SLICE-RIBBON CONJECTURE

SOME RESULTS ON THE SLICE-RIBBON CONJECTURE

By

HOMAYUN KARIMI, M.SC.

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AUTHOR:	Homayun Karimi, B.Sc.(Amirkabir University of Technology)
SUPERVISOR(S):	Dr. Hans U. Boden
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MCMASTER UNIVERSITY
DEPARTMENT OF
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The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance of a thesis entitled "**Some Results On the Slice-Ribbon Conjecture**" by **Homayun Karimi** in partial fulfillment of the requirements for the degree of **Master of Science**.

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Research Supervisor: ______________________________
Dr. Hans U. Boden

Examing Committee: ______________________________
Dr. Ian Hambleton

____
Dr. Andrew J. Nicas

To Samaneh

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Chapter 1

Introduction

The original motivation for the study of slice knots, and the first definition, was made by Ralph H. Fox and John W. Milnor in 1958; they were interested in smoothing PL singularities of surfaces in 4-space which arise naturally when considering complex hypersurfaces. The definition of a slice knot first appeared in a paper by Fox and Milnor [6].

Slice knots are also intimately related with the failure of the Whitney trick in 4-dimensions. The Whitney trick is used to remove intersections of submanifolds which cancel algebraically: if there are paths between the intersection points in each submanifold which form a loop, and if this loop can be made to bound an embedded disk, then by isotoping across the disk, the intersections can be removed. This works in higher dimensions, but in dimension 4 the disks can only be immersed generically. The question of improving these to embeddings is like trying to slice a knot.

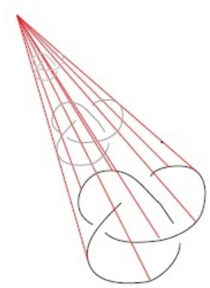
Finally, slice knots are interesting because they enable us to make the set of all knots into a group.

In this section we have some preliminaries, and some obstructions for a knot being slice.

Definition 1.1. A knot $K \subset S^3$ is a topologically slice knot, if there is a flat disk D^2 contained in D^4, such that $K = \partial D^2 = D^2 \cap S^3$. Such a disk is called a slicing disk for K.

Here flat means that D^2 has a neighborhood N that is a copy of $D^2 \times I^2$ meeting S^3 in $\partial D^2 \times I^2$ (of course, $I^2 = I \times I$, and this is just another disk).

Flatness is essential. Any knot $K \subset S^3$ is the boundary of a disk D^2 embedded in D^4, which can be seen by taking the cone over the knot. See [24, page 2].



Definition 1.2. A knot $K \subset S^3$ is a smoothly slice knot if there is a smoothly embedded disk $D^2 \subset D^4$ such that $K = \partial D^2 = D^2 \cap S^3$.

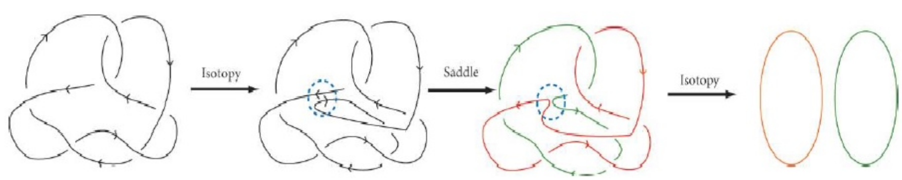
All the smoothly slice knots are topologically slice, because smoothly embedded D^2 is flat. But we can find examples of topologically slice knots, which are not smoothly slice. All the knots with Alexander polynomial equal to 1 are topologically slice (M. Freedman, 1980). See Example 3.5.

From now on, if we say a given knot K is slice, we will take that to mean K is smoothly slice.

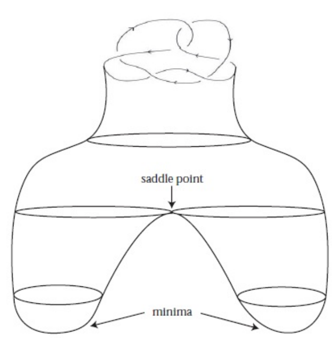
Example 1.3. If K is a knot which is symmetric with respect to a plane $\mathbb{R}^2 \subset \mathbb{R}^3$, then K is slice because we can spin it through $\mathbb{R}^4_+$ about the axis $\mathbb{R}^2$ to produce the desired flat disk. (We can spin a point $x = (x_1, x_2, x_3, 0)$ of $\mathbb{R}^3_+$ about $\mathbb{R}^2$ according to the formula $x_\theta = (x_1, x_2, x_3 \cos\theta, x_3 \sin\theta)$. The spin $K^* = \{x_\theta : x \in K, 0 \leq \theta \leq 2\pi\}$ is a 2-sphere in $\mathbb{R}^4$.) So if K is a knot in S^3 and $r : S^3 \longrightarrow S^3$ is an orientation-reversing homeomorphism, then $K\#\overline{rK}$ is a slice knot. Here rK means the knot K with orientation reversed, and $\overline{rK}$ means the mirror image of rK.

