GAME THEORETICAL MODELS OF COMPETITION IN TIME-SENSITIVE MARKETS

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by

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Abstract

This study focuses mainly on situations of time-based competition. Three problems in this context will be studied in three different parts.

In the first part, we will examine the *promised delivery time* (PDT) competition for firms whose production processes consist of more than one stage. We study three games; a) when each firm consists of two stages and has identical production rates in both stages, b) when each firm consists of k stages and has identical production rate in all stages and, c) when each firm consists of two stages and has different production rates in each stage.

In the second part, we focus on a duopolistic market where the firms compete against each other by determining their PDT. The firms try to win the business of a single customer who is sensitive to PDT but will also penalize the winning firm through tardiness costs. This situation may emerge when the production duration is too long and the product is expensive as in the aviation industry.

The third part of this study deals with situations of investment competition in the presence of incomplete information in the market. The investment decision will affect the time to production (speed) and determines the probability of winning the business. The notion of incompleteness in information is projected when firms are not fully certain about each other's objective function.

In each chapter, we will find the equilibrium of the game and determine the players' optimal strategies. At the end of each chapter, a numerical analysis is presented, where numerous numerical examples are solved. Based on the numerical examples, a sensitivity analysis is also presented for each model that would capture the sensitivity of the Nash equilibria and the firms' optimal strategies towards changes in parameters in the market or the competitor's operations.

Dedication

I would like to dedicate this doctoral thesis to Kamran and Mahboubeh, my wonderful parents, and to Bahar and Rasa, my lovely sisters. Everything that I have done and accomplished from the first day of my life to this very day is because of their love, support and caring. I am grateful to them forever.

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Contents

List of Tables

List of Figures

Chapter 1

Introduction

This thesis studies time-based competition. As the nature of the word "competition" suggests, time-based competition analyzes situations of conflict, where players compete with each other based on time. Therefore, game theory will be the natural methodology for this study. We will present cases and literature reviews to exhibit the attention given to time-based competition by both practitioners as well as academicians. However, there is still considerable room and opportunity for further research in the context of game theory for studying of situations of time-based competition. We believe that this thesis will contribute to the body of literature in this context.

Following this short introduction, we will introduce time-based competition followed by a section explaining the importance of time-based competition in the literature. After that, we will discuss the relationship between time-based competition and supply chain management. This chapter will conclude by introducing the structural organization of this study.

1.1 Time-Based Competition

Today, time is known to be an important factor in determining the success of businesses. The notion of time that is important to businesses and will be referred to in the rest of this text has been defined by Kumar and Motwani [41] as the "totality of time required to perform all activities on a critical path that commences with the identification of a market need and terminates with the delivery of a matching product to the customer". In the 21st century, it is universally accepted that time plays an important role in the advancement of businesses and is an indispensable part of every operation. Every day we see companies that try to compete and win a greater share of the market by focusing on the time aspect of their operations. Numerous examples are at hand. For instance, e-retailers (online retailers) such as Bestbuy.com and Amazon.com use time as a means of competition and promise delivery times that are competitive with that of their rivals and the market rewards them by giving them more demand as explained in Maltz et al. [48]. Other examples of companies that compete based on the time aspect of their business includes UPS, Federal Express, etc.

Although a clear competitive advantage today, time has been enjoying the attention only in the last couple of decades. The notion of time as a competitive advantage was explicitly introduced in the literature at the end of the '80s by Stalk [38]. In this work, he explained how the competitive advantages of companies have been evolving through time and how time has become important in shaping companies' competitive advantages. According to Stalk [38], "As a strategic weapon, time is the equivalent of money, productivity, quality, even innovation". In this work, he explains the evolution that led to the identification of time and speed as competitive advantages. He argues that companies always used low-cost labour as a competitive advantage up to the early years of 1960s, when the wages started to increase. Industries responded to the rising wages by investing in technology and shifting production to an era of large scale manufacturing taking advantage of large volumes in order to bring the unit cost of production down. By the mid 1970s, industries sought competitive advantage by focusing on specific operations involved in production and splitting the manufacturing processes into smaller factories. This led to focused factories. The focused factories provided companies with competitive advantage by keeping the cost per unit of production down but they limited companies' ability to capture the ever-growing demand for variety. Industries responded to the growing awareness in the variety-seeking consumers by the introduction of flexible factories. With flexible manufacturing, companies produced low-cost products with greater variability. With the majority of the market focusing on producing lower cost products with a great degree of variability, companies that did so faster certainly stood out from the rest of the players. This gave rise to the concept of time-based competition and time-based manufacturing.

Stalk [38] argues that companies started incorporating time competition in every process from production, service delivery and new product development to sales, logistics and transportation, etc. According to him, some companies that took active part in time-based competition include Sony, Sharp, Toyota, Hitachi, Toshiba, The Limited (women's clothing manufacturer), Federal Express, Domino's Pizza, Wilson Art and McDonald's. Stalk and Hout [60] studied the issue of time-based competition more thoroughly and presented many examples of time-based competitors and provided insights to businesses on how to become time-based competitive and how it will help customers.

Time-based competition today is a strategical and managerial approach that addresses all units and sections across the organization and seeks to shrink time in every step of the production/service delivery throughout the organization or across the supply chain (see Bozarth and Chapman [10], Hum and Sim [37], Rich and Hines [51], and Sapkauskiene and Leitoniene [52])

1.2 Benefits of Time-Based Competition

Time-based practices will not only help the companies involved in time-based competition serve more customers and create more satisfaction and better reputation, they will also help the organizations internally in making their processes more efficient, reducing their costs and creating and sharing more information. Kumar and Motwani [41] identify three main sources, from which the strategic value of time originates. These three sources include; (a) the price premium as a result of faster response, (b) brand loyalty and higher market share as a result of faster delivery and customized products, (c) lower production and logistical costs resulting in higher profitability. Bozarth and Chapman [10] argue that a number of benefits of the application of time-based competition in a company includes; (a) the benefits to the customer (shorter delivery time, improved customer support, higher-quality goods and services, etc.), (b) benefits to the organization (lower development costs, simplified production control, higher efficiencies, etc.), and (c) tactics (system-wide measure of time, continuous improvement efforts, integrated information systems, etc.). Hum and Sim [37] generalized the benefits of time-based competition as "They [benefits] include increase in productivity, ability to command a price premium, market share gain, customer loyalty and the ability to shut out competition through planned obsolescence-a process of planning one's own products to become obsolete by introducing new products rapidly to replace them."

Sapkauskiene and Leitoniene [52] take a holistic view of time-based competition and look at it from a management theory perspective. They argue that various concepts, methods and tools should be involved in making the organization time-based competitive. They propose the concept of time-based management, that is required to manage and integrate all these processes in order for the organization to achieve time-based competition. According to them, implementing time-based management will help the organization enjoy greater profitability through other advantages like higher efficiency, higher productivity, lower cost, higher price premiums, etc. The diagram presented in [52] explaining the advantages provided by time-based competition can be seen in Figure 1.1.

Sim and Curatola [55] do an empirical study on the benefits of time-based competition. They study the performance of 83 electronic plants (such as semiconductor producers) in the United States in dealing with time-based performance and report that companies that manage their time-based performances effectively, usually have a greater market share and can reduce their manufacturing and warranty costs. For a thorough review on the benefits of time-based competition, see Blackburn [9], Stalk [38], and Stalk and Hout [60].

This chapter briefly describes game theory and supply chain management (SCM), and explains the organizational structure of this thesis. Our brief description indicates that the theory of games has broad applications in diverse fields, and especially plays an increasingly important role in analyzing various game-related SCM problems.

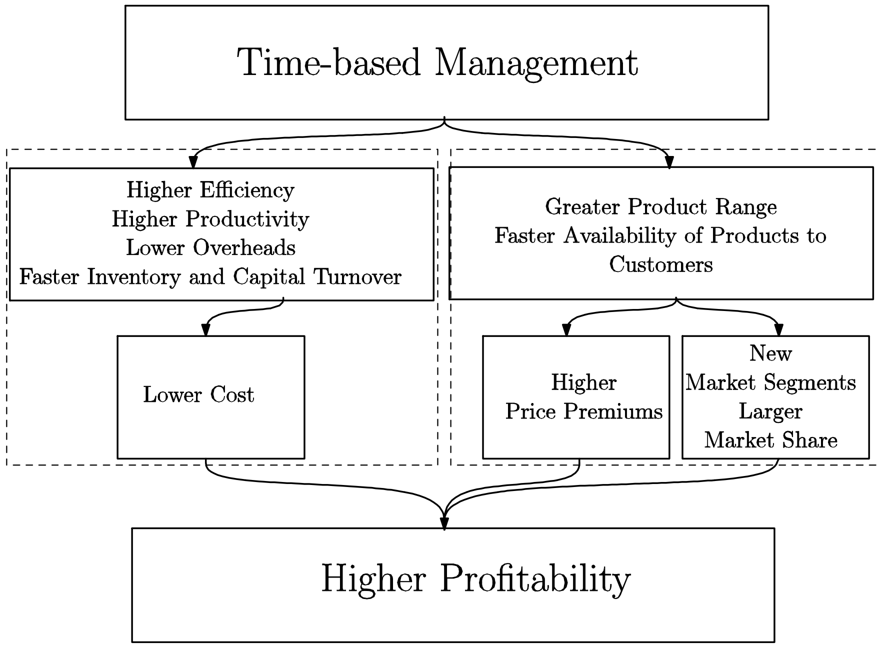

Figure 1.1: The basic advantages provided by time-based competition presented in Sapkauskiene and Leitoniene [52].

We also show that both academics and practitioners have paid growing attention to SCM in the recent years. Therefore, it is worthwhile for us to propose this study concerned with game theoretical concepts and their applications, especially in SCM. The structure of this thesis is introduced at the end of this chapter.

1.3 Time-Based Competition and Supply Chain Management

Globalization and new trade laws in the past decade have had important impacts on the nature of business environment. This has contributed to the field of supply chain management in different ways. With companies now operating through the co-operation of various previously individual business entities, the knowledge of supply chain management has become richer in the past decade. The competing or cooperating entities in the business world are no longer individual companies, but chains of interrelated companies who are all geared towards a mutual goal, which is defined for the whole chain. The move towards chain-based business entities (supply chains hereafter) has had numerous merits and has also contributed to the emergence of new complications and problems that never existed before. Moreover, the evaluation of the performance of supply chains has opened up doors to new research opportunities and managerial decision making challenges. It is an interesting issue for researchers and practitioners to evaluate the performance of supply chains.

The Supply Chain Council has published a number of reports by the name Supply Chain Operations Reference model (SCOR model), in which a number of performance metrics have been introduced as performance attributes. Through these performance attributes, the performance of supply chains can be evaluated and compared against different benchmarks. Performance attributes are measured by indicators called *key performance indicators* (KPI). One of these performance attributes is responsiveness. The SCOR model 10 published by the supply chain council [19] defines responsiveness as follows; "The *responsiveness* attribute describes the speed at which tasks are performed. Examples include cycle-time metrics. The SCOR KPI is Order Fulfillment Cycle Time. Responsiveness is a customer-focused attribute."

As introduced in the statement, the key performance indicator of this attribute is order fulfillment cycle time, which measures responsiveness from the customer's point of view. As mentioned before, firms seek competitive advantage by quoting delivery times to customers. It is important for businesses to find out what is the optimal delivery time to quote to the customer. This problem is mostly relevant in Make-to-Order (MTO) and Assemble-to-Order (ATO) environments, where the goods are not purchased off the shelf. In these environments, the production or the preparation for the good/service delivery is triggered by the customer's order.

It should be noted that deriving a conclusion about an optimal or close to optimal delivery time quote could sometimes be quite difficult. This is due to the fact that the completion time at each stage of the supply chain is usually a random variable and involves uncertainties. Therefore, the uncertainty resulting from each stage will further increase the uncertainties involved in the delivery time of the whole chain. As a result, any valid statement of a quoted delivery time should always be coupled with the probability of achieving that time by the supply chain. In other words, the rate of satisfied demand or reliability should also be considered in determining the market share gained by the firm. The satisfied demand in this context refers to customers whose orders are delivered at or before the promised delivery time.

1.4 Organization and Overview

This thesis is organized as follows. Chapter 2 presents a brief introduction to the application of game theory in time-based competition. Later in that chapter, a survey of the literature in the field of game theory and time-based competition is presented. This section is categorized into three parts based on the decision variables, according to which the firms compete in the context of time. In Section 2.2 of this chapter, we will present the building blocks of modelling for studying time-based competition using game theory. The demand models are being discussed and the notion of quality of service is introduced and presented. Next we discuss supply chain design and explain how the quality of service is affected by the design of the supply chain.

In Chapter 3 we will present and solve three models of time-based competition between firms that have more than one stage of production (supply chains). First, a model of duopoly with supply chains that are competing based on their PDT is presented. In this section, it is assumed that each firm has two serial stages of production with identical production rates in both stages. The next section deals with a problem with the same structure but with firms that have k serial stages of production with identical production rates at all stages. Next, we will study a similar problem of a duopolistic market, where each firm has two serial stages of production but the production rate at each stage is different. The Nash equilibrium and optimal strategies for the firms are derived in each section. Following each section, we present numerical examples based on the analytical results of that section. Using the numerical examples, we construct a sensitivity analysis for each game and study the effect of changing parameter values on the outcome of the game as well as players' optimal strategies. The chapter ends with a conclusion and suggestions for future research in that context.

In Chapter 4, we will study a game in a duopolistic market with two players that are competing for the business of one customer. The firms compete by quoting a promised delivery time to the customer. Only one firm will win the business of the customer. Once the business is won by a firm, the other firm is out of the game and the winning firm will be rewarded for selling the product. In addition to that, the winning firm will now incur production and tardiness costs. Quoting smaller PDTs will increase the chance of winning the business but will increase expected tardiness costs. On the other hand, quoting larger PDTs will decrease the chance of winning the business but will also decrease tardiness costs if the business is won. This trade-off situation, along with the uncertainty involved in determining the winning firm creates an interesting model that will be discussed in this chapter. We will characterize the best-response curves of the firms and will present the Nash equilibrium of the game. This chapter concludes with a numerical sensitivity analysis as well as conclusions based on the chapter findings.

The games introduced in all the chapters so far were games with complete information. In Chapter 5, however we will study a game with incomplete information. In this chapter, we build upon a model that was introduced by Gerchak and Parlar [24]. However, here we assume that one of the players has incomplete information about the objective function of its opponent. Games with complete and incomplete information will be explained in more detail in this chapter. This chapter will also include the presentation of the best-response curves as well as the Nash equilibrium of the game. It will conclude with numerical examples, sensitivity analysis and conclusions and suggestions for future research topics in this regard.

Finally, in Chapter 6, we will summarize all the work and findings of the thesis and will present the future research opportunities for all the three problems that will be studied in this work.

Chapter 2

Time-Based Competition and Game Theory

This chapter consists of a literature review for models of time-based competition studied with game theoretical tools and methodologies. In this chapter we will study markets, where firms compete by deciding upon different attributes including promised delivery time, performance, capacity (rate of production), price, etc. In Section 2.1 we will explore the body of literature in the areas of time-based competition and game theory and the results are presented.

This chapter also prepares the ground work for the following chapters by focusing on the introduction of issues and ideas that will be useful in model building in those chapters. In Section 2.2, we will discuss issues such as order completion rules and supply chain design, the relationship between promised delivery time and quality of service as well as the demand model. In this section, two examples of modelling the market demand from the literature will be presented and explained.

2.1 Introduction and Literature Review

Game theory is defined by Straffin [61, p. 1] as the "*logical analysis of situations of conflict and cooperation*". He defines a game as a situation, in which,

- There are at least two *players*. A player may be an individual, a company, a supply chain, or a nation
- Each player has a set of actions or *strategies* to choose from
- The strategies chosen by all players will determine the *outcome* of the game
- Each outcome is associated with a set of *payoffs* to each player in the game

Game theory then determines how players should rationally play the games considering their strategies, payoff and game outcomes. For complete reviews of the application of game theory in supply chain management we would refer the reader to the reviews performed by Cachon and Netessine [13] and Leng and Parlar [44].

In the concept of time-based competition, the players could be individual production facilities, service providers, an entire supply chain or a stage in a supply chain, offering a product/service to a time-sensitive market. It is important to note that the market may be sensitive to other factors as well (like price or quality), but in this study we will focus on markets that are time-sensitive. We will refer to this market as a *time-based market*. Based on the nature of the firm, it may use different strategies to position itself in the time-based market. It may concentrate on capacity allocation (like in Kalai et al. [39] and Shang and Liu [53]), price (like in Armony and Haviv [4]), on a combination of capacity and price (like in Allon and Federgruen [1] and Cachon and Harker [12]), on their inventory management (like in Li [45]), or any other tool that would serve as their strategy in the time-based market. The outcome of the game will be determined by the fraction of demand or the market share that each server or firm serves. Each market share translates into a level of profitability for the firm that is considered to be the firm's payoff in that outcome.

There is a rich body of literature that studies the strategies and outcomes of games played between two or more firms in a time-sensitive market. This stream of research seeks to find the optimal strategies (price, capacity, inventory, etc.) for the firms that are competing against each other for a greater market share in a market that is sensitive to time.

2.1.1 Competing with Capacity Decisions

One of the earliest papers that studied the problem of capacity decisions for competing firms/servers in a market, where time is a competitive advantage, was the work of Kalai et al. [39]. In this paper, the authors introduce the two new concepts of *competitive game of servers* and *the market share*. They study a situation with two servers serving customers who are sensitive to the speed of service. The servers have control over their service rates at a cost with a convex cost function. The authors give a closed-form expression for the market share of each server depending on the service rates of both servers and prove the conditions for three types of Nash equilibria. They show that symmetric and non-symmetric Nash equilibria may exist depending on the revenue for serving each customer as well as the cost of maintaining the service rates. From this paper, a stream of research originates that looks at the competition of servers/firms in markets with time-sensitive customers.

Ching et al. [15] extend the work of Kalai et al. [39] by studying the same model but allowing for multiple servers competing in the system. They only look at the situation where the queueing system is stable or the total service capacity is greater than the mean rate of demand into the system. They show that when the marginal cost of serving the $1/n$ fraction of the customers is low enough, then there is a unique and symmetric Nash equilibrium $(\mu^*, \mu^*, ..., \mu^*)$, where μ^* is the service rate of each server.

They also present a numerical example for a system with three competing servers and derive the value of μ^* as a function of the Poisson customer arriving rate. Gilbert and Weng [26] build upon Kalai et al. [39] by studying a case, where two independent servers compete for profit under the coordination of an agent. The servers decide upon their service rate and the coordinating agent determines the amount to pay to each server for serving each customer, as well as the rule for allocating customers to the servers. The coordinating agent is interested in minimizing its cost while keeping the service time below an exogenously defined level. The rules for customer allocation is either to have a common queue for the whole system or to have two separate queues for each server. The authors find the equilibrium service rates for each case and compare them against each other. They find that the rates in equilibrium for the case with separate queues are higher than that of the common queue system. They argue that this is due to an intensified competition between servers to increase their service rates in the case of separate queues, which is weakened when there is one common queue for both servers. By coordinating the customer allocation in the system, the agent ensures that the waiting time for customers is within an exogenously given bound.

Christ and Avi-Itzhak [16] extend the work of Kalai et al. [39] by studying the same model and assuming that the customers may not enter the waiting line with probability $1-\theta_n$, when there are n customers in the system. The term θ_n depends on the number of customers in the system waiting or getting served. The authors show that the proportion of arrivals to each server is strictly increasing and concave in the server's own rate and also is decreasing in the other server's rate. They also show that there is a unique Nash equilibrium in the system which is symmetric under the assumption that the cost function is convex and increasing. In other words, they show that when the cost function c is convex and increasing on $[0, \infty)$, there exists a μ^* such that (μ^*, μ^*) is the unique Nash equilibrium for the game (rates for both servers are equal in equilibrium). Avi-Itzhak et al. [5] extend the work of Christ and Avi-Itzhak [16] by showing that a globally optimal solution is strictly superior to the Nash equilibrium found in their model. They use a linear penalty term for each server to force the Nash equilibrium to move to a new symmetric point yielding smaller service rates for each server. They show that the whole system will incur losses due to a longer waiting line and leaving customers. However, this loss is compensated by the servers maintaining a smaller service rate. Although their model produces smaller revenues for each server, the overall system is better off and generates higher profits. They also show that when the service rates are not observable by an external entity, it is still possible to force them into a Nash equilibrium that is globally optimal. This is done by imposing a linear penalty (or reward) on observable random signals received from the servers. For examples of using linear reward/penalty functions in decentralized supply chains to make a global optimal become a Nash equilibrium see Golany and Rothblum [27].

Other studies of competition with service rates (capacity level) include the works of Ho and Zheng [36] and the recent paper of Shang and Liu [53]. In both of these works, the authors assume that firms compete by determining their performance level or their service rates. Other than that, firms announce their promised delivery time. Therefore, we will discuss these two papers in more detail later in this study.

Some firms may also manipulate their inventory design in order to become time competitive. For example Li [45] studies an oligopolistic game, in which customers react to early delivery. In his model, customers can place orders with multiple firms and then complete the purchase with the firm that delivers the products at the earliest time. This model compares the make-to-order (MTO) and the make-to-stock (MTS) design in their effect on the response time in duopolistic markets, monopolistic markets and demand sharing markets. In the demand sharing market, the placing of multiple orders is not allowed and customers cannot differentiate between the delivery performances of the firms. They find formulas similar to the newsvendor model formulas and show that companies have large incentives to produce in make-to-stock fashion in oligopolistic time-sensitive markets. This incentive decreases in monopolistic and demand sharing markets, in the same order.

2.1.2 Competition with a Combination of Price, Performance and Other Attributes

Some authors include the price of the services or goods as a decision variable taken by the firms. Sometimes price is the only decision variable, and sometimes it is combined with other factors that can be controlled by the firms. Li and Lee [46] show the importance of having a more responsive system in achieving higher profits and a greater market share. They study a system with customers that react to delivery speed as well as price and quality. They study a two-firm system and assume that customers join a firm that would maximize their utility value function that takes into account the price, quality as well as the delivery speed provided by that firm. This utility function is referred to in the literature as the full price, incorporating price and other factors into a single value full price (see the two papers by Allon and Federgruen [1] and [3]). They find the demand rate (throughput rate) of each firm and find conditions for the existence of Nash equilibria. They report that if a firm has a faster delivery speed, in equilibrium the firm will serve a greater share of the market and will also charge higher prices than its competitor. This translates into higher profitability for the faster firm.

Lederer and Li [42] study a system of firms that compete in a market with delay sensitive customers. The firms are modeled as $M/G/1$ queues and have control over their prices, production rates as well as their scheduling policies to specify how the jobs are sequenced. They study the effects of time-based competition on prices, demands and companies' profitability, and report that a unique equilibrium exists. They show that firms with faster delivery and lower variability and costs always enjoy a larger market share, higher utilization of capacity and higher profits. They also study the behavior of different types of customers with regard to their delay sensitivity and show that in equilibrium, customers that are more delay sensitive receive faster services but will also pay higher prices.

Many researchers combine price with other attributes and present the term *full price*, that aggregates all those attributes into one. Cachon and Harker [12] study a duopolistic market with demand that is sensitive to a full price. The full price of the firms are determined by their price (explicit fee) and expected operational performance (here measured by total time in system). Also, the firms face scale economies, where each firm's cost per unit of demand will decrease as demand increases. In other words, the cost function of the firms is concave. The firms determine their prices and their expected operational performance as their decision variables. Considering the firms as $M/M/1$ queueing entities, the authors find the equilibrium full price for the game. They also find that in the equilibrium full price, the firm that has lower costs may serve a greater share of the market and also charge higher prices. They also consider the case, where firms will seek opportunities to weaken the price competition. They suggest that one of these strategies is outsourcing their operations to a supplier. They add a supplier to the model that can take responsibility for the operations at each firm (with dedicated resources for each firm) and charge a price for each customer that is served. They define a two-stage game with outsourcing allowed and show that when scale economies exists, firms have strong incentives to outsource.

Ha et al. [29] consider delivery frequency decisions as a competitive advantage in time-based competition, similar to delivery speed (e.g., see, Kalai et al. [39]). They highlight two important roles of delivery frequency in time-based competition. By delivering more frequently to the customers, the suppliers can find competitive advantage in their business. Also, since a more frequent delivery by the suppliers reduces the inventory of the customer, the suppliers will be able to offer complementary services to the customers and improve their profitability. The authors define two three-stage non-cooperative games between two suppliers and a customer. Based on their models, the customer could be sensitive to price or delivery frequency. In their first game, they assume that the suppliers handle logistics and compete only on delivery frequencies. In their second model, they assume that the customer handles logistics (e.g., comodity-like goods) and thus the suppliers compete only on price. The authors compare their results in their models and show that high frequency delivery is a source of competitive advantage in time-based competition. Based on their model, the customer is better off when the suppliers handle logistics and the suppliers are better off when the customer takes that role.

Armony and Haviv [4] study a duopolistic market, where customers are only sensitive to the full price. Full price in their model includes the service cost plus the expected waiting costs. They assume two firms that offer a homogenous service to a market of customers that belong to one of two groups based on their patience level in receiving service (time-sensitivity). Upon arrival, customers can join the queue at one of the two servers, or alternatively leave and avoid getting served (balk). The authors define a two-stage game for studying this model. In the first stage, the customers compete by observing the prices (and not the queue lengths) and join one of the queues, or leave. The decision made by every customer to join each queue has an effect on the service experience all other customers receive. In the next stage, the firms observe the allocation of customers and decide on the prices to charge. The authors study Nash equilibria in the model and find that the first level of the game (played between customers) has mixed strategies but the game played between the firms has pure strategies. For a review on the definition of pure and mixed strategies we refer the reader to Straffin [61, pp. 3–21]. In the game between the customers, their findings show that the full price equilibrium paid by the less time-sensitive (more patient) customers is less than or equal to that paid by impatient customers. In the game played between the firms, the authors cannot predict what prices the firms will charge because of the existence of multiple equilibria or occasionally continuous asymmetric equilibria. When finding multiple Nash equilibria, the optimal equilibrium is one that is *Pareto optimal*. This concept is thoroughly explained and discussed in Straffin [61, pp. 65–71].

Allon and Federgruen [1] study the situation where N firms are competing in the same market, where customers are sensitive to the price level of the industry and the steady-state waiting time standard. The waiting time standard is announced by the firm and committed to by adjusting the capacity level at each firm. The firms position themselves in the market by choosing their prices and service levels. Service level is defined as the difference between a given upper-bound benchmark for waiting-time standard and the actual waiting time that customers experience. They model the firms as $M/M/1$ queueing facilities and study three types of competition between firms. In the first type (service-level first or SF), they define a two-stage game, where the firms select their waiting-time standards and announce them and then will select their prices in the second stage. In the second type (price first or PF), the firms will reverse these decisions in the two stages. Finally, the third type (simultaneous competition or SC) studies the situation, where the firms select their waiting time standards and prices simultaneously. Their research is different from other non-cooperative models before in that, they do not assume a full price that is an aggregate of price and waiting time the customers experience. They find the equilibrium in each type and compare them against each other. They show that (PF) and (SC) games have the same set of equilibria. Therefore, announcing the prices first does not have an effect on their selection of waiting time standard in equilibrium. They also show that the (SF) model results in higher prices, lower waiting times and higher demand for all firms. This shows that the firms can enjoy better profitability in equilibrium if they announce their waiting time standards and then choose their prices at a later stage.

The work of Allon and Federgruen [1] is generalized upon by Allon and Federgruen [3]. In their model, the customers are sensitive to the prices as well as the waiting times. However, the customers are assigned to different classes. Different classes are offered different prices and waiting times by the firms. The firms choose their prices and waiting times for all customer classes as well as their capacity levels and priority discipline, which will enable them to meet their promised waiting time. They assume firms are $M/M/1$ queueing facilities and the demand streams according to a Poisson process. They define three games of competition based on price only, waiting time standard only and price and waiting time together. They show that in each game a Nash equilibrium exists under minor conditions regarding demand volumes and minimum waiting time standards and show how the equilibria vary as a function of cost and other given parameters. Finally, they compare the equilibria with the situation, where the firms service each customer class with dedicated service facilities as opposed to pooling services. Based on their model, the firms will always benefit from service pooling in equilibrium.

In her recent paper, Parra-Frutos [50] studies the problem of firms that are time competitors and compete in a perfect competition market offering a homogeneous service to customers who are sensitive to price and waiting time. She models a firm as an $M/M/1$ queue and assumes that customer arrivals react to a combination of waiting time and price (a full price). Based on this reaction, in equilibrium, the profit-maximizing firms will reach a full price that is dictated by the market and is equal for all firms in the perfect competition market. She argues that firms may then use different combinations of price and waiting time to assume the full price dictated by the market. She concludes that in case of a convex service capacity cost function, an equilibrium may exist. But an equilibrium does not exist if the cost function is concave. Other papers that study the effect of a combination of price, service rate and other attributes include the papers of So [56] and Allon and Federgruen [2]. These studies also consider price in addition to other service attributes in determining the outcome of the game. Since a promised delivery time is also involved in these studies, we will discuss them in more detail later in this chapter.

2.1.3 Competition with Promised Delivery Time

Some firms seek to find competitive advantage in the competition in time-sensitive markets by promising (quoting) delivery times (service completion times) to customers. They usually promise to reward the customer or compensate her for the delay if the delivery point is beyond their promised delivery time. Firms are also aware of the fact that although an attractive promised delivery time could work as a strong competitive advantage, it can as much hurt the reputation of the company if it fails to commit to it. For example, Hanson [30] mentions in his paper that Domino's Pizza guaranteed that any order will be delivered within 30 minutes or it will be free of charge, and used this as a powerful marketing strategy for many years . Nevertheless, the company had to drop the advertisement in 1993 for fear of reckless driving by its delivery crew following an accident involving a delivery agent of the company. The strategy, however, gave the company great competitive advantage over its competitors when it was in effect. Another example is the time guarantees made by the Federal

Express. FedEx offers promised delivery times including *next-day-delivery* or *next-flight-delivery* with a promise to refund the full payment in case of a late delivery. We quote from the Federal Express website "If your FedEx Express package is delivered even 60 seconds later than we promise, you get your money back. It's that simple.[1]".

The choice of delivery time commitment is a balance strategy between the marketing related issues (customer's perceived quality) and the operations related issues (capacity and process variability). Zeithaml et al. [69] explain that customer's expected delivery time can be determined by many factors. These factors include the price of the product or service, the reputation of the company, the marketing and communication between the firm and the customer, other service experiences of the customer from the firm, word of mouth, etc.

Some authors limit the customer's expected delivery time to the quoted delivery time announced by the firms (like in Ho and Zheng [36] and Shang and Liu [53]). They assume that the only channel of communication between the firm and the customer is done through an expected maximal delivery time [53], or an expected delivery time [36]. Ho and Zheng [36] argue that the customer's perceived delivery time is positively affected by the customer's actual delivery time, which is determined by both the demand rate and the capacity of the firm. Figure 2.1 depicts the diagram presented in the paper by Ho and Zheng [36]. Based on this diagram, the demand rate is also affected by the expected delivery time, and the delivery (service) quality. The perceived delivery time and the expected delivery time will mutually affect the quality of service.

Problems involving quoting delivery time (also referred to quoting leadtime or planned leadtime) have been studied by many authors. For example, Yano [68] looks at the problem of determining the optimal leadtime in serial production systems with stochastic procurement and processing times. In her study, she minimizes holding and tardiness costs, where a certain delivery time is quoted to the customer. In this work, Yano does not include tardiness costs in intermediate levels in her study. With a somehow different setting of costs, Yano [67] looks at the problem of finding optimal planned leadtimes in a serial production system where the leadtimes themselves are stochastic. Yano minimizes the sum of holding costs as well as tardiness costs and also rescheduling costs for intermediate stages in the production system that fall behind schedule. Yano also comes up with an algorithm that finds nearly closed form functions for the planned leadtime at each stage as a function of the planned leadtime at successor stages as well as the cost parameters. She also discusses the importance of rescheduling costs and how they would affect the models. For other examples of single-firm studies involving promised delivery time see Elhafsi [21], Hnaien [35], Ould-Louly and Dolgui [49], Song et al. [58], Song and Yao [59] and the references therein.

Several studies have been performed in the competitive environment, where two or more firms are competing in a time-sensitive market and use promised delivery time

[1] http://fedex.com/ca_english/services/moneyback.html

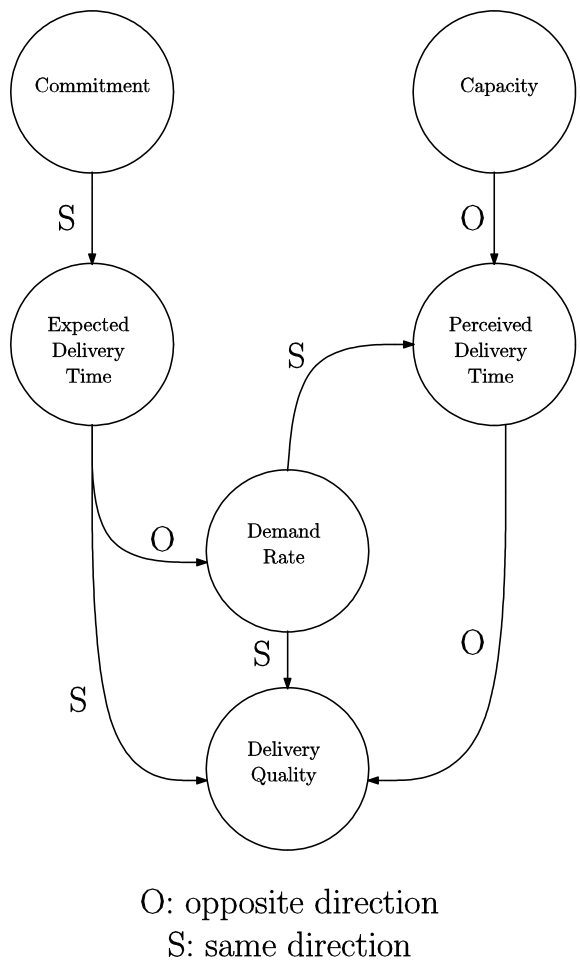

Figure 2.1: The integrative framework presented by Ho and Zheng [36].

as a competitive strategy.

So [56] studies the effect of price as well as delivery time guarantees in time-based competition in an oligopolistic situation. He assumes that customers are sensitive to both the price and the delivery time guarantee that firms offer. He extends the work of So and Song [57], in which the authors study the effect of delivery time guarantees in a single firm setting, into a multiple firm setting. He studies the optimization problem in the single firm setting first and proves that a unique Nash equilibrium exists in the multiple firm setting. This Nash equilibrium can be computed using an iterative procedure suggested by the author. Based on his work, the unique Nash equilibrium in the multiple firm setting behaves similarly to the optimal solution in the optimization problem when all of the firms are homogeneous. The profit of the firms in equilibrium, however, does not behave similarly to the optimal profit in the single firm setting and in the work of So and Song [57]. Finally, the author exploits numerical experiments and derives managerial insights in situations, where the firms are not homogeneous and can differentiate themselves by their capacity (size) and unit operating cost (efficiency). He shows that the higher capacity firms can provide lower time guarantees, whereas the firms with lower unit operating costs are able to offer lower prices to the customer.

In their paper, Ho and Zheng [36] look at a duopolistic market, where customers are sensitive to both the expected maximal delivery time and service quality. Service quality is defined as the conformance of the expected delivery time to the customer's perceived delivery time. The expected maximal delivery time is announced by the firms to attract customers into their service. Customers are assumed to be satisfied if their perceived delivery time is less than the expected maximal delivery time. The authors develop a model to study the effect of delivery time commitment and service quality on the demand share of the firm. They find a closed-form solution for the optimal delivery time commitment when the firms have ample capacity and congestion does not exist. The model also captures the effect of the demand share of the firm and its process variability on delivery quality in the existence of congestion for a duopolistic market. The authors prove the existence of more than one Nash equilibria and show that this game is similar to the well-known prisoners' dilemma game.

In a recent paper Shang and Liu [53] studied an oligopoly market with customers sensitive to both the promised delivery time as well as the rate of on-time delivery made by the firms. They refer to the rate of on-time delivery as the quality of service and study the trade-off between the quality of service and the promised delivery time. For allocation of demand, they use the multinomial logit (MNL) attraction model. Basuroy and Nguyen [6] argue that the multinomial logit model is able to capture customers' behavior for equilibrium analyses for marketing decisions. Shang and Liu [53] assume that firms position themselves in the market based on their promised delivery time as well as their investment in capacity (their capacity). They assume there are two kinds of competition; at the marketing level and at the strategic level. Firms compete at the marketing level based on their promised delivery time in the existence of a quality of service that is exogenously determined. In another model, the authors define a two-stage game, in which the firms compete in capacity in the first stage and in promised delivery time in the second stage. They find a unique Nash equilibrium in their first model and show the existence of two different types of Nash equilibria in the second model. Another important contribution of this paper is the introduction of an index of time-based competitive advantage (ITCA), which is defined as the residual capacity at the marketing level competition or the ratio of unit revenue to unit capacity cost at the strategic level game. They show that usually firms with higher ITCA have a larger share of the market. In terms of the customer's sensitivity to both the promised delivery time and the quality of service, their model is very similar to that of Ho and Zheng [36].

Some authors assume that firms may announce a price as well as a promised delivery time. Allon and Federgruen [2] investigate the effects of the servers' queueing facility design on the service industry's competitive behavior. They study an oligopolistic market, where the customers are sensitive to the waiting time as well as the price. The firms position themselves in the market by announcing a price as well as a service level (or the waiting time standard). They define three games; (a) firm's competing on price only when the service level is exogenously given, (b) firms competing on the service level only under exogenously given prices, and (c) firms competing on both price and service levels. They propose an approach to study different single-stage queueing service designs from $M/M/1$ systems to the general form $G/GI/s$ servers. They show that the capacity cost function always exactly, or with a good approximation, belongs to a four-parameter class of functions. They prove the existence of a Nash equilibrium in these models when the cost functions are convex under different settings for the demand function.

Some researchers have investigated the performance of supply chains in markets, where time competition plays a role. For example Liu et al. [47] study a supply chain with two players (a supplier and a retailer) performing in a market that is both price and lead time sensitive. Both players in the supply chain are independent entities and try to maximize their profit. The authors structure a Stackelberg game with the supplier being the leader and the retailer being the follower. The promised delivery time (PDT) and the price announced to the customer are determined by the supplier and the retailer, respectively. They find the unique time-price equilibrium and provide closed-form formulas that calculate the optimal PDT and price. They also show that the supply chain performance when decentralized is usually poorer than that in a centralized supply chain with similar market conditions.

