COVERING DIMENSION AND THE MODELING DISTRIBUTION

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By

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ABSTRACT

This thesis deals with paracompact spaces with the covering dimension of Lebesgue. A paracompact Hausdorff space with finite covering dimension is characterized by sequences of covers, as an inverse limit of finite dimensional metric spaces, and in terms of a single finite dimensional metric space. In connection with non-deterministic mathematics we introduce the modeling distribution and it is proved (under suitable assumptions) that a modeling distribution preserves paracompactness, complete paracompactness, strong paracompactness, compactness, and final compactness, and lowers covering dimension.

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INTRODUCTION

The concept of modeling distribution was communicated to me by R. G. Lintz either orally or through unpublished notes. This concept generalizes the concept of homeomorphism in that instead of mapping points to points in a one to one correspondence, a modeling distribution is a collection of "special" continuous non-deterministic functions called modeling functions, each of which maps open sets to open sets. The concepts of continuous non-deterministic function, modeling function, and modeling distribution have been applied by R. G. Lintz to several questions in topology ([1], [2]).

The present work consists of three parts designated by the last three Chapters. In II we make characterizations of paracompact spaces with finite covering dimension. In III we investigate the effect of a modeling distribution on paracompactness, covering dimension, and related covering properties. Finally, in IV, we apply the previous work to begin investigation specifically of compact Hausdorff spaces under a modeling distribution; and we make several conjectures.

In this thesis I use the convention that my results are indexed by two numerals while other results are indexed by a numeral and a letter.

I

Fundamental Concepts

The following are some concepts and results from topology and dimension theory to be used in this thesis. For a comprehensive review of these subjects, the reader is referred to ([3], [4], [5]).

1. *A cover of a set X* is a collection of subsets of X whose union is all of X. An open cover (almost open cover) of a space X is a cover of X where each member of the cover is open (where the collection of interiors of members of the cover is a cover of X). The *closure of a subset V of a space X* is denoted $\overline{V}$ and if $\mathcal{V}$ is a collection of subsets of X then the collection of closures of members of $\mathcal{V}$ is denoted $\overline{\mathcal{V}}$. We say X is Hausdorff (*regular*, *normal*) if for any two distinct points $x, y \in X$ there are open sets U, V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$ (for any point $x \in X$ and open set U with $x \in U$ there is an open set V with $x \in V \subseteq \overline{V} \subseteq U$, for any closed set F and open set U with $F \subseteq U$ there is an open set V with $F \subseteq V \subseteq \overline{V} \subseteq U$). We say X is *compact* (*finally compact*) if each open cover has a finite (countable) subcover; X is *separable* if it has a countable dense subset; X is *connected* if it cannot be written as $X = U \cup V$ where U, V are open, both non empty,

$\emptyset$. We say X is a pseudometric space if there

$\rho(x,y) \geq 0$, $\rho(x,y) = \rho(y,x)$, $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$, and $x = y$ implies $\rho(x,y) = 0$. If also $\rho(x,y) = 0$ implies $x = y$, then X is a metric space. If (X,ρ) is a pseudometric space then $S_\varepsilon(x) = \{y \in X \mid \rho(x,y) < \varepsilon\}$ for $\varepsilon>0$ and $x \in X$ denote the spheres of radius ε about x, and if $U \subseteq X$ then the diameter of U is $\rho(U) = \sup\{\rho(x,y) \mid x, y \in U\}$ while the mesh of a collection $\mathcal{U}$ of subsets of X is $\text{mesh } \mathcal{U} = \sup\{\rho(U) \mid U \in \mathcal{U}\}$. A space X is metrizable if there is a metric ρ on X such that the topology of ρ is the topology of X. A space X is called discrete if each point of X is open.

As usual covering dimension is denoted dim and $\dim X \leq n$ means that for each finite open cover $\mathcal{U}$ there is an open cover $\mathcal{V}$ such that $\mathcal{V}$ refines $\mathcal{U}$ and $\text{ord } \mathcal{V} \leq n + 1$. If $\mathcal{U}$, $\mathcal{V}$ are collections of subsets of a set X and each member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$ then we say $\mathcal{V}$ refines $\mathcal{U}$ and write $\mathcal{V} << \mathcal{U}$. The order of a cover $\mathcal{V}$ is $\text{ord } \mathcal{V} = \sup\{\text{ord}_x \mathcal{V} \mid x \in X\}$ where $\text{ord}_x\mathcal{V}$ is the number of members of $\mathcal{V}$ containing x.

A collection $\mathcal{V}$ of subsets of a space X is locally finite (point finite, closure preserving) if for each $x \in X$ there is an open nbhd. of x which has a non empty intersection with at most a finite number of members of $\mathcal{V}$ (each $x \in X$ is contained in at most a finite number of members of $\mathcal{V}$, for each subcollection $\mathcal{U}$ of $\mathcal{V}$ we have $\bigcup\{\overline{V} \mid V \in \mathcal{U}\} = \overline{\bigcup\{V \mid V \in \mathcal{U}\}}$). A space X is paracompact if for each open cover $\mathcal{U}$ there is a [illegible] $\mathcal{V}$ [illegible] that $\mathcal{V} << \mathcal{U}$.

Lemma 1.a: If $\mathcal{U}$ is a locally finite collection of subsets of X, then $\mathcal{U}$ is closure preserving ([4], page 126).

Lemma 1.b: If X is paracompact and either Hausdorff or regular then X is normal ([6]).

Lemma 1.c: A space X is normal if and only if for each point finite open cover $\mathcal{U} = \{U_\gamma \mid \gamma \in \Gamma\}$ there is an open cover $\mathcal{V} = \{V_\gamma \mid \gamma \in \Gamma\}$ such that $V_\gamma \subseteq \bar{V}_\gamma \subseteq U_\gamma$ for each $\gamma \in \Gamma$ ([6]).

Lemma 1.d: A normal space X has dim X $\leq$ n if and only if for each locally finite open cover $\mathcal{U}$ there is a locally finite open cover $\mathcal{V}$ such that $\mathcal{V} << \mathcal{U}$ and ord $\mathcal{V} \leq n + 1$ ([7]).

Using Lemmas 1.b, 1.c, 1.d we have:

Lemma 1.e: For a Hausdorff or regular space X the following are equivalent:

1) The space X is paracompact with dim X $\leq$ n

2) For each open cover $\mathcal{U}$ there is a locally finite open cover $\mathcal{V}$ such that $\mathcal{V} << \mathcal{U}$ and ord $\mathcal{V} \leq n + 1$.

3) For each open cover $\mathcal{U}$ there is an open cover $\mathcal{V}$ such that $\bar{\mathcal{V}}$ is locally finite $\mathcal{V} << \mathcal{U}$, and ord $\bar{\mathcal{V}} \leq n + 1$.

2. We shall also use the following notions and conventions. Let $\mathcal{U}$, $\mathcal{V}$ be collections of subsets of a set X. If $\mathcal{U}$, $\mathcal{V}$ are such that if $V_1, V_2, V_3 \in \mathcal{V}$ with $V_1 \cap V_2 \neq \emptyset$ and $V_2 \cap V_3 \neq \emptyset$ then there is $U \in \mathcal{U}$ with $V_1 \cup V_2 \cup V_3 \subseteq U$, then we say that $\mathcal{V}$ 3-chain refines $\mathcal{U}$ and write $\mathcal{V} <<_{3c} \mathcal{U}$.

$\mathcal{U}_{k+1} <<_{3c} \mathcal{U}_k$ for each k: If V is a subset of X then $St(V, \mathcal{V})$ denotes the union of all members of $\mathcal{V}$ which have a non empty intersection with V; and in case $V = \{x\}$ for some $x \in X$, then we write $St(x, \mathcal{V})$ for $St(V, \mathcal{V})$. If $\{St(x, \mathcal{V}) \mid x \in X\} << \mathcal{U}$ ($\{St(V, \mathcal{V}) \mid V \in \mathcal{V}\} << \mathcal{U}$) we say <u>$\mathcal{V}$ star (strong star) refines</u> $\mathcal{U}$ and write $\mathcal{V} <<^* \mathcal{U}$ ($\mathcal{V} <<^{**} \mathcal{U}$). A sequence $\{\mathcal{U}_k \mid k \in Z\}$ of covers is called a starring (strong starring) sequence if $\mathcal{U}_{k+1} <<^* \mathcal{U}_k$ ($\mathcal{U}_{k+1} <<^{**} \mathcal{U}_k$) for each k. We denote $\{U \cap V \mid U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}$ by $\mathcal{U} \wedge \mathcal{V}$ and extend this to any finite number of collections. If $x, y \in X$, we shall write the collection $\{\{x\}, \{y\}\}$ simply as $\{x, y\}$.

Let Λ be a directed set. If Γ is a subset of Λ such that for each $\alpha \in \Lambda$ there is $\beta \in \Gamma$ with $\alpha < \beta$, then we say that <u>Γ is cofinal in Λ</u>. If Γ is a subset of the collection of all covers of a space X where for each open cover $\mathcal{U}$ of X there is $\mathcal{V} \in \Gamma$ such that $\mathcal{V} << \mathcal{U}$, then we say that <u>$\Gamma$ is cofinal in X</u>.

<u>Lemma 1.f</u>: If X is a compact pseudometric space then for any sequence $\{t_k \mid k \in Z\}$ of positive real numbers with $\lim_{k \to \infty} t_k = 0$, the sequence $\{\{S_{t_k}(x) \mid x \in X\} \mid k \in Z\}$ is cofinal in X ([4], page 154).

<u>Lemma 1.g</u>: Let $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ be collections of subsets of a set X. If $\mathcal{V} <<^* \mathcal{U}$ and $\mathcal{U} <<^* \mathcal{W}$, then $\mathcal{V} <<^{**} \mathcal{W}$.

<u>Proof</u>: Let V be a fixed member of $\mathcal{V}$. Since $\mathcal{V} <<^* \mathcal{U}$ for each $x \in V$ choose $U_x \in \mathcal{U}$ such that $St(x, \mathcal{V}) \subseteq U_x$. Let y be some point of V. Then $\bigcup\{U_x \mid x \in V\} \subseteq St(y, \mathcal{U})$ since $y \in V \subseteq St(x, \mathcal{V}) \subseteq U_x$ for

each $x \in V$. Hence,

$St(V,\mathcal{V}) = \cup\{St(x,\mathcal{V}) \mid x \in V\} \subseteq \cup\{U_x \mid x \in V\} \subseteq St(y,\mathcal{U})$ and since $\mathcal{U} <<^* \mathcal{W}$, $St(v, \mathcal{V})$ is contained in some member of $\mathcal{W}$ so that $\mathcal{V} <<^{**}\mathcal{W}$.

The notion of star refinement is due to J. W. Tukey, who proved:

<u>Lemma 1.h</u>: In a metric space, for each open cover $\mathcal{U}$ there is an open cover $\mathcal{V}$ such that $\mathcal{V} <<^* \mathcal{U}$ ([8], page 53).

It was A. H. Stone, who proved:

<u>Lemma 1.i</u>: A regular space is paracompact if and only if for each open cover $\mathcal{U}$ there is an open cover $\mathcal{V}$ such that $\mathcal{V} <<^* \mathcal{U}$ ([4], page 170 and 171).

Thus, by Lemma 1.h he obtained the following famous result:

<u>Lemma 1.j</u>: A metric space is paracompact.

As a consequence of Lemma 1.g we have:

<u>Lemma 1.k</u>: A regular space is paracompact if and only if for each open cover $\mathcal{U}$ there is an open cover $\mathcal{V}$ such that $\mathcal{V} <<^{**} \mathcal{U}$.

If $\mathcal{V}$ is a collection of subsets of a set X and V is a subset of X we define

$St^1(V, \mathcal{V}) = St(V,\mathcal{V})$ and $St^k(V, \mathcal{V}) = St(St^{k-1}(V, \mathcal{V}), \mathcal{V})$ for $k \geq 2$. Let us note that given a fixed positive integer k, we can restate Lemma 1.k by saying: A regular space X is paracompact if and only if for each open cover $\mathcal{U}$ there is an open cover $\mathcal{V}$ such that $\{St^k(V, \mathcal{V}) \mid V \in \mathcal{V}\} << \mathcal{U}$.

This is because, if X is paracompact and regular and $\mathcal{U}$ is an open cover of X, by Lemma 1.k we can choose open covers $\mathcal{V}$, $\mathcal{V}_i$ $1 \leq i \leq k$ such that

$\mathcal{V} <<^{**} \mathcal{V}_k <<^{**} \mathcal{V}_{k-1} <<^{**}, \ldots, <<^{**} \mathcal{V}_2 <<^{**} \mathcal{V}_1$.

If $v \in \mathcal{V}$ then $St(V, \mathcal{V}) \subseteq V_k \in \mathcal{V}_k$,

$St(St(V, \mathcal{V}), \mathcal{V}) = St^2(V, \mathcal{V}) \subseteq St(V_k, \mathcal{V}_k) \subseteq V_{k-1} \in \mathcal{V}_{k-1}$,

$St(St(St(v,\mathcal{V}),\mathcal{V}),\mathcal{V}) = St^3(V,\mathcal{V}) \subseteq St(V_{k-1},\mathcal{V}_{k-1}) \subseteq V_{k-2} \in \mathcal{V}_{k-2}$,

$\ldots, St^k(V,\mathcal{V}) \subseteq V_1 \in \mathcal{V}_1$.

The following result, essentially due to C. H. Dowker, is basic to our work since it enables us to obtain metric spaces from 3-chain sequences of covers.

<u>Lemma 1.ℓ</u>: If $\{\mathcal{U}_k \mid k \in Z\}$ is a 3-chain sequence of covers of a set X, then there is a pseudometric ρ on X such that $St(x, \mathcal{U}_{k+1}) \subseteq S_{\frac{1}{2^{k-1}}}(x) \subseteq St(x, \mathcal{U}_k)$ for each $x \in X$ and k.

<u>Proof</u>: Define a function h on X x X as follows:

For each $(x,y) \in X \times X$,

$$h(x,y) = \begin{cases} 0 \text{ if for each } k, \{x,y\} \text{ refines } \mathcal{U}_k \\ 2 \text{ if for each } k, \{x,y\} \text{ does not refine } \mathcal{U}_k \\ \frac{1}{2^{k-1}} \text{ if } \{x,y\} \text{ refines } \mathcal{U}_k \text{ and does not refine } \mathcal{U}_{k+1} \end{cases}$$

For each $(x,y) \in X \times X$ define $\rho(x,y) = \inf B(x,y)$ where $B(x,y) = \{h(x,x_2, \ldots, x_{k-1}, y) \mid x_2, \ldots, x_{k-1} \in X\}$ and $h(x,x_2, \ldots, x_{k-1}, y)$ denotes $h(x,x_2)+h(x_2,x_3)+ \ldots +h(x_{k-1},y)$

Then ρ is a pseudometric on X. Clearly $\rho(x,y) \geq 0$,

$\rho(x,y) = \rho(y,x)$, and $\rho(x,x) = 0$ for all $x,y \in X$. Given $x, y, z \in X$, by definition of $\rho(x,y)$ there is $h(x_1, \ldots, x_m) \in B(x,y)$ such that $h(x_1, \ldots, x_m) \leq \rho(x,y)+\varepsilon/2$; and similarly there is $h(x_m, \ldots, x_{m+q}) \in B(y,z)$ such that $h(x_m, \ldots, x_{m+q}) \leq \rho(y,z) + \varepsilon/2$. Hence

$$\begin{aligned}\rho(x,z) &\leq h(x_1, \ldots, x_m, \ldots, x_{m+q}) \\ &= h(x_1, \ldots, x_m) + h(x_m, \ldots, x_{m+q}) \\ &\leq \rho(x,y) + \varepsilon/2 + \rho(y,z) + \varepsilon/2 = \rho(x,y) + \rho(y,z) + \varepsilon\end{aligned}$$

Since $\varepsilon>0$ was arbitrary, $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$.

For each x and k, $St(x, \mathcal{U}_{k+1}) \subseteq S_{\frac{1}{2^{k-1}}}(x)$ since if $y \in St(x, \mathcal{U}_{k+1})$ then $\{x,y\} << \mathcal{U}_{k+1}$ so that $h(x,y) \leq \frac{1}{2^k}$ and since $\rho(x,y) \leq h(x,y)$, we have $\rho(x,y) \leq \frac{1}{2^k} < \frac{2}{2^k} = \frac{1}{2^{k-1}}$ and $y \in S_{\frac{1}{2^{k-1}}}(x)$.

<u>Claim:</u> For each p, for any $x_1, \ldots, x_p \in X$, if $h(x_1, \ldots, x_p) < \frac{1}{2^{k-1}}$ then $\{x_1, x_p\} << \mathcal{U}_k$. If $p = 1$ or if $p = 2$ the Claim is true and suppose the Claim is true for each i with $i \leq p - 1$. Consider $h(x_1, \ldots, x_p) < \frac{1}{2^{k-1}}$.

<u>Case one:</u> Suppose for some $1 \leq j < p$ we have $h(x_j, x_{j+1}) = \frac{1}{\ }$. Then, $\{x_j, x_{j+1}\} << \mathcal{U}_{k+1}$ and

so that $h(x_1,\ldots,x_j) + h(x_{j+1},\ldots,x_p) < \frac{1}{2^{k-1}} - \frac{1}{2^k} = \frac{1}{2^k}$.

Since $h(x_1,\ldots,x_j) < \frac{1}{2^k}$ and $h(x_{j+1},\ldots,x_p) < \frac{1}{2^k}$, by the induction hypothesis $\{x_1,x_j\} << \mathcal{U}_{k+1}$, $\{x_{j+1},x_p\} << \mathcal{U}_{k+1}$. Since $\mathcal{U}_{k+1} <<_{3c} \mathcal{U}_k$, we have that $\{x_1,x_p\} << \mathcal{U}_k$.

<u>Case two</u>: Suppose for each $1 \leq j < p$ we have $h(x_j,x_{j+1}) \neq \frac{1}{2^k}$. If $h(x_j,x_{j+1}) > \frac{1}{2^k}$ for some $1 \leq j < p$ then $h(x_j,x_{j+1}) \geq \frac{1}{2^{k-1}}$ and $h(x_1,\ldots,x_p) \geq \frac{1}{2^{k-1}}$. Hence $h(x_j,x_{j+1}) < \frac{1}{2^k}$ for each $1 \leq j < p$. Let s be the largest integer with $s \leq p$ such that $h(x_1,\ldots,x_s) < \frac{1}{2^k}$.

<u>Subcase one</u>: Suppose $p-1 \leq s$. Then $h(x_1,\ldots,x_{p-1}) < \frac{1}{2^k}$ and $h(x_{p-1},x_p) < \frac{1}{2^k}$ so that $\{x_{p-1},x_p\} << \mathcal{U}_{k+1}$ and by the induction hypothesis $\{x_1,x_{p-1}\} << \mathcal{U}_{k+1}$. Since $\mathcal{U}_{k+1} <<_{3c} \mathcal{U}_k$, we have $\{x_1,x_p\} << \mathcal{U}_k$.

<u>Subcase two</u>: Suppose $s < p-1$. Let t be the largest integer with $s < t \leq p$ such that $h(x_s,\ldots,x_t) < \frac{1}{2^k}$. If $p-1 \leq t$ then $s < p-1 \leq t$ and $h(x_s,\ldots,x_{p-1}) < \frac{1}{2^k}$, $h(x_1,\ldots,x_s) < \frac{1}{2^k}$, and $h(x_{p-1},x_p) < \frac{1}{2^k}$. Hence $\{x_{p-1},x_p\} << \mathcal{U}_{k+1}$ and by the induction hypothesis $\{x_s,x_{p-1}\} << \mathcal{U}_{k+1}$, $\{x_1,x_s\} << \mathcal{U}_{k+1}$. Since $\mathcal{U}_{k+1} <<_{3c} \mathcal{U}_k$ we

have that $\{x_1, x_p\} << \mathcal{U}_k$. On the other hand, if $t < p-1$ then $s < t < p-1$. Now,

$$h(x_1,\ldots,x_p) = h(x_1,\ldots,x_t) + h(x_t,\ldots,x_p) < \frac{1}{2^{k-1}} .$$

If $h(x_t,\ldots,x_p) \geq \frac{1}{2^k}$ then

$$h(x_1,\ldots,x_t) < \frac{1}{2^{k-1}} - h(x_t,\ldots,x_p) < \frac{1}{2^{k-1}} - \frac{1}{2^k} = \frac{1}{2^k}$$ where

$s < t$. But s is the largest integer with $s \leq p$ such that $h(x_1,\ldots,x_s) < \frac{1}{2^k}$. Hence $h(x_t,\ldots,x_p) < \frac{1}{2^k}$, $h(x_s,\ldots,x_t) < \frac{1}{2^k}$, $h(x_1,\ldots,x_s) < \frac{1}{2^k}$ and by the induction hypothesis, $\{x_t, x_p\} << \mathcal{U}_{k+1}$, $\{x_s, x_t\} << \mathcal{U}_{k+1}$, $\{x_1, x_s\} << \mathcal{U}_{k+1}$. Since $\mathcal{U}_{k+1} <<_{3c} \mathcal{U}_k$ we have $\{x_1, x_p\} << \mathcal{U}_k$. So the Claim holds.

Now consider $S_{\frac{1}{2^{k-1}}}(x)$ for some x and k and let $y \in S_{\frac{1}{2^{k-1}}}(x)$ so that $\rho(x,y) < \frac{1}{2^{k-1}}$. By definition of ρ there is $h(x_1,\ldots,x_p) \in B(x,y)$ such that $h(x_1,\ldots,x_p) < \frac{1}{2^{k-1}}$. By the Claim we have $\{x_1, x_p\} = \{x,y\} << \mathcal{U}_k$ so that $y \in St(x, \mathcal{U}_k)$. Hence $S_{\frac{1}{2^{k-1}}}(x) \subseteq St(x, \mathcal{U}_k)$ and this completes the proof.

3: We shall call the pseudometric and the pseudometric space obtained in Lemma 1.ℓ, the <u>pseudometric and pseudometric space generated by $\{\mathcal{U}_k \mid k \in Z\}$</u>. On the pseudometric space (X,ρ) generated by $\{\mathcal{U}_k \mid k \in Z\}$ we can define an equivalence relation by calling two points x, y of X related if $\rho(x,y) = 0$. Then $\rho^*(x^*,y^*) = \rho(x,y)$, where x^*, y^* are the equivalence classes represented by x,y, is a metric on the set X^* of equivalence classes and we shall call ρ^* and (X^*,ρ^*) the <u>metric and metric space associated with ρ</u>. The surjection defined by assigning x^* to $x \in X$ we shall call the <u>canonical map</u> and denote it by $*$, and if U is any subset of X we denote the image of U under $*$ by U^*. Then, for each $x^* \in X^*$ and $\varepsilon > 0$, the spheres of X^* are $S_\varepsilon^*(x)$ and $*^{-1}[S_\varepsilon^*(x)] = S_\varepsilon(x)$. From this it follows that $*^{-1}[U^*] = U$ for each open set U of (X,ρ). Finally, let us note that in Lemma 1.ℓ, if X is a space and each $\mathcal{U}_k$ is an open cover of X, then the topology of ρ is contained in the topology of X so that $*$ is continuous on X.

II

Paracompact Spaces with Finite Covering Dimension

Our main purpose here is to write a paracompact space whose covering dimension is bounded by an integer n as an inverse limit of metric spaces each of which also has its covering dimension bounded by n.

1. Towards this end we consider the following definition and the following result by P. Vopenka ([9]).

Definition 2.1: Let $\mathcal{U}$, $\mathcal{V}$ be collections of subsets of a set X, let $P(\mathcal{V})$ be the power set of $\mathcal{V}$, and let $Q: X \to P(\mathcal{V})$ be a function such that if $V \in Q(x)$ then $x \in V$ and $\bigcup\{Q(x) \mid x \in X\} = \mathcal{V}$. If for each $x \in X$ there is $V \in Q(x)$ and $U \in \mathcal{U}$ such that $V \subseteq U$, then we say $\mathcal{V}$ partially refines $\mathcal{U}$ with respect to Q(x) and we write $\mathcal{V} <<_p \mathcal{U}$ with respect to Q(x).

The author is aware that with this notion in mind and using Lemma 1.ℓ, S. L. Gulden has obtained, but not published, characterizations of paracompactness which are similar to Theorems 2.1 and 2.3 in the sequel.

In Definition 2.1, if X is a space and $\mathcal{V}$ is a base and $\mathcal{U}$ is an open cover, then $\mathcal{V} <<_p \mathcal{U}$ with respect to Q(x) where $Q: X \to P(\mathcal{V})$ is defined by $Q(x) = \{V \in \mathcal{V} \mid x \in V\}$. However, it can easily happen that $\mathcal{V}$ does not refine $\mathcal{U}$.

X has X if and

that $\mathcal{U}_{k+1} << \mathcal{U}_k$ and ord $\mathcal{U}_k \leq n + 1$ for each k, and $\lim_{k\to\infty} \text{mesh } \mathcal{U}_k = 0$.

