INVESTMENT BEHAVIOUR OF CANADIAN LIFE INSURANCE COMPANIES

To RONI, NATALIE

and MY PARENTS.

MEAN-VARIANCE UTILITY FUNCTIONS AND THE INVESTMENT BEHAVIOUR OF CANADIAN LIFE INSURANCE COMPANIES

By

ITZHAK KRINSKY, B.A., M.A.

A Thesis
Submitted to the School of Graduate Studies
in Partial Fulfilment of the Requirements
for the Degree of
Doctor of Philosophy.

McMaster University

October, 1982

DOCTOR OF PHILOSOPHY (1982)
(Economics)

MCMASTER UNIVERSITY
Hamilton, Ontario

TITLE: Investment Behaviour of Canadian Life Insurance Companies: A Mean-Variance Approach

AUTHOR: Itzhak Krinsky, B.A. (Tel-Aviv University)
M.A. (Tel-Aviv University)

SUPERVISORY COMMITTEE: Professors V.A. Aivazian (Chairman)
J.L. Callen
A.L. Robb

NUMBER OF PAGES: x, 162

ABSTRACT

In recent years, considerable effort has been directed toward establishing the nature of the investment behaviour of life insurance companies. In this dissertation an extended portfolio analysis model was developed for the simultaneous determination of the efficient composition of insurance and investment activities of a life insurance company. This was done within a model that takes advantage of the existing finance foundations and the concepts and techniques of modern demand system analysis.

Unlike current models which used quadratic programming techniques and are interested in the construction of efficient sets, we have used a utility maximization approach. A two parameter portfolio model was constructed utilizing elements of utility theory and of the theory of insurance. The model provided us with the proportion of assets held in the balance sheet as well as which liabilities are used to raise the necessary capital.

The model developed has sufficient empirical content to yield hypotheses about life insurance portfolio behaviour and thus was tested using appropriate econometric techniques. A comparative static analysis yielded elasticities of substitution between financial assets and liabilities. The estimation of these elasticities in the context of a flexible functional form model, forms a central part of this dissertation. More specifically, by utilizing a mean-variance portfolio framework and a general Box-Cox utility function we were able to model the demand for assets and liabilities by an insurance company.

On empirical grounds we found that, in general, the square root quadratic utility function best fits the data. We also tried to evaluate the square root quadratic approximation by showing that, broadly speaking, it yields signs for elasticities of substitution which are consistant with the theory.

A by-product of the model developed is the ability to compare stock and mutual life insurance companies. The common belief that mutual companies follow a riskier path in the way they conduct their business was supported by the results in this study.

The results obtained from the study are of significant importance since life insurance companies have substantial obligations to millions of households in the economy. Furthermore, despite the extraordinary decline in the importance of the life insurance industry in the bond and mortgage markets during the sixties and the seventies, the industry is still a major supplier of funds to those markets.

ACKNOWLEDGEMENTS

In the process of writing a dissertation one necessarily incurs many debts. Intellectual debts are the most important and the hardest to quantify but the influence, work and example of the professors in the department of economics at McMaster University have had great effects on the way I have come to approach the subject.

First and foremost, I would like to thank the members of the dissertation committee for their advice, support and patience. In particular, I am indebted to Professors V. Aivazian and J. Callen who introduced me to finance and gave me insightful comments and encouragement throughout. Professor Les Robb bore a particularly heavy burden of supervisory duties after Professor Aivazian left for the University of Toronto and Professor Callen's sabbatical leave at Hebrew University. He devoted countless hours and helped me in more ways than expected. Without the effort and the generous assistance of the committee, this dissertation would have never been completed.

A vote of thanks is also due to Professors F.T. Denton and E.H. Oksanen who allowed me to share their time and knowledge.

I would also like to thank Betty May Lamb for her superb typing of numerous drafts of the manuscript.

The financial support through various McMaster scholarships and the Ontario Government scholarship is gratefully acknowledged.

Finally, the contribution of my wife, Roni, was of inestimable importance. No perfunctory uxorial reference of "cheerful encouragement" can do justice to her contribution.

Needless to say, I alone am responsible for remaining errors. No matter what time and effort is put into a task, perfection is hard to come by.

TABLE OF CONTENT

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

"Everything's got a moral if
only you can find it"

Lewis Carroll,
Alice's Adventures in Wonderland

I.1 Introductory Remarks

There is almost a consensus among economists that risk is the essential problem to which insurance provides an answer. Human beings recognize the existence of certain kinds of risk and show an anxiety for finding means of protecting against losses risk can provide. The danger of a fire, a storm at sea or an untimely death, have been recognized for centuries as risks for which protection must be found. Among the protective steps suggested have been the development of fire insurance, marine insurance, life insurance,etc. The characterization of risk as a human problem and insurance as a means for reducing the seriousness of the problem is well known in the literature.

The fundamental principle upon which insurance has been based for several centuries is a simple one; pooling of risk. When a group of people that are all subject to the same unpredictable event (for example, a fire) pool a sum of funds, it becomes possible for those who actually do suffer the event to recover from the pool. Those who do not suffer the event within a specific time period have in effect purchased a conditional promise of protection, the condition being the occurrence of the event. For life insurance the unpredictable risky event is the time of death not death itself. Thus, a person who participates in a life insurance pool receives protection against the investment of this pool of funds, around which, this thesis concentrates.

The number of companies which provide life insurance coverage in Canada has increased steadily in the past century. There were 10 Canadian federally registered companies in 1885, 22 in 1905 and 147 in 1979. At the end of 1979 there were also 25 companies registered under provincial laws. Of the 172 active life insurance companies at the end of 1979, 83 were Canadian incorporated, 63 United States, 10 British and 10 from continental Europe. About 78 percent of the 381 billion dollars of life insurance owned by Canadians at the end of 1979, was with Canadian incorporated companies, 17 percent with United States companies, 5 percent with British companies and under 1 percent with other European companies. (see Table I.1)

Table I.2, contains data on gross insurance premiums and annuity income received by Canadian companies. It is evident from the data that annuity premiums as a fraction of all premiums have increased from 14% in 1950 to 46% in 1979. The introduction of Registered Retirement Savings Plans (RRSP's) in 1957 (which come under the annuities heading), encouraged the rapid growth of annuities and pension plans, since one is allowed to deduct from taxable income, premiums paid into pension plans before earnings are assessed for income tax.[1]

As Table I.2 shows, the proportion of total gross premiums plus annuity income obtained from individual life policies has fallen substantially from 85% in 1950 to 62% in 1979. The change in the

---

[1]The 1972 Act which raised maximum tax deductable contributions, provided additional encouragement.

TABLE I.1: Life Insurance Owned In Canada by Type of Company ($000,000 omitted)

END OF YEAR	CANADIAN Federally Registered	BRITISH and other EUROPEAN	UNITED STATES	TOTAL Federally Registered Companies	CANADIAN Provincial Companies	TOTAL
1900	267	40	124	431	-	431
1910	566	48	242	856	-	856
1920	1,664	77	916	2,657	18	2,675
1930	4,319	117	2,056	6,492	96	6,588
1940	4,609	146	2,220	6,975	66	7,041
1945	6,442	183	3,126	9,751	213	9,964
1950	10,756	343	4,647	15,746	483	16,229
1955	17,401	695	7,355	25,451	1,173	26,624
1960	30,418	1,629	12,602	44,649	2,218	46,867
1965	47,900	3,504	18,252	69,656	4,419	74,075
1970	76,775	6,340	28,001	111,116	7,706	118,822
1971	84,946	7,097	29,839	121,822	8,060	129,942
1972	96,292	7,856	32,256	136,404	8,772	145,176
1973	109,505	8,699	35,399	153,603	9,689	163,292
1974	128,179	9,969	38,972	177,120	11,551	188,671
1975	151,816	12,206	43,700	207,722	12,190	219,912
1976	179,083	13,800	49,807	242,690	14,462	257,152
1977	210,962	15,510	51,667	278,139	15,581	293,720
1978	239,801	17,046	56,586	313,433	19,303	332,736
1979	273,393	19,718	64,434	357,545	23,527	381,072

Source: Canadian Life Insurance Association, (1980).

TABLE I.2: Life Insurance and Annuity Premium Income*
($000,000 omitted)

	INSURANCE		ANNUITIES		
YEAR	INDIVIDUAL	GROUP	INDIVIDUAL	GROUP	TOTAL
1900	15	0	-	0	15
1910	30	0	-	0	30
1920	89	1	-	0	90
1930	215	5	2	1	223
1940	193	9	10	4	216
1945	252	14	15	14	295
1950	353	29	21	39	442
1955	487	55	27	81	650
1960	660	109	33	151	953
1965	846	181	64	262	1,353
1970	1,128	325	126	311	1,890
1971	1,185	357	184	419	2,145
1972	1,213	406	273	499	2,391
1973	1,292	456	433	575	2,756
1974	1,404	535	542	618	3,099
1975	1,543	604	634	735	3,516
1976	1,670	684	772	884	4,010
1977	1,799	755	880	1,039	4,473
1978	1,972	801	1,066	1,188	5,027
1979	2,143	870	1,278	1,267	5,558

Source: Canadian Life Insurance Association (1980).

* Figures for provincial companies are included from 1940 on.

product mix is dominated by the decline in individual insurance and increase in group contracts which contributed to the reduction in the proportion of financial assets held by the life insurance industry relative to other financial institutions.[2]

In Table I.3, we present an overview of two aspects of the portfolio of assets held by the insurance industry in Canada namely, annual holdings and net acquisitions. The table illustrates the variability of net acquisitions and indicates that life insurance companies are highly diversified in terms of the type of financial assets held. The proportion of assets held in the form of bonds declined from 67 percent in 1950 to 36 percent at the end of 1979 while mortgage loans went up from 20% to 39% in the same period.[3] The data also reveals a rapid growth of stocks relative to other assets from 3 percent to approximately 11 percent in less than 30 years. Thus, it would appear that the investment decision process is not solely one of directing new funds but also one of reviewing the holdings of the existing portfolio. Those decisions, which are taken by individual companies are examined in this thesis.

The results obtained from this study are of significant

---

[2]Another factor which contributed to this phenomena is the shift from Whole life (a policy with a high saving component) to term insurance (a product with no saving component). For an extensive discussion on the contribution of those contracts and others to the funds available for investments see Chapter III.

[3]The proportion of mortgages increased until 1966 (45 percent), levelled off, and then declined during the seventies when other financial institutions entered into this market. (Mortgage and Loan and Trust Companies).

TABLE I.3: Holdings and Net Acquisition of Assets by Life Insurance Companies ($000,000 omitted)

a. Holdings

End of Year	Bonds		Stocks		Mortgage Loans		Real Estate		Policy Loans		Cash		Other Assets		Total
	$	%	$	%	$	%	$	%	$	%	$	%	$	%	$
1950	2,744	67.3	138	3.4	798	19.6	63	1.5	180	4.5	50	1.2	103	2.5	4,076
1960	3,990	48.3	257	3.1	3,105	37.6	294	3.6	356	4.3	49	0.6	209	2.5	8,260
1970	5,697	36.3	1,107	7.1	6,873	43.9	737	4.7	799	5.1	139	0.9	321	2.0	15,673
1971	6,082	36.3	1,447	8.6	7,012	41.8	913	5.4	821	4.9	141	0.8	355	2.1	16,771
1972	6,531	35.5	2,040	11.1	7,348	40.0	1,012	5.5	854	4.6	211	1.1	389	2.1	18,385
1973	7,121	35.5	2,241	11.2	7,969	39.8	1,112	5.5	926	4.6	184	0.9	493	2.5	20,046
1974	7,817	36.1	2,052	9.5	8,705	40.2	1,215	5.6	1,114	5.1	168	0.8	585	2.7	21,656
1975	8,710	36.1	2,535	10.5	9,486	39.4	1,298	5.4	1,205	5.0	273	1.1	590	2.4	24,097
1976	9,681	36.2	2,848	10.6	10,521	39.3	1,421	5.3	1,293	4.0	318	1.2	677	2.5	26,759
1977	10,991	36.7	2,983	10.0	11,909	39.7	1,504	5.0	1,352	4.5	444	1.5	781	2.6	29,964
1978	12,406	36.5	3,694	10.8	13,255	39.1	1,549	4.6	1,443	4.3	438	1.3	1,141	3.4	33,926
1979	13,891	36.1	4,382	11.4	14,876	38.7	1,709	4.4	1,648	4.3	568	1.5	1,355	3.5	38,429

b. Net Acquisition*

	Bonds		Stocks		Mortgage Loans		Real Estate & Ground Rents		Policy Loans		Cash and Collateral Loans		Other **		Total
	$	%	$	%	$	%	$	%	$	%	$	%	$	%	$
1972	266	27.4	228	23.4	226	23.2	78	8.1	16	1.6	37	3.8	122	12.5	973
1973	414	31.6	207	15.8	517	39.5	98	7.5	54	4.1	30	2.3	- 10	-0.8	1,310
1974	349	23.5	110	7.4	550	37.0	108	7.3	156	10.5	16	1.1	196	13.2	1,485
1975	661	40.1	242	14.7	561	34.1	87	5.3	81	4.9	-71	-4.3	86	5.2	1,647
1976	878	43.3	232	11.4	704	34.7	102	5.0	62	3.1	34	1.7	17	0.8	2,030
1977	857	34.8	69	2.8	1,097	44.5	83	3.4	50	2.0	-26	-1.1	334	13.6	2,464
1978	1,206	45.9	194	7.4	1,024	38.9	97	3.7	64	2.4	43	1.6	2	0.1	2,630
1979	1,182	41.1	-126	-4.4	1,317	45.8	116	4.0	154	5.4	37	1.3	195	6.8	2,875

Source: Canadian Life Insurance Association (various years).

* This data shows the transactions of Canadian assets by a group of 16 companies representing 80 percent of total life insurance assets held in Canada.

** Including Treasury Bills and Short Term Commercial Papers.

importance since life insurance companies have substantial obligations to millions of households in the economy. Furthermore, despite the extraordinary decline in the importance of the life insurance industry in the bond and especially the mortgage market during the sixties and the seventies, the industry is still a major supplier of funds to those markets. Thus, any change in the behaviour of life insurance companies will affect those markets. Table I.4 below, reveals the trends described.

TABLE I.4: Life Insurance Investment in Mortgages, Bonds and Stocks as a Percentage of the Total Amount Held by All Sectors

End of Year	Mortgages	Bonds	Stocks
1961	28.6	11.3	1.1
1965	27.3	10.5	1.6
1970	20.3	8.1	2.7
1971	18.4	7.7	3.3
1972	16.8	7.6	4.2
1973	15.3	7.9	4.3
1974	14.2	7.6	3.5
1975	13.3	7.3	3.7
1976	13.0	7.2	3.8
1977	11.9	7.1	3.1
1978	11.2	7.2	3.2
1979	11.5	7.4	2.6
1980	11.8	7.5	2.8

Source: Financial Flow Accounts, end of year outstandings, Statistics Canada.

I.2 Synopsis of the Dissertation

The dissertation consists of eight chapters including the present, and two appendices. With the present chapter serving as an introduction, the contents of the following chapters are now summarized.

The primary purpose of life insurance is to protect the beneficiary against the risk of financial loss through the death of the individual on whose life the policy is issued. Thus, life insurance companies have developed a large number of "products" which are designed to match the particular needs of different groups of consumers. The major insurance "products" from which the industry derives its funds are life insurance contracts and annuities. Each category in turn can be divided into individual contracts, which are purchased by individuals, and group contracts, which are purchased by a third party to cover a group of lives. The main purpose of the discussion in Chapter II is to highlight the principles underlying the basic contracts written by life insurance companies in Canada, with special attention devoted to the three basic contracts; term, whole life and endowment. As will be seen all contracts are written on the level premium basis and thus contain provisions for policy reserves, which in turn are an important source of funds for investments.

In Chapter III, a description of the creation of funds for investment by life insurance companies is provided. A simple model is developed to explain the pricing policy of an insurance company. It is then shown that, in general, in the earlier years the premium income paid by a group of policyholders will exceed current claims and an insurance fund, available for investment, will be created. Those funds play a vital role in life insurance and therefore the factors which determine their size are considered.

Various investment fund profiles are considered pointing out that policies with larger savings components are the major contributors of investment funds.

Chapter IV surveys part of the existing literature dealing with the independency between investment decisions and insurance portfolio. This chapter is organized in two subsections. The first (IV.1), reviews the hedging hypothesis in the context of life insurance companies. Following this review, it is shown that under certain conditions hedging can safeguard against insolvency. It is argued that when dealing with life insurance companies and because the treatment of surplus provides so much protection of this sort, it is not necessary for firms to closely match maturities. The second subsection (IV.2), deals with studies of a portfolio nature, attempting to explain the optimal balance sheet proportions. Those studies evaluate the risk and return of every asset simultaneously with the risk and cost of every liability. The idea is that life insurance companies should be viewed as managers of a portfolio of insurance policies in addition to the handling of an investment portfolio.

While Chapters II, III and IV provide an important perspective on the phenomenon of portfolio behavior of life insurance companies, the three following chapters constitute the main contributions of the dissertation.

In Chapter V, a model is developed for the simultaneous determination of the efficient composition of insurance and investment activities of a life insurance company. Since the two parameter portfolio model is used as a benchmark for developing our model, its major features are summarized (Section V.1). In Section V.2, the model is constructed while utilizing elements of utility theory and of the theory of insurance. A comparative static analysis of portfolio adjustment (Section V.3) and some further theoretical implications conclude this chapter.

While Chapter V offers a theoretical framework within which we can attempt to organize and interpret the data, in Chapter VI we show how the theory can be made specific and useful in empirical work. The use of flexible functional forms as a tool will be explained and particular results will be referred to the general theory (Section VI.1). Furthermore, share equations and expressions for marginal utilities and elasticities of substitution are derived within the context of flexible functional forms (Section VI.2). This chapter is technical in nature and it was decided not to put it as an appendix because of its importance in computing the empirical results.

The objective of the empirical work, presented in Chapter VII, is to illustrate the portfolio optimization in the presence of some relevant constraint. To do so, we use data for eight Canadian life insurance companies. The empirical results are then tested and analysed.

Since many important themes are not addressed by this dissertation and since substantial detail is lost by the aggregate nature of the data available, Chapter VIII provides suggestions for further research and identifies issues thought to be important for potential implementation of this model to other intermediaries. In addition to this, Chapter VIII summarizes and concludes the dissertation.

Finally, two appendices are provided. Appendix A contains a detailed discussion on the regulations imposed on Canadian life insurance companies. Appendix B presents the data used in the statistical analysis. Footnotes are provided at the bottom of each page and the bibliography is a separate section at the end of the dissertation.

CHAPTER II

THE INSURANCE CONTRACT

"Grow old along with me!
The best is yet to be,
The last of life for which the first was made"

Rabbi Ben-Ezra

As part of the attempt to remain competitive both within the savings market generally, and in the life insurance industry in particular, life insurance companies have developed a large number of products, each designed to meet the particular needs of different groups of consumers. The existing range of products are likely to be enlarged and modified over the future, as a result of changes in government restrictions,[1] interest rates, inflation, and other socio-economic factors.

The major insurance "products" from which the industry derives its funds are life insurance contracts and annuities, Each category in turn can be divided into individual contracts, which are purchased by individuals, and group contracts, which are purchased by a third party to cover a group of lives.

II.1 Life Insurance

II.1.1 Life Insurance - Individual Life Policies

The essence of private life insurance (and life annuity plans) is that they are risk sharing arrangements. With life insurance the risk is that of dying prematurely, while with life annuities it is of exhausting wealth before death. An individual who purchases an

---

[1]For example, government restrictions on the type of life insurance which qualifies for tax relief.

insurance plan pools his own risk with the risk of other individuals. Each receives protection at a cost which depends primarily on the expected cost of benefits for the entire pool. For those who choose to participate in the pool, an assessment is made of the degree of risk which each person adds to the risk pool. The assessment involves identifying those characteristics which significantly affect longevity. Factors taken into account in assessing longevity i.e., in estimating the risk that a particular person brings to the risk pool are: age, sex, medical history, physical condition, occupation, avocation and life style. Life insurance policies are of three major types, term, whole (or permanent) life and endowment. These will be discussed in turn.

II.1.1.1 Term Insurance

A term insurance policy provides life insurance protection for a specified period. The death benefit under the policy is payable only if the insured dies within the specified term. None of the premiums are returned once the period of the policy has expired. It is therefore clear, that no saving options are included in term policies, and their sole purpose is to provide insurance against risk.[2] Life insurance companies employ term insurance to provide the risk cover in other types of policies - whole life or endowment - and to provide the lump sum death benefits in some group contracts.

The term may be for a specific number of years, usually 1 to 25, or it may be to a specific age, usually 65 to 70. Premium payments are typically level throughout the term period. If the policy is

---

[2] Term insurance is also known under the name of "pure insurance".

"renewable", premiums increase at the end of each term period throughout the life of the policy. With decreasing term insurance, the amount of insurance protection[3] decreases each year (often after some age, say 70) until the insurance expires.

Term insurance provides the cheapest protection of any form of individual life insurance. One reason is that it does not provide any insurance protection after the end of the term when mortality rates are higher. Another reason is the lack of a saving component in this policy.

A Convertible Term Policy is one that may be converted to a whole life or endowment insurance during a specified period. This type of policy is of benefit to young people who may want high coverage, but cannot afford the higher premium for endowment insurance. After a period of time, when either their income or needs have changed, they can take the option to switch to another policy without any further medical evidence being required.

II.1.1.2 Whole or Permanent Life Insurance

Whole or permanent life insurance pays the face amount, also called the sum insured, on the death of the life insured. With level premiums, the insured makes excess payments with regard to mortality expectations in the early years. The excess payments accumulate as savings by policy holders and provide insurance companies with a pool of funds for investment purposes.[4] These funds grow with each premium

---

[3]The decrease in protection relates to the amount payable on death of the insured.

[4]A detailed discussion on the funds is presented below.

payment and with investment earnings. They are reduced by the insurers' charges for expenses, contingencies and profit.

Premia for a whole life policy are payable throughout life unless special arrangements are made for payments to cease at a given age. Whole life often has the option to convert to an endowment policy (described below) within a certain period at a fixed premium rate. Other options available are policy loans, automatic premium loans, reduced paid-up insurance, and extended term insurance.[5]

II.1.1.3 Endowment Policies

When a fixed sum is payable at the end of a specified period if the insured is still living, or at death, the life insurance plan is called endowment insurance. The endowment period may be a specified number of years 10, 15 or 25, or may extend to a specified age such as 60, 65 or 70. Premiums are typically level throughout the endowment

---

[5] The Definitions of these new terms are as follows:

Policy loan - Ordinary life insurance policies, in general, contain a privilege enabling the insured to borrow from the life insurance company on the security of the policy an amount within the guaranteed cash surrender value of the policy.
Automatic premium loan - Overdue premiums may be paid automatically by means of a loan on the policy. The policy is continued in force without change or cancellation of any benefits. When the loan, with interest, equals the cash surrender value, the policy terminates.
Reduced paid up insurance - Where the cash surrender value is applied in exchange for a paid up policy (policy bought with a single premium), the new benefit is referred to as the reduced paid up policy. The amount of the new policy will be whatever the cash surrender value will buy as a single premium. The amount of paid up insurance is guaranteed and stated in the policy.
Extended term insurance - when at any point of time the net cash surrender value under a life insurance policy is applied as a net single premium to purchase paid-up term insurance for an amount equal to the sum insured at that time, the resulting insurance is called extended term insurance. It is clear that the policy will be for a shorter period of time. This period will be the number of years and days that the cash surrender value will provide when applied as a single premium.

period. Whole life insurance might actually be considered as endowment insurance to age 100 or some very high age. The endowment as in whole life, combines decreasing term insurance (risk cover) with a savings plan. The saving element in the policy necessitates the collection of higher premiums than required for the term insurance. Compound interest is then added to the accumulated premia to produce a lump sum amount payable at the end of the term (in endowments) and/or on death of the insured. At any time during the validity of the policy, a capital sum will have accrued to the credit of the insured. But, this sum may not be sufficient to pay the face amount of the policy in the event of death. Therefore, the decreasing term insurance part of the policy is used and is sufficient at any time to make up to the sum assured.

II.1.2 Life Insurance - Group Contracts

Group life insurance has experienced more rapid growth than the other competing systems of insurance, industrial and ordinary. Unlike individual insurance it insures the lives of a number of persons by a single contract. The single contract is called the master policy. The individuals insured are not actually parties to the contract. However, the law requires that they be given a certificate describing their coverage rights and privileges under the contract. Typically, an employer takes out such a contract for the benefit of his employees. Other groups to whom life insurance is available are: trade associations, associations of individuals (the most common - professional associations), unions and creditor's group insurance.[6] Other unique characteristics of

[6]Plans of this type provide insurance to individual borrowers of a particular creditor.

group insurance are:

a. With few exceptions, group insurance is issued without medical examination or other evidence of individual insurability. Therefore, it reaches a market that might otherwise go uninsured if forced to rely on individual forms of insurance.

b. Group insurance is low-cost insurance because of savings in administrative expenses, the emphasis on group underwriting, and lower commission rates.

c. Premiums are usually subject to experience rating. The policy may give the insurer the right to change the monthly premium at any time after the first policy year, but not more than once a year.

d. Group insurance is of a continuing nature, in that the contract and the plan may last long beyond the lifetime, or membership in the plan, of any one individual.

e. There are size specifications relating to the minimum number of persons and the minimum proportion of the entire group covered.[7]

II.1.2.1 Group Term Life Insurance

Group term life insurance is one year renewable term insurance. In its most common form it is issued to a policyholder who is an employer for purposes of protecting his employees. The insurance is referred to non-contributing if the policyholder pays the whole cost and contributory if the person insured pays part of the cost. The amount

---

[7]Originally group insurance was made available only to groups of 100 lives or more. In recent years this has been reduced to a minimum of 10 lives or more. Some limit is necessary to give the insurer some protection against the insurance being taken primarily for the purpose of covering one or more individuals with incipient losses and claims.

of insurance to which the insured is entitled is indicated in a schedule in the master contract. The amount may be the same for all the lives insured or it may be based on such factors (for employees) as salary, position, length of service, size of family etc. Coverage may be extended to cover the insured's spouse and eligible children. Most group plans include a disability waiver provision and a conversion privilege. The conversion privilege allows the insured to convert his group insurance within 31 days of terminating membership in the group to an individual policy issued without evidence of insurability.

II.1.2.2 Group Ordinary Life Insurance

Group ordinary life insurance is provided under a rider attached to a conventional group term life insurance policy, which enables an eligible insured individual to have all or portion of his scheduled group term life insurance coverage changed to one or more forms of permanent life insurance. Level annual premiums are charged based on the insured age at the effective date of his insurance.[8]

II.1.2.3 Other Group Contracts

Group Creditor Life Insurance - Issued under a master contract to institutions lending money or financing the sale of goods. If the borrower or credit customer dies, the debt is paid and the family of the deceased is relieved of the financial burden of an unpaid debt. Note the creditor is both policyholder and beneficiary.

Group Survivor Income Benefit - Similar to one year renewable term insurance, but it provides a monthly income benefit to eligible

---

[8] Unlike the regular group insurance some sort of risk classification based on age exists in group permanent contracts.

survivors rather than a specified lump sum.

Group Paid Up life insurance is a combination of annually increasing, single-premium, permanent whole life insurance with cash values and decreasing one-year renewable term insurance.

II.2 Annuities

In its simplest form an annuity involves the insurance company paying out a given amount at a regular interval, usually monthly, to the annuitant during his lifetime[9] in exchange for an initial single premium. The payments begin at a specified date and extend for a designated period of years or for the remainder of the annuitant's life.[10] The premium becomes immediately the property of the company.

While life insurance protects the insured against the financial difficulties resulting from dying prematurely, an annuity contract protects the annuitant against financial difficulties consequent upon "living too long". The following discussion will deal separately with individual annuities and group annuities or pension plans.

II.2.1 Annuities - Individual Contracts

Annuities may be classified on at least four different bases

a. Method of paying premiums

b. Disposition of proceeds

c. Number of lives covered

d. Date benefits begin

These classifications are not mutually exclusive since every annuity will

---

[9] A series of annual premiums can also be paid.

[10] The usual term is for the life of the annuitant with a fixed number of years guaranteed.

fall into all four classes. The last method will be used here.

II.2.1.1 Immediate Annuities

An immediate annuity is bought with a single lump-sum payment. The income payments usually begin one month later, if they are to be paid monthly, or one year later, if they are to be paid annually. Immediate annuities are typically bought by people at retirement with funds that have been accumulated over the years in mutual funds, banks, trust companies or life insurance companies.

II.2.1.2 Deferred Annuities

Under this contract income payments start at the end of some specified period of time or at a specific age. The policy may call for regular periodic payments or may allow the policyholder some flexibility as to the amount and timing of the contributions. The annuity income will depend upon the contributions made over deferred period.

A fixed dollar annuity guarantees the annuitant a fixed, minimum number of dollars during each pay-out period. If the amount of monthly income payment is not fixed, the annuity is called a variable annuity. The variable annuity was developed as a response to inflation and it is based on units of value rather than units of currency. The amount payed each period depends on the investment performance of a block of assets.[11]

Life insurance companies have also been permitted to set up Segregated Funds which are invested in common stock and other eligible assets where the market value reflects changes in economic activity and the

---

[11] In some contracts the payments depends on mortality and expense experience as well.

price level. Premiums are used to acquire "units" in the fund during the accumulation period. The number acquired for any premium depends on the current market values of a unit. At the end of the deferred period the current value of all the units acquired determines the periodical income.

II.2.2 Pension Plans

A pension plan means an arrangement by which a program is established to provide for the payment of specific amounts to employees after their retirement. At normal retirement age a pension becomes payable for life, usually with a guaranteed period. The amount of such pension will depend on the terms of the plan which may be classified as follow:

a. Defined Contribution Plan.[12] The benefit is determined by the amount of contributions. Contributions can be made either by the employee or by the employer or both.

b. Flat Benefit Plans. Provides a fixed benefit, say $10 monthly for each year of service.

c. Unit Benefit Plans. The benefit is a function of years of employee service and/or earnings.

There are two methods whereby life insurance companies in Canada have extended their services to pension funds: individual policies,[13] and group annuity contracts.

---

[12] Also known as "defined benefit plan".

[13] Usually these are in the form of a deferred annuity (purchased by an employer for the benefit of an employee) discussed before. The pension plan then consists of a number of individual policies under the terms of a trust agreement for the purpose of carrying out the conditions of the plan.

II.2.2.1 Group Annuity Contract

A group annuity is a contract between the plan sponsor and a life insurance company. It provides for the purchase each year of a deferred retirement annuity (fixed or variable) for each of the employees covered by the pension plan. Every year an additional deferred annuity in the appropriate amount will be purchased for the employee's account. At retirement, the employee will be entitled to the income from a series of deferred annuities purchased. The naming of beneficiaries, leave of absence provisions, portability, resting provisions,etc. are intergral parts of every plan.

II.3 Participating And Non-Participating Contracts

A policy providing for payment of dividends is called a participating policy. Conversely, a policy which does not provide for payment of dividends is a non-participating policy. Mutual companies which are owned by their policy owners normally write only participating policies. A stock company writes primarily non-participating policies, although many stock companies offer participating policies as well.

Life insurance deals mainly with long-term contracts. When calculating its premiums the insurance company does not know the interest rates, mortality rates and expenses it is going to experience in the future. These must be estimated based on what has happened in the past and making allowance for any conditions which are expected to change in the future. A dividend is a refund of that portion of the premium paid that is in excess of the amount necessary for current benefit payments, expenses, and reserves required to cover future policy guarantees.

According to Canadian federal insurance law as given in Section 83 of the Canadian and British Insurance Companies Act

> "Every company ... shall keep separate and distinct accounts for participating and non-participating business."

From all net income on participating contracts (premiums and interest earnings less benefit payments and expenses) the company must set aside any additional funds required to cover future claims (increase actuarial reserves) and funds to allow for future unexpected fluctuations (contingency or surplus funds). The remaining funds will be available for dividend distribution to policy owners.[14]

II.4 Summary

The principles underlying the three basic plans,term, whole life and endowment,are very important since all the other plans can be seen as variations or combinations of them.

The principles underlying the three basic plans are very important since all the other plans can be seen as variations or combinations of them.

As was explained above, all the three basic types of policies are written on the level premium basis thus contain provisions for a policy reserve. In the whole life and endowment policies, the revenue must eventually equal the face value of the contract because, at some point payments become certain. It is therefore clear, that an important source of funds for investment occurs as a result of level premiums. In what follows, a description of the creation of funds for investments by life insurance companies is provided.

---

[14] For further details see R.T. Jackson (1959).

CHAPTER III

SOURCES OF FUNDS FOR INVESTMENTS

"How to have your cake and eat it too:
Lend it out at interest."

Anonymous

At the end of 1979, life insurance companies had about 38.5 billion dollars invested in Canada. This is an increase of about 150 percent since the beginning of the 1970's.[1] At the same time, the price level more than doubled so that in real terms, the assets fell over this period.

The main sources of funds available for investment have been:

a. Premia collected in early policy years under the level premium plans that are in excess of those needed to pay claims and expenses for those years.

b. The accumulation of funds under annuity contracts and pension plans.

c. Funds left with the company under dividend options (Participating contracts).

d. Funds left with the company under policy settlement options. For example, proceeds of policies which have become payable can be left with the company at interest. It also includes dividends similarly left at interest and prepaid future premiums.