2.2 Time-Based Competition with Homogeneous Firms

Consider a market with M firms competing for customers' demand. The firms offer the same product or service to a homogenous market with random demand. A firm i is assumed to be a supply chain, where i denotes the index for the firm. Let the overall potential market size or the overall demand rate be denoted by Λ. Assume that λ_i is the demand rate that is captured by firm i. Therefore, firm i's long-run market share is equal to λ_i/Λ. It is important to note that λ_i is actually the mean arrival rate into the supply chain i.

We build upon the model introduced by Shang and Liu in [53] and Ho and Zheng [36] by assuming that each player is a centralized serial supply chain instead of a single firm. By serial supply chain, we mean a supply chain whose stages are aligned in a serial fashion and the flow of goods passes through every single stage of the chain.

2.2.1 Order Completion and Supply Chain Design

As mentioned above, each firm is a supply chain. For the sake of clarity, we emphasize that the firms are producers of a certain product and their products are assumed to be completely substitutable. The process of order submission and order completion are as follows. Orders are placed by the customers to the firm. Once the order is placed, the production is triggered and the order will traverse through the supply chain in the form of work-in-process until it is ready to be delivered at the last stage of the chain.

The production duration at each stage of the supply chain is assumed to be exponential with rate μ_{ij}, where i is the index for the supply chain and j is the index for the stage. For example, μ_{24} refers to the rate of production at the fourth stage of the second supply chain. Maintaining a production rate at each stage of the supply chain is costly. We define the cost function $c_{ij}(\mu_{ij})$ as an increasing and convex function in μ_{ij}. Also, the total cost for supply chain i will be defined as $C_i = \sum_j c_{ij}(\mu_{ij})$.

We are also assuming that the production rates for firms are fixed and will not change. The choice of production rates are strategical decisions and thus are part of firms' long term plans. This study focuses on decisions on time, which are made at the operational level. Therefore, strategical choice of production rates are beyond the scope of this study.

2.2.2 Quoted Leadtime and Quality of Service

Each supply chain quotes a delivery leadtime to the customer as the promised delivery time (PDT). Let's define T_i as the PDT of the supply chain i. Also, suppose that W_i is the random variable representing the actual waiting time that the supply chain i's customer experiences. This time period is actually the real time that elapses from the order submission until the order completion with firm i. This random time is denoted by the random variable W_i with p.d.f. q_i and c.d.f. Q_i. The quality of service (QoS) or the fraction of on-time delivery is defined as the probability that the actual delivery time is less than or equal to the promised delivery time and is denoted by the function $Q_i(T_i, \lambda_i) = \Pr\{W_i \leq T_i\}$, where λ_i is the demand rate captured by firm i. Also, assume that X_i is the random variable representing the actual production time of the supply chain with p.d.f. $f_{X_i}(t)$ and c.d.f. $F_{X_i}(t)$. Obviously, with the presence of congestion at the system, we expect the total time spent in the system to be at least equal to the total production time of the firm. In addition to the total production time, the total time spent in the system includes all the time spent in the queue before entering the system, as well as all the time spent in the queues before each stage at the supply chain. Therefore, we have,

$$E[W_i] \geq E[X_i].$$

Taking into account the congestion effect, we assume that $Q = Q_i(T_i, \lambda_i)$ is non-increasing in λ_i. We assume that Q is non-increasing in λ_i because in presence of congestion, more customers in the system will never lead to a more responsive system when capacity remains unchanged. In other words, the quality of service (Q) will never increase when demand (λ) increases when congestion exists and capacity (μ) is the same. If we assume that Q is independent of λ, the service is defined as uncongested. This means that the demand will not have an effect on performance and responsiveness of the system. For any given demand rate λ_i, the term $Q_i(\cdot, \lambda_i)$ is defined as the distribution function (c.d.f.) of the delivery time. This time duration includes the time spent in the queue as well as the time spent in the service delivery process itself.

2.2.3 The Demand Model

In their paper, Allon and Federgruen [2] propose two different types of demand into their system as the mostly frequently used models. These demand types are the *separable demand functions* and the demand functions given by an *attraction model.* Here, we will briefly introduce these two types.

The Separable Demand Functions

Referring to the work of Allon and Federgruen [1], the authors argue that a suitable demand model should be able to incorporate the following attributes; (a) the price of service, (b) the waiting-time standard, and (c) other attributes. For this to happen, the demand function will be a general system of functions that characterize the behavior of demand towards any of these attributes. With each of these functions representing an attribute in the demand model, the authors refer to this type as separable demand functions. The demand function presented in Allon and Federgruen

[2] is as follows,

$$\lambda_i = a_i(\theta_i) - \sum_{i \neq j} \alpha_{ij}(\theta_j) - b_i p_i + \sum_{i \neq j} \beta_{ij} p_j,\ i = 1, ..., M, \tag{2.1}$$

where λ_i is the demand rate of firm i, p_i is the price of firm i, and θ_i is the service level of firm i. Also, $a_i(\theta_i)$ and $\alpha_{ij}(\theta_j)$ are positive functions relating firm i's demand to its own and the other firms' service levels, respectively. Similarly, b_i and β_{ij} are positive parameters relating firm i's demand rate to its own and the other firms' prices, respectively. It should be noted that the separable demand function in (2.1) cannot be negative. We assume that it gets the value zero for parameters that result in a negative demand value as follows,

$$\lambda_i = \begin{cases} \delta_i(\theta_i),\ i = 1, ..., M, \text{ if } \delta_i(\theta_i) \geq 0, \\ 0, \text{ otherwise,} \end{cases}$$

where $\delta_i(\theta_i) = a_i(\theta_i) - \sum_{i \neq j} \alpha_{ij}(\theta_j) - b_i p_i + \sum_{i \neq j} \beta_{ij} p_j$, $i = 1, ..., M$.

Example 2.1 *Assume we have two firms in the market. The following is an example of separable demand functions, when λ_1 and λ_2 are demand rates for companies one and two,*

$$\lambda_1 = 25 + 2\theta_1 - 4\theta_2 - 5p_1 + 4p_2$$
$$\lambda_2 = 25 + 4\theta_2 - 2\theta_1 - p_2 + 2p_1.$$

It is important to note that the demand rates will be zero for λ_1,$\lambda_2 < 0$.

The Demand Functions Based on Attraction Models

A very popular and frequently used type of demand models is the attraction models. As explained in Bell et al. [7], an attraction model is based on a relationship that has the structure (us)/(us+them). In this notion, "us" and "them" represent the attractiveness of each firm and that of all the other firms in the market. As mentioned by the authors, a common approach is to relate the attractiveness to the amount of marketing effort or the service level or quality. Bell et al. [7] present four assumptions that should be satisfied in order for the attraction model to be valid. Assuming that **a** represents the attraction vector and there are M firms in the market, these assumptions introduced by Bell et al. [7] are as follows,

- The attraction vector is non-negative and non-zero

$$\mathbf{a} \geq 0 \text{ and } \sum_{i=1}^{M} a_i > 0,$$

where the condition $\mathbf{a} \geq 0$ ensures non-negativity and $\sum_{i=1}^{M} a_i > 0$ guarantees that the vector is non-zero. The components of a non-negative vector are either zero or positive. A non-zero vector is a vector with at least one non-zero component.

- A firm with zero attraction has no market share or demand rate

$$a_i = 0 \Rightarrow \lambda_i = 0,$$

where λ_i is the demand rate of the market of firm i.

- Two firms with equal attraction have equal market shares

$$a_i = a_j \Rightarrow \lambda_i = \lambda_j.$$

- If the attraction of any firm j is increased by a fixed amount, the attraction of another firm i, where $i \neq j$, will decrease.

Then according to Bell et al. [7], the market share (demand rate) that will satisfy the four assumptions, is given as follows,

$$\lambda_i = \frac{a_i}{\sum_{j=1}^{M} a_j},\ i = 1, ..., M,$$

where there are M firms in the market.

For a more detailed analysis of varieties of market share models and their applications see Cooper and Nakanishi [17] and Leeflang et al. [43].

A number of recent papers including the works of Shang and Liu [53] and Ho and Zheng [36] use the multinomial logit (MNL) model. This model as presented in Ho and Zheng [36] can be introduced for a market with M non-cooperative firms as follows,

$$S_i = \frac{e^{U_i}}{\sum_{j=1}^{M} e^{U_j}},$$

where S_i is the market share of firm i and U_i is the customer's utility for firm i's service. Also using this model, we assume that the customer population is homogeneous in its utility function.

In order for the customers to select between the firms, we should devise a utility function that assesses the attractiveness of the firms to the customer. We assume that PDT and QoS are two important factors in this process. Therefore, the utility function should depend on both of them. We assume that price is not involved in the decision making of the customer. Shang and Liu [53] argue that in many industries, where the production or service provision contains routine processes and the technology is mature, price is usually not the focus of competition. They assume that price is exogenously given and is not determined by the firms. In this model, we follow their reasoning and restrict our study to time-based markets, where price is not a competitive factor. Similar to the model in [53], we will aggregate price with other attributes of the product that may affect the customer's decision and call them the passive attribute of the utility function. Similarly, we will define the customer's utility function as follows,

$$U_i(T_i, \lambda_i) = \beta_{0i} - \beta_T T_i + \beta_Q Q_i(T_i, \lambda_i), \tag{2.2}$$

where β_{0i} is the utility from the passive attributes of firm i, β_T measures the sensitivity of the customer to the promised delivery time and β_Q measures her sensitivity to the quality of service of firm i. Also, we assume that β_T and β_Q are positive and independent parameter values. Therefore, the utility of the customer is monotone decreasing in its promised delivery leadtime and monotone increasing in its quality of service. Note that from the coefficients of the utility function (β_{0i}, β_T and β_Q), only β_{0i} changes from firm to firm. This is because the other attributes of the firm (that are represented by β_{0i}) can be different from one firm to another. This is thus, a firm-based attribute. The other two coefficients, β_T and β_Q, represent the degree of customer sensitivity towards the quoted leadtime and the quality of service, respectively. These preferences will not be different from one firm to another. Therefore, we would refer to these preferences as customers-based attributes.

The utility function presented in (2.2) measures the attractiveness of a firm for customers. This function determines the market share of each firm. The higher this value is, the more attractive that firm is for the market. As a result, firms would desire to have as high a utility function as possible. We are assuming that this function has the range $[0, \infty)$, where a utility function of zero means that the firm attracts no customers. We are also assuming that utility function cannot be negative, and that negative values for the expression result in a utility function of zero. Therefore, we have,

$$U_i(T_i, \lambda_i) = \begin{cases} \beta_{0i} - \beta_T T_i + \beta_Q Q_i(T_i, \lambda_i), \text{ if } \beta_{0i} - \beta_T T_i + \beta_Q Q_i(T_i, \lambda_i) \geq 0, \\ 0, \text{ otherwise.} \end{cases}$$

Assume that Λ is the total demand rate of the market. Assuming that $\lambda_i \equiv \lambda_i(\mathbf{T})$ is firm i's demand rate, then firm i's long-run market share is represented by λ_i/Λ. Also, $\mathbf{T}$ is the vector containing the promised delivery times from all firms. The MNL model will allocate the following demand rate to firm i when there are M non-cooperating firms in the market,

$$\lambda_i = \Lambda \frac{e^{U_i(T_i, \lambda_i)}}{\sum_{j=1}^{M} e^{U_j(T_j, \lambda_j)}},$$

yielding,

$$S_i \equiv S_i(U_i) = \frac{e^{U_i(T_i,\lambda_i)}}{\sum_{j=1}^{M} e^{U_j(T_j,\lambda_j)}},$$

where S_i is the market share for firm i. It is important to mention that since the total demand rate of the market Λ is fixed, an increase in the demand rate of one firm translates into a decrease in the demand rate of at least one other firm in the market. So we have the following as the relationship between the market share and the demand share,

$$\lambda_i = \Lambda S_i(U_i).$$

Chapter 3

Duopoly with Homogeneous Supply Chains Competing with Promised Delivery Time

In Chapter 2, we explained how the quality of service is calculated and showed how it will be different for different designs of supply chains. In this chapter we will discuss a duopolistic market with two firms competing based on their promised delivery time (PDT). Three different settings for the design of serial supply chains will be introduced in Sections 3.2, 3.3 and 3.4. In Section 3.2, two serial supply chains with two stages and identical production rates are studied. Section 3.3 looks at the design of the supply chains by introducing k stages into each firm. Finally, Section 3.4 considers two supply chains with two stages, where each stage has a different production rate. In each section, the outcome of the game, along with the closed-form optimal solutions by each player and the payoffs in equilibrium are presented. Each section is followed by a section of numerical examples, where the findings of the previous sections will be confirmed through numerical examples. This section will also include sensitivity analysis, where a number of numerical examples with different attributes are solved using the model to construct a sensitivity analysis that will give us managerial insight into the behavior of the systems.

Finally, Section 3.5 concludes the chapter by summarizing the findings and also presenting areas for future research in this context.

The models studied in this chapter are based on the models introduced by Ho and Zheng [36] and Shang and Liu [53]. In both mentioned works, the models study firms that have only one production (service) stage. This chapter contributes to the literature by extending their works and analyzing firms with multiple production stages. As mentioned above, models with two- and k-stage supply chains with the same production rate and a model with a two-stage supply chain with different production rates are explored in this chapter. This study is different from those of Ho and Zheng [36] and Shang and Liu [53] in that the quality of service assumes more complicated forms when the firms have more than one production stage. Regarding a constraint for the quality of service in this chapter, we are taking an approach similar to the one in Ho and Zheng [36] and include no external QoS constraint (no market entrance requirement regarding the QoS). The more complicated expressions for the quality of service result in more complicated expressions for the Nash equilibria and impose new conditions on the problem for the existence of a unique Nash equilibrium. The problems studied in this chapter are also a better reflection of the reality than the previous models as in reality, firms rarely have only one stage of production. An example for this system is the delivery of a package by UPS. The process of sending a package includes multiple stages that are independent of each other and take different amounts of time to complete. Therefore, for a company like UPS it would be more beneficial to have a model that studies such processes where the procedure of service delivery (production) takes place through more than one stage with difference rates at each stage.

In addition to all of these contributions that we made from the previous works, we will also perform numerical sensitivity analyses and report on the behavior of the optimal strategies and Nash equilibria with respect to different parameter values in the problem.

3.1 Model Building

In this chapter, we are assuming that each firm is a supply chain producing a perfectly substitutable product. Let's assume each supply chain consists of two stages that are aligned in a serial fashion. We also assume that for firm i, the service time at each of the stages is exponential with parameter μ_i. Each supply chain quotes a delivery leadtime T_i to the customer as the promised delivery leadtime (PDT). Also, suppose that W_i is the random variable representing the actual time that the supply chain i's customer experiences. This total time includes all the time products spend in the queues as well as the time they spend in production. Also, assume that X_i is the random variable representing the production time for the supply chain. Not considering congestion, based on the assumption that the supply chain consists of two serial stages with exponential service times, we conclude that the total production time for the firm is Erlangian with parameters $(2, \mu)$ as explained in Kao [40, p. 21] and thus, its p.d.f. can be written as $f_{X_i}(t) = \mu^2 te^{-\mu t}$. Also, the expected duration of production and its variance will be $E[X_i] = 2/\mu_i$ and $var[X_i] = 2/\mu_i^2$.

The quality of service or the fraction of on-time delivery is defined as the probability that the actual delivery time is less than or equal to the promised delivery time and is denoted by the function $Q_i(T_i, \lambda_i) = \Pr\{W_i \leq T_i\}$, where λ_i is the demand rate captured by firm i ($Q_i(T_i, \lambda_i) \equiv Q_i(T_i, \lambda_i(\mathbf{T}))$). For any given demand rate λ_i, the term $Q_i(\cdot, \lambda_i)$ is defined as the distribution function (c.d.f.) of the delivery time in firm i. This time duration includes the time spent in the queue as well as the time spent in the service delivery process itself. Also, taking into account the congestion

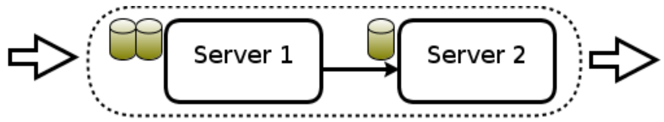

Figure 3.1: This picture shows the supply chain with boundaries depicted by the dashed line. The queues form before each stage of the chain.

effect, we know that $Q = Q_i(T_i, \lambda_i)$ is non-increasing in λ. This is because as the capacity is constant, a busier system will never become more responsive. In this case, a busier system is one with a higher demand rate λ.

To derive the expression for Q, we should emphasize that the total time spent in the system will be the determining factor for Q. This total time is affected by the assumptions we make in our modelling including the production times distributions. In our case, we are assuming that both stages have exponential production times with a common parameter μ. In addition to that, we are assuming that we are allowing queues to form in between the stages. This assumption makes the model more realistic as in reality, queues form before each stage of the supply chain. The queue before the first stage is the orders that are entering the system. The queue before the second stage is for the work-in-process (WIP hereafter) and has the unfinished products that have finished the process at the first stage and are waiting to enter the second stage of production or service facility. This is shown in Figure 3.1. In this picture, the two stages of the supply chain have a dashed line boundary. The two larger arrows on the left and right of the supply chain represent the arrival of orders and the departure of completed products, respectively. Each stage is a production facility of the chain and the items before each stage represents queues waiting to be served by that stage.

The expression for Q, is basically the distribution function of the total time spent in the system. In the supply chain under study, we may consider each stage separately to derive Q.

Table 3.1 summarizes the notations and symbols definitions used in the following chapters.

Lemma 3.1 *The distribution of total time in an $M/M/1$ system is exponential with parameter $(\mu - \lambda)$, where μ and λ are the exponential parameters for the service time and interarrival times, respectively.*

Proof. This has been shown in Gross and Harris [28, pp. 66–68] and Hillier and Lieberman [34, pp. 779–780] and has been used in many papers. For example, Ho and Zheng [36] introduce it in their paper as the distribution of total time in an $M/M/1$ queueing system. ■

Example 3.1 *Assume that the supply chain has only one stage with exponential service time with parameter μ. The demand rate to the supply chain is according to a*

Notation	Description
T_i :	The promised delivery time quoted by firm i
λ_i :	The demand rate into firm i
Λ :	The total demand rate of the market
μ_i :	Production rate at each stage of firm i
μ_{ij} :	Production rate at stage j of firm i
W_i :	Total time spent in firm i with p.d.f. q_i and c.d.f. Q_i
$Q(T_i, \lambda_i)$:	Quality of service at firm i, equivalent to $\Pr\{W_i \leq T_i\}$
β_{0i} :	Customers' sensitivity to passive attributes of firm i
β_T :	Customers' sensitivity to the promised delivery time of firm i
β_Q :	Customers' sensitivity to the quality of service at firm i
$U_i(T_i, \lambda_i)$:	Customers' utility function for firm i
ℓ :	The common allocation parameter between the firms
k :	Number of stages of production at each firm
$W(x)$:	The Lambert W function

Table 3.1: Summary of notations and symbols.

Poisson process with parameter λ. Then the expression for the quality of service will be readily computed as $Q(T, \lambda) = 1 - \exp\{-(\mu - \lambda)T\}$ as the c.d.f. of the total waiting time in the system W according to Hillier and Lieberman [34, p. 780].
Defining X as the total service time, we can see that the expression for $F_X(T) = \Pr\{X \leq T\}$ will be equal to $F_X(T) = 1 - \exp\{-\mu T\}$. Intuitively, we expect the probability $F_X(T)$ to be greater than $Q(T, \lambda)$ as the random variable W includes waiting in the queue in addition to the service time. This intuition is confirmed in this example as we have $F_X(T) > Q(T, \lambda)$ for $\lambda > 0$.

Lemma 3.2 *The steady-state distribution of interdeparture times in an $M/M/1$ system is exponential with parameter λ, with the condition that the system reaches the steady-state or $\mu > \lambda$. The interarrival times are exponential with parameter λ and the service times are exponential with parameter μ.*

Proof. The proof of this lemma appears in Gross and Harris [28, pp. 168–169] and Burke [11]. This result was originally proven by Burke [11]. For a simple proof see Gross and Harris [28, pp. 168–169]. Based on their proofs, the interdeparture times in an $M/M/1$ system is also exponential with parameter λ, with the condition that the system reaches the steady-state or $\mu > \lambda$, where λ the parameter for interarrival times and μ is the parameter for service times. In other words, the interarrival times and interdeparture times at steady-state in the $M/M/1$ queueing system are both exponential with the mean time $1/\lambda$. ∎

Proposition 3.1 *The c.d.f. for the total time spent in the system in the supply chain under consideration is equal to $Q(T, \lambda)$, where it is defined as follows,*

$$Q(T, \lambda) = 1 - e^{-(\mu-\lambda)T} - \mu T e^{-(\mu-\lambda)T} + \lambda T e^{-(\mu-\lambda)T}. \tag{3.1}$$

Proof. As explained before, the supply chain under study has two stages with the same service time distributions. Based on Lemma 3.1, the distribution of total time spent in the system at stage one of the supply chain is exponential with parameter $(\mu - \lambda)$. Also, from Lemma 3.2 we know that the arrival rate into stage two of the supply chain is exactly the same as the arrival rate into the first stage of the chain. Therefore, the distribution of total time spent in both stages are the same, exponential with parameter $(\mu - \lambda)$. Assume Y_1 and Y_2 are random variables for the total times spent in stages one and two (including the queue times), respectively. Let f_{Y_1} and f_{Y_2} be the p.d.f. of Y_1 and Y_2, respectively. Now we have,

$$f_{Y_1}(t) = f_{Y_2}(t) = (\mu - \lambda)e^{-(\mu-\lambda)t}.$$

Based on the assumption, if W represents the total time spent in the system, we should have $W = Y_1 + Y_2$. To find the probability function of W we can use the Laplace transform. Since W has the c.d.f. Q, using the Laplace transform we can write,

$$\widetilde{Q}(s) = \frac{1}{s}\widetilde{f}_{Y_1}(s)\widetilde{f}_{Y_2}(s),$$

where $\widetilde{Q}(s)$, $\widetilde{f}_{Y_1}(s)$ and $\widetilde{f}_{Y_2}(s)$ are the Laplace transforms of Q, f_{Y_1} and f_{Y_2}, respectively. We have,

$$\widetilde{Q}(s) = \frac{1}{s}\left(\frac{\mu - \lambda}{\mu - \lambda + s}\right)^2,$$

which after the inverse transformation will give,

$$Q(t) = 1 + (-1 - t\mu + t\lambda)e^{-(\mu-\lambda)t},$$

which is equivalent to $Q(T, \lambda) = 1 - \exp\{-(\mu - \lambda)T\} - \mu T \exp\{-(\mu - \lambda)T\} + \lambda T \exp\{-(\mu - \lambda)T\}$. Note that since Q changes with λ, we have brought it into the brackets as well. ■

Example 3.2 *Consider a supply chain with two stages with exponential service times both with parameter μ such that the mean time is $1/\mu = 1/3$ days. Also, assume that customers place orders at the firm according to a Poisson process with mean rate $\lambda = 2$ order/day. The quality of service for a quoted leadtime T has the expression $Q(T) = 1 - Te^{-T} - e^{-T}$. The graph of Q is depicted in Figure 3.2.*

Lemma 3.3 *For any given T, the quality of service $Q(T, \lambda)$ shown in (3.1) is strictly decreasing in λ.*

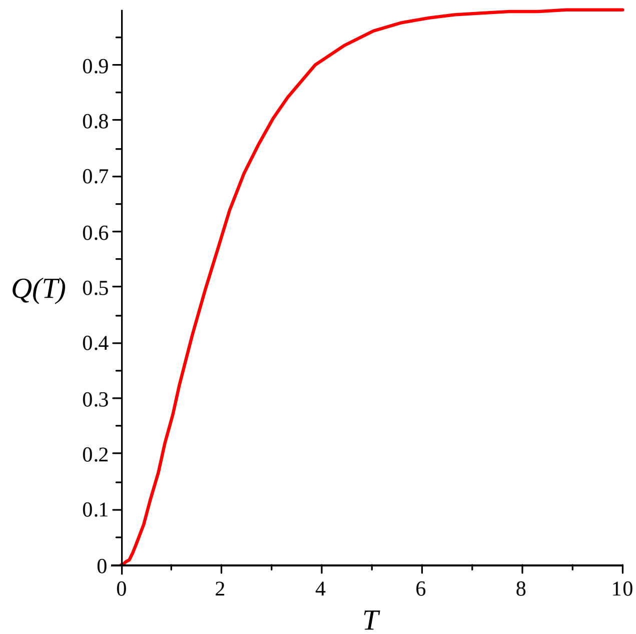

Figure 3.2: The graph for quality of service for a supply chain with two exponential stages with $\mu = 3$ and $\lambda = 2$.

Proof. Taking the derivative w.r.t. λ from (3.1) we have,

$$\frac{\partial Q(T,\lambda)}{\partial \lambda} = -\mu T^2 e^{-(\mu-\lambda)T} + \lambda T^2 e^{-(\mu-\lambda)T},$$

which after simplification is,

$$\frac{\partial Q(T,\lambda)}{\partial \lambda} = \left[T^2 e^{-(\mu-\lambda)T}\right](\lambda - \mu).$$

Since we assume $\mu > \lambda$, then $\partial Q(T,\lambda)/\partial\lambda < 0$ and for a given T, the quality of service is strictly decreasing in λ. ■

Having found the expression for the quality of service we have the demand rate of firm i as follows,

$$\lambda_i = \Lambda S_i(U_i),$$

where the market share is,

$$S_i(U_i) = \frac{e^{U_i}}{\sum_{j=1}^{2} e^{U_j}},$$

with the utility function,

$$U_i(T_i, \lambda_i) = \beta_{0i} - \beta_T T_i + \beta_Q Q_i(T_i, \lambda_i),$$

and the quality of service,

$$Q_i(T_i, \lambda_i) = 1 - e^{-(\mu_i - \lambda_i)T_i} - \mu_i T_i e^{-(\mu_i - \lambda_i)T_i} + \lambda_i T_i e^{-(\mu_i - \lambda_i)T_i}. \tag{3.2}$$

Next, we will study the PDT competition between the two firms.

3.2 Duopolistic PDT Competition with Two-Stage Centralized Supply Chains with Identical Production Rates

As mentioned before the only works in the literature that deal with determining the optimal promised delivery time in the existence of competition between multiple firms are the works of Ho and Zheng [36] and Shang and Liu [53]. Ho and Zheng [36] study a duopolistic market and find the sufficient conditions for the existence of a Nash equilibrium in promised delivery time (PDT). Shang and Liu [53] build upon their work by extending the scope of their study into an oligopolistic market and also finding the closed-form solution for the Nash equilibrium. In this section, we will build upon those works by extending the model of Shang and Liu [53] and Ho and Zheng [36] into serial supply chains instead of single firms. At this stage, we are considering a duopolistic market, where each firm is a centralized supply chain of two stages. This can be interpreted as a centralized supply chain with two players (e.g., a manufacturer and a wholesaler) that has to quote a certain PDT to its customers (the retailers). In this part of the study, we are assuming that the service time at both stages of each supply chain is exponential with a common parameter (service rate). But the service rate from one supply chain to the other is different. Later in this study, we will relax these assumption to allow more stages at each firm as well as different production rates at stages of a firm.

As mentioned before, we are considering a market with two players that compete based on their promised delivery time (PDT). We know that the firms manipulate their PDT in competing against each other. At this stage, we assume that the capacities of the firms are fixed and they compete solely on their promised delivery time. Since the capacities are fixed, the profitability of the firms are an increasing function of their market share. It is important to note that in equilibrium, the following should hold,

$$\lambda_i = \Lambda S_i(U_i). \tag{3.3}$$

Ho and Zheng [36] argue that before any analysis, the existence of a market equilibrium should be established. A market equilibrium is reached when equation (3.3) holds. It is important to note that λ_i is endogenous and appears at both sides of equation (3.3). Ho and Zheng [36] refer to λ_i as "today's demand rate" and to $\Lambda S_i(U_i)$ as "tomorrow's demand rate", which is the result of a demand rate of λ_i, "today". They argue that a market equilibrium is reached only when "today's" and "tomorrow's" demand rates become equal. In other words, in equilibrium, the demand rate into each firm does not fluctuate anymore and has reached a stable state. In this state of market equilibrium, equation (3.3) holds.

Since we are not taking into account the capacity cost and other production costs of the firms, maximizing the demand rate into the firm is the desired outcome of the game for each firm. After the market equilibrium is reached and the stable state is established, we can formulate the optimization problem of firm i in order to maximize its market share (demand rate) as follows,

$$\begin{aligned} &\max_{T_i} \lambda_i(\mathbf{T}) \\ &\text{subject to } \lambda_i = \Lambda S_i(U_i), \end{aligned} \tag{3.4}$$

where $S_i(U_i) = e^{U_i} / \sum_{j=1}^{2} e^{U_j}$ and Λ is fixed such that

$$\Lambda = \lambda_1 + \lambda_2, \tag{3.5}$$

and thus, we are assuming that potential customers do not balk, i.e., they cannot refuse to get service. Also the term $\lambda_i = \Lambda S_i(U_i)$ ensures that the analysis is being done when the market has reached the equilibrium state in its demand rate into firms. We are also assuming that customers do not renege or jockey in the system. The act of reneging refers to the behavior of leaving the queue as the waiting time grows and customer becomes so impatient that she leaves the queue. Also jockeying between the queues refers to the behavior of switching between the queues (of different firms).

We are interested in finding the Nash equilibrium of the game played between the two firms (supply chains). The Nash equilibrium has the property that, given the decision by the other firm, the firm under consideration cannot increase its demand rate by deviating from the Nash equilibrium point. We are specifically interested in proving the existence of the Nash equilibrium, or to derive conditions for the existence of it. As shown in the optimization problem (3.4), the strategies that firms employ is their promised delivery time, PDT. Firm i's strategy set is then equal to $\{T_i \mid T_i \geq 0\}$.

Here, we will follow the arguments presented in Shang and Liu [53] for their oligopolistic case with M firms with $M/M/1$ queueing settings and adapt it to our supply chain setting.

Shang and Liu [53] propose an alternate way to express the demand equation by introducing the function $L_i(T_i, \lambda_i)$ as follows.

We know that

$$\lambda_i = \Lambda \frac{e^{U_i(T_i,\lambda_i)}}{\sum_{j=1}^{2} e^{U_j(T_j,\lambda_j)}},$$

and thus we have,

$$\frac{\lambda_i}{\lambda_j} = \frac{e^{U_i(T_i,\lambda_i)}}{e^{U_j(T_j,\lambda_j)}}.$$

After some algebraic manipulation we have,

$$\ln \lambda_i - U_i(T_i, \lambda_i) = \ln \lambda_j - U_j(T_j, \lambda_j).$$

We define the demand function $L_i(T_i, \lambda_i) = \ln \lambda_i - U_i(T_i, \lambda_i)$ and we have,

$$L_i(T_i, \lambda_i) = L_j(T_j, \lambda_j). \tag{3.6}$$

Shang and Liu [53] introduce the function $\ell(\mathbf{T})$ as the *common allocation parameter* and define it as follows,

$$L_i(T_i, \lambda_i) = \ell(\mathbf{T}), \tag{3.7}$$

where $\mathbf{T} = \{T_i, T_j\}$. Below, we will show that for any set of decision variables $\mathbf{T}$, there is always a unique λ_i that would satisfy the equation (3.7).

Lemma 3.4 *For any given T_i, the function $L_i(T_i, \lambda_i)$ is increasing in λ_i.*

Proof. Taking the derivative of $L_i(T_i, \lambda_i)$ w.r.t. λ_i we have,

$$\begin{aligned}\frac{\partial L_i(T_i, \lambda_i)}{\partial \lambda_i} &= \frac{1}{\lambda_i} - \frac{\partial U_i(T_i, \lambda_i)}{\partial \lambda_i} \\ &= \frac{1}{\lambda_i} - \beta_Q \frac{\partial Q_i(T_i, \lambda_i)}{\partial \lambda_i},\end{aligned}$$

From Lemma 3.3 we know $\partial Q_i(T_i, \lambda_i)/\partial \lambda_i < 0$. Therefore, we will have $\partial L_i(T_i, \lambda_i)/\partial \lambda_i > 0$ and it proves the lemma.

■

For a given set of decision variables $\mathbf{T}$, at $\lambda_i = 0^+$ we have,

$$\lim_{\lambda_i \to 0^+} L_i(T_i, \lambda_i) = \lim_{\lambda_i \to 0^+} (\ln \lambda_i - U_i(T_i, \lambda_i)) = -\infty, \tag{3.8}$$

as when the demand rate into firm i goes to zero, the customers' utility function for that firm will be a positive value. Also, when the demand rate into firm i, λ_i goes to infinity we have,

$$\lim_{\lambda_i \to \infty} L_i(T_i, \lambda_i) = \lim_{\lambda_i \to \infty} (\ln \lambda_i - U_i(T_i, \lambda_i)) = \infty, \tag{3.9}$$

as practically, when the demand rate into firm i goes to infinity, the customers' utility function for that firm should tend to zero.

Based on (3.8) and (3.9), as well as Lemma 3.4, we can conclude that for any set of decision variables $\mathbf{T}$, $L_i(T_i, \lambda_i)$ is increasing in λ_i going from $-\infty$ to ∞ and thus, there is always a unique λ_i that would satisfy the equation (3.7).

In fact, since we have $L_i(T_i, \lambda_i) = L_j(T_j, \lambda_j)$ from equation (3.6), for a given set of decision variables $\mathbf{T}$, we can be certain that there are unique λ_i and λ_j such that the following holds,

$$L_i(T_i, \lambda_i) = L_j(T_j, \lambda_j) = \ell(\mathbf{T}).$$

The function $\ell(\mathbf{T})$ has been introduced by Shang and Liu [53] and we will use it again here. To explain the reason behind using this function we would like to refer the reader to the optimization problem in (3.4). Notice how λ_i is present in both sides of the equation in the constraint $\lambda_i = \Lambda S_i(U_i)$. This is because $S_i(U_i)$ depends on λ_i as well. As a result, taking the derivative from the objective function (λ_i) w.r.t. T_i becomes difficult. Therefore, Shang and Liu [53] introduced the intermediary function $\ell(\mathbf{T})$ to find relationships between the derivatives of the objective function as well as this function w.r.t. T_i. This relationship helps us to find a closed-form solution for the optimal decision variable. Lemma 3.5, along with similar lemmas in the following sections of this chapter illustrate this relationship in more detail.

Shang and Liu [53] introduce a number of lemmas in order to prove the existence and uniqueness of the Nash equilibrium in their case. We will follow the same procedure to find the results in our own supply chain setting.

Lemma 3.5 *For any given PDT T_j quoted by firm j, the following holds,*

$$\partial\lambda_i(\mathbf{T})/\partial T_i \begin{cases} > \\ < \; 0, \\ = \end{cases} \iff \partial\ell(\mathbf{T})/\partial T_i \begin{cases} < \\ > \; 0, \\ = \end{cases} \iff Q_i \begin{cases} < \\ > \; \omega_i(T_i, \lambda_i), \\ = \end{cases}$$

where

$$\omega_i(T_i, \lambda_i) = 1 - \frac{\beta_T \left[1 + (\mu_i - \lambda_i)\, T_i\right]}{\beta_Q T_i (\mu_i - \lambda_i)^2}.$$

Lemma 3.6 *The first-order condition for the demand rate into company i ($\lambda_i(\mathbf{T})$) is satisfied at the point $\widehat{T}_i$ where,*

$$\widehat{T}_i = -\frac{1}{(\mu_i - \lambda_i)} W\left(-\frac{\beta_T}{\beta_Q(\mu_i - \lambda_i)}\right), \tag{3.10}$$

that would impose the condition $\beta_T/\beta_Q \le e^{-1}(\mu_i - \lambda_i)$ on the parameter values. Also, the function W in (3.10) is the Lambert W function.

The proof of the above two lemmas (Lemmas 3.5 and 3.6) can be found in Appendix A. Now we will prove the existence of a unique interior Nash equilibrium in the following proposition.

Proposition 3.2 *An interior unique Nash equilibrium exists for the promised delivery time (PDT) when we have two symmetric supply chains, each with two stages with $M/M/1$ queueing settings and a common parameter and queues forming before each* *stage and when the quality of service is greater than* 26.4%*. Also, the parameters should satisfy the condition,*

$$\frac{\beta_T}{\beta_Q} \leq e^{-1}(\mu_i - \lambda_i). \tag{3.11}$$

The proof of Proposition 3.2 can be found in Appendix A.

It should be noted that if the conditions of the above theorem are not satisfied, the game under study will not have a unique interior Nash equilibrium.

3.2.1 Numerical Examples and Sensitivity Analysis

In this section we will solve a number of numerical examples to get better insights into the results of this model. First, we will confirm the results derived by the closed-form solution in equation (A.7) through a numerical example. Later, we will use the results from the analytical study and solve a numerical example. We will conclude this section by presenting a sensitivity analysis based on the numerical results.

Example 3.3 *In this example we are trying to confirm the validity of the closed-form solution,*

$$\widehat{T}_i = -\frac{1}{\mu_i - \lambda_i} W \left\{ -1, -\frac{\beta_T}{\beta_Q(\mu_i - \lambda_i)} \right\}.$$

This solution is the result of the following equation that is based on the values of the quality of service Q,

$$\omega_i(T_i, \lambda_i) = 1 - \frac{\beta_T \left[1 + (\mu_i - \lambda_i) T_i\right]}{\beta_Q T_i (\mu_i - \lambda_i)^2} = 1 - e^{-(\mu_i - \lambda_i)T_i} \left(1 + (\mu_i - \lambda_i) T_i\right) = Q_i(T_i, \lambda_i), \tag{3.12}$$

where the left-hand side of the equation is the value that Q_i assumes when the first-order condition is satisfied ($\omega_i(T_i, \lambda_i)$) and the right-hand side of it is the expression for the quality of service in the proposed system in this section.