<u>Theorem 2.1</u>: A regular space X is paracompact with dim $X \leq n$ if and only if for each open cover $\mathcal{U}$ there is a 3-chain sequence $\{\mathcal{U}_k \mid k \in Z\}$ of open covers such that ord $\mathcal{U}_k \leq n + 1$ for each k and $\{St(x,\mathcal{U}_k) \mid x \in X, k \in Z\} <<_p \mathcal{U}$ with respect to $Q(x) = \{St(x,\mathcal{U}_k) \mid k \in Z\}$ ([10]).

\ <u>Proof</u>: To prove the necessity, let $\mathcal{U}$ be an open cover of X. Since X is paracompact with dim $X \leq n$ by Lemma 1.e there is an open cover $\mathcal{U}_o = \{U_\gamma \mid \gamma \in \Gamma\}$ such that $\mathcal{U}_o << \mathcal{U}$, ord $\overline{\mathcal{U}}_o \leq n + 1$, and $\overline{\mathcal{U}}_o$ is locally finite. For each $x \in X$ choose $U_{\gamma(x)} \in \mathcal{U}_o$ with $x \in U_{\gamma(x)}$ and an open nbhd. V_x of x which has a non empty intersection with at most finitely many members of $\overline{\mathcal{U}}_o$. Then $U_x = (V_x - \bigcup\{\bar{U}_\gamma \mid \bar{U}_\gamma \cap V_x \neq \emptyset \text{ and } x \notin \bar{U}_\gamma\}) \cap U_{\gamma(x)}$ is an open nbhd. of x which has a non empty intersection with at most n + 1 members of $\overline{\mathcal{U}}_o$. By Lemma 1.k there is an open cover $\mathcal{W}$ such that $\mathcal{W} <<^{**} \{U_x \mid x \in X\}$. Applying Lemma 1.e again let $\mathcal{U}_1$ be an open cover such that $\mathcal{U}_1 << \mathcal{W}$, ord $\overline{\mathcal{U}}_1 \leq n + 1$, and $\overline{\mathcal{U}}_1$ locally finite. Then $\mathcal{U}_1, \mathcal{U}_o$ are open covers of X such that $\mathcal{U}_1 <<_{3c} \mathcal{U}_o$; and for each $x \in X$, $St(x,\mathcal{U}_1)$ has a non empty intersection with at most n + 1 members of $\mathcal{U}_o$. Now carrying out the above construction with $\mathcal{U}_1$ in place of $\mathcal{U}_o$, continue inductively to obtain a 3-chain sequence $\{\mathcal{U}_k \mid k \in Z\}$

nd k, ,) has a

non empty intersection with at most n + 1 members of $\mathcal{U}_k$. Then for each k, ord $\{St(U,\mathcal{U}_{k+1}) \mid U \in \mathcal{U}_k\} \leq n + 1$ and since $\mathcal{U}_1 <<* \mathcal{U}$, trivially $\{St(x,\mathcal{U}_k) \mid x \in X,\ k \in Z\} <<_p \mathcal{U}$ with respect to $Q(x) = \{St(x,\mathcal{U}_k) \mid k \in Z\}$. Hence $\{\mathcal{U}_k \mid k \in Z\}$ is the required sequence.

To prove the sufficiency let $\mathcal{U} = \{U_\gamma \mid \gamma \in \Gamma\}$ be an open cover and $\{\mathcal{U}_k \mid k \in Z\}$ a 3-chain sequence of open covers such that $\{St(x,\mathcal{U}_k) \mid x \in X,\ k \in Z\} <<_p \mathcal{U}$ with respect to $Q(x) = \{St(x,\mathcal{U}_k) \mid k \in Z\}$ and ord $\mathcal{U}_k \leq n + 1$ for each k. Since ord $\mathcal{U}_{(n+1)+1} \leq n + 1$ and $\mathcal{U}_{i+1} <<_{3c} \mathcal{U}_i$ for $1 \leq i \leq n + 1$, we have $\mathcal{U}_{(n+1)+1} <<* \mathcal{U}_1$. Since ord $\mathcal{U}_{2(n+1)+1} \leq n + 1$ and $\mathcal{U}_{i+1} <<_{3c} \mathcal{U}_i$ for $(n+1)+1 \leq i \leq 2(n+1)$, we have $\mathcal{U}_{2(n+1)+1} <<* \mathcal{U}_{(n+1)+1}$ so that by Lemma 1.g, $\mathcal{U}_{2(n+1)+1} <<** \mathcal{U}_1$. Similarly, $\mathcal{U}_{3(n+1)+1} <<* \mathcal{U}_{2(n+1)+1}$ and $\mathcal{U}_{4(n+1)+1} <<* \mathcal{U}_{3(n+1)+1}$ so that $\mathcal{U}_{4(n+1)+1} <<** \mathcal{U}_{2(n+1)+1}$; and continuing in this way we can obtain a strong starring sequence $\{\mathcal{V}_k \mid k \in Z\}$ of open covers where $\mathcal{V}_k = \mathcal{U}_{(k-1)(2)(n+1)+1}$ so that the sequence $\{\mathcal{V}_k \mid k \in Z\}$ has all the properties of the sequence $\{\mathcal{U}_k \mid k \in Z\}$. Thus, we simply assume that $\{\mathcal{U}_k \mid k \in Z\}$ is a strong starring sequence. By Lemma 1.ℓ, that $St(x,\mathcal{U}_{k+1}) \subseteq S_{\frac{1}{2^{k-1}}}(x) \subseteq St(x,\mathcal{U}_k)$ for each $x \in X$ and k; and let Y be the metric space associated with ρ. For each k, let $\mathcal{S}_k = \{S_U^k \mid U \in \mathcal{U}_k\}$ where

$S_U^k = \bigcup\{S_{\frac{1}{2^k}}(x) \mid St(x,\mathcal{U}_{k+1}) \subseteq U\}$. Since $S_U^k \subseteq U$ for each $U \in \mathcal{U}_k$ and since $\mathcal{U}_k <<^* \{S_{\frac{1}{2^{k-2}}}(x) \mid x \in X\}$, we have

ord $\mathcal{S}_k \leq n + 1$ and mesh $\mathcal{S}_k \leq \frac{1}{2^{k-2}}$ for each k. Also for any k, if $U \in \mathcal{U}_{k+1}$, then $S_U^{k+1} \subseteq S_V^k$ for some $V \in \mathcal{U}_k$ since $\mathcal{U}_{k+1} <<^{**} \mathcal{U}_k$. Then, $\{\mathcal{S}_k^* \mid k \in Z\}$, where for each k $\mathcal{S}_k^* = \{(S_U^k)^* \mid U \in \mathcal{U}\}$, is a sequence of open covers of Y such that $\mathcal{S}_{k+1}^* << \mathcal{S}_k^*$ and ord $\mathcal{S}_k^* \leq n + 1$ for each k, and $\lim_{k\to\infty}$ mesh $\mathcal{S}_k^* = 0$; and by Lemma 2.a we have dim $Y \leq n$. For each x, choose an integer $k(x)$ and $U \in \mathcal{U}$ with $St(x,\mathcal{U}_{k(x)}) \subseteq U$, and for each $\gamma \in \Gamma$ let

$W_\gamma = \bigcup\{S^*_{\frac{1}{2^{k(x)-1}}}(x) \mid St(x,\mathcal{U}_{k(x)}) \subseteq U_\gamma\}$. Then

$\mathcal{W} = \{W_\gamma \mid \gamma \in \Gamma\}$ is an open cover of Y, and since by Lemma 1.j, Y is paracompact, by Lemma 1.e there is a locally finite open cover $\mathcal{V}$ of Y such that $\mathcal{V} << \mathcal{W}$ and ord $\mathcal{V} \leq n + 1$. Then $\{*^{-1}[V] \mid V \in \mathcal{V}\}$ is a locally finite open cover which refines $\mathcal{U}$ and ord $\{*^{-1}[V] \mid V \in \mathcal{V}\} \leq n + 1$. Thus, by Lemma 1.e, X is paracompact with dim $X \leq n$. This completes

Corollary 2.1: A regular space X is paracompact with dim X $\leq$ n if and only if for each open cover $\mathcal{U}$ there is a strong starring sequence $\{\mathcal{U}_k \mid k \in Z\}$ of open covers such that ord $\mathcal{U}_k \leq n + 1$ for each k and $\{St(x, \mathcal{U}_k) \mid x \in X, k \in Z\} <<_p \mathcal{U}$ with respect to $Q(x) = \{St(x, \mathcal{U}_k) \mid k \in Z\}$.

Corollary 2.2: A regular space X is paracompact with dim X $\leq$ n if and only if for each open cover $\mathcal{U}$ there is a 3-chain sequence $\{\mathcal{U}_k \mid k \in Z\}$ of open covers such that ord $\{St(U, \mathcal{U}_{k+1}) \mid U \in \mathcal{U}_k\} \leq n + 1$ for each k, and $\{St(x, \mathcal{U}_k) \mid x \in X, k \in Z\} <<_p \mathcal{U}$ with respect to $Q(x) = \{St(x, \mathcal{U}_k) \mid k \in Z\}$.

It is clear from Lemma 1.k that if $\mathcal{U}$ is an open cover of a regular paracompact space X then there is a sequence $\{\mathcal{U}_k \mid k \in Z\}$ of open covers of X such that $\mathcal{U}_{k+1} <<^{**} \mathcal{U}_k$ for each k and $\mathcal{U}_1 <<^{**} \mathcal{U}$. As in the proof of the sufficiency of Theorem 2.1 we can consider the pseudometric space generated by the sequence $\{\mathcal{U}_k \mid k \in Z\}$, the metric space Y associated with it, the canonical map *, and obtain an open cover $\mathcal{W}$ of Y such that $\{*^{-1}[W] \mid W \in \mathcal{W}\} << \mathcal{U}$. Since * is continuous on X we have:

Corollary 2.a: A regular space X is paracompact if and only if for each open cover $\mathcal{U}$ there is a continuous surjection f from X to a metric space Y and an open cover $\mathcal{W}$ of Y such that $\{f^{-1}[W] \mid W \in \mathcal{W}\} << \mathcal{U}$.

In a similar way, from the proof of the sufficiency of Theorem 2.1, we have:

Corollary 2.3: A regular space X is paracompact with dim X $\leq$ n if and only if for each open cover $\mathcal{U}$ there is a continuous surjection f from X to a metric space Y with dim Y $\leq$ n and an open cover $\mathcal{W}$ of Y such that $\{f^{-1}[W] \mid W \in \mathcal{W}\} << \mathcal{U}$.

2. Now we shall use the notion of a full inverse limiting system.

Definition 2.2: Let $\{Y_\alpha, p_\alpha^\beta \mid \alpha \in \Lambda\}$ be an inverse limiting system and Π_α be the projection restricted to $Y = \text{inv lim } (Y_\alpha, p_\alpha^\beta \mid \alpha \in \Lambda)$. The system is full if for each open cover $\mathcal{U}$ of Y there is $\alpha \in \Lambda$ and an open cover $\mathcal{W}$ of Y_α such that $\{\Pi_\alpha^{-1}[W] \mid W \in \mathcal{W}\} << \mathcal{U}$. The system is surjective if Π_α is surjective for each $\alpha \in \Lambda$.

Theorem 2.2: A Hausdorff space X is paracompact (paracompact with dim X $\leq$ n) if and only if X is homeomorphic to the limit space of a full inverse limiting system of metric spaces (metric spaces Y_α with dim $Y_\alpha \leq$ n for each α) ([10]).

Proof: By Lemmas 1.j and 1.e the sufficiency is ... To ... the ... first note that since X

According to Lemma 1.k (Corollary 2.1) there is a collection Λ of all strong starring sequences $\alpha = \{\mathcal{U}_k^\alpha \mid k \in Z\}$ of open covers of X (with ord $\mathcal{U}_k^\alpha \leq n + 1$ for each k). By Lemma 1.ℓ let (X, ρ_α) be the pseudometric space generated by $\{\mathcal{U}_k^\alpha \mid k \in Z\}$ such that $St(x, \mathcal{U}_{k+1}^\alpha) \subseteq S^\alpha_{\frac{1}{2^{k-1}}}(x) \subseteq St(x, \mathcal{U}_k^\alpha)$ for each $x \in X$ and k where $S^\alpha_{\frac{1}{2^{k-1}}}(x)$ are the spheres in ρ_α.

Let Y_α be the metric space associated with ρ_α (let Y_α be the metric space associated with ρ_α with dim $Y_\alpha \leq n$ according to the proof of the sufficiency of Theorem 2.1), and let $*_\alpha$ be the canonical map.

Define order between two members $\alpha = \{\mathcal{U}_k^\alpha \mid k \in Z\}$, $\beta = \{\mathcal{U}_k^\beta \mid k \in Z\}$ of Λ by $\alpha < \beta$ if and only if $\mathcal{U}_k^\beta << \mathcal{U}_k^\alpha$ for each k. If $\alpha, \beta \in \Lambda$ by Lemma 1.k (Lemma 1.k and Lemma 1.e) choose an open cover $\mathcal{U}_1^\delta$ such that $\mathcal{U}_1^\delta <<** \mathcal{U}_1^\delta \wedge \mathcal{U}_1^\beta$ ($\mathcal{U}_1^\delta <<** \mathcal{U}_1^\alpha \wedge \mathcal{U}_1^\beta$ and ord $\mathcal{U}_1^\delta \leq n + 1$). Again by Lemma 1.k (Lemma 1.k and Lemma 1.e) choose an open cover $\mathcal{U}_2^\delta$ such that $\mathcal{U}_2^\delta <<** \mathcal{U}_2^\alpha \wedge \mathcal{U}_2^\beta \wedge \mathcal{U}_1^\delta$ ($\mathcal{U}_2^\delta <<** \mathcal{U}_2^\alpha \wedge \mathcal{U}_2^\beta \wedge \mathcal{U}_1^\delta$ and ord $\mathcal{U}_2^\delta \leq n + 1$). Continuing in this way we have a strong starring sequence $\{\mathcal{U}_k^\delta \mid k \in Z\}$ of open covers of X such that $\mathcal{U}_k^\delta << \mathcal{U}_k^\alpha \wedge \mathcal{U}_k^\beta$ for each k. (with ord $\mathcal{U}_k^\delta \leq n + 1$ for each k). Hence

is a set

Suppose $\alpha < \beta$ so that $\mathcal{U}_k^\beta \ll \mathcal{U}_k^\alpha$ for each k. If $x \in X$ and $y \in x^{*\beta}$ then $\rho_\beta(x, y) = 0$ so that $y \in S^\beta_{\frac{1}{2^{k-1}}}(x)$ for each k. Since $S^\beta_{\frac{1}{2^{k-1}}}(x) \subseteq St(x, \mathcal{U}_k^\beta) \subseteq St(x, \mathcal{U}_k^\alpha) \subseteq S^\alpha_{\frac{1}{2^{k-2}}}(x)$ for each k, $\rho_\alpha(x, y) = 0$ and $y \in x^{*\alpha}$. Hence $x^{*\beta} \subseteq x^{*\alpha}$ for each $x \in X$ so that assigning $x^{*\alpha}$ to each $x^{*\beta}$, we have a surjection p^β_α from Y_β to Y_α. Also, for each $x \in X$ and k we have $(p^\beta_\alpha)^{-1}[(S^\alpha_{\frac{1}{2^k}}(x))^{*\alpha}] = (*_\alpha^{-1}[(S^\alpha_{\frac{1}{2^k}}(x))^{*\alpha}])^{*\beta} = (S^\alpha_{\frac{1}{2^k}}(x))^{*\beta}$. Since $S^\alpha_{\frac{1}{2^k}}(x)$ is open in ρ_β, $(S^\alpha_{\frac{1}{2^k}}(x))^{*\beta}$ is open in Y_β so that p^β_α is continuous. Thus we have an inverse limiting system $\{Y_\alpha, p^\beta_\alpha \mid \alpha \in \Lambda\}$ of metric spaces Y_α (with $\dim Y_\alpha \leq n$ for each $\alpha \in \Lambda$).

Let $Y = \text{inv lim }(Y_\alpha, p^\beta_\alpha \mid \alpha \in \Lambda)$ and define $f\colon X \to Y$ by $f(x) = (x^{*\alpha} \mid \alpha \in \Lambda)$ for each $x \in X$. Then f is continuous since $\Pi_\alpha \circ f = *_\alpha$ is continuous for each $\alpha \in \Lambda$ where Π_α are the projections.

Let $z \doteq (z_\alpha^{*\alpha} \mid \alpha \in \Lambda) \in Y$ and $\mathcal{F} = \{F \subseteq X \mid z_\alpha^{*\alpha} \subseteq F$ for some $\alpha \in \Lambda\}$. If $F, G \in \mathcal{F}$ let $z_\alpha^{*\alpha} \subseteq F$ and $z_\beta^{*\beta} \subseteq G$ and choose $\delta \in \Lambda$ with $\alpha, \beta < \delta$. Then $\Pi_\alpha(z) = p_\alpha^\delta o \Pi_\delta(z)$, $\Pi_\beta(z) = p_\beta^\delta o \Pi_\delta(z)$ and $z_\alpha^{*\alpha} = p_\alpha^\delta(z_\delta^{*\delta}) = z_\delta^{*\alpha}$, $z_\beta^{*\beta} = p_\beta^\delta(z_\delta^{*\delta}) = z_\delta^{*\beta}$ so that $z_\delta \in z_\alpha^{*\alpha} \cap z_\beta^{*\beta} \subseteq F \cap G \neq \emptyset$ and $\mathcal{F}$ is a filter on X. By Lemma 1.k (Corollary 2.1) for each open cover of X there is a strong starring sequence $\beta = \{\mathcal{U}_k^\beta \mid k \in \mathbb{Z}\}$ of open covers of X where $\{St(x, \mathcal{U}_k^\beta) \mid x \in X, k \in \mathbb{Z}\}$ partially refines the open cover with respect to $Q(x) \doteq \{St(x, \mathcal{U}_k^\beta) \mid k \in \mathbb{Z}\}$ (with ord $\mathcal{U}_k^\beta \leq n + 1$ for each k). So for each open cover of X there is $\beta \in \Lambda$ and integer $k(z_\beta)$ with $z_\beta^{*\beta} \subseteq S^\beta_{\frac{1}{2^{k(z_\beta)}}}(z_\beta)$ where $S^\beta_{\frac{1}{2^{k(z_\beta)}}}(z_\beta)$ is contained in some member of the open cover. Hence, there is $x \in X$ such that each open nhbd. of x is contained in $\mathcal{F}$. Since $\mathcal{F}$ is a filter, if $\alpha \in \Lambda$ and U is open with $x \in U$, then $U \cap z_\alpha^{*\alpha} \neq \emptyset$ so that $x \in \overline{z_\alpha^{*\alpha}}$. But since $*_\alpha$ is continuous on X, $x \in z_\alpha^{*\alpha}$. Hence, $x \in \bigcap_{\alpha \in \Lambda} z_\alpha^{*\alpha}$ and $f(x) = (x^{*\alpha} \mid \alpha \in \Lambda) = z$ so that f is surjective.

Let x, y ε X and $x \neq y$. Since X is regular let V be open with $y \varepsilon V \subseteq \overline{V} \subseteq X - \{x\}$. Then $\{X - \{x\}, X - \overline{V}\}$ is an open cover of X and by Lemma 1.k (Corollary 2.1) there is $\beta \varepsilon \Lambda$ and integer k(y) with $S^{\beta}_{\frac{1}{2^{k(y)}}}(y) \subseteq X - \{x\}$. If $f(x) = f(y)$ then $\Pi_\beta \circ f(x) = \Pi_\beta \circ f(y)$ so that $x^{*\beta} = y^{*\beta} \varepsilon (S^{\beta}_{\frac{1}{2^{k(y)}}}(y))^{*\beta}$. But this is impossible, since $*^{-1}_{\beta}[(S^{\beta}_{\frac{1}{2^{k(y)}}}(y))^{*\beta}] = S^{\beta}_{\frac{1}{2^{k(y)}}}(y)$. Hence, $f(x) \neq f(y)$ and f is injective.

Let U be open in X and $z = f(x) \varepsilon f[U]$ for some $x \varepsilon U$. As before let V be open with $x \varepsilon V \subseteq \overline{V} \subseteq U$ so that $\{U, X - \overline{V}\}$ is an open cover of X and there is $\beta \varepsilon \Lambda$ and integer k(x) with $S^{\beta}_{\frac{1}{2^{k(x)}}}(x) \subseteq U$. Now $Y \cap \Pi^{-1}_{\beta}[(S^{\beta}_{\frac{1}{2^{k(x)}}}(x))^{*\beta}]$ is an open nbhd. of z in Y. If $z' \varepsilon Y \cap \Pi^{-1}_{\beta}[(S^{\beta}_{\frac{1}{2^{k(x)}}}(x))^{*\beta}]$ then $z' = f(y)$ for some $y \varepsilon X$ since f is surjective, so that $\Pi_\beta(z') = y^{*\beta} \varepsilon (S^{\beta}_{\frac{1}{2^{k(x)}}}(x))^{*\beta}$ and $y \varepsilon S^{\beta}_{\frac{1}{2^{k(x)}}}(x)$. Hence $z \varepsilon Y \cap \Pi^{-1}_{\beta}[(S^{\beta}_{\frac{1}{2^{k(x)}}}(x))^{*\beta}] \subseteq f[U]$ and f is open.

If $\mathcal{U}$ is an open cover of Y, then $\{f^{-1}[U] \mid U \varepsilon \mathcal{U}\}$ is an open cover of X and again by Lemma 1.k (Corollary 2.1) there are integers k(x) for $x \varepsilon X$ such that $\{S^{\beta}_{\frac{1}{\cdots}}(x) \mid x \varepsilon X\} << \{f^{-1}[U] \mid U \varepsilon \mathcal{U}\}$ for some $\beta \varepsilon \Lambda$. So

$\mathcal{W} = \{(S^{\beta}_{\frac{1}{2^{k(x)}}}(x))^{*\beta} \mid x \in X\}$ is an open cover of Y_β and $\mathcal{W} << \{(f^{-1}[U])^{*\beta} | U \in \mathcal{U}\}$. But then $\{\Pi_\beta^{-1}[W] | W \in \mathcal{W}\} << \mathcal{U}$ where Π_β now is the projection restricted to Y. Thus the system is full and this completes the proof.

Corollary 2.4: A Hausdorff space X is paracompact with dim X = 0 if and only if X is homeomorphic to the limit space of a full inverse limiting system of discrete spaces.

Proof: Since every discrete space is paracompact Hausdorff with zero covering dimension by Lemma 1.e the sufficiency is evident. To prove the necessity consider the collection of disjoint open covers of X; which, by Lemma 1.e, is cofinal in X. If $\mathcal{U}^\alpha$ is a disjoint open cover, for each k let $\mathcal{U}^\alpha_k = \mathcal{U}^\alpha$ so that we have a collection Λ of strong starring sequences $\alpha = \{\mathcal{U}^\alpha_k \mid k \in Z\}$ of disjoint open covers of X. By Lemma 1.ℓ let (X, ρ_α) be the pseudometric space generated by $\{\mathcal{U}^\alpha_k \mid k \in Z\}$ such that $St(x, \mathcal{U}^\alpha_{k+1}) \subseteq S^\alpha_{\frac{1}{2^{k-1}}}(x) \subseteq St(x, \mathcal{U}^\alpha_k)$ for each $x \in X$ and k where $S^\alpha_{\frac{1}{2^{k-1}}}(x)$ are the spheres in ρ_α. Let Y_α be the metric space associated with ρ_α and let $*_\alpha$ be the canonical map. If $x^{*\alpha} \in Y_\alpha$ then $x \in U$ for some $U \in \mathcal{U}^\alpha$ and $U = S^\alpha_{\frac{1}{2^{k-1}}}(x)$,

$U^{*\alpha} = (S^{\alpha}_{\frac{1}{2^{k-1}}}(x))^{*\alpha}$ for each k. Hence $U^{*\alpha} = \{x^{*\alpha}\}$ which is open in Y_α so that Y_α is a discrete space. As in Theorem 2.2 define order $<$ on Λ and if $\alpha, \beta \in \Lambda$ with $\alpha < \beta$ define the continuous surjection p^{β}_{α} from Y_β to Y_α, so that we have an inverse limiting system $\{Y_\alpha, p^{\beta}_{\alpha} \mid \alpha \in \Lambda\}$ of discrete spaces.

Let $Y = \text{inv lim } (Y_\alpha, p^{\beta}_{\alpha} \mid \alpha \in \Lambda)$ and define $f\colon X \to Y$ by $f(x) = (x^{*\alpha} \mid \alpha \in \Lambda)$ for each $x \in X$. Then f is continuous since $\Pi_\alpha \circ f = *_\alpha$ is continuous for each $\alpha \in \Lambda$ where Π_α are the projections. In Theorem 2.2, to show that a homeomorphism was defined and that the inverse limiting system was full, Lemma 1.k was applied and the collection of all strong starring sequences of open covers was considered. Here, similarly, by applying Lemma 1.e and considering Λ we have that f is a homeomorphism and that the inverse limiting system is full. This completes the proof.