Among the sources mentioned above, the first two are of major importance and therefore, will be discussed in detail.

---

[1] Canadian Life Insurance Facts, (1980).

It was indicated earlier, that different types of policies give different mixtures of protection and savings. It is the saving element that generates funds for long-term investments. As a result, policies with larger savings component are the major contributors of investment funds.

If all life insurance had to be purchased and premiums calculated on a year by year basis, the cost of life insurance would be higher each year with advancing age. This is a result of the increased probability of death as the insured ages. In order to avoid the problem of adjusting premiums annually, life insurance companies often use level premium policies. That is, policyholders pay the same premium each and every year, overpaying in the early years when mortality is low, and underpaying in later years when mortality is high, as shown in Figure III.1.[2] Calculations of the necessary amount of level premiums are based on the nature of mortality tables and the present value concept. A simple model is developed below explaining the calculation of premiums for the three basic types of plans. This model is based on a verbal discussion and examples in A. Pedoe and C.E. Jack (1978),[3] and is presented here for the benefit of the reader.

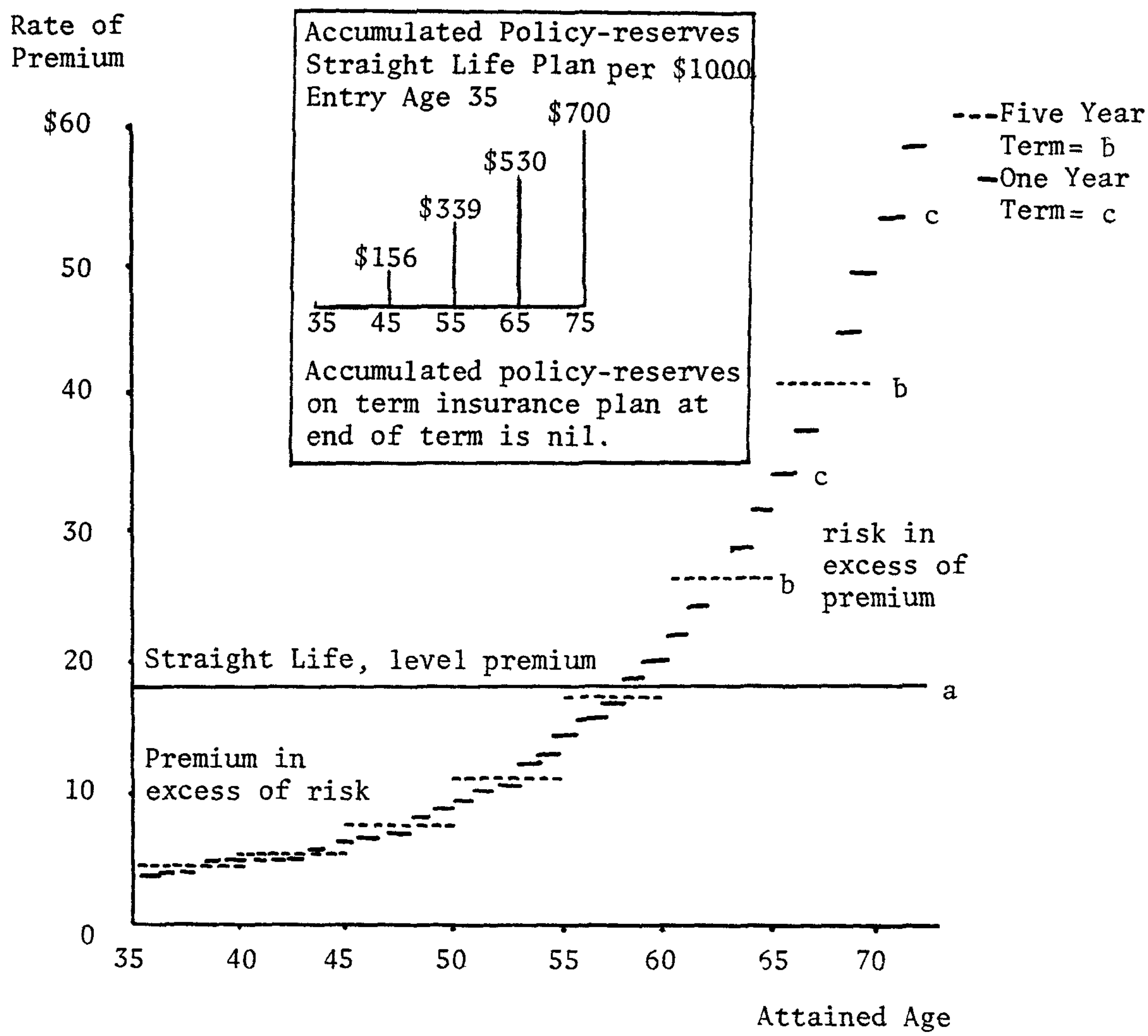
Assume that all annual premiums are paid in advance in the sense that the first annual premium is paid at the beginning of the first year, that for the second year at the beginning of the second

---

[2]Source: Pedoe and Jack. (1978), p. 68.

[3]See there pp. 59-84.

FIGURE III.1: Comparison of Net Level Annual Premiums: a) Straight Life, Age 35 at Entry, b) One-Year Term, c) Five-Year Term, CM(5) 3% per $1,000 Sum Insured



year, and so on. It is further assumed that death claims are payable at the end of the policy year in which death occurs.[4]

Define

AP_i - The annual premium paid to the insurance company in the i^{th} year.

P_i - Probability that the insured will die during the i^{th} year given that he "survived" until that year.

SI - The sum insured.

r - Expected rate of interest based on the expected earnings of the company on its investments.

AR_i - Actuarial (i.e., expected) reserves in period i.[5]

Consider first a five year term contract written on a level premium basis. We obtain the following five equations:

(1) $$AP_1 = P_1[\frac{SI}{1+r}] + AR_1$$

(2) $$AP_2 = P_2[(\frac{SI}{1+r}) - AR_1(1+r) - AR_2] + AR_2$$

(3) $$AP_3 = P_3[(\frac{SI}{1+r}) - AR_1(1+r)^2 - AR_2(1+r) - AR_3] + AR_3$$

(4) $$AP_4 = P_4[(\frac{SI}{1+r}) - AR_1(1+r)^3 - AR_2(1+r)^2 - AR_3(1+r) - AR_4] + AR_4$$

(5) $$AP_5 = P_5(\frac{SI}{1+r}) - AR_1(1+r)^4 - AR_2(1+r)^3 - AR_3(1+r)^2 - AR_4(1+r)$$

---

[4] In practice this is not so as claims are payable immediately following proof of death.

[5] In the above model AP_i represents "net premiums". The same model can be used when "gross premiums" are considered. In such a case, all overhead including agent's commissions and administration costs are included in r. It is then possible to consider r as the rate of return of the insured.

where

$$AP_i = AP - \text{for all } i$$

This system has a solution since we have five equations and five unknowns namely AP, AR_1, AR_2, AR_3, AR_4. Equation (5) is different from the other four equations as a result of the nature of the last year in a term contract. In the last year all funds accumulated must equal the expected claims.

If a five years endowment contract is considered the same model can be used. Under this contract, as previously explained, the sum insured is paid on the maturity date (at the end of the 5^{th} year in our example) if the insured is then living, or, on his or her previous death. Therefore P_5 would be unity,[6] and equation (5) will be written as:

$$\text{(5A)} \qquad AP = \frac{SI}{(1+r)} - AR_1(1+r)^4 - AR_2(1+r)^3 - AR_3(1+r)^2 - AR_4(1+r)$$

From the above model it follows, that any excess above the one year term rate paid in early years[7] is accumulated to meet claims payments in later years. Therefore, a major part of the early premiums can be put into a fund which grows by virtue of interest payments. The invested income earned reduces the size of the premium

---

[6]When a straight life contract is considered P_i will be equal to unity when i = last year of actuarial life (for example 100).

[7]For example, $AP - P_1(\frac{SI}{1+r}) = R_1$ for the first year.

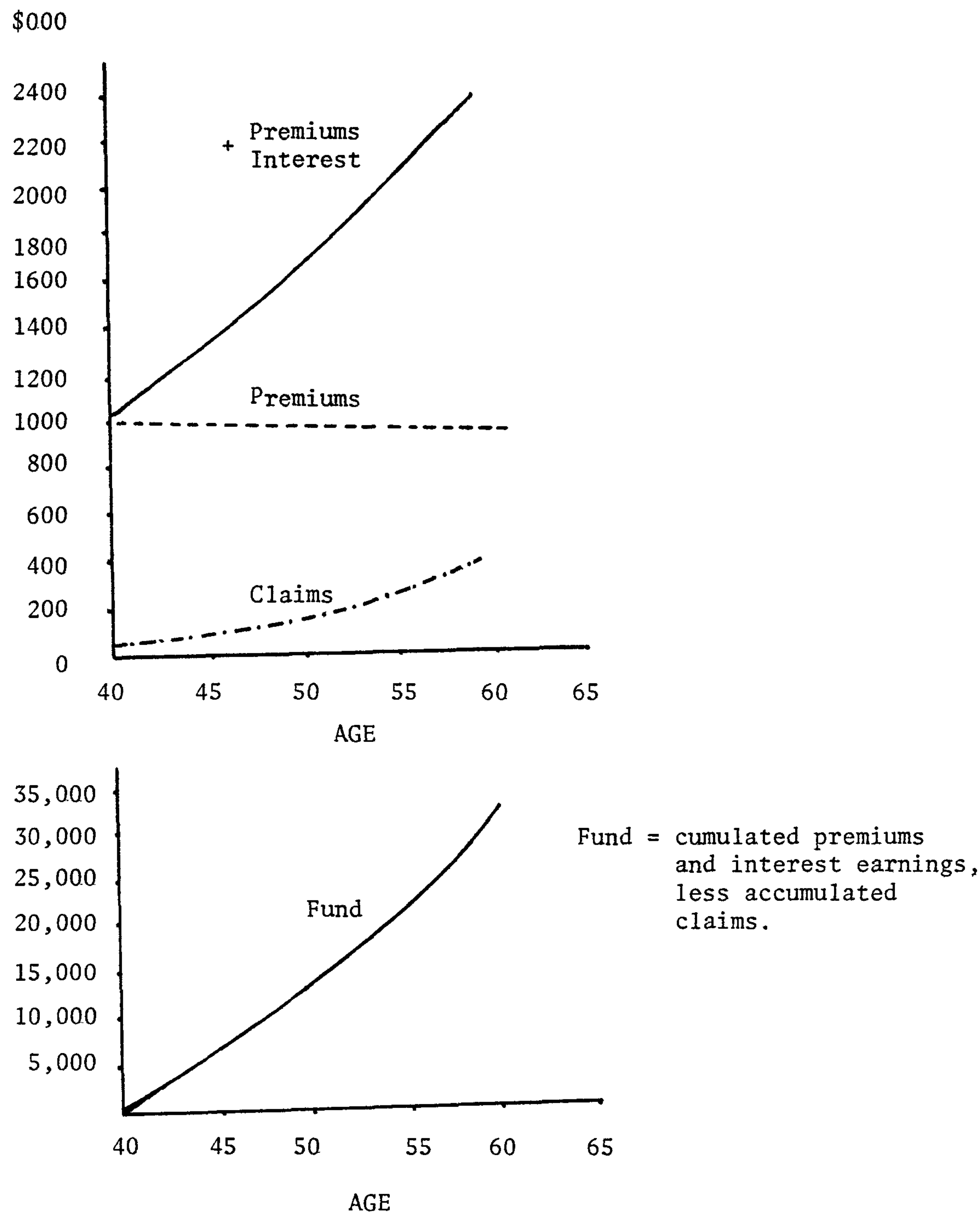
required to make up the face amount of the policy.[8]

Consider now the behaviour of investment funds accumulated under different contracts. In the case of endowment contracts, the fund available for investment increases constantly until the policy matures at the selected period. The combination of premium income and interest earnings on the fund are sufficient to pay the claims at maturity. The following diagram graphs the investment pool curve resulting from the sale of twenty years endowment contracts.

---

[8]Gross premiums and annuity cash inflow do not directly reveal the net amount provided for investment. Operating expenses (including sales commissions and taxes), claims and annuity benefits must be subtracted from gross premiums and annuity consideration while investment income on existing assets must be added before the net source of fund for investment is found.

FIGURE III.2: Accumulation of Funds for Endowment Contracts



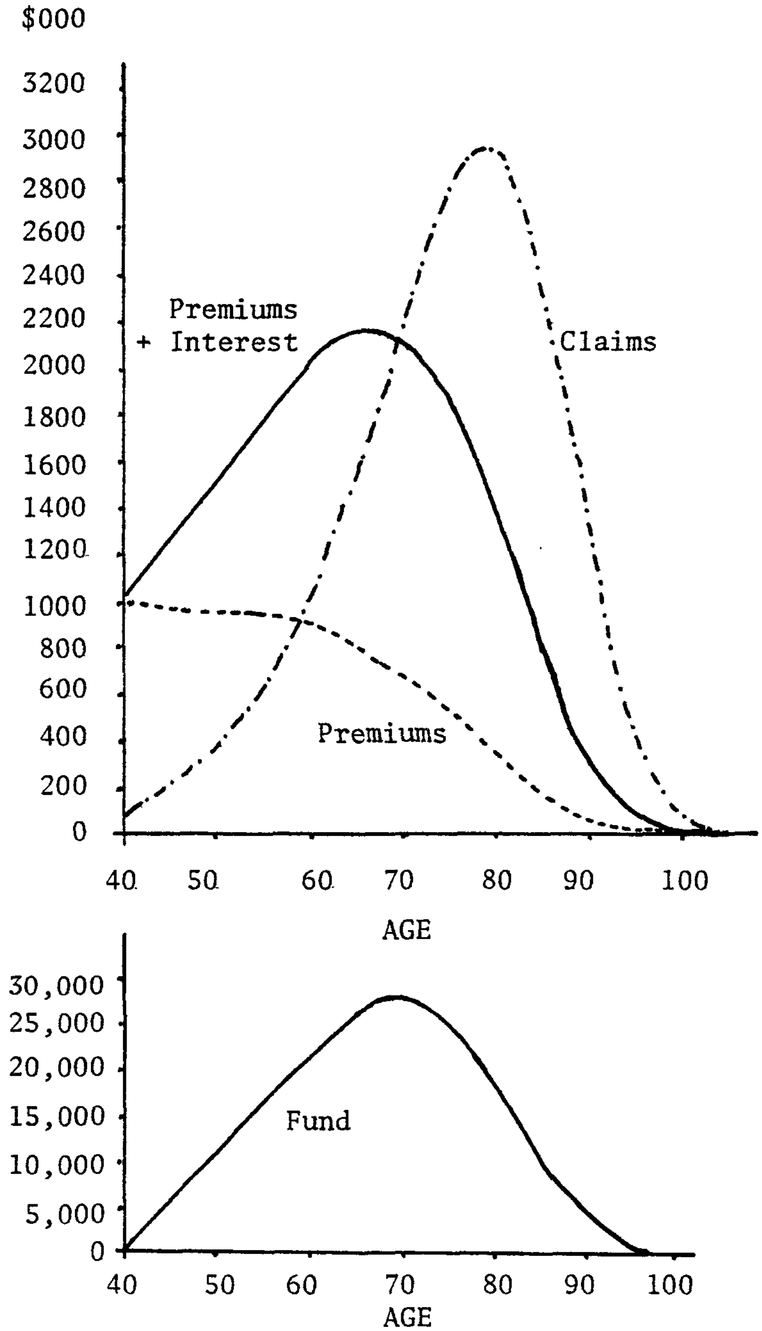
Source: J.C. Dodds (1979) "The Investment Behaviour of British Life Insurance Companies", Croom-Helm London.

Some explanation of the diagram in Figure III.2, is required. The three curves drawn in the upper part are influenced by mortality rates. As a result of increasing death rates (policy holders are getting older), the total amount of claims paid out each year increases and the total premiums collected every year decreases slightly. The curve representing interest earnings plus premiums, however, has a positive slope.[9] In the lower part of the diagram a cumulative curve is drawn representing the total amount of money in the fund. When the policy-holders reach maturity age, say 60, and the policies mature, the total value of the fund is distributed.

Figure III.3 describes the behaviour of a fund for whole life contracts sold to a group of people at the age of 40. The saving element for one policy, never terminated by death, typically grows until it reaches the full amount of the policy at the limiting age of the mortality table. The fund for a group of policies presented in the lower part of Figure III.3 rises, reaches a maximum and then starts to decline when claims exceeds interest earnings and assets have to be realized.

[9]Interest earning are increasing every year as a result of accumulation of investable funds in past years.

FIGURE III.3:[10] Accumulation of Funds for Whole Life Contracts



Source: J.C. Dodds (1979). "The Investment Behaviour of British Life Insurance Companies", Croom-Helm, London.

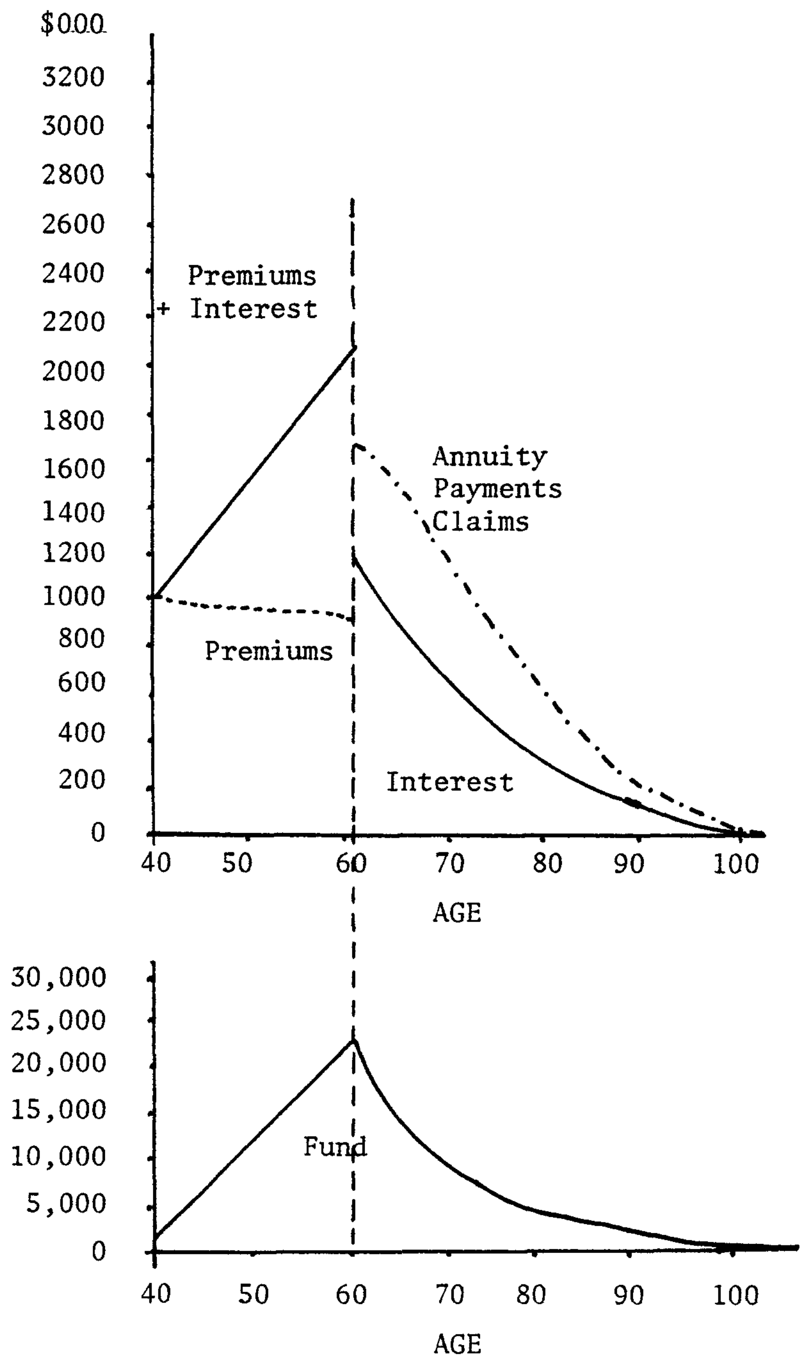
---

[10] The explanation of the curves provided for Figure III.2 is relevant here.

One can see the similarities when the policyholders are between the ages 40-60. After the age of 60 as a result of high mortality rates, there is a sharp drop in premium collection and an increase in claims every year. The premium plus interest curve reaches a peak (similar to the endowment case) and then starts to decline as a result of a drop in premiums and interest earnings (smaller fund).

As might be expected, a deferred annuity has a pattern similar to that of the endowment contract, due to the large savings element in these types of policies. As shown in the lower part of Figure ~~III~~.4. the fund is created at the beginning, reaches a maximum and then ~~is reduced~~ reduced by the annual payments.

FIGURE III.4: Accumulation of Funds for Deferred (40-60) Annuities.



Up to the age of sixty, policyholders are paying premiums and interest earnings plus premiums grow constantly. After the age of sixty, no premiums are collected (thus we have a non-continuous Premiums + Interest curve) and interest earnings fall.[11]

Unlike the other types of policies, term insurance is characterized by a very small saving element. Thus, term policies provide only a small amount of investment funds. Therefore, little investment income can be anticipated and the full cost of insurance protection must be collected as premiums from the term policy owner.

In general, an insurance company sells several types of policies and of different maturities. As a result, the various investment fund profiles will be mixed together. It is also clear that any trend toward term insurance, will decrease the pool of funds administrated by the life insurance companies.

[11]The drop in interest earnings is a result of the decline in the funds invested. The "cash outflow" (Annuity payments) exceed "cash inflow" (interest earnings).

CHAPTER IV

THE INTERDEPENDENCY BETWEEN INVESTMENT AND INSURANCE PORTFOLIOS

> "Man should always divide his wealth into three parts: one-third in land, one-third in commerce and one-third retained in his own hands."
>
> Babylonian Talmud

Life insurance companies play the role of financial intermediaries in financial markets. Financial intermediaries are business institutions which issue claims on themselves and use the proceeds to purchase other financial assets. In the process of conducting their business, they perform various brokerage functions in financial markets. Some of these institutions intermediate in the transfer of funds from surplus to deficit sectors. They intervene in the borrowing-lending process by offering wealth holders obligations which are more attractive than debt and equity issued by borrowers directly. Among the features intermediaries can offer are liquidity, reduced portfolio risk, lower investment costs, and insurance services for property, health or life protection.

IV.1 The Hedging Hypothesis

It is often mentioned in the literature that financial institutions manage their investments so that the maturity composition of the asset portfolio matches, to some extent, the maturity composition of liabilities in order to reduce a variety of intermediaries risks.[1]

---

[1] See for example, F. Modigliani and R. Sutch (1966), or W.T. Terrell and W.J. Frazer Jr. (1972).

For example, it is argued that life insurance companies and other lenders with long term liabilities prefer assets with longer maturities. This is the market segmentation or hedging hypothesis.

If an insurance company owns assets with a shorter term than its liabilities, there is a risk that the funds yielded by the asset at maturity could not be re-invested at the same yield. This makes it risky to promise a given yield to the original long term lenders (policyholders). If, on the contrary, the assets have a longer term than the liabilities, as the liabilities mature and are repaid, the longer term assets may lack liquidity and/or may have fallen in capital value. Thus, if the yields on investments made by the insurance company are less than those assumed in the original premiums, the company could be insolvent and not all claims could be met. Alternatively, where estimates of the future rates of interest are conservative, then after all claims are met, a surplus would exist.

As pointed out earlier, these problems can be solved at least theoretically. By matching asset and liability maturities, such that the maturity value of assets plus interest income equal claims less premiums expected any year, the life insurance company can guarantee that all sums assured can be met.[2] This form of matching can be referred to as absolute matching. But, a life insurance company that only hedges anticipated outlays, runs the risk of involuntarily liquidating assets due to

---

[2]In practice, it is not clear that the life insurance company will be able to find fixed interest redeemable securities with term as long as the longest contract. Furthermore, the guarantee can only be as good as the forecast of expected claims.

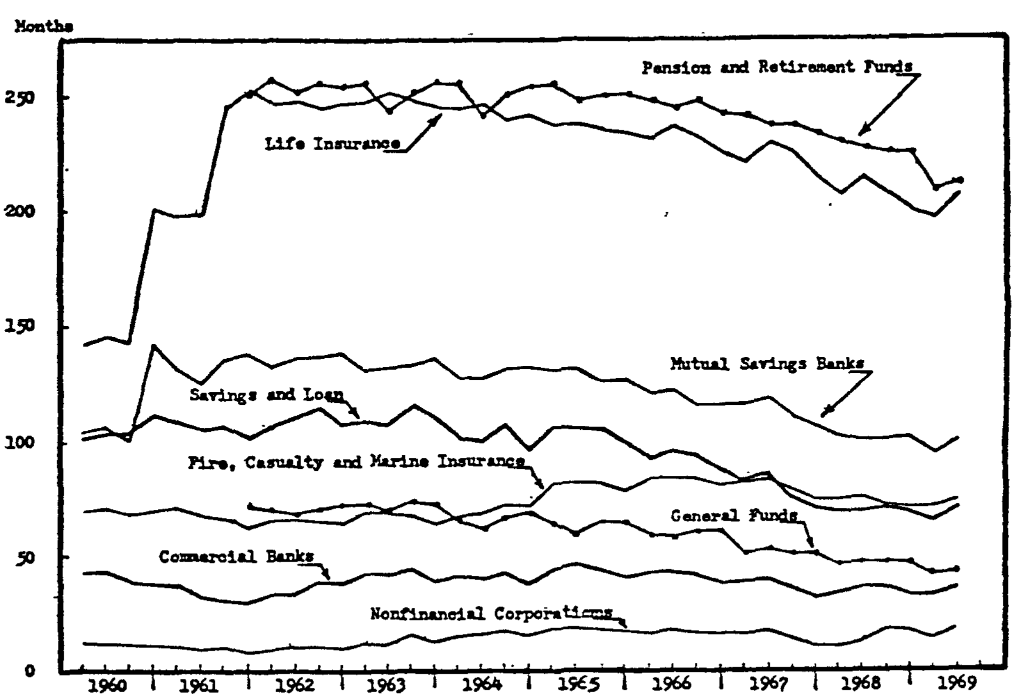
unexpected cash requirements. When consideration is given to hedging along with liquidity, one can expect that the maturity distribution of the assets would be weighted more heavily by short-dated instrument. W.T. Terrell and W.J. Frazer, Sr. (1972) have investigated the maturity distribution of public debt held by institutional investors. They conclude that the distribution of

> "the various groups (of public debt) are shown to be influenced by the liability structures of the institutions in question. Apparently funds are being held in store for meeting an anticipated need and for increasing the certainty of the availability of funds on prescribed terms. Thus, there is support for the presence of a liquidity-hedging motive." (p. 11)

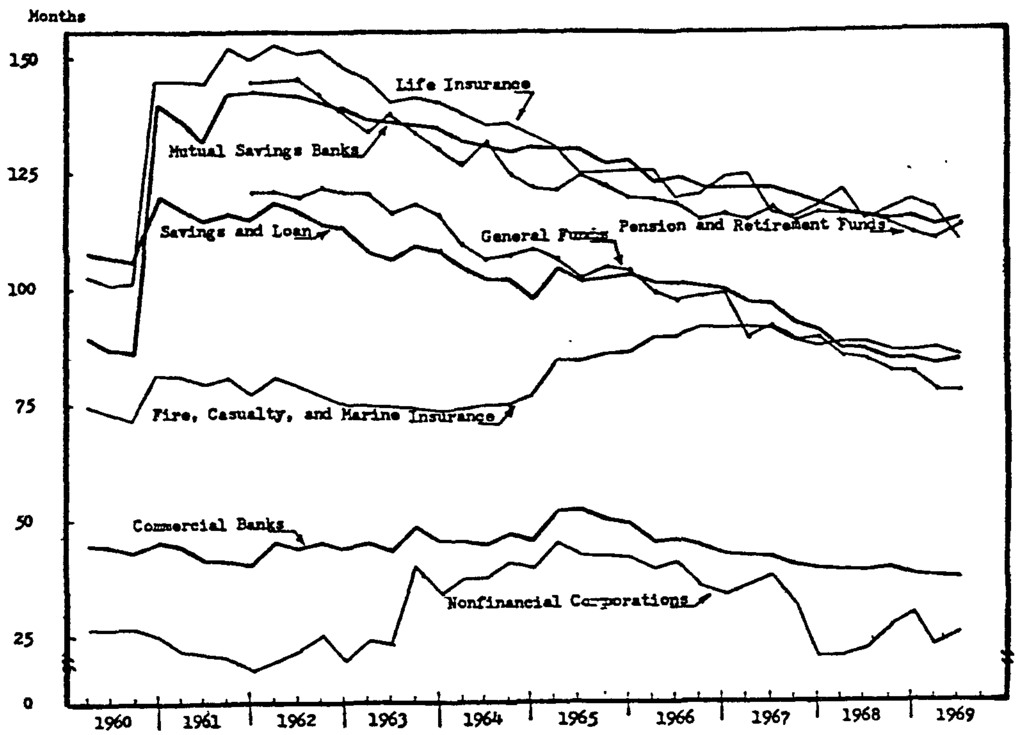
Figure IV.1 below supports their conclusions for U.S. data.

<u>FIGURE IV.1</u>: Marketable Interest-Bearing Debt: Measures of Maturity Profiles, Quarterly I, 1960 to II, 1969

a. Mean Maturity by Institution



b. Standard Deviation of Maturity by Institution



Source: W.T. Terrell and W.J. Frazer, Jr. (1972) (p. 17)

The formal links between hedging and solvency were first explored by A.T. Haynes and R.J. Kirton (1952) who analysed the case where a change in interest rates has no effect on the solvency of life insurance companies. Under strict assumptions they proved that a change in interest rates has no effect on the solvency of a life insurance company that has a standard asset distribution.[3] A major weakness of their model is that it includes only redeemable fixed interest securities

---

[3]The characteristics of a standard asset distribution are that the present value of the fund's net liabilities (where net liabilities are defined to be equal to the present value of future claims less future premiums) is just equal to the fund's assets and, that in the aggregate at each date, over the life of the fund, the sum of the bonds maturity, plus interest income is just equal to the claims coming due.

without recognizing the possibility that other types of assets may be preferred if it is expected they will provide the fund with a higher return. (for example, common stocks).

A later study by B.J. Michaelsen and R.C. Goshay (1967) regard intermediaries as perpetual funds, with assets and liabilities maturing and being replaced continuously. They argue that:

> "To focus on the uncertainty of terminal value is to neglect the opportunity cost of foregoing gains that might occur from successful forecasting of interest rate movements"
>
> Because ...
>
> "long-term assets may be purchased just prior to unanticipated increase in interest rate." (p. 168)

A more recent study by P.J. Franklin and C. Woodhead (1980) points out two instruments (in addition to hedging) to prevent insolvency of life insurance companies:

a. Shareholders capital and undistributed profits.

b. Participating contracts.

The second point may require some elaboration. In previous chapters it was pointed out that in dealing with participating contracts one is concerned with the determination of the amounts of overpayments and the distribution of these overpayments known as surplus. At the end of each year life insurance companies determine their liabilities (i.e., policy reserves and amounts owing by the company including amounts on deposits by policyholders) and their assets. The deduction of the liabilities from assets yields the surplus of each company at the date specified.[5]

Then, a decision has to be made as to the apportionment of this surplus between (1) participating policyholders (2) shareholders' funds (in a stock company, and this is limited by law as described previously) (3) special reserves (4) amounts carried forward to increase the existing free surplus. It is clear from the above discussion that the life insurance company has many degrees of freedom in distributing its surplus and thus preventing insolvency.[6] D.J. Franklin and C. Woodhead quote one of the most distinguished actuaries in the U.K., R.S. Sherman saying:

> " ... for most insurers in the United Kingdom, the proportion of with-profits business on the books is so high that the demonstration of solvency presents no problem." (p. 140)

The previous discussion represents only one aspect of the relation between assets and liabilities debated in the literature namely, hedging. It was explained that hedging can safeguard against insolvency and thus there may be motives to match maturities. However, because the treatment of surplus provides so much protection of this sort, it is not clear that it is necessary for firms to closely match maturities. It is this line of argument that has lead us to look at other models of insurance companies and, in particular, to portfolio choice models.

---

[5]A company may be accumulating special reserves to strengthen its policy-reserves or have a reason for setting aside special investment and contingency reserves and these special reserves would be added to total liabilities.

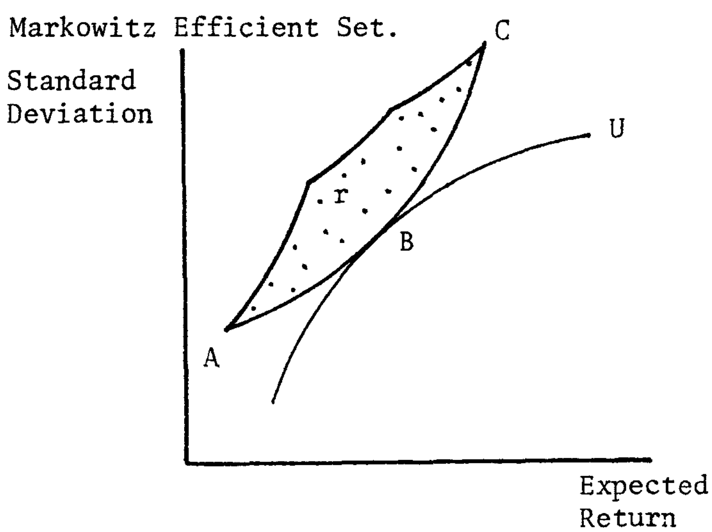
[6]The State of New York limits the amount of surplus the life insurance company can hold. For companies underwriting participating contracts the figure is 10 percent of the policy reserves and liabilities. In the case of Canadian companies surplus funds and special reserves equalled 7.5 percent of total assets at the end of 1977.