To confirm the results from the analytical solution, we will calculate and plot the best-response curve by both firms and observe their point of intersection. Assuming that we do not have the analytical closed-form solution given by (A.7), given any T_2, we would also have to look for the T_1, for which equation (3.12) would hold. The resulting T_1 would be the best response for that specific T_2. In other words, what follows is a two-stage analysis that will determine the best response to any given T_2 in its first stage. In the second stage, it will plot the best-response curve by changing T_2 and finding the best-response T_1 accordingly. Stage one of this process should confirm the closed-form solution given by (A.7). As an example, we are assuming that T_2 is chosen by company 2 to be 1.229*. The closed-form solution (A.7) suggests the value* 1.482 *for T_1. Now by changing T_1 we observe the two sides of equation (3.12) to find the T_1, for which the equation would hold. The resulting T_1 should match (A.7). We*

T_1	λ_1	λ_2	$\omega_1(T_1, \lambda_1)$	$Q_1(T_1, \lambda_1)$
0.2	0.2025	0.7975	0.9389	0.1767
0.4	0.2898	0.7102	0.9548	0.4368
0.6	0.3694	0.6306	0.9598	0.6401
0.8	0.4254	0.5746	0.9622	0.7788
1.0	0.4592	0.5408	0.9637	0.8683
1.2	0.4766	0.5234	0.9649	0.9237
1.4	0.4830	0.5170	0.9657	0.9569
1.482	**0.4835**	**0.5165**	**0.9661**	**0.9661**
1.5	0.4835	0.5165	0.9661	0.9678
1.8	0.4782	0.5218	0.9671	0.9870

Table 3.2: Summary of results for the numerical analysis for changing T_1, where $\mu_1 = 4$ and $\mu_2 = 5$ and T_2 is fixed at 1.229.

demonstrate the results for $T_2 = 1.229$ in Table 3.2. We assume the parameter values $\mu_1 = 4$, $\mu_2 = 5$, $\beta_{oi} = 0$, $\beta_T = 0.2$, $\beta_Q = 2$ and $\Lambda = 1$.

As it is shown on Table 3.2, the difference between the values for $\omega_1(T_i, \lambda_i)$ and $Q_1(T_i, \lambda_i)$ starts to shrink as T_1 grows, with $\omega_1(T_i, \lambda_i)$ being strictly below $Q_1(T_i, \lambda_i)$ until T_1 reaches 1.482. At $T_1 = 1.482$, the left-hand side and right-hand side of equation (3.12) match and the solution derived from the closed-form result given by (A.7) indicating $T_1 = 1.482$ is confirmed as $\omega_1(1.482, \lambda_1) = Q_1(1.482, \lambda_1)$. Therefore, $T_1 = 1.482$ is firm one's best response to the decision $T_2 = 1.229$ taken by the second firm. From Table 3.2 we can also see that at this point the value of the demand rate for the first firm reaches a local maximum and this in turn confirms the analytical results of this section.

Figure (3.3) shows the intersection of the two lines with the parameter values presented in the example. The curves are the result of 20 points for each function. The dotted line represents $\omega_1(T_1, \lambda_1)$ and the solid line is $Q_1(T_1, \lambda_1)$.

Now, by changing T_2, we will find the best response of firm 1 (T_1) and show them in Table 3.3.

To find the best response of firm 2 towards the choice of firm 1, T_1, we will do the same process as above. Table 3.4 summarizes the best response of firm 2, T_2^*, for a given decision by firm 1, T_1.

Now, plotting the best-response curves from both companies together, we can see that they intersect each other at the point $T_1 = 1.48$ and $T_2 = 1.22$. This is the point, where both companies are on their best-response curve and thus their market share is maximized. The best-response curves are shown in Figure 3.4. This figure confirms the points derived by the analytical solution.

The analytical results for the Nash equilibrium of the game would also suggest that the equilibrium is at $\mathbf{T} = \{1.482, 1.229\}$.

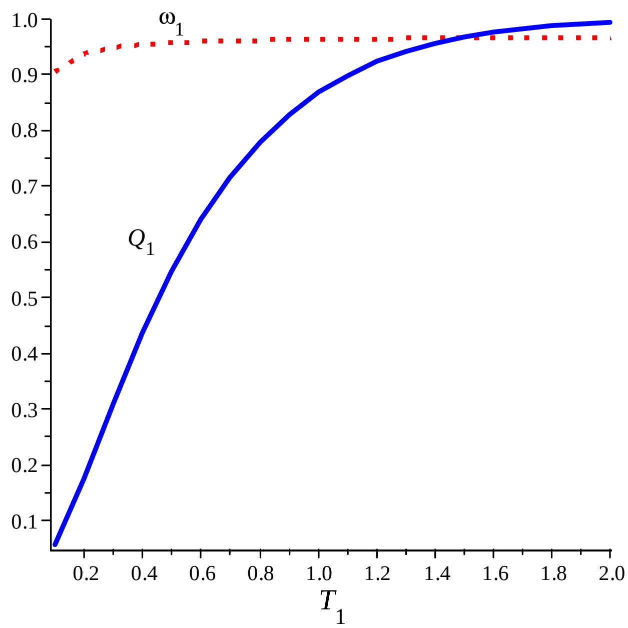

Figure 3.3: The functions $\omega_1(T_1, \lambda_1)$ (with the dotted line) and $Q_1(T_1, \lambda_1)$ (with the solid line) intersect around the point $T_1 = 1.48$.

T_1^*	T_2	λ_1	λ_2
1.518	0.5	0.5959	0.4041
1.491	0.8	0.5108	0.4892
1.483	1	0.4896	0.5104
1.481	1.2	0.4836	0.5163
1.483	1.5	0.4884	0.5116
1.487	1.8	0.4999	0.5001
1.490	2	0.5090	0.4910
1.493	2.2	0.5184	0.4816
1.496	2.4	0.5280	0.4720
1.505	3	0.5571	0.4429
1.513	3.5	0.5811	0.4189
1.521	4	0.6047	0.3953
1.529	4.5	0.6279	0.3721

Table 3.3: Summary of results for the best response of firm 1, T_1^*, given the decision taken by firm 2, T_2.

T_2^*	T_1	λ_1	λ_2
1.262	0.5	0.3321	0.6679
1.241	0.8	0.4254	0.5746
1.234	1	0.4592	0.5408
1.230	1.2	0.4766	0.5234
1.228	1.5	0.4835	0.5165
1.229	1.8	0.4782	0.5218
1.231	2	0.4713	0.5287
1.233	2.2	0.4631	0.5369
1.235	2.4	0.4541	0.5459
1.241	3	0.4258	0.5742
1.246	3.5	0.4020	0.5980
1.251	4	0.3785	0.5215
1.256	4.5	0.3556	0.6444

Table 3.4: Summary of results for the best response of firm 2, T_2^*, given the decision taken by firm 1, T_1.

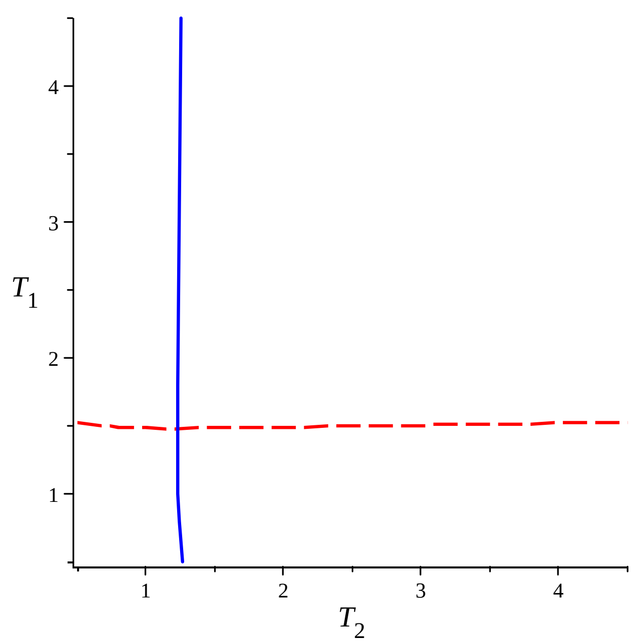

Figure 3.4: The best-response curves of both firms plotted implicitly. The intersection point is the only point in the decision set, where both firms are on their best-response curve. The dashed line is the best response of firm 1 and the solid line represents best response of firm 2.

In the following section, we will present a numerical example and solve for the optimal PDT values.

Example 3.4 *Assume we let $\beta_{oi} = 0$, $i = 1, 2$, $\beta_T = 0.2$, $\beta_Q = 2$ and $\Lambda = 1$. Also, assume that the production rate at the stages of the first supply chain is $\mu_1 = 2$ and the production rate at stages for the second supply chain is $\mu_2 = 6$. Solving the problem for each supply chain we will get the following optimal demand rates at equilibrium for each supply chain. First we present the demand rates as the market equilibrium should hold first,*

$$\lambda_1^N \cong 0.39,$$

which means that the first firm serves 39% of the market. Also, for the second firm we have,

$$\lambda_2^N \cong 0.61,$$

which is equivalent to the second firm serving 61% of the market. Also, the optimal promised delivery times for both supply chains at the equilibrium point are as follows,

$$T_1^N \cong 2.63,$$

and,

$$T_2^N \cong 1.1.$$

Also, we calculated the supply chains' quality of service or the probability of meeting the demand at or before the promised delivery time at the equilibrium point. These are as follows,

$$Q_1^N \cong 92\%,$$

and,

$$Q_2^N \cong 98\%.$$

As it can be seen from the solutions, the second supply chain that has a higher production rate enjoys a greater demand rate and a greater market share. The firm with the higher production rate (second firm) also does better at marketing level by promising a shorter delivery time to the customers. Finally, the second firm has a better probability of sticking to its promised delivery time and thus has a better quality of service.

We have calculated the optimal demand rate as well as the optimal promised delivery times and quality of service expressions for values of μ_1 ranging from 2 to 6 with increments of 0.5 when the production rate of the second firm is fixed at $\mu_2 = 6$. The results are presented in Table 3.5.

As it is shown in Table 3.5, when the processing capabilities of the firms differ, their market shares will be always on both sides of 0.5. Their market shares will

μ_1	λ_1^N	λ_2^N	T_1^N	T_2^N	Q_1^N	Q_2^N
2	0.3957	0.6043	2.6264	1.0628	0.9229	0.9782
2.5	0.4253	0.5747	2.1915	1.0583	0.9412	0.9784
3	0.4459	0.5541	1.8838	1.0551	0.9527	0.9784
3.5	0.4610	0.5390	1.6550	1.0528	0.9606	0.9785
4	0.4725	0.5275	1.4783	1.0511	0.9662	0.9785
4.5	0.4816	0.5184	1.3377	1.0497	0.9705	0.9785
5	0.4889	0.5111	1.2230	1.0486	0.9738	0.9786
5.5	0.4949	0.5051	1.1275	1.0477	0.9765	0.9786
6	0.5000	0.5000	1.0469	1.0469	0.9787	0.9787

Table 3.5: Summary of results for the numerical analysis for changing the production rate μ_1, where μ_2 is fixed at 6.

	λ_1^N	λ_2^N	T_1^N	T_2^N	Q_1^N	Q_2^N
$\mu_1 \uparrow$ and μ_2 fixed and $\mu_1 < \mu_2$	$\uparrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\uparrow$	$\uparrow$

Table 3.6: Direction of change in system performance with changing production rate μ_1.

become 0.5 when they have similar production rates. When μ_2 is fixed, by increasing μ_1, the values for λ_1^N will increase towards 0.5 and thus λ_2^N will decrease towards 0.5. This is because the first firm is becoming better capable of processing orders faster and thus will capture a larger market share as its capabilities increase. Also, by increasing μ_1, both T_1^N and T_2^N show a decrease but the decrease is more noticeable in T_1^N. When the firms have similar production rates, they will both promise a common time to the customers. However, when the processing capabilities of the firms differ, both will quote a greater leadtime than that common promised delivery time. When both firms have similar production rates, their quality of service will be the same. If their production rates differ, both firms will suffer from a lower quality of service but this decrease is more noticeable for the firm, whose production capacity is lower. Fixing μ_2, as we increase μ_1 from a value less than μ_2, the quality of service of both firms will increase towards the common QoS until when their production capabilities become similar. This is summarized with direction arrows in Table 3.6 for a fixed μ_2 and an increasing μ_1.

Next we will examine the effect of changing the parameters β_T and β_Q. We will hold β_0 at zero as it depends on other parameters that are out of the scope of this work. We find the values for λ_1^N, λ_2^N, T_1^N, T_2^N, Q_1^N and Q_2^N for the case where $\mu_1 = 4$ and $\mu_2 = 6$ and present the results in Table 3.7.

From Table 3.7, we can see that when the first firm has a lower production rate than the second firm, with an increase in β_T and all other factors held constant, the values for λ_1^N decreases while λ_2^N increases. Therefore, as customers become more

β_T	β_Q	λ_1^N	λ_2^N	T_1^N	T_2^N	Q_1^N	Q_2^N
0.01	2	0.4977	0.5023	2.4889	1.6779	0.9984	0.9990
0.06	2	0.4894	0.5106	1.8966	1.3081	0.9902	0.9938
0.11	2	0.4827	0.5173	1.6888	1.1799	0.9817	0.9884
0.17	2	0.4757	0.5243	1.5361	1.0863	0.9714	0.9819
0.20	2	0.4725	0.5275	1.4783	1.0511	0.9662	0.9785
0.2	**3**	0.4701	0.5299	1.6181	1.1404	0.9779	0.9859
0.2	**4**	0.4684	0.5316	1.7157	1.2030	0.9835	0.9895
0.2	**5**	0.4672	0.5328	1.7905	1.2512	0.9869	0.9916
0.2	**6**	0.4661	0.5339	1.8512	1.2904	0.9891	0.9930
0.2	**7**	0.4653	0.5347	1.9022	1.3233	0.9907	0.9940

Table 3.7: Summary of results for the numerical analysis with changing parameter values β_T and β_Q, where $\mu_1 = 4$ and $\mu_2 = 6$.

	λ_1^N	λ_2^N	T_1^N	T_2^N	Q_1^N	Q_2^N
$\beta_T \uparrow$ and β_Q fixed and $\mu_1 < \mu_2$	$\downarrow$	$\uparrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\beta_Q \uparrow$ and β_T fixed and $\mu_1 < \mu_2$	$\downarrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$

Table 3.8: Direction of change in system performance with changing parameter values β_T and β_Q with $\mu_1 = 4$ and $\mu_2 = 6$ where the conditions of the problem are satisfied.

sensitive to the exact promised delivery time, the market share of the firm with lower processing capability will decrease, while the firm with the higher processing capability will enjoy a greater market share. Also, when customers become more aware of the exact promised delivery time, the optimal promised delivery time for both firms (lower and higher production capacity) will decrease. Since the production rates of the firms are held constant, it is expected to observe a decline in the quality of service of both firms when they promise shorter times to customers. This is indeed obvious from the results in the Table 3.7.

By changing the parameter β_Q with all other factors held constant, we observe a decrease in λ_1^N and an increase in λ_2^N. In other words, when customers become more sensitive to the quality of service, the market share of the firm with lower production rate will decrease, while the firm with the higher production rate will enjoy a higher market share. Intuitively, when customers become more aware of the degree to which, firms can stick to their promised delivery time, firms should tend to quote longer leadtimes in order to move to a safer zone. In fact, this can be seen in the results of the table as both T_1^N and T_2^N will be larger as β_Q increases. As expected, with an increase in customers' sensitivity to the quality of service of firms, their values for quality of service will increase. This is shown in the table, where both Q_1^N and Q_2^N increase as β_Q increases. The direction of change in system performance indicators with changing parameters is depicted in Table 3.8.

3.3 Duopolistic PDT Competition with k-Stage Centralized Supply Chains with Identical Production Rates

As mentioned in the chapter introduction, the contribution of this section is again another extension on the work done by Ho and Zheng [36] and Shang and Liu [53]. In the previous section, we built upon their model by extending each player into a supply chain. We assumed that each supply chain had two stages and the production rates of both stages at each firm are the same. In this section, we relax the constraint on the number of stages at the firms and allow for the general case of k stages at each supply chain. It should be noted that both firms have the same number of stages k. Also, the production rate at all k stages of each firm is the same. We are assuming that all stages of the supply chain i have exponential service times with an identical parameter μ_i. We are also assuming that the service provided by the supply chain is continuous and therefore, as soon as a stage becomes idle, a new customer (order) may enter that stage. We are also allowing queues to form before each stage. Obviously, for the system to reach the steady state, we are assuming that $\mu_i > \lambda_i$, where λ_i is the demand rate for firm i.

We are interested in finding the Nash equilibrium of the game played by the two firms. We are specifically interested in the optimal promised delivery time of each firm in equilibrium. Again, because of symmetry we focus on one firm only. Also, we know that at equilibrium, the following should hold,

$$\lambda_i = \Lambda S_i(U_i). \tag{3.13}$$

As in the previous section, we would like to establish market equilibrium before starting the analysis. Market equilibrium is reached when "today's" demand rate (λ_i) is equal to "tomorrow's" demand rate ($\Lambda S_i(U_i)$) and equation (3.13) holds and the demand rates do not fluctuate over time.

After market equilibrium is reached, firm i is interested in solving the following optimization problem,

$$\max_{T_i} \lambda_i(\mathbf{T}) \tag{3.14}$$
$$\text{subject to } \lambda_i = \Lambda S_i(U_i),$$

where $S_i(U_i) = e^{U_i} / \sum_{j=1}^{2} e^{U_j}$ and Λ is fixed such that

$$\Lambda = \lambda_i + \lambda_j,$$

and

$$U_i(T_i, \lambda_i) = \beta_{0i} - \beta_T T_i + \beta_Q Q_i(T_i, \lambda_i). \tag{3.15}$$

where the term $\lambda_i = \Lambda S_i(U_i)$ ensures that the market has reached the equilibrium state. Also, $\mathbf{T}$ is the vector containing the decision variables of both firms.

As in the previous section, the strategy set of firm i as shown in optimization problem (3.14) is $\{T_i \mid T_i \geq 0\}$. What is different between the analysis in this section and Section 3.2 is the function for the quality of service.

Proposition 3.3 *The expression for the quality of service, where the supply chain has k stages with exponential stage durations with the common parameter k is equal to*

$$Q_i^{(k)}(T_i,\lambda_i) = Q_i(T_i,\lambda_i) = 1 - \left[1 + \sum_{n=1}^{k-1} \frac{1}{n!} T_i^n (\mu_i - \lambda_i)^n\right] e^{-(\mu_i-\lambda_i)T_i}, \qquad (3.16)$$

where k is integer and $k \geq 2$.

Proof. The proof for Proposition 3.3 is done similarly to the proof of Proposition 3.1. We use the Laplace transform to find the probability distribution of the total time spent in the system. It is easily verified that this expression is equal to,

$$\begin{aligned} Q_i(T_i,\lambda_i) = 1 + \frac{1}{(k-1)!}[&\frac{(k-1)!}{1!}(-1 - T(\mu-\lambda)) - \frac{(k-1)!}{2!}T_i^2(\mu_i-\lambda_i)^2 \\ &- \frac{(k-1)!}{3!}T_i^3(\mu_i-\lambda_i)^3 - \cdots - \frac{(k-1)!}{(k-1)!}T_i^{k-1}(\mu_i-\lambda_i)^{k-1}]e^{-(\mu_i-\lambda_i)T_i} \end{aligned}$$

which after simplification becomes

$$\begin{aligned} Q_i(T_i,\lambda_i) = 1 + [&-1 - \frac{1}{1!}T_i(\mu_i-\lambda_i) - \frac{1}{2!}T_i^2(\mu_i-\lambda_i)^2 \\ &- \frac{1}{3!}T_i^3(\mu_i-\lambda_i)^3 - \cdots - \frac{1}{(k-1)!}T_i^{k-1}(\mu_i-\lambda_i)^{k-1}]e^{-(\mu_i-\lambda_i)T_i} \end{aligned}$$

or

$$Q_i(T_i,\lambda_i) = 1 - \left[1 + \sum_{n=1}^{k-1} \frac{1}{n!} T_i^n (\mu_i - \lambda_i)^n\right] e^{-(\mu_i-\lambda_i)T_i}.$$

■

Example 3.5 *Consider a supply chain with 5 stages that have exponential service times with mean 1/3. Also assume that customers place orders at the supply chain according to a Poisson process with at rate 2. Then the expression for the quality of service at this supply chain will be equal to*

$$Q(t) = 1 - \frac{1}{24}e^{-t}(24 + 24t + 4t^3 + 12t^2 + t^4).$$

Lemma 3.7 *For any given T, the quality of service is strictly decreasing in the firm's demand rate λ.*

Proof. Taking the derivative w.r.t. λ from the quality of service in (3.16) we have,

$$\frac{\partial Q_i(T_i, \lambda_i)}{\partial \lambda_i} = -\frac{1}{(k-1)!} T_i^k e^{-(\mu_i - \lambda_i)T_i} (\mu_i - \lambda_i)^{k-1}.$$

Now, since we know from the assumptions that $\mu_i > \lambda_i$, then the expression $\partial Q_i(T_i, \lambda_i)/\partial \lambda_i$ is negative and therefore, quality of service is strictly decreasing in the demand rate. ■

The demand function $L(T_i, \lambda_i)$ and the common allocation parameter $\ell(\mathbf{T})$ are defined similarly to the Section 3.2 as introduced in Shang and Liu [53] as follows,

$$L_i(T_i, \lambda_i) = \ln \lambda_i - U_i(T_i, \lambda_i).$$

Also, we have $L_i(T_i, \lambda_i) = L_j(T_j, \lambda_j)$ as in equation (3.6). And from equation (3.7) we have,

$$L_i(T_i, \lambda_i) = \ell(\mathbf{T}). \tag{3.17}$$

Lemma 3.8 *For any given T_i, the function $L_i(T_i, \lambda_i)$ is increasing in λ_i.*

Proof. Taking the derivative w.r.t. λ_i from $L_i(T_i, \lambda_i)$ we have,

$$\frac{\partial L_i(T_i, \lambda_i)}{\partial \lambda_i} = \frac{1}{\lambda_i} - \frac{\partial U_i(T_i, \lambda_i)}{\partial \lambda_i} \tag{3.18}$$

We know from (3.15) that $U_i(T_i, \lambda_i) = \beta_{0i} - \beta_T T_i + \beta_Q Q_i(T_i, \lambda_i)$. Therefore, 3.18 becomes,

$$\frac{\partial L_i(T_i, \lambda_i)}{\partial \lambda_i} = \frac{1}{\lambda_i} - \beta_Q \frac{\partial Q_i(T_i, \lambda_i)}{\partial \lambda_i}.$$

We already know from Lemma 3.7 that $\partial Q_i(T_i, \lambda_i)/\partial \lambda_i < 0$. Therefore, we will have,

$$\frac{\partial L_i(T_i, \lambda_i)}{\partial \lambda_i} > 0,$$

or for any given T_i, $L_i(T_i, \lambda_i)$ is strictly increasing in λ_i. ■

For a given set of decision variables $\mathbf{T}$, at $\lambda_i = 0^+$ we have,

$$\lim_{\lambda_i \to 0^+} L_i(T_i, \lambda_i) = \lim_{\lambda_i \to 0^+} (\ln \lambda_i - U_i(T_i, \lambda_i)) = -\infty, \tag{3.19}$$

because with the demand rate into firm i approaching zero, the market's utility function for that firm will be a positive value. Also, at $\lambda_i = \infty$ we have,

$$\lim_{\lambda_i \to \infty} L_i(T_i, \lambda_i) = \lim_{\lambda_i \to \infty} (\ln \lambda_i - U_i(T_i, \lambda_i)) = \infty, \tag{3.20}$$

as with the demand rate into firm i approaching infinity, the market's utility function for that firm tends to zero.

Based on (3.19) and (3.20), as well as Lemma 3.8, we can conclude that for any set of decision variables $\mathbf{T}$, there is always a unique λ_i that will satisfy equation (3.17).

In Section 3.2 we presented Lemma 3.5 that led to the proof of the existence of a Nash equilibrium in the PDT competition for two-stage supply chain. As mentioned before, this method was originally used in Shang and Liu [53] in order to prove the existence of an equilibrium in their study. We will follow the same procedure here as well but will use the expressions corresponding to our own supply chain setting.

Lemma 3.9 *For any given promised delivery time for the other firm T_j, the following holds,*

$$\partial\lambda_i(\mathbf{T})/\partial T_i \begin{cases} > \\ < \\ = \end{cases} 0, \iff \partial\ell(\mathbf{T})/\partial T_i \begin{cases} < \\ > \\ = \end{cases} 0, \iff Q_i \begin{cases} < \\ > \\ = \end{cases} \omega_k(T_i, \lambda_i) \qquad (3.21)$$

where,

$$\omega_k(T_i, \lambda_i) = 1 - \frac{\beta_T \left[1 + \sum_{n=1}^{k-1} \frac{1}{n!} T_i^n (\mu_i - \lambda_i)^n\right]}{\beta_Q \frac{1}{(k-1)!} T_i^{k-1} (\mu_i - \lambda_i)^k}.$$

Lemma 3.10 *The first-order condition for the demand rate into firm i ($\lambda_i(\mathbf{T})$) is satisfied at the point $\widehat{T}_i$ where,*

$$\widehat{T}_i = -\frac{k-1}{\mu_i - \lambda_i} W \left\{ -\frac{1}{k-1} \left(\frac{\beta_T\,(k-1)!}{\beta_Q\,(\mu_i - \lambda_i)} \right)^{\frac{1}{k-1}} \right\},$$

that would impose the condition on the parameter values,

$$\frac{\beta_T}{\beta_Q} \leq \frac{(\mu_i - \lambda_i)\left[e^{-1}(k-1)\right]^{k-1}}{(k-1)!}.$$

The proofs of the above two lemmas (Lemmas 3.9 and 3.10) can be found in Appendix A.

Proposition 3.4 *A unique interior Nash equilibrium exists for the promised delivery time (PDT) when we have two symmetric supply chains, each with k stages with $M/M/1$ queueing settings and a common parameter and queues forming before each stage and the quality of service is greater than the quality of service threshold $\delta(k)$, where this threshold is defined as follows,*

$$\delta(k) = 1 - e^{-(k-1)} \left[1 + \sum_{n=1}^{k-1} \frac{(k-1)^n}{n!}\right],$$

T_1^*	T_2	λ_1	λ_2	T_2^*	T_1	λ_1	λ_2
5.418	0.5	0.6106	0.3894	1.856	2	0.4075	0.5925
5.125	0.8	0.4998	0.5002	1.854	2.2	0.4151	0.5849
4.958	1	0.4303	0.5697	1.852	2.4	0.4195	0.5805
4.844	1.2	0.3801	0.6199	1.853	3	0.4186	0.5813
4.756	1.5	0.3397	0.6603	1.854	3.2	0.4147	0.5153
4.731	1.8	0.3274	0.6726	1.856	3.5	0.4067	0.5933
4.731	2	0.3278	0.6722	1.861	4	0.3892	0.6108
4.740	2.2	0.3318	0.6682	1.863	4.2	0.3813	0.6187
4.752	2.4	0.3378	0.6622	1.871	4.8	0.3562	0.6438
4.801	3	0.3603	0.6397	1.873	5	0.3477	0.6523
4.845	3.5	0.3807	0.6193	1.879	5.5	0.3262	0.6738
4.892	4	0.4014	0.5986	1.886	6	0.3052	0.6748

Table 3.9: Summary of results for the best response of each firm based on the decision made by the other firm.

and we should have,

$$\frac{\beta_T}{\beta_Q} < \frac{(\mu_i - \lambda_i)\,[e^{-1}(k-1)]^{k-1}}{(k-1)!}.$$

It is interesting to note that the quality of service threshold for $k = 2$ is found to be %26 based on the expression in (A.28) is the same threshold that were found in Section 3.2, where there were two stages with the same production rate. It should be noted that if condition $Q_i(T_i, \lambda_i) > \delta(k)$ is not satisfied, a unique interior Nash equilibrium will not exist.

3.3.1 Numerical Examples and Sensitivity Analysis

In this section we will solve a number of numerical examples to get better insights into the results of this model. First we will find the best-response curve of each company in a pointwise manner and plot the curves. Then we will solve a numerical example and finally we will perform a numerical sensitivity analysis.

Example 3.6 *Assume we let $\beta_{oi} = 0$, $i = 1, 2$, $\beta_T = 0.2$, $\beta_Q = 2$, $k = 5$ and $\Lambda = 1$. Also, assume that the production rate at the stages of the first supply chain is $\mu_1 = 2$ and the production rate at stages for the second supply chain is $\mu_2 = 6$. Now using the closed-form solution given by equation (A.20) we will find the best-response curves of both firms. Like in the previous section, the best-response curves are only available in a pointwise manner. The points of the best-response curves for both companies are summarized in Table 3.9.*

Plotting the results we illustrate the best-response curves of both firms plotted implicitly in the same graph. The intersection of the best-response curves is a Nash

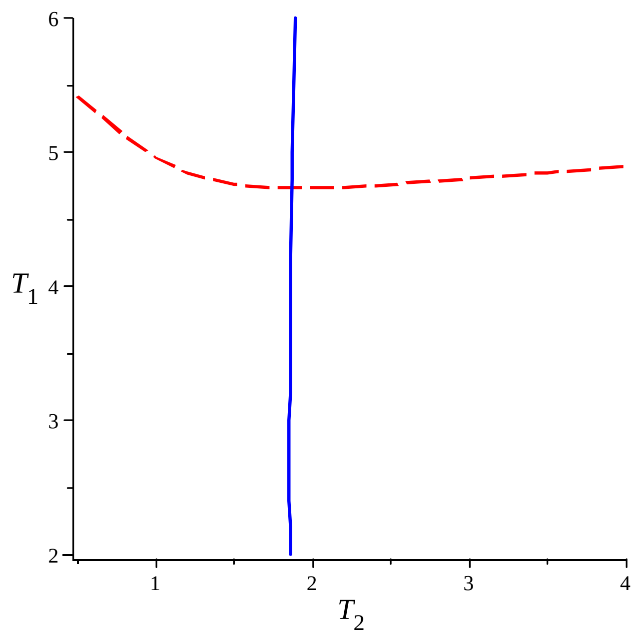

Figure 3.5: The best-response curves of both firms shown together. The point of their intersection is a Nash equilibrium for the game.

equilibrium for the game. The best-response curves are shown in Figure 3.5. It is interesting to mention that the reason why the second firm's best-reponse curve is almost a vertical line is because its production rate is three times the production rate of the first firm and therefore, it does not get changed considerably based on decisions made by the first firm.

The best-response curves intersect once and this point is around $T_1^N = 4.8$ *and* $T_2^N = 1.8$. *In fact, in the next example we show that this point is the Nash equilibrium derived by the analytical results of the section.*

Example 3.7 *This problem is defined with the parameter values assumed in the previous example. We namely assume* $\beta_{oi} = 0,\ i = 1, 2,\ \beta_T = 0.2,\ \beta_Q = 2,\ k = 5$ *and* $\Lambda = 1$. *We also assume* $\mu_1 = 2$ *and* $\mu_2 = 6$. *Solving the problem for each supply chain we will get the following optimal demand rates at equilibrium for each supply chain,*

$$\lambda_1^N \cong 0.327,$$

which means that the first firm serves 32.7% *of the market. Also for the second firm we have,*

$$\lambda_2^N \cong 0.673,$$

which is equivalent to the second firm serving 67.3% *of the market. Also, the optimal promised delivery times (quoted leadtimes) for both supply chains at the equilibrium*

k	λ_1^N	λ_2^N	T_1^N	T_2^N	Q_1^N	Q_2^N
2	0.3956	0.6044	2.6264	1.0628	0.9228	0.9782
3	0.3699	0.6301	3.3812	1.3507	0.9123	0.9755
4	0.3473	0.6527	4.0755	1.6204	0.9033	0.9731
5	0.3270	0.6730	4.7304	1.8796	0.8953	0.9709
6	0.3085	0.6915	5.3560	2.1315	0.8878	0.9689
7	0.2914	0.7086	5.9603	2.3782	0.8810	0.9670
8	0.2755	0.7245	6.5465	2.6208	0.8745	0.9652
9	0.2607	0.7393	7.1182	2.8604	0.8683	0.9634
10	0.2469	0.7531	7.6780	3.0976	0.8624	0.9618

Table 3.10: Summary of results for the numerical analysis for changing the number of stages at each firm k, where $\mu_1 = 2$ and $\mu_2 = 6$.

point are as follows,

$$T_1^N \cong 4.73,$$

and,

$$T_2^N \cong 1.88.$$

The reason the PDTs are so different is because of the difference in the production rate of the two firms. In the next step, we calculate the firms' quality of service or the probability of meeting the demand at or before the promised delivery time at the equilibrium point. These are as follows,

$$Q_1^N \cong 89\%,$$

and,

$$Q_2^N \cong 97\%.$$

As it can be seen from the solutions, when each supply chain has 5 *stages, the second supply chain that has a higher production rate enjoys a greater demand rate and a greater market share. The firm with the higher production rate (second firm) also does better at marketing level by promising a shorter delivery time to the customers. Finally, the second firm has a better probability of sticking to its promised delivery time and thus has a better quality of service.*

We have calculated the optimal demand rate as well as the optimal promised delivery times and quality of service expressions for values of k ranging from 2 to 10 when the production rate of the first firm is fixed at $\mu_1 = 2$ and that of the second firm is fixed at $\mu_2 = 6$. The results are presented in Table 3.10.

	λ_1^N	λ_2^N	T_1^N	T_2^N	Q_1^N	Q_2^N
$k \uparrow$, μ_1 and μ_2 fixed and $\mu_1 < \mu_2$	$\downarrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\downarrow$	$\downarrow$

Table 3.11: Direction of change in system performance with changing the number of stages at both firms k, where μ_1 and μ_2 are fixed.

β_T	β_Q	λ_1^N	λ_2^N	T_1^N	T_2^N	Q_1^N	Q_2^N
0.01	2	0.4839	0.5161	8.3796	2.6488	0.9953	0.9987
0.06	2	0.4274	0.5726	6.3504	2.1998	0.9704	0.9920
0.11	2	0.3850	0.6150	5.5624	2.0419	0.9444	0.9848
0.17	2	0.3444	0.6556	4.9620	1.9247	0.9120	0.9757
0.20	2	0.3270	0.6730	4.7304	1.8796	0.8953	0.9709
0.2	**3**	0.3159	0.6841	5.1708	2.0073	0.9343	0.9811
0.2	**4**	0.3084	0.6915	5.4596	2.0954	0.9524	0.9861
0.2	**5**	0.3030	0.6970	5.6736	2.1624	0.9628	0.9890
0.2	**6**	0.2987	0.7013	5.8440	2.2164	0.9696	0.9909
0.2	**7**	0.2952	0.7048	5.9832	2.2616	0.9743	0.9922

Table 3.12: Summary of results for the numerical analysis with changing parameter values β_T and β_Q, where $\mu_1 = 2$ and $\mu_2 = 6$ and $k = 5$.

As it is shown in Table 3.10, as the number of stages in both firms increase, the firm with the lower production capacity will suffer from a decline in its market share, which translates directly into a higher market share for the firm with the higher production capacity. As the number of stages increases, we observe that the promised delivery time of both firms increase. This is intuitive because with more stages, the products have to travel through a higher number of stages and thus, the PDT should be greater. Also, as we increase the number of stages in the firms, both firms suffer from a decline in their quality of service. It should be noted, however that the increase in PDT and decline in quality of service of the lower capacity firm is more remarkable than those in the firm with the higher production capacity. This is summarized with direction arrows in Table 3.11 for a fixed μ_2 and an increasing μ_1.

Next we will examine the effect of changing the parameters β_T and β_Q. We will hold β_0 at zero as it depends on other parameters that are out of the scope of this work. We find the values for λ_1^N, λ_2^N, T_1^N, T_2^N, Q_1^N and Q_2^N for the case where $\mu_1 = 2$, $\mu_2 = 6$ and $k = 5$, and present the results in Table 3.12.

Similar to the results derived from Table 3.7, we can see from Table 3.12 that when customers' sensitivity to the announced PDT increases, the market share for the lower capacity firm will increase and that of the higher capacity firm increases. In this case, both firms will quote shorter delivery times in an attempt to capture higher market shares. Doing so, they will both assume the risk of unresponsiveness and show a decline in their quality of service. The decrease in PDT and quality of service is

	λ_1^N	λ_2^N	T_1^N	T_2^N	Q_1^N	Q_2^N
$\beta_T \uparrow$ and β_Q fixed and $\mu_1 < \mu_2$	$\downarrow$	$\uparrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\beta_Q \uparrow$ and β_T fixed and $\mu_1 < \mu_2$	$\downarrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$

Table 3.13: Direction of change in system performance with changing parameter values β_T and β_Q, where μ_1, μ_2 and k are fixed.

again more remarkable for the firm with the lower capacity compared to those from the higher capacity firm.

With β_T held constant and increasing customers' sensitivity to quality of service, the market share of the lower capacity firm decreases again, while that of the higher capacity firm will increase. Also, both firms will quote longer PDTs in order to be in the safe side when customers are more sensitive to how well the firms keep to their commitment. Longer PDTs will translate into better quality of service for both firms. The direction of change in system performance indicators with changing parameters is depicted in Table 3.13. These results are identical to the results in Table 3.8.

3.4 Duopolistic PDT Competition with Two-Stage Centralized Supply Chains with Non-Identical Production Rates

The contribution of this section is yet another extension on the model proposed by Shang and Liu [53]. Consider the duopolistic system with identical production rates introduced in Section 3.2. Now we generalize that system by assuming that the production rates at each firm could be different. This makes the model more general as production rates at different stages of a supply chain are usually different in reality. Similar to the models presented in previous sections, we are again assuming that the service provided by the supply chain is continuous and therefore, as soon as a stage becomes idle, a new customer (order) may enter that stage. We are also allowing queues to form before each stage. This is shown in Figure 3.6. Obviously, for the system to reach the steady state, we are assuming that $\mu_{ij} > \lambda_i$, where λ_i is the demand rate for firm i and μ_{ij} is the production rate at stage j of the firm i. A realistic example for the model proposed in this section could be a centralized supply chain consisting of a manufacturer and a third party shipping company, where a single PDT is quoted to the customer by the manufacturer. The times for the manufacturer to produce the product and for the shipping company to ship it are both random variables with different parameters. In an MTO environment and such settings, the time that spans from the point when the order is placed until it is shipped to the customer (including all the queue times) will be the total wait time that the customer experiences. An example for this system could be a supplier (as stage one) that

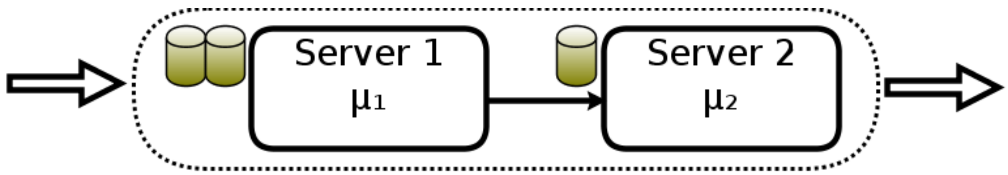

Figure 3.6: A supply chain consisting of two stage, where each stage has a different production rate.

supplies the material to the manufacturer (stage two) that will produce the product. These two stages work in a serial fashion and have different production (delivery) rates.

Next we will present the expression for the quality of service in the new system.

