NUMERICAL RANGES OF ELEMENTS OF TOPOLOGICAL ALGEBRAS

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BY

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ABSTRACT

Giles and Husain have extended the concept of numerical range of an element of a normed algebra to locally multiplicatively convex algebra and have studied. We make a further study of it. We show that for a normal element, a, of a complete unital lmc*-algebra the numerical range of a, $V(A,\{p_\alpha\};a)$ is equal to the convex hull of Sp(A,a)and we apply it to compute the numerical ranges of certain elements of certain complete lmc*-algebras. The relation between the numerical range $V(A,\{p_\alpha\};x)$ and various growth conditions on the resolvent $(x-\lambda)^{-1}$ is discussed.

We also extend the concept of numerical range of elements from Banach algebras to pseudo-banach algebras introduced by Allan,Dales and Mcclure. We show that the numerical range of an element of a pseudo-banach algebra is a compact convex subset of the complex plane containing the spectrum of that element and that the spectral radius is equal to the numerical radius.

We also study certain extreme positive maps of B*-algebras, Bp*-algebras and show them to be equal to the nonzero multiplicative linear functionals of the respective algebras. It is shown that the boundary of the numerical range of an element is the spectrum of that element of these topological algebras.

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INTRODUCTION

The numerical range W(T,(,)) of a linear operator T acting on a finite dimensional Hilbert space (H,(,)) was introduced by Toeplitz in the year 1918. In the year 1919 Hausdorff proved that W(T,(,)) is convex. The Hausdorff-Toeplitz theorem that W(T,(,)) is convex, was extended to linear operators on infinite dimensional Hilbert spaces by M.H.Stone. Winter in 1930 and M.H.Stone in 1932 discussed the relationship between the Sp(T) of a bounded linear operator on a Hilbert space and its numerical range. Halmos,Berberian,Filmore, Stampfli, Williams, Kato, Schechter, Anderson, Orland, Berger, Vidav, Pearcy etc. have contributed to numerical ranges of operators on Hilbert space.

In the year 1961 Lumer defined the concept of semi-inner-product on a linear space and generalised the definition of numerical range of linear operators on Hilbert spaces to Banach spaces. He also introduced the numerical range of elements of normed algebras. Bonsall, Duncan, Bollobás Mcgregor, Crabb, Sinclair, Stampfli, Williams etc. have further contributed to numerical range of linear operators on Banach spaces and of elements of normed algebras.

In the year 1972 Giles and Husain have extended the concept of numerical range of an element from Banach algebras to lmc-algebras and have characterized elements with bounded numerical ranges. Also Giles and Koehler have generalised the Vidav-Palmer theorem to give a characterization of b*-algebras by numerical ranges. Moore has studied numerical ranges of operators on locally convex spaces. Giles,Goseph, Koehler and Sims have also studied numerical range of operators on locally convex spaces.

The contents of chapters of this thesis are arranged as follows:

ChapterI is introductory in nature wherein we collect some of the available results on numerical ranges of linear operators on Hilbert and Banach spaces and also of elements of normed algebras. Some of them are generalised to elements of lmc-algebras and pseudo-banach-algebras in latter chapters.

In chapter II an example of a Fréchet algebra $(A, (\|\cdot\|_n))$ with identity such that $V(A, (\|\cdot\|_n); a) = \mathbb{C}$ for a certain a belonging to A is given. Also the normed algebra numerical ranges $V(A, \|\cdot\|_n; a)$ are studied. It is shown that the numerical range of an element of a complete unital lmc-algebra is 'invariant'. It is proved that $D(A, (q_\lambda)_\Lambda; 1)$ is the projective limit of $D_\alpha(A, q_\alpha; 1)$. Various expressions for the numerical and spectral radius and the supremum of the real part of the numerical range of a complete unital lmc-algebra are obtained. Hermitian elements are discussed. It is shown that $V(A,(q_\alpha);a)$ is equal to the convex hull of $Sp(A,a)$ for a normal element, a, of a complete unital lmc*-algebra. Using this result, the numerical ranges of certain normal elements of certain Function-algebras are computed. For an element x of a complete unital lmc-algebra, the relation between the numerical range $V(A,(q_\alpha);x)$ and various growth conditions on the resolvent $(x-\lambda)^{-1}$ is discussed. Some examples of differential and integral operators whose numerical range lie in certain regions of the complex plane are given.

In chapter III, the concept of numerical range of elements is extended to pseudo-banach algebras from banach algebras. It is shown that the numerical range of an element of a pseudo-Banach algebra is a compact convex subset of C containing the spectrum of that element and the spectral radius is equal to the numerical radius.

In chapter IV, certain extreme positive maps of B*, BP* algebras are studied. It is shown that the boundary of the numerical range of an element is the spectrum of that element for these topological algebras.

CHAPTER I

PRELIMINARIES

1. Numerical ranges of opérators on a Hilbert Space

In this section we collect some well known results from the theory of numerical ranges of linear operators on a Hilbert space.

First, the concept of the numerical range $W(T,(,)_o)$ and the spectrum $Sp(T)$ of a linear operator T on a Hilbert space $(H,(,)_o)$ are defined. Even when T is closed (bounded) the numerical range of T is neither open nor closed, but it is a convex subset of C, the closure of which contains the $Sp(T)$. When the Hilbert space is finite dimensional, the numerical range of any linear operator acting on it is a compact convex subset of C. Some examples are given to show the comparison between the numerical range and the spectrum of certain operators. For example, the numerical range may be very big where as the spectrum may be very small consisting of a singleton. If T is a normal operator, then the numerical range of T is equal to the convex hull of $Sp(T)$. Further, $W(T,(,)_o)$ is contained in R, if T is self-adjoint. Some properties of numerical radius are given.

1.1 Notation: By a Hilbert space H or $(H,(,)_o)$, we mean a complex Hilbert space H with a fixed reference inner-product $(,)_o$. We call another inner-product $(,)$ equivalent to $(,)_o$ if and only if the norms $\|u\| = (u,u)^{\frac{1}{2}}$ and $\|u\|_o = (u,u)_o^{\frac{1}{2}}$

induce the same topology on H.

1.2 Definition: Let T be a linear operator acting on the Hilbert space $(H,(,)_0)$. The Hausdorff set or the numerical range of the operator T is defined as the set of complex numbers: $\{(Tx,x)_0 ; (x,x)_0 = 1\} = \{(Tx,x)_0 ; \|x\|_0 = 1\}$ and is denoted by $W(T,(,)_0) = W(T)$ for short, whenever there is no confusion about the inner-product under reference. The set of complex numbers: $\{\lambda : (T-\lambda I)$ is not invertible$\}$ is called the spectrum of T and is denoted by $Sp(T)$.

1.3 Definition : Let A be a bounded linear operator acting on the Hilbert space H. The positive real number $w(A,(,)_0) = \text{Sup}(|\lambda|; \lambda \in W(A,(,)_0))$ is called the numerical radius of A, for shortness it is denoted by $w(A)$. The positive real number $r(A) = \text{Sup}(|\lambda|; \lambda \in Sp(A))$ is called the spectral radius of A. A complex number λ is called an eigenvalue of A if there exists x belonging to H, $x \neq 0$ such that $Ax = \lambda x$.

1.4 Proposition : Let T be a linear operator acting on the Hilbert space H. The numerical range of T, W(T) contains all the eigenvalues of T.

Proof;: Let λ be an eigenvalue of T. Then there exists a u belonging to H, $\|u\|_0 = 1$ with $Tu = \lambda u$. Hence $\lambda = \lambda(u,u)_0$, which belongs to $W(T,(,)_0)$.

1.5 Definition; Let T be a bounded linear operator acting on the Hilbert space H. T is called unitary if $T^*T = TT^* = I$; self-adjoint if $T = T^*$; normal if $T^*T = TT^*$.

1.6 Proposition: Let T be a bounded linear operator acting on the Hilbert space H. Then $Sp(T) = Sp(UTU^{-1})$ and $W(T) = W(UTU^{-1})$, where U is an unitary transformation.

Proof: See M.H.Stone , (48) page 130.

1.7 Theorem: Let T be a linear operator acting on the Hilbert space H. In general W(T) is neither open nor closed, even when T is a bounded linear operator. However W(T) is a convex subset of C and the closure of W(T) contains the Sp(T).

Proof: See M.H.Stone, (48) page 131 ; also Halmos, (27) page 317 and Berberian, (7).

Remark: The theorem about the convexity of W(T) is called the Hausdorff-Toeplitz theorem.

We give below some examples from Halmos, (27) :

1.8 Examples:

I. Let A be an operator acting on C^2 and let $(,)_0$ be the usual scalar product.

(i) If $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $W(A) = [0,1]$ and $Sp(A) = \{1\}$.

(ii) If $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$; then $W(A) = \{\lambda \in C, |\lambda| \leq \frac{1}{2}\}$ and $Sp(A) = \{0\}$.

(iii) If $A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, then $W(A) = \{(x,y);\ 2(x-\frac{1}{2})^2 + 4y^2 \leq 1\}$

and $Sp(A) = \{(1 \pm \sqrt{5})/2\}$.

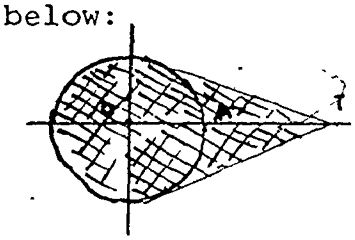
II. Let A be a linear operator acting on C^3, and let $(,)_0$ be the usual scalar product.

(i) If $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, then $Sp(A) = \{1, w, w^2\}$, where $w = (-1+\sqrt{3}\, i)/2$

and W(A) is the equilateral triangle with vertices 1, w, w^2 (boundary and interior).

(ii) If $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \lambda & 0 \end{pmatrix}$, $\frac{1}{2} > \lambda > 0$,

then $Sp(A) = \{0,1\}$ and W(A) is the region described in the graph below:



, $OA = \lambda/2$.

1.9 Theorem: Let T be a linear operator acting on the Hilbert space H. Then,

(i) $\overline{W}(T)$ = convex hull of Sp(T), if T is normal;

(ii) W(T) is contained in R, if T is self-adjoint;

(iii) $W(\alpha T) = \{\alpha\}$, if and only if T = I.

Proof: (i) See Halmos, (27) page 172, or Berberian (7).

(ii) See M.H.Stone, (48), page 133.

(iii) If part,

$$W(\alpha I) = \{ (\alpha I x, x) ; \|x\|_0 = 1 \}$$
$$= \{ \alpha (x,x)_0 ; \|x\|_0 = 1$$
$$= \{\alpha\} .$$

Only if part: Let $W(T) = \{\alpha\}$.

Then $\{(Tx,x)_0 ;\ \|x\|_0 = 1\} = \alpha$. Now for any $(x,x)_0 = 1$, we have $\alpha = \alpha(x,x)_0 = (\alpha x,x)_0 = (Tx,x)_0$. Hence $T = \alpha I$.

As an application of (i) we have:

Let X be a closed unit disc and μ be the two dimensional Lebesgue measure, then $L^2(X,\mu)$ is a Hilbert space. Let A be the normal eigenfree operator $(Af)(z) = zf(z)$. Clearly Sp(A) is the closed unit disc, A being normal, it is equal to the closure of the numerical range of A. Hence the numerical range of A is exactly the open unit disc.

<u>1.10 Theorem</u>: Let T be an operator acting on the Hilbert space H and let U be any open set containing the closed convex hull of Sp(T), Cocl Sp(T). Then there exists an inner-product (,) on H equivalent to $(,)_0$ such that $Sp(T) \subset \overline{W}(T,(,)) \subset U$. Also we have

$\text{Cocl } Sp(T) = \bigcap \{\overline{W}(T,(,));\ (,) \text{ equivalent to } (,)_0\}$

<u>Proof</u>: See R.T.Moore, (35).

<u>1.11 Theorem</u>: Let A,B be linear operators acting on the Hilbert space H. Then we have the following:

(i) $w(A) \geqslant 0$ and $w(A) = 0$ if and only if $A = 0$;

(ii) $w(\alpha A) = |\alpha| w(A), \alpha \in C$;

(iii) $w(A+B) \leq w(A) + w(B)$;

(iv) $r(A) \leq w(A) \leq \|A\|_0$;

(v) $w(AB) \nleq w(A)\, w(B)$ in general;

(vi) $w(A^n) \leq w^n(A)$, where n is a positive integer;

(vii) If $w(A) \leq 1$, then $w(A^n) \leq 1$, where n is a positive integer;

(viii) $w(A^*) = w(A)$;

(ix) $w(A^*A) = \|A\|_o^2$.

Proof: (i) to (iv) are obvious and easy to check.
For (v) to (vii); see Pearcy (39).
For (vii) to (ix) ; see Halmos (27), page 114 and 116.

1.12 Theorem; Let T be an operator acting on a Hilbert space H. Then T is unitary if either i) or ii) holds.

i) $W(T^{\pm 1}) \subset \{z: |z| \leq 1\}$.

ii) $\|T^{-1}\| \leq 1$ and $W(T) \subset \{z; |z| \leq 1\}$.

Proof: See Stampfli (46).

1.13 Theorem : Let f(z) be a rational function with $f(\infty) = \infty$. Let E' be a compact convex set in the complex plane. Let $E = f^{-1}(E')$ and let K be the convex hull of E. If A is a bounded linear operator on the Hilbert space H with W(A) contained in K, then W(f(A)) is contained in E'.

Proof: See Kato (30).

<u>1.14 Definition</u> : Let T be a bounded linear operator acting on the Hilbert space H. We say that T has circular symmetry if T and $e^{i\theta}T$ are unitarily equivalent for all θ.

<u>1.15 Proposition</u>: (A.L.Shields) Let T be a bounded linear operator acting on the Hilbert space H.
Let $A = (T + T^*)/2$, $B = (T - T^*)/2i$. Then A and B are self-adjoint and T = A + Bi. If T has circular symmetry, then

i) $\{z: |z| < w(T)\} \subset W(T) \subset \{z: |z| \leq w(T)\}$;

ii) $\overline{W}(A) = \overline{W}(B) = \{t \in R;\ -w(T) \leq t \leq w(T)\}$;

iii) $w(T) = \|A\| = \|B\|$.

<u>Proof</u>: A.L.Shields (43), page 72.

2. Numerical ranges of operators on normed linear spaces

The results of this section are also known. we collect them here for our future use.

The spatial numerical range $V(X,\|\cdot\|;T)$ and semi-inner-product numerical range $W(X,(,);T)$ of a linear operator T on a normed linear space are defined, and it is shown that $W(X,(,);T)$ is contained in $V(X,\|\cdot\|;T)$; $V(X,\|\cdot\|;T)$ is neither closed nor convex, but it is connected; $Sp(T) \subset \overline{W}(X,(,);T)$;

$\overline{Co}V(X,\|\cdot\|;T) = \overline{Co}\, V(X,\|\cdot\|;T^*)$;

$V(X,\|\cdot\|;T)$ is contained in $V(X,\|\cdot\|;T^*)$. Formulas for spatial numerical radius $v(X,\|\cdot\|;T)$ and semi-inner-product numerical radius $w(X,(,);T)$ are provided. The upper semi-continuity of the set mappings $x \to D(X,\|\cdot\|;x)$ and $x \to V(X,\|\cdot\|;T)$ are discussed.

<u>2.1 Notation</u>: By a normed linear space X or $(X,\|\cdot\|)$ we mean a normed linear space over the real or complex field. We denote the dual space of continuous linear functionals on X by X' and the unit sphere of $(X,\|\cdot\|)$ by $S(X,\|\cdot\|)$, which is the subset $\{x \in X;\ \|x\| = 1\}$ of X. We denote by $B(X,\|\cdot\|)$ the normed algebra of bounded linear operators on $(X,\|\cdot\|)$.

<u>2.2 Definition</u>: Let $(X,\|\cdot\|)$ be a normed linear space. For $x \in S(X,\|\cdot\|)$ and $T \in B(X,\|\cdot\|)$, we define the following:

i) $D(X,\|\cdot\|;x) = \{f \in X';\ \|f\| = f(x) = 1\}$,

ii) $V_x(X,\|\cdot\|;T) = \{f(Tx);\ f \in D(X,\|\cdot\|;x)\}$, and

iii) the <u>spatial numerical range of T</u>, denoted by

$$V(X,\|\cdot\|;T) = \cup\{V_x(X,\|\cdot\|;T);\ x \in S(X,\|\cdot\|)\}$$
$$= \{f(Tx);\ f \in X',\ \|f\| = f(x) = \|x\| = 1\}.$$

Observe that $D(X,\|\cdot\|;x) \neq \emptyset$ by the Hahn Banach theorem.

<u>2.3 Remark</u>: In the special case when X is a Hilbert space, $D(X,\|\cdot\|;x)$ can be identified with $\{x\}$. $V(X,\|\cdot\|;T)$ is neither closed nor convex. Bonsall etc., (14), have proved that $V(X,\|\cdot\|;T)$ is connected.

<u>2.4 Definition</u>: (Lumer) Let X be a complex (real) vector space. We shall say that a complex(real) <u>semi-inner-product</u> is defined on X, if to any x,y belonging to X, there corresponds a complex(real) number (x,y) and the following properties hold:

i) $(x+y,z) = (x,z) + (y,z)$,

$(px,y) = p(x,y)$, for $x,y,z \in X$, p complex(real);

ii) $(x,x) > 0$ for $x \neq 0$;

iii) $|(x,y)|^2 \leq (x,x)(y,y)$.

Then (X,(,)) is called a complex(real) <u>semi-inner-product space</u>.

<u>2.5 Theorem</u>: (Lumer) A semi-inner-product(s.i.p) space (X,(,)) is a normed linear space with the norm $\|\cdot\|$ given by $\|x\| = (x,x)^{\frac{1}{2}}, x \in X$. Every normed linear space can be made into a s.i.p space (in general, in infinitely many different ways).

<u>Proof</u>: See Lumer (33).

<u>2.6 Definition</u>: (Lumer) Given a s.i.p space (X,(,)) and $T \in B(X,(,))$, the <u>numerical range</u> W(X,(,);T) is defined as a set of complex numbers given by:

$$W(X,(,);T) = \{(Tx,x);\ (x,x) = 1\}.$$

2.7 Theorem: Let $(X, \|\cdot\|)$ be a normed linear space, $(,)$ be a s.i.p on X satisfying $\|x\| = (x,x)^{\frac{1}{2}}, x \in X$ and $T \in B(X, \|\cdot\|)$. Then we have the following:

i) $W(X,(,);T) \subset V(X,\|\cdot\|;T)$;

ii) $\operatorname{Sup}\{\operatorname{Re} z;\ z \in W(X,(,),T)\} = \begin{matrix}\operatorname{Lim} p \to 0 \\ \operatorname{Inf} p > 0\end{matrix} \left\{\frac{(\|T+pI\| - 1)}{p}\right\}$;

iii) $\operatorname{Sup}\{|z| : z \in W(X,(,);T)\} = \operatorname{Sup}\{|z|;\ z \in V(X,\|\cdot\|;T)\}$;

iv) $Sp(T) \subset \overline{W}(X,(,);T)$.

Proof: For i), ii) and iii), See (13), page 86, also Lumer (33).

For (iv) See Williams (50).

2.8 Theorem: Let $(X, \|\cdot\|)$ be a normed linear space, $T \in B(X, \|\cdot\|)$ and T^* be the adjoint of T. Then we have the following:

i) $V(X,\|\cdot\|;T) \subset V(X',\|\cdot\|';T^*) \subset \{z \in C;\ |z| \leq \|T\|\}$;

ii) $\overline{Co}\,V(X,\|\cdot\|;T) = \overline{Co}\,V(X',\|\cdot\|':T^*)$;

iii) $v(X,\|\cdot\|;T) = v(X',\|\cdot\|';T^*)$.

Proof; See (13), page 85.

2.9 Theorem: Let $(X, \|\cdot\|_0)$ be a normed linear space and $T \in B(X, \|\cdot\|_0)$. Then we have the following:

i) $Sp(T) \subset \overline{Co}\,V(X,\|\cdot\|_0;T)$;

ii) $Co\,Sp(T) = \bigcap\{\overline{Co}\,V(X,\|\cdot\|;T)$; $\|\cdot\|$ is equivalent to $\|\cdot\|\}$.

Proof: For i) and ii) See (13) page 88,91 respectively.

2.10 THEOREM .... Let $(x,\|\cdot\|)$ be a normed linear space and $T \in B(X,\|\cdot\|)$. Then we have,

$$\begin{aligned} \text{Sup Re } V(X,\|\cdot\|;T) &= \text{Sup}\{\text{Re } z;\ z \in V(X;\|\cdot\|;T)\} \\ &= \text{Sup}\{c:\ c \in R, \text{ there exists } x = x(c), \\ &\quad x \neq 0, \text{ such that } \|(1-rc+rT)x\| \geq \|x\| \\ &\quad \text{for all } r \geq 0\}. \end{aligned}$$

Proof: See Bollobás (9), page 376.

2.11 Definition: We denote by $\mathcal{P}(U)$, the set of all subsets of a set U. If E is a topological space, let $\{\mathcal{P}(U) : U \subseteq E, U \text{ is open}\}$ be a basis for the τ-topology on $\mathcal{P}(E)$. A mapping $x \longrightarrow A(x)$ of a topological space F into the set of subsets of a topological linear space E is upper-semicontinuous (u.s.c) on F if for every x belonging to F and every neighbourhood U of 0 in E there exists a neighbourhood V of x such that for all y belonging to V, A(y) is contained in A(x) + U. (See Berge (8), pages 35-36).

2.12 Theorem: (Bonsall etc. (14)) Let F be a topological vector space. If the set valued mapping $x \longrightarrow A(x)$ is τ-continuous, then it is u.s.c. If for every x in F, A(x) is a compact subset of E, then the function A is τ-continuous if and only if it is u.s.c.

Proof: Assume A is τ-continuous. If x belongs to F and U is an open neighbourhood of 0 in E, then since $\wp(A(x)+U)$ is a τ-open subset of $\wp(E)$, it follows that $A^{-1}(\wp(A(x)+U)) = \{y \in F;\ A(y) \subseteq A(x)+U\}$ is an open subset of F. Hence $A^{-1}(\wp(A(x)) + U)$ is a V whose existence is required in the definition of u.s.c.

Assume that A is u.s.c. Let $\wp(U)$, with U an open subset of E, be a basic τ-open set. If x belongs to $A^{-1}(\wp(U))$, then $A(x) \subseteq U$. The compactness hypothesis now provides a neighbourhood G of 0 in E, such that $A(x) + G \subseteq U$ (See Berge (8), pages 35-36). Since A is u.s.c there is a neighbourhood V of x such that for each y belonging to V, A(y) is contained A(x) + G, which is contained in U. That is A(y) belongs to $\wp(U)$. Thus $x \in V \subseteq A^{-1}(\wp(U))$. So $A^{-1}(\wp(U))$ is open and τ-continuous.

2.13 Proposition: Let $(X,\|\cdot\|)$ be a normed linear space. If x belongs to $S(X,\|\cdot\|)$, $D(X,\|\cdot\|;x)$ is a weak*-closed subset of the (solid) unit ball in X' and hence is w*-compact by the Alaoglu's theorem. $D(X,\|\cdot\|;x)$ is connected in any topology which makes X' a topological linear space. Also $D(X,\|\cdot\|;x)$ is convex.

Proof: See Bonsall etc. (14).

2.14 Theorem: Let $(X,\|\cdot\|)$ be a normed linear space, $S(X,\|\cdot\|)$ have the norm topology and X' have the w*-topology. Then the mapping $x \longrightarrow D(X,\|\cdot\|;x)$ is τ-continuous and u.s.c. If $T:S(X,\|\cdot\|) \longrightarrow (X,\|\cdot\|)$ is a continuous linear mapping, then the mapping $x \longrightarrow V(X,\|\cdot\|;T)$ is a τ-continuous and u.s.c mapping of $S(X,\|\cdot\|)$ with the norm topology into the set of subsets of the scalar field.

Proof: See Bonsall etc. (14), page 88.

3. Numerical range of an element of a normed algebra

The results of this section are also known and are given here for future use. The numerical range $V(A,\|\cdot\|;a)$ of an element of a normed algebra $(A,\|\cdot\|)$ is defined to be the image of the w*-compact subset $D(A,\|\cdot\|;1)$ of the unit sphere of A', under the weak*-continuous mapping $f \longrightarrow f(a)$ of $D(A,\|\cdot\|;1)$ into C. $V(A,\|\cdot\|;a)$ is a compact convex subset of C containing the spectrum of a. The set mappings $a \longrightarrow Sp(A,a)$ and $a \longrightarrow V(A,\|\cdot\|;a)$ are u.s.c. The numerical range is'invariant', in some sense to be specified in the text. Several expressions for the numerical and spectral radius are written. Estimates of best positive constants f(n), g(n) satisfying $v(A,\|\cdot\|;a^n) \leq f(n)\ v^n(A,\|\cdot\|;a)$ and $\|a^n\| \leq g(n)\ v^n(A,\|\cdot\|;a)$ are given. Finally a characterization of B*-algebras interms of numerical ranges is stated.

<u>3.1 Definition</u>: Let $(A,\|\cdot\|)$ be a normed algebra over a field F (F is either the real field R or the complex field C). If A has an identity 1, such that $\|1\| = 1$, then A is called a unital normed algebra. Let A' be the dual space of A, that is the set of all continuous linear functionals on A. Let $S(A,\|\cdot\|) = \{x \in A;\ \|x\| = 1\}$. For $x \in S(A,\|\cdot\|)$, let $D(A,\|\cdot\|;x) = \{f \in A';\ f(x) = 1 = \|f\|\}$. Then by the Hahn-Banach theorem $D(A,\|\cdot\|;x) \neq \emptyset$. Elements of $D(A,\|\cdot\|;x)$ are called the <u>support functionals</u> for $S(A,\|\cdot\|)$ at x and those of $D(A,\|\cdot\|;1)$ are called the <u>normalized states</u>.

<u>Note</u>: All topological algebras are commutative and over the field of complex numbers unless otherwise stated.

<u>3.2 Definition</u>: Let $(A,\|\cdot\|)$ be an unital normed algebra. For a belonging to A and x an element of $S(A,\|\cdot\|)$, write $V(A,\|\cdot\|;a,x) = \{f(ax);\ f \in D(A,\|\cdot\|;x)\}$ and define the <u>numerical range</u> of a as $V(A,\|\cdot\|;a) = \cup\{V(A,\|\cdot\|;a,x);\ x \in S(A,\|\cdot\|)\}$. The <u>spectrum of a</u>, denoted by Sp(A,a) is defined by $Sp(A,a) = \{z \in F;\ (z-a) \text{ is not invertible in } A\}$.

<u>3.3 Remark</u>: From the above definition we observe that the numerical range of an element of a normed algebra depends on the norm and the spectrum depends on the algebraic structure and is independent of the norm.

<u>3.4 Proposition</u>: Let $(A,\|\cdot\|)$ be an unital normed algebra and let $a \in A$. If $y, y^{-1} \in S(A,\|\cdot\|)$, then

$V(A,\|\cdot\|;a) = V(A,\|\cdot\|;a,y)$.

In particular, $V(A,\|\cdot\|;a) = V(A,\|\cdot\|;a,1)$.

Proof: See Bollobás (9), page 378, theorem 6.

3.5 Proposition: Let $(A,\|\cdot\|)$ be an unital normed algebra and let a be an element of A. Then we have the following:

i) $D(A,\|\cdot\|;1)$ is a convex weak*-compact subset of A' and $V(A,\|\cdot\|;a)$ is a nonempty convex compact subset of F.

ii) $Sp(A,a)$ is a non-empty compact subset of F.

iii) The set functions $a \to Sp(A,a)$ and $a \to V(A,\|\cdot\|;a)$ are u.s.c.

Proof: (i) $D(A,\|\cdot\|;1) = \{f \in A' ; \|f\| \leq 1 \text{ and } f(1) = 1\}$. is a closed subset of the solid unit ball in A' and hence it is a weak*-compact subset of A' by the Alaoglu's theorem. The fact that it is convex is easy to check. The set $V(A,\|\cdot\|;a,1)$ is the image of $D(A,\|\cdot\|;1)$ under the weak*-continuous linear mapping $f \to f(a)$, and so is a compact convex subset of F. Since $V(A,\|\cdot\|;a,1) = V(A,\|\cdot\|;a)$, the result follows. Also $V(A,\|\cdot\|;a) \neq \emptyset$, because $D(A,\|\cdot\|;1) \neq \emptyset$ by prop.1.1.

ii) It is well known. See (41), page 354, 18.6.

iii) For the upper-semi-contunity of $a \to Sp(A,a)$, See Rickart (40), 1.6.16, page 35.

For the u.s.c of $a \to V(A,\|\cdot\|;a)$, see 2.14.

3.6 Proposition: Let $(A,\|\cdot\|)$ be an unital normed algebra and let B be a sub-algebra of A containing the identity element 1. Then for any $a \in B$, $V(B,\|\cdot\|:a)$ is equal to $V(A,\|\cdot\|;a)$ and $Sp(B,a) \supset Sp(A,a)$. Also $r(B,a) = r(A,a)$.

Proof: If $f \in D(A,\|\cdot\|;1)$ then $f/B \in D(B,\|\cdot\|;1)$ and so $V(A,\|\cdot\|;a) \subseteq V(B,\|\cdot\|;a)$. On the other hand if f belongs to $D(B,\|\cdot\|;1)$, by the Hahn-Banach theorem, f can be extended to an element of $D(A,\|\cdot\|;1)$. Therefore $V(B,\|\cdot\|;a)$ is contained in $V(A,\|\cdot\|;a)$. Hence $V(A,\|\cdot\|;a) = V(B,\|\cdot\|;a)$.

Since for $z \in C$, $(z1-a)$ is not invertible in A implies $(z1-a)$ is not invertible in B, we have $Sp(B,a)$ contains $Sp(A,a)$. Also $r(B,a) = r(A,a) = \lim_{n\to\infty} \|a^n\|^{1/n}$ by prop. 3.12.

3.7 Corollary: Let $(A,\|\cdot\|)$ be an unital normed algebra and let a be an element of A. Then $V(A,\|\cdot\|;a)$ is the same as $V(\bar{A},\|\cdot\|;a)$, where $(\bar{A},\|\cdot\|)$ is the Banach algebra obtained by completing the normed algebra $(A,\|\cdot\|)$.

3.8 Theorem: Let $(A,\|\cdot\|)$ be an unital Banach algebra and let a be an element of A. Then $Sp(A,a) \subseteq V(A,\|\cdot\|;a)$.

Proof: See (13), page 19, theorem 6.

<u>3.9 Definition</u> : Let $(A,\|\cdot\|)$ be an unital Banach algebra and let a be an element of A. Then,

i) $v(A,\|\cdot\|;a) = \text{Sup}\{|z|;\ z \in V(A,\|\cdot\|;a)\}$ is called the <u>numerical radius</u> of a;

ii) $r(A,a) = \text{Sup}\{|z|;\ z \in Sp(A,a)\}$ is called the <u>spectral radius</u> of a.

<u>3.10 Theorem</u>: Let $(A,\|\cdot\|)$ be an unital Banach algebra and let a be an element of A. Then the following numbers are equal.

i) $\text{Sup Re } V(A,\|\cdot\|;a)$,

ii) $\inf\limits_{z>0} \dfrac{(\|1+za\|-1)}{z}$,

iii) $\lim\limits_{z\to 0^+} \dfrac{(\|1+za\|-1)}{z}$,

iv) $\sup\limits_{z>0} \dfrac{\log\|\exp(za)\|}{z}$,

v) $\lim\limits_{z\to 0^+} \dfrac{\log\|\exp(za)\|}{z}$,

vi) $\text{Sup}\ (c : c \in R, \|1-rc+ra\| \geq 1 \text{ for all } r \geq 0)$,

vii) $\text{Inf}\ (c; c \in R$, there exists an $\epsilon > 0$, such that $\|1-\epsilon c+\epsilon a\| \leq 1)$.

<u>Proof</u>: For the equality of ii), iii), iv) and v) to i), see (13). For the equality of vi) and vii) to i) see Bollobás (9), Page 379 ,theorem7; page380, prop. 2.

<u>3.11 Theorem</u>: Let $(A,\|\cdot\|)$ be an unital Banach algebra and let a be an element of A. The following positive

real numbers are equal.

i) $v(A,\|\cdot\|;a)$,

ii) $\sup_{|p|=1} \lim_{q\to 0^+} \frac{(\|1 + pqa\| - 1)}{q}$,

iii) $\sup_{|p|=1} \lim_{q\to 0^+} \frac{(\log\|\exp(pqa)\|)}{q}$,

iv) $\sup_{z\in C} \frac{\log\|\exp(za)\|}{z}$, $z \neq 0$,

v) Inf (c; c > 0, there exists a positive p such that, if $w\in C, |w|=1$, then $\|1p+wpa/c\| \leq 1$).

Proof: For the equality of v) and i) see (10), page 412, prop.1.

For the equality of ii), iii) and iv) to i), see (44), page 20, 3.4.

3.12 Theorem: Let $(A,\|\cdot\|)$ be an unital Banach algebra and let a be an element of A. Then the following positive real numbers are equal:

i) $r(A,a)$,

ii) $\lim_{n\to\infty} \|a^n\|^{1/n}$,

iii) Inf ($\|a^n\|^{1/n}$; n=1,2,3,.....) ,

iv) Sup ($|f(a)|$; $f\in M(A)$) ,

v) $\max_{|z|=1} \lim_{p\to\infty} \frac{\log\|\exp(pza)\|}{p}$.

Proof: For the equality of ii), iii) and iv) to i), see Rickart (40).

For the equality of i) and v), see (13), page 32 to corollary 9.

3.13 Theorem: Let $(A,\|\cdot\|)$ be an unital Banach algebra and let a be an element of A. The following real numbers are equal:

i) $\operatorname{Max} \operatorname{Re} Sp(A,a)$,

ii) $\inf_{z>0} \frac{\log\|\exp(za)\|}{z}$,

iii) $\lim_{z\to\infty} \frac{\log\|\exp(za)\|}{z}$.

Proof: See (13), page 32, theorem 8.

3.14 Theorem: Let $(A,\|\cdot\|)$ be an unital Banach algebra and let a be an element of A. Let f(n), g(n) be positive real numbers such that $v(A,\|\cdot\|;a^n) \leq f(n)\; v^n(A,\|\cdot\|;a)$ and $\|a^n\| \leq g(n)\; v^n(A,\|\cdot\|;a)$. Then $g(n) = n!(e/n)^n$ is the best constant satisfying the above inequality and $f(n) \leq g(n)$.

Proof: See (10), page 413, theorem 3.

3.15 Theorem; Let $(A,\|\cdot\|)$ be an unital Banach algebra and let a be an element of A. Then we have $\|a\| \geq v(A,\|\cdot\|;a) \geq e^{-1}\|a\|$.

Proof: See (13), page 34, theorem 1.

3.16 Theorem: Let J be a closed two sided ideal of a Banach algebra $(A,\|\cdot\|)$, a be an element of A and let a' denote the J-coset of a. Then $V(A/J,\|\cdot\|';a') = \bigcap\{V(A,\|\cdot\|;a+j);\ j\in J\}$, where $\|\cdot\|'$ is the quotient norm.

Proof: See (15), page 53.

3.17 Definition: Let $(A, \|\cdot\|)$ be a Banach algebra and let a be an element of A. The element a is called dissipative if $V(A, \|\cdot\|; a) \subseteq (z \in C;\ \mathrm{Re}\ z \leq 0)$, hermitian if $V(A, \|\cdot\|; a)$ is a subset of the reals.

3.18 Theorem: Let $(A, \|\cdot\|)$ be a Banach algebra and let a be an element of A. The element a is dissipative if and only if $\| \exp(za) \| \leq 1 \quad (z \geq 0)$, and is hermitian if and only if $\|\exp(iza)\| = 1 \ (z \in R)$. If a is normal, then $V(A, \|\cdot\|; a) = \mathrm{Co}\ \mathrm{Sp}(A, a)$. If a is hermitian, then $r(A, a) = v(A, \|\cdot\|; a) = \| a \|$.

Proof: See (13), page 55.

3.19 Theorem: (Bonsall and Duncan (15))

Let $(A, \|\cdot\|)$ be an unital Banach*-algebra such that $D(A, \|\cdot\|; 1) \subset \mathrm{Sym}(A') = \{ f \in A';\ f = f^* \}$, then A is a B*-algebra.

Proof: Let h be an element of Sym(A). Then,
$V(A, \|\cdot\|; h) = \{ f(h);\ f \in D(A, \|\cdot\|;\ 1) \} \subset \{ f(h);\ f \in \mathrm{Sym}(A') \} \subseteq R$.

3.20 Theorem: Let $(A, \|\cdot\|)$ be a B*-algebra and let J be a closed two sided ideal of A. Then $J^* = J$ and A/J is a B*-algebra.

Proof: See (15), page 213.

Proof: Let A be unital. We write (a) for the canonical image of a in A/J. Given $f \in D(A,\|\cdot\|;1)$, let $f(a) = f(\ (a)\ )$ $(a \in A)$. Then $f \in D(A,\|\cdot\|;1)$.

Given $h \in Sym(A)$, $f(\ (h)\ ) = \tilde{f}(h) \in R$. Since $A/J = \{ (h) + i(k);\ h,k \in Her\ A \}$, by the Vidav-palmer theorem A/J is a B*-algebra with the canonical norm and involution. Also $a \rightarrow (a)$ is a star homomorphism and $J^* = J$.

3.21 Theorem: If g is a norm decreasing homomorphism of a complex unital normed algebra $(A,\|\cdot\|)$ into another complex unital normed algebra $(B,\|\cdot\|')$ and $g(1) = 1$, then $V(B,\|\cdot\|';g(a)) \subset V(A,\|\cdot\|;a)$ for all a belonging to A.

Proof: Obvious.

3.22 Theorem: Let $(A,\|\cdot\|)$ be an unital Banach algebra and let x,y be elements of A. Then we have the following:

i) $0 \leq r(A,x) \leq \|x\|$,

ii) $r(A,xy) \leq r(A,x)\ r(A,y)$,

iii) $r(A,x+y) \leq r(A,x) + r(A,y)$,

iv) $r(A,x^n) = r^n(A,x)$.

Proof: See Rickart (40), page 10.

3.23 Theorem: Let T be a bounded linear operator on a normed linear space $(X,\|\cdot\|)$. Then we have,

$\overline{Co}\ V(X,\|\cdot\|;T) = V(\ B(X),\|\cdot\|;T)$, where $V(\ B(X),\|\cdot\|;T)$ is the normed algebra numerical range of T, T being considered as an element of the normed algebra ($B(X), \|\cdot\|$) of all bounded linear operators on $(X, \|\cdot\|)$.

<u>Proof</u>: See (13), page 84, theorem 4.

<u>Remark</u>: $W(X,(,);T) \subset V(X,\|\cdot\|;T) \subset V(B(X),\|\cdot\|;T)$

and $\overline{Co}\, W(X,(,);T) = \overline{Co}\, V(X,\|\cdot\|;T) = V(B(X),\|\cdot\|;T)$,

where the s.i.p. (,) determines the norm of X.

4. Topological Algebras

In this section we define topological algebra over a field F. Various special cases of topological algebras viz. Banach algebra, locally multiplicatively convex algebra (lmc-algebra), locally convex algebra, Fréchet algebra are defined. A lmc-algebra is shown to be isomorphic with a sub-algebra of a cartesian product of Banach algebras. Some examples of topological algebras are collected.

<u>4.1 Definition</u>: A <u>topological algebra</u> is an algebra A which is at the same time a topological linear space over a field (which may be either the real field R or the complex field C) such that the mapping $(x,y) \longrightarrow xy$ of AxA into A is continuous.

<u>4.2 Definition</u>: Let A be an (topological) algebra with unit 1. We write $G(A) = (x \in A;\ x^{-1} \in A)$. If $G(A) \setminus \{o\} = A$, then A is called a (topological) <u>division</u>

algebra.

Note: All topological algebras in the sequel are Hausdorff and commutative unless otherwise stated.

4.3 Definition: A locally convex algebra is a topological algebra which is at the same time a locally convex linear space. A subset U of an algebra is called multiplicatively convex (m-convex) if U is convex and idempotent, ie. $U.U \subset U$. A topological algebra is called locally m-convex if there exists a basis $\{U_\alpha\}_\Lambda$ for the neighbourhoods of the origin consisting of m-convex symmetric sets. Clearly a normed algebra is a locally m-convex algebra. An F-algebra is a complete metrizable lmc-algebra.

4.4 Proposition: The topology of a locally convex algebra can be given by means of a family of continuous seminorms $\{p_\alpha\}_\Lambda$ such that for every continuous seminorm p_α on A there exists a continuous seminorm p_β such that $p_\alpha(xy) \leq p_\beta(x)p_\beta(y)$, for all x,y belonging to A. A lmc-algebra can be topologized by a family $\{p_\alpha\}_\Lambda$ of submultiplicative seminorms p_α, ie. $p_\alpha(xy) \leq p_\alpha(x)p_\alpha(y)$, for all x,y belonging to A. A normed algebra is topologized by a single norm, $\|\cdot\|$. An F-algebra can be topologized by a countable family $\{p_n\}_1^\infty$ of sub-multiplicative seminorms p_n and the sequence $\{p_n\}_1^\infty$ may be assumed to be increasing ie. $p_n(x) \leq p_{n+1}(x)$, for all $n \geq 1$ and $x \in A$. A is Hausdorff if and only if $p_n(x) = 0$ for $n \geq 1$ implies $x = 0$.

<u>4.5 Theorem</u> ; Let A be a lmc-algebra. Then A is isomorphic with a sub-algebra of a cartesian product of Banach algebras. If A is complete then the sub-algebra in question is closed.

<u>Proof</u>: Let $\{p_\alpha\}_\Lambda$ be a system of submultiplicative seminorms in A giving its topology. If we set $N_\alpha = \{x \in A \; ; \; p_\alpha(x) = 0\}$, $\alpha \in \Lambda$. We obtain a system of closed ideals in A.

Let A_α designate the Banach algebra obtained by completion of the quotient normed algebra $(A/N_\alpha ; \|\cdot\|_\alpha)$, where $\|\cdot\|_\alpha$ is the quotient norm. We denote by π_α the natural homomorphism of A into A_α. Let $\tilde{A} = \prod_{\alpha \in \Lambda} A_\alpha$, with the cartesian product topology and coordinate wise operations. The mapping $x \xrightarrow{\pi} (\pi_\alpha(x))$ of A into $\tilde{A}$ is the desired isomorphism. It is clearly 1 to 1, because A is Hausdorff and it is a topological isomorphism, since the topology of A is identical with the topology of its image in $\tilde{A}$. If A is complete, then clearly $\pi(A)$ is closed in $\tilde{A}$.

<u>4.6 Corollary</u>: If A is a lmc-algebra with unit 1, then its topology can be given by means of a system $\{p_\alpha\}_{\alpha \in \Lambda}$ of submultiplicative seminorms, such that $p_\alpha(1) = 1$, for all $\alpha \in \Lambda$.

<u>Notation</u>: lmc = locally m-convex.

<u>4.7 Theorem</u>: Let A be a complete lmc-algebra. Then A is isomorphic with an inverse limit of Banach algebras A_α, with mappings $\pi_{\alpha,\beta}$ of A_β into A_α $(\beta > \alpha)$.

Proof: See Michael (35).

4.8 Definition : Let A be an unital lmc-algebra. We call by Q(A) the class of all families of submultiplicative seminorms generating the topology of A. By $(A, \{q_\alpha\}_\Lambda)$ we mean the algebra A, with the particular family of seminorms $\{q_\alpha\}_\Lambda \in Q(A)$, which may be assumed to be directed by Λ, $\alpha, \beta \in \Lambda$, $\alpha \leq \beta$ implies that $q_\alpha(x) \leq q_\beta(x)$ for each $x \in A$. It is easy to see that there exists a directed family $\{q_\alpha\}_\Lambda$ in Q(A).

4.9 Definition: A topological algebra E is called an algebra with involution if there is a mapping $x \longrightarrow x^*$ of E into itself satisfying the following conditions, for x,y belonging to E and $\beta \in C$:

i) $x^{**} = x$,

ii) $(x + y)^* = x^* + y^*$,

iii) $(px)^* = \bar{p}x^*$ ($\bar{p}$ is the complex conjugate of p),

iv) $(xy)^* = y^* x^*$.

E is called a b*-algebra if it is an algebra with involution admitting a set $\{p_\alpha\}_\Lambda$ of seminorms satisfying

i)' $p_\alpha(xy) \leq p_\alpha(x)p_\alpha(y)$, for $x,y \in E$,

ii)' the sets $\{x \in E;\ p_\alpha(x) < t\}$ for $\alpha \in \Lambda$, $t > 0$ form a base of o-neighbourhoods in E,

iii)' $p_\alpha(xx^*) = [p_\alpha(x)]^2$, for x in E. In otherwords, a b*-algebra is a lmc-algebra with an involution such that the seminorms $\{p_\alpha\}_\Lambda$ satisfy iii)'.

A Banach algebra with involution in which the norm satisfies the condition $\|x^*x\| = \|x\|^2$ is called a B*-algebra.

4.10 Examples of topological algebras:

1) Let X be a completely regular space, then $C(X) = \{ f;\ f$ continuous : X into $C \}$ is a lmc-algebra by simply taking the seminorms : $\{p_K;\ K$ a compact subset of $X\}$, where p_K is defined by $p_K(f) = \operatorname{Sup}\{ |f(x)|\ ;\ x \text{ in } K\}$ for f in C(X).

2) The algebra A of continuous complex valued functions on the real line, in the compact-open topology is a complete unital lmc-algebra.

3) The algebra A of entire functions on the complex plane T, in the compact-open topology is a complete unital lmc-algebra.

4) Consider $L^{\omega}[0,1]$, the equivalence classes of measurable functions on the unit interval $[0, 1]$ such that $\|x\|_p = (\int_0^1 |x(t)|^p dt)^{1/p} < \infty$, p = 1,2, ... This is an algebra under pointwise operations, as it follows from Schwartz inequality:

$$\|xy\|_p = (\int_0^1 |x^p(t)y^p(t)|\, dt)^{1/p} \leq (\int_0^1 |x(t)|^{2p} dt)^{1/2p} (\int_0^1 |y(t)|^{2p} dt)^{1/2p}.$$

$$L^{\omega}[0, 1] = \bigcap_{p \geq 1} L^p[0, 1].$$

$L^{\infty} \subset L^{\omega} \subset L^p \subset L^n$, $1 \leq p \leq n$. (See Arens (4)).

The identity mappings $L^{\infty} \to L^{\omega} \to L^p$ are continuous, but their inverses are not. L^{∞} is dense in L^{ω}, L^{ω} is dense in each L^p.

L^{ω} is a commutative locally convex algebra which is metrizable but is not locally m-convex, because if U is a convex open set in L^{ω} containing 0, and if $U \cdot U \subset U$, then U coincides with the whole space L^{ω}.

5) The algebra B(X) of bounded linear operators on a normed linear space X, with composition for the product is a normed algebra with the operator norm, and is a Banach algebra if X is complete.

6) The algebra C[a, b] of continuous functions on a closed interval [a, b] with the sup norm and pointwise product $fg(x) = f(x)\,g(x)$ is a Banach algebra.

7) The algebra A of functions analytic in the open unit disc in the complex plane and continuous in the closed unit disc, with the norm $\|f\| = \sup_{|z| \leq 1} |f(z)|$ and pointwise product $fg(z) = f(z)\,g(z)$ is a Banach algebra.

8) The algebra of continuous functions on a compact Hausdorff space with pointwise operations and sup norm is a Banach algebra.

CHAPTER II

NUMERICAL RANGE THEORY FOR UNITAL lmc-ALGEBRAS

Giles and Husain have defined, for a unital lmc-algebra $(A, \{q_\alpha\}_\Lambda)$, the sets $D_\alpha(A, q_\alpha; 1)$, $\alpha \in \Lambda$ and $D(A,\{q_\alpha\}_\Lambda; 1) = \bigcup_{\alpha\in\Lambda} D_\alpha(A, q_\alpha; 1)$ which are subsets of A' and also have defined the numerical range of $a \in A$ as $V(A,\{q_\alpha\}_\Lambda; a) = \{f(a);\ f \in D(A,\{q_\alpha\}_\Lambda; 1)\} = \bigcup_{\alpha\in\Lambda} V_\alpha(A, q_\alpha; a)$ $= \bigcup_{\alpha\in\Lambda} V(A_\alpha, \|\cdot\|_\alpha; a)$. It is shown that for $\alpha, \beta \in \Lambda, \alpha \le \beta$; $D_\alpha(A, q_\alpha; 1) \subseteq D_\beta(A, q_\beta; 1)$ and $V(A_\alpha, \|\cdot\|_\alpha; a_\alpha) \subseteq V(A_\beta, \|\cdot\|_\beta; a_\beta)$, and that if $(A, \{q_\alpha\}_\Lambda)$ is complete and for each $x \in A$ and $\alpha \in \Lambda$, $q_\alpha(x) < \infty$, then $V(A, \{q_\alpha\}_\Lambda; a) = V(A, q; a)$ where $q(x) = \sup_\alpha q_\alpha(x)$. We give an example of a Fréchet algebra $(A, \{\|\cdot\|_n\})$ with identity such that $V(A, \{\|\cdot\|_n\}; a) = C$, for a certain $a \in A$.

1. Numerical range of an element of a lmc-algebra

1.1 Definition : (Giles and Husain) Let $(A, \{q_\alpha\}_\Lambda)$ be an unital lmc-algebra. For each fixed $\alpha \in \Lambda$ define

$D_\alpha(A, q_\alpha; 1) = \{f \in A' : f(1) = 1 \text{ and } |f(x)| \le q_\alpha(x),\ \forall\ x \text{ in } A\}$,

and let $D(A, \{q_\alpha\}_\Lambda; 1) = \bigcup_\alpha D_\alpha(A, q_\alpha; 1)$. For each α, we write

$V_\alpha(A, q_\alpha; a) = \{f(a);\ f \in D_\alpha(A, q_\alpha; 1)\}$. The subset

$V(A, \{q_\alpha\}_\Lambda; a) = \bigcup_\alpha V_\alpha(A, q_\alpha; a)$ of the complex numbers is called the numerical range of $a \in A$.

1.2 Proposition: Let $(A, \{q_\alpha\}_\Lambda)$ be a lmc-algebra. Then for $\alpha \le \beta$, $\alpha, \beta \in \Lambda$; $D_\alpha(A, q_\alpha; 1) \subseteq D_\beta(A, q_\beta; 1)$.

Proof: Let $f \in D_\alpha(A, q_\alpha; 1)$, then $f \in A'$, $f(1) = 1$ and $|f(x)| \leq q_\alpha(x)$, for all x in A. But $q_\alpha(x) \leq q_\beta(x)$ for $\alpha \leq \beta$ and for all x in A, since the family of seminorms $\{q_\alpha\}_\Lambda$ may assumed to be directed by Λ. Hence $|f(x)| \leq q_\beta(x)$, which shows that f belongs to $D_\beta(A, q_\beta; 1)$. Thus $D_\alpha(A, q_\alpha; 1)$ is contained in $D_\beta(A, q_\beta; 1)$ $\forall\ \alpha, \beta \in \Lambda$, $\alpha \leq \beta$.

1.3 Remark: We define $D(A, \{q_\alpha\}_\Lambda; 1) = \bigcup_\alpha D_\alpha(A, q_\alpha; 1)$. Since for each $\alpha \in \Lambda$, $D_\alpha(A, q_\alpha; 1) = D(A_\alpha, \|\cdot\|_\alpha; 1) \neq \emptyset$ by prop.3.1, of Ch.I, $V(A, \{q_\alpha\}_\Lambda; a) \neq \emptyset$.

1.4 Proposition: Let $(A, \{q_\alpha\}_\Lambda)$ be an unital lmc-algebra, and let a be an element of A. Then we have,

$V(A, \{q_\alpha\}_\Lambda; a) = \bigcup_\alpha V(A_\alpha, \|\cdot\|_\alpha; a_\alpha) = \bigcup_\alpha V(\pi_\alpha A, \|\cdot\|_\alpha; a_\alpha)$.

Proof: (Giles and Husain (24)) To each linear functional f on $(A, \{q_\alpha\}_\Lambda)$ which annihilates N_α, we can define the linear functional F on A_α by $F(x_\alpha) = f(x)$ and to each linear functional F on A_α, we can define the linear functional f on (A, q_α) by $f(x) = F(x_\alpha)$. From the definition of the norm in A_α we see that $D_\alpha(A, q_\alpha; 1)$ is isomorphic to $D(A_\alpha, \|\cdot\|_\alpha; 1_\alpha)$ and for a in A, $V_\alpha(A, q_\alpha; a) = V(A_\alpha, \|\cdot\|_\alpha; a_\alpha) = V(\pi_\alpha A, \|\cdot\|_\alpha; a_\alpha)$ Hence $V(A, \{q_\alpha\}_\Lambda; a) = \bigcup_\alpha V_\alpha(A, q_\alpha; a) = \bigcup_\alpha V(A_\alpha, \|\cdot\|_\alpha; a_\alpha)$.

1.5 Remark : Let $(A, \{q_\alpha\}_\Lambda)$ be an unital lmc-algebra. Then $D(A, \{q_\alpha\}_\Lambda; 1)$ and $V(A, \{q_\alpha\}_\Lambda; a)$ depend upon the particular family of seminorms $\{q_\alpha\}_\Lambda$ belonging to $Q(A)$.

1.6 Proposition: Let $(A, \{q_\alpha\}_\Lambda)$ be an unital lmc-algebra and let a be an element of A. Then for α, β in Λ, $\alpha \leq \beta$, $V(A_\alpha, \|\cdot\|_\alpha; a_\alpha)$ is contained in $V(A_\beta, \|\cdot\|_\beta; a\ )$.

Proof: Since $D_\alpha(A, q_\alpha; 1) \subseteq D_\beta(A, q_\beta; 1)$ for $\alpha \leq \beta$, by prop.1.2., we have:

$$\begin{aligned} V(A_\alpha, \|\cdot\|_\alpha; a_\alpha) &= V_\alpha(A, q_\alpha; a) \\ &= \{f(a);\ f \in D_\alpha(A, q_\alpha; 1)\} \\ &\subseteq \{f(a);\ f \in D_\beta(A, q_\beta; 1)\} \\ &= V_\beta(A, q_\beta; a) \\ &= V(A_\beta, \|\cdot\|_\beta; a_\beta). \end{aligned}$$

In particular we have:

1.7 Corollary: Let $(A, \{q_n\}_1^\infty)$ be an unital Fréchet algebra (F-algebra) and let a be an element of A. Then for all $n \geqslant 1$, $V(A_n, \|\cdot\|_n; \pi_n a)$ is contained in $V(A_{n+1}, \|\cdot\|_{n+1}; \pi_{n+1} a)$.

1.8 Corollary: Let $(A, \{q_\alpha\}_\Lambda)$ be a complete unital lmc-algebra such that for each x in A and each α in Λ, $q_\alpha(x) < \infty$. Let $q(x) = \sup_\alpha q_\alpha(x)$; x in A. Then $\overline{V}(A, \{q_\alpha\}_\Lambda; a) = V(A, q; a)$.

Proof: Since, $V(A, \{q_\alpha\}_\Lambda; a) = \bigcup_\alpha V(A_\alpha, \|\cdot\|_\alpha; a_\alpha)$, $V(A_\alpha, \|\cdot\|_\alpha; a\ )$ is contained in $V(A_\beta, \|\cdot\|_\beta; a_\beta)$ for $\alpha \leq \beta$, by prop.1.6, and $q(x) = \sup_\alpha q_\alpha(x)$, we have: $\overline{V}(A, \{q_\alpha\}_\Lambda; a) = V(A, q; a)$.

Now we give an example of a Frechet algebra $(A, \{\|\ \|_n\})$ with identity such that $V(A, \{\|\ \|_n\}_N; a) = C$, for a certain a in A.

1.9 Example :

1) Let $T = \{z \in C \ ; \ 1<|z|<2\} \cup \{0\}$ ..

T is a locally compact Hausdorff space and A = (C(T), compact -open topology) is a complete lmc-algebra with identity; (See Michael (36), Ex D.3 and lemma D.5). Also M(A) = T ((36), Ex 7.6). Define a : T into C by

$a(0) = 0, \quad a(t) = ((|t| - 1)/(2 - |t|))t$ for $t \neq 0$.

Now a is in A and $Sp(A,a) = \hat{a}(M(A)) = \hat{a}(T) = C$.

Since V(A,a) contains Sp(A,a), which is the whole complex plane, we have V(A,a) = C.

2) Again we consider C(T) of Example 1 and an increasing sequence $\{\|\cdot\|_n ; \ n = 1,2,\ldots\ldots\}$ of seminorms for A and study numerical ranges $V(A_n,\|\cdot\|_n;a_n)$ and $Sp(A_n,a_n)$.

Let $\{r_n ; \ n = 1,2,3,\ldots\}$ be a decreasing sequence such that r_1 is less than 2 and $\operatorname{Lim} r_n = 1$, as n tends to ∞; $\{R_n ; \ n = 1,2,\ldots\}$ be an increasing sequence such that $1 < R_1 < 2$ and $\operatorname{Lim} R_n = 2$. Let $K_n = \{0\} \cup \{t; \ r_n \leq |t| \leq R_n\}$, for each n. Define $\|\cdot\|_n$ to be the supremum on K_n.

Clearly $a_n(t) = a(t)/K_n$. Then $\{\|\ \|_n ; \ n = 1,2,\ldots\}$ is an increasing sequence of seminorms for A, and for each n in N we have $Sp(A_n, a_n) = \hat{a}_n(M_n(A_n)) = \hat{a}_n(K_n) =$

$$\left\{z \ ; \ \frac{r_n - 1}{2 - r_n} r_n < |z| < \frac{R_n - 1}{2 - r_n} R_n \right\} \cup \{0\},$$

a compact annulus plus the isolated point 0, where $A_n = A/\{x; \|x\|_n = 0\}$. But $V(A_n,\|\cdot\|_n;a_n)$ is contained in $\{z; |z| \leq \|a_n\|_n\}$, which is a compact convex subset of C containing $Sp(A_n, a_n)$.

Hence $V(A_n, \|\cdot\|_n ; a_n) = \{ z ; |z| \leq (R_n - 1/2 - R_n) R_n \}$

and so $V(A, \{\|\cdot\|_n\}_{n \in \Lambda} ; a) = \bigcup_n V(A_n, \|\cdot\|_n ; a_n)$

$$= \bigcup_{R_n} \{ z ; |z| \leq (R_n - 1/2 - R_n) R_n \}$$

$$\lim R_n = 2$$

$$= C .$$

2. Convexity and invariance of the numerical range

Let $(A, \{q_\alpha\}_\Lambda)$ be an unital lmc-algebra and let a be an element of A. We show that $V(A, \{q_\alpha\}_\Lambda ; a)$ is a convex subset of C and $D(A, \{q_\alpha\}_\Lambda ; 1)$ is a weak*-compact convex subset of A'. We show that $V(A, \{q_\alpha\}_\Lambda ; a)$ is invariant in contrast to $Sp(A,a)$ which is not invariant.

2.1 Theorem ; Let $(A, \{q_\alpha\}_\Lambda)$ be an unital lmc-algebra. Then $D(A, \{q_\alpha\}_\Lambda ; 1)$ is a convex subset of A', and $V(A, \{q_\alpha\}_\Lambda ; a)$ is a convex subset of C, where a is in A.

Proof: Let $f_1, f_2 \in D(A, \{q_\alpha\}_\Lambda ; 1)$, then there exist α, β in Λ such that $f_1 \in D_\alpha(A, q_\alpha ; 1)$ and $f_2 \in D_\beta(A, q_\beta ; 1)$.

Hence $f_1, f_2 \in A'$, $f_1(1) = f_2(1) = 1$ and $|f_1(x)| \leq q_\alpha(x)$, $|f_2(x)| \leq q_\beta(x)$ for all x belonging to A.

Since $\{q_\alpha\}_\Lambda$ is directed there exists γ such that $q_\alpha(x) \leq q_\gamma(x)$, $q_\beta(x) \leq q_\gamma(x)$, for all x in A. Let $0 \leq \theta \leq 1$ and let $g = \theta f_1 + (1 - \theta) f_2$.

Clearly g in A', also $g(1) = 1$ and $|g(x)| \leq \theta |f_1(x)| + (1-\theta)|f_2(x)|$

$\leq \theta q_\gamma(x) + (1 - \theta) q_\gamma(x) = q_\gamma(x)$, which implies that g is in $D_\gamma(A, q_\gamma ; 1)$, which is contained in $D(A, \{q_\alpha\}_\Lambda ; 1)$ and hence $D(A, \{q_\alpha\}_\Lambda ; 1)$ is a convex subset of A'.

Now we prove the convexity of $V(A, \{q_\alpha\}_\Lambda ; a)$.

Let z_1, z_2 be elements of $V(A, \{q_\alpha\}_\Lambda ; a)$. Then there exist f_1, f_2 belonging to $D(A, \{q_\alpha\}_\Lambda ; 1)$ such that $f_1(a) = z_1$ and $f_2(a) = z_2$. Let $0 \leq \theta \leq 1$, and put $f = \theta f_1 + (1 - \theta) f_2$. Consider $\theta z_1 + (1 - \theta) z_2 = (\theta f_1 + (1 - \theta) f_2)(a) = f(a)$. By the convexity of $D(A, \{q_\alpha\}_\Lambda ; 1)$, $f = \theta f_1 + (1 - \theta) f_2$ belongs to $D(A, \{q_\alpha\}_\Lambda ; 1)$. Therefore $\theta z_1 + (1 - \theta) z_2 = f(a)$ belongs to $V(A, \{q_\alpha\}_\Lambda ; 1)$. Hence $V(A, \{q_\alpha\}_\Lambda ; a)$ is a convex subset of C.

<u>2.2 Theorem</u>: Let $(A, \{q_\alpha\}_\Lambda)$ be a complete unital lmc-algebra, then $D(A, \{q_\alpha\}_\Lambda ; 1)$ is the projective limit of $D_\alpha(A, q_\alpha ; 1)$.

<u>Proof</u> : By prop. 1.2, $D_\alpha(A, q_\alpha ; 1)$ is contained in $D_\beta(A, q_\beta ; 1)$, α, β in Λ, $\alpha \leq \beta$. Therefore there are

continuous canonical mappings :

i) $i_{\alpha\beta} : f_\beta \longrightarrow f_\alpha$ mapping $D_\beta(A, q_\beta ;1)$ onto $D_\alpha(A, q_\alpha ;1)$,

ii) $i_\alpha : f \longrightarrow f_\alpha$ mapping $D(A,\{q_\alpha\}_\Lambda ;1)$ onto $D_\alpha(A, q_\alpha ;1)$ such that, for α, β, γ in Λ, $\alpha \le \beta \le \gamma$, $i_{\alpha\gamma} = i_{\alpha\beta} \circ i_{\beta\gamma}$, $i_{\alpha\alpha}$ is the identity map on $D_\alpha(A, q_\alpha ; 1)$ and $i_\alpha = i_{\alpha\beta} \circ i_\beta$.

Hence $D(A,\{q_\alpha\}_\Lambda; 1) = \varprojlim D_\alpha(A, q_\alpha ; 1)$.

<u>2.3 Corollary</u> : Let $(A,\{q_\alpha\}_\Lambda)$ be a complete unital lmc-algebra, then $D(A,\{q_\alpha\}_\Lambda ;1)$ is a non-empty convex subset of A' which is weak*-compact.

<u>Proof</u>: The mapping $g_\alpha : f \longrightarrow f_\alpha$ of $D_\alpha(A, q_\alpha ;1)$ onto $D(A_\alpha, \|\cdot\|_\alpha ;1)$ is clearly a homomorphism, and if $f_\alpha = 0$ then $f(x) = f_\alpha(x_\alpha) = 0$, for each x belonging to A. Thus the map g_α of $D_\alpha(A, q_\alpha ;1)$ onto $D(A_\alpha, \|\cdot\|_\alpha; 1_\alpha)$ is an isomorphism and is bicontinuous with respect to the relative weak*-topologies on the respective spaces. But $D(A_\alpha, \|\cdot\|_\alpha ;1_\alpha)$ is $\sigma(A_\alpha, A'_\alpha)$ compact, by prop. 3.6 of Ch.I, hence $D_\alpha(A, q_\alpha ;1)$ is weak*-compact.
Thus we have $(D_\alpha(A, q_\alpha ;1), i_{\alpha\beta})$ as an inverse system of weak*-compact spaces indexed by the directed set Λ , such that $i_{\alpha\alpha}$ is the identity mapping for each α in Λ. Also by prop. 2.2, $D(A,\{q_\alpha\}_\Lambda;1) = \varprojlim D_\alpha(A, q_\alpha ;1_\alpha)$.
By prop. 8, page 89, Bourbaki (16) we have:
$D(A,\{q_\alpha\}_\Lambda ;1)$ is weak*-compact and for each α in Λ we have

$i_\alpha(D(A,\{q_\alpha\}_\Lambda ;1)) = \bigcap_{\beta \geq \alpha} i_{\alpha\beta}(D_\beta(A , q_\beta ;1))$.
Also since $D_\alpha(A, q_\alpha ;1_\alpha)$ are all nonempty $D(A,\{q_\alpha\}_\Lambda ;1)$ is nonempty.

$D(A,\{q_\alpha\}_\Lambda ;1)$ is a convex subset of A' by prop.2.1. Hence $D(A,\{q_\alpha\}_\Lambda ;1)$ is a nonempty convex weak*-compact subset of A'.

2.4 Proposition: We note that if B is a closed subalgebra containing 1 of a complete unital lmc-algebra $(A,\{q_\alpha\}_\Lambda)$, then $(B,\{q_\alpha\}_\Lambda)$ is also a complete unital lmc-algebra and for each α in Λ, $\overline{B}_\alpha = \overline{\Pi_\alpha B} \subset \overline{A}_\alpha$.

2.5 Definition : Let A be a lmc-algebra and let a be an element of A, then the space of multiplicative linear functionals is defined by : $M(A) = \{ f \in A';\ f(xy) = f(x)f(y)\}$. M(A) is also called the maximal ideal space of A

2.6 Proposition: Let $(A,\{q_\alpha\}_\Lambda)$ be a commutative lmc-algebra and a be an element of A. Then ,
$Sp(A,a) = \{f(a);\ f \text{ in } M(A)\}$.

Proof ; First we observe that $M(A) \neq \emptyset$, see Zelazko (53), page 95, 12.6. For $Sp(A,a) = \{f(a);f \in M(A)\}$ see, Zelazko (53), page 95, 12.8.

2.7 Definition : Let A be a complete unital lmc-algebra. For an element a belonging to A, we say that Sp(A,a)

is "invariant" if Sp(A,a) = Sp(B, a) for every closed subalgebra B of A containing a and 1.
If $V(A,\{q_\alpha\}_\Lambda;a) = V(B,\{q_\alpha\}_\Lambda;a)$ for every closed subalgebra B of A containing a and 1, then $V(A,\{q_\alpha\}_\Lambda;a)$ is said to be "invariant".

<u>2.8 Theorem:</u> Let A be a complete unital lmc-algebra and a be an element of A. Then $V(A,\{q_\alpha\}_\Lambda;a)$ is invariant.

<u>Proof</u>: Let B be a closed subalgebra of A containing a and 1. Clearly the restriction mapping $f \longrightarrow f/B$ maps $D(A,\{q_\alpha\}_\Lambda;1)$ onto $D(B,\{q_\alpha\}_\Lambda;1)$. Therefore we have $V(A,\{q_\alpha\}_\Lambda;a) = V(B,\{q_\alpha\}_\Lambda;a)$ by definition.

<u>2.9 Proposition</u> : Let $(A,\{q_\alpha\}_\Lambda)$ be a complete unital lmc-algebra and a be an element of A. Then the closure in A of the algebra $P(a) = \{p(a);\ p \in P\}$, where P is the family of all polynomials in one variable with complex coefficents, is a closed lmc-subalgebra of $(A,\{q_\alpha\}_\Lambda)$.

<u>2.10 Corollary</u> : Let $(A,\{q_\alpha\}_\Lambda)$ be a complete unital lmc-algebra. Then for each a belonging to A, we have $V(A,\{q_\alpha\}_\Lambda;a) = V(P(a),\{q_\alpha\}_\Lambda;a)$.

<u>2.11 Corollary</u>: Let $(A,\|\cdot\|)$ be an unital Banach algebra and a be an element of A. Then $V(A,\|\cdot\|;a) = V(P(a),\|\cdot\|;a)$.

2.12 Remark: Let A be a topological algebra and let a be an element of A. Then Sp(A, a) is in general not invariant.

Example: Let C(T) be the algebra of all continuous complex valued functions on the unit circle T (with pointwise addition and multiplication and the sup norm), and let A be the set of all f belonging to C(T) which can be extended to a continuous function F on the closure of the unit open disc $U = \{(x,y) ; x^2 + y^2 < 1\}$, such that F is holomorphic in U. It is easily seen that A is a subalgebra of C(T). If f_n belongs to A and (f_n) converges uniformly on T, the maximum modulus theorem forces the associated sequence (F_n) to converge uniformly on the closure of U. This shows that A is a closed subalgebra of C(T), and so A is itself a Banach algebra.

Define the function f_o by $f_o(e^{i\theta}) = e^{i\theta}$. Then the corresponding extension is $F_o(z) = z$. The spectrum of f_o, as an element of A, consists of the closed unit disc; with respect to C(T), the spectrum of f_o consists only of the unit circle. However we note that the two spectral radii coincide by 3.6 of Ch.I.

2.13 Definition: If K is a compact subset of C, the polynomially convex hull $\hat{K}$ of K is the set $\{z \in C; |p(z)| \leq \|p\|_K$ for each p belonging to $P\}$, where $\|p\|_K = \sup_{z \in K} |p(z)|$.

2.14 Theorem: Let A be a complete unital lmc-algebra and let a be an element of A. Then Sp(A;a) is invariant, if and only if, there exists a directed family $\{q_\alpha\}_\Lambda$ of seminorms for A such that for each α in Λ, we have $\widehat{Sp}(A_\alpha, \pi_\alpha a) \subseteq Sp(A, a)$.

Proof: See Brooks (17), theorem 3.3.

3. Relation between the spectrum and the numerical range

Various inclusion relations between the spectra and numerical ranges of elements of a complete unital lmc-algebra are discussed.

3.1 Definition: Let $(A, \{q_\alpha\}_\Lambda)$ be a complete unital lmc-algebra and a be an element of A. Then $v(A, \{q_\alpha\}_\Lambda; a) = \text{Sup}\{|z|;\ z \in V(A, \{q_\alpha\}_\Lambda; a)\}$ is called the numerical radius of a. $r(A, a) = \text{Sup}\{|z|;\ z \in Sp(A, a)\}$ is called the spectral radius of a.

3.2 Remark: Let $(A, \{q_\alpha\}_\Lambda)$ be a complete unital lmc-algebra and let a be an element of A. Since, $V(A, \{q_\alpha\}_\Lambda; a) = \bigcup_\alpha V_\alpha(A, q_\alpha; a)$, clearly $v_\alpha(A, q_\alpha; a) \leq v(A, \{q_\alpha\}_\Lambda; a)$ and $v(A, \{q_\alpha\}_\Lambda; a) = \text{Sup}_\alpha\, v_\alpha(A, q_\alpha; a)$. Also $v_\alpha(A, q_\alpha; a) \leq \|a\|_\alpha \leq q_\alpha(a)$.

3.3 Proposition: Let $(A, \{q_\alpha\}_\Lambda)$ be a complete unital lmc-algebra and a be an element of A. Then

$v(A,\{q_\alpha\}_\Lambda;a) \leq \sup_\alpha q_\alpha(a)$.

<u>Proof</u>: $v(A,\{q_\alpha\}_\Lambda;a) = \operatorname{Sup}\{|z|;\ z \in V(A,\{q_\alpha\}_\Lambda;a)\}$

$= \sup_\alpha v_\alpha(A,\ q_\alpha;a) \leq \sup_\alpha \|a_\alpha\|_\alpha \leq \sup_\alpha q_\alpha(a)$.

<u>3.4 Proposition</u> : Let $(A,\{q_\alpha\}_\Lambda)$ be a complete unital lmc-algebra and let a,b be elements of A and $\lambda,\beta \in C$. Let us denote $V(A,\{q_\alpha\}_\Lambda;a)$ by $V(A;a)$ and $v(A,\{q_\alpha\}_\Lambda;a)$ by $v(A;a)$; $D(A,\{q_\alpha\}_\Lambda;1)$ by $D(A;1)$ for convenience. Then we have the following:

1) $V(A;\ a+b)$ is contained in $V(A;a) + V(A;b)$,

2) $V(A;\lambda+\beta a) = \lambda + \beta V(A;a)$ and $v(A;\lambda+\beta a) \leq |\lambda|+|\beta| v(A;a)$,

3) $v(A;a+b) \leq v(A;a) + v(A;b)$,

4) $v(A;\lambda a) = |\lambda| v(A;a)$,

5) $r(A;a+b) \leq r(A;a) + r(A;b)$,

6) $r(A;ab) \leq r(A;a)\ r(A;b)$,

7) $r(A;a^n) = r^n(A;a)$.

<u>Proof</u>: $V(A;\ a+b) = \{f(a+b);\ f \in D(A;\ 1)\}$

$= \{f(a) + f(b);\ f \in D(A;1)\}$

$\subseteq \{f(a); f \in D(A;1)\} + \{g(b); g \in D(A;1)\}$

$= V(A;a) + V(A;b)$.

2) $V(A;\lambda+\beta a) = \{f(\lambda 1+\beta a);\ f \in D(A;1)\}$

$= \{\lambda + \beta f(a);\ f \in D(A;1)\}$

$= \lambda + \beta\{f(a);\ f\ \ D(A;1)\}$

$= \lambda + \beta V(A;a)$,

$v(A;\lambda+\beta a) = \sup\{|\lambda+\beta f(a)|;\ f \in D(A;1)\}$

$\leq |\lambda| + |\beta| \operatorname{Sup}\{|f(a)|;\ f \in D(A;1)\}$

$= |\lambda| + |\beta| v(A;a)$.

3) $v(A;a+b) = \text{Sup}\{|f(a+b)| ; f \in D(A;1)\}$

$\leq \text{Sup}\{|f(a)| ; f \in D(A;1)\} + \text{Sup}\{|f(b)|; f \in D(A;1)\}$

$= v(A;a) + v(A;b)$.

4) $v(A;\lambda a) = \text{Sup}\{|f(\lambda a)|; f \in D(A;1)\}$

$= |\lambda| \text{Sup}\{|f(a)|; f \in D(A;1)\}$

$= |\lambda| v(A;a)$.

5) $r(A;a+b) = \text{Sup}\{|f(a+b)| ; f \in M(A)\}$

$\leq \text{Sup}\{|f(a)| ; f \in M(A)\} + \text{Sup}\{|f(b)| ; f \in M(A)\}$

$= r(A,a) + r(A,b)$.

6) $r(A;ab) = \text{Sup}\{|f(ab)| ; f \in M(A)\}$

$\leq \text{Sup}\{|f(a)| ; f \in M(A)\} \text{Sup}\{|g(b)| ; g \in M(A)\}$

$= r(A;a)\ r(A;b)$.

7) $r(A;a^n) = \text{Sup}\{|f(a^n)| ; f \in M(A)\}$

$= \text{Sup}\{|f(a)|^n ; f \in M(A)\}$

$= r^n(A;a)$.

<u>3.5 Theorem</u> : Let $(A, \{q_\alpha\}_\Lambda)$ be a complete unital lmc-algebra and a be an element of A. Then $Sp(A;a)$ is contained in the numerical range of a.

<u>Proof</u>: (Giles and Husain (24), page4, theorem 1). By 5.3(a) Michael (35), we have $Sp(A;a) = \bigcup_\alpha Sp(A_\alpha;\pi_\alpha a)$. But by prop.3.8 of Ch.I, $Sp(A_\alpha,\pi_\alpha a)$ is contained in $V(\pi_\alpha A, \|\cdot\|_\alpha; a_\alpha)$. Hence $Sp(A,a) = \bigcup_\alpha Sp(A_\alpha,\pi_\alpha a)$ which is contained in $\bigcup_\alpha V(A_\alpha, \|\cdot\|_\alpha; a_\alpha) = V(A, \{q_\alpha\}_\Lambda; a)$.

<u>3.6 Theorem</u>: Let A be a complete unital lmc-algebra and a be an element of A. Then,

the convex hull of $Sp(A,a) \subseteq \bigcap\{V(A,\{q_\alpha\}_\Lambda;a) ; \{q_\alpha\}_\Lambda \in Q(A)\}$

$$\subseteq \overline{Co}\ Sp(A;a).$$

Proof: See Giles and Husain (24), page 12, theorem 5.

3.7 Theorem: Let A be a complete unital lmc-algebra and let a,b be elements of A and $0 \notin V(A,\{q_\alpha\}_\Lambda;a)$, then a^{-1} exists and $Sp(A,a^{-1}b) \subseteq V(A,\{q_\alpha\}_\Lambda;b)/V(A,\{q_\alpha\}_\Lambda;a)$.

Proof: For convenience let us denote $V(A,\{q_\alpha\}_\Lambda;a)$ by $V(A;a)$ and $D(A,\{q_\alpha\}_\Lambda;1)$ by $D(A;1)$.
clearly $a^{-1}b - 1 = a^{-1}(b - a)$. If $\lambda \in Sp(A,a^{-1}b)$, then $0 \in Sp(A; b - \lambda a)$ which is contained $V(A; b-\lambda a)$.

There exists f belonging to $D(A;1)$ such that $f(b-\lambda a) = 0$ ie $f(b) = \lambda f(a)$,. Hence $\lambda = f(b)/f(a)$ belongs to $V(A;b)/V(A;a)$. This being true for each λ in $Sp(A;a^{-1}b)$, we have $Sp(A; a^{-1}b) \subseteq V(A;b) / V(A;a)$.

4. Numerical radius and the spectral radius

Various expressions for the numerical and spectral radius and the sup. of real part of the numerical ranges of elements of a complete unital lmc algebra are obtained.

4.1 (a) Theorem: Let A be a complete unital lmc-algebra and a be an element of A. Then each of the following is equal to $v(A;a)$, where $v(A;a) = v(A,\{q_\alpha\}_\Lambda;a)$.

i) $\displaystyle \sup_{|p|=1} \lim_{q \to 0^+} \left(\sup_{\beta \in \Lambda} \|(a + pqa)_\beta\|_\beta - 1 \right) / q$,

ii) $\displaystyle \sup_{|z|=1} \lim_{p \to 0^+} \left(\log \sup_{\beta \in \Lambda} \|\exp(za)_\beta\|_\beta \right) / p$;

iii) $\displaystyle \sup_{0 \neq p \in C} \left(\log \sup_{\beta \in \Lambda} \|\exp(pa)_\beta\|_\beta \right) / p$.

<u>Proof</u>: i) $\displaystyle v(A;a) = \sup_{\beta \in \Lambda} v(A_\beta, \|\cdot\|_\beta ; a_\beta)$ by 3.2

$\displaystyle = \sup_{\beta \in \Lambda} \sup_{|p|=1} \lim_{q \to 0^+} (\|(1+pqa)_\beta\|_\beta - 1)/q$

(by 3.11(ii), Ch.I)

$\displaystyle = \sup_{|p|=1} \lim_{q \to 0^+} \left(\sup \|(1+pqa)_\beta\|_\beta - 1 \right)/q$.

ii) $\displaystyle v(A;a) = \sup_{\beta \in \Lambda} v(A_\beta, \|\cdot\|_\beta ; a_\beta)$.

$\displaystyle = \sup_{\beta \in \Lambda} \sup_{|z|=1} \lim_{p \to 0^+} \frac{\log \|\exp(pza)_\beta\|_\beta}{p}$ (by 3.11,ChI)

$\displaystyle = \sup_{|z|=1} \lim_{p \to 0^+} \left(\log \sup_{\beta \in \Lambda} \|\exp(za)_\beta\|_\beta\right) / p$.

iii) $\displaystyle v(A;a) = \sup_{\beta \in \Lambda} v(A_\beta, \|\cdot\|_\beta ; a_\beta)$

$\displaystyle = \sup_{\beta \in \Lambda} \sup_{0 \neq z \in C} \left(\log \|\exp(za)_\beta\|_\beta \right) / |z|$ (by 1.1(iv))

$\displaystyle = \sup_{0 \neq z \in C} \left(\log \sup \|\exp(za)_\beta\|_\beta\right) / |z|$.

<u>4.1(b) Theorem</u>: Let $(A, \{q_\alpha\}_\Lambda)$ be an unital lmc-algebra and let a be an element of A. Then each of the following is equal to Sup Re V(A;a):

i) $\displaystyle \sup_{\alpha \in \Lambda} \sup \operatorname{Re} V_\alpha(A, q_\alpha : a)$,

ii) $\displaystyle \inf_{p > 0} \left(\sup_{\beta \in \Lambda} \|(1+pa)_\beta\|_\beta - 1 \right)/p$,

iii) $\displaystyle \lim_{p \to 0^+} \left(\sup_{\beta \in \Lambda} \|(1+pa)_\beta\|_\beta - 1 \right)/p$,

iv) $\sup_{p>0} \left(\log \sup_{\beta} \|\exp(pa)_{\beta}\|_{\beta} \right)/p$,

v) $\lim_{p\to\infty} \left(\log \sup_{\beta} \|\exp(pa)_{\beta}\|_{\beta} \right)/p$,

Proof: Similar to that of the Banach algebra theory of numerical ranges; see prop. 3.10 of ChI.

4.2 Theorem: Let $(A,\{q_{\alpha}\}_{\Lambda})$ be a complete unital lmc-algebra and a be an element of A. Then we have:

$$r(A;a) = \max_{|z|=1} \inf_{p>0} \left(\log \sup_{\beta} \|\exp(zpa)_{\beta}\|_{\beta} \right)/p$$

$$= \max_{|z|=1} \lim_{p\to\infty} \left(\log \sup_{\beta} \|\exp(pza)_{\beta}\|_{\beta} \right)/p .$$

Proof : $r(A;a) = \sup_{\beta} r(\bar{A}_{\beta};a_{\beta})$ (by (36), page22 cor.5.3 (b))

$$= \sup_{\beta} \max_{|z|=1} \left\{ \begin{matrix} \inf_{p>0} \\ \lim_{p\to\infty} \end{matrix} \right\} (\log \|\exp(pza)_{\beta}\|_{\beta})/p$$

$$= \max_{|z|=1} \left\{ \begin{matrix} \inf_{p>0} \\ \lim_{p\to\infty} \end{matrix} \right\} (\log \sup \exp(pza))/p$$

We need the following results:

4.3 Theorem: Let A be a complete unital lmc-algebra, then given $(A,\{q_{\alpha}\}_{\Lambda})$ for each a belonging to A, we have: $v(A;a) \geq e^{-1}(q_{\alpha}(a))$ for all α .

Proof: See Giles and Husain (24), Theorem 2.

4.4 Proposition: Let $(A, \{q_\alpha\}_\Lambda)$ be an unital lmc-algebra and let $B = \{x \in A;\ \sup_\alpha p_\alpha(x) < \infty\}$.
Then (B,p), where $p(x) = \sup_\alpha p_\alpha(x)$ is a normed algebra. If A is complete, then (B,p) is a Banach algebra.

Proof: See, Giles and Husain (24), page8.

4.5 Proposition: Let $(A, \{q_\alpha\}_\Lambda)$ be a complete unital lmc-algebra. Then $B = \{a \in A;\ V(A, \{q_\alpha\}_\Lambda; a) \text{ is bounded}\}$, where B is defined in prop.6.4.

Proof: See, Giles and Husain (24), page 9.

4.6 Remark: Let $(A, \{q_\alpha\}_\Lambda)$ be a complete unital lmc-algebra and a be an element of B. Then we have:
$V(A, \{q_\alpha\}_\Lambda; a) = V(B,p;a)$, by 1.8 and 4.4. Thus many of the theorems on numerical ranges of Banach algebras are true for elements of (B,p) considered as elements of $(A, \{q_\alpha\}_\Lambda)$.
For example we have the following theorem:

4.7 Theorem: Let $(A, \{q_\alpha\}_\Lambda)$ be a complete unital lmc-algebra and a be an element of (B,p).
Let $v(A, \{q_\alpha\}_\Lambda; a^n) \leq f(n) v^n(A, \{q_\alpha\}_\Lambda; a)$, $f(n)$ be a positive real number. Then the best value of $f(n)$ is $n!\,(e/n)^n$.

Proof: By proposition 4.6, $V(A, \{q_\alpha\}_\Lambda; a) = V(B,p;a)$.
Hence $v(A, \{q_\alpha\}_\Lambda; a) = v(B,p;a)$. Now by prop. 3.14,
$v(B,p;a^n) \leq f(n)\, v^n(B,p;a)$ implies $f(n) = n!\,(e/n)^n$ is

the best constant satisfying the above inequality.

Hence if $v(A,\{q_\alpha\}_\Lambda;a^n) \leq f(n)v^n(A,\{q_\alpha\}_\Lambda;a)$, $f(n)$ is a real positive constant. Then the best value of $f(n)$ is $n!(e/n)^n$.

5. Hermitian elements and joint numerical range

A characterization of b*-algebras which is due to Giles and Koehler (25) is stated. Conditions for an element to be hermitian in terms of numerical radius are discussed. In the real locally convex space $H(A,\{p_\alpha\}_\Lambda)$, the set $K(A,\{p_\alpha\}_\Lambda)$ of positive elements is a normal closed cone in which 1 is an interior point. $J(A,\{p_\alpha\}_\Lambda) = \{h + ik;\ h,k \in H(A,\{p_\alpha\})\}$, with the restricted seminorms of A is a complex locally convex space and $h+ik \longrightarrow h-ik$ is a continuous involution on $J(A,\{p_\alpha\})$. Joint spectrum and joint numerical range of elements $x_1, x_2, \ldots x_n$ belonging to A are discussed.

5.1 Definition: An F*-algebra is a Fréchet algebra with a continuous involution, for which there is an ascending sequence $\{\|\cdot\|_n\}_1^\infty$ of seminorms for A each of which has the B*-property $\|a^*a\|_n = \|a\|_n^2$, $(a \in A,\ n \in N)$.

5.2 Definition: If A is a lmc-algebra, then the set H(A) of elements of A having real numerical range is called the hermitian elements of A.

5.3 Proposition : Let A be a complete unital lmc-algebra, given $(A,\{p_\alpha\}_\Lambda)$, the set $H(A,\{p_\alpha\}_\Lambda)$ is closed.

Proof: See Giles and Koehler (25), lemma 3.

5.4 Theorem: Let A be a complete unital lmc-algebra. Given $(A,\{p_\alpha\}_\Lambda)$, the following statements are equivalent:

i) $A = H(A,\{p_\alpha\}_\Lambda) + iH(A,\{p_\alpha\}_\Lambda)$, a direct sum;

ii) There is an involution * on A, such that A is a lmc*-algebra, where $S(A) = H(A,\{p_\alpha\}_\Lambda)$; $S(A) = \{x \in A;\ x = x^*\}$;

iii) There is an involution * on A such that A is *-isomorphic to a *-subalgebra of a product of B*-algebras $(\bar{A}_\alpha, \|\cdot\|_\alpha)$;

iv) There is an involution * on A such that A is a b*-algebra;

v) There is an involution on B such that (B,p) is a dense B*-subalgebra of A.

Proof: See Giles and Koehler (25), Theorem 4.6.

5.5 Remark: In particular we have theorem 5.4 for Frechet algebras.

5.6 Theorem: Let A be a complete unital lmc-algebra and a be an element of A. If 0 belongs to $V(A,\{p_\alpha\}_\Lambda;a)$, then $|z| \leq v(A,\{p_\alpha\}_\Lambda;\ a-z1)$, for all $z \in C$.

Proof: If 0 belongs to $V(A;a)$ then there exists an f belonging to $D(A;1)$ such that $f(a)=0$. Let z belong to C, then $|z|=|f(a-z)| \leq v(A; a-z1)$. Hence the result.

5.7 Theorem: Let $(A,\{p_\alpha\}_\Lambda)$ be a complete unital lmc*algebra such that $D(A,\{p_\alpha\}_\Lambda;1)$ is contained in $Sym(A')=\{f \in A';\ f = \overline{f}\}$. Then A is a b*-algebra.

Proof: Let h be an element of A and $h=h^*$, then $V(A,\{p_\alpha\}_\Lambda;h) = \{f(h);\ f\in D(A,\{p_\alpha\}_\Lambda;1)\} \subset \{f(h);\ f \in Sym(A')\} \subset R$. By 5.4(ii) A is a b*-algebra.

5.8 Theorem: Let $(A,\{q_\alpha\}_\Lambda)$ be a complete unital lmc-algebra and a be an element of A. The following are equivalent:

i) a is hermitian ;

ii) $$\lim_{p\to 0} \frac{\sup_\beta \|1+ipa\|_\beta - 1}{p} = 0 ;$$

iii) $\sup_\beta \|\exp(ipa)\|_\beta = 1$, for all real p.

Proof: i) implies ii) since a is hermitian, by definition, $V(A;a)$ is real. Also $V(A;ia) = iV(A;a)$, shows that $V(A;ia)$ is purely imaginary. Therefore $\sup \mathrm{Re}\, V(A;ia) = \sup \mathrm{Re}\, V(A;-ia) = 0$. Hence by prop. 4.1(a) we have :

$$\begin{matrix}\text{Inf}\\ p>0\\ \text{Lim}\\ p\to 0\end{matrix}\left\{\frac{\text{Sup}_\alpha \|1 \pm ipa\|_\alpha - 1}{p}\right\} = 0 .$$

ii) implies iii) : From 4.1(b), we have:

$$\begin{matrix}\text{Inf}\\ p>0\\ \text{Lim}\\ p\to 0^+\end{matrix}\left\{\frac{\text{Sup}_\alpha \|1 \pm ipa\|_\alpha - 1}{p}\right\} =$$

$$= \begin{matrix}\text{Sup}\\ p>0\\ \text{Lim}\\ p\to 0\end{matrix}\left\{\frac{\log \text{Sup}_\alpha \|\exp(ipa)\|_\alpha}{p}\right\} = 0.$$

Therefore $\text{Sup}_\beta \|\exp(ipa)\|_\beta = 1$, for all p.

iii) implies i) : Proof is simple and follows immediately.

<u>5.9 Definition</u>: Let $(A, \{q_\alpha\}_\Lambda)$ be a complete unital lmc-algebra and a be an element of A. Define

$$\exp(a) = 1 + \sum_{n=1}^{\infty}(1/n!)\, a^n = \sum_{n=0}^{m} a^n/n! + \sum_{m+1}^{\infty} a^n/n!$$

$$= s_m + \sum_{m+1}^{\infty} a^n/n! .$$

<u>5.10 Proposition</u>: Let $(A, \{q_\alpha\}_\Lambda)$ be a complete unital lmc-algebra and a be an element of A. The exponential function exp(a) is well defined in A.

<u>Proof</u>: For each α in Λ, we have:

$\|\exp(a) - s_m\|_\alpha \leq \sum_{m+1}^{\infty} \|a\|_\alpha^n / n! < \epsilon$ for all $m \geq m_0(\alpha)$. Hence for each α, there exists an M_0 such that $\|\exp(a) - s_m\|_\alpha < \epsilon$, for all $m \geqslant M_o$. Therefore exp(a) is well defined in A.

5.11 Proposition: Let $(A, \{q_\alpha\}_\Lambda)$ be a complete unital lmc-algebra and let a,b be elements of A such that ab = ba. Then we have:

i) exp(a+b) = exp(a) exp(b) ;

ii) exp(a) exp(-a) = 1 ;

iii) exp(a) is always invertible in A.

Proof: Straight forward application of propositions 5.9 and 5.10.

5.12 Definition: We say that an element k belonging to $(A, \{p_\alpha\}_\Lambda)$ is positive if $V(A, \{p_\alpha\}_\Lambda; k)$ is contained in the positive reals. We denote by $K(A, \{p_\alpha\}_\Lambda)$ the set of all positive elements of A.

5.13 Definition: A convex cone of vertex 0 is a subset of a vector space E such that C + C is contained in C and for all $p > 0$, pC is contained in C. A subset $K(x_o)$ of a vector space E is called a cone with vertex x_o if $K(x_o)$ contains every point $x_o + p(x-x_o)$, $p > 0$, whenever it contains x. If E is a locally convex space a cone C of E is called a normal cone if there exists a generating family $\{p_\alpha\}_\Lambda$ of seminorms on E such that $p_\alpha(x) \leq p_\alpha(x+y)$, whenever $x, y \in C$ and for each α in Λ.

5.14 Proposition: Let $(A, \{p_\alpha\}_\Lambda)$ be a complete unital lmc-algebra. In the real locally convex space $H(A,\{p_\alpha\}_\Lambda)$, the set $K(A,\{p_\alpha\}_\Lambda)$ is a normal closed cone in which 1 is an interior point.

<u>Proof</u>: Clearly $K(A,\{p_\alpha\}_\Lambda)$ is a closed cone in $H(A,\{p_\alpha\}_\Lambda)$. Let h,k belong to $K(A,\{p_\alpha\}_\Lambda)$. Then we have, $\|h+k\|_\alpha \geq v(A_\alpha,\|\cdot\|_\alpha;(h+k)_\alpha) \geq v(A_\alpha,\|\cdot\|_\alpha;h_\alpha) \geq e^{-1}\|h_\alpha\|_\alpha$ for each α. Therefore $K(A,\{p_\alpha\}_\Lambda)$ is a normal cone of the real locally convex space $H(A,\{p_\alpha\}_\Lambda)$. Let h belong to $H(A,\{p_\alpha\}_\Lambda)$ with $\sup_\alpha\|(1-h)_\alpha\|_\alpha \leq 1$. Then $v(A,\{p_\alpha\}_\Lambda;\ 1-h) = \sup_\alpha v_\alpha(A,\ p_\alpha;1-h) \leq 1$ by prop. 3.2, Also by prop. 3.4(2) we have $V(A,\{p_\alpha\}_\Lambda;1-h) = 1 - V(A,\{p_\alpha\}_\Lambda;h)$. Therefore we have $V(A,\{p_\alpha\}_\Lambda;h)$ is contained in R^+. Hence h belongs to $K(A,\{p_\alpha\}_\Lambda)$. Also we have $v(A,\{p_\alpha\}_\Lambda;\ 1-h) = \sup_\alpha\|(1-h)_\alpha\|_\alpha$, for hermitian elements. Therefore 1 is an interior point of $K(A,\{p_\alpha\}_\Lambda)$ in $H(A,\{p_\alpha\}_\Lambda)$.

<u>5.15 Proposition</u>: Let $(A,\{p_\alpha\}_\Lambda)$ be a complete unital lmc-algebra and let $J(A) = \{h + ik;\ h,k \in H(A,\{p_\alpha\}_\Lambda)\}$. If x belongs to J(A), then the representation is unique.

<u>Proof</u>: We know that $H(A,\{p_\alpha\}_\Lambda)$ is a closed real locally convex space. Let $a = h+ik = h' + ik'$; h,k $H(A,\{p_\alpha\}_\Lambda)$. Then $h-h' = -i(k-k')$. Therefore $V(A;h-h') = -iV(A;k-k')$, which is impossible because $V(A;h-h')$ is real. Hence $h=h'$ and $k=k'$.

<u>5.16 Definition</u>: Let $(A,\{q_\alpha\}_\Lambda)$ be a complete unital

lmc algebra. Then the mapping $* : J(A) \longrightarrow J(A)$ defined by $(h+ik)^* = h-ik$, is a linear involution.

5.17 Proposition: Let $(A,\{p_\alpha\}_\Lambda)$ be a complete unital lmc-algebra. Then $J(A)$ with the restricted seminorms of A is a complex locally convex space and * is a continuous involution on J(A).

Proof: Let h,k be elements of $H(A,\{p_\alpha\}_\Lambda)$.
Then $v(A_\alpha, \|\cdot\|_\alpha; h) \leq v(A_\alpha, \|\cdot\|_\alpha; h+ik) \leq \| h+ik \|_\alpha$, $\|h\|_\alpha \leq e\| h+ik \|_\alpha$ and $\| k \|_\alpha \leq e\| h+ik \|_\alpha$.
Since $\| (h+ik)^* \|_\alpha \leq \| h \|_\alpha + \| k \|_\alpha \leq 2e\| h+ik \|_\alpha$, * is continuous. Also J(A) is a complete locally convex space.

5.18 Theorem: Let $(A,\{p_\alpha\}_\Lambda)$ be a complete unital lmc*-algebra and a be an element of A which is normal. Then $V(A,\{p_\alpha\}_\Lambda;a) = \text{Co}\ Sp(A;a)$.

Proof: By prop.3.5, we have, Sp(A;a) is contained in $V(A,\{p_\alpha\}_\Lambda;a)$. By prop. 2.1, $V(A,\{p_\alpha\}_\Lambda;a)$ is a convex subset of C. Therefore we have $\text{Co}\ Sp(A;a) \subset V(A,\{p_\alpha\}_\Lambda;a)$. Also we have by prop.1.4., $V(A,\{p_\alpha\}_\Lambda;a) = \bigcup_{\alpha\in\Lambda} V(A_\alpha,\|\cdot\|_\alpha;a_\alpha)$. Hence

$$V(A,\{p_\alpha\}_\Lambda;a) = \bigcup V(A_\alpha,\|\cdot\|_\alpha;a_\alpha)$$
$$= \bigcup \text{Co}\ Sp(A_\alpha;a_\alpha)$$ (By prop.3.18 of Ch.I, for the normal element a_α of the Banach algebra $(A_\alpha,\|\cdot\|_\alpha)$, $V(A_\alpha,\|\cdot\|_\alpha;a_\alpha) = \text{Co}\ Sp(A_\alpha, a_\alpha)$.)

Clearly $\cup$ Co $Sp(A_\alpha, a_\alpha)$ is contained in Co $\cup$ $Sp(A_\alpha, a_\alpha)$.

Therefore $V(A, \{p_\alpha\}_\Lambda; a) \subset \text{Co} \bigcup_\alpha Sp(A_\alpha; a_\alpha) = \text{Co } Sp(A;a)$.

Hence $V(A, \{p_\alpha\}_\Lambda; a) = \text{Co } Sp(A;a)$.

5.19 Application :

1) Let $T = [0, 2\pi)$ and let $A = (C(T)$, compact-open topology). Then A is a complete unital lmc*-algebra.

Let $a(t) = \exp(it)$. Then $a \in A$.

$Sp(A;a) = \hat{a}(T) =$ unit circle.

If $\{t_n, n=1,2,\ldots\}$ is an increasing sequence in $[0, 2\pi)$ with limit 2π, then $\{\|\cdot\|_n ; n = 1,2,\ldots\}$ is an increasing sequence of seminorms for A, where $\|\cdot\|_n =$ Sup on $[0, t_n]$. Clearly $a_n(t) = a(t)/[0, t_n)$.

Then $Sp(A_n; a_n) = \{\exp(it) ; 0 \leq t \leq t_n\}$.

Since a_n is normal in the Banach algebra A_n,

$$V(A_n, \|\cdot\|_n; a_n) = \text{Co } Sp(A_n; a_n)$$
$$= \text{Co}\{\exp(it);\ 0 \leq t \leq t_n\}.$$

$$V(A, \{\|\cdot\|_n\}_N; a) = \bigcup_n V(A_n, \|\cdot\|_n; a_n)$$
$$= \bigcup_{\substack{t_n \\ \lim t_n = 2\pi}} \text{Co}\{\exp(it);\ 0 \leq t \leq t_n\}$$
$$= \{z;\ |z| \leq 1\} = \text{Co } Sp(A;a).$$

Remark: We can easily show that, for $p > 0$,

$$V(A, \{\|\cdot\|_n\}_N; e^{ipt}) = \{z; |z| \leq 1\}.$$

2) Let T denote the open unit disc and let $A = (C(T)$, compact-open topology).

Then A is a complete unital lmc*-algebra. Let $\{r_n ; n \in N\}$ be an increasing sequence of positive numbers with limit 1. Let $K_n = \{t; |t| \leq r_n\}$, $\Gamma_n = \{t; |t| = r_n\}$, $F_n = K_n \cup \Gamma_{n+1}$; $\|\cdot\|_n$ the sup norm on K_n and $\|\cdot\|'_n$ the sup norm on F_n for each $n \in N$. Let $a(t) = t$, for $t \in T$. We use primes to denote objects related to the sequence $\{\|\cdot\|'_n ; n \geq 1\}$. For each $n \geq 1$, we have

$a_n(t) = a(t)/K_n$. $Sp(A_n; a_n) = \hat{a}_n(K_n) = K_n$.

Since a_n is normal in the Banach algebra A_n,

$$V(A_n, \|\cdot\|_n ; a_n) = \text{Co } Sp(A_n ; a_n) = K_n,$$

$$\begin{aligned} V(A, \{\|\cdot\|_n\}; a) &= \bigcup_n V(A_n, \|\cdot\|_n ; a_n) \\ &= \bigcup_{\substack{r_n \\ \text{Lim } r_n = 1}} \{t; |t| \leq r_n\} \\ &= \{z; |z| \leq 1\} = \text{CoSp}(A; a). \end{aligned}$$

Similarly we can show that,

$$V(A, \{\|\cdot\|'_n\}; a) = \bigcup_{\substack{n \geq 1 \\ \text{Lim } r_n = 1}} F_n = \{z; |z| \leq 1\}.$$

Now we consider joint numerical range and joint spectrum of a set of elements of an unital lmc-algebra. It is an easy extension of the concept of numerical range of an element.

5.19 Definition: Let $(A, \{p_\alpha\}_\Lambda)$ be a complete commutative unital lmc-algebra and let $x_1, x_2, \ldots x_n$ be elements of A. Then the joint spectrum is defined as
$Sp(A; x_1, x_2, \ldots, x_n) = \{ (f(x_1), f(x_2), \ldots f(x_n)) \in C^n; f \in M(A)\}$.
The joint numerical range is defined as follows:
$V(A; x_1, x_2, \ldots, x_n) = \{ (f(x_1), f(x_2), \ldots, f(x_n)) \in C^n; f \in D(A, \{p_\alpha\}; 1)\}$.

5.20 Proposition: Let $(A, \{p_\alpha\}_\Lambda)$ be a complete commutative unital lmc-algebra and let $x_1, x_2, \ldots x_n$ be elements of A. Then we have the following:

i) $Sp(A; x_1, x_2, \ldots x_n) = \bigcup_\alpha Sp(A_\alpha; \pi_\alpha x_1, \ldots \pi_\alpha x_n)$;

ii) $V(A, \{p_\alpha\}_\Lambda; x_1, \ldots x_n) = \bigcup_\alpha V(A_\alpha, \|\cdot\|_\alpha; \pi_\alpha x_1, \ldots \pi_\alpha x_n)$;

iii) $V(A, \{p_\alpha\}_\Lambda; x_1, \ldots x_n)$ is a non-empty convex subset of C^n;

iv) $Sp(A, x_1, \ldots, x_n) \subset V(A, \{p_\alpha\}_\Lambda; x_1, \ldots x_n)$;

v) $Co\ Sp(A, x_1, \ldots x_n) = \bigcap_{\{p_\alpha\} \in P(A)} \{V(A, \{p_\alpha\}_\Lambda; x_1, \ldots x_n)\}$.

Proof: Follows from 5.3(a) of (36); 1.4 ; 2.1; 3.5 ; 3.6 and 5.19.

Note: All the theorems on numerical range extend to joint numerical range.

6. Growth conditions and the numerical range

For an element x of a complete unital lmc-algebra the relation between the numerical range $V(A, \{p_\alpha\}_\Lambda; x)$ and various growth conditions on the resolvent $(x-\lambda)^{-1}$ is discussed. The corresponding results for unital Banach algebra and Hilbert space operators are stated as corollaries, with references for separate proofs in some cases. Some examples of differentials and integral operators whose numerical range lie inside some sectors and inside $\{z \in C; |z| > 1\}$ are given.

6.1 Proposition: Let $(A, \{p_\alpha\}_\Lambda)$ be an unital lmc-algebra and x be an element of A. If $\overline{Co}\, V(A, \{p_\alpha\}_\Lambda; x)$ is not the whole complex plane, it is contained in a half-plane say H, because it is a convex subset of C. In this case we can prove that $p_\alpha(x-\lambda)^{-1} \leq d^{-1}(\lambda, H)$ for all λ not belonging to H and for all p_α. Conversely if $p_\alpha(x-\lambda)^{-1} \leq d^{-1}(\lambda, H)$ for all $\lambda \notin H$ and for all p_α; then we prove that $\overline{Co}\, V(A, \{p_\alpha\}_\Lambda; x)$ is contained in a half-plane.

Before proving the above proposition, we prove first the case in which $\overline{Co}\, V(A, \{p_\alpha\}_\Lambda; x)$ is contained in the right half-plane: $H_0 = \{z;\ Re\ z \geq 0\}$.

6.2 Theorem: Let $(A, \{p_\alpha\}_\Lambda)$ be an unital lmc-algebra and x be an element of A. The following statements are equivalent:

i) $\overline{Co}\ V(A,\{p_\alpha\}_\Lambda;x) \subset H_0 = \{z;\ \text{Re}\ z \geq 0\}$;

ii) $p_\alpha(x-\lambda)^{-1} \leq d^{-1}(\lambda, H)$ for all $\lambda \notin H$ and for all p_α .

Proof: i) implies ii) :

Let $\overline{Co}\ V(A,\{p_\alpha\}_\Lambda;x)$ be contained in H_0, and let $\lambda \notin H$. . Since $Sp(A;x)$ is contained in $V(A,\{p_\alpha\}_\Lambda;x)$, we have $\lambda \notin Sp(A;x)$ and therefore $(x-\lambda 1)^{-1}$ exists, by the definition of spectrum.

First we show that for any p_α, $p_\alpha[(x-\lambda)y] \geq d(\lambda,H_0)\ p_\alpha(y)$ for all y belonging to A.

If $p_\alpha(y) = 0$, the above is clearly true.

If $p_\alpha(y) \neq 0$, without loss of generality we may take $p_\alpha(y) = 1$.

Choose $g \in D(A,\{p_\alpha\}_\Lambda;1)$ such that $g(y) = 1$.

Let $f(u) = g(uy)$ for $u \in A$.

Then f belongs to $D(A,\{p_\alpha\}_\Lambda;1)$, because

$|f(u)| = |g(uy)| \leq p_\alpha(uy) \leq p_\alpha(u)\ p_\alpha(y) = p_\alpha(u)$, and

$f(1) = g(y) = 1$.

Also $d(\lambda, H_0)\ p_\alpha(y) \leq |\lambda - f(x)| = |f(x-\lambda)|$

$$= |g((x-\lambda)y)|$$

$$\leq p_\alpha((x-\lambda)y).$$

This implies that $p_\alpha(x-\lambda)^{-1} \leq d^{-1}(\lambda,H_0)$, by letting $y = (x-\lambda 1)^{-1}$, because $p_\alpha(1) = 1$.

ii) implies i) ; Let $p_\alpha(x-\lambda)^{-1}$ be less than or equal to $d^{-1}(\lambda, H_o)$ for all λ not belonging to H_o.

Let $\lambda < 0$, take $\lambda = -1/t$ $(t > 0)$.

Then for any p_α, $p_\alpha[(x + 1/t)^{-1}] \leq t$ or

$p_\alpha[(x + t^{-1})^{-1}] \leq t$.

Now $p_\alpha[(1 + tx)^{-1}] = t^{-1} p_\alpha[(t^{-1} + x)^{-1}] \leq t^{-1} t = 1$.

Consider $f((1 + tx)^{-1})$ with $f \in D(A, \{p_\alpha\}_\Lambda ; 1)$.

We have $\text{Re } f((1 + tx)^{-1}) \leq p_\alpha((1 + tx)^{-1}) \leq 1 = f(1)$.

Hence $\text{Re } f(1 - (1 + tx)^{-1}) \geq 0$,

ie $\text{Re } f(tx(1 + tx)^{-1}) \geq 0$.

Therefore $\text{Re } f(x/ 1 + tx) \geq 0$, $t > 0$.

By letting $t \longrightarrow 0$, we get $\text{Re } f(x) \geq 0$.

Thus we have proved that $\text{Re } f(x) \geq 0$, for $f \in D(A, \{p_\alpha\}_\Lambda ; 1)$.

Hence $V(A, \{p_\alpha\}_\Lambda ; x)$ is contained in H_o and so

$\overline{Co}\, V(A, \{p_\alpha\}_\Lambda ; x)$ is contained in H_o.

<u>6.3 Theorem</u>: Let H be a closed half-plane. Let $(A, \{p_\alpha\}_\Lambda)$ be a complete unital lmc-algebra and x be an element of A. Then the following are equivalent:

i) H contains $\overline{Co}\, V(A, \{p_\alpha\}_\Lambda ; x)$;

ii) $p_\alpha((x - \lambda)^{-1}) \leq d^{-1}(\lambda, H)$ for all $\lambda \notin H$ and for all p_α.

Proof: We can find $\alpha, \beta \in C$, $|\alpha| = 1$ such that $\alpha H + \beta = H_0$.

Now $z \longrightarrow \alpha z + \beta$ maps the complement of H into the complement of H_0.

We have $\overline{Co}\, V(A, \{p_\alpha\}_\Lambda; x)$ is contained in H, if and only if

$$\overline{Co}\, V(A, \{p_\alpha\}_\Lambda; \lambda x + \beta) = \lambda \overline{Co}\, V(A, \{p_\alpha\}_\Lambda; x) + \beta$$
$$\subset \lambda H + \beta = H_0 .$$

Hence $z \notin H$ if and only if $\lambda z + \beta \notin H_0$.

We have shown in the previous theorem $\overline{Co}V(A, \{p_\alpha\}_\Lambda; x) \subset H_0$ if and only if $p_\alpha((x - \lambda)y) \geq d(\lambda, H_0)\, p_\alpha(y)$ for each p_α, for all y belonging to A and for all λ not in H.

Hence $\overline{Co}\, V(A, \{p_\alpha\}_\Lambda; \lambda x + \beta) \subset H_0$, if only if

$d(\alpha z + \beta, H_0)\, p_\alpha(y) \leq p_\alpha((\alpha x + \beta) - (\alpha z + \beta)y)$,

for all $\alpha z + \beta$ not in H_0 and for all p_α and y.

ie. $d(\alpha z + \beta, \alpha H + \beta)\, p_\alpha(y) \leq |\alpha| p_\alpha((x - z)y)$

for all y belonging to A, $\alpha z + \beta$ not in H and for all p_α;

ie. $d(\alpha z, \alpha H)\, p_\alpha(y) \leq |\alpha| p_\alpha((x - z)y)$, for all y in A and for all p_α;

ie, $|\alpha|\, d(z, H)\, p_\alpha(y) \leq |\alpha|\, p_\alpha((x - z)y)$, for all y in A, for all p_α and for all z not in H.

Hence $p_\alpha((x - z)^{-1}) \leq d^{-1}(z, H)$.

Therefore $\overline{Co}V(A, \{p_\alpha\}_\Lambda; x) \subset H$ if and only if

$p_\alpha((x - z)^{-1}) \leq d^{-1}(z, H)$, for all z not in H and for all p_α.

<u>6.4 Corollary</u>: Let $(A, \{p_\alpha\}_\Lambda)$ be a complete unital lmc-algebra and x be an element of A. Let S be a closed convex set in the complex plane. Then,

S contains $\overline{Co}V(A, \{p_\alpha\}_\Lambda; x)$ if and only if $p_\alpha((x-\lambda 1)^{-1}) \leq d^{-1}(\lambda, S)$ for all λ not in S and for all p_α.

<u>Proof</u>: Since any closed convex set in the complex plane is the intersection of closed half-planes,

$p_\alpha((x-\lambda)^{-1}) \leq d^{-1}(\lambda, H)$, for all λ not in H and for all p_α and for every closed half-plane H, such that

$H \supseteq S \supseteq \overline{Co}\, V(A, \{p_\alpha\}_\Lambda; x)$.

Therefore $S \supseteq \overline{Co}\, V(A, \{p_\alpha\}_\Lambda; x)$ if and only if

$p_\alpha((x-\lambda)^{-1}) \leq d^{-1}(\lambda, S)$, for all λ not in S and for all p_α.

<u>6.5 Corollary</u>: Let $(A, \{p_\alpha\}_\Lambda)$ be a complete unital lmc-algebra and x be an element of A. Then,

$\overline{Co}\, V(A, \{p_\alpha\}_\Lambda; x) = Sp(A; x)$ if and only if

$p_\alpha((x-\lambda)^{-1}) \leq d^{-1}(\lambda, Sp(A; x))$ for all λ not in $Sp(A; x)$ and for all p_α.

<u>6.6 Corollary</u>: Let $(A, \{p_\alpha\}_\Lambda)$ be a complete unital lmc-algebra and x be an element of A. Then,

$p_\alpha((x-\lambda)^{-1}) \leq d^{-1}(\lambda, \overline{Co}\, V(A, \{p_\alpha\}_\Lambda; x))$, for all λ not in $\overline{Co}\, V(A, \{p_\alpha\}_\Lambda; x)$ and for all p_α.

6.7 Corollary: Let $(A, \|\cdot\|)$ be an unital Banach algebra and let H be any closed hlaf plane. Let x be an element of A. Then the following statements are equivalent:

i) H contains $V(A, \|\cdot\|; x)$;

ii) $\|(x - \lambda)^{-1}\| \leq d^{-1}(\lambda, H)$, for all $\lambda \notin H$.

Proof: This is a particular case of 6.3.

6.8 Corollary: Let $(A, \|\cdot\|)$ be an unital Banach algebra and x be an element of A. If S is any closed convex set in the complex plane, then, S contains $V(A, \|\cdot\|; x)$ if and only if $\|(x - \lambda)^{-1}\| \leq d^{-1}(\lambda, S)$, $\forall\, \lambda$ not in S.

Proof: Follows from 6.4.
For a separate proof, see Stampfli and Williams (46).

6.9 Corollary: Let $(A, \|\cdot\|)$ be an unital Banach algebra and x be an element of A. Then,

$$\|(x - \lambda)^{-1}\| \leq d^{-1}(\lambda, V(A, \|\cdot\|; x)), \quad \forall\, \lambda \text{ not in } V(A, \|\cdot\|; x).$$

Note: $V(A, \|\cdot\|; x)$ is a compact convex set.

6.10 Corollary: Let $(A, \|\cdot\|)$ be an unital Banach algebra and a be an hermitian element of A. Then,

$$\|(a - \lambda)^{-1}\| \leq |\operatorname{Im}\lambda|^{-1}, \text{ for all } \lambda \text{ such that } \operatorname{Im}\lambda \neq 0.$$

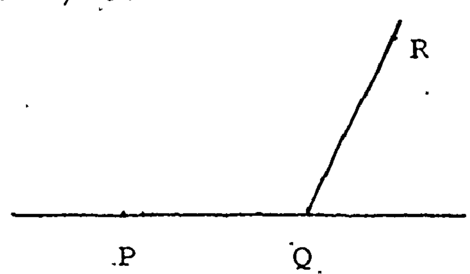
Proof: Since a is hermitian $V(A, \|\cdot\|; a) \subset R$.

Also $V(A,\|\cdot\|;a)$ is a convex compact subset of C. Therefore it is an interval of the Real Line R. Let it be PQ. Let R be any point in C represented by $\lambda = x + iy$ ($y \neq 0$).

Now by the previous theorem ,

$\|(a-\lambda)^{-1}\| \leq 1/d(\lambda, PQ) \leq 1/|RQ| \leq 1/|y|$.

Hence $\|(a-\lambda)^{-1}\| \leq |\mathrm{Im}\,\lambda|^{-1}$ for all $\lambda \in C$ such that $\mathrm{Im}\,\lambda \neq 0$.



<u>6.11 Corollary</u>: Let K be a closed half-plane. Let T be a bounded linear operator acting on a Hilbert space H. Then the following statements are equivalent:

i) K contains $W(T,(,)_0)$;

ii) $\|(T-\lambda)^{-1}\| \leq d^{-1}(\lambda, K)$, for all λ not in K.

Berberian and Orland have proved 6.15.

<u>6.12 Corollary</u>: Let T be a bounded linear operator acting on a Hilbert space H. Then $\overline{W}(T,(,)_0)$ is the intersection of closed half-planes containing $W(T,(,)_0)$.

<u>6.13 Corollary</u>: Let T be a bounded linear operator acting on a Hilbert space H. Aclosed set S contains $W(T,(,)_0)$, if and only if $\|(T-\lambda)^{-1}\| \leq d^{-1}(\lambda, S)$, $\forall \lambda \notin S$.

6.14 Corollary: Let T be a bounded linear operator acting on a Hilbert space H. Then $Sp(T) = \overline{W}(T,(,)_o)$, if and only if $\|(T-\lambda)^{-1}\| \leq d^{-1}(\lambda, Sp(T))$ for all λ not belonging to $Sp(T)$.

Proof: For a separate proof see (55), Theorem 2.

6.15 Theorem: Let K be a closed half-plane and let T be a linear operator acting on T. Then, $W(T,(,)_o)$ is contained in K, if and only if,

$\|(T-\lambda I)^*(T-\lambda I)\| \geq (\mathrm{dis}(\lambda,H))^2$, $\forall$ λ belonging to C.

Proof: See Berberian and Orland (5), Theorem 1.

6.16 Corollary: Let T be a bounded linear operator acting on a Hilbert space H. Then $r(T) = \|T\|$, if and only if $w(T,(,)_o) = \|T\|$. If λ belongs to $W(T,(,)_o)$ and $|\lambda| = \|T\|$, then $\lambda \in Sp(T)$.

Proof: See, Orland (55), theorem 4.

6.17 Theorem: (Lumer) Let $(A,\|\cdot\|)$ be an unital Banach algebra and x be an element of A. Then,

$$\mathrm{Sup}\ \mathrm{Re}\ V(A,\|\cdot\|; x) = \lim_{t\to\infty} \|x + t\| - t .$$

Proof: See, Lumer (33), Lemma2.

<u>6.18;Theorem</u> : Let $(A,\{p_\alpha\}_\Lambda)$ be a complete unital lmc-algebra and x be an element of A. Then,

$$\operatorname{Sup} \operatorname{Re} V(A,\{p_\alpha\}_\Lambda;x) = \operatorname*{Sup}_{\alpha} \lim_{t\to\infty} p_\alpha(x+t) - t .$$

<u>Proof</u>: $\operatorname{Sup} \operatorname{Re} V(A,\{p_\alpha\}_\Lambda;x) = \operatorname*{Sup}_{\alpha} \operatorname{Sup} \operatorname{Re} V(A_\alpha, p_\alpha;x)$

$$= \operatorname*{Sup}_{\alpha} \lim_{t\to\infty} p_\alpha(x+t) - t \quad (\text{By } 6.17).$$

<u>6.19 Theorem</u> : Let $(A,\{p_\alpha\}_\Lambda)$ be a complete unital lmc-algebra and x be an element of A. Let β belong to C. Then β belongs to $V(A,\{p_\alpha\}_\Lambda;x)$, if and only if $|\beta-\lambda| \leq \operatorname*{Sup}_{\alpha} p_\alpha(x-\lambda)$, for all λ belonging to C

Hence $V(A,\{p_\alpha\}_\Lambda;x) = \bigcap_{\lambda} \{z;\ |z-\lambda| \leq \operatorname*{Sup}_{\alpha} p_\alpha(x-\lambda)\}$.

<u>Proof</u>: Forward implication:

If $\beta = f(x)$ belongs to $V(A,\{p_\alpha\}_\Lambda;x)$, for $f \in D(A,\{p_\alpha\}_\Lambda;1)$, then $|\beta-\lambda| = |f(x-\lambda)| \leq \operatorname*{Sup}_{\alpha} p_\alpha(x-\lambda)\ \ \forall\ \lambda \in C$.

Reverse implication: We show that if $\beta \in C$ and whenever $|\beta-\lambda| > \operatorname*{Sup}_{\alpha} p_\alpha(x-\lambda)$, for all $\lambda \in C$, then $\beta \notin V(A,\{p_\alpha\}_\Lambda;x)$.

Since $V(A,\{p_\alpha\};x)$ is convex, we may assume that $V(A,\{p_\alpha\};x)$ is contained in the half-plane $\{z;\ \operatorname{Re} z \geq 0\}$.

If $\beta > 0$, then $\operatorname*{Sup}_{\alpha} (p_\alpha(x+t) - t) < \beta$, for large positive t, which implies that $\operatorname*{Sup}_{\alpha} p_\alpha(x+t) < |\beta - t|$, while $\beta \notin V(A,\{p_\alpha\};x)$.

Hence $|\beta-\lambda| \leq \operatorname*{Sup}_{\alpha} p_\alpha(x-\lambda)$, for all $\lambda \in C$.

<u>6.20 Corollary</u>: Let $(A,\{p_\alpha\}_\Lambda)$ be a complete unital lmc-algebra and x be an element of A. Then,

$$v(A, \{p_\alpha\}_\Lambda ; x) = \underset{\theta}{\text{Max}} \lim_{t \to \infty} (\sup_\alpha p_\alpha(x + te^{i\theta}) - t).$$

<u>Proof</u>: Follows from 6.19 and 6.18.

Now we give some examples of differential and integral operators whose numerical range lie inside some sectors and inside $\{ z \in C;\ |z| > 1 \}$.

<u>6.21 Examples</u> :

i) Consider the differential operator :

$L : L^2[a, b] \longrightarrow L^2[a, b]$, defined by

$$L(u) = p_0(x)\, u'' + p_1(x)\, u' + p_2(x)\, u ,$$

on a finite interval $[a, b]$, where $p_1(x)$, $p_2(x)$ are real valued differentiable functions and $p_0(x) < 0$.

Let T_1 be defined as L with the boundary condition $u(a) = u(b) = 0$.

Then $W(T_1, (,)_o)$ is contained in a sector.

<u>Proof</u>: $(T_1u, u) = \int (p_0u'' + p_1u' + p_2u)\, \bar{u}\, dx$ $(u \in D(T_1))$

$$= - \int_a^b p_0|u'|^2\, dx + \int_a^b ((p_1 - p_0') u' + p_2 u)\, \bar{u}\, dx .$$

Since $-\sup p_0(x) \geq m_0 > 0$, $\sup_{x \in [a,b]} |p_1(x) - p_0'(x)| \leq M_1$, $\sup|p_2(x)| \leq M_2$, where the sup is taken over $[a, b]$, for some positive constants m_0, M_1 , M_2 .

We have: $\text{Re}\, (T_1u, u) \geq m_0 \int |u'|^2\, dx - M_1 \int |u'|\, |u|\, dx - M_2 \int |u|^2\, dx$.

$|\mathrm{Im}\ (T_1u,\ u)| = \mathrm{Im}\ (p_1'-p_0')u'\bar{u}\ dx \quad M_1\ u'\ u\ dx$.

Hence for any given $k > o$,

$\mathrm{Re}\ (T_1u,u) - k\ |\mathrm{Im}\ (T_1u,u)|$

$$\geq (m_0 - \varepsilon(1+k)\ M_1)\int|u'|^2\,dx - ((1+k)M_1/4\varepsilon + M_2)\int|u|^2dx,$$

where $\varepsilon > 0$ is arbitrary. If ε is chosen in such a way that $m_0 - \varepsilon(1+k)M_1 \geq 0$, we have:

$\mathrm{Re}\ (T_1u,u) - k|\mathrm{Im}(T_1u,u)| \geq \gamma(u,u)$, for some negative number γ.

In other words $|\mathrm{Im}(T_1u,u)| \leq k^{-1}\ \mathrm{Re}\ ((T_1-\gamma)u,u)$.

This means that $W(T_1,(,))$ is contained in a sector with vertex γ and semi-angle $\theta = \arctan(k^{-1})$.

<u>Example 2</u> ; Let E be a compact topological space and let $H = L^2(E)$. Let $(Au,v) = \int_E f(x)\ u(x)\ \overline{v}(x)dx$, where $f(x)$ is a complex measurable function on E .

Domain of $A = (\ u \in H; \int|f(x)|\ |u(x)|^2\ dx < \infty)$.

A is densely defined. A is hermitian, if $f(x)$ is real valued. $W(A,\ (,)_o)$ lie inside a sector with vertex γ and semi-angle θ , if the values of $f(x)$ lie in that sector.

<u>Example 3</u> : Let $T : L^2(-\infty,\infty) \longrightarrow L^2(-\infty,\infty)$ be defined by:

$$(Tu,u) = \int_{-\infty}^{\infty}(p|u'|^2 + q|u|^2)\ dx;\quad p(x) > 0,\ q(x) \geq 1.$$

Then $(Tu,\ u) > \int_{-\infty}^{\infty} q|u|^2\ dx > \int_{-\infty}^{\infty}|u|^2\ dx = \|u\|$.

Therefore $W(T,(,)_o) \subset \{z \in C;\ |z| > 1\}$.

CHAPTER III

Numerical range theory for pseudo-Banach algebras

1. Pseudo-Banach algebras

In this section we collect the definition of pseudo-Banach algebra and some relevant results due to Allan, Dales and Mcclure (3), which are needed for our use in subsequent sections. In (3), a pseudo-Banach algebra $(A, \mathcal{B})$ is defined as a commutative algebra A with identity 1 together with a bound structure $\mathcal{B}$ which satisfies some conditions. They have proved that a pseudo-Banach algebra A is algebraically an inductive limit of Banach algebras $(A_\alpha, \|\cdot\|_\alpha)$. We observe that a linear functional f on a pseudo-Banach algebra $(A, \mathcal{B})$ with the inductive limit topology is continuous, if and only if for each α, $f \circ i_\alpha = f_\alpha$ is a continuous linear functional on the Banach algebra $(A_\alpha, \|\cdot\|_\alpha)$, where $i_\alpha : A_\alpha \longrightarrow A$ is the injection map. We also cite some examples of pseudo-Banach algebras from (3).

1.1 Definition: (Allan, Dales and Mcclure (3))
Let A be a commutative algebra with identity 1. A bound structure for A is a non-empty collection $\mathcal{B}$ of subsets of A,

such that,

i) B_α is absolutely convex, $B_\alpha^2 \subset B$ and $1 \in B_\alpha$, for each B_α belonging to $\mathcal{B}$,

ii) Given $B_1, B_2 \in \mathcal{B}$, there exists B_3 in $\mathcal{B}$ and $\lambda > 0$ such that $B_1 \cup B_2 \subset \lambda B_3$.

. ($A, \mathcal{B}$) is then a <u>Bound algebra</u> .

For B_α in $\mathcal{B}$, let $A(B_\alpha) = \{\lambda b : \lambda \in C,\ b \in B_\alpha\}$. In view of i), $A(B_\alpha)$ is the subalgebra of A generated by B_α. The Minkowski functional of B_α defines a submultiplicative seminorm $\|\cdot\|_{B_\alpha}$ on $A(B_\alpha)$. If each $\|\cdot\|_{B_\alpha}$ is a norm, and if $A(B_\alpha)$ is a Banach algebra with respect to $\|\cdot\|_{B_\alpha}$, then ($A; \mathcal{B}$) is complete in the inductive limit topology.

From ii) $A_0 = \cup \{A(B_\alpha);\ B_\alpha \in \mathcal{B}\}$ is a subalgebra of A. If A is complete and if $A = A_0$, then A is called a <u>pseudo-Banach algebra</u> .

<u>1.2 Proposition</u>: (Allan etc. (3)) An algebra A is pseudo-Banach with respect to some bound structure, if and only if A is isomorphic with the inductive limit of an inductive system ($A_\alpha;\ i_{\beta\alpha}: \alpha, \beta \in \Lambda,\ \alpha \leq \beta$), of Banach algebras with identity and continuous unital monomorphisms.

Proof: Let $(A, \mathcal{B})$ be a pseudo-Banach algebra and let the bound structure be indexed by the set Λ. The set Λ is directed upwards by the relation $\leq$ defined by $\alpha \leq \beta$ if and only if $B_\alpha \subset \lambda B_\beta$ for some $\lambda > 0$. Write A_α for $A(B_\alpha)$ and $\|\cdot\|_\alpha$ for the norm on A_α. For $\alpha \leq \beta$, $A_\alpha \subset A_\beta$ and the inclusion map $i_{\beta\alpha}$ is a continuous monomorphism. It is clear that $(A_\alpha; i_{\beta\alpha})$ is the required inductive system.

Conversely, if $(A_\alpha, i_{\beta\alpha})$ is such an inductive system, the unit balls of the algebras A_α (when identified with subalgebras of A) can be taken for the members of a bound structure with respect to which A is pseudo-Banach.

Remark: Observe that apriori a pseudo-Banach algebra does not carry the inductive limit topology. It carries a topology which is in general coarser than the inductive limit topology.

1.3 Proposition: Let $(A, \mathcal{B})$ be a pseudo-Banach algebra with identity 1. Let i_α be the inclusion map of A_α into A. For $\alpha \leq \beta$, let $i_{\alpha\beta}$ be the inclusion map of A_α into A_β. If f is a continuous linear functional on A in the inductive limit topology, then $f \cdot i_\alpha = f_\alpha$ is a continuous linear functional on the Banach-algebra $(A_\alpha, \|\cdot\|_\alpha)$.

1.4 Proposition : Let $(A, \mathcal{B})$ be a pseudo-Banach algebra with the inductive limit topology. Then the linear functional f on $(A, \mathfrak{B})$ is continuous, if and only if for each α, $f.i_\alpha = f_\alpha$ is a continuous linear functional on the Banach algebra $(A_\alpha, \|\cdot\|_\alpha)$.

<u>1.5 Examples</u>: (Allan, Dales and Mcclure (3))

1) <u>Banach algebras</u>: Of course every commutative Banach algebra with identity is a pseudo-Banach algebra with respect to the bound structure consisting of the unit ball of the algebra.

2) <u>Locally convex algebras</u>: Let (A, τ) be a Hausdorff locally convex algebra. Denote by $\mathfrak{B}$ the collection of all subsets of A such that,

i) B is absolutely convex, $B^2 \subset B$, $1 \in B$,

ii) B is closed and bounded.

If (A, τ) is complete (or sequentially complete or quasi-complete), then $(A; \mathfrak{B})$ is a complete bound algebra. The algebra $(A, \mathfrak{B})$ is pseudo-Banach if and only if every element is bounded, ie , for each element a of A, there is a non-zero complex number λ for which the set $\{(\lambda a)^n ; n \geq 1\}$ is a bounded subset of A. In this case we call it a <u>bounded pseudo-complete algebra</u> .

3) <u>Complete unital lmc-algeb...</u>: Let A be a complete

unital lmc-algebra. Then there exists a bound structure with respect to which A is a pseudo-Banach algebra, if and only if the character space χ_A is compact. If M(A) is compact, then χ_A is compact, and if A is also Fréchet, then $\chi_A = M(A)$.

4) p-normed algebras: A Banach algebra in which the norm $\|\cdot\|$ satisfies the condition $\|\lambda x\| = |\lambda|^p \|x\|$, λ a scalar, $x \in X$, $0 < p \leq 1$, instead of $\|\lambda x\| = |\lambda| \|x\|$ is called a complete p-normed algebra or a p-Banach algebra. An example of a p-Banach algebra is $L^p(z)$ with convolution as multiplication with the p-norm $\|x\| = \sum_{n=1}^{\infty} |x_n|^p$ $(0 < p \leq 1)$.

Let A be a p-Banach algebra $(0 < p \leq 1)$.

Suppose that $a_1, a_2, \ldots a_n$ are elements of A with $0 < \|a_i\| < 1$ $(i = 1, 2 \ldots n)$.

Let $B(a_1, \ldots a_n) = \Delta\{a_1^{i_1} \ldots a_n^{i_n};\ i_1, \ldots i_n \geq 0\}$, the absolutely convex combinations of monomials in $a_1, \ldots a_n$ (where $a_i^0 = 1$).

Let $\mathcal{B}$ consist of the collection of the closures of the sets $B(a_1, \ldots a_n)$, $0 < \|a_i\| < 1$, $i = 1, 2, \ldots$

Then $(A, \mathcal{B})$ is a pseudo-Banach algebra.

5) A-Holomorphic functions : Let A be a uniform algebra on the compact space X and let $\|\cdot\|_\infty$ be the uniform norm on X. A continuous function f on X is A- holomorphic on X if, for each x in X, f can be

approximated uniformly on some fixed neighbourhood of x in X by functions in A. Write H for the algebra of A-holomorphic functions on X. In general, H is not uniformly closed. H is a pseudo-Banach algebra.

6) <u>Germs of analytic functions</u>: If U is an open set in C^n, write $\theta(U)$ for the algebra of functions analytic on U and $H^{\infty}(U)$ for the algebra of functions analytic and bounded on U, with the compact-open topology. $\theta(U)$ is a Frechet algebra and $H\ (U)$ is a Banach algebra with respect to the uniform norm on U. Let K be a compact set in C^n and write θ_K for the algebra of germs on K of functions analytic in neighbourhoods of K. Then $\{\theta(U)\ ; \gamma_{VU} : U,V$ open neighbourhoods of K, $V \subset U\}$ is an inductive system of Fréchet algebras and continuous homomorphisms, where $\gamma_{VU} : \theta(U) \longrightarrow \theta(V)$ is the restriction map. We can identify θ_K algebraically with the inductive limit of this system, and θ_K can be given the locally convex inductive limit topology determined by the spaces $\theta(U)$. This is the inductive compact-open topology. An explicit representation of θ_K as a pseudo-Banach algebra is given as follows:

Let $\mathcal{U}$ be the set of open neighbourhoods U of K such that $\bar{U}$ is compact and each component of U meets K. Then $\{H^{\infty}(U)\ ; \gamma_{VU} : U,V \in \mathcal{U},\ V \subset U\}$ is an inductive system of Banach algebras with identity and continuous unital monomorphisms, whose inductive limit is θ_K. If B_U be the closed unit ball of $H^{\infty}(U)$, then the family $\{B_U; U \in \mathcal{U}\}$ is a bound structure in θ_K with respect to which θ_K is a pseudo-Banach algebra. Note that each of the Banach algebras $H^{\infty}(U)$ is semi-simple, but that θ_K is not always semi-simple.

2. Numerical range of an element of a pseudo-Banach algebra

For the pseudo-Banach algebra $(A, \mathcal{B}) = \varinjlim (A_\alpha, \|\cdot\|_\alpha)$ we define the sets of functions as follows:

$D_\alpha(A, B_\alpha; 1) = \{f \in A^* ; f_\alpha \in D(A_\alpha, \|\cdot\|_\alpha; 1)\}$;

$D(A, \mathcal{B}; 1) = \{f \in A' ; f(1) = 1, \|f_\alpha\|_\alpha \leq 1, \forall \alpha \in \Lambda\}$,

Where as usual, A^* is the algebraic dual and A' is the topological dual of A.

We show that $\bigcap_{\alpha \in \Lambda} D_\alpha(A, B_\alpha; 1) = D(A, \mathcal{B}; 1) = \varprojlim D_\alpha(A, B_\alpha; 1)$.

We define the numerical range of an element a in A, as the set of complex numbers: $V(A, \mathcal{B}; a) = \{f(a); f \in D(A, \mathcal{B}; 1)\}$.

If a is an element of the pseudo-Banach algebra $(A, \mathcal{B})$, we define $V_\alpha(A, B_\alpha; a) = \{f(a); f \in D_\alpha(A, B_\alpha; 1)\}$, $a \in A_\alpha$.

Using the fact that $V_\alpha(A, B_\alpha; a) = V(A_\alpha, \|\cdot\|_\alpha; a)$, $a \in A_\alpha$, we show that $V(A, \mathcal{B}; a) = \bigcap_{\substack{A_\alpha \\ a \in A_\alpha}} V(A_\alpha, \|\cdot\|_\alpha; a)$;

$r(A;a) = \inf_{\substack{A_\alpha \\ a \in A_\alpha}} r(A_\alpha, a)$ and $Sp(A,a) \subseteq V(A, \mathcal{B}; a)$.

<u>2.1 Definition:</u> Let $(A, \mathcal{B})$ be a pseudo-Banach algebra with identity 1. Recall $\mathcal{B} = \{B_\alpha ; \alpha \in \Lambda\}$, where each B_α is an absolutely convex bounded set satisfying the condition in 1.1. We write $A_\alpha = A(B_\alpha) \subset A$. We define $D(A, \mathcal{B}; 1) = \{ f \in A' : f(1) = 1, \|f_\alpha\|_\alpha \leq 1, \forall \alpha \in \Lambda\}$, and $D_\alpha(A, B_\alpha; 1) = \{ f \in A^* ; f/A_\alpha = f \in D(A_\alpha, \|\cdot\|_\alpha ; 1)\}$, where $D(A_\alpha, \|\cdot\|_\alpha; 1) = \{ f_\alpha \in A_\alpha' ; \|f_\alpha\|_\alpha = 1 = f_\alpha(1)\}$. Observe that for $f_\alpha \in A_\alpha'$, by the Hahn-Banach theorem there exists a $g \in A^*$ such that $g/A_\alpha = f_\alpha$.

<u>2.2 Theorem</u>: Let $(A, \mathcal{B})$ be a pseudo Banach algebra with identity 1 and with the inductive limit topology. Then $\bigcap_{\alpha \in \Lambda} D_\alpha(A, B_\alpha; 1) = D(A, \mathcal{B}; 1)$.

<u>Proof:</u> Let f belong to $\bigcap_{\alpha \in \Lambda} D_\alpha(A, B_\alpha; 1)$, then f is a linear functional on A such that $i_\alpha . f = f_\alpha$ is a continuous linear functional on A_α and $\|f_\alpha\|_\alpha \leq 1$; $f_\alpha(1) = 1$ for each α in Λ. The continuity of f_α on A_α for all α in Λ implies the continuity of f on A by prop. 1.4. Also $f(1) = 1$, $\|f_\alpha\|_\alpha \leq 1$, for all α in Λ.
Hence $f \in D(A, \mathcal{B}; 1)$, beacause A has the inductive limit topology.
Conversely if $f \in D(A, \mathcal{B}; 1)$, clearly for each α in Λ, f_α in A', $f_\alpha(1) = 1$ and $\|f_\alpha\|_\alpha \leq 1$.

Hence $f \in \bigcap_{\alpha \in \Lambda} D_\alpha(A, B_\alpha; 1)$.

Therefore $\bigcap_{\alpha \in \Lambda} D_\alpha(A, B_\alpha; 1) = D(A, \mathcal{B}; 1)$.

2.3 Theorem: Let $(A, \mathcal{B})$ be a pseudo-Banach algebra with identity 1 and with the inductive limit topology. Then,

$D(A, \mathcal{B}; 1) = \varprojlim D_\alpha(A, B_\alpha; 1)$.

Proof: First we show that for α, β in Λ; $\alpha \leq \beta$,

$D_\alpha(A, B_\alpha; 1) \supset D_\beta(A, B_\beta; 1)$.

If $f \in D_\beta(A, B_\beta; 1)$, then f in A^* and f_β is in $D(A_\beta, \|\cdot\|_\beta; 1)$.

But $f_\alpha = f/A_\alpha \in A'_\alpha$. Since $\alpha \leq \beta$ implies $B_\alpha \subset B_\beta$,

it follows that $f_\beta / A_\alpha = f_\alpha \in A'_\alpha$.

Since $f_\beta \in D(A_\beta, \|\cdot\|_\beta; 1)$, $f_\alpha \in D(A_\alpha, \|\cdot\|_\alpha; 1)$ and so

$f \in D_\alpha(A, B_\alpha; 1)$.

Thus $D_\alpha(A, B_\alpha; 1) \supset D_\beta(A, B_\beta; 1)$.

Now $\{D_\alpha(A, B_\alpha; 1)\}_\Lambda$ is a family of subsets of A^* indexed by the directed set Λ, such that for $\alpha \leq \beta$, $\alpha, \beta \in \Lambda$,

$D_\alpha(A, B_\alpha; 1) \supset D_\beta(A, B_\beta; 1)$.

For each α in Λ, $\|\cdot\|_\alpha$ topology on $D_\alpha(A, B_\alpha; 1)$ is such that $\|\cdot\|_\beta$ topology is finer than the topology induced on $D_\beta(A, B_\beta; 1)$ by $\|\cdot\|_\alpha$, whenever $\alpha \leq \beta$. Take $i_{\alpha\beta}$ to be the canonical injection $D_\beta(A, B_\beta; 1) \longrightarrow D_\alpha(A, B_\alpha; 1)$ for $\alpha \leq \beta$, then $\varprojlim D_\alpha(A, B_\alpha; 1)$ may be identified canonically with $\bigcap_{\alpha \in \Lambda} D_\alpha(A, B_\alpha; 1)$ (See Bourbaki (16), page 50). But by prop.2.2,

$$\bigcap_{\alpha \in \Lambda} D_\alpha(A, B_\alpha; 1) = D(A, \mathcal{B}; 1),$$

Hence $D(A, \mathcal{B}; 1) = \varprojlim D_\alpha(A, B_\alpha; 1)$.

2.4 Definition: Let $(A, \mathcal{B})$ be a pseudo-Banach algebra and a be an element of A. Since $A = \bigcup_\alpha \{A(B_\alpha); B_\alpha \in \mathcal{B}\} = \bigcup_\alpha A_\alpha$, a belongs to A_α for some α. Define,

$V_\alpha(A, B_\alpha; a) = \{f(a);\ f \in D_\alpha(A, B_\alpha; 1)\}$, and

define the numerical range of a to be,

$$V(A, \mathcal{B}; a) = \{f(a);\ f \in D(A, \mathcal{B}; 1)\}.$$

Remark: Since a locally convex algebra in which every element is bounded is a pseudo-Banach algebra (Prop.1.5, Eg.2), the results of this chapter hold good for those locally convex algebras.

2.5 Theorem: Let $(A, \mathcal{B})$ be a pseudo-Banach algebra and a be an element of A. If $a \in A_\alpha$, then,

$V_\alpha(A, B_\alpha; a) = V(A_\alpha, \|\cdot\|_\alpha; a)$ and

$$V(A, \mathcal{B}; a) = \bigcap_{A_\alpha, a \in A} V(A_\alpha, \|\cdot\|_\alpha; a).$$

$$v(A, \mathcal{B}; a) = \operatorname{Inf}_\alpha v(A_\alpha, \|\cdot\|_\alpha; a) \leq \|a\|_\alpha, \quad a \in A_\alpha.$$

Proof:

$$\begin{aligned} V_\alpha(A, B_\alpha; a) &= \{f(a);\ f \in D_\alpha(A, B_\alpha; 1)\} \\ &= \{g(a);\ g \in A^*,\ g_\alpha \in D(A_\alpha, \|\cdot\|_\alpha; 1)\} \\ &= \{g_\alpha(a);\ g_\alpha \in D(A_\alpha, \|\cdot\|_\alpha; 1)\} \\ &\qquad (\text{Since } g(a) = g_\alpha(a), \text{ for } a \in A_\alpha) \\ &= V(A_\alpha, \|\cdot\|_\alpha; a). \end{aligned}$$

$$V(A, \mathcal{B}; a) = \{f(a);\ f \in D(A, \mathcal{B}; 1)\}$$

$$= \{g(a);\ g \in \bigcap_{\alpha \in \Lambda} D_\alpha(A, B_\alpha : 1)\}$$

$$= \bigcap_{A_\alpha, a \in A_\alpha} \{g(a);\ g \in D_\alpha(A, B_\alpha : 1)\}$$

$$= \bigcap_\alpha \{V_\alpha(A, B_\alpha ; a);\ a \in A_\alpha\} .$$

$$= \bigcap_\alpha \{V(A_\alpha, \|\cdot\|_\alpha ; a);\ a \in A_\alpha\}.$$

Since $V(A, \mathcal{B} ; a) = \bigcap V(A_\alpha, \|\cdot\|_\alpha ; a)$, we have,

$$v(A, \mathcal{B} ; a) = \operatorname*{Inf}_{A_\alpha} \{v(A_\alpha, \|\cdot\|_\alpha ; a);\ a \in A_\alpha\} \leq \|a\|_\alpha .$$

<u>2.6 Theorem</u>: Let A be a commutative pseudo-Banach algebra with identity 1 and a be an element of A. Then,

i) $Sp(A;a) = \bigcap_{A_\alpha} \{Sp(A_\alpha;a);\ a \in A_\alpha\}$.

ii) $r(A;a) = \operatorname*{Inf}_{A_\alpha} \{r(A_\alpha;a);\ a \in A_\alpha\}$.

<u>Proof</u>: i) We have $Sp(A;a) = \{\lambda;\ (\lambda 1 - a) \notin G(A)\}$.

If $\lambda \in Sp(A;a)$, then $(\lambda - a) \notin G(A)$.

Hence $(\lambda - a) \notin G(A_\alpha)$ for each A_α, such that $a \in A_\alpha$; for otherwise $(\lambda - a) \in G(A_\alpha) \subset G(A)$, for some A_α, such that $a \in A_\alpha$, which will imply $(\lambda - a) \in G(A)$, contradicting $(\lambda - a) \notin G(A)$.

Hence $SP(A;a) \subset Sp(A_\alpha;a)$, for $a \in A_\alpha$; and so

$$Sp(A;a) \subset \bigcap_{A_\alpha} \{Sp(A_\alpha;a);\ a \in A_\alpha\}.$$

Conversely, let $\lambda \in \bigcap_{A_\alpha} Sp(A_\alpha;a)$.

Then $(\lambda - a) \notin G(A_\alpha)$ for all A_α such that $a \in A_\alpha$.

But the algebra A being the union of the subalgebras A_α, which are outer directed by inclusion, we see that $(\lambda - a) \notin G(A)$; and therefore $\lambda \in Sp(A;a)$.

Hence $\bigcap_{A_\alpha, a \in A_\alpha} Sp(A_\alpha;a) \subset Sp(A;a)$,

and so $Sp(A;a) = \bigcap_{A_\alpha} \{Sp(A_\alpha;a);\ a \in A_\alpha\}$.

ii) From definition, $r(A;a) = \text{Sup}\{|\lambda|; \lambda \in Sp(A;a)\}$.

$$= \text{Sup}\{|\lambda|; \lambda \in \bigcap_{A_\alpha} Sp(A_\alpha;a)\} \text{ by (i)}$$

$$= \inf_\alpha \{r(A_\alpha;a);\ a \in A_\alpha = A(B_\alpha)\}.$$

<u>2.7 Theorem</u> : Let $(A, \mathcal{B})$ be a pseudo-Banach algebra and a be an element of A. Then $Sp(A;a)$ is contained in $V(A, \mathcal{B};a)$.

<u>Proof</u>: We have $Sp(A_\alpha;a)$ is contained in $V(A_\alpha, \|\cdot\|_\alpha;a)$, for $a \in A_\alpha$, because A_α is a Banach algebra and so prop.3.8 of Ch I, applies,

Hence $Sp(A;a) = \bigcap_{A_\alpha} \{Sp(A_\alpha;a);\ a \in A_\alpha\}$

$$\subset \bigcap_{A_\alpha} \{V(A_\alpha, \|\cdot\|_\alpha;a);\ a \in A_\alpha\}$$

$$= V(A, \mathcal{B};a) \quad \text{by theorem 2.5.}$$

<u>3. Some properties of the numerical range and spectrum</u>

We show that for a pseudo-Banach algebra $(A, \mathcal{B})$ and $a \in A$, $Sp(A;a)$ is a compact subset of $\mathbb{C}$; $V(A, \mathcal{B};a)$ is a convex compact subset of $\mathbb{C}$; $r(A;a) = v(A, \mathcal{B};a)$; $V(A, \mathcal{B};a) = V(F, \mathcal{B};a)$, for every closed subalgebra F of A such that $a \in F$.

<u>3.1 Proposition</u>: Let $(A, \mathcal{B})$ be a pseudo-Banach algebra and a be an element of A. Then $Sp(A;a)$ is a compact subset of C and $V(A, \mathcal{B}; a)$ is a convex compact subset of C.

<u>Proof</u>: We have $Sp(A;a) = \bigcap_{A_\alpha} \{Sp(A_\alpha;a);\ a \in A_\alpha\}$ (by Th.2.6); where each set $Sp(A_\alpha;a)$ is a compact subset of C, by Prop.3.5 of Ch I. Since $Sp(A;a)$ is the intersection of compact subsets of C, it is compact.

Further, since $V(A, \mathcal{B}; a) = \bigcap_{A_\alpha} \{V(A_\alpha, \|\cdot\|_\alpha; a);\ a \in A_\alpha\}$, by theorem 2.5, where each $V(A_\alpha, \|\cdot\|_\alpha; a)$, $a \in A_\alpha$ is a convex compact subset of C by prop. 3.5 of Ch I, $V(A, \mathcal{B}; a)$ is a convex compact subset of C.

<u>3.2 Theorem</u>: Let $(A, \mathcal{B})$ be a pseudo-Banach algebra with the inductive limit topology and a be an element of A. Then, $\operatorname{Inf} r(A_\alpha;a) = \operatorname{Inf}_{A_\alpha}\{\|a\|_\alpha;\ a \in A_\alpha\}$ $= r(A;a)$. Also $r(A;a) = v(A, \mathcal{B}; a)$.

<u>Proof:</u> Let $\beta(a) = \operatorname{Inf}\{\|a\|_{B_\alpha},\ B_\alpha \in \mathcal{B};\ a \in A_\alpha = A(B_\alpha)\}$.

We have shown that
$$r(A;a) = \operatorname{Sup}\{|\lambda|;\ \lambda \in Sp(A;a)\}$$
$$= \operatorname{Inf}_{A_\alpha}\{r(A;a);\ a \in A_\alpha\}$$
(Prop. 2.6 (ii))

Now we prove that $\beta(a) \leq r(A;a)$.

If a belongs to A, then $a \in A(B_\alpha)$ for some $B_\alpha \in \mathcal{B}$.

If $z \notin Sp(A;a)$ and $|z| > \|a\|_{B_\alpha}$, then $(z-a)^{-1} \in A(B_\alpha)$ and

$$(z-a)^{-1} = z^{-1} + z^{-2}a + z^{-3}a^2 + \ldots\ldots ,$$

which converges in $A(B_\alpha)$.

Thus if $f \in A'$ and $g(z) = f((z-a)^{-1})$, then

$$g(z) = z^{-1}f(1) + z^{-2}f(a) + z^{-3}f(a^2) + \ldots\ldots \quad (*),$$

for $z \notin Sp(A;a)$ and $|z| > \|a\|_{B_\alpha} \geq \beta(a)$.

g is holomorphic for $|z| > r(A;a)$ and has a Laurent expansion which must coincide with (*).

Thus $\displaystyle\limsup_{n\to\infty} |f(a^n)|^{1/n} \leq r(A;a)$, $(f \in A')$,

and $\beta(a) \leq r(A;a)$.

Now $r(A;a) = \displaystyle\operatorname*{Inf}_{A_\alpha} r(A_\alpha;a) \leq \operatorname*{Inf}_{A_\alpha} v(A_\alpha;\|\cdot\|_\alpha;a)$

$\leq \displaystyle\operatorname*{Inf}_{B_\alpha} \|a\|_{B_\alpha}$; $a \in A(B_\alpha)$

$= \beta(a)$, together with $\beta(a) \leq r(A;a)$

imply $r(A;a) = v(A,\mathcal{B};a) = \displaystyle\operatorname*{Inf}_{\substack{B_\alpha \\ a \in A(B_\alpha)}} \|a\|_{B_\alpha} = \beta(a)$.

<u>3.3 Theorem</u> : Let $(A,\mathcal{B})$ be a pseudo-Banach algebra and a be an element of A. Then,

$Co\ Sp(A;a) = V(A,\mathcal{B};a)$.

<u>Proof</u>: We have proved (Prop. 3.1) that $Sp(A;a)$ is a compact subset of C and $V(A,\mathcal{B};a)$ is a convex compact subset of C such that $Sp(A;a) \subseteq V(A,\mathcal{B};a)$ and also by the previous

theorem $r(A;a) = v(A,\mathfrak{B};a)$.

Hence it can be shown that $\text{Co Sp}(A;a) = V(A,\mathfrak{B};a)$.

<u>3.4 Theorem</u>: Let F be a closed , under the inductive topology, subalgebra of the pseudo-Banach algebra $(A,\mathfrak{B})$. Let $(F,\mathfrak{B})$ denote F with the bound structure $\mathfrak{B}$ restricted to F and let a be an element of F. Then $V(A,\mathfrak{B};a) = V(F,\mathfrak{B};a)$.

<u>Proof</u>: We have for each α, $A(B_\alpha) \supseteq F(B_\alpha)$, ie $A_\alpha \supseteq F_\alpha$.

Also by Banach-algebra numerical range theory (prop.3.6, Ch.I) we have for each α such that $a \in F_\alpha$,

$$V(A_\alpha, \|\cdot\|_\alpha; a) = V(F_\alpha, \|\cdot\|_\alpha; a) \text{ or}$$

$$V_\alpha(A, B_\alpha; a) = V_\alpha(F, B_\alpha; a), \text{ for } a \in F_\alpha.$$

Hence
$$V(A,\mathfrak{B};a) = \bigcap_{A_\alpha} \{V_\alpha(A, B_\alpha; a) ; a \in A_\alpha\}$$
$$= \bigcap_{F_\alpha} \{V_\alpha(F, B_\alpha; a) ; a \in F_\alpha\}$$
$$= V(F,\mathfrak{B};a).$$

<u>3.5 Corollary</u>: Let $(A,\mathfrak{B})$ be a pseudo-Banach algebra and a be an element of A. Then,

$$V(A,\mathfrak{B};a) = V(P(a),\mathfrak{B};a).$$

<u>Proof</u>: This is an immediate consequence of 2.11, Ch.II and 3.4.

<u>3.6 Theorem</u>: Let $(A, \mathcal{B})$ be a pseudo -Banach algebra with the inductive limit topology and let a,b be elements of A; p, q $\in$ C. Let us denote $V(A, \mathcal{B}; a)$ by $V(A;a)$ for convenience. Then we have the following:

i) $V(A, a+b) \subset V(A;a) + V(A;b)$,

ii) $V(A; p+qa) = p + q\, V(A;a)$,

and $v(A; p+qa) \leq |p| + |q|\, v(A;a)$,

iii) $v(A;pa) = |p|\, v(A;a)$,

iv) $v(A; a+b) \leq v(A;a) + v(A;b)$,

v) $r(A; a+b) \leq r(A;a) + r(A;b)$,

vi) $r(A;ab) \leq r(A;a)\, r(A;b)$ and

$v(A;ab) \leq v(A;a)\, v(A;b)$,

vii) $v(A;a^n) = v^n(A;a)$ and

$r(A;a^n) = r^n(A;a)$.

<u>Proof</u>: Let us denote $D(A, \mathcal{B}; 1)$ by $D(A;1)$ for short.

i)
$$\begin{aligned} V(A; a+b) &= \{f(a+b);\ f \in D(A;1)\} \\ &= \{f(a) + f(b);\ f \ D(A;1)\} \\ &\subset \{f(a) + g(b);\ f, g \in D(A;1)\} \\ &= \{f(a);\ f \in D(A;1)\} + \{g(b);\ g \in D(A;1)\} \\ &= V(A;a) + V(A;b). \end{aligned}$$

ii)
$$\begin{aligned} V(A; p+qa) &= \{f(p+qa);\ f \in D(A;1)\} \\ &= p\, f(1) + q\{f(a);\ f \in D(A;1)\} \\ &= p + q\, V(A;a). \end{aligned}$$

Since $V(A; p+qa) = p + qV(A;a)$, we have

$v(A; p+qa) \leq |p| + |q|\, v(A;a)$.

iii) Since $V(A; pa) = p\,V(A;a)$,

we have $v(A;pa) = |p|\,v(A;a)$.

iv) Follows from i).

v) Follows from iv), since $v(A;a) = r(A;a)$,

by theorem 3.2.

vi) $$r(A;ab) = \operatorname*{Inf}_{A_\alpha}\{r(A_\alpha;ab);\ ab \in A_\alpha\}$$

$$\leq \operatorname*{Inf}_{A_\alpha} r(A_\alpha;a)\,r(A_\alpha;b)$$ (By 3.22(ii) chapter I)

$$= r(A;a)\,r(A;b),$$

$$v(A;ab) = v(A;a)v(A;b)$$ (Since $r(A;a) = v(A;a)$ for $a \in A$ by 3.2).

vii) $$r(A;a^n) = \operatorname*{Inf}_{A_\alpha}\{r(A_\alpha;a^n);\ a^n \in A_\alpha\}$$

$$= \operatorname*{Inf}_{A_\alpha}\{r^n(A_\alpha;a);\ a^n \in A_\alpha\}$$ (Prop.3.22, Ch.I)

$$= r^n(A;a).$$

Hence $v(A;a^n) = v^n(A;a)$, because

$v(A;a) = r(A;a)$, by 3.2.

<u>3.7 Theorem</u> : Let $(A\ \mathcal{B})$ be a pseudo-Banach algebra and a be an element of A. Then,

$$\text{Max Re } V(A;\mathcal{B};a) = \operatorname*{Inf}_{p>0} \frac{\operatorname*{Inf}_{\beta} \|1 + pa\|_\beta - 1}{p}, \quad a \in A_\beta$$

$$= \operatorname*{Lim}_{p \to 0} \frac{\operatorname*{Inf}_{\beta} \|1 + pa\|_\beta - 1}{p}.$$

Proof: Using 2.5, we have,

$$\text{Max Re } V(A,\mathcal{B};a) = \text{Max Re} \bigcap_{A_\gamma} \{V(A_\gamma,\|\cdot\|_\gamma;a);\ a \in A_\gamma\}$$

$$= \inf_{A_\gamma} \{\text{Max Re } V(A_\gamma,\|\cdot\|_\gamma;a);\ a \in A_\gamma\}$$

$$= \inf_{A_\gamma} \inf_{p>0} \frac{1}{p} (\|1 + pa\|_\gamma - 1)$$
$$\lim_{p \to 0^+} \quad \text{(By 3.10 of Ch.I).}$$

$$= \inf_{p>0} \lim_{p \to 0^+} \frac{1}{p} \left(\inf_{A_\gamma} \|1 + pa\|_\gamma - 1 \right), \quad a \in A_\gamma$$

3.8 Theorem : Let $(A,\mathcal{B})$ be a pseudo-Banach algebra and a be an element of A. Then $a \longrightarrow V(A,\mathcal{B};a)$ is upper semicontinuous.

Proof: Observe that $a \longrightarrow V(A,\mathcal{B};a)$ is continuous w.r.t the inductive limit topology on A. Also for every $a \in A$, $V(A,\mathcal{B};a)$ is a convex compact subset of A, by 3.1. Hence by 2.12 of Ch.I, the set valued mapping $a \longrightarrow V(A,\mathcal{B};a)$ is upper semi-continuous.

3.9 Theorem: Let $(A,\mathcal{B})$ be a pseudo-Banach algebra and let a be an element of A. Then,

$$\text{Max Re } V(A,\mathcal{B};a) = \sup_{p>0} \lim_{p \to 0^+} \frac{1}{p} \log \left\{ \inf_{A_\gamma} \|\exp(pa)\|_\gamma;\ a \in A_\gamma \right\}$$

Proof: $\operatorname{Max} \operatorname{Re} V(A,\mathcal{B};a) = \operatorname{Max} \operatorname{Re} \bigcap_{\substack{\gamma \\ a \in A_\gamma}} V(A_\gamma, \|\cdot\|_\gamma; a)$

$$= \operatorname*{Inf}_{\substack{\gamma \\ a \in A_\gamma}} \operatorname{Max} \operatorname{Re} V(A_\gamma, \|\cdot\|_\gamma; a)$$

$$= \operatorname*{Inf}_{\substack{\gamma \\ a \in A_\gamma}} \begin{matrix} \operatorname*{Sup}_{p>0} \\ \operatorname*{Lim}_{p \to 0^+} \end{matrix} \left\{ \frac{1}{p} \log \|\exp(pa)\| \right\}$$

$$= \begin{matrix} \operatorname*{Sup}_{p>0} \\ \operatorname*{Lim}_{p \to 0^+} \end{matrix} \left\{ \frac{1}{p} \log \operatorname*{Inf}_{\substack{\gamma \\ a \in A}} \|\exp(pa)\|_\gamma \right\}.$$

3.10 Definition: An element of a pseudo-Banach algebra is said to be dissipative, if $\operatorname{Re} z \leq 0$, for all $z \in V(A,\mathcal{B};a)$.

3.11 Theorem: Let $(A,\mathcal{B})$ be a pseudo-Banach algebra and let a be an element of A. Then a is dissipative, if and only if $\operatorname*{Inf}_{\substack{\gamma \\ a \in A_\gamma}} \|\exp(ta)\|_\gamma \leq 1 \quad (t > 0)$.

Proof: Applying 3.9, a is dissipative, if and only if $\log \operatorname*{Inf}_{\substack{\gamma \\ a \in A_\gamma}} \|\exp(ta)\| \leq 0$,

ie. if and only if $\operatorname*{Inf}_{\substack{\gamma \\ a \in A_\gamma}} \|\exp(ta)\| \leq 1 \quad (t > 0)$.

3.12 Theorem : Let $(A,\mathcal{B})$ be a pseudo-Banach algebra and let a be an element of A. Then,

$$\operatorname{Max} \operatorname{Re} Sp(A;a) = \mathop{\operatorname{Inf}}_{\substack{p>0 \\ \operatorname{Lim} \\ p\to 0}} \left\{ \frac{1}{p} \log \mathop{\operatorname{Inf}}_{\substack{\gamma \\ a\in A_\gamma}} \|\exp(pa)\|_\gamma \right\}$$

<u>Proof</u>: $\operatorname{Max} \operatorname{Re} Sp(A;a) = \operatorname{Max} \operatorname{Re} \bigcap_{\substack{\gamma \\ a\in A_\gamma}} Sp(A_\gamma;a)$

$$= \operatorname{Max} \operatorname{Re} \left\{ z \,;\, z \in Sp(A_\gamma;a);\ a \in A_\gamma \right\}$$

$$= \mathop{\operatorname{Inf}}_{\gamma} \left\{ \operatorname{Max} \operatorname{Re} Sp(A_\gamma;a);\ a \in A_\gamma \right\}$$

$$= \mathop{\operatorname{Inf}}_{\substack{\gamma \\ a\in A_\gamma}} \mathop{\operatorname{Inf}}_{\substack{p>0 \\ \operatorname{Lim} \\ p\to+\infty}} \frac{1}{p} \log \|\exp(pa)\|_\gamma ,$$

$$= \mathop{\operatorname{Inf}}_{\substack{p>0 \\ \operatorname{Lim} \\ p\to\infty}} \frac{1}{p} \log \mathop{\operatorname{Inf}}_{\gamma} \left\{ \|\exp(pa)\|_\gamma \,;\, a \in A_\gamma \right\} .$$

CHAPTER IV

Certain positive linear maps of topological algebras

1. Extreme positive maps of B*-algebras

We show that for an unital B*-algebra $(A,\|\cdot\|)$, Ext.D$(A,\|\cdot\|;1)$ coincides with $M(A)$ and that the boundary of $V(A,\|\cdot\|;a)$ is the spectrum of a, $Sp(A;a)$ for $a \in A$.

<u>1.1 Proposition</u>: Let $(A,\|\cdot\|)$ be an unital Banach algebra. Then the set $M(A)$ of non-zero multiplicative linear functionals of A, is contained in $D(A,\|\cdot\|;1)$.

<u>Proof</u>: Let $f \in M(A)$, we may assume that $f(1) = 1$, if not we consider the functional $g(x)=f(x)/f(1)$. Clearly $g(1) = 1$. But then $\|f\| = \sup_{\|x\|\leq 1} f(x) = 1$.

If not, then $\|f\| > 1$, and $f(x) = 1$ for some x with $\|x\| < 1$. But this is imposible because $x^n \longrightarrow 0$ and $f(x^n) = 1$ contradicting the contunity of f, since any multiplicative linear functional defined on a Banach algebra is continuous.

Thus $f \in M(A)$ implies $\|f\| = 1 = f(1)$ and $f \in A'$.

Hence $f \in D(A,\|\cdot\|;1)$ and we have shown that

$M(A) \subset D(A,\|\cdot\|;1)$.

<u>Remark</u>: From the above theorem, for an unital Banach algebra $(A,\|\cdot\|)$ and $a \in A$, we have,

$$Sp(A;a) = \{f(a);\ f \in M(A)\} \subset \{f(a);\ f \in D(A,\|\cdot\|;1)\}$$
$$= V(A,\|\cdot\|;a).$$

This gives a different proof of prop. 3.8 of Ch.I.

Now we are going to prove that for an unital B*-algebra the set $Ext.D(A,\|\cdot\|;1)$, of extreme points of $D(A,\|\cdot\|;1)$ coincides with $M(A)$. But first we state and prove for completeness the following known result.

<u>1.2 Theorem</u>: Let f be a linear functional on the unital B*-algebra $(A,\|\cdot\|)$. The following conditions on f are equivalent:

i) $f(a^*a) \geq 0$, for all $a \in A$,

ii) f is continuous and $\|f\| = f(1)$.

<u>Proof</u>: (See Berberian (6), prop.6.19)

<u>i) implies ii)</u> ; By the hypothesis on f, the mapping $(a,b) \longrightarrow f(ab^*)$ is a positive sesquilinear form (linear in the first variable and conjugate linear in the second variable). Also the sesqulinear form is hermitian and satisfies the cauchy-schwarz inequality;

expressed interms of f, this means that,

$f(ba^*) = \bar{f}(ab^*)$, $|f(ab^*)|^2 \leq f(aa^*)f(bb^*)$,

for all $a,b \in A$.

Setting $b = 1$, we have :

i) $f(a^*) = \bar{f}(a)$, ii) $|f(a)|^2 \leq f(aa^*)f(1)$,

for all a belonging to A.

The first of these relations shows that f(a) is real when a is self adjoint; the contunity of f will be derived from the second. We observe that for every self-adjoint $b \in A$, we have $|f(b)| \leq \|b\| f(1)$.

Writing $B = \{ b \}''$ the bicommutant of b in A, by the Gelfand theory we have $B = C(T)$, for a suitable compact space T; then b is a real valued function on T with $-\|b\| \leq b(t) \leq \|b\|$ for all $t \in T$.

Since $\|b\| - b$ is a continuous function on T with non-negative values, we may define $c = (\|b\| - b)^{\frac{1}{2}}$ belongs to $C(T) = B$; then $f(\|b\| - b) = f(c^2) = f(c^*c) \geq 0$.

Thus $f(b) \leq \|b\| f(1)$. This shows that f is continuous and $\|f\| \leq f(1)$. On the other hand, $\|f\| \geq f(1)$ leads to $\|f\| = f(1)$.

<u>ii) implies i)</u> ; Suppose f is a continuous linear form on A such that $f(1) = \|f\| > 0$. Replacing f by $f/f(1)$, we can assume that $\|f\| = f(1) = 1$.

Let $a \in A$. It is to be shown that $f(a^*a) \geq 0$. Writing I for the closed interval $[0, \|a^*a\|]$, $Sp(A, a^*a) \subset I$, It will suffice to show that $f(a^*a) \in I$. Let D be any closed disc containing I, say $D = \{z;\ |z - z_0| \leq p\}$. Since I is the intersection of all such discs, it will suffice to show that $f(a^*a) \in D$. Since,

$$Sp(A, aa^* - z_0 1) = Sp(A; a^*a) - z_0$$
$$\subset D - z_0 = \{z \in C;\ |z| \leq p\},$$

$$\| a^*a - z_0 \| = r(a^*a - z_0) \leq p\ ;$$

then
$$|f(a^*a) - z_0| = |f(a^*a) - z_0 f(1)|$$
$$= |f(a^*a - z_0)|$$
$$\leq \|f\| \|a^*a - z_0\|$$
$$\leq p\ .$$

This proves that $f(a^*a) \in D$.

<u>1.3 Proposition:</u> Let $(A, \|\cdot\|)$ be an unital B^*-algebra and let,

$$P(A, \|\cdot\|) = \{f\ ;\ f(a^*a) \geq 0,\ \forall a \in A,\ f(1) = 1\}.$$

Then $P(A, \|\cdot\|) = D(a, \|\cdot\|; 1)$.

<u>Proof</u>: Follows from Prop.1.2.

<u>1.4 Theorem</u>: Let $(A, \|\cdot\|)$ be an unital Banach algebra with involution $*$. If $f, g \in P(A, \|\cdot\|)$, then there exists a sequence $\{y_n\} \subset A$ such that for every $x \in A$, $g(x) = \lim_{n \to \infty} f(y_n^* x)$, for $g \leq f$.

<u>Proof</u>: (See, Bucy and Maltese (20), Theorem 1).

1.5 Theorem: Let $(A, \|\cdot\|)$ be an unital B^*-algebra. Then $\text{Ext.}D(A, \|\cdot\|; 1) = M(A)$.

Proof: Let $f \in \text{Ext.}D(A, \|\cdot\|; 1)$ and let $z \in A$, $\|z\| < 1$, $f(z^*z) \neq 0$.

Consider the functionals $f_1(x) = f(z^*zx)/f(z^*z)$,
and $f_2(x) = f((1 - z^*z)x)/(1 - f(z^*z))$.
If $f(z^*z) = 1$, we may take $f_2(x) = 0$.
Clearly $f_1(x^*x) \geq 0$ and $f_1(1) = 1$.
Also $f_2(x^*x) \geq 0$, since $1 - z^*z = y^*y$ for some $y \in A$, and $f_2(1) = 1$.
Therefore $f_1, f_2 \in P(A, \|\cdot\|) = D(A, \|\cdot\|; 1)$.
However since, $f = f(z^*z)f_1 + (1 - f(z^*z))f_2$ is a convex combination of elements of $P(A, \|\cdot\|)$, it follows that $f = f_1 = f_2$ and hence $f(z^*zx) = f(x)\, f(z^*z)$ for all $x \in A$, which implies that $f \in M(A)$.

Conversely suppose that $f \in M(A)$. Then $\|f\| = 1$ and $f(1) = 1$. Hence $f \in D(A, \|\cdot\|; 1)$.
Suppose that $g \in P(A, \|\cdot\|)$ is such that $g \leq f$. Then by prop.1.4, there exists a sequence $\{y_n\} \subset A$ such that $g(x) = \lim\limits_{n \to \infty} f(y_n^*x)$, $\forall\ x \in A$.
Since $g(x) = cf(x)$, where $c = \lim\limits_{n \to \infty} f(y_n^*)$ and since $g(x) = \lim\limits_{n \to \infty} f(y_n^*x)$ for every $g \in P(A, \|\cdot\|)$ with $g \leq f$,

we have f to be an extreme point of $D(A, \|\cdot\|; 1)$.

(See Bourbaki, Espaces vectoriels topologiques, P 82-83).

Corollary : For an unital B*-algebra $(A, \|\cdot\|)$ and $a \in A$, the boundary of $V(A, \|\cdot\|; a)$ coincides with $Sp(A; a)$.

Proof:
$$\begin{aligned} Sp(A;a) &= \{f(a);\ f \in M(A)\} \\ &= \{f(a);\ f \in \text{Ext } D(A, \|\cdot\|; 1)\} \\ &= \text{Boundary of } V(A, \|\cdot\|; a). \end{aligned}$$

2. Certain normalized positive linear maps of BP*-algebras

We show that $\text{Ext.}D(A, \|\cdot\|; 1) = M(A)$, for a BP*-algebra A.

2.1 Definition: (Allan (1)) Let A be a locally convex algebra. An element x of A is bounded if and only if, for some nonzero complex number λ, the set $\{(\lambda x)^n;\ n = 1, 2, \ldots\ldots\}$ is a bounded subset of A. The set of all bounded elements of A will be denoted by A_0.

Let $\mathcal{B}_1$ denote the collection of all subsets of A such that,

i) B is absolutely convex and $B^2 \subset B$;

ii) B is bounded and closed.

For each $B \in \mathcal{B}_1$, $A(B) = \{\lambda x \,;\, \lambda \in C,\ x \in B\}$ is a subalgebra of A generated by B.

With the norm $\|x\|_B = \operatorname{Inf}\{\lambda > 0;\ x \in \lambda B\}$ ($x \in A(B)$),

A(B) is a normed algebra.

The locally convex algebra is called <u>Pseudo-complete</u> if and only if each of the normed algebras A(B), $B \in \mathcal{B}_1$ is a Banach algebra.

A sub-collection $\mathcal{B}_2$ of $\mathcal{B}_1$ is said to be basic if for every $B_1 \in \mathcal{B}_1$, there is some B_2 in $\mathcal{B}_2$ such that $B_1 \subseteq B_2$.

<u>2.2 Proposition</u>; Let A be a locally convex algebra and let $\mathcal{B}_2$ be any basic subcollection of $\mathcal{B}_1$. Then $A_0 = \cup \{A(B);\ B \in \mathcal{B}_2\}$.

<u>Proof</u>: See Allan (1), Prop.2.4.

<u>2.3 Definition</u> : (Warsi (49)) Let A be a commutative pseudo-complete locally convex algebra with a continuous involution * . A is called a <u>BP*-algebra</u> if $A = A_0$, ie. every element of A is bounded.

<u>Remark</u>: Let $\mathcal{B}^* = \{B \in \mathcal{B}_1 \,;\ B = B^*\}$.

A locally convex*-algebra A will be called <u>*-pseudo-complete</u> if A(B) is complete for every $B \in \mathcal{B}^*$. Since $\mathcal{B}^* \subset \mathcal{B}_1$ clearly if A is pseudo-complete, then A is *-pseudo-

-complete. Consequently every BP*-algebra is *-pseudo-complete. Hence in a BP*-algebra every A(B) will be a *-subalgebra of A for each $B \in \mathcal{B}^*$.

Note 1: Throughout this section we assume that $\mathcal{B}^*$ is a basic subcollection of $\mathcal{B}_1$ so that by proposition 2.2, $A = \cup \{A(B);\ B \in \mathcal{B}^*\}$, where each A(B) is a B*-algebra.

Note 2 : By example 2 of Prop. 1.5, Ch III, every BP*-algebra is a pseudo-Banach algebra and hence we can define the numerical range for elements of BP*-algebras with the inductive limit topology, as in Ch. III.

2.3 Theorem: Let A be a BP*-algebra with the inductive limit topology. Then $f \in \text{Ext.}D(A, \mathcal{B}; 1)$ if and only if $f \in \text{Ext } D_\alpha(A, B_\alpha : 1)$, for each α.

Proof: Since $D(A, \mathcal{B}; 1) = \bigcap_\alpha D_\alpha(A, B_\alpha; 1)$, clearly $f \in \text{Ext } D(A, \mathcal{B}; 1)$ iff $f \in \text{Ext } D_\alpha(A, B_\alpha : 1)$, for each α.

2.4 Theorem: If for any $f \in D(A, \mathcal{B}; 1)$, $f_\alpha \in \text{Ext.}D(A_\alpha, \|\cdot\|_\alpha; 1)$ for each α, then f is an extreme point of $D(A, \mathcal{B}; 1)$.

Proof: Let $f \in D(A, \mathcal{B}; 1)$, then $f \in D_\alpha(A, B_\alpha; 1)$ and $f_\alpha \in D(A_\alpha, \|\cdot\|_\alpha; 1)$ for each α.

Let $f = \lambda f_1 + (1-\lambda) f_2$, $f_1, f_2 \in D(A, \mathcal{B}; 1)$.

Then $f/A_\alpha = \lambda f_1/A_\alpha + (1-\lambda) f_2/A_\alpha$ for all α.

where f_1/A_α and f_2/A_α belong to $D(A_\alpha, \|\cdot\|_\alpha; 1)$ for each α. Since f/A_α is an extreme point of $D(A_\alpha, \|\cdot\|_\alpha; 1)$, clearly $f/A_\alpha = f_1/A_\alpha = f_2/A_\alpha$ for all α.

This shows that $f \in \text{Ext}.D(A, \mathcal{B}; 1)$.

2.5 Theorem: $\text{Ext}\, D(A, \mathcal{B}; 1) = M(A)$, where A is a BP*-algebra as in note 2 of 2.2.

Proof: Let $f \in D(A, \mathcal{B}; 1)$, clearly $f \in \text{Ext-}D(A, \mathcal{B}; 1)$ iff $f \in \text{Ext}\, D_\alpha(A, B_\alpha; 1)$ for each α, ie. iff $f \in \text{Ext}.D(A_\alpha, \|\cdot\|_\alpha; 1)$ for each α.

Since A is a B*-algebra by hypothesis, we have

$\text{Ext}.D(A_\alpha, \|\cdot\|_\alpha; 1) = M(A_\alpha)$.

Therefore we for $f \in D(A, \mathcal{B}; 1)$, $f \in \text{Ext}.D(A, \mathcal{B}; 1)$ iff $f_\alpha \in M(A_\alpha)$, for each α.

Hence $\text{Ext}.D(A, \mathcal{B}; 1) = M(A)$.

2.6 Theorem: Let $(A, \{p_\alpha\}_\Lambda)$ be an unital b*-algebra. Then, $\text{Ext}.\, D(A, \{p_\alpha\}_\Lambda; 1) = \{f \in M(A);\ f \in A'\}$.

Proof: Since A is a projective limit of B*-algebras, the result follows in a way similar to that of the one we have described for BP*-algebras.

Note: Boundary of the numerical range of an element is the spectrum of that element for the topological algebras of this section.

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8 In line 10 for Exactly, read is a dense open subset of the closed unit disc.

41 In 3.2 Remark $v(A,q_\alpha;a)$ is defined to be $\mathrm{Sup}\{|z|;\ z \in V(A,q_\alpha;a)\}$.

66 In 6.19 Theorem: Reverse implication I para should read if $\beta \notin V(A,\{p_\alpha\};x)$, then for some $\lambda \in C$, $|\lambda - \beta| > \mathrm{Sup}_\alpha\ p_\alpha(x-\lambda)$.

23 In 3.18 Theorem: normal element is defined as follows: $a = h + ik$, $h, k \in H(A)$ and $hk = kh$, $i = \sqrt{-1}$.

While referring by proposition (...) we mean article (...).

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