IV.2. The Portfolio Approach to the Relation Between Assets and Liabilities

Finance researchers have been interested in the portfolio problems of intermediaries, abstracting from the important problems of liquidity and transactions demand. They see the insurance company as the management of a portfolio of insurance policies, in addition to the handling of an investment portfolio. Various models were suggested to determine simultaneously the optimal composition of insurance claims against the firm and its investment portfolio.

Using the technique of portfolio analysis, it is possible to derive the set of what Markowitz (1952) calls "efficient" portfolios - portfolios which satisfy the requirement that no combination of assets can produce a higher expected return without also producing a greater variability of return. The efficient frontier is drawn in Figure IV.2 to be the line ABC. Markowitz provided the "proof" of the advantage of portfolio diversification. He showed that portfolio's risk depends not only on the risks (measured by the standard deviation) associated with the individual assets, but also on the nature of the relationships among

FIGURE IV.2 Markowitz Efficient Set.



assets. Assuming that investors desire a high expected return and a low variability of return, they will select a portfolio from among the set of efficient combinations. The specific portfolio selection will be determined by each investor's preferences as between risk and return. Often it is postulated that these preferences can be formalized in terms of a utility function and that the investor can be assumed to select portfolios which will maximize his expected utility. For example, if U (in Figure IV.2) represents an indifference curve, the utility maximizing efficient portfolio will be at point B.

The usual portfolio analysis[7] assumes the absolute level of funds available for investment is fixed, and concerns itself only with the distribution of that given amount over the candidate opportunities. In a wide variety of applications this restriction is not desirable. Financial intermediaries can, presumably, benefit from an extension of the portfolio techniques to accommodate liabilities as having variable returns, especially if some correlation between the returns on investments and on underwriting activities exists. In those cases, Markowitz type portfolio analysis can be utilized to delineate the optimal balance sheet proportions by evaluating the risk and return of every asset simultaneously with the risk and cost of every liability. The solution tells which assets should be on the balance sheet, and in what proportions, as well as which liabilities should be used to raise the necessary capital. The analysis encompasses the capital allocation <u>and</u> financial leverage decisions. <u>The firm is the portfolio</u> and the

---

[7]See for example: H.M. Markowitz (1952) or W. Sharpe (1963).

efficient frontier obtained is a set of balance sheets for the firm which have the maximum rate of return on equity at each level of equity risk and/or the minimum risk at each rate of return. R.A. Haugen and C.O. Kroncke (1970) have applied this approach to life insurance companies. The following discussion reviews their model.[8]

The insurance company can hold different proportions of its liabilities in the various insurance policies. The expected rate of cost for a portfolio of policies is equal to E(C) with standard deviation G(C) which are defined:[9]

$$(1) \qquad E(C) = \sum_{i=1}^{n} X_i E(C_i)$$

$$(2) \qquad G(C) = \left[\sum_{i=1}^{n} \sum_{j=1}^{n} X_i X_j \; G(C_i, C_j)\right]^{1/2}$$

where:

X_j = porportion of total capital raised by the j^{th} policy

n = number of policies in the portfolio

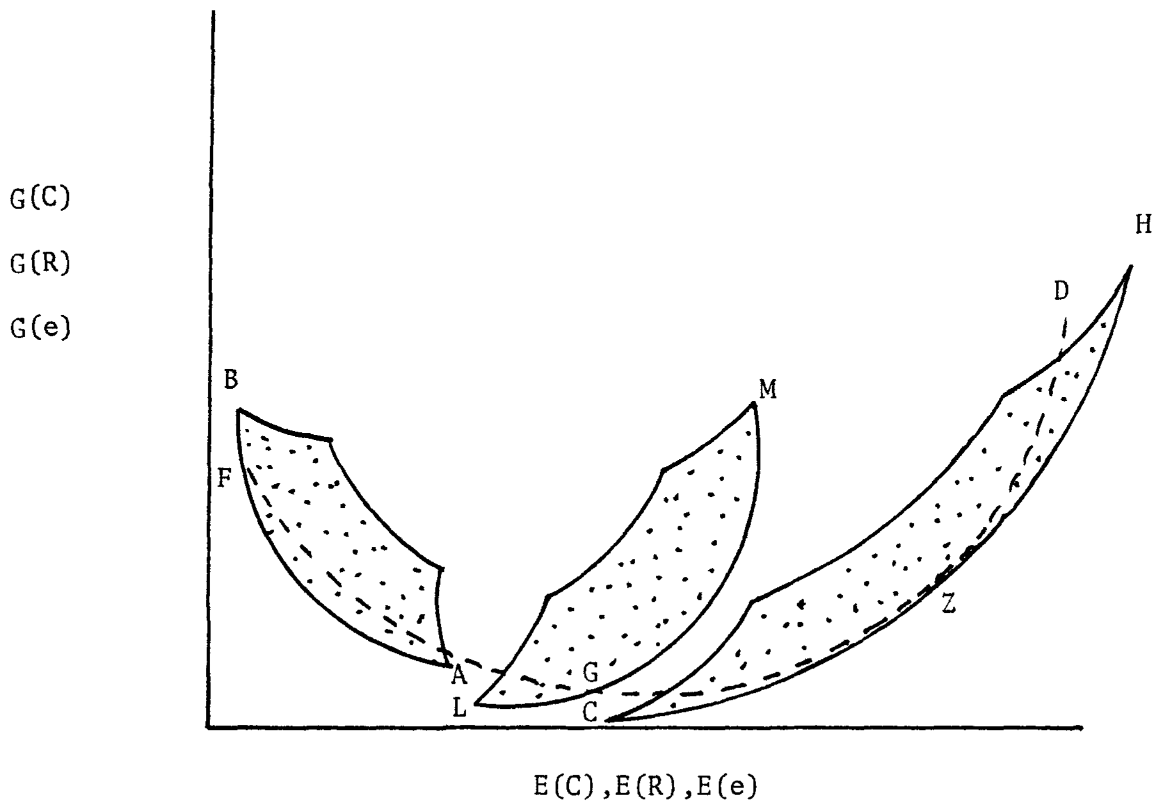
In Figure IV.3 the efficient frontier of the expected cost is given by the boundary AFB. It is efficient in the sense that for any given expected costs, the standard deviation is minimized. Similarly, the mean and variance of the insurance company's return from its investments are:

---

[8]A similar discussion can be found in D.J. Cummins (1975) pp. 17-27, and R.E. Graham (1974).

[9]All premiums are known and presumed to be paid at the inception of the planning period. A discussion on this point can be found in R.A. Haugen and C.O. Kroncke (1970) p. 42.

FIGURE IV.3: The Efficient Set For an Insurance Company



Source: R.A. Haugen and C.O. Kroncke (1970), p. 43.

(3) $$E(R) = \sum_{i=1}^{m} Y_i E(R_i)$$

(4) $$G(R) = [\Sigma\Sigma Y_i Y_j G(R_i, R_j)]^{1/2}$$

where Y_i = The proportion of total assets held in the i^{th} asset.

m = Number of assets held.

Line MGL is the set of efficient investments opportunities in the sense that for any given risk level G(R), no other portfolio offers a greater rate of return.

An insurance company generates funds by selling insurance policies. Together with shareholders capital these funds are invested. If the ratio of total assets to equity is represented by ℓ, $\ell \geq 1$, the mean and variance of rate of return on equity are defined to be:

(5) $$E(e) = \ell E(R) + (1-\ell)E(C)$$

(6) $$G(e) = [\ell^2 G^2(R) + (1-\ell)^2 G^2(C) + 2\ell(1-\ell)G(R,C)]^{1/2}$$

where: $G(R,C)$ = is the covariance between rates of return on the investment portfolio and rates of cost of the portfolio of insurance policies.

Note that a positive $E(C)$ (insurance costs) results in a lower $E(e)$. This result is intuitively reasonable and is due to the negative coefficient of $E(C)$ in equation (5).

Using quadratic programming technique, it is possible to generate an efficient frontier of risk and return combinations on equity. Referring back to Figure IV.3, Line HZC is the efficient set of attainable levered positions.[10]

By defining $Q = \lambda E(e) - G(e)$ and maximizing this expression with respect to Y_i and X_i for all λ, $0 \leq \lambda \leq \alpha$

subject to: $0 \leq Y_i \leq 1$

$0 \leq X_i \leq 1$

$\ell \geq 1$

the efficient frontier can be generated.

Two comments are warranted:

---

[10]For any given risk level no other position offers a greater rate of return to equity.

a. The insurance company can reduce $G(e)$ by investing in securities with returns positively correlated with the cost of underwriting insurance ($G(R,C)$ has a negative coefficient in equation (6)).

> b. "The attainable levered equity positions (line HZC in Figure IV.3.) will not necessarily fall to the right of the set of available investment portfolios ... . Indeed, if the insurance policies sold by the company were priced in competition with all other financial assets, the levered equity positions would all fall within the set of available investment opportunities." (Haugen and Kroncke, p. 45).

In perfect competition policy holders will get a rate of return on their policy (rate of cost for the insurance company) equal to the rate of return on any other financial asset with the same risk. Thus, the rate of return on the investment portfolio and the rates of cost of the portfolio of insurance policies will be perfectly and positively correlated. If we define:

$$G(R,C) = r(R,C) \cdot G(R) \cdot G(C)$$

where

$r(R,C)$ - is the correlation coefficient between R and C

under the assumption of perfect competition we obtain:

$$\left.\begin{array}{ll} (7) & r(R,C) = 1 \\ (8) & G(R) = G(C) \end{array}\right\} \Rightarrow G(R,C) = G^2(R)$$

and thus[11] $G(e) = G(R)$ and $E(R) = E(e)$.

Using similar approaches, an extensive literature for balance sheet optimization of insurance companies has developed in the

---

[11] Substitute equations (7) and (8) into (6) and equation (8) into (5).

last decade (C.G. Krouse (1970). D.J. Cummins (1975), Y. Kahane and D.J. Nye (1975), Y. Kahane (1977a), D. Stowe (1978), etc.). The models discussed by these authors and others can be used to generate mean-variance efficient frontiers for insurance company decision-making. Some of them provide guidance with regard to the selection of an operating point on the efficient frontier. This point can be selected through:

a. Utility theory (C.G. Krouse (1970), D.H. Pyle (1971))

b. Ruin theory-safety first (A. Roy (1952), R.E. Graham (1974))

c. Chance constrained models (H.E. Thompson, J.P. Matthews, and B.C.L. Li (1974), D. Stowe (1978)).

A rigorous discussion on the relations between the above decision rules can be found in the literature and will not be reviewed here.[12] However, it is important to note that ruin theory and chance constrained models were developed as part of the attempt to depart from the utility framework. Because of the arbitrary nature of a utility function, researchers tried to invoke criteria based on more objective concepts that would be easy to use in empirical work and would not require any assumptions about the preference function of the insurance company. As will be shown in the following chapter, by taking advantage of the existing finance foundations and the concepts and techniques of modern demand system analysis, utility theory can be used to provide guidance

---

[12] See for example, D.H. Pyle and S.J. Turnovsky (1970) on the relation between safety first and expected utility maximization. On safety first and chance constrained models, see R.E. Graham (1974). J.D. Cummins and D.J. Nye (1981) explored the linkages among quadratic programming and each of the above rules.

with regard to the selection of an operating point for a life insurance company.

In his pioneering paper "On the Theory of Financial Intermediation", D.H. Pyle (1971) explored the conditions under which a firm, maximizing a preference function, will sell a given deposit liability and use the proceeds to purchase a given type of financial asset. The following discussion reviews his results within a mean - variance framework of financial intermediation.[13]

Suppose a firm selects a portfolio of securities on the basis of a preference function F, defined over the mean and variance of the firm's terminal wealth. Assume for simplicity that there are only three securities defined as:

X_0 - Intermediary's holdings of the riskless security.

X_1 - Intermediary's holdings of loans.

X_2 - Intermediary's holdings of deposits.

and i_j - The holding period yields on security j such that

$m_j \equiv E(i_j)$

$S_{jk} = E[(i_j-m_j)(i_k-m_k)]$

The firm's decision problem after substituting the balance sheet constraint $\sum_{j=0}^{2} X_j = K$ into the preference function may be written as:

$$\max \quad F(\mu, G^2)$$

where

$$\mu = \bar{m}_1 X_1 + \bar{m}_2 X_2 + m_0 K$$

---

[13]See D.H. Pyle (1971) pp. 742-745.

$$G^2 = X_1{}^2 S_{11} + 2X_1X_2S_{12} + X_2{}^2 S_{22}$$

and

$\bar{m}_j = m_j - m_0$, the so-called risk premia.

The first order conditions for a maximum lead to the following optimal holdings of loans and deposits:

$$(9) \qquad X^*_1 = \left[\frac{\bar{m}_1 S_{22} - \bar{m}_2 S_{12}}{S_{11}S_{22} - S_{12}^2}\right] \emptyset$$

$$(10) \qquad X^*_2 = \left[\frac{\bar{m}_2 S_{11} - \bar{m}_2 S_{12}}{S_{11}S_{22} - S_{12}^2}\right] \emptyset$$

where

$$\emptyset = -\frac{1}{2}\left[\frac{\partial F}{\partial \mu} \Big| \frac{\partial F}{\partial G^2}\right]$$

Since $\emptyset$ is positive for $\frac{\partial F}{\partial \mu} > 0$ and $\frac{\partial F}{\partial G^2} < 0$, the signs of X^*_1 and X^*_2 are determined by the terms in brackets in (9) and (10). The signs of X^*_1 and X^*_2 will depend on the numerators of the two ratios since the denominator of the terms in brackets is positive as long as loans and deposits yields are not perfectly and positively correlated. Therefore the firm will engage in intermediation if and only if:

$$(11) \qquad \bar{m}_1 S_{22} - \bar{m}_2 S_{12} > 0 \qquad \text{i.e., } X^*_1 > 0$$

$$(12) \qquad \bar{m}_2 S_{11} - \bar{m}_2 S_{12} < 0 \qquad \text{i.e., } X^*_2 < 0$$

Pyle has shown that for positive risk premia $(\bar{m}_j)$ on loans and on deposits, (11) and (12) can be reduced to:

$$(5) \qquad \frac{\bar{m}_1 S_2}{\bar{m}_2 S_1} > \frac{1}{r_{12}}$$

where S_j - standard deviation of yield on security j.

r_{jk} - the correlation coefficient between the yields on securities j and k.

"Inequality (5) implies that intermediation will be more likely:

1. the smaller the risk premium on deposits ($\bar{m}_2$) and the larger the risk premiums on loans ($\bar{m}_1$),
2. the greater the positive dependence between loan and deposit yields (r_{12}), and
3. the larger the standard deviation of deposit yields (S_2) and the smaller the standard deviation on loan yields (S_1)." (pp. 765-6)

In his concluding remarks Pyle says:

> "By explicitly considering the dependence between the securities bought and sold by financial intermediaries, it has been shown that asset (liabilities) portfolios cannot, in general, be chosen independently of the parameters of liability (asset) yields." (p. 766)

These theoretical results are supported by the empirical work carried out for the insurance industry.[14] R. Daines (1968), in a statistical analysis of 131 multiple-line insurance companies showed the interaction between investments and underwritings arguing that "... as the insurance exposure increases the common stock ratio tends to decrease" (p. 363). Y. Kahane and D. Nye (1975) have provided similar results. Their study,

---

[14] F. Black (1975) and J.C. Francis (1978) extended the Pyle model and tested it for commercial banks.

> "... has utilized a model which recognizes the correlations among underwriting profits in various insurance lines, among investment profits, and also between insurance and investment activities. In some cases, the correlation coefficients were highly negative, which emphasizes the importance of minimizing the variance of the return on the capital invested in the industry by diversifying simultaneously both the insurance and investment portfolios." (p. 596)

Similar results (highly negative correlation coefficient between rates of cost of underwriting life insurance and rates of return on investments) were observed using data for Canadian life insurance companies. These results are summarized in Table IV.1.

It would appear from the above discussion that underwriting and investment returns are dependent and one can only determine the optimal position by considering the insurance and the investment decision simultaneously. In what follows, a model is developed for a life insurance company, in which both investment and underwriting decisions are taken into account.

TABLE IV.1:Correlation Coefficient Matrix for Aggregate Data (1945-1977) - Canadian Life Insurance Companies[15]

1	1.00000														
2	.99512	1.00000													
3	.98749	.99612	1.00000												
4	.96836	.98189	.99182	1.00000											
5	.96785	.98005	.99011	.99752	1.00000										
6	.96694	.97803	.98736	.99441	.99867	1.00000									
7	.96079	.97284	.98340	.99225	.99689	.99760	1.00000								
8	.96172	.97259	.98042	.98913	.99333	.99460	.99608	1.00000							
9	.92869	.94580	.96283	.97979	.98416	.98624	.98999	.98926	1.00000						
10	-.33097	-.31634	-.31120	-.31050	-.32970	-.34313	-.32837	-.32730	-.32106	1.00000					
11	.98110	.97127	.96111	.94547	.94185	.94113	.93822	.94282	.90573	-.33158	1.00000				
12	.91630	.91620	.92311	.93781	.93627	.93887	.94014	.94632	.94547	.38662	.92511	1.00000			
13	-.47500	-.47498	-.46906	-.45392	-.46069	-.46658	-.44799	-.43480	-.42968	.80759	-.44312	-.50242	1.00000		
14	.93577	.95377	.96918	.98648	.99168	.99121	.99396	.98949	.98869	-.32164	.90659	.92803	-.44542	1.00000	
15	-.72566	-.72358	-.72235	-.69072	-.67687	-.65867	-.64559	-.63470	-.58648	.12008	-.71684	-.55460	.30210	-.61735	1.00000
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

where:

1 - Canadian Government Bonds 1-3 years
2 - Canadian Government Bonds 3-5 years
3 - Canadian Government Bonds 5-10 years
4 - Canadian Government Bonds 10+
5 - Provincial bonds
6 - Municipal bonds
7 - Industrial bonds
8 - Conventional Mortgage
9 - NHA Mortgage
10 - TSE yields
11 - 3 Month Treasury Bills
12 - Prime Rate
13 - NYSE yields
14 - Moody's Composite index
15 - Cost of underwriting

[15] The underwriting costs were approximated using the formula presented in Chapter VII.

CHAPTER V

THE SIMULTANEOUS DETERMINATION OF INSURANCE AND INVESTMENT ACTIVITIES

As was explained earlier, in collecting premiums a life insurance company accumulates substantial amounts of reserve funds that can be used to meet unknown future claims. The funds are invested in financial instruments or real property. The role of the company management is to select from the available sets of opportunities those that are optimal with respect to its objectives.

In this section, a model for the simultaneous determination of the efficient composition of insurance and investment activities of a life insurance company is constructed. Unlike some current models which use the quadratic programming technique to construct efficient sets, we use a utility maximization approach. As will be shown later this approach can be operationalized using flexible functional forms.

In a 1978 paper S.D. Stowe argues that:

> "In the Markowitz model, the tangency between the efficiency frontier and the highest indifference curve is used to find the optimal portfolio that maximizes the economic unit's utility. This model was not used because no explicit relationship exists between the amount and cost of the life insurance company's liabilities and its portfolio choices. Consequently, this model does not yield explicit testable hypotheses." (p. 435)

In a recent article D.J. Cummins and D.J. Nye (1981) investigate the linkages between quadratic programming, ruin theory and utility theory, and developed efficient frontiers, ruin probabilities and implicit risk aversion coefficients for an insurance company. They point out that an operating point on the efficient frontier can be selected through utility theory and that

> " . . . in some instances (utility theory) permits the analyst to infer the insurance company's risk aversion parameter." (p. 417)

It is argued below that a utility dependent approach is indeed applicable to portfolio analysis of life insurance companies and can potentially lead to more powerful results than those that have been found in the past.

V.1 Some Notes on the Mean-Variance Approach

In Chapter IV, the two parameter portfolio model (mean and standard deviation) was introduced. Beginning with this section and for the balance of this study, the two parameters model is adopted. Therefore, we now consider the model in more detail. We first give a general treatment of its major features and then discuss some of its different aspects.

It is assumed that the representative investor's utility function is of the form:

$$U_\tau = U_\tau \; (C_t, \; \ldots \; C_{t+T}; \; X_t, \; \ldots, \; X_{t+T})$$

in which

C_τ is an m vector of quantities consumed in period $(\tau = t, \ldots, t+T)$

X_τ is an n vector of quantities of assets held at the end of τ.

and

T is the number of periods in the unit's planning horizon.

The investor is assumed to maximize his utility subject to a set of appropriate budget equations.[1]

J. Hadar (1971) has argued that:

> "If a consumer maximizes a multiperiod utility function which is constrained by a set of appropriate budget equations and if optimal plans are subject to revision after every period, then there exists a one period utility function which, when maximized subject to a single budget constraint, yields a set of dynamical demand functions that trace out the time paths of the actual amounts consumed and held by the consumer in question." (p. 225)

The above theorem known as the Collapsibility theorem is useful in transforming our problem from a multiperiod one into a single period maximization problem. Thus, the investors' problem can be summarized as one of,

maximizing $U_t = U(C_1, \ldots, C_m, X_1, \ldots X_n)$

subject to

$$\sum_{j=1}^{m} P_j C_j + \sum_{i=1}^{n} \pi_i X_i = K_t$$

where

P_j's - are the commodity prices

π_i's - are the asset prices

K_t - is total capital available to the investor

One more simplification is necessary in order to obtain the final formulation that will lead us to a two-parameter model. This is known as the multistage maximization procedure. To permit the construction

---

[1] This problem is known as an intertemporal utility maximization problem. For a detailed discussion see, for example, H.A.J. Green (1976), E. Malinvaud (1972) or J. Hadar (1971).

of demand systems involving only opportunity costs and quantities of assets, the utility function is assumed to be functionally separable between consumption goods and assets,in the assets. The investor choice of the assets can, therefore, be viewed as the second stage of a two stage maximization. The problem facing the investor at stage one is to allocate his capital between consumption and investment in some portfolio. In the second stage, the investor chooses the mixture of assets (X_i's) in his portfolio. He is supposed to be interested only in the terminal value that the portfolio will attain at the end of the decision period, the terminal value being subject to uncertainty.[2] Thus, the individual can be considered as an expected utility maximizer. This result is so general, however, that it yields nothing about observable behaviour. E.F. Fama (1976) states that:

> "... We would like to simplify the decision problem so it involves only a few potentially measurable parameters and yields some simple proposition about how the typical investor behaves with respect to these parameters." (p. 214)

The first two moments of the distribution of the uncertain asset income seems to satisfy the properties laid by Fama and thus, may serve as the objects for investors' decisions. Those two moments (the mean, a measure of the central tendency of income,and the standard deviation, serving as a measure of variability or riskiness) are utilized in the two-parameter portfolio model. The exact relationship between the mean

---

[2]There are two assumptions inherent in such an approach

a. Portfolio transactions are carried out only at the beginning of each period.

b. Interest earnings will not be available for spending before the end of the period.

standard deviation approach and the expected utility approach to portfolio selection is to be explored below. We will examine under what conditions an expected utility maximizer is able to neglect all the features of the expected returns on his portfolio other than the mean and variance of their probability distribution.

Let the initial wealth be symbolized by W_o.

Define

$\tilde{W}_1$ - terminal value of the portfolio

$\tilde{R}$ - the proportionate portfolio yield

$\tilde{Z} = (1+\tilde{R})$ - the proportionate portfolio return

Referring to the previous discussion W_o was defined by $K_o - C_o = W_o$. Thus, the wealth available for portfolio investment is what was left of the original capital after subtracting current period consumption. The future wealth W_1 can be regarded as standing for K_1 - total capital one period from now. Thus we obtain that:[3]

$$\tilde{W}_1 = W_o\ (1+\tilde{R}) \equiv W_o\tilde{Z}$$

Thus the expected utility rule can be applied to W_1 (representing potential consumption in the future period)

$$U \equiv EU\ (\tilde{W}_1)$$

The k^{th} central moment of $\tilde{W}_1$ is:

$$E[(W_1-\bar{W}_1)^k] = E[W_o{}^k(\tilde{Z}-\bar{Z})^k] \equiv m_k W_o{}^k$$

where

$m_k \equiv E[(\tilde{Z}-\bar{Z})^k]$ is the k^{th} central moment of Z.

---

[3]A tilde indicates a random variable.

By a Taylor's series expansion we can write the expected utility of the end of the period wealth to be:

$$EU(\tilde{W}_1) = E\{U(W_0\bar{Z})+U'(W_0\bar{Z})W_0(\tilde{Z}-\bar{Z})+U''\frac{(W_0\bar{Z})}{2} W_0{}^2(\tilde{Z}-\bar{Z})^2 + \dots\}$$

$$= E\{\sum_{i=0}^{\infty} \frac{U^{(i)}(W_0\bar{Z})}{i\,!} W_0{}^i(\tilde{Z}-\bar{Z})^i\} = \sum_{i=0}^{\infty} \frac{U^{(i)}(W_0\bar{Z})}{i\,!} W_o{}^i m_i$$

Where $U^{(i)}$ is the i^{th} derivative of U and ! means factorial. In general, one can assume that the expected utility maximizer would take into account all the moments of the distribution. The key question is: Under what circumstances, if ever, are we allowed to say that U is a function of mean $(W_0\bar{Z})$ and the variance $E[W_0{}^2(\tilde{Z}-\bar{Z})^2]$?Possible justification can be found in the properties we are willing to assume for the utility function $U(\tilde{W}_1)$ or in the properties of the probability distribution of $\tilde{W}_1$.

Unless U is a second order polynomial, all the moments are required to evaluate alternative portfolios. As was explained before, for simplicity of analysis and because some general results can be derived and checked empirically, we would prefer to work with only the first two moments. Note from that, if U is linear, the decision depends only on the mean; if U is a second-degree polynomial, then the decision depends only on the mean and the variance. We call this the <u>quadratic case</u>. So, in general, for an investor to consider only the mean and the variance of his portfolio in making his decision and to be consistent with the expected utility hypothesis, he must have a quadratic utility function. If this was the only rationale for the mean-variance analysis, then it would be of little importance since not only is the quadratic utility a special case, but its behavioural implications are not realistic.[4]

The second condition under which the mean variance and the expected utility approaches can be reconciled concerns the probability distribution of $\tilde{W}_1$. Using the Central Limit Theorem it is possible to show that:

> "... the distribution of the sum of a large number N of random variables tends toward the normal probability distribution as N increases. Now portfolio income $\tilde{W}_1$ can be regarded as a random variable which is the sum of the individual random variables represented by returns on the component securities. Since the normal is a two-parameter distribution, the mean μ and standard deviation G of the distribution summarize all the relevant information ... The tendency to normality is stronger, the closer to normal are the individual random variables and the more independently distributed they are one from another." 5

Furthermore, the normal distribution is the only two-parameter distribution (of finite variance) which is also stable. The stability of the distribution guarantees that the sum or other linear combinations of normal variables remains normal. Thus, the assumption that the joint

---

[4]For example, the quadratic utility function exhibits increasing absolute risk aversion (ARA) i.e., $\frac{d(ARA)}{dW} > 0$, which implies that the investor's risk aversion increases with wealth. This property does not make much sense intuitively and was rejected empirically (see for example: Fried and Blume (1975)).

However, it gains importance when the intertemporal (multiperiod) portfolio selection problem is examined. While beyond the interest of this discussion, roughly the reason is that if the time interval between portfolio revisions is small, then the variations in wealth over the interval will be small and we may think of approximating his utility function by a quadratic function (i.e., three terms of the Taylor series). See for example R.C. Merton (1968).

[5]See T. Hirshleifer (1970), p. 282.

distribution of security returns is multivariate normal assures that the probability distributions of the portfolio returns are normal.[6]

A mean-variance model is one where the choice among alternative portfolios depends only on the expected value (mean) and variance of the portfolio (or terminal wealth). Thus, the criterion function for choice for a "mean-variance investor" can be written as U [expected wealth, variance of wealth]. If such an investor is risk-averse then higher expected wealth is preferred and lower variance of wealth is preferred, i.e. the mean is a "good", $\frac{\partial V}{\partial \text{mean}} > 0$ and the variance is a "bad", $\frac{\partial V}{\partial \text{variance}} < 0$. This criterion implies that the only characteristic of the distributions of individual securities' return that affect portfolio choice are the means, variances and covariances (between securities) of the returns.

It can be concluded that expected utility can be expressed exactly in terms of mean and variance (standard-deviation) of wealth as arguments only if, as we have seen earlier in this section, either: a. the elementary-utility function is quadratic, or b. all the opportunities available have approximately normal probability distributions. The mean-variance model represents the first step in quantitatively introducing uncertainty into the ranking of portfolios. Classical methods of ranking portfolios use a single parameter measure such as (expected) rate of return or (expected) present discounted value.

---

[6] Those results were fiercely debated in the literature after J. Tobin has published his article "Liquidity Preference as Behavior Towards Risk", (1958). The interested reader is referred to M.S. Feldstein (1969), J. Tobin (1969), K. Borch (1963), P.A. Samuelson (1970) and more recently S.C. Tsiang (1972), H. Levy (1974), G.O. Bierwag (1974), K. Borch (1974) and D.P. Baron (1977).

Although a one-parameter measure makes ranking quite easy (highest to lowest), it clearly does not reflect differences in portfolios due to uncertainty. With a two-parameter ranking, there is no simple ranking. The purpose of the mean-variance model is to show how to determine optimal portfolios and to make explicitly clear, the trade off between risk and return.

V.2 The Utility Maximization Model

A fundamental assumption of micro-economics theory is that economics units are engaged in some optimization process. As shown in the previous section, it is possible under certain conditions, to make investment decision in financial assets independently of the overall consumption-investment decision and the amount invested in non-financial assets. This means that the total amount of wealth to be invested in financial assets is exogenous to the model of portfolio behaviour so that the only decision of consequence is the proportion of wealth to be invested in each financial asset.

The optimization activity of a life insurance company is more complicated than the one described above for the consumer. In this case it is necessary to evaluate the risk and return of every asset simultaneously with the risk and cost of capital of every liability. The selection of an appropriate combination of policies to issue (liabilities) and investments to make is required. The solution should indicate which assets should be on the balance sheet, and in what proportions, as well as which liabilities should be used to raise the necessary capital.

The problem may be thought of as one of choosing a particular mix of risky investments so as to maximize an objective function defined over mean and standard deviation subject to a set of constraints. The simultaneous optimization of both the insurance and investment portfolio is required, as explained in previous chapters, because of the correlation between the returns on investment and underwriting activities.

Assume that there are n possible types of investment and m-n types of life insurance contracts.

Define the following variable[7]

W_o = initial wealth

$$\tilde{\mu}_{it} = \begin{cases} \text{Rate of return on the } i^{th} \text{ investment} \\ i=1\ldots n, \text{ in period } t. \\ \text{Rate of return on the } i^{th} \text{ type of insurance contract} \\ i=n+1 \ldots m \text{ in period } t. \end{cases}$$

$$\ell_{it} = \begin{cases} \text{Investment in the } i^{th} \text{ asset in period } t,\ i=1\ldots n \\ \text{Actuarial reserves for the } i^{th} \text{ type of insurance contract} \\ \text{in period } t, \\ i = n+1 \ldots m \end{cases}$$

K_{ot} - Policy holder's surplus plus shareholder's equity in period t.

(1) $$K_{ot} = S_{ot} + E_{ot}$$

---

[7] Random variables are denoted by tildes (∿).

where

S_{ot} - Policyholder's surplus in period t

E_{ot} - Shareholder's equity at the beginning of period t

The profit of the company in period t, $\tilde{\pi}_t$ is a linear combination of the random variables given by

(2) $$\tilde{\pi}_t = \sum_{i=1}^{m} \ell_{it} \tilde{\mu}_{it}$$

In order to get the rate of return on equity in period t, $\tilde{\pi}_{et}$ we divide both sides of equation (2) by E_{ot} to obtain.[8]

(3) $$\frac{\tilde{\pi}_t}{E_{ot}} = \tilde{\pi}_{et} = \Sigma\left(\frac{\ell_{it}}{E_{ot}}\right)\tilde{\mu}_{it}.$$

Equation (3) can be rewritten as follows:

(4) $$\tilde{\pi}_{et} = \sum_{i=1}^{m} w_{it} \tilde{\mu}_{it}$$

where

$$w_{it} = \begin{cases} \text{The } i^{th} \text{ asset to equity ratio in period } t,\ i=1, \ldots, n. \\ \text{Actuarial reserves to equity ratio for insurance line } i \\ \text{in period } t, i = n+1, \ldots, m. \end{cases}$$

The investment leverage of the company is introduced into the model through the balance sheet (solvency) constraint. For each and every period the ℓ_i's (i = 1, ..., m) must be determined such as to equate

---

[8]An alternative approach is to treat the surplus as retained earnings and to divide equation (3) by K_{ot}. For comparison with shareholders in other industries, the present approach seems more reasonable.

total assets with the sum of liabilities. Thus, the following relationship must hold:

$$(5) \qquad \sum_{i=n+1}^{m} \ell_{it} + K_{ot} = \sum_{i=1}^{n} \ell_{it}.$$

Equation (5) represents the balance sheet constraint; i.e.,

total liabilities = sum of actuarial reserves + equity + surplus = total assets.

Some explanation on the balance sheet constraint is in order. Unlike most models which utilize a similar approach,[9] premiums collected from underwriting different types of insurance do not appear in our model. The discussion in Chapter III shows explicitly that the funds raised for investment (known in the literature as the funds generating factors) differ among types of insurance. It has been suggested that "The reserves/premiums ratios are not unreasonable as a first approximation to the funds generating factor."[10] To avoid the need for any assumption about the relationship between premiums collected and investable funds, the actuarial reserves for each type of contract are utilized in our model.[11] Furthermore, life insurance companies have

---

[9] See for example Y. Kahane (1977); J.D. Cummins and D.J. Nye (1981).

[10] Cummins and Nye (1981) p. 422.

[11] Using the reserves/premiums ratios as an approximation to the funds generating factor leads to our model. The balance sheet would take the following form

$$\sum_{i=n+1}^{m} g_i p_i + K_o = \sum_{i=1}^{n} \ell_i$$

where p_i are premiums collected in line i and g_i is the funds generating factor. Substituting reserves/premiums for g_i yields equation (5). It is important to note that p_i represents the total amount of premiums collected in the past for different type of contracts currently in force. To avoid adding up $ collected in different years, actuarial reserves were introduced in the model.

limited control over the funds generating factor. Interest rates and mortality rates do influence the actuarial reserves, the premiums collected and thus the funds available for investment. It has already been mentioned that the main responsibility of government supervision of the life insurance industry, is to ensure that the life insurance companies hold assets to cover the policy-reserves according to the official government basis.[12] The higher the interest rate assumed in calculating the premium the lower is the premium. The lower the premium the lower the actuarial reserves and vice-versa. This follows from common-sense reasoning that the higher the rate of interest at which a company may expect to accumulate the funds in its hands, the lower need these funds be to meet definite future liabilities. A similar argument can be presented for mortality rates. Lower mortality means lower premiums, and it is clear that lower premiums and lower death claims offset each other in the accumulation which gives the actuarial reserves. But it is not known as a rule, what actuarial reserves will be necessary for a new mortality table until they are calculated.[13] In short, actuarial reserves were used in our model in order to avoid the problems which arise when using premiums as a variable representing the activities of the company in different types of contracts, though this variable, too, is not without its problems.

Dividing equation (5) by E_{ot} yields:

$$\sum_{i=n+1}^{m} w_{it} + 1 + \frac{S_{ot}}{E_{ot}} = \sum_{i=1}^{n} w_{it} \tag{6}$$

---

[12] There are, of course, other liabilities which must be covered by assets.

[13] The variation of the actuarial-reserves between different mortality tables depends on the slope of the mortality curve.

Further, define,

$$\frac{S_{ot}}{E_{ot}} = W_{m+1,t} \quad \text{(surplus to equity ratio)}$$

Substituting the above definition into equation (6) enables us to write the solvency constraint as:[14]

$$(7) \qquad \sum_{i=1}^{n} w_i - \sum_{i=n+1}^{m+1} w_i = 1$$

or

$$\sum_{i=1}^{m+1} X_i = 1 \quad \text{where} \begin{cases} X_i = W_i & i=1,\dots, n \\ X_i = -W_i & i=n+1, \dots, m+1 \end{cases}$$

The insurance company's investment preferences are assumed to be captured by an institutional utility function of the form:

$$(8) \qquad U(E,V)$$

where E is the expected (end of the period) net worth and V is its standard deviation.

$$E = W_o[1+E(\tilde{\pi}_e)]$$

$$V = [W_o^2 \sum_{i=1}^{m+1} \sum_{j=1}^{m+1} X_i X_j G_{ij}]^{1/2}$$

Where G_{ij} is the covariance of end of the period expected return on asset i and j. This model is specified in expected returns and standard deviation of returns space. It is the analogue to a model Aivazian (1976) constructed in expected prices and variance of prices spaces.

---

[14] Equation (7) holds for each and every period of time and thus the subscripts t were dropped.

[15] For a discussion on institutional utility functions see: J.B. Michaelsen and R.C. Goshay (1967).

The relation between his specification and the above model is shown in Appendix V.A. at the end of this chapter. It seems to us that the utilization of expected returns (as opposed to prices) is preferred when empirical work is to be carried out since information on prices of different financial assets is not available.

The utility function is assumed to be continuous and twice differentiable with $U_E > 0$ and $U_V < 0$. The subscript E denotes the partial derivative of U with respect to E and similarly, the V subscript denotes the partial derivative of U with respect to V. In other words, the insurance company is assumed to be risk averse with indifference curves in the E-V space which are upward sloping and convex from below.[16]

The insurance company is assumed to choose those proportions to invest in each financial asset and the proportions of underwriting in each type of insurance contract so as to maximize the utility function (8) subject to the constraint in (7) and the non-negativity conditions

(9) $\quad X_i \geq 0, i=1, \ldots, m+1$[17]

The solution may then be expressed in terms of the ratios $\frac{X_i}{\sum_i X_i}$, $(i=1, \ldots, n)$ which represents the ratio of the i^{th} asset to total assets and the ratios $\frac{X_i}{\sum_i X_i}$, $(i=n+1, \ldots, m)$ which represent the actuarial reserves for insurance contract of type i as a proportion of total actuarial reserves. Furthermore, X_{m+1}, the surplus to equity ratio which can be interpreted as the "cushion" of the firm is also obtained.

---

[16]The convexity conditions are not implied by $U_E > 0$, $U_V < 0$ but are explained later in the text.

[17]In fact, as we observe all the companies to hold all the assets chosen, we expect the strict inequality to hold.

The specification of the model in this form[18] allows us to capture another solvency consideration which relates to the amount of new business sold. The sale of new life insurance and annuity business causes a strain on the insurance company's surplus. That is because the actuarial reserves the company is legally required to set up in the year the policies are issued normally exceed the amount of money the company has available after payment of expenses involved in issuing the policies.[19] A dramatic increase in the sale of new contracts could conceivably force a company to temporarily suspend sales.

V.3 A Comparative Static Analysis Of Portfolio Adjustment

In his 1976 paper Aivazian points out the relation between our problem and traditional consumer and producer theory. Even though his model is specified in prices space many of his comments are valid here. He states that:

> "In traditional consumer theory the preference function is defined directly over commodities, i.e., the underlying properties of commodities are only implicitly taken into account in the preference function. With no explicit information of how the properties of commodities enter into the preference function, one can make no restriction within such a framework on how demand would be affected by changes in these properties. On the other hand, in portfolio theory an asset is demanded indirectly because of its contributions to various well-defined (desirable and undesirable) portfolio characteristics and not for other intrinsic properties. Thus the demand for an asset can be more properly regarded as analogous to the demand for a factor of production." (p. 2)

---

[18] Specifying the surplus as a separate category.

[19] For example, high commission to the agent in the first year.

It is shown below that analysis of assets and liabilities demands which explicitly takes into account the underlying productive characteristics, expected returns and standard deviations, provides more meaningful interpretations of the nature of portfolio adjustment of life insurance companies.

Formally, the insurance company's assets (liabilities) choice framework can be described by the program:[20]

(10)
$$\underset{X_1, \ldots, X_{m+1}}{\text{Maximize}}\ U(E,V)$$

$$\text{Subject to } \sum_{i=1}^{m+1} X_i = 1$$

$$X_i \geqslant 0$$

where
$$E = W_o[1 + \sum_{i=1}^{m+1} X_i \tilde{\mu}_i]$$

$$V = [W_o^2 \sum_{i=1}^{m+1} \sum_{j=1}^{m+1} X_i X_j G_{ij}]^{1/2}$$

Solving the utility maximization program in (10) yields the first-order conditions:

(11)
$$\frac{\partial L}{\partial X_i} = W_o U_E \mu_i + W_o^2 U_V V^{-1} \sum_j X_j G_{ij} = \lambda$$

$$\frac{\partial L}{\partial \lambda} = 1 - \sum_i X_i = 0 \qquad (i, j = 1, \ldots, m+1)$$

where λ is the lagrange multiplier and V^{-1} is $1/V$.

The second-order conditions for a maximum require that the principal minors of the determinant D, obtained by totally differentiating

---

[20] A similar specification of the insurance company's utility function can be found in D.J. Cummins and J.D. Nye (1981) even though the analysis which follows differs substantially from theirs. A similar discussion can be found in V. Aivazian, J. Callen, I. Krinsky and C.C.Y. Kwan (1981); though their specification of the utility function is somewhat different.

the first-order conditions in (11) with respect to the X_i's, alternate in sign.

In particular,

$$(12) \qquad D = \begin{bmatrix} Z_{11} & \cdots\cdots\cdots & Z_{1,\, m+1} & 1 \\ \vdots & & \vdots & \vdots \\ Z_{m+1,1} & \cdots\cdots\cdots & Z_{m+1,\, m+1} & 1 \\ 1 & \cdots\cdots\cdots\cdots & 1 & 0 \end{bmatrix}$$

and the elements Z_{ij} are given by:

$$(13) \qquad \frac{\partial^2 L}{\partial X_i \partial X_j} = Z_{ij} = W_o^{\,2} U_{EE} \mu_i \mu_j + V^{-1} W_o^{\,3} U_{EV} (\mu_i \sum_{i=1}^{m+1} X_i G_{ji} + \mu_j \sum_{j=1}^{m+1} X_j G_{ij})$$

$$+ W_o^{\,4} (U_{VV} V^{-2} - U_V V^{-3}) \sum_{i=1}^{m+1} X_i G_{ji} \sum_{j=1}^{m+1} X_j G_{ij} + V^{-1} W_o^{\,2} U_V G_{ij}$$

We assume these second-order conditions hold, though later we report on tests on whether they, in fact, do.

(i) The effect of a change in expected return.

The quantities held by an insurance company will always satisfy Equation (11). Changes in expected returns and the standard deviations of returns will normally alter its assets[21] mix, but the new quantities must still satisfy (11). In order to find the magnitude of the effect

---

[21]The same is true when liabilities are considered, i.e., i = n+1 ..., m+1. For simplifying reasons, assets are used in the discussion unless different results are obtained when liabilities are involved.

of a change in the j^{th} asset expected return on the quantity demanded of asset i, holding the G_{ij} elements constant, one can totally differentiate (11) with respect to μ_r. Thus we obtain the matrix equation:

$$(14) \qquad \begin{bmatrix} Z_{11} & \cdots\cdots & Z_{1,m+1} & 1 \\ \vdots & & \vdots & \vdots \\ Z_{j,1} & \cdots\cdots & Z_{j,m+1} & 1 \\ \vdots & & \vdots & \vdots \\ Z_{m+1,1} & \cdots\cdots & Z_{m+1,m+1} & 1 \\ 1 & \cdots\cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial X_1}{\partial \mu_j} \\ \\ \frac{\partial X_j}{\partial \mu_j} \\ \\ \frac{\partial X_{m+1}}{\partial \mu_j} \\ -\frac{\partial \lambda}{\partial \mu_j} \end{bmatrix} = \begin{bmatrix} T_1 \\ \vdots \\ T_j \\ \vdots \\ T_{m+1} \\ 0 \end{bmatrix}$$

where:

$$T_i = -W_o U_E \delta_{ir} - X_r [W_o^2 U_{EE} \mu_i + V^{-1} W_o^3 U_{EV} \Sigma X_j G_{ij}]$$

Where $\delta_{ir} = 1$ for $i = r$

$= 0$ for $i \neq r$

Solving for $\frac{\partial X_k}{\partial \mu_r}$ yields:

$$(15) \qquad \frac{\partial X_k}{\partial \mu_r} = - W_o U_E \frac{D_{rk}}{D} - X_r \, [W_o^2 U_{EE} \sum_{i=1}^{m+1} \mu_i \frac{D_{ik}}{D} + V^{-1} W_o^3 U_{EV} \sum_i \sum_j X_j G_{ij} \frac{D_{ik}}{D}] .$$

Similar equations have been obtained by Aivazian (1976) and others.[22]

---

[22] See Aivazian (p. 7) equation (10) or Aivazian, Callen, Krinsky and Kwan (1981), (p. 4) equation (5).

Equation (15) enables us to calculate the demand elasticity of asset i with respect to the expected return on asset j (i, j=1, ... m+1).

(16) $$\eta(X_i,\mu_j) = \frac{\partial X_i}{\partial \mu_j} \frac{\mu_j}{X_i}$$

(ii) The effect of a change in the variance-covariance matrix.

Following an analogous procedure it is possible to derive the effect of a change in G_{rf} on X_k. By differentiating the first-order conditions in (11) with respect to G_{rf} and solving for $\frac{\partial X_k}{\partial G_{rf}}$ we get

(17) $$\frac{\partial X_k}{\partial G_{rf}} = -V^{-1} U_V W_o^{\,2} (X_r \frac{D_{rk}}{D} + X_f \frac{D_{fk}}{D}) - V^{-1} X_r X_f W_o^{\,3} [U_{EV} \sum_i \mu_i \frac{D_{ik}}{D}$$

$$+ W_o V^{-1} U_{VV} \sum_i \sum_j X_j G_{ij} \frac{D_{ik}}{D}]$$

and the demand elasticities of asset i with respect to the change in the variance covariance matrix will be

(18) $$\eta(X_i,G_{ij}) = \frac{\partial X_i}{\partial G_{ij}} \cdot \frac{G_{ij}}{X_i}$$

(iii) On the Existence of Some Exogenous Variables.

The purpose of any economic model is to provide answers to a series of if - then statements: what would happen to the set of endogenous variables (the X_i's in our model) for any assumed change in the values of the exogenous variables (the μ_i's and G_{ij}'s).

The life insurance company's asset choice framework presented in equation (10) assumes that all the assets on its balance sheet are endogenous variables. That is, the insurance company wishes to determine the optimal holdings of cash, government securities, mortgages, equities, etc. given a set of expected returns and a variance-covariance matrix. It is often argued that the holdings of some assets are determined exogenously.[23] Policy loans, for example, constitute an investment made not by the decision of company management, but by the actions of a policyholder.[24] Thus, we consider a variant of the basic model in which one element in the balance sheet is exogenously determined.

Assume $\bar{X}_p$ is determined exogenously. Thus the maximization program is:[25]

$$\text{(10a)} \qquad \underset{\substack{X_i \\ i = p}}{\text{Maximize}}\; U(E,V)$$

$$\text{Subject to } \sum_{\substack{i=1 \\ i \neq p}}^{m+1} X_i = 1 - \bar{X}_p$$

$$X_i > 0 \; (i \neq p)$$

where, as before

$$E = W_0 \, [1 + \Sigma X_i \mu_i],$$

and $$V = [W_0^{\,2} \sum_i \sum_j X_i X_j G_{ij}]^{1/2}.$$

The first-order conditions, assuming a regular interior maximum are:

---

[23]See for example A.I. Brodt (1982) p. 6.

[24]The policy loan is granted upon the security of the cash value already built up in the policy. Sections G3 (1) and (5) of the Canadian and British Insurance Act provides for them.

[25]For a discussion on this specification see Appendix V.B at the end of this chapter.

$$\text{(11a)}\quad \begin{aligned} \frac{\partial L}{\partial X_i} &= W_0 U_E \mu_i + W_0{}^2 V^{-1} U_V \left(\sum_j^{m+1} X_j G_{ij} + \bar{X}_p G_{ip} \right) = \lambda \qquad i,j \neq p \\ \frac{\partial L}{\partial \lambda} &= 1 - \bar{X}_p - \sum_i^{m+1} X_i = 0 \end{aligned}$$

Here only m+1 first-order conditions exist, one fewer than in equation set (11).

As before the quantities held by an insurance company will always satisfy equation (11a), and changes in $\bar{X}_p$ will normally alter the asset mix. In order to find the effect of a change in $\bar{X}_p$ on the quantity demanded of asset i, holding G_{ij} and μ_i $(i, j = 1, \ldots, m+1)$ constant, totally differentiate (11a) with respect to $\bar{X}_p$.

$$\text{(19)}\quad \begin{bmatrix} Z_{1,1} & \cdots & Z_{1,m+1} & 1 \\ \vdots & & \vdots & \\ Z_{j,1} & \cdots & Z_{j,m+1} & 1 \\ \vdots & & \vdots & \\ Z_{m+1,1} & \cdots & Z_{m+1,m+1} & 1 \\ 1 & \cdots & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \dfrac{\partial X_1}{\partial \bar{X}_p} \\ \vdots \\ \dfrac{\partial X_j}{\partial \bar{X}_p} \\ \vdots \\ \dfrac{\partial X_{m+1}}{\partial \bar{X}_p} \\ \dfrac{\partial \lambda}{\partial \bar{X}_p} \end{bmatrix} = \begin{bmatrix} S_1 \\ \vdots \\ S_j \\ \vdots \\ S_{m+1} \\ 0 \end{bmatrix}$$

where:[26]

$$S_i = -W_0 \mu_i \mu_p U_{EE} \delta_{ip} - W_0{}^2\, V^{-1} G_{ip} [W_0 U_{EV} \mu_i + U_V + W_0{}^2 V^{-1} (U_{VV} - V^{-1} U_V) \sum_j X_j G_{ij}]$$

[26]Note, however, that the bordered Hessian is of dimension (m+1) by (m+1) since column p and row p were dropped.

and $\delta_{ip} = 1$ for $i = p$

$= 0$ for $i \neq p$

$$(20) \qquad \frac{\partial X_r}{\partial X_p} = -W_0 U_{EE} \mu_p \sum_{i=1}^{m} \mu_i \frac{D_{ir}}{D} - W_0{}^2 V^{-1} [W_0 U_{EV} \sum_{i=1}^{m} \mu_i G_{ip} \frac{D_{ir}}{D} +$$

$$U_V \sum_{i}^{m} G_{ip} + W_0{}^2 V^{-1} (U_{VV} - V^{-1} U_V) \sum_{i=1}^{m} \sum_{j=1}^{m} X_j G_{ij} G_{ip} \frac{D_{ir}}{D}]$$

The signs of equations (15), (17) and (20) are impossible to determine since no information is available on whether U_{EE}, U_{EV}, U_{VV}, G_{ij} etc. are positive or negative. In the following chapter, specific utility functions are introduced and the equations to be estimated are derived. The estimated coefficient will enable us to compute the above relations and thus obtain their signs.

APPENDIX V.A

The relations between prices and expected return in a utility framework

Define:

W_o - Initial wealth

H_i - Desired quantity of asset i

P_i - Initial price of i

F_i - Terminal price of i (end of the period)

V_{ij} - Covariance between end of the period prices of asset i and j

λ - lagrange multiplier

E_p - expected wealth

V_p - variance of wealth

The demand for a risky asset is derived by maximizing the investor's utility function subject to the investor's initial wealth available for portfolio investment, W_o.

The maximization program will be:

(1.A) $$L = U(E_p, V_p) - \lambda \Sigma P_i H_i$$

where: $$E_p = \sum_{i=1}^{n} H_i F_i ; \quad V_p = \sum_{i=1}^{n} \sum_{j=1}^{n} H_i H_j v_{ij}$$

Define the expected return on asset i to be

(2.A) $$\mu_i = \frac{F_i - P_i}{P_i} \rightarrow F_i = P_i \mu_i + P_i$$

If we define $X_i = \frac{H_i P_i}{W_o}$ to be the portion of wealth invested in asset i we obtain that

$$\text{(3.A)} \qquad H_i = \frac{W_o X_i}{P_i}$$

Substitute (2.A) and (3.A) into the definition of E_p we obtain:

$$\text{(4.A)} \qquad E_p = \sum_i H_i F_i = \Sigma \frac{W_o X_i}{P_i} (P_i \mu_i + P_i) = W_o [1 + \sum_{i=1}^{n} X_i \mu_i]$$

$$(i=1,\ldots,n)$$

$$\text{(5.A)} \qquad V_{ij} = COV(F_i, F_j) = COV(P_i \mu_i + P_i,\ P_j \mu_j + P_j)$$

$$= P_i P_j \ COV(\mu_i, \mu_j) = P_i P_j G_{ij}$$

where G_{ij} = covariance between expected return on asset i and j.

When substituting (5.A) and (3.A) into the definition of Vp yields:

$$\text{(6.A)} \qquad V_p = \Sigma\Sigma H_i H_j \ V_{ij} = \Sigma\Sigma \frac{W_o X_i}{P_i} \cdot \frac{W_o X_j}{P_j} P_i P_j \ G_{ij} = W_o^{\ 2} \ \Sigma\Sigma X_i X_j \ G_{ij}$$

Using the results obtained in (4.A) and (6.A) the utility maximization program defined in (1.A) will get the following form:

$$\max \ U[w_o(1+\Sigma X_i \mu_i),\ w_o^{\ 2} \sum_i \sum_j X_i X_j \ G_{ij}]$$

s.t.

$$\sum_{i=1}^{n} X_i = 1$$

APPENDIX V.B

The maximization problem and the presence of exogenous variables

Equation (6) in Chapter V was obtained by dividing through the balance sheet constraint by equities. When one of the assets on the balance sheet is determined exogenously, there are two ways to specify the maximization program.

A.

$$\underset{\substack{X_i \\ i \neq p}}{\text{Maximize}}\ U\ (E,V)$$

S.T.

$$\sum_{i \neq p} X_i = 1-\bar{X}_p$$

where,

$X_i = \frac{A_i}{\ell}$ in the ratio of asset (liability) i to equities

$X_p = \frac{A_p}{\ell}$ and where A_p is exogenous.

$$E = W_o[1+\sum_{i \neq p} X_i\mu_i+\bar{X}_p\mu_p]$$

$$V = [W_o^{\,2}(\sum_{\substack{i \\ i,j \neq p}}\sum_{j} X_iX_jG_{ij} + 2\bar{X}_p \sum_j X_jG_{pj})]^{1/2}$$

The first-order conditions of the above maximization program are:

$$\frac{\partial L}{\partial X_i} = W_oU_E\mu_i + W_o^{\,2}V^{-1}U_V\ (\sum_{\substack{j \\ i,j \neq p}} X_jG_{ij} + \bar{X}_pG_{pj}) = \lambda \qquad (1)$$

$$(i \neq p)$$

(2) $$\frac{\partial L}{\partial \lambda} = 1 - \bar{X}_p - \sum_{i \neq p} X_i = 0$$

Using equation (1) we obtain that for each asset pair i,r ($i \neq r$ and $i,r \neq p$):

$$U_E(\mu_i - \mu_r) + W_o V^{-1} U_V(\sum_j X_j (G_{ij} - G_{rj}) + \bar{X}_p G_{pj} - \bar{X}_p G_{pj}) = 0$$

$$i,j \neq p$$

or

(3) $$U_E(\mu_i - \mu_r) + W_o V^{-1} U_V(\sum_j X_j (G_{ij} - G_{rj})) = 0$$

Equation (3) is identical to equation (17) in Chapter VI which is used to derive the share equation.

B.

The second specification for the maximization problem is:

$$\underset{S_i}{\text{Maximize}}\ U(E,V)$$

S.T.

$$\sum_{i \neq p} S_i = 1$$

where:

$S_i = \frac{A_i}{\ell - A_p}$ the ratio of asset (liability) i ($i \neq p$) to equity plus asset p.

$$E = [(\frac{\ell - A_p}{\ell}) \sum_{i \neq p} S_i \mu_i + \frac{A_p}{\ell} \mu_p + 1] W_o$$

$$V = [W_o^2 (T^2 \sum_i \sum_j S_i S_j G_{ij} + (\frac{A_p}{\ell})^2 G_p^2 + \frac{A_p}{\ell} T \sum_{j \neq p} S_j G_{pj}]^{1/2}$$

$$i,j \neq p$$

where $T = \dfrac{\ell - A_p}{\ell}$

and the first-order conditions are:

$$(4) \qquad \frac{\partial L}{\partial S_i} = W_o U_E T \mu_i + W_o^{\,2} V^{-1} U_V \Big(T^2 \sum_{\substack{j \\ i,j \neq p}} S_j G_{ij} + \frac{A_p}{\ell} T\, G_{pj}\Big) = \lambda$$

$$(5) \qquad \frac{\partial L}{\partial \lambda} = \sum_{\substack{i \\ i \neq p}} S_i = 1$$

In what follows it will be shown that the share equations obtained from (4) and (5) are identical to the share equations obtained in the previous problem.

Proof:

Using the definitions for X_i, $\bar{X}_p$ and S_i we get:

$$(6) \qquad \ell X_i = A_i, \quad \bar{X}_p = \frac{A_p}{\ell}$$

$$(7) \qquad S_i(\ell - A_p) = \ell X_i \quad \Rightarrow \quad S_i = \frac{X_i}{1 - \bar{X}_p}$$

$$(8) \qquad \frac{X_i}{S_i} = \frac{\ell - A_p}{\ell} = T \quad \Rightarrow \quad \frac{X_j}{T} = S_j$$

Substituting (8) into (4) yields:

$$(9) \qquad W_o U_E \frac{X_i}{S_i} \mu_i + W_o^{\,2} V^{-1} U_V \Big(\frac{X_i}{S_i} \sum X_j G_{ij} + \frac{A_p}{\ell} \frac{X_i}{S_i} G_{pj}\Big) = \lambda$$

or,

$$(10) \qquad W_o U_E X_i \mu_i + W_o^{\,2} V^{-1} U_V \Big(X_i \sum_{j, i \neq p} X_j G_{ij} + \bar{X}_p X_i G_{pj}\Big) = \lambda S_i = \Big[\frac{\lambda}{T}\Big] X_i$$

Since T is a constant, $\frac{\lambda}{T}$ can be viewed as the lagrange multiplier.

By utilizing equation (5) and (8) we get:

(11) $$\sum_{i \neq p} S_i = \frac{X_i}{T} = 1 \Rightarrow \sum_{i \neq p} X_i = T \Rightarrow \sum_{i \neq p} X_i = 1 - \bar{X}_p$$

Equations (10) and (11) are identical to equations (2) and (3) utilized in order to obtain the share equations.

QED

CHAPTER VI

THE THEORY AT WORK

The theory presented in the previous chapter offers a framework within which we can attempt to organize and interpret the data. In this chapter, we try to show how the theory can be made specific and check its usefulness in empirical work. In what follows, the use of Flexible Functional Forms as a tool will be explained and the particular results of each case will be referred to the general theory.

Following the important paper by Diewert (1971) many researchers[1] are now using what have come to be known as "Flexible Functional Forms" method (hereafter FFF method). Deaton and Muellbauer (1980b) state that this method is used

> "... to approximate the direct utility function, or the cost function by some specific functional form that has enough parameters to be regarded as a reasonable approximation to whatever the true unknown function may be." (p. 74)

FFF provide a second-order local approximation, but their approximated global properties are usually unknown.[2]

Recently, Honohan (1980) has tested the standard theory of portfolio selection for the U.K. life insurance industry. In his empirical estimation he used Constant Elasticity of Substitution (CES) utility functions (defined over mean and standard deviation) in order to

---

[1]See for example, A. Deaton and J. Muellbauer (1980b), P. Simmon and D. Weiserbs (1979), A.D. Woodland (1979) and E.R. Berndt, M.N. Darrough and W.E. Diewert (1977).

[2]See Simmon and Weiserbs.

obtain among other things, "reliable estimates of the risk-return trade off."[3] We, on the other hand, instead of restricting ourselves to the CES case, will explore a more general flexible functional form. Our functions are more "flexible" in the sense that they do not a priori constrain the elasticity of substitution between mean and variance to be constant.

VI.1. Flexible Functional Forms and Risky Assets

In a paper which analysed the demand for risky assets by the personal sector in the U.K., Aivazian, Callen, Krinsky and Kwan (1981) utilized the FFF method in their empirical estimation. Following their approach we specify a general Box-Cox utility function which takes on the generalized square root quadratic, the generalized Leontief, the generalized quadratic and the translog utility functions as special cases.

The insurance company is assumed to maximize an "institutional utility function", defined over mean and standard deviation, of the following form:[4]

(1) $$U(\delta) = \alpha_0 + \alpha_1 E(\lambda) + \alpha_2 V(\lambda) + \tfrac{1}{2}\alpha 3[E(\lambda)]^2 + \tfrac{1}{2}\alpha_4 [V(\lambda)]^2 + \alpha_5 E(\lambda)V(\lambda).$$

where $E(\lambda)$, $V(\lambda)$ and $U(\delta)$ are the Box-Cox transformation[5] functions defined as:

(2) $$U(\delta) = (U^{2\delta} - 1)/2\delta$$

---

[3]See p. 29.

[4]We use standard deviation as a risk measure. A similar specification can be applied when the variance is used.

[5]For further details on the properties of the transformation see G.E.P. Box and D.R. Cox (1964) and P. Zarembka (1974).

(3) $\quad E(\lambda) = (E^{\lambda} - 1)/\lambda$

(4) $\quad V(\lambda) = (V^{\lambda} - 1)/\lambda$

Four alternative cases of the general transformation will be considered. In each case the parameters λ and δ take different values and thus the institutional utility function specified in (1) takes on different flexible functional forms.

CASE I. $\quad \delta,\lambda \to 0 \quad U(\delta) = \ln U;\ E(\lambda) = \ln E$

$V(\lambda) = \ln V.$

(Since $\lim_{\lambda \to 0} (X^{\lambda} - 1)/\lambda = \ln X$ by l'Hospital's rule).

This case yields the translog utility function

(5) $$\ln U = \alpha_0 + \alpha_1 \ln E + \alpha_2 \ln V + 1/2\alpha_3(\ln E)^2 + 1/2\alpha_4(\ln V)^2 + \alpha_5(\ln E)(\ln V)$$

CASE II. $\quad \delta,\lambda \to 1/2 \quad U(\delta) = U-1;\ E(\lambda) = 2(E^{1/2} - 1)$

$V(\lambda) = 2(V^{1/2} - 1)$

The utility function will then assume the form:

(6) $$U = 2\alpha_3 E + 2\alpha_4 V + 4\alpha_5 E^{1/2}V^{1/2} + (2\alpha_1 - 4\alpha_3 - 4\alpha_5)E^{1/2} + (2\alpha_2 - 4\alpha_4 - 4\alpha_5)V^{1/2} + 2\alpha_3 + 2\alpha_4 + 4\alpha_5 - 2\alpha_1 - 2\alpha_2 + 1.$$

This is known as the generalized Leontief utility function.

CASE III. $\quad \delta,\lambda = 1 \quad U(\delta) = (U^2 - 1)/2;\ E(\lambda) = E - 1$

$V(\lambda) = V - 1$

In this case a square rooted quadratic utility function is obtained

(7) $$U = [\alpha_3 E^2 + \alpha_4 V^2 + 2\alpha_5 E\cdot V + 2(\alpha_1 - \alpha_3 - \alpha_5)E + 2(\alpha_2 - \alpha_4 - \alpha_5)V + 2\alpha_5 + \alpha_3 + \alpha_4 - 2\alpha_1 - 2\alpha_2 + 1]^{1/2}.$$

CASE IV.

The quadratic function itself (rather than the square rooted one) can be obtained by setting $\delta = 1/2$ and $\lambda = 1$. Appelbaum (1979) points out that the quadratic function yields share equations identical to the square rooted quadratic. Therefore, no special treatment is required in the empirical section since it is share equations that are estimated. This follows, as will be shown later, since δ does not appear in the estimating system (the share equations) in any of the cases described. Thus we have,

$$\delta = 1/2 \quad \lambda = 1; \qquad U(\delta) = U - 1$$

$$E(\lambda) = E-1; \qquad V(\lambda) = V-1.$$

Substituting the above into equation (1) yields the following utility function:

$$U = 1/2[\alpha_3 E^2 + \alpha_4 V^2 + 2\alpha_5 EV + 2(\alpha_1 - \alpha_3 - \alpha_5)E + 2(\alpha_2 - \alpha_4 - \alpha_5)V + 2\alpha_5 + \alpha_3 + \alpha_4 - 2\alpha_1 - 2\alpha_2 + 1]. \tag{8}$$

The similarities between equations (7) and (8) require no further explanation.