Proposition 3.5 *The expression for the quality of service, where the supply chain has two stages with exponential stage durations with different parameters is equal to*

$$Q_i(T_i, \lambda_i) = 1 + \frac{1}{\mu_{i1} - \mu_{i2}} \left[(\mu_{i2} - \lambda_i) e^{-(\mu_{i1} - \lambda_i) T_i} - (\mu_{i1} - \lambda_i) e^{-(\mu_{i2} - \lambda_i) T_i} \right], \tag{3.22}$$

where μ_{i1} and μ_{i2} are the production rates of the first and second stages of firm i.

Proof. Based on Lemmas 3.1 and 3.2, the distribution of total time in the system for stage one is exponential with parameter $(\mu_{i1} - \lambda_i)$ and is $(\mu_{i2} - \lambda_i)$ for stage two. Assume that Y_1 represents the total time spent in stage one of the supply chain and has the p.d.f. f_{Y_1} and c.d.f. F_{Y_1}. Similarly, let Y_2 be the random variable representing the time spent in the second stage of the supply chain and has the p.d.f. f_{Y_2} and c.d.f. F_{Y_2}. Then we have,

$$f_{Y_1}(t) = (\mu_{i1} - \lambda_i) e^{-(\mu_{i1} - \lambda_i) t},$$
$$f_{Y_2}(t) = (\mu_{i2} - \lambda_i) e^{-(\mu_{i2} - \lambda_i) t}.$$

Based on the assumptions, if W represents the total time spent in the system, we should have $W = Y_1 + Y_2$. If W has the c.d.f. $Q_W = Q$, then using the Laplace transform we can write,

$$\widetilde{Q}(s) = \frac{1}{s} \widetilde{f}_{Y_1}(s) \widetilde{f}_{Y_1}(s),$$

where $\widetilde{Q}(s)$ is the Laplace transform of Q and $\widetilde{f}_{Y_1}(s)$ and $\widetilde{f}_{Y_1}(s)$ are the Laplace transforms of $f_{Y_1}(t)$ and $f_{Y_2}(t)$, respectively. Therefore, we have,

$$\widetilde{Q}(s) = \frac{(\mu_{i1} - \lambda_i)(\mu_{i2} - \lambda_i)}{s(\mu_{i1} - \lambda_i + s)(\mu_{i1} - \lambda_i + s)}. \tag{3.23}$$

Using the inverse of the Laplace transform in equation (3.23) to find $Q(t)$ we will have,

$$Q(t_i) = 1 + \frac{1}{\mu_{i1} - \mu_{i2}} \left[(\mu_{i2} - \lambda_i) e^{-(\mu_{i1} - \lambda_i) t_i} - (\mu_{i1} - \lambda_i) e^{-(\mu_{i2} - \lambda_i) t_i} \right]. \tag{3.24}$$

But since the demand rate also determines the quality of service, we can write (3.24) as follows,

$$Q_i(T_i, \lambda_i) = 1 + \frac{1}{\mu_{i1} - \mu_{i2}} \left[(\mu_{i2} - \lambda_i) e^{-(\mu_{i1} - \lambda_i) T_i} - (\mu_{i1} - \lambda_i) e^{-(\mu_{i2} - \lambda_i) T_i} \right].$$

■

Since Q is the c.d.f. of the random variable W, the following should hold,

$$0 \leq Q < 1. \tag{3.25}$$

or,

$$0 \leq 1 + \frac{1}{\mu_{i1} - \mu_{i2}} \left[(\mu_{i2} - \lambda_i) e^{-(\mu_{i1} - \lambda_i) T_i} - (\mu_{i1} - \lambda_i) e^{-(\mu_{i2} - \lambda_i) T_i} \right] < 1.$$

The right-hand side of inequality (3.25) simplifies to,

$$\frac{1}{\mu_{i1} - \mu_{i2}} \left[(\mu_{i2} - \lambda_i) e^{-(\mu_{i1} - \lambda_i) T_i} - (\mu_{i1} - \lambda_i) e^{-(\mu_{i2} - \lambda_i) T_i} \right] < 0. \tag{3.26}$$

Assuming $\mu_{i1} > \mu_{i2}$, based on inequality (3.26), we should have,

$$(\mu_{i2} - \lambda_i) e^{-(\mu_{i1} - \lambda_i) T_i} - (\mu_{i1} - \lambda_i) e^{-(\mu_{i2} - \lambda_i) T_i} < 0,$$

which results in the condition,

$$T_i > \frac{1}{\mu_{i2} - \mu_{i1}} \ln \left(\frac{\mu_{i1} - \lambda_i}{\mu_{i2} - \lambda_i} \right), \tag{3.27}$$

which is always satisfied as $\mu_{i2} - \mu_{i1} < 0$ and $\ln((\mu_{i1} - \lambda_i)/(\mu_{i2} - \lambda_i)) > 0$, resulting in the right-hand side of inequality (3.27) assuming the negative sign. Therefore, when $\mu_{i1} > \mu_{i2}$, any positive T will satisfy condition (3.27).

Now assuming $\mu_{i1} < \mu_{i2}$, based on inequality (3.26), we should have,

$$(\mu_{i2} - \lambda_i) e^{-(\mu_{i1} - \lambda_i) T_i} - (\mu_{i1} - \lambda_i) e^{-(\mu_{i2} - \lambda_i) T_i} > 0,$$

which results in the same condition (3.27). Assuming $\mu_{i1} < \mu_{i2}$, this condition is always satisfied again as the denominator is always positive and the numerator is always negative. Therefore, any positive T will satisfy that condition for the case $\mu_{i1} < \mu_{i2}$. As a result, the right-hand side of the condition (3.25) is always satisfied.

Focusing our attention to the left-hand side of condition (3.25) we get,

$$\frac{1}{\mu_{i2}-\mu_{i1}}\left[(\mu_{i2}-\lambda_i)e^{-(\mu_{i1}-\lambda_i)T_i}-(\mu_{i1}-\lambda_i)e^{-(\mu_{i2}-\lambda_i)T_i}\right]\leq 1.$$

This condition is equivalent to the following,

$$\begin{cases} (\mu_{i2}-\lambda_i)e^{-(\mu_{i1}-\lambda_i)T_i}-(\mu_{i1}-\lambda_i)e^{-(\mu_{i2}-\lambda_i)T_i}\geq \mu_{i2}-\mu_{i1}, \text{ if } \mu_{i1}>\mu_{i2} \\ (\mu_{i2}-\lambda_i)e^{-(\mu_{i1}-\lambda_i)T_i}-(\mu_{i1}-\lambda_i)e^{-(\mu_{i2}-\lambda_i)T_i}\leq \mu_{i2}-\mu_{i1}, \text{ if } \mu_{i1}<\mu_{i2} \end{cases}.$$

Here we will follow the same procedure as was used by Shang and Liu [53] but with the difference that we will apply them to our own supply chain settings and will introduce new conditions on the problem.

Lemma 3.11 *For any given T, the quality of service is strictly decreasing in the demand rate into the firm, λ, as long as we have $T(\min\{\mu_1,\mu_2\}-\lambda)>1$.*

Proof. For the sake of simplicity, we drop the index i and take the derivative of Q w.r.t. λ. We have,

$$\frac{\partial Q(T,\lambda)}{\partial\lambda}=\frac{e^{-(\mu_1-\lambda)T}((\mu_2-\lambda)T-1)-e^{-(\mu_2-\lambda)T}((\mu_1-\lambda)T-1)}{\mu_1-\mu_2}. \tag{3.28}$$

Now let's assume that $\mu_1>\mu_2$. Then we have $0<\exp\{-(\mu_1-\lambda)T\}<\exp\{-(\mu_2-\lambda)T\}$. Also, we have $((\mu_2-\lambda)T-1)<((\mu_1-\lambda)T-1)$. We know that as long as $(\mu_1-\lambda)T-1>0$, the following will hold,

$$e^{-(\mu_1-\lambda)T}((\mu_2-\lambda)T-1)-e^{-(\mu_2-\lambda)T}((\mu_1-\lambda)T-1)<0$$

and thus,

$$\frac{\partial Q(T,\lambda)}{\partial\lambda}<0.$$

Obviously, the above results are always valid if we have

$$0<((\mu_2-\lambda)T-1)<((\mu_1-\lambda)T-1).$$

Also, the condition $((\mu_2-\lambda)T-1)<0<((\mu_1-\lambda)T-1)$ will also ensure its validity. For the case $((\mu_2-\lambda)T-1)<((\mu_1-\lambda)T-1)<0$, the condition holds as long as $\left|e^{-(\mu_1-\lambda)T}((\mu_2-\lambda)T-1)\right|>\left|e^{-(\mu_2-\lambda)T}((\mu_1-\lambda)T-1)\right|$. Now, in order to avoid this cumbersome condition, we sacrifice a small part of the solution set and apply the sufficient condition $((\mu_1-\lambda)T-1)>0$.

Now if we have $\mu_1<\mu_2$ we get $0<\exp\{-(\mu_2-\lambda)T\}<\exp\{-(\mu_1-\lambda)T\}$ and also $((\mu_1-\lambda)T-1)<((\mu_2-\lambda)T-1)$. Therefore, for the numerator of (3.28) we have,

$$e^{-(\mu_1-\lambda)T}((\mu_2-\lambda)T-1)-e^{-(\mu_2-\lambda)T}((\mu_1-\lambda)T-1)>0$$

but since $\mu_1 < \mu_2$, we have,

$$\frac{\partial Q(T,\lambda)}{\partial \lambda} < 0.$$

Similarly, here we apply the sufficient condition $(\mu_2 - \lambda)\,T - 1 > 0$ for the condition to hold.

This proves that as long as $T\,(\min\{\mu_1, \mu_2\} - \lambda) > 1$ holds, the quality of service is strictly decreasing in the demand rate in the firm. ■

We are again interested in finding the optimal promised delivery time in equilibrium in a duopolistic market with two firms with both having two stages but different production rates at each stage. Again, because of symmetry we focus on one firm only. Firm i solves the following optimization problem,

$$\max_{T_i} \lambda_i(\mathbf{T})$$
$$\text{subject to } \lambda_i = \Lambda S_i(U_i),$$

where $S_i(U_i) = \frac{e^{U_i}}{\sum_{j=1}^{2} e^{U_j}}$ and Λ is fixed such that

$$\Lambda = \lambda_i + \lambda_j,$$

and

$$U_i(T_i, \lambda_i) = \beta_{0i} - \beta_T T_i + \beta_Q Q_i(T_i, \lambda_i).$$

Again, the demand function $L(T_i, \lambda_i)$ and the common allocation parameter ℓ are defined similarly to the Section 3.2 as introduced in Shang and Liu [53] as follows,

$$L_i(T_i, \lambda_i) = \ln \lambda_i - U_i(T_i, \lambda_i). \tag{3.29}$$

Also, we have $L_i(T_i, \lambda_i) = L_j(T_j, \lambda_j)$ as in equation (3.6). And from equation (3.7) we have,

$$L_i(T_i, \lambda_i) = \ell(\mathbf{T}). \tag{3.30}$$

Lemma 3.12 *For any given T_i, the function $L_i(T_i, \lambda_i)$ is always increasing in λ_i as long as $T\,(\min\{\mu_1, \mu_2\} - \lambda) > 1$ holds.*

Proof. Taking the derivative w.r.t. λ_i from both sides of equation (3.29) we have,

$$\begin{aligned}\frac{\partial L_i(T_i, \lambda_i)}{\partial \lambda_i} &= \frac{1}{\lambda_i} - \frac{\partial U_i(T_i, \lambda_i)}{\partial \lambda_i} \\ &= \frac{1}{\lambda_i} - \beta_Q \frac{\partial Q_i(T_i, \lambda_i)}{\partial \lambda_i}\end{aligned}$$

but since we know from Lemma (3.11) that as long as $T\,(\min\{\mu_1, \mu_2\} - \lambda) > 1$ we have $\partial Q_i(T_i, \lambda_i)/\partial \lambda_i < 0$, then we have,

$$\frac{\partial L_i(T_i, \lambda_i)}{\partial \lambda_i} > 0$$

■

For any given set of decision variables $\mathbf{T}$, at $\lambda_i = 0^+$ we have,

$$\lim_{\lambda_i \to 0^+} L_i(T_i, \lambda_i) = \lim_{\lambda_i \to 0^+} \{\ln \lambda_i - U_i(T_i, \lambda_i)\} = -\infty, \tag{3.31}$$

as at a demand rate of $\lambda_i = 0^+$, customers will have a positive utility function for firm i. Also, at $\lambda_i = \infty$ we have,

$$\lim_{\lambda_i \to \infty} L_i(T_i, \lambda_i) = \lim_{\lambda_i \to \infty} \{\ln \lambda_i - U_i(T_i, \lambda_i)\} = \infty, \tag{3.32}$$

as with $\lambda_i = \infty$, the customers' utility function for firm i approaches zero.

Therefore, under the minor condition $T(\min\{\mu_1, \mu_2\} - \lambda) > 1$, based on (3.31) and (3.32) as well as Lemma 3.12, for any given set of decision variables $\mathbf{T}$, there is always a unique demand rate λ_i that would satisfy equation (3.30).

As in previous sections, and also used by Shang and Liu [53], we proceed with finding the relationship between $\partial \lambda_i/\partial T_i$, $\partial \ell(\mathbf{T})/\partial T_i$ and Q_i. This is dealt with in Lemma 3.13.

Lemma 3.13 *For any given promised delivery time for the other firm T_j, as long as we have $T(\min\{\mu_1, \mu_2\} - \lambda) > 1$, the following holds,*

$$\partial \lambda_i(\mathbf{T})/\partial T_i \begin{cases} > \\ < \\ = \end{cases} 0, \iff \partial \ell(\mathbf{T})/\partial T_i \begin{cases} < \\ > \\ = \end{cases} 0, \iff Q_i \begin{cases} < \\ > \\ = \end{cases} \omega_i(T_i, \lambda_i) \tag{3.33}$$

where,

$$\omega_i(T_i, \lambda_i) = 1 - \frac{\beta_T \left[e^{-(\mu_{i1}-\lambda_i)T_i}(\mu_{i2} - \lambda_i) - e^{-(\mu_{i2}-\lambda_i)T_i}(\mu_{i1} - \lambda_i)\right]}{\beta_Q (\mu_{i1} - \lambda_i)(\mu_{i2} - \lambda_i)\left(e^{-(\mu_{i1}-\lambda_i)T_i} - e^{-(\mu_{i2}-\lambda_i)T_i}\right)}. \tag{3.34}$$

The proof of Lemma 3.13 can be found in Appendix A.

From Lemma (3.13) we can see that sufficient conditions for the existence of one or multiple points that satisfy the first-order condition $\partial \lambda_i/\partial T_i = 0$ is when $Q = \omega_i(T_i, \lambda_i)$ as well as $T(\min\{\mu_1, \mu_2\} - \lambda) > 1$ where,

$$\omega_i(T_i, \lambda_i) = 1 - \frac{\beta_T \left[e^{-(\mu_{i1}-\lambda_i)T_i}(\mu_{i2} - \lambda_i) - e^{-(\mu_{i2}-\lambda_i)T_i}(\mu_{i1} - \lambda_i)\right]}{\beta_Q (\mu_{i1} - \lambda_i)(\mu_{i2} - \lambda_i)\left(e^{-(\mu_{i1}-\lambda_i)T_i} - e^{-(\mu_{i2}-\lambda_i)T_i}\right)},$$

Now we will proceed with presenting the next proposition that will find sufficient conditions for the second-order conditions to be satisfied.

Proposition 3.6 *At least one interior Nash equilibrium exists for the promised delivery time (PDT) when we have two symmetric supply chains, each with two stages with $M/M/1$ queueing settings and different production rates. For the existence of the interior Nash equilibria we need to have the following conditions,*

$$T\left(\min\{\mu_1, \mu_2\} - \lambda\right) > 1, \tag{3.35}$$

and,

$$T_i > \frac{\ln\left[(\mu_{i2} - \lambda_i) \,/\, (\mu_{i1} - \lambda_i)\right]}{(\mu_{i2} - \mu_{i1})}. \tag{3.36}$$

The proof of Proposition 3.6 can be found in Appendix A.

Unfortunately, a closed-form solution for this part could not be found. As a result, we conclude our analysis for this part with the sufficient conditions that were found in Proposition 3.6. Remember in the previous sections, the point $\mathbf{T} = \{T_1, T_2\}$ that satisfied the first-order condition was the point that satisfied the following equation,

$$\omega_i(T_i, \lambda_i) = Q_i(T_i, \lambda_i), \tag{3.37}$$

where from (3.34) we have,

$$\omega_i(T_i, \lambda_i) = 1 - \frac{\beta_T \left[e^{-(\mu_{i1}-\lambda_i)T_i}\left(\mu_{i2} - \lambda_i\right) - e^{-(\mu_{i2}-\lambda_i)T_i}\left(\mu_{i1} - \lambda_i\right)\right]}{\beta_Q\left(\mu_{i1} - \lambda_i\right)\left(\mu_{i2} - \lambda_i\right)\left(e^{-(\mu_{i1}-\lambda_i)T_i} - e^{-(\mu_{i2}-\lambda_i)T_i}\right)},$$

and from (3.22) we have,

$$Q_i(T_i, \lambda_i) = 1 + \frac{1}{\mu_{i1} - \mu_{i2}}\left[(\mu_{i2} - \lambda_i)e^{-(\mu_{i1}-\lambda_i)T_i} - (\mu_{i1} - \lambda_i)e^{-(\mu_{i2}-\lambda_i)T_i}\right].$$

Equation (3.37) does not produce any closed-form solution for $\mathbf{T}$. However, we can always try to find the Nash equilibrium of the game numerically for a specific setting of the problem. This is done in the next part of this section.

3.4.1 Numerical Example

In this section we will define a specific setting for the problem and try to find the Nash equilibrium of the game by enumeration and also by using the best-response curves. Since no closed-form solution is in hand, to find every best-response point we have to try all the possibilities until the left-hand side and right-hand side of equation (3.37) are equal. This has also been explained briefly in Section 3.2.1. We will show the process in more depth in the next example.

Example 3.8 *Let $\beta_{oi} = 0$, $i = 1, 2$, $\beta_T = 0.2$, $\beta_Q = 2$ and $\Lambda = 1$. Also, assume the following production rates for the firms $\mu_{11} = 2$, $\mu_{12} = 4$, $\mu_{21} = 3$ and $\mu_{22} = 5$. Now to find the best-response of firm 1 (T_1^*) to any decision made by firm 2 (T_2), we have to*

T_1	λ_1	λ_2	$\omega_1(T_1, \lambda_1)$	$Q_1(T_1, \lambda_1)$
1.000	0.5388	0.4611	0.9253	0.6214
2.000	0.6168	0.3832	0.9269	0.8944
2.200	0.6190	0.3810	0.9270	0.9193
2.250	0.6192	0.3808	0.9270	0.9247
2.260	0.6192	0.3808	0.9271	0.9257
2.270	0.6192	0.3808	0.9271	0.9267
2.274	**0.6192**	**0.3808**	**0.9271**	**0.9271**
2.280	0.6192	0.3808	0.9271	0.9277
2.300	0.6192	0.3808	0.9271	0.9297
2.400	0.6187	0.3813	0.9272	0.9387

Table 3.14: Summary of results for the numerical analysis for changing T_1, where $\mu_{1,1} = 2$, $\mu_{1,2} = 4$, $\mu_{2,1} = 3$, $\mu_{2,2} = 5$ and T_2 is fixed at 0.5. The row that is highlighted is the point where ω_1 and Q_1 are equal, and thus the best response of firm 1, $T_1^* = 2.274$.

change T_1 until both sides of equation (3.37) are equal. Let's assume the second firm has chosen $T_2 = 0.5$. To find the best response of firm 1 (T_1^*), we should change T_1 and find the corresponding values for the expressions $\omega_1(T_1, \lambda_1)$ and $Q_1(T_1, \lambda_1)$ until the two become equal. Once the two expressions are equal, then the equation (3.37) is satisfied and the T_1 chosen is the best response of firm 1 (T_1^*) to the choice of firm 2 (0.5). In Table 3.14, we show different values of T_1 in response to the decision made by firm 2 ($T_2 = 0.5$) as well as the values for $\omega_1(T_1, \lambda_1)$ and $Q_1(T_1, \lambda_1)$.

The values of $\omega_1(T_1, \lambda_1)$ and $Q_1(T_1, \lambda_1)$ are shown in Figure 3.7.

Doing this process for different values of T_2, we can find the best responses of firm 1 to firm 2. The same procedure can be also done for firm 2 to find its best response to a decision made by firm 1. The results are summarized in Table 3.15.

Based on the points found, we will plot the best-response curves of the firms. The best-response curves are shown in Figure 3.8.

We can observe from Figure 3.8 that the Nash equilibrium should be around the point $(2.2, 1.6)$. Working with the functions around that area and using trial and error, we found the Nash equilibrium of the game. This information is summarized in Table 3.16.

Therefore, the Nash equilibrium of the game occurs at the point $\mathbf{T} = \{2.138, 1.611\}$.

3.5 Conclusion and Suggestions for Future Research

In this chapter we investigated the solution to a single problem using three settings for the supply chains. We built upon the models introduced by Ho and Zheng [36]

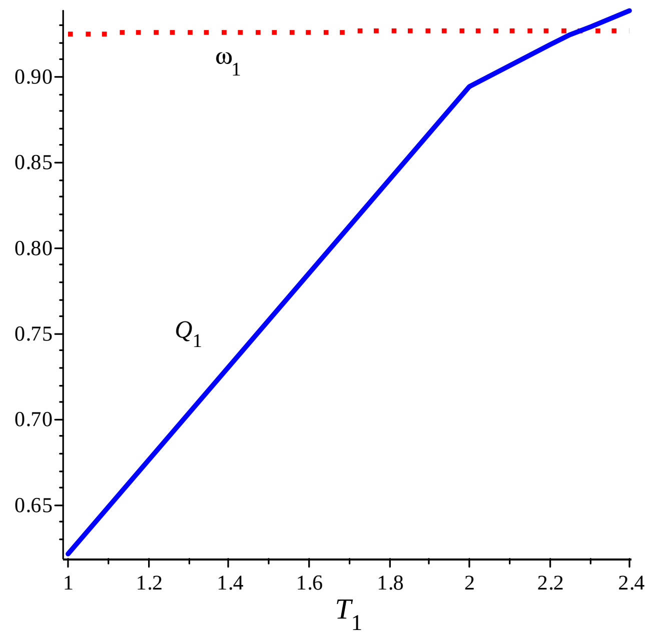

Figure 3.7: The values for the functions ω_1 and Q_1 when T_1 changes and $T_2 = 0.5$. The function ω_1 is shown in dots and Q_1 is the shown with the solid line. The point of intersection is $T_1 = 2.274$.

T_1^*	T_2	λ_1	λ_2	T_2^*	T_1	λ_1	λ_2
2.274	0.5	0.6192	0.3808	1.692	0.5	0.2736	0.7264
2.194	0.8	0.5288	0.4712	1.659	0.8	0.3482	0.6518
2.164	1	0.4941	0.5059	1.642	1	0.3864	0.6136
2.143	1.2	0.4741	0.5259	1.630	1.2	0.4150	0.5850
2.139	1.5	0.4623	0.5377	1.619	1.5	0.4428	0.5572
2.140	1.8	0.4633	0.5367	1.613	1.8	0.4570	0.5430
2.144	2	0.4679	0.5321	1.611	2	0.4609	0.5391
2.149	2.2	0.4743	0.5257	1.611	2.2	0.4614	0.5386
2.155	2.4	0.4819	0.5181	1.611	2.4	0.4595	0.5405
2.176	3	0.5080	0.4920	1.618	3	0.4436	0.5564
2.196	3.5	0.5310	0.4690	1.627	3.5	0.4243	0.5757
2.216	4	0.5542	0.4457	1.636	4	0.4027	0.5973

Table 3.15: Summary of results for the best response of each firm based on the decision made by the other firm.

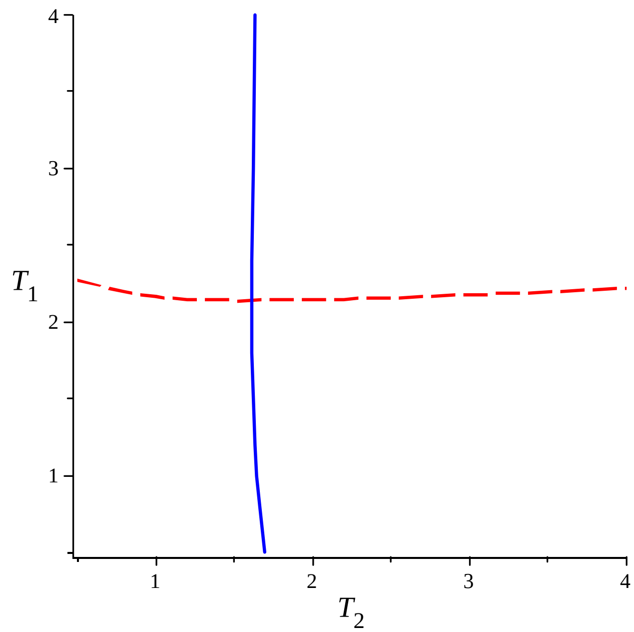

Figure 3.8: The best-response curves of the two firms. The dashed line is the best-response curve of firm 1 and that of the second firm is shown with a solid line. The intersect at around $T_1 = 2.2$ and $T_2 = 1.6$.

T_1^*	T_2	λ_1	λ_2
2.138	1.611	0.5288	0.4712

T_2^*	T_1	λ_1	λ_2
1.611	2.138	0.5288	0.4712

Table 3.16: Summary of results for the Nash equilibrium of the game. Notice that the response of firms result in the same solution. Also notice that no matter which firm is responding to the other player's decision, the demand rates into firms stays the same.

and Shang and Liu [53] by changing the settings of their models from a single stage firm to multiple stage firms with the same or different production rates. We followed the same procedures that were used by Shang and Liu [53] and used our own supply chain settings. These settings included the following cases; (a) when each firm consists of two stages and has identical production rates in both stages, (b) when each firm consists of k stages and has identical production rates in all stages and, (c) when each firm consists of two stages and has different production rates in each stage. The first two settings were studied in Sections 3.2 and 3.3, where we found closed-form solutions for the optimal strategies (PDTs) for firms at the Nash equilibrium of the game. We proved that under mild conditions, there is a unique interior Nash equilibrium in this game. The last setting was studied in Section 3.4, where a closed-form solution for the optimal strategies for the firms could not be derived. We derived sufficient conditions for the existence of at least one interior Nash equilibrium. In this section, we came up with an equation that under the sufficient conditions will yield the optimal strategies and the interior Nash equilibrium of the game.

This work has improved the results of Ho and Zheng [36] and Shang and Liu [53] because it has expanded the scope of the problem to a wider range of situations. This is because the process of delivering a product to the customers rarely depends on production at only one stage. Usually firms (supply chains) consist of different stages where the production or service delivery rates at each stage are different. The model presented in this chapter aims to represent these kinds of processes where serial stages have to be used for delivering a product or a service to the ultimate customer. As an example, the process of delivering a package at the UPS consists of more than one stage with different production (service delivery) rates. The models presented by Ho and Zheng [36] and Shang and Liu [53] cannot be used there because they only study firms with one stage of production (service delivery). The results from this chapter help companies like the UPS to determine the optimal PDT in competition with their rivals.

At the end of each section, we confirmed our analytical results using different numerical examples. Using the numerical examples, we also performed numerical sensitivity analyses. These analyses provided insight into the behavior of the objective functions as well as the optimal strategies in reaction to changes in parameter values of the problems.

For future research opportunities in this context, other settings of supply chain design may be explored. For instance, we have assumed that supply chains are both serial. In other words, the product needed to travel through all stages of the supply chain in order to be deemed completed. In reality, many processes do not require the product to go through all stages. Occasionally, products may be sent back to a previous stage for rework or correction as well. Studying the problem with a general phase-type distribution that captures these characteristics will be a great addition to the literature. As another area for future research, we suggest adding other decision variables to the problem. Price for instance, may serve as a good candidate to be used as a decision variable in addition to promised delivery time. In this kind of competition, firms will be quoting their determined promised delivery time as well as a suggested price to the customers. The customers' utility function will capture the behavior of the customers towards the quoted price and PDT and will determine the market share for each firm.

Chapter 4

A Duopolistic Competition Based on Promised Delivery Times with Tardiness Costs

The duopoly game studied in the previous chapter was a competition game based on promised delivery time. Firms tried to attract as many customers as possible by manipulating their promised delivery time. Based on a utility function that took into account the promised delivery time as well as congestion level at each firm, the market share of each firm was determined. In this chapter, we will look at another duopolistic game that is played based on the promised delivery time. In this game, firms compete for the business of a single customer. They try to win the business of the customer and earn a profit by manipulating their promised delivery time.

The structure of this chapter is as follows. In Section 4.1 we will introduce the problem by giving an overview on the nature of the competition. We will also mention relevant papers from the literature. In Section 4.2, we will introduce the notation used in the model and will build the model. In Section 4.3 we will find the Nash equilibrium of the game. Based on the results from the analysis we will solve multiple numerical examples and will perform a sensitivity analysis whose results will be presented in Section 4.4. Finally, in Section 4.5 we will give a conclusion to the findings in the chapter and will present some managerial insights into the problem.

4.1 Introduction and Problem Overview

The issue of selecting a manufacturer (supplier) has been discussed in the operations management literature for decades. The notion of *time* has always been present as a criterion for selecting the best manufacturer. According to a review of 74 articles discussing the issue of supplier selection, Weber et al. [64] have concluded that delivery performance ranks in the top three most important criteria for supplier selection in the literature along with cost and quality. These criteria were chosen as the top three out of 23 criteria first introduced by Dickson [20] in his article discussing supplier selection. Time-based activities of the manufacturers can be captured in the delivery performance criterion in this list. These studies show the importance of delivery performance (time-based) in the body of literature.

A more recent study captures the importance of time-based performance of the suppliers from the perspective of practitioners. Verma and Pullman [62] showed that based on the 58 completed surveys that they received from managers, they ranked "quality" as the most important criterion evaluated by managers in their process of choosing a supplier. Quality was followed by "on-time delivery" (quality of service) as the second most important factor for selecting a supplier. "Leadtime" (promised delivery time) came up as the fourth most important criterion after "unit cost". The results of this study show how important time-based activities of manufacturers are to ensure their success in the market.

In this chapter, we will also focus on the effect of time on the selection of a manufacturer by a customer. We will consider a duopolistic market with two firms (manufacturers) and only one customer. The firms are competing against each other to win the business of the customer. In other words, the customer will choose one of the firms based on certain criteria that are reflected in her utility function. Only one of the two firms will win the business of the customer and the other one will be out of the game. An example of such market can be found in businesses, where a usually expensive product with a long production time is manufactured, where the market size is very small and the suppliers are not plentiful. An instance of this market is the aircraft purchasing market, where the size of the market is limited and the number of products sold are very few compared to other markets like the auto industry.

In this chapter, we are assuming that the two firms will announce their promised delivery time (PDT hereafter) to the customer simultaneously. Based on that, the customer will choose the winning firm. The losing firm will leave the game and the winning firm will have to start production to deliver the product to the customer. The customer gives the business to the firm that announces the shorter PDT. The trade-off situation that appears in this situation leads to an interesting problem that we intend to analyze in this study. On the one hand, firms would prefer to quote a short leadtime to win the business. On the other hand, they are leaning towards a longer quoted leadtime to minimize tardiness costs.

We are aware that in similar situations in real life, customers tend to pay attention to a greater variety of criteria in making their decisions (for a review see, Weber et al. [64]). However, for the sake of simplicity and ease of analysis, this model assumes that the only criterion observed by the customer is the promised delivery time.

It may be tempting to suggest that the best strategy for any firm playing this game would be to quote as short a PDT as possible. However, it is important to note that winning the business of the customer is not the ultimate goal of the firms. The firms are trying to maximize their profit. Since the production of the product involves procedures that are costly, it becomes important for firms to quote a PDT that, if won, will yield as big a profit for them as possible. For instance, if a firm quotes an unrealistically short PDT, it will increase its chances of winning the business. However, once the business is won, the company is committed to delivering the product and that is when other costs involved in the model kick in. One of these costs will be tardiness cost. This type of cost will be incurred by the firm in proportion to the time interval between the realized delivery time and the PDT if the delivery was late. This cost is designed as a penalty to discourage the firms from quoting unrealistically short PDTs with the mere purpose of winning customer's business. Tardiness cost and penalties have been used in many articles in the literature that discuss the issue of setting quoted leadtimes to maximize expected profit, e.g., see, Bertrand and van Ooijen [8], Chatterjee et al. [14] and Wu et al. [66].

This chapter contributes to the literature by studying a model, where the customer will choose between suppliers taking use of PDTs that are quoted to her. The suppliers will have to consider the selecting process as well as their internal production and tardiness costs. The contribution of the results to the literature will be in the context of supplier selection and maximization of profit in the presence of tardiness costs for suppliers in the context of time-based competition where the competition is based on promised delivery times.

In the following section, we will introduce the sources of revenue and cost for the firms and demonstrate the behavior of their profit functions in this game.

4.2 The Model

In this section, we will introduce the notations used in the model and present the model to capture the firms' expected profit functions. To incorporate customer's choice (the winning firm), we are defining a binary variable I_i, $i = 1, 2$ as follows,

$$I_i(\mathbf{T}) = \begin{cases} 0, & \text{if } T_i > T_j + \Delta, \\ 1, & \text{otherwise,} \end{cases}$$

where $\mathbf{T} = \{T_i, T_j\}$ and T_i and T_j are the PDT values quoted by the first and the second firms, respectively. The value Δ is assumed to be customer's threshold with regards to her sensitivity to time. This value is assumed to be relatively small to the values of the PDTs. For instance, for a product that has a leadtime of two years, this Δ can be assumed to be one week. Any two PDTs that are closer to each other than one week, will be the same in the eyes of the customer. This value is the difference in time that is visible to the customer. The customer is basically unable to see the difference between any two PDTs that have a difference less than Δ. It should also be noted that the value Δ is exogenously determined and may be different for each single customer. It is not in any ways related to any other parameters in the model. This value will be determined in consultation with the customer and varies from one customer to another.

The customer pays firm i the value $R_i(\mathbf{T})$ based on the following condition,

$$R_i(\mathbf{T}) = \begin{cases} Pe^{-\gamma T_i}, & \text{if } I_i(\mathbf{T}) = 1, \\ 0, & \text{if } I_i(\mathbf{T}) = 0, \end{cases}$$

where P is the full price that will be paid to the firm that quotes an immediate delivery ($T = 0$). In this problem, we are assuming that the full price P is exogenously determined by the forces of the market. Also, the positive parameter γ is the depreciation rate for the full price and is the rate at which the full price is depreciated. Obviously, if $I_i(\mathbf{T}) = 0$ (firm i loses the business), the firm does not get paid anything. Also note, assuming firm i wins the business, for $T_i = 0$ the firm will be paid the full price P. Also note that the reward to the winning firm (here firm i) goes to zero as T_i goes to infinity.

Let's assume that X_i, $i = 1, 2$ is the random variable representing time to completion for firm i with c.d.f. F_i. In this part, we are assuming that the time to completion is exponential with parameter λ_i for firm i. The parameter λ_i can be interpreted as the production rate for the firm. Therefore, the c.d.f. of firm i will be defined as,

$$F_i(T_i) = 1 - e^{-\lambda_i T_i},\ \lambda_i > 0 \text{ and } T_i \geq 0.$$

Although the assumption for exponential manufacturing times has been made to make the analysis more tractable, this assumption is not completely without empirical support. Shanthikumar and Sumita [54] show that the time spent in a large class of dynamic job shops can be approximated using an exponential random variable. They also claim that their results are applicable to other manufacturing models like some flexible manufacturing systems.

It should be noted that the winning firm will be announced right after the PDT values are quoted and the winning firm should start the production right away to deliver the product. There are two costs associated with production and delivery, the production cost $C_i(\lambda_i)$ and the delay cost $K_i(T_i)$. The production cost for firm i is assumed to be quadratic based on the production rate and can be defined as follows,

$$C_i(\lambda_i) = a_i \lambda_i^2,\ a_i, \lambda_i > 0.$$

where a_i is a parameter value to adjust the effect of production rate on the production cost. A quadratic cost function has been used by many authors. For example, Eliashberg and Steinberg [23] and [22] discuss various justifications for quadratic cost functions. However, any other form of production cost, that depends on the production rate only, can replace it in this study. The reason is that since the production cost is not a function of the promised delivery time (T), it does not affect the characteristics of the best-response functions and the Nash equilibrium of the game. Therefore, wherever the production cost $a_i \lambda_i^2$ appears in the conditions, it can be replaced by any other production cost $C_i(\lambda_i)$. It is important to mention that in this model, we do not

Parameter Values	Description
P :	Full price for the item
T_i :	Promised delivery time quoted by firm i
γ :	Depreciation rate for the full price
λ_i :	Production rate for firm i
a_i :	Production cost rate for firm i
θ :	Rate of penalization for delayed delivery
Δ :	Customer's threshold w.r.t. sensitivity to time
$C_i(\lambda_i)$:	Production cost of firm i
$K_i(T_i)$:	Delay cost of firm i
$I_i(\mathbf{T})$:	Binary variable indicating the winning status
$J_i(\mathbf{T})$:	Expected profit function for firm i

Table 4.1: Summary of notations and symbols.

study the case, where a higher PDT quoted by the firm would trigger an increase in capacity. We believe that a change in capacity falls under the category of strategical decisions made by firms, whereas the competition based on PDT is performed more at the marketing level. As a result, changing the capacity based on PDT will be out of the scope of this chapter.

The other cost for the winning firm is the cost of delay. If the firm delivers the product after the promised delivery time, it will incur a cost of delay and has to pay the customer the fine. This cost denoted by $K_i(T_i)$ is defined as follows,

$$K_i(T_i) = \theta \int_{T_i}^{\infty} (t - T_i) dF_i(t) = \theta \frac{1}{\lambda_i} e^{-\lambda_i T_i},\ i = 1, 2, \tag{4.1}$$

where θ is the rate at which the firm is penalized for the duration of delay in delivering the product. This cost is decreasing and convex in T_i.

The objective function of firm i is defined as its expected profit function and is shown in equation (4.2).

$$J_i(\mathbf{T}) = R_i(\mathbf{T}) - I_i(\mathbf{T}) \left[C_i(\lambda_i) + K_i(T_i) \right],\ i = 1, 2. \tag{4.2}$$

Substituting the values for each function in (4.2), we get the following,

$$J_i(\mathbf{T}) = I_i(\mathbf{T}) \left[P e^{-\gamma T_i} - a_i \lambda_i^2 - \theta \frac{1}{\lambda_i} e^{-\lambda_i T_i} \right],\ i = 1, 2. \tag{4.3}$$

Table 4.1 summarizes all the parameter values for this model.

