leave λ invariant. Thus, a λ-invariant automorphism has its associated matrix equal to the product between a matrix of type 1.1 and a matrix $s = \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}$ Thus the automorphism that corresponds to $g \in G$, $a(g)$, is the product $s(g)r(g)$. But $a(g)^m = a(g^m) = 1 = r(g^m) = r(g)^m$ and we must have $s(g)^m = 1$. As m is odd, the only possibility is $s(g) = 1$. Thus the λ-invariant automorphisms induced by G are permutations of the basis $\{e_i\}$. ■

Remark 1.12. *A cyclotomic representation matrix as in 1.2 does not leave λ invariant.*

Definition 1.13. *Let S be an arbitrary G-set and let $\mathbb{Z}S$ be the free abelian group generated by S. The G-module $(\mathbb{Z}S, G)$ obtained by the $\mathbb{Z}$-linear extension of the action on S is called a* **permutation module**.

Let $S = \bigsqcup_{H \leq G} S_H$ be the disjoint union of the isotypical subsets of S. Then

$$\mathbb{Z}S = \bigoplus_{H \leq G} \mathbb{Z}S_H$$

is a direct sum of permutation modules and

$$\mathbb{Z}S_H = \bigoplus_x \mathbb{Z}[G/G_x] \ ,$$

where $G_x \simeq H$ and x ranges over a set of representatives for the G-orbits in S_H. In our case $S = \{e_1, \ldots, e_n\}$ and $\mathbb{Z}S = H_2(X)$. By Proposition 1.11 $H_2(X)$ is a permutation module and

$$H_2(X) = \bigoplus_{H \leq G} \mathbb{Z}[G/H]^{m_H}, \tag{1.3}$$

where the integers m_H show how many times the representations $\mathbb{Z}[G/H]$ occur.

The purpose of this chapter is to provide a proof for the following result which can also be inferred from part 1. and part 2. of [9].

Theorem 1.14. *If $G = C_m$, m odd and X is a closed, smooth, compact, oriented, simply connected, positive definite 4-manifold, then the fixed set X^G is non-empty and consists of a union of isolated points and 2-spheres.*

Definition 1.15. *Let (X, G) be a topological locally linear action, with $G = C_m$ and m odd. For any element $g \in G$ the* **Lefschetz number** *of g is*

$$L(g, X) = \sum_{j=0}^{4} (-1)^j trace(g_*)|_{H_j(X;\mathbb{Q})},$$

where g_* is the induced homomorphism of g in homology.

Because of Poincaré duality we observe that we can define $L(g, X)$ in the same way using the rational cohomology groups of X. A well known result is the Lefschetz Fixed-Point Formula:

$$L(g, X) = e(X^g) \ , \tag{1.4}$$

where $e(X^g)$ is the Euler characteristic of the fixed set of g. Lefschetz Formula has a direct proof using the transfer map and several other results on the G action on homology as pointed out in [9], Prop. 1.2., or, when (X, G) is smooth, it can be regarded as the DeRham complex of X case of the more general formula given in [3] for $L(g, E)$, where E is an elliptic complex.

Corollary 1.16. *If $G = C_m$, m odd, then $X^G \neq \emptyset$.*

Proof. Let g be a generator of G. Because m is odd, g acts preserving the orientation, i.e. g_* acts trivially on $H_0(X, \mathbb{Z})$ and $H_4(X, \mathbb{Z})$. By 1.4, $e(X^G) = e(X^g) = 1 + trace(g_*)|_{H_2(X;\mathbb{Q})} + 1$, and by (1.3), we obtain $e(X^G) \geq 2$, since $trace(g_*)$ on any regular representation $\mathbb{Z}[G/H]$ is zero, for $H \neq G$, and equal to 1, otherwise. Therefore X^G cannot be empty. ∎

Actually the preceding proof says more about X^G:

Corollary 1.17. *The Euler characteristic of the fixed set is given by*

$$e(X^G) = t + 2$$

where $t = m_G$ in formula (1.3) represents the multiplicity of the trivial subrepresentation.

Lemma 1.18. *Let $G = C_m$, m odd. The fixed set X^G consists of a union of isolated points and smooth surfaces.*

Proof. In general the connected components of the fixed set of a smooth G-manifold are G-fixed submanifolds.The lemma follows from the types of possible representations at the fixed points. Let's prove the lemma first for $G = C_p$, p a prime number. If $x \in X^G$, then the G-slice through x is G-diffeomorphic to a G-invariant ball in T_xX, according to Proposition

0.20 and Proposition 0.14. The action on T_xX is the differential of the action on X. Then, because $\dim T_xX = 4$, the complex slice representation at x is given by the matrix $\begin{pmatrix} \xi^a & 0 \\ 0 & \xi^b \end{pmatrix}$, where, by notation $\xi^a = e^{2\pi i a/p}$, $a \in \{0, \ldots, p-1\}$. Referring to 0.28, we distinguish for a and b the following cases:

1. $a = b = 0$
2. $a = 0$, $b \neq 0$ or $a \neq 0$, $b = 0$
3. $a \neq 0$, $b \neq 0$

The case 1. cannot occur because X is connected and the action is non-trivial.

In case 2. we have a two-dimensional G-invariant space that is the tangent space of a fixed connected surface. Notice that due to Theorem 0.40, the representations at two points in the same connected component of X^G are equivalent if we regard them as trivial G-bundles over each point and we consider a G-fixed path connecting the two points as a G-equivariant homotopy.

For the case 3., the action on the slice is free and we have only isolated fixed points. For the general case, $G = C_m$, m odd, notice that X^G is fixed by C_p, $p|m$. Therefore X^G consists again of isolated fixed points and surfaces. ■

Lemma 1.19. *Each component of X^G is orientable.*

Proof. The statement is true for G finite of odd order, acting locally smoothly on the n-dimensional manifold X (compare [7], Theorem IV. 2.1). In our case, $\dim X = 4$, the argument is easier because of the complex structure on the tangent spaces which induces orientations on each surface. ■

The next results apply for $G = C_p$ with p a prime integer. A result of Reiner (see [27], pp. 508) gives a complete classification of finitely generated $\mathbb{Z}[G]$-modules for G cyclic of prime order.

Lemma 1.20 (Reiner). *Let M be a torsion free $\mathbb{Z}$-module and let's assume that M has a structure of finitely generated G-module with G cyclic of prime order p. Then M admits a decomposition in a direct sum $M_1 \oplus \cdots \oplus M_r$ of G-modules, where each of them is of the following three types:*

type t: $\mathbb{Z}$ *(as G-module with trivial G-action).*

type c: J *as a non-trivial ideal of* $\mathbb{Z}[\xi]$ *with the G-module structure*

$$\mathbb{Z}[\mathbb{G}]/(1 + g + \cdots + g^{p-1})$$

i.e. a chosen generator $g \in G$ acts on $\mathbb{Z}[\xi]$ by multiplication with ξ, where ξ is a primitive root of 1.

type r: $\hat{J}$ *such that there exists a extension of G-modules*

$$0 \to J \to \hat{J} \to \mathbb{Z} \to 0$$

which is not split, and J is of type c.

If the non-negative integers t, c, r are the corresponding multiplicities of each of the three types of G-modules in the decomposition of M, then M is determined up to isomorphism by t, c, r and the ideal class group $C_0(\mathbb{Z}[\xi])$.

Proof. Let's observe that the modules in the direct sum are indecomposable but not necessarily irreducible. Detailed proofs can be found in [27], page 508, or in [31], page 74. We highlight here the key facts of the proof in [31]. For a module M and a group $G = C_p$ as in the statement of Lemma 1.20, one considers the G-submodules $M^G = \{m \in M \mid (g-1)m = 0\}$ and $M^N = \{m \in M | (1+g+\ldots,\ldots,g^{p-1})m = 0\}$, where g is a generator of G. Let ξ be a primitive p-root of 1 and let $I = (g-1)$ and $N = (1+g+\ldots,\ldots,g^{p-1})$ principal ideals in $\mathbb{Z}[G]$. Then $\mathbb{Z}[G]/I \simeq \mathbb{Z}$, $\mathbb{Z}[G]/N \simeq \mathbb{Z}[\xi]$, and we obtain the torsion free $\mathbb{Z}$-modules $A = M/M^G$ and $B = M/M^N$. Moreover B has a $\mathbb{Z}[\xi]$-module structure. If C is the G-module $M/(M^G + M^N)$ and D is the G-module $M/M^G \cap M^N$ we recall the well known exact sequence $0 \to D \to A \oplus B \to C \to 0$. But we notice $M^G \cap M^N = \{0\}$. Thus $0 \to M \to A \oplus B \to C \to 0$ is exact, i.e. M is obtained by the pull-back of the diagram

$$\begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle r} \\ B & \xrightarrow[s]{} & C \end{array} \tag{1.5}$$

A is a free $\mathbb{Z}$-module and we have $A = A_1 \oplus \cdots \oplus A_a$, where $A_i \simeq \mathbb{Z}$. Also $C = C_1 \oplus \cdots \oplus C_c$, where $C_i \simeq \mathbb{Z}[G]/(I+N) \simeq \mathbb{Z}/p$. Because for any ideals I_1 and I_2 of $\mathbb{Z}[\xi]$, $I_1 \oplus I_2 \simeq \mathbb{Z}[\xi] \oplus I_1 I_2$, we have $B = B_1 \oplus \cdots \oplus B_b$ where $B_i \simeq \mathbb{Z}[\xi]$, for $i \geq 2$, and $B_1 = J$ is a non-zero ideal of $\mathbb{Z}[\xi]$. These decompositions can be chosen such that $s(B_i) = \begin{cases} C_i & \text{if } i \leq c \\ 0 & \text{if } i > c, \end{cases}$ and $r(A_i) = \begin{cases} C_i & \text{if } i \leq c \\ 0 & \text{if } i > c. \end{cases}$

Then the diagram (1.5) is a direct sum of diagrams of the following forms

$$\begin{array}{ccc} & & A_i \\ & & \downarrow \\ 0 & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccc} & & 0 \\ & & \downarrow \\ B_i & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccc} & & A_i \\ & & \downarrow \\ B_i & \longrightarrow & C_i \end{array} \tag{1.6}$$

The pullbacks of the first two diagrams in (1.6) yield obviously A_i and B_i as summands of M,i.e. types t and c respectively. Then it is shown that the pull-back P of the third diagram gives $\mathbb{Z}[G]$, for $i \geq 2$, or an ideal of $\mathbb{Z}[G]$ of type r, for $i = 1$ and the isomorphism class of P depends only on B_1, and is independent on the epimorphisms shown in the diagram. ■

Applying the result [31], Theorem 3.3., we obtain

$$M = \mathbb{Z}^t \oplus \mathbb{Z}[\xi]^c \oplus \mathbb{Z}[G]^r \oplus J \tag{1.7}$$

for J an ideal of $\mathbb{Z}[G]$ obtained by the pullback of the third diagram of (1.6) when $B_i = B_1$ is a non-principal ideal of $\mathbb{Z}[\xi]$. Following the terminology adopted in [9], we call such an ideal J **exotic**. Our interest is in the types of representations we obtain via Lemma 1.20 for $M = H_2(X;\mathbb{Z})$. It is known that $\mathbb{Z}[\xi]$ is a principal ring when $p < 23$. Therefore we don't obtain exotic ideals when $\beta_2(X) < 22$ or when $p < 23$. Otherwise, we prove the lemma:

Lemma 1.21. *If $G = C_p$ acts smoothly on X, then $H_2(X)$ has the standard decomposition $\mathbb{Z}^t \oplus \mathbb{Z}[\xi]^c \oplus \mathbb{Z}[G]^r$.*

To prove 1.21 we take up the approach of Swan as in [31]. We denote by $K_0(G)$ the Grothendieck group of the category of finitely generated projective $\mathbb{Z}[G]$-modules. This is the abelian group generated by generators $[P]$, where P is a finitely generated projective $\mathbb{Z}[G]$-module, subject to relations $[P] = [P'] + [P'']$ whenever there is an exact sequence of finitely generated projective $\mathbb{Z}[G]$-modules $0 \to P' \to P \to P'' \to 0$. Notice that the sequence is split. For the abelian category of finitely generated $\mathbb{Z}[G]$-modules the Grothendieck group $G_0(\mathbb{Z}[G])$ is the abelian group generated by the finitely generated $\mathbb{Z}[G]$-modules $[M]$, with relations $[M] = [M'] + [M'']$

whenever there is an exact sequence (not necessarily split) of finitely generated $\mathbb{Z}[G]$-modules $0 \to M' \to M \to M'' \to 0$. We write $K_0(G)$, respectively $G_0(G)$, when the ring R is the group-ring $\mathbb{Z}[G]$, or even K_0 and G_0 if the group is understood. The Cartan map is defined to be the group homomorphism $K_0 \ni [M] \mapsto [M] \in G_0$. On K_0 one introduces the equivalence relation: $[P_1] \sim [P_2]$ when there exists F_1 and F_2 free G-modules such that $P_1 \oplus F_1 \simeq P_2 \oplus F_2$. The quotient $C_0(\mathbb{Z}[G]) = K_0(\mathbb{Z}[G])/\sim$ can be viewed here as $K_0/\hat{K}_0$, where $\hat{K}_0$ is the Grothendieck group of the abelian category of finitely generated free $\mathbb{Z}[G]$-modules. In [28], theorems 6.19 and 6.24, Rim shows that $C_0(\mathbb{Z}[G])$ is finite, isomorphic to the ideal class group $C_0(\mathbb{Z}[\xi])$. According to Swan ([31], Corollary 4.21), the following $\mathbb{Z}[G]$-modules isomorphism holds:

$$G_0(\mathbb{Z}[G]) = \mathbb{Z} \oplus \mathbb{Z} \oplus C_0(\mathbb{Z}[\xi]) \tag{1.8}$$

Proof of Lemma. 1.21: The key fact in the proof is that, by Proposition 1.4, the simplicial chain complex $C_*(X)$ inherits a G-simplicial action. In fact $C_*(X) = C_*(X, X^G) \oplus C_*(X^G)$, where G acts by permuting cells freely on $C_*(X, X^G)$ or by fixing cells on $C_*(X^G)$. We define $\overline{G_0} = G_0/\{[M] \mid M \text{ is } G\text{−}free\}$. If $C_* = C_*(X)$, $H_* = H_*(X)$, $C_* \xrightarrow{\partial_*} C_{*-1}$, $Z_* = Ker\partial_*$ and $B_* = \operatorname{Im}\partial_{*+1}$, then it is evident that $0 \to B_* \to Z_* \to H_* \to 0$ and $0 \to Z_* \to C_* \to B_{*-1} \to 0$ are exact sequences. Thus, in $\overline{G_0}$, $[Z_*] = [B_*] + [H_*]$ and $[C_*] = [Z_*] + [B_{*-1}]$. Then the $\overline{G_0}$-Euler characteristics $E(C_*) = \sum^4{}_{i=1}(-1)^i[C_i]$ and $E(H_*) = \sum^4{}_{i=1}(-1)^i[H_i]$ are equal. But $C_*(X, X^G)$ is G-free and we have then

$$E(C_*(X^G)) = E(H_*(X)) \, .$$

Let's assume that we have q points and s surfaces in X^G, and $H_2(X) =$

$\mathbb{Z}^t \oplus \mathbb{Z}[\xi]^c \oplus \mathbb{Z}[G]^r \oplus J$, where J is a non-principal G-ideal. Then $q[\mathbb{Z}] + b_1[\mathbb{Z}] + 2s[\mathbb{Z}] = E(C_*(X^G)) = E(H_*(X)) = (t+2)[\mathbb{Z}] + c[\mathbb{Z}[\xi]] + r[\mathbb{Z}[G]] + [J]$. But $[\mathbb{Z}[G]] = [\mathbb{Z}] + [\mathbb{Z}[\xi]]$ in G_0 and therefore $0 = [\mathbb{Z}] + [\mathbb{Z}[\xi]]$ in $\overline{G_0}$. We obtain $q[\mathbb{Z}] + b_1[\mathbb{Z}] + 2s[\mathbb{Z}] = (t + 2 - c)[\mathbb{Z}] + [J]$ which means $[J] = 0$ in $C_0(\mathbb{Z}[\xi])$. ∎

Proposition 1.11 showed that a $\mathbb{Z}[G]$-summand of type $\mathbb{Z}[\xi]$ cannot occur either. Thus we have

$$H_2(X) = \mathbb{Z}^t \oplus \mathbb{Z}[G]^r, \qquad G = C_p \tag{1.9}$$

1.2 The fixed point set

Because the fixed set X^G is nonempty, we can use the following important result from [9] (Prop. 2.4.)

Proposition 1.22 (Edmonds). *If $G = C_p$, p an odd prime, and $\beta_i(X^G)$ is the rank of $H_i(X^G; \mathbb{Z}_p)$, then*

$$\beta_1(X^G) = c$$

$$\beta_0(X^G) + \beta_2(X^G) = t + 2 \quad ,$$

where t, c, and r are the positive integers from Lemma 1.20 *applied to $M = H_2(X)$.*

The relation (1.9) shows that $c = 0$. Therefore

Corollary 1.23. *$\beta_1(X^G) = 0$ and the oriented surfaces in X^G are homology 2-spheres, i.e. they are diffeomorphic to spheres, when $G = C_p$, with p odd prime.*

We return to the case $G = C_m$, m odd integer and we finish the

proof of Theorem. 1.14. The first part, that the G-fixed set is non-empty, is due to the strong input: the G-invariant intersection form λ is the diagonal form $n\langle 1\rangle$ and this is due on its turn to the hypothesis that our G-action is smooth. This was shown by Corollary 1.16. Thus, according to (1.3) we have $H_2(X) = \oplus_i \mathbb{Z}[G/G_i]^{r_i}$, where G_i are the isotropy groups of the action of G on $H_2(X)$. For the second part, let $g \in G$ be such that $G = \langle g\rangle$,and let p be a prime dividing m. Then $H = \langle g^{m/p}\rangle \simeq \mathbb{Z}/p$ is a subgroup of G and X^H is a union of isolated points and spheres by Corollary 1.23. The non-empty fixed set X^G is a substratum of this H-stratum and X^G is the G-fixed-point set of X^H. Since X^G is non-empty, G cannot permute the points and the spheres of X^H freely. Also a cyclic group of odd order cannot act smoothly and freely on a 2-sphere, it fixes points or the 2-sphere. Therefore X^G is a union of points and 2-spheres. ∎

Chapter 2

The Equivariant Moduli Space

For a very good account regarding the definitions and properties of stratifications and Whitney stratifications we refer the reader to [24]. What we call here a stratification is defined in [24] to be a prestratification whereas the notion of stratification is assigned to an equivalence class of prestratifications in a certain sense (see [24], pp. 200).

Definition 2.1. *A* **stratification** *of a topological space X is a partition $\mathcal{S}$ of X into subsets called strata such that the following conditions hold:*

i. *Each stratum U is locally closed, i.e. U is the intersection of a closed set with an open set or, equivalently, $\forall u \in U$, $\exists N$ a neighbourhood of u in X such that $U \cap N$ is closed in N.*

ii. *$\mathcal{S}$ is locally finite.*

iii. *(axiom of frontier) If U_1 and U_2 are strata of $\mathcal{S}$ and $\overline{U_1} \cap U_2 \neq \emptyset$, then $U_2 \subset \overline{U_1}$.*

For a triple (U, V, x) with U, V analytic submanifolds of a smooth manifold W and $x \in V$ we present what's called **Whitney's Condition (a)**

Definition 2.2. *If* $\{x_i\}_i \subset U$ *is a sequence of points converging to* $x \in V$ *and* TU_{x_i} *converges in the Grassmannian of* $(\dim U)$*-planes in* TW *to* τ, *then* $TV_x \subset \tau$.

If W is an open set in $\mathbb{R}^n$, for any $x, y \in \mathbb{R}^n$ we denote by $\overline{x,y}$ the vector line determined by x and y. Then we can introduce **Whitney's Condition (b)**:

Definition 2.3. *For any sequences* $\{x_i\}$ *of points in* V *and* $\{y_i\}$ *of points in* U, *such that* $x_i \to x$, $y_i \to x$, $x_i \neq y_i$, $\overline{x_i y_i}$ *converges (in the projective space* P^{n-1}*), and* TUy_i *converges (in the Grassmannian of* $(\dim U)$*-planes in* $\mathbb{R}^n$*), we have* $l \subset \tau$ *,where* $l = \lim \overline{x_i y_i}$, $\tau = limTU_{y_i}$.

One can show that Whitney's Condition (b) is invariant under diffeomorphisms of W and therefore, using manifold charts, it can be defined for a triple (U, V, x), with U and V analytic submanifolds of a manifold W. Also one can see that Whitney's Condition (a) is a consequence of Whitney's Condition (b).

Definition 2.4. *Let* $Y \subset X$ *be a subset of a smooth manifold* X. *If* $\mathcal{S}$ *is a stratification of* Y *we say it is a* **Whitney stratification** *if each stratum is a smooth manifold and any two strata* U *and* V *with* $V \subset \overline{U}$ *satisfy condition (b) for any* x *in* V.