VI.2. Demand Functions

Before solving the utility maximization problem that leads to the share equations, it is convenient to find expressions for U_E, U_V, U_{EE}, U_{VV} and U_{EV}. Those are derived below from the general utility function (1).

$$U_E = \frac{\partial U}{\partial E} = [\alpha_1 + \alpha_3 E(\lambda) + \alpha_5 V(\lambda)]E^{\lambda-1} \tag{9}$$

$$(10) \qquad U_V = \frac{\partial U}{\partial V} = [\alpha_2 + \alpha_4 V(\lambda) + \alpha_5 E(\lambda)]V^{\lambda-1}$$

$$(11) \qquad U_{EE} = \frac{\partial^2 U}{\partial E^2} = \alpha_1(\lambda-1)E^{\lambda-2} + \alpha_3 E^{2\lambda-2} + \alpha_3(\lambda-1)E(\lambda)E^{\lambda-2} + \alpha_5(\lambda-1)V(\lambda)E^{\lambda-2}$$

$$(12) \qquad U_{VV} = \frac{\partial^2 U}{\partial V^2} = \alpha_2(\lambda-1)V^{\lambda-2} + \alpha_4(\lambda-1)V(\lambda)V^{\lambda-2} + \alpha_4 V^{2\lambda-2} + \alpha_5(\lambda-1)V^{\lambda-2}E(\lambda)$$

$$(13) \qquad U_{EV} = \frac{\partial^2 U}{\partial E \partial V} = \alpha_5 E^{\lambda-1} V^{\lambda-1}$$

In Chapter V demand elasticities for different assets were calculated by maximizing a general utility function. Explicit demand equations were not obtained since no assumptions were made about the particular functional form of the utility function. For any particular FFF (as an approximation to the utility function), it is possible to derive the demand for assets (liabilities) by an insurance company as follows:

The first order conditions corresponding to the utility function (1) can be found by substituting equations (10) to (13) to equation (11) of the last chapter. Hence,

$$(14) \qquad \frac{\partial L}{\partial X_i} = \alpha_1 W_0 \mu_i [W_0(1+\sum_i X_i \mu_i)]^{\lambda-1}$$

$$+ \alpha_2 W_0^2 \; [W_0^2 \sum_i \sum_j X_i X_j G_{ij}]^{\lambda/2\,-1} \cdot (\sum_j X_j G_{ij})$$

$$+ \alpha_3 W_0 \mu_i \; \left\{\frac{[W_0(1+\sum_i X_i \mu_i)]^{\lambda}-1}{\lambda}\right\} \cdot [W_0(1+\sum_i X_i \mu_i)]^{\lambda-1}$$

$$+ \alpha_4 W_o^2 \; \left\{\frac{[W_0^2 \sum_i \sum_j X_i X_j G_{ij}]^{\lambda/2} - 1}{\lambda}\right\} \cdot [W_0^2 \sum_i \sum_j X_i X_j G_{ij}]^{\lambda/2-1} \cdot [\sum_j X_j G_{ij}]$$

$$+ \alpha_5 W_0 \mu_i [W_0(1+\sum X_i \mu_i)]^{\lambda-1} \left\{\frac{[W_0^2 \sum_i \sum_j X_i X_j G_{ij}]^{\lambda/2} - 1}{\lambda}\right\}$$

$$+ \alpha_5 W_0^2 \, [W_0^2 \sum_{ij}\sum X_i X_j G_{ij}]^{\lambda/2-1} . \, \{\frac{[W_0(1+\sum_i X_i \mu_i)]^{\lambda}-1}{\lambda}\} \, . (\sum_j X_j G_{ij}) - \gamma = 0$$

(15) $$\frac{\partial L}{\partial \gamma} = \sum_i X_i - 1 = 0 \qquad\qquad i,j = 1 \ldots m+1$$

where L is the lagrangian and γ is the lagrange multiplier,

$$E = W_0 \, [1 + \sum_{i=1}^{m+1} X_i \mu_i]$$

$$V = [W_0{}^2 \sum_i \sum_j X_i X_j G_{ij}]^{1/2}$$

Using the above definitions for E and V and substituting into equation (15) yield:[6]

(16) $$W_0[\alpha_1 + \alpha_3 E(\lambda) + \alpha_5 V(\lambda)] E^{\lambda-1} \, \mu_i + W_0^2 \, [\alpha_2 + \alpha_4 V(\lambda)$$

$$+ \alpha_5 E(\lambda)] . V^{\lambda-2} . (\sum X_j G_{ij}) = \gamma$$

As a practical matter, we can use the first-order conditions in order to derive the "reduced form" of the model that will enable us to obtain "good" estimates of the parameters of the utility function. The basic characteristic of a reduced form is that the original system has been solved to express the values of the endogenous variables as functions of all the other variables in the system, so that each equation of the reduced form contains only one current endogenous variable.

Using (16) we obtain that for each asset pair i, r ($i \neq r$):

---

[6]Using the definitions for U_E (equation (9)) and U_V (equation (10)) and substituting them into equation (16) one can obtain.

$W_o U_E \mu_i + W_0^2 V^{-1} U_V (\sum_j X_j G_{ij})$ = which is similar to equation (11) in Chapter V.

$$U_E\ (\mu_i - \mu_r) + W_0 U_V V^{-1} [\sum_{j=1}^{m+1} X_j (G_{ij} - G_{rj})] = 0. \tag{17}$$

The interpretation of (17) is that the marginal contribution of one asset relative to another to utility through both mean and standard deviation of the portfolio must be just offsetting.

By rearranging (17) one can get:

$$\sum_{m=1}^{m+1} X_j (G_{ij} - G_{rj}) = - \frac{U_E(\mu_i - \mu_r)}{W_0 U_V V^{-1}} \tag{17a}$$

Since there are m+1 assets in the model, there are m equations like (17) which together with equation (16) can be written using matrix notation as:

$$X = k \cdot Z^{-1} \cdot \mu^* \tag{18}$$

where: $X = \begin{bmatrix} X_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_{m+1} \end{bmatrix}$ (mx1)

$$k = \frac{U_E}{W_0 U_V V^{-1}} = \frac{[\alpha_1 + \alpha_3 E(\lambda) + \alpha_5 V(\lambda)] E^{\lambda-1}}{W_0 [\alpha_2 + \alpha_4 V(\lambda) + \alpha_5 E(\lambda)] V^{\lambda-2}}$$

$$Z = \begin{bmatrix} G^*_{21}, & G^*_{22}, & G^*_{23}, & \ldots, G^*_{2m+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ G^*_{m+1,1}, & G^*_{m+1,2}, & G^*_{m+1,3}, & \ldots, G^*_{m+1,m+1} \\ 1, & 1, & 1, & , \ 1 \end{bmatrix} \quad \text{(m+1 x m+1)}$$

$$\mu^* = \begin{bmatrix} \mu_1 - \mu_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \mu_1 - \mu_{m+1} \\ 1/k \end{bmatrix} \quad (m+1 \times 1)$$

and

$$G^*_{rj} = G_{ij} - G_{rj} \qquad (r = 2, \ldots, m+1)$$
$$(j = 1, \ldots, m+1)$$

By adding to (18) ν, a serially uncorrelated multivariate normal disturbance, we obtain the demand system to be estimated:[7,8]

(18a) $\quad X = k \cdot Z^{-1} \cdot \mu^* + \nu$

where

$$\nu = \begin{bmatrix} \nu_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \nu_{m+1} \end{bmatrix} \quad (m+1 \times 1)$$

---

[7]For further discussion on the stochastic specification of the system see Chapter VII.

[8]The system in (18a) is not a true reduced form since U_E and U_V are functions of E and V which themselves are functions of the asset shares - the exogenous variables. This specification however is common in the literature. See for example, P. Honohan (1980), Levy, H., and H.M. Markowitz (1979).

Rewriting (18) element-by-element without the errors we have

$$X_i^* = k \sum_{j=1}^{m+1} \eta_{ij}^* \mu_i^* \qquad i = 1, \ldots, m+1,$$

where the * indicates optimal values. Thus, the relative proportion of asset i to asset k held is:

$$\frac{X_i^*}{X_k^*} = \frac{\sum_{j=1}^{m+1} \eta_{ij}^* \mu_i}{\sum_{j=1}^{m+1} \eta_{kj}^* \mu_k} \qquad i, k = 1, \ldots, m+1$$

where:

η_{ij} are defined as the elements of Z^{-1}, i.e., $Z^{-1} \equiv [\eta_{ij}]$.

This ratio is independent of k and hence independent of initial wealth or the particular utility function being maximized. This means that <u>all</u> risk averse utility maximizers whose utility function can be expressed as solely a function of the mean and variance of terminal wealth and who face the same sets of securities (defined by their mean, variance and covariance) will hold the same <u>relative</u> proportions of assets (liabilities) in their optimal portfolios. This result occurs frequently in the portfolio literature and is known as a separation theorem.

The share equations corresponding to the translog, the square rooted quadratic and the general Leontief utility functions can be obtained from equation (18) by selecting the appropriate λ. Since δ does not appear in the above equations, it is clear that the share equations relating to the quadratic and the square rooted quadratic utility functions are identical since in both cases $\lambda=1$.

The second-order conditions for a maximum are obtained by differentiating (14) and (15) with respect to the Xj's.

$$(19)\qquad \frac{\partial^2 L}{\partial X_i \partial X_j} = \alpha_1 W_0^2 \mu_i \mu_j (\lambda-1) E^{\lambda-2}$$

$$+ \alpha_2 [W_0^4 (\lambda-2) V^{\lambda-4} (\sum_i X_i G_{ji})(\sum_j X_j G_{ij}) + W_0^2 V^{\lambda-2} G_{ij}]$$

$$+ \alpha_3 W_0^2 \mu_i \mu_j [E^{2\lambda-2} + (\lambda-1) E(\lambda) E^{\lambda-2}]$$

$$+ \alpha_4 W_0^2 [W_0^2 V^{2\lambda-4} (\sum_j X_j G_{ij})(\sum_i X_i G_{ji}) + W_0^2 V(\lambda)(\lambda-2) V^{\lambda-4} (\sum_i X_i G_{ji})(\sum_j X_j G_{ij})$$

$$+ V(\lambda) V^{\lambda-2} G_{ij}]$$

$$+ \alpha_5 W_0^2 \mu_i \mu_j (\lambda-1) E^{\lambda-2} V(\lambda) + \alpha_5 W_0^3 \mu_i E^{\lambda-1} V^{\lambda-2} (\sum_i X_i G_{ji})$$

$$+ \alpha_5 W_0^4 (\lambda-2) V^{\lambda-4} E(\lambda) (\sum_j X_j G_{ij})(\sum_i X_i G_{ji})$$

$$+ \alpha_5 W_0^3 V^{\lambda-2} E^{\lambda-1} \mu_j (\sum_j X_j G_{ij}) + \alpha_5 W_0^2 V^{\lambda-2} E(\lambda) G_{ij}$$

Rearranging equation (19) yields (19a)

$$(19a)\qquad \frac{\partial 2_L}{\partial X_i \partial X_j} = W_0^2 \mu_i \mu_j [\alpha_1 (\lambda-1) E^{\lambda-2} + \alpha_3 (\lambda-1) E(\lambda) E^{\lambda-2} + \alpha_3 E^{2\lambda-2}$$

$$+ \alpha_5 (\lambda-1) E^{\lambda-2} V(\lambda)] + W_0^3 [\alpha_5 E^{\lambda-1} V^{\lambda-2} (\mu i \sum_i X_i G_{ji} + \mu_j \sum_j X_j G_{ij})]$$

$$+ W_0^4 (\sum_i X_i G_{ji})(\sum_j X_j G_{ij}) \{[\alpha_2 (\lambda-1) V^{\lambda-4} + \alpha_4 V^{2\lambda-4} + \alpha_4 (\lambda-1) V(\lambda) V^{\lambda-4}$$

$$+ \alpha_5 (\lambda-1) V^{\lambda-4} E(\lambda)] - [\alpha_2 + \alpha_4 V(\lambda) + \alpha_5 E(\lambda)] V^{\lambda-1} V^{-3}\} + W_0^2 [\alpha_2 + \alpha_4 V(\lambda)$$

$$+ \alpha_5 E(\lambda)] V^{\lambda-2} G_{ij}$$

Using the definitions of U_V, V_{EE}, U_{VV} and U_{EV} in equations (10), (11), (12) and (13) respectively one can get:

$$(20) \qquad \frac{\partial^2 L}{\partial X_i \partial X_j} = Z_{ij} = W_0^2 U_{EE}\mu_i\mu_j + W_0^3 V^{-1} U_{EV}(\mu_i \sum_i X_i G_{ji} + \mu_j \sum X_j G_{ij})$$

$$+ W_0^4 (V^{-2} U_{VV} - U_V V^{-3})(\sum_i X_i G_{ji})(\sum_j X_j G_{ij}) + V^{-1} W_0^2 U_V G_{ij}$$

Equation (20) is identical to equation (12) in Chapter V , derived from the general model.

In order to compute the comparative static results, one must examine the effect of changes in μ_r and G_{ij} on the X_i's. By differentiating the first-order conditions with respect to μ_j we get:[9]

$$(21) \qquad \frac{\partial^2 L}{\partial X_i \partial \mu r} = \alpha_1 W_0^2 \mu_i (\lambda-1) E^{\lambda-2} (\frac{\partial X_j}{\partial \mu_r} \mu_j + X_r) + \alpha_1 W_0 E^{\lambda-1} \delta_{ir}$$

$$+ \alpha_2 W_0^2 [1/2(\lambda-2) V^{\lambda-4} W_0^2 (\frac{\partial X_i}{\partial \mu_r} \sum_j X_j G_{ij} + \frac{\partial X_j}{\partial \mu_r} \sum_i X_i G_{ij})(\sum_j X_j G_{ij})$$

$$+ \frac{\partial X_j}{\partial \mu_r} G_{ij} \; V^{\lambda-2}]$$

$$+ \alpha_3 W_0 [E^{2\lambda-2} W_0 (\frac{\partial X_j}{\partial \mu_r} \mu_j + X_r)\mu_i + E(\lambda) E^{\lambda-1} \delta_{ir}$$

$$+ E(\lambda)(\lambda-1) E^{\lambda-2} W_0 (\frac{\partial X_j}{\partial \mu_r} \mu_j + X_r)\mu_i]$$

$$+ \alpha_4 W_0^2 [1/2 V^{2\lambda-4} W_0^2 (\frac{\partial X_i}{\partial \mu_r} \sum_j X_j G_{ij} + \frac{\partial X_j}{\partial \mu_r} \sum_i X_i G_{ji})(\sum_j X_j G_{ij})$$

---

[9] The following results are derived under the assumption that G_{ij} is not changing, i.e., the covariance between the return on asset i and the return on asset j is constant for all i, j=1, ..., m+1.

$$+V(\lambda)\frac{1}{2}(\lambda-2)V^{\lambda-4}W_0^2(\frac{\partial X_i}{\partial\mu_r}\sum_j X_jG_{ij}+\frac{\partial X_j}{\partial\mu_r}\sum_i X_iG_{ji})(\sum_j X_jG_{ij})$$

$$+V(\lambda)V^{\lambda-2}\frac{\partial X_j}{\partial\mu_r}G_{ij}]$$

$$+\alpha_5W_o[(\lambda-1)E^{\lambda-2}V(\lambda)W_0(\frac{\partial X_j}{\partial\mu_r}\mu_j+X_r)\mu_i+V(\lambda)E^{\lambda-1}\delta_{ir}$$

$$+1/2E^{\lambda-1}V^{\lambda-2}W_0^2(\frac{\partial X_i}{\partial\mu_r}\sum_j X_jG_{ij}+\frac{\partial X_j}{\partial\mu_r}\sum_i X_iG_{ji})\mu_i]$$

$$+\alpha_5W_0^2[1/2(\lambda-2)V^{\lambda-4}E(\lambda)W_0^2(\frac{\partial X_i}{\partial\mu_r}\sum_j X_jG_{ij}+\frac{\partial X_j}{\partial\mu_r}\Sigma X_iG_{ji})(\Sigma X_jG_{ij})$$

$$+E^{\lambda-1}V^{\lambda-2}W_0(\frac{\partial X_j}{\partial\mu_r}\mu_j+X_r)(\sum_j X_jG_{ij})+V^{\lambda-2}E(\lambda)(\frac{\partial X_j}{\partial\mu_r}G_{ij})]$$

$$+\frac{\partial\gamma}{\partial\mu_r}=0.$$

where $\delta_{ir}=1$ for $i=r$

$=0$ for $i\neq r$

Equation (21) can be written as:

$$W_0^2\,\mu_i\mu_j[\alpha_1(\lambda-1)E^{\lambda-2}+\alpha_3(\lambda-1)E(\lambda)E^{\lambda-2}+\alpha_3E^{2\lambda-2}]\frac{\partial X_j}{\partial\mu_r}\tag{22}$$

$$+W_0^3[\alpha_5E^{\lambda-1}V^{\lambda-2}(\mu_i\Sigma X_iG_{ji}+\mu_j\Sigma X_jG_{ij})]\frac{\partial X_j}{\partial\mu_r}$$

$$+W_0^4(\sum_i X_iG_{ji})(\sum_j X_jG_{ij})[\alpha_2(\lambda-2)V^{\lambda-4}+\alpha_4V^{2\lambda-4}+\alpha_4V(\lambda)(\lambda-2)V^{\lambda-4}$$

$$+\alpha_5(\lambda-2)V^{\lambda-4}E(\lambda)]\frac{\partial X_j}{\partial\mu r}+W_0^2V^{\lambda-2}G_{ij}[\alpha_2+\alpha_4V(\lambda)+\alpha_5E(\lambda)]\frac{\partial X_j}{\partial\mu_r}+\frac{\partial\Upsilon}{\partial\mu_r}$$

$$=-W_0[\alpha_1+\alpha_3E(\lambda)+\alpha_5V(\lambda)]E^{\lambda-1}\delta_{ir}-W_0^2X_r\mu_i[\alpha_1(\lambda-1)E^{\lambda-2}$$

$$+ \alpha_3(\lambda-1)E^{\lambda-2}E(\lambda)+\alpha_5(\lambda-1)E^{\lambda-2}V(\lambda)] - X_r W_0^3(\alpha_5 E^{\lambda-1}V^{\lambda-2}\sum_j X_j G_{ij})$$

Using the definition of Z_{ij} in (20) and equations (9), (11) and (13) we obtain:

$$(23) \qquad Z_{ir}\frac{\partial X_j}{\partial \mu_r} + \frac{\partial \gamma}{\partial \mu_r} = - W_0 U_E \delta_{ir} - X_r[W_0^2 U_{EE}\mu_i + W_0^3 U_{EV}V^{-1}\sum_j X_j G_{ij}]$$

One can carry the same operation as performed in (21) for all $i, j = 1,\ldots m+1$ to obtain the matrix equation (14) in Chapter V and the solution for $\frac{\partial X_i}{\partial \mu_r}$ will be identical to the one derived for the general model.[10]

Using an analogous procedure, it is possible to find the effect of a change in G_{ij} on the holdings of any asset i, i = 1, ..., m+1. Differentiating the first-order conditions with respect to G_{rf} yields:

$$(24) \qquad \frac{\partial 2_L}{\partial X_i \partial G_{rf}} = \alpha_1 W_0^2 \mu_i \mu_j(\lambda-1)E^{\lambda-2}\frac{\partial X_j}{\partial G_{rf}}$$

$$+ \alpha_2 W_0^2[1/2(\lambda-2)V^{\lambda-4}W_0^2(\frac{\partial X_i}{\partial G_{rf}}\sum_j X_j G_{ij} + \frac{\partial X_j}{\partial G_{rf}}\sum_i X_i G_{ji} + 2X_r X_f)$$

$$\cdot (\sum_j X_j G_{ij}) + V^{\lambda-2}(\frac{\partial X_j}{\partial G_{rf}} \; G_{ij} + X_r + X_f)]$$

$$+ \alpha_3 W_0^3 \mu_i \mu_j(E^{2\lambda-2}+E(\lambda)(\lambda-1)E^{\lambda-2})\frac{\partial X_j}{\partial G_{rf}}$$

$$+ \alpha_4 W_0^2[1/2V^{2\lambda-4}W_0^2(2X_r X_f + \frac{\partial X_i}{\partial G_{rf}}\sum_j X_j G_{ij} + \frac{\partial X_j}{\partial G_{rf}}\sum_i X_i G_{ji})(\sum_j X_j G_{ij})$$

---

[10] See equation (15) in Chapter V .

$$+ V(\lambda)1/2(\lambda-2)V^{\lambda-4}W_0^2(2X_rX_f + \frac{\partial X_i}{\partial G_{rf}} \sum_j X_jG_{ij} + \frac{\partial X_j}{\partial G_{rf}} \sum_i X_iG_{ji})$$

$$(\sum_j X_jG_{ij}) + V(\lambda)V^{\lambda-2}(\frac{\partial X_j}{\partial G_{rf}} G_{ij}+X_r+X_f)$$

$$+ \alpha_5W_0^2\mu_i[(\lambda-1)E^{\lambda-2}V(\lambda)(\frac{\partial X_j}{\partial G_{rf}}\mu_j)+1/2E^{\lambda-1}V^{\lambda-2}W_0(2X_rX_f +$$

$$\frac{\partial X_i}{\partial G_{rf}} \sum_j X_jG_{ij} + \frac{\partial X_j}{\partial G_{rf}} \sum_i X_iG_{ji}]$$

$$+ \alpha_5W_0^2[1/2(\lambda-2)V^{\lambda-4}E(\lambda)W_0^2(2X_rX_f + \frac{\partial X_i}{\partial G_{rf}} \sum_j X_jG_{ij}+ \frac{\partial X_j}{\partial G_{rf}} \sum_i X_iG_{ji})$$

$$\cdot (\sum_i X_iG_{ji})+V^{\lambda-2}E^{\lambda-1}W_0 \frac{\partial X_j}{\partial G_{rf}} \mu_j(\sum_j X_jG_{ij})$$

$$+ V^{\lambda-2}E(\lambda)(\frac{\partial X_j}{\partial G_{rf}} G_{ij}+X_r+X_f)] - \frac{\partial \gamma}{\partial G_{rf}} = 0$$

Rearranging equation (24) and substituting for Z_{ij}, U_V, U_{EV}, and U_{VV} from equations (19a), (10), (13) and (12) respectively yields:

$$(25) \qquad Z_{ij} \frac{\partial X_i}{\partial G_{rf}} + \frac{\partial \gamma}{\partial G_{rf}} = -W_0^2V^{-1}U_V(X_r\delta_{ir}+X_f\delta_{if})$$

$$-V^{-1}X_rX_fW_0^3[U_{EV}\mu_i+W_0V^{-1}U_{VV} \sum_j X_jG_{ij}]$$

where $\delta_{ir} = 1$ for $i = r$ $\qquad$ $\delta_{if} = 1$ for $i = f$

$= 0$ for $i \neq r$ $\qquad$ $= 0$ for $i \neq f$

Equation (25), when written in matrix form,is identical to equation (17) in Chapter V , and can be written as follows:

$$(26) \qquad \frac{\partial X_k}{\partial G_r^{\,2}} = -2\,W_0^2 V^{-1} U_V X_r \frac{D_{rk}}{D} - X_r^{\,2} W_0^{\,3} V^{-1} [U_{EV} \sum_i \mu_i \frac{D_{ik}}{D} + W_0 V^{-1} U_{VV} \sum_i \sum_j X_j G_{ij} \frac{D_{ik}}{D}]$$

In the following Chapter, the share equations specified in equation (18a) are estimated. Based on the parameters obtained from the regression, marginal utilities and elasticities of substitution are calculated.

CHAPTER VII

EMPIRICAL RESULTS

"The results of a portfolio analysis
are no more than the logical consequence
of its information concerning securities."

Harry Markowitz

Although we may have cleared away some of the undergrowth surrounding the investment process of life insurance companies, we are still left with the task of attempting to capture as much of these problems as possible in our empirical analysis. In this chapter, we wish to start the discussion of the estimating of our model, taking into account the various considerations stressed elsewhere in this study.

In carrying out our empirical work we have essentially two choices (though not mutually exclusive) in the sample selection. We can utilize an "industry" level approach using aggregate time series data and/or we can adopt a "micro" strategy by investigating the investment behaviour of individual companies. Ideally, to follow both strategies together is to be preferred in that they are complementary. Micro studies in general permit a more in depth analysis of investment intentions. However, these studies suffer from extrapolation difficulties when one tries to draw industry-wide conclusions. Macro studies, however, do offer an opportunity to understand the activities of the whole life insurance sector within the financial markets, but unfortunately, aggregation brings about sacrifices in terms of data sources, investment attitudes and the like.

Our brief, as we believe is already evident from previous chapters, is to follow a "micro" type approach which is consistent with the theoretical model developed. It is the objective of the empirical work presented below, to illustrate the portfolio optimization of financial intermediaries in the presence of some relevant constraints. Unlike most prior research,[1] this model is implemented using data for individual life insurance companies rather than data for the entire industry. In so doing, we avoid making assumptions (like symmetry, linear homogeneity) required to justify the aggregation over individual companies, and thus avoid any bias which could result from imposing restrictions that may be inappropriate. Furthermore, a "micro" type approach allows us to test whether or not significant differences exist between companies in their attitudes toward risk.

VII.1 The Data

The annual data used in the empirical part of the thesis was obtained from the reports of the Superintendent of Insurance for Canada (1955-1977).[2]

A sample comprising the 8 largest Canadian life insurance companies (ranked by asset size) on December 31, 1977, was taken. The total asset value of the sampled companies was 21.7 billion dollars which represents about 70 percent of total industry assets.

---

[1] See for example P. Honohan (1980), Y. Kahane and D. Nye (1975).

[2] Canada, Department of Insurance, Report of The Superintendent of Insurance, (Ottawa; Queen's Printer, various years).

The companies included in the sample were:

a. Canada Life

b. Confederation Life

c. Great West Life

d. Imperial Life

e. London Life

f. Manufacturers' Life

g. Mutual Life of Canada

h. Sun Life

We categorized the financial holdings of these life insurance companies into five asset and two liabilities types. The groups of assets are:[3]

a. Canadian Bonds (CB)

b. Foreign Bonds (FB) - mainly U.K. and U.S. bonds

c. Canadian Stocks (CS)

d. Foreign Stocks (FS)

e. Mortgage Loans and Real Estate (MLRE)

On the liabilities side we calculated the "net actuarial reserves" and surplus. These are defined as:

f. Net Actuarial Reserves (NAR).

Reserves for contracts in force (the major component) plus claims under consideration, plus amount on deposits, plus provisions for profit to policy-

---

[3]Since segregated funds come under different regulations, the assets (liabilities) included in them were not accounted for, in our analysis.

holders, plus other liabilities, less policy loans, less cash, less investment income due and accrued, less outstanding insurance premiums and annuity consideration, less other assets.

Policy loans were transferred from the asset side to liability side in order to obtain discretionary assets at the end of each period. By doing so, we actually calculated the net actuarial reserves that the company should hold in order to fulfill its contractual obligations.

g. Surplus or Retained Earnings (SR).

This consists of capital stock paid, shareholders funds and insurance funds.

Firms are maximizing their objective function subject to their beliefs about future rates of return. Therefore, in estimating expected rates of return in order to utilize them in our regression analysis, the implicit assumption is that the firms' beliefs about future rates coincide with our estimates. In a real world environment, the best approach would have been to combine objective and subjective information in estimating the parameters of the rate of return distribution. Since insurance companies' subjective data was not available, those parameters were estimated using historical time series for the period 1945-1977.[4] Different market indices were used in order to approximate the distribution of the rates of return on the above investments. By doing so, it was implicitly assumed

---

[4]The index method suggested by Y. Kahane (1977a) or the innovative approach of estimating the subjective covariances of assets yield along Bayesian lines suggested by P. Honohan (1980) provide alternative solutions for this problem.

that life insurance companies' stocks, bonds, mortgages, etc., were well diversified and included a representative cross section of the securities available in the market. This assumption appears tenable but it is possible that the investment portfolio of life insurance companies is of different composition than the market portfolio due to the portfolio effect itself, i.e., correlation between the investment and insurance portfolio or because of regulation affecting the insurance company investment. Although portfolio differences could bias the results, no substantial differences are believed to exist.

Annual holding period returns for Canadian bonds were calculated using a weighted average of past annual yields on 5-10 years Canadian Government Bonds, Canadian Provincial Bonds and Canadian Industrial Bonds.

The annual yields on foreign bonds were obtained in a similar way, after being adjusted for exchange-rate fluctuations. Yields on British Government Stocks (medium dated) were used as an approximation for the annual yields on U.K. bonds, while the return on U.S. bonds was based upon Moody's composite yields index.[5] A weighted average was calculated using the actual holdings of U.K. and U.S. bonds by each insurance company.

For Canadian stocks, the relevant measure is the return due on both dividends and capital gains. As a proxy for this measure, the Toronto Stock Exchange (TSE) index adjusted for dividends was used

---

[5] In order to obtain "real yields" on those assets, the rate of change in the exchange rate was added to the rate of change in those indices. The same mechanism was utilized when yields on foreign stocks were calculated.

which implies that the Canadian stock portfolio of each life insurance company can be represented by the TSE index.

Data from the Bank of England Quarterly Bulletin on U.K. stock yields, and the rate of change in the Standard and Poor's (S&P) 500 Stock Index for the New York Stock Exchange plus the S&P dividend/price ratio for common stocks, were utilized in computing the annual return on foreign stocks.[6]

Conventional mortgages rates and NHA mortgages were used in calculating the annual return on mortgage loans and real estate.

The total expected profit for an insurance company, as previously explained, is the sum of its investment and underwriting profits. The underwriting profit (cost) for a life insurance company is difficult to estimate. Since life insurance policies tend to be long term, any estimate of underwriting profits must make some assumptions about future interest rates, mortality rates and expenses. If one assumes that actuarial reserves provide a good measure of expected future claims and expenses, then it is possible to estimate the underwriting profits as:[7]

Underwriting profit = revenues - costs.

where:[8]

revenues = premiums + annuity payments collected.

---

[6]Among the foreign bonds (stocks) held by insurance companies in Canada, one can find bonds issued by countries other than the U.K. or the U.S. It is assumed that the yields on those bonds (stocks) were identical to those on U.S. bonds (stocks).

[7]In calculating underwriting profits we used aggregate data for all the Canadian companies and thus obtained an average rate of profit.

[8]A similar approach was used by S. Kellner and G.F. Mathewson (1980).

and,

costs = claims paid + change in actuarial reserves
+ taxes, licences and fees + commissions and
general expenses + policy dividends.

The expenses of life insurance companies for a particular policy tend to be high in the first year and then drop significantly in subsequent years. In order to allow for this uneven distribution, the actual first year expenses were averaged over a ten year period which is the average length of a policy.

For each of the categories mentioned, a series of observations was constructed for the period 1945-1977. (33 data points). Then, ten years of yield data are employed in order to calculate mean returns and variances for each asset as well as sample covariances between asset yields. These sample estimates were then used to calculate the expected return and variance of the portfolio held by each of the eight companies at the end of 1955. The shares used (X_i's) for 1955 were the actual proportions of every asset (liability) held by each company at the end of that year. The calculated portfolio sample mean and variance for 1955 represents one data point to be utilized in estimating the share equations. The second and subsequent data points are calculated using the "rolling sample" updating technique. The sample means, variances and covariances are recalculated dropping the 1945 data and adding the data for 1956. These new estimates together with the asset proportions held by the insurance company at the end of 1956 provided the 1956 portfolio mean, variance and covariance and, hence, another data point. By this updating procedure a time series of 23 data points was generated and utilized to estimate the utility function parameters. Table VII.1 contains an example of the actual holdings and their associated expected returns for one company.[9]

It was assumed in our model that life insurance companies measure the expected utility of choices among risky assets by looking at the mean and variance of their portfolio provided by combinations of those assets. By looking only at mean and variance, we are necessarily assuming that no other statistics are necessary to describe the distribution of end of period wealth. Unless investors have a special type of utility function (quadratic utility function), it is necessary to assume that returns have a normal distribution, or at least one which can be completely described by the mean and variance. To determine whether the normal distribution is reasonable to use, Kolmogorov-Smirnov tests of goodness of fit were conducted on the expected rates of return series for the companies in the sample.[10]

In this test, it is assumed that a sample of size n is drawn from a population with a cumulative distribution F(X). Define the empirical distribution function $F_n(X)$ to be the step function

$$F_n(X) = \frac{k}{n} \quad \text{for } X_{(i)} \leq X \leq X_{(i+1)}$$

where k is the number of observations not greater than X and the sample values $(X_{(1)}, \ldots, X_{(n)})$ are arranged in ascending order. Under the null hypothesis that the sample has been drawn from the specified

---

[9]Appendix B contains the data utilized in the estimations for each of the eight companies in our sample.

[10]For further discussion on the test see R.V. Mises (1964) or B.V. Gnedenko (1962). S.S. Shapiro and M.B. Wilk (1965) offer another test for normality.