Depending on the parameter values, the expected profit functions for the firms may take different forms. For instance, it may start with a negative value at $T = 0$ and never become positive for any chosen value of T. Or alternatively, it may start with a positive value and drop from the point $T = 0$ instantly.

We believe that for this game to be meaningful in business terms, it should have two criteria. These criteria for both firms are as follows,

1. The profit function should be strictly increasing at $T = 0$.

2. The profit function should become positive in at least one region.

In the next section, we will present justifications for the above criteria and introduce mathematical conditions that will address these criteria for the problem.

4.2.1 Problem Conditions, Unimodality and Concavity Analysis at the Mode

In this section we will introduce the problem conditions (PC hereafter) that will satisfy the criteria introduced above. We will also prove the unimodality of the profit function and prove the concavity at its mode. We believe that none of these two conditions are restrictive. The first condition assures that the firms are discouraged from quoting the time $T = 0$ or unrealistically small PDTs. This condition is not restrictive because in the context of the problem in hand, we are usually talking about one-time purchases that involve big and costly products (ships, aeroplanes etc.). Quoting a promised delivery time of zero does not make any practical sense in these type of business environments. For this to happen, we would like our profit function to increase from the point $T = 0$. In Lemma 4.1 we find the mathematical condition that will address the first condition.

Lemma 4.1 *To address the first criterion for the problem introduced in Section 4.2, the following should hold,*

$$P\gamma < \theta. \tag{4.4}$$

Proof. Assuming only firm i is bidding in the game, we take the first derivative from the profit function in (4.3). We get,

$$\frac{\partial J_i(T_i)}{\partial T_i} = -P\gamma e^{-\gamma T_i} + \theta e^{-\lambda_i T_i}.$$

By substituting the value zero for T_i we get,

$$\left.\frac{\partial J_i(T_i)}{\partial T_i}\right|_{T_i=0} = \theta - P\gamma\ .$$

For the profit function to be increasing from the point $T = 0$, we need this expression to be positive. Therefore, we will have the following as the condition that ensures criterion number 1.

$$P\gamma < \theta.$$

■

Lemma 4.2 *The profit function (for the winning firm) has the negative sign when T goes to infinity.*

Proof. Assuming that only firm i is bidding at the game, we take the limit from equation 4.3 when T_i goes to infinity. We will have,

$$\lim_{T_i \to \infty} J_i(T_i) = -a_i \lambda_i^2 < 0.$$

The profit function for the losing firm is equal to zero in infinity. ■

Next, we will prove the unimodality of the profit function in Lemma 4.3.

Lemma 4.3 *Assuming that only firm i is bidding to the customer, the profit function for this firm is unimodal and the mode is at the point $\widehat{T}_i > 0$.*

Proof. Let's assume that firm i is the only firm bidding to the customer. Taking the first derivative from the expression in (4.3) and solving for T_i we get,

$$\widehat{T}_i = \frac{1}{\gamma - \lambda_i} \ln\left(\frac{P\gamma}{\theta}\right). \tag{4.5}$$

Also, since the function has only one internal mode, we conclude that the function is unimodal. We already know from Lemma 4.1 that the profit function is increasing at the beginning and from Lemma 4.2 that it is negative in $T_i = \infty$. Based on these results and the above equation for $\widehat{T}_i$, we can conclude that the profit function has only one mode $\widehat{T}_i$ and it is greater than zero. This results shows that $\gamma < \lambda_i$ should also always hold as we have $\widehat{T}_i > 0$. ■

Apart from proving Lemma 4.3, the proof also presented us with a condition on λ_i and γ. Based on the proof, the condition $\gamma < \lambda_i$ should always hold in this problem so that the PDT is greater than zero.

From Lemma In the next proposition, we will prove that the expected profit function is concave at its mode.

Proposition 4.1 *Assuming that only one firm is bidding to the customer, the expected profit function reaches its maximum value at its internal mode and not at a boundary point.*

Proof. Let's assume that firm i is the only firm bidding to the customer. We already know from Lemma 4.3 that the profit function is at its mode at the point $\widehat{T}_i = \ln(P\gamma/\theta)/(\gamma - \lambda_i)$. Taking the second derivative of the function in (4.3) we get,

$$\frac{\partial^2 J_i(\mathbf{T})}{\partial T_i^2} = P\gamma^2 e^{-\gamma T_i} - \theta \lambda_i e^{-\lambda_i T_i}. \tag{4.6}$$

Substituting the mode from (4.5) for T_i in (4.6) we get,

$$\frac{\partial^2 J_i(\mathbf{T})}{\partial T_i^2} = P\gamma^2 \exp\left(-\frac{\gamma}{\gamma-\lambda_i}\ln\left(\frac{P\gamma}{\theta}\right)\right) - \theta\lambda_i \exp\left(-\frac{\lambda_i}{\gamma-\lambda_i}\ln\left(\frac{P\gamma}{\theta}\right)\right). \quad (4.7)$$

Now in order to derive a contradiction, let's assume that the RHS of equation (4.7) is positive and thus the function is not reaching its maximum at its mode. Therefore, we will have,

$$P\gamma^2 \exp\left(-\frac{\gamma}{\gamma-\lambda_i}\ln\left(\frac{P\gamma}{\theta}\right)\right) - \theta\lambda_i \exp\left(-\frac{\lambda_i}{\gamma-\lambda_i}\ln\left(\frac{P\gamma}{\theta}\right)\right) > 0,$$

which gives,

$$\frac{P\gamma^2}{\theta\lambda_i} > \frac{\exp\left(-\lambda_i \ln\left(\frac{P\gamma}{\theta}\right)/(\gamma-\lambda_i)\right)}{\exp\left(-\gamma \ln\left(\frac{P\gamma}{\theta}\right)/(\gamma-\lambda_i)\right)}. \quad (4.8)$$

From (4.8) and after some simplifications we get,

$$\ln\left(\frac{P\gamma^2}{\theta\lambda_i}\right) > \ln\left(\frac{P\gamma}{\theta}\right),$$

which simplifies to,

$$\frac{\gamma}{\lambda_i} > 1. \quad (4.9)$$

We already know from Lemma 4.3 that $\gamma < \lambda_i$ should hold. This is a contradiction and thus proves that under the PC, the profit function is concave at its mode and also reaches its maximum at that point. This completes the proof. ■

Now we are ready to introduce the condition that satisfies criterion number 2 introduced in the previous section. Criterion number 2 is not restrictive either. The reason is, if there is no such region with a positive profit for the firm, there will be no incentive for that firm to take part in the game and place a bid to win the business. Therefore, we would naturally assume that we want the second criterion to hold as well. In Lemma 4.4, we will find the condition that should hold to satisfy criterion number 2.

Lemma 4.4 *To address the second criterion for the problem introduced at the end of Section 4.2, the following should hold for firm i,*

$$\text{if } P < a_i\lambda_i^2 + \frac{\theta}{\lambda_i}, \text{ then we should have } \frac{P\lambda_i}{\theta} < \frac{1-e^{-\lambda_i\beta_i}}{1-e^{-\gamma\beta_i}}, \quad (4.10)$$
$$\text{where } \beta_i = \frac{1}{\gamma-\lambda_i}\ln\left(\frac{P\gamma}{\theta}\right).$$

Proof. We assume that only firm i is bidding for the business of the customer. For criterion number 2 to hold we would consider two cases; (a) when the profit function starts with a positive value at $T_i = 0$, and (b) when the profit function starts with a negative value at $T_i = 0$. For case (a), we already have criterion number 2 satisfied. Therefore, we focus our attention on case (b). In case (b), we know that the function value is negative at $T_i = 0$. Therefore, we will have the following,

$$J_i(0) < 0,$$

which results in,

$$P < a_i\lambda_i^2 + \frac{\theta}{\lambda_i}. \tag{4.11}$$

We know from Lemma 4.3 and Proposition 4.1 that the function reaches its maximum value at the point $\widehat{T}_i = \ln(P\gamma/\theta)/(\gamma - \lambda_i)$. For criterion number 2 to hold, it is sufficient for us to introduce a condition that ensures that the function has a positive value at point $\widehat{T}_i$. Replacing the value of this point into the profit function we get,

$$J_i(\widehat{T}) = Pe^{-\gamma\beta_i} - a\lambda_i^2 - \frac{\theta}{\lambda_i}e^{-\lambda_i\beta_i},$$

where,

$$\beta_i = \frac{1}{\gamma - \lambda_i}\ln\left(\frac{P\gamma}{\theta}\right).$$

We also know from Lemmas 4.1 and 4.3 that β_i is always positive.

Imposing the condition $J_i(\widehat{T}) > 0$ we get,

$$Pe^{-\gamma\beta_i} - a_i\lambda_i^2 - \frac{\theta}{\lambda_i}e^{-\lambda_i\beta_i} > 0,$$

or,

$$Pe^{-\gamma\beta_i} > a_i\lambda_i^2 + \frac{\theta}{\lambda_i}e^{-\lambda_i\beta_i}. \tag{4.12}$$

By combining conditions (4.11) and (4.12) we get,

$$Pe^{-\gamma\beta_i} + a_i\lambda_i^2 + \frac{\theta}{\lambda_i} > P + a_i\lambda_i^2 + \frac{\theta}{\lambda_i}e^{-\lambda_i\beta_i},$$

which simplifies to,

$$\frac{P\lambda_i}{\theta} < \frac{1 - e^{-\lambda_i\beta_i}}{1 - e^{-\gamma\beta_i}},$$

where,

$$\beta_i = \frac{1}{\gamma - \lambda_i}\ln\left(\frac{P\gamma}{\theta}\right).$$

This completes the proof. ■

So far, we have proven unimodality of the profit function. We have also introduced the two PC in (4.4) and (4.10) that will ensure that the criteria we have defined for the problem hold. From conditions in (4.4) and (4.10), we can conclude that the first condition is determined by the market/customer and is the same for both firms. The reason is that the parameter value price P is assumed to be exogenously determined by the market. Also, the two parameters γ and θ are determined by the customer and the firms have no influence on them. The second condition defined in (4.10), however involves factors that are related to the structure of the firms. The existence of the parameter λ in the second condition, makes it a structure-related condition that captures the capacity of the firm and is different for each firm.

It is important to note that in the presence of competition, the expected profit functions for the firms will not always be continuous. This is due to the fact that for each firm, the decision taken by the opponent firm will affect the form and structure of its expected profit function.

Next, we will observe how the profit function will look like in the presence of a competitor in the game. We will demonstrate this through an example. In Example 4.1, we will solve a numerical problem with some parameter values and show how the profit function for firm i can be truncated at some point based on the decision taken by firm j.

Example 4.1 *Assume firm j has quoted a PDT of* 30*. Now with parameter values $P = 25$, $\gamma = 0.1$, $\lambda_i = 0.2$, $a_i = 2$ and $\theta = 8$, the expected profit function for firm i will be shown in Figure 4.1. Note that the PC apply here.*

Before we start any analysis, let us make sure that the PC hold in this problem and the criteria are satisfied. According to the first condition of the PC in (4.4), we should have $P\gamma < \theta$. Since $P\gamma = 2.5$ and $\theta = 8$, the first condition is satisfied.

For the second condition introduced in (4.10), note that we have $P < a_i\lambda_i^2 + \theta/\lambda_i$ or $25 < 0.4 + 40$. Therefore, the profit function starts at $T_i = 0$ in the negative region. Now we investigate if the condition holds. In other words, we need to make sure that the following holds,

$$\frac{P\lambda_i}{\theta} < \frac{1 - e^{-\lambda_i \beta_i}}{1 - e^{-\gamma\beta_i}}, \text{ where } \beta_i = \frac{1}{\gamma - \lambda_i} \ln\left(\frac{P\gamma}{\theta}\right).$$

Based on the parameter values, we will have $\beta_i = 11.63$. Since $P\lambda_i/\theta = 0.63$ and $1 - e^{-\lambda_i\beta_i}/1 - e^{-\gamma\beta_i} = 1.33$, we conclude that both the PC apply in this example.

In this example, the point where the expected profit reaches its maximum is $T_i = 11.6$ and at that point an expected profit equal to $J_i(11.6, 30) = 3.8$ will be generated for firm i. Also, note that the profit function is in the negative area in the beginning. It intersects with the T_i axis at point $T_i = 4.8$.

In this example, we can see that the profit function is truncated at point $T_i = 30$. This is because whether each firm wins or loses the business of the customer is also a

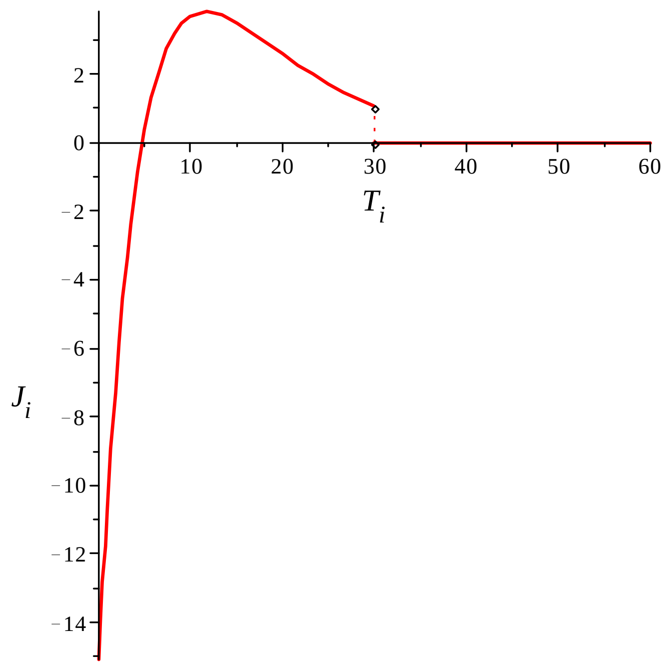

Figure 4.1: The profit function for firm i is shown in this picture, where firm j has quoted a PDT equal to 30. Note that after the function reaches the point $T = 30$, firm i will lose the business. After this point, there will be no production triggered at firm i and therefore, its profit function will be zero.

function of the decision taken by the other firm. In this example, it can be observed that for any PDT quoted by firm i that is greater than that of firm j (30), firm i will lose the business and therefore, its profit function becomes zero.

In the next section we will prove the existence and uniqueness of a Nash equilibrium in this game.

4.3 Analysis of the Nash Equilibrium

In this section we will introduce propositions that will capture the best-response functions as well as the Nash equilibrium of the game. We will also prove uniqueness of the Nash equilibrium found. Before we start with the best-response functions, we need to define the point T_i^E for firm i. It should be noted that all the results that have been found for firm i are interchangeably valid for firm j as well.

Definition 4.1 *In order to find the Nash equilibrium, we will first find the best response functions. In order to do that let us define* T_i^E *as follows,*

$$T_i^E = \begin{cases} 0, \textit{ if } T_i^{c(1)} = T_i^{c(2)} \\ \min\{T_i^{c(1)}, T_i^{c(2)}\}, \textit{ otherwise} \end{cases}, \tag{4.13}$$

where $T_i^{c(1)} > 0$ *and* $T_i^{c(2)} > 0$ *are the points where the profit function assumes the value zero. In other words, based on the conditions introduced in (4.13), if the profit function assumes zero only once, then* T_i^E *is the point zero. If the profit function becomes zero for two positive values* $T_i^{c(1)}$ *and* $T_i^{c(2)}$*, then* T_i^E *is the first point, for which the profit function is zero (*$\min\{T_i^{c(1)}, T_i^{c(2)}\}$*).*

Having defined T_i^E, we can now introduce the best-response function in Proposition 4.2.

Proposition 4.2 *For any given* $T_j = \overline{T}_j > 0$*, the best response for firm* i *is* T_i^* *and can be defined as follows,*

$$T_i^* = \begin{cases} T_i^E, \textit{ if } \overline{T}_j \leq T_i^E, \\ \overline{T}_j - \Delta, \textit{ if } T_i^E < \overline{T}_j \leq \widehat{T}_i, \\ \widehat{T}_i, \textit{ if } \widehat{T}_i < \overline{T}_j, \end{cases} \tag{4.14}$$

where $\widehat{T}_i$ *is calculated in (4.5) and* Δ *is the customer's sensitivity to time.*

Proof. To prove the first line in (4.14) we can argue that if $T_i^E = 0$ (from (4.13)) then $\overline{T}_j$ has to be zero and in this case, both companies are quoting a zero leadtime and the customer will randomly choose one. Note that quoting any other leadtime by firm i will only lead to his loss of business. On the other hand, if $T_i^E = \min\{T_i^{c(1)}, T_i^{c(2)}\}$

(from (4.13)), then based on PC conditions in (4.4) and (4.10), we can say that the profit function for firm i is negative before T_i^E. Therefore, quoting anything below T_i^E will yield a negative profit for firm i. The only value that firm i can quote and still hope that it may win the business is T_i^E itself. Also note that if firm i quotes any values above T_i^E, there is no chance for it to win the business as the other firm has quoted a value below T_i^E.

To prove the second line in (4.14) we can argue that if firm j quotes a leadtime between T_i^E and $\widehat{T}_i$ (firm i's profit maximizer), then it is in firm i's best interests to always quote a leadtime only slightly lower than $\overline{T}_j$ in an effort to win the business and still yield the highest possible profit. This is due to the fact that in this region the profit function is increasing in T_i according to PC conditions in (4.4) and (4.10). Therefore, it's optimal for firm i to quote $\overline{T}_j - \Delta$.

To prove the third line in (4.14) we can argue that after the point $\widehat{T}_i$, the profit function for firm i is decreasing according to Proposition 4.1. Therefore, as long as firm j is quoting a leadtime greater than the maximizer leadtime for firm i, it is in firm i's best interest to keep quoting $\widehat{T}_i$ that will not only win it the business, but it will also yield the highest possible profit for it. This completes the proof. ■

It is important to mention that the value of Δ has not been specified in Proposition 4.2 and has been left to be decided by the user. The reason for this is that the value of Δ depends very much on the sensitivity of the customer to time. Remember that in the absence of any other factors for the customer to decide upon (including brand name, company reputation etc.), the only point of interaction between the firms and the customer will be their PDT. Therefore, the customer makes her choice solely based on the PDTs quoted to her. In this case, any positive value for Δ will affect customer's choice as long as it makes one PDT smaller than the other. In the real life, however this may not always be the case. Even if we eliminate all other factors affecting customer's choice (like brand name etc.), the customer may still be insensitive towards very small values for Δ. Therefore, customer's sensitivity to time should also be a factor in determining the value of Δ. It should be noted again that in (4.14), firm i wants Δ to be as small as possible. The reason is that in the region $\left[T_i^E, \widehat{T}_i\right]$, the profit function is increasing in T_i and the smaller the Δ is, the higher the profit for firm i will be. We would like to suggest to the reader to determine a small positive value for Δ in the beginning and use that value throughout the study.

In Figure 4.3 you can see the best-response function for firm i with respect to the decision made by firm j. The parameter values for this function have been chosen based on the values chosen for Example 4.1, albeit without a fixed T_j. Figure 4.2 shows the profit function for firm i without considering firm j. The value for Δ in this example has been chosen as 0.3.

As it is shown in Figure 4.3, Firm i will quote a PDT equal to 4.75 for all the PDTs chosen by firm j that are below 4.75 (remember $T_i^E = 4.75$ and $\widehat{T}_i = 11.63$). For PDTs quoted by firm j that are between 4.75 and 11.63, firm i will quote a leadtime equal to $T_j - \Delta$. In this example, we have given the value 0.3 to Δ. For PDTs quoted

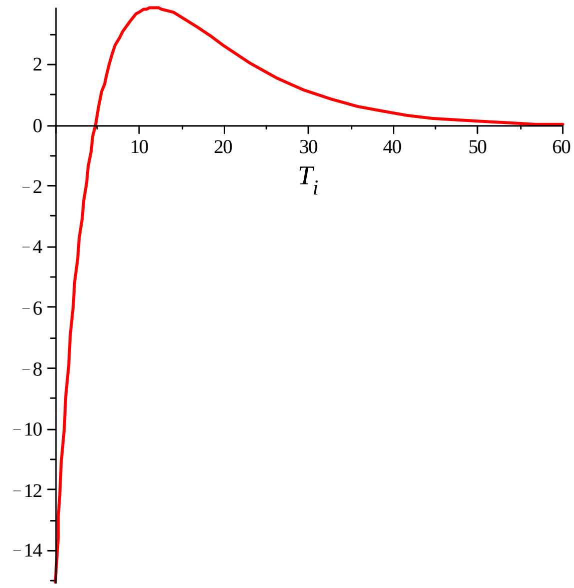

Figure 4.2: The profit function for firm i without considering the choice for firm j.

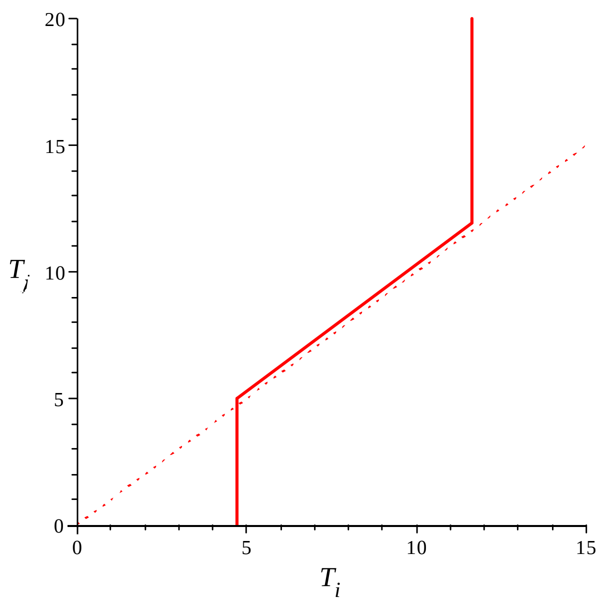

Figure 4.3: The best-response curve for firm i based on the decision taken by firm j. In this picture break points for the line occur at the two points $T_i = 4.75$ and $T_i = 11.63$. Also, we have assumed that $\Delta = 0.3$. The dotted line represents the line $T_i = T_j$.

by firm j that are greater than 11.63, firm i will quote a PDT equal to 11.63 (its $\widehat{T}_i$).

Having introduced the best-response function, we can now find the Nash equilibrium of the game. This is introduced in Proposition 4.3.

Proposition 4.3 *Under PC conditions introduced in (4.4) and (4.10), the game has a unique Nash equilibrium* (T_i^N, T_j^N) *that is calculated as follows,*

$$(T_i^N, T_j^N) = \begin{cases} (T_i^E, \widehat{T}_j), \text{ if } \widehat{T}_j < T_i^E, \\ (T_i^E, T_i^E - \Delta), \text{ if } T_j^E < T_i^E \leq \widehat{T}_j, \\ (T_i^E, T_j^E), \text{ if } T_j^E = T_i^E, \\ (T_j^E - \Delta, T_j^E), \text{ if } T_i^E < T_j^E \leq \widehat{T}_i, \\ (\widehat{T}_i, T_j^E), \text{ if } T_j^E > \widehat{T}_i, \end{cases} . \tag{4.15}$$

Proof. To prove the first line in (4.15), we argue that for firm j it is not optimal to quote anything greater than $\widehat{T}_j$ because the profit function will be decreasing after that point. Also, based on the best-response function for firm i introduced in (4.14), it should quote T_i^E if the opponent is quoting a smaller PDT. Note that firm i will not quote anything smaller than T_i^E.

To prove the second line in (4.15), we argue that based on the best-response function for firm i introduced in (4.14), as long as firm j is quoting a PDT smaller than its T_i^E, it should quote its T_i^E. For firm j, since it has $T_j^E < T_i^E$ and since its profit function is increasing in T_j, it will quote a PDT that is as big as possible and is still smaller than T_i^E by Δ to win the business.

To prove the third line in (4.15), we argue that since both firms have $T_j^E = T_i^E$, each firm knows that the other firm will at least quote its T^E and thus both firms will quote the same thing.

To prove the fourth line in (4.15), we can say that based on the best-response function for firm i introduced in (4.14), since the profit function is still increasing in T_i for firm i, it will quote a PDT that is as big as possible but still smaller than the PDT firm j is quoting by Δ, which is $T_j^E - \Delta$. For firm j, since what firm i is quoting $(T_j^E - \Delta)$ is still less than its T_j^E, based on its best-response function, it should still quote T_j^E.

To prove the last line in (4.15), we argue that for firm j it is not optimal to quote anything bigger than its T_j^E because firm i's profit maximizer is still greater than T_j^E and that is what firm i will be quoting. Based on firm i's best-response function in (4.14), as long as firm j is quoting a PDT greater than its $\widehat{T}_i$, it should quote its profit maximizer, which is $\widehat{T}_i$.

To prove the uniqueness of the Nash equilibrium, note that we are dealing with two best-response functions that consist of three straight lines. Studying the best-response functions in the same coordinate that has its horizontal axis labeled as T_i and its vertical one labeled as T_j we can use line slopes to prove uniqueness for the Nash equilibrium. For firm i, the slopes for this curve are infinity, slope equal to 1,

Parameter Values	Description
$P = 25$	Full price
$\gamma = 0.1$	Depreciation rate for the full price
$\lambda_1 = 0.2$	Production rate for the first firm
$\lambda_2 = 0.18$	Production rate for the second firm
$a_1 = 1$	Production cost rate first firm
$a_2 = 2$	Production cost rate second firm
$\theta = 8$	Rate of penalization for delayed delivery
$\Delta = 0.3$	Customer's threshold w.r.t. sensitivity to time

Table 4.2: Parameter values for the example.

and infinity again. For firm j, this curve has the slopes zero, slope equal to 1, and zero again. Based on these facts, there will only be one and only one point, where these two functions can intersect. Therefore, there will be only one Nash equilibrium in this game. This completes the proof. ■

It is worth mentioning that the value of Δ is fixed and determined for each customer separately. The results we presented here for the Nash equilibrium of the game will only produce a unique Nash equilibrium if the value of Δ is fixed and is positive.

In the next section, we will present an example with two firms with different parameter values and find their Nash equilibrium based on the findings in Proposition 4.3. We will then plot the best-response functions for both firms based on the rules in (4.14) to confirm the results of the proposition. We will also present a sensitivity analysis based on the numerical examples presented.

4.3.1 Numerical Example

In this section, we will solve a numerical example and present the results both numerically and with pictures. In Example 4.2, we will define parameter values for two firms and based on the best-response functions and the Nash equilibrium rules introduced in the previous section, we will find the Nash equilibrium of the game. This will also be illustrated graphically for clarity.

Example 4.2 *We solve a small numerical example to observe the behavior of the function. We are assuming that there are two players and they are competing to win the business of a single customer. Both firms are assumed to have exponential delivery times. Let's assume the values shown in Table 4.2 for the parameters.*

Based on the parameter values defined above, we can plot the expected profit curves for both firms, regardless of the choice taken by their opponent. The expected profit functions for firms 1 *and* 2 *are depicted in Figures 4.4 and 4.5, respectively.*

Note that the parameter values defined for firm 1 *are exactly the same as those defined for firm i in Example 4.1. The PC conditions for this firm have been examined and approved in that example. Therefore, we will just examine the PC conditions for*

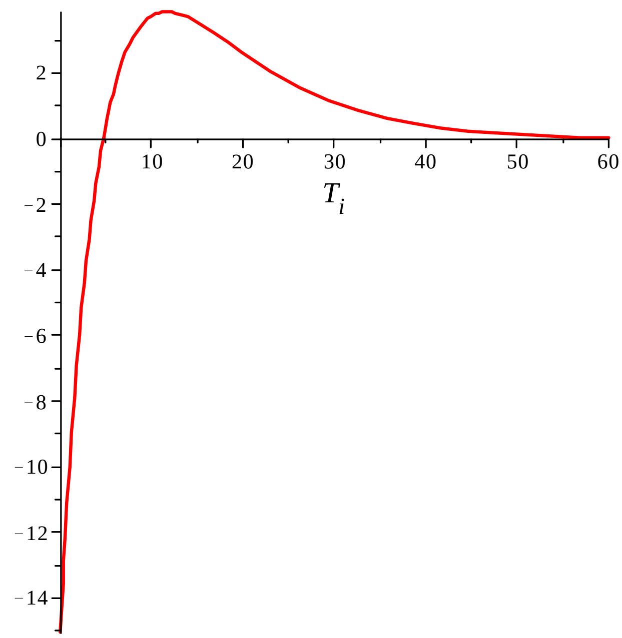

Figure 4.4: The profit function for firm i based on the parameter values given in Example 4.2 without considering the choice of firm j.

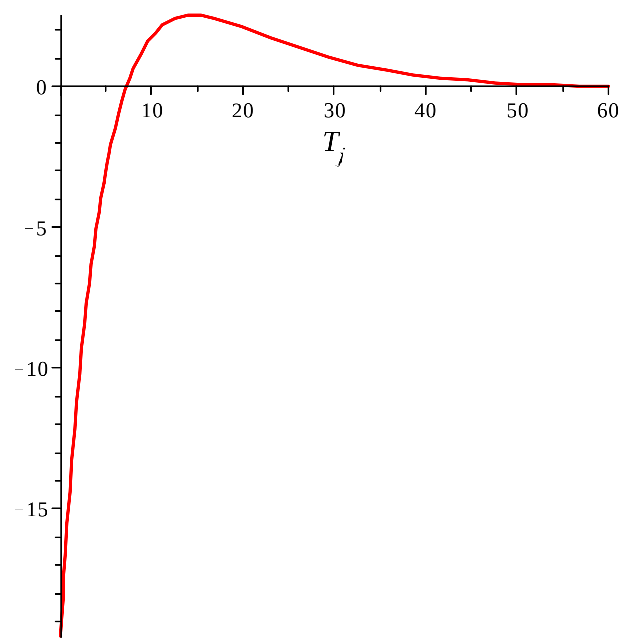

Figure 4.5: The profit function for firm j based on the parameter values given in Example 4.2 without considering the choice of firm i.

firm 2. The first PC condition defined in (4.4) is the same for both firms (defined by the market/customer) and has already been approved. The second PC condition for firm 2 requires us to first examine whether the profit function for this firm is negative at $T_2 = 0$*. Since we have* $P = 25$ *and* $a\lambda_2^2 + \theta/\lambda_2 = 3.6 + 44.5$*, the second firm's profit function starts with a negative value (this can also be observed in Figure 4.5). Therefore, we need to find out whether the following condition holds,*

$$\frac{P\lambda_2}{\theta} < \frac{1 - e^{-\lambda_2\beta_2}}{1 - e^{-\gamma\beta_2}}, \text{ where } \beta_2 = \frac{1}{\gamma - \lambda_2}\ln(\frac{P\gamma}{\theta}).$$

For this example $\beta_2 = 14.54$*. Since* $P\lambda_2/\theta = 0.56$ *and* $1 - e^{-\lambda_2\beta_2}/1 - e^{-\gamma\beta_2} = 1.21$*, the second condition in (4.10) is also approved and the PC conditions for the problem are satisfied. Now, we will proceed with the analysis of the example.*

Based on the parameter values determined in Table 4.2, we can find the values for T_i^E*,* $\widehat{T}_i$*,* T_j^E *and* $\widehat{T}_j$*. These values are shown below,*

$$T_1^E = 4.73 \text{ and } \widehat{T}_1 = 11.61,$$

and,

$$T_2^E = 7.25 \text{ and } \widehat{T}_2 = 14.53.$$

According to the rules presented in (4.15), we can see that $T_1^E < T_2^E < \widehat{T}_1$ *and therefore, the Nash equilibrium should be* $(T_2^E - \Delta, T_2^E)$ *or* $(7.25 - 0.3, 7.25)$*, which is equal to* $(6.95, 7.25)$*. The Nash equilibrium* $(T_i^N, T_j^N) = (6.95, 7.25)$ *will yield profits equal to* 2.47 *and* 0 *for firms* 1 *and* 2*, respectively. Obviously, firm* 1 *is the winner in this example.*

Now we will plot the best-response curves to observe if the realized Nash equilibrium will be confirmed by the curves. The best-response curves for both firms are plotted implicitly together in Figure 4.6.

As it is obvious in Figure 4.6, the only Nash equilibrium for the game occurs at the point $(T_1^N, T_2^N) = (6.95, 7.25)$*. As a matter of fact, the Nash equilibrium shown in this figure confirms the Nash equilibrium that was calculated numerically in this example using the rules in (4.15).*

In the next section, we will perform a sensitivity analysis based on the numerical results we obtain from solving examples and changing parameter values to observe the effect of these changes on the firms' optimal policies and the Nash equilibrium of the game.

4.4 Numerical Sensitivity Analysis

In this part we will introduce a sensitivity analysis based on the numerical example that we solved in the previous section. In the following sections we will change the

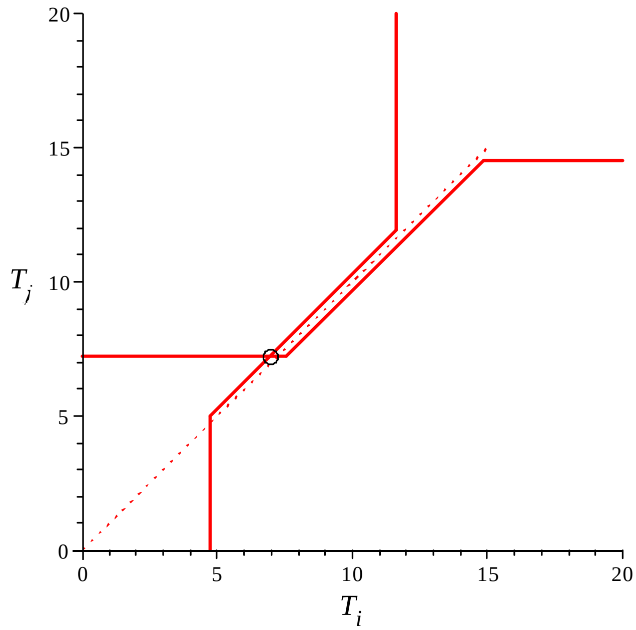

Figure 4.6: The best-response curves for firms 1 and 2 are depicted in this picture. The Nash equilibrium of the game is pointed out with a circle around that point.

parameter values one by one and observe the effect of changing these parameter values on the optimal strategies as well as the values of the objective function. In this section, we will assume that the customer's sensitivity to time is set to $\Delta = 0.3$.

In Table 4.3 we are studying the effect of changing the parameter values for the full price P and the depreciation rate for full price γ.

As it is obvious from the figures in Table 4.3, when we increase the full price P, the optimal value for the PDT quoted by both firms will decrease. As expected, by increasing the full price, the expected profit for the winning firm will also increase.

In Table 4.4, we will study the effect of changing the parameter values for the firms' production rates λ_1 and λ_2.

From the information shown in Table 4.4, we can see the effect of changing λ_1 and λ_2 on the optimal value of the decision variables as well as the objective functions. When we increase λ_1 and keep λ_2 constant, we observe that the values for the decision variables for both firms actually decrease. As expected, when we increase the production rate for the first firm λ_1 and keep production rate for the second firm λ_2 constant, we see the profit value for the first firm increase and that of the second firm decrease. By increasing the production rate for the second firm λ_2 and keeping that of the first firm λ_1 constant, we can see that the optimal decision variables for both firms actually decrease. Also, we can see that the profit value for the first firm will decrease and that of the second firm will increase.

P	γ	T_1^E	$\widehat{T}_1$	T_2^E	$\widehat{T}_2$	(T_1^N, T_2^N)	(J_1, J_2)
5	0.1	21.5	27.7	32.2	34.6	$(27.7, 32.2)$	$(0.1, 0)$
10	0.1	14	20.8	19.2	26	$(18.9, 19.2)$	$(0.6, 0)$
20	0.1	7	13.9	10	17.3	$(9.7, 10)$	$(1.8, 0)$
25	0.1	4.7	11.6	7.3	14.5	$(7, 7.3)$	$(2.5, 0)$
30	0.1	2.9	9.8	4.9	12.3	$(4.6, 4.9)$	$(2.9, 0)$
35	0.1	1.3	8.3	3	10.3	$(2.7, 3)$	$(3.4, 0)$
25	**0.07**	3.6	11.7	5.3	13.8	$(5, 5.3)$	$(2.9, 0)$
25	**0.08**	3.9	11.5	5.8	13.9	$(5.5, 5.8)$	$(2.7, 0)$
25	**0.09**	4.3	11.5	6.4	14	$(6.1, 6.4)$	$(2.6, 0)$
25	**0.10**	4.7	11.6	7.3	14.5	$(7, 7.3)$	$(2.5, 0)$
25	**0.11**	5.3	11.9	8.3	15.2	$(8, 8.3)$	$(2.2, 0)$
25	**0.12**	5.9	12.2	9.7	16.3	$(9.4, 9.7)$	$(1.9, 0)$

Table 4.3: Sensitivity analysis capturing the effect of changing the full price P and γ on the firms' optimal policies and objective function values with parameters $\lambda_1 = 0.2$, $\lambda_2 = 0.18$, $a_1 = 1$, $a_2 = 2$ and $\theta = 8$.

λ_1	λ_2	T_1^E	$\widehat{T}_1$	T_2^E	$\widehat{T}_2$	(T_1^N, T_2^N)	(J_1, J_2)
0.15	0.18	15.2	23.2	7.6	14.5	$(15.2, 14.5)$	$(0, 2.5)$
0.17	0.18	9	16.6	7.3	14.5	$(9, 8.7)$	$(0, 1.1)$
0.18	0.18	7.2	14.5	7.3	14.5	$(7.2, 7.3)$	$(0, 0)$
0.20	0.18	4.7	11.6	7.3	14.5	$(7, 7.3)$	$(2.5, 0)$
0.21	0.18	3.8	10.6	7.3	14.5	$(7, 7.3)$	$(3.6, 0)$
0.27	0.18	1	6.8	7.3	14.5	$(6.8, 7.3)$	$(7.9, 0)$
0.2	**0.14**	4.7	11.6	21	29	$(11.6, 21)$	$(3.9, 0)$
0.2	**0.16**	4.7	11.6	11.7	19.3	$(11.6, 11.7)$	$(3.9, 0)$
0.2	**0.18**	4.7	11.6	7.3	14.5	$(7, 7.03)$	$(2.5, 0)$
0.2	**0.2**	4.7	11.6	4.8	11.6	$(4.7, 4.8)$	$(0, 0)$
0.2	**0.25**	4.7	11.6	1.7	7.8	$(4.7, 4.4)$	$(0, 5.3)$
0.2	**0.30**	4.7	11.6	0.4	5.8	$(4.7, 4.4)$	$(0, 5.3)$

Table 4.4: Sensitivity analysis capturing the effect of changing firms' production rates λ_1 and λ_2 on the firms' optimal policies and objective function values with parameters $P = 25$, $\gamma = 0.1$, $a_1 = 1$, $a_2 = 2$ and $\theta = 8$.

a_1	a_2	T_1^E	$\widehat{T}_1$	T_2^E	$\widehat{T}_2$	(T_1^N, T_2^N)	(J_1, J_2)
0.1	2	4.7	11.6	7.3	14.5	$(7, 7.3)$	$(2.5, 0)$
1	2	4.7	11.6	7.3	14.5	$(7, 7.3)$	$(2.5, 0)$
2	2	4.8	11.6	7.3	14.5	$(7, 7.3)$	$(2.5, 0)$
20	2	5.3	11.6	7.3	14.5	$(7, 7.3)$	$(2.5, 0)$
80	2	8	11.6	7.3	14.5	$(8, 7.7)$	$(0, 0.4)$
1	2	1	6.8	7.3	14.5	$(6.8, 7.3)$	$(7.9, 0)$
1	**0.1**	4.7	11.6	7.2	14.5	$(7, 7.3)$	$(2.5, 0)$
1	**2**	4.7	11.6	7.3	14.5	$(7, 7.3)$	$(2.5, 0)$
1	**20**	4.7	11.6	7.9	14.5	$(7, 7.3)$	$(2.5, 0)$
1	**30**	4.7	11.6	8.4	14.5	$(7, 7.3)$	$(2.5, 0)$
1	**40**	4.7	11.6	8.9	14.5	$(7, 7.3)$	$(2.5, 0)$
1	**50**	4.7	11.6	9.5	14.5	$(7, 7.3)$	$(2.5, 0)$

Table 4.5: Sensitivity analysis capturing the effect of changing firms' production cost rates a_1 and a_2 on the firms' optimal policies and objective function values with parameters $P = 25$, $\gamma = 0.1$, $\lambda_1 = 0.2$, $\lambda_2 = 0.18$ and $\theta = 8$.