Definition 2.5. *Let* X *be a manifold,* $Y \subset X$ *and* $\mathcal{S}$ *a Whitney stratification of* Y. *Let* $\mathcal{S}_y$ *denote a stratum of* $\mathcal{S}$ *containing* y. *Then*

$$Y_k = \{y \in Y \mid \dim \mathcal{S}_y \leq k\}$$

make the **filtration by dimension** *associated with* $\mathcal{S}$.

Definition 2.6. *If $\{Y_k\}$ and $\{Y'_k\}$ are filtrations by dimension associated with the Whitney stratifications $\mathcal{S}$ and $\mathcal{S}'$ respectively, we say $\mathcal{S} < \mathcal{S}'$ if there is an integer k such that $\{Y_k\} \subset \{Y'_k\}$ and $\{Y_l\} = \{Y'_l\}$, for $l > k$.*

$\mathcal{S} = \mathcal{S}'$ if and only if $\{Y_k\} = \{Y'_k\}$ for all k.

Definition 2.7. *A Whitney stratification is called* **minimum** *if it is minimal relative to the order relation defined above.*

2.1 General position of equivariant maps

Definition 2.8. *If G is a compact Lie group, M, N are G-manifolds and P is a G-submanifold of N with the same action, then a smooth equivariant map $f\colon M \to N$ is G-transverse to P at $x \in M$ if either $f(x) \notin P$ or*

$$df_x TM_x + TP_{f(x)} = TN_{f(x)}$$

is satisfied.

In the non-equivariant case we have the well known transversality theorem:

Theorem 2.9. *Let M, N, P be smooth manifolds and let $f\colon M \to N$ and $g\colon P \to N$ be smooth maps. Then there exists a smooth manifold S and a smooth map $F\colon S \times M \to N$, with $F(s_0, x) = f(x)$, for a fixed $s_0 \in S$, and for any $x \in M$, so that F is transverse to $g(P)$. Moreover, defining $F_s\colon M \to N$, $F_s(m) = F(s, m)$ for any $s \in S$, the set $\{s \mid F_s \text{is transverse to } g(P)\}$ is a Baire set in S.*

This theorem asserts that the transversality condition is generic, i.e. any two smooth maps can achieve general position by an arbitrarily small perturbation of at least one of them. In the equivariant case, transversality does not ensure this result. We would have to perturb one map equivariantly and we may encounter obstructions. In [5], E. Bierstone introduced a definition for G-equivariant general position for G-equivariant smooth maps, separate from G-transversality, that provides the desired result: the set of smooth G-equivariant maps between the G-manifolds M and N, that are in general position with respect to a G-submanifold P of N is a Baire set with respect to C^∞ or compact-open topology. We will use Bierstone's notations in the discussion bellow. Let V and W be G-linear spaces and let $\mathcal{C}_G^\infty(V,W)$ denote the module of smooth G-equivariant maps over the ring $\mathcal{C}_G^\infty(V)$ of smooth real valued functions on V invariant under the action of G. The Malgrange preparation theorem (see [23]) shows that there is a finite number of G-equivariant polynomials, F_1,...,F_k, generating $\mathcal{C}_G^\infty(V,W)$ over $\mathcal{C}_G^\infty(V,\mathbb{R})$. Therefore any map F in $\mathcal{C}_G^\infty(V,W)$ can be written:

$$F(x) = \sum_{i=1}^{k} h_i(x)F_i(x) \ ,$$

with $h_i \in \mathcal{C}_G^\infty(V)$. If $h(x) = (x, h_1(x), ..., h_k(x))$ and $U(x,y) = \sum_{i=1}^k y_i F_i(x)$ then $F(x) = U \circ graph(h)$.

Definition 2.10. ([24], pp. 206-207) *A subset of* $\mathbb{R}^n$ *is called* **semialgebraic** *if it is in the smallest family of subsets of the form*

$$\{f > 0, f \textit{real polynomial in } n \textit{ variables } \}$$

of $\mathbb{R}^n$ *which is closed under taking finite intersection, finite union, and complements.*

Definition 2.11. ([5], pp. 462) *A* **strongly stratified** *set is a Hausdorf space* H *such that for each point in* H *there is a local presentation of*

H as the transversal intersection of a semialgebraic subset A of $\mathbb{R}^n$ by a diffeomorphism $g\colon \mathbb{R}^m \to \mathbb{R}^n$.

In [24], Thm.(4.9), Mather shows that there exists a canonical minimum Whitney stratification of a semialgebraic set and thus a strongly stratified set has a minimum Whitney stratification as well.

Definition 2.12 ([5], pp. 456). *Let X be a smooth manifold and let $f\colon X \to \mathbb{R}^q$ be a smooth map. Then f is* **transversal to an algebraic subvariety** *E of $\mathbb{R}^q$ if f is transversal* (see Definition 2.8) *to each stratum of the minimum Whitney stratification of E.*

Definition 2.13 ([5], def.1.1). *An equivariant map $F\colon V \to W$ is in general position with respect to $0 \in W$ at $0 \in V$ if $graph(h)\colon V \to V \times \mathbb{R}^k$ is transverse to the minimum Whitney stratification of the affine algebraic variety $\{U(x,y) = 0\}$, at $0 \in V$.*

By [5], Prop.6.1., Definition 2.13 extends to maps $f\colon \mathcal{O} \subset V \to W$, with $\mathcal{O}$ an open invariant neighbourhood of $0 \in V$. To extend it further to G-manifolds we need an intermediate step:

Definition 2.14. *If V, W_1 and W_2 are G-representation spaces, then a G-equivariant map $f\colon V \to W_1 \times W_2$ is in G-equivariant general position with respect to $W_1 \times \{0\}$ at $0 \in V$ if and only if $pr_2 \circ f\colon V \to W_2$ is in G-equivariant general position with $0 \in W_2$ at $0 \in V$. $pr_2\colon W_1 \times W_2 \to W_2$ is the projection map, obviously G-equivariant.*

Notice that in the non-equivariant case we similarly have: $f\colon V \to W_1 \times W_2$ is transverse with $W_1 \times \{0\}$ if and only if $pr_2 \circ f\colon V \to W_2$ is transverse with $0 \in W_2$. Let G be a compact Lie group, M and N smooth G-manifolds and let $P \subset N$ be a G-submanifold of N. If $F\colon M \to N$ is a G-equivariant smooth map, $x \in F^{-1}(P)$ and S is a G_x-equivariant slice through x, then $dF_x\colon TS_x \to TN_{F(x)}$ is a G_x-equivariant map between G-vector spaces. G_x is here the isotropy group of x. An equivariant map sends orbits to orbits and slices to slices. Thus $F(S)$ is a slice through $F(x)$ and we have

$$TN_{F(x)} = TF(S)_{F(x)} \oplus T(G \cdot F(x))_{F(x)} \ ,$$

$$TN_{F(x)} = TP_{F(x)} \oplus \nu.$$

But $F(x) \in P$ implies that the orbit $G \cdot F(x)$ is included in P. So $T(G \cdot F(x))$ is a linear subspace of $TP_{F(x)}$. Then Bierstone states:

Definition 2.15. *$F\colon M \to N$ is in G-equivariant general position with respect to P at $x \in M$ if either $F(x) \notin P$ or $F(x) \in P$ and for any slice S of the orbit $G \cdot x$ at x, the G_x-equivariant map $dF_x\colon TS_x \to TN_{f(x)}$ is in G_x-general position with respect to $TP_{F(x)}$, in the sense of the preceding definition. F is in equivariant general position with P if it is in equivariant general position with P at any point of M.*

Bierstone shows that the definition is not dependent on the slice chosen, or on the choice of F_i and h_i. Thus G-equivariant general position at the level of manifolds reduces naturally to G_x-equivariant general position at the level of slice representations: In our above notations we obtain that $dF_x\colon TS_x \to \nu$ is in G_x-general position with $0 \in \nu$ at $0 \in TS_x$. Notice that this is equivalent to the statement $dF_x\colon TS_x \to TF(S)_{F(x)}$ is in G_x-general position with $0 \in TF(S)_{F(x)}$ at $0 \in TS_x$, since $T(G \cdot F(x)) \subset TP_{F(x)}$. The following results from [5] are essential for the construction of an equivariant moduli space:

Theorem 2.16 ([5], Thm. 1.3. & Thm. 1.4.). *Let P be a closed G-submanifold of N. Then the set of smooth equivariant maps $F\colon M \to N$ which are in G-equivariant general position with respect to P at each point of a compact subset of M is open in the C^∞ topology and is a countable intersection of open dense sets.*

Proposition 2.17. ([5], Prop. 6.4.) *Let $F\colon M \to N$ be a smooth equivariant map in G-general position with respect to a G-submanifold P of N. Then F is* **stratumwise transverse** *to P, i.e. for any isotropy subgroup $H \leq G$, $F|_{M_H}\colon M_H \to N^H$ is transverse to P^H. If $F^{-1}(P) \neq epmtyset$ then $F^{-1}(P)_H$ is a manifold of dimension* $\dim M_H - \dim N^H + \dim P^H$.

Proposition 2.18. ([5], Prop. 6.5.) *Let $F\colon M \to N$ be a smooth equivariant map in G-general position with respect to a G-submanifold P of N. Then $F^{-1}(P)$ is a strongly stratified set and therefore has a minimum Whitney stratification.*

2.2 The equivariant moduli space

Let π be a compact Lie group acting smoothly, orientation preserving and by isometries on a smooth compact closed oriented simply connected Riemannian four-manifold X, and let $E \xrightarrow{p} X$ be an $SU(2)$-vector bundle with the second Chern number denoted by $-k$. We also denote by $\mathcal{A}$ and $\mathcal{G}$ the affine space of connections and the **gauge group** of E respectively. The gauge group of E is defined to be the group of smooth bundle automorphisms of E that cover the identity on X and whose restrictions to each fiber E_x are elements of $SU(2)$. If we consider instead P, the underlying principal $SU(2)$-bundle of E, $\mathcal{G}$ can be viewed as the space of smooth sections of the $SU(2)$-bundle of groups $Ad(P) = P \times_{SU(2)} SU(2)$, where the action of $SU(2)$ on the fibers of $Ad(P)$ is by conjugation. $\mathcal{G}$ is an infinite dimensional Lie group and it can be shown that it is connected when the intersection form of X is odd (see [14], pp 78-80). The **Lie algebra** of $\mathcal{G}$ is $\mathfrak{g}_E = \{v \in Hom(E, E) \mid v|_{E_x} \in su(2)\}$. A connection on E is a horizontal distribution in the tangent space of the underlying principal bundle of E (see for example [8], pp. 31-33). The covariant derivative associated with a connection A is a smooth $\mathbb{R}$-linear operator $d_A \colon \Gamma(E) \longrightarrow \Gamma(T^*X \otimes E)$ with the Leibniz property: $d_A(f \otimes s) = df \otimes s + f \otimes d_A s$, for any smooth function $f \colon X \longrightarrow \mathbb{R}$ and any $s \in \Gamma(E)$, where $\Gamma(E)$ denotes the space of the smooth sections of the bundle E. We often do not distinguish between a connection and its covariant derivative. Let $\Omega^k(E)$ be $\Gamma((\bigwedge^k T^*X) \otimes E)$. Then d_A extends inductively to a smooth operator $d_A \colon \Omega^k(E) \longrightarrow \Omega^{k+1}(E)$ by $d_A(s_1 \wedge s_2) = ds_1 \otimes s_2 + (-1)^l s_1 \wedge d_A s_2$, where $s_1 \in \Omega^0(E)$ and $s_2 \in \Omega^k(E)$. Also A induces a connection on $\mathfrak{g}_E$, still denoted A, and defined by $d_A \colon \Omega^0(\mathfrak{g}_E) \longrightarrow \Omega^1(\mathfrak{g}_E)$, $(d_A v)(s) = d_A(v(s)) - v(d_A s)$, for all $s \in \Omega^0(E)$. In the same inductive manner is defined $d_A \colon \Omega^k(\mathfrak{g}_E) \longrightarrow \Omega^{k+1}(\mathfrak{g}_E)$: $d_A(\omega_1 \wedge \omega_2) = d(\omega_1) \oplus \omega_2 + (-1)^k \omega_2 \wedge d_A \omega_1$, $\omega_1 \in \Omega^0(\mathfrak{g}_E), \omega_2 \in \Omega^k(\mathfrak{g}_E)$. When X is a Riemannian manifold, the metric on X induces a metric $\langle , \rangle$ on the spaces of smooth forms $\Omega^k(\mathfrak{g}_E)$. Let ω_1 and ω_2 be in $\Omega^{k+1}(\mathfrak{g}_E)$. Then $\omega_1 = \alpha_1 \otimes M$ and $\omega_2 = \alpha_2 \otimes N$, where α_1, α_2 are real forms in $\bigwedge^k(X)$ and M, N are sections of $\mathfrak{g}_E$. Then for any $x \in X$

$$\langle \omega_1, \omega_2 \rangle_x = (\alpha_1, \alpha_2)_x Trace({}^T M_x N_x) \tag{2.1}$$

where $(,)$ is the metric on real forms defined by the Hodge operator $*$:

$$(\alpha_1, \alpha_2)dV = \alpha_1 \wedge *\alpha_2 \tag{2.2}$$

with dV the volume form on X. Let's remark that the covariant derivative d_A associated with a connection A is a natural generalization of the ordinary derivative on scalar forms d. In fact, when A is the trivial connection we can identify between d_A and d.

The metric spaces $\mathcal{A}$, $\mathcal{G}$ and $\Omega^k(E)$ are only Frechet spaces with the C^∞-norm and thus they may not be complete. To ensure good convergence properties we consider, keeping the same notations, their completions in a Sobolev l-norm. Let A_0 be a base point connection in the affine space $\mathcal{A}$, $l \geq 0$. Then we define

$$\|u\|_l{}^2 = \int_X (\|u\|^2 + \|d_{A_0}u\|^2 + \ldots \|(d_{A_0})^l u\|^2), \tag{2.3}$$

for u an element in any of the spaces $\mathcal{A}$, $\mathcal{G}$ or $\Omega^k(E)$. With this norm, for a good choice of l, we benefit now from analysis on Hilbert spaces. For a more precise description, see ([14], pp. 92-96).

The gauge group $\mathcal{G}$ acts on the space of connections $\mathcal{A}$:

$$\mathcal{G} \times \mathcal{A} \longrightarrow \mathcal{A}$$

$$(u, A) \rightarrow u \cdot A$$

$$d_{u \cdot A}s = u d_A u^{-1} s, \quad \forall s \in \Omega^0(E) \tag{2.4}$$

Let $\mathcal{B} = \mathcal{A}/\mathcal{G}$ be the orbit space of the action. Let $Aut(E)$ be the group of bundle automorphisms of E that cover $Diff(X)$, the diffeomorphisms of X. Clearly we have the left-exact sequence of groups

$$1 \rightarrow \mathcal{G} \rightarrow Aut(E) \rightarrow Diff(X) .$$

Also π can be regarded a subgroup of $Diff(X)$. We'd like to find conditions so that π acts naturally on $\mathcal{A}$ and $\mathcal{B}$. It would be enough to construct a lift of the π-action to E, i.e. a group homomorphism $\pi \longrightarrow Aut(E)$, (see Definition 0.33). Let $g \in \pi$. The bundles E and its pull-back g^*E are equivalent since the π-action is orientation preserving (see the proof of Proposition 1.11), i.e. the second Chern class $c_2(E) \in H_4(X)$ is π-fixed. We define then $\mathcal{G}(\pi)$ to be the group of all bundle automorphisms $\hat{g} \colon E \longrightarrow E$ that cover some $g \in \pi$. We can see that the natural action $\pi \times \mathcal{A} \longrightarrow \mathcal{A}$, $(g, A) \longrightarrow \hat{g} \cdot A$, given by $d_{\hat{g}\cdot A} = \hat{g} d_A \hat{g}^{-1}$, is well defined up to conjugacy with gauge transformations u and we have thus an action of π on $\mathcal{B}$. Moreover, for any $u \in \mathcal{G}$ and any $g \in \pi$, the bundle automorphism $\hat{g}u\hat{g}^{-1}$ covers the identity, i.e. it is a gauge transformation. Therefore $\mathcal{G}$ is normal in $\mathcal{G}(\pi)$ and π is the quotient $\mathcal{G}(\pi)/\mathcal{G}$. We summarize with the exact sequence

$$1 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}(\pi) \longrightarrow \pi \longrightarrow 1 \tag{2.5}$$

On $\Omega^k(\mathfrak{g}_E)$ we have the natural action: $(g, v) \mapsto \hat{g} v \hat{g}^{-1}$, $g \in \pi$, $v \in \Omega^k(\mathfrak{g}_E)$. Again, the action is well defined modulo $\mathcal{G}$. Let h and h' be elements of $\mathcal{G}(\pi)$ covering the same $g \in G$ and acting trivially on a connection A. Then $h' = hu$ for some $u \in \mathcal{G}$. This implies u acts trivially on A. Thus we have the exact sequence of isotropy groups of A analogous to (2.5),

$$1 \longrightarrow \mathcal{G}_A \longrightarrow \mathcal{G}(\pi)_A \longrightarrow \pi_A \longrightarrow 1 \tag{2.6}$$

and let us note that the groups involved are finite dimensional compact Lie groups. In the non-equivariant case the self-dual moduli space $\mathcal{M}$ is the space (possibly empty) of the gauge classes of solutions A of the SD Yang-Mills equations:

$$F_A{}^- = 0 \tag{2.7}$$

where $F_A^- = \frac{1}{2}(F_A - *F_A)$, $F_A = d_A d_A \in \Omega^2(\mathfrak{g}_E)$ is the curvature operator of A, and $*$ is the Hodge operator associated with a metric on X.

Proposition 2.19. (see [13], Prop.5.1 & Prop.5.2) *Let A be a connection on E. Then the maps $d_A\colon \Omega^k(\mathfrak{g}_E) \to \Omega^{k+1}(\mathfrak{g}_E)$, are $\mathcal{G}(\pi)_A$-equivariant, $F_A \in \Omega^2(\mathfrak{g}_E)^{\pi_A}$ and, for any A, A' in $\mathcal{A}$ with $\pi_A = \pi_{A'}$, we have $A - A' \in \Omega^1(\mathfrak{g}_E)^{\pi_A}$.*

Therefore $\mathcal{M}$ is π-invariant in $\mathcal{B}$ if and only if the equations (2.7) are π-invariant, i.e. the metric on X is π-invariant. That is why we require that π acts on X by isometries.

Definition 2.20. *If π is a compact Lie group acting smoothly, orientation preserving and by isometries on the smooth compact oriented simply-connected closed Riemannian manifold $(X, \langle,\rangle)$ with a real analytic metric, then the self-dual* **equivariant moduli space** *$(\mathcal{M}, \pi)$ of an $SU(2)$-bundle $E \to X$ is the solution space of the π-equivariant elliptic PDE* (2.7) *in $(\mathcal{B}, \pi)$.*

Let A be a connection. The dual covariant operator associated with A is $d_A{}^*\colon \Omega^k(\mathfrak{g}_E) \to \Omega^{k-1}(\mathfrak{g}_E)$ defined by $\int_X \langle d_A\omega_1, \omega_2\rangle = \int_X \langle \omega_1, d_A{}^*\omega_2\rangle$, where $\omega_1 \in \Omega^{k-1}(\mathfrak{g}_E)$, $\omega_2 \in \Omega^k(\mathfrak{g}_E)$ and $\langle,\rangle$ is the metric (2.1).
For a SD-connection A we have the $\mathcal{G}(\pi)$-equivariant elliptic complex

$$\Omega^0(\mathfrak{g}_E) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_E) \xrightarrow{d^-{}_A} \Omega_-(\mathfrak{g}_E) \qquad (2.8)$$

The dimension δ of the free stratum $\mathcal{M}^*$ in the self-dual moduli space, when transversality conditions are satisfied, is provided by the Atyiah-Singer index formula (see [8], pp. 137):

$$\delta = 8(-c_2(E)) - 3(1 - b_1(X) + b_-(X)) \qquad (2.9)$$

where $b_1(X)$ is the first Betti number of X, $b_-(X)$ is the negative part of the second Betti number, and δ is obtained as the index of the elliptic complex (2.8).

Remark 2.21. *Let $\sigma\colon \mathcal{A} \to \mathcal{A} \times \Omega_-(\mathfrak{g}_E)$ be the map defined by $\sigma(A) = (A, {F_A}^-)$. By Proposition 2.19, σ is a $\mathcal{G}(\pi)$-equivariant map of infinite dimensional Hilbert spaces. We can identify the moduli space just defined with the space $(\sigma^{-1}(0)/\mathcal{G}, \pi)$.*

Let A be an arbitrary self-dual connection in $\mathcal{A}$. The **gauge slice** through A is given by T_A, the normal space to the orbit $\mathcal{G} \cdot A$ (see [14], pp. 48-49):

$$T_A = \{a \in \Omega^1(\mathfrak{g}_E) \mid {d_A}^* a = 0\} \tag{2.10}$$

The reduction of σ to a gauge slice through A is

$$\psi_A\colon T_A \to \Omega_-(\mathfrak{g}_E)$$
$$\psi_A(a) = {d^-}_A(a) + (a \wedge a)^- \tag{2.11}$$

Notice that T_A becomes a $\mathcal{G}(\pi)_A$-representation space by the result of Proposition 2.19. The derivative of ψ_A at A is the linear map ${d^-}_A$ restricted to T_A. ${d^-}_A$ is an elliptic operator and thus ψ_A is Fredholm. By a similar slice theorem as Theorem 0.11 (see [8], Proposition 4.2.9), we have that $(T_A/\mathcal{G}_A, \pi)$ is equivariantly diffeomorphic to a π-invariant neighbourhood $\mathcal{U}_{[A]} \subset \mathcal{B}$ The disadvantage of having to work with equivariant maps between infinite dimensional spaces is overcome by the Kuranishi method (see [14], Lemma 4.7):

Lemma 2.22. (Kuranishi's method) *Let U and V be G-Hilbert spaces and let $\psi\colon U \to V$ be a G-Fredholm map so that $\psi(0) = 0$. If $\delta = d\psi_0\colon TU_0 \to$*

TV_0 is the differential at the origin, then there are orthogonal splittings $U = \mathrm{Ker}(\delta) \oplus U'$, $V = \mathrm{Im}(\delta) \oplus V'$ and an equivariant map $F\colon U \to V'$, $F(0) = 0$, $dF_0 = 0$ such that $\psi = (\delta + F)(\alpha)$, where α is an equivariant diffeomorphism of U with $\alpha(0) = 0$.

Essential in [17] is the following fact:

Proposition 2.23. ([17], (1.11))
Let M and N be finite dimensional G-representations and let $f\colon M \to N$ be a map in G-general position with respect to $0 \in N$. Then we can perturb f to f' such that the representations $\mathrm{Ker}(df'_0)$ and $\mathrm{Coker}(df'_0)$ have no representations in common.

Then Lemma 2.22 and Proposition 2.23 prove

Proposition 2.24. *Let $Fred_G(U, V)$ be the space of G-Fredholm maps. Then the subset of the maps in $Fred_G(U, V)$ which are in G-equivariant general position with respect to $0 \in V$, is open and dense.*

Proposition 2.24 is the equivariant version of Sard-Smale theorem (see [8], pp. 145) and it makes the following definition possible:

Definition 2.25. *Let ψ be in $Fred_G(U, V)$ so that $\psi(0) = 0$. Let*

$$\phi\colon \mathrm{Ker}(d\psi_0) \to \mathrm{Coker}(d\psi_0)$$

be its finite dimensional reduction map: $\phi = pr \circ \psi|_{\mathrm{Ker}(d\psi_0)}$. Then ψ is in **general position** *with respect to $0 \in V$ if and only if ϕ is in general position with respect to $0 \in \mathrm{Coker}(d\psi_0)$.*

Let us note that through the regularity theorem of PDE's, since our PDE (2.7) is elliptic with real analytic coefficients, the solution $(\mathcal{M}, \pi)$, when non-empty, supports a real analytic structure and, according to [20], it is triangulable. In order to place $(\mathcal{M}, \pi)$ in equivariant general position, Hambleton and Lee developed in [17] a procedure so that one can equivariantly perturb σ, a $\mathcal{G}(\pi)$-equivariant map of infinite dimensional spaces, (see Remark 2.21) to an equivariant general position with the zero section. The map σ is not Fredholm. To place σ in general position means to place its slice reductions ψ_A, which are Fredholm maps, in general position with $0 \in \Omega_-(\mathfrak{g}_E)$ according to Definition 2.25. Consider $\mathcal{U}$ a locally finite $\mathcal{G}(\pi)$-equivariant open covering of a neighbourhood of $\sigma^{-1}(0) \subset \mathcal{A}$. Using a partition of unity associated with the covering U (see [8], pp. 143-144, 155), due to Proposition 2.24, one can perturb σ to an arbitrarily close (in the compact-open topology) equivariant map σ_ϵ sequentially, on each open set of $\mathcal{U}$, such that eventually σ_ϵ is in equivariant general position with respect to the zero section. The equivariant moduli space in general position that is thus obtained is $({\sigma_\epsilon}^{-1}(0)/\mathcal{G}, \pi)$. The Kuranishi method applied to $\psi = \psi_A$ implies obtaining general position with respect to 0 for the finite dimensional map $\phi_A\colon \operatorname{Ker}(d\psi_0) \to \operatorname{Coker}(d\psi_0)$, and we can use Proposition 2.13. From general elliptic theory, we have the orthogonal splittings in the Sobolev norms:

$$\Omega^1(\mathfrak{g}_E) = \operatorname{Ker} {d_A}^* \oplus \operatorname{Im} d_A, \quad \Omega^0(\mathfrak{g}_E) = \operatorname{Ker} d_A \oplus \operatorname{Im} {d_A}^* \tag{2.12}$$

Then, for a fixed connection A, we have

$$H^1_A = \operatorname{Ker}(d\phi)_0, \quad H^2_A = \operatorname{Coker}(d\phi)_0$$

where $H^1_A = \frac{\operatorname{Ker} d^-{}_A}{Imd_A}$, $H^2_A = \operatorname{Coker} d^-{}_A$, and $H^0_A =$ the Lie algebra of $\mathcal{G}_A$ are the cohomology groups of the $\mathcal{G}(\pi)_A$-equivariant elliptic complex (2.8) and thus we have an induced $\mathcal{G}(\pi)_A$-action on H^*_A, (see [8], pp. 135-139, and [17], pp. 27). Moreover

Proposition 2.26. *For any connection $A \in \mathcal{A}$, a neighbourhood of $[A]$ in $(\mathcal{M}, \pi)$ is modelled on $({\phi_A}^{-1}(0)/\mathcal{G}_A, \pi_A)$, where $\phi_A \colon H^1_A \to H^2_A$ is the $\mathcal{G}(\pi)_A$-general position map of finite dimensional spaces obtained as in* Definition 2.25 *by applying Kuranishi's method to a $\mathcal{G}(\pi)_A$-equivariant perturbation of the map* (2.11).

In view of Proposition 2.18 and Proposition 2.26 we have:

Theorem 2.27. *The π-equivariant moduli space $(\mathcal{M}, \pi)$ is a Whitney stratified space with open manifold strata $\mathcal{M}^*_{(\pi')}$, where π' is a subgroup of π and $\mathcal{M}^* \subset \mathcal{M}$ is the set of irreducible connections. Each such stratum has a smoothly locally trivial equivariant cone bundle neighbourhood in $(\mathcal{M}, \pi)$ (see [16] pp. 720).*

By Proposition 2.17 the map ϕ_A in Proposition 2.26, when in general position, is stratumwise transverse to $0 \in H^2_A$ and one obtains (see also Proposition 3.9):

Proposition 2.28. *The formal dimension of $\mathcal{M}_{\pi'}$ is the index of the π'-fixed fundamental complex*

$$\Omega^0(\mathfrak{g}_E)^{\pi'} \xrightarrow{d_A} \Omega^1(\mathfrak{g}_E)^{\pi'} \xrightarrow{d^-{}_A} \Omega_-(\mathfrak{g}_E)^{\pi'}$$

for A a connection in $\mathcal{M}_{\pi'}$.

Even when in general position, the moduli space $(\mathcal{M}, \pi)$ just obtained may not be compact. In the non-equivariant case the Uhlenbeck compactification method consists in taking the closure of $\mathcal{M}_{\pi'}$ in the space

$I\mathcal{M}_k = \mathcal{M} \cup \bigcup_{l=1}^{k}(\mathcal{M}_{k-l} \times Sym^l(X)$, where $Sym^l(X) = \underbrace{X \times \cdots \times X}_{l \text{ times}} / \sim$, with $(x_1, \ldots, x_l) \sim (y_1, \ldots, y_l)$ if there exists a permutation $s \in \Sigma_l$ such that $s(x_1, \ldots, x_l) = (y_1, \ldots, y_l)$. The elements $([A], (x_1, \ldots, x_l))$, $[A] \in \mathcal{M}_{k-l}$, of $I\mathcal{M}_k$ are called **ideal connections** and to each such an element is assigned a **curvature density** given by

$$|F_A|^2 + 8\pi^2 \sum_{r=1}^{l} \delta_{x_r},$$

where δ_{x_r} denotes the Dirac distribution in x_r. Then the topology of $I\mathcal{M}_k$ is given by a certain convergence as measures for the curvature densities of a sequence of ideal connections (see [8], 4.4.1). The main result is:

Theorem 2.29. *The closure of $\mathcal{M}_k$ in $I\mathcal{M}_k$ is compact.*

In [16] it was shown that one can equivariantly perturb $\mathcal{M}_{\pi'}$ such that the closure $\overline{\mathcal{M}_k}$ has a natural π-action and is compact. We briefly describe the process here as well by restricting ourselves to the case of our interest: $c_2(E) = -1$ and $X = \#\mathbb{C}P^2$. The group π may still be any finite group with the required action.

In the non-equivariant setting one constructs a collar in $\mathcal{M}$ which is diffeomorphic to $X \times (0, 1)$. By adding to $\mathcal{M}$ the closure of this collar, the moduli space becomes compact. The collar is a subset of the set of concentrated connections $\mathcal{CC}$ which is defined in ([14], pp. 129), as a subset of the set of connections $[A]$ whose energies are bounded: $\int_X \omega(A) \leq 9\pi^2$, $\omega(A) = |F_A|^2 \star 1$. In [14], Thm. 8.31, it is shown that $\mathcal{M} \cap \mathcal{CC}$ is non-empty. By [14], Thm. 8.28., the system of nonlinear equations in $\lambda \in (0, \infty)$ and

$x \in X$, parameterized by $\mathcal{CC}$ and depending on a fixed metric on X

$$\begin{aligned} R(\lambda, x, \omega(A)) &= 4\pi^2 \\ \frac{\partial R}{\partial x}(\lambda, x, \omega(A) &= 0 \end{aligned} \tag{2.13}$$

has a unique solution $(x(A), \lambda(A))$ depending smoothly on $A \in \mathcal{CC}$ and on the metric, i.e. there is a well defined map

$$(x, \lambda) \colon \mathcal{CC} \to X \times (0, \infty). \tag{2.14}$$

Also, for some λ_0, we have $H_A^2 = 0$ if $\lambda(A) < \lambda_0$. Then $\mathcal{M}_{\lambda_0} = \lambda^{-1}(0, \lambda_0)$ is smooth. By [14], Thm. 8.36. we know that $\mathcal{M} \setminus \mathcal{M}_{\lambda_0}$ is compact.

For the equivariant setting, in [6],(1.8), it is proved that the system 2.13 is π-invariant and thus the map (x, λ) becomes π-equivariant with respect to the product action on $(X \times (0, \lambda_0), \pi)$. The fact that $H_A^2 = 0$ for $A \in \mathcal{M}_{\lambda_0}$ ensures that the section σ (see Remark 2.21) is in general position (actually equivariantly transverse) with the zero section on the part of $\mathcal{A}$ that projects over $\mathcal{M}_{\lambda_0}$. We equivariantly perturb σ on the complementary part as above and we obtain an equivariant compact moduli space with a collar equivariantly diffeomorphic to $(X \times (0, \lambda_0), \pi)$ with the product action. The inverse map of (x, λ) is denoted by τ and is called Taubes embedding. The immediate consequences of this construction that we are going to use in the sequel are:

Corollary 2.30. ([16], Cor. 3.2.) *Let π' be a subgroup of π and let S_x be the normal slice representation of π' at $x \in X$. Then there exists a* 5-*dimensional manifold $\mathcal{H} \subset \mathcal{M}^*$ such that a connected component $\mathcal{C}$ of $\mathcal{H}_{\pi'}$ has normal slice representation S_x and $\mathcal{C} \cap \mathcal{M}_{\lambda_0} = X_{\pi'} \times (0, \lambda_0)$.*

and

Corollary 2.31. ([16], Cor 3.3.) *If x is an isolated fixed point in X^π such that its slice representation S_x is not equivalent via an orientation reversing isomorphism to the slice representation S_y at any other fixed point y, then there exist a π-fixed path $\gamma_x\colon [0,1] \to \mathcal{M}$, $\gamma_x((0,1)) \in \mathcal{M}^*$, such that $\gamma_x(0) = x$, $\gamma_x \cap \mathcal{M}_{\lambda_0} = \{x\} \times (0, \lambda_0)$ and $\gamma_x(1)$ is a reducible connection.*

Illman answered in general a question asked by Hilbert (see [19]): When the smooth action of a Lie group on a smooth manifold X can be made into a real analytic action on a compatible analytic structure on X?

Definition 2.32. *Let G be an arbitrary Lie group and let M be a locally compact G-space. Then the action of G on X is* **proper** *if the map $G \times M \to M \times M$, $(g, x) \mapsto (gx, x)$ is proper or, equivalently,for any $N \subset M$ compact, the set $G_{[N]} = \{g \in G \mid gN \cap N \neq \emptyset\}$ is compact. The action is* **Cartan** *if for any point of X there is a compact neighbourhood Y such that $G_{[Y]}$ is compact.*

Notice that the action of any finite group π is proper on a locally compact space.

Theorem 2.33 ([19], Thm. 7.1.). *If M is a Cartan G-manifold, then there exist a real analytic structure β on M such that the action of G on M^β is real analytic.*

This generalizes the earlier result of Palais ([26]) for G a compact Lie group. In our case, Palais' result still suffices to state:

Proposition 2.34. *The π action on $\mathcal{M}$ is effective, i.e. if a subgroup π' of π acts trivially on an open set of $\mathcal{M}^*$, then $\pi' = 0$.*

This is due to the fact that, by Theorem 2.33, we can choose a real analytic equivariant metric on X which induces a real analytic structure on $\mathcal{M}^*$. We know that the non-equivariant moduli space is orientable (see [8], Chp.5). For $(\mathcal{M}, \pi)$ we have:

Lemma 2.35. ([16], Lemma 8) *Let $\pi = C_p$ with p an odd prime. If $\mathcal{C} \subset \mathcal{M}^*{}_\pi$ is a connected component, then $\mathcal{C}$ is an orientable manifold.*

Remark 2.36. *The moduli space might be disconnected but there is a connected component whose closure contains the Taubes boundary X and the set of reducible connections. From now on $\mathcal{M}$ will denote that component. Also one can see that the closure of $\mathcal{M}$ is the same as the closure of $\mathcal{M}^*$. We'll employ the latter notation to emphasize that the reducible connections appear in the closure of the manifold of irreducible connections $\mathcal{M}^*$.*

Chapter 3

Reducible Connections

In this chapter we denote by π the cyclic group of order m, with m an odd positive integer. The group π is considered to act smoothly and effectively on $X = \overset{n}{\underset{1}{\#}}\mathbb{C}P^2$. The connected sum X is simply connected. We concentrate on the case $n > 1$. Also, because we are making use of the π equivariant moduli space presented in Chapter 2, π acts on X by isometries relative to a real analytic metric on X. Let's recall also that the action of π preserves the orientation on X, since m is odd.

Definition 3.1. *If X and Y are G-spaces, A is a G-invariant closed subset of X, and $f\colon X \longrightarrow Y$ is an equivariant map, then we say that the space denoted by $X \cup_f Y$ and equal to $X \cup Y/\sim$, where $a \sim f(a)$ for all $a \in A$, is obtained by* **equivariant attaching** *via the map f.*

Notice that the equivariant attaching process provides a way to construct new G-spaces from given ones.

Definition 3.2. *Let X and Y be connected G-manifolds of the same dimension and let x and y be fixed points in X^G and Y^G, respectively. If the* *tangential representations of the group G at the fixed points on TX_x and TY_y are equivalent to a representation V of G, and there exist the G-equivariant embeddings*

$$f_x \colon (\overline{D(V)}, 0) \longrightarrow (X, x) \quad ,$$

$$f_y \colon (\overline{D(V)}, 0) \longrightarrow (Y, y) \quad ,$$

*where $D(V)$ denotes an open disk in V centered in 0 and f_y reverses orientation, then the G-***equivariant connected sum** *of X and Y is defined by the equivariant attaching*

$$X\#Y = (X \setminus D(V)) \cup_f (Y \setminus D(V)) \ ,$$

with f identifying f_x and f_y on the sphere boundaries of the two $D(V)$'s.

We present first a short account of results and notions about finite group actions on $\mathbb{C}P^2$ or on a homology $\mathbb{C}P^2$ that we are going to use. We can regard $\mathbb{C}P^2 = \{[z_0, z_1, z_2] \mid z_i \in \mathbb{C}, [t_0, t_1, t_2] = [z_0, z_1, z_2] \Leftrightarrow \exists z \in \mathbb{C} \setminus \{0\}$ such that $z_i = zt_i, i \in \{0, 1, 2\}\} \subset \mathbb{C}^3$. Then we define

Definition 3.3. *Let X be a smooth manifold acted locally linearly by the finite group G. The* **fixed point data** *of the action consists in the fixed point set X^G together with the tangential representations of G at every x in X^G.*

Definition 3.4. *The finite group G* **acts linearly** *on $\mathbb{C}P^2$ if the action is induced from a faithful complex representation $\pi \longrightarrow Gl_3(\mathbb{C})$.*

Definition 3.5. *Given the manifold $X = \mathbb{C}P^2$ acted on by a finite cyclic group G and $x \in X^G$ we say that the fixed point data of x is linear if the tangential representation at x is the same as the one of a linear action.*

A description of the linear fixed point data is offered by the **rotation numbers**: Let $G = \pi = C_m$ and let t be a generator of π. Then, in a proper complex basis, the associated matrix of $t\colon \mathbb{C}P^2 \to \mathbb{C}P^2$ describes the action by $t[z_0, z_1, z_2] = [z_0, \xi^a z_1, \xi^b z_2]$, where $\xi = \exp^{2\pi i/m}$ and a, b are integers $\mod m$. Let's observe that $t[z_0, z_1, z_2] = [\xi^{-a} z_0, z_1, \xi^{b-a} z_2] = [\xi^{-b} z_0, \xi^{a-b} z_1, z_2]$. Thus the fixed point set of the action can be:

1. three isolated fixed points represented by $[1,0,0]$, $[0,1,0]$ and $[0,0,1]$, when $a \neq b$;
2. one isolated fixed point represented by $[1,0,0]$ and a 2-sphere represented by $[0, z_1, z_2]$, when $a = b$.

The pairs of rotation numbers (a, b), $(-a, b-a)$, $(-b, a-b)$ describe the linear fixed point data, each pair defining the complex representation at a fixed point. If one rotation number is zero then we have a fixed sphere. If both rotation numbers were zero then $\mathbb{C}P^2$ would be fixed by the group G and the action would be trivial, i. e. non-effective, in contradiction to our assumption on the action of π in this chapter. In fact we have:

Proposition 3.6. *The linear action of $\pi = C_m = \langle t \rangle$ on $\mathbb{C}P^2$ is effective if and only if the representation of the generator* $t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \xi^a & 0 \\ 0 & 0 & \xi^b \end{pmatrix}$ *is such that $g.c.d(g.c.d.(a, m), g.c.d(b, m)) = 1$, where $g.c.d(x, y)$ denotes the greatest common divisor of the integers x and y.*

Proof. Let's assume that there exists π' a non-trivial subgroup of π that acts trivially. This is equivalent to: there is k a divisor of m, such that t^k

generates $\pi' = C_{m/k}$ and t^k acts trivially on $\mathbb{C}P^2$. Since, for any $x \neq 0$, ξ^x acts freely on $\mathbb{C} \setminus \{0\}$, we must have $m|ak$ and $m|bk$. Then $mm_1 = ak$, $mm_2 = bk$, $m = kk_1$, and $k_1m_1 = a$, $k_1m_2 = b$. Thus $k_1|g.c.d.(a,m)$, $k_1|g.c.d.(b,m)$ and thus $k_1 = 1$. But $k_1 = m/k$ is the order of π', a contradiction with the assumption π' non-trivial subgroup. ■

If the tangential representation at one fixed point is given by the pair (a, b), then all the other pairs of rotation numbers giving an equivalent representation are (b, a), $(-a, -b)$ and $(-b, -a)$.

The P. A. Smith theory shows that if $G = C_p$, with p prime, acts smoothly on $\mathbb{C}P^2$, then the fixed point set consists of either three isolated fixed points or a sphere and one isolated fixed point (compare [7], VII., Thm. 3.2.). Wilczynski in [35] and Hambleton - Lee in [15] have shown that the only finite groups that can act locally linearly on $\mathbb{C}P^2$ are the groups that act linearly. Wilczynski and Edmonds - Ewing (see [10]) proved that the fixed point data of the action of $G = C_p$ on $\mathbb{C}P^2$ is linear too. The following theorem which we are going to prove in Chapter 5, is central to our discussion:

Theorem 3.7. *The fixed-point data and tangential isotropy representations of a smooth π-action on X are the same as those of a π-equivariant connected sum of linear π-actions on the components $\mathbb{C}P^2$.*

The same result, in the case when the induced π action in the homology of X is trivial was already proved in [16] and we are following the arguments presented there closely.

3.1 The equivariant structure of the set of reducible connections

By Proposition 1.11 we have an induced action by permutations in homology. We remark that the hypothesis that π acts smoothly is crucial in Theorem 3.7 (see [16], Thm.21). As in [16], the smooth π-action enables us to give a proof of Theorem 3.7 using the π-equivariant self-dual Yang-Mills moduli space of a π-equivariant principal $SU(2)$-bundle of instanton number $c_2(P) = -1$. For the next results we consider X and π in the more general assumptions of Chapter 2.

It is classical that the equivalence classes of $U(1)$-complex bundles over the simply connected manifold X are in bijection to the space of homotopy classes of maps $[X, BU(1)]$ which is isomorphic to $H^2(X;\mathbb{Z})$. In particular, to any class in $H^2(X;\mathbb{Z})$ corresponds an equivalence class of a complex line bundle over X. Let $A \in \mathcal{A}$ be a reducible connection. Then its gauge isotropy group $\mathcal{G}_A$ is the centralizer in $SU(2)$ of the holonomy group of A which is S^1. This implies $\mathcal{G}_A = S^1$, (see [14], Thm. 3.1.), and

$$E = L \oplus L^{-1}, \tag{3.1}$$

where L is an S^1-bundle and L^{-1} is L with the reverse orientation. Consequently

$$\mathfrak{g}_E \simeq \tau \oplus L^{\otimes 2}, \tag{3.2}$$

where τ is the trivial real line bundle over X and $L^{\otimes 2}$ is the bundle obtained from L by tensoring it with itself. Following ([14], pp. 82), the complex

(2.8) splits

$$\begin{array}{ccccc} \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d^-} & \Omega_- \\ \oplus & & \oplus & & \oplus \\ \Omega^0(L^{\otimes 2}) & \xrightarrow{d_A} & \Omega^1(L^{\otimes 2}) & \xrightarrow{d^-{}_A} & \Omega_-(L^{\otimes 2}) \end{array} \tag{3.3}$$

and accordingly we have also an orthogonal splitting in cohomology:

$$\begin{aligned} H^0_A(\mathfrak{g}_E) &\simeq \mathcal{H}^0(X), \\ H^1_A(\mathfrak{g}_E) &\simeq H^1_A(L^{\otimes 2}) \oplus \mathcal{H}^1(X) \\ H^2_A(\mathfrak{g}_E) &\simeq H^2_A(L^{\otimes 2}) \oplus \mathcal{H}^2_-(X) \end{aligned} \tag{3.4}$$

where $\mathcal{H}^*(X)$ is the cohomology of the de Rham complex which is the first summand complex in (3.3). Since A is reducible we have $\mathcal{G}_A = S^1$ isotropy group of $A \in \mathcal{A}$ and an action of S^1 on the $\mathcal{G}$-slice T_A and on $\Omega^*(\mathfrak{g}_E)$ such that the map ψ_A of (2.11) is S^1-equivariant. Then around A, the moduli space is modelled by

$$\phi_A{}^{-1}(0)/S^1 \ , \tag{3.5}$$

where $\phi_A \colon H^1_A(\mathfrak{g}_E) \to H^2_A(\mathfrak{g}_E)$ is the finite dimensional S^1-equivariant map obtained out of ψ_A by Kuranishi method: i.e.

$$\phi_A \colon H^1_A(L^{\otimes 2}) \oplus \mathcal{H}^1(X) \to H^2_A(L^{\otimes 2}) \oplus \mathcal{H}^2_-(X) \tag{3.6}$$

The action of π on $\mathcal{B}$ induces a stratification of $\mathcal{B}_{(S^1)}$.

Proposition 3.8. *The group $\mathcal{G}_A = S^1$ acts trivially on $\mathcal{H}^1(X)$ and $\mathcal{H}^2_-(X)$ and it acts freely except in the origin (i.e. by complex multiplication) on the finite dimensional complex spaces $H^1_A(L^{\otimes 2})$ and $H^2_A(L^{\otimes 2})$. If the complex dimensions of $H^1_A(L^{\otimes 2})$ and $H^2_A(L^{\otimes 2})$ are q and p, respectively, and if $\mathcal{H}^1(X) = 0$ and $\mathcal{H}^2_-(X) = 0$, then $q - p = 3$*

Proof. (see [14], pp. 67-68). ∎

Let $\mathcal{B}_{red}$ be the subset classes of reducible connections. The local structure in $[A] \in \mathcal{B}_{red}$ of $\mathcal{M} \cap \mathcal{B}_{red}$ is given by the zero set of the Fredholm map $(\psi_A)^{\mathcal{G}_A}$ which is the restriction of the map ψ_A to the $\mathcal{G}_A$-fixed set in the slice T_A (see also Remark 0.17). Then the corresponding restriction of the map ϕ_A at (3.6) is

$$(\phi_A)^{\mathcal{G}_A} \colon \mathcal{H}^1(X) \to \mathcal{H}^2_-(X) \tag{3.7}$$

and we obtain a new feature of the equivariant moduli space $(\mathcal{M}, \pi)$:

Proposition 3.9. *The gauge classes of reducible connections form a π-equivariant substratified space $\mathcal{M}_{red} = \mathcal{M} \backslash \mathcal{M}^*$ in $(\mathcal{M}, \pi)$. For any reducible SD-connection A and any subgroup $\pi' \subset \pi_A$ the dimension of the fixed point set ${\mathcal{M}^{\pi'}}_{red}$ is given by*

$$\delta = \dim \mathcal{H}^1(X)^{\pi'} - \dim \mathcal{H}^2_-(X)^{\pi'}$$

with ${\mathcal{M}^{\pi'}}_{red} = \emptyset$ if $\delta < 0$.

Proposition 3.10. ([17], Prop. 2.14) *If $\mathcal{H}^1(X) = \mathcal{H}^2_-(X) = 0$, then we can perturb $\mathcal{M}$ such that $\mathcal{M}_{red}$ is a union of isolated points and a neighbourhood of such a point A has a cone structure obtained by factoring the zero set of $\mathcal{G}(\pi)_A$-general position map by $\mathcal{G}_A = S^1$. In addition the dimension of a π_A-stratum is determined by the index of the subcomplex*

$$\Omega^0(L^{\otimes 2})^{\pi_A} \xrightarrow{d_A} \Omega^1(L^{\otimes 2})^{\pi_A} \xrightarrow{d^-{}_A} \Omega_-(L^{\otimes 2})^{\pi_A}$$

We return to the hypothesis on X and π made at the beginning of this chapter.

Lemma 3.11. *Let $A \in \mathcal{B}$ be a reducible connection. Then A is a self-dual connection.*

Proof. This is a consequence of X being positive definite ($b^-(X) = 0$). Let $E = L \oplus L^{-1}$ be a corresponding bundle reduction. Then A is a connection on L, i.e. $A \in \Omega^1(\mathfrak{g}_E) = \Omega^1(X) \oplus \Omega^1(\mathfrak{g}_L)$. We can therefore identify the curvature F_A with $2\pi i\alpha$, where α is a real 2-form on X . The Bianchi identity $d_A F_A = 0$ ensures that α is closed and therefore it represents a cohomology class $[\alpha] \in H^2(X, \mathbb{R})$. Through the Hodge and DeRham theorems $\Omega^2(X) \simeq H^2(X,\mathbb{R}) \oplus Imd \oplus Imd^*$, where $d\colon \Omega^1(X) \to \Omega^2(X)$ is the differential operator on real forms on X, we can see that $H^2(X,\mathbb{R}) = \mathrm{Ker}(dd^* + d^*d)$ is invariant under the $*$ operator which interchanges $\mathrm{Ker}\, d$ and $\mathrm{Ker}\, d^*$. This implies that, for a fixed metric on X, we have the decomposition $H^2(X,\mathbb{R}) = \mathcal{H}_+ \oplus \mathcal{H}_-$ in self-dual and antiselfdual harmonic forms. But $0 = b^-(X) = \dim \mathcal{H}_-$. Thus $[\alpha]$ is self-dual and therefore A is self-dual. ■

Lemma 3.12. (compare [8], Prop. 4.1.15) *The set of the classes $[A] \in \mathcal{M}$ of reducible SD connections A is in bijection with the set $\{\pm c \mid \lambda(c, c) = 1\} \subset H^2(X)$, where λ is the quadratic intersection form of X.*

Proof. Let A be reducible and let $E = L \oplus L^{-1}$ be its corresponding bundle reduction. The choice of the unordered pair of the S^1-bundles (L, L^{-1}) is unique modulo the automorphisms of $\mathcal{G}_A$. Then $[A]$ corresponds uniquely to the chosen representatives (L, L^{-1}), i.e. to $\pm c$, where $c = c_1(L) \in H^2(X)$ is the classifying Chern class of L. But the bundle splitting implies $c_2(E) = -c^2$. Since in our case $c_2(E) = -1$ and since $H^2(X)$ classifies the S^1-bundles on X, by Lemma 3.11, the result follows. ■

For the equivariant case we have: Let A be a reducible connection and let π_A be the isotropy group of $[A] \in (\mathcal{B}, \pi)$. Let $\pm c \in H^2(X)$ be the unique pair of cohomology classes corresponding to $[A]$ by Lemma 3.12. Then

Lemma 3.13. *There exists a splitting $E = L \oplus L^{-1}$ as in (3.1), i. e. given by the reduction coming from the holonomy of A, so that L is a π_A-equivariant S^1-bundle with the first Chern class $c_1(L) = c$.*

Corollary 3.14. *If A is a reducible connection, then the action of $\mathcal{G}(\pi)_A$ on the gauge slice T_A is the product action $S^1 \times \pi_A$.*

Proof. We know that $\mathcal{G}_A \simeq S^1$ is a normal subgroup in $\mathcal{G}(\pi)_A$ (compare (2.5)) and $\pi_A = \mathcal{G}(\pi)_A/\mathcal{G}_A$. Because $\mathcal{G}_A$ is abelian, there exists $\phi\colon \pi_A \to Aut(\mathcal{G}_A)$ defining map as in Definition 0.30and the complex line bundle L is a $(\pi_A, \phi, \mathcal{G}_A)$-bundle in the sense of Definition 0.31. By Lemma 3.13, L is a π_A-equivariant bundle, i.e. ϕ is trivial. Hence $\mathcal{G}(\pi_A)$ is a product. ∎

The following lemma is the main result in [22]:

Lemma 3.15 (Lashof, May, Segal). *Let π be a compact Lie group and let G be a compact abelian Lie group. Then the space of π-homotopy classes of π-equivariant maps* $[X, B(\pi, G)]^\pi$ (see Theorem 0.42), which classifies the π-equivariant G-bundles over the G-manifold X, is isomorphic to $[E\pi \times_\pi X, BG]$.

Proof of Lemma. 3.13 Let $\pi' = \pi_A$ be the isotropy group of $[A] \in (\mathcal{B}, \pi)$. Since S^1 is a compact abelian Lie group, by Lemma 3.15 we have $[X, BS^1]^{\pi'} \simeq [E\pi' \times_{\pi'} X, BS^1]$ and, by the classification of S^1 bundles, $[X, BS^1]^{\pi'} \simeq H^2(E\pi' \times_{\pi'} X, \mathbb{Z})$. Therefore the π'-equivariant S^1-bundles over X are classified by the π'- Borel cohomology group of X, $H^2(E\pi' \times_{\pi'} X, \mathbb{Z})$. According to Lemma 3.12 there is a splitting $E = \mathcal{L} \oplus \mathcal{L}^{-1}$ such that $c_1(\mathcal{L}) = c$ and $g^*\mathcal{L} \simeq \mathcal{L}$ as S^1-bundles for any $g \in \pi'$. Therefore $c \in H^2(X)^{\pi'}$. We have to show that there is a π'-equivariant S^1-bundle L with $c_1(L) = c$, i.e. the map

$$H^2(E\pi' \times_{\pi'} X) \xrightarrow{i^*} H^2(X)^{\pi'}$$

is surjective, where i^* is the induced map in cohomology by the natural inclusion $i\colon X \xrightarrow{i} E\pi' \times_{\pi'} X$.

Considering the Lerray-Serre spectral sequence ${E_*}^{i,j}$ of the fibration $E\pi' \times_{\pi'} X \to B\pi'$ we have

$${E_2}^{i,j} = H^i(\pi'; \{H^j(X;\mathbb{Z})\}).$$

From group cohomology, ${E_2}^{0,2} = H^0(\pi'; H^2(X;\mathbb{Z}) = H^2(X;\mathbb{Z})^{\pi'}$ and, because π' is cyclic, $H^3(\pi') = 0$. Since ${E_*}^{i,j}$ converges to $H^*(E\pi' \times_{\pi'} X)$, we are done if we show that $E_2^{0,2}$ survives to E_∞. Let's observe that the differential $d_2\colon E_2^{0,2} \to E_2^{2,1}$ is zero because X simply connected implies $E_2^{2,1} = 0$ and then $d_3\colon E_3^{0,2} \to E_3^{3,0}$. But $E_3^{3,0} = H^3(\pi') = 0$. Therefore all the possible differentials d_r, $r \geq 2$ vanish which means that ${E_2}^{0,2} = H^2(X;\mathbb{Z})^{\pi'}$ survives to E_∞. ■

Lemma 3.12 proves more: the set of reducible SD connections is an invariant of the bundle $E \to X$ and, since $\lambda = n\langle 1\rangle$, we have a finite number of classes of reducible SD connections in $\mathcal{B}$, one for each element e_i of the standard $\mathbb{Z}$-basis of $H^2(X)$. We have showed (Chp.1, (1.9)) that

$$H_2(X) = \bigoplus_k \mathbb{Z}[\pi/\pi_k] ,$$

where $\{\pi_k\}_k$ are the isotropy subgroups of the action of π on $H_2(X)$. By 3.13 and by identifying the Poincaré dual spaces $H_2(X)$ and $H^2(X)$ we can now summarize:

Corollary 3.16. *Let $\{e_1, e_2, \dots, e_n\}$ be the set of the standard $\mathbb{Z}$-basis of the free $\mathbb{Z}$-module $H_2(X)$ and let $\{[A_1], [A_2], \dots, [A_n]\}$ be the set of the classes of reducible connections. Then we can rearrange them if necessary such that the map $e_i \mapsto A_i$ is a π-equivariant bijection.*

Remark 3.17. *The isotropy types in the stratum of reducible connections in $(\mathcal{M}, \pi)$ might not be all the isotropy types in $(H^2(X;\mathbb{Z}), \pi)$. For example, if $\pi = \mathbb{Z}/p$ acts on $H^2(X;\mathbb{Z})$ by permuting the standard basis $\{e_1, \dots, e_p\}$, then the isotropy of each e_i is $\{1\}$ but the isotropy of $e_1 + \dots + e_p$ is π.*

3.2 Cones over linear actions on $\mathbb{C}P^2$'s

The remainder of this chapter is dedicated to proving the following theorem which is the analogue of [16], Thm.15.:

Theorem 3.18. *If π acts smoothly on X then the classes of reducible connections form a discrete singular subset of the equivariant moduli space $(\mathcal{M}, \pi)$ and each reducible connection class $[A]$ has a π_A-invariant neighbourhood $\mathcal{N}_{[A]}$ in the equivariant moduli space with the properties:*

1. $\mathcal{N}_{[A]}$ is π_A-homeomorphic to the cone of a linear action over $\mathbb{C}P^2$.

2. $\mathcal{N}_{[A]}$ is smooth away from the corresponding cone point.

3. $\mathcal{N}_{[A]}$ and the cone of a linear π_A-action over $\mathbb{C}P^2$ are equivariantly diffeomorphic away from the cone point.

Let $\mathcal{N}^* = \mathcal{N} \cap \mathcal{B}^*$, where $\mathcal{N} = T/\Gamma_A$ is a neighbourhood of a reducible connection A in the non-equivariant case ($\pi = 0$).

Lemma 3.19. ([8], Lemma 5.1.18) *The weak homotopy type of* $\mathcal{N}^*$ *is that of* $\mathbb{C}P^\infty$.

Proposition 3.20. ([14], Thm. 4.11 and Cor. 4.10) If $b^-(X) = 0$ (our case)and $E \to X$ is a non-trivial $SU(2)$-bundle of second Chern number -1,then there exists a perturbation of $\mathcal{M}$ such that the cohomology groups H_A^2 are trivial for all A reducible SD-connections on E, and a neighbourhood of A in the SD-moduli space $\mathcal{M}$ is homeomorphic to an open cone over $\mathbb{C}P^d$, and diffeomorphic off the vertex point, where $d = \frac{1}{2}(\delta - 1)$ and the virtual dimension of the stratum $\mathcal{M}^*$ in $\mathcal{M}$ is $\delta = 8(-c_2(E)) - 3$.

In our case $\delta = 5$, $d = 2$ and Prop. 3.20 proves Theorem 3.18 for any A reducible with $\pi_A = 0$, if any, the treatment being the same as in the non-equivariant case. It is left to show Theorem 3.18 for $A \in \mathcal{A}_{red}$ with $\pi_A \neq 0$. Let M be a stratified space (Definition 2.1) with finitely many disjoint strata M_i, $i = 0, \dots, l$, $M = \bigsqcup_{i=0,\dots,l} M_i$ such that each stratum M_i is a manifold modelled by a Hilbert space V_i. Let x be a point in $M_j \subset M$. Then there exists a manifold chart h such that $h \colon \overline{D}(0, \epsilon) \subset V_0 \to M$ is a homeomorphism on its image and such that $h(0) = x$, where $D(0, \epsilon)$ is the disk in $V_0 = V$ centered in the origin and of radius ϵ.

Definition 3.21. *The* **link of the singularity** $x \in M_j$ *in the stratum* M_i *is*

$$lk_\epsilon(x) = h(\partial D(0, \epsilon)) \cap M_i$$

.

According to Theorem 2.27, $(\mathcal{M}, \pi)$ is a Whitney stratified space with respect to the action of π and, for any π' isotropy subgroup, $\mathcal{M}^*_{\pi'}$ is an open manifold stratum. Let $[A]$ be the gauge class of a reducible SD-connection whose isotropy group π_A is non-trivial. For what follows we need to describe the link of $[A]$ in the closure of the free stratum of $(\mathcal{M}^*, \pi)$.

Let $\phi_A\colon H^1_A(\mathfrak{g}_E) \to H^2{}_A(\mathfrak{g}_E)$ be the $\mathcal{G}(\pi)_A$-equivariant chart that describes the moduli space around A. Because X is a simply connected positive definite manifold $\mathcal{H}^1(X) = \mathcal{H}^2_-(X) = 0$ and by (3.4), $\phi_A\colon H^1_A(L^{\otimes 2}) \to H^1_A(L^{\otimes 2})$. By Corollary 3.14, $\mathcal{G}(\pi)_A = \mathcal{G}_A \times \pi_A$. Thus ϕ_A is $S^1 \times \pi_A$-equivariant.

We denote $V = H^1_A(L^{\otimes 2})$, $W = H^2_A(L^{\otimes 2})$, $\phi = \phi_A$ and we place ϕ in $S^1 \times \pi_A$-equivariant general position with respect to $0 \in W$. Then (Proposition 2.26) a neighbourhood of $[A]$ in $(\mathcal{M}, \pi)$ is modelled on $(\phi^{-1}(0)/S^1, \pi_A)$ and, because equivariant general position implies stratumwise transversality, we have

$$\dim \phi^{-1}(0)_{\pi'_A} = \dim V_{\pi'_A} - \dim W^{\pi'_A}$$

for any $\pi_A{}'$ subgroup of π_A when $\phi^{-1}(0) \neq \emptyset$. For $\epsilon > 0$ let's consider $\phi^{-1}(0)_\epsilon = \{a \in \phi^{-1}(0) \mid \|a\| = \epsilon\}$. Because $[A]$ is isolated in $\mathcal{B}$, we have $\phi^{-1}(0)_\epsilon/S^1 \subset \mathcal{M}^*$ for a small enough ϵ. Then the link $lk(A)$ of $[A]$ in the closure of the free stratum of $(\mathcal{M}^*, \pi)$ is the intersection between $\phi^{-1}(0)_\epsilon/S^1$ and the closure of the free stratum of $(\mathcal{M}^*, \pi)$.

Let π be a cyclic group of odd order m. Let V and W be finite dimensional complex representations of π. Then S^1 acts as multiplication with scalars on V and W. Obviously the action of S^1 is free except at the origins which are fixed. Let $\chi_0, \ldots, \chi_{m-1}$ be the set of irreducible characters of the π-action. Then (Proposition 0.25) the character of the representation V is $\chi(V) = \sum m_k \chi_k$ and

$$V = \bigoplus_{k=0,\ldots,m-1} V(\chi_k)$$

where $V(\chi_k) = C(\chi_k)^{m_k}$, $C(\chi_k)$ is the degree two complex irreducible representation of character χ_k (see Proposition 0.28), and $m_k = (\chi_k, \chi(V))$ is the multiplicity. In similar notation, $W = \bigoplus_{k=0,\dots,m-1} W(\chi_k)$, $W(\chi_k) = C(\chi_k)^{l_k}$, $l_k = (\chi_k, \chi(W))$.

Let g be a generator of π_A. Then choose the representatives $\chi_k = \chi_k(g) = e^{2\pi_A ik/n}$, $i^2 = -1$. Then the representation $C(\chi_k)$ is given by $gv = \chi_k v$. If $v \neq 0$ in $V(\chi_k)$, then ${\chi_k}^{-1}gv = v$. It follows that the isotropy group of $v \in V(\chi_k)$ is $(\pi)_v = \pi(\chi_k) = \langle {\chi_k}^{-1} g\rangle$ and therefore $\pi(\chi_k, \chi_{k'}) = \pi(\chi_k) \cap \pi(\chi_{k'})$ is the isotropy of $v \in V(\chi_k) \oplus V(\chi_{k'})$. Then we have the isotropy structure of V:

Proposition 3.22. ([17], pp. 22)

$$V^{\pi(\chi_{k_1},\dots,\chi_{k_r})} = \bigoplus_{j=1,\dots,r} V(\chi_{k_j})$$

$$V_{\pi(\chi_{k_1},\dots,\chi_{k_r})} = \bigoplus_{j=1,\dots,r} V(\chi_{k_j}) - \bigcup \{V(\chi_{k_j}), \pi(\chi_{k_1}, \dots, \chi_{k_r}) \subsetneq \pi(\chi_{k_j})\}$$

and in the notations above

Proposition 3.23.

$$\phi^{-1}(0)^{\pi_A} = 0$$

$$\phi^{-1}(0)_{\pi(\chi_{k_1},\dots,\chi_{k_r})} =$$

$$= \begin{cases} \emptyset, \text{ if } \sum_{1\le j\le r} m_j < \sum_{1\le j\le r} l_j, \\ \text{a submanifold of real dimension } 2\sum_{1\le j\le r} m_j - 2\sum_{1\le j\le r} l_j, \text{ otherwise.} \end{cases}$$

Remark 3.24. *Following* Proposition 3.23, *the free stratum* $\phi^{-1}(0)_0$ *is either empty or of dimension* $2\dim V - 2\dim W$. *So far nothing rules out the* *possibility that, for a sequence of characters* $\chi_{k_1}, \ldots, \chi_{k_r}$, $\sum_{1\leq j\leq r} m_j - \sum_{1\leq j\leq r} l_j \geq \dim V - \dim W$. *If so then* $\dim \phi^{-1}(0)_{\pi(\chi_{k_1},\ldots,\chi_{k_r})} = \dim \phi^{-1}(0)_0$ *which means* (see Definition 2.4) *that the stratum* $\phi^{-1}(0)_{\pi(\chi_{k_1},\ldots,\chi_{k_r})}$ *is not contained in the closure of the free stratum* $\phi^{-1}(0)_0$ *but the closures of these two strata do intersect.*

However in [16], Lemma 9, it was shown:

Lemma 3.25. *The closure of the free stratum in* $\mathcal{M}^* \cap \phi^{-1}(0)/S^1$ *contains all singular strata of dimension* < 5. *The link of any singular point of* $(\mathcal{M}^* \cap \phi^{-1}(0)/S^1, \pi_A)$ *in the free stratum is connected.*

In particular $lk(A)$ is connected and, for any singular point $p \in lk(A)$, the link $lk(p)$ in the free stratum of $(lk(A), \pi_A)$ is connected.

Lemma 3.26. ([16], Lemma 10.) *The link* $lk(A)$ *is a homotopy 4-dimensional Poincaré complex.*

Corollary 3.27. ([16], Cor.11.) *The link* $lk(A)$ *is a homotopy* $\mathbb{C}P^2$.

Let $p = \dim_{\mathbb{C}} H^1_A$ and let $q = \dim_{\mathbb{C}} H^2_A$. From Theorem 2.28 we have the dimension δ of the free stratum in $\mathcal{M}^*$:

$$\delta = -\dim H^0_A + \dim H^1_A - \dim H^2_A = -\dim(L(\mathcal{G}_A)) + 2p - 2q = -1 + 2p - 2q$$

By (2.9), $\delta = 5$. Therefore $p - q = 3$.

Remark 3.28. *The difference in real dimension between two different isotropy strata in* $(\mathcal{M}, \pi)$ *is even and possibly zero. The dimension of any nonempty isotropy stratum of* $(\mathcal{M}, \pi)$ *is odd. The dimension of the free stratum is 5.*

Proof. This is a direct consequence of the dimension formula $\delta^{\pi'\times S^1} = -\dim(H^0_A)^{\pi'\times S^1} + \dim(H^1_A)^{\pi'\times S^1} - \dim(H^2_A)^{\pi'\times S^1}$, for $\pi' \leq \pi$-isotropy stratum and the fact that the action of $\pi_A \times S^1$ on H^0_A, the Lie algebra of S^1, is trivial and $V = H^1_A$ and $W = H^2_A$ are complex representations (Proposition 3.23). ∎

Let C be a non-compact connected component of the stratum$\mathcal{M}^*{}_{\pi'}$, for $\pi' \leq \pi$ a nontrivial subgroup. Following Remark 3.24 and Remark 3.28 we have $\dim C$ odd and possibly $\dim C \geq 5$.

Because the determinant line bundle $\Lambda(E)$ associated with the elliptic complex (2.8) has a canonical trivialization over $\mathcal{B}$, it induces an orientation on $\mathcal{M}$ which induces the orientation on X if X is identified with the Taubes boundary of $\mathcal{M}$. By fixing $[A]$ in $C \subset \mathcal{M}^*_{(\pi)}$, where $\pi = \mathbb{Z}/p$, p an odd prime, the elliptic complex (2.8) splits in a π-fixed complex and its orthogonal complex and the line bundle $\Lambda(E)$ is the tensor product of the line bundles of complexes in which (2.8) splits. Then we obtain:

Lemma 3.29. ([16], Lemma 8.) *If $\pi = \mathbb{Z}/p$ with p an odd prime and if $C \subset \mathcal{M}^*_{(\pi)}$ is a connected component, then C is an orientable manifold.*

We investigate the intersection properties of C in the moduli space and we need to introduce Donaldson's μ-map (see [8], Chp. V).

Let $P \to X$ be a $SU(2)$-principal bundle and let $\mathcal{P}$ be the pull-back bundle of P via the projection map $T \times X \to X$, where T is a parameterizing topological space.

Definition 3.30. *A* **family of connections** *in P parameterized by T is a family $\{A\}_{t\in T}$ so that A_t is a connection in $P_t = \mathcal{P}_{\{t\}\times X}$ and P_t is isomorphic to P.*

Remark 3.31. *$\mathcal{P}$ might not be isomorphic to $T \times P$.*

We consider now $\mathcal{P}$ when $T = \mathcal{A}^*$. The gauge group $\mathcal{G}$ does not act trivially on $\mathcal{A}^* \times X$ (the action is diagonal and the gauge isotropy is isomorphic to the centralizer $C(SU(2)) = \mathbb{Z}/2$). The quotient bundle obtained from $\mathcal{P}$ dividing with the action of $\mathcal{G}$ is denoted by $\mathbb{P}$ and is a $S0(3)$-bundle over $\mathcal{B}^* \times X$. Let $p = p_1(\mathbb{P}) \in H^4(\mathcal{B}^*)$ be the first Pontryagin class. Using the slant product operation $/\colon H^4(\mathcal{B}^* \times X) \times H_2(X) \to H^2(\mathcal{B}^*)$ we have the map

$$\begin{aligned} \mu\colon H_2(X;\mathbb{Q}) &\to H^2(\mathcal{B}^*;\mathbb{Q}) \\ \mu(\Sigma) &= \frac{-1}{4} p(\mathbb{P})/\Sigma \end{aligned} \tag{3.8}$$

and the following result:

Lemma 3.32 ([8], Prop. 5.1.21). *For any $\Sigma \in H_2(X;\mathbb{Z})$, the restriction of $\mu(\Sigma)$ to the copy of $\mathbb{C}P^\infty$ which links the reducible connection A* (see Prop. 3.20) *is given by*

$$\mu(\Sigma)|_{\mathbb{C}P^\infty} = -\langle c_1(L), \Sigma\rangle \cdot h \ ,$$

where L is the line bundle from (3.1) *and $h \in H^2(\mathbb{C}P^\infty)$ is the positive generator.*

The restriction to $\mathbb{C}P^\infty$ of $\mu(\Sigma)$ is defined as follows. The rational cohomology sequence

$$H^2(\mathcal{B}) \to H^2(\mathcal{B}^*) \to H^3(\mathcal{B};\mathcal{B}^*) \simeq H^3(\mathcal{U},\partial\mathcal{U}) \simeq H^2(\partial\mathcal{U})$$

is exact, where $\mathcal{U}$ is the union of all cone over $\mathbb{C}P^\infty$ neighbourhoods around every reducible. Let $i\colon \mathbb{C}P^\infty \to \mathcal{B}^*$ be the inclusion map. Then $\mu(\Sigma)|_{\mathbb{C}P^\infty}$ is by definition $i^*\mu(\Sigma) \in H^2(\partial\mathcal{U})$

Lemma 3.33. ([16], Lemma 12.) *If* $dimC \geq 5$ *then* C *is empty. If* $dimC = 3$, *then the closure* $\overline{C} \subset \overline{\mathcal{M}}$ *must intersect the Taubes boundary* $\partial\overline{\mathcal{M}} = X \times \{0\} \subset \overline{\mathcal{M}}$.

Proof. The non-compactness condition on C implies that its closure must intersect the set of reducible connections or the Taubes boundary X. Recall that $(\mathcal{M}^*, \pi)$ is a Whitney stratified space. If $\dim C \geq 5$, then by Whitney's Condition (a), the closure of the free stratum (of dimension 5) does not intersect C (see Remark 3.24). However, the closure $\overline{C}$ may intersect the closure of the free stratum only at reducible connections of isotropy π'.

The next possibility is $\dim C = 3$. Let's assume that, in this situation as well, $\overline{C}$ does not intersect $\overline{\mathcal{M}}$ but at reducible connections and let $(\mathcal{M}, \mathbb{Z}/p)$ be in $\mathbb{Z}/p$-equivariant general position for $\mathbb{Z}/p$ subgroup of π'. Then the deformed stratum C' of C is fixed by $\mathbb{Z}/p$. Let $[A_1], \ldots, [A_r]$ be the reducible classes which are limit points of C' in $\mathcal{B}^*$. By removing from $\overline{C'}$ small open neighbourhoods around each $[A_i]$, we obtain an oriented manifold W whose boundary components $\partial_i W$ are homotopic to $\mathbb{C}P^d$, where the real dimension of W is $2d+1$, $d \geq 1$. Let L_i be the line bundle coming from the reduction induced by $[A_i]$, let $e_i \in H_2(X)$ be the dual of $c_1(L_i)$, and let $h_j \in H^2(\partial_j W) \simeq H^2(\mathbb{C}P^d)$ be the positive generator. Let $[W] \in H_{2d+1}(W, \partial W)$ and $[\partial W] \in H_{2d}(\partial W)$ denote the fundamental classes (in this proof the homology and cohomology are considered with $\mathbb{Z}/2$-coefficients). We have a natural homomorphism $\partial\colon H_{2d+1}(W, \partial W) \to H_{2d}(\partial W)$ such that $\partial[W] = [\partial W]$ and we have the exact cohomology sequence:

$$H^{2d}(W) \xrightarrow{i^*} H^{2d}(\partial W) \xrightarrow{\delta} H^{2d+1}(W, \partial W)\ .$$

Then

$$\langle \hat{\mu}(e_i)^d, [\partial W] \rangle = \langle \delta(\hat{\mu}(e_i)^d), [W] \rangle \ ,$$

for a cocycle representative $\hat{\mu}(e_i)^d$ of $\mu(e_i)$. From 3.32,

$$\mu(e_i)|_{\partial_j W} = -\langle c_1(L_j), e_i \rangle h_j = \delta_{ij} h_j \tag{3.9}$$

Also $\mu(e_i)|_{\partial_j W} = i^*(\mu(e_i)|_W)$. Therefore $\langle \hat{\mu}(e_i)^d, \partial W \rangle = 0$. But

$$\langle \hat{\mu}(e_i)^d, [\partial W] \rangle = \sum_{j=1}^{r} \langle \hat{\mu}(e_i)^d, [\partial_j W] = \langle h_i{}^d, [\partial W] \rangle = 1 \ .$$

This is a contradiction coming from the assumption that $\overline{C}$ is nonempty and intersects $\overline{\mathcal{M}}$ only at reducible connections. ∎

We set up new notation: let $\phi \colon V \to W$ be the π_A-equivariant map $\phi_A \colon H^1_A \to H^2_A$ in π_A general position from Proposition 2.26. Then a π_A-invariant neighbourhood in $(\mathcal{M}, \pi)$ is diffeomorphic to $\phi^{-1}(0)/S^1$. Let χ_j be the irreducible characters of the π_A-representation V. Then $V = \bigoplus_i V(\chi_j)$ with $V(\chi_j)$ irreducible representation. Let $V_0 = \mathrm{Ker}(d\phi_0)$ and $W_0 = \mathrm{Coker}(d\phi_0)$. Then we have the finite dimensional π_A-representations decompositions: $V = V_0 \oplus V'$ and $W = W_0 \oplus W'$ with V' and W' isomorphic as complex finite dimensional spaces.

We restrict to π_A a cyclic group of odd order m. Then the representations V and W enjoy the properties described in Proposition 3.22 and Proposition 3.23. Also, by [17], (1.11), V_0 and W_0 can be assumed to have no representations in common. Let χ_j be an irreducible character of the representation V and let $m_0(\chi_j) = \langle \chi_j, \chi(V_0) \rangle$ be its multiplicity in V_0. Then, in real dimensions, $\dim \phi^{-1}(0)_{\pi_A(\chi_j)} = \dim V_0(\chi_j) = 2m_0(\chi_j)$ Recall that $lk(A)$ denotes the link of $[A]$ in the closure of the free stratum of $(\mathcal{M}^*, \pi)$ and is described by the intersection of the unit sphere in V with the closure of the free stratum in $(\phi^{-1}(0)/S^1) \cap \mathcal{M}^*$. The singular stratum associated with $V(\chi_j)$ in $lk(A)$ is $lk(A)_j = lk(A) \cap S(V(\chi_j))/S^1$, where $S(\chi_j)$ is the unit sphere in $V(\chi_j)$. Thus, any isotropy subspace of V_0 corresponds to a cone over an isotropy stratum in $lk(A)$ and $\{lk(A)_j\}_j$ are all disjoint strata. By lemma 3.25, $\mathcal{U}^* \cap (\phi^{-1}(0)/S^1)$ contains all the isotropy strata of dimension less then 5. Therefore $m_0(\chi_j) \leq 3$. If $m_0(\chi_j) = 3$ then we have a projectively trivial representation of V_0 and a trivial action of π_A on an open neighbourhood of $\mathcal{M}^*$, in contradiction to the effectiveness of the π action on $\mathcal{M}$ (see [16], pp. 724). Therefore $m_0(\chi_j) \leq 2$ and

Proposition 3.34. *Any isotropy stratum, different from the free stratum, in* $(\phi^{-1}(0)/S^1, \pi_A)$ *has real dimension less or equal to* 3.

In view of Proposition 3.34, the fixed point sets in $lk(A)$ are isolated points or 2-spheres. The next lemma, which can be proved in the more general case of π_A arbitrary finite group (see [16], Lemma 14.), establishes how many different singular strata intersect $lk(A)$.

Lemma 3.35. *There are at most* 3 *nonempty singular strata* $lk(A)_j$ *of real dimension* $2m_0(\chi_j)-2$ *in the link* $lk(A)$, *and* $\sum_{\chi_j \text{ irred. character of } V_0} m_0(\chi_j) \leq 3$.

Proof of Theorem. 3.18 By Lemma 3.35 we have the real dimension $\dim V_0 = \sum_{\chi_j \text{ irred. character of } V_0} 2m_0(\chi_j) \leq 6$, and by Proposition 3.8, $6 = \dim V - \dim W = \dim V_0 - \dim W_0$. Therefore $\dim W_0 = 0$ and $\dim V_0 = 6$. This means that ϕ is π_A-equivariantly transverse to $0 \in W$, $H^1_A \simeq (\mathbb{C}^3, \pi_A)$ near the origin, and $(\phi^{-1}(0)/S^1, \pi_A)$ is equivariantly diffeomorphic to the cone of a linear action on $\mathbb{C}P^2$ away from the origin. ∎

Chapter 4

Linear Models

4.1 Connected sums of linear actions on $\mathbb{C}P^2$ with trivial action on homology

As usual we restrict to $\pi = C_m$ cyclic group of odd order m. We investigate the orbit structure and the number of possible orbit types that can arise on an equivariant connected sum of linear π-actions on $\mathbb{C}P^2$ when π acts trivially on $H_*(X)$, with $X = \overset{n}{\underset{1}{\#}}\mathbb{C}P^2$. The connected sum is such that X is simply connected.

Let t be one generator of π. Recall that a linear action on $\mathbb{C}P^2$ corresponds to π acting as a subgroup of $PGL_3(\mathbb{C}) = Gl_3(\mathbb{C})/(\mathbb{C} \setminus \{0\})$. The matrix representation of t in $PGl_3(\mathbb{C})$ is the conjugacy class of $Gl_3(\mathbb{C})$ matrices $t = \begin{pmatrix} 1 & & \\ & \xi^a & \\ & & \xi^b \end{pmatrix}$ given by the rotation numbers (a, b) which are integers modulo m (see Proposition 3.6).

Proposition 4.1. *The group generator t has an equivalent representa-*

tion in $SO_6(\mathbb{R})$ *when we replace* (a,b) *with the equivalent rotation numbers* $(-a,-b)$, (b,a), $(-b,-a)$.

Proof. We can see this if we pass to the real matrix representation $r(\xi^a) = \begin{pmatrix} \cos(2\pi a/m) & \sin(2\pi a/m) \\ -\sin(2\pi a/m) & \cos(2\pi a/m) \end{pmatrix}$ of the complex representation ξ^a. If $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $r(\xi^a) = A \cdot r(\xi^{-a}) \cdot A^{-1}$ and thus $r(\xi^a)$ and $r(\xi^{-a})$ are equivalent real representations. Notice that ξ^a and ξ^{-a} are not equivalent complex representations. Hence we showed that (a,b) and $(-a,-b)$ are equivalent rotation numbers describing the same real representation of t. (a,b) and (b,a) give even equivalent complex representations for t:

$$\begin{pmatrix} 1 & & \\ & \xi^a & \\ & & \xi^b \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \xi^b & \\ & & \xi^a \end{pmatrix} \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}.$$

■

One can see that the $GL_3(\mathbb{C})$ matrices $\begin{pmatrix} 1 & & \\ & \xi^a & \\ & & \xi^b \end{pmatrix}$, $\begin{pmatrix} \xi^{-a} & & \\ & 1 & \\ & & \xi^{b-a} \end{pmatrix}$ and $\begin{pmatrix} \xi^{-b} & & \\ & \xi^{a-b} & \\ & & 1 \end{pmatrix}$ which fix the coordinate complex subspaces of $\mathbb{C}^3$, are among the representatives of that class in $PGL_3(\mathbb{C})$ represented by t. Therefore t (i.e. π) fixes $[1,0,0]$, $[0,1,0]$, $[0,0,1]$ in $\mathbb{C}P^2$. Thus any linear action on $\mathbb{C}P^2$ has at least three fixed points and at most three isolated fixed points.

Proposition 4.2. *Let $\pi = C_m$ be a finite cyclic group and let $\pi' = C_l$ be a subgroup. If π acts linearly and effectively on $\mathbb{C}P^2$ and t is one generator given by the rotation numbers (a, b), then π' fixes a 2-sphere if and only if one and only one of the following cases happens:*

1. $a \equiv b \mod l$
2. $a \equiv 0 \mod l$
3. $b \equiv 0 \mod l$

Otherwise π' fixes only $\{x_1, x_2, x_3\}$, where, in the homogeneous coordinates for which the matrix of the generator of π' takes a diagonal form, $x_1 = [1, 0, 0]$, $x_2 = [0, 1, 0]$, $x_3 = [0, 0, 1]$.

Proof. π' fixes a 2-sphere is equivalent to the fact that the generator of π', $t^{m/l} = \begin{pmatrix} 1 & & \\ & \xi^{am/l} & \\ & & \xi^{bm/l} \end{pmatrix}$, fixes a 2-sphere and this comes to having two equal diagonal entries. This happens exactly when $m | am/l$ or $m | bm/l$ or $am/l \equiv bm/l \mod m$. $m | am/l \Leftrightarrow l | a$; $m | bm/l \Leftrightarrow l | b$; $am/l \equiv bm/l \mod m \Leftrightarrow l | a - b$. Notice that the occurrence of more than one of the cases above is equivalent to a non-effective action and each case corresponds to one distinct fixed 2-sphere that contains two of the fixed points x_i. If none of the three cases happens than indeed the fixed point set of $t^{m/l}$ is $\{x_1, x_2, x_3\}$. ∎

Regarding the orbit types that can appear in a linear action of $\pi = C_m$ on $\mathbb{C}P^2$ we give the following example:

Example 4.3. We consider $m = 3.5.7 = 105$ and the generator t given by the rotation numbers $(a, b) = (10, 3)$. In view of Proposition 3.6, the action is effective and the fixed point set of the action consists of three isolated points represented by $x_1 = [1, 0, 0]$, $x_2 = [0, 1, 0]$ and $x_3 = [0, 0, 1]$. By Proposition 4.2 we see that each $\mathbb{Z}/3$, $\mathbb{Z}/5$, $\mathbb{Z}/7$ fixes a different 2-sphere and the other two subgroups act on it linearly with the same two fixed points belonging to $\{x_1, x_2, x_3\}$. We thus have five orbit types: (0), $(\mathbb{Z}/3)$, $(\mathbb{Z}/5)$, $(\mathbb{Z}/7)$ and $(\mathbb{Z}/105)$.

Proposition 4.4. *The maximum number of orbit types that can arise from a linear action of an odd order cyclic group on* $\mathbb{C}P^2$ *is* 5.

Proof. The group of the action is $\pi = C_m$ with m odd. We have remarked that always $x_1 = [1, 0, 0]$, $x_2 = [0, 1, 0]$ and $x_3 = [0, 0, 1]$ are part of the total fixed set or the whole fixed set of the action. Therefore any subgroup π' of $\pi = C_m$ acts fixing these three points. Let $x = [z_1, z_2, 0]$ be a point on the 2-sphere S_{12} of poles x_1 and x_2, different from x_1 and x_2, and let π_{12} be the isotropy group of x. Then all the other points on S_{12} except the poles have the isotropy π_{12} (we can easily see this from the representation of the linear action on S_{12}). We set similar notations: S_{23}, π_{23}, S_{13}, π_{13}. Any other point away $S_{12} \cup S_{23} \cup S_{13}$ must be part of the free orbit. Thus, in case π_{ij} are all different, we count the maximum number of orbit types, five. ∎

Remark 4.5. *The isotropy subgroups* π_{ij} *cannot be all the same either because the* π*-action is effective (see Proposition 4.2).*

A natural question is (see [11]) if there exists a maximum number of orbit types that cannot be exceeded by any action of $\pi = C_m$ on $X = \#_1^n \mathbb{C}P^2$ for all odd integers m and all integers $n > 1$.

We'll show that the answer to this question is negative providing a procedure to construct an equivariant connected sum of linear actions on $\mathbb{C}P^2$'s and an integer $m > 0$ such that we obtain all the orbits of type $(\mathbb{Z}/p)$, with p a prime dividing m.

Proposition 4.6. *Let x and y be relatively prime odd integers, such that $|x| < |y|$. Let $\pi = C_m$ be a cyclic group of odd order m such that $m = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$ is the decomposition of m in prime factors, all of the form $x + ky$ with k any integer. Then there exists $n > 1$ and there exists $X = \overset{n}{\underset{1}{\#}}\mathbb{C}P^2$ equivariant connected sum of linear π-actions on the components such that we obtain $r + 2$ orbits of types (0), $(\mathbb{Z}/p_1)$, $\dots$, $(\mathbb{Z}/p_r)$ and (π).*

We will describe a way to generate all possible pairs of rotation numbers appearing in a simply connected manifold which is an equivariant connected sum of linear $\pi = C_m$-actions on 4-dimensional complex projective spaces with trivial action on homology. Let $t = \begin{pmatrix} 1 & & \\ & \xi^a & \\ & & \xi^b \end{pmatrix}$ be the representation of one generator of π, with $a, b < m$ and such that the action is effective. A 2-sphere in $\mathbb{C}P^2$ is fixed by π when $a = 0$, $b = 0$ or $a = b$. Let x_1, x_2, x_3 be the fixed points $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$, respectively. We maintain these notations in this paragraph. The rotation numbers corresponding to the tangential representations at these points are (a, b) at x_1, $(-a, b - a)$ at x_2, and $(-b, a - b)$ at x_3. An equivariant connected sum of two copies of $\mathbb{C}P^2$ can be constructed (see Definition 3.2) only at two fixed points with opposite oriented tangential representations, i.e. complementary rotation numbers, (a, b) and $(a, -b)$. If we work out an equivariant connected sum at x_1 we obtain four fixed points corresponding to four pairs of rotation numbers: the initial ones, $(-a, b-a)$, $(-b, a-b)$ and two new pairs, $(-a, -a-b)$ and $(b, a+b)$. Working out a connected sum at x_2 and then at x_3, we obtain the new rotation numbers $(a, 2a-b)$, $(-a+b, -2a+b)$, and $(b, 2b-a)$, $(a-b, a-2b)$, respectively. Let's remark that the new six pairs just produced together with the three initial pairs are the all possible pairs of rotation numbers that can arise from an equivariant connected sum of two linear π actions on $\mathbb{C}P^2$, modulo equivalent rotation numbers . Notice also that these rotation numbers include the case of an initial fixed sphere as well. More $\mathbb{C}P^2$'s can be connected using the fixed points of the initial fixed sphere, but no new pairs of rotation numbers are produced because the points on the sphere correspond to the same pair of the form $(a, 0)$. We remark that an equivariant multiple connected sum cannot be constructed at points along orbits of a type different from (π) because the action was assumed trivial in $H_*(X)$.

We can organize the analysis by constructing the following infinite tree $T = (V, E)$ of valence three as follows: each **vertex** in V is one copy of $\mathbb{C}P^2$, each **edge** in E corresponds to two cancelling pairs of rotation numbers present in two vertices. The **valence** of a vertex is the number of edges coming out of it and which we call **incident edges**. It is clear that each vertex has valence three. Any finite equivariant connected sum of linear actions corresponds to a finite subtree T' of T and the fixed point data of the action is read from the rotation numbers corresponding to the removed edges in T that were incident to the terminal vertices of T'. Without the loss of generality we can assume that the vertex of rotation numbers (a, b), $(-a, -a+b)$, $(-b, a-b)$ is the center vertex of T, i.e. the vertex from which we start "growing" all the other possible rotation numbers of any connected sum.

In order to have a better control over these rotation numbers we consider 2-dimensional square matrices that transform the initial rotation numbers (a, b) in rotation numbers corresponding to each vertex of T. We call **depth** of one vertex the length of the connected path of edges connecting the vertex to the center. The **length** of a path in a tree is the number of edges present in that path. Thus, to generate the rotation numbers of the depth 0 vertex (the center), we need the matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, S = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, D = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \tag{4.1}$$

to be applied to (a, b). To generate the rotation numbers of the reversed oriented tangential representation we need

$$K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and we need to apply the matrices KI, KS, KD. To generate the depth 1 rotation numbers we apply to (a, b) the matrices SKI, DKI, SKS, DKS, SKD, DKD, to generate the depth 2 rotation numbers we apply to (a, b) the matrices $SKSKI$, $DKSKI$, $SKRKI$, $DKDKI$, $SKSKS$, $DKSKS$, $SKDKS$, $DKDKS$, $SKSKD$, $DKSKD$, $SKDKD$, $DKDKD$, and so on. By making the notation $L = KS$ for the matrix that "travels" to the left along the tree T, and $R = KD$ for the matrix that "travels" to the right, we obtain the Figure 4.1. Thus we obtain $3 \cdot 2^n$ rotation numbers of depth n, of which 2^n are given by a reduced word of length $2n$, and the others by a word of length $2n + 1$. The letters of these words are the matrices K, S

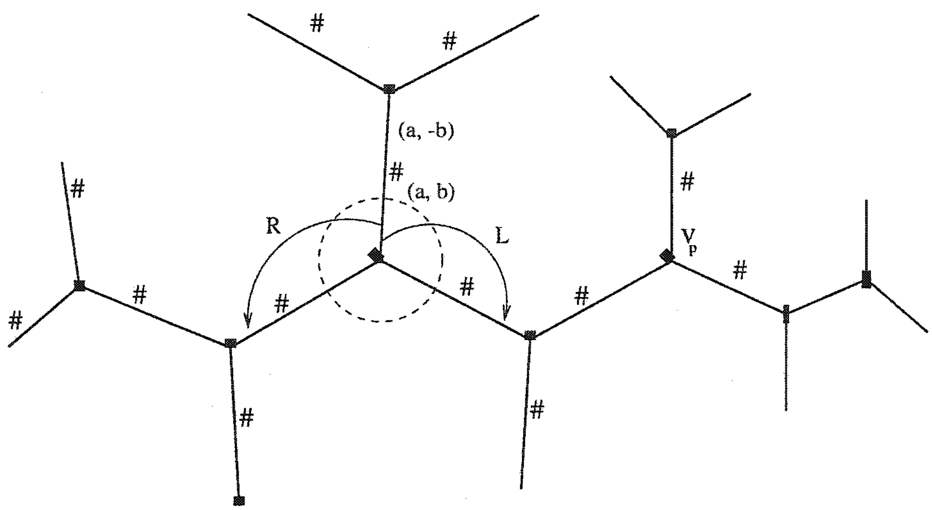

Figure 4.1: The fixed point data tree of a connected sum

and D. A word is called reduced when the adjacent letters of any letter are different from it.

Lemma 4.7 (Dirichlet). *Let (x, y) be a pair of relatively prime integers. Then there are infinitely many primes of the form $x + ny$, with n an integer.*

Proof. See ([1], pp. 146) ■

We are now ready for the

Proof of Proposition 4.6. For any $k > 0$, $L^k = (-1)^k \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}$. When (a, b) are the initial rotation numbers $\mod m$ from the tree T above, by travelling always to the left in T along a path γ of length k we arrive at the rotation numbers $L^k \begin{pmatrix} a \\ b \end{pmatrix} = (-1)^k \begin{pmatrix} a \\ -ka+b \end{pmatrix}$. According to Lemma 4.7, for any prime p dividing m we can find k_p such that $p = -k_p a + b$. Then we consider the connected sum with components the vertices of the path γ. By Proposition 4.2, we have then a 2-sphere fixed by $\mathbb{Z}/p$ on the component of depth k_p, call it v_p. By comparing the rotation numbers at the parent vertex v_{p-1} of v_p we see that v_{p-1} contains the same isotropy $\mathbb{Z}/p$-sphere (see Figure 4.2). We apply this procedure for $p = p_1$ and we

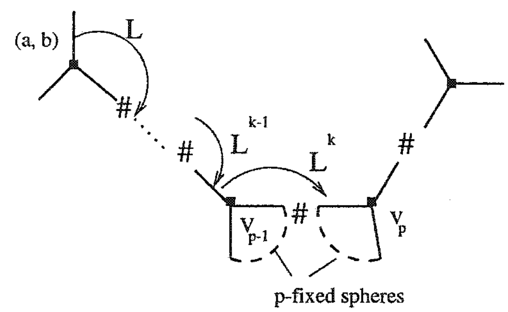

Figure 4.2: The rotation numbers along the tree

find k_1 for the depth of v_{p_1}. Now we bring in p_2 and we apply the same procedure starting with the rotation numbers at the $\mathbb{Z}/p_1$-isolated fixed point of v_p. Therefore we apply the procedure above for $p = p_2$ to the rotation numbers $L^{k_1+1} \begin{pmatrix} a \\ b \end{pmatrix} = (-1)^{k_1+1} \begin{pmatrix} a \\ -(k_1+1)a+b \end{pmatrix}$. We will find k_2 such that $p_2 = -k_2 a + b$ and the equivariant connected sum will be made along a path of length $k_1 + k_2 + 1$ with the depth $k_1 + k_2 + 1$ terminal vertex v_{p_2} which has a $\mathbb{Z}/p_2$-fixed sphere not fixed by $\mathbb{Z}/p_1$. We go on by bringing in one by one all the other $r - 2$ remaining prime numbers applying the same algorithm for each of them. The final path of the connected sum will have length $k_1 + \cdots + k_r + r - 1$. Because the p_i's are prime integers, on any $\mathbb{Z}/p_i$-fixed sphere obtained as before, $\mathbb{Z}/p_j$ acts freely except at the π-fixed poles, for $p_i \neq p_j$. Thus any $\mathbb{Z}/p_i$-fixed sphere contains the isotropy $\mathbb{Z}/p_i$-stratum of the π-action and the connected sum will have then the desired properties. ∎

Example 4.8. We construct an equivariant connected sum of linear π-actions on $\mathbb{C}P^2$'s with 6 orbit types, for $\pi = C_{3315}$.
$3315 = 3 \cdot 5 \cdot 13 \cdot 17$ and all these prime numbers are of the form $4k - 3$ with $k = 0$, $k = 2$, $k = 4$, $k = 5$, respectively (the signs are not important). The tree of the connected sum and the total fixed point set of the π-action are illustrated below (see Figure 4.3).

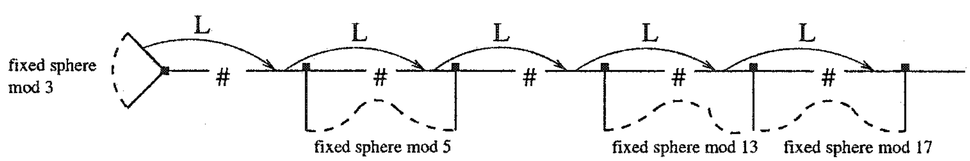

Figure 4.3: An action with six orbit types

We remark that the algorithm above does not address the issue of the maximum number of orbit types that can appear in a linear equivariant connected sum of a given number of $\mathbb{C}P^2$'s, or the issue of the minimum number of $\mathbb{C}P^2$ components necessary to produce a given number of orbit types, but it may provide the set-up to answer these questions too.

4.2 Equivariant connected sums of linear actions on $\mathbb{C}P^2$ with non-trivial action on homology

We keep the same notations as above. This time π does not act necessarily trivially on the homology of X. From Proposition 1.11 we know that π acts trivially on $H_0(X)$ and $H_4(X)$, and permutes the basis elements e_i of $H_2(X) = \bigoplus_{i=1}^{n} \mathbb{Z}e_i$ inducing on $H_2(X)$ the π-module structure $H_2(x) = \bigoplus_k \mathbb{Z}[\pi/\pi_k]^{m_k}$, with π_k subgroup of π. Let t be a generator of π. The Lefschetz number of t is $L(t) = \sum_{i=0}^{4}(-1)^i Tr(t)|_{H_i(X)} \geq 1+0+0+0+1 = 2$, with equality only when the representation of t on $H_2(X)$ is regular or a direct sum of regular representations because only in these cases the trace $Tr(t)$ of the representation on $H_2(X)$ is 0 (see 1.1). According to Lefschetz theorem the fixed set $Fix(X,\pi)$ is non-empty having Euler characteristic at least 2. It's obvious that the manifold X remains diffeomorphically unchanged if we attach 4-spheres, i.e. $X \simeq X\#S^4$. But we can use linear π-actions on these spheres to construct the actions to which this section is dedicated.

Proposition 4.9. *The linear action of π on a 4-sphere S^4 has at most four orbit types.*

Proof. We look at S^4 as embedded in $\mathbb{R}\oplus\mathbb{C}^2$ acted by the generator t whose $SO(5)$-representation is $\begin{pmatrix} 1 & & \\ & \xi^a & \\ & & \xi^b \end{pmatrix}$, where $\xi^m = 1$ with ξ a primitive root of unity. This linear action is determined by the rotation numbers (a, b) and we assume effectiveness by requiring $g.c.d.(g.c.d.(a, m), g.c.d.(b, m)) = 1$ (see Proposition 3.6). Let $x = (u, z_1, z_2)$ be on S^4, i.e. $u^2 + |z_1|^2 + |z_2|^2 = 1$. Then $tx = x \Leftrightarrow u = \pm 1$ and $z_1 = z_2 = 0$. Let $\pi' = C_l$ be a proper subgroup of $\pi = C_m$ and let assume that x has isotropy π'. As in the proof of Proposition 4.2, the generator $t^{m/l}$ fixes x is equivalent to $u = \pm 1$ and ($l|a$ or $z_1 = 0$) and ($l|b$ or $z_2 = 0$). By the effectiveness condition, l cannot divide both a and b. Therefore the π'-isotropy set may consist of one of the 2-spheres: $\{z_1 = 0\}$ or $\{z_2 = 0\}$, less the poles $(\pm 1, 0, 0)$ or it may be empty. Depending on (a, b) and m we obtain maximum 4 distinct isotropy groups corresponding to points sitting on $\{(\pm 1, 0, 0)\}$, $\{z_1 = 0\} \setminus \{(\pm 1, 0, 0)\}$, $\{z_2 = 0\} \setminus \{(\pm 1, 0, 0)\}$, or on free orbits. ■

The connected sums we considered in the first section of this chapter are constructed by an iterated procedure, each step being described by Definition 3.2. For each iteration it was essential to select two π-fixed points x and y, one in each simply connected and oriented manifold component, such that the tangential isotropy representations at x and y are given by the rotation numbers (a, b) and $(a, -b)$, respectively; i.e. there exists a π-equivariant diffeomorphism between two disks D_x and D_y, centered in x and y, which reverses the orientation. The resulting connected sum manifold will be simply connected as well. Essential in Definition 3.2 is the fact that x and y are points of the same isotropy on each manifold allowing us to identify disks centered in the points of the same π-orbits. Let (M, π) and (N, π) be oriented π-manifolds of the same dimension such that M is simply connected and $N = \pi \times_{\pi'} P$ with π' subgroup of π and P simply connected π'-manifold. As π is a discrete group, N is a disconnected manifold diffeomorphic to a disjoint union of $ord(\pi/\pi')$ many P's. Let $x \in M_{\pi'}$, $y \in N_{\pi'}$ and let $D_x \subset M$ and $D_y \subset N$ be open disks centered in x, and y such that $D_x \subset S_x$ and $D_y \subset S_y$, with S_x and S_y the linear slices of the π-action in x and y (see Definition 0.10). Because both x and y have the same stabilizer, π', there exists a π'-equivariant orientation reversing diffeomorphism $f\colon D_x \to D_y$. The equivariant connected sum we construct is obtained by the equivariant attaching that identifies via the diffeomorphism $F\colon \pi \times_{\pi'} D_x \to \pi \times_{\pi'} D_y$, $F = \pi \times_{\pi'} f$, the π-invariant sets $\pi \times_{\pi'} A_1$ and $\pi \times_{\pi'} A_2$ where A_1 and A_2 are annuli in D_x and D_y, respectively.

Example 4.10. Let $\pi = \mathbb{Z}/15$. Let $t = \begin{pmatrix} 1 & & \\ & \xi^a & \\ & & \xi^b \end{pmatrix}$ be the $SO_6(\mathbb{R})$-representation of a generator of π, with ξ an order 15 primitive root of unity. Let $(M, \pi) = (S^4, \pi)$ and let $N = \pi \times_{\pi'} \mathbb{C}P^2$, where $\pi' = \mathbb{Z}/5$ and the actions are linear. Then $\pi/\pi' = \mathbb{Z}/3$ acts freely on M_π and on $N_\pi = (\pi/\pi') \times \mathbb{C}P^2{}_{\pi'}$ (see Corollary 0.16). We choose (a, b) such that the actions of π on M and N are effective and such that $b \equiv 0 \mod 5$. Then π' fixes a 2-sphere on S^4 and on each $\mathbb{C}P^2$ (see Proposition 4.9 and Proposition 4.2). Then we have $M^{\pi'} = S^2$ and $N^{\pi'} = \pi' \times S^2$. We pick $x \in M_{\pi'}$, S_x slice in x, $D_x \subset S_x$ a disk centered in x, and we pick in the same manner y and D_y in a connected component of $N_{\pi'}$. A diffeomorphism f as above exists due to the choice of the rotation numbers. Then we construct the connected sum just described identifying via $\pi \times_{\pi'} f$ the annuli centered in the points of each free $\mathbb{Z}/3$-orbit in each $\mathbb{Z}/5$-fixed set. The resulting manifold X is a connected sum of three $\mathbb{C}P^2$'s on which $\mathbb{Z}/15$ acts by fixing only two isolated fixed points (the poles of S^4), and by permuting the elements of the base $\{e_1, e_2, e_3\}$ of $H_2(X)$, i.e. $H_2(X) = \mathbb{Z}[\mathbb{Z}/3]$.

Example 4.11. Let $\pi = \mathbb{Z}/3$ acting linearly on $M = \mathbb{C}P^2$. Let $N = \pi \times P$, $P = \mathbb{C}P^2$. We take $D_x \subset S_x \subset M$ and $D_y \subset S_y \subset P$ disks centered in x and y respectively, and with S_x and S_y π-slices. Let $h\colon D_x \to D_y$ be an orientation reversing map. Then $\pi \times h\colon \pi \times D_x \to \pi \times D_y$ is the orientation reversing π-equivariant map that we use in the construction of an equivariant connected sum. The resulting manifold X is an equivariant connected sum of four copies of $\mathbb{C}P^2$ on which π acts producing the same fixed point set as of a linear action of π on $\mathbb{C}P^2$. The action of π on homology is $H_2(X) = \mathbb{Z} \oplus \mathbb{Z}[\mathbb{Z}/3]$.

Given $H_2(X) = \bigoplus\limits_k \mathbb{Z}[\pi/\pi_k]$ as possible representation on homology of the action of $\pi = C_m$ on $X = \#\mathbb{C}P^2$, we can ask if there exists a π-equivariant connected sum of linear actions on $\mathbb{C}P^2$ which realizes this action on X.

The answer is negative and in [11], Thm 5.1., A. Edmonds showed that for $m = p^k$, $k \geq 3$ and p prime, there is no locally linear action on X with a representation in homology of the form $\mathbb{Z}[\mathbb{Z}/p] \oplus \mathbb{Z}[\mathbb{Z}/p^2]$. However, we can construct an equivariant connected sum of linear actions that produces as many regular summands in $H_2(X)$ we want, by considering a combination of the two procedures presented in Section 1. and Section 2. Let $\pi = C_m$ with $m = p_1 \dots p_r$, and (a, b) be as in Proposition 4.6. Let M denote the manifold X whose existence is ensured in Proposition 4.6 and let n denote the necessary number of $\mathbb{C}P^2$'s to realize X. Let π_i be $\mathbb{Z}/p_i$, for $i = 1, \dots, r$. We choose $x_i \in M_{\pi_i} \subset M^{\pi_i} \simeq S^2$. Let N_i be the π-manifold $\pi \times_{\pi_i} \mathbb{C}P^2$ and let y_i be a point in $\mathbb{C}P^2{}_{\pi_i}$ such that $\mathbb{C}P^{2^{\pi_i}} \simeq S^2$. For $i = 1, \ldots, r$ we apply the construction described in Section 2. We obtain an equivariant connected sum of $n + m/p_1 + m/p_2 + \cdots + m/p_r$ many $\mathbb{C}P^2$'s with a representation on homology given by $H_2(X) = \mathbb{Z}^n \oplus \bigoplus_i \mathbb{Z}[\pi/\pi_i]$.

Chapter 5

The Singular Set

The following conjecture appears in [11]:

Conjecture 1 (Edmonds). *If $\pi = C_m$ is an odd order cyclic group acting locally linearly and pseudofreely (see* Definition 0.9*) on a simply connected, positive definite smooth* 4*-manifold, inducing a representation by permutations on homology, then π acts semifreely (see* Definition 0.8*).*

In the same article (see [11], Thm. 5.4.) Edmonds shows that Conjecture 1 is false in this generality by providing a counterexample: C_{25} can act locally linearly and pseudofreely on $X = \#_{10}\mathbb{C}P^2$, inducing the representation $H_2(X) = \mathbb{Z}[\mathbb{Z}/5]^2$, but not semifreely. The construction of this action is based on equivariant attaching of handles and is not smooth. On the other hand Edmonds suggests as very likely the possibility that Conjecture 1 holds true in the category of smooth actions. We will show that this is the case.

Let π be a cyclic group of odd order m acting smoothly on the simply connected 4-manifold X. As before, due to Theorem 1.1 and Proposition 1.11, we don't lose generality if we restrict to the case of $X = \overset{n}{\underset{1}{\#}}\mathbb{C}P^2$.

In our notation $H_2(X) = \bigoplus_{i=1}^{n} \mathbb{Z}e_i$, where $\mathcal{E} = \{e_1, \ldots, e_n\}$ is the standard basis of $H_2(X)$. If $\mathcal{R} = \{A_1, \ldots, A_n\}$ denotes the set of classes of reducible connections, from Corollary 3.16 we know that $\mathcal{E}$ and $\mathcal{R}$ are equivalent π-sets. Let's recall that the π-representation on homology is given by (1.3):

$$H_2(X) = \mathbb{Z}^t \oplus \bigoplus_i \mathbb{Z}[\pi/\pi_i]^{r_i},\ \pi_i \underset{\neq}{<} \pi\ . \tag{5.1}$$

5.1 The connectivity of the fixed point sets and the proof of Conjecture 1.

Let π' be a nontrivial subgroup of π. We continue the work done in Chapter 3 toward the description of the singular set of $(\overline{\mathcal{M}^*}, \pi')$ and $(\overline{\mathcal{M}^*}, \pi)$. Let's recall that the map (x, λ) from (2.14) is π-equivariant and the set $\mathcal{M} \backslash \mathcal{M}_{\lambda_0}$ is compact ([14], Thm. 8.36.), where $\mathcal{M}_{\lambda_0} = \lambda^{-1}((0, \lambda_0))$ is the π-equivariant end given by $\lambda_0 > 0$. As a direct observation, the closure of any noncompact subset in the moduli space must intersect the set of reducible connections or the Taubes boundary X or both. By Theorem 2.27 and Remark 3.28, $\mathcal{M}^*_{\pi'}$ is an odd dimensional open manifold in $\mathcal{M}^*$ of maximum dimension 5. One result towards our goal is Lemma 3.33: if $\mathcal{C}_{\pi'}$ is a noncompact nonempty connected component of $\overline{\mathcal{M}^*}_{\pi'}$ and $\pi' \neq 0$, then $\dim \mathcal{C}_{\pi'} \leq 3$. When $\dim \mathcal{C}_{\pi'} = 3$, $\mathcal{C}_{\pi'} \cap \mathcal{R} \neq \emptyset$ and $\mathcal{C}_{\pi'} \cap X \neq \emptyset$. To this result we add

Proposition 5.1. (see also [16], Thm. 16.) *No nonempty collection of 2-dimensional π'-fixed sets in the links of reducible connections bounds a compact fixed set in $\mathcal{M}^*$.*

Proof. Let's assume that there exists $W \subset \mathcal{M}^*$, compact 3-dimensional π'-fixed set, such that the boundary ∂W is a nonempty collection of π'-fixed sets in the links of π'-fixed reducibles. Because the action of π' on the links is a linear action on $\mathbb{C}P^2$'s, we may write $\partial W = \bigsqcup \partial W_i$, where ∂W_i is a $\mathbb{C}P^1$ standardly embedded in the link $\mathbb{C}P^2$, for all i. Then W is exactly the manifold constructed in the proof of Lemma 3.33 by trimming around each reducible the closure of a noncompact connected component of a π'-stratum $\mathcal{C}_{\pi'}$. Thus W must intersect the boundary X, in contradiction with the assumption made. ∎

So far we have eliminated configurations for 3-dimensional π'-fixed components of the singular set. We take up now π'-fixed 1-dimensional components. Let's recall Corollary 2.31.

Remark 5.2. *The arc emerging from an isolated π'-fixed point on X and ending to a reducible is unique. The existence of another π'-fixed arc would create a singularity on the manifold ${\mathcal{M}^*}_{\pi'}$ because the two arcs would intersect the equivariantly embedded collar $X \times [0, \lambda_0]$ along the same segment starting at the π'-fixed point too.*

Lemma 5.3. *(*[16], Lemma 17. *) If $\mathcal{R}^{\pi'} \neq \emptyset$, then the closure of each noncompact 1-dimensional π'-fixed set in $\mathcal{M}^*$ contains at least one reducible connection.*

Proof. We remark that in our case all the reducible connections may have trivial stabilizer. Then it is possible to have a fixed arc in $\overline{\mathcal{M}^*}$ with endpoints two isolated fixed points on X. In the presence of a π'-fixed reducible, we have $e(X^{\pi'}) \geq 3$ and this ensures the existence of a third isolated fixed point on X when a π'-fixed arc with isolated endpoints on the boundary X is assumed to exist. We arrive in the same way at the same contradiction as in [16]. ∎

Definition 5.4. *For any π'-fixed reducible connection class $[A]$ (if any), we call a π'-***incident stratum** *any connected component $C_{\pi'}$ of ${\mathcal{M}^*}_{\pi'}$ such that $[A] \in \overline{C_{\pi'}}$. Restricted to the cone $\mathcal{N}_A$ of the π'-linear action over the $\mathbb{C}P^2$-link of $[A]$* (Theorem 3.18), *a π'-incident stratum is a connected component of $(\mathcal{N}_A \setminus [A])_{\pi'}$.*

Notice that, according to Lemma 3.35, there are at most three disjoint incident strata to a connection A. More precisely, by Theorem 3.18, there are either three 1-dimensional π_A-fixed strata intersecting the $\mathbb{C}P^2$-link $lk(A)$ in three π_A-fixed isolated points, or there are one 1-dimensional and one 3-dimensional π_A-fixed strata intersecting $lk(A)$ in a π_A-fixed isolated point and a π_A-fixed 2-sphere, respectively. Thus

Remark 5.5. *The Euler characteristic $e(\mathrm{Fix}(\pi_A, lk(A)))$ of the fixed set of the link around each reducible is always equal to* 3.

Proposition 5.6. *The closure $\bar{C}$ of any incident stratum can be one of the following types:*

1. *$\bar{C}$ contains only reducibles and it must be* 1*-dimensional.*
2. *$\bar{C}$ contains only one reducible and a subset of X. It must be either* 1 *or* 3*-dimensional.*
3. *$\bar{C}$ contains more than one reducible and a subset of X. It must be* 3*-dimensional.*

Proof. 1. According to Proposition 5.1 or Lemma 3.33, any connected component of a non-compact singular stratum whose closure contains only reducibles cannot be 3-dimensional. Then it must be 1-dimensional.

2. Both these types of incident strata may exist.

3. If $\bar{\mathcal{C}}$ is 1-dimensional, contains more then one reducible and intersects X , then the stratum $\mathcal{C}$ which is obtained by removing the limit points (reducibles and subsets of X) will not be connected, a contradiction. ■

Lemma 5.7. *For any subset $\mathcal{R}' \subset \mathcal{R}$ of connections fixed by π' there is no union of incident one-dimensional singular strata such that its closure becomes a loop containing $\mathcal{R}'$.*

Proof. The argument is, as in the proof of [16], Thm. C., the orientability of connected manifold components of π'-strata in the moduli space (see Lemma 2.35): Suppose the contrary, i.e. there exists a closed path γ which is the closure of a union of one-dimensional incident strata. Then γ inherits an orientation. The intersection of γ with the cone $\mathcal{N}_A$ of one of the connections A of isotropy group π_A consists of two incident strata l_1 and l_2 which are distinct and therefore must have distinct isotropies $\pi(l_1) \leq \pi_A \times S^1$ and $\pi(l_2) \leq \pi_A \times S^1$. The complex structure of ${H^1}_A$ induces the same orientation on l_1 and l_2. Then one of l_i must have the opposite orientation of γ, a contradiction. ■

We recall some notions and results from the theory of graphs that we are going to use in sequel.

Definition 5.8. ([30], 2.1, Def. 1.) *A graph $\Gamma(V, E)$ consists of two sets V and E called* **vertices**, *respectively* **edges** *with the following structure:*

there exists a map $E \to V \times V$, $e \to (0(e), t(e))$, *and a map* $E \to E$, $e \to \bar{e}$, *such that* $\bar{\bar{e}} = e$, $\bar{e} \neq e$ *and* $o(e) = t(\bar{e})$. $o(e)$ *is called the* **origin** *of the edge* e *and* $t(e)$ *is called* **terminus** *of the edge* e. Γ *is* **oriented** *if one sets an orientation, i.e. a subset* $E_+ \subset E$ *such that we have the disjoint union* $E = E_+ \sqcup \bar{E}_+$.

Definition 5.9. *A* **path** *of length* n *in the oriented graph* $\Gamma(V, E)$ *is a subgraph given by edges* $\{e_1, \ldots, e_n\} \subset E_+$ *such that* $t(e_i) = t(e_{i+1})$, *for* $i = 1, \ldots, n-1$. *A* **circuit** *of length* n *in* $\Gamma(V, E)$ *is a path of length* n *given by the edges* $\{e_1, \ldots, e_n\}$ *which in addition satisfy* $t(e_n) = o(e_1)$.

Definition 5.10. *A graph* $\Gamma(V, E)$ *is connected if for any two distinct vertices* v_1 *and* v_2 *there exists a path* $\{e_1, \ldots, e_n\}$ *such that* $o(e_1) = v_1$ *and* $t(e_n) = v_2$, *or* $V = \{v\}$ *and* $E = \emptyset$, *or* Γ *is empty.*

Definition 5.11. *A connected graph is a tree if no subset of edges is a circuit.*

Definition 5.12. *The group* G *acts on the graph* $\Gamma(V, E)$ *if there are defined actions* (V, G) *and* (E, G) *such that the maps* o *and* t *are* G*-equivariant.*

Remark 5.13. *If* G *is a finite group of odd order, then, for any edge* e *there is no element* g *such that* $ge = \bar{e}$ *because* G *has no subgroups of order* 2.

Definition 5.14. *A group* G *acts freely on a graph* $\Gamma(V, E)$ *if* $ge \neq \bar{e}$ *for any* $g \in G$ *and any* $e \in E$, *and* $gv = v \Leftrightarrow g = 1$.

Theorem 5.15. ([30], 3.3, Thm. 4.) *Only a free group can act freely on a tree.*

The next result gives a more precise description of the fixed set $\mathrm{Fix}(\pi', \overline{\mathcal{M}^*})$.

Theorem 5.16. *In the presence of at least one π'-fixed reducible connection, no nonempty collection of 2-dimensional π'-fixed sets in X bounds a fixed set in $\mathcal{M}^* \cup X$. When there are no π'-fixed reducible connections, no more than one 2-dimensional π'-fixed sphere can bound a π'-fixed set in $\mathcal{M}^* \cup X$. In addition, $\mathrm{Fix}(\pi', \overline{\mathcal{M}^*})$ is connected.*

Proof. Firstly we prove the theorem in the case of $\pi' = \pi = \mathbb{Z}/p$ with p an odd prime integer and we place $\mathcal{M}$ in equivariant general position with respect to π'.

Case a): The set of π'-fixed reducibles, $\mathcal{R}^{\pi'}$, is nonempty. Let $r = |\mathcal{R}^{\pi'}|$. By Corollaries 3.16 and 1.17, the Euler characteristic of the fixed set is $e(X^{\pi'}) = r + 2$.

We assume the existence of a collection of 2-dimensional π'-fixed spheres, $\{F_i\}$, in X which bounds a fixed set in $\mathcal{M}^* \cup X$. This means that the bounded set does not contain any reducible and $e(\{F_i\}) \geq 2$. If all the incident strata to all fixed connections intersected X, by Remark 5.5, we would count an Euler characteristic of the fixed set on X equal to $3r$. By comparison to $e(X^{\pi'}) = r+2$, this is possible only when $r = 1$ and $\{F_i\} = \emptyset$. Therefore not all the incident strata intersect X (e.g. there are strata of types *1.* and *3.*, according to Proposition 5.6). Let's denote by χ the charge representing the number of units due to incident strata that do not intersect the boundary such that the Euler characteristic of $X^{\pi'}$ is $r + 2$, as required, i.e.

$$3r - \chi \leq r + 2. \tag{5.2}$$

Notice that, as χ was defined, we have equality in 5.2 only when $\{F_i\} = \emptyset$. Therefore Case a) is proved if 5.2 is an equality.

We reduce now our topological problem to a combinatorial one by considering the following graph $\Gamma_r(V, E)$: The set of vertices V is $\mathcal{R}^{\pi'}$. An edge in E corresponding to two vertices v_1 and v_2 exists when:

i. there is a type *1.* stratum whose closure contains v_1 and v_2 (one fixed arc with endpoints in the reducibles v_1 and v_2, see Proposition 5.6)

ii. there is a type *3.* stratum whose closure contains v_1 and v_2 and, if all the reducibles contained in the closure of the type *3.* stratum are $v_1, \ldots, v_k$, then the edges defined are (v_i, v_{i+1}), $i = 1, \ldots, k-1$, for a succession of vertices such that the condition of Remark 5.5 is satisfied.

To each edge so defined we attach a weight of 2, thus one type *1.* stratum contributes 2 units to χ and a type *3.* stratum with k reducibles contributes $2(k-1)$ units to χ. Notice that type *2.* strata do not contribute to χ because their closures intersect X. The proof reduces now to showing that $\Gamma_r(V, E)$ is a tree (i.e. a connected graph with no cycles) and χ, which represents now the sum of the weights, equals $2(r-1)$. That Γ_r contains no cycles we could see from Lemma 5.7. We proceed by induction about r. For $r = 1$, $\chi = 0 = 2(r-1)$ (no type *1.* and no type *3.* strata, so $E = \emptyset$) and Γ_1 is a point. The induction hypothesis is that Γ_l is connected, $\chi(\Gamma_l) = 2(l-1)$ for all $l = 1, \ldots, r$. We show that this is true for $l = r+1$. Let's suppose that

Γ_{r+1} is disconnected, i.e. $\Gamma_{r+1} = \Gamma_k \bigsqcup \Gamma_s$ as a disjoint union of graphs which by the induction hypothesis are trees and for which $\chi(\Gamma_k) = 2(k-1)$ and $\chi(\Gamma_s) = 2(s-1)$. The inequality 5.2 translates into $\chi(\Gamma_{r+1}) \geq 2r$. Moreover $r+1 = k+s$ and $\chi(\Gamma_{r+1}) = \chi(\Gamma_k) + \chi(\Gamma_s) = 2(k+s) - 4 = 2r-2$, a contradiction. Therefore Γ_r is connected and $\chi(\Gamma_r) = 2(r-1)$ for any r. This implies that $\mathrm{Fix}(\pi', \overline{\mathcal{M}^*})$ is connected and $\chi = 2r$, i.e. $\{F_i\} = \emptyset$.

Case b): $\mathcal{R}^{\pi'} = \emptyset$. Then $e(X^{\pi'}) = 2$. By Theorem 1.14, the intersection of a π'-fixed set in $\overline{\mathcal{M}^*}$ with X must be two isolated fixed points or a fixed sphere. For the general case, when π' is a nontrivial subgroup of π, let's choose $\mathbb{Z}/p$ a subgroup of π'. Then $\mathrm{Fix}(\pi', \overline{\mathcal{M}^*}) \subset \mathrm{Fix}(\mathbb{Z}/p, \overline{\mathcal{M}^*})$ and we have just shown that $\mathrm{Fix}(\mathbb{Z}/p, \overline{\mathcal{M}^*})$ is connected. Let's assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ are two distinct connected components of $\mathrm{Fix}(\pi', \overline{\mathcal{M}^*})$, and let $A_1 \in \mathcal{M}_1$ and $A_2 \in \mathcal{M}_2$ be two reducible connections in $\mathcal{R}^{\pi'} \subset \mathcal{R}^{\mathbb{Z}/p}$. We choose A_1 and A_2 such that they correspond to terminal vertices in the graphs Γ_1 and Γ_2 constructed as above with the reducibles in $\mathcal{M}_1$ respectively $\mathcal{M}_2$ as vertices. Because $\mathrm{Fix}(\mathbb{Z}/p, \overline{\mathcal{M}^*})$ and its associated graph Γ are connected there exists γ an arc in Γ with endpoints in the vertices A_1 and A_2, and such that $\gamma \cap \mathrm{Fix}(\pi', \overline{\mathcal{M}^*}) = \{A_1, A_2\}$. Let g be the generator of $\pi/\mathbb{Z}/p$. Then $g\gamma \cap \gamma = \{A_1, A_2\}$ and $g\gamma \cup \gamma$ is a circuit in Γ. But this is in contradiction to the fact that Γ is a tree as it results from Lemma 5.7. ∎

Lemma 5.17. *Let $A \in \mathcal{R}$ with nontrivial stabilizer $\pi_A \leq \pi$. Let $(\mathcal{N}_A, \pi_A)$ be the cone over the linear action of π_A on the $\mathbb{C}P^2$-link $lk(A)$ of A in the closure of the free stratum of $\mathcal{M}^*$. If $\mathcal{I}$ is a 1-dimensional incident π_I-stratum to A, with $\pi_I \neq \{e\}$ then $\pi_I = \pi_A$.*

Proof. By Corollary 1.17 applied to the linear action of π_A on $lk(A)$, we have $e(lk(A)^{\pi_A}) = 3$ and the linear action gives the following possibilities: 1. $lk(A)^{\pi_A}$ is a set of three isolated π_A-fixed points; 2. $lk(A)^{\pi_A}$ consists of an isolated π_A-fixed point and a π_A-fixed 2-sphere.

Case 1. Let x_1, x_2, x_3 be the three isolated fixed points. The cone structure of $\mathcal{N}_A$ implies that $(\mathcal{N}_A \setminus A, \pi_A)$ is π_A-diffeomorphic to the linear action $(\mathbb{C}P^2 \times [0,1), \pi_A)$, where the action is diagonal and trivial on the second factor of the product (Theorem 3.18). Therefore there are three π_A-fixed one dimensional strata in $\mathcal{N}_A$ given by $x_i \times [0,1)$, $i = 1,2,3$. Let's assume that $\mathcal{I}$ is another 1-dimensional stratum of isotropy $\pi_I < \pi_A$, $\pi_I \neq \{e\}$ such that $x = lk(A) \cap \mathcal{I}$ is an isolated point of the π_I-stratum of $lk(A)$. This implies that x is not in the interior of any π_I-fixed 2-sphere because $\mathcal{I}$ is 1-dimensional and defined as the connected component of a π_I-stratum in $\mathcal{M}^*$. If x is distinct from the x_i's, by considering that $lk(A)^{\pi_A} \subset lk(A)^{\pi_I}$, we get in conflict with the condition of Corollary 1.17 because $e(lk(A)^{\pi_I})$ would be 4. Therefore x must be one of the x_i's and $\mathcal{I}$ is a π_A-stratum.

Case 2. Assume that an $\mathcal{I}$ as in Case 1. exists. If x is outside the fixed set, the same Corollary 1.17 is contradicted. ∎

Proposition 5.18. *If $\pi' \neq 0$ is a subgroup of π and if the π'-fixed set* $\mathrm{Fix}(\pi', \overline{\mathcal{M}^*})$ *is one-dimensional, then* $\mathrm{Fix}(\pi', \overline{\mathcal{M}^*}) = \mathrm{Fix}(\pi, \overline{\mathcal{M}^*})$.

Proof. We place $(\mathcal{M}, \pi')$ in π'-equivariant general position.

Case a): $\mathcal{R}^{\pi'} = \emptyset$. Then Corollary 1.17 implies $e(X^{\pi'}) = 2$, i.e. $X^{\pi'}$ consists either of two isolated π'-fixed points or of a π'-fixed 2-sphere. Using the equivariant Taubes boundary (see Corollary 2.30) we obtain that $\mathrm{Fix}(\pi', \overline{\mathcal{M}^*})$ is either a 1-dimensional π'-fixed arc joining the two isolated fixed points on X, or a connected 3-dimensional manifold with boundary the π'-fixed 2-sphere. The hypothesis of the lemma shows that $\mathrm{Fix}(\pi', \overline{\mathcal{M}^*})$ is the arc. Since $e(X^\pi) \geq 2$ and $X^\pi \subseteq X^{\pi'}$ we have $X^\pi = X^{\pi'}$ and $\mathrm{Fix}(\pi, \overline{\mathcal{M}^*})$ is an arc with boundary X^π. But $\mathrm{Fix}(\pi, \overline{\mathcal{M}^*}) \subseteq \mathrm{Fix}(\pi', \overline{\mathcal{M}^*})$ and by Remark 5.2, $\mathrm{Fix}(\pi, \overline{\mathcal{M}^*}) = \mathrm{Fix}(\pi', \overline{\mathcal{M}^*})$.

Case b) $\mathcal{R}^{\pi'} \neq \emptyset$. Then $e(X^{\pi'}) > 2$ and the assumed one-dimensionality of $\mathrm{Fix}(\pi', \overline{\mathcal{M}^*})$ implies that $X^{\pi'}$ is discrete and contains at least three points (see Corollary 1.17). Let $A \in \mathcal{R}^{\pi'}$. Then the π'-fixed incident strata to A are of type *1.* or 1-dimensional type *2.* (see Proposition 5.6). By Remark 5.5, for the types of all incident strata to A we have the following possibilities:

i. three incident strata of type *2.*

ii. one incident stratum of type *1.* and 2 incident strata of type *2.*

iii. two incident strata of type *1.* and 1 incident stratum of type *2.*

iv. three incident strata of type *1.*

Theorem 5.16 shows that $\mathrm{Fix}(\pi', \overline{\mathcal{M}^*})$ is connected. In fact we can regard $\mathrm{Fix}(\pi', \overline{\mathcal{M}^*})$ as the topological realization of a finite tree with the π'-fixed reducibles as vertices, the π'-fixed points as terminal vertices, and with the closures of the incident strata as edges. Note that each non-terminal vertex belongs to exactly 3 edges. Because of the connectivity, by Lemma 5.17, all vertices and all edges must have the same stabilizer, say $\pi'' \geq \pi'$ and $\mathrm{Fix}(\pi', \overline{\mathcal{M}^*}) = \mathrm{Fix}(\pi'', \overline{\mathcal{M}^*}) = \overline{\mathcal{M}^*}_{\pi''}$. Therefore π/π'' must act freely on the tree, i.e., by Theorem 5.15, π/π'' is a free group. But obviously π/π'' is finite. Therefore $\pi'' = \pi$ and $\mathrm{Fix}(\pi', \overline{\mathcal{M}^*})$ is actually $\mathrm{Fix}(\pi, \overline{\mathcal{M}^*})$. ∎

In the case of a smooth action, Conjecture 1. becomes:

Corollary 5.19. *Let π be a cyclic group of odd order acting smoothly and pseudofreely on the smooth, closed, simply connected, positive definite, oriented 4-manifold X. Then π acts semifreely.*

Proof. Since the action of π is pseudofree, for any π' nontrivial subgroup of π, $X^{\pi'}$ consists of isolated points. If there are no π'-fixed reducibles, then the two π'-fixed points on X must be in the closure of a one dimensional π'-fixed stratum of $\mathcal{M}^*$. If there are π'-fixed reducibles, from Theorem 2.27 it follows that $\mathcal{M}^{*\pi'} = \bigsqcup_{\pi'' \geq \pi'} \mathcal{M}^*{}_{\pi''}$ is a union of 1-dimensional manifolds. By Proposition 5.18, we have that $\mathrm{Fix}(\pi', \overline{\mathcal{M}^*})$ equals $\mathrm{Fix}(\pi, \overline{\mathcal{M}^*})$. Thus any π'-isotropy stratum is a π-fixed stratum and this shows that we have only one singular orbit type. ■

5.2 The structure of the singular set

We are now set for the proof of the main result, Theorem 3.7. Let's recall some notation: The set of classes of reducible connections is $\mathcal{R} = \{A_1, \ldots, A_n\}$. $\mathcal{R}$ is in a π-equivariant bijection to the standard base $\mathcal{E} = \{e_1, \ldots, e_n\}$ of $H_2(X)$ (Corollary 3.16). If $\pi_1, \ldots, \pi_k$ denote all the stabilizer subgroups of all connections in $(\mathcal{R}, \pi)$, then $H_2(X) = \bigoplus_{i,\ldots,k} \mathbb{Z}[\pi/\pi_i]^{r_i}$ and $|\mathcal{R}_{\pi_i}| = |\pi/\pi_i| \cdot r_i$, $n = \sum_{i=1,\ldots,k} |\pi/\pi_i| \cdot r_i$. The multiplicities r_i are positive integers.

Proof of Theorem 3.7: The set of stabilizers $\{\pi_1, \ldots, \pi_k\}$ is ordered with the inclusion and we consider all totally ordered subsets, e.g. chains $\pi_i > \pi_j > \ldots$. We distinguish two cases:

Case 1.: there are connections in $\mathcal{R}$ with stabilizer π i.e. there are π-fixed reducible connections. Let $\pi > \pi_i > \ldots$ be a descending chain of stabilizers that starts with π. If the chain contains only π and the trivial subgroup $\{e\}$, the π-representation (5.1) becomes $H_2(X) = \mathbb{Z}^t \oplus \mathbb{Z}[\pi]^{r_0}$. We have then only free orbits and singular orbits of type π in $(\mathcal{M}, \pi)$. The proof in this case is the proof of [16], Theorem C., as the connections in $\mathcal{R} \setminus \mathcal{R}^\pi$ are not contained in the singular set of $(\overline{\mathcal{M}^*}, \pi)$. If the chain contains proper subgroups, for any $A \in \mathcal{R}_{\pi_A}$, let $(\mathcal{N}_A, \pi_A)$ be the neighbourhood of A which is, according to Theorem 3.18, a cone over a linear action on the $\mathbb{C}P^2$-link of A in $\mathcal{M}^*$, and let π_i be the successor of π in the chain. Then $\mathrm{Fix}(\pi_i, \overline{\mathcal{M}^*}) = \overline{\mathcal{M}^*{}_{\pi_i}} \cup \mathrm{Fix}(\pi, \overline{\mathcal{M}^*})$. Because $\mathrm{Fix}(\pi_i, \overline{\mathcal{M}^*})$ is connected (Theorem 5.16) and $\mathrm{Fix}(\pi, \overline{\mathcal{M}^*}) \subsetneqq \mathrm{Fix}(\pi_i, \overline{\mathcal{M}^*})$, there must be an isotropy π_i-stratum $\mathcal{I}$ of type *1.* or type *3.* which is incident to A and to a connection $B \in \mathcal{R}_{\pi_i}$. By Lemma 5.17, $\mathcal{I}$ cannot be 1-dimensional. So $\mathcal{I}$ is of type *3.* Since $\pi_A = \pi$, $\mathcal{I} \cap \mathcal{N}_A$ is π-invariant. Then there exists an open 3-dimensional set $\mathcal{U} \subset \mathcal{I}$ which is π-invariant and such that $\mathcal{U} \cap \mathcal{N}_B \neq \emptyset$. Let g be a generator of π/π_i. We have $g \cdot \mathcal{U} = \mathcal{U}$ and $g \cdot \mathcal{N}_B = \mathcal{N}_{g \cdot B}$. Therefore $\emptyset \neq g \cdot (\mathcal{U} \cap \mathcal{N}_B) = \mathcal{U} \cap \mathcal{N}_{g \cdot B}$. By Proposition 5.1 it follows that the closure $\overline{\mathcal{U}}$ in $\overline{\mathcal{M}}$ contains the orbit of reducibles $\pi \cdot B$. Therefore we have just shown that once we have a π_B-stratum $\mathcal{I}$ incident to a connection A of stabilizer π_A, and to a connection B with stabilizer π_B such that $\pi_A > \pi_B$, with π_A and π_B consecutive in a chain of stabilizers, then $\mathcal{I}$ must be 3-dimensional and incident to the reducible connections of $\pi_A/\pi_B \cdot B$.

Let's assume now that we have a chain $\pi > \pi_i > \pi_j > \ldots$ which contains at least 3 nontrivial stabilizers. Because $\overline{\mathcal{M}^*{}_{\pi_j}}$ is connected, we find again a connection A with stabilizer $\pi_A > \pi_j$, and a 3-dimensional π_j-stratum $\mathcal{I}$

incident to A, such that $\overline{\mathcal{I}}$ contains a connection B with stabilizer $\pi_B = \pi_j$. If $\pi_A = \pi$, by the arguments from above, $\overline{\mathcal{I}}$ must contain the π_j-orbit $\pi \cdot B$ which is the set $\pi/\pi_j \times B$. If $\pi_A = \pi_i$, $\overline{\mathcal{I}}$ is π_i-invariant and must contain $\pi_i \cdot B \simeq \pi_i/\pi_j \times B$. But in this case A is freely permuted by π/π_i. Let $\mathcal{C}_A$ be the connected component of A in $\overline{\mathcal{M}^*}_{\pi_j}$. Then $\mathcal{C}_A$ is disjoint from $\mathcal{C}_{A'}$, for any other connection $A' \in \pi/\pi_i \times A$. This fact is due to Lemma 5.7. Moreover $\mathcal{C}_{A'}$ must be the identical copy of $\mathcal{C}_A$ and the orbit $\pi \cdot B$ is contained in $\pi \times_{\pi_i} \mathcal{C}_A$ (see the picture below). The description of the sin-

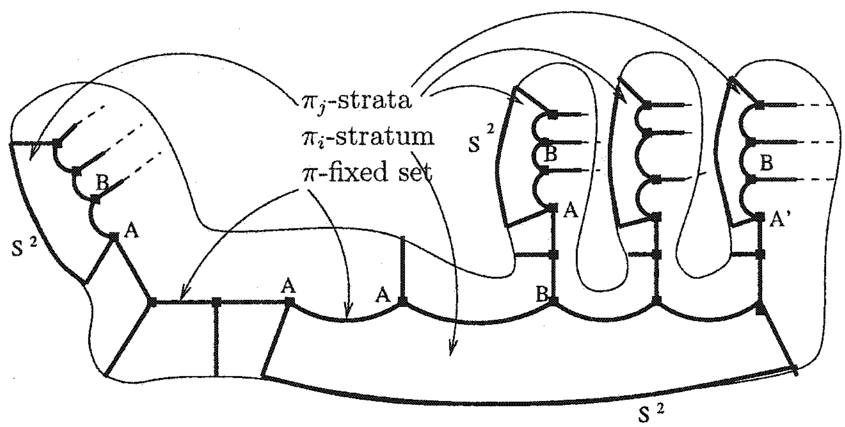

Figure 5.1: The singular set of $(\overline{\mathcal{M}^*}, \pi)$

gular set goes step by step until we exhaust all the stabilizers of the chain. For another chain which must start with π, we have the same description: the connected components of connections in $\mathcal{R}_{\pi_i}$ for one chain are disjoint from the connected components of connections in $\mathcal{R}_{\pi_i}$ for another chain that contains π_i. This is due again to Lemma 5.7.

Case 2.: there are no π-fixed reducible connections. Then $e(X^\pi) = 2$ and X^π consists of either two isolated fixed points, or a fixed 2-sphere and $\text{Fix}(\pi, \overline{\mathcal{M}^*})$ is either an arc with the endpoint set equal to X^π or a 3 dimensional manifold with boundary equal to the fixed 2-sphere, respectively. If there are connections in $\mathcal{R}$ with nontrivial stabilizers, we can single out chains of stabilizers as in *Case 1.* with the only difference that they do not start with π. Let π_i be a maximal stabilizer encountered and let $\pi_i > \pi_j > \ldots$ be one of its chains. By Theorem 5.16, $\text{Fix}(\pi_i, \overline{\mathcal{M}^*})$ is connected and contains $\text{Fix}(\pi, \overline{\mathcal{M}^*})$. Let's assume that $\text{Fix}(\pi_i, \overline{\mathcal{M}^*})$ is 1-dimensional. Then there are π_i-fixed arcs which start at the isolated π-fixed points and end at π_i-fixed reducible connections. But the same π-fixed points are joined by a π-fixed arc that does not contain reducibles. Then we are in contradiction with Remark 5.2, and therefore $\text{Fix}(\pi_i, \overline{\mathcal{M}^*})$ must contain 3-dimensional manifold components. Among these, there is one whose π_i-fixed sphere boundary on X contains X^π, otherwise $\text{Fix}(\pi_i, \overline{\mathcal{M}^*})$ would not be connected and we would contradict Theorem 5.16. If X^π were a sphere, i.e. $\text{Fix}(\pi, \overline{\mathcal{M}^*})$ were 3-dimensional, the intersections of the 3 dimensional π-isotropy component of $\text{Fix}(\pi_i, \overline{\mathcal{M}^*})$ and respectively $\text{Fix}(\pi, \overline{\mathcal{M}^*})$ with the Taubes collar would coincide and would be equivariantly diffeomorphic to $S^2 \times [0, 1)$ (see Corollary 2.30). Then $\mathcal{M}^*_{\pi_i} \cap \mathcal{M}^*_\pi \neq \emptyset$ which is a contradiction. Hence $\text{Fix}(\pi, \overline{\mathcal{M}^*})$ is an arc which joins the two fixed points of X^π and is embedded in $\text{Fix}(\pi_i, \overline{\mathcal{M}^*})$. Let $\mathcal{I}$ be the 3-dimensional π_i-stratum which is incident to a π_i-fixed connection A and whose closure is the π-fixed sphere that contains the two points of X^π. The stratum $\mathcal{I}$ is π-invariant and its closure contains at least one orbit $\pi \cdot A$ by the same arguments as in *Case 1.* Now the description of the singular set follows the same steps and arguments as in *Case 1.*

5.3 The stratified cobordism and the rotation numbers

In order to complete the proof of Theorem 3.7, we must show that the rotation numbers of the tangential isotropy representations at the fixed points of the π_i-fixed sets of X are the same as those of some equivariant connected sum of linear actions on complex projective spaces, for all stabilizers π_i of reducible connections. As usual, we place $(\mathcal{M}, \pi)$ in equivariant general position. The π_i-fixed set of X is the intersection with the Taubes boundary of $\text{Fix}(\pi_i, \overline{\mathcal{M}^*})$, for any $\pi_i \leq \pi$. Let's recall (see Theorem 2.27) that the equivariant moduli space $(\mathcal{M}, \pi)$ is a Whitney stratified space in which each singular stratum has a locally trivial equivariant cone bundle structure in $\mathcal{M}$. In Figure 5.2 is illustrated the situation of one-dimensional π-strata in $\mathcal{M}$. The π-fixed arc γ from the fixed point x to the π-fixed

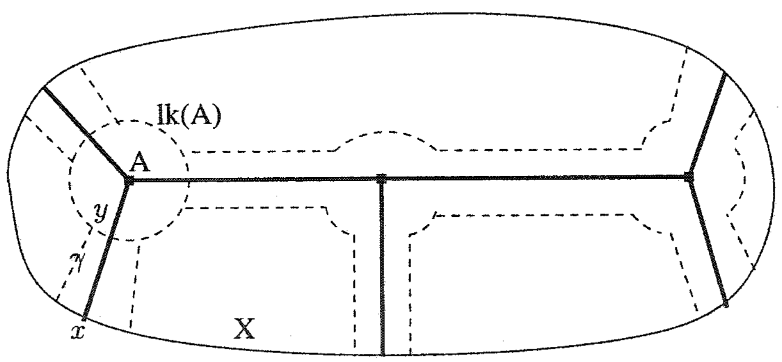

Figure 5.2: The stratified cobordism

reducible connection A intersects the the $\mathbb{C}P^2$-link $lk(A)$ in y. Since the portion $[x, y]$ of γ is a contractible path, the equivariant cone bundle structure over $[x, y]$ is trivial. Therefore the representations of π on the fibers at

x and y are equivalent. But these fibers are the tangential representations at x to X and at y to $lk(A)$, which means that the rotation numbers at x and y coincide. By Theorem 3.18, the rotation numbers at y are those of a linear action on $lk(A)$. Thus, we have the picture of an equivariant cobordism between the 4-manifold X and the equivariant connected sum of linear actions on the links of the fixed reducibles. The same is true when 3-dimensional strata are present in $(\mathcal{M}, \pi)$, and we present the arguments in what follows. For simplicity we describe the rotation numbers for the π and π_i-fixed sets, where $\pi > \pi_i > \dots$ is a chain of stabilizers as above. The description goes on similarly following the steps from *Case 1.* and *Case 2.*. Let A denote connections stabilized by π and let B denote connections stabilized by π_i. As we could see in Figure 5.1, the π_i-fixed set comprises incident π and π_i-strata of type *1.*, type *2.*, and type *3.*, which are defined in Proposition 5.6. Let $\mathcal{I}$ be an incident 3-dimensional π or π_i-stratum of type *2.* or type *3.* There exist arcs from the links $lk(A) \cap \mathcal{I}$ or $lk(B) \cap \mathcal{I}$ to $\overline{\mathcal{I}} \cap X$, equivariantly embedded in $\overline{\mathcal{I}}$. There exist trivial equivariant cone bundle neighbourhoods around these arcs inside the moduli space. The π or π_i-representations at each fiber of such a bundle are equivalent to the tangential representations at the points of intersection of the arc with the $\mathbb{C}P^2$-link $lk(A)$ and with the Taubes boundary X. For a 1-dimensional $\mathcal{I}$ of type 2, the same argument works by considering an arc as above to be $\mathcal{I}$ itself. Therefore the rotation numbers of linear actions at fixed points on the links coincide with rotation numbers at fixed points on X. We show that these rotation numbers are those of an equivariant connected sum of linear actions on $\mathbb{C}P^2$. We proceed as follows: for each type *1.* incident π or π_i-stratum $\mathcal{I}$, there exists an equivariant tubular neighbourhood inside

$\mathcal{M}^*$ with trivial equivariant normal bundle structure over $\mathcal{I}$, such that it contains as fibers the tangential representations at the points of intersection of $\mathcal{I}$ with the links of the connections that form the endpoints of $\mathcal{I}$ (see Figure 5.3). The orientation on $\mathcal{I}$ and the orientations induced by the

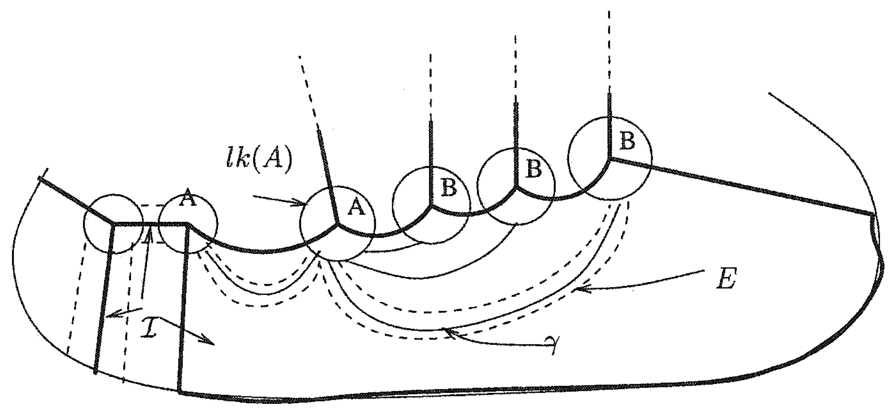

Figure 5.3: The singular set and the linear model

complex structure of the links imply that we obtain orientation reversing rotation numbers (a, b) and $(a, -b)$.

When $\mathcal{I}$ is a 3-dimensional π_i-stratum of type *3.* whose closure contains both types of connections, A and B, let $\pi \cdot x \simeq \pi/\pi_i \times \{x\}$ be the orbit of x on the sphere $lk(A) \cap \mathcal{I}$. There exists $\gamma \subset \mathcal{M}^*$ an arc joining x and a point y on $lk(B) \cap \mathcal{I}$. By an equivariant perturbation we can make γ equivariantly embedded in $\mathcal{I}$. Again, by Theorem 0.45, there exists an equivariant neighbourhood around γ whose restriction E to $\mathcal{I}$ has the structure of a trivial equivariant normal bundle $D^1 \times D^2$ over γ containing as fibers the tangential representations on the links at the endpoints of γ. We obtain the equivariant structure $(lk(A) \sqcup (\pi \times_{\pi_i} lk(B))) \backslash [\pi \times_{\pi_i} (S^0 \times D^2)] \sqcup_{\pi \times_{\pi_i} (S^0 \times S^1)} [\pi \times_{\pi_i} (D^1 \times S^1)]$,

i. e. an equivariant spherical modification of $lk(A) \sqcup (\pi \times_{\pi_i} lk(B))$ which is equivalent to the equivariant connected sum of linear π actions between $lk(A)$ and $\pi \times_{\pi_i} lk(B)$ along the orbit $\pi \cdot x$ as in Chapter 4. ∎

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