We can visualise a slice disk by making movies. If a knot is slice then it bounds a disk $D^2 \subset D^4$ so that concentric 3-spheres move through (intersect) it to produce either an ordinary nonsingular knot or link or a knot or link with singularities corresponding to one of simple maximum or minimum or saddle point.

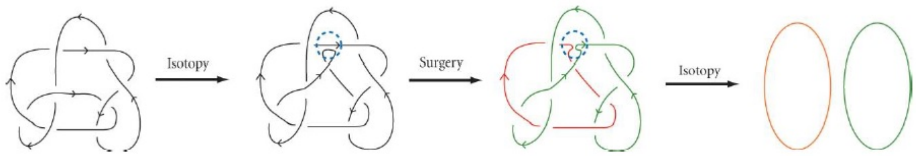
Example 1.4. Stevedore's knot, otherwise known as 6_1 in the standard knot tables, is the simplest slice knot (other than the unknot). The following movie shows how 3-spheres move through the slice disk (see [24, page 3]):



The slice disk is shown schematically below, of course in reality this is a knotted disk in 4-space ([24, page 4]):



Example 1.5. Another example of a slice knot is the 8-crossing knot 8_8. Here is the corresponding slice movie ([24, page 4]):



Recall, if V is a Seifert matrix for a knot K, and $\omega \in \mathbb{C}$, with $|\omega| = 1$, then

$$\begin{aligned} \sigma_\omega(K) &= \operatorname{sign}\left((1-\omega)V + (1-\omega^{-1})V^T\right) \\ \Delta_K(t) &= \det(V - tV^T) \end{aligned}$$

And 4-ball genus of a knot K, denoted by $g_4(K)$, is the minimum genus of an oriented surface $F \subset D^4$, with $\partial F = K$.

We know that, a knot K is slice if and only if $g_4(K) = 0$.

Also we know that if K is slice, its signature function $\sigma_\omega(K)$ is identically zero. And its Alexander polynomial factors as $f(t)f(t^{-1})$, for some polynomial $f(t) \in \mathbb{Z}[t]$ [17, page 90]. So the determinant of slice knot $\det(K) = |\Delta_K(-1)|$ must be a square integer.

Now we want to use these facts to investigate which one of torus knots cannot be a slice knot.

Proposition 1.6. *The Alexander polynomial of a torus knot of type (p, q), is given by the following formula:*

$$\Delta_{T(p,q)}(t) = \frac{(t^{|pq|}-1)(t-1)}{(t^{|p|}-1)(t^{|q|}-1).}$$

Proof. [17, pages 118–119] □

Theorem 1.7. *The torus knot $T(p, q)$ is not slice unless $p = \pm 1$ or $q = \pm 1$.*

Proof. Assume $T(p, q)$ is slice. Then its signature function is identically zero. It follows then, $\Delta_{T(p,q)}(t)$ has no roots on unit circle. But roots of $\Delta_{T(p,q)}(t)$ are $|pq|$-th roots of unity that are not $|p|$-th or $|q|$-th roots of unity. Because

$\gcd(p,q) = 1$, $|p|$-th roots of unity are different from $|q|$-th roots of unity, unless the root equals 1. Now, the number of $|pq|$-th roots of unity without 1, must be equal to the number of $|p|$-th and $|q|$-th roots of unity without 1, so we can write:

$$|pq| - 1 = |p| - 1 + |q| - 1 \Rightarrow |pq| - |p| - |q| + 1 = 0 \Rightarrow$$
$$(|p| - 1)(|q| - 1) = 0 \Rightarrow p = \pm 1 \textit{ or } q = \pm 1$$

$\square$

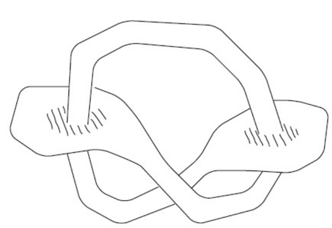
Corollary 1.8. *The only slice torus knot is the unknot.*

Definition 1.9. A knot K is called algebraically slice if for some Seifert surface F of K, the Seifert pairing $\theta : H_1(F;\mathbb{Z}) \times H_1(F;\mathbb{Z}) \to \mathbb{Z}$ vanishes on a submodