SENSITIVITY OF OPTIMAL CONTROL SYSTEMS

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By

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SCOPE AND CONTENTS: The sensitivity of performance to variations in plant parameters is considered for optimal open-loop linear control as well as optimal closed-loop linear control with one or two degrees of freedom. It has been shown that, in general, the sensitivity of performance to finite variations in a plant parameter, is smaller for closed-loop optimal control than for the open-loop case.

A second order linear plant has been considered as an example, to further illustrate the theory, which is contrary to some recently published work.

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CHAPTER I

INTRODUCTION

The sensitivity of a control system to the variations of its plant parameters plays an important role in the analysis and synthesis of automatic control systems. The study of this aspect is necessary, firstly, because of the fact that the system plant parameter values differ, as a rule, from the computed ones either because of inaccuracies in the computed data, or because they vary with time, or because it is impossible to achieve an exact realization of the controlling device. Secondly, information on how the system characteristics depend upon the variations of its parameters may be utilized to improve its performance as, for example, in adaptive systems.

In the past, researchers have studied the sensitivity problem in classical feedback control systems in great detail, some aspects of which have been presented in Chapter IV of this thesis. However, little work has been published on the sensitivity problem in the more recent theory of optimal control systems. The theory of optimal control systems has been treated extensively in Chapters II and III of this thesis.

The mathematical solution to the optimal control will depend upon the parameters of the vector differential equation $\dot{X} = f\left[X(t), u(t), t\right]$. This differential equation, in general, will represent an idealization of the behaviour of a physical plant. Due to the fact that the assumed mathematical model will never be an exact replica of the physical plant and due to the fact that certain plant parameters are either different or will deviate slowly from their assumed values, it becomes imperative to investigate the effects of parameters before one implements the mathematical solution of the posed optimization problem.

Dorato[25] called attention to this problem of sensitivity of optimal control systems for the first time in 1963, which is concerned with the change in the value of the performance index with infinitesimal plant parameter variations.

The primary motivation for introducing feedback around a system, rather than relying upon an open-loop control system, is that the feed back system is much less sensitive to parameter variations than the open-loop system.

However, Pagurek[26] has shown that the sensitivity of the performance index, to a first order approximation, due to an infinitesimal change in the plant parameter is the same for both open-loop and closed-loop optimal control of linear systems with respect to a quadratic performance index. These results were extended by Weitsenhausen[27] in a recent note.

This apparent contradiction can be explained by the fact that both Pagurek and Weitsenhausen have considered infinitesimal variations in plant parameters, with the result that their sensitivity functions are only first-order approximations. On the other hand, in every practical system, the variations in the plant parameters are of finite amounts, and due to this fact it is not possible to approximate the sensitivity to a first-order only.

In this thesis the sensitivity of the index of performance to variations in plant parameters is considered for optimal open-loop linear control as well as optimal closed-loop linear control with one or two degrees of freedom. It is here shown that, in general, the sensitivity of the index of performance to finite variations in a plant parameter is smaller for closed-loop optimal, particularly for a closed-loop optimal control with two degrees of freedom, than for the open-loop case.

A second order linear plant has been considered as an example. Both the open-loop and closed-loop optimal control systems with one or two degrees of freedom have been designed. The sensitivities of index of performance to a slight variation in the location of a pole of the plant, for open-loop optimal control as well as closed-loop optimal control with one or two degrees of freedom are obtained, to further illustrate the theory.

CHAPTER II

AN INTRODUCTION TO OPTIMAL CONTROL THEORY

2.1 INTRODUCTION

Optimization has been considered as the number one problem in automation. Automation may be defined as the branch of science and technology concerned with the development of devices, plants and systems that operate without direct human intervention and assume responsibility for performing certain types of human mental work.

Considerable interest in optimal control theory has been developed over the past decade and a broad general theory based on a combination of variational techniques, conventional servomechanism theory and high speed computation has been the result of this interest.

In this chapter, we set the context by discussing the system design problem and by specifying the particular type of system design problem which generates the control problem. We then discuss the historical development of the optimal control theory, thus putting a temporal perspective.

2.2 THE SYSTEM DESIGN PROBLEM

A system design problem begins with the statement of a job to be accomplished either by an existing physical process or by a physical process which is to be constructed.

The engineer will be provided with:

(1) a set of goals or objectives which broadly describe the desired performance of the physical process.

(2) a set of constraints which represent limitations that either are inherent in the physics of the situation or are artificially imposed.

The development of a system which accomplishes the desired objectives and meets the imposed constraints is, in essence, the system design problem.

There are basically two ways in which the system design problem can be approached; the "direct" approach and the "usual standard" approach.

In the direct approach, the engineer combines experience, know-how, ingenuity and the results of experimentation to produce a prototype of the required system. He deals normally with specific components and does not develope mathematical modes or resort to simulation.

Unfortunately, this direct approach becomes inadequate for complicated systems. Also the risks and cost involved in experimentation are too great. Thus the direct approach, no doubt it may lead to a sharpening of an engineers intuition, it fails to provide broad, general design principles which can be applied to a variety of problems.

At this point the second method, namely the "usual standard" approach, begins with the replacement of the real world problem by a problem involving mathematical relationships. That is to say, the first step consists in formulating a suitable model of the physical process, the system objectives and the imposed constraints.

Once the system design problem is formulated in terms of a mathematical model, the system engineer then seeks a pencil-and-paper design which represents the solution to the mathematical version of the design problem. Then the engineer simulates these results to obtain insight into the operation of the system and to test the behaviour of the model under ideal conditions. Conclusions about whether the mathematical model will lead to a reasonable physical system can be drawn, and the sensitivity of the model due to parameter variations can be studied. Various alternative pencil-and-paper designs can be compared and evaluated.

Then the engineer goes to the job of building the prototype.

But very often the engineer is not satisfied with the system which fulfills the task, he will seek to improve or optimize his design. The process of optimization in the pencil-and-paper stage is quite useful in providing broad insight into the problem and a basis for comparison, while on the other hand the process of optimization in the prototype building stage is mostly concerned with the choice of best components. The role of optimization in the control system design problem will be examined now.

2.3 CONTROL PROBLEM

The translation of control-system design objectives into the mathematical language of the pencil-and-paper design stage gives rise to what will be called the control problem.

The essential factors of the control problem are:

1. A mathematical model (system) to be "controlled"
2. A desired output of the system
3. A set of admissible inputs or "controls"
4. A performance index or cost functional which measures the effectiveness of a given "control action"

The mathematical model, which represents the physical system, consists of a set of relationships which describe the response or output of the system to its various inputs. Constraints based upon the physical situation are incorporated in this set of equations. Normally in translating the design problem into a control problem the objective of the system is translated into a requirement of the output.

As "control" signals in physical systems are usually obtained from equipment which can provide only a limited amount of force or energy, constraints are imposed upon the inputs to the system. These constraints lead to a set of admissible inputs.

Often, the desired objective can be attained by many admissible inputs and so the engineer seeks a measure of performance index or cost functional which will allow him to choose the "best input".

When a performance index or cost junctional has been decided, the engineer formulates the control problem: Determine the (admissible) inputs which generate the desired output, and which, in so doing minimizes (optimize) the chosen performance criterion. At this point optimal-control theory enters the picture to aid the engineer in finding a solution to his problem. Such a solution when it exists is called an "optimal control".

2.4 Historical Perspective

In the early fifties, minimum time control laws (in terms of switch curves and surfaces) were obtained for a variety of second and third order systems. Proofs of optimality were, however, more or less heuristic and geometric in nature. The very idea of determining an optimum system with respect to a specific performance measure, the response time, was very appealing and hence attracted the interest of the mathematician.

The time-optimal control problem was extensively studied by mathematicians both in the United States and Soviet Union. In the period from 1953-1957, Bellman, Gramkrelidze, Krasovskii, and La Salle developed the basic theory of minimum-time problems and presented results concerning the existence, uniqueness, and general properties of the time-optimal control. Then engineers and mathematicians recognized that optimal control problems were essentially problems in the calculus of variations which was founded as an independent mathematical discipline by Euler, about 150 years ago.

But calculus of variations theory could not readily handle the more complicated control problems with "hard" constraints imposed on them. This difficulty lead Pontryagin[1] to first conjecture his celebrated "maximum principle" and together with Boltyanskii , and Gamkrelidze[2] to provide a proof of it. The maximum principle was first announced at the International Congress of Mathematicians held at Edinburgh in 1958.

While Pontryagin's maximum principle may be viewed as an outgrowth of Hamiltonian approach to variational problems, the method of "dynamic programming", which was developed by Bellman [3] around 1953-1957, maybe viewed as an outgrowth of Hamilton-Jacobi [4] approach to variational problems.

The method of maximum principle was extensively studied and applied for various types of problems by Athans [5], Merriam [6], Tow [7], Rozoner [8] and Roberts [9] to mention a few.

Kalman [10] utilized the classical tools of the calculus of variations, in particular the Hamilton-Jacobi equation in finding an optimal control law for a certain class of control problems and studied the stability properties of the matrix Ricatti equation, which arises as a special case of the Hamilton-Jacobi equation.

At present the optimal control theory is primarily a design aid which provides the engineer with insight into the structure and properties of solutions to the optimal-control problems. Specific design procedures and rules of thumb are rather few in number.

Finally although the optimal designs may rarely be implemented, the theory has expanded the horizon of the engineer and has thus allowed the engineer to takle complex and difficult problems.

CHAPTER III

METHODS FOR SOLVING THE OPTIMAL CONTROL PROBLEM

Among many techniques for solving problems in optimization, two methods are generally regarded as most promising for the solution of complex problems. They are the maximum principle of Pontryagin and the method of dynamic programming developed by Bellman.

The Pontryagins maximum principle is chosen as the method for solving the optimal control law for the given system.

The optimal control problem is stated precisely in the first section of this chapter. In the sections that follow, the Pontryagin's maximum principle and its applications are presented.

3.1 STATEMENT OF THE OPTIMAL CONTROL PROBLEM

The basic continuous-time optimal control problem will be stated in order to establish a quantitative basis for discussion for the results that follow.

The basic ingredients of a well formulated optimization problem are:

1. The equations of motion, in state variable form, of the dynamical system to be controlled

2. A set of constraints on the control variables

3. A set of boundary conditions on the state variables at the initial time and at the terminal time

4. A cost functional or performance index which is to be minimized.

The dynamical system which is controlled is assumed to satisfy the following vector differential equation.

$$\dot{X}(t) = f[X(t), u(t), t] \tag{3-1}$$

where X(t) is a vector with n components representing the state of the system at time t and where u(t) is a vector with r-components representing the control input to the system at time t. In equation (3-1) f is a vector valued function of the state X(t) and of the control u(t).

Let Ω be a set on the r-dimensional space of the control variables. Usually Ω is a closed, bounded and convex set, and it is called the control constraint set. In general, Ω represents the mathematical model which takes care of tiny physical bounds upon the magnitudes of the control. The fact that the control vector u(t) must satisfy any posed constraints is written as

$$u(t) \in \Omega \text{ for all } t \tag{3-2}$$

and any control that satisfies the constraint relation given by Eq. (3-2) is called admissible.

Let t_o be the initial time. Then it is assumed that the state of the dynamical system given by Eq. (3-1) is known at time $t = t_o$; this is specified by the relation

$$X(t_o) = X_o \tag{3-3}$$

where X_o is the known vector of initial conditions.

Let t_f be the terminal time. In general t_f may be specified a prior or not; in the latter case the determination of t_f is part of the optimization problem. Let S be a surface, or manifold or point in the n-dimensional state space. Usually S is called the target set. As part of the boundary conditions for the optimization problem, one may demand that the state vector at the terminal time t_f belongs to the target set S, i.e.,

$$X(t_f) \in S \tag{3-4}$$

The target S represents the mathematical model of any requirements upon the desired values of the state variables at the terminal time t_f; for tolerance requirements upon the state variables can be incorporated in the algebraic equations which describe the target set S.

The general control problem consists in finding one or more controls which satisfy the constraint $u(t) \in \Omega$ and which force the dynamical system given by Eq. (3-1) from the initial state X_o in such a way so that $X(t_f) \in S$. In general there are many controls that accomplish this objective. In order to find the "best" control, in some sense, one must decide upon a measure of performance. Thus, depending upon the physical nature of the control problem, one must decide what is important and then transform these physical criterion of goodness into a mathematical cost functional or performance index.

The types of cost functionals or performance indices that one considers are scalar valued and take the form

$$J = \int_{t_o}^{t_f} L\ [X(t),\ u(t),\ t] \tag{3-5}$$

where L is a scalar-valued function of the state vector X(t) and of the control vector u(t). In an optimal control problem, one must find admissible controls that satisfy the posed boundary conditions and in addition, minimize the cost functional or performance index J.

Now a precise statement of the optimization problem can be given as follows:

Given the system $\dot{X}(t) = f[X(t), u(t), t]$. Given the boundary condition $\dot{X}(t_o) = X_o$. Given the control constraint set Ω. Given the target set S. Given the cost functional or performance index

$$J = \int_{t_o}^{t_f} L\ [X(t),\ u(t),\ t]\ dt$$

Then find the control that

(a) satisfies the constraint $u(t) \in \Omega$

(b) transfers the state of the system given by Eq. (3-1) from $X(t_o) = X_o$ to $X(t_f)$ so that $X(t_f) \in S$, and

(c) minimizes the cost functional J

In general, the solution to the above optimization problem involves the following topics:

1. Does a solution exist?

2. If a solution exists, is it unique?

3. Are there any properties of optimal solution which enable the designer to find it either analytically or by using a digital computer?

4. How sensitive are the solutions as well as the minimum value of the cost functional or performance index to variations in the parameters of the plant and/or of the initial conditions?

The answers to such questions provide the engineer with significant qualitative and quantitative information regarding the optimal system.

3.2 PONTRYAGIN'S MAXIMUM PRINCIPLE

Consider an nth order control process controlled by Eq. (3-1). In terms of the components of the state vector, Eq. (3-1) may be written as

$$\dot{x}_i = f_i \; (X(t),\, u(t),\, t) \qquad i = 1, 2, \ldots n \tag{3-6}$$

The optimum design problem requires the minimization of the integral criterion function given as

$$J = \int_{t_o}^{t_f} L\,[X(t),\, u(t),\, t]\; dt$$

with respect to u(t), the control vector.

By introducing a new state variable $x_{n+1}(t)$ defined by

$$x_{n+1}(t) = \int_{t_o}^{t_f} L\,[X(t),\, u(t),\, t]\; dt \tag{3-7}$$

$$x_{n+1}(t_o) = 0 \tag{3-8}$$

and

$$\dot{x}_{n+1}(t) = L\ [X(t),\ u(t),\ t]\ \ dt \tag{3-9}$$

The problem of minimization of the integral given by Eq. (3-5) becomes the problem of minimizing the (n+1) st co-ordinate, $x_{n+1}(t_f)$, at the terminal of the trajectory $t = t_f$. The derivatives of the other co-ordinates are given by Eq. (3-6).

In general, an optimum control problem can be transformed into the problem of minimizing or maximizing a Pontryagin function such as

$$\mathcal{P} = (b,\ X(t_f)) = b'\ X(t_f) \tag{3-10}$$

subject to certain constraining functionals. The control strategy which minimizes or maximizes the Pontryagin function is referred to as the optimum-control strategy. In Eq. (3-10) X is a state vector of nth order control process under constrain, and b is a column vector which depends upon the co-ordinates to be minimized or maximized.

In terms of components of the state vector X, The Pontryagin function $\mathcal{P}$ may be expressed as

$$\mathcal{P} = \sum_{1}^{n} b_i\ x_i(t_f) \tag{3-11}$$

Intuitively, the Pontryagin junction may be minimized by maximizing the energy or power in the system. This physical intuition leads to the speculation that there may exist an energy function such that its maximization implies the minimization of the Pontryagin function. Here the concept of the Hamiltonian comes into the picture. The Hamiltonian is defined as the sum of the kinetic energy and the potential energy and is expressed as the inner product of the momentum vector and the coordinate vector of the system. The very nature of Hamiltonian leads to the fact that maximization of the Hamiltonian imply minimization of the Pontryagin function. Pontryagin used the natures secret weapon and formulated his celebrated maximum principle.

The maximum (or minimum) principle states that, if the control vector u is optimum i.e., if it minimizes (or maximizes) the Pontryagin function $\wp$, then the Hamiltonian H (X, P, u, t) is maximized (or minimized) with respect to u over the control interval. The Hamiltonian is defined as

$$H\,[X(t);\ \ P(t);\ \ u(t);\ t] \ = \ (P,\ f) = \sum_{j=1}^{n} p_j f_j \qquad (3\text{-}12)$$

where X(t) is the state vector, P(t) is the momentum vector to be defined later, and the vector function is as given in Eq. (3-1). The obove statement points out that maximum H implies minimum $\wp$ and minimum H implies maximum $\wp$. Thus a necessary condition for the control vector u(t) to minimize the Pontryagin function is the fulfilment of the maximum condition for u(t).

The design of optimum control aims at the determination of an optimum control law $u^*(X)$ or an optimum-control sequence $u^*(t)$. However, direct application of the maximum principle yields the optimum-control vector u^* as a function of the momentum vector P. In order to find u^* as a function of X or t, some equations providing the relationships between u and P must be established. The momentum vector P is defined on the solution to the differential equation:

$$\dot{p}_i = -\sum_{j=1}^{n} p_j \frac{\partial f_j}{\partial x_i} \qquad i = 1, 2, \ldots . n \tag{3-13}$$

where

$$p_i(t_f) = -b_i \tag{3-14}$$

b_i being some known constant specified in the Pontryagin function $\mathcal{P}$, and:

$$\dot{x}_i = f_i \; [X(t), u(t), t] \tag{3-15}$$

Differentiating Eq. (3-12) with respect to p_i yields:

$$\frac{\partial H}{\partial p_i} = f_i \; [X(t), u(t), t] \tag{3-16}$$

Differentiating Eq. (3-12) with respect to x_i gives:

$$\frac{\partial H}{\partial x_i} = \sum_{j=1}^{n} p_j \frac{\partial f_j}{\partial x_i} \qquad i = 1, 2, \ldots n \tag{3-17}$$

Making use of these two equations reduces Eqs. (3-13) and (3-15) to the Hamilton canonical form:

$$\dot{x}_i = \frac{\partial H}{\partial p_i} \tag{3-18}$$

$$\dot{p}_i = \frac{\partial H}{\partial x_i} \tag{3-19}$$

These canonical equations are subject to be boundary conditions on $x_i(t_o)$ and $p_i(t_f)$; that is:

$$x_i(t_o) = x_i^{\,o} \tag{3-20}$$

and

$$p_i(t_f) = -b_i \,, \quad i = 1, 2, \ldots n \tag{3-21}$$

The physical interpretation of the maximum principle may be stated as follows: The Hamiltonian H is the inner product of P and f, or that of P and $\dot{X}$, which represents the power when P is identified as the momentum. Thus, to minimize $\mathcal{P}$, the power is maximized, and when is minimum, H is a maximum.

The design procedure is to maximize H with respect to u, substitute the optimum-control vector $u^*(P)$ into the Hamiltonian canonical equations given in Eqs. (3-18) and (3-19) and solve the resulting boundary-value proble m for the optimum trajectory X(t) and the momentum vector P(t) subject to boundary conditions given in Eqs. (3-20) and (3-21) with the optimum trajectory X(t) and the momentum vector P(t) known, the optimum control strategy u^* can be determined.

The Pontryagin function given in Eq. (3-10) and the boundary condition given by Eq. (3-21) are valid for control process with free final state. When the final state of the control process is constrained by:

$$R_k\,[X(t_f)] = 0 \qquad k = 1, 2, \ldots n \tag{3-22}$$

The pontryagin function takes the form:

$$= b'\,X(t_f) + \lambda'\,R\,[X(t_f)] \tag{3-23}$$

where λ' is a vector Lagrange multiplier[7]. The canonical equations (3-18) and (3-19) are now subject to boundary conditions:

$$x_i(t_o) = x_i^o \qquad (3\text{-}24)$$

and

$$p_i(t_f) = -\left[b_i + \sum_{k=1}^{n} \lambda_k \frac{\partial R_k}{\partial x_i(t_f)}\right] \qquad (3\text{-}25)$$

with the final state $X(t_f)$ constrained by Eq. (3-21).

In general, the maximum principle provides a necessary condition for system optimization. However, if the control process is linear and subject to an additive control function, i.e., when the process dynamics is characterized by:

$$\dot{x}_i(t) = \sum_{k=1}^{n} a_{ik}(t)\, x_k(t) + m_i(u_1, u_2 \ldots u_r)$$

$$i = 1, 2, \ldots .n \qquad (3\text{-}26)$$

or in vector notation:

$$\dot{X}(t) = A(t)\, X(t) + m(u) \qquad (3\text{-}27)$$

the maximum principle provides the necessary and sufficient condition for optimum control.

3.3 CLOSED LOOP OPTIMAL-CONTROL

The closed loop optimal control has greater advantages on the open loop optimal control, especially from the point of view of sensitivity due to plant parameter variations.

As has been shown in the previous Section 3.3, one can always find an optimum control function $u^*(t)$, i.e., the open loop solution to the problem. However, the disadvantages of open loop over closed loop are well known in engineering circles. In the closed loop optimal control we seek that the optimum control variable u^* must be obtained as function of current values of state variables, i.e., we seek a control function $u^*(x)$.

One of the most powerful design techniques that has been developed to-date deals with the design of the optimal closed loop system for a linear, possibly time varying, plant with respect to a quadratic performance index. The pioneering work in the area was done by Kalman[10]. He utilized the well known Hamilton-Jacobi equation of calculus of variations as the method of attack. The basic results for this problem are as follows: Consider the linear system:

$$\dot{X}(t) = A(t)\,X(t) + B(t)\,u(t) \tag{3-28}$$

the cost functional or performance index is:

$$J = \frac{1}{2}\,X^T(t_f)F\,X(t_f) + \frac{1}{2}\int_{t_o}^{t_f}\left[X^T(t)\,Q(t)\,X(t) + u^T(t)\,R(t)\,u(t)\right]dt \tag{3-29}$$

with the assumptions that the matrix F is positive semidefinite and what the matrices Q(t) and R(t) are positive definite. Then, the optimal con-control $u^*(t)$ which minimizes the cost functional or performance index given by Eq. (3-29) exists, is unique, and it is given by the equation:

$$u^*(t) = -R^{-1}(t)\,B^T(t)\,K(t)\,X(t) \tag{3-30}$$

where $K(t)$ is symmetric and positive definite matrix which is the solution of the matrix Riccati differential equation:

$$\frac{d}{dt} K(t) = -K(t)\, A(t) - A^T(t)\, K(t) + K(t)\, B(t)\, R^{-1}(t)\, B^T(t)\, K(t) - Q(t) \tag{3-31}$$

subject to boundary condition:

$$K(t_f) = F$$

Figure 1. whows the structure of the optimal closed loop system. Since the optimal control is $u^*(t) = -R^{-1}(t)\, B^T(t)\, K(t)\, X(t)$, the state $X(t)$ is operated on by the linear transformation $K(t)$ and then by the linear transform $-R^{-1}(t)\, B^T(t)$ to generate the control. The feed back system is thus time-varying. Since $R(t)$ and $B(t)$ are known matrices, it follows that the matrix $K(t)$ governs the behaviour of the system $K(t)$ is called "gain" matrix.

The response $X(t)$ of the optimal system is the solution of the differential equation:

$$\dot{X}(t) = G(t)\, X(t) \tag{3-32}$$

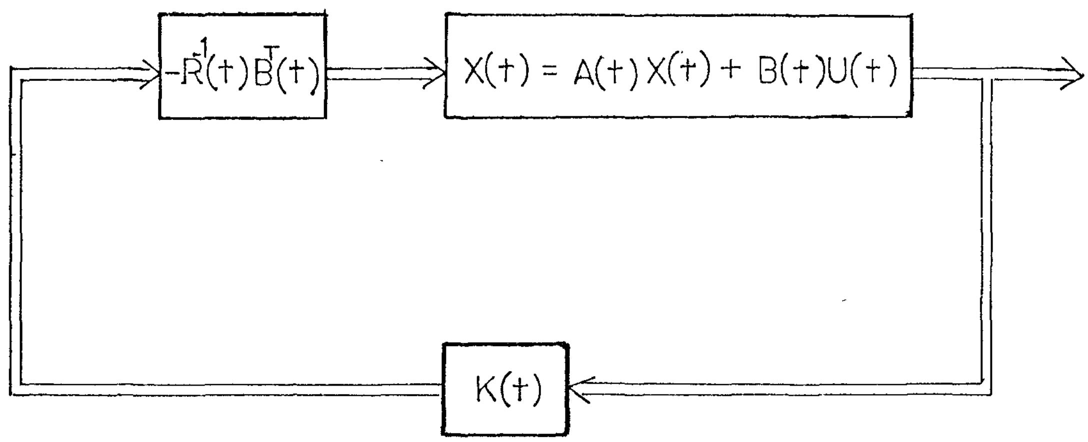


Fig 1. The Structure of the Optimal Feedback System

CHAPTER IV

SENSITIVITY ANALYSIS OF A SYSTEM

4.1 INTRODUCTION

The study of automatic control system sensitivity is started with the origins of feed back system theory. Feed back is used for two primary reasons[11].

1) Feed back "may" decrease the effects of parameter variations upon the system theory.

2) Feed back "may" improve rejection of disturbance signals[12].

The study of sensitivity in a control system is necessary firstly because of the fact that the system parameter values differ, as a rule, from the computed one either because of inaccuracies or because they vary with time or because it is impossible to achieve an exact realization of controlling device. Secondly, information on how the system characteristics depend upon the variations of its parameters may be utilized to improve its performance, as for example in adaptive system.

In this chapter some of the basic definitions of sensitivity in a feed back system are given in the first section. In the following sections, some methods of finding the sensitivity are presented, both in time-domain and frequency-domain.

4.2 BASIC DEFINITION OF BODE

The basic concepts of feed back system sensitivity appeared in the fundamental work of H. W. Bode[11].

Bode's definition: In a feed back circuit, the sensitivity, S, for an element, W, is given by:

$$S = \frac{1}{\dfrac{\partial \theta}{\partial \log W}} \tag{4-1}$$

where θ represents the gain through the complete circuit. Now, if it is expressed in terms of the logarithm of the output E_R, and replace the partial derivative by ordinary differentiation, on the assumption that W is the only element in the circuit which varies. This allows Eq. (4-1) to be written as:

$$S = \frac{dW/W}{dE_R/E_R} \tag{4-2}$$

4.3 MODIFICATIONS FOR APPLICATIONS TO CONTROL SYSTEMS

Horowitz[13] modified Bodes difinition by defining the sensitivity of a system as:

$$S_\alpha^T = \frac{d\ T/T}{d\ \alpha/\alpha} \tag{4-3}$$

T is the over-all transfer function and α is a plant parameter.

S_α^T as defined by Horowitz is normally called as the "classical sensitivity".

Another extended estimate for the sensitivity is the logarithmic derivative ("the logarithmic sensitivity")

$$S_{\alpha}^{T} = \frac{\partial \ln T}{\partial \ln \alpha} \tag{4-4}$$

A more elegant, mathematical method is used by Haykim[14] to evaluate sensitivity in transistor feed back amplifiers which is equally applicable to control system analysis. The sensitivity $S_k^{K_c}$ of the closed loop gain K_c to variations in a specified parameter k can be determined from:

$$S_k^{K_c} = \frac{d\, K_c/K_c}{d\, k/k}$$

$$= \frac{1}{F} - \frac{1}{F_N} \tag{4-5}$$

where F is the return difference with reference to k; evaluated under the condition of zero source signal. F_N is the null return difference which results when the source is adjusted so as to produce a zero load signal. In terms of the circuit determinant Δ, we have $F = \frac{\Delta}{\Delta^o}$ where Δ^o is the special value assumed by Δ when the parameter k is reduced to zero. Let node 1 of a feed back amplifier refers to the source and node 2 refers to the load, we find that $F_N = \frac{\Delta_{12}}{\Delta^o_{12}}$ by deleting its first row and second column. Δ^o_{12} is the value that Δ_{12} reduces to when k = 0.

The concept of Logarithmic sensitivity has also been extended to multidimensional linear systems. The following approach is used by Cruz and Perkins[12]. A sensitivity matrix S(s) is introduced as:

$$\Delta K_o(s)\ S(s) = \Delta K_c(s) \tag{4-6}$$

where $\Delta K_o(s)$ and $\Delta K_c(s)$ are the variations in the transfer functions, respectively, of the open and closed loop systems.

However, Hung[15] pointed out that the definitions given by Bode, Horowitz[13] and others are troublesome when used as a measure of a pole sensitivity under the following conditions:

(i) When the pole of the closed-loop transfer function, whose sensitivity is desired, has multiplicity greater than one.

(ii) When the pole of the closed-loop transfer function, whose sensitivity is desired, is at the origin.

(iii) When the pole, or zero, of the open-loop transfer function, with respect to which the closed-loop sensitivity is desired, is at the origin.

The first two conditions have been shown to yield infinite sensitivity yield the third condition yields zero sensitivity.

In almost every practical system, the change of the parameter α is an increment $\Delta\alpha$ rather than a differential. Therefore, Hung used the definition :

$$S_{\alpha}^{T} = \frac{\Delta \ln T}{\Delta \ln \alpha} = \frac{\Delta T/T}{\Delta\alpha/\alpha} = \frac{\Delta T}{\Delta\alpha} \quad \frac{\alpha}{T} \tag{4-7}$$

4.4 SENSITIVITY IN FREQUENCY-DOMAIN

Out of all the methods developed, the root locus or algebraic methods of estimating sensitivity are important[16, 17]. To these are related the sensitivity coefficients of the poles and zeros of the systems transfer function.

$$K(s) = k \frac{\prod_{r}(S-z_{\mu})}{\prod_{\gamma}(S-p_{\gamma})} \tag{4-8}$$

and with respect to the parameter α_j, the sensitivities are given as:

$$S^{p}_{\gamma,j}(s) = \frac{\partial p_i{}^{j}}{\partial \ln \alpha_j} \tag{4-9}$$

$$S^{z}_{\mu,j}(s) = \frac{\partial z_\mu}{\partial \ln \alpha_j} \tag{4-10}$$

In general, these quantities are complex coefficients characterizing the trends of the movement of the poles p_γ in the S-plane as a function of the small relative variations of the parameter α; for example:

$$\Delta p_{\gamma,j} \approx S^{p}_{\gamma,j} \frac{\Delta\alpha_j}{\alpha_j} \tag{4-11}$$

another definition such as $\partial p_i/\partial\alpha_j$ were also used. With the help of the estimates $S^{p}_{\gamma,j}$ and $S^{z}_{\mu,j}$ we can obtain expressions for the relative increments in the transfer function:

$$\frac{\Delta_j K(s)}{Ks} = \left[\frac{\partial \ln k}{\partial \ln\alpha_j} - \sum_{\mu=1}^{i} \frac{1}{s-z_\mu} S^{z}_{\mu,j} + \sum_{\gamma=1}^{n} \frac{1}{S-P_\gamma} S^{P}_{\gamma,j}\right]\frac{\Delta\alpha_j}{\alpha_j} \tag{4-12}$$

which graphically connects the frequency and the algebraic estimates of the sensitivity.

Methods of finding the variations of the closed-loop roots of a system due to variations of the open loop parameters (such as gain, pole or zero) have been developed[15, 16, 17]. The sensitivity relating to this is termed as "root sensitivity".

Following Ur[16], the root sensitivity is defined as:

$$S^{q_j}_{\alpha} = \frac{dq_j}{d\alpha/\alpha} \tag{4-13}$$

q_j is the closed loop root and "α" is an open loop parameter (such as gain, pole, or zero of open loop system) whereas Haung[17], Truxal used the definition:

$$S_{\alpha}^{q_j} = \frac{dq_j / q_j}{d\alpha/\alpha} \tag{4-14}$$

yet, another definition is given by Hung[15], as:

$$S_{K}^{q_j} = \frac{(\Delta q_j)^m}{\Delta K/K} \tag{4-15}$$

q_j is the closed loop pole, M is the multiplicity of the pole, K is the open loop gain.

And also,

$$S_{p}^{q_j} = \frac{(\Delta q_j)^m}{\Delta p} \tag{4-16}$$

p is the open loop pole, or

$$S_{z}^{q_j} = \frac{(\Delta q_j)^m}{\Delta z} \tag{4-17}$$

z is the open loop zero.

For a system with unity feed back as in Figure 2, the open loop transfer function is:

$$G(s) = \frac{K\,q(s)}{p(s)} \tag{4-18}$$

K is the open-loop gain and q(s) and p(s) are polynomials in s. The closed-loop transfer function is:

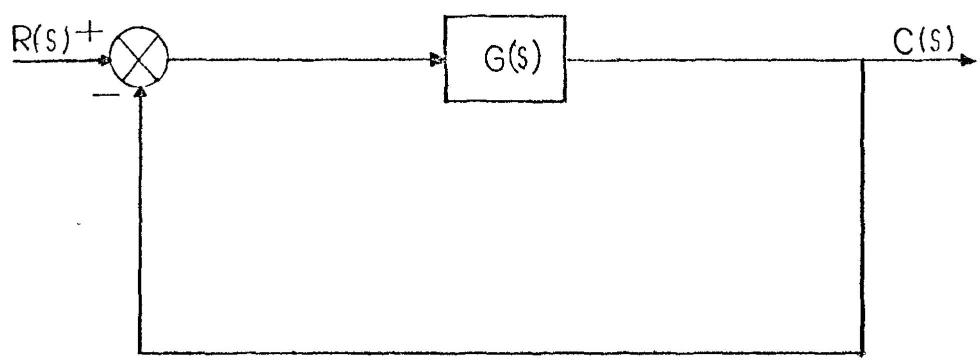


Fig. 2. Control System with Unity Feedback.

$$T(s) = \frac{C}{R}(s) = \frac{Kq(s)}{p(s) + Kq(s)} \quad (4\text{-}19)$$

The above equations (4-15), (4-16) and (4-17) can be written as:

$$S_K^{q_j} = \frac{(\Delta q_j)^m}{\Delta K/K} = -[(s-q_j)^m \, T(s)]_{s \to q_j} \quad (4\text{-}20)$$

$$S_{p_i}^{q_j} = \frac{(\Delta q_j)^m}{\Delta p} = \left[\frac{(s - q_j)^m \, p(s)}{[p(s) + Kq(s)](s-p_i)}\right]_{s \to q_j} . \quad (4\text{-}21)$$

and

$$S_{z_i}^{q_j} = \frac{(s - q_j)^m}{(s - z_i)} \, T(s) \qquad s \to q_j \quad (4\text{-}22)$$

Also sensitivity was analyzed graphically by various authors. They use the well known "root-locus" techniques in the frequency-domain studies. The graphical techniques used to find sensitivity are termed as "root-contour" methods[19]; to distinguish the fact that in root-locus the closed-loop poles are plotted when K, the gain, is variable parameter, where as in "root-contour", the closed poles are plotted when K is held constant, but the open-loop poles and zeros are varied (due to the variation of the open loop parameters of the system other than K, the open-loop gain). The single-degree-of-freedom system, and the two-degree-of-freedom systems are now studied from the sensitivity point of view. Consider a single-degree-of-freedom structure as shown in Figure 3 [13].

P is the transfer function of the a priori given plant. An important feature of this structure, from that feedback theory point of view, is its single degree of freedom L = GP.

The sensitivity to the plant is given as:

$$S_P^T = \frac{\Delta T/T}{\Delta P/P} \tag{4-23}$$

$$= \frac{1}{1 + GP}$$

$$= \frac{1}{1 + L}$$

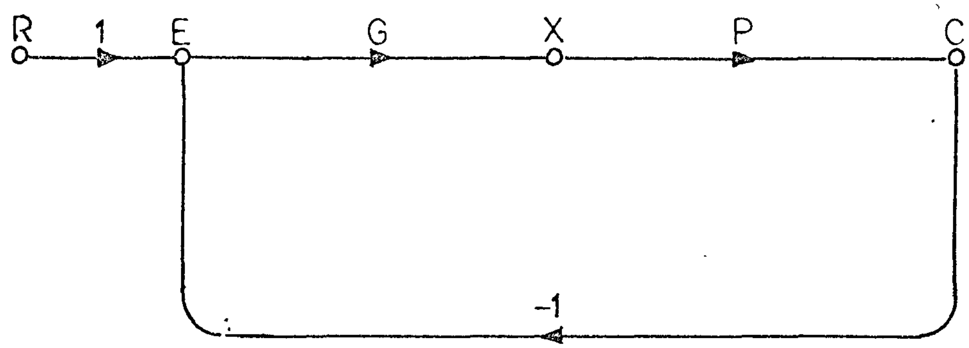
$$= 1 - T \tag{4-24}$$

where

$$T = \frac{GP}{1 + GP}$$

The one-degree freedom system, is inherently more sensitive to parameter variation than the two-degree freedom system.

Consider next, a two-degree-of-freedom structure as in Figure 4. The significant feature of this type of system is that the system sensitivity (to plant) function S_P^T, and the system transmission function T can be independently realized. These two functions T and S_P^T, fix the values of G and H functions in the configuration shown in Figure 4.



L = GP   T = C/R

Fig 3. Single-degree-of-freedom Structure

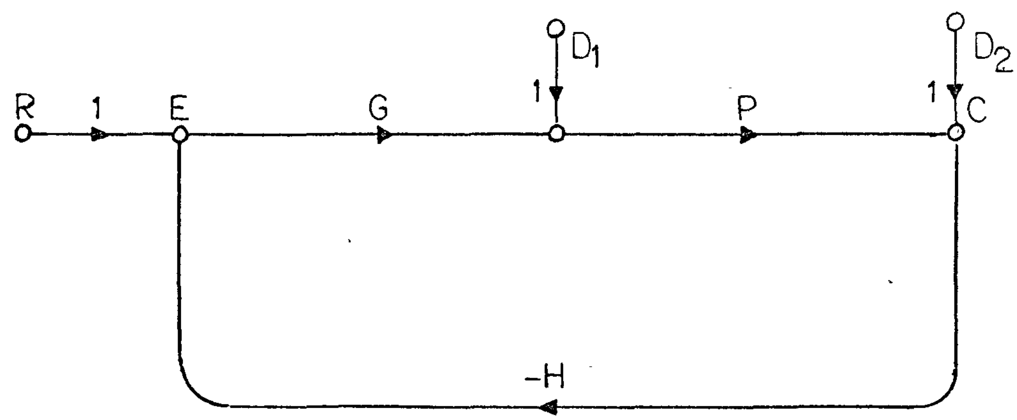


Fig 4. Two-degree-of-freedom Structure

$$T = \frac{GP}{1 + GPH} = \frac{GP}{1 + L} \qquad (4\text{-}25)$$

where

$$L = GPH$$

so

$$S_T^P = \frac{1}{1 + L} \qquad (4\text{-}26)$$

By adding a zero or pole to L, it is possible to improve the system, from the sensitivity point of view. But in one-degree freedom systems, by adding a zero to L means, it appears as a zero in T also, so significantly changes T as well as L. There is no sufficient freedom to independent by control T and the sensitivity of the poles of T.

In the case of a two-degree freedom system, zeros (and to a lesser extent, poles) can be assigned to L without worrying about their effect on T. Thus L is chosen primarily to attain the desired sensitivity of T, and there still is one-degree of freedom to achieve the desired T.

Thus, it is possible to achieve better sensitivity values, using a two-degree freedom system configuration, rather than a one-degree freedom system configuration.

4.5 SENSITIVITY IN THE TIME DOMAIN

The sensitivity of the system in the time domain is defined as the variation of the state variables due to a parameter variation[18,20].

By parameter variations we mean any deviations from the values initially taken by them. These may be static, time-invariant or time-variant.

Consider a multi-variable system, whose dynamic response may be represented by a set of first order equations as follows:

$$\dot{x}_i = f_i\,[X,\ t,\ \alpha] \qquad i = 1,\ 2,\ \ldots\ldots n \tag{4-27}$$

Where x_i are the set of independent state variables which describe at any time t, the state of the system.

$$\alpha = (\alpha_1,\ \alpha_2 \ldots) \text{ are system parameters.}$$

with initial conditions $x_i(0) = x_i^0$.

Let the parameters change by $\Delta\alpha$, the varied motion then is described by the system as:

$$\dot{\tilde{x}}_i = f_i\,(\tilde{X},\ t,\ \alpha+\Delta\alpha) \tag{4-28}$$

Then $$\Delta X_i = \tilde{X}_i(t) - X_i(t) \qquad i = 1,\ 2,\ \ldots n \tag{4-29}$$

$$= \sum_{j=1}^{M} \left(\frac{\partial x_i}{\partial \alpha_i} \right) \Bigg|_{\substack{\Delta\alpha = 0 \\ \Delta\alpha_m = 0}} \Delta\alpha_j \tag{4-30}$$

by Taylor series expansion, to a first order approximation, let us denote

$$w_{ij}(t) = \frac{\partial x_i(t)}{\partial \alpha_j} \tag{4-31}$$

when $\Delta\alpha_1 = 0,\ \ldots\ldots\ \Delta\alpha_m = 0$

The function $w_{ij}(t)$ is called the "sensitivity function" of the co-ordinate X_i relative to the parameter α_j[21, 31].

The standard procedure to obtain the sensitivity function w_{ij} follows the pattern proposed by Murray and Miller[21]. The method is based on the differentiation of equation (4-27); followed by a Taylor series development in $\Delta\alpha$ to obtain equations (4-30). This method reduces the problem of calculating the sensitivity function $w_{ij}(t)$ to the solution of the so-called sensitivity differential equations, obtained from equation (4-27). For each component of state vector x_i

$$\dot{w}_{kj} = \sum_{1}^{n} \frac{\partial f_k}{\partial x_i} w_{ij} + \frac{\partial f_k}{\partial \alpha_i}$$

$$w_{kj}(0) = 0 \qquad (i, k = 1, 2, \ldots .n \quad j = 1, 2, \ldots .m) \qquad (4\text{-}32)$$

Equations (4-32) are called the "sensitivity differential equations" or simply "sensitivity equations".

Also note $\dot{w}_{ij}(t) = \dfrac{dw_{ij}(t)}{dt} = \dfrac{\partial \dot{x}_i(t)}{\partial \alpha}$. The sensitivity functions $w_{ij}(t)$ are obtained as a result of solving the "m" equations given by Eq. (4-32). In general these are linear equations with variable coefficients $\dfrac{\partial f_K}{\partial x_i}$.

The general method of solving the equations (4-32) on anolog computors are given by Meissinger[22]. However, the above method, unfortunately, requires that the function of f in equation (4-27) be regular in α, which is by no means always the case in practice.

There are methods, including the method of undetermined coefficients, difference equations, and asymptotic expansions which do not have this limitation. Kokotović[23], has demonstrated the advantage of such a broader approach to sensitivity analysis.

A method, developed by Richard Dorf[20] using partial derivatives, to solve sensitivity equations is given:

$$\dot{w}_{Kj} = \sum_{i=1}^{n} \left(\frac{\partial f_K}{\partial x_i}\right) w_{ij} + \frac{\partial f_K}{\partial \alpha_j}$$

$$w_{Kj}(0) = 0 \qquad (i,\ K = 1,\ 2,\ .\ .\ .\ .n$$

$$j = 1,\ 2,\ .\ .\ .\ .m)$$

or

$$\frac{\partial \dot{x}_K}{\partial \alpha_i} = \sum_{1}^{n} \left(\frac{\partial f_K}{\partial \alpha_i}\right) \frac{\partial x_i}{\partial \alpha_j} + \frac{\partial f_K}{\partial \alpha_j} \tag{4-33}$$

or in matrix form we have:

$$\begin{bmatrix} \frac{\partial \dot{x}_1}{\partial \alpha_r} \\ \frac{\partial \dot{x}_2}{\partial \alpha_r} \\ \vdots \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial \alpha_r} \\ \frac{\partial x_2}{\partial \alpha_r} \\ \vdots \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial \alpha_r} \\ \frac{\partial f_2}{\partial \alpha_r} \\ \vdots \end{bmatrix} \tag{4-34}$$

As it is desired to obtain the variation in each state variable with respect to the parameter α_r that is $\frac{\partial X(t)}{\partial \alpha_r}$ or denoting the gradient of f with respect to X by a Jacobian matrix J and $\frac{\partial X}{\partial \alpha_r}$ by a vector V and:

$$Z = \frac{\partial f}{\partial \alpha_r}, \text{ equation (4-34) can be written as}$$

$$\dot{V} = JV + Z \tag{4-35}$$

Iterating the solution of Eq. (4-35) is[24] using transition matrix.

$$V(t) = \Phi(t)\ V(0) + \int_0^t \Phi\ (t-\tau)\ Z(\tau)d\tau \tag{4-36}$$

So the change in the state vector due to a change in a simple parameter is:

$$V(t) = \frac{\Delta X(t)}{\Delta \alpha_r} = \Phi(t)\ V(0) + \int_0^t \Phi(t-\tau)\ Z(\tau)\ d\tau \tag{4-37}$$

The simultaneous solution of Eqs. (4-35) and (4-37) can be obtained by computor solution[22]. The following example illustrates the procedure to obtain the resulting change in each state variable for a linear system with parameter variations.

Consider the system with transfer functions:

$$G(s) = \frac{1}{(s+1)(s+2)} \quad \text{and the system matrix} \quad A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

The system transition matrix is:

$$\Phi(t) = \begin{bmatrix} (2e^{-t} - e^{-2t}) & (e^{-t} - e^{-2t}) \\ (-2e^{-t} + 2e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix} \tag{4-38}$$

Assume that the parameter $\alpha_{21} = p_1 = -2$ increases by a small amount so that $\alpha_{21} = -2\ (1 + \epsilon)$ and $V(0) = 0$. Then we get using Eq. (4-36):

$$V(t) = \int_0^t \Phi(t - \tau)\ Z(\tau)\ d\tau \tag{4-39}$$

where

$$Z(\tau) = \begin{bmatrix} 0 \\ x_1(\tau) \end{bmatrix}$$

Therefore, we find that,

$$V_1(t) = \frac{x_1}{p_1} = \int_0^t \Phi_{12}(t - \tau)\ Z(\tau)\ d\tau \tag{4-40}$$

where

$$Z(\tau) = \begin{bmatrix} 0 \\ x_1(\tau) \end{bmatrix}$$

Therefore, we find that,

$$V_1(t) = \frac{\partial x_1}{\partial p_1} = \int_0^t \Phi_{12}(t - \tau)\ \Phi_{11}(\tau)\ d\tau \tag{4-40}$$

$$V_2(t) = \frac{\partial x_2}{\partial p_1} = \int_0^t \Phi_{22}(t - \tau)\ \Phi_{11}(\tau)\ d\tau \tag{4-41}$$

Φ_{11}, Φ_{12}, Φ_{21}, Φ_{22} are elements of the transition metrix.

when $X^T(0) = [1,\ 0]$ and thus $X_1(\tau) = \Phi_{11}(\tau)$

Evaluating Eqs. (4-40) and (4-41) we get:

$$V_1(t) = e^{-t}\ (2t - 3) + e^{-2t}\ (t + 3) \tag{4-42}$$

$$V_2(t) = e^{-t}(5 - 2t) - e^{-2t}(5 + 2t) \quad (4\text{-}43)$$

The time response of the sensitivity variables, $V_1(t)$ and $V_2(t)$ is given in Table I and is shown in Figure 5.

It would be desirable to compare the accuracy of the sensitivity variables with the exact change of the state variables due to a change in α_{21} so that $\alpha_{21} = -2(1 + \varepsilon)$. The actual change in $x_1(t)$ was calculated for $\varepsilon = 0.05$ by obtaining $x_1(t)$ when $\alpha_{21} = -2$ and $\alpha_{21} = -2.1$. Several values of the actual $\Delta x_1(t)$ are shown in Figure 5.

Since $V_1(t) = \dfrac{\Delta x_1}{\Delta p_1}$, we have,

$$\begin{aligned} \Delta x_1 &= V_1(t)\ \Delta p_1 \\ &= V_1(t)\ (\varepsilon\,\alpha_{21}) \\ &= -0.1\ V_1(t) \end{aligned} \quad (4\text{-}44)$$

The curve of the change in $x_1(t)$ obtained from the sensitivity coefficient is shown in Figure 6. The agreement is very good for this 5% increasement in the parameter.

Sensitivity analysis by using perturbation techniques were also developed[24]. The system equation given by (4-27) can be written as:

$$\dot{X} = f\ (X, u, t) \quad (4\text{-}45)$$

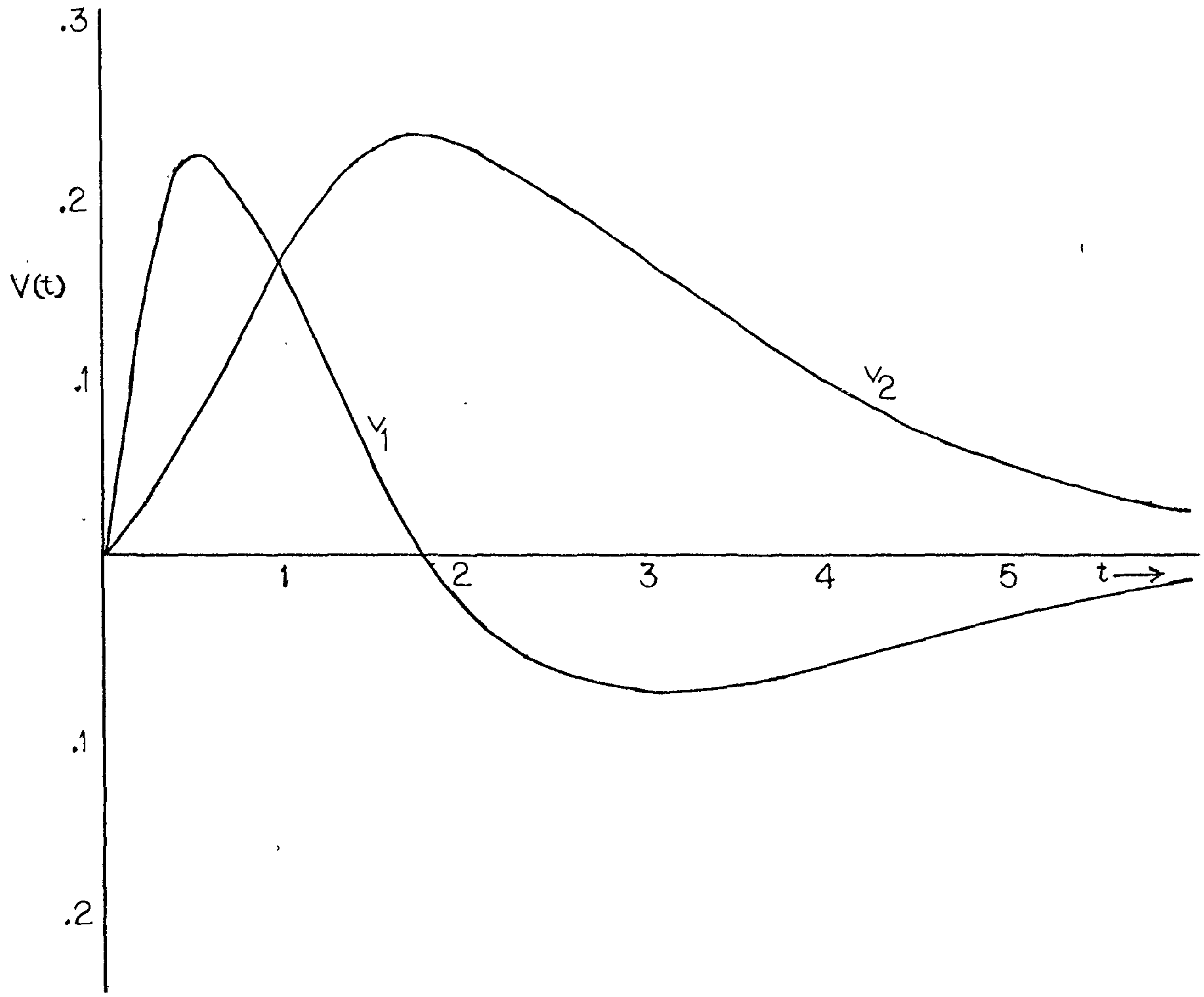


Fig.5. The Time Response of the Sensitivity Variables

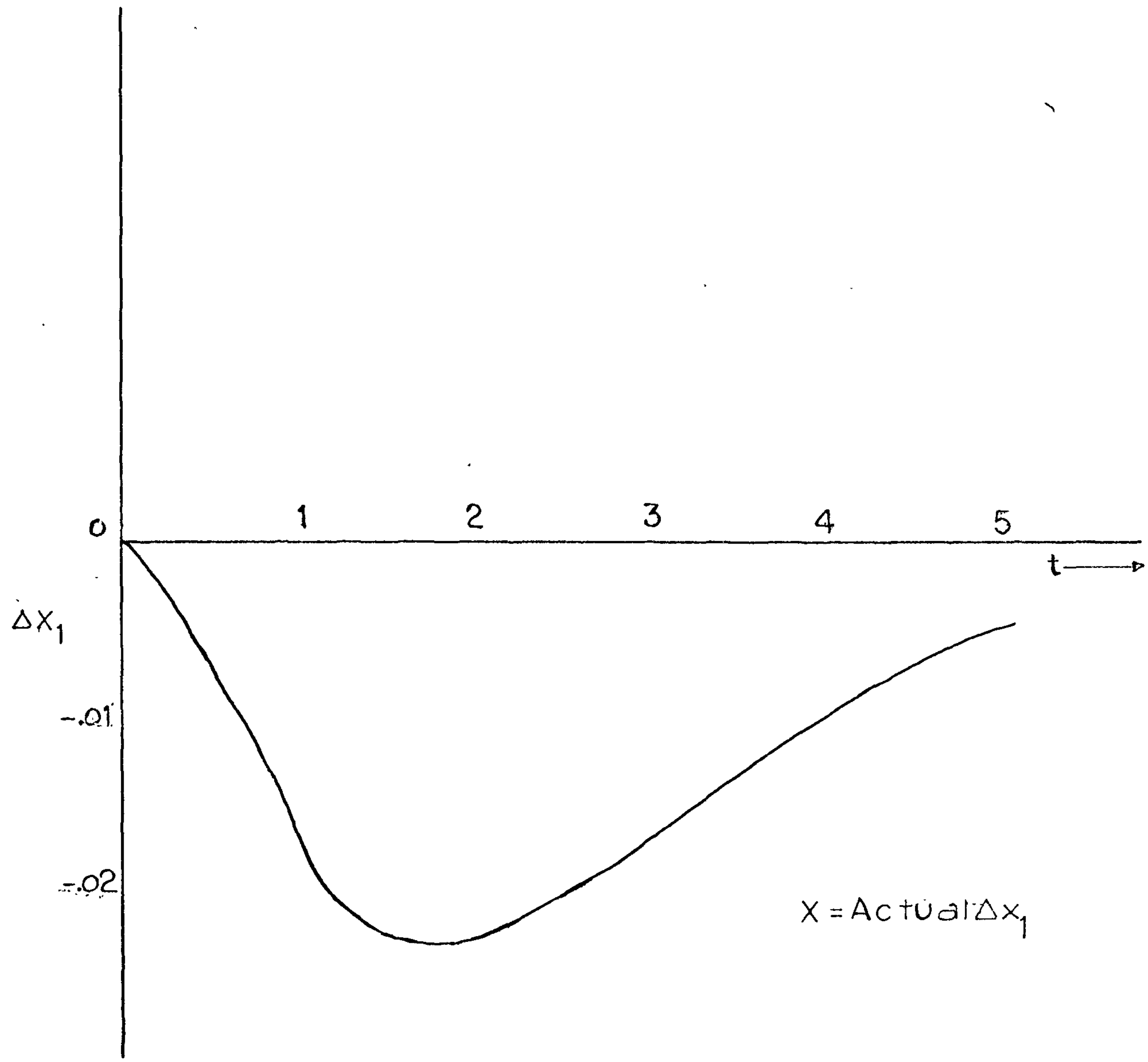


Fig. 6. The Change in the State Variable $x_1(t)$ Obtained from the Sensitivity Coefficient

Linearizing the system equation about the operating point, the equation can be written as:

$$\dot{X} = A X + B u \tag{4-46}$$

A is the Jacobian matrix so that $a_{ij} = \frac{\partial f_i}{\partial x_i}$ and B is $b_{ij} = \frac{\partial f_i}{\partial u_j}$.

If there is a change in the system parameters, then the system matrix A will change, and if the change in system matrix is represented by εD. Then the problem can be considered to be represented by the perturbation equation[24] as:

$$\dot{X} = (A + \varepsilon D) X \tag{4-47}$$

The solution of Eq. (4-47) obtained by the perturbation methods is written as:

$$X(t) = \Phi(t)\ X(0) + \varepsilon \int_0^t \Phi(t - \tau)\ D\ X(\tau)\ d\tau + O(\varepsilon^2) \tag{4-48}$$

$\Phi(f) = e^{At}$ is the systems transition matrix, and X(0) is the initial value of the state vector.

Iterating equation (4-48) can be written as (Since $X(t) = \Phi(f)$ $X(0)$, then $X(\tau) = \Phi(\tau)\ X(0)$).

$$X(t) = \Phi(t)\ X(0) + \varepsilon \int_0^t \Phi(t - \tau)\ D\ \Phi(\tau)\ X(0)\ d\tau + O(\varepsilon^2) \tag{4-49}$$

Therefore, the change in state vector due to a change in system matrix is given by:

$$\Delta X(t) = \varepsilon \int_0^t \Phi(t - \tau)\ D\ \Phi(\tau)\ X(0)\ d\tau \tag{4-50}$$

neglecting the terms containing second and higher order powers of ϵ.

For a stable system, the elements of $\Phi(t)$ are decaying exponentials and this equation (4-50) converges to zero as time increases without limit.

Here in this method, we assumed that the non-linearities in the system are not large, so that we can linearize the system equation, but if the non-linearities are of large nature it is impossible to use the linearized system equation (4-46). In that case we may have to resort to computer solutions or variable gain approximation[20] methods. This technique can also be applied to time varying systems as well.

Sensitivity analysis utilizing the time-domain matrix has been developed by Richard Dorf[20].

CHAPTER V

SENSITIVITY OF OPTIMAL CONTROL SYSTEMS

5.1 INTRODUCTION

In the past, researchers have studied the sensitivity problem in classical feed back control systems in great detail, some aspects of which have been presented in the previous chapter. However, very little work has been published on the sensitivity problem in the more recent theory of multi-variable optimal control systems.

Dorato[25] in 1963 has called attention to this problem, for the first time, which is concerned with the change in value of the performance index with infinitesimal parameter variations.

Pagurek[26] in 1965 has shown the sensitivity of cost functional, to a first order approximation due to an infinitesimal change in the plant parameter is same for both open-loop and closed-loop control of linear systems with respect to a quadratic performance index. These results were extended by Witsenhausen[27] in a recent note.

Inclusion of sensitivity function as an integral part of the performance index has been considered by Rohrer and Sobral [28], also by Holtzman and Horing[29].

Typical results of sensitivity analysis which deal with the effects of perturbations on the optimal control have been reported by

Belanger[30] in 1966.

In the first section of this chapter the results of Pagurek and Witsenhausen are presented and some of their practical limitations are studied. In the next section, for a typical second order system, the results concerned with optimal control law and sensitivity in both open-loop and closed-loop optimal control systems are obtained for a definite variation in a plant parameter. Finally, the sensitivity with finite variations of plant parameters is considered.

5.2 RESULTS OBTAINED BY PAGUREK AND WITSENHAUSEN AND THEIR PRACTICAL LIMITATIONS

Suppose that the system to be controlled is described by vector differential equation.

$$\dot{X}(t) = f(X(t),\ u(t),\ t,\ \alpha)\ ,\ X(t_o) = X_o \tag{5-1}$$

The vector $\alpha = (\alpha_1, \alpha_2 \ldots \ldots \alpha_m)$ represents a set of m plant parameters. The optimization problem is to choose the control input over an interval (t_o, t_f) such that the index of performance:

$$J = \int_{t_o}^{t_f} L\left[X(t),\ u(t),\ t,\ \alpha\right] \tag{5-2}$$

is minimized.

The optimal closed loop control law is assumed as:

$$u^*(t) = \Psi\left[X(t),\ t,\ \alpha\right] \tag{5-3}$$

or open-loop control law:

$$u^*(t) = \theta \quad [X(t), t, \alpha] \tag{5-4}$$

as the case may be, has been determined by employing the available techniques of optimization, as presented in chapter 3.

In synthasizing the optimal controller, a number of components are used, whose values we denote by the vector:

$$b = (b_1, b_2 : \ldots \ldots b_p)$$

These components are related to the system (plant) parameter by:

$$b_i = g^*{}_i \, (\alpha_1, \alpha_2 \ldots \ldots \alpha_m) \quad i = 1, 2 \ldots p$$

or in vector form:

$$b = g^*(\alpha) \tag{5-5}$$

So that the output of the optimal controller (assuming closed-loop control):

$$u^*(t) = \Psi_c[X(t), t, b]$$

is related to the optimal control law given by Equation (5-2), by:

$$\Psi_c \; [X(t), t, g^*(\alpha)] = \Psi \quad [X(t), t, \alpha]$$

Unfortunately in actual practice the plant parameter vector α, which appears in Equation (5-2), seldom corresponds to the value of α used in the controller (5-3). This is due to such things as component inaccuracies, environmental effects, aging etc. The problem then is to determine the effect of such variations on the performance index.

In general, controller components are less subject to variations than plant parameter. With this in mind, we shall consider only the case of a fixed controller which is designed for a plant, whose parameters vary, i.e., when we say parameter variations, we are speaking of plant parameter variations.

Assume that the plant parameters have drifted from the above nominal values to values given by a vector $\alpha \neq \alpha_o$. Again, when the plant parameters are given by α, and the controller component values are given by $b_o = g^*(\alpha_o)$, we denote the corresponding values of the performance index by $J(b_o, \alpha)$. Here optimal operation requires that $\alpha = \alpha_o$ with the corresponding value of the index of performance $J(b_o, \alpha_o)$.

Due to the difference $\Delta\alpha_i = \alpha_i - \alpha_{oi}$, the change in the value of performance index is:

$$\Delta J = J(b_o, \alpha) - J(b_o, \alpha_o) \tag{5-6}$$

If the parameter variations are infinitesimal, Equation (5-6) can be written by Taylor series expansion of $J(b_o, \alpha)$ about $\alpha = \alpha_o$,

$$\Delta J = \sum_i \left(\frac{\partial J(b_o, \alpha)}{\partial \alpha_i} \bigg|_{\alpha = \alpha_o} \right) \Delta\alpha_i \tag{5-7}$$

the derivative:

$$\frac{\partial J(b_o, \alpha)}{\partial \alpha_i} \bigg|_{\alpha = \alpha_o}$$

is denoted as the performance index sensitivity function for the parameter

α_i. It should be noted that $J(b_o, \alpha)$ is not necessarily the minimum with respect to α at $\alpha = \alpha_o$, and so these sensitivity functions need not vanish.

It should also be noted that the actual structure of the controller and the controller component values do not appear in this problem at all although they do enter in the first case considered. Here, the results depend only on the nominal plant parameters α_o, and the actual plant parameters given by α. This follows because $b_o = g^*(\alpha_o)$ and so:

$$\Psi_c \; [t, X(t), b_o] = \Psi \; [t, X(t), \alpha_o] \tag{5-8}$$

Following Dorato[25], the direct computation of the performance index sensitivity function, assuming an index of performance of the form given by Equation (5-2). Recalling Equation (5-8), we substitute either the open-loop control law given by Equation (5-4) or the closed-loop control law given by Equation (5-3), with $\alpha = \alpha_o$ into Equations (5-1) and (5-2) to get:

$$\dot{X} = f \; [t, X(t), u^*(t), \alpha] = f_1 \; [t, X(t), \alpha_o, \alpha] \tag{5-9}$$

and

$$J(b_o, \alpha) = \int_{t_o}^{t_f} L[t, X(t), u^*(t)] \, dt = \int_{t_o}^{t_f} L_1[t, X(t), \alpha_o, \alpha] \, dt \tag{5-10}$$

Therefore, from Eq. (5-10):

$$\left. \frac{\partial J(b_o, \alpha)}{\partial \alpha_i} \right|_{\alpha = \alpha_o} = \int_{t_o}^{t_f} \frac{\partial L_1}{\partial x}_{\alpha = \alpha_o} \left(\left. \frac{\partial L_1}{\partial \alpha_1} \right|_{\alpha = \alpha_o} \right) dt \tag{5-11}$$

and from (5-9):

$$\frac{d}{dt}\left(\frac{\partial X}{\partial \alpha_i}\bigg|_{\alpha = \alpha_o}\right) = \frac{\partial f_1}{\partial X}\bigg|_{\alpha = \alpha_o}\left(\frac{\partial X}{\partial \alpha_i}\bigg|_{\alpha = \alpha_o}\right) + \frac{\partial f_1}{\partial \alpha_i}\bigg|_{\alpha = \alpha_o} \tag{5-12}$$

Here $\frac{\partial L_1}{\partial X}$ is an n-dimensional row vector, $\frac{\partial X}{\partial \alpha_i}$ is an n-dimensional column vector, $\frac{\partial f_1}{\partial X}$ is an n by n matrix such as:

$$\frac{\partial f_1}{\partial X} = \begin{bmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial X_1} & \cdots & \frac{\partial f_n}{\partial X_n} \end{bmatrix}, \tag{5-13}$$

$\frac{\partial f_1}{\partial \alpha_i}$ is an n-dimensional column vector.

Except, for certain degenerate cases the boundary condition for Equation (5-12) is:

$$\left(\frac{\partial X_i}{\partial \alpha_i}\bigg|_{\alpha = \alpha_o}\right) = 0 \text{ at } t = t_o \ (i = 1, 2, \ldots m) \tag{5-14}$$

Since Equation (5-12) is linear though time-varying, a solution of Equation (5-12) may be written explicitly in terms of the transition motion $\Phi(t, \tau)$ (Zadeh and Desoer 1963[33]) for the system:

$$\frac{d}{dt}\left(\frac{\partial X}{\partial \alpha_i}\bigg|_{\alpha = \alpha_o}\right) = \frac{\partial f_1}{\partial x}\bigg|_{\alpha = \alpha_o}\left(\frac{\partial X}{\partial \alpha}\bigg|_{\alpha = \alpha_o}\right) \tag{5-15}$$

Thus one can write:

$$\left.\frac{\partial X}{\partial \alpha_i}\right|_{\alpha = \alpha_o} = \int_{t_o}^{t_f} \Phi(t, \tau) \left.\frac{\partial f_i}{\partial \alpha}\right|_{\alpha = \alpha_o} d\tau \tag{5-16}$$

which completes the evaluation of:

$$\left(\left.\frac{\partial J(b_o, \alpha)}{\partial \alpha_i}\right|_{\alpha = \alpha_o} \right) \quad \text{given by eq. (5-11).}$$

In general, the solution of Eq. (5-12) and the integration in eq. (5-11) are sufficiently complex as to require a computer solution.

The procedure must be repeated for every set of initial conditions $X^*(t_o) = X_o$. But Pagurek developed a method which eliminates the necessity of repeating the computations for each set of initial conditions, which is presented now. Consider a linear system of the form:

$$\left.\begin{aligned} \dot{X}(t) &= A_o(t)\, X(t) + B_o(t)\, u(t),\ X(t_o) = X_o \\ y(t) &= H_o(t)\, X(t) \end{aligned}\right\} \tag{5-17}$$

A, B, are n by n and n by r-dimensional matrices respectively, H_o is a p by n matrix, where $p < n$. The index of performance is of the form;

$$J(t_o, X(t), t_f) = \frac{1}{2} \int_{t_o}^{t_f} [\langle y(t), Q_o y(t)\rangle + \langle u(t), R_o(t)u(t)\rangle]dt + \frac{1}{2}\ X(t_f), M_o X(t_f) \quad , \tag{5-18}$$

where Q and R are symmetric positive definite p by p and r by r-dimentional matrices respecitvely and M_o is a constant symmetric non-negative definite n by n-dimentional matrix. We use the notation:

$$\langle y, Qy \rangle = \sum_i \sum_j q_{ij} y_i y_j = y^T Qy \tag{5-19}$$

for quadratic forms.

It has been given in chapter III, Equation (3-30), using Kalman's method, the optimal control law as:

$$u^*(t) = -R_o^{-1}(t)\, B_o^T(t)\, M_o(t, t_f)\, X(t) \tag{5-20}$$

where $M_o(t, t_f)$ is the solution of the matrix Ricatti equation:

$$\frac{dM_o}{dt} + A_o^T M_o + M_o A_o - M_o B_o R_o^{-1} B_o^T M_o + H_o^T Q_o H_o = 0, M_o(t_f, t_f) = K \tag{5-21}$$

The optimal value of index of performance, when $x(t_o) = x_o$ can be written as:

$$J^*(t_o, X_o, t_f) = \frac{1}{2} \langle X_o, M_o(t, t_f), X_o \rangle \tag{5-22}$$

we assume here, that the nominal system is given by matrices A_o, B_o, Q_o, R_o and H_o, also that α_o represents the set of parameters in these matrices, constant over (t_o, t_f), the optimal time variable feed back control input based on the nominal parameters α_o is:

$$u^*(t) = \Psi_c[t, X(t), g^*(\alpha_o)] = \Psi[t, X(t), \alpha_o] = -R_o^{-1} B_o^T M_o(t, t_f)\, X(t). \tag{5-23}$$

Now suppose that the actual system is given by the matrices A, B, Q, R and H, with the true value of the parameter vector being α. When the control input given by Equation (5-23) is used with this plant, there results the closed-loop system:

$$\dot{X}(t) = F(t)X(t), \; X(t_o) = X_o \; , \tag{5-24}$$

where

$$F(t) = A - B\, R_o^{-1} B_o^T M_o(t, t_f) \tag{5-25}$$

The closed-loop response is then:

$$X(t) = \Phi(t, t_o)\, X_o \; , \tag{5-26}$$

where $\Phi(t, t_o)$ is the transition matrix of the free system given by equation (5-24). On substituting Equation (5-25) and Equation (5-23) into the index of performance we obtian:

$$J\,(t_o, X_o, b_o, \alpha, t_f) = \frac{1}{2} X_o^T \left[\int_{t_o}^{t_f} \Phi^T(t, t_o) Q_1(t) \Phi(t, t_o)\, dt + \Phi^T(t_f, t_o)\, K\, \Phi(t_f, t_o) \right] X_o \; , \tag{5-27}$$

where

$$Q_1(t) = M_o B_o R_o^{-1} R\, R_o^{-1} B_o^T M_o + H^T Q\, H.$$

Clearly it is seen from Equation (5-27), that $J(t, X(t), b_o, \alpha, t_f)$ is of the form:

$$J(t, X(t), b_o, \alpha, t_f) = \frac{1}{2} \left\langle X, M(t, t_f)\, X \right\rangle \; ,$$

where

$$M(t,\ t_f) = \int_{t_o}^{t_f} \Phi^T(\tau,t)\ Q_1(\tau)\ \Phi(\tau,t)\ d\tau + \Phi^T(t_f,t)K\ \Phi(t_f,\ t) \qquad (5\text{-}28)$$

Differentiating Equation (5-28) with respect to t using Leibnitz's rule, and using the properties of transition matrices (Zadeh and Desoer 1963)[33] we get:

$$\frac{dM}{dt} + F^T M + MF + Q_1 = 0; \quad M(T,\ T) = K \qquad (5\text{-}29)$$

This equation is a special case of the matrix Ricatti equation, and is well known in Liapunov stability theory.

To calculate the performance index sensitivity function:

$$\left.\frac{\partial J(t_o,\ X_o,\ b_o,\ \alpha,\ t_f)}{\partial \alpha_i}\right|_{\alpha = \alpha_i} \qquad i = 1,\ 2\ .\ .\ .\ .m.$$

differentiating Equation (5-29) with respect to t, there results:

$$\frac{d}{dt}\ \frac{\partial M}{\partial \alpha_i} + F^T\ \frac{\partial M}{\partial \alpha_i} + \frac{\partial M}{\partial \alpha_i} F + \frac{\partial F^T}{\partial \alpha_i} M + M\,\frac{\partial F}{\partial \alpha_i} + \frac{\partial Q_1}{\partial \alpha_i} = 0 \qquad (5\text{-}30)$$

with

$$\frac{\partial M}{\partial \alpha_i}\ (t_f,\ t_f) = 0$$

Now let

$$P_i(t,\ t_f) = \left.\frac{\partial M(t,\ t_f)}{\partial \alpha_i}\right|_{\alpha = \alpha_o}$$

Hence letting $\alpha = \alpha_o$ in Equation (5-30):

$$\frac{dP_i}{dt} + F_o^T P_i + P_i F_o + Q_2 = 0; \quad P_i(t_f, t_f) = 0 \tag{5-31}$$

where

$$Q_2 = \left[\frac{\partial F^T}{\partial \alpha_i} M_o + M_o \frac{\partial F}{\partial \alpha_i} + \frac{\partial Q_1}{\partial \alpha_i} \right]_{\alpha = \alpha_o} \tag{5-32}$$

In practice, an open-loop controller may be easier to synthesize than a closed-loop controller. In this case for the system given by Equation (5-17) and performance index given by Equation (5-18), the open-loop controller based on nominal parameters α_o, A_o, B_o etc., produces a control input:

$$u^*(t) = -R_o^{-1} B_o^T M_o(t, t_f) \, X^*(t) \, , \tag{5-33}$$

where

$$\dot{X}^*(t) = (A_o - B_o R_o^{-1} B_o^T M_o) \, X^*(t) \qquad x^*(t_o) = x_o$$

letting

$$F_o(t) = A_o - B_o R_o^{-1} B_o^T M_o(t, t_f)$$

and

$$\dot{\Phi}(t, t_o) = F_o(t) \, \Phi_o(t, t_o), \; \Phi_o(t_o, t_o) = 1$$

we obtain

$$X^*(t) = \Phi_o(t, t_o) \, X_o,$$

The dynamics of the perturbed system are then given by:

$$\dot{X} = Ax - BR_o^{-1}B_o^T M_o(t, t_f)\ \Phi_o(t, t_o)\ X_o\ .$$

Defining $\Pi(t, t_o)$, such that:

$$\dot{\Pi}(t, t_o) = A\ \Pi(t, t_o),\ \Pi(t_o, t_o) = 1$$

we have

$$X(t) = G(t, t_o)\ X_o$$

where

$$G(t, t_o) = \Pi(t, t_o) - Y(t, t_o)$$

and

$$Y(t, t_o) = \int_{t_o}^{t} \Pi(t, \tau)\ B\ R_o^{-1}\ B_o^T M_o(\tau, t_f)\ \Phi(\tau, t_o)\ d\tau$$

We define also:

$$F(t) = A - B\ R_o^{-1}\ B_o^T M_o(t, T)$$

Substituting $u^*(t)$ and $X(t)$ into Equation (5-18), we get:

$$J(t_o, X_o, b_o, \alpha, t_f) = \frac{1}{2} X_o^T \left[\int_{t_o}^{T} G^T(t, t_o)\ H^T\ Q\ H\ G(t, t_o)\ dt \right.$$

$$\left. + G^T(t_f, t_o)\ K\ G(t_f, t_o) \right] X_o + \frac{1}{2} X_o^T \int_{t_o}^{T} \left[\Phi_o^T(t, t_o)\ [Q_1 - H^T Q\ H] \Phi_o(t, t_o) dt \right] X_o \qquad (5\text{-}34)$$

where

$$Q_1(t) = M_o B_o R_o^{-1} R\, R_o^{-1} B_o^T M_o + H^T Q\, H.$$

let

$$J(t, X, b_o, \alpha, t_f) = \frac{1}{2} \left\langle X\, M(t, t_f)\, X \right\rangle \tag{5-35}$$

where

$$M(t, t_f) = M_1(t, t_f) + M_2(t, t_f)$$

with

$$M_1(t, t_f) = \int_{t_o}^{t_f} G^T(\tau,t) H^T Q\, H\, G(\tau, t)\, d\tau + G^T(t_f, t)\, K\, G(t_f,t) \tag{5-36}$$

and

$$M_2(t, t_f) = \int_{t_o}^{t_f} \Phi_o^T(\tau, t) \left[Q_1(\tau) - H^T Q\, H\right] \Phi_o(\tau, t)\, d\tau \tag{5-37}$$

Differentiating Equations (5-36) and (5-37) with respect to t and adding we get as:

$$\frac{dM}{dt} + F_o^T M + MF_o + \left[Q_1 + (F-F_o)^T (C_1-C_2) + (C_1-C_2)^T (F-F_o)\right] = 0,\ M(t_f, t_f) = K \tag{5-38}$$

where C_1 and C_2 satisfies:

$$\frac{dC_1}{dt} + A^T C_1 + \dot{C}_1 + C_1 A + H^T Q\, H = 0; \quad C_1(t_f, t_f) = K \tag{5-39}$$

and

$$\frac{dC_2}{dt} + A^T C_2 + C_2 A + C_1 B\, R_o^{-1} B_o^T M_o = 0; \quad C_2(t_f, t_f) = 0 \tag{5-40}$$

Note that of $\alpha = \alpha_o (A = A_o, B = B_o$, etc.) then M becomes M_o and also $C_o = (C_1 = C_2)\big|_{\alpha = \alpha_o} = M_o$.

Again let

$$P_i(t, t_f) = \left.\frac{\partial M(t, t_f)}{\partial \alpha_i}\right|_{\alpha = \alpha_o}$$

Proceeding in α similar as in the closed-loop case, there results:

$$\frac{dP_i}{dt} + F_o^T P_i + P_i F_o + \left[\frac{\partial F^T}{\partial \alpha_i} C_o + C_o \frac{\partial F}{\partial \alpha_i} + \frac{\partial Q_1}{\partial \alpha_i}\right]_{\alpha = \alpha_o} = 0 \tag{5-41}$$

Using Equation (5-32) we get:

$$\frac{\partial P_i}{\partial t} + F_o^T P_i + P_i F_o + Q_2 = 0$$

with

$$P_i(t_f, t_f) = 0$$

Therefore, comparing Equations (5-31) and (5-41) it is seen that the performance index sensitivity function to a first degree approximations are turned out to be identical in both open-loop and closed-loop optimal control systems, for small (infinitesimal) variations in plant parameters.

These results were extended by Weitsenhausen [27], who showed that the first order sensitivity of the performance index was identical for both open-loop and closed-loop optimal control for a broad class of non-linear systems and performance criteria, as follows.

Any sufficiently smooth non-linear problem is linear-quadratic to first order approximation. This is the case even in the presence of smooth constraints and for any sufficiently smooth controller optimal just for the one nominal system condition for which the sensitivity is sought.

Consider three real Banach spaces [28]:

an "input space" Σ_u

an "output space" Σ_x

a "parameter space" Σ_α

Let S be a function with domain D(S) in $\Sigma_u \oplus \Sigma_\alpha$ and range Σ_x. Let B be a real function with domain D(V) in $\Sigma_u \oplus \Sigma_\alpha$. Let F be a function with domain D(F) in Σ_x and range in Σ_u. Then, by implicit function theorem [28], in a neighbourhood of (u_o, x_o, α_o) the system:

$$x = S(u, \alpha) \tag{5-42}$$

$$u = F(x) \tag{5-43}$$

defines u and x uniquely as differentiable functions of α.

$x = X(\alpha)$ with derivative X_α' at α_o

$$u = U(\alpha) \text{ with derivative } U_\alpha' \text{ at } \alpha_o$$

and to first order approximation.

$$dx = S_u' \, du + S_\alpha' \, d\alpha$$

$$du = F_u' \, dx \tag{5-44}$$

from which

$$X_\alpha' = (I - S_u' F_x')^{-1} S_\alpha' \tag{5-45}$$

$$U_\alpha' = F_x'(I - S_u' F_x')^{-1} S_\alpha' \tag{5-46}$$

I is the identity operator on Σ_u.

Now define $J(\alpha)$, a real function in a neighbourhood of α_o in Σ_α, by:

$$J(\alpha) = V\,[U(\alpha), X(\alpha), \alpha] \tag{5-47}$$

<u>Weitsenhausen's Theorem:</u>†

The Frichet derivative J_α' of J at α_o, a linear operator from $\Sigma_\alpha \to R$, exists and is given by:

$$J_\alpha' = V_x' S_\alpha' + V_\alpha' \tag{5-48}$$

an expression independent of F.

The theorem may be made intuitively obvious by considering the

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† The proof of this theorem can be found in Weitsenhaussen's paper

feed back loop cut at the input to S. Then, J is a function of α and u with zero partial with respect to u at $\alpha = \alpha_o$, $u = u_o$ by stationarity. When the loop is closed u changes with α, but this change has no first-order effect on J.

The input u_o need not be optimal for $\alpha = \alpha_o$, only stationarity is required.

The open-loop case corresponds to $F(x)$ = the constant u_o for all x. The content of theorem is that the same sensitivity J_α' is obtained for any F which:

1) maps x_o into u_o and is smooth in a neighbourhood of x_o.

2) makes sense in closed-loop operation (i.e., the linear operation $I - F_x' S_u'$ on Σ_u is invertible, which implies that $I - S_u' F_x'$ is investible on Σ_x.)

3) maps a neighbourhood of x_o into the constriant set if one is present.

Thus F need not be external for any condition other than (u_o, x_o, α_o). In particular it need not be related to what may happen for a different S. No similar results hold for the second sensitivity derivative.

Weitsenhassen applied his theorem to problems of 1) fixed nominal initial condition 2) arbitrary initial conditions.

In his results Pagurek considered an infitesimal plant parameter variation. Due to this, it became possible to approximate the performance index sensitivity function to a first order as:

$$\Delta J = \sum_i \left(\frac{\partial J(b_o, \alpha)}{\partial \alpha_i} \bigg|_{\alpha = \alpha_o} \right) \Delta\alpha_i$$

But in every practical system, the plant parameter variation is a finite quantity. So, in every practical system, the sensitivity function cannot be approximated to a first order, besides the second order and maybe the higher order terms also have to be included to get a fairly good knowledge of sensitivity. So expanding ΔJ given by Eq. (5-6) by Taylor series,

$$\Delta J = \sum_i \left(\frac{\partial J(b_o, \alpha)}{\partial \alpha_i} \bigg|_{\alpha = \alpha_o} \right) \Delta\alpha_i + \frac{1}{2} \sum_i \left(\frac{\partial^2 J(b_o, \alpha)}{\partial \alpha^2_i} \bigg|_{\alpha = \alpha_o} \right) (\Delta\alpha_i)^2 + \cdots \tag{5-49}$$

including the higher order terms also

Then the result of Pagurek may not hold good with this new identity which is obtained by considering a finite plant parameter variation.

Also Weitsenhausen pointed out that the results of (Pagurek and Weitsenhausen) will not hold for a second order approximation in his recent note[27].

Thus, although the results of Pagurek are very true for an infitesimal plant parameter variation, in a practical system, where the variations of plant parameter are finite amounts, the closed-loop (particularly if we consider a two-degree-of-freedom structure) optimal system is always advantageous than the open-loop optimal system, from the point of view of sensitivity of performance.

Following Dorato[25], we consider the sensitivity in an optimal control system as that concern with a variation in the performance index for a variation in a plant parameter.

Thus,

$$\Delta J = \sum_i \left(\frac{\partial J}{\partial \alpha_i}(b_o, \alpha) \Bigg|_{\alpha = \alpha_o} \right) \Delta\alpha_i + \frac{1}{2} \sum_i \left(\frac{\partial^2 J}{\partial \alpha^2_i}(b_o, \alpha) \Bigg|_{\alpha = \alpha_o} \right) (\Delta\alpha_i)^2 + \text{-----}$$

We define the sensitivity of performance in an optimal system as:

$$S = \frac{\Delta J}{\Delta \alpha}$$

where ΔJ = the variation in the performance index due to a variation in plant parameter.

$\Delta\alpha$ = the variation in plant parameter.

So from Equation (5-49), we get:

$$\frac{\Delta J}{\Delta \alpha} = \sum_i \left(\frac{\partial J}{\partial \alpha_i}(b_o, \alpha) \Bigg|_{\alpha = \alpha_o} \right) + \frac{1}{2} \sum_i \left(\frac{\partial^2 J}{\partial \alpha^2_i}(b_o, \alpha) \Bigg|_{\alpha = \alpha_o} \right) (\Delta\alpha_i) + \text{-----} \tag{5-50}$$

It can be seen from Eq. (5-50), that by defining the sensitivity of performance as $\frac{\Delta J}{\Delta \alpha}$ all the higher order terms are included. So we will get a fairly correct measure of sensitivity of performance,in a optimal control system, by calculating $\frac{\Delta J}{\Delta \alpha}$.

5.3 AN EXAMPLE OF SENSITIVITY OF OPEN-LOOP AND CLOSED-LOOP OPTIMAL SYSTEMS FOR A SECOND ORDER SYSTEM

Statement of the Problem:

Given a linear system as shown in Figure 10a with plant transfer function:

$$P(s) = \frac{1}{s(s+1)} \tag{5-51}$$

Given the boundary conditions:

$$X(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5-52}$$

Given the constraint on the control signal (input):

$$\int_0^{t_f} u^2 \, dt \leqslant K \tag{5-53}$$

Given the target:

$$X(t_f) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{5-54}$$

Given the performance index:

$$J = \int_0^{t_f} x_1^2 \, dt \tag{5-55}$$

t_f is not pre-specified.

Then, to find the control that:

(1) satisfies the constraint on u(t)

(2) transfers the state of the system from $X(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $X(t_f) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and

(3) minimizes the performance index.

Then to investigate,

(1) how sensitive is the minimum value of performance index to variations in the parameters of the plant.

(2) how sensitive is the minimum value of performance index to variations in the parameter of the plant, in an optimal closed-loop linear control system.

(3) comparison of the sensitivities of performance in optimal open-loop and closed-loop linear control systems.

SOLUTION:

$$P(S) = \frac{1}{s(s+1)}$$

The differential equation describing the system is:

$$\frac{d^2C}{dt^2} + \frac{dC}{dt} = u(t) \qquad (5\text{-}56)$$

The state variables are chosen as:

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} C \\ dC/dt \end{bmatrix} \qquad (5\text{-}57)$$

Then the set of first order differential equations describing the system is:

$$\frac{dx_1}{dt} = x_2 \qquad (5\text{-}58)$$

$$\frac{dx_2}{dt} = -x_2 + u \qquad (5\text{-}59)$$

This equation can be represented in matrix form as:

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad (5\text{-}60)$$

The index of performance is:

$$J = \int_{o}^{t_f} x_1^2 \quad dt$$

and the constraint on u is:

$$\int_{o}^{t_f} u^2 \, dt \leqslant K$$

Applying the Lagrange multiplier results in the criterion functions:

$$J_1 = \int_{o}^{t_f} (x_1^2 + \lambda u^2) \, dt \qquad (5\text{-}61)$$

let X_3 be the new co-ordinate given by:

$$x_3 = \int_{o}^{t_f} (x_1^2 + \lambda u^2) \quad dt \qquad (5\text{-}62)$$

or

$$\dot{x}_3 = (x_1^2 + \lambda u^2) \qquad (5\text{-}63)$$

So we have the three first differential equations as:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 + u$$

$$\dot{x}_3 = x_1^2 + \lambda u^2 \qquad (5\text{-}64)$$

subject to the boundary conditions:

$$X(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad X(t_f) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (5\text{-}65)$$

The Pontryagin function $\wp$ is:

$$\wp = x_{n+1} (t_f) \tag{5-66}$$

with the coefficients b_i given as:

$$b_i = 0 \qquad i = 1, 2, \ldots n$$

$$b_{n+1} = 1 \tag{5-67}$$

The Hamiltonian H is given by:

$$H = p_1\dot{x}_1 + p_2\dot{x}_2 + p_3\dot{x}_3$$

$$= p_1x_2 + p_2 (-x_2 + u) + p_3 (x_1^2 + \lambda u^2) \tag{5-68}$$

Taking the partial derivative of H with respect to u and equating the derivative to zero yields:

$$0 = p_2 + 2 \lambda u^* p_3 \tag{5-69}$$

Therefore the optimal control law is:

$$u^* = - \frac{p_2}{2 \lambda p_3}$$

The Hamilton canonical equations are:

$$\dot{x}_i = \frac{\partial H}{\partial P_i} \tag{5-71}$$

$$\dot{P}_i = - \frac{\partial H}{\partial x_i} \tag{5-72}$$

so,

$$\dot{p}_1 = - \frac{\partial H}{\partial x_i} \tag{5-73}$$

$$\dot{p}_2 = -\frac{\partial H}{\partial x_2} = -(p_1 - p_2) = p_2 - p_1 \qquad (5\text{-}74)$$

$$\dot{p}_3 = -\frac{\partial H}{\partial x_3} = 0 \qquad (5\text{-}75)$$

so,

$$p_3 = -1 \qquad (5\text{-}76)$$

Hence, substituting Eq. (5-76) in Eq. (5-73), we get:

$$\dot{p}_1 = 2\,x_1 \qquad (5\text{-}77)$$

and

$$\dot{p}_2 = p_2 - p_1$$

subject to boundary conditions on auxilary variables p_i:

$$\left.\begin{aligned} p_1(t_f) &= 0 \\ \text{and} \quad p_2(t_f) &= 0 \end{aligned}\right. \qquad (5\text{-}78)$$

Substituting Eq. (5-70) in Eq. (5-64) we get:

$$\dot{x}_2 = -x_2 + \frac{p_2}{2\lambda} \qquad (5\text{-}79)$$

Thus we have the following first order equations:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 + \frac{p_2}{2\lambda} \\ \dot{p}_i &= 2\,x_1 \\ \dot{p}_2 &= p_2 - p_1 \end{aligned} \qquad (5\text{-}80)$$

These equations are subject to the boundary conditions:

$$X(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$X(t_f) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$p_1(t_f) = 0 \qquad (5\text{-}81)$$

$$p_2(t_f) = 0$$

The best way of solving these equations is working backwards in time, starting from t_f since $p_1(t_f) = 0$, $p_2(t_f) = 0$ for all optimum trajectories. So, the equations given by Eq. (5-80) are modified as:

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_2 - \frac{p_2}{2\lambda} \\ \dot{p}_1 &= -2\,x_1 \\ \dot{p}_2 &= p_1 - p_2 \end{aligned} \qquad (5\text{-}82)$$

and now, the boundary conditions will be:

$$X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad X(t_f) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$p_1(0) = 0; \qquad p_2(0) = 0 \qquad (5\text{-}83)$$

let

$$\frac{1}{2\lambda} = k \tag{5-84}$$

By successive differentiation of the identities given by Eq. (5-82),

$$\frac{d^4 x_2}{dt^4} - \frac{d^2 x_2}{dt^2} + 2kx_2 = 0$$

or

$$(D^4 - D^2 + 2k)\, x_2 = 0 \tag{5-85}$$

where $D = \frac{d}{dt}$ is the operator.

The roots of Eq. (5-85) are:

$$D = \pm\beta_1 \text{ and } \pm\beta_2$$

where

$$\beta_1 = \sqrt{\alpha_1} \quad \text{and} \quad \beta_2 = \sqrt{\alpha_2}$$

where

$$\alpha_1 \text{ and } \alpha_2 = \frac{1 \pm \sqrt{1 - 8k}}{2} \tag{5-86}$$

Hence the solution of Eq. (5-85) is:

$$x_2 = C_1\, e^{\beta_1 t} + C_2\, e^{-\beta_1 t} + C_3\, e^{\beta_2 t} + C_4\, e^{-\beta_2 t} \tag{5-87}$$

where C_1, C_2, C_3 and C_4 are constants, to be evaluated.

From Eq. (5-82), we get, substituting Eq. (5-87),

$$p_2 = \frac{1}{k}\left[C_1 e^{\beta_1 t}(1-\beta_1) + C_2 e^{-\beta_1 t}(1+\beta_1) + C_3 e^{\beta_2 t}(1-\beta_2) + C_4 e^{-\beta_2 t}(1+\beta_2)\right] \tag{5-88}$$

$$p_1 = \frac{1}{2k}\left[C_1 e^{\beta_1 t}(1-\beta_1^2) + C_2 e^{-\beta_1 t}(1-\beta_1^2) + C_3 e^{\beta_2 t}(1-\beta_2^2) + C_4 e^{-\beta_2 t}(1-\beta_2^2)\right] \tag{5-89}$$

$$x_1 = \frac{1}{2k}\left[C_1 e^{\beta_1 t}\beta_1(1-\beta_1^2) + C_2 e^{-\beta_2 t}\beta_1(p_1^2 - 1)\right.$$

$$\left. + C_3 e^{\beta_2 t}{}_2(1-\beta_2^2) + C_4 e^{-\beta_2 t}\beta_2(\beta_2^2-1)\right] \tag{5-90}$$

The boundary conditions,

$X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $p(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and now substituted in the Eqs. (5-87), (5-88), (5-89) and (5-90), so we get:

$$\begin{aligned}
&C_1 + C_2 + C_3 + C_4 = x_2(0) \\
&C_1(1-\beta_1) + C_2(1+\beta_1) + C_3(1-\beta_1) + C_4(1+\beta_2) = 0 \\
&C_1(1-\beta_1^2) + C_2(1-\beta_1^2) + C_3(1-\beta_2^2) + C_4(1-\beta_2^2) = 0 \\
&C_1\beta_1(1-\beta_1^2) + C_2\beta_1(\beta_1^2-1) + C_3\beta_2(1-\beta_2^2) + C_4\beta_2(\beta_2^2-1) = 2kx_1(0)
\end{aligned} \tag{5-91}$$

where

$x_1(0) = 1$ and $x_2(0) = 0$

The above equations are solved to get C_1, C_2, C_3 and C_4 and the values are:

$$
\left.
\begin{aligned}
C_1 &= x_2(0)\,\frac{(1+\beta_1)(1-\beta_2^2)}{2\beta_1(\beta_1^2-\beta_2^2)} - x_1(0)\,\frac{2k}{2\beta_1\,(\beta_1^2-\beta_2^2)} \\
C_2 &= -x_2(0)\,\frac{(1-\beta_1)(1-\beta_2^2)}{2\beta_1\,(\beta_1^2-\beta_2^2)} + x_1(0)\,\frac{2k}{2\beta_1\,(\beta_1^2-\beta_2^2)} \\
C_3 &= x_2(0)\,\frac{(1-\beta_1^2)(1+\beta_2)}{2\beta_2\,(\beta_2^2-\beta_1^2)} - x_1(0)\,\frac{2k}{2\beta_2\,(\beta_2^2-\beta_1^2)} \\
C_4 &= -x_2(0)\,\frac{(1-\beta_1^2)(1-\beta_2)}{2\beta_2\,(\beta_2^2-\beta_1^2)} + x_1(0)\,\frac{2k}{2\beta_2\,(\beta_2^2-\beta_1^2)}
\end{aligned}
\right\} \qquad (5\text{-}92)
$$

The value of λ is taken as 0.1. We are justified in assuming this value because the given constraint on control input

$$\int_0^{t_f} u^2\,dt \leqslant k$$

will be satisfied with λ as 0.1, when k assumes a corresponding value.

$$\lambda = 0.1$$

so,

$$k = \frac{1}{0.2} = 5$$

Substituting in Eq. (5-86) we get,

$$
\begin{aligned}
\alpha_1 &= 3.16\ \underline{/80^\circ 54'} = 0.5 + j\,3.123 = \beta_1^2 \\
\alpha_2 &= 3.16\ \underline{/-80^\circ 54'} = 0.5 - j\,3.123 = \beta_2^2
\end{aligned}
$$

$$\beta_1 = 1.776 \angle 40^\circ 27' = 1.35 + j\ 1.15$$

$$\beta_2 = 1.776 \angle -40^\circ 27' = 1.35 + j\ 1.15$$

$$1-\beta_1 = -1.202 \angle 73^\circ 5' = -0.35 - j\ 1.15$$

$$1-\beta_2 = -1.202 \angle -73^\circ 5' = -0.35 + j\ 1.15 \qquad (5\text{-}93)$$

$$1+\beta_1 = 2.61 \angle 26^\circ 7' = 2.35 + j\ 1.15$$

$$1+\beta_2 = 2.61 \angle -26^\circ 7' = 2.35 - j\ 1.15$$

$$\beta_1^2-\beta_2^2 = 6.246 \angle 90^\circ$$

Now $x_2(t_f) = 0 = C_1 e^{\beta_1 t_f} + C_2 e^{-\beta_1 t_f} + C_3 e^{\beta_2 t_f} + C_4 e^{-\beta_2 t_f}$

Putting in the values of C_1, C_2, C_3 and C_4 and using $x_2(0) = 0$ and $x_1(0) = 1$ we get:

$$x_2(t_f) = 0 = \frac{2k}{2\beta_1(\beta_1^2-\beta_2^2)} \left[e^{-\beta_1 t_f} - e^{\beta_1 t_f} \right]$$

$$+ \frac{2k}{2\beta_2\ (\beta_2^2-\beta_1^2)} \left[e^{-\beta_2 t_f} - e^{\beta_2 t_f} \right]$$

substituting values of β_1, β_2 and k we get

$$0 = \Big[2.5952 \cos (1.15 t_f) \sinh (1.35 t_f)$$

$$- 3.04388 \sin (1.15 t_f) \cosh (1.35 t_f) \Big] \qquad (5\text{-}94)$$

by the method of iteration, the value of t_f is computed as:

$$t_f = 3.3456296$$

Figure 7 shows the plot of x_2. Since $\dot{x}_1 = -x_2$, we get:

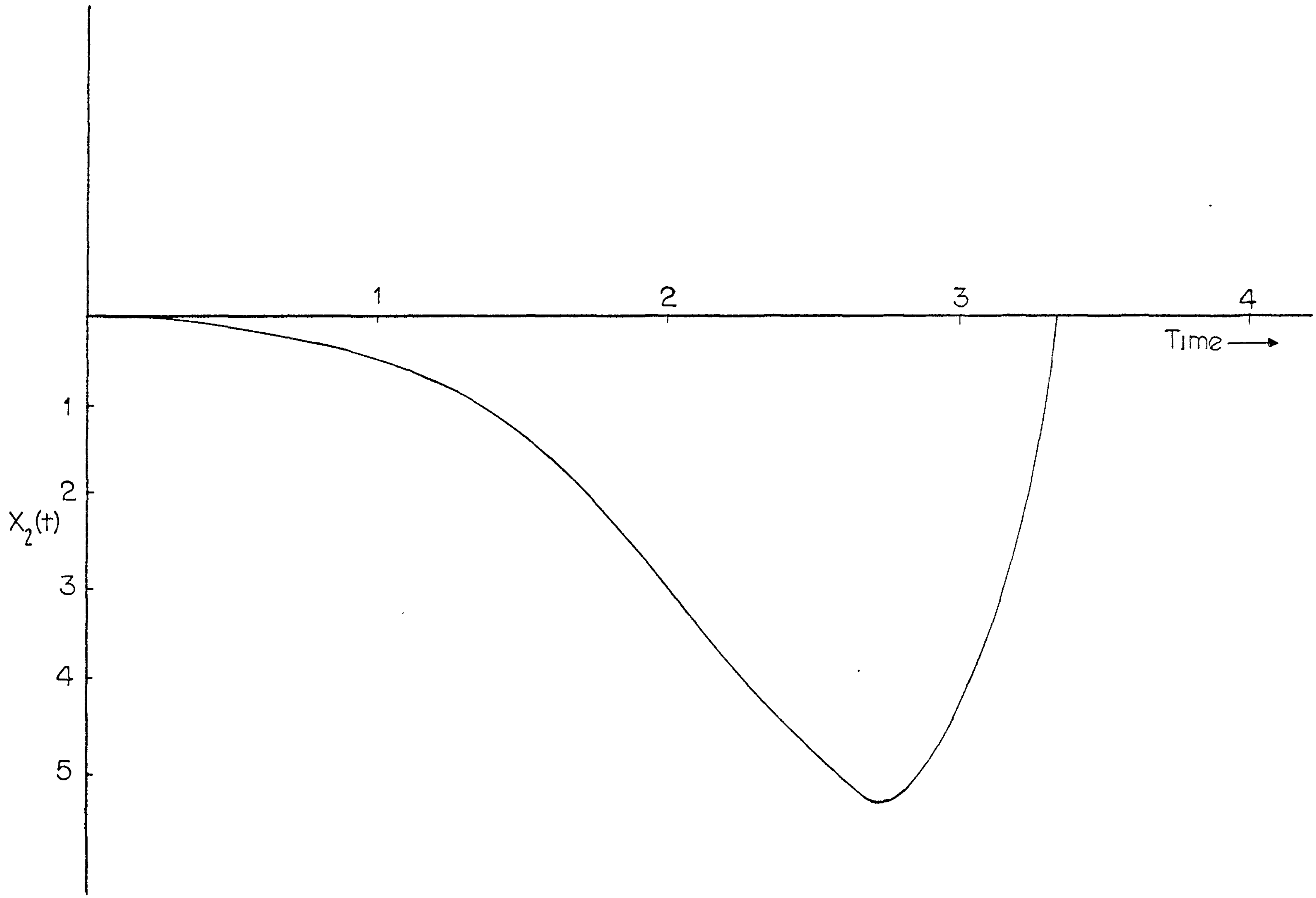


Fig.7. The Response of the State variable $X_2(t)$

$$x_1(t_f) = \left[\cos(1.15t_f)\cosh(1.35t_f) - 0.16\sin(1.15t_f)\sinh(1.35t_f)\right] \tag{5-95}$$

The optimum control input is given by Eq. (5-70)

$$u^* = -\frac{p_2}{2\lambda}$$

Substituting for p_2 from Eq. (5-88) and using λ as 0.1 we get:

$$u^* = -\left[C_1 e^{\beta_1 t}(1-\beta_1) + C_2 e^{-\beta_1 t}(1+\beta_1) + C_3 e^{\beta_2 t}(1-\beta_2) + C_4 e^{-\beta_2 t}(1+\beta_2)\right]$$

Simplifying with values of C_1 and C_2, C_3, C_4 and $x_1(0) = 1$ and $x_2(0) = 0$.

$$u^* = -\frac{2k(1-\beta_1)}{2\beta_1(\beta_1^2-\beta_2^2)} e^{\beta_1 t} + \frac{2k(1+\beta_1)}{2\beta_1(\beta_1^2-\beta_2^2)} e^{-\beta_1 t}$$

$$-\frac{2k(1-\beta_2)}{2\beta_2(\beta_2^2-\beta_1^2)} e^{\beta_2 t} + \frac{2k(1+\beta_2)}{2\beta_2(\beta_2^2-\beta_1^2)} e^{-\beta_2 t}$$

using values of β_1, β_2 from Eq. (5-93) we get:

$$\begin{aligned} u^* = {} & e^{-1.35t}\left[0.5814\cos(1.15t) + 2.27956\sin(1.15t)\right] \\ & -e^{1.35t}\left[0.5814\cos(1.15t) + 0.912509\sin(1.15t)\right] \end{aligned} \tag{5-96}$$

Now going back to Eq. (5-95) calculating $x_1(t_f)$ at $t_f = 3.345629$ we get:

$$x_1(t_f) = -29.64$$

But actually $x_1(t_f)$ should be equal to zero, so scaling the $u^*(t)$ by $(29.64+1) = +30.64$, the value of $x_1(t_f)$ will be zero.

$$u^* = \frac{1}{30.64}\left[e^{-1.35t}\left[0.5814 \cos(1.15t) + 2.27956 \sin(1.15t)\right] - e^{1.35t}\left[0.5814 \cos(1.15t) + 0.912509 \sin(1.15t)\right]\right]$$

Changing the time co-ordinate from t to $t_f - t$, we will get the solution of the optimum control input starting from t = 0.

$$u^* = \frac{1}{30.64}\left[e^{-1.35(t_f-t)}\left[0.5814 \cos 1.15\,(t_f-t) + 2.27956 \sin 1.15\,(t_f-t)\right] - e^{1.35(t_f-t)}\left[0.5814 \cos 1.15(t_f-t) + 0.912509 \sin 1.15(t_f-t)\right]\right] \tag{5-97}$$

using $t_f = 3.345629$, and simplifying we get the optimum control law as:

$$u^* = e^{-1.35t}\left[3.104 \cos 1.15t - 0.9517 \sin 1.15t\right] - e^{1.35t}\left[0.00677 \cos 1.15t - 0.004498 \sin 1.15t\right] \tag{5-98}$$

we summarize the results starting from t = 0 as follows:

$$x_1(t) = \left[\cos 1.15(t_f-t) \cosh 1.35(t_f-t) - 0.16 \sin 1.15(t_f-t) \sinh 1.35(t_f-t)\right] \tag{5-99}$$

$$x_2(t) = 0.45\left[2.5952 \cos 1.15(t_f-t) \sin\ 1.35(t_f-t) - 3.04388 \sin 1.15(t_f-t) \cosh 1.35(t_f-t)\right] \tag{5-100}$$

$$u^*(t) = e^{-1.35t}\left[3.104 \cos(1.15t) - 0.9517 \sin(1.15t)\right] - e^{1.35t}\left[0.006777 \cos(1.15t) - 0.00449 \sin(1.15t)\right] \tag{5-101}$$

Checking the optimal control law $u^*(t)$ satisfies the Hamilton Canonical equations.

The Figure 8, shows the plot of $x_1(t)$. It is seen that the

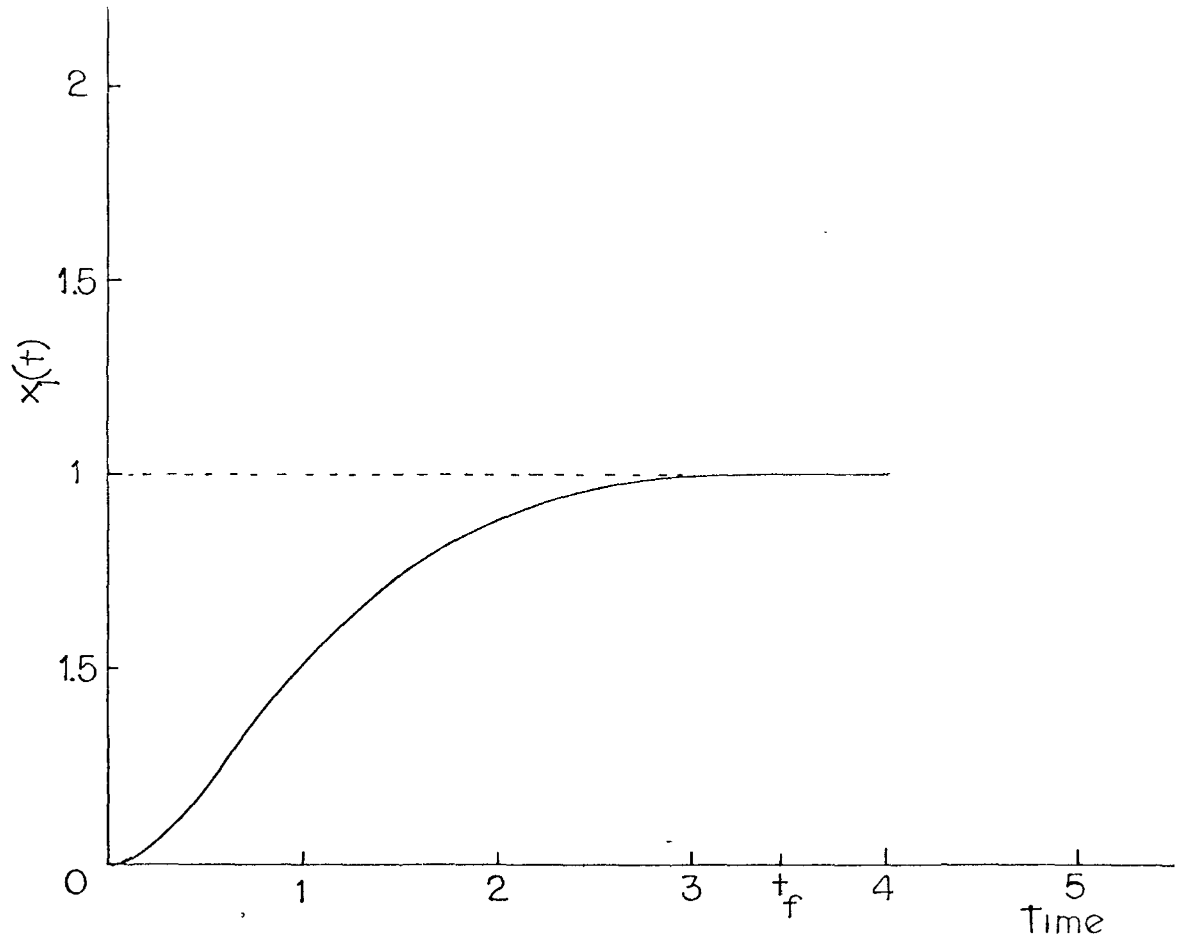


Fig.8. The Response of $x_1(t)$ for an Optimum Input

the target $x_1(t_f) = 1$, and $x_1(0) = 0$ are satisfied.

Also $x_2(0) = 0$ and $x_2(t_f) = 0$. The Figure 9 shows the plot of $u^*(t)$. Also the performance index is minimized. Thus the calculated optimal control law given by Eq. (5-101) satisfies the constraint on u(t), transfers the state of the system from $X(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $X(t_f) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and minimizes the performance index.

Hence, $u^*(t)$ is the optimal control law.

Optimal open-loop linear control:

The optimal open-loop linear control system is shown in Figure 10b representing the solution in a open-loop manner, the controller G(s) is calculated when a step input is applied.

P is the plant, and G is the controller.

R is the step input.

Analyzing the system:

$$U^*(s) = G(s)\, R(s)$$

$$P(s)\, U^*(s) = X_1(s)$$

so
$$X_1(s) = P(s)\, G(s)\, R(s) \tag{5-102}$$

and
$$G(s) = \frac{U^*(s)}{R(s)} \tag{5-103}$$

Since R(s) = 1/s, taking Laplace transform of $u^*(t)$ and substituting in Eq. (5-103) we get:

$$U^*(s) = \frac{3.0972\, s^3 - 5.388\, s^2 + 1.6866\, s + 9.471}{\left[(s + 1.35)^2 + (1.15)^2\right]\left[(s - 1.35)^2 + (1.15)^2\right]} \tag{5-104}$$

$$G(s) = \frac{s\,(3.0972\, s^3 - 5.388\, s^2 + 1.6866\, s + 9.471}{\left[(s + 1.35)^2 + (1.15)^2\right]\left[(s - 1.35)^2 + (1.15)^2\right]} \tag{5-105}$$

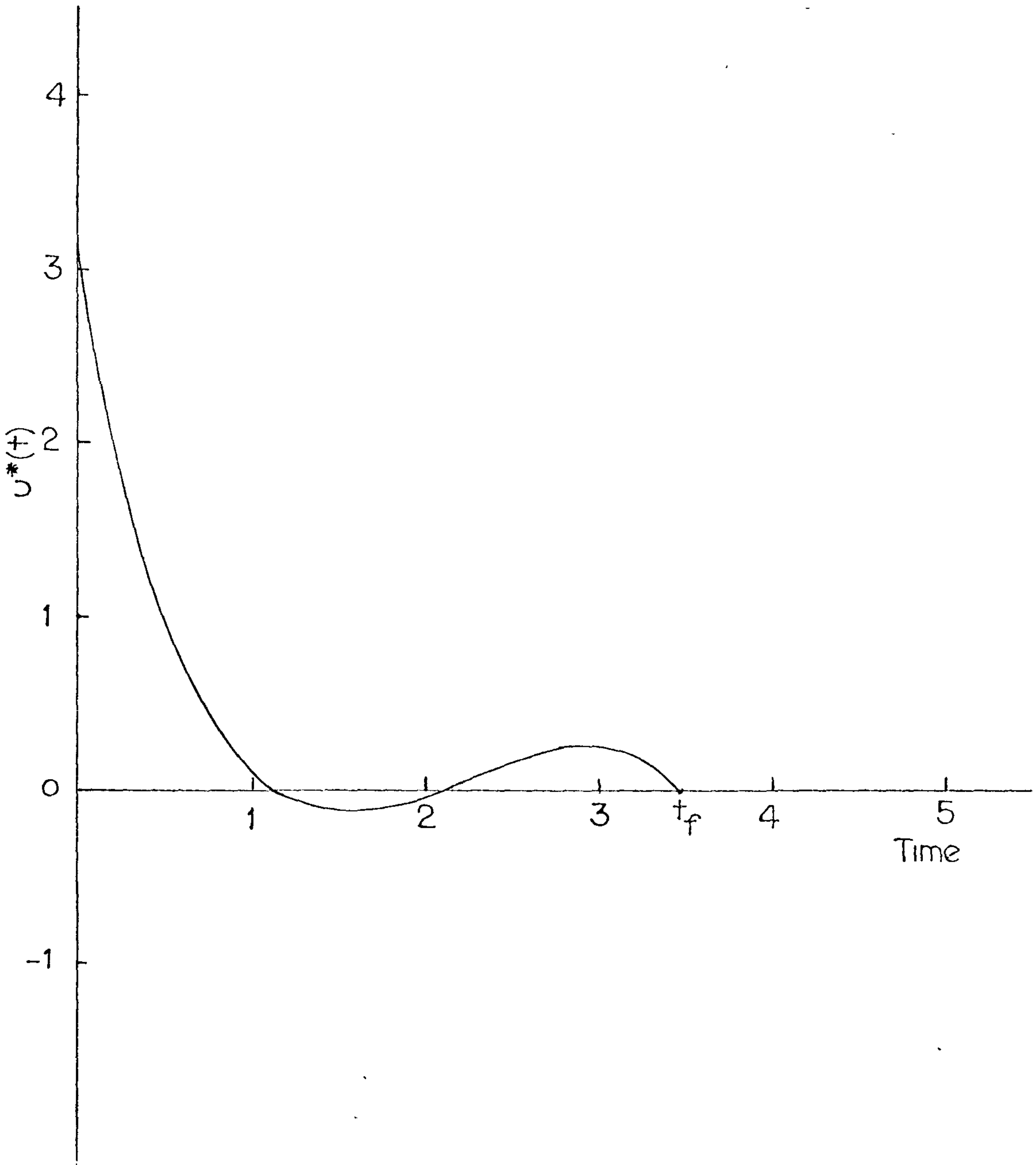


Fig. 9. Behavior of Optimum Control law $u^*(t)$ with time

Using Eq. (5-102) $x_1(s)$ is calculated.

$$X_1(s) = \frac{(3.0972\ s^3 - 5.388\ s^2 + 1.6866\ s + 9.471)}{s(s+1)\left[(s + 1.35)^2 + (1.15)^2\right]\left[(s - 1.35)^2 + (1.15)^2\right]} \tag{5-106}$$

Taking inverse Laplace transform of Eq. (5-106) we get:

$$\begin{aligned} x_1(t) = &-e^{-1.35t}\left[0.946597 \cos (1.15t) + 1.0517 \sin (1.15t)\right] \\ &-e^{1.35t}\left[0.0013557 \cos (1.15t) + 0.00087688 \sin (1.15t)\right] \\ &+ 0.8814 + 0.06646\ e^{-t} \end{aligned} \tag{5-107}$$

Now let the plant parameter vary by 0.01. Here we take the plant parameter as the plant open-loop pole. The open-loop system is shown in Figure 10a. The controller is assumed to be constant.

so,
$$P'(s) = \frac{1}{s\ (s + 1.01)} \tag{5-108}$$

$$U^*(s) = R(s)\ G(s) \tag{5-109}$$

$$X_1'(s) = U^*(s)\ P'(s) = R(s)\ G(s)\ P'(s) \tag{5-110}$$

where P' is the new plant and X' is the output with this plant.

$$X_1'(s) = \frac{3.0972\ s^2 - 5.388\ s^2 + 1.6866\ s + 9.471}{s(s+1.01)\left[(s + 1.35)^2 + (1.15)^2\right]\left[(s - 1.35)^2 + (1.15)^2\right]} \tag{5-111}$$

$x_1'(t)$ is calculated taking the inverse Laplace transform of $X_1'(s)$, and is

$$\begin{aligned} x_1'(t) = &-e^{-1.35t}\left[1.0319 \cos (1.15t) + 1.132 \sin (1.15t)\right] \\ &-e^{1.35t}\left[0.00147898 \cos (1.15t) + 0.0009543 \sin (1.15t)\right] \\ &+ 0.94771 + 0.09656\ e^{-1.01t}. \end{aligned} \tag{5-112}$$

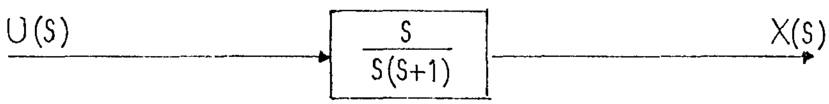
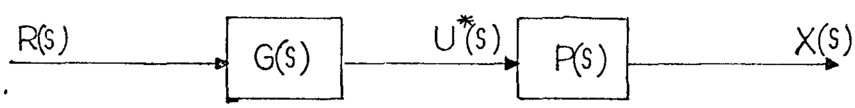


Fig. 10a. A Second Order System



Plant $P(S) = \dfrac{S}{S(S+1)}$

Controller $\dfrac{S(3.0972\,S^3 - 5.388\,S^2 + 1.6866\,S + 9.471)}{(S^4 - .998\,S^2 + 9.897)}$

Fig. 10b. Open-loop Optimal Control System.

Sensitivity in open-loop linear optimal control system:

The sensitivity of performance in the open-loop optimal control system is now calculated. We will define the sensitivity of performance as:

$$\frac{\Delta J}{\Delta \alpha}$$

where ΔJ is change in performance index due to a change in a plant parameter, J^* is the optimum performance index,

$$\Delta J = J' - J^* \tag{5-113}$$

$$J^* = \int_0^{3.345629} x_1^2 \, dt \tag{5-114}$$

$$J' = \int_0^{3.345629} x_1'^2 \, dt \tag{5-115}$$

Where x_1 and x_1' are given by Equations (5-107) and (5-112) respectively, By the method of numerical integration the values of J^* and J' are computed as:

$$J^* = 1.9094351$$

$$J' = 2.2289334$$

So sensitivity performance in optimal open-loop linear control system is:

$$\frac{\Delta J}{\Delta \alpha} = \frac{0.3194983}{0.01} = 31.94983 \tag{5-116}$$

Two degrees of freedom closed-loop linear optimal control:

The optimal closed-loop linear control system is shown in Figure 11. We use here two-degree-of-freedom structure.

P(s) is the plant transfer function; G(s) is the controller transfer function. The feedback element is (1 + sa) with variable "a". Analyzing the system we have:

$$R(s) - X_1(s)\ (1 + sa) \quad G(s) = U^*(s) \tag{5-117}$$

or

$$G(s) = \frac{U^*(s)}{R(s) - X_1(s)\ (1 + sa)}$$

also

$$U^*(s)\ P(s) = X_1(s)$$

so

$$G(s) = \frac{U^*(s)}{R(s) - U^*(s)\ P(s)\ (1 + sa)} \tag{5-118}$$

Now when the plant parameter is varied, let the new plant be P'(s). The controller G(s) is kept fixed, and the output with plant P'(s) be X"(s), and the optimum input be $U^{*\prime}(s)$. Then,

$$R(s) - (1 + sa)\ X_1''(s) \quad G(s) = U^{*\prime}(s)$$

also,

$$P'(s)\ U^{*\prime}(s) = X_1''(s)$$

so,

$$R(s) - (1 + sa)\ X_1''(s) = \frac{X_1''(s)}{P'(s)}$$

therefore,

$$X_1''(s) = \frac{P'(s)\ R(s)\ G(s)}{1 + G(s)\ P'(s)\ (1 + as)}$$

and G(s) is given by

$$G(s) = \frac{U^*(s)}{R(s) - U^*(s)\ P(s)(1 + a\ s)}$$

therefore, $$X_1''(s) = \frac{P'(s)\,R(s)\,U^*(s)}{R(s) + U^*(s)(1 + a\,s)(P'(s) - P(s))} \tag{5-119}$$

So we have the controller:

$$G(s) = \frac{U^*(s)}{R(s) - U^*(s)\,P(s)(1+s\,a)}$$

and

$$X_1'' = \frac{P'(s)\,R(s)\,U^*(s)}{R(s) + U^*(s)\,(1 + a\,s)(P'(s) - P(s))}$$

Let $a = 2$: The optimal closed-loop control system is shown in Figure 11 with $a = 2$. We have,

$$P' = \frac{1}{s(s + 1.01)}$$

$$R = \frac{1}{s}$$

and

$$U^*(s) = \frac{3.0972\,s^3 - 5.388\,s^2 + 1.6866\,s + 9.471}{\left[(s + 1.35)^2 + (1.15)^2\right]\left[(s - 1.35)^2 + (1.15)^2\right]}$$

Substituting $a = 2$ in Equations (5-118 and (5-119) we get the controller;

$$G(s) = \frac{U^*(s)}{R(s) - U^*(s)\,P(s)(1 + 2\,s)} \tag{5-120}$$

and

$$X_1'' = \frac{P'(s)\,R(s)\,U^*(s)}{R(s) + U^*(s)(1 + 2\,s)(P'(s) - P(s))} \tag{5-121}$$

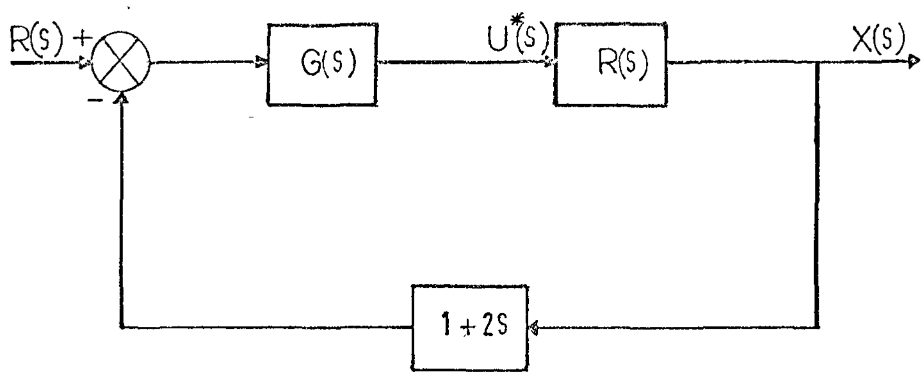


Fig.11 .Two-degree-of-freedom Closed loop Optimal Control System.

$$\text{Plant} \quad P(S) = \frac{S}{S(S+1)}$$

$$\text{Controller} \quad G(S) = \frac{S(S+1)(3.0972\,S^3 - 5.388\,S^2 + 1.6866\,S + 9.471)}{(S^5 - 5.1944\,S^4 + 6.46828\,S^3 + 1.0168\,S^2 - 9.7316\,S + 0.426)}$$

Using the values of $P'(s)$, $R(s)$ $U^*(s)$, and $P(s)$ we get;

$$G(s) = \frac{s(s+1)(3.0972\ s^3 - 5.388\ s^2 + 1.6866\ s + 9.471)}{(s^5 - 5.1944\ s^4 + 6.46828\ s^3 + 1.0168\ s^2 - 9.7316\ s + 0.426)} \tag{5-122}$$

The output $X_1''(s)$ is now calculated from Equation (5-121)

$$X_1''(s) = \frac{(s+1)(3.0972\ s^3 - 5.388\ s^2 + 1.6866\ s + 9.471)}{S(s^6 + 2.01\ s^5 - 0.049944\ s^4 - 1.92921\ s^3 + 8.90995\ s^2 + 19.686714\ s + 9.90219)} \tag{5-123}$$

The roots of the denominator of Equation (5-123) are computed, so $X_1''(s)$ is written as:

$$X_1''(s) = \frac{(s+1)(3.0972\ s^3 - 5.388\ s^2 + 1.6866\ s + 9.471)}{s(s+0.988)(s+1.0471)(s+1.338 - j\ 1.1198)(s+1.388 + j\ 1.1198)(s - 1.35 + j\ 1.15)(s - 1.35 - j\ 1.15)} \tag{5-124}$$

Using partial fractions, and taking inverse Laplace transform, we get:

$$\begin{aligned} x_1''(t) &= 0.095592 + 0.0085859\ e^{-0.988\ t} + 0.1234\ e^{-1.0471\ t} \\ &\quad -e^{1.35\ t}\left[0.0009033\ \cos(1.15t) + 0.00062132(\sin 1.15t)\right] \\ &\quad -e^{-1.338\ t}\left[(1.0109\ \cos(1.1198t) + 1.223\ \sin(1.1198t)\right] \end{aligned} \tag{5-125}$$

So for the two-degree-of-freedom closed-loop optimal system with $a = 2$ we have, the output x_1 as given in Equation (5-107)

$$\begin{aligned} x_1(t) &= -e^{-1.35\ t}\left[0.946597\ \cos(1.15t) + 1.0517\ \sin(1.15t)\right] \\ &\quad -e^{1.35\ t}\left[0.0013557\ \cos(1.15t) + 0.00087688\ \sin(1.15t)\right] \\ &\quad + 0.8814 + 0.06646\ e^{-t} \end{aligned}$$

and the output x_1'', when the plant parameter (open-loop pole) is varied by 0.01, is from Equation (5-125).

$$x_1''(t) = -e^{1.35\ t}\left[0.0009033\ \cos(1.15 \quad + 0.00062132\ \sin(1.15t)\right]$$
$$-e^{-1.35\ t}\left[1.0109\ \cos(1.1195 \quad + 1.223\ \sin(1.1198\right]$$
$$+ 0.95592 + 0.0085859\ e^{-0.988\ t} + 0.1234\ e^{-1.0471\ t}$$

Sensitivity in two-degree-of-freedom closed-loop linear optimal control system as given in Figure 11.

The performance indices

$$J^* = \int_0^{3.345629} x_1^2\, dt$$

and

$$J'' = \int_0^{3.345629} x_1''^2\, dt$$

are computed, using Numerical Integration methods. The results are as shown below.

$$J^* = 1.9094351$$
$$J'' = 2.1730306 \tag{5-126}$$

So the sensitivity of performance in the closed-loop linear optimal system with a a = 2 is:

$$S = \frac{\Delta J}{\Delta\alpha} = \frac{J''-J^*}{\Delta\alpha} = \frac{0.2645955}{0.01} = \underline{26.45955} \tag{5-127}$$

Now let a = 1.

The optimal closed-loop control system is shown in Figure 12, with a = 1. Substituting a = 1 in Eqs. (5-118) and (5-119), and simplifying we get:

$$G(s) = \frac{S\ (3.0972\ s^3 - 5.388\ s^2 + 1.6866\ s + 9.471)}{(S^4 - 3.0972\ s^3 + 4.390\ s^2 - 1.6866\ s + 0.426)} \qquad (5\text{-}128)$$

and

$$X_1''(s) = \frac{P'(s)\ R(s)\ U^*(s)}{R(s) + U^*(s)\ (1 + s)(P'(s) - P(s))} \qquad (5\text{-}129)$$

and

$$P(s) = \frac{1}{S(S + 1)}\ , \quad P'(s) = \frac{1}{S(s + 1.01)}$$

$$u^*(s) = \frac{3.0972\ s^3 - 5.388\ s^2 + 1.6866\ s + 9.471}{(S + 1.35)^2 + (1.15)^2\ \ (s - 1.35)^2 + (1.15)^2} \qquad (5\text{-}129)$$

Substituting these values in Equation (5-129) we get;

$$X_1''(s) = \frac{3.0972\ s^3 - 5.388\ s^2 + 1.688\ s + 9.471}{S(S^5 + 1.018^4 - 1.02897\ s^3 - 0.954\ s^2 + 9.8801\ s + 9.9013)} \qquad (5\text{-}130)$$

The roots of the denominator of Equation (5-130) are computed, so $X_1''(s)$ is written as:

$$X_1''(s) = \frac{3.0972\ s^3 - 5.388\ s^2 + 1.688\ s + 9.471}{(S + 1.3595 - j1.1367)(S + 1.3495 + j1.1367)\ (S - 1.35 + j1.15)(S - 1.35 - j1.15)(S + 1.0197)} \qquad (5\text{-}131)$$

Using partial fractional method, and taking inverse Laplace transform of Equation (5-131) we get:

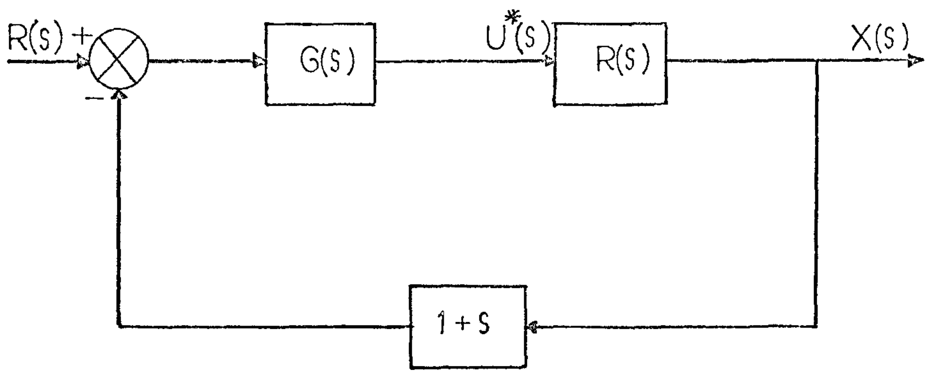


Fig.12. Two degree of freedom Closed loop Optimal Control System.

$$\text{Plant} \quad P(S) = \frac{S}{S(S+1)}$$

$$\text{Controller} \quad G(S) = \frac{S\,(3.0972\,S^3 - 5.388\,S^2 + 1.6866\,S + 9.471)}{(S^4 - 3.0972\,S^3 + 4.390\,S^2 - 1.6866\,S + 0.426)}$$

$$x_1''(t) = 0.94831 + 0.11453\ e^{-10.0197t}$$
$$+ e^{1.35t}\left[-0.0007497 \cos 1.15t + 0.00094576 \sin 1.15t\right]$$
$$-e^{-1.3495t}\left[1.016172 \cos 1.136t + 1.155 \sin 1.136t\right] \quad (5\text{-}132)$$

so for the two-degree-of-freedom closed-loop optimal control system with $a = 1$, the output is as given in Equation (5-107)

$$x_1(t) = -e^{-1.35t}\left[0.946597 \cos 1.15t + 1.0517 \sin 1.15t\right]$$
$$-e^{1.35t}\left[0.0013557 \cos 1.15t + 0.00087688 \sin 1.15t\right]$$
$$+ 0.8814 + 0.06646\ e^{-t}$$

and the output x_1'' when the plant parameter (open-loop pole) is varied by 0.01 is from Equation (5-132)

$$x_1''(t) = e^{1.35t}\left[-0.0007497 \cos 1.15t + 0.00074576 \sin 1.15t\right]$$
$$-e^{-1.3495t}\left[1.016172 \cos 1.136t + 1.155 \sin 1.136t\right]$$
$$+ 0.94831 + 0.11453\ e^{-1.0197t}$$

Sensitivity in two-degree-of-freedom closed-loop linear optimal control system given in Figure 12. The performance indices,

$$J^* = \int_0^{3.345629} x_1^2\ dt$$

and

$$J'' = \int_0^{3.345629} x_1''^2\ dt$$

are computed. The results are,

$$J^* = 1.9094351 \qquad (5\text{-}133)$$
$$J'' = 2.1485611$$

To the sensitivity of performance in the closed-loop linear optimal system with a = 1 is,

$$S = \frac{\Delta J}{\Delta\alpha} = \frac{J''-J^*}{\Delta\alpha} = \frac{0.2391260}{0.01} = 23.91260 \qquad (5\text{-}134)$$

Single-degree-of-freedom closed-loop linear optimal control:

Now, let us consider a single-degree-of-freedom closed-loop optimal control system as given in Figure 13. Analyzing we get,

$$G(s) = \frac{U^*(s)}{R(s) - U(s)\ P(s)} \qquad (5\text{-}135)$$

and

$$X(s) = P(s)\ U(s)$$

and when the plant parameter (a open-loop pole) is varied, let $P'(s)$ be the new plant, $U^{*\prime}(s)$ be the optimal input, and $X_1''(s)$ be the output. Keeping the controller $G^*(s)$ fixed, we get;

$$(R(s) - X_1'(s))\ G(s) = U^*(s)$$

$$P'(s)\ U^{*\prime}(s) = X_1'(s)$$

$$X_1''(s) = \frac{R(s)\ G(s)}{(1/P'(s) + G(s))} = \frac{P'(s)\ R(s)\ G(s)}{1 + P'(s)\ G(s)}$$

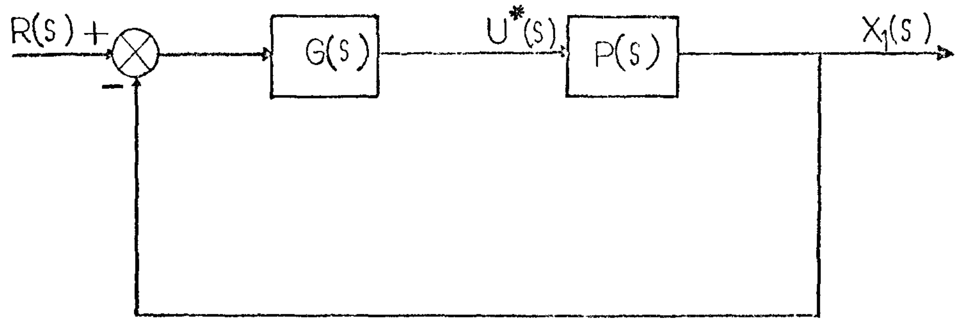


Fig. 13. Single-degree-of-freedom Closed loop Optimal Control System.

$$\text{Plant } P(S) = \frac{S}{S(S+1)}$$

$$\text{Controller } G(S) = \frac{S(S+1)(3.0972S^3 - 5.388S^2 + 1.6866S + 9.471)}{(S^5 + S^4 - 4.0952S^3 + 4.390S^2 + 8.2104S + 0.426)}$$

Substituting for G(s) from Equation (5-135),

$$X_1''(s) = \frac{P'(s)\ R(s)\ U^*(s)}{R(s) + U(s)\ (P'(s) - P(s))} \tag{5-136}$$

$$P'(s) = \frac{1}{s(s + 1.01)} \qquad R(s) = \frac{1}{s} \quad P(s) = \frac{1}{s(s + 1)}$$

$$U^*(s) = \frac{(3.0972\ s^3 - 5.388\ s^2 + 1.6866\ s + 9.471)}{(S + 1.35)^2 + (1.15)^2 \quad (S - 1.35)^2 + (1.15)^2}$$

Using these values and calculating $X_1''(s)$ we get;

$$x_1''(s) = \frac{(S - 1)(3.0972\ s^3 - 5.388\ s^2 + 1.688\ s + 9.471)}{S(s^6 + 2.01\ s^5 + 0.012\ s^6 - 2.037\ s^3 + 8.943\ s^2 + 19.876\ s + 9.9013)} \tag{5-137}$$

The roots of demoninator of Equation (5-137) are computed, so $X_1''(s)$ is written as:

$$x_1''(s) = \frac{(S + 1)(3.0972\ s^3 - 5.388\ s^2 + 1.688\ s + 9.471)}{s(s + 0.994 - j0.0264)(S + 0.994 + j0.0264)(s - 1.35 - j1.15)(s - 1.35 + j1.15)(s + 1.361 - j1.1541)(s + 1.361 + j1.1154)} \tag{5-138}$$

using partial fractional method, and taking inverse Laplace transform of Equation (5-138) we get;

$$x_1''(t) = 0.956312 + \bar{e}^{0.994t}\left[0.041154 \cos 0.0264t - 0.071555 \sin 0.0264t)\right] + e^{1.35t}\left[-0.00884932 \cos 1.15t + 0.0066681 \sin 1.15t\right]$$

$$-e^{-1.361t}\left[0.91234 \cos 1.154t + 0.62283 \sin 1.154t\right]$$

so for the single-degree-of-freedom closed-loop control system we have the output $x_1(t)$

$$x_1(t) = -e^{-1.35t}\left[0.945597 \cos 1.15t + 1.0517 \sin 1.15t\right]$$
$$-e^{+1.35t}\left[0.0013557 \cos 1.15t + 0.00087688 \sin 1.15t\right]$$
$$+ 0.8814 + 0.06646\ e^{-t}$$

and when the plant parameter (open-loop pole) is varied by 0.01, the output x_1' is:

$$x_1''(t) = 0.9565312 + \bar{e}^{0.994t}\left[0.041154 \cos 0.0264t - 0.071555 \sin 0.0264t)\right]$$
$$+ e^{1.35t}\left[-0.0884932 \cos 1.15t + 0.0066681 \sin 1.15t\right]$$
$$-e^{-1.361t}\left[0.91234 \cos 1.154t + 0.62283 \sin 1.154\ t)\right]$$

<u>Sensitivity in the single-degree-of-freedom- closed-loop linear optimal control system as given in Figure 13.</u>

The performance indices

$$J^* = \int_0^{3.345629} x_1^2\ dt$$

and

$$J'' = \int_0^{3.345629} x_1''^2\ dt$$

are computed, and the results are:

$$J^* = 1.9094351; \qquad J'' = 2.2062123 \tag{5-139}$$

So the sensitivity in the single-degree-of-freedom closed-loop linear optimal control system is:

$$\frac{\Delta J}{\Delta \alpha} = \frac{J^{n}-J^{*}}{\Delta \alpha} = 29.6772 \tag{5-140}$$

Summarizing the sensitivity results:

SYSTEM	SENSITIVITY
Open-loop optimal control	31.94983
Two-degree-of-freedom closed-loop optimal control system with a = 2	26.45955
Two-degree-of-freedom closed-loop optimal control system with a = 1	23.91260
Single-degree-of-freedom closed-loop optimal control system	29.6772

Thus the sensitivities of performance in optimal open-loop and closed-loop linear systems is calculated. Both single degree-of-freedom feed back structures are considered.