TABLE VII.1: Asset Holdings By Imperial Life of Canada and Their Associated Expected Returns

a. Holdings (000 $)

End of Year	CB	FB	CS	FS	MLRE	NAR	SR
1955	83367.	30604.	7211.	5056.	73813.	-189280.	10771.
1956	79109.	31959.	7465.	5861.	86841.	-198730.	12505.
1957	83477.	33612.	7841.	6447.	90951.	-208619.	13709.
1958	86332.	36705.	8585.	6137.	97956.	-220760.	14955.
1959	92137.	42157.	8097.	6880.	102513.	-234455.	17329.
1960	106150.	45259.	8269.	7002.	101292.	-248194.	19778.
1961	115744.	48200.	8494.	7513.	105740.	-263402.	22289.
1962	113902.	48899.	8985.	7905.	125887.	-280984.	24594.
1963	112314.	53339.	10209.	8524.	142694.	-300213.	26867.
1964	114826.	54970.	14230.	9031.	156932.	-320457.	29532.
1965	96334.	60019.	29427.	8069.	180036.	-341429.	32456.
1966	90463.	64676.	31125.	9413.	196828.	-357227.	35278.
1967	96340.	68049.	31082.	11171.	201909.	-370857.	37694.
1968	95962.	63096.	34150.	14059.	207633.	-355340.	49560.
1969	84518.	65154.	39023.	15337.	212577.	-367154.	49455.
1970	100498.	58546.	40050.	14317.	208214.	-366443.	55182.
1971	106643.	63772.	42581.	16817.	215378.	-387763.	57428.
1972	106443.	67192.	46978.	20405.	226884.	-411434.	56468.
1973	99232.	61208.	51838.	21761.	243343.	-420762.	56620.
1974	98355.	62442.	51447.	22547.	260506.	-438183.	57114.
1975	118529.	52266.	55215.	23681.	281926.	-475228.	56389.
1976	115084.	50838.	61125.	23598.	297740.	-497262.	51123.
1977	136948.	54855.	49804.	23952.	317229.	-529154.	53634.

b. Expected Returns (percentage point)

1955	.0335	.0082	.1378	.1784	.0514	.0260
1956	.0345	.0094	.1349	.1793	.0519	.0254
1957	.0361	.0112	.1146	.1628	.0525	.0256
1958	.0375	.0176	.1170	.1580	.0533	.0259
1959	.0396	.0184	.1233	.1780	.0541	.0270
1960	.0417	.0235	.1180	.1818	.0551	.0304
1961	.0436	.0393	.1225	.1729	.0561	.0312
1962	.0451	.0465	.0940	.1496	.0571	.0324
1963	.0463	.0550	.0984	.1580	.0581	.0336
1964	.0474	.0573	.1223	.1659	.0589	.0349
1965	.0492	.0589	.1196	.1474	.0598	.0378
1966	.0517	.0607	.0893	.1124	.0608	.0399
1967	.0538	.0619	.0806	.1072	.0619	.0417
1968	.0562	.0572	.0908	.1250	.0632	.0432
1969	.0596	.0586	.1069	.1177	.0648	.0451
1970	.0624	.0605	.0842	.0789	.0667	.0460
1971	.0644	.0618	.0974	.0902	.0635	.0453
1972	.0666	.0603	.0898	.0894	.0703	.0448
1973	.0691	.0586	.0986	.0865	.0722	.0447
1974	.0730	.0604	.0774	.0445	.0745	.0437
1975	.0769	.0672	.0608	.0520	.0772	.0434
1976	.0809	.0580	.0557	.0566	.0801	.0429
1977	.0836	.0666	.0592	.0736	.0828	.0434

Definitions: Canadian Bonds (CB)
Foreign Bonds (FB)
Canadian Stocks (CS)
Surplus (SR)
Foreign Stocks (FS)
Mortgage Loans and Real Estate (MLRE)
Net Actuarial Reserves (NAR)

Sources: Bank of England Quarterly Bulletin, various issues
Bank of Canada Review, various issues
Report of the Superintendent of Insurance, various years.

distribution (normal in our case), $F_n(X)$ should be fairly close to $F(x)$, the theoretical cumulative function.

Define

$$D = \max \mid F_n(X) - F(X) \mid .$$

For a two-tailed test we reject the hypothetical distribution if D exceeds the critical values for the Kolmogorov-Smirnov test.

Table VII.2 contains the results of the test for the rates of return series as shown in Table VI.1.

TABLE VII.2: The Kolmogorov-Smirnov Test Results* (Imperial Life)

ASSET	COMPUTED D	CRITICAL VALUES (N=23) Significance Level .20	.10	.05	.02	.01
CB	.112454					
FB	.306176					
CS	.129206	.216	.247	.275	.307	.330
FS	.168938					
MLRE	.117837					
NAR	.210525					

* Reject the hypothetical distribution F(X) if $D_n = \max|F_n(X)-F(X)|$ exceeds the tabulated (critical) value.

Since the computed D's are generally less than the critical values (at the 10 percent level) for the asset returns, one would normally fail to reject the normality hypothesis for this company. Similar results were obtained for the other companies in the sample.[11] On the basis of the above results it was concluded that it is reasonable to assume that all the distributions of returns are normal.

VII.2 Method of Estimation

Equation (18a) in Chapter VI represents a company's system of demand equations. In order to estimate this system, which is non-linear in the parameters (in k), one needs a non-linear estimation method. The best known and well-established method of estimation to deal with, is the maximum-likelihood method. Maximum-likelihood (ML) estimators are consistent and asymptotically efficient. Furthermore, if the researcher is willing to assume that the disturbances are normally distributed, the estimators are also asymptotically normally distributed.[12] Another advantage of the ML method is that it

---

[11] The fact that FB's return violates the assumption (at the 20 percent level) will have only a marginal effect because it represents a small proportion of the companies' portfolio. The test results for all the companies in the sample are tabulated in Appendix VII.A at the end of this chapter.

[12] This specification ignores the requirement that budget shares must be between zero and one by giving positive probability to shares outside this range. The Dirichlet distribution, for example, which limits shares to the unit simplex, would have been a more appropriate stochastic specification. However, Woodland (1979) provides justification for the continued use of the normal distribution specification in the estimation of share equations by showing that there are no substantial differences in empirical results using the normal model estimator rather than the Dirichlet model. He explains that;

> "Application of the two estimators to three different actual data sets resulted in the estimates very close to each other for each set. Moreover, the calculated standard errors were very close. These results together with those arising from the sampling experiments suggest that the normal model is rather robust with respect to stochastic specification. While further evidence from alternative sampling experiments is

continued ...

is linked up with the likelihood ratio test, to test overall hypotheses about the system. The likelihood ratio test uses the likelihood values that are a simple by-product of the method of estimation.[13]

In the system to be estimated

$X = k \cdot Z^{-1}\mu^* + v$

We define the column vector of disturbances at time t as:

$v(t) = \{ v_1(t), v_2(t), \ldots, v_{m+1}(t)\} \qquad t = 1, \ldots T.$

---

desirable, the results of this paper suggest that, while the normal model may not be a theoretically appropriate specification for share equations, it may, for a large number of data sets, yield valid results." (pp. 381-382)

[13]The algorithm used in our study is a Quazi-Newton method. The basis of this iterative technique may be outlined as follows:

Let $\Pi(h)$, a scalar function of a vector of parameters h, be the function to be maximized. Then we seek some solution to the first order condition

$$(1) \qquad \frac{\partial \Pi}{\partial h_i} = 0 \qquad (i = 1, \ldots, n)$$

and we assume that (1) is non-linear in the h's. By choosing some arbitrary point h_o, hopefully close to the maximum, we are able to write the following approximation:

$$(2) \qquad \frac{\partial \Pi}{\partial h} = \frac{\partial \Pi}{\partial h_o} + \frac{\partial^2 \Pi}{\partial h\, \partial h'} \; \delta h$$

If we define H_o to be the Hessian, and setting the left hand side of (2) equal to zero, we obtain a "step" δh given by:

$$(3) \qquad \partial h = -H_o^{-1} \frac{\partial \Pi}{\partial h_o} = -H_o^{-1} g_o$$

where g_o is the gradient of the function at the point. An iterative procedure can be built upon this basis. If the starting point (h_o) chosen is in the complex region, the steps, δh, will yield to succesively higher values of $\Pi(h)$.

and the associated (assumed constant) disturbance variance-covariance matrix as Ω.

Since the expenditure shares sum to unity, the m+1 components in v(t) add up to zero at each annual observation. Thus, Ω in each of our models is singular and nondiagonal. As Barten (1969) points out this is a reflection of the fact that:

> "...shifts in one direction for some commodities have to be compensated by ... shifts in the opposite direction for other commodities to stay within the limits of the budget." (p. 16)

If the estimation procedure is to be efficient, the disturbance covariances must be taken into account. Because the (m+1) by (m+1) estimated disturbances covariance matrix is also singular, it is not possible to estimate the full system of m+1 equations by the traditional ML method.[14] To avoid this problem, one equation is arbitrarily dropped in each of our models. Thus, we define a new vector v*(t) as v(t) with one element (assume the m+1) deleted. We assume that the "new" random vector v*(t) is independently joint normally distributed with mean vector zero and a nonsingular variance-covariance matrix Ω^*, t=1, ... T. The density of the "new" vector v*(t) can be written as:

$$f(v^*(t)) = 2\Pi^{-1/2m} \cdot |\Omega^*|^{-1/2} \exp - 1/2(v^*(t))'\Omega^{*-1}v^*(t) \tag{1}$$

and the likelihood function:

$$L(v^*) = \Pi_t\, f(v^*(t)) = 2\Pi^{-1/2\ Tm}|\Omega^*|^{-1/2T}$$

$$\exp - 1/2 \sum_t (v^*(t))'\Omega^{*-1}v^*(t)$$

---

[14]The density of v(t) is not defined.

Barten (1969) has shown that the value of the right-hand side of (1):

> " ... does not depend on the index of the deleted component of v(t). In other words, for the purpose of maximization of the likelihood function it is completely irrelevant what component is deleted or, equivalently, what equation is dropped from the system." (p. 25) [15] [16]

The disadvantages of the use of ML procedure are its computational difficulties in dimensions higher than one, and it may not be surprising, therefore, that considerable difficulty was encountered in obtaining clear cut global maxima. Furthermore, the possibility of nonconvergance of the iterative procedure exists. However, in our model and data, the likelihood surfaces appeared to provide clearly defined and easily approachable peaks.

VII.3 Choosing the Best Utility Function

In our previous discussion, it was pointed out that one of the main purposes of our paper is to compare and discriminate among three "flexible" functional forms. It is impossible to discriminate between the three forms on pure economic grounds since each of the forms can represent arbitrary well-behaved preferences in the

---

[15] The results reported hereafter confirm Barten theoretical proof. The parameters estimated and the value of the log likelihood function in each one of the models estimated and for each one of the companies in the sample, were invariant to the equation omitted. Furthermore, it was found very helpful to omit different equations in the process of searching for the global maximum.

[16] The Barten proof relates only to Full Information Maximum Likelihood (FIML) parameter estimates. Independently, S. Kmenta and R.F. Gilbert (1968) showed that iterated OLS converged to FIML using Monte Carlo techniques and P. Dhrymes (1973) proved this convergence analytically; that is, he proved that iterated Seemingly Unrelated Regression (SUR) is asymptotically equivalent to FIML. It is this later technique that is, in fact, used in this thesis.

neighbourhood of a given point with an accuracy of the second order. A priori, we are also unable to choose among the forms on econometric grounds. The estimation of each one of the forms, suggested in Chapter VI, involves the same dependent variable, the same number of free parameters and the maximization of a similar likelihood function. In order to use traditional tests, a fourth form is estimated, namely the unrestricted system where λ is a free parameter. Thus, the three "original" forms are nested (i.e., they are a special case of the unrestricted λ case). Therefore, four different budget share models were estimated for each of the companies in the sample; the translog ($\lambda=0$), the generalized Leontief ($\lambda=1/2$), the square root quadratic ($\lambda=1$), and the unrestricted system where λ is a free parameter. Since the share equations are homogeneous of degree zero in the α_i parameters, these parameters were normalized with respect to α_1.

Table VII.3, summarizes the results for all the companies and each of the estimated systems. The unrestricted λ model involves nonlinear estimation of five free parameters. In all the other versions only four free parameters must be estimated.

In two out of the eight companies in the sample the unrestricted system yielded a parameter estimate for λ of -.35644 (Confederation Life) and -.21243 (Imperial Life) which are close to the value of the translog. For all the other companies the estimated λ was above one (i.e., close to the square root quadratic case). More rigorously, it can be shown that $-2\ \ell nL$ is asymptotically distributed $\chi^2(1)$ where L is the ratio of the value of the unrestricted likelihood

(Asymptotic t Statistic

Functional Form \ Company	Canada Life	Confederation Life	Great West Life	Imperial Life	London Life	Manufacturers Life	Mutual Life	Sun Life
Translog $\lambda=0$								
$\frac{\alpha 2}{\alpha 1}=$	.27722	3.8176 (2.8607)	-1535.5	-8.4822 (-8.5305)	-.13497	11.438	.66413	91.360
$\frac{\alpha 3}{\alpha 1}=$	.79269E-02	.014994 (.5542)	16.735	.13615 (1.3736)	.083799	-.16732	-.18877	-.48689
$\frac{\alpha 4}{\alpha 1}=$	.19279	-.20157 (-1.3632)	221.98	1.2541 (5.9170)	-1.0439	-1.2807	-.27084	-8.4536
$\frac{\alpha 5}{\alpha 1}=$	-.12187	-.13224 (-3.3786)	-23.442	-.32559 (-2.2773)	.95045E-02	.12970	.16084	.52820
Log likelihood function	-31.12020	-38.96212	-50.88252	-48.21299	46.12006	-61.68767	-1.594262	10.14806
Generalized Leontief $\lambda=1/2$								
$\frac{\alpha 2}{\alpha 1}=$	7.2368	26.536	-61.239	-82.842	-18.000	85.951	41.706	8528.0
$\frac{\alpha 3}{\alpha 1}=$	.59502E-04	.28619E-04	.12081E-02	.12632E-02	-.19558E-02	-.13324E-02	-.43970E-02	-.14787E-01
$\frac{\alpha 4}{\alpha 1}=$	.65741E-01	-.59687E-01	.63705	.37849	.12939	-.21722	-.23589	-10.254
$\frac{\alpha 5}{\alpha 1}=$	-.58278E-02	-.52400E-02	-.13386E-01	-.18460E-01	-.85631E-02	.52606E-02	.26577E-01	.28348E-01
Log likelihood function	-30.27213	-39.3622	-50.57360	-48.40701	50.82598	-62.18549	.7784979	10.20590

TABLE VIII.3 (continued)

Functional Form \ Company	Canada Life	Confederation Life	Great West Life	Imperial Life	London Life	Manufacturers Life	Mutual Life	Sun Life
Square root quadratic λ=1								
$\frac{\alpha 2}{\alpha 1}$=	111.93	229.37	-189.10	766.50	-294.92	452.58	440.63	4694.4
	(2.1735)		(-.82216)		(-1.3657)	(7.5307)	(2.8973)	(.07169)
$\frac{\alpha 3}{\alpha 1}$=	.10246E-06	-.58599E-06	.32624E-05	.30565E-05	-.85646E-05	-.23563E-05	-.12885E-04	-.45928E-06
	(.27085)		(3.1815)		(-1.9268)	(-9.6562)	(-2.9545)	(-.073028)
$\frac{\alpha 4}{\alpha 1}$=	.90072E-02	-.83549E-02	.76979E-01	-.31013	.99077E-01	-.10198E-01	-.24414E-01	.15500E-01
	(1.7781)		(1.6725)		(7.0301)	(-7.5017)	(-2.3241)	(.11216)
$\frac{\alpha 5}{\alpha 1}$=	-.11671E-03	-.81381E-04	-.27042E-03	-.49869E-03	.39424E-03	.54272E-04	.55426E-03	-.12032E-03
	(3.2846)		(-5.0454)		(1.0743)	(5.8542)	(2.3496)	(-.11267)
Log likelihood function	-29.56254	-39.94553	-49.86024	-48.44756	54.87833	-60.59928	3.230978	10.26184
Unrestricted								
λ	2.1399	-.35644	4.2465	-.21243	2.3987	2.1417	2.5113	1.3750
$\frac{\alpha 2}{\alpha 1}$=	.17705E+05	5.0588	-200.71	-.27070E+08	-.18294E+08	.19938E+05	.29672E+06	4694.5
$\frac{\alpha 3}{\alpha 1}$=	-.61006E-12	.91655	.21296E-23	.63702E+07	-.62196E-02	-.55568E-12	-.66896E-13	.11367E-08
$\frac{\alpha 4}{\alpha 1}$=	.19672E-04	-.51072	.56692E-07	.14036E+08	.29668E+08	-.42099E-05	-.28900E-05	.39564E-02
$\frac{\alpha 5}{\alpha 1}$=	-.40612E-08	-1.3168	-.79778E-16	-.7152E+07	348.28	.71643E-09	.10529E-08	-.24123E-05
Log likelihood function	-27.75845	-38.75193	-46.82290	-48.13855	58.35373	-57.86796	8.140257	10.28788

function to the value of the restricted likelihood function.[17] Table VII.4 contains the test statistics ($-2\ell nL$) for all the companies in the sample. For example, the likelihood ratio test statistics for the three restricted models, when Manufacturers' Life data is used are: 7.639 (TLOG), 8.635 (GL) and 5.463 (SRQ), while the .01 chi-square critical value is 6.635. This implies that with our annual data for Manufacturers' Life we cannot reject the square root quadratic as our utility function but we can reject the other two at the 1% significance level.[18] For two companies, London Life and Mutual Life, we rejected all the three functional forms at the 1% significance level. However, at least for London Life we cannot reject (SRQ) as our utility function at the 0.5% significance level. To summarize:

a. For 7 out of 8 companies we could not reject the square root quadratic utility function at 0.5% significance level and for 6 out of 8 at 1% significance level.

b. When the three functional forms were ranked based on the test results (i.e., $-2\ \ell nL$) for 6 out of 8 companies the SRQ came out first while the TLOG came out first twice. The GL form was never chosen as our utility function. It seems from the results reported above that the SRQ utility function best represents the preferences of Canadian life insurance companies.

---

[17] See Berndt, Hall, Hall and Hausman (1974) on this point.

[18] Some caution must be used in interpreting our test results, since we are using an asymptotic test with only 23 observations.

TABLE VII.4: Likelihood Ratio Test Results for Three Functional Forms

Company	Test Statistic* $\lambda=0$ $-2\ell nL$	$\lambda=1/2$ $-2\ell nL$	$\lambda=1$ $-2\ell nL$	Number of Restrictions	.01 Chi-Square Critical Value
Canada Life	6.723	5.027	3.608	1	6.635
Confederation Life	0.420	1.221	2.387	1	6.635
Great West Life	8.119	7.501	6.075	1	6.635
Imperial Life	0.149	0.537	.309	1	6.635
London Life	24.467	15.056	6.951	1	6.635
Manufacturers Life	7.639	8.635	5.463	1	6.635
Mutual Life	19.469	14.724	9.779	1	6.635
Sun Life	0.280	0.164	0.052	1	6.635

* $-2\ell nL \sim \chi^2(1)$. In this case $\chi^2_{.95}(1) = 3.84$; $\chi^2_{.99}(1) = 6.635$; $\chi^2_{.995}(1) = 7.879$

TABLE VII.5: The Sign of U_E/U_V for the Companies in the Sample

Company	Sign
Canada Life	<0
Confederation Life	<0
Great West Life	<0
Imperial Life	>0
London Life	>0
Manufacturers Life	<0
Mutual Life	>0
Sun Life	<0

VII.4 The Validation of the Chosen Forms

From the theory of asset demand, we expected the following:

a. the sign of U_E/U_V to be negative.

b. the own elasticity with respect to expected return to be positive for all assets and negative for liabilities. (i.e., an increase in the expected return on an asset will increase the proportion of this asset held by the company, while an increase in the expected cost of a liability will reduce the proportion of this liability held by the company - the company will underwrite less policies of a particular insurance contract.)[19]

c. the own elasticity with respect to risk (as measured by the standard deviation) to be negative for all assets and liabilities.

d. the principal minors of the bordered Hessian (Chapter V, equation (12)) to alternate in sign. Second-order conditions for a maximum require that the principle minors of the determinant D, obtained by totally differentiating the equation system (11) in Chapter V with respect to the X_i's, alternate in sign.

Table VII.5, lists the sign of U_E/U_V for 1977 derived from the "best" utility function. We estimated the U_E/U_V ratio for each year of our sample. The results reported for 1977 are reasonably representative and the generalizations which follow apply to the

[19] Note however that these results are not obtained directly from the theory and our expectations with regard to their signs is based on previous studies. See for example, Barret, Gray and Parkin (1975), Kahane and Nye (1975) Bhattacharyya (1978) etc. For a theoretical discussion on the signs see, for example, Beirwag and Grove (1968) or Aivazian (1976).

entire period 1955-1977.[20,21] In five out of eight companies U_E and U_V were of opposite signs as predicted by the theory. Interestingly enough, two out of the three companies for which U_E and U_V have the same sign are London Life and Mutual Life. These companies are the ones for which all the functional forms were rejected at a 1 percent significance level. The parameters used for computing U_E and U_V for those two companies, were obtained from the forms with the lowest value of $-2\ell nL$. Imperial Life is the third company for which U_E/U_V is positive. In this case we computed U_E/U_V using the estimated parameters obtained from the other two forms as well. It turns out that U_E and U_V are of the same sign, no matter what form is used.

The signs of U_E and U_V play a major role in determining the elasticities of demand with respect to expected returns and variances. This point requires some elaboration. The mean, or expected return, of a portfolio is just a weighted average of the expected returns on the securities in the portfolio. The contribution of a security to the expected return on a portfolio is $X_i\tilde{\mu}_i$, the expected return on the security weighted by the proportion of portfolio funds invested in the security. Equation (15) in Chapter V shows the effect on X_k of a change in the productivity of asset r (i.e., how the weight of asset k will change as a result of a change in the expected return on asset r). Formally[22]

---

[20] In the case of Mutual Life where all the forms were rejected, we used the form with the lowest test statistic (i.e., SRQ).

[21] The same comment applies to all the results reported hereafter.

[22] Equations (1) and (2) were derived in Chapter V and are identical to equations (15) and (17) there.

$$(1) \qquad \frac{\partial X_K}{\partial \mu_r} = -W_o U_E \frac{D_{rk}}{D} - X_r \left[W_o^2 U_{EE} \sum_{i=1}^{m+1} \mu_i \frac{D_{ik}}{D} + V^{-1} W_o^2 U_{EV} \sum_i \sum_j X_j G_{ij} \frac{D_{ik}}{D}\right]$$

A similar expression can be derived to show the effect of a change in the variance of the expected return of asset r on the holdings of asset k. In particular:

$$(2) \qquad \frac{\partial X_k}{\partial G_r^2} = -2V^{-1} U_V W_o^2 X_r \frac{D_{rk}}{D} - V^{-1} X_r^2 W_o^3 \left[U_{EV} \sum_i \mu_i \frac{D_{ik}}{D} + W_o V^{-1} U_{VV} \sum_i \sum_j G_{ij} \frac{D_{ik}}{D}\right]$$

V. Aivazian (1976) identifies the last term in equations (1) and (2) as the average productivity effect and the first term on the right-hand side as the pure marginal effect.[23] In our case the marginal productivity effects are larger in magnitude than the average productivity effects and hence the signs of $\frac{\partial X_k}{\partial \mu_r}$ and $\frac{\partial X_k}{\partial G_r^2}$ are determined by the first terms on the right-hand side of equations (1) and (2). These marginal productivity terms are obviously of opposite sign if U_E and U_V are of opposite sign as indeed they were in five out of eight companies in the sample.

Table VII.6 contains the own elasticities of substitution with respect to expected return and variance. For the companies with opposite signs for U_E and U_V, all assets (liability) own elasticities of substitution with respect to expected return are positive (negative). As expected, the opposite holds for the own elasticities with respect

---

[23]See pp. 6-9 of Aivazian.

TABLE VII.6: Own Elasticities*

Own Interest Elasticity	Canada Life	Confederation Life	Great West Life	Imperial Life	London Life	Manufacturers Life	Mutual Life	Sun Life
CB	1.9770	6.1236	.96787	-2.2593	-.15893	2.6145	-.75981	.90802
FB	.03476	.05503	.03055	-.01348	-1.3981	.00998	-3.0260	.02341
CS	.03257	.07899	.012919	-.00946	-.01149	.05728	-.01355	.01398
FS	.01647	.06253	.00704	-.00681	-.31959	.00842	-.01344	.00799
MLRE	1.1946	3.2935	.34041	-.96275	-.07712	1.2341	-.47335	.51912
NAR	-.08472	-.000843	-.02115	-.000101	.016831	-.05743	-.09302	-.05102
Own Elasticity with respect to Variance								
CB	-18.527	-19.570	-20.664	-2.0000	-12.974	-27.795	-15.017	-19.222
FB	-2.5911	-2.3387	-2.8288	-2.1104	-2.8828	-2.2260	-2.8718	-4.4521
CS	-5.4661	-4.9190	-6.5958	-4.0835	-6.4886	-5.9134	-6.0208	-5.9305
FS	-5.1921	-4.0720	-6.0543	-3.5137	-5.9735	-5.4341	-5.6035	-8.4747
MLRE	-14.791	-15.741	-16.898	-15.382	-10.519	-23.374	-11.810	-15.959
NAR	-1.0429	-.40323	-.88672	-.39761	-1.0554	-.71845	-.02424	-1.7195

* These elasticities were calculated for 1977 and are based on the parameters obtained from the "best" utility function.

to the variance. Therefore, the utility functions chosen for those companies yields signs which are consistent with the theory. Furthermore, the necessary conditions of curvature are satisfied.[24] Interestingly enough, it can be seen that the last row in the upper part of Table VII.6 contains negative numbers for seven out of the eight companies in the sample. This row contains the own interest elasticities of the net actuarial reserves, thus suggesting that when expected costs of underwriting insurance goes up the insurance company reduces the proportion of net actuarial reserves it holds on its balance sheet. Those results are consistant with our intuition. The purpose of a life insurance company (like all other financial intermediaries) is to make a profit by purchasing various securities yielding more than the return they must pay on their liabilities and on operations. An increase in the expected cost of underwriting Ceteris Paribus, will reduce expected profits. Thus, for example, those policies which generate less reserves will be less profitable since they are expected to generate less investment profits. Therefore, the insurance company will consider those policies "undesired", reducing or limiting their sales.[25]

The own elasticities with respect to variance reveal a clear-

---

[24] We also checked the signs of the principle minors of the bordered Hessian but unfortunately these were ambiguous. This result does not contradict the theory. It just means that the sufficient conditions for a maximum were not obtained.

[25] It should be noted that the desirability of a particular policy (from the insurance company's point of view) is determined according to the "performance" of each type of policy with all other ones, and not only on the basis of the expected return on this activity in isolation.

cut pattern for all companies. An increase in the variance of the expected return (cost) of an asset (liability) will lead to a reduction in the proportion of it held in the balance sheet of the company. Those elasticities are smaller in magnitude for FB, FS, and CS in comparison with the elasticities of CB and MLRE. These results suggest that in the margin when a life insurance company invests in FB, FS and CS, it expects to face a relatively higher risk (as measured by the standard deviation) then it expects when it buys CB or MLRE which presumably play the role of the "safer" asset on the balance sheet. The higher risk on FS, FB and CS is a result of the nature of the market (stock markets) and exchange rate fluctuation (FB and FS). Thus an increase in the standard deviation of expected return on those assets will lead to a lesser effect (reducing holdings) than the same change in the standard deviation of expected return on CB or MLRE. Finally, the own variance elasticities of NAR are the smallest for all companies. After all, even if costs of underwriting insurance vary, the life insurance company must continue underwriting insurance as long as it wants to stay in business.

Before referring to the cross elasticities some elaboration on the own elasticities with respect to the variance is required. From inspecting equation (8) in Chapter V, it is clear that the contribution of a security to the variance of a portfolio's return is a complicated matter. One important point, emphasized by writing V as in equation (8), is that when the number of securities in the portfolio is large, individual security return variances are much less numerous in V than are covariances. In particular, if we have n securities in our portfolio, V will contain n terms for the security return variances, where there are n(n-1) covariances. In computing the above elasticities, only the elasticities with respect to the variances were computed. Those elasticities reflect the change in holdings of an asset as a result of a change in the variance of the asset's expected return, Ceteris Paribus. From the information in Table IV.1 in Chapter IV, it is evident that any change in one asset's expected return on variance will have an effect on its correlation with other assets and thus affect the optimal portfolio allocation. This kind of an argument can explain the high magnitude of the own elasticities with respect to the variance obtained.

Theoretically, the off diagonal elasticities with respect to expected returns can be of any sign. Intuition provides no help in predicting those signs because of the many variables influencing them. Table VII.7 lists the signs of the cross elasticities.[26] For the five companies with the opposite sign[27] for U_E and U_V we obtain that an increase in the expected return on CB will lead to a reduction in the proportion of MLRE held. Furthermore, those elasticities which represent a substitutability relationship are fairly high. Over all, with the exception of NAR, the elasticities of substitution between CB and FB,

---

[26] Appendix VII.C at the end of this chapter contains the estimated elasticities of substitution with respect to variances and expected returns.

[27] As mentioned earlier the companies are: Canada Life, Confederation Life, Great West Life, Manufacturer's Life and Sun Life. The discussion here often relates only to those companies.

TABLE VII.7: Signs of the Cross Elasticities with Respect to Expected Return for the End of 1977 (by Company)

1. Canada Life

	CB	FB	CS	FS	MLRE	NAR
CB	+	+	+	+	-	-
FB	+	+	+	+	-	-
CS	+	+	+	-	-	-
FS	+	+	-	+	-	+
MLRE	-	-	+	+	+	+
NAR	+	-	-	-	-	-

2. Confederation Life

	CB	FB	CS	FS	MLRE	NAR
CB	+	+	+	-	-	-
FB	+	+	+	-	-	+
CS	+	+	+	-	-	-
FS	+	-	-	+	-	+
MLRE	-	-	-	-	+	+
NAR	+	+	+	+	-	-

3. Great West Life

	CB	FB	CS	FS	MLRE	NAR
CB	+	+	+	+	-	-
FB	+	+	+	-	-	-
CS	+	+	+	-	-	-
FS	+	-	-	+	-	+
MLRE	-	+	+	+	+	+
NAR	+	-	-	-	-	-

4. Imperial Life

	CB	FB	CS	FS	MLRE	NAR
CB	-	-	-	-	+	+
FB	-	-	+	+	+	-
CS	-	+	-	+	+	+
FS	-	+	+	-	+	-
MLRE	+	+	+	+	-	-
NAR	-	-	-	-	-	-

5. London Life

	CB	FB	CS	FS	MLRE	NAR
CB	-	-	-	-	+	+
FB	-	-	-	+	+	+
CS	-	-	-	+	+	+
FS	-	+	+	-	+	-
MLRE	+	-	-	-	-	-
NAR	+	+	+	+	+	+

TABLE VII.7: continued

6. Manufacturer's Life

	CB	FB	CS	FS	MLRE	NAR
CB	+	+	+	+	-	-
FB	+	+	+	-	-	+
CS	+	+	+	-	-	-
FS	+	-	-	+	-	+
MLRE	-	-	-	-	+	+
NAR	+	-	-	-	-	-

7. Mutual Life

	CB	FB	CS	FS	MLRE	NAR
CB	-	+	+	+	+	+
FB	-	-	-	+	+	+
CS	-	-	-	+	+	+
FS	-	+	+	-	+	-
MLRE	+	+	+	+	-	+
NAR	-	-	-	-	-	-

8. Sun Life

	CB	FB	CS	FS	MLRE	NAR
CB	+	+	+	+	-	-
FB	+	+	+	+	-	-
CS	+	+	+	-	-	-
FS	+	+	+	+	-	-
MLRE	-	-	+	+	+	+
NAR	+	-	-	-	-	-

Definitions:

CB - Canadian Bonds
FB - Foreign Bonds
CS - Canadian Stock
FS - Foreign Stock
MLRE - Mortgage Loans and Real Estate
NAR - Net Actuarial Reserves

FS and CS are positive which means that foreign bonds, foreign stocks and Canadian stocks are complementary to Canadian bonds.[28]

As to be expected the elasticity of substitution between CS and FS are fairly low but represent substitutability relationship. Other than this there is evidence of strong substitutability between MLRE and all the other assets in the balance sheet.

Three out of the eight companies in the sample were stock companies at the end of 1977. The three are: Great West Life, Imperial Life and London Life. The other five companies were mutuals.[29] The small sample used in this study (in terms of the number of companies) makes the comparison between mutuals and stock companies difficult. Moreover, only for Great West Life the ratio U_E/U_V turned out to be negative.[30] Table VII.8 contains the own elasticities of assets with respect to expected returns and variances for Great West Life and Canada Life. Canada Life was chosen for the comparison since its size (in terms of total assets) is similar to Great West Life.[31]

---

[28] There is however one exception. Foreign stocks appear to be substitutes for Canadian bonds in the case of Confederation Life.

[29] Four out of the five mutual life insurance companies in the sample were mutualized after section 91 of the Canadian and British Insurance Companies Act was passed in 1957. The companies and the dates of their mutualization are: Canada Life and Sun Life,(1962) Confederation and Manufacturers (1968). The reasons for mutualization can be found in Pedoe and Jack (1978) pp. 97-101. Mutual Life was a mutual company since 1870.

[30] That two out of the three stock companies fail to support the theoretical prediction is somewhat disturbing.

[31] Total assets owned by Canada Life and Great West Life at the end of 1977 was 2.2 and 2.8 billion dollars respectively.

TABLE VII.8: Great West Life (S) and Canada Life (M) - A Comparison of Own Elasticities*

ASSET	Own Elasticities With Respect to					
	Variance			Expected Return		
	Canada Life (M)	Great West (S)	$(3)=\frac{(1)-(2)}{(1)}x100$	Canada Life (M)	Great West (S)	$(6)=\frac{(4)-(5)}{(4)}x100$
	(1)	(2)	(3)	(4)	(5)	(6)
CB	-18.527	-20.664	-11.5%	1.97700	0.96787	51.0%
FB	- 2.5911	- 2.8288	- 9.2%	0.03476	0.03553	-2.2%
CS	- 5.4661	- 6.5958	-20.7%	0.03257	0.01292	60.3%
FS	- 5.1921	- 6.0543	-16.6%	0.01647	0.00704	57.3%
MLRE	-14.791	-16.898	-14.2%	1.19460	0.34041	71.5%
NAR	- 1.0429	- .88672	15.0%	-.08472	-.02115	75.0%

* (M) and (S) are included to remind the reader which company is a Mutual and which is a Stock company.