Corollary 2.4 has already been obtained by K. Nagami in ([11]) and without reference to pseudometric spaces generated by sequences of covers.

<u>Corollary 2.5</u>: A Hausdorff space X is compact (compact with dim $X \leq n$) if and only if X is homeomorphic to the limit space of an inverse limiting system of spaces (compact metric spaces Y_α with

Proof: The necessity follows since a compact space is trivially paracompact, the inverse limiting system constructed in Theorem 2.2 is surjective, and the continuous image of a compact space is again compact.

Let X be homeomorphic to Y = inv lim $(Y_\alpha, p_\alpha^\beta | \alpha \in \Lambda)$ where each Y_α is a compact metric space. We can assume that the system $\{Y_\alpha, p_\alpha^\beta | \alpha \in \Lambda\}$ is full since it can be shown using ([3], pages 428 and 429) that any inverse limiting system of compact metric spaces is full. Let $\mathcal{U}$ be an open cover of Y and $\{\Pi_\alpha^{-1}[W] \mid W \in \mathcal{W}\} << \mathcal{U}$ for some $\alpha \in \Lambda$ and open cover $\mathcal{W}$ of Y_α, where Π_α is the projection restricted to Y. Then $\mathcal{W}$, hence $\{\Pi_\alpha^{-1}[W] \mid W \in \mathcal{W}\}$, hence $\mathcal{U}$, has a finite subcover and X is compact. If also dim $Y_\alpha \leq n$ for each α then it is evident from Lemmas 1.j and 1.e that dim $X \leq n$ and this completes the proof.

A compact Hausdorff space has already been obtained as an inverse limit of compact metric spaces by S. Mardešić in ([12]), with a method entirely different from the method of Corollary 2.5, and relying heavily on induction.

Definition 2.3: A collection $\mathcal{V}$ of subsets of a space X is discrete (σ - discrete) if for each $x \in X$ there is an open nbhd. of x which has a non empty intersection with at most one member of $\mathcal{V}$ ($\mathcal{V}$ can be written as the countable union of discrete collections).

E. Michael has proved the following ([4], page 156).

Lemma 2.b: A regular space X is paracompact if and only if for each open cover $\mathcal{U}$ there is a σ - discrete open cover $\mathcal{V}$ such that $\mathcal{V} << \mathcal{U}$.

Trivially, a collection with only one member is discrete so that we have:

Lemma 2.c: A regular finally compact space is paracompact.

Lemma 2.d: A metric space is separable if and only if it is finally compact ([3], page 187).

Corollary 2.6: A Hausdorff space X is regular and finally compact (regular and finally compact with dim $X \leq n$) if and only if X is homeomorphic to the limit space of a full inverse limiting system of separable metric spaces (separable metric spaces Y_α with dim $Y_\alpha \leq n$ for each α).

Proof: The necessity follows from Lemma 2.c, from the fact that the inverse limiting system constructed in Theorem 2.2 is surjective, and from the fact that the continuous image of a finally compact space is finally compact.

If X is homeomorphic to the limit space of a full inverse limiting system of separable metric spaces then X is regular since a product of regular spaces is regular and a subspace of a regular space is regular. Because of Lemma 2.d the proof of the sufficiency is similar to the proof of the sufficiency of Corollary 2.5. This completes the proof.

Since the full inverse limiting system constructed ... ve we easily obtain:

Corollary 2.7: A Hausdorff space X is compact (regular and finally compact) with dim X = 0 if and only if X is homeomorphic to the limit space of a full inverse limiting system of finite (countable) discrete spaces.

Corollary 2.8: A Hausdorff space X is paracompact and connected (paracompact and connected with dim $X \leq n$) if and only if X is homeomorphic to the limit space of a surjective full inverse limiting system of connected metric spaces (connected metric spaces Y_α with dim $Y_\alpha \leq n$ for each α).

Proof: The necessity follows since the inverse limiting system constructed in Theorem 2.2 is surjective and since the continuous image of a connected space is again connected.

Let X be homeomorphic to $Y = \text{inv lim } (Y_\alpha, p_\alpha^\beta | \alpha \varepsilon \Lambda)$ where the system $\{Y_\alpha, p_\alpha^\beta \mid \alpha \varepsilon \Lambda\}$ is surjective and full and each Y_α is a connected metric space. Suppose X is not connected so that there is a disjoint open cover $\{U_1, U_2\}$ of X of non empty sets. Choose $\alpha \varepsilon \Lambda$ and an open cover $\mathcal{W}$ of Y_α such that $\{\Pi_\alpha^{-1}[W] | W \varepsilon \mathcal{W}\} << \{U_1, U_2\}$ where Π_α is the projection restricted to Y. Since Π_α is surjective, $\{\bigcup\{W \varepsilon \mathcal{W} | \Pi_\alpha^{-1} [W] \subseteq U_1\}, \bigcup\{W \varepsilon \mathcal{W} | \Pi_\alpha^{-1} [W] \subseteq U_2\}\}$ is a disjoint open cover of Y_α of non empty sets and this contradicts that Y_α is connected. Hence X is connected. If also dim $Y_\alpha \leq n$

$X < n$. This

A <u>continuum</u> is a connected compact space and combining the proofs of Corollaries 2.5, 2.8, we have:

<u>Corollary 2.9</u>: A Hausdorff space X is a continuum (continuum with dim $X \leq n$) if and only if X is homeomorphic to the limit space of a surjective full inverse limiting system of metrizable continua (metrizable continua Y_α with dim $Y_\alpha \leq n$ for each α).

3. Briefly let us turn to sequences of closed covers. In the proof of the next characterization we shall refer to the following result by K. Morita ([13]).

<u>Lemma 2.e</u>: A metric space Y has dim $Y \leq n$ if and only if Y has a sequence $\{F_k \mid k \in Z\}$ of locally finite closed covers satisfying the following conditions: each F_k is of the form $F_k = \{F\alpha_1,\ldots,\alpha_k \mid \alpha_i \in \Omega\ 1 \leq i \leq k\}$ where for $k > 1$, $F\alpha_1,\ldots,\alpha_{k-1} = \bigcup\{F\alpha_1,\ldots,\alpha_{k-1},\beta \mid \beta \in \Omega\}$ and $F\alpha_1,\ldots,\alpha_{k-1}$ may be empty; ord $F_k \leq n + 1$ for each k; for each nbhd. U of every point $y \in Y$, there is k such that $St(y, F_k) \subseteq U$.

Theorem 2.3: A regular space X is paracompact with dim $X \leq n$ if and only if for each open cover $\mathcal{U}$ there is a sequence $\{\mathcal{F}_k \mid k \in Z\}$ of locally finite closed covers satisfying the following conditions:

1) each $\mathcal{F}_k$ is of the form

$\mathcal{F}_k = \{F\alpha_1, \ldots, \alpha_k \mid \alpha_i \in \Omega \; 1 \leq i \leq k\}$ where for $k > 1$,

$F\alpha_1, \ldots, \alpha_{k-1} = \bigcup\{F\alpha_1, \ldots, \alpha_{k-1}, \beta \mid \beta \in \Omega\}$ and $F\alpha_1, \ldots, \alpha_{k-1}$ may be empty.

2) ord $\mathcal{F}_k \leq n + 1$ for each k.

3) $\{St(x, \mathcal{F}_k) \mid x \in X, k \in Z\} <<_p \mathcal{U}$ with respect to $Q(x) = \{St(x, \mathcal{F}_k) \mid k \in Z\}$.

4) for each $x \in X$ and k, if $St(x, \mathcal{F}_k) \subseteq U$ for some $U \in \mathcal{U}$ then there is p with

$St(x, \mathcal{F}_p) \subseteq V_k(x) = X - \bigcup\{F \in \mathcal{F}_k \mid x \notin F\}$ ([10]).

Proof: To prove the necessity let $\mathcal{U}$ be an open cover of X and by Corollary 2.3 let $f: X \to Y$ be a continuous surjection where Y is a metric space with dim $Y \leq n$ and let $\mathcal{W}$ be an open cover of Y such that $\{f^{-1}[W] \mid W \in \mathcal{W}\} << \mathcal{U}$. Let $\{\mathcal{F}_k \mid k \in Z\}$ be a sequence of locally finite closed covers of Y given by Lemma 2.e. Then for each k,

$\mathcal{G}_k = \{f^{-1}[F\alpha_1, \ldots, \alpha_k] \mid \alpha_i \in \Omega \; 1 \leq i \leq k\}$ is a closed cover of X such that $f^{-1}[F\alpha_1, \ldots, \alpha_{k-1}] = f^{-1}[\bigcup\{F\alpha_1, \ldots, \alpha_{k-1}, \beta \mid \beta \in \Omega\}]$

$= \bigcup\{f^{-1}[F\alpha_1, \ldots, \alpha_{k-1}, \beta] \mid \beta \in \Omega\}$ for $k > 1$ where

$f^{-1}[F\alpha_1, \ldots, \alpha_{k-1}, \beta]$ may be empty. If $x \in X$, there is a nbhd. U of f(x) such that U has a non empty intersection with at most finitely many members of $\mathcal{F}_k$, so that $f^{-1}[U]$ is a nbhd. of x which has a non empty intersection with at most finitely many members of $\mathcal{G}_k$. Hence each $\mathcal{G}_k$ is locally finite and similarly we have ord $\mathcal{G}_k \leq n + 1$ for each k. Suppose $St(x, \mathcal{G}_k) \subseteq U$ for some $x \in X$, k, and $U \in \mathcal{U}$ and let $y = f(x)$. Now

$y \in Y - \bigcup\{F \in \mathcal{F}_k \mid y \notin F\}$ and since $\mathcal{F}_k$ is closed and closure preserving by Lemma 1.a, this set is an open nbhd. of y. Hence, there is p such that $St(y, \mathcal{F}_p) \subseteq Y - \bigcup\{F \in \mathcal{F}_k \mid y \notin F\}$. For each $F \in \mathcal{F}_k$, $y \in F$ if and only if $x \in f^{-1}[F]$ and letting $V_k(x) = X - \bigcup\{f^{-1}[F] \mid f^{-1}[F] \in \mathcal{G}_k \text{ and } x \notin f^{-1}[F]\}$ we have $St(x, \mathcal{G}_p) = f^{-1}[St(y, \mathcal{F}_p)] \subseteq V_k(x)$. If $x \in X$ and $f(x) = y$, since $\mathcal{W}$ is an open cover of Y there is k such that $St(y, \mathcal{F}_k) \subseteq W$ for some $W \in \mathcal{W}$. Hence $St(x, \mathcal{G}_k) = f^{-1}[St(y, \mathcal{F}_k)] \subseteq f^{-1}[W] \subseteq U$ for some $U \in \mathcal{U}$ so that $\{St(x, \mathcal{G}_k) \mid x \in X, k \in Z\} <<_p \mathcal{U}$ with respect to $Q(x) = \{St(x, \mathcal{G}_k) \mid k \in Z\}$. Thus, for the sequence $\{\mathcal{G}_k \mid k \in Z\}$ conditions 1) to 4) are satisfied.

Now suppose that for $\mathcal{U}$ there is a sequence $\{\mathcal{F}_k \mid k \in Z\}$ of locally finite closed covers of X satisfying conditions

If $x \in X$ and $k \in Z$ by conditions 1) and 3) choose $i > k$ and $U \in \mathcal{U}$ with $St(x, F_i) \subseteq U$. By conditions 1) and 4) choose $p > i$ such that $St(x, F_p) \subseteq V_i(x)$ and again by 1) and 4), since $V_i(x) \subseteq St(x, F_i)$, choose $(x, k) > p$ such that $St(x, F_{(x,k)}) \subseteq V_p(x)$. Now, if $St(x, F_{(x,k)}) \cap St(y, F_{(x,k)}) \neq \emptyset$ then $V_p(x) \cap St(y, F_p) \neq \emptyset$ and $V_p(x) \cap G \neq \emptyset$ for some $G \in F_p$ with $y \in G$. If $x \notin G$ then $G \subseteq \cup\{F \in F_p \mid x \notin F\}$ and $V_p(x) \cap G = \emptyset$. Hence $x \in G$ so that $y \in G \subseteq St(x, F_p) \subseteq V_i(x)$. If $z \in St(y, F_i)$ let $z \in F \in F_i$ with $y \in F$. If $x \notin F$ then $F \cap V_i(x) = \emptyset$ so that $x \in F$ and $z \in F \subseteq St(x, F_k)$. Thus, $St(y, F_i) \subseteq St(x, F_k)$, and since $(x, k) > i$, $St(y, F_{(x,k)}) \subseteq St(x, F_k)$. So for each $x \in X$ and k, there is $(x, k) \in Z$ such that if $St(x, F_{(x,k)}) \cap St(y, F_{(x,k)}) \neq \emptyset$ then $St(y, F_{(x,k)}) \subseteq St(x, F_k)$. For each $x \in X$ this allows us to choose a sequence of integers $1(x) < 2(x) = (x, 1(x)) < 3(x) = (x, 2(x)) < \ldots$, where $(k + 1)(x) = (x, k(x))$ means that if $St(x, F_{(k+1)(x)}) \cap St(y, F_{(k+1)(x)}) \neq \emptyset$ then $St(y, F_{(k+1)(x)}) \subseteq St(x, F_{k(x)})$.

For each k let $\mathcal{W}_k = \{St(x, F_{(2k-1)(x)}) \mid x \in X\}$. Suppose we have three members $St(x, F_{(2k+1)(x)})$, $St(y, F_{(2k+1)(y)})$

$St(x, F_{(2k+1)(x)}) \cap St(y, F_{(2k+1)(y)}) \neq \emptyset$ and

$St(y, F_{(2k+1)(y)}) \cap St(z, F_{(2k+1)(z)}) \neq \emptyset$. Consider

$St(x, F_{(2k+1)(x)}) \cap St(y, F_{(2k+1)(y)}) \neq \emptyset$ and the case

where $(2k + 1)(x) \leq (2k + 1)(y)$. Then, by 1)

$St(y, F_{(2k+1)(y)}) \subseteq St(y, F_{(2k+1)(x)})$ so that

$St(x, F_{(2k+1)(x)}) \cap St(y, F_{(2k+1)(x)}) \neq \emptyset$ and

$St(y, F_{(2k+1)(x)}) \subseteq St(x, F_{(2k)(x)})$. Hence

$St(y, F_{(2k+1)(y)}) \subseteq St(x, F_{(2k)(x)})$. But since

$(2k + 1)(x) > (2k)(x)$, by 1) we also have

$St(x, F_{(2k+1)(x)}) \subseteq St(x, F_{(2k)(x)})$. So altogether we

have $St(x, F_{(2k+1)(x)}) \cup St(y, F_{(2k+1)(y)}) \subseteq St(x, F_{(2k)(x)})$.

On the other hand suppose $(2k +1)(y) \leq (2k +1)(x)$. Then

by 1) $St(x, F_{(2k+1)(x)}) \subseteq St(x, F_{(2k+1)(y)})$ so that

$St(y, F_{(2k+1)(y)}) \cap St(x, F_{(2k+1)(y)}) \neq \emptyset$

and $St(x, F_{(2k+1)(y)}) \subseteq St(y, F_{(2k)(y)})$. Hence,

$St(x, F_{(2k+1)(x)}) \subseteq St(y, F_{(2k)(y)})$. But since

$(2k + 1)(y) > (2k)(y)$, by 1) we also have

$St(y, F_{(2k+1)(y)}) \subseteq St(y, F_{(2k)(y)})$. So altogether we have

$St(x, F_{(2k+1)(x)}) \cup St(y, F_{(2k+1)(y)}) \subseteq St(y, F_{(2k)(y)})$.

In any case, $St(x, F_{(2k+1)(x)}) \cup St(y, F_{(2k+1)(y)}) \subseteq St(a, F_{(2k)(a}$

where a is either x or y. Now in the same way we can consider

$St(y, F_{(2k+1)(y)}) \cap St(z, F_{(2k+1)(z)}) \neq \emptyset$ to obtain

$St(y, F_{(2k+1)(y)}) \cup St(z, F_{(2k+1)(z)}) \subseteq St(b, F_{(2k)(b)})$

where b is either y or z. Since

$St(a, F_{(2k)(a)}) \cap St(b, F_{(2k)(b)}) \neq \emptyset$ we can repeat the argument above to obtain

$St(a, F_{(2k)(a)}) \cup St(b, F_{(2k)(b)}) \subseteq St(c, F_{(2k-1)(c)})$

for some $c \in X$ where $St(c, F_{(2k-1)(c)}) \in \mathcal{W}_k$. Thus, $\{\mathcal{W}_k | k \in Z\}$ is a 3-chain sequence of covers of X.

Let ρ be the pseudometric generated by $\{\mathcal{W}_k | k \in Z\}$ such that $St(x, \mathcal{W}_{k+1}) \subseteq S_{\frac{1}{2^{k-1}}}(x) \subseteq St(x, \mathcal{W}_k)$ for each $x \in X$ and k, according to Lemma 1.ℓ; let Y be the metric space associated with ρ; and let * be the canonical map. For each $x \in X$ and k, since $F_{(2k+1)(x)}$ is closed and locally finite, by Lemma 1.a, $V_{(2k+1)(x)}(x)$ is an open nbhd. of x. Hence for each x and k we have

$V_{(2k+1)(x)}(x) \subseteq St(x, F_{(2k+1)(x)}) \subseteq St(x, \mathcal{W}_{k+1}) \subseteq S_{\frac{1}{2^{k-1}}}(x)$

$= *^{-1}[S^*_{\frac{1}{2^{k-1}}}(x)]$ so that * is continuous on X.

For each k, let $\mathcal{G}_k = \{(F\alpha_1,\ldots,\alpha_k)^* | \alpha_i \in \Omega\ 1 \leq i \leq k\}$ If $k \in Z$ and x is in the closure of $F\alpha_1,\ldots,\alpha_k$ in (X, ρ) by 3) choose $i > k$ such that $St(x, F_i) \subseteq U$ for some $U \in \mathcal{U}$ and by 1) and 4) choose $p > i$ with $St(x, F_p) \subseteq V_i(x)$. Then choosing $(k-1)(x) > p$ we have

$S_{\frac{1}{2^{k(x)}}}(x) \subseteq St(x, \mathcal{U}_{k(x)+1}) \subseteq St(x, F_{(k-1)(x)}) \subseteq V_i(x)$ since

if $y \in St(x, \mathcal{U}_{k(x)+1})$ then $y \in St(z, F_{(2k(x)+1)(z)})$

where $x \in St(z, F_{(2k(x)+1)(z)})$ so that

$z \in St(y, F_{(2k(x)+1)(z)}) \cap St(x, F_{(2k(x)+1)(z)})$ (by $2k(x)$ here we mean $k(x)$ doubled). Since $(2k(x)+1)(z) > 2k(x)+1$, we have $z \in St(y, F_{k(x)}) \cap St(x, F_{k(x)}) \neq \emptyset$ so that

$St(y, F_{k(x)}) \subseteq St(x, F_{(k-1)(x)})$ and $y \in St(x, F_{(k-1)(x)})$.

Since $S_{\frac{1}{2^{k(x)}}}(x) \cap F\alpha_1, \ldots, \alpha_k \neq \emptyset$ we have $V_i(x) \cap F\alpha_1, \ldots, \alpha_k \neq \emptyset$

and by 1) $V_i(x) \cap F\alpha_1, \ldots, \alpha_k, \alpha_{k+1} \neq \emptyset$ for some $\alpha_{k+1} \in \Omega$

with $F\alpha_1, \ldots, \alpha_k, \alpha_{k+1} \subseteq F\alpha_1, \ldots, \alpha_k$. Similarly

$V_i(x) \cap F\alpha_1, \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_i \neq \emptyset$ for some

$\alpha_{k+1}, \ldots, \alpha_i \in \Omega$ where $F\alpha_1, \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_i \subseteq F\alpha_1, \ldots, \alpha_k$,

so that $x \in F\alpha_1, \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_i \subseteq F\alpha_1, \ldots, \alpha_k$. Thus,

$F\alpha_1, \ldots, \alpha_k$ is closed in (X, ρ) so that $\mathcal{G}_k$ is a closed cover of Y.

Again consider F_k and $x \in X$ and as before let

$(k-1)(x) > i > k$ such that

$S_{\frac{1}{2^{k(x)}}}(x) \subseteq St(x, \mathcal{U}_{k(x)+1}) \subseteq St(x, F_{(k-1)(x)}) \subseteq V_i(x)$.

members of F_k, say $F\alpha_1^t, \ldots, \alpha_k^t$ for n + 2 distinct indices $(\alpha_1^t, \ldots, \alpha_k^t)\ 1 \leq t \leq n + 2$. Then $V_i(x) \cap F\alpha_1^t, \ldots, \alpha_k^t \neq \emptyset$ for $1 \leq t \leq n + 2$ and for each t choose $\alpha_{k+1}^t, \ldots, \alpha_i^t \in \Omega$ such that $V_1(x) \cap F\alpha_1^t, \ldots, \alpha_k^t, \alpha_{k+1}^t, \ldots, \alpha_i^t \neq \emptyset$. Then the indices $(\alpha_1^t, \ldots, \alpha_k^t, \alpha_{k+1}^t, \ldots, \alpha_i^t)$ are distinct where $x \in F\alpha_1^t, \ldots, \alpha_k^t, \alpha_{k+1}^t, \ldots, \alpha_i^t$ for $1 \leq t \leq n + 2$. This contradicts ord $F_i \leq n + 1$. Hence, $S_{\frac{1}{2^{\overline{k(x)}}}}(x)$ has a non empty intersection with at most n + 1 members of F_k. This means F_k is locally finite in (x, ρ) so that G_k is locally finite in Y and also ord $G_k \leq n + 1$. Given $x \in X$ and $S^*_{\frac{1}{2^{\overline{k-1}}}}(x)$ we have

$St(x, F_{(2k+1)(x)}) \subseteq St(x, W_{k+1}) \subseteq S_{\frac{1}{2^{\overline{k-1}}}}(x)$ so that

$(St(x, F_{(2k+1)(x)}))^* = St(x^*, G_{(2k+1)(x)}) \subseteq S^*_{\frac{1}{2^{\overline{k-1}}}}(x)$ and the sequence $\{G_k \mid k \in Z\}$ satisfies all the conditions in Lemma 2.e. Hence dim $Y \leq n$.

Since $\{St(x, W_k) \mid x \in X, k \in Z\} <<_p U$ with respect to $Q(x) = \{St(x, W_k \mid k \in Z\}$, form an open cover W of Y such that $\{*^{-1}[W] \mid W \in W\} << U$. Then, by Corollary 2.3, X is paracompact with dim $X \leq n$. This completes the proof.

Paracompact Spaces and the Modeling Distribution

Our main purpose here is to show that a modeling distribution preserves paracompactness in regular spaces and lowers covering dimension in regular paracompact spaces.

1. First consider the following concepts due to R. G. Lintz ([1]). Let X, Y be spaces.

Definition 3.1: A non-deterministic function is a pair of collections $\mathcal{V}$, $\mathcal{V}'$ of open covers of X, Y respectively, with a function $r\colon\ \mathcal{V} \to \mathcal{V}'$ and a collection of functions $\{f_V\colon\ V \to r(V) \mid V \in \mathcal{V}\}$.

Here, as in the literature, it is implicit in this definition that for each $V \in \mathcal{V}$, if $V \in V$ and $V \neq \emptyset$ then $f_V(V) \neq \emptyset$.

We denote a non-deterministic function as defined above as $f\colon\ (X, \mathcal{V}) \to (Y, \mathcal{V}')$ or simply as f.

Definition 3.2: A non-deterministic function is called cofinal if the image of r is cofinal in Y.

Definition 3.3: A non-deterministic function is said to be continuous if for each pair V_1, V_2 of members of $\mathcal{V}$ where $V_1 << V_2$, if $V_1 \in V_1$ and $V_2 \in V_2$ with $V_1 \subseteq V_2$

<u>Definition 3.4</u>: A non-deterministic function is surjective (injective) if for each $\mathcal{V} \in \mathbb{V}$, $f_{\mathcal{V}}$ is surjective (injective).

We shall index a non-deterministic function by some index, say $\mathcal{W}$, by letting $\mathbb{V} = \mathbb{V}^{\mathcal{W}}$, $\mathbb{V}' = \mathbb{V}'^{\mathcal{W}}$, $r = r_{\mathcal{W}}$, and $f_{\mathcal{V}} = f_{\mathcal{V}}^{\mathcal{W}}$ for each $\mathcal{V} \in \mathbb{V}^{\mathcal{W}}$ and writing $f^{\mathcal{W}} : (X, \mathbb{V}^{\mathcal{W}}) \to (Y, \mathbb{V}'^{\mathcal{W}})$ or simply $f^{\mathcal{W}}$. The index $\mathcal{W}$ will often be an open cover of Y.

<u>Definition 3.5</u>: A modeling function from X to Y is a continuous, surjective, non-deterministic function $f: (X, \mathbb{V}) \to (Y, \mathbb{V}')$ where $\mathbb{V}$ is cofinal in X and for each $\mathcal{V} \in \mathbb{V}$, if $V_1, V_2 \in \mathcal{V}$ with $f_{\mathcal{V}}(V_1) \cap f_{\mathcal{V}}(V_2) \neq \emptyset$ then $V_1 \cap V_2 \neq \emptyset$.

<u>Definition 3.6</u>: A modeling distribution from X to Y is a collection of modeling functions from X to Y where for each open cover $\mathcal{W}$ of Y there is a member $f^{\mathcal{W}}: (X, \mathbb{V}^{\mathcal{W}}) \to (Y, \mathbb{V}'^{\mathcal{W}})$ of the collection such that $r_{\mathcal{W}}(\mathcal{V}) << \mathcal{W}$ for each $\mathcal{V} \in \mathbb{V}^{\mathcal{W}}$. If there is a modeling distribution from X to Y we say that Y is a model of X.

Any usual function f which is open, continuous, and surjective, with the property that $f^{-1} f [U] = U$ for each open set U, always induces a cofinal injective modeling function. In particular, if X is a regular paracompact space, $\{\mathcal{U}_k | k \in Z\}$ is a 3-chain sequence of open covers of X obtained by Lemma 1.k, and (X, ρ)

k Z} wh

X^* is the metric space associated with ρ, then the canonical map * induces a cofinal injective modeling function from (X, ρ) to X^*. However (x, ρ) and X^* may not be homeomorphic for if X is regular and compact but not Hausdorff (for example, if X has at least two points and the only open sets are X and $\emptyset$) then (X, ρ) cannot be Hausdorff whereas X^* is Hausdorff.

2. Using Corollary 2.3 we easily generalize the following result due to A. Ostrand ([14]) to regular paracompact spaces.

Lemma 3.a: If X is a metric space then $\dim X \leq n$ if and only if for each open cover $\mathcal{U}$ and $i \in Z$, $i \geq n + 1$, there are i discrete collections of open sets $\mathcal{V}_k$ $1 \leq k \leq i$ such that the union of any $n + 1$ of the $\mathcal{V}_k$ is an open cover which refines $\mathcal{U}$.

Corollary 3.1: A regular space X is paracompact with $\dim X \leq n$ if and only if for each open cover $\mathcal{U}$ and $i \in Z$, $i \geq n + 1$, there are i discrete collections of open sets $\mathcal{V}_k$ $1 \leq k \leq i$ such that the union of any $n + 1$ of the $\mathcal{V}_k$ is an open cover which refines $\mathcal{U}$.

Theorem 3.1: Let X,Y be regular. If X is paracompact (paracompact with $\dim X \leq n$) and Y is a model of X, then Y is paracompact (paracompact with $\dim Y \leq n$) ([10]).

Proof: If $\mathcal{W}$ is an open cover of Y let

$\mathcal{W}$ $(X, \mathsf{V}^{\mathcal{W}}) \to (Y, \mathsf{V}'^{\mathcal{W}})$ be a modeling function and $\mathcal{V} \in \mathsf{V}^{\mathcal{W}}$

Since X is regular and paracompact by Lemma 1.k let $\mathcal{A}$ be an open cover of X such that $\mathcal{A} <<^{**} \mathcal{V}$, and since $\mathcal{V}^{\mathcal{W}}$ is cofinal in X choose $\mathcal{U} \in \mathcal{V}^{\mathcal{W}}$ such that $\mathcal{U} << \mathcal{A}$ so that $\mathcal{U} <<^{**} \mathcal{V}$. Now, if $f^{\mathcal{W}}_{\mathcal{U}}(U_1) \cap f^{\mathcal{W}}_{\mathcal{U}}(U_2) \neq \emptyset$ for some

$U_1, U_2 \in \mathcal{U}$ then $U_1 \cap U_2 \neq \emptyset$ so that $U_1 \subseteq St(U_2, \mathcal{U}) \subseteq V$ for some $V \in \mathcal{V}$. By the continuity of $f^{\mathcal{W}}$, $f^{\mathcal{W}}_{\mathcal{U}}(U_1) \subseteq f^{\mathcal{W}}_{\mathcal{V}}(V)$ so that $St(f^{\mathcal{W}}_{\mathcal{U}}(U_2), r_{\mathcal{W}}(\mathcal{U})) \subseteq f^{\mathcal{W}}_{\mathcal{V}}(V)$ and $r_{\mathcal{W}}(\mathcal{U}) <<^{**} r_{\mathcal{W}}(\mathcal{V})$. Since $f^{\mathcal{W}}$ is surjective, $r_{\mathcal{W}}(\mathcal{U})$ is an open cover of Y such that $r_{\mathcal{W}}(\mathcal{U}) <<^{**} \mathcal{W}$, and by Lemma 1.k, Y is paracompact.

If also dim X $\leq$ n, then by Corollary 3.1, we can further assume that $\mathcal{U} << \bigcup_{k=1}^{n+1} \mathcal{V}_k << \mathcal{V}$ where $\mathcal{V}_k = \{V_\gamma \mid \gamma \in \Gamma_k\}$ $1 \leq k \leq n+1$ are n + 1 discrete collections of open sets. If $1 \leq k \leq n+1$, for each $\gamma \in \Gamma_k$ let

$S_\gamma = \bigcup\{f^{\mathcal{W}}_{\mathcal{U}}(U) \mid U \in \mathcal{U} \text{ and } U \subseteq V_\gamma\}$ and let $\mathcal{S}_k = \{S_\gamma \mid \gamma \in \Gamma_k\}$.

Since $f^{\mathcal{W}}$ is surjective, $\mathcal{S} = \bigcup_{k=1}^{n+1} \mathcal{S}_k$ is an open cover of Y; and by the continuity of $f^{\mathcal{W}}$, $\mathcal{S} << \mathcal{W}$. If we suppose that there are distinct indices $\gamma_1, \gamma_2 \in \Gamma_k$ with $S_{\gamma_1} \cap S_{\gamma_2} \neq \emptyset$ then we have $f^{\mathcal{W}}_{\mathcal{U}}(U_1) \cap f^{\mathcal{W}}_{\mathcal{U}}(U_2) \neq \emptyset$ for some $U_1, U_2 \in \mathcal{U}$ with $U_1 \subseteq V_{\gamma_1}$, $U_2 \subseteq V_{\gamma_2}$. Then $U_1 \cap U_2 \neq \emptyset$ implies that $V_{\gamma_1} \cap V_{\gamma_2} \neq \emptyset$ which contradicts that $\mathcal{V}_k$ is a disjoint

$\mathcal{S}_k$ 1 for k. us. $\mathcal{S}$ is an open cover of Y, $\mathcal{S} << \mathcal{W}$, and ord $\mathcal{S} \leq n + 1$ so that dim $Y \leq n$. This completes the proof.

Corollary 3.2: Let X be regular. If X is finally compact (finally compact with dim $X \leq n$) and Y is a model of X then Y is finally compact (finally compact with dim $Y \leq n$).

Proof: If $\mathcal{W}$ is an open cover of Y let $f^{\mathcal{W}} : (X, \mathcal{V}^{\mathcal{W}}) \rightarrow (Y, \mathcal{V}'^{\mathcal{W}})$ be a modeling function and $\mathcal{V} \in \mathcal{V}^{\mathcal{W}}$ such that $r_{\mathcal{W}}(\mathcal{V}) << \mathcal{W}$. If X is finally compact, let $\mathcal{A}$ be a countable subcover of $\mathcal{V}$ and choose $\mathcal{U} \in \mathcal{V}^{\mathcal{W}}_{\mathcal{W}}$ such that $\mathcal{U} << \mathcal{A}$. By the continuity of $f^{\mathcal{W}}$, $r_{\mathcal{W}}(\mathcal{U}) << \{f_{\mathcal{V}}(V) \mid V \in \mathcal{A}\}$ so that the latter is a countable cover which refines $\mathcal{W}$. Hence, we can choose a countable subcover of $\mathcal{W}$ so that Y is finally compact. If also dim $X \leq n$ then Theorem 3.1 can be applied to obtain dim $Y \leq n$ since by Lemma 2.c X is paracompact. This completes the proof.

Now, in the above proof, considering X to be compact and $\mathcal{A}$ to be a finite subcover of $\mathcal{V}$, we have:

Corollary 3.3: Let X be regular. If X is compact (compact with dim $X \leq n$) and Y is a model of X then Y is compact (compact with dim $Y \leq n$).

Theorem 3.2: Let X be regular and paracompact. If Y is a model of X then X is compact (finally compact) if and only if Y is compact (finally compact).

Proof: We have the necessity from Corollaries 3.2, 3.3. To prove the sufficiency, let $f: (X, \mathcal{V}) \to (Y, \mathcal{V}')$ be any member of the modeling distribution from X to Y. Let $\mathcal{V}$ be any member of $\mathcal{V}$ and since X is regular and paracompact by Lemma 1.k choose $\mathcal{U} \in \mathcal{V}$ such that $\mathcal{U} <<^{**} \mathcal{V}$. Assuming Y is compact, let $\{f_{\mathcal{U}}(U_i) \mid 1 \leq i \leq s\}$ be a finite subcover of $r(\mathcal{U})$.

If $x \in X$, then $x \in U$ for some $U \in \mathcal{U}$ so that $U \neq \emptyset$ implies that $f_{\mathcal{U}}(U) \neq \emptyset$ and $f_{\mathcal{U}}(U) \cap f_{\mathcal{U}}(U_i) \neq \emptyset$ for some $1 \leq i \leq s$. Hence, $U \cap U_i \neq \emptyset$ so that $x \in U \subseteq St(U_i, \mathcal{U})$. Thus, $\{St(U_i, \mathcal{U}) \mid 1 \leq i \leq s\}$ covers X and we can choose a finite subcover of $\mathcal{V}$. Since $\mathcal{V}$ is cofinal in X, X is compact. Similarly, assuming Y is finally compact and considering countable subcovers in place of finite ones, we have X finally compact. This completes the proof.

Now using Theorem 3.2 and Lemmas 1.j, 2.d, we have:

Corollary 3.4: Let X, Y be metric spaces. If Y is a model of X, then X is separable if and only if Y is separable.

Theorem 3.3: Let X be Hausdorff. If Y is a model of X, then X is connected if and only if Y is connected.

Proof: Suppose Y is not connected so that there is a disjoint open cover $\mathcal{W} = \{Y_1, Y_2\}$ of Y of non empty sets. Choose a modeling function $f^{\mathcal{W}}: (X, \mathcal{V}^{\mathcal{W}}) \to (Y, \mathcal{V}'^{\mathcal{W}})$ and $\mathcal{V} \in \mathcal{V}^{\mathcal{W}}$ such that $r_{\mathcal{W}}(\mathcal{V}) << \mathcal{W}$, and let

$X_i = \bigcup \{V \in \mathcal{V} \mid \overset{\omega}{\underset{\mathcal{V}}{f}}(V) \subseteq Y_i\}$ for i = 1, 2 so that $\{X_1, X_2\}$ is an open cover of X. Choose $V \in \mathcal{V}$ with $\overset{\omega}{\underset{\mathcal{V}}{f}}(V) \subseteq Y_1$ and $\overset{\omega}{\underset{\mathcal{V}}{f}}(V) \neq \emptyset$. Then $V \neq \emptyset$ and $X_1 \neq \emptyset$. Similarly , $X_2 \neq \emptyset$. If $X_1 \cap X_2 \neq \emptyset$ then $V_1 \cap V_2 \neq \emptyset$ where $\overset{\omega}{\underset{\mathcal{V}}{f}}(V_1) \subseteq Y_1$ and $\overset{\omega}{\underset{\mathcal{V}}{f}}(V_2) \subseteq Y_2$ and let $x \in V_1 \cap V_2$. Since X is Hausdorff, $\{V_1 \cap V_2, X - \{x\}\}$ is an open cover of X and choose $\mathcal{U} \in \mathbb{V}^{\omega}$ such that $\mathcal{U} << \{V_1 \cap V_2, X - \{x\}\}$. If $x \in U \in \mathcal{U}$ then $U \subseteq V_1 \cap V_2$ and by the continuity of f^{ω}, $\overset{\omega}{\underset{\mathcal{U}}{f}}(U) \subseteq \overset{\omega}{\underset{\mathcal{V}}{f}}(V_1) \cap \overset{\omega}{\underset{\mathcal{V}}{f}}(V_2)$. Since $U \neq \emptyset$ implies $\overset{\omega}{\underset{\mathcal{U}}{f}}(U) \neq \emptyset$, we contradict that Y_1, Y_2 are disjoint. Hence $\{X_1, X_2\}$ is a disjoint open cover of X of non empty sets and X is not connected.

Now suppose X has this cover; choose a modeling function f: $(X, \mathbb{V}) \to (Y, \mathbb{V}')$ and some $\mathcal{V} \in \mathbb{V}$ such that $\mathcal{V} << \{X_1, X_2\}$, and let $Y_i = \bigcup \{\underset{\mathcal{V}}{f}(v) \mid V \subseteq X_i\}$ for i = 1, 2. Since $\mathcal{V}$ covers X there is $V_1 \in \mathcal{V}$, $V_1 \neq \emptyset$ and $V_1 \subseteq X_1$ so that $\underset{\mathcal{V}}{f}(V_1) \subseteq Y_1$. Since $V_1 \neq \emptyset$ implies $\underset{\mathcal{V}}{f}(V_1) \neq \emptyset$, $Y_1 \neq \emptyset$. Similarly $Y_2 \neq \emptyset$. If $Y_1 \cap Y_2 \neq \emptyset$ so that $\underset{\mathcal{V}}{f}(V_1) \cap \underset{\mathcal{V}}{f}(V_2) \neq \emptyset$ where $V_1 \subseteq X_1$ and $V_2 \subseteq X_2$, then we contradict that X_1, X_2 are disjoint. Hence, $\{Y_1, Y_2\}$ is a ... of Y of non empty sets so that Y is not connected. This completes the proof.

3. Let us turn briefly to two of the most important modifications of paracompactness, strong paracompactness and complete paracompactness.

Definition 3.7: A collection V of subsets of a space X is star finite if each member of V has a non empty intersection with at most finitely many members of V.

Definition 3.8: A space X is strongly paracompact if for each open cover U there is a star finite open cover V such that $V << U$.

Theorem 3.4: Let X be regular. If X is strongly paracompact (strongly paracompact with dim X $\leq$ n) and Y is a model of X then Y is strongly paracompact (strongly paracompact with dim Y $\leq$ n).

Proof: Let W be an open cover of Y and f^{W}: $(X, V^{W}) \to (Y, V'^{W})$ a modeling function and $V \in V^{W}$ such that $r_{W}(V) << W$. Since X is strongly paracompact let $A = \{A_{\gamma} | \gamma \in \Gamma\}$ be a star finite open cover of X such that $A << V$ and choose $U \in_{W} V^{W}$ such that $U << A$. For each $\gamma \in \Gamma$ let $S_{\gamma} = \bigcup\{f_{U}(U) | U \in U \text{ and } U \subseteq A_{\gamma}\}$. Then $S = \{S_{\gamma} | \gamma \in \Gamma\}$ is an open cover of Y and $S << W$ by the continuity of f^{W}. If we suppose there are distinct indices $\gamma_1, \gamma_2 \in \Gamma$ with $S_{\gamma_1} \cap S_{\gamma_2} \neq \emptyset$ then we have

$\underset{\mathcal{U}}{\overset{\mathcal{W}}{f}}(U_1) \cap \underset{\mathcal{U}}{\overset{\mathcal{W}}{f}}(U_2) \neq \emptyset$ for some $U_1, U_2 \in \mathcal{U}$ with $U_1 \subseteq A_{\gamma_1}$, $U_2 \subseteq A_{\gamma_2}$. Then $U_1 \cap U_2 \neq \emptyset$ implies $A_{\gamma_1} \cap A_{\gamma_2} \neq \emptyset$. Hence, $\mathcal{S}$ is star finite, since otherwise $\mathcal{A}$ would not be star finite. If also dim $X \leq n$, then since strong paracompactness implies paracompactness, Theorem 3.1 gives dim $Y \leq n$. This completes the proof.

<u>Definition 3.9</u>: A space X is completely paracompact if for each open cover $\mathcal{V}$ there is an open cover $\mathcal{U}$ such that $\mathcal{U} << \mathcal{V}$ where $\mathcal{U} \subseteq \bigcup_{k=1}^{\infty} \mathcal{U}_k$ and each $\mathcal{U}_k$ is a star finite open cover.

Using Lemma 2.b and a result by M. Smirnov ([15], page 256), in ([16], page 1535) A. Zarelua proves:

<u>Lemma 3.b</u>: A regular completely paracompact space is paracompact.

<u>Theorem 3.5</u>: Let X be regular. If X is completely paracompact (completely paracompact with dim $X \leq n$) and Y is a model of X, then Y is completely paracompact (completely paracompact with dim $Y \leq n$).

<u>Proof</u>: If X is regular and completely paracompact let $\mathcal{W}$ be an open cover of Y. Let $f^{\mathcal{W}}: (X, \mathbb{V}^{\mathcal{W}}) \to (Y, \mathbb{V}'^{\mathcal{W}})$ be a modeling function with $\mathcal{V} \in \mathbb{V}^{\mathcal{W}}$ such that $\underset{\mathcal{W}}{r}(\mathcal{V}) << \mathcal{W}$; and let $\mathcal{U}$ be an open cover such that $\mathcal{U} << \mathcal{V}$ and $\mathcal{U} \subseteq \bigcup_{k=1}^{\infty} \mathcal{U}_k$ where $\mathcal{U}_k = \{U_{k\alpha} \mid \alpha \in \Lambda_k\}$ is a star finite open cover for

By Lemma 3.b, let $A_1 = \{A_\gamma \mid \gamma \in \Gamma_1\}$ be a locally finite open cover such that $A_1 << U_1 \wedge U$ and choose $V_1 \in V^\omega$ such that $V_1 << A_1$; for each $\gamma \in \Gamma_1$ let $S_\gamma = \bigcup\{V \in V_1 \mid V \subseteq A_\gamma\}$ and let $S_1 = \{S_\gamma \mid \gamma \in \Gamma_1\}$ so that $V_1 << S_1 << A_1 << U_1 \wedge U$. Let $A_2 = \{A_\gamma \mid \gamma \in \Gamma_2\}$ be a locally finite open cover such that $A_2 << U_2 \wedge S_1$ and choose $V_2 \in V^\omega$ such that $V_2 << A_2$; for each $\gamma \in \Gamma_2$ let $S_\gamma = \bigcup\{V \in V_2 \mid V \subseteq A_\gamma\}$ and let $S_2 = \{S_\gamma \mid \gamma \in \Gamma_2\}$ so that $V_2 << S_2 << A_2 << U_2 \wedge S_1$. Continuing in this way, for each $k \geq 2$ we have a locally finite open cover $A_k = \{A_\gamma \mid \gamma \in \Gamma_k\}$ such that $A_k << U_k \wedge S_{k-1}$, and $V_k \in V^\omega$ such that $V_k << S_k << A_k$ where $S_k = \{S_\gamma \mid \gamma \in \Gamma_k\}$ and for each $\gamma \in \Gamma_k$, $S_\gamma = \bigcup\{V \in V_k \mid V \subseteq A_\gamma\}$.

For each k, if $1 \leq i \leq k$, let $S_{ki} = \{S_{k\alpha i} \mid \alpha \in \Lambda_i\}$ where $S_{k\alpha i} = \bigcup\{\overset{\omega}{\underset{V_k}{f}}(V) \mid V \subseteq A_\gamma$ for some $\gamma \in \Gamma_k$ and $S_\gamma \subseteq U_{i\alpha}\}$ for each $\alpha \in \Lambda_i$. Consider a fixed S_{ki}. If $y \in Y$ then $y \in \overset{\omega}{\underset{V_k}{f}}(V)$ for some $V \in V_k$. Since $V_k << A_k$, $V \subseteq A_\gamma$ for some $\gamma \in \Gamma_k$ and since $S_k << U_i$, $S_\gamma \subseteq U_{i\alpha}$ for some $\alpha \in \Lambda_i$. Hence $y \in \overset{\omega}{\underset{V_k}{f}}(V) \subseteq S_{k\alpha i}$ so that S_{ki} covers Y. Suppose α_1, α_2 are distinct indices in Λ_i and $S_{k\alpha_1 i} \cap S_{k\alpha_2 i} \neq \emptyset$. Then,

$\underset{\mathcal{V}_k}{\overset{\mathcal{W}}{f}}(V_1) \cap \underset{\mathcal{V}_k}{\overset{\mathcal{W}}{f}}(V_2) \neq \emptyset$ where $V_1 \subseteq S_{\gamma_1}$ for some $\gamma_1 \in \Gamma_k$ and and $S_{\gamma_1} \subseteq U_{i\alpha_1}$, and $V_2 \subseteq S_{\gamma_2}$ for some $\gamma_2 \in \Gamma_k$ and $S_{\gamma_2} \subseteq U_{i\alpha_2}$. So $V_1 \cap V_2 \neq \emptyset$ and $V_1 \cap V_2 \subseteq S_{\gamma_1} \cap S_{\gamma_2} \subseteq U_{i\alpha_1} \cap U_{i\alpha_2} \neq \emptyset$. This means $\mathcal{S}_{ki}$ must be star finite, since otherwise $\mathcal{U}_i$ would not be star finite. So we have the countable union $\bigcup_{k=1}^{\infty} (\bigcup_{i=1}^{k} \mathcal{S}_{ki})$ where each $\mathcal{S}_{ki}$ is a star finite open cover of Y.

Let $x \in X$ and consider some $\mathcal{S}_p$. Let S_{γ_j} $1 \leq j \leq \ell$ be all the members of $\mathcal{S}_p$ containing x (this number is finite since, in particular, $\mathcal{S}_p$ is point finite). Since $\mathcal{S}_p << \mathcal{U}$, for each j let $S_{\gamma_j} \subseteq U_{q_j\alpha_j}$ where $U_{q_j\alpha_j} \in \mathcal{U} \cap \mathcal{U}_{q_j}$. Let k be an integer with $k > \max\{p, q_j | 1 \leq j \leq \ell\}$ and let $x \in S_\gamma \in \mathcal{S}_k$. Since $\mathcal{S}_k << \mathcal{S}_p$ we must have $S_\gamma \subseteq S_{\gamma_j} \subseteq U_{q_j\alpha_j}$ for some $1 \leq j \leq \ell$. Now, $1 \leq q_j \leq k$ and denote the set $S_{k\alpha_j q_j} \in \mathcal{S}_{kq_j}$ as S_x. Doing this for each $x \in X$, we have the collection
$\mathcal{S} = \{S_x | x \in X\} \subseteq \bigcup_{k=1}^{\infty} (\bigcup_{i=1}^{k} \mathcal{S}_{ki})$.

If $S_x \in \mathcal{S}$ then $S_x = S_{k\alpha i}$ for some k, some $1 \leq i \leq k$ and $\alpha \in \Lambda_i$; and $U_{i\alpha} \in \mathcal{U}$

V' $\mathcal{V}$. If $V \in \mathcal{V}_k$ - V $\subset A_\gamma$

for some $\gamma \in \Gamma_k$ where $S_\gamma \subseteq U_{1\alpha}$, then $V \subseteq V'$ so that $f\overset{\mathcal{W}}{\underset{\mathcal{V}_k}{(V)}} \subseteq f\overset{\mathcal{W}}{\underset{\mathcal{V}}{(V')}}$. Hence $S_x \subseteq f\overset{\mathcal{W}}{\underset{\mathcal{V}}{(V')}}$ so that $\mathcal{S} << r\underset{\mathcal{W}}{(\mathcal{V})} << \mathcal{W}$. For each $x \in X$, x is contained in the union of the members of the collection

$C_x = \{V \in \mathcal{V}_k \mid V \subseteq A_\gamma$ for some $\gamma \in \Gamma_k$ and $S_\gamma \subseteq U_{i\alpha}$ where $S_x = S_{k\alpha i}\}$. Choose $\mathcal{T} \in \mathcal{V}^{\mathcal{W}}$ such that $\mathcal{T} << \bigcup_{x \in X} C_x$. If $y \in Y$ then $y \in f\overset{\mathcal{W}}{\underset{\mathcal{T}}{(T)}}$ for some $T \in \mathcal{T}$, $T \subseteq V \in C_x$ for some x, and $f\overset{\mathcal{W}}{\underset{\mathcal{T}}{(T)}} \subseteq f\overset{\mathcal{W}}{\underset{\mathcal{V}_k}{(V)}}$ for some k where $f\overset{\mathcal{W}}{\underset{\mathcal{V}_k}{(V)}} \subseteq S_x$; so that $\mathcal{S}$ covers Y.

Thus, $\mathcal{S}$ is an open cover of Y such that $\mathcal{S} << \mathcal{W}$, and $\mathcal{S} \subseteq \bigcup_{k=1}^{\infty} (\bigcup_{i=1}^{k} \mathcal{S}_{ki})$; and Y is completely paracompact. If also $\dim X \leq n$, then from Lemma 3.b and Theorem 3.1, we have $\dim Y \leq n$. This completes the proof.

4. Two other important concepts of dimension are large inductive dimension (denoted Ind) and small inductive dimension (denoted ind). For a space X, $\operatorname{Ind} X \leq n$ ($\operatorname{ind} X \leq n$) if for each closed set F (for each point $x \in X$) and each open set U with $F \subseteq U$ (with $x \in U$) there is an open set V such that $F \subseteq V \subseteq U$ ($x \in V \subseteq U$) and $\operatorname{Ind} \operatorname{bd}_X(V) \leq n - 1$ ($\operatorname{ind} \operatorname{bd}_X(V) \leq n - 1$) where $\operatorname{bd}_X(V) = \overline{V} \cap \overline{X - V}$ is the boundary of V in X.

It was M. Katětov ([17]) who first proved that covering dimension and large inductive dimension are equivalent in metric spaces. As shown by A. Zarelua ([16]), covering dimension is bounded above by small inductive dimension in regular completely paracompact Hausdorff spaces. Since small inductive dimension is bounded above by large inductive dimension in any Hausdorff space, this means that all three dimensions are identical in completely paracompact metric spaces. N. B. Vedenissoff ([18]) has shown that covering dimension is bounded above by large inductive dimension in normal Hausdorff spaces.

From these remarks and Lemmas 1.b, 3.b, utilizing Theorems 3.1, 3.5 we have:

Corollary 3.5: Let X be paracompact Hausdorff and Y metrizable. If Ind X $\leq$ n and Y is a model of X then Ind Y $\leq$ n.

Corollary 3.6: Let X be regular completely paracompact Hausdorff and Y metrizable. If ind X $\leq$ n and Y is a model of X then ind Y $\leq$ n.

Since a regular finally compact Hausdorff space is strongly paracompact ([15]), Corollaries 3.5, 3.6 hold when X is regular finally compact Hausdorff.

5. Briefly let us consider a weakening of the concept of modeling distribution.

If we do not assume that for each $\mathcal{V} \in \mathbb{V}$, if $V \in \mathcal{V}$ and $V \neq \emptyset$ then $f_{\mathcal{V}}(V) \neq \emptyset$, then Definition 3.1 describes what we call a *weak non-deterministic function*, denoted $f: (X, \mathbb{V}) \to (Y, \mathbb{V}')$. Definitions 3.2, 3.3, 3.4 apply to a weak non-deterministic function; Definition 3.5 describes a weak modeling function from X to Y if the non-deterministic function is replaced by a weak non-deterministic function; and Definition 3.6 describes a weak modeling distribution from X to Y if the collection of modeling functions is replaced by a collection of weak modeling functions. If there is a weak modeling distribution from X to Y then we say *Y is a weak model of X.*

We note that Theorems 3.1 and 3.5 are valid if we consider Y to be a weak model of X rather than a model of X.

Applying the following results by M. Katětov ([17]) and J. Nagata ([19]) respectively, we obtain yet another characterization of a regular paracompact space with finite covering dimension.

Lemma 3.c: If X is a space which has a locally finite closed cover $\{F_\gamma \mid \gamma \in \Gamma\}$ such that each F_γ is metrizable (metrizable with $\dim F_\gamma \leq n$) then X is metrizable (metrizable with $\dim X \leq n$).

Theorem 3.6: A regular space X is paracompact (paracompact with $\dim X \leq n$) if and only if X is a weak model of a metric space (metric space Y with $\dim Y \leq n$).

Proof: Lemma 1.j and Theorem 3.1 give the sufficiency. To prove the necessity let Λ index the collection of metric spaces Y_α obtained as follows: If $\mathcal{U}_\alpha$ is an open cover of X, by Corollary 2.a (Corollary 2.3) let Y_α be a metric space (with dim $Y_\alpha \leq n$), let $f_\alpha: X \to Y_\alpha$ be a continuous surjection, and let $\mathcal{W}_\alpha$ be an open cover of Y_α such that $\{f_\alpha^{-1}[W] \mid W \in \mathcal{W}_\alpha\} << \mathcal{U}_\alpha$. We can assume that the collection of $\{Y_\alpha \mid \alpha \in \Lambda\}$ is disjoint, and consider $Y = \bigcup\{Y_\alpha \mid \alpha \in \Lambda\}$ to have the sum topology; that is, a subset of Y is open if and only if its intersection with Y_α is open in Y_α for each $\alpha \in \Lambda$. Since $\{Y_\alpha \mid \alpha \in \Lambda\}$ is a locally finite closed cover of Y where each Y_α is metrizable (metrizable with dim $Y_\alpha \leq n$), by Lemma 3.c, Y is metrizable (metrizable with dim $Y \leq n$).

For each $\alpha \in \Lambda$, $\mathcal{S}_\alpha = \mathcal{W}_\alpha \cup \{Y_\beta \mid \beta \neq \alpha\}$ is an open cover of Y and $\{f_\alpha^{-1}[S] \mid S \in \mathcal{S}_\alpha\} << \mathcal{U}_\alpha$; and if $\mathcal{V}^\alpha$ is the collection of open covers of Y which refine $\mathcal{S}_\alpha$ and $\mathcal{V}'^\alpha$ is the collection of inverse images under f_α of the members of $\mathcal{V}^\alpha$, then each member of $\mathcal{V}'^\alpha$ refines $\mathcal{U}_\alpha$. For each $\alpha \in \Lambda$, defining the function $r_\alpha: \mathcal{V}^\alpha \to \mathcal{V}'^\alpha$ by $r_\alpha(\mathcal{V}) = \{f_\alpha^{-1}[V] \mid V \in \mathcal{V}\}$ for each $\mathcal{V} \in \mathcal{V}^\alpha$ and the collection of functions $\{f_\mathcal{V}^\alpha: \mathcal{V} \to r_\alpha(\mathcal{V}) \mid \mathcal{V} \in \mathcal{V}^\alpha\}$

by $f^{\alpha}_{V}(V) = f^{-1}_{\alpha}[V]$ for each $V \varepsilon \mathcal{V}$, we see that X is a weak model of Y. This completes the proof.

As yet we do not know if Theorem 3.6 is valid when "weak model" is replaced by "model".

<u>Definition 3.10</u>: A space is strongly metrizable if it is regular, Hausdorff and has a base $\mathcal{B} = \bigcup_{k=1}^{\infty} \mathcal{B}_k$ where each $\mathcal{B}_k$ is a star finite open cover.

The relationship between Definition 3.9 and 3.10 is, as A. Zarelua observes in ([16]), that a strongly metrizable space is simply a completely paracompact metrizable space.

<u>Theorem 3.7</u>: A Hausdorff space X is regular and completely paracompact (regular and completely paracompact with dim X $\leq$ n) if and only if X is homeomorphic to the limit space of a full inverse limiting system of strongly metrizable spaces (strongly metrizable spaces Y_α with dim $Y_\alpha \leq n$ for each α).

<u>Proof</u>: Let X be homeomorphic to the limit space of the full inverse limiting system $\{Y_\alpha, p^\beta_\alpha \mid \alpha \varepsilon \Lambda\}$ where each Y_α is strongly metrizable (with dim $Y_\alpha \leq n$). Since the system is full, X is completely paracompact (with dim X $\leq$ n). Since a product of regular spaces is regular and since a subspace of a regular space is regular, X is regular.

To prove the necessity first note that by Lemma 3.b, X is regular paracompact (with dim $X \leq n$). Repeatedly applying Lemma 1.k and 1.e let Λ be the collection of all strong starring sequences $\alpha = \{ \mathcal{U}_k^\alpha | k \in Z\}$ of open covers of X obtained as follows:

Let $\mathcal{W}_1$ be an open cover of X (with ord $\mathcal{W}_1 \leq n + 1$) and choose an open cover $\mathcal{U}_1^\alpha$ such that $\mathcal{U}_1^\alpha <<** \mathcal{W}_1$ and $\mathcal{U}_1^\alpha \subseteq \bigcup_{i=1}^{\infty} \mathcal{U}_{1,i}^\alpha$ where $\mathcal{U}_{1,i}^\alpha$ is a star finite open cover for each i. Let $\mathcal{W}_2$ be an open cover (with ord $\mathcal{W}_2 \leq n + 1$) such that $\mathcal{W}_2 <<** \mathcal{U}_1^\alpha \wedge \mathcal{U}_{1,1}^\alpha$ and choose open cover $\mathcal{U}_2^\alpha$ such that $\mathcal{U}_2^\alpha <<** \mathcal{W}_2$ and $\mathcal{U}_2^\alpha \subseteq \bigcup_{i=1}^{\infty} \mathcal{U}_{2,i}^\alpha$ where $\mathcal{U}_{2,i}^\alpha$ is a star finite open cover for each i. Let $\mathcal{W}_3$ be an open cover (with ord $\mathcal{W}_3 \leq n + 1$) such that $\mathcal{W}_3 <<** \mathcal{U}_2^\alpha \wedge \left(\bigwedge_{\substack{1 \leq p \leq 2 \\ 1 \leq q \leq 2}} \mathcal{U}_{p,q}^\alpha \right)$

and choose open cover $\mathcal{U}_3^\alpha$ such that $\mathcal{U}_3^\alpha <<** \mathcal{W}_3$ and $\mathcal{U}_3^\alpha \subseteq \bigcup_{i=1}^{\infty} \mathcal{U}_{3,i}^\alpha$ where each $\mathcal{U}_{3,i}^\alpha$ is a star finite open cover.

Continuing inductively, we have strong starring sequences $\{\mathcal{W}_k | k \in Z\}$, $\{\mathcal{U}_k^\alpha | k \in Z\}$ of open covers of X; where for each k, $\mathcal{U}_k^\alpha <<** \mathcal{U}_{k-1}^\alpha \wedge \left(\bigwedge_{\substack{1 \leq p \leq k-1 \\ 1 \leq q \leq k-1}} \mathcal{U}_{p,q}^\alpha \right)$ where

$\mathcal{U}_k^\alpha \subseteq \bigcup_{i=1}^{\infty} \mathcal{U}_{k,i}^\alpha$ and $\mathcal{U}_{k,i}^\alpha$ is a star finite open cover for each i, and $\mathcal{U}_{k+1}^{\alpha} << \mathcal{W}_k << \mathcal{U}_k^{\alpha}$ (and ord $\mathcal{W}_k \leq n+1$). By Lemma 1.ℓ, let (X, ρ_α) be the pseudometric space generated by $\{\mathcal{U}_k^{\alpha} \mid k \in Z\}$ such that $St(x, \mathcal{U}_{k+1}^{\alpha}) \subseteq S_{\frac{1}{2^{k-1}}}^{\alpha}(x) \subseteq St(x, \mathcal{U}_k^{\alpha})$ for each $x \in X$ and k where $S_{\varepsilon}^{\alpha}(x)$ for $x \in X$ and $\varepsilon > 0$ denote the spheres in ρ_α. Let Y_α be the metric space associated with ρ_α and $*_\alpha$ be the canonical map. Since $\mathcal{U}_{k+1}^{\alpha} << \mathcal{W}_k << \mathcal{U}_k^{\alpha}$ for each k, (X, ρ_α) has the same topology as the pseudometric space generated by $\{\mathcal{W}_k \mid k \in Z\}$. So if ord $\mathcal{W}_k \leq n+1$ for each k then dim $Y_\alpha \leq n$ according to the proof of the sufficiency of Theorem 2.1.

For each k, if $1 \leq p, q \leq k-1$, let $S_{p,q}^{k} = \{(S_U^k)^{*\alpha} \mid U \in \mathcal{U}_{p,q}^{\alpha}\}$ where $S_U^k = \bigcup\{S_{\frac{1}{2^{k-1}}}^{\alpha}(x) \mid St(x, \mathcal{U}_k^{\alpha}) \subseteq U\}$.

Then we have the countable union $\mathcal{B} = \bigcup_{k=1}^{\infty} \left(\bigcup_{\substack{1 \leq p \leq k-1 \\ 1 \leq q \leq k-1}} S_{p,q}^{k} \right)$ of star finite open covers of Y_α. Let O be open in Y_α and $y^{*\alpha} \in O$ so that $y \in *_\alpha^{-1}[O]$ which is open in (X, ρ_α) and there is k such that $St(y, \mathcal{U}_{k+1}^{\alpha}) \subseteq S_{\frac{1}{2^{k-1}}}^{\alpha}(y) \subseteq *_\alpha^{-1}[O]$. Since $\mathcal{U}_{k+2}^{\alpha} <<** \mathcal{U}_{k+1}^{\alpha}$, $St(y, \mathcal{U}_{k+2}^{\alpha}) \subseteq U$ for some $U \in \mathcal{U}_{k+1}^{\alpha}$ and in particular let $U \in \mathcal{U}_{k+1,s}^{\alpha}$. Choose $j > \max\{k+1, s\}$ so that $\mathcal{U}_{j+1}^{\alpha} <<** \mathcal{U}_{k+1,s}^{\alpha}$, $St(y, \mathcal{U}_{j+1}^{\alpha}) \subseteq St(y, \mathcal{U}_{k+2}^{\alpha})$, $y \in \bigcup\{S_{?}^{\alpha}(x) \mid St(x, \mathcal{U}_{j+1}^{\alpha}) \subseteq U\}$, and $y^{*\alpha} \in (S_U^{j+1})^{*\alpha} \in S_{k+1,s}^{j+1}$

where $(S_U^{j+1})^{*\alpha} \subseteq O$. Hence, $\mathcal{B}$ is a base for Y_α and Y_α is strongly metrizable.

Similarly as in Theorem 2.1 it can be shown that Λ is directed by the order defined there, and similarly forming the inverse limiting system indexed by Λ, we have the desired result.

If $\mathcal{U}$ is an open cover of a regular completely paracompact space X (with dim $X \leq n$) as in the proof of Theorem 3.7 there is a strong starring sequence $\{\mathcal{U}_k \mid k \in Z\}$ of open covers of X such that $\mathcal{U}_1 <<* \mathcal{U}$ and the metric space Y associated with the pseudometric space generated by $\{\mathcal{U}_k \mid k \in Z\}$ is strongly metrizable (with dim $Y \leq n$). We can form an open cover $\mathcal{W}$ of Y such that $\{*^{-1}[W] \mid W \in \mathcal{W}\} << \mathcal{U}$ where * is the canonical map; and since * is continuous on X we have:

<u>Corollary 3.7</u>: A regular space X is completely paracompact (completely paracompact with dim $X \leq n$) if and only if for each open cover $\mathcal{U}$ there is a continuous surjection f from X to a strongly metrizable space Y (strongly metrizable space Y with dim $Y \leq n$) and an open cover $\mathcal{W}$ of Y such that $\{f^{-1}[W] \mid W \in \mathcal{W}\} << \mathcal{U}$.

<u>Theorem 3.8</u>: A regular space X is completely paracompact (completely paracompact with dim $X \leq n$) if and only if X is a weak model of a strongly metrizable space (strongly metrizable space Y with dim $Y \leq n$).

Proof: Since a strongly metrizable space is a completely paracompact metric space, Theorem 3.5 gives the sufficiency. To prove the necessity, apply Corollary 3.7 and just as in Theorem 3.6, form the metric space $Y = \bigcup\{Y_\alpha \mid \alpha \in \Lambda\}$ to show that X is a weak model of Y.

We need only observe that if each Y_α is strongly metrizable then so is Y. If for each $\alpha \in \Lambda$, $\mathcal{B}_\alpha = \bigcup_{k=1}^{\infty} \mathcal{B}_{k\alpha}$ is a base for Y_α where each $\mathcal{B}_{k\alpha}$ is a star finite open cover of Y_α, simply let $\mathcal{B} = \bigcup_{k=1}^{\infty} \mathcal{B}_k$ where $\mathcal{B}_k = \bigcup_{\alpha \in \Lambda} \mathcal{B}_{k\alpha}$ for each k. Then $\mathcal{B}$ is a base for Y and since the spaces Y_α are disjoint, each $\mathcal{B}_k$ is a star finite open cover of Y. This completes the proof.

Compact Spaces and the Modeling Distribution

Our purpose here is to assist further investigation of spaces under a modeling distribution. We restrict ourselves to compact spaces and link the concept of modeling distribution and the notion of inverse limiting system. This is expressed precisely in Theorem 4.2.

1. For our purposes here it is convenient to generalize the concept of modeling function. Let X, Y be spaces.

Definition 4.1: A crude non-deterministic function is a pair of collections $\mathcal{V}, \mathcal{V}'$ of almost open covers of X, Y respectively, with a function $r: \mathcal{V} \to \mathcal{V}'$ and a collection of functions $\{f_V: V \to r(V) \mid V \in \mathcal{V}\}$

As in the case of a non-deterministic function; it is assumed that for each $V \in \mathcal{V}$, if $V \in V$ and $V^{o} \neq \emptyset$ then $f_V(V) \neq \emptyset$; a crude non-deterministic function as defined is denoted $f: (X, \mathcal{V}) \to (Y, \mathcal{V}')$ or simply as f; and Definitions 3.2, 3.3, 3.4 apply to a crude non-deterministic function.

Definition 4.2: A crude modeling function from X to Y is a continuous, surjective, crude non-deterministic function $f\colon (X, \mathcal{V}) \to (Y, \mathcal{V}')$ where $\mathcal{V}$ is cofinal in X and for each $\mathcal{V} \in \mathcal{V}$, if $V_1, V_2 \in \mathcal{V}$ with $f_{\mathcal{V}}(V_1) \cap f_{\mathcal{V}}(V_2) \neq \emptyset$ then $St(V_1, \mathcal{V}) \cap St(V_2, \mathcal{V}) \neq \emptyset$.

Lemma 4.1: Let X be regular and $f\colon (X, \mathcal{V}) \to (Y, \mathcal{V}')$ a continuous crude non-deterministic function where $\mathcal{V}$ is cofinal in X. If $V_1 \in \mathcal{V}_1 \in \mathcal{V}$ and $V_2 \in \mathcal{V}_2 \in \mathcal{V}$ where V_1, V_2 are nbhds. of a common point of X, then $f_{\mathcal{V}_1}(V_1) \cap f_{\mathcal{V}_2}(V_2) \neq \emptyset$.

Proof: Let V_1, V_2 be nbhds. of a point $x \in X$ so that there are open sets U_1, U_2 with $x \in U_1 \subseteq V_1$, $x \in U_2 \subseteq V_2$. Since X is regular let V be open with $x \in V \subseteq \bar{V} \subseteq U_1 \cap U_2$. Then $\{U_1 \cap U_2, X - \bar{V}\}$ is an open cover of X and since $\mathcal{V}$ is cofinal in X choose $\mathcal{U} \in \mathcal{V}$ such that $\mathcal{U} \ll \{U_1 \cap U_2, X - \bar{V}\}$. If $x \in U \in \mathcal{U}$ then $U \subseteq U_1 \cap U_2 \subseteq V_1 \cap V_2$ and by the continuity of f, $f_{\mathcal{U}}(U) \subseteq f_{\mathcal{V}_1}(V_1) \cap f_{\mathcal{V}_2}(V_2)$. Since $U \neq \emptyset$ implies $f_{\mathcal{U}}(U) \neq \emptyset$, we have $f_{\mathcal{V}_1}(V_1) \cap f_{\mathcal{V}_2}(V_2) \neq \emptyset$. This completes the proof.

The following is an adaptation of a proof by A. Jansen in ([20], page 8) where the existence of a continuous function under conditions similar to the ones is proved.

Lemma 4.a: Let X, Y be regular and paracompact and f: $(X, \mathcal{V}) \to (Y, \mathcal{V}')$ a cofinal, crude non-deterministic function where $\mathcal{V}$ is cofinal in X. If Y is Hausdorff then there is a continuous function from X to Y realized by f in a manner made precise below.

Proof: Claim: For each $x \in X$ there is $y \in Y$ such that for each nbhd. N_y of y and $V \in \mathcal{V}$ there is a nbhd. V of x with $V \in V$ such that $f_V(V) \cap N_y \neq \emptyset$.

Suppose that for some fixed point $x \in X$, for each $y \in Y$ there is a nbhd. N_y of y and $V_y \in \mathcal{V}$ such that for each nbhd. V of x with $V \in V_y$, we have $f_{V_y}(V) \cap N_y = \emptyset$. Choose

$U \in \mathcal{V}$ such that $r(U) << \{N_y | y \in Y\}$, and choose a nbhd. U of x with $U \in U$ and $f_U(U) \subseteq N_{y'}$ for some y'. If $V \in \mathcal{V}$, choose a nbhd. V of x with $V \in V$. Then by Lemma 4.1, $f_U(U) \cap f_V(V) \neq \emptyset$, so that $f_V(V) \cap N_{y'} \neq \emptyset$, and this contradicts the choice of $N_{y'}$. Hence the Claim holds.

Suppose that for some fixed $x \in X$ there are two distinct points y_1, y_2 of Y given by the Claim. Since Y is Hausdorff let N_1, N_2 be disjoint open nbhds. of y_1, y_2 respectively. By Lemma 1.b, Y is regular and let U_1, U_2 be open with $y_1 \in U_1 \subseteq \overline{U}_1 \subseteq N_1$, $y_2 \in U_2 \subseteq \overline{U}_2 \subseteq N_2$. Choose $V \in \mathcal{V}$ such that $r(V) << \{N_1, N_2, Y - \overline{U_1 \cup U_2}\}$ and choose nbhds. V_1, V_2 of x with $V_1, V_2 \in V$ such that $f_V(V_1) \cap U_1 \neq \emptyset$

and $\underset{\mathcal{V}}{f}(V_2) \cap U_2 \neq \emptyset$. Then, $\underset{\mathcal{V}}{f}(V_1) \subseteq N_1$ and $\underset{\mathcal{V}}{f}(V_2) \subseteq N_2$ so that $\underset{\mathcal{V}}{f}(V_1) \cap \underset{\mathcal{V}}{f}(V_2) = \emptyset$, and this contradicts Lemma 4.1.

Define $g: X \to Y$ by $g(x) = y$ where for each nbhd. N_y of y and $\mathcal{V} \in \mathbb{V}$ there is a nbhd. V of x with $V \in \mathcal{V}$ such that $\underset{\mathcal{V}}{f}(V) \cap N_y \neq \emptyset$.

Let N_y be an open nbhd. of $g(x) = y$. Choose open sets U_1, U_2, U_3 such that $y \in U_1 \subseteq \overline{U}_1 \subseteq U_2 \subseteq \overline{U}_2 \subseteq U_3 \subseteq \overline{U}_3 \subseteq N_y$. Then $\mathcal{U} = \{U_2, U_3 - \overline{U}_1, N_y - \overline{U}_2, Y - \overline{U}_3\}$ is an open cover of Y and choose $\mathcal{V} \in \mathbb{V}$ such that $r(\mathcal{V}) << \mathcal{U}$. Since $g(x) = y$, choose a nbhd. V of x with $V \in \mathcal{V}$ such that $\underset{\mathcal{V}}{f}(V) \cap U_1 \neq \emptyset$. Then we must have $\underset{\mathcal{V}}{f}(V) \subseteq U_2$, and let M be an open set with $x \in M \subseteq V$. Let $y' \in g[M]$ so that $y' = g(x')$ for some $x' \in M$. Let N be any open nbhd. of y'. Since $g(x') = y'$, choose a nbhd. V' of x' with $V' \in \mathcal{V}$ such that $\underset{\mathcal{V}}{f}(V') \cap N \neq \emptyset$. Since V, V' are both nbhds. of x', by Lemma 4.1, $\underset{\mathcal{V}}{f}(V) \cap \underset{\mathcal{V}}{f}(V') \neq \emptyset$, and since $\underset{\mathcal{V}}{f}(V) \subseteq U_2$, we have $\underset{\mathcal{V}}{f}(V') \cap U_2 \neq \emptyset$. So we must have $\underset{\mathcal{V}}{f}(V') \subseteq U_3 - \overline{U}_1$ or $\underset{\mathcal{V}}{f}(V') \subseteq U_2$, and in either case, $\underset{\mathcal{V}}{f}(V') \subseteq U_3$. Thus, $N \cap U_3 \neq \emptyset$ so that $y' \in \overline{U}_3$ and $y' \in N_y$. Thus, M is an open nbhd. of x with $g[M] \subseteq N_y$ and g is continuous. This completes the proof.

Lemma 4.2: Let X, Y be regular paracompact and f: $(X, \mathcal{V}) \to (Y, \mathcal{V}')$ a cofinal, crude modeling function. If Y is Hausdorff (X and Y are Hausdorff) then there is a continuous surjection (a homeomorphism) from X to Y.

Proof: Let $g: X \to Y$ be the function defined in Lemma 4.a. Consider a fixed point $y \in Y$ and for each $V \in \mathcal{V}$ choose $V_V \in V$ with $y \in f_V(V_V)$. Consider any pair V, U with $V, U \in \mathcal{V}$. By Lemma 1.k let $S \in \mathcal{V}$ such that $S <<^{**} U \wedge V$. Then $St(V_S, S) \subseteq U \cap V$ for some $U \in U$ and $V \in V$. Hence $V_S \subseteq U \cap V$ so that $y \in f_S(V_S) \subseteq f_V(V)$ and $y \in f_S(V_S) \subseteq f_U(U)$. Thus, $f_V(V) \cap f_V(V_V) \neq \emptyset$, $f_U(U) \cap f_U(V_U) \neq \emptyset$ so that $St(V, V) \cap St(V_V, V) \neq \emptyset$ and $St(U, U) \cap St(V_U, U) \neq \emptyset$. But then $V \cap St^2 (V_V, V) \neq \emptyset$ and $U \cap St^2 (V_U, U) \neq \emptyset$ so that $St(V_S, S) \subseteq U \cap V \subseteq St^3(V_V, V) \cap St^3(V_U, U)$. Since $f_S(V_S) \neq \emptyset$ implies that $V_S \neq \emptyset$, we have $St^3(V_V, V) \cap St^3(V_U, U) \neq \emptyset$. Thus, the collection $F = \{F \subseteq X \mid St^3(V_V, V) \subseteq F$ for some $V \in \mathcal{V}\}$ is a filter on X.

Suppose that for each $x \in X$ there is a nbhd. O_x of x such that $O_x \notin F$. Then $\{O_x \mid x \in X\}$ covers X and since X is regular and paracompact, by Lemma 1.k, choose $V \in \mathcal{V}$ such that $\{St^3(V, V) \mid V \in V\} << \{O_x \mid x \in X\}$. But then $St^3(V_{\mathcal{V}}, \mathcal{V}) \subseteq O_{x'}$ for some $x' \in X$ and $O_{x'} \in F$. This means there is a point of X, call it x, such that each nbhd. of x is in F.

Let N_y be any nbhd. of y and choose $\mathcal{U} \in \mathbb{V}$ such that $r(\mathcal{U}) <<^{**} \{N_y, Y - \{y\}\}$ so that $St(f_{\mathcal{U}}(V_{\mathcal{U}}), r(\mathcal{U})) \subseteq N_y$. By Lemma 1.k, choose $\mathcal{V} \in \mathbb{V}$ such that $\{St^2(V, \mathcal{V}) \mid V \in \mathcal{V}\} << \mathcal{U}$ and choose a nbhd. V' of x with $V' \in \mathcal{V}$. Since $V' \in F$, $V' \cap St(V_{\mathcal{V}}, \mathcal{V}) \neq \emptyset$ and so $V' \subseteq St^2(V_{\mathcal{V}}, \mathcal{V}) \subseteq U$ for some $U \in \mathcal{U}$ where U is also a nbhd. of x. Since $V_{\mathcal{V}} \subseteq U$, $y \in f_{\mathcal{V}}(V_{\mathcal{V}}) \subseteq f_{\mathcal{U}}(U)$ so that $f_{\mathcal{U}}(U) \cap f_{\mathcal{U}}(V_{\mathcal{U}}) \neq \emptyset$ and $f_{\mathcal{U}}(U) \subseteq St(f_{\mathcal{U}}(V_{\mathcal{U}}), r(\mathcal{U})) \subseteq N_y$. Now consider any $\mathcal{V} \in \mathbb{V}$ and choose a nbhd. V of x with $V \in \mathcal{V}$. By Lemma 4.1, $f_{\mathcal{V}}(V) \cap f_{\mathcal{U}}(U) \neq \emptyset$ so that $f_{\mathcal{V}}(V) \cap N_y \neq \emptyset$. So, for any nbhd. N_y of y and any $\mathcal{V} \in \mathbb{V}$ there is a nbhd. V of x with $V \in \mathcal{V}$ such that $f_{\mathcal{V}}(V) \cap N_y \neq \emptyset$. Thus, $g(x) = y$ and g is surjective.

Now suppose X is also Hausdorff. Let x_1, x_2 be distinct points of X and suppose $g(x_1) = g(x_2) = y$. Let O_1, O_2 be disjoint open nbhds. of x_1, x_2 respectively, so that $\mathcal{O} = \{O_1, O_2, X - \{x_1, x_2\}\}$ is an open cover of X and by Lemma 1.k, choose $\mathcal{V} \in \mathbb{V}$ such that

$\{St^3(V, \mathcal{V}) \mid V \in \mathcal{V}\} << \mathcal{O}$. Choose $V \in \mathcal{V}$ such that $f_{\mathcal{V}}(V)$ is an nbhd. of y. Since $g(x_1) = g(x_2) = y$ there is a nbhd. V_1 of x_1 with $V_1 \in \mathcal{V}$ such that $f_{\mathcal{V}}(V_1) \cap f_{\mathcal{V}}(V) \neq \emptyset$ and there is a nbhd. V_2 of x_2 with $V_2 \in \mathcal{V}$ such that $f_{\mathcal{V}}(V_2) \cap f_{\mathcal{V}}(V) \neq \emptyset$. Then $St(V_1, \mathcal{V}) \cap St(V, \mathcal{V}) \neq \emptyset$ and $St(V_2, \mathcal{V}) \cap St(V, \mathcal{V}) \neq \emptyset$ so that $V_1 \cup V_2 \subseteq St^3(V, \mathcal{V})$. But this contradicts that $St^3(V, \mathcal{V})$ is contained in some member of $\mathcal{O}$. Hence $g(x_1) \neq g(x_2)$ and g is injective.

So we have h: $Y \to X$, the inverse of g defined $h(y) = x$ where $g(x) = y$.

<u>Claim</u>: If $h(y) = x$, then for each nbhd. N_x of x and $r(\mathcal{V})$ for $\mathcal{V} \in \mathbb{V}$, there is a nbhd. $f_{\mathcal{V}}(V)$ of y for some $V \in \mathcal{V}$ such that $V \cap N_x \neq \emptyset$.

If $h(y) = x$, let N_x be a nbhd. of x, consider some $r(\mathcal{V})$ for $\mathcal{V} \in \mathbb{V}$, choose $\mathcal{U} \in \mathbb{V}$ such that $\{St^3(U, \mathcal{U}) \mid U \in \mathcal{U}\} << \{N_x, X - \{x\}\} \wedge \mathcal{V}$, and choose $U \in \mathcal{U}$ such that $f_{\mathcal{U}}(U)$ is a nbhd. of y. Since $g(x) = y$ there is is a nbhd. U' of x with $U' \in \mathcal{U}$ such that $f_{\mathcal{U}}(U') \cap f_{\mathcal{U}}(U) \neq \emptyset$. Hence, $St(U', \mathcal{U}) \cap St(U, \mathcal{U}) \neq \emptyset$ and $U \subseteq St^3(U', \mathcal{U}) \subseteq N_x$. Also, $St^3(U', \mathcal{U}) \subseteq V$ for some $V \in \mathcal{V}$ so that $f_{\mathcal{U}}(U) \subseteq f_{\mathcal{V}}(V)$

which means $f_{\mathcal{V}}(V)$ is also a nbhd. of y where $V \cap N_x \neq \emptyset$ and this proves the Claim.

Now let N_x be an open nbhd. of $h(y) = x$, choose open sets U_1, U_2, U_3 such that

$x \in U_1 \subseteq \bar{U}_1 \subseteq U_2 \subseteq \bar{U}_2 \subseteq U_3 \subseteq \bar{U}_3 \subseteq N_x$ so that

$\mathcal{U} = \{U_2, U_3 - \bar{U}_1, N_x - \bar{U}_2, X - \bar{U}_3\}$ is an open cover of X, and choose $\mathcal{V} \in \mathbb{V}$ such that $\mathcal{V} <<** \mathcal{U}$. Since $h(y) = x$, by the Claim there is a nbhd. $f_{\mathcal{V}}(V)$ of y for some $V \in \mathcal{V}$ such that $V \cap U_1 \neq \emptyset$. Then we must have $St(V, \mathcal{V}) \subseteq U_2$ and let M be an open set with $y \in M \subseteq f_{\mathcal{V}}(V)$. Let $x' \in h[M]$ so that $x' = h(y')$ for some $y' \in M$. Let N be any open nbhd. of x'. Since $h(y') = x'$, by the Claim let $f_{\mathcal{V}}(V')$ be a nbhd. of y' for some $V' \in \mathcal{V}$ such that $V' \cap N \neq \emptyset$. Since $f_{\mathcal{V}}(V) \cap f_{\mathcal{V}}(V') \neq \emptyset$ we have $St(V, \mathcal{V}) \cap St(V', \mathcal{V}) \neq \emptyset$ and since $St(V, \mathcal{V}) \subseteq U_2$ we have $U_2 \cap St(V', \mathcal{V}) \neq \emptyset$. So we must have $St(V', \mathcal{V}) \subseteq U_3$. Thus, $N \cap U_3 \neq \emptyset$ so that $x' \in \bar{U}_3$ and let $x' \in N_x$. Thus, M is an open nbhd. of y with $h[M] \subseteq N_x$ and h is continuous.

So g is a homeomorphism and this completes the proof.

Corollary 4.1: If X, Y are paracompact Hausdorff and there is a cofinal modeling function from X to Y, then X and Y are homeomorphic.

Definition 4.3: Let X, Y be spaces. An injective modeling distribution from X to Y is a modeling distribution from X to Y where each member of the modeling distribution is injective. If there is an injective modeling distribution from X to Y then we say Y is an injective model of X.

In general Corollary 4.1 does not remain valid (even when X, Y are compact Hausdorff) if an injective modeling distribution is substituted for the cofinal modeling function. For example in ([21]) R. G. Lintz shows that a compact subset of the product of two generalized arcs is an injective model of a closed subset of the product of two unit intervals

2. Now we shall return to the inverse limiting systems constructed in II.

Lemma 4.b: Let $\{X_\sigma, p_{\sigma_1}^{\sigma_2} \mid \sigma \in \Gamma\}$ and $\{Y_\omega, q_{\omega_1}^{\omega_2} \mid \omega \in \Omega\}$ be inverse limiting systems where there is a function $Q\colon \Gamma \to \Omega$ such that $Q[\Gamma]$ is cofinal in Ω and if $\sigma_1, \sigma_2 \in \Gamma$ with $\sigma_1 < \sigma_2$ then $Q(\sigma_1) < Q(\sigma_2)$, and for each $\sigma \in \Gamma$ there is a homeomorphism $f_\sigma\colon Y_{Q(\sigma)} \to X_\sigma$ such that if $\sigma_1 < \sigma_2$ then $p_{\sigma_1}^{\sigma_2} \circ f_{\sigma_2} = f_{\sigma_1} \circ q_{Q(\sigma_1)}^{Q(\sigma_2)}$. Then $\{Y_\omega, q_{\omega_1}^{\omega_2} \mid \omega \in Q[\Gamma]\}$ is an inverse limiting system and inv lim $(Y_\omega, q_{\omega_1}^{\omega_2} \mid \omega \in Q[\Gamma])$,

im $(Y\ \ q_{\omega_1}^{\omega_2} \mid \omega \in \Omega)$

inv lim$(X_\sigma, p_{\sigma_1}^{\sigma_2} | \sigma \in \Gamma)$ are all homeomorphic.

([3], page 430 and 431).

<u>Lemma 4.3</u>: If $\{X_\sigma, p_{\sigma_1}^{\sigma_2} | \sigma \in \Gamma\}$ and $\{Y_\omega, q_{\omega_1}^{\omega_2} | \omega \in \Omega\}$ are inverse limiting systems then there is a set Λ and there are inverse limiting systems $\{X_\alpha, p_\alpha^\beta | \alpha \in \Lambda\}$, $\{Y_\alpha, q_\alpha^\beta | \alpha \in \Lambda\}$ such that inv lim $(X_\sigma, p_{\sigma_1}^{\sigma_2} | \sigma \in \Gamma)$ is homeomorphic to inv lim $(X_\alpha, p_\alpha^\beta | \alpha \in \Lambda)$ and inv lim $(Y_\omega, q_{\omega_1}^{\omega_2} | \omega \in \Omega)$ is homeomorphic to inv lim $(Y_\alpha, q_\alpha^\beta | \alpha \in \Lambda)$.

<u>Proof</u>: On $\Gamma \times \Omega = \Lambda$ define order by calling $(\sigma_1, \omega_1) < (\sigma_2, \omega_2)$ if $\sigma_1 < \sigma_2$ and $\omega_1 < \omega_2$. Since Γ, Ω are directed sets, Λ is also a directed set.

For each $(\sigma, \omega) \in \Lambda$ define $X_{(\sigma, \omega)} = X_\sigma$ and if $(\sigma_1, \omega_1) < (\sigma_2, \omega_2)$ define the continuous function $p_{(\sigma_1, \omega_1)}^{(\sigma_2, \omega_2)}: X_{(\sigma_2, \omega_2)} \to X_{(\sigma_1, \omega_1)}$ by $p(x)_{(\sigma_1, \omega_1)}^{(\sigma_2, \omega_2)} = p(x)_{\sigma_1}^{\sigma_2}$ for each $x \in X_{\sigma_2}$. If $(\sigma_1, \omega_1) < (\sigma_2, \omega_2) < (\sigma_3, \omega_3)$ then we have $p_{(\sigma_1, \omega_1)}^{(\sigma_2, \omega_2)} \circ p_{(\sigma_2, \omega_2)}^{(\sigma_3, \omega_3)} = p_{\sigma_1}^{\sigma_2} \circ p_{\sigma_2}^{\sigma_3} = p_{\sigma_1}^{\sigma_3} = p_{(\sigma_1, \omega_1)}^{(\sigma_3, \omega_3)}$ and $\{X_{(\sigma, \omega)}, p_{(\sigma_1, \omega_1)}^{(\sigma_2, \omega_2)} | (\sigma, \omega) \in \Lambda\}$ is an inverse limiting system.

Now, $Q: \Lambda \to \Gamma$ by $Q(\sigma, \omega) = \sigma$ defines a surjective function such that if $(\sigma_1, \omega_1) < (\sigma_2, \omega_2)$ then $Q(\sigma_1, \omega_1) < Q(\sigma_2, \omega_2)$ and for each $(\sigma, \omega) \in \Lambda$ there is a homeomorphism $f_{(\sigma, \omega)}: X_\sigma \to X_{(\sigma, \omega)}$, namely the identity. If $(\sigma_1, \omega_1) < (\sigma_2, \omega_2)$ then

$$p^{(\sigma_2, \omega_2)}_{(\sigma_1, \omega_1)} \circ f(x)_{(\sigma_2, \omega_2)} = p(x)^{(\sigma_2, \omega_2)}_{(\sigma_1, \omega_1)} = p(x)^{\sigma_2}_{\sigma_1} = f_{(\sigma_1, \omega_1)} \circ p(x)^{\sigma_2}_{\sigma_1}$$

for each $x \in X_{\sigma_2}$. Hence, by Lemma 4.b, inv lim $(X_\sigma, p^{\sigma_2}_{\sigma_1} | \sigma \in \Gamma)$ is homeomorphic to inv lim $(X_{(\sigma, \omega)}, p^{(\sigma_2, \omega_2)}_{(\sigma_1, \omega_1)} | (\sigma, \omega) \in \Lambda)$.

Similarly , for each $(\sigma, \omega) \in \Lambda$ define $Y_{(\sigma, \omega)} = Y_\omega$ and if $(\sigma_1, \omega_1) < (\sigma_2, \omega_2)$ define the continuous function $q^{(\sigma_2, \omega_2)}_{(\sigma_1, \omega_1)}: Y_{(\sigma_2, \omega_2)} \to Y_{(\sigma_1, \omega_1)}$ by $q(y)^{(\sigma_2, \omega_2)}_{(\sigma_1, \omega_1)} = q(y)^{\omega_2}_{\omega_1}$ for each $y \in Y_{\omega_2}$, to obtain, using Lemma 4.b, an inverse limiting system $\{Y_{(\sigma, \omega)}, q^{(\sigma_2, \omega_2)}_{(\sigma_1, \omega_1)} | (\sigma, \omega) \in \Lambda\}$ with inv lim $(Y_\omega, q^{\omega_2}_{\omega_1} | \omega \in \Omega)$ homeomorphic to inv lim $(Y_{(\sigma, \omega)}, q^{(\sigma_2, \omega_2)}_{(\sigma_1, \omega_1)} | (\sigma, \omega) \in \Lambda)$.

For brevity, denote the members of Λ by α, β. This completes the proof.

Lemma 4.4: In Lemma 4.b, if $\{Y_\omega, q_{\omega_1}^{\omega_2} | \omega \in \Omega\}$ is a full inverse limiting system, then $\{X_\sigma, p_{\sigma_1}^{\sigma_2} | \sigma \in \Gamma\}$ is a full inverse limiting system.

Proof: Let f be the homeomorphism from inv lim $(Y_\omega, q_{\omega_1}^{\omega_2} | \omega \in Q[\Gamma])$ to inv lim $(X_\sigma, p_{\sigma_1}^{\sigma_2} | \sigma \in \Gamma)$ defined by $f(y_{Q(\sigma)} | Q(\sigma) \in Q[\Gamma]) = (f_\sigma(y_{Q(\sigma)}) | \sigma \in \Gamma)$ and let g be the homeomorphism from inv lim $(Y_\omega, q_{\omega_1}^{\omega_2} | \omega \in \Omega)$ to inv lim $(Y_\omega, q_{\omega_1}^{\omega_2} | \omega \in Q[\Gamma])$ defined by $g(y_\omega | \omega \in \Omega) = (y_\omega | \omega \in Q[\Gamma])$.

Let $\mathcal{U}$ be an open cover of inv lim $(X_\sigma, p_{\sigma_1}^{\sigma_2} | \sigma \in \Gamma)$ so that $\{g^{-1} f^{-1}[U] | U \in \mathcal{U}\}$ is an open cover of inv lim $(Y_\omega, q_{\omega_1}^{\omega_2} | \omega \in \Omega)$. Choose $\omega_0 \in \Omega$ and an open cover $\mathcal{V}_{\omega_0}$ of Y_{ω_0} such that $\{\Pi_{\omega_0}^{-1}[V] | V \in \mathcal{V}_{\omega_0}\} << \{g^{-1} f^{-1}[U] | U \in \mathcal{U}\}$ where Π_{ω_0} is the projection restricted to inv lim $(Y_\omega, q_{\omega_1}^{\omega_2} | \omega \in \Omega)$. Since $Q[\Gamma]$ is cofinal in Ω, choose $\sigma_0 \in \Gamma$ with $Q(\sigma_0) > \omega_0$. Then $\{f_{\sigma_0}[(q_{\omega_0}^{Q(\sigma_0)})^{-1}[V]] | V \in \mathcal{V}_{\omega_0}\}$ is an open cover of X_{σ_0} and let Π_{σ_0} be the projection restricted to inv lim $(X_\sigma, p_{\sigma_1}^{\sigma_2} | \sigma \in \Gamma)$.

If $V \in \mathcal{V}_{\omega_0}$ choose $U \in \mathcal{U}$ such that $\Pi_{\omega_0}^{-1}[V] \subseteq g^{-1} f^{-1}[U]$.

If $f(y_{Q(\sigma)} \mid Q(\sigma) \in Q[\Gamma]) \in \Pi_{\sigma_0}^{-1} f_{\sigma_0}[(q_{\omega_0}^{Q(\sigma_0)})^{-1}[V]]$, then

$f_{\sigma_0}(y_{Q(\sigma_0)}) \in f_{\sigma_0}[(q_{\omega_0}^{Q(\sigma_0)})^{-1}[V]]$ and $y_{Q(\sigma_0)} \in (q_{\omega_0}^{Q(\sigma_0)})^{-1}[V]$.

Let $(y_\omega \mid \omega \in \Omega)$ be a point in inv lim $(Y_\omega, q_{\omega_1}^{\omega_2} \mid \omega \in \Omega)$ with

$g(y_\omega \mid \omega \in \Omega) = (y_\omega \mid \omega \in Q[\Gamma]) = (y_{Q(\sigma)} \mid Q(\sigma) \in Q[\Gamma])$ so

that $q_{\omega_0}^{Q(\sigma_0)}(y_{Q(\sigma_0)}) = y_{\omega_0} \in V$. Hence,

$(y_\omega \mid \omega \in \Omega) \in g^{-1} f^{-1}[U]$, $(y_{Q(\sigma)} \mid Q(\sigma) \in Q[\Gamma]) \in f^{-1}[U]$, and

$(f_\sigma(y_{Q(\sigma)}) \mid \sigma \in \Gamma) \in U$. Thus, $\Pi_{\sigma_0}^{-1} f_{\sigma_0}[(q_{\omega_0}^{Q(\sigma_0)})^{-1}[V]] \subseteq U$;

$\{\Pi_{\sigma_0}^{-1} f_{\sigma_0}[(q_{\omega_0}^{Q(\sigma_0)})^{-1}[V]] \mid V \in \mathcal{V}_{\omega_0}\} << \mathcal{U}$; and $\{X_\sigma, p_{\sigma_1}^{\sigma_2} \mid \sigma \in \Gamma\}$

is a full inverse limiting system. This completes the proof.

From Lemma 4.b and 4.4 we conclude that in Lemma 4.3, if $\{X_\sigma, p_{\sigma_1}^{\sigma_2} \mid \sigma \in \Gamma\}$ and $\{Y_\omega, q_{\omega_1}^{\omega_2} \mid \omega \in \Omega\}$ are full inverse limiting systems then $\{X_\alpha, p_\alpha^\beta \mid \alpha \in \Lambda\}$ and $\{Y_\alpha, q_\alpha^\beta \mid \alpha \in \Lambda\}$ are full inverse limiting systems.

Suppose X is a paracompact Hausdorff space with $\dim X = 0$ and we have everything as in the proof of Corollary 2.4 (with the exception that it is assumed now that each member of each $\mathcal{U}^\alpha$ is not empty). Let $\Lambda' = \{\mathcal{U}^\alpha \mid \alpha \in \Lambda\}$, define order between two members $\mathcal{U}^\alpha$, $\mathcal{U}^\beta$ of Λ' by $\mathcal{U}^\alpha < \mathcal{U}^\beta$ if and only if $\mathcal{U}^\beta << \mathcal{U}^\alpha$, and if $\mathcal{U}^\alpha < \mathcal{U}^\beta$ let $p_{\mathcal{U}^\alpha}^{\mathcal{U}^\beta} = p_\alpha^\beta$.

From the proof of Corollary 2.4 we have that $Y_\alpha = \mathcal{U}^\alpha$ for each α and that $\{\mathcal{U}^\alpha, p_{\mathcal{U}^\alpha}^{\mathcal{U}^\beta} \mid \mathcal{U}^\alpha \in \Lambda'\}$ is a surjective full inverse limiting system of discrete spaces whose limit space is homeomorphic to X. In case X is compact (regular and finally compact) Hausdorff with $\dim X = 0$ then by Lemma 2.c X is still paracompact and since a continuous surjection preserves compactness (final compactness) we can assume that each $\mathcal{U}^\alpha$ is a finite (countable) discrete space.

<u>Theorem 4.1</u>: Let X, Y be paracompact (compact; regular and finally compact) Hausdorff spaces with $\dim X = 0$. If Y is a model of X then X is homeomorphic to the limit space of a full inverse limiting system $\{X_\alpha, p_\alpha^\beta \mid \alpha \in \Lambda\}$ of discrete (finite discrete; countable discrete) spaces and Y is homeomorphic to the limit space of a full inverse limiting system $\{Y_\alpha, q_\alpha^\beta \mid \alpha \in \Lambda\}$ of discrete (finite discrete; countable discrete) spaces where Y_α is homeomorphic to X_α for $\alpha \in \Lambda$.

Proof: By Lemmas 1.b, 2.c, in each case X is regular and paracompact so that by Theorem 3.1 dim Y = 0. According to the above remarks and Lemma 4.3, X is homeomorphic to the limit space of a full inverse limiting system $\{X_\alpha, p_\alpha^\beta \mid \alpha \in \Lambda'\}$ of discrete (finite discrete; countable discrete) spaces and Y is homeomorphic to the limit space of a full inverse limiting system $\{Y_\alpha, q_\alpha^\beta \mid \alpha \in \Lambda'\}$ of discrete (finite discrete; countable discrete) spaces; with Λ' being all pairs $\alpha = (\mathcal{V}, \mathcal{W})$ where $\mathcal{V}$ is a disjoint open cover of X and $X_\alpha = \mathcal{V}$, and where $\mathcal{W}$ is a disjoint open cover of Y and $Y_\alpha = \mathcal{W}$. And if $\alpha = (\mathcal{V}, \mathcal{W})$, $\beta = (\mathcal{V}', \mathcal{W}') \in \Lambda'$ then $\alpha < \beta$ means $\mathcal{V}' << \mathcal{V}$ and $\mathcal{W}' << \mathcal{W}$. Consider a fixed pair $\alpha = (\mathcal{V}, \mathcal{W}) \in \Lambda'$.

Let $f^{\mathcal{W}}$: $(X, \mathbb{V}^{\mathcal{W}}) \to (Y, \mathbb{V}'^{\mathcal{W}})$ be a modeling function with $\mathcal{O} \in \mathbb{V}^{\mathcal{W}}$ such that $\mathcal{O} << \mathcal{V}$ and $r_{\mathcal{W}}(\mathcal{O}) << \mathcal{W}$. Since X is paracompact (compact; regular and finally compact) Hausdorff with dim X = 0, by Lemma 1.e let $\mathcal{A} = \{A_\gamma \mid \gamma \in \Gamma\}$ be a disjoint (finite disjoint; countable disjoint) open cover of X of non empty sets such that $\mathcal{A} << \mathcal{O}$ and choose $\mathcal{U} \in \mathbb{V}^{\mathcal{W}}$ such that $\mathcal{U} << \mathcal{A}$. For each $\gamma \in \Gamma$ let $S_\gamma = \bigcup\{U \in \mathcal{U} \mid U \subseteq A_\gamma\}$ and $T_\gamma = \bigcup\{\overset{\mathcal{W}}{f}_{\mathcal{U}}(U) \mid U \in \mathcal{U} \text{ and } U \subseteq A_\gamma\}$. Then $\mathcal{S} = \{S_\gamma \mid \gamma \in \Gamma\}$ is a disjoint open cover of X and $\mathcal{S} << \mathcal{V}$. If we suppose there are distinct indices $\gamma_1, \gamma_2 \in \Gamma$ with $T_{\gamma_1} \cap T_{\gamma_2} \neq \emptyset$ then we have $\underset{\mathcal{U}}{\overset{\mathcal{W}}{f}}(U_1) \cap \underset{\mathcal{U}}{\overset{\mathcal{W}}{f}}(U_2) \neq \emptyset$ for some $U_1, U_2 \in \mathcal{U}$ with $U_1 \subseteq A_{\gamma_1}, U_2 \subseteq A_{\gamma_2}$. Then $U_1 \cap U_2 \neq \emptyset$ implies that $A_{\gamma_1} \cap A_{\gamma_2} \neq \emptyset$ which contradicts that A is a disjoint collection. Hence, since $f^{\mathcal{W}}$ is surjective, $\mathcal{T} = \{T_\gamma \mid \gamma \in \Gamma\}$ is a disjoint open cover of Y; and since $f^{\mathcal{W}}$ is continuous, $\mathcal{T} << \mathcal{W}$. Given $\gamma \in \Gamma$, there is $U \in \mathcal{U}$ with $U \neq \emptyset$ and $U \subseteq A_\gamma$ so that $S_\gamma \neq \emptyset$, $\underset{\mathcal{U}}{\overset{\mathcal{W}}{f}}(U) \neq \emptyset$, and $T_\gamma \neq \emptyset$. Hence $\delta = (S, \mathcal{T}) \in \Lambda'$ with $\alpha < \delta$, and $X_\delta = \{S_\gamma \mid \gamma \in \Gamma\}$, $Y_\delta = \{T_\gamma \mid \gamma \in \Gamma\}$ so that X_δ, Y_δ are homeomorphic.

Thus, $\Lambda = \{\delta \in \Lambda' \mid Y_\delta$ is homeomorphic to $X_\delta\}$ is cofinal in Λ', so that by Lemmas 4.b, 4.4, X is homeomorphic to inv lim $(X_\alpha, p_\alpha^\beta \mid \alpha \in \Lambda)$ and Y is homeomorphic to inv lim $(Y_\alpha, q_\alpha^\beta \mid \alpha \in \Lambda)$ and for each $\alpha \in \Lambda$, X_α, Y_α are homeomorphic discrete (finite discrete; countable discrete) spaces, and the systems are full. This completes the proof.

__Lemma 4.5__: Let $\mathcal{U}_i$ $1 \leq i \leq j$, $\mathcal{V}_i$ $1 \leq i \leq j$ be covers of a set X. If $\mathcal{U}_i <<** \mathcal{V}_i$ for $1 \leq i \leq j$, then

$$\bigwedge_{1 \leq i \leq j} \mathcal{U}_i <<** \bigwedge_{1 \leq i \leq j} \mathcal{V}_i.$$

__Proof__: Consider $St(\bigcap_{1 \leq i \leq j} U_i, \bigwedge_{1 \leq i \leq j} \mathcal{U}_i)$ where

$U_i \in \mathcal{U}_i$. Since $\mathcal{U}_i <<^{**} \mathcal{V}_i$ for each i, choose $V_i \in \mathcal{V}_i$ with $St(U_i, \mathcal{U}_i) \subseteq V_i$. Consider $\bigcap_{1 \leq i \leq j} U_i'$ where $U_i' \in \mathcal{U}_i$. If $\bigcap_{1 \leq i \leq j} (U_i \cap U_i') \neq \emptyset$ then $U_i \cap U_i' \neq \emptyset$ for each i so that $U_i' \subseteq St(U_i, \mathcal{U}_i) \subseteq V_i$ for each i and $\bigcap_{1 \leq i \leq j} U_i' \subseteq \bigcap_{1 \leq i \leq j} V_i$. Hence,

$St(\bigcap_{1 \leq i \leq j} U_i, \bigwedge_{1 \leq i \leq j} \mathcal{U}_i) \subseteq \bigcap_{1 \leq i \leq j} V_i$ so that

$$\bigwedge_{1 \leq i \leq j} \mathcal{U}_i <<^{**} \bigwedge_{1 \leq i \leq j} \mathcal{V}_i .$$

<u>Lemma 4.6</u>: Let X, Y be compact metric spaces, $\{\mathcal{W}_j \mid j \in Z\}$ a sequence of covers of Y cofinal in Y such that $\mathcal{W}_{j+1} << \mathcal{W}_j$ for each j, and $\{\mathcal{S}_i \mid i \in Z\}$ a sequence of covers of X cofinal in X such that $\mathcal{S}_{i+1} << \mathcal{S}_i$ for each i. Then Y is an injective model of X, if for each j there is an injective, surjective, non-deterministic function $f^j: (X, \mathcal{V}^j) \to (Y, \mathcal{V}'^j)$ satisfying the following conditions:

1) $\mathcal{V}^j = \{\mathcal{S}_i \mid i \geq j\}$ and $r_j(\mathcal{S}_j) << \mathcal{W}_j$.

2) for each $i \geq j$, if $S_1, S_2 \in \mathcal{S}_i$ with $\underset{\mathcal{S}_i}{\overset{j}{f}}(S_1) \cap \underset{\mathcal{S}_i}{\overset{j}{f}}(S_2) \neq \emptyset$ then $S_1 \cap S_2 \neq \emptyset$.

3) if $q \geq s \geq j$ and $S_1 \in \mathcal{S}_q$ and $S_2 \in \mathcal{S}_s$ with $S_1 \subseteq S_2$, then $\underset{\mathcal{S}_q}{\overset{j}{f}}(S_1) \subseteq \underset{\mathcal{S}_s}{\overset{j}{f}}(S_2)$.

Proof: First suppose X is discrete. Since X is compact, X must be finite, say $X = \{x_\ell \mid 1 \le \ell \le m\}$ where $\{x_\ell\}$ is open in X. Choose j such that $S_j << \{\{x_\ell\} \mid 1 \le \ell \le m\}$ so that $S_i = \{\{x_\ell\} \mid 1 \le \ell \le m\}$ for each $i \ge j$.

For each $t \in Z$ let $A_t = \{B_{\frac{1}{2^t}}(y) \mid y \in Y\}$ where $B_\varepsilon(y)$ denote the spheres in Y for $y \in Y$ and $\varepsilon > 0$. Choose $j_1 > j$ such that $\mathcal{W}_{j_1} << A_1$ so that $r_{j_1}(S_{j_1}) << A_1$. Choose $j_2 > j_1$ such that $\mathcal{W}_{j_2} << A_2 \wedge r_{j_1}(S_{j_1})$ so that $r_{j_2}(S_{j_2}) << A_2$ and $r_{j_2}(S_{j_2}) << r_{j_1}(S_{j_1})$. In this way choose a strictly increasing sequence $\{j_t \mid t \in Z\}$ such that $r_{j_{t+1}}(S_{j_{t+1}}) << r_{j_t}(S_{j_t})$ and $r_{j_t}(S_{j_t}) << A_t$ for each t. By 2) for each t we have a disjoint open and closed cover $r_{j_t}(S_{j_t}) = \{f^{j_t}_{S_{j_t}}(\{x_\ell\}) \mid 1 \le \ell \le m\}$ of Y.

Since $r_{j_2}(S_{j_2}) << r_{j_1}(S_{j_1})$, for $1 \le \ell \le m$ we have $f^{j_2}_{S_{j_2}}(\{x^2_\ell\}) \subseteq f^{j_1}_{S_{j_1}}(\{x_\ell\})$ where $\{x^2_\ell \mid 1 \le \ell \le m\}$ is just a permutation of $\{x_\ell \mid 1 \le \ell \le m\}$. Similarly, $r_{j_3}(S_{j_3}) << r_{j_2}(S_{j_2})$ and for

$1 \leq \ell \leq m$ we have $f^{j_3}_{S_{j_3}}(\{x^3_\ell\}) \subseteq f^{j_2}_{S_{j_2}}(\{x^2_\ell\})$ where $\{x^3_\ell \mid 1 \leq \ell \leq m\}$

is just a permutation of $\{x^2_\ell \mid 1 \leq \ell \leq m\}$. In this way we can continue, to obtain for each ℓ with $1 \leq \ell \leq m$, a decreasing sequence

$\ldots, f^{j_t}_{S_{j_t}}(\{x^t_\ell\}) \subseteq, \ldots, \subseteq f^{j_3}_{S_{j_3}}(\{x^3_\ell\}) \subseteq f^{j_2}_{S_{j_2}}(\{x^2_\ell\}) \subseteq f^{j_1}_{S_{j_1}}(\{x^1_\ell\})$, of closed

sets whose diameters approach 0. Since Y is compact metric, for each ℓ, $\bigcap_{t=2}^{\infty} f^{j_t}_{S_{j_t}}(\{x^t_\ell\})$ is one point, say y_ℓ. Thus,

$\{y_\ell \mid 1 \leq \ell \leq m\}$ is a set of distinct points and for each t, $\{B_{\frac{1}{2^t}}(y_\ell) \mid 1 \leq \ell \leq m\}$ covers Y. If $y \in Y$ then for some y_ℓ there is a subsequence of $\{B_{\frac{1}{2^t}}(y_\ell) \mid t \in Z\}$ such that each member of the subsequence contains y. This means that the distance between y and y_ℓ can be made arbitrarily small so that $y = y_\ell$. Hence $Y = \{y_\ell \mid 1 \leq \ell \leq m\}$ so that X and Y are homeomorphic.

Now suppose X is not discrete and let $S_\varepsilon(x)$ for $x \in X$ and $\varepsilon > 0$ denote the spheres in X. Consider any integer i_1 and suppose S_{i_1} refines more than a finite number of members of V^{i_1}. Then $S_{i_1} << \{S_\varepsilon(x) \mid x \in X\}$ for each $\epsilon > 0$ so that the sets in S_{i_1} are point sets. But this is impossible since X is not discrete. So let i_2 be the smallest integer such that S_{i_1} does not refine any member of V^{i_2}. Similarly, S_{i_2} can refine at most a finite number of members of V^{i_2} and let i_3 be the smallest such that S_{i_2} does not refine any member of V^{i_3}. In this way choose a strictly increasing sequence $\{i_t \mid t \in Z\}$ such that S_{i_t} does not refine any member of $V^{i_{t+1}}$ for each t. Then $\{W_{i_t} \mid t \in Z\}$ is cofinal in Y. For each t let $V^{W_{i_t}} = \{S_{i_\ell} \mid \ell \geq t\}$ and let $r_{W_{i_t}}$ be the retriction of r_{i_t} to $V^{W_{i_t}}$ and let $V'^{W_{i_t}}$ be the image of $r_{W_{i_t}}$. We also have the collection of functions $\{f^{W_{i_t}}_{S_{i_\ell}}: S_{i_\ell} \to r_{W_{i_t}}(S_{i_\ell}) \mid \ell \geq t\}$ defined by $f(S)^{W_{i_t}}_{S_{i_\ell}} = f(S)^{i_t}_{S_{i_\ell}}$ for $S \in S_{i_\ell}$. Suppose $\ell_1, \ell_2 \geq t$, $S_{i_{\ell_1}} << S_{i_{\ell_2}}$, $S_1 \in S_{i_{\ell_1}}$, $S_2 \in S_{i_{\ell_2}}$ and $S_1 \subseteq S_2$. Now $S_{i_{\ell_1}}$ does not refine any member of $V^{i_{\ell_1+1}}$. If $\ell_2 > \ell_1$ then $V^{i_{\ell_2}} \subseteq V^{i_{\ell_1+1}}$ and $S_{i_{\ell_1}}$ does not refine any member of $V^{i_{\ell_2}}$

and this contradicts that $S_{i_{\ell_1}} \ll S_{i_{\ell_2}}$. Hence $t \leq \ell_2 \leq \ell_1$

so that by 3), $f(S_1)^{W_{i_t}}_{S_{i_{\ell_1}}} \subseteq f(S_2)^{W_{i_t}}_{S_{i_{\ell_2}}}$. So by condition 2), for

each t there is an injective modeling function
$f^{W_{i_t}}: (X, V^{W_{i_t}}) \to (Y, V'^{W_{i_t}})$ such that $r_{W_{i_t}}(S_{i_\ell}) = W_{i_t}$

for each $\ell \geq t$.

Thus, whether X is discrete or not, Y is an injective model of X. This completes the proof.

<u>Theorem 4.2</u>: Let X, Y be compact Hausdorff spaces. If Y is a model of X then X is homeomorphic to the limit space of an inverse limiting system $\{X_\alpha, p_\alpha^\beta | \alpha \in \Lambda\}$ of compact metric spaces and Y is homeomorphic to the limit space of an inverse limiting system $\{Y_\alpha, q_\alpha^\beta | \alpha \in \Lambda\}$ of compact metric spaces where Y_α is an injective model of X_α for each $\alpha \in \Lambda$.

<u>Proof</u>: According to Theorem 2.2, Corollary 2.5, and Lemma 4.3, X is homeomorphic to the limit space of an inverse limiting system $\{X_\alpha, p_\alpha^\beta | \alpha \in \Lambda'\}$ of compact metric spaces and Y is homeomorphic to the limit space of an inverse limiting system $\{Y_\alpha, q_\alpha^\beta | \alpha \in \Lambda'\}$ of compact metric spaces; with Λ' being all pairs
$\alpha = (\{V_k | k \in Z\}, \{U_k | k \in Z\})$ where $\{V_k | k \in Z\}$ is a

strong starring sequence of open covers of X and X_α is the metric space associated with the pseudometric space generated by $\{V_k \mid k \in Z\}$, and where $\{U_k \mid k \in Z\}$ is a strong starring sequence of open covers of Y and Y_α is the metric space associated with the pseudometric space generated by $\{U_k \mid k \in Z\}$. And if $\alpha = (\{V_k \mid k \in Z\}, \{U_k \mid k \in Z\}), \beta = (\{V_k' \mid k \in Z\}, \{U_k' \mid k \in Z\}) \in \Lambda'$ then $\alpha < \beta$ means $V_k' << V_k$ and $U_k' << U_k$, for each k. Consider a fixed pair $\alpha = (\{V_k \mid k \in Z\}, \{U_k \mid k \in Z\}) \in \Lambda'$.

Throughout the following inductive procedure we shall repeatedly apply Lemma 1.k and the properties of a modeling function in order to choose the desired strong starring sequences. Let $W_1 = U_1$ and let $f^{W_1}: (X, V^{W_1}) \to (Y, V'^{W_1})$ be a modeling function where each member of the image of r_{W_1} refines W_1. Choose a strong starring sequence $\{V_k^{1,1} \mid k \in Z\} \subseteq V^{W_1}$ such that $V_k^{1,1} <<** V_k^{0,1}$ and $r_{W_1}(V_k^{1,1}) <<** W_1$ for each k ($V_k^{0,1} = V_k$ for each k).

Let $W_2 = U_2 \wedge r_{W_1}(V_1^{1,1})$ and let $f^{W_2}: (X, V^{W_2}) \to (Y, V'^{W_2})$ be a modeling function where each member of the image of r_{W_2}

refines $\mathcal{W}_2$. Choose a strong starring sequence $\{\mathcal{V}_k^{2,2} \mid k \in Z\} \subseteq \bigvee^{\mathcal{W}_2}$ such that $\mathcal{V}_k^{2,2} <<** \mathcal{V}_k^{1,1}$ and $r_{\mathcal{W}_2}(\mathcal{V}_k^{2,2}) <<** \mathcal{W}_2$ for each k and choose a strong starring sequence $\{\mathcal{V}_k^{1,2} \mid k \in Z\} \subseteq \bigvee^{\mathcal{W}_1}$ such that $\mathcal{V}_k^{1,2} <<** \mathcal{V}_k^{2,2}$ for each k.

We have $f^{\mathcal{W}_1}, f^{\mathcal{W}_2}$ and let $\mathcal{W}_3 = \mathcal{U}_3 \wedge \left(\bigwedge_{\substack{1 \le k \le 2 \\ 1 \le j \le 2}} r_{\mathcal{W}_j}(\mathcal{V}_k^{j,2}) \right)$, and let $f^{\mathcal{W}_3}$: $(X, \bigvee^{\mathcal{W}_3}) \to (Y, \bigvee^{'\mathcal{W}_3})$ be a modeling function where each member of the image of $r_{\mathcal{W}_3}$ refines $\mathcal{W}_3$. Choose a strong starring sequence $\{\mathcal{V}_k^{3,3} \mid k \in Z\} \subseteq \bigvee^{\mathcal{W}_3}$ such that $\mathcal{V}_k^{3,3} <<** \mathcal{V}_k^{1,2}$ and $r_{\mathcal{W}_3}(\mathcal{V}_k^{3,3}) <<** \mathcal{W}_3$ for each k and for $1 \le j \le 2$ choose strong starring sequences $\{\mathcal{V}_k^{j,3} \mid k \in Z\} \subseteq \bigvee^{\mathcal{W}_j}$ such that $\mathcal{V}_k^{j,3} <<** \mathcal{V}_k^{j+1,3}$ for each k.

Now we have $f^{\mathcal{W}_1}, f^{\mathcal{W}_2}, f^{\mathcal{W}_3}$ and let $\mathcal{W}_4 = \mathcal{U}_4 \wedge \left(\bigwedge_{\substack{1 \le k \le 3 \\ 1 \le j \le 3}} r_{\mathcal{W}_j}(\mathcal{V}_k^{j,3}) \right)$, and let $f^{\mathcal{W}_4}$: $(X, \bigvee^{\mathcal{W}_4}) \to (Y, \bigvee^{'\mathcal{W}_4})$ be a modeling function where each member of the image of $r_{\mathcal{W}_4}$ refines $\mathcal{W}_4$. Choose a strong starring sequence $\{\mathcal{V}_k^{4,4} \mid k \in Z\} \subseteq \bigvee^{\mathcal{W}_4}$ such that $\mathcal{V}_k^{4,4} <<** \mathcal{V}_k^{1,3}$ and $r_{\mathcal{W}_4}(\mathcal{V}_k^{4,4}) <<** \mathcal{W}_4$ for each k and for

$1 \leq j \leq 3$ choose strong starring sequences $\{V_k^{j,4} \mid k \in Z\} \subseteq \mathcal{V}^{W_j}$ such that $V_k^{j,4} <<** V_k^{j+1,\,4}$ for each k.

Continuing in this way, for a given i we have modeling functions f^{W_j} $1 \leq j \leq i-1$ where each member of the image of r_{W_j} refines $W_j = U_j \wedge (\bigwedge_{\substack{1 \leq k \leq j-1 \\ 1 \leq \ell \leq j-1}} r_{W_\ell}(V_k^{\ell,\,j-1}))$,

and we have chosen a strong starring sequence $\{V_k^{j,j} \mid k \in Z\} \subseteq \mathcal{V}^{W_j}$ such that $V_k^{j,j} <<** V_k^{1,\,j-1}$ and $r_{W_j}(V_k^{j,j}) <<** W_j$ for each k and for $1 \leq \ell \leq j-1$ we have chosen strong starring sequences $\{V_k^{\ell,j} \mid k \in Z\} \subseteq \mathcal{V}^{W_\ell}$ such that $V_k^{\ell,j} <<** V_k^{\ell+1,\,j}$ for each k. And we have

$W_i = U_i \wedge (\bigwedge_{\substack{1 \leq k \leq i-1 \\ 1 \leq j \leq i-1}} r_{W_j}(V_k^{j,\,i-1}))$, a modeling function

$f^{W_i}: (X, \mathcal{V}^{W_i}) \to (Y, \mathcal{V}'^{W_i})$ where each member of the image of r_{W_i} refines W_i, a strong starring sequence $\{V_k^{i,i} \mid k \in Z\} \subseteq \mathcal{V}^{W_i}$ such that $V_k^{i,i} <<** V_k^{1,\,i-1}$ and $r_{W_i}(V_k^{i,i}) <<** W_i$ for each k, and for $1 \leq j \leq i-1$ strong starring sequences $\{V_k^{j,i} \mid k \in Z\} \subseteq \mathcal{V}^{W_j}$ such that $V_k^{j,i} <<** V_k^{j+1,\,i}$ for each k.

By this inductive procedure for each i we have $\mathcal{V}_k^{i+1,\,i+1} <<^{**} \mathcal{V}_k^{1,i} <<^{**} \mathcal{V}_k^{2,i} <<^{**},\ldots$ $\ldots, <<^{**} \mathcal{V}_k^{i-1,\,i} <<^{**} \mathcal{V}_k^{i,i}$ and $\mathcal{V}_k^{i,i} <<^{**} \mathcal{V}_k^{1,\,i-1} <<^{**} \mathcal{V}_k^{2,\,i-1} <<^{**},\ldots$ $\ldots, <<^{**} \mathcal{V}_k^{i-2,\,i-1} <<^{**} \mathcal{V}_k^{i-1,\,i-1}$ for each k. Thus, $\mathcal{V}_k^{i+1,\,i+1} <<^{**} \mathcal{V}_k^{i,i}$ for each k and if $1 \leq j \leq i-1$ then $\mathcal{V}_k^{j,i} <<^{**} \mathcal{V}_k^{i,i}$ and $\mathcal{V}_k^{j,i} <<^{**} \mathcal{V}_k^{j,\,i-1}$ for each k. Furthermore, if $1 \leq j \leq \ell \leq i-1$ then $\mathcal{V}_k^{j,i} <<^{**} \mathcal{V}_k^{j,\ell}$ for each k since $1 \leq j \leq \ell < \ell+1,\ldots, < i-2 < i-1$ and repeating the above argument a finite number of times we have $\mathcal{V}_k^{j,i} <<^{**}\mathcal{V}_k^{j,i-1} <<^{**}\mathcal{V}_k^{j,i-2} <<^{**},\ldots,<<^{**}\mathcal{V}_k^{j,\ell+1} <<^{**}\mathcal{V}_k^{j,\ell}$ for each k.

For each pair j, i consider the modeling function $f^{\mathcal{W}_j}$, $\{\mathcal{V}_k^{j,i} \mid k \in Z\} \subseteq \mathcal{V}^{\mathcal{W}_j}$, and $\{r_{\mathcal{W}_j}(\mathcal{V}_k^{j,i}) \mid k \in Z\}$. Let $X_{j,i}$ $(Y_{j,i})$ be the pseudometric space generated by the strong starring sequence $\{\mathcal{V}_k^{j,i} \mid k \in Z\}$ of open covers of X $(\{r_{\mathcal{W}_j}(\mathcal{V}_k^{j,i}) \mid k \in Z\}$ of open covers of Y) according to Lemma 1.ℓ so that $St(x, \mathcal{V}_{k+1}^{j,i}) \subseteq S^{j,i}_{\frac{1}{2^{k-1}}}(x) \subseteq St(x, \mathcal{V}_k^{j,i})$ for each $x \in X$ and k where $S^{j,i}_{\varepsilon}(x)$ for $x \in X$ and $\varepsilon > 0$ denote the spheres in $X_{j,i}$ $(St(y, r_{\mathcal{W}_j}(\mathcal{V}_{k+1}^{j,i})) \subseteq B^{j,i}_{\frac{1}{2^{k-1}}}(y) \subseteq St(y, r_{\mathcal{W}_j}(\mathcal{V}_k^{j,i}))$ for each $y \in Y$ and k where $B_{\varepsilon}^{j,i}(y)$ for $y \in Y$ and $\varepsilon > 0$ denote the spheres in $Y_{j,i}$). Let $X_{j,i}^{*j,i}$ ($Y_{j,i}^{*j,i}$) be the metric space associated with $X_{j,i}$ ($Y_{j,i}$). No confusion arises if we denote by $*j,i$ the canonical map from $X_{j,i}$ to $X_{j,i}^{*j,i}$ and the canonical map from $Y_{j,i}$ to $Y_{j,i}^{*j,i}$. Since the topology of $X_{j,i}(Y_{j,i})$ is contained in the topology of $X(Y)$, $X_{j,i}$ and $X_{j,i}^{*j,i}$ ($Y_{j,i}$ and $Y_{j,i}^{*j,i}$) are compact.

Consider a fixed pair j,i. For each k, $\mathcal{V}_{k+1}^{j,i} <<** \mathcal{V}_{k}^{j,i}$ so that $\{S^{j,i}_{\frac{1}{2^k}}(x) \mid x \in X\} << \mathcal{V}_{k}^{j,i} << \{S^{j,i}_{\frac{1}{2^{k-2}}}(x) \mid x \in X\}$,

$r_{\mathcal{W}_j}(\mathcal{V}_{k+1}^{j,i}) <<** r_{\mathcal{W}_j}(\mathcal{V}_{k}^{j,i})$, and

$\{(B^{j,i}_{\frac{1}{2^k}}(y))^{*j,i} \mid y \in Y\} << r_{\mathcal{W}_j}(\mathcal{V}_{k}^{j,i})^{*j,i} << \{(B^{j,i}_{\frac{1}{2^{k-2}}}(y))^{*j,i} \mid y \in Y\}$

where $r_{\mathcal{W}_j}(\mathcal{V}_{k}^{j,i})^{*j,i} = \{(f^{\mathcal{W}_j}_{\mathcal{V}_k^{j,i}}(V))^{*j,i} \mid V \in \mathcal{V}_{k}^{j,i}\}$. Since

$X_{j,i}^{*j,i}$ ($Y_{j,i}^{*j,i}$) is a compact pseudometric space, by Lemma 1.f, $\mathcal{V} = \{\mathcal{V}_{k}^{j,i} \mid k \in Z\}$ ($\mathcal{V}' = \{r_{\mathcal{W}_j}(\mathcal{V}_{k}^{j,i})^{*j,i} \mid k \in Z\}$) is a collection of covers of $X_{j,i}^{*j,i}$ ($Y_{j,i}^{*j,i}$) cofinal in $X_{j,i}^{*j,i}$ ($Y_{j,i}^{*j,i}$). Since $f^{\mathcal{W}_j}$ is a modeling function, $r\colon \mathcal{V} \to \mathcal{V}'$ defined by $r(\mathcal{V}_{k}^{j,i}) = r_{\mathcal{W}_j}(\mathcal{V}_{k}^{j,i})^{*j,i}$ and the collection of functions

$\{f_{\mathcal{V}_k^{j,i}} : \mathcal{V}_k^{j,i} \to r(\mathcal{V}_k^{j,i}) \mid \mathcal{V}_k^{j,i} \in \mathcal{V}\}$ where

$f(V)_{\mathcal{V}_k^{j,i}} = \left(f(V)_{\mathcal{V}_k^{j,i}}^{\mathcal{W}_j}\right)^{*j,i}$ for $V \in \mathcal{V}_k^{j,i}$, defines a continuous,

surjective, cofinal, crude non-deterministic function. If

$V_1, V_2 \in \mathcal{V}_k^{j,i}$ and $f(V_1)_{\mathcal{V}_k^{j,i}} \cap f(V_2)_{\mathcal{V}_k^{j,i}} \neq \emptyset$ let

$y^{*j,i} \in \left(f(V_1)_{\mathcal{V}_k^{j,i}}^{\mathcal{W}_j}\right)^{*j,i} \cap \left(f(V_2)_{\mathcal{V}_k^{j,i}}^{\mathcal{W}_j}\right)^{*j,i}$ so that $y^{*j,i} = a^{*j,i}$

for some $a \in f(V_1)_{\mathcal{V}_k^{j,i}}^{\mathcal{W}_j}$ and $y^{*j,i} = b^{*j,i}$ for some $b \in f(V_2)_{\mathcal{V}_k^{j,i}}^{\mathcal{W}_j}$.

By definition of the pseudometric on $Y_{j,i}$ we have that

$y \in St(a, r(\mathcal{V}_k^{j,i})_{\mathcal{W}_j}) \cap St(b, r(\mathcal{V}_k^{j,i})_{\mathcal{W}_j})$ so that there are

$V_3, V_4 \in \mathcal{V}_k^{j,i}$ such that $f(V_1)_{\mathcal{V}_k^{j,i}}^{\mathcal{W}_j} \cap f(V_3)_{\mathcal{V}_k^{j,i}}^{\mathcal{W}_j} \neq \emptyset$,

$f(V_3)_{\mathcal{V}_k^{j,i}}^{\mathcal{W}_j} \cap f(V_4)_{\mathcal{V}_k^{j,i}}^{\mathcal{W}_j} \neq \emptyset$, $f(V_4)_{\mathcal{V}_k^{j,i}}^{\mathcal{W}_j} \cap f(V_2)_{\mathcal{V}_k^{j,i}}^{\mathcal{W}_j} \neq \emptyset$. Hence, $V_1 \cap V_3 \neq \emptyset$,

$V_3 \cap V_4 \neq \emptyset$, $V_4 \cap V_2 \neq \emptyset$, $V_3 \subseteq St(V_1, \mathcal{V}_k^{j,i})$, $V_4 \subseteq St(V_2, \mathcal{V}_k^{j,i})$

and $St(V_1, \mathcal{V}_k^{j,i}) \cap St(V_2, \mathcal{V}_k^{j,i}) \neq \emptyset$. Thus, we have a cofinal crude modeling function from $X_{j,i}$ to $Y^{*j,i}_{j,i}$ and let $g_{j,i}$ be the continuous surjection from $X_{j,i}$ to $Y^{*j,i}_{j,i}$ given by Lemma 4.2. Consider $y^* \in Y^{*j,i}_{j,i}$ and let $g(x)_{j,i} = y^{*j,i}$ for some $x \in X$. If $c \in x^{*j,i}$ then by definition of the pseudometric on $X_{j,i}$, each nbhd. of x in $X_{j,i}$ is a nbhd. of c in $X_{j,i}$, so that $g(x)_{j,i} = g(c)_{j,i}$ and $c \in g^{-1}_{j,i}(y^{*j,i})$.

On the other hand suppose $g(c)_{j,i} = y^{*j,i}$. If $c \notin x^{*j,i}$ then for some $\varepsilon > 0$ $S_\varepsilon^{j,i}(x) \cap S_\varepsilon^{j,i}(y) = \emptyset$ and

$\mathcal{O} = \{S_\varepsilon^{j,i}(x), S_\varepsilon^{j,i}(c), X - (x^{*j,i} \cup c^{*j,i})\}$ is an open cover of $X_{j,i}$. By Lemma 1.k choose k such that

$\{St^3(V, \mathcal{V}_k^{j,i}) \mid V \in \mathcal{V}_k^{j,i}\} \ll \mathcal{O}$ and choose $V \in \mathcal{V}_k^{j,i}$ where

$(f(V)_{\mathcal{V}_k^{j,i}})^{*j,i}_{\mathcal{W}_j}$ is a nbhd. of $y^{*j,i}$. Since $g(x)_{j,i} = g(c)_{j,i}$

choose $V_x, V_c \in \mathcal{V}_k^{j,i}$ with $x \in V_x$, $c \in V_c$ and

$$(f(V_x)_{\mathcal{V}_k^{j,i}}^{\mathcal{W}_j})^{*j,i} \cap (f(V)_{\mathcal{V}_k^{j,i}}^{\mathcal{W}_j})^{*j,i} \neq \emptyset,$$

$$(f(V_c)_{\mathcal{V}_k^{j,i}}^{\mathcal{W}_j})^{*j,i} \cap (f(V)_{\mathcal{V}_k^{j,i}}^{\mathcal{W}_j})^{*j,i} \neq \emptyset.$$ Then

$$St(V_x, \mathcal{V}_k^{j,i}) \cap St(V, \mathcal{V}_k^{j,i}) \neq \emptyset$$ and

$$St(V_c, \mathcal{V}_k^{j,i}) \cap St(V, \mathcal{V}_k^{j,i}) \neq \emptyset$$

so that $V_x \cup V_c \subseteq St^3(V, \mathcal{V}_k^{j,i})$ and this contradicts that

$St^3(V, \mathcal{V}_k^{j,i})$ is contained in some member of $\mathcal{O}$. Hence

$c \in x^{*j,i}$. So for each $x \in X$, if

$g(x)_{j,i} = y^{*j,i}$ then $x^{*j,i} = g_{j,i}^{-1}(y^{*j,i})$. Thus, for each pair j,i; $f_{j,i}: X_{j,i}^{*j,i} \to Y_{j,i}^{*j,i}$ defined by $f(x_{j,i}^{*j,i}) = g(x)_{j,i}$ for each $x \in X$, is a homeomorphism.

Now choose a strictly increasing sequence $\{p_i \mid i \in Z\}$ of integers such that

$\{*_{i,i}^{-1} f_{i,i}[S^{*i,i}] \mid S \in S_i\} << \mathcal{W}_i$ where $S_i = \{S^{i,i}_{\frac{1}{2^{p_i-1}}}(x) \mid x \in X\}$.

(If p_{i-1} has already been chosen, since $r_{\mathcal{W}_i}(\mathcal{V}_1^{i,i}) <<* \mathcal{W}_i$, choose an open cover $\mathcal{W}$ of $Y_{i,i}$ such that $\mathcal{W} << \mathcal{W}_i$ and by Lemma 1.f choose an integer $p_i > p_{i-1}$ such that $S_i = \{S^{i,i}_{\frac{1}{2^{p_i-1}}}(x) \mid x \in X\} << \{*_{i,i}^{-1} f_{i,i}^{-1} [W^{*i,i}] \mid W \in \mathcal{W}\}$ so that $\{*_{i,i}^{-1} f_{i,i}[S^{*i,i}] \mid S \in S_i\} << \mathcal{W}_i$). Given i, since $p_{i+1} > p_i$, $\mathcal{V}_{p_{i+1}}^{i+1,\ i+1} <<** \mathcal{V}_{p_i}^{i+1,\ i+1}$ and since $\mathcal{V}_{p_i}^{i+1,\ i+1} <<** \mathcal{V}_{p_i}^{i,i}$ we have that $\mathcal{V}_{p_{i+1}}^{i+1,\ i+1} <<** \mathcal{V}_{p_i}^{i,i}$. And, since $p_i \geq i$, $\mathcal{V}_{p_i}^{i,i} << \mathcal{V}_i^{1,1} << \mathcal{V}_i^{0,1} = \mathcal{V}_i$. If $1 \leq j \leq i-1$ then $\mathcal{V}_k^{j,i} <<** \mathcal{V}_k^{j,i-1}$ for each k and for the modeling function $f^{\mathcal{W}_j}$, $\{\mathcal{V}_k^{j,i} \mid k \in Z\} \subseteq V^{\mathcal{W}_j}$ and $\{\mathcal{V}_k^{j,i-1} \mid k \in Z\} \subseteq V^{\mathcal{W}_j}$. Hence $r_{\mathcal{W}_j}(\mathcal{V}_k^{j,i}) <<** r_{\mathcal{W}_j}(\mathcal{V}_k^{j,i-1})$ for each k and in particular for

$1 \le k \le i - 1$. By Lemma 4.5

$$\mathcal{U}_{i+1} \wedge \Big(\bigwedge_{\substack{1 \le k \le i-1 \\ 1 \le j \le i-1}} r(\mathcal{V}_k^{j,i}\ \mathcal{W}_j) \Big) <^{**} \mathcal{U}_i \wedge \Big(\bigwedge_{\substack{1 \le k \le i-1 \\ 1 \le j \le i-1}} r(\mathcal{V}_k^{j,i-1}\ \mathcal{W}_j) \Big)$$

so that $\mathcal{W}_{i+1} <^{**} \mathcal{W}_i$.

Hence $\delta = (\{\mathcal{V}_{p_i}^{i,i} \mid i \in Z\}, \{\mathcal{W}_i \mid i \in Z\}) \in \Lambda'$, $\alpha < \delta$,

and X_δ (Y_δ) is the compact metric space associated with

the compact pseudometric space X_0 (Y_0) generated by

$\{\mathcal{V}_{p_i}^{i,i} \mid i \in Z\}$ ($\{\mathcal{W}_i \mid i \in Z\}$). No confusion arises if we

denote by * the canonical map from X_0 to X_δ and the canonical

map from Y_0 to Y_δ.

<u>Claim</u>: Y_δ is an injective model of X_δ.

We shall show that for these spaces we have the situation

described in Lemma 4.6.

For each i, let $S_i^* = \{S^* \mid S \in S_i\}$. Given i and

$x \in X$ choose $j > \max\{k+2, i\}$ so that $p_j > k+2$ and

$S^{i,i}_{\frac{1}{2^{p_j-2}}}(x) \subseteq S^{i,i}_{\frac{1}{2^k}}(x)$. Since $\mathcal{V}_{p_j}^{j,j} << \mathcal{V}_{p_j}^{i,i}$, we have

$St(x, \mathcal{V}_{p_j}^{j,j}) \subseteq St(x, \mathcal{V}_{p_j}^{i,i}) \subseteq S^{i,i}_{\frac{1}{2^{p_j-2}}}(x)$; and according to Lemma 1.ℓ,

$St(x, \mathcal{V}_{p_j}^{j,j})$ contains an open set of X_0 containing x. Hence the topology of $X_{i,i}$ is contained in the topology of X_0.

Since $V^{i+1,\, i+1}_{p_{i+1}} << V^{i,i}_{p_{i+1}}$, for each $x \in X$ we have

$$S^{i+1,\, i+1}_{\frac{1}{2^{p_{i+1}-1}}}(x) \subseteq St(x, V^{i+1,\, i+1}_{p_{i+1}}) \subseteq St(x, V^{i,i}_{p_{i+1}}) \subseteq S^{i,i}_{\frac{1}{2^{p_{i+1}-2}}}(x)$$

and since $p_{i+1} > p_i$, $S^{i,i}_{\frac{1}{2^{p_{i+1}-2}}}(x) \subseteq S^{i,i}_{\frac{1}{2^{p_i-1}}}(x) \subseteq St(x, V^{i,i}_{p_i})$ so that

$S_{i+1} << S_i << \{St(x, V^{i,i}_{p_i}) \mid x \in X\}$. By Lemmas 1.f, 1.ℓ,

the collection $\{\{St(x, V^{i,i}_{p_i}) \mid x \in X\} \mid i \in Z\}$ is cofinal in

X_0 so that $\{S^*_i \mid i \in Z\}$ is a sequence of open covers of X_δ

cofinal in X_δ such that $S^*_{i+1} << S^*_i$ for each i.

For each j, let $V^j = \{S^*_i \mid i \geq j\}$ and let

$V'^j = \{\underset{j}{r}(S^*_i) \mid i \geq j\}$ where

$\underset{j}{r}(S^*_i) = \{(*^{-1}_{j,i}\, f_{j,i}[S^{*j,i}])^* \mid S \in S_i\}$ and for $i \geq j$ let

$f^j_{S^*_i} : S^*_i \to \underset{j}{r}(S^*_i)$ be defined by $\underset{S^*_i}{f^j}(S^*) = (*^{-1}_{j,i}\, f_{j,i}[S^{*j,i}])^*$

for each $S \in S_i$.

If $i \geq j$ then $\mathcal{V}_k^{j,i} << \mathcal{V}_k^{i,i}$ for each k so that the topology of $X_{i,i}$ is contained in the topology of $X_{j,i}$.

Hence $*_{j,i}^{-1} f_{j,i}[S^{*j,i}]$ is open in $Y_{j,i}$ for each $S \in S_i$. If $y \in Y$ choose ℓ such that $\ell - 1 > \max\{i, j, k-1\}$. Then

$\mathcal{W}_\ell << r(\mathcal{V}_{k-1}^{j,\ \ell-1})_{\mathcal{W}_j}$ so that $St(y, \mathcal{W}_\ell) \subseteq St(y, r(\mathcal{V}_{k-1}^{j,\ \ell-1})_{\mathcal{W}_j})$.

Since $1 \leq j \leq i \leq \ell-2$, $\mathcal{V}_{k-1}^{j,\ \ell-1} << \mathcal{V}_{k-1}^{j,i}$; and since

$\mathcal{V}_{k-1}^{j,\ \ell-1}$, $\mathcal{V}_{k-1}^{j,i} \in \mathbb{V}^{\mathcal{W}_j}$ for the modeling function $f^{\mathcal{W}_j}$,

$r(\mathcal{V}_{k-1}^{j,\ \ell-1})_{\mathcal{W}_j} << r(\mathcal{V}_{k-1}^{j,i})_{\mathcal{W}_j}$. So

$St(y, r(\mathcal{V}_{k-1}^{j,\ \ell-1})_{\mathcal{W}_j}) \subseteq St(y, r(\mathcal{V}_{k-1}^{j,i})_{\mathcal{W}_j}) \subseteq B_{\frac{1}{2^{k-3}}}^{j,i}(y)$ where, according to Lemma 1.ℓ, $St(y, \mathcal{W}_\ell)$ contains an open set of Y_o containing y. Hence the topology of $Y_{j,i}$ is contained in the topology of Y_o and $(*_{j,i}^{-1} f_{j,i}[S^{*j,i}])^*$ is open in Y_δ for each $S \in S_i$. Thus, $\mathbb{V}'^j$ is a collection of open covers

Let q, s be integers with $q \geq s \geq j$ and let

$S_1^* \in \mathcal{S}_q^*$, $S_2^* \in \mathcal{S}_s^*$ with $S_1^* \subseteq S_2^*$ so that $S_1 \subseteq S_2$. If $q > s$

let $y^* \in \underset{S_q^*}{f^j}(S_1^*)$ so that $y^{*j,q} = f_{j,q}(x^{*j,q}) = g_{j,q}(x)$ for some

$x \in S_1$. Let $N_{y^{*j,s}}$ be a nbhd. of $y^{*j,s}$ in $Y_{j,s}^{*j,s}$. Since

$1 \leq j \leq s \leq q-1$, $\mathcal{V}_k^{j,q} << \mathcal{V}_k^{j,s}$ for each k and for the modeling

function f^{ω_j}, $\{\mathcal{V}_k^{j,q} \mid k \in Z\} \subseteq \mathcal{V}^{\omega_j}$ and $\{\mathcal{V}_k^{j,s} \mid k \in Z\} \subseteq \mathcal{V}^{\omega_j}$

so that $\underset{\omega_j}{r}(\mathcal{V}_k^{j,q}) << \underset{\omega_j}{r}(\mathcal{V}_k^{j,s})$ for each k. Hence the topology

of $Y_{j,s}$ is contained in the topology of $Y_{j,q}$ so that

$*_{j,s}^{-1}[N_{y^{*j,s}}]$ is a nbhd. of y in $Y_{j,q}$ and

$M_{y^{*j,q}} = (*_{j,s}^{-1}[N_{y^{*j,s}}])^{*j,q}$ is a nbhd. of $y^{*j,q}$ in $Y_{j,q}^{*j,q}$.

Consider $\mathcal{V}_k^{j,s}$ for a fixed k. Since $g_{j,q}(x) = y^{*j,q}$, there is

a nbhd. V of x in $X_{j,q}$ with $V \in \mathcal{V}_{k+1}^{j,q}$ such that

$(f(V)^{\omega_j}_{\mathcal{V}^{j,q}_{k+1}})^{*j,q} \cap M_{y^{*j,q}} \neq \emptyset$. So let $a \in f(V)^{\omega_j}_{\mathcal{V}^{j,q}_{k+1}}$ and

$b \in *^{-1}_{j,s}[N_{y^{*j,s}}]$ where $a^{*j,q} = b^{*j,q}$. Since

$\mathcal{V}^{j,q}_{k+1} << \mathcal{V}^{j,s}_{k+1} <<* \mathcal{V}^{j,s}_{k}$, $V \subseteq St(x, \mathcal{V}^{j,s}_{k+1}) \subseteq U$ for some $U \in \mathcal{V}^{j,s}_{k}$

so that U is a nbhd. of x in $X_{j,s}$. By the continuity of

f^{ω_j}, $a \in f(V)^{\omega_j}_{\mathcal{V}^{j,q}_{k+1}} \subseteq f(U)^{\omega_j}_{\mathcal{V}^{j,s}_{k}}$ and $a^{*j,s} \in (f(U)^{\omega_j}_{\mathcal{V}^{j,s}_{k}})^{*j,s}$. But the

topology of $Y_{j,s}$ is contained in the topology of $Y_{j,q}$,

so that $a^{*j,q} \subseteq a^{*j,s}$ and $b^{*j,q} \subseteq b^{*j,s}$ and $a^{*j,s} = b^{*j,s}$.

Hence $a^{*j,s} \in N_{y^{*j,s}}$ so that $(f(U)^{\omega_j}_{\mathcal{V}^{j,s}_{k}})^{*j,s} \cap N_{y^{*j,s}} \neq \emptyset$. By

definition of $f_{j,s}$, $f_{j,s}(x^{*j,s}) = g(x)_{j,s} = y^{*j,s}$ and since

$x^{*j,s} \in S_2^{*j,s}$, $y^{*j,s} = f_{j,s}(x^{*j,s}) \in f_{j,s}[S_2^{*j,s}]$ so that

$y \in *^{-1}_{j,s}\ f_{j,s}\ [S_2^{*j,s}]$ and $y^* \in \underset{S_s^*}{f}(S_2^{j\,*})$. Thus

$\underset{S_q^*}{f}(S_1^{j\,*}) \subseteq \underset{S_s^*}{f}(S_2^{j\,*})$.

For each, j let $\mathcal{W}_j^* = \{W^* \mid W \in \mathcal{W}_j\}$. Then $\{\mathcal{W}_j^* \mid j \in Z\}$ is a sequence of covers of Y_δ cofinal in Y_δ such that $\mathcal{W}_{j+1}^* << \mathcal{W}_j^*$ for each j. Furthermore, by the choice of p_j, $\underset{j}{r}(S_j^*) << \mathcal{W}_j^*$ for each j.

Since $f_{j,i}$ is a homeomorphism for each pair j,i; for each j we have an injective, surjective, non-deterministic function f^j: $(X_\delta, \mathcal{V}^j) \to (Y_\delta, \mathcal{V}'^j)$ satisfying the conditions in Lemma 4.6. So the Claim holds.

Thus, $\Lambda = \{\delta \in \Lambda' \mid Y_\delta$ is an injective model of $X_\delta\}$ is cofinal in Λ', so that by Lemma 4.b, X is homeomorphic to inv lim $\{X_\alpha, p_\alpha^\beta \mid \alpha \in \Lambda\}$ and Y is homeomorphic to inv lim $\{Y_\alpha, q_\alpha^\beta \mid \alpha \in \Lambda\}$ and for each $\alpha \in \Lambda$, X_α, Y_α are compact metric spaces where Y_α is an injective model of X_α. This completes the proof.

In the proof of Theorem 4.2, using Corollary 2.8 in place of Corollary 2.5, we have:

Corollary 4.2: Let X, Y be Hausdorff continua. If Y is a model of X then X is homeomorphic to the limit space of an inverse limiting system $\{X_\alpha, p_\alpha^\beta | \alpha \in \Lambda\}$ of metrizable continua and Y is homeomorphic to the limit space of an inverse limiting system $\{Y_\alpha, q_\alpha^\beta | \alpha \in \Lambda\}$ of metrizable continua where Y_α is an injective model of X_α for each $\alpha \in \Lambda$.

3. Let us conclude with some questions related to our work.

Conjecture: A regular space X is paracompact with dim $X \leq n$ if and only if for each open cover $\mathcal{U}$ there is a sequence $\{\mathcal{U}_k | k \in Z\}$ of open covers such that $\mathcal{U}_{k+1} << \mathcal{U}_k$ and ord $\mathcal{U}_k \leq n + 1$ for each k, and $\{St(St(x, \mathcal{U}_k), \mathcal{U}_k) | x \in X, k \in Z\} <<_p \mathcal{U}$ with respect to $Q(x) = \{St(St(x, \mathcal{U}_k), \mathcal{U}_k) | k \in Z\}$.

Conjecture: If X, Y are compact metric spaces and Y is an injective model of X then X and Y are homeomorphic.

If this is true then Theorems 2.2, 3.1, 4.2 can be used to show that covering dimension is invariant under a modeling distribution in compact Hausdorff spaces.

Conjecture: If X, Y are paracompact Hausdorff spaces and Y is a model of X then dim X = dim Y.

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