It is clearly seen from the results above that the optimal closed-loop linear control system is advantageous to optimal open-loop linear control systems, from the sensitivity point of view.

From the results above it can be seen that the optimal closed-loop control system is less sensitive than the open-loop optimal control system, for a finite amount of variation in a plant parameter. (Here in the above calculations an open-loop pole is taken as plant parameter.)

Also comparing the sensitivity values in the three types of optimal closed-loop systems considered, it can be seen that the two-degrees-of-freedom closed-loop optimal system is less sensitive than the one-degree-of-freedom closed-loop optimal system.

The sensitivity in the two-degree-of-freedom closed-loop optimal control system shown in Figure 12 is 23.91260, which is less than the other systems considered. So this type of two-degree-of-freedom closed-loop optimal control system is very less sensitive than the other systems considered.

5.4 SENSITIVITY OF PERFORMANCE OF OPTIMAL LINEAR CONTROL SYSTEM TO FINITE VARIATIONS IN PLANT PARAMETERS

Linear continuous optimal systems with Quadratic performance criteria

We shall consider linear systems of the form

$$\begin{aligned} \dot{X} &= A_o X(t) + B_o u(t) \\ X(t_o) &= X_o \end{aligned} \tag{5-141}$$

where A_o, B_o, are n by n and n by r-dimensional metrices respectively. The index of performance is of the form:

$$J = \frac{1}{2}\int_{t_o}^{t} \left[\langle X(t),\ Q_o X(t)\rangle + \langle u(t),\ R_o u(t)\rangle\right] dt + \frac{1}{2}\ \langle X(t_f),\ KX(t_f)\rangle \tag{5-142}$$

where Q_o and R_o are symmetric positive definite p by p and r by r-dimentional matrix. We use the notation:

$$\langle X, QX\rangle = \sum_t \sum_j q_{ij} X_i X_j = X^T QX \text{ for quadratic forms.}$$

It follows (from Kalman[32]) that the optimal control input is given by:

$$u^*(t) = -R_o^{-1} B_o^T M_o X(t) \tag{5-143}$$

where M_o is the solution of the matrix Ricatti equation

$$\frac{dM_o}{dt} + A_o^T M_o - M_o A_o - M_o B_o R_o^{-1} B_o^T M_o + Q_o = 0$$
$$M_o(t_f, t_f) = K \tag{5-144}$$

The optimal value of the index of performance is given by, following Kalman[10, 32],

$$J^*\left[X(t), t_f\right] = \frac{1}{2}\langle X(t), M_o(t), X(t)\rangle \tag{5-145}$$

For a time invariant system, A_o, B_o, Q_o and R_o are time-invariant, and so is M_o. Hence $\frac{dM_o}{dt} = 0$ and so Equation (5-144) becomes:

$$A_o^T M_o + M_o A_o - M_o B_o R_o^{-1} B_o^T M_o + Q_o = 0 \tag{5-146}$$

<u>Derivation of performance index for optimal linear closed-loop systems:</u>

Consider the system given by Eq. (5-141) with the performance index given by Eq. (5-142). If the nominal system is given by the matrices A_o, B_o, Q_o, R_o, and α_o represents the set of parameters in these matrices, then the optimal input based on nominal parameters α_o, is

$$u^*(t) = -R_o^{-1}B_o^TM_oX(t)$$

The configuration for the closed-loop optimal control system is given in Figure 14.

Suppose that the actual system is given by the matrices A, B, Q, R, with the true value of the parameter vector being α, now as the matrix A_o is varied to A due to the parameter variation from α_o to α, in the closed-loop case, the optimal input $u^*(t)$ is no longer optimum and let the new input to plant be $u^{*\prime}(t)$, so due to this change in the input, let the state vector X(t) change to $X''(t)$, hence the system equation becomes:

$$\dot{X}''(t) = AX''(t) + B_ou^{*\prime}(t) \tag{5-147}$$

where $u^{*\prime}(t)$ is given by,

$$u^{*\prime}(t) = -R_o^{-1}B_o^TM_oX''(t) \tag{5-148}$$

so substituting Equation (5-148) in to Equation (5-147) we get:

$$\begin{aligned}\dot{X}''(t) &= AX''(t) - B_oR_o^{-1}B_o^TM_oX''(t)\\ &= (A - B_oR_o^{-1}B_o^TM_o)\ X''(t)\end{aligned} \tag{5-149}$$

The performance index given by Eq. (5-142) will now be written, using the new input $u^{*\prime}$ and the new state vector $X''(t)$, as

$$J'' = \frac{1}{2}\int_{t_o}^{t_f}\left[\langle X''(t),\ Q_oX''(t)\rangle + \langle u^{*\prime}(t),\ R_ou^{*\prime}(t)\rangle\right]dt + \frac{1}{2}\langle X''(t_f),\ KX''(t_f)\rangle \tag{5-150}$$

where J'' is the performance index for the closed-loop case. Now, substituting Eq. (5-148) and Eq. (5-149) in to (5-150), the performance of index for the closed-loop case is:

$$J'' = \frac{1}{2} \int_{t_o}^{t_f} \left[\langle X''(t),\ Q_o X''(t) \rangle + \langle -R_o^{-1} B_o^T M_o X''(t),\ -B_o^T M_o X''(t) \rangle \right] dt + \frac{1}{2} \langle X''(t_f),\ K X''(t_f) \rangle$$

Deviation of performance index for optimal open-loop systems:

The optimal control input based on the nominal parameters α_o, is:

$$u^*(t) = -R_o^{-1} B_o^T M_o X(t)$$

substituting this value in Equation (5-141) we get,

$$\dot{X}(t) = (A_o - BR_o^{-1} B_o^T M_o)\ X(t) \qquad (5\text{-}152)$$

$$= P\ X(t)$$

Let the transition matrix of equations (5-152) be given by $\Phi(t,\ t_o)$. Hence, we have: $X(t) = \Phi\ (t,\ t_o)\ X_o$. The configuration of the open-loop optimal controller is, therefore, as shown in Figure 15. For a time-invariant system $\Phi(t) = e^{Pt}$. Now, consider the nominal parameter of the system plant α_o, be varied to α. Due to this let the matrix A_o change to A. Here in this open-loop the optimum input to the plant $u^*(t)$ is not altered. Due to the change in the system matrix A_o, let the state vector X(t) change to X'(t), so from equation (5-141), the state vector X'(t) is given by,

$$\dot{X}'(t) = A\ X'(t) + B_o u^*(t)$$

From equation (5-142), the performance of index is written using the new state vector X'(t), as

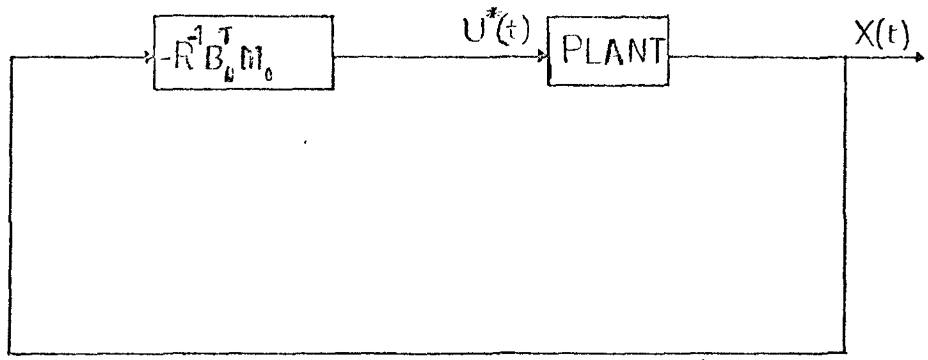


Fig.14 Closed-loop Optimal Control System.

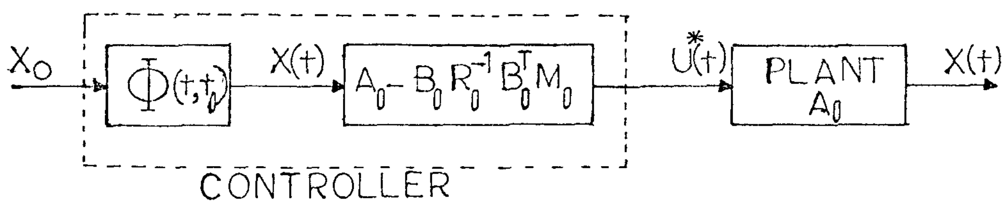


Fig.15. Open-loop Optimal Control System

$$J' = \frac{1}{2}\int_{t_o}^{t_f} \left[\langle X'(t) Q_o X'(t)\rangle + \langle u^*(t) R_o u^*(t)\rangle\right] dt + \frac{1}{2}\langle X'(t_f), KX'(t_f)\rangle \tag{5-153}$$

where J' is the performance index for the open-loop optimal control system.

Determination of the sensitivity of performance for optimal closed-loop linear systems:

As has been stated previously, we define sensitivity of performance of optimal control system as: $S = \frac{\Delta J}{\Delta\alpha}$, where ΔJ is the change in the performance index due to a $\Delta\alpha$ variation in a plant parameter. From equation (5-142), the optimum performance of index for nominal parameters α_o, with optimum input $u^*(t)$ is,

$$J^* = \frac{1}{2}\int_{t_o}^{t_f} \left[\langle X(t), Q_o X(t)\rangle + \langle u^*(t), R_o u^*(t)\rangle\right] dt + \frac{1}{2}\langle X(t_f), KX(t_f)\rangle \tag{5-154}$$

or substituting for $u^*(t)$ from Eq. (5-143), we get:

$$J^* = \frac{1}{2}\int_{t_o}^{t_f} \left[\langle X(t), Q_o X(t)\rangle + \langle -R_o^{-1}B_o^T M_o X(t), -B_o^T M_o X(t)\rangle\right] dt + \frac{1}{2}\langle X(t_f), KX(t_f)\rangle \tag{5-155}$$

The performance index when the plant parameters α_o vary to α from Equation (5-151) is

$$J'' = \frac{1}{2}\int_{t_o}^{t_f} \left[\langle X''(t), Q_o X''(t)\rangle + \langle -R_o^{-1}B_o^T M_o X''(t), -B_o^T M_o X''(t)\rangle\right] dt + \frac{1}{2}\langle X''(t_f), KX''(t_f)\rangle \tag{5-156}$$

so, the change in the performance index due to a variation in the plant perameters $\Delta J = J''-J^*$ is obtained by substracting Eq. (5-155) from

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Eq. (5-156) as J^* is optimum, and hence is minimum.

$$\text{so} \quad \Delta J = \frac{1}{2} \int_{t_o}^{t_f} \left[\langle X''(t),\, Q_o X''(t) \rangle - \langle X(t),\, Q_o X(t) \rangle \right] dt$$

$$+ \frac{1}{2} \int_{t_o}^{t_f} \left[\langle -R_o^{-1} B_o^T M_o X''(t),\, -B_o^T M_o X''(t) \rangle - \langle -R_o^{-1} B_o^T M_o X(t),\, -B_o^T M_o X(t) \rangle \right] dt$$

$$+ \frac{1}{2} \left[\langle X''(t_f),\, KX''(t_f) \rangle - \langle X(t_f),\, KX(t_f) \rangle \right] \qquad (5\text{-}157)$$

A special case of the above problem is the one in which $t_f = \infty$ and $K = 0$, then ΔJ becomes:

$$\Delta J = \frac{1}{2} \int_{t_o}^{t_f} \left[\langle X''(t),\, Q_o X''(t) \rangle - \langle X(t),\, Q_o X(t) \rangle \right] dt$$

$$+ \frac{1}{2} \int_{t_o}^{t_f} \left[\langle R_o^{-1} B_o^T M_o X''(t),\, -B_o^T M_o X''(t) \rangle - \langle -R_o^{-1} B_o^T M_o X(t),\, -B_o^T M_o X(t) \rangle \right] dt \qquad (5\text{-}158)$$

so the sensitivity of performance for closed-loop optimal control system is $S = \frac{\Delta J}{\Delta \alpha}$,

$$S = \frac{1}{\Delta \alpha} \left[\frac{1}{2} \int_{t_o}^{t_f} \left[\langle X''(t),\, Q_o X''_o(t) \rangle - \langle X(t),\, Q_o X(t) \rangle \right] dt \right.$$

$$+ \frac{1}{2} \int_{t_o}^{t_f} \left[\langle -R_o^{-1} B_o^T M_o X''(t),\, -B_o^T M_o X''(t) \rangle - -R_o^{-1} B_o^T M_o X(t),\, -B_o^T M_o X(t) \right. dt$$

$$\left. + \frac{1}{2} \left[\langle X''(t_f),\, KX''(t_f) \rangle - \langle X(t_f),\, KX(t_f) \rangle \right] \right] \qquad (5\text{-}159)$$

and for the case when $t_f = \infty$ and $K = 0$, we have

$$S = \frac{1}{\Delta\alpha}\left[\frac{1}{2}\int_{t_o}^{t_f}\left[\langle X''(t),\ Q_o X''(t)\rangle - \langle X(t),\ Q_o X(t)\rangle\right] dt\right.$$

$$- \frac{1}{2}\int_{t_o}^{t_f}\left[\langle -R_o^{-1}B_o^T M_o X''(t),\ -B_o^T M_o X''(t)\rangle\right.$$

$$\left.\left. - \langle -R_o^{-1}B_o^T M_o X(t),\ -B_o^T M_o X(t)\rangle\right] dt\right] \tag{5-160}$$

Determination of the sensitivity of performance for optimal open-loop

Linear systems:

The optimum performance index for nominal parameters α_o, is

$$J^* = \frac{1}{2}\int_{t_o}^{t_f}\left[\langle X(t),\ Q_o X(t)\rangle + \langle u^*(t),\ R_o u^*(t)\rangle\right] dt$$

$$+ \frac{1}{2}\langle X(t_f), KX(t_f)\rangle$$

The performance index when the plant parameters α_o are changed to α is,

from Equation (5-153),

$$J' = \frac{1}{2}\int_{t_o}^{t_f}\left[\langle X'(t),\ Q_o X'(t)\rangle + \langle u^*(t),\ R_o u^*(t)\rangle\right] dt$$

$$+ \frac{1}{2}\langle X'(t_f), KX'(t_f)\rangle$$

so $\Delta J = J' - J^*$ is obtained as,

$$\Delta J = \frac{1}{2}\int_{t_o}^{t_f}\left[\langle X'(t),\ Q_o X'(t)\rangle - \langle X(t),\ Q_o X(t)\rangle\right] dt$$

$$+ \frac{1}{2}\left[\langle X'(t_f),\ KX'(t_f)\rangle - \langle X(t_f),\ KX(t_f)\rangle\right] \tag{5-161}$$

Again, when $t_f = \infty$ and $K = 0$, we have:

$$\Delta J = \frac{1}{2}\int_{t_o}^{t_f}\left[\langle X'(t),\ Q_oX'(t)\rangle - \langle X(t),\ Q_oX(t)\rangle\right] dt \tag{5-162}$$

so the sensitivity of performance for open-loop optimal control system is:

$$S = \frac{\Delta J}{\Delta\alpha} = \frac{1}{\Delta\alpha}\left[\frac{1}{2}\int_{t_o}^{t_f}\left[\langle X'(t),\ Q_oX(t)\rangle - \langle X(t),\ Q_oX(t)\rangle\right] dt + \frac{1}{2}\left[\langle X'(t_f),\ KX'(t_f)\rangle - \langle X(t_f),\ KX(t_f)\rangle\right]\right] \tag{5-163}$$

or for the case when $T = \infty$,

$$S = \frac{1}{\Delta\alpha}\ \frac{1}{2}\int_{t_o}^{t_f}\left[\langle X'(t),\ Q_oX'(t)\rangle - \langle X(t),\ Q_oX(t)\rangle\right] dt \tag{5-164}$$

Thus from equations (5-159) and (5-163) we have the sensitivity of performance of indices in both the closed-loop and open-loop optimal control systems. Rewriting them again we have; in the closed-loop optimal system,

$$S = \frac{1}{\Delta\alpha}\left[\frac{1}{2}\int_{t_o}^{t_f}\left[\langle X''(t),\ Q_oX''(t)\rangle - \langle X(t),\ Q_oX(t)\rangle\right] dt\right.$$

$$+ \frac{1}{2}\int_{t_o}^{t_f}\left[\langle -R_o^{-1}B_o^TM_oX''(t),\ -B_o^TM_oX''(t)\rangle - \langle -R_o^{-1}B_o^TM_oX(t),\ -B_o^TM_oX(t)\rangle\right] dt$$

$$\left. + \frac{1}{2}\left[\langle X''(t_f),\ KX''(t_f)\rangle - \langle X(t_f),\ KX(t_f)\rangle\right]\right]$$

In open-loop optimal control system

$$S = \frac{1}{\Delta\alpha}\left[\frac{1}{2}\int_{t_o}^{t_f}\left[\langle X'(t),\ Q_oX'(t)\rangle - \langle X(t),\ Q_oX(t)\rangle\right] dt\right.$$

$$\left. + \frac{1}{2}\left[\langle X'(t_f),\ KX'(t_f)\rangle - \langle X(t_f),\ KX(t_f)\rangle\right]\right]$$

It can be clearly seen that from the above two equations that the sensitivity of closed-loop optimal control system is different from that of open-loop optimal control system, because of the fact that $X'(t)$ and $X''(t)$ are different and also due to an additional term $\frac{1}{2}\int_{t_o}^{t_f}\left[\langle -R_o^{-1}B_o^T M_o X''(t), -B_o^T M_o X''(t)\rangle - \langle -R_o^{-1}B_o^T M_o X(t), -B_o^T M_o X(t)\rangle\right]dt$ in the sensitivity of closed-loop optimal control system.

Also due to the fact that the value of the performance index for optimal closed-loop control system given by Eq. (5-151) is less than that for optimal open-loop control system given by Eq. (5-153), we can say that the sensitivity of performance for optimal closed-loop control systems is less than that for optimal open-loop systems, in general.

This above theory can be further illustrated by the results of the example of second order system considered in the last section.

CHAPTER IV

CONCLUSION

In this Thesis the practical limitations of the results of some recently published work on sensitivity of optimal control systems have been examined and discussed. A new definition of sensitivity of performance of optimal control systems has been proposed. In essence, with the proposed definition of sensitivity it is possible to calculate the sensitivity without approximating it to a first order. Thus the proposed definition is significant and quite useful for almost every practical system.

It has been shown that the sensitivity of performance for a closed-loop optimal control is less than that for an open-loop optimal control system. A second order system has been considered as an example, to illustrate the merits and the realiability of the proposed definition of sensitivity, and to further illustrate the theory.

The following problems may also be approached using similar techniques.

1. Sensitivity of performance index to controller component variations.

2. Sensitivity analysis in discrete-time linear optimal systems.

3. The theory may be generalized to include non-linear systems.

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