The own elasticities with respect to variances are larger in magnitude for the stock company than for the mutual life insurance company for all the assets under consideration, and smaller for net actuarial reserves, (i.e., Canada Life will reduce the proportion of an asset held by less than Great West Life for the same change in the variance of the assets expected return). However, when elasticities with respect to expected return are considered, Canada Life's elasticities are larger in magnitude than those of Great West Life. Those results may indicate that Great West Life is following a more conservative approach in conducting its business. An examination of the elasticities reported in Table VII.6 shows that this is also true of most of the other mutual companies.

It is hard to draw firm conclusions from the above exercise, to what extent mutual and stock life insurance companies are similar or not similar in their business practice. Nevertheless, the common belief that, in general, mutual companies follow a riskier path in the way they conduct their business is supported by the results in this study.[32]

---

[32] See for example Klemkosky (1973), Mcdonald (1974) and Kon and Jen (1978).

APPENDIX VII.A

The Kolmogorov-Smirnov Test Results*

	COMPUTED D'S BY COMPANY (N=23)						
ASSET	Canada Life	Confederation Life	Great-West Life	London Life	Manufacturer's Life	Mutual Life	Sun Life
CB	.112454	.112454	.112454	.112454	.112454	.112454	.112454
FB	.290196	.299262	.247237	.247194	.274116	.254381	.269921
CS	.129206	.129206	.129206	.129206	.129206	.129206	.129206
FS	.167527	.143707	.197242	.197242	.197598	.197242	.188448
MLRE	.129681	.123283	.129552	.158721	.121799	.154767	.133274
AR	.210525	.210525	.210525	.210525	.210525	.210525	.210525

*Reject the hypothetical distribution F(X) if $D_n = \max|F_n(X) - F(X)|$ exceeds the tabulated values. The critical values (n=23) are: .330 for 1 percent significance level, .307 for 2 percent, .275 for 5 percent .247 for 10 percent and .216 for 20 percent significance level.

APPENDIX VII.B
Estimated Elasticites of Substitution (End of 1977) With Respect to Expected Return and Variance

1. Canada Life

A. Expected Return

(1)	(2)	(3)	(4)	(5)	(6)
.19770E+01	.62877E-01	.47854E-01	.42654E-01	-.17084E+01	-.13271E+00
.14052E+00	.34760E-01	.87500E-02	.76744E-02	-.13450E+00	-.10019E-01
.21394E+00	.10316E-01	.32570E-01	-.60759E-02	-.11714E+00	-.61246E-01
.13287E+00	.59309E-02	-.73073E-02	.16468E-01	-.12701E+00	.16683E-02
-.11335E+01	-.75179E-03	.11763E-01	.12365E-01	.11946E+01	.40341E-01
.18963E-01	-.63382E-01	-.51055E-01	-.62754E-01	-.10251E+00	-.84716E-01

B. Variance

(1)	(2)	(3)	(4)	(5)	(6)
-.18527E+02	-.33252E+01	-.46910E+01	-.67460E+01	.21846E+02	-.31450E+01
-.13113E+01	-.25911E+01	-.66032E+00	-.72590E+00	.16511E+01	-.45617E+00
-.20425E+01	-.72234E+00	-.54661E+01	.40180E+01	.14288E+01	-.14218E+01
-.12769E+01	-.25186E+00	.17383E+01	-.51521E+01	.15457E+01	-.18616E+00
.11004E+02	.21705E+01	.16575E+01	.41418E+01	-.14791E+02	.33793E+00
-.98864E+00	-.36819E+00	-.10362E+01	-.31568E+00	.15191E+00	-.10429E+01

2. Confederation Life

A. Expected Return

(1)	(2)	(3)	(4)	(5)	(6)
.61236E+01	.38414E-01	.60764E-01	-.21270E-01	-.56248E+01	-.45344E+00
.16388E+00	.55028E-01	.28137E-02	-.15865E-01	-.35293E+00	.50488E-01
.80149E+00	.27712E-01	.78992E-01	-.36988E-01	-.71061E+00	-.11119E+00
.36035E+00	-.70763E-01	-.63942E-01	.62528E-01	-.42565E+00	.85803E-01
-.34806E+01	-.10410E+00	-.54974E-01	-.20753E-01	.32935E+01	.19194E+00
.33437E+00	.39170E-01	.53387E-01	.55443E-01	-.13290E+00	.84349E-03

B. Variance

(1)	(2)	(3)	(4)	(5)	(6)
-.19570E+02	-.29807E+01	-.56940E+01	-.14906E+01	.26318E+02	-.31312E+01
-.59109E+00	-.23387E+01	-.57104E+00	.52977E+00	.15187E+01	.19784E+00
-.25939E+01	-.12925E+01	-.49150E+01	.20629E+01	.32379E+01	-.91465E+00
-.11951E+01	.21237E+01	.36165E+01	-.40720E+01	.18878E+01	.41286E+00
.10951E+02	.32075E+01	.29592E+01	.99013E+00	-.15741E+02	.11905E+01
-.90759E+00	.27825E+00	-.59225E+00	.14507E+00	.80632E+00	-.40323E+00

3. Great West Life

A. Expected Return

(1)	(2)	(3)	(4)	(5)	(6)
.96787E+00	.64839E-02	.13576E-01	.99932E-02	-.88798E+00	-.49887E-01
.89781E-02	.30553E-01	.44031E-02	-.95029E-03	-.32464E-01	-.11743E-02
.68875E-01	.77665E-02	.12919E-01	-.52334E-02	-.40510E-01	-.21630E-01
.51188E-01	-.47718E-02	-.57576E-02	.70398E-02	-.57478E-01	.80412E-02
-.31118E+00	.71733E-02	.64011E-02	.55127E-02	.34041E+00	.10685E-01
.31282E-02	-.18931E-01	-.12981E-01	-.14628E-01	-.25930E-01	-.21154E-01

B. Variance

(1)	(2)	(3)	(4)	(5)	(6)
-.20664E+02	-.29250E+00	-.57137E+01	-.67535E+01	.45251E+02	-.35804E+01
-.18264E+00	-.28288E+01	-.17239E+01	.19314E+01	.15429E+01	-.35424E+00
-.14839E+01	-.71225E+00	-.65958E+01	.52837E+01	.19186E+01	-.16244E+01
-.11043E+01	.50730E+00	.33208E+01	-.60543E+01	.27805E+01	.28004E+00
.68990E+01	.38234E+00	.11302E+01	.26047E+01	-.16898E+02	.92310E-01
-.53837E+00	-.84350E-01	-.94814E+00	.25255E+00	.65906E-01	-.88672E+00

4. Imperial Life

A. Expected Return

(1)	(2)	(3)	(4)	(5)	(6)
-.22593E+01	-.21499E-01	-.24235E-01	-.66774E-02	.21981E+01	.83314E-01
-.87232E-01	-.13481E-01	.47228E-03	.29380E-02	.13702E+00	-.15331E-01
-.11595E+00	.59535E-03	-.94551E-02	.67131E-02	.11675E+00	.65730E-02
-.91413E-01	.39959E-02	.93504E-02	-.68066E-02	.12483E+00	-.21333E-01
.96169E+00	.24832E-01	.18244E-01	.14944E-01	-.96275E+00	-.97195E-02
-.59060E-01	-.13156E-01	-.14707E-01	-.15702E-01	-.39649E-02	-.10086E-03

B. Variance

(1)	(2)	(3)	(4)	(5)	(6)
-.20000E+02	-.39807E+01	-.11230E+02	-.65709E+01	.34385E+02	-.20773E+01
-.80706E+00	-.21104E+01	-.47362E+00	.54228E+00	.20291E+01	.24418E+00
-.10605E+01	-.21631E+00	-.40835E+01	.21735E+01	.17106E+01	-.27925E+00
-.83778E+00	.33725E+00	.29165E+01	-.35137E+01	.18461E+01	.37588E+00
.83797E+01	.24437E+01	.44226E+01	.35473E+01	-.15382E+02	.20677E+00
-.37389E+00	.21093E+00	-.55652E+00	.52778E+00	-.13229E+00	-.39761E+00

5. London Life

A. Expected Return

(1)	(2)	(3)	(4)	(5)	(6)
-.15893E+00	-.65679E-02	-.58651E-02	-.49278E-02	.13388E+00	.56909E-02
-.85951E+00	-.13981E+01	-.15073E+00	.92376E-01	.21429E+01	.10454E+00
-.50800E-01	-.69560E-02	-.11455E-01	.53289E-02	.25405E-01	.20392E-01
-.19109E+01	.24584E+00	.32119E+00	-.31959E+00	.22974E+01	-.37783E+00
.35214E-01	-.16737E-01	-.13419E-01	-.12818E-01	-.77121E-01	-.88669E-02
.20539E-01	.23576E-01	.17379E-01	.17534E-01	.22469E-01	.16831E-01

B. Variance

(1)	(2)	(3)	(4)	(5)	(6)
-.12974E+02	-.34108E-02	-.12470E+01	-.25829E-01	.23983E+02	-.20243E+01
-.72959E+02	-.28828E+01	-.85934E+02	.17289E+01	.37142E+03	-.24851E+02
-.43118E+01	-.14011E-01	-.64886E+01	.10185E+00	.41623E+01	-.44820E+01
-.16077E+03	.50714E+00	.18274E+03	-.59735E+01	.40094E+03	.76547E+02
.45421E+01	.33464E-02	.22769E+00	.12232E-01	-.10519E+02	-.52167E+00
-.46006E+00	-.23473E-03	-.29643E+00	.28411E-02	-.66622E+00	-.10554E+01

6. Manufacturers' Life

A. Expected Return

(1)	(2)	(3)	(4)	(5)	(6)
.26145E+01	.10760E+00	.90057E-01	.94481E-01	-.21369E+01	-.16074E+00
.17544E-01	.99767E-02	.11000E-02	-.15477E-02	-.42102E-01	.66344E-02
.31377E+00	.23061E-01	.57279E-01	-.21182E-01	-.17183E+00	-.99201E-01
.11222E+00	-.89580E-02	-.90200E-02	.84211E-02	-.12339E+00	.90105E-02
-.13734E+01	-.56040E-01	-.30574E-01	-.38396E-01	.12341E+01	.65862E-01
.38533E-01	-.67539E-01	-.47141E-01	-.53770E-01	-.14522E+00	-.57427E-01

B. Variance

(1)	(2)	(3)	(4)	(5)	(6)
-.27795E+02	-.38574E+01	-.21674E+01	-.12603E+02	.41096E+02	-.63786E+01
-.21552E+00	-.22250E+01	-.13656E+00	.71113E+00	.71365E+00	.83351E-01
-.34247E+01	-.37262E+01	-.59134E+01	.15039E+02	.32041E+01	-.31016E+01
-.12645E+01	.12313E+01	.95981E+00	-.54341E+01	.22020E+01	.15622E+00
.14776E+02	.45160E+01	.73060E+00	.78944E+01	-.23374E+02	.23614E+01
-.12360E+01	.24778E+00	-.38252E+00	.28574E+00	.12587E+01	-.71845E+00

7. Mutual Life

A. Expected Return

(1)	(2)	(3)	(4)	(5)	(6)
-.75981E+00	.55884E-01	.40216E-01	.40768E-01	.86019E+00	.59150E-01
-.19729E+01	-.30260E+01	-.30342E+00	.20321E+00	.49846E+01	.12739E+00
-.70648E-01	-.37315E-02	-.13547E-01	.18286E-01	.87287E-01	.16903E-01
-.10369E+00	.19659E-01	.25261E-01	-.13443E-01	.19396E+00	-.51261E-01
.66353E+00	.69538E-01	.62137E-01	.62689E-01	-.47335E+00	.11785E-01
-.19256E+00	-.15388E+00	-.11692E+00	-.11032E+00	-.12835E+00	-.93020E-01

B. Variance

(1)	(2)	(3)	(4)	(5)	(6)
-.15017E+02	-.77231E-02	-.25780E+01	-.21178E+01	.16560E+02	-.10107E+01
-.36104E+02	-.29718E+01	-.89688E+02	.63653E+02	.10433E+03	-.52137E+01
-.15513E+01	-.11660E-01	-.60208E+01	.36976E+01	.14689E+01	-.63331E+00
-.20875E+01	.13713E-01	.60185E+01	-.56035E+01	.37858E+01	.14117E+01
.10403E+02	.14279E-01	.15352E+01	.24226E+01	-.11810E+02	.68284E+00
-.46324E+00	-.41864E-03	-.49387E+00	.65542E+00	.47056E+00	-.24244E-01

8. Sun Life

A. Expected Return

(1)	(2)	(3)	(4)	(5)	(6)
.90902E+00	.35061E-01	.20860E-01	.21272E-01	-.75206E+00	-.88332E-01
.41777E-01	.23414E-01	.77520E-02	.80495E-02	-.14201E-01	-.14482E-01
.90177E-01	.13326E-01	.13980E-01	-.16389E-02	-.39082E-01	-.37843E-01
.47723E-01	.73564E-02	.32914E-03	.79916E-02	-.22299E-01	-.13609E-01
-.50002E+00	-.13455E-02	.33356E-02	.41201E-02	.51912E+00	.15016E-01
.21624E-01	-.23505E-01	-.21592E-01	-.26662E-01	-.43508E-01	-.51019E-01

B. Variance

(1)	(2)	(3)	(4)	(5)	(6)
-.19222E+02	-.63787E+01	-.55950E+01	-.16198E+02	.23844E+02	-.47830E+01
-.75332E+00	-.44521E+01	-.11157E+01	-.27770E+01	.63258E+00	-.10035E+01
-.19168E+01	-.32111E+01	-.59305E+01	.41286E+01	.12295E+01	-.19938E+01
-.98449E+00	-.14273E+01	.72725E+00	-.84747E+01	.73544E+00	-.83669E+00
.10914E+02	.24974E+01	.16510E+01	.56258E+01	-.15559E+02	.42294E+00
-.12375E+01	-.22053E+01	-.15151E+01	-.35824E+01	.21970E+00	-.17195E+01

Definitions:

(1) = Canadian Bonds
(2) = Foreign Bonds
(3) = Canadian Stocks
(4) = Foreign Stocks
(5) = Mortgage Loans and Real Estate
(6) = Net Actuarial Reserves

CHAPTER VIII

SUMMARY AND CONCLUSIONS

> "The Theory of Economics does not furnish a body of settled conclusions immediately applicable to a policy. It is a method rather than a doctrine, an apparatus of the mind, a technique of thinking, which helps its possessor to draw correct conclusions." [1]
>
> J.M. Keynes

VIII.1 Summary and Conclusions

In this chapter we wish to offer an overview on our study as well as to indicate areas for further research.

In recent years, considerable effort has been directed toward establishing the nature of the investment behaviour of life insurance companies. In this dissertation an extended portfolio model was developed for the simultaneous determination of the efficient composition of insurance and investment activities of a life insurance company. The model takes advantage of the existing finance foundation and the concepts and techniques of modern demand system analysis.

Unlike the current models which use quadratic programming techniques and are interested in the construction of efficient asset choice sets, we use a utility maximization approach that determines the optimal choice on the efficient set. A two parameter portfolio model was constructed utilizing elements of utility theory and of the theory of insurance. The model provided us with the proportion of assets

---

[1] J.M. Keynes, Editor's Introduction, Cambridge Economic Handbooks, Cambridge University Press, 1922.

held in the balance sheet as well as which liabilities are used to raise the necessary capital. Legal quantitative and qualitative restrictions on portfolio composition, the accounting procedures imposed by the Superintendent of Insurance, Federal and provincial tax laws, risk, expected costs and expected returns are all elements that could be dealt simultaneously within this model.

The model developed had sufficient empirical content to yield hypotheses about life insurance portfolio behaviour and thus could be tested using the appropriate econometric techniques. A comparative static analysis which yielded elasticities of substitution between financial assets and liabilities was provided. The estimation of those elasticities in the context of a flexible functional form model, forms the central part of this dissertation. More specifically, by utilizing a mean-variance portfolio framework and a general Box-Cox utility function we were able to model the demand for assets and liabilities by an insurance company. In particular, we have compared three "flexible" functional forms - the translog, the generalized Leontief and the square root quadratic. On empirical grounds we found that, in general, the square root quadratic best fits the data. For 7 out of the 8 companies in our sample we could not reject the square root quadratic utility function at 0.5% significance level. Furthermore, when the three functional forms were ranked based on a likelihood test (i.e., on $-2 \ell nL$) for six out of eight companies the square root quadratic came first while the translog came out first twice.

We tried also to validate the square root quadratic approximation by showing that broadly speaking, it yields signs for the elasticities of substitution with respect to expected return and variances, which are consistent both with the theory and with previous empirical findings. For the companies with opposite signs for U_E and U_V, all assets (liabilities) own elasticities of substitution with respect to expected return were found to be positive (negative). As expected the opposite held for the own elasticities with respect to the variance. For the same companies we found substitutability relationship between Canadian bonds and Mortgage loans. On the other hand, foreign bonds, foreign stocks and Canadian stocks were found to be complementary to Canadian bonds.

A by-product of the model developed is the ability to compare stock and mutual life insurance companies. It was hard to draw firm conclusions from the comparison because of the small sample used in the study. Nevertheless, the common belief that in general mutual companies follow a riskier path in the way they conduct their business was supported by the results in this study.

In their role as one of the largest financial intermediaries, life insurance companies have substantial obligations to millions of households and are major suppliers of funds to several financial markets. Consequently, the life insurance industry directly affects these sectors of the economy with which it deals and is of interest to those who regulate and tax the industry. Thus we believe that the results obtained in this study will benefit future financial research and will provide considerable insight into life insurance investment behaviour. The present study is, therefore, a first step in attempting to analyze the behaviour of life insurance companies by using some of the modern tools commonly used by economists.

VIII.2 Suggestions for Further Work

There are four groups of suggestions for further work that follow from this thesis. The first is concerned with the specification of the liability side of the insurance company's portfolio. The liabilities in our study were divided into two types. Ideally, one should recognize at least five classes of liabilities based on the potential contract types that the firm writes each year: a) ordinary life insurance, b) group life insurance written for groups of consumers, c) ordinary annuities sold to individual consumers, d) group annuities and e) surplus. Unfortunately, from the data obtained from the Superintendent of Insurance for Canada, it was impossible to derive the costs associated with underwriting each of these types of contracts and thus we were forced to consider only two broad types of liabilities insurance on the one hand and surplus on the other. Any extension in this direction should reveal more information about the degree of substitutability between financial assets and different types of liabilities.

The second group of suggestions for further work lies in the specification and calculation of expected returns. Many different formulations have been used in the literature; rational expectations, distributed lag models, perfect foresight, or the innovative approach of estimating subjective covariances of assets' yields along Baysian lines, to mention a few. It should be determined whether these alternative specifications affect the results reached based on the simple adaptive expectations used here.

The third group of suggestions relates directly to the second and is concerned with inflation. In our formulation, nominal interest rates have been used. Thus we have implicitly assumed that all financial assets and liabilities are equally affected by inflation. Since inflation is an important phenomena in modern society a more explicit treatment of it seems desirable. One alternative specification that might be employed is to use expected real interest rates, calculated based on expected inflation.

Finally, the effect of taxation, which we considered marginal for the period under consideration could be easily introduced into the model and this also seems desirable.

APPENDIX A

GOVERNMENT REGULATION AND INVESTMENT

The significant feature of supervision of life insurance in Canada is the dual control exercised by the federal and provincial governments. In general, it can be said that the provinces have exclusive jurisdiction in the area of the insurance contract, as well as in the area of licensing insurance agents, brokers and adjusters.[1] The federal government concerns itself mainly with the financial soundness of the insurance companies.

The regulations affecting the investment policies of life insurance companies are outlined in the Canadian and British Insurance Companies Act and the Foreign Insurance Companies Act. These regulations apply to all Canadian and foreign insurance companies doing business in Canada and are designed to ensure that companies continue to be solvent.

Why government supervision? The following example reveals the answer.

Define a life insurance contract using the following two variables:[2]

p - the premium collected by the company under the contract.

c - a stochastic variable which the company pays to settle insurance claims made under the contract.

---

[1]It is also the provinces' role to parallel the supervision of the federal government for those companies operating under provincial charter exclusively.

[2]For simplicity we abstract from time consideration.

Assume the company has acquired a portfolio of insurance constracts such that:

$P = \sum_i p_i$ - total amount of premiums collected.

$C = \sum_i c_i$ - total amount paid to settle claims, where $F(C)$ is the distribution of C.

If the expost claim payments exceed the premium collected i.e., $C > P$, the company will suffer a loss. The probability of such an event is:

$$Pr\ (C > P) = 1 - F(P)$$

If the company has no reserves, it will be unable to meet its obligations if the underwriting loss would occur. When the above probability is significantly different from zero, the life insurance contract sold will not give adequate protection. Life insurance and annuity policies may have a period of existance of forty years or more. To ensure the payment of the amounts due during and at the end of such long periods it is usual to require that the company will set aside certain amounts, reserve funds or equity capital, that can be drawn upon to cover underwriting losses. If those reserves amount to R, the probability that the company will be unable to meet its obligations and declared insolvent is:

$$Pr\ (C > R + P) = 1 - F(R + P)$$

In order to prevent insolvency R has to be large enough so that the public will have confidence in the company and thus will buy the contracts it sells. No individual policyholder can be expected to be in a position to ascertain that the proper reserves are accumulated. Therefore, in most countries the government stepped in to protect the insurance-buying public.[3]

Since it is practically impossible to set reserves requirements so high that the probability of insolvency will be zero, the objective of the supervision can be achieved if the following inequality is satisfied by the company:

(1) $$Pr\ (C > R + P) = 1 - F(R + P) \leq \alpha$$

where $1 - \alpha$ can be considered as the "minimum quality" contract offered by the company.

In his 1962 submission to the Royal Commission on Banking and Finance, the Superintendent of Insurance wrote:

> "the main purposes of the legislation and of the Departmental examination of companies in implementation of that legislation, have been to ensure that each and every company licensed or registered with the Department is in a sound condition ... In the main, these purposes have been attained by requiring (1) the maintenance in Canada by all out-of-Canada companies of adequate assets and of records and accounts of their transactions; (2) the placing of sound values on the assets of all companies; (3) the proper determination of the liabilities of all companies; (4) the regular examination of the records and accounts of companies to see that these requirements are met by companies on a continuing basis; and (5) the publication of a detailed annual report on all companies, giving full information for the insuring public and affording a basis for informed criticism within the industry itself." (pp. 41-42)

In order to satisfy equation (1), the superintendent tries to influence α (by setting up qualities of asset that will be permitted) and R (by providing standards for the valuation of assets and amounts of reserve liabilities).

---

[3]Often, government supervision has been established at the request of the insurance companies since they found it difficult to do business without some official approval.

The investments standards are set in sections 63 to 68 of the Canadian and British Insurance Act, and schedule I of the Foreign Insurance Companies Act. We outline briefly below the authorized investments according to the Federal Insurance Act.[4]

A. <u>Bonds:</u>

Government, provincial and municipal bonds issued or guaranteed by the Government of: Canada, the United Kingdom and specified countries of, or formerly of the British Commonwealth, or the United States of America, or of a country where the company is carrying on business may be purchased with no restrictions as to the quality or quantity.

Corporate bonds may be purchased without limits provided they are fully secured by mortgages or real estate, plant or by equipment used in the transaction of its business, or securities which would qualify for a life insurance investment.

Corporation debentures may be purchased without limit, subject to certain tests based on the dividend record of the corporation or its guarantor.

B. <u>Mortgages:</u>

There is no limit on the proportion of assets which may be invested in mortgage loans on real estate, provided that a particular mortgage in which the investment is made will not exceed three quarters

[4] For a rigorous discussion on government supervision and the early history of the legislation see E.P. Neufeld (1972) pp. 232-242, D.J. Baum (1973), pp. 95-108, J.W. Burns (1973), pp. 1075-1077, A. Pedoe and C.E. Jack (1978), pp. 371-397, or in the appropriate act. The above review refers to the legislation in force up to the end of 1977, since the sample used in the empirical part is for the period 1955 - 1977.

of the value of the real estate. The loan to value ratio was restricted to 60 per cent until 1961 and to 66 2/3 per cent until 1964. Mortgage loans in excess of the three quarters limit may be purchased in certain cases where the excess is guaranteed by the government or by a policy of mortgage insurance. This permits investments under the National Housing Act with ratio up to 95% of the value of the real estate.

C. Real Estate for the Production of Income:

Up to 1948, Canadian life insurance companies were only allowed to own real estate for their own use and occupancy. Since then, the law permits it to own incoming producing real estate under the following restrictions: "(a) a lease of the real estate is made to or guaranteed by a corporation whose preferred or common shares would be an eligible investment as outlined below, and (b) 85 per cent of the amount invested is to be repaid within 30 years or the period of the lease if less. The amount of the investment in any one parcel of real estate is limited to 20 per cent of the total assets of the company."[5]

D. Stocks:

Preferred shares of a corporation are permitted without limit subject to certain tests based on the dividend payed on these shares in the last five years.

Common shares of corporation that paid in each of the five years including the last year of the period dividend upon its common shares of at least 4 per cent, or earned during that period an equal

---

[5]A. Pedoe and C.E. Jack (1978) p. 403.

amount for the payment of dividend may be purchased. However, not more than 30 per cent of the outstanding common shares of one corporation may be held. Common shares of other life insurance corporations transacting business in Canada cannot be acquired. Finally, the amount of holding of common shares permitted (book value) is limited to 25 per cent of the company's total assets (book value). Up to 1932 there were no regulations relating to the quantity of common shares holdings. The depression and the stock market crash of 1925 influenced the decision to limit the common shares holdings. The act at first stipulated that investments in common shares be limited to 25 per cent of total assets, but at the request of the life insurance companies this was changed to 15 per cent. In 1965, after the recommendation of the Royal Commission on Banking and Finance[6] the limit was increased back to 25 per cent.

E. "Basket Clause":

In 1948 the "basket clause" was first introduced, which permitted Canadian life insurance companies to acquire investments not permitted otherwise. The total of such investments was limited first to 3 per cent then to 5 per cent and since 1965 to 7 per cent of the company's total assets. The purpose of the 1948 amendment was to enable the companies to invest within a narrow limit income-producing real estate and in bonds and stocks which were considered desirable by the life insurance industry and yet were not previously permitted by law. Under this clause, the upper limit on any one parcel

---

[6]Canada, Laws, Statutes, etc. An Act to amend certain Acts administered in the Department of Insurance, 1965, 13 ELIZ. II (Ch. 40, p. 351).

of real estate is 1 per cent of the total assets of the company.

The legal solvency of life insurance companies is determined by the excess of valued assets over liaiblities. The difference between the two determines the free surplus available, which indicates the financial strength of the company. Since the determination of the actuarial reserves is based on an inflexible formula related to the cash surrender values guaranteed in the policy, any fluctuation in the value of assets owned will affect the surplus and indirectly the legal solvency of life insurance companies.

Two methods are used in Canada in evaluating debt type assets: amortized value and market cost.

A. Amortized Value:

In 1950, Canadian life insurance companies were permitted to use, for redeemable securities, the amortized value instead of the market value. Section 73 (1) in the Canadian and British Insurance Companies Act define amortized value to be:

> " ... a value so determined that if the security were purchased at that date and at that value, the yield would be the same as the yields with reference to the original purchase price."

The 1950 amendment insulates the valuation of mortgages and government bonds holdings of life insurance companies from the volatile influence of interest rates.

B. Market Value:

Up to 1965 assets not carried on an amortized value basis including stocks were evaluated by "market value" as published by the Superintendent of Insurance. "When the market values are unduly depressed, the Minister of Finance is permitted to authorize values in excess of the market value but not greater than those used in the last proceeding financial statement or the book value for securities acquired in the inter m."[7] After 1965 this section of the act was amended to allow a three year average of market value to be used for non-amortized securities. "With this change, together with the use of amortized values as outlined above, and the use in a serious emergency of "authorized" values, it can no longer be said that asset valuation imposes a significant handicap on life insurance companies operating in Canada."[8]

---

[7] A. Pedoe and C.E. Jack (1978) p. 408.

[8] E.P. Neufeld (1972) p. 268.

APPENDIX B

ASSET HOLDINGS BY COMPANY, AND THEIR ASSOCIATE EXPECTED RETURNS
(End of the year, 1955-1977)

1. Canada Life

A. Holdings (000$)

(1)	(2)	(3)	(4)	(5)	(6)	(7)
122265.	115688.	6626.	25539.	211854.	-449399.	32573.
124926.	115496.	7664.	27153.	240391.	-480130.	35500.
123333.	121155.	8653.	30192.	268122.	-513252.	38203.
137308.	129139.	10034.	27074.	290665.	-539088.	55132.
127819.	124475.	10524.	31346.	313025.	-561019.	46170.
139075.	127370.	12900.	36559.	328111.	-596834.	47181.
149438.	134227.	16409.	39049.	350933.	-641315.	48741.
163794.	138514.	18727.	42472.	376284.	-689631.	50160.
178817.	146958.	21393.	44752.	398031.	-734615.	55336.
206933.	157042.	25949.	47992.	416592.	-793390.	61118.
216647.	177092.	30594.	50907.	449829.	-857728.	67341.
213768.	186125.	34357.	44542.	477132.	-884533.	71391.
223705.	182745.	36459.	51917.	489682.	-907587.	76921.
221482.	188718.	40756.	54852.	496511.	-919885.	82434.
208657.	185197.	46785.	55529.	525995.	-938933.	83230.
222900.	177633.	46889.	52720.	535582.	-950724.	85000.
238053.	186186.	57256.	61596.	545338.	-1001122.	87307.
237823.	188116.	64695.	64742.	557367.	-1022672.	90071.
256558.	194309.	68593.	69213.	611441.	-1107110.	93004.
269519.	179361.	68215.	77634.	647475.	-1143285.	98919.
319745.	187235.	72973.	75998.	670283.	-1226009.	100225.
397866.	181608.	78423.	78080.	683585.	-1312289.	107273.
476933.	156545.	80822.	80510.	720372.	-1397849.	117333.

B. Expected Returns (percentage point)

(1)	(2)	(3)	(4)	(5)	(6)
.0335	.0108	.1378	.1752	.0513	.0260
.0345	.0118	.1349	.1761	.0518	.0254
.0361	.0133	.1146	.1598	.0524	.0256
.0375	.0190	.1170	.1558	.0531	.0259
.0396	.0197	.1233	.1754	.0539	.0270
.0417	.0247	.1180	.1777	.0548	.0304
.0436	.0387	.1225	.1709	.0558	.0312
.0451	.0457	.0940	.1481	.0568	.0324
.0463	.0541	.0984	.1570	.0577	.0336
.0474	.0561	.1223	.1650	.0586	.0349
.0492	.0572	.1196	.1473	.0595	.0378
.0517	.0584	.0893	.1118	.0605	.0399
.0538	.0600	.0806	.1067	.0616	.0417
.0562	.0579	.0908	.1213	.0628	.0432
.0596	.0596	.1069	.1140	.0644	.0451
.0624	.0612	.0842	.0755	.0662	.0460
.0644	.0620	.0974	.0881	.0681	.0453
.0666	.0602	.0898	.0835	.0699	.0448
.0691	.0586	.0986	.0834	.0719	.0447
.0730	.0604	.0774	.0460	.0743	.0437
.0769	.0676	.0608	.0474	.0770	.0434
.0809	.0606	.0557	.0550	.0800	.0429
.0836	.0696	.0592	.0663	.0827	.0434

2. Confederation Life

A. Holdings (000$)

(1)	(2)	(3)	(4)	(5)	(6)	(7)
115882.	87019.	4244.	11224.	87839.	-279157.	27051.
121337.	86983.	4214.	13143.	99445.	-296033.	29089.
128356.	93036.	4511.	12914.	106096.	-312357.	32556.
133800.	98189.	3942.	16217.	113459.	-332920.	32687.
137468.	107746.	3323.	16026.	124690.	-355188.	34065.
154610.	110650.	3136.	17833.	133410.	-381064.	38575.
175339.	109427.	4190.	16640.	136513.	-397505.	44604.
171356.	109981.	5213.	17939.	154778.	-414467.	44800.
175919.	118738.	4968.	16886.	176067.	-443644.	48934.
165535.	123778.	7487.	20000.	206761.	-469408.	54153.
146831.	122904.	8770.	15636.	253256.	-497396.	50001.
133719.	123095.	8713.	17416.	286180.	-517316.	51807.
140197.	121465.	12087.	19576.	295947.	-533656.	55616.
140367.	116171.	32122.	26825.	302123.	-541191.	76417.
109710.	127307.	27232.	19960.	323211.	-549503.	57917.
123612.	149943.	19718.	14654.	334997.	-581238.	61686.
175864.	158664.	21321.	17571.	330357.	-618863.	84914.
206724.	177147.	27922.	18881.	339992.	-677248.	93418.
253036.	163967.	32162.	27024.	359748.	-739224.	96713.
283200.	146010.	32426.	24527.	383851.	-782590.	87424.
306991.	155560.	35975.	21928.	444168.	-867270.	97352.
340011.	175209.	50826.	21219.	484627.	-963335.	108557.
361715.	195337.	59960.	24535.	582306.	-1104660.	119193.