In Table 4.5, we will change the values for production cost rates for both firms and observe the effect of this change on the optimal values for the decision variables of both firms, as well as their objective function values.

From the figures in Table 4.5, we saw that by increasing the production cost rate for the first firm a_1 and keeping that of the second firm a_2 constant, we do not see significant changes in the optimal decision variables, as well as optimal objective function values. The reason is that by increasing a firm's production cost rate, the only value that changes is the point T^E for that firm, and the change will not be really significant. What we saw in this table was that by increasing the production cost rate for the first firm, the optimal decision variable increased and then the objective function value for the profit function decreased (went from the winning firm into the losing firm). The same effect can be seen in the optimal decision variable values and objective function values of the second firm when we increase its production cost rate a_2 and keep that of the first firm a_1 constant.

Finally, in Table 4.6, we will be observing the effect of changing customer's sensitivity to delay on the values of the optimal decision variables and optimal objective function values.

Based on the figures in Table 4.6, we can see that by increasing the customer's sensitivity to delay, the optimal value for the decision variables by both firms increase. The value for the objective function for the winning firm will actually decrease as well.

θ	T_1^E	$\widehat{T}_1$	T_2^E	$\widehat{T}_2$	(T_1^N, T_2^N)	(J_1, J_2)
8	4.8	11.6	7.3	14.5	$(7, 7.3)$	$(2.5, 0)$
10	7	13.9	10	17.3	$(9.7, 10)$	$(2.2, 0)$
12	8.8	15.7	12.3	19.6	$(12, 12.3)$	$(2, 0)$
14	10.3	17.2	14.3	21.5	$(14, 14.3)$	$(1.9, 0)$
20	14	20.8	18.9	26	$(18.6, 18.9)$	$(1.4, 0)$

Table 4.6: Sensitivity analysis capturing the effect of changing customer's sensitivity to delay θ on the firms' optimal policies and objective function values with parameters $P = 25$, $\gamma = 0.1$, $\lambda_1 = 0.2$, $\lambda_2 = 0.18$, $a_1 = 1$ and $a_2 = 2$.

4.5 Conclusion and Managerial Insight

In this study, we analyzed a game played between two firms under certain conditions. The firms compete by determining their promised delivery leadtime to win the business of a customer, who gives the business to the firm that quotes the shorter leadtime. The losing firm will be eliminated from the game and the winning firm will be bound to start production immediately and deliver the product. However, winning the business is not the ultimate goal of firms. Since the winning firm will incur production costs as well as delay costs (cost that the firm incurs by delivering the product past its promised delivery time), it may not always be profitable for firms to win the business at any cost. Therefore, on one hand firms prefer to quote as short a leadtime as possible to increase their chance of winning the business and on the other hand, they would like to quote as long a leadtime as possible to minimize their costs.

We studied the competition between the two firms and introduced conditions, based on which the study is meaningful in terms of its representativeness of the problems that surface in the real world. We then found the firms' best-response functions under these conditions. We also proved the existence and uniqueness of a Nash equilibrium for this game. Finally, we solved a fair number of numerical examples and performed a numerical sensitivity analysis to capture the effect of changing parameter values for the firms and the customer on the firms' optimal policies as well as the Nash equilibrium of the game.

For managers, the model in this chapter will be beneficial in their bidding wars with their competitors. The instructions presented in the analysis will help managers to make sure that their expected profit function will never become negative. In the worst case scenario, if the instructions of this chapter are followed, will end up with a profit of zero (never negative). As a result, managers can benefit from the findings in this chapter to maximize their expected profit in the long run.

We believe that the model and analysis in this chapter can be extended in a number of directions. It will be worthwhile to investigate the changes in the model when the number of players are increased from two to M. This will project a more realistic reflection of reality as we usually observe more than two competitors competing in the market. For instance, a relatively small market like the aircraft market usually sees more than two manufacturers bidding to win the business of an airline customer that requires more airplanes.

This model can also be extended by considering price to be a decision variable. In the current model in hand, we have assumed that price is exogenously determined by external forces (i.e. the market collectively). Therefore, this model eliminates the cases, where each firm can individually include its price along with its PDT in the bid submitted to the customer. This additional extension to the model will present a more complicated but more realistic case of the problem, where firms compete not only based on PDT but also on the price they charge to the customer.

Chapter 5

A Duopolistic Competition with Incomplete Information

In the models of time-based competition that we have reviewed so far in this study, players were completely aware of each other's payoff functions. In this chapter, however we are going to study a competitive model using game theory, where the objective function of at least one player is not common knowledge to all other players. In the following sections, we will observe how this asymmetry in access to information affects modelling and solving game theoretical models.

This chapter will focus on the application of games with incomplete information in a model that has been previously studied by Gerchak and Parlar [24]. In this model, firms compete based on the amount of the investment in two markets. Their decisions will impact the market delivery time by the firms, which in turn will impact the probability of a firm capturing a market. This chapter follows the same line of study that has been pursued throughout the thesis, which is time-based competition. Although time is not the decision variable in this chapter, the decision variable of the firms (investment amount) will directly affect time. Other decision variables (e.g., price) which do not directly affect the time-based performance of firms have not been included in this chapter.

This chapter is structured as follows. In Section 5.1, we will give a brief overview on games with incomplete information. Section 5.2.1 focuses on the introduction of the original model presented in the paper by Gerchak and Parlar [24]. Then in Section 5.2.2, we will add assumptions to the model to change it into a game of incomplete information. We will extend their model by assuming that one of the players does not have full knowledge about the other player's objective function. In Section 5.2.3, we will introduce the best-response curves of both players and illustrate them. Section 5.2.4 introduces and analyzes the Nash equilibrium of the game and a numerical sensitivity analysis is performed in Section 5.2.5. We will conclude the chapter with a conclusion and managerial insights into the problem in Section 5.3.

5.1 An Introduction to Games with Incomplete Information

As stated in the previous section, game theoretic models can be divided into two categories based on players' access to information; games of *complete information* and games of *incomplete information*. Gibbons [25, p. 1] refers to games with complete information as games, in which all players' payoff (objective) functions are common knowledge to every player in the game. Game theory in its infancy dealt exclusively with games of this type (e.g., see von Neumann and Morgenstern [63]).

Although extensively studied, games with complete information are rarely found in reality. There are numerous occasions of conflict and cooperation between players in the real world (games) that do not follow the same pattern. Games played between individuals, firms, etc., usually involve some levels of uncertainty when it comes to information. We usually encounter games, where players are either completely unsure of their opponents' payoff functions, or have only partial information of them. An example is the game played by firms who are competing by introducing substitutable products to the same market. Although firms usually have some ideas about the performance of their competitors, this knowledge is rarely accurate. Firms are usually kept in the dark by their competitors on specific features in the performances of their sales, production or other departments. In fact, the problem studied in this chapter involves a firm that conceals some information regarding its R&D department from its competitor. Based on Gibbons [25, p. 143], games in which at least one player is uncertain about another player's objective function are called games with incomplete information (also referred to as *Bayesian games*). Since its introduction in the literature by Harsanyi ([31], [32], and [33]), games with incomplete information have attracted the attention of operations researchers as well as mathematicians.

In this chapter, we will present a game of static incomplete information, where players both make decisions simultaneously. We will try to find the Nash equilibrium of this game. The Nash equilibrium derived from such game (a static game of incomplete information) is referred to as the *Bayesian Nash equilibrium* by Gibbons [25, p. 143]. Also, to learn more about static games of incomplete information, the reader can consult Gibbons [25, pp. 143–173]. For a recent example of a study involving static and dynamic games of incomplete information we can refer to the paper by Wu and Parlar [65]. In this paper, the authors study a game of incomplete information with two competing newsvendors and illustrate the Nash equilibria in the static and dynamic game settings.

5.2 A Duopolistic Game with Incomplete Information

The game studied in Gerchak and Parlar [24] involves two firms that are competing to maximize their expected profit by investing in the same markets. The decision variables of the firms are the fraction of their budget that they are investing in each of the two markets (R&D projects) available to both firms. Each market (R&D project) has different profitability for each firm. Also firms capture markets based on a probability that depends on how much they invest in each market. Ultimately, each firm is interested in determining the optimal fraction of its budget to invest in each market that will maximize its final expected profit. It is assumed that the first firm that delivers to market i will capture that whole market. This is therefore, a time-based competition where firms compete using their investment amount to shorten the time to market.

The game studied by Gerchak and Parlar [24] is a game of complete information. In other words, the objective function (expected profit function) of each firm was known to the other firm. In this chapter, however we are investigating the result of the game if one of the firms did not have complete information about the objective function of the other firm. In the following sections, we will study the effect of the uncertainty added to the model on the optimal strategies of the firms and the equilibrium of the game.

5.2.1 The Model

In this section, we will present the original model that was initially introduced in Gerchak and Parlar [24]. We will explain the notations and present the objective functions of each firm. We start with explaining the model that was used in the paper by Gerchak and Parlar [24].

We assume there are two potential R&D projects and the budget allocated to activity i by each of the two competitors is x_i and y_i, respectively, $i = 1, 2$.

The budget constraints are therefore, $\sum_{i=1}^{2} x_i = B_1$ and $\sum_{i=1}^{2} y_i = B_2$. Let $f_i(x_i, y_i)$ be the probability that the first firm will capture market i. This probability is increasing in x_i and decreasing in y_i. If we define C_i as the event that the first firm captures market i, the binary variable I_i will be,

$$I_i = \begin{cases} 1, & \text{if } C_i, \\ 0, & \text{otherwise.} \end{cases}$$

Then $E(I_i) = \Pr(C_i) = f_i(x_i, y_i)$ will be the probability that the first firm captures market i. Each firm wants to maximize the profit that it draws from the markets that it captures. For example, the first firm wishes to maximize $F(\mathbf{x}, \mathbf{y}) \equiv \sum_{i=1}^{2} r_i f_i(x_i, y_i)$, where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ and the positive value r_i is the relative importance

Parameter Values	Description
x_i :	Budget allocated to market i by the first firm, $i = 1, 2$
y_i :	Budget allocated to market i by the second firm, $i = 1, 2$
B_j :	Total budget for firm j, $j = 1, 2$
$f_i(x_i, y_i)$:	Probability that the first firm captures market i, $i = 1, 2$
$g_i(x_i, y_i)$:	Probability that the second firm captures market i, $i = 1, 2$
$F(x, y)$:	Expected profit for the first firm
$G(x, y)$:	Expected profit for the second firm
r :	Relative importance of the first market for the first firm
s :	Relative importance of the first market for the second firm

Table 5.1: Summary of notations and symbols in the complete information setting.

(profit) of capturing market i to the first firm. The equivalent of the $f_i(x_i, y_i)$, r_i and $F(\mathbf{x}, \mathbf{y})$ to the first firm are $g_i(x_i, y_i)$, s_i and $G(\mathbf{x}, \mathbf{y})$ to the second firm, respectively. Table 5.1 summarizes the notation used in the complete information model.

For now, we will assume that $B_1 = B_2$. In this case, we can claim that without loss of generality, we can say that $B_1 = B_2 = 1$. Even if we have $B_1 \neq B_2$, we can still adjust f_i's so that one of the "currencies" will be in different units such that its total budget will sum to one. Therefore, we can simply say that without loss of generality we have $B_1 = B_2 = 1$.

Let's assume we define random variables X_i and Y_i to denote the time it takes firms one and two to complete their activities and deliver to market i. We have also assumed that the first firm that delivers to market i captures the whole market. As a result, the first firm captures market i with probability $\Pr(X_i < Y_i)$. For instance, if X_i and Y_i are assumed to be exponential with parameters, $\lambda_i(x_i)$ and $\mu_i(y_i)$, respectively, then this probability becomes,

$$f_i(x_i, y_i) = \Pr(X_i < Y_i) = \frac{\lambda_i(x_i)}{\lambda_i(x_i) + \mu_i(y_i)}.$$

The authors have replaced this probability with the following form, assuming $\lambda_i(x_i) = x_i^{a_i}$ and $\mu_i(y_i) = y_i^{b_i}$,

$$f_i(x_i, y_i) = \frac{x_i^{a_i}}{x_i^{a_i} + y_i^{b_i}},$$

where a_i and b_i moderate the effect of the relative budget allocation. They later propose the values $a_i = b_i = 1$ to create a linear-fractional model for the firms' profit functions.

The authors define the expected profit of the first firm in the linear-fractional model as follows,

$$F(x,y) = \frac{rx}{x+y} + \frac{1-x}{(1-x)+(1-y)}, \tag{5.1}$$

over $0 \leq x \leq 1$ for any y. In expression (5.1), x is the amount of expenditure spent into the development for the first market by the first firm. Since the budget is assumed to be equal to one, the amount of expenditure spent into the development for the second market by the first firm will be equal to $(1-x)$. Also, $r = r_1$ is the relative importance (profit) to the first firm for the first market and without loss of generality, r_2 is assumed to be equal to one. The second firm's objective function (expected profit) is also structured the same way in that paper as follows,

$$G(x,y) = \frac{sy}{x+y} + \frac{1-y}{(1-x)+(1-y)}, \tag{5.2}$$

over $0 \leq y \leq 1$ for any x. Also, $s = s_1$ is the relative importance (profit) of the first market to the second firm and s_2 is assumed to be equal to one.

5.2.2 A Game of Incomplete Information

In this section, we add a new assumption that will change the game presented in the previous section into a game of incomplete information. The incompleteness in information is regarding the relative importance (profit) for the firms for the markets, namely r and s. In this case, we are assuming that the relative importance for the first firm is common knowledge between the two firms, i.e., both firms know the value of r. The value of s, however, is not known to either firms before the game starts. Upon the start of the game, the value of s will be known to the second firm only. In other words, the second firm will know what the value of its relative importance towards the first market will be, but the first firm will be barred from knowing that information. However, the first firm will not be completely kept in the dark as he knows that s may be of two types only, s_1 and s_2. It also knows that this value will be s_1 with probability θ and s_2 with probability $1-\theta$. An example for this model could be two manufacturers of electronic devices that are engaged in R&D and production of tablet computers as well as laptops (two projects). One of these manufacturers may be more successful in keeping its R&D information from its rival. Therefore, the rival will have access to incomplete information regarding the investments the company is making in either of the projects (tablets vs. laptops).

Table 5.2 includes the new and additional notations used in this section for the model in the incomplete information setting.

As stated above, upon the start of the game, the second firm will know for sure what its relative importance for the first market will be. Therefore, based on equation (5.2) its two objective functions for the cases where $s = s_1$ and $s = s_2$ will be,

Parameter Values	Description
x :	Budget for the first market by the first firm
y_k :	Budget for the first market by the second firm type k, $k = 1, 2$
r :	Relative importance of market 1 to firm 1
s_k :	Relative importance of market 1 to firm 2 type k, $k = 1, 2$
θ :	Probability that $s = s_1$
$F(x, y_1, y_2)$:	Expected profit for the first firm
$G(x, y_1)$:	Expected profit for the second firm of type 1
$G(x, y_2)$:	Expected profit for the second firm of type 2

Table 5.2: Summary of notations and symbols in the incomplete information setting.

respectively,

$$G(x, y_1) = \frac{s_1 y_1}{x + y_1} + \frac{1 - y_1}{(1 - x) + (1 - y_1)}, \tag{5.3}$$

and,

$$G(x, y_2) = \frac{s_2 y_2}{x + y_2} + \frac{1 - y_2}{(1 - x) + (1 - y_2)}, \tag{5.4}$$

where y_1 and y_2 will be second firm's choice of expenditure in the first market for the cases, where $s = s_1$ and $s = s_2$, respectively.

The first firm will not know for sure which type the second firm will be, therefore, based on equation (5.1) its objective function will be as follows,

$$F(x, y_1, y_2) = \theta F(x, y_1) + (1 - \theta) F(x, y_2),$$

or,

$$F(x, y_1, y_2) = \theta \left(\frac{rx}{x + y_1} + \frac{1 - x}{(1 - x) + (1 - y_1)} \right) + (1 - \theta) \left(\frac{rx}{x + y_2} + \frac{1 - x}{(1 - x) + (1 - y_2)} \right). \tag{5.5}$$

In order to find the Nash equilibrium of the game, we will need to find the values for x^*, y_1^* and y_2^* that are the optimal expenditure values of the first firm, and the second firm for its two cases, respectively. It is important to note that the first firm will solve the following optimization problem,

$$\max_x F(x, y_1^*, y_2^*) = \theta \left(\frac{rx}{x + y_1^*} + \frac{1 - x}{(1 - x) + (1 - y_1^*)} \right) + (1 - \theta) \left(\frac{rx}{x + y_2^*} + \frac{1 - x}{(1 - x) + (1 - y_2^*)} \right). \tag{5.6}$$

The second firm will solve the following two optimization problems,

$$\max_{y_1} G(x^*, y_1) = \frac{s_1 y_1}{x^* + y_1} + \frac{1 - y_1}{(1 - x^*) + (1 - y_1)}, \tag{5.7}$$

and,

$$\max_{y_2} G(x^*, y_2) = \frac{s_2 y_2}{x^* + y_2} + \frac{1 - y_2}{(1 - x^*) + (1 - y_2)}. \tag{5.8}$$

5.2.3 Best-Response Curves

In this section we will develop the best-response functions for the firms in this competition. We will show the closed-form solution for a best-response rule whenever one is available and if not, we will look at special numerical cases to give an overview on each firm's optimal response.

Looking at the objective function of the second firm of type 1, we take the first and second derivatives to realize how the objective function looks like. This has been done exactly in Gerchak and Parlar [24]. Differentiating (5.3) with respect to y_1 we get,

$$\frac{\partial G(x, y_1)}{\partial y_1} = \frac{s_1 y_1}{(x + y_1)^2} + \frac{y_1 - 1}{(2 - x - y_1)^2}, \tag{5.9}$$

and,

$$\frac{\partial^2 G(x, y_1)}{\partial y_1^2} = \frac{-2 s_1 y_1}{(x + y_1)^3} + \frac{2(y_1 - 1)}{(2 - x - y_1)^3} < 0. \tag{5.10}$$

Therefore, $G(x, y_1)$ is strictly concave in y_1 for any given value of x. The objective function for the second firm of type 2 has a similar behavior that follows,

$$\frac{\partial G(x, y_2)}{\partial y_2} = \frac{s_2 y_2}{(x + y_2)^2} + \frac{y_2 - 1}{(2 - x - y_2)^2}, \tag{5.11}$$

and,

$$\frac{\partial^2 G(x, y_2)}{\partial y_2^2} = \frac{-2 s_2 y_2}{(x + y_2)^3} + \frac{2(y_2 - 1)}{(2 - x - y_2)^3} < 0, \tag{5.12}$$

which shows that $G(x, y_2)$ is also concave in y_2 for any given value of x.

Therefore, the two functions $G(x, y_1)$ and $G(x, y_2)$ reach their unique maximum value at an interior point. Now we take the first and second derivatives from the objective function of the first firm in (5.5).

$$\frac{\partial F(x, y_1, y_2)}{\partial x} = \theta \left(\frac{s_2 y_2}{(x + y_2)^2} + \frac{y_2 - 1}{(2 - x - y_2)^2} \right) + (1 - \theta) \left(\frac{s_1 y_1}{(x + y_1)^2} + \frac{y_1 - 1}{(2 - x - y_1)^2} \right), \tag{5.13}$$

and,

$$\frac{\partial^2 F(x,y_1,y_2)}{\partial x^2} = \theta\left(\frac{-2s_2y_2}{(x+y_2)^3} + \frac{2(y_2-1)}{(2-x-y_2)^3}\right) + (1-\theta)\left(\frac{-2s_1y_1}{(x+y_1)^3} + \frac{2(y_1-1)}{(2-x-y_1)^3}\right) < 0, \tag{5.14}$$

which shows that $F(x,y_1,y_2)$ is also concave in x for any given values of y_1 and y_2.

Based on the above analysis, we can now find the best-response functions for the two firms.

Both objective functions for the second firm are similar to those studied in Gerchak and Parlar [24]. Therefore, we will just present the best-response found in their work. The best-response functions of the second firm are found by making $\partial G(x,y_1)/\partial y_1$ and $\partial G(x,y_2)/\partial y_2$ equal to zero and solving for y_1 and y_2 for any given value of x. Solving these equations, we will find,

$$y_1^*(x) = \frac{(1+2s_1)x - (1+s_1)x^2 \pm \sqrt{s_1x(1-x)}}{(1+s_1)x-1}.$$

Let's call the two solutions $y_1^+(x)$ and $y_1^-(x)$. Using numerical investigations we can show that $y_1^+(x)$ is not an accepted answer as it is always negative. Using the same kind of investigation, we can show that $y_1^-(x)$ is actually an accepted solution. Therefore, we have,

$$y_1^*(x) = \frac{(1+2s_1)x - (1+s_1)x^2 - \sqrt{s_1x(1-x)}}{(1+s_1)x-1},$$

as the best-response function of the second firm (of type 1) for any given value of x. Similarly, the second firm's best-response function (of type 2) can be derived and shown as follows,

$$y_2^*(x) = \frac{(1+2s_2)x - (1+s_2)x^2 - \sqrt{s_2x(1-x)}}{(1+s_2)x-1},$$

for any given value of x.

In Gerchak and Parlar [24], the authors have illustrated the behavior of this best-response function using different values of the relative importance (here s_1 and s_2) and the value of the opponent's decision variable (here x). We refer the reader to Gerchak and Parlar [24] to observe the best-response curves for the second firm.

The best-response surface for the first firm can be derived by equating the expression in (5.13) to zero and solving for x. Since a closed-form solution for the best-response surface of the first firm could not be derived, we have used multiple points across the axes for y_1 and y_2 and have found the value of $x^*(y_1,y_2)$ for each

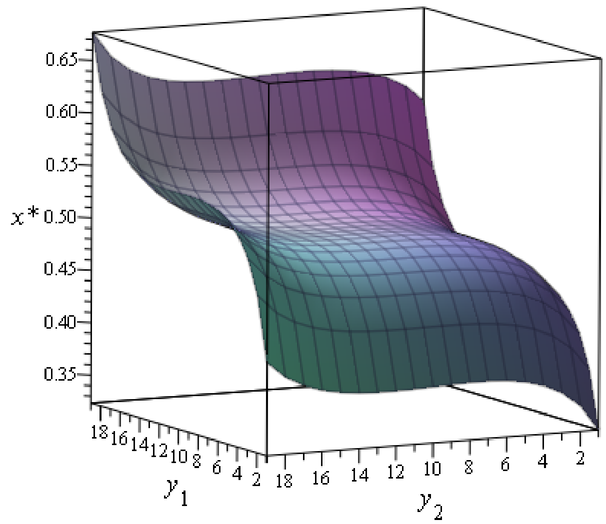

Figure 5.1: The best-response surface for the first player $x^*(y_1, y_2)$, for different values of y_1 and y_2 where $r = 3$ and $\theta = 0.7$.

value. Figure 5.1 shows the values of $x^*(y_1, y_2)$ for different values of y_1 and y_2 and for $r = 3$ and $\theta = 0.7$.

In Figure 5.1 the row and column axes correspond to the values of y_1 and y_2, respectively, ranging from 0.05 to 0.95 with step sizes of 0.05. The vertical axis corresponds to the value of $x^*(y_1, y_2)$.

Figure 5.2 shows the best-response surface for the first company for three different values of r. This pictures shows how $x^*(y_1, y_2)$ is lowered as r decreases from a different angle.

We have also looked at the first firm's best response for different values of its parameters. Figure 5.3 shows the best-response curve $x^*(\theta)$ for $y_1 = 1/2$ and $y_2 = 1/3$ and different values of r. As it is shown in this figure, the higher r is (the more important the first market becomes to the first firm), the higher the curve will be. A higher x indicates that the first firm will invest a higher fraction of its budget in this market.

For $y_1 = 1/2$ and $y_2 = 1/3$ and different values of θ the function moves very insignificantly. Therefore, we have plotted the first firm's best-response curve for the value of $\theta = 0.9$ only. Figure 5.4 shows this function with the above mentioned values illustrated with respect to r.

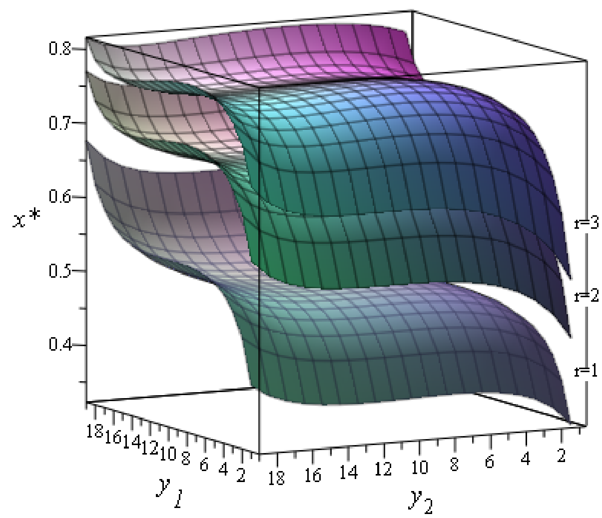

Figure 5.2: Three best-response surfaces for the first player for $r = 1$, $r = 2$ and $r = 3$ and $\theta = 0.7$.

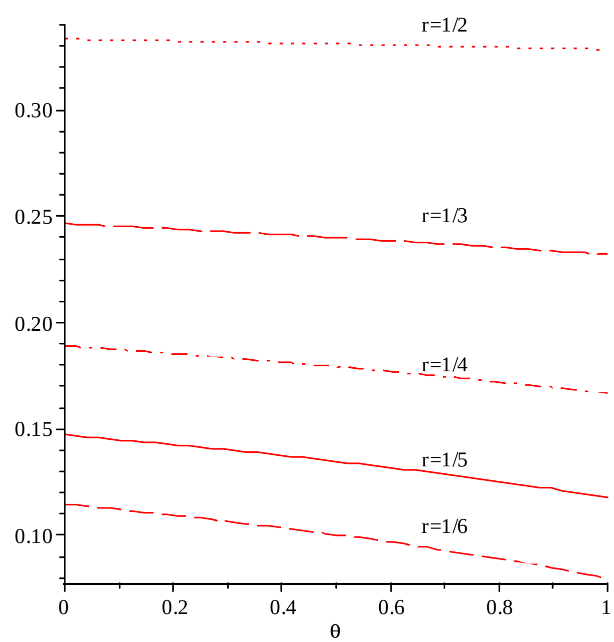

Figure 5.3: The best-response curve $x^*(\theta, y_1, y_2)$ assuming $y_1 = 1/2$ and $y_2 = 1/3$ for different values of r.

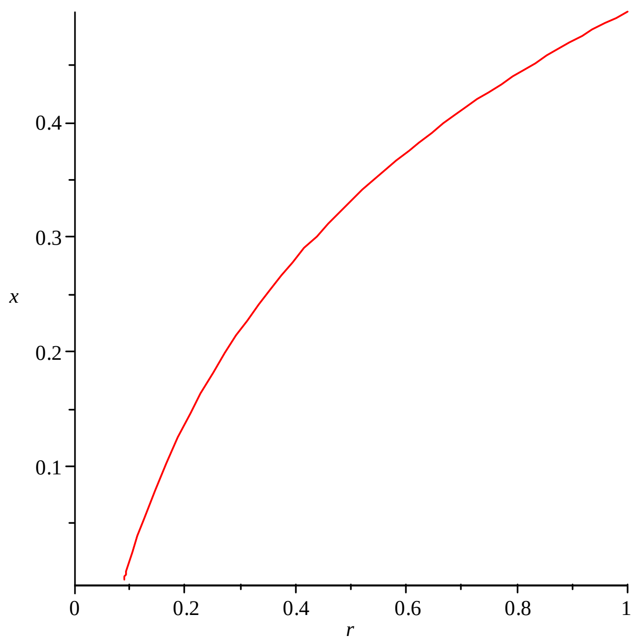

Figure 5.4: The best-response curve $x^*(r, y_1, y_2)$ assuming $y_1 = 1/2$ and $y_2 = 1/3$ for $\theta = 0.9$.

5.2.4 Nash Equilibrium

In a game that has a Nash equilibrium, we have to know that both firms make their decisions simultaneously. To find this equilibrium, we need to solve the following system of three equations for the unknowns x, y_1 and y_2,

$$\frac{\partial F(x, y_1, y_2)}{\partial x} = \frac{\partial G(x, y_1)}{\partial y_1} = \frac{\partial G(x, y_2)}{\partial y_2} = 0. \tag{5.15}$$

The resulting values (x_N, y_{1N}, y_{2N}) will be the interior Nash equilibrium of the game.

We consider a special case where $r = s_1 = 1$ and $s_2 = 2$. We are also assuming that there is a 50% chance that the second firm will be of type 2. In this case, there will be two Nash equilibria for the game. The equilibria are shown below,

$$x = 0.50,\ y_1 = 0.50,\ y_2 = 0.67, \tag{5.16}$$

and,

$$x = 0.60,\ y_1 = 10.30,\ y_2 = 0.67.$$

However, only the first equilibrium is accepted as we know $0 \leq x \leq 1$, $0 \leq y_1 \leq 1$, and $0 \leq y_2 \leq 1$. Therefore, the only accepted Nash equilibrium for this game will be the first one. The values of the objective functions at the equilibrium will be,

$$\begin{aligned} F(x, y_1, y_2) &= 1.02, \\ G(x, y_1) &= 1.00, \\ G(x, y_2) &= 1.54. \end{aligned}$$

It is worth noting that because of the existence of additional parameters in this model aimed at studying a situation of incomplete information, it is not easy to compare the results of this chapter with those presented by Gerchak and Parlar [24]. However, a similar example to the above example has been also studied by them where $r = s_1 = 1$. Obviously, their models does not include information on s_2 or θ as it is a model in complete information. Comparing the objective function values between these two models indicate that in the current model with incomplete information, the second firm of type 2 improved its results from the objective function value in the game in complete information. While we believe that the choice of the inclusion of s_2 and θ may have affected the results, it is interesting to notice that the second firm has actually benefited from keeping information from the first firm. In fact, based on our numerical investigations, increasing the value of s_2 will result in yet higher profits for the second firm of type 2.

Figure 5.5 shows the three best-response planes in this problem. The point where these planes intersect will be the Nash equilibrium of the game. As we can see in the figure, the point is where the Nash equilibrium in (5.16) happens.

As we can see in Figure 5.5, there is only one Nash equilibrium for the game with the given parameter values.

5.2.5 Sensitivity Analysis

In this section we will change the parameter values of the objective functions numerically and observe the Nash equilibria of the game as well as the value of the objective functions at equilibria. We take the parameter values $r = 3$, $s_1 = 4$, $s_2 = 5$, and $\theta = 0.5$ as the base model (shown as a row in bold numbers in tables) and will vary the values from that point. Table 5.3 looks at the effects of changing the relative importance of the first market to the first firm (r) on the Nash equilibrium of the game as well as the objective functions, where $F \equiv F(x_N, y_{1N}, y_{2N})$, $G_1 \equiv G(x_N, y_{1N})$ and $G_2 \equiv G(x_N, y_{2N})$.

In Table 5.3 we can see that by raising the value of r the importance of the investment of the first market (compared to the second market) for the first firm increases and therefore, the fraction of the budget by the first firm invested in the first market also increases (x_N). No consistent change was observed in the values of y_{1N} and y_{2N} after varying the value of r. Also, no consistent change was seen in the behavior of $G(x_N, y_{1N})$ and $G(x_N, y_{2N})$. However, the value of the first firm's objective function $F(x_N, y_{1N}, y_{2N})$ rose consistently as r grew larger.

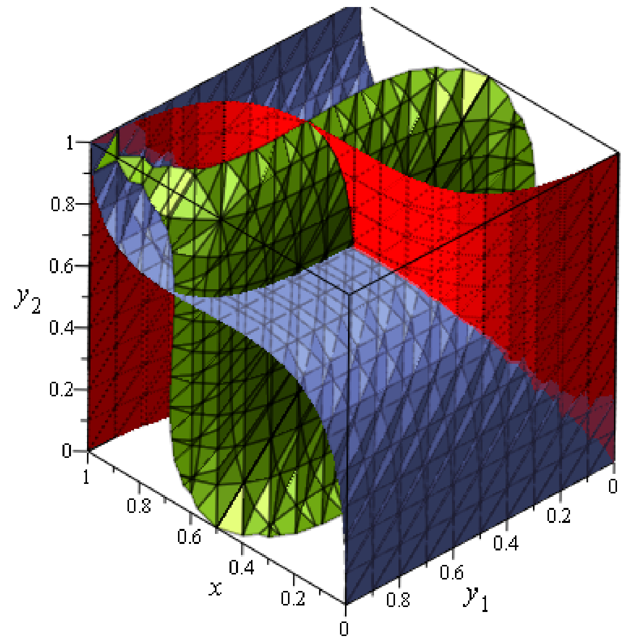

Figure 5.5: An illustration of the three best-response surfaces when $r = s_1 = 1$ and $s_2 = 2$ with a 50% chance for player 2 to be of type 2.

r	x_N	y_{1N}	y_{2N}	F	G_1	G_2
0.1	0.01	0.37	0.41	0.62	4.24	5.21
1	0.56	0.83	0.87	1.14	2.67	3.27
1.5	0.63	0.82	0.86	1.34	2.59	3.15
2	0.69	0.81	0.85	1.55	2.54	3.09
2.5	0.72	0.80	0.84	1.78	2.52	3.05
3	**0.75**	**0.80**	**0.84**	**2.02**	**2.51**	**3.03**
3.5	0.78	0.80	0.84	2.26	2.50	3.01
4	0.80	0.80	0.83	2.50	2.50	3.00
4.5	0.82	0.80	0.84	2.75	2.50	3.00
5	0.83	0.80	0.83	3.00	2.50	3.00
10	0.92	0.82	0.85	5.57	2.58	3.05

Table 5.3: Sensitivity analysis based on varying the value of the parameter r with $s_1 = 4$, $s_2 = 5$ and $\theta = 0.5$.

s_1	x_N	y_{1N}	y_{2N}	F	G_1	G_2
0.5	0.76	0.35	0.84	2.17	0.89	3.02
1	0.76	0.52	0.84	2.07	1.07	3.02
1.5	0.76	0.61	0.84	2.03	1.29	3.03
2	0.76	0.67	0.84	2.02	1.51	3.03
2.5	0.75	0.72	0.84	2.01	1.75	3.03
3	0.75	0.75	0.84	2.01	2.00	3.03
3.5	0.75	0.78	0.84	2.01	2.25	3.03
4	**0.75**	**0.80**	**0.84**	**2.02**	**2.51**	**3.03**
4.5	0.76	0.82	0.84	2.02	2.77	3.03
5	0.76	0.84	0.84	2.02	3.03	3.03
5.5	0.76	0.85	0.84	2.03	3.29	3.03
10	0.78	0.93	0.84	2.08	5.69	3.01

Table 5.4: Sensitivity analysis based on varying the value of the parameter s_1 with $r = 3$, $s_2 = 5$ and $\theta = 0.5$.

Table 5.4 looks at the effects of varying the parameter value s_1 on the Nash equilibrium of the game as well as the value of the objective functions.

Table 5.4 shows that increasing the value of the relative importance of the first market to the second firm of type 1 (s_1) will indeed have an effect on the fraction of budget the second firm of type 1 will invest in the first market. This value y_{1N} rose consistently as the value of s_1 grew. The other consistent change in the table was the value of the objective function of the second firm of type 1 ($G(x_N, y_{1N})$), which rose as we increased s_1. Other values in the table did not show any consistent change.

In Table 5.5 we will investigate the effect of changing the value of s_2 on the Nash equilibrium and the objective function values.

As shown in Table 5.5, increasing the relative importance of the first market to the second firm of type 2 will have an effect on the fraction of the budget it invests in the first market. As we would expect, increasing s_2 will make the value of y_{2N} also increase. In addition to that, the value of the objective function for the second firm of type 2 ($G(x_N, y_{2N})$) also rises as we increase s_2. Other values in the table do not show any consistent change when varying s_2.

Finally, Table 5.6 shows the effects of changing the likelihood that the second firm will be of type 1 or 2 (θ) on the Nash equilibrium of the game as well as the values of the objective functions. To show the effect of changing the θ, we are showing other parameter values with four decimal digits.

As θ increases, the likelihood that the second firm will be of type 1 also increases. From the data in Table 5.6, we can see that an increase in θ translates into a consistent decrease in the value of x_N. In other words, the more the likelihood that the second firm is of type 1, the smaller the fraction of budget that the first firm invests in the first market. This could be due to the fact that the second firm of type 1 places less

s_2	x_N	y_{1N}	y_{2N}	F	G_1	G_2
0.5	0.76	0.80	0.35	2.16	2.51	0.89
2	0.75	0.80	0.67	2.01	2.51	1.51
2.5	0.75	0.80	0.72	2.01	2.51	1.75
3	0.75	0.80	0.75	2.00	2.51	2.00
3.5	0.75	0.80	0.78	2.00	2.51	2.25
4	0.75	0.80	0.80	2.01	2.51	2.51
4.5	0.75	0.80	0.82	2.01	2.51	2.77
5	**0.75**	**0.80**	**0.84**	**2.02**	**2.51**	**3.03**
5.5	0.76	0.80	0.85	2.02	2.51	3.29
6	0.76	0.80	0.87	2.03	2.51	3.56
6.5	0.76	0.80	0.88	2.03	2.51	3.82
10	0.78	0.80	0.93	2.07	2.50	5.69

Table 5.5: Sensitivity analysis based on varying the value of the parameter s_2 with $r = 3$, $s_1 = 4$ and $\theta = 0.5$.