B. Expected Returns (percentage point)

(1)	(2)	(3)	(4)	(5)	(6)
.0335	.0081	.1378	.1712	.0514	.0260
.0345	.0092	.1349	.1709	.0519	.0254
.0361	.0110	.1146	.1553	.0526	.0256
.0375	.0173	.1170	.1522	.0533	.0259
.0396	.0181	.1233	.1731	.0542	.0270
.0417	.0234	.1180	.1760	.0551	.0304
.0436	.0391	.1225	.1696	.0562	.0312
.0451	.0462	.0940	.1470	.0572	.0324
.0463	.0546	.0984	.1566	.0581	.0336
.0474	.0567	.1223	.1629	.0589	.0349
.0492	.0578	.1196	.1442	.0597	.0378
.0517	.0591	.0893	.1104	.0607	.0399
.0538	.0605	.0806	.1068	.0617	.0417
.0562	.0593	.0908	.1232	.0629	.0432
.0596	.0606	.1069	.1152	.0645	.0451
.0624	.0621	.0842	.0772	.0664	.0460
.0644	.0625	.0974	.0910	.0682	.0453
.0666	.0596	.0898	.0938	.0700	.0448
.0691	.0578	.0986	.0896	.0721	.0447
.0730	.0593	.0774	.0470	.0747	.0437
.0769	.0668	.0608	.0563	.0775	.0434
.0809	.0631	.0557	.0603	.0806	.0429
.0836	.0719	.0592	.0769	.0833	.0434

3. Great West Life

A. Holdings (000$)

(1)	(2)	(3)	(4)	(5)	(6)	(7)
163216.	107051.	1928.	13098.	224936.	-480766.	29463.
163156.	115711.	2213.	15059.	252883.	-516355.	32667.
167272.	125061.	1877.	16402.	270683.	-544490.	36805.
177529.	132426.	1997.	17463.	295474.	-582906.	41983.
177346.	143889.	2317.	18443.	335216.	-631221.	45990.
183058.	132588.	2022.	20849.	373970.	-660383.	52104.
204973.	133533.	3009.	24056.	397488.	-704617.	58442.
217232.	128724.	3608.	27279.	442344.	-753624.	65563.
226281.	123434.	6353.	31242.	481357.	-796395.	72272.
251042.	121510.	12377.	34163.	520473.	-859519.	80046.
280470.	115217.	17409.	40525.	545920.	-911645.	87896.
315730.	124465.	15207.	50921.	557817.	-969571.	94569.
350004.	120593.	19434.	59294.	568785.	-1015764.	102346.
367851.	127282.	28151.	56924.	590071.	-1058935.	111344.
317169.	131054.	42617.	61303.	691686.	-1130777.	113052.
336607.	137647.	35148.	51415.	706455.	-1151987.	115285.
344422.	142314.	40623.	66399.	754054.	-1227720.	120092.
366318.	148685.	42663.	74200.	815362.	-1322624.	124604.
387415.	152459.	49871.	80348.	871481.	-1410212.	131362.
408375.	147129.	69872.	80341.	922738.	-1492103.	136352.
425006.	179089.	91999.	93822.	1015617.	-1663192.	142341.
436861.	203999.	105859.	104745.	1149910.	-1853018.	148356.
493870.	186017.	108445.	100959.	1360251.	-2093670.	155872.

B. Expected Returns (percentage point)

(1)	(2)	(3)	(4)	(5)	(6)
.0335	.0317	.1378	.1870	.0513	.0260
.0345	.0314	.1349	.1886	.0517	.0254
.0361	.0309	.1146	.1714	.0523	.0256
.0375	.0330	.1170	.1650	.0530	.0259
.0396	.0335	.1233	.1818	.0539	.0270
.0417	.0361	.1180	.1792	.0548	.0304
.0436	.0369	.1225	.1731	.0558	.0312
.0451	.0437	.0940	.1512	.0568	.0324
.0463	.0518	.0984	.1578	.0577	.0336
.0474	.0528	.1223	.1694	.0586	.0349
.0492	.0532	.1196	.1561	.0595	.0378
.0517	.0532	.0893	.1175	.0605	.0399
.0538	.0562	.0806	.1109	.0616	.0417
.0562	.0601	.0908	.1226	.0629	.0432
.0596	.0621	.1069	.1164	.0647	.0451
.0624	.0631	.0842	.0783	.0667	.0460
.0644	.0625	.0974	.0933	.0686	.0453
.0666	.0597	.0898	.0808	.0705	.0448
.0691	.0581	.0986	.0829	.0726	.0447
.0730	.0596	.0774	.0497	.0751	.0437
.0769	.0679	.0608	.0433	.0778	.0434
.0809	.0692	.0557	.0505	.0809	.0429
.0836	.0785	.0592	.0576	.0837	.0434

4. London Life

A. Holdings (000$)

(1)	(2)	(3)	(4)	(5)	(6)	(7)
180526.	497.	2054.	136.	303950.	-450360.	36803.
161379.	497.	2061.	136.	364644.	-487410.	41307.
166108.	498.	2509.	165.	403832.	-527203.	45909.
172449.	498.	2308.	169.	432310.	-556836.	50898.
195443.	498.	2309.	169.	474432.	-615784.	57067.
204850.	498.	2502.	204.	519449.	-665234.	62269.
210150.	498.	2925.	204.	585937.	-731456.	68258.
219808.	499.	4173.	400.	643716.	-795369.	73227.
227709.	499.	3096.	359.	707568.	-860312.	78919.
240592.	499.	3048.	359.	782821.	-942777.	84942.
241880.	499.	5625.	359.	860775.	-1016944.	92194.
242504.	983.	7195.	402.	948792.	-1097106.	102770.
248117.	1475.	7412.	402.	1023231.	-1168894.	111743.
245322.	1813.	8525.	553.	1112432.	-1244602.	124043.
233560.	1314.	13257.	553.	1172100.	-1292459.	128325.
239076.	1315.	14831.	548.	1203325.	-1325624.	133471.
260827.	1316.	17394.	417.	1225855.	-1368265.	137544.
286607.	1218.	19524.	416.	1282821.	-1450045.	140741.
333133.	1.	26862.	416.	1383503.	-1603807.	140108.
371948.	1.	29028.	698.	1499432.	-1757309.	143798.
427013.	1.	29757.	698.	1564619.	-1869735.	152353.
526523.	1.	30454.	1973.	1623160.	-2019877.	162234.
665138.	1362.	40039.	698.	1752205.	-2293204.	166238.

B. Expected Returns (percentage point)

(1)	(2)	(3)	(4)	(5)	(6)
.0335	.0317	.1378	.1870	.0511	.0260
.0345	.0314	.1349	.1886	.0515	.0254
.0361	.0310	.1146	.1714	.0520	.0256
.0375	.0330	.1170	.1650	.0527	.0259
.0396	.0336	.1233	.1818	.0534	.0270
.0417	.0361	.1180	.1792	.0542	.0304
.0436	.0369	.1225	.1731	.0552	.0312
.0451	.0437	.0940	.1512	.0561	.0324
.0463	.0518	.0984	.1578	.0571	.0336
.0474	.0528	.1223	.1694	.0580	.0349
.0492	.0532	.1196	.1561	.0588	.0378
.0517	.0532	.0893	.1175	.0598	.0399
.0538	.0562	.0806	.1109	.0609	.0417
.0562	.0601	.0908	.1226	.0621	.0432
.0596	.0621	.1069	.1164	.0635	.0451
.0624	.0631	.0842	.0783	.0652	.0460
.0644	.0625	.0974	.0933	.0669	.0453
.0666	.0597	.0898	.0808	.0687	.0448
.0691	.0581	.0986	.0829	.0705	.0447
.0730	.0596	.0774	.0497	.0727	.0437
.0769	.0679	.0608	.0433	.0751	.0434
.0809	.0692	.0557	.0505	.0778	.0429
.0836	.0785	.0592	.0576	.0805	.0434

5. Manufacturer's Life

A. Holdings (000$)

(1)	(2)	(3)	(4)	(5)	(6)	(7)
161283.	254052.	5039.	34663.	144572.	-550055.	49554.
164011.	263084.	3437.	38385.	183045.	-597428.	54534.
158715.	273715.	2556.	31362.	228350.	-627666.	67032.
153962.	302284.	808.	24413.	258719.	-668068.	72118.
149369.	326649.	787.	35972.	289612.	-725040.	77349.
148750.	355340.	902.	47997.	321053.	-791433.	82609.
156347.	342361.	4353.	68786.	375417.	-855173.	92091.
174395.	344287.	4969.	86944.	423732.	-936105.	98222.
151596.	361995.	6250.	88665.	489799.	-1031467.	106838.
204499.	388750.	13269.	96917.	557139.	-1145442.	115132.
236967.	397877.	14289.	99523.	613629.	-1234077.	128208.
251248.	407229.	20833.	112203.	686593.	-1336354.	141752.
259365.	425329.	27728.	134067.	734201.	-1426700.	153990.
260656.	436960.	43285.	147526.	785959.	-1507856.	166530.
260618.	420666.	87884.	165046.	790923.	-1556446.	168691.
266353.	462767.	87248.	167068.	820024.	-1644401.	179059.
327359.	531712.	94172.	156860.	843922.	-1772497.	181528.
385411.	523169.	103932.	165615.	863498.	-1852010.	189615.
452958.	507847.	94265.	175072.	968286.	-2004491.	193937.
461207.	524289.	90901.	163761.	1046620.	-2107110.	199668.
546264.	591023.	93480.	189302.	1128551.	-2313915.	234705.
634736.	693411.	72821.	242797.	1149456.	-2544423.	248798.
751509.	856791.	66889.	183706.	1223055.	-2801622.	280328.

B. Expected Returns (percentage point)

(1)	(2)	(3)	(4)	(5)	(6)
.0335	.0210	.1378	.1809	.0512	.0260
.0345	.0214	.1349	.1818	.0516	.0254
.0361	.0220	.1146	.1654	.0522	.0256
.0375	.0259	.1170	.1605	.0530	.0259
.0396	.0266	.1233	.1788	.0538	.0270
.0417	.0303	.1180	.1775	.0548	.0304
.0436	.0380	.1225	.1729	.0558	.0312
.0451	.0449	.0940	.1512	.0568	.0324
.0463	.0532	.0984	.1589	.0578	.0336
.0474	.0548	.1223	.1697	.0587	.0349
.0492	.0557	.1196	.1553	.0596	.0378
.0517	.0565	.0893	.1181	.0607	.0399
.0538	.0587	.0806	.1125	.0620	.0417
.0562	.0595	.0908	.1252	.0634	.0432
.0596	.0612	.1069	.1182	.0652	.0451
.0624	.0625	.0842	.0797	.0673	.0460
.0644	.0624	.0974	.0948	.0693	.0453
.0666	.0595	.0898	.0849	.0713	.0448
.0691	.0578	.0986	.0852	.0735	.0447
.0730	.0593	.0774	.0495	.0765	.0437
.0769	.0671	.0608	.0462	.0796	.0434
.0809	.0662	.0557	.0524	.0830	.0429
.0836	.0751	.0592	.0620	.0859	.0434

6. Mutual Life

A. Holdings (000$)

(1)	(2)	(3)	(4)	(5)	(6)	(7)
222496.	7454.	9760.	8713.	202870.	-418544.	32749.
218160.	6809.	11179.	8888.	234434.	-447291.	32179.
222071.	8417.	11769.	8889.	255202.	-471547.	34801.
236920.	7253.	11884.	9446.	276090.	-503765.	37828.
253073.	8333.	11347.	7687.	297852.	-535985.	42307.
268578.	9823.	10647.	7537.	319260.	-570621.	45224.
267151.	8238.	11054.	7988.	347962.	-613600.	48793.
314626.	7118.	13427.	7119.	368601.	-659153.	51738.
347435.	7352.	15344.	7697.	387670.	-710763.	54735.
374986.	6916.	18166.	9557.	412012.	-762852.	58785.
389217.	6078.	21147.	10588.	449168.	-813827.	62371.
382198.	6316.	25861.	9906.	501163.	-859156.	66288.
398664.	5918.	30657.	13490.	533951.	-911894.	70786.
392395.	4957.	38363.	16431.	568969.	-945261.	75854.
391265.	4716.	53054.	18414.	587123.	-975312.	79260.
399174.	4584.	58079.	19841.	600920.	-999828.	82770.
425917.	4387.	65112.	23809.	603479.	-1035911.	86793.
466002.	3501.	72757.	25167.	623551.	-1098323.	92655.
492093.	3263.	82303.	28153.	669746.	-1174646.	100912.
506548.	3106.	87802.	30110.	719049.	-1244312.	104303.
578073.	3124.	93055.	30462.	769875.	-1364598.	109991.
631257.	2534.	92361.	47368.	844690.	-1498399.	119811.
678301.	2973.	97669.	56490.	941994.	-1646998.	130429.

B. Expected Returns (percentage point)

(1)	(2)	(3)	(4)	(5)	(6)
.0335	.0307	.1378	.1870	.0511	.0260
.0345	.0305	.1349	.1886	.0515	.0254
.0361	.0301	.1146	.1714	.0521	.0256
.0375	.0324	.1170	.1650	.0527	.0259
.0396	.0329	.1233	.1818	.0534	.0270
.0417	.0355	.1180	.1792	.0543	.0304
.0436	.0371	.1225	.1731	.0553	.0312
.0451	.0439	.0940	.1512	.0562	.0324
.0463	.0521	.0984	.1578	.0571	.0336
.0474	.0534	.1223	.1694	.0580	.0349
.0492	.0538	.1196	.1561	.0589	.0378
.0517	.0539	.0893	.1175	.0599	.0399
.0538	.0567	.0806	.1109	.0610	.0417
.0562	.0604	.0908	.1226	.0622	.0432
.0596	.0624	.1069	.1164	.0638	.0451
.0624	.0635	.0842	.0783	.0656	.0460
.0644	.0629	.0974	.0933	.0674	.0453
.0666	.0599	.0898	.0808	.0692	.0448
.0691	.0583	.0986	.0829	.0712	.0447
.0730	.0597	.0774	.0497	.0735	.0437
.0769	.0678	.0608	.0433	.0761	.0434
.0809	.0691	.0557	.0505	.0790	.0429
.0836	.0783	.0592	.0576	.0817	.0434

7. Sun Life

A. Holdings (000$)

(1)	(2)	(3)	(4)	(5)	(6)	(7)
406660.	830808.	8245.	131747.	429631.	-1649200.	157891.
415591.	789123.	9518.	144404.	504732.	-1692516.	170852.
426231.	833974.	11372.	139012.	548560.	-1769366.	189783.
446196.	832738.	14678.	149404.	574284.	-1818230.	199070.
460363.	849196.	19922.	164694.	612122.	-1895153.	211144.
498264.	817902.	22624.	177470.	674479.	-1969505.	221234.
534708.	821705.	29873.	187815.	720152.	-2061655.	232598.
568833.	830378.	29944.	200517.	761976.	-2141500.	250148.
592612.	865965.	33284.	194492.	808526.	-2225372.	269507.
601124.	882330.	48763.	203375.	873535.	-2321789.	287338.
597891.	895384.	56635.	212874.	963903.	-2417575.	309112.
598720.	872282.	66133.	225353.	1057772.	-2486272.	333988.
614132.	886332.	75179.	223395.	1121306.	-2561668.	358676.
618096.	874835.	101874.	245365.	1172155.	-2628867.	383458.
604391.	861800.	117573.	243910.	1233806.	-2654815.	406665.
605922.	859085.	133773.	272008.	1272058.	-2709654.	433192.
618547.	833516.	150129.	295188.	1308587.	-2759387.	446580.
647329.	842314.	160107.	311735.	1330810.	-2830974.	461321.
665986.	838111.	172014.	333998.	1432001.	-2983197.	478913.
755679.	750187.	168761.	330145.	1512835.	-3001864.	515743.
846936.	803018.	178778.	322467.	1565591.	-3162307.	554483.
959857.	852336.	195671.	321110.	1613979.	-3367901.	575052.
1138267.	890601.	217483.	334458.	1717474.	-3678558.	619725.

B. Expected Returns (percentage point)

(1)	(2)	(3)	(4)	(5)	(6)
.0335	.0228	.1378	.1814	.0512	.0260
.0345	.0231	.1349	.1826	.0516	.0254
.0361	.0234	.1146	.1660	.0522	.0256
.0375	.0272	.1170	.1609	.0529	.0259
.0396	.0278	.1233	.1802	.0537	.0270
.0417	.0313	.1180	.1816	.0546	.0304
.0436	.0378	.1225	.1729	.0556	.0312
.0451	.0447	.0940	.1500	.0566	.0324
.0463	.0530	.0984	.1578	.0576	.0336
.0474	.0545	.1223	.1670	.0585	.0349
.0492	.0553	.1196	.1507	.0594	.0378
.0517	.0559	.0893	.1143	.0604	.0399
.0538	.0583	.0806	.1086	.0615	.0417
.0562	.0596	.0908	.1240	.0628	.0432
.0596	.0614	.1069	.1170	.0643	.0451
.0624	.0627	.0842	.0782	.0661	.0460
.0644	.0627	.0974	.0909	.0680	.0453
.0666	.0601	.0898	.0862	.0698	.0448
.0691	.0584	.0986	.0863	.0718	.0447
.0730	.0600	.0774	.0488	.0743	.0437
.0769	.0678	.0608	.0493	.0772	.0434
.0809	.0657	.0557	.0572	.0803	.0429
.0836	.0748	.0592	.0691	.0831	.0434

Definitions:
(1) Canadian Bonds
(2) Foreign Bonds
(3) Canadian Stocks
(4) Foreign Stocks
(5) Mortgage Loans and Real Estate
(6) Net Actuarial Reserves
(7) Surplus

Sources: Bank of England Quarterly Bulletin, various issues.
Bank of Canada Review, various issues.
Report of The Superintendent of Insurance, various issues.

BIBLIOGRAPHY

Aivazian, V.A. (1976): "On the Comparative Statics of Assets Demand", Research and Working Paper Series No. 124, Faculty of Business, McMaster University.

Aivazian, V.A., J.L. Callen, I. Krinsky, and C.C.Y. Kwan (1981): "Mean-Variance Utility Functions and the Demand for Risky Assets: An Empirical Analysis Using Flexible Functional Forms", Working Paper No. 81-25, Department of Economics, McMaster University.

Appelbaum, E. (1975): "On the Choice of Functional Forms", International Economic Review.

Baily, A. (1862): "On the Principles on Which the Funds of Life Assurance Societies Should Be Invested", Journal of The Institute of Actuaries.

Bank of Canada, Bank of Canada Review, various years.

Bank of England, Bank of England Quarterly Bulletin, various years.

Baron, D.P. (1977): "On the Utility Theoretic Foundation of Mean-Variance Analysis", Journal of Finance.

Barret, R.J., M.R. Gray, and J.M. Parkin (1975): "The Demand for Financial Assets by the Personal Sector of the U.K. Economy", in G.A. Renton (ed.), Modelling the Economy, Heinemann Educational Books, London.

Barten, A.P. (1969): "Maximum Likelihood Estimation of a System of Demand Equations", European Economic Review.

Baum, D.J. (1973): The Investment Function of Canadian Financial Institutions, Prueger Publishers Inc., New York.

Berndt, E.R., B.H. Hall, R.E. Hall and J.A. Hausman (1974): "Estimation and Inference in Non-Linear Structural Models", Annals of Economic and Social Measurement.

Berndt, E.R., M.N. Darrough, and W.E. Diewert (1977): "Flexible Functional Forms and Expenditure Distributions: An Application to Canadian Consumer Demand Functions", International Economic Review.

Bhattacharyya, D.K. (1978): Demand for Financial Assets: An Econometric Model of U.K. Personal Sector, Saxon House, Westmead, England.

Bierwag, G.O. (1974): "The Rationale of the Mean-Standard Deviation Analysis: Comment", American Economic Review.

Bierwag, G.O., and M. Grove (1968): "Slutsky Equations for Assets", Journal of Political Economy.

Black, F. (1975): "Bank Funds Management in an Efficient Market", Journal of Financial Economics.

Borch, K. (1969): "A Note on Uncertainty and Indifference Curves", Review of Economic Studies.

Borch, K. (1974): "The Rationale of the Mean-Standard Deviation Analysis: Comment", American Economic Review.

Box, G.E.P. and D.R. Cox (1964): "An Analysis of Transformations", Journal of the Royal Statistical Society, series B.

Brimmer, A.F. (1962): Life Insurance Companies In the Capital Market, MSU Business Studies, Bureau of Business and Economic Research, Graduate School of Business Administration, Michigan State University, East Lansing.

Brodt, A.I. (1982): "MID- MAD Life: A Multi-Period Optimization Model for Life Insurance Company Investment Decisions", Rapport de Recherche No. 82-05 Ecole des Hautes Etudes Commerciales, Montréal, Canada.

Burns, J.W. (1973): "Canadian Life Insurance - Important Variation", in D.W. Gregg and V.B. Lucas (ed.), Life and Health Insurance Handbook, Richard D. Irwin Inc., New York.

Chiang, Hui-Shyong (1977): "Functional Forms and the Demand For Meat in the United States", Review of Economics and Statistics.

Christensen, L.R., D.W. Jorgenson and L.G. Lau (1975): "Transcendental Logarithmic Utility Functions", American Economic Review.

Cohn, R.A. (1974): "Mutual Life Insurer's Portfolio and Policyholder Utility Function", Journal of Risk and Insurance.

Cummins, D.J. (1975): An Econometric Model of the Life Insurance Sector of the U.S. Economy, Lexington Books, D.C. Heath and Company, Lexington, Massachusetts.

Cummins, D.J. (ed. 1977): Investment Activities of Life Insurance Companies, Richard D. Irwin Inc., Illinois.

Cummins, D.J. and D.J. Nye (1981): "Portfolio Optimization Models For Property-Liability Insurance Companies: An Analysis and Some Extension", Management Science.

Daines, R. (1968): "Analysis of Impact of Underwriting Function on Investment in Common Stocks", The Journal of Risk and Insurance.

Deaton, A. (1975): Models and Projections of Demand in Post-War Britain, Chapman and Hall Ltd., London.

Deaton, A. and J. Muellbauer (1980a): "An Almost Ideal Demand System", American Economic Review.

Deaton, A. and J. Muellbauer (1980b): Economic and Consumer Behavior, Cambridge University Press, Cambridge.

Department of Insurance (various years): Report of the Superintendent of Insurance, Queen's Printer, Ottawa.

Dhrymes, P. (1973): "Small Sample and Asymptotic Relations Between Maximum Likelihood and Three Stage Least Square", Econometrica.

Diewert, W.E. (1971): "An Application of the Shephard Duality Theorem: A Generalized Leontief Production Function", Journal of Political Economy.

Dodds, J.C. (1975): The Investment Behaviour of British Life Insurance Companies, Croom Helm Ltd., London.

Ewis, N.A., and D. Fisher (1982): "The Translog Utility Function and the Demand For Money in the United States", Working Paper No. 11, Department of Economics and Business, North Carolina State University.

Fama, E.F. (1971): "Risk, Return, and Equilibrium", Journal of Political Economy.

Fama, E.F. (1976): Foundation of Finance, Basic Books Inc. Publishers, New York.

Farrell, J.L. (1977): "Modern Portfolio Theory and the Common Stock Investment Process", in D.J. Cummins (ed.) Investment Activities of Life Insurance Companies, Richard D. Irwin Inc., Illinois.

Feldstein, M.S. (1969): "Mean-Variance Analysis in the Theory of Liquidity Preference", Review of Economic Studies.

Flath, D. and C.R. Knober (1980): "Taxes, Failure Costs and Optimal Industry Capital Structure: An Empirical Test", Journal of Finance.

Francis, J.C. (1978): "Portfolio Analysis of Asset and Liability Management in Small-, Medium-, and Large Size Banks", Journal of Monetary Economics.

Franklin, P.J. and C. Woodhead (1980): The U.K. Life Assurance Industry, Croon Helm Ltd., London.

Geehan, R. (1977): "Returns to Scale in the Life Insurance Industry", Bell Journal of Economics.

Gentry, J. and J. Pike (1970): "An Empirical Study of the Risk-Return Hypothesis Using Common Stock Portfolio of Life Insurance Companies", Journal of Financial and Quantitative Analysis.

Gnedenko, B.V. (1962): The Theory of Probability, Chelsea Publishing Co., New York.

Graham, R.E. (1974): The Effect of Regulation On the Common Stock Holdings of Canadian Life Insurance Companies, Unpublished Ph.D. dissertation, University of Toronto.

Green, J.H.A. (1976): Consumer Theory, The Macmillan Press Ltd., London.

Gregg, D.W. and V.B. Lucas (ed. 1973): Life and Health Insurance Handbook, Richard D. Irwin Inc., New York.

Hadar, J. (1971): Mathematical Theory of Economic Behaviour, Addison-Wesley Publishing Company, Reading, Mass.

Hart, O. and O. Jaffe (1974): "On the Application of Portfolio Theory to Depository Financial Intermediaries", Review of Economic Studies.

Haugen, R.A. and C.O. Kroncke (1970): "A Portfolio Approach to Optimizing the Structure of Capital Claims and Assets of a Stock Insurance Company", Journal of Risk and Insurance, and "Reply" (1972).

Haugen, R.A. and C.O. Kroncke (1971): "Rate Regulation and the Cost of Capital and Insurance Industry", Journal of Financial and Quantitative Analysis.

Haynes, A.T. and R.J. Kirton (1952): "The Financial Structure of A Life Office", Transactions of the Faculty of Actuaries.

Hirschleifer, J. (1970): Investment and Capital, Prentice-Hall Inc., Englewood Cliffs, N.J.

Honohan, P. (1980): "Testing a Standard Theory of Portfolio Selection", Oxford Bulletin of Economics and Statistics.

Humphrey, D.B. (1981): "Intermediation and Cost Determinants of Large Bank Liability Composition", Journal of Banking and Finance.

Ibrahim, I.B. (1974): "Risk in the Macro Financial Investment Equation of Pension Funds and Life Insurance Companies", Journal of Risk and Insurance.

Jackson, R.T. (1959): "Some Observation on Ordinary Dividends", Transactions of the Society of Actuaries.

Jones, L.D. (1968): Investment Policies of Life Insurance Companies, Division of Research, Graduate School of Business Administration, Harvard University, Boston, Mass.

Kahane, Y. and D.J. Nye (1975): "A Portfolio Approach to the Property Liability Insurance Industry", Journal of Risk and Insurance.

Kahane, Y. (1977a): "Determination of the Product Mix and the Business Policy of An Insurance Company - A Portfolio Approach", Management Science.

Kahane, Y. (1977b): "Insurance Exposure and Investment Risks: A Comment on the Chance-Constrained Programming", Operation Research.

Kane, E. (1972): "Book Review, Risk Return and Equilibrium", Journal of Finance.

Kellner, S. and F.G. Mathewson (1980): "Entry, Size Distribution, Scale and Scope Economics in the Life Insurance Industry", Working Paper No. 8006, Institute for Policy Analysis, University of Toronto.

Klemkosky, R. (1973): "The Bias in Composite Performance Measures", Journal of Financial and Quantitative Analysis.

Kmenta, J. and R.F. Gilbert (1968): "Small Sample Properties of Alternative Estimators of Seemingly Unrelated Regressions", Journal of The American Statistical Association.

Kon, S.J. and F.C. Jen (1978): "Mutual Fund Portfolios: An Application of Switching Regression", Journal of Finance.

Krouse, C.G. (1970): "Portfolio Balancing, Corporate Assets and Liabilities With Special Application to Insurance Management", Journal of Financial and Quantitative Analysis.

Lambert, E.W. and A.E. Hofflander (1966): "Impact of New Multiple Line Underwriting on Investment Portfolios of Property-Liability Insurers", The Journal of Risk and Insurance.

Lancaster, K.J. (1966): "A New Approach to Consumer Theory", Journal of Political Economy.

Levy, H. (1974): "The Rationale of the Mean-Standard Deviation Analysis: Comment", American Economic Review.

Levy, H. and H.M. Markowtiz (1979): "Approximating Expected Utility by a Function of Mean and Variance", American Economic Review.

Lluch, C., A.A. Powell, and R.A. Williams (1977): Patterns in Household Demand and Saving, Oxford University Press, Oxford.

Madala, G.S. (1977): Econometrics, McGraw-Hill Inc., New York.

Malinvaud, E. (1972): Lectures in Microeconomic Theory, American Elsevier, New York.

Markowitz, H.M. (1952): "Portfolio Selection", The Journal of Finance.

Markowitz, H.M. (1959): Portfolio Selection, John Wiley and Sons, Inc., New York.

Mayers, D. and C.W. Smith Jr. (1981): "Contractual Provisions Organization Structure and Conflict Control in Insurance Markets", Journal of Business.

McDonald, J. (1974): "Objectives and Performance of Mutual Funds", Journal of Financial and Quantitative Analysis.

Merton, R.C. (1969): "Lifetime Portfolio Selection Under Uncertainty: The Continuous Time Case", Review of Economics and Statistics.

Michaelsen, J.B. and R.C. Goshay (1967): "Portfolio Selection in Financial Intermediaries: A New Approach", Journal of Financial and Quantitative Analysis.

Mises, R. Von (1964): Mathematical Theory of Probability and Statistics, Academic Press, New York.

Modigliani, F. and R. Sutch (1966): "Innovations in Interest Rate Risk", American Economic Review.

Montgomery, J.O. (1976): "Regulation Test in the Life and Health Insurance Industry", Journal of Risk and Insurance.

Moody's International Manual (1981): Moody's Investors Services, Inc., New York.

Neufeld, E.P. (1972): The Financial System in Canada, Its Growth and Development, St. Martin's Press, New York.

Pedoe, A. and C.E. Jack, (1978): Life Insurance Annuities and Pensions, University of Toronto Press, Toronto.

Pyle, D.H. (1971): "On the Theory of Financial Intermediation", Journal of Finance.

Pyle, D.H. and S.T. Turnovsky (1970): "Safety First and Expected Utility Maximization in Mean-Standard Deviation Portfolio Analysis", The Review of Economics and Statistics.

Quirin, D.G. et al. (1974): Competition, Economic Efficiency and Profitability in the Canadian Property and Casualty Insurance Industry, prepared for the Insurance Bureau of Canada, Toronto.

Rennie, R.A. (1977): "Investment Strategy for the Life Insurance Company", in D.J. Cummins (ed.), Investment Activities of Life Insurance Companies, Richard D. Irwin, Inc., New York.

Roberts, G.S. (1975): "Lancaster's New Demand Theory: Its Application in Portfolio Analysis", Journal of Economic Literature.

Roy, A.D. (1952): "Safety First in the Holding of Assets", Econometrica.

Ryan, T.M. (1973): "The Demand for Financial Assets by the British Life Funds", Oxford Bulletin of Economics and Statistics.

Samuelson, P.A. (1970): "The Fundamental Approximation Theorem of Portfolio Analysis in Term of Means Variances and Higher Moments", Review of Economic Studies.

Shapiro, S.S. and M.B. Wilk (1965): "An Analysis of Variance Test for Normality (complete samples)", Biometrica.

Sharpe, W.F. (1963): "A Simplified Model for Portfolio Analysis", Management Science.

Simmons, P. and D. Weiserbs (1979): "Translog Flexible Functional Forms and Associated Demand Systems", American Economic Review.

Stowe, J.D. (1978): "Life Insurance Company Portfolio Behavior", Journal of Risk and Insurance.

Superintendent of Insurance for Canada (1962): Submission to the Royal Commission on Banking and Finance, Queen's Printer, Ottawa.

Szego, G.P. (1980): Portfolio Theory: With Application to Bank Asset Management, Academic Press, New York.

Terrell, W.T. and W.J. Frazer Jr. (1972): "Interest Rates Portfolio Behavior and Marketable Government Securities", Journal of Finance.

The Canadian Life Insurance Officers Association (1962): Submission to the Royal Commission on Banking and Finance, Queen's Printer, Ottawa.

The Canadian Life Insurance Association (various years): Canadian Life Insurance Facts, Canadian Life Insurance Assocation, Toronto.

The Canadian Life Insurance Association (1979): Submission to the Ontario Select Committee on Company Law.

Theil, H. (1971): Principles of Econometrics, John Wiley and Sons, Inc., New York.

Thompson, H.E., J.P. Mattews and B.C.L. Li (1974): "Insurance Exposure and Investment Risks: An Analysis Using Chance-Constrained Programming", Operations Research.

Tobin, J. (1958): "Liquidity Preference as Behavior Toward Risk", Review of Economic Studies.

Tobin, J. (1965): "Comment on Borch and Feldstein", Review of Economic Studies.

Tsiang, S.C. (1972): "The Rationale of the Mean-Standard Deviation Analysis Skewness Preference and the Demand for Money", American Economic Review.

Weisbrod, S.R. (1978): The Regulation of The Securities Portfolio of Life Insurance Companies, Unpublished Ph.D. dissertation, University of Chicago.

Winkleross, H.E. and R.A. Zelten (1973): "An Empirical Analysis of Mutual Life Insurance Company Surplus", Journal of Risk and Insurance.

Winter, R.A. (1980): "Regulation and Response of the U.S. Life Insurance Industry to Rising Interest Rates", Working Paper No. 8001, Institute for Policy Analysis, University of Toronto, Toronto.

Woodland, A.D. (1979): "Stochastic Specification and the Estimation of Share Equations", Journal of Econometrics.

Yawitz, J.B. et. al (1976): "A Risk-Return Approach to the Selection of Optimal Government Bond Portfolios", Financial Management.

Zarembka, P. (1974): "Transformation of Variables in Econometrics", Ch. 3 in P. Zarembka (ed.): Frontiers in Econometrics, Academic Press, New York and London.