θ	x_N	y_{1N}	y_{2N}	F	G_1	G_2
0.01	0.7572	0.8015	0.8387	2.0241	2.5067	3.0268
0.1	0.7567	0.8015	0.8387	2.0226	2.5068	3.0271
0.2	0.7562	0.8015	0.8388	2.0209	2.5069	3.0275
0.3	0.7558	0.8016	0.8388	2.0193	2.5071	3.0278
0.4	0.7553	0.8016	0.8389	2.0177	2.5072	3.0281
0.5	**0.7548**	**0.8016**	**0.8390**	**2.0160**	**2.5074**	**3.0285**
0.6	0.7543	0.8017	0.8390	2.0144	2.5076	3.0288
0.7	0.7537	0.8017	0.8391	2.0127	2.5077	3.0292
0.8	0.7532	0.8018	0.8392	2.0111	2.5079	3.0295
0.9	0.7527	0.8018	0.8392	2.0094	2.5081	3.0298
0.99	0.7523	0.8019	0.8392	2.0079	2.5082	3.0302

Table 5.6: Sensitivity analysis based on varying the value of the parameter θ with $r = 3$, $s_1 = 4$ and $s_2 = 5$.

	x_N	y_{1N}	y_{2N}	F	G_1	G_2
$r \uparrow$, s_1, s_2 and θ fixed and $s_1 < s_2$	$\uparrow$	–	–	$\uparrow$	–	–
$s_1 \uparrow$, r, s_2 and θ fixed	–	$\uparrow$	–	–	$\uparrow$	–
$s_2 \uparrow$, r, s_1 and θ fixed	–	–	$\uparrow$	–	–	$\uparrow$
$\theta \uparrow$, s_1, s_2 and r fixed and $s_1 < s_2$	$\downarrow$	$\uparrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\uparrow$

Table 5.7: Direction of change in system performance with changing different parameter values.

relative importance on the first market than the second firm of type 2 (i.e. invests less in it). This could mean that the second firm of type 1 places more relative importance on the second market than the second firm of type 2. As a result, the first firm focuses more on capturing the second market as the likelihood of the second firm becoming of type 1 increases. Therefore, the first firm will divert more of its budget into investing in the second market as θ grows.

With increasing θ, both types of the second firm will see a rise in the fraction of budget that they invest in the first market, y_{1N} and y_{2N}. This can be interpreted as a result of less investment by the first firm in the first market, which will increase the second firm's chance of capturing that market. Therefore, both types of the second firm tend to invest more in the first market as the likelihood of the second firm becoming of type 1 increases.

By increasing θ, we also observe a consistent decline in the objective function value of the first firm and a consistent gain in the objective function values of the two types of the second firm.

It is worth noting that x_N, y_{1N} and y_{2N} are fractions of a total budget that could be a large sum. Therefore, even a small change in them may result in significant changes in the amount invested by firms in different markets. Table 5.7 summarizes the results of the sensitivity analysis performed above, where $F \equiv F(x_N, y_{1N}, y_{2N})$, $G_1 \equiv G(x_N, y_{1N})$ and $G_2 \equiv G(x_N, y_{2N})$.

5.3 Conclusion and Suggestions for Future Research

In this chapter we studied a model of game theory with two firms competing by investing in two markets. The firm that reaches a market first will capture that whole market. The speed at which a firm reaches a market depends on the amount of budget invested in that market by the firm. This problem has been previously studied by Gerchak and Parlar [24]. In their model, they assume that the game is played in the complete information setting. This chapter relaxes this assumption and assumes that the problem is an incomplete information game.

In this chapter we assumed that one of the firms (the first firm) does not have complete information about the structure of the objective function of the other firm (second firm). All the first firm knows is that the second firm may be of two types with respect to the relative importance that it places on capturing the first market. The first firm also knows the probability, based on which the second firm will be of either type. The second firm, however knows the structure of the first firm's objective function. Therefore, the incompleteness of information is asymmetrical in the game.

We proved the concavity of the objective functions and illustrated the best-response planes of the firms. We also demonstrated the Nash equilibrium of the game in different special cases of the game. Finally, we performed a numerical sensitivity analysis and studied the effects of varying parameter values on the firms' optimal strategies, the Nash equilibrium of the game as well as the objective function values at equilibrium.

The results from this chapter extended the results derived by Gerchak and Parlar [24] by allowing one of the firms to have incomplete information. The new settings created interesting results in terms of completely new forms of best-response planes and Nash equilibria. Also, the sensitivity analysis presented at the end of the chapter provided insight into the behavior of the profit functions as a reaction to changes in parameter values.

For future studies, we suggest to look at the problem in situations, where both firms have incomplete information about their opponent's objective function. In this case, each firm will have two or more types depending on a certain characteristic of their objective function (e.g., the relative importance of the first market in this chapter). We believe that this assumption will create even more realistic results as in reality, firms rarely have complete information regarding the objective function of their opponents.

We have used linear profit functions for the firms in this chapter. Studying the problem in an incomplete information setting with nonlinear profit functions will be another area that we suggest requires further research. Another area that could be interesting to look at regarding this problem is the area of cooperative games. We believe that analyzing the problem under the cooperative game assumption will result in interesting findings.

Chapter 6

Concluding Remarks

6.1 Thesis Summary and our Contributions

Time-based competition has become an important subject both in manufacturing and service industries. We have presented examples of companies using *time* as a competitive advantage in their businesses and we encounter more examples every day. A very useful tool that is used in the field of management science to analyze situations of competition is game theory. In this thesis we have made contributions to the literature in using game theoretical models in situations where companies compete based on the notion of time. In some situations, time has been the direct decision variable (e.g. in Chapters 3 and 4) and in some other situations it has been the determining factor in the problem that gets directly affected by our decisions (e.g. in Chapter 5).

In Chapter 1, we introduced the concept of time-based competition. We explained the importance of time in today's business world making references to different companies benefiting from time-based competition such as Sony, Sharp, Toyota, Hitachi, Toshiba, The Limited (women's clothing manufacturer), Federal Express, Domino's Pizza, Wilson Art and McDonald's. All of these companies have one thing in common, they have improved their time-based performance in their competition against their rivals and have benefitted from this endeavor. In this chapter, we also made references to works in the literature to present benefits of time-based competition that were mentioned in the literature. We also touched upon the concept of time-based competition in the study of supply chains and presented *responsiveness* as an important factor in analyzing time-based competition. The term responsiveness surfaces in later chapters of the thesis as the probability that a firm delivers the product at or before its promised delivery time.

In Chapter 2, we did a literature review on the subject of time-based competition and game theory. In the first section of this chapter, we explored different settings of time-based competition in the literature. We showed that the research in this area can be divided into different categories based on the decision variable under study.

All of the studies in this section deal with time-based competition but the decision variables utilized by the firms may be different. We looked at firms that compete based on capacity decisions only, firms that compete based on a combination of price, performance and other attributes, and finally firms that compete based on promised delivery time. Promised delivery time, as the duration that is quoted to the customer, within which a service or product will be delivered to him, is an important notion in this thesis. The decision variable under study in two of the chapters of this thesis (Chapters 3 and 4) is the promised delivery time.

The next section in Chapter 2 prepared the ground for Chapter 3, where time-based competition with homogeneous firms is studied. In this section, we presented concepts that would be used in the next chapter. We introduced different supply chain settings and explained how quality of service (responsiveness) changes based on the supply chain design. We also introduced two different demand functions (separable demand functions, and demand functions based on attraction models) that are used in the literature and explained their properties. In doing so, we also demonstrated how the utility function of the customers towards each firm is affected based on different factors and how that determines the market share of firms.

In this thesis, we studied three different problems that deal with time-based competition and utilize game theory. These problems are presented in Chapters 3, 4 and 5. In Chapter 3 we built upon a model that was previously introduced by Ho and Zheng [36] and Shang and Liu [53]. In this chapter, we build upon their problem and introduce a model that would incorporate new settings for firms. Ho and Zheng [36] and Shang and Liu [53] study models that consist of two firms that compete in a market with homogeneous customers. The firms offer substitutable products and compete to maximize their total market share (demand rate) λ. The decision variables used by the firms are the leadtimes that they quote to the customers. We have used their idea and developed a similar model that has the same characteristic with the difference that the firms are not individual entities any more. The firms in our model are centralized supply chains consisting of two or more stages or entities of production. The market share of each firm depends on the utility function of the market (customers) towards that firm. The utility function of the customers depends on the promised delivery time (quoted time) as well as the quality of service. The quality of service is the probability that each firm can deliver the product at or before its promised delivery time. This probability depends on the production rate of each stage at each firm as well as the promised delivery time quoted by the firm. Based on this probability and the utility function of the customers, the total market share (demand rate) of each firm is determined. Each firm tries to maximize its market share by manipulating its promised delivery time. This problem is studied with game theory as the decision taken by the firms affect each other's objective function. Each firm's promised delivery time affects his own market share and therefore, affects the market share of the other firm.

We looked at three settings for the firms in this chapter. In the first section of the chapter, we assume that the firms are like centralized supply chains consisting of two stages. We assumed that these two stages have the same production rate. In the second section of the chapter, we assumed that firms are like supply chains consisting of M stages and all the M stages have the same production rate. Finally, in the last section of the chapter we assume that the firms have only two production stages but that the stages have different production rates. In the first two sections of this chapter, we could solve the problems and derive a closed-form solution for the optimal strategies of the firms as well as the Nash equilibrium of the game. At the end of the sections, we confirmed our analytical results using numerical examples. In the last section, however we could not derive a closed-form solution for the Nash equilibrium of the game. But we developed sufficient conditions and an equation, through which the optimal strategies of the firms could be derived. We also confirmed the conditions as well as the equation using numerical examples. At the end of each section, we also performed a numerical sensitivity analysis and observed the effects of changing different parameters of the problem on the optimal strategies of the firms as well as the Nash equilibrium of the game.

The solutions provided in this chapter have improved the results of Ho and Zheng [36] and Shang and Liu [53]. This is because the process of delivering a product to the customers rarely depends on production at only one stage. Usually firms (supply chains) consist of different stages where the production or service delivery rates at each stage are different. The model presented in this chapter aims to represent these kinds of processes where serial stages have to be used for delivering a product or a service to the ultimate customer. As an example, the process of delivering a package at the UPS consists of more than one stage with different production (service delivery) rates. The models presented by Ho and Zheng [36] and Shang and Liu [53] can not be used there because they only study firms with one stage of production (service delivery). The results from this chapter help companies like the UPS to determine the optimal PDT in competition with their rivals.

Like in Chapter 3, the decision variable under study in Chapter 4 is the promised delivery time. In this chapter, we studied a duopolistic market, where two firms compete for the business of a single customer by manipulating the promised delivery time that they quote to the customer. We believe that this model is applicable in situations, where the manufacturers and customers for a product are not plentiful and the leadtime for the production is relatively long. An example for this kind of market can be found in the aviation industry. In this chapter, the two manufacturers simultaneously quote a leadtime to the single customer. The customer then decides to give her business to only one of them. We have assumed that the customer has a simple rule to choose between the firms; she will give her business to the firm that quotes the shorter leadtime. The firm that has lost the bid will exit the problem. The firm that has won the deal has to start production and deliver the product to the customer. However, the winning firm incurs production costs as well as delay costs. The delay cost is incurred when the product is delivered later than the promised delivery time that was quoted previously to the customer. The production cost is fixed and will not change based on the promised delivery time. The delay cost, however decreases in the promised delivery time. For the winning firm, the longer the leadtime is, the less its delay cost would be. Therefore, a longer leadtime is desirable to the winning firm. However, firms would also like to quote a leadtime as short as possible so that they can win the business. This creates an interesting situation of trade-off between shorter and longer leadtimes that we studied in this chapter. This problem is studied with game theory because the decision taken be each firm affects the objective function of the other firm.

We developed the model for this problem and created two criteria that need to hold for the problem to be meaningful. These criteria were enforced to make sure that the objective functions of the firms were similar to those of firms in similar situations in the real world. After that, we derived the best-response functions of each firm. Using the best-response functions, we came up with the Nash equilibrium of the game. The Nash equilibrium was confirmed using multiple numerical examples that were solved at the end of the chapter. Finally, we concluded the chapter with a numerical sensitivity analysis to study the effects of changing different parameter values on the optimal promised delivery times for the firms as well as the Nash equilibrium of the game.

The last problem of this thesis was studied in Chapter 5. In this chapter, we built upon a model that was previously studied by Gerchak and Parlar [24]. In their paper, the authors had studied a game theoretical model of competition between two firms that competed based on the budget that they invest in each of two markets that are available to both firms. Each market has different profitability for each firm and the firms are maximizing their expected profit from their investments in these markets. This is also a time-based competition as the amount that is invested in each market determined the speed, at which the firm will deliver to that market. This speed in turn changes the probability that the firm will capture that market, as the firm that delivers to a market faster will capture the whole market. Each firm is interested in knowing what fraction of its budget it should invest in each market to increase its expected profit derived from its investments. The more a firm invests in a market, the shorter the time will be for it to deliver to that market and therefore, the higher its probability of capturing that market will be. Based on these rules, the expected profit for each firm can be studied.

In Gerchak and Parlar [24], the authors study this problem and derive a number of solutions for special cases of the problem. In these special situations, they derive the optimal strategies for the firms and present the Nash equilibrium of the game. The authors in that paper have assumed that the firms know about the structure of the objective function of each other. This makes the game-theoretical model that they study a model of complete information. In the real wold, however firms rarely know about the structure and parameter values of their rivals. We built upon the model of Gerchak and Parlar [24] by studying this model in a situation of incomplete information games. We changed the model such that one of the firms does not have a clear knowledge about the objective function of the other firm. We assumed that one of the firms does not know the relative importance of the markets to its rivals. However, this firm is not completely kept in the dark regarding this piece of information. It knows that the importance of the markets to its rival could be of two types. Based on the new settings for the problem, we modeled the problem and derived the objective functions of both firms. We considered special cases and found the best-response functions of the firms. We also found the optimal strategies for the firms as well as the Nash equilibrium of the game in these cases. Like the other chapters, we solved many numerical examples of the problem and performed a sensitivity analysis to observe the effect of changing parameter values on the optimal strategies of the firms and the Nash equilibrium of the game.

6.2 Thoughts for Future Work

We have studied three different problems in the area of time-based competition in this thesis. We believe that this the literature in this area can be furthered in a number of directions. In the following, we will present our ideas for the possible opportunities for future research in the context of each problem.

In Chapter 3, we studied new settings for the firms. We changed the structure of single-stage firms into multiple-stage centralized supply chains. A possible direction for furthering this study is to consider supply chains that consist of multiple stages, where the product does not need to travel through all stages of the chain. A product may jump back and forth between different stages of the chain based on certain probabilities. This may happen due to re-processing or correction at any stage of the production. This change will affect the quality of service of the firm and creates an interesting problem to study. The natural methodology that comes to mind to analyze the quality of service of firms in this situation would be using the phase-type distributions.

We have assumed that the only decision variable that firms have in this problem is their promised delivery leadtime. Another possible extension to this model is considering other decision variables as well as the promised delivery time. Price, for instance is a possible decision variable that can be added to the model to create a more realistic reflection of the competition between firms in the real world. In this case, the promised delivery time and the price will both be quoted to the customer and the utility function of the market will be affected by both factors at the same time. This will change the problem into a more interesting model to study.

In Chapter 4, we studied a duopolistic market with firms competing based on their promised delivery leadtime. We suggest to consider a market that consists of M firms instead of only two firms. A case that studies more than two firms at the market will present a better reflection of the reality and will create interesting situations in the optimal strategies of the firms and the Nash equilibrium of the game. Like in the previous chapter, we would like to suggest to consider price as an additional decision variable in this model. In the model in hand, we have assumed that the price is exogenously determined by the market. This may be the fact in many industries, but price still plays an important factor in determining the winner of business deals in most of the industries. Therefore, we believe that adding the price as an addition decision variable is an important extension to this model. Firms can quote their promised delivery time as well as their price to the customer and the customer can then choose the winning firm based on both criteria. We believe, adding price to the decision making process of the customer creates an interesting problem that is valuable to study.

Extending the model studied in Chapter 4 by assuming asymmetric information (incomplete information) is also another area for future research. To do this, we can assume that one or both firms do not have complete information regarding the T^E or $\widehat{T}$ of the other firm. This will make the model more realistic as complete information is rarely present in the business world.

In Chapter 5, we have assumed that one of the firms has incomplete information regarding the objective function of its rival. In the real world, we usually observe situations, where none of the firms competing knows about the structure of the objective function from the other firm. Therefore, to reflect the reality better, we suggest to study a model where both firms have incomplete information regarding the objective function of their competitor. For this, we can assume that each firm has two or more types regarding a parameter in their objective function. This will create a more complicated and yet more realistic problem and it will be interesting to analyze it.

In the model in hand, the profit function of the firms have been assumed to be linear. Studying the problem with non-linear objective function is another possible extension to the model that will present a more realistic case of the problem. In addition to that, we believe that this problem can be studied in a cooperative setting as well. To study this problem with cooperating firms will create an interesting problem in the context of time-based competition.

Appendix A

Proof of lemmas and propositions in Chapter 3

In this appendix we present the proofs to lemmas and propositions from Chapter 3.

A.1 Proof of Lemma 3.5

Considering the equation $L_i(T_i, \lambda_i(\mathbf{T})) = \ln \lambda_i(\mathbf{T}) - U_i(T_i, \lambda_i(\mathbf{T}))$, taking the derivative from both sides of equation (3.7) w.r.t. T_i we have,

$$\begin{aligned}
&\left[\frac{1}{\lambda_i} + \beta_Q \mu_i T_i^2 \exp\{-(\mu_i - \lambda_i)T_i\} - \beta_Q \lambda_i T_i^2 \exp\{-(\mu_i - \lambda_i)T_i\}\right] \frac{\partial \lambda_i(\mathbf{T})}{\partial T_i} \\
&+ \beta_T - \beta_Q \mu_i^2 T_i \exp\{-(\mu_i - \lambda_i)T_i\} + 2\beta_Q \lambda_i \mu_i T_i \exp\{-(\mu_i - \lambda_i)T_i\} \\
&- \beta_Q \lambda_i^2 T_i \exp\{-(\mu_i - \lambda_i)T_i\} = \frac{\partial \ell(\mathbf{T})}{\partial T_i},
\end{aligned}$$

which can be rearranged to get the following,

$$\begin{aligned}
&\left[\frac{1}{\lambda_i} + \beta_Q \mu_i T_i^2 \exp\{-(\mu_i - \lambda_i)T_i\} - \beta_Q \lambda_i T_i^2 \exp\{-(\mu_i - \lambda_i)T_i\}\right] \frac{\partial \lambda_i(\mathbf{T})}{\partial T_i} \\
&= -\beta_T + \beta_Q \mu_i^2 T_i \exp\{-(\mu_i - \lambda_i)T_i\} - 2\beta_Q \lambda_i \mu_i T_i \exp\{-(\mu_i - \lambda_i)T_i\} \\
&+ \beta_Q \lambda_i^2 T_i \exp\{-(\mu_i - \lambda_i)T_i\} + \frac{\partial \ell(\mathbf{T})}{\partial T_i}
\end{aligned}$$

After simplification we have,

$$\begin{aligned}
&\left[\frac{1}{\lambda_i} + \left[\beta_Q T_i^2 \exp\{-(\mu_i - \lambda_i)T_i\}\right](\mu_i - \lambda_i)\right] \frac{\partial \lambda_i(\mathbf{T})}{\partial T_i} \\
&= -\beta_T + \left[\beta_Q T_i \exp\{-(\mu_i - \lambda_i)T_i\}\right](\mu_i - \lambda_i)^2 + \frac{\partial \ell(\mathbf{T})}{\partial T_i}.
\end{aligned} \tag{A.1}$$

Bringing the expression for the quality of service (Q_i) from (3.2) into the equation we have,

$$\left[\frac{1}{\lambda_i} + \left[\beta_Q T_i^2 \exp\{-(\mu_i - \lambda_i)T_i\}\right](\mu_i - \lambda_i)\right]\frac{\partial \lambda_i(\mathbf{T})}{\partial T_i} \\ = -\beta_T + \frac{\beta_Q T_i (\mu_i - \lambda_i)^2 (1 - Q_i)}{1 + (\mu_i - \lambda_i) T_i} + \frac{\partial \ell(\mathbf{T})}{\partial T_i} \tag{A.2}$$

Now from the assumption in equation (3.5), we know that $\sum_{i=1}^{2} \lambda_i = \Lambda$. Therefore, we have,

$$\sum_{j=1}^{2} \frac{\partial \lambda_j(\mathbf{T})}{\partial T_i} = \frac{\partial \Lambda}{\partial T_i} = 0. \tag{A.3}$$

To prove the first line of the lemma, we assume that $\partial \lambda_i(\mathbf{T})/\partial T_i > 0$. Then from the equation (A.3), we know that $\partial \lambda_j/\partial T_i < 0$. We also know that T_j does not depend on T_i. Taking these two into account and taking the derivative w.r.t. T_i from both sides of equation (3.7) for the firm j we have,

$$\frac{\partial L_j(T_j, \lambda_j)}{\partial \lambda_j}\frac{\partial \lambda_j(\mathbf{T})}{\partial T_i} = \frac{\partial \ell(\mathbf{T})}{\partial T_i} \tag{A.4}$$

We already know from Lemma 3.4 that $\partial L_j(T_j, \lambda_j)/\partial \lambda_j > 0$. Also, based on our assumption here we have $\partial \lambda_j(\mathbf{T})/\partial T_i < 0$. Therefore, from equation (A.4), $\partial \ell(\mathbf{T})/\partial T_i < 0$ has to hold true. For the opposite direction, assume $\partial \ell(\mathbf{T})/\partial T_i < 0$, then since based on Lemma 3.4, $\partial L_j(T_j, \lambda_j)/\partial \lambda_j > 0$, then from equation (A.4), $\partial \lambda_j(\mathbf{T})/\partial T_i$ should be negative. Then, from the equation (A.3), we will have $\partial \lambda_i(\mathbf{T})/\partial T_i > 0$.

To prove the first row of the conditions regarding Q_i, we again assume that $\partial \lambda_i(\mathbf{T})/\partial T_i > 0$. We also know that the term in the big brackets in the left-hand side of equation (A.2) is always positive. Also, we have $\partial \ell(\mathbf{T})/\partial T_i < 0$ from the previous arguments here. Then, for equation (A.2) to hold true, we should have,

$$-\beta_T + \frac{\beta_Q T_i (\mu_i - \lambda_i)^2 (1 - Q_i)}{1 + (\mu_i - \lambda_i) T_i} > 0,$$

After some rearrangements, this becomes,

$$Q_i < 1 - \frac{\beta_T \left[1 + (\mu_i - \lambda_i)T_i\right]}{\beta_Q T_i (\mu_i - \lambda_i)^2}, \tag{A.5}$$

which is equivalent to $Q_i < \omega_i(T_i, \lambda_i)$.

For the opposite direction, we can say if the equation (A.5) holds, then assuming $\partial \ell(\mathbf{T})/\partial T_i < 0$, we should have $\partial \lambda_i(\mathbf{T})/\partial T_i > 0$. This is because the term in the big brackets in the left-hand side of equation (A.2) is always positive.

The second and third rows of the lemma will be proven similarly.

A.2 Proof of Lemma 3.6

At this point $\widehat{T}_i$, the first order condition $\partial\lambda_i(\mathbf{T})/\partial T_i = 0$ should hold. From Lemma 3.5 we know that this is equivalent to $Q_i = 1 - \beta_T [1 + (\mu_i - \lambda_i) T_i] / \beta_Q T_i(\mu_i - \lambda_i)^2$. We also know from equation (3.2) that $Q_i = 1 - \exp\{-(\mu - \lambda)T_i\} (1 + (\mu_i - \lambda_i) T_i)$. Therefore, for the first-order conditions to exist, the following should have a solution for T_i,

$$1 - \frac{\beta_T [1 + (\mu_i - \lambda_i) T_i]}{\beta_Q T_i(\mu_i - \lambda_i)^2} = 1 - e^{-(\mu-\lambda)T_i} (1 + (\mu_i - \lambda_i) T_i),$$

which is equivalent to,

$$\frac{\beta_T}{\beta_Q T_i(\mu_i - \lambda_i)^2} = e^{-(\mu_i-\lambda_i)T_i},$$

or,

$$-\frac{\beta_T}{\beta_Q(\mu_i - \lambda_i)} = -(\mu_i - \lambda_i)\widehat{T}_i e^{-(\mu_i-\lambda_i)\widehat{T}_i}. \tag{A.6}$$

where $\widehat{T}_i$ is the T_i that solves the above equation.

According to Corless et al. [18], the product logarithm or the Lambert W function $W(z)$ is the function satisfying $z = W(z)e^{W(z)}$. Then equation (A.6) has the following solution,

$$-(\mu_i - \lambda_i)\widehat{T}_i = W\left(-\frac{\beta_T}{\beta_Q(\mu_i - \lambda_i)}\right),$$

which is equivalent to,

$$\widehat{T}_i = -\frac{1}{(\mu_i - \lambda_i)} W\left(-\frac{\beta_T}{\beta_Q(\mu_i - \lambda_i)}\right), \tag{A.7}$$

as the solution for $\partial\lambda_i(\mathbf{T})/\partial T_i = 0$. The W function is the product logarithm function, which is also referred to as the Lambert W function. This function has two main real-valued branches according to Corless et al. [18]. It is worth noting that $\widehat{T}_i$, if it exists, cannot be unique. This is because the argument inside the curly brackets for the W function has the negative sign and therefore, it is in the area where the Lambert W function has two real-valued solutions. The reason for this is, in this range, the Lambert W function has two real-valued branches. This is shown in Figure A.1, where the branches are identifiable by their line style. Since the term inside the brackets in (A.7) is always negative, then for the Lambert W function to have a solution, the term inside has to be between 0 and $-e^{-1}$. According to Corless et al. [18], if x is real and has the negative sign, it has to be between $-e^{-1} \leq x < 0$ to have a real-valued

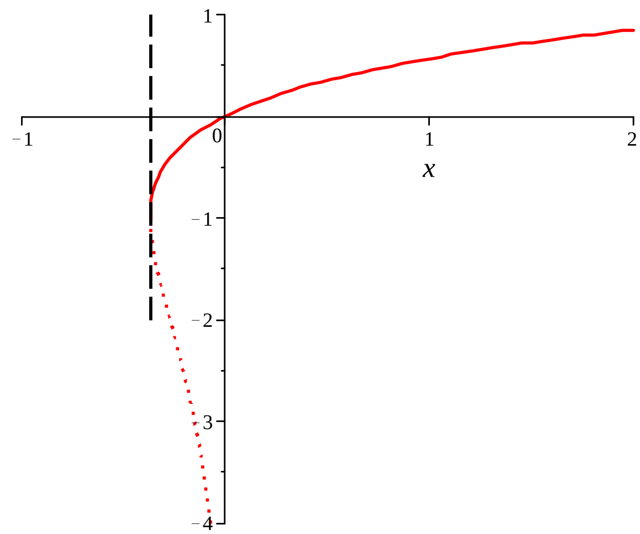

Figure A.1: The function $W(x)$ with two real-valued branches for $-e^{-1} \leq x < 0$. The vertical dashed line represents the line $x = -e^{-1}$.

solution for $W(x)$. From this argument, it is obvious that for $\widehat{T}_i$ to have a real-valued solution, the following should hold,

$$-e^{-1} \leq -\frac{\beta_T}{\beta_Q(\mu_i - \lambda_i)} < 0,$$

or,

$$0 < \frac{\beta_T}{\beta_Q(\mu_i - \lambda_i)} \leq e^{-1}.$$

Since this term is always positive we can reduce the condition to,

$$\frac{\beta_T}{\beta_Q} \leq e^{-1}(\mu_i - \lambda_i).$$

If this condition is not satisfied, the Lambert W function does not return any values. But if the conditions imposed on the parameters is satisfied, we will have two real-valued results for $\widehat{T}_i$. Let's call them $\widehat{T}_i^{(1)}$ and $\widehat{T}_i^{(2)}$, where $\widehat{T}_i^{(2)} > \widehat{T}_i^{(1)}$. Since $\lambda_i(\mathbf{T})$ is continuous in T_i, having two points, for which $\partial\lambda_i(\mathbf{T})/\partial T_i = 0$ means that the function $\lambda_i(\mathbf{T})$ is convex in some range and concave in some other range in T_i.

A.3 Proof of Proposition 3.2

In order for the game to have an interior Nash equilibrium, the first-order conditions should hold and also the function $\lambda_i(\mathbf{T})$ should be concave in T_i at that point. We have the two points $\widehat{T}_i^{(1)}$ and $\widehat{T}_i^{(2)}$, for which the first-order condition holds. To find out about the behavior of $\lambda_i(\mathbf{T})$ at its initial point we look at the sign of $\partial\lambda_i(\mathbf{T})/\partial T_i$ at $T_i = 0$. From equation (A.1), we can find the equation for $T_i = 0$. This is shown below,

$$\frac{1}{\lambda_i}\frac{\partial\lambda_i(\mathbf{T})}{\partial T_i} = -\beta_T + \frac{\partial\ell(\mathbf{T})}{\partial T_i}. \tag{A.8}$$

Now if we assume that $\partial\lambda_i(\mathbf{T})/\partial T_i > 0$ at $T_i = 0$, then based on Lemma 3.5, we should have $\partial\ell(\mathbf{T})/\partial T_i < 0$. But then the sign of the equation on the left hand-side will be positive and it will be negative on the right hand-side, which is a contradiction and thus $\partial\lambda_i(\mathbf{T})/\partial T_i > 0$ at $T_i = 0$ is not a possibility. Considering the fact that $\beta_T > 0$, then Assuming $\partial\lambda_i(\mathbf{T})/\partial T_i < 0$ at $T_i = 0$, which based on Lemma 3.5 will result in $\partial\ell(\mathbf{T})/\partial T_i > 0$ is the only possibility. Therefore, at $T_i = 0$ we have $\partial\lambda_i(\mathbf{T})/\partial T_i < 0$ and the function $\lambda_i(\mathbf{T})$ is strictly decreasing in T_i at $T_i = 0$. This means that the first stationary point $\widehat{T}_i^{(1)}$ is not a local maximizer since $\widehat{T}_i^{(1)} = \min\{\widehat{T}_i^{(1)}, \widehat{T}_i^{(2)}\}$. The point $\widehat{T}_i^{(1)}$ is either a local minimum or a point of inflection. In any case, we now draw our attention to $\widehat{T}_i^{(2)}$. In order to make sure that $\widehat{T}_i^{(2)}$, which is derived from the second branch of the W function, is our local maximizer we should study the second derivative of $\lambda_i(\mathbf{T})$ w.r.t. T_i. For the point $\widehat{T}_i^{(2)}$ to be the firm i's strategy in the Nash equilibrium, the value of the expression $\partial^2\lambda_i(\mathbf{T})/\partial T_i^2$ has to be negative at this point so that the function $\lambda_i(\mathbf{T})$ is concave at $\widehat{T}_i^{(2)}$. Taking the derivative w.r.t. T_i from both sides of equation (A.1) and writing the equation at $\widehat{T}_i^{(2)}$, for which $\partial\lambda_i(\mathbf{T})/\partial T_i = 0$, we get,

$$\begin{aligned}
&\left[\frac{1}{\lambda_i} + \beta_Q\mu_i T_i^2 \exp\{-(\mu_i-\lambda_i)T_i\} - \beta_Q\lambda_i T_i^2 \exp\{-(\mu_i-\lambda_i)T_i\}\right]\frac{\partial^2\lambda_i(\mathbf{T})}{\partial T_i^2} \qquad \text{(A.9)}\\
&= \beta_Q\mu_i^2\exp\{-(\mu_i-\lambda_i)T_i\} - 2\beta_Q\lambda_i\mu_i T_i\exp\{-(\mu_i-\lambda_i)T_i\}\\
&+ \beta_Q\lambda_i^2 T_i\exp\{-(\mu_i-\lambda_i)T_i\} - \beta_Q\mu_i^3 T_i\exp\{-(\mu_i-\lambda_i)T_i\} + \beta_Q\lambda_i^3 T_i\exp\{-(\mu_i-\lambda_i)T_i\}\\
&- 3\beta_Q\lambda_i^2\mu_i T_i\exp\{-(\mu_i-\lambda_i)T_i\} + 3\beta_Q\lambda_i\mu_i^2 T_i\exp\{-(\mu_i-\lambda_i)T_i\} + \frac{\partial\ell(\mathbf{T})}{\partial T_i},
\end{aligned}$$

In equation (A.9) and what comes next in the proof, we are assuming $T_i = \widehat{T}_i^{(2)}$, which has been used to avoid unnecessary complexity in writing. After simplifications, equation (A.9) becomes,

$$\left[\frac{1}{\lambda_i} + \beta_Q T_i^2 \exp\{-(\mu_i - \lambda_i)T_i\}(\mu_i - \lambda_i)\right] \frac{\partial^2 \lambda_i(\mathbf{T})}{\partial T_i^2} = \tag{A.10}$$
$$\beta_Q \exp\{-(\mu_i - \lambda_i)T_i\}(\mu_i - \lambda_i)^2 - \beta_Q T_i \exp\{-(\mu_i - \lambda_i)T_i\}(\mu_i - \lambda_i)^3 + \frac{\partial^2 \ell(\mathbf{T})}{\partial T_i^2}.$$

Now if we assume that $\partial^2 \lambda_i(\mathbf{T})/\partial T_i^2 > 0$, then the right-hand-side of the equation should also be positive. This is because the term in the big brackets is always positive. The term $\beta_Q \exp\{-(\mu_i - \lambda_i)T_i\}(\mu_i - \lambda_i)^2 + \beta_Q T_i \exp\{-(\mu_i - \lambda_i)T_i\}(\lambda_i - \mu_i)^3$ may be negative or positive depending on the parameter values. In order for us to result in the contradiction in this proof, we need this term to be negative. For this term to be negative we have to have

$$\beta_Q \exp\{-(\mu_i - \lambda_i)T_i\} \left[(\mu_i - \lambda_i)^2 + T_i(\lambda_i - \mu_i)^3\right] < 0, \tag{A.11}$$

or,

$$\beta_Q \exp\{-(\mu_i - \lambda_i)T_i\}(\mu_i - \lambda_i)^2 \left[1 + T_i(\lambda_i - \mu_i)\right] < 0,$$

which results in $1 + T_i(\lambda_i - \mu_i) < 0$ or $T_i(\mu_i - \lambda_i) > 1$. This inequality also appears in Ho and Zheng [36] in their proof as a condition for concavity of $\lambda_i(\mathbf{T})$ or to show $\partial^2 \lambda_i(\mathbf{T})/\partial T_i^2 < 0$. So if $T_i(\mu_i - \lambda_i) > 1$ it means that the quality of service is greater than 26.4%. This is because we have,

$$T_i(\mu_i - \lambda_i) > 1, \tag{A.12}$$

therefore,

$$-e^{-(\mu_i - \lambda_i)T_i} < -e^{-1}, \tag{A.13}$$

multiplying the positive term $1 + (\mu_i - \lambda_i)T$ by both sides of the inequality (A.13) and because of the inequality (A.12) we have,

$$-e^{-(\mu_i - \lambda_i)T_i} \left[1 + (\mu_i - \lambda_i)T\right] > -2e^{-1},$$

or,

$$1 - e^{-(\mu_i - \lambda_i)T_i} \left[1 + (\mu_i - \lambda_i)T\right] > 1 - 2e^{-1}.$$

The left hand-side of the inequality is the expression for the quality of service in our system. The right hand-side of the equation is equal to 0.264. Therefore, for the inequality (A.11) to hold, the quality of service in our system should be greater than 26.4%.

With the QoS being greater than 26.4%, for the right hand-side of equation (A.10) to be positive, we should have $\partial^2 \ell(\mathbf{T})/\partial T_i^2 > 0$. Also, from equation (A.4) for firm j

we get,

$$\frac{\partial L_j(T_j,\lambda_j)}{\partial \lambda_j}\frac{\partial^2 \lambda_j(\mathbf{T})}{\partial T_i^2} = \frac{\partial^2 \ell(\mathbf{T})}{\partial T_i^2}.$$

So again knowing that $\partial L_j(T_j,\lambda_j)/\partial\lambda_j > 0$ and $\partial^2\ell(\mathbf{T})/\partial T_i^2 > 0$, we should have $\partial^2\lambda_j(\mathbf{T})/\partial T_i^2 > 0$. This will result in

$$\frac{\partial^2 \lambda_i(\mathbf{T})}{\partial T_i^2} + \frac{\partial^2 \lambda_j(\mathbf{T})}{\partial T_i^2} > 0.$$

But from equation (A.3) we know that $\sum_{j=1}^{2} \partial^2\lambda_j(\mathbf{T})/\partial T_i^2 = \partial^2\Lambda/\partial T_i^2 = 0$. This is a contradiction. Therefore, we should have $\partial^2\lambda_i(\mathbf{T})/\partial T_i^2 < 0$ for $T_i = \widehat{T}_i^{(2)}$. As a result, the value of $\partial^2\lambda_i(\mathbf{T})/\partial T_i^2$ is negative at $\widehat{T}_i^{(2)}$ and this shows that $\widehat{T}_i^{(2)}$ is indeed the unique maximizer of $\lambda_i(\mathbf{T})$ when the quality of service is greater than 26.4% and the conditions 3.5 and 3.6 hold. Therefore, with the above mentioned conditions holding, the point $\widehat{T}_i^{(2)}$ will be the local maximizer of the demand rate function and from Lemma 3.6 we have,

$$T_i^N = \widehat{T}_i^{(2)} = -\frac{1}{\mu_i-\lambda_i} W\left\{-1, -\frac{\beta_T}{\beta_Q(\mu_i-\lambda_i)}\right\},$$

where the added -1 in the W function denotes the lower branch as suggested in Corless et al. [18]. This branch is with dotted curve in Figure A.1. It should be noted that this branch produces the greater T_i, which in our case is the desired $\widehat{T}_i^{(2)}$. The point $T_i^N = \widehat{T}_i^{(2)}$ is firm i's strategy at the unique interior Nash equilibrium.

Also, from Lemmas 3.5 and 3.6 we know that $\beta_T/\beta_Q \le e^{-1}(\mu_i - \lambda_i)$ should hold. Since this condition is necessary for the existence of the first-oder conditions, it should hold for an interior Nash equilibrium. This confirms the condition (3.11).

A.4 Proof of Lemma 3.9

We start with taking the derivative w.r.t. T_i from both sides of equation (3.17). We get,

$$\frac{1}{\lambda}\frac{\partial \lambda_i(\mathbf{T})}{\partial T_i} + \beta_T - \beta_Q \frac{\partial Q_i(T_i,\lambda_i)}{\partial T_i} = \frac{\partial \ell(\mathbf{T})}{\partial T_i}$$

then we have

$$\left[\frac{1}{\lambda_i} + \beta_Q \frac{1}{(k-1)!} T_i^k (\mu_i-\lambda_i)^{k-1} e^{-(\mu_i-\lambda_i)T_i}\right] \frac{\partial \lambda_i(\mathbf{T})}{\partial T_i}$$
$$= -\beta_T + \beta_Q \frac{1}{(k-1)!} T_i^{k-1} (\mu_i-\lambda_i)^k e^{-(\mu_i-\lambda_i)T_i} + \frac{\partial \ell(\mathbf{T})}{\partial T_i}$$

Bringing in the term $(1 - Q_i(T_i, \lambda_i))$ we have,

$$\left[\frac{1}{\lambda_i} + \beta_Q \frac{1}{(k-1)!} T_i^k (\mu_i - \lambda_i)^{k-1} e^{-(\mu_i - \lambda_i)T_i}\right] \frac{\partial \lambda_i(\mathbf{T})}{\partial T_i}$$
$$= -\beta_T + \frac{\beta_Q \frac{1}{(k-1)!} T_i^{k-1} (\mu_i - \lambda_i)^k (1 - Q_i)}{\left[1 + \sum_{n=1}^{k-1} \frac{1}{n!} T_i^n (\mu_i - \lambda_i)^n\right]} + \frac{\partial \ell(\mathbf{T})}{\partial T_i}$$

or,

$$\left[\frac{1}{\lambda_i} + \beta_Q \frac{1}{(k-1)!} T_i^k (\mu_i - \lambda_i)^{k-1} e^{-(\mu_i - \lambda_i)T_i}\right] \frac{\partial \lambda_i(\mathbf{T})}{\partial T_i} \tag{A.14}$$
$$= \frac{-\beta_T - \beta_T \sum_{n=1}^{k-1} \frac{1}{n!} (T_i\mu_i - T_i\lambda_i)^n + \beta_Q \frac{1}{(k-1)!} T_i^{k-1} (\mu_i - \lambda_i)^k (1 - Q_i)}{\left[1 + \sum_{n=1}^{k-1} \frac{1}{n!} T_i^n (\mu_i - \lambda_i)^n\right]} + \frac{\partial \ell(\mathbf{T})}{\partial T_i}$$

It should be noted that the first row for the relationship between $\partial\lambda_i(\mathbf{T})/\partial T_i$ and $\partial\ell(\mathbf{T})/\partial T_i$ is proven exactly similarly to the proof of Lemma 3.5. To find the relationship of the two with Q_i, we first assume that $\partial\lambda_i(\mathbf{T})/\partial T_i > 0$. Knowing that $\partial\lambda_i(\mathbf{T})/\partial T_i > 0$ implies $\partial\ell(\mathbf{T})/\partial T_i < 0$, for equation (A.14) to hold we should have the following,

$$\frac{-\beta_T - \beta_T \sum_{n=1}^{k-1} \frac{1}{n!} (T_i\mu_i - T_i\lambda_i)^n + \beta_Q \frac{1}{(k-1)!} T_i^{k-1} (\mu_i - \lambda_i)^k (1 - Q_i)}{\left[1 + \sum_{n=1}^{k-1} \frac{1}{n!} T_i^n (\mu_i - \lambda_i)^n\right]} > 0,$$

which means,

$$Q < \frac{-\beta_T - \beta_T \sum_{n=1}^{k-1} \frac{1}{n!} (T_i\mu_i - T_i\lambda_i)^n + \beta_Q \frac{1}{(k-1)!} T_i^{k-1} (\mu_i - \lambda_i)^k}{\beta_Q \frac{1}{(k-1)!} T_i^{k-1} (\mu_i - \lambda_i)^k},$$

or,

$$Q < 1 - \frac{\beta_T \left[1 + \sum_{n=1}^{k-1} \frac{1}{n!} (T_i\mu_i - T_i\lambda_i)^n\right]}{\beta_Q \frac{1}{(k-1)!} T_i^{k-1} (\mu_i - \lambda_i)^k}, \tag{A.15}$$

where the right-hand side of the inequality is equal to $\omega_k(T_i, \lambda_i)$ as introduced in in the lemma.

For the opposite direction we can say that if inequality (A.15) holds, assuming that $\partial\ell(\mathbf{T})/\partial T_i < 0$, we should have $\partial\lambda_i(\mathbf{T})/\partial T_i > 0$ so that the equation (A.14) holds. The reason is that the term in the brackets on the left-hand side of equation (A.14) is always positive.

The second and third rows of the relationships in (3.21) can be proven similarly.

A.5 Proof of Lemma 3.10

For the demand rate into firm i ($\lambda_i(\mathbf{T})$) to have its first-order condition satisfied, we should solve the condition $\partial\lambda_i(\mathbf{T})/\partial T_i = 0$. From Lemma 3.9 we know that $\partial\lambda_i(\mathbf{T})/\partial T_i = 0$ implies $\partial\ell(\mathbf{T})/\partial T_i = 0$. Also, in this case the quality of service will be equal to the following,

$$Q_i(T_i,\lambda_i) = 1 - \frac{\beta_T\left[1+\sum_{n=1}^{k-1}\frac{1}{n!}(T_i\mu_i - T_i\lambda_i)^n\right]}{\beta_Q\frac{1}{(k-1)!}T^{k-1}(\mu_i-\lambda_i)^k}. \tag{A.16}$$

We know from (3.16) that $Q_i(T_i,\lambda_i)$ is also equal to,

$$Q_i(T_i,\lambda_i) = 1 - \left[1+\sum_{n=1}^{k-1}\frac{1}{n!}T_i^n(\mu_i-\lambda_i)^n\right]e^{-(\mu_i-\lambda_i)T_i}. \tag{A.17}$$

Setting (A.16) and (A.17) equal to each other we get $\widehat{T}_i$ that would satisfy the first-order condition $\partial\lambda_i(\mathbf{T})/\partial T_i = 0$,

$$1 - \frac{\beta_T\left[1+\sum_{n=1}^{k-1}\frac{1}{n!}\widehat{T}_i^n(\mu_i-\lambda_i)^n\right]}{\beta_Q\frac{1}{(k-1)!}\widehat{T}_i^{k-1}(\mu_i-\lambda_i)^k} = 1 - \left[1+\sum_{n=1}^{k-1}\frac{1}{n!}\widehat{T}_i^n(\mu_i-\lambda_i)^n\right]e^{-(\mu_i-\lambda_i)\widehat{T}_i},$$

which after simplification becomes,

$$\frac{\beta_T}{\beta_Q\frac{1}{(k-1)!}\widehat{T}_i^{k-1}(\mu_i-\lambda_i)^k} = e^{-(\mu_i-\lambda_i)\widehat{T}_i}. \tag{A.18}$$

which is equivalent to,

$$\left(\frac{\beta_T(k-1)!}{\beta_Q(\mu_i-\lambda_i)}\right)^{\frac{1}{k-1}} = \widehat{T}_i(\mu_i-\lambda_i)e^{-\frac{1}{k-1}(\mu_i-\lambda_i)\widehat{T}_i},$$

or,

$$-\frac{1}{k-1}\left(\frac{\beta_T(k-1)!}{\beta_Q(\mu_i-\lambda_i)}\right)^{\frac{1}{k-1}} = -\frac{1}{k-1}(\mu_i-\lambda_i)\widehat{T}_i e^{-\frac{1}{k-1}(\mu_i-\lambda_i)\widehat{T}_i}. \tag{A.19}$$

The solution to equation (A.19) is the following,

$$-\frac{1}{k-1}(\mu_i-\lambda_i)\widehat{T}_i = W\left\{-\frac{1}{k-1}\left(\frac{\beta_T(k-1)!}{\beta_Q(\mu_i-\lambda_i)}\right)^{\frac{1}{k-1}}\right\},$$

where according to Corless et al. [18], W is the Lambert W function. Therefore, the solution to equation (A.18) is the following,

$$\widehat{T_i} = -\frac{k-1}{\mu_i - \lambda_i} W \left\{ -\frac{1}{k-1} \left(\frac{\beta_T (k-1)!}{\beta_Q (\mu_i - \lambda_i)} \right)^{\frac{1}{k-1}} \right\}. \tag{A.20}$$

Since the term inside the braces is always negative, we are looking at the area, where the Lambert W function is defined over a negative set. In this area, the function has two real-valued solutions.

For $x < 0$, the Lambert W function $W(x)$ is defined only where $-\exp(-1) \leq x$. This implies that,

$$-\frac{1}{k-1} \left(\frac{\beta_T (k-1)!}{\beta_Q (\mu_i - \lambda_i)} \right)^{\frac{1}{k-1}} \geq -e^{-1},$$

which after algebraic simplifications becomes,

$$\frac{\beta_T}{\beta_Q} \leq \frac{(\mu_i - \lambda_i) \left[e^{-1}(k-1) \right]^{k-1}}{(k-1)!}.$$

A.6 Proof of Proposition 3.4

In order for the game to have an interior Nash equilibrium, a strategy T_i^N that satisfies the first-order conditions, should also satisfy the second-order condition $\partial^2 \lambda_i(\mathbf{T}) / \partial T_i^2 \leq 0$.

Now for the point $\widehat{T_i}$ to be firm i's strategy at a Nash equilibrium, it should satisfy the second-order condition $\partial^2 \lambda_i(\mathbf{T}) / \partial T_i^2 < 0$ as well. Noting that the value of $\partial \lambda_i(\mathbf{T}) / \partial T_i$ is zero at point $\widehat{T_i}$, we take the second derivative w.r.t. T_i from both sides of equation (3.17). After some algebraic simplification, we get the following,

$$\begin{aligned} &\left[\frac{1}{\lambda_i} + \beta_Q \frac{1}{(k-1)!} T_i^k (\mu_i - \lambda_i)^{k-1} e^{-(\mu_i - \lambda_i) T_i} \right] \frac{\partial \lambda_i^2(\mathbf{T})}{\partial T_i^2} \\ &= \beta_Q \frac{1}{(k-2)!} T_i^{k-2} (\mu_i - \lambda_i)^k e^{-(\mu_i - \lambda_i) T_i} - \beta_Q \frac{1}{(k-1)!} T_i^{k-1} (\mu_i - \lambda_i)^{k+1} e^{-(\mu_i - \lambda_i) T_i} + \frac{\partial^2 \ell(\mathbf{T})}{\partial T_i^2}. \end{aligned}$$

or,

$$\begin{aligned} &\left[\frac{1}{\lambda_i} + \beta_Q \frac{1}{(k-1)!} T_i^k (\mu_i - \lambda_i)^{k-1} e^{-(\mu_i - \lambda_i) T_i} \right] \frac{\partial \lambda_i^2(\mathbf{T})}{\partial T_i^2} \\ &= \beta_Q \frac{1}{(k-2)!} T_i^{k-2} (\mu_i - \lambda_i)^k e^{-(\mu_i - \lambda_i) T_i} \left[1 - \frac{1}{k-1} T_i (\mu_i - \lambda_i) \right] + \frac{\partial^2 \ell(\mathbf{T})}{\partial T_i^2}. \end{aligned} \tag{A.21}$$

In the above equations and those following in this proof, we replaced $\widehat{T_i}$ by T_i for the sake of simplicity in writing. In fact, by T_i we are referring to $\widehat{T_i}$ that satisfies the first-order condition.

If we assume that $\partial\lambda_i^2(\mathbf{T})/\partial T_i^2$ is positive, the left-hand side of equation (A.21) becomes positive. For that equation to hold, the right-hand side of it should also be positive. The term $[\beta_Q/(k-2)!]T_i^{k-2}(\mu_i-\lambda_i)^k \exp\{-(\mu_i-\lambda_i)T_i\}\,[1 - T_i(\mu_i - \lambda_i)/k - 1]$ may be positive or negative depending on its parameter values. In this proof, we are interested in finding the conditions for this term to be negative. In other words we want,

$$\beta_Q \frac{1}{(k-2)!} T_i^{k-2}(\mu_i - \lambda_i)^k e^{-(\mu_i-\lambda_i)T_i} \left[1 - \frac{1}{k-1} T_i(\mu_i - \lambda_i)\right] < 0. \tag{A.22}$$

Since the expression outside of the brackets is always positive, then from (A.22) we have,

$$1 - \frac{1}{k-1} T_i(\mu_i - \lambda_i) < 0,$$

or,

$$\frac{1}{k-1} T_i(\mu_i - \lambda_i) > 1, \tag{A.23}$$

which means,

$$1 - e^{-(\mu_i-\lambda_i)T_i} > 1 - e^{-(k-1)}. \tag{A.24}$$

From (A.23) we have,

$$(T_i\mu_i - T_i\lambda_i)^r > (k-1)^r, \text{ for any integer } r \geq 1. \tag{A.25}$$

Then from equations (A.23), (A.24) and (A.25) we can form the following inequality,

$$\begin{aligned}
&1 - e^{-(\mu_i-\lambda_i)T_i} - \frac{T_i(\mu_i - \lambda_i)}{1!} e^{-(\mu_i-\lambda_i)T_i} - \frac{T_i^2(\mu_i - \lambda_i)^2}{2!} e^{-(\mu_i-\lambda_i)T_i} \\
&- \frac{T_i^3(\mu_i - \lambda_i)^3}{3!} e^{-(\mu_i-\lambda_i)T_i} - \dots - \frac{T_i^{k-1}(\mu_i - \lambda_i)^{k-1}}{(k-1)!} e^{-(\mu_i-\lambda_i)T_i} \\
&= 1 - e^{-(\mu_i-\lambda_i)T_i} - e^{-(\mu_i-\lambda_i)T_i} \left[\frac{T_i(\mu_i - \lambda_i)}{1!} + \frac{T_i^2(\mu_i - \lambda_i)^2}{2!} + \dots + \frac{T_i^{k-1}(\mu_i - \lambda_i)^{k-1}}{(k-1)!}\right] \\
&> 1 - e^{-(k-1)} - e^{-(k-1)} \left[\frac{(k-1)}{1!} + \frac{(k-1)^2}{2!} + \dots + \frac{(k-1)^{k-1}}{(k-1)!}\right].
\end{aligned} \tag{A.26}$$

But the left-hand side of inequality (A.26) is the equivalent of the quality of service expression as introduced in (3.16). Therefore, for condition (A.22) to hold, the quality of service should satisfy the following condition,

$$Q_i(T_i, \lambda_i) > 1 - e^{-(k-1)} - e^{-(k-1)} \sum_{n=1}^{k-1} \frac{(k-1)^n}{n!}. \tag{A.27}$$

k	$\delta(k)$
2	0.264
3	0.323
4	0.352
5	0.371

Table A.1: Values for the quality of service threshold for different number of stages.

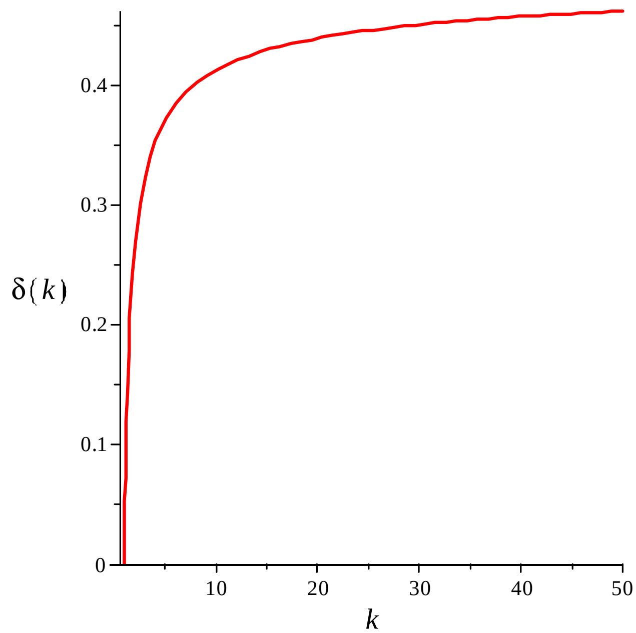

Figure A.2: The threshold for the quality of service for values of k from 1 to 50.

We call the right-hand side of condition (A.27) the quality of service threshold as follow,

$$\delta(k) = 1 - e^{-(k-1)} \left[1 + \sum_{n=1}^{k-1} \frac{(k-1)^n}{n!} \right], \tag{A.28}$$

and present this value for different values of k in Table A.1. The quality of service threshold for $1 \leq k \leq 50$ is plotted in Figure A.2. It is important to note that the function is only plotted continuously for the sake of illustration. Obviously, the quality of service threshold is a discrete function of k.

Assuming that condition (A.22) is always satisfied, with the assumption that $\partial \lambda_i^2(\mathbf{T})/\partial T_i^2$ is positive, equation (A.21) is satisfied only if $\partial^2 \ell(\mathbf{T})/\partial T_i^2 > 0$.

Also, from equation (3.17) for firm j we get,

$$\frac{\partial L_j(T_j, \lambda_j)}{\partial \lambda_j} \frac{\partial^2 \lambda_j(\mathbf{T})}{\partial T_i^2} = \frac{\partial^2 \ell(\mathbf{T})}{\partial T_i^2}.$$

So again knowing that $\partial L_j(T_j, \lambda_j)/\partial \lambda_j > 0$ and $\partial^2 \ell(\mathbf{T})/\partial T_i^2 > 0$, we should have $\partial^2 \lambda_j(\mathbf{T})/\partial T_i^2 > 0$. This will result in

$$\frac{\partial^2 \lambda_i(\mathbf{T})}{\partial T_i^2} + \frac{\partial^2 \lambda_j(\mathbf{T})}{\partial T_i^2} > 0,$$

or,

$$\sum_{j=1}^{2} \frac{\partial^2 \lambda_j(\mathbf{T})}{\partial T_i^2} > 0,$$

but since $\lambda_i(\mathbf{T}) + \lambda_j(\mathbf{T}) = \Lambda$ and is fixed, we know that $\sum_{j=1}^{2} \partial^2 \lambda_j(\mathbf{T})/\partial T_i^2 = \partial^2 \Lambda/\partial T_i^2 = 0$. This is a contradiction. Therefore, we should have $\partial^2 \lambda_i(\mathbf{T})/\partial T_i^2 < 0$ for $\widehat{T}_i$. As a result, the value of $\partial^2 \lambda_i(\mathbf{T})/\partial T_i^2$ is negative at $\widehat{T}_i$ and this shows that $T_i^N = \widehat{T}_i$ is indeed the unique maximizer of $\lambda_i(\mathbf{T})$ and is firm i's strategy at the Nash equilibrium.

But $\widehat{T}_i$ that satisfies the first-order condition is not unique. According to Lemma 3.10, if the conditions imposed on the parameters is satisfied, we will have two real-valued results for $\widehat{T}_i$. Let's call them $\widehat{T}_i^{(1)}$ and $\widehat{T}_i^{(2)}$, where $\widehat{T}_i^{(2)} > \widehat{T}_i^{(1)}$. The point $\widehat{T}_i^{(2)}$ is derived from the lower branch of the Lambert W function and $\widehat{T}_i^{(1)}$ is derived by the upper branch of that function making it a smaller value than $\widehat{T}_i^{(2)}$. The two branches of the Lambert W function can be seen in figure A.1. Having two points, for which $\partial \lambda_i(\mathbf{T})/\partial T_i = 0$ means that the function $\lambda_i(\mathbf{T})$ is convex in some range and concave in some other range in T_i.

To find out at which point the function $\lambda_i(\mathbf{T})$ is concave in T_i, we look at the sign of $\partial \lambda_i(\mathbf{T})/\partial T_i$ at $T_i = 0$. From equation (A.14), we can find the equation for $T_i = 0$. This is shown below,

$$\frac{1}{\lambda_i} \frac{\partial \lambda_i(\mathbf{T})}{\partial T_i} = -\beta_T + \frac{\partial \ell(\mathbf{T})}{\partial T_i}. \tag{A.29}$$

Now if we assume that $\partial \lambda_i(\mathbf{T})/\partial T_i > 0$ at $T_i = 0$, then based on Lemma 3.9, we should have $\partial \ell(\mathbf{T})/\partial T_i < 0$. But then the sign of the equation on the left hand-side will be positive and it will be negative on the right hand-side, which is a contradiction and thus $\partial \lambda_i(\mathbf{T})/\partial T_i > 0$ at $T_i = 0$ is not a possibility. Assuming $\partial \lambda_i(\mathbf{T})/\partial T_i < 0$ at $T_i = 0$, based on Lemma 3.9 we will have $\partial \ell(\mathbf{T})/\partial T_i < 0$. Considering the sign of $-\beta_T$, the sign on both sides of equation (A.29) will be negative and this is the only possibility. Therefore, at $T_i = 0$ we have $\partial \lambda_i(\mathbf{T})/\partial T_i < 0$ and the function $\lambda_i(\mathbf{T})$ is strictly decreasing in T_i at $T_i = 0$. This means that the first stationary point $\widehat{T}_i^{(1)}$ is a local minimum since $\widehat{T}_i^{(1)} = \min\{\widehat{T}_i^{(1)}, \widehat{T}_i^{(2)}\}$. Therefore, $\widehat{T}_i^{(2)}$ which is derived from the second branch of the W function and is greater than $\widehat{T}_i^{(1)}$ is our unique strategy in the Nash equilibrium and is denoted by T_i^N. Then from Lemma 3.10 and equation (A.20) we have,

$$T_i^N = \widehat{T}_i^{(2)} = -\frac{k-1}{\mu_i - \lambda_i} W\left\{-1, -\frac{1}{k-1}\left(\frac{\beta_T\,(k-1)!}{\beta_Q\,(\mu_i - \lambda_i)}\right)^{\frac{1}{k-1}}\right\},$$

where as in the previous section, the added -1 in the W function denotes the lower branch as suggested in Corless et al. [18]. This result is also confirmed through our numerical analysis, where $\widehat{T}_i^{(1)}$ results in a quality of service that is always less than the required threshold and $T_i^N = \widehat{T}_i^{(2)}$ results in the quality of service that satisfies equation (A.27) and is greater than the threshold.

There are two conditions for the point $\widehat{T}_i^{(2)}$ to produce the Nash equilibrium of the game. We already know from Lemma 3.10 that the following condition should hold,

$$\frac{\beta_T}{\beta_Q} \le \frac{(\mu_i - \lambda_i)\left[e^{-1}(k-1)\right]^{k-1}}{(k-1)!}.$$

Therefore, for Proposition 3.4 to be valid, this condition should still hold. In addition to that, the quality of service should also be greater than $\delta(k)$ or,

$$Q_i(T_i, \lambda_i) > 1 - e^{-(k-1)}\left[1 + \sum_{n=1}^{k-1}\frac{(k-1)^n}{n!}\right].$$

A.7 Proof of Lemma 3.13

We take the derivative w.r.t. T_i from both sides of equation (3.30). Then we get,

$$\frac{1}{\lambda}\frac{\partial \lambda_i(\mathbf{T})}{\partial T_i} + \beta_T - \beta_Q \frac{\partial Q_i(T_i, \lambda_i)}{\partial T_i} = \frac{\partial \ell(\mathbf{T})}{\partial T_i}$$

By replacing $Q_i(T_i, \lambda_i)$ with its corresponding expression we get,

$$\left[\frac{1}{\lambda_i} - \frac{\beta_Q}{\mu_{i1} - \mu_{i2}}\left\{e^{-(\mu_{i1}-\lambda_i)T_i}\left[(\mu_{i2} - \lambda_i)\,T_i - 1\right] - e^{-(\mu_{i2}-\lambda_i)T_i}\left[(\mu_{i1} - \lambda_i)\,T_i - 1\right]\right\}\right]\frac{\partial \lambda_i(\mathbf{T})}{\partial T_i}$$
$$= -\beta_T - \frac{\beta_Q}{\mu_{i1} - \mu_{i2}}\{e^{-(\mu_{i1}-\lambda_i)T_i}\,(\mu_{i1} - \lambda_i)\,(\mu_{i2} - \lambda_i) - e^{-(\mu_{i2}-\lambda_i)T_i}\,(\mu_{i1} - \lambda_i)\,(\mu_{i2} - \lambda_i)\}$$
$$+ \frac{\partial \ell(\mathbf{T})}{\partial T_i}.$$

which after simplification becomes,

$$\left[\frac{1}{\lambda_i} + \frac{\beta_Q}{\mu_{i2} - \mu_{i1}} \left\{ e^{-(\mu_{i1}-\lambda_i)T_i} \left[(\mu_{i2} - \lambda_i) T_i - 1 \right] - e^{-(\mu_{i2}-\lambda_i)T_i} \left[(\mu_{i1} - \lambda_i) T_i - 1 \right] \right\} \right] \frac{\partial \lambda_i(\mathbf{T})}{\partial T_i}$$
$$= -\beta_T + \frac{\beta_Q (\mu_{i1} - \lambda_i)(\mu_{i2} - \lambda_i)}{\mu_{i2} - \mu_{i1}} \left(e^{-(\mu_{i1}-\lambda_i)T_i} - e^{-(\mu_{i2}-\lambda_i)T_i} \right) + \frac{\partial \ell(\mathbf{T})}{\partial T_i}. \tag{A.30}$$

Bringing the term $(1 - Q_i(T_i, \lambda_i)) = 1 - Q_i$ into the expression we will have,

$$\left[\frac{1}{\lambda_i} + \frac{\beta_Q}{\mu_{i2} - \mu_{i1}} \left\{ e^{-(\mu_{i1}-\lambda_i)T_i} \left[(\mu_{i2} - \lambda_i) T_i - 1 \right] - e^{-(\mu_{i2}-\lambda_i)T_i} \left[(\mu_{i1} - \lambda_i) T_i - 1 \right] \right\} \right] \frac{\partial \lambda_i(\mathbf{T})}{\partial T_i}$$
$$= -\beta_T + \beta_Q \frac{(\mu_{i1} - \lambda_i)(\mu_{i2} - \lambda_i)\left(e^{-(\mu_{i1}-\lambda_i)T_i} - e^{-(\mu_{i2}-\lambda_i)T_i} \right)}{e^{-(\mu_{i1}-\lambda_i)T_i}(\mu_{i2} - \lambda_i) - e^{-(\mu_{i2}-\lambda_i)T_i}(\mu_{i1} - \lambda_i)} (1 - Q_i) + \frac{\partial \ell(\mathbf{T})}{\partial T_i}. \tag{A.31}$$

The part of the lemma that deals with the relationship between $\partial\lambda_i/\partial T_i$ and $\partial\ell(\mathbf{T})/\partial T_i$ is proven exactly as in the proof of Lemma 3.5. Now to prove the part dealing with Q_i, assuming $\partial\lambda_i/\partial T_i > 0$, we will have $\partial\ell(\mathbf{T})/\partial T_i < 0$. Considering $T(\min\{\mu_1, \mu_2\} - \lambda) > 1$, the left-hand side of equation (A.31) is always positive (see the proof for Lemma (3.11)). For both sides of this equation to have the positive sign, it is required that $\Omega_i(T_i, \lambda_i) > 0$ where,

$$\Omega_i(T_i, \lambda_i) = -\beta_T + \beta_Q \frac{(\mu_{i1} - \lambda_i)(\mu_{i2} - \lambda_i)\left(e^{-(\mu_{i1}-\lambda_i)T_i} - e^{-(\mu_{i2}-\lambda_i)T_i} \right)}{e^{-(\mu_{i1}-\lambda_i)T_i}(\mu_{i2} - \lambda_i) - e^{-(\mu_{i2}-\lambda_i)T_i}(\mu_{i1} - \lambda_i)} (1 - Q_i).$$

Now for $\Omega_i(T_i, \lambda_i) > 0$ to hold after simplifications, we should have,

$$Q_i < \frac{1}{-\beta_Q (\mu_{i1} - \lambda_i)(\mu_{i2} - \lambda_i)(e^{-(\mu_{i1}-\lambda_i)T_i} - e^{-(\mu_{i2}-\lambda_i)T_i})}$$
$$\{\beta_T \left[e^{-(\mu_{i1}-\lambda_i)T_i}(\mu_{i2} - \lambda_i) - e^{-(\mu_{i2}-\lambda_i)T_i}(\mu_{i1} - \lambda_i) \right]$$
$$- \beta_Q \left(e^{-(\mu_{i1}-\lambda_i)T_i} - e^{-(\mu_{i2}-\lambda_i)T_i} \right)(\mu_{i1} - \lambda_i)(\mu_{i2} - \lambda_i)\}.$$

After simplifications, this becomes,

$$Q_i < 1 - \frac{\beta_T \left[e^{-(\mu_{i1}-\lambda_i)T_i}(\mu_{i2} - \lambda_i) - e^{-(\mu_{i2}-\lambda_i)T_i}(\mu_{i1} - \lambda_i) \right]}{\beta_Q (\mu_{i1} - \lambda_i)(\mu_{i2} - \lambda_i)(e^{-(\mu_{i1}-\lambda_i)T_i} - e^{-(\mu_{i2}-\lambda_i)T_i})}, \tag{A.32}$$

whose right-hand side is equal to the term $\omega_i(T_i, \lambda_i)$ in (3.34) confirming the condition in (3.33). For the opposite direction, we can say that if inequality (A.32) holds, then assuming that $\partial\ell(\mathbf{T})/\partial T_i < 0$ holds, we should have $\partial\lambda_i/\partial T_i > 0$. Therefore, for the equation (A.31) to hold, we should have $\Omega_i(T_i, \lambda_i) > 0$. This is because knowing $T(\min\{\mu_1, \mu_2\} - \lambda) > 1$, the left-hand side of (A.31) is always positive, and so should be its right-hand side. This will result in inequality (A.32) confirmed. The next two rows of the set of relationships (3.33) are proven accordingly.

A.8 Proof of the Proposition 3.6

For the game to have an interior Nash equilibrium, first-order conditions should be satisfied. From Lemmas (3.11) and (3.13) we know that the sufficient condition for first-order conditions to hold is the conditions (3.35). Now we proceed to find the remaining conditions.

From Lemma (3.13) we can see that for both situations of $T = 0$ and $T = +\infty$, we have,

$$\frac{1}{\lambda_i}\frac{\partial \lambda_i(\mathbf{T})}{\partial T_i} = -\beta_T + \frac{\partial \ell(\mathbf{T})}{\partial T_i}. \tag{A.33}$$

Now assuming $\partial\lambda_i(\mathbf{T})/\partial T_i > 0$, from Lemma (3.13) we know that this will imply $\partial\ell(\mathbf{T})/\partial T_i < 0$. Therefore, the left-hand side of equation (A.33) will be positive and its right-hand side will be negative. As a result, at both $T = 0$ and $T = +\infty$ we cannot have $\partial\lambda_i(\mathbf{T})/\partial T_i > 0$. Also, from Lemma (3.13) $\partial\lambda_i(\mathbf{T})/\partial T_i = \partial\ell(\mathbf{T})/\partial T_i = 0$ is not also possible as equation (A.33) will not hold. Therefore, the only possibility for $\partial\lambda_i(\mathbf{T})/\partial T_i$ at $T = 0$ and $T = +\infty$ is to have the negative sign. In other words, the demand rate into company i is decreasing at both $T = 0$ and $T = +\infty$. Now remember equation (A.31) as shown below,

$$\left[\frac{1}{\lambda_i} + \frac{\beta_Q}{\mu_{i2}-\mu_{i1}}\left\{e^{-(\mu_{i1}-\lambda_i)T_i}\left[(\mu_{i2}-\lambda_i)T_i - 1\right] - e^{-(\mu_{i2}-\lambda_i)T_i}\left[(\mu_{i1}-\lambda_i)T_i - 1\right]\right\}\right]\frac{\partial\lambda_i(\mathbf{T})}{\partial T_i}$$
$$= -\beta_T + \beta_Q\frac{(\mu_{i1}-\lambda_i)(\mu_{i2}-\lambda_i)\left(e^{-(\mu_{i1}-\lambda_i)T_i} - e^{-(\mu_{i2}-\lambda_i)T_i}\right)}{e^{-(\mu_{i1}-\lambda_i)T_i}(\mu_{i2}-\lambda_i) - e^{-(\mu_{i2}-\lambda_i)T_i}(\mu_{i1}-\lambda_i)}(1-Q_i) + \frac{\partial\ell(\mathbf{T})}{\partial T_i}.$$

Assuming $\mu_{i2} > \mu_{i1}$, it can be easily verified that there exist parameter values μ_{i1}, μ_{i2}, β_T, β_Q as well as T_i, for which $\partial\lambda_i(\mathbf{T})/\partial T_i > 0$ holds. In other words, there exists parameter values as well as T_i, where the demand rate into firm i is increasing in T_i. Based on this results and the fact that $\partial\lambda_i(\mathbf{T})/\partial T_i$ is negative for $T = 0$ and $T = +\infty$, we can conclude that at some point(s) T_i, we have $\partial\lambda_i(\mathbf{T})/\partial T_i = 0$. Also, based on the sign of $\partial\lambda_i(\mathbf{T})/\partial T_i$ at $T = 0$ and $T = +\infty$ and the continuity property of $\partial\lambda_i(\mathbf{T})/\partial T_i$, we can be certain that there is at least one T_i, where the first-order condition, as well as the second-order condition $\partial^2\lambda_i(\mathbf{T})/\partial T_i^2 < 0$ is satisfied.

To find conditions for the second-order conditions to be satisfied, we take the derivative w.r.t. T_i from both sides of equation (A.30). Also, since we are only looking at the points, where the first-order condition is satisfied, the expressions containing

$\partial\lambda_i(\mathbf{T})/\partial T_i$ are set to zero. The result is shown below,

$$\left[\frac{1}{\lambda_i}+\frac{\beta_Q}{\mu_{i1}-\mu_{i2}}\left[e^{-(\mu_{i2}-\lambda_i)T_i}\left(\left(\mu_{i1}-\lambda_i\right)T_i-1\right)-e^{-(\mu_{i1}-\lambda_i)T_i}\left(\left(\mu_{i2}-\lambda_i\right)T_i-1\right)\right]\right]\frac{\partial^2\lambda_i(\mathbf{T})}{\partial T_i^2}$$
$$=\frac{\beta_Q}{\mu_{i1}-\mu_{i2}}\left[e^{-(\mu_{i2}-\lambda_i)T_i}\left(\lambda_i-\mu_{i2}\right)^2\left(\lambda_i-\mu_{i1}\right)-e^{-(\mu_{i1}-\lambda_i)T_i}\left(\lambda_i-\mu_{i1}\right)^2\left(\lambda_i-\mu_{i2}\right)\right]+\frac{\partial^2\ell(\mathbf{T})}{\partial T_i^2}. \tag{A.34}$$

We know that the expression in the brackets on the left-hand side of equation (A.34) is always positive. This is because we have the condition $T\left(\min\{\mu_1,\mu_2\}-\lambda\right)>1$. The positive sign of the expression inside the brackets can be verified by looking at the proof for Lemma (3.11).

Now, in order to derive a contradiction, let's assume that at this point, where the first-order condition is satisfied, we have $\partial^2\lambda_i(\mathbf{T})/\partial T_i^2>0$. As a result, the left-hand side of equation (A.34) will always be positive. For the sake of the proof we also need the following to hold true as a condition,

$$\frac{\beta_Q}{\mu_{i1}-\mu_{i2}}\left[e^{-(\mu_{i2}-\lambda_i)T_i}\left(\lambda_i-\mu_{i2}\right)^2\left(\lambda_i-\mu_{i1}\right)-e^{-(\mu_{i1}-\lambda_i)T_i}\left(\lambda_i-\mu_{i1}\right)^2\left(\lambda_i-\mu_{i2}\right)\right]<0.$$

The above condition can be summarized as follows,

$$\begin{cases} e^{-(\mu_{i1}-\lambda_i)T_i}\left(\lambda_i-\mu_{i1}\right)<e^{-(\mu_{i2}-\lambda_i)T_i}\left(\lambda_i-\mu_{i2}\right), \text{ if } \mu_{i1}<\mu_{i2} \\ e^{-(\mu_{i1}-\lambda_i)T_i}\left(\lambda_i-\mu_{i1}\right)>e^{-(\mu_{i2}-\lambda_i)T_i}\left(\lambda_i-\mu_{i2}\right), \text{ Otherwise} \end{cases},$$

which is equivalent to,

$$T_i>\frac{\ln\left[\left(\mu_{i2}-\lambda_i\right)/\left(\mu_{i1}-\lambda_i\right)\right]}{\left(\mu_{i2}-\mu_{i1}\right)},$$

which is similar to the condition (3.36). With that condition satisfied, we now focus our attention on the sign of $\partial^2\ell(\mathbf{T})/\partial T_i^2$.

We already know that $\partial^2\lambda_i(\mathbf{T})/\partial T_i^2+\partial^2\lambda_j(\mathbf{T})/\partial T_i^2=0$, as $\lambda_i+\lambda_j=\Lambda$. Therefore, since we assumed that $\partial^2\lambda_i(\mathbf{T})/\partial T_i^2>0$, we should have $\partial^2\lambda_j(\mathbf{T})/\partial T_i^2<0$. Now, taking two derivatives of the equation $L_j(T_j,\lambda_j)=\ell(\mathbf{T})$ w.r.t. T_i we will have,

$$\frac{\partial L_j(T_j,\lambda_j)}{\partial\lambda_j}\frac{\partial^2\lambda_j(\mathbf{T})}{\partial T_i^2}=\frac{\partial^2\ell(\mathbf{T})}{\partial T_i^2}. \tag{A.35}$$

We know already from Lemma (3.12) that $\partial L_j(T_j,\lambda_j)/\partial\lambda_j>0$. Also, we are assuming that $\partial^2\lambda_i(\mathbf{T})/\partial T_i^2$ is positive. This requires for $\partial^2\lambda_j(\mathbf{T})/\partial T_i^2$ to be negative. Now, based on equation (A.35), this will result in $\partial^2\ell(\mathbf{T})/\partial T_i^2<0$. With a negative $\partial^2\ell(\mathbf{T})/\partial T_i^2$, the right-hand side of equation (A.34) will be negative, which is a contradiction. From this argument, we can conclude that based on the above generated conditions, at the point, where the first-order condition of the problem is satisfied, the second-order condition ($\partial^2\lambda_i(\mathbf{T})/\partial T_i^2 < 0$) is guaranteed to be satisfied with the condition,

$$T_i > \frac{\ln\left[(\mu_{i2} - \lambda_i)\,/\,(\mu_{i1} - \lambda_i)\right]}{(\mu_{i2} - \mu_{i1})},$$

that proves the condition (3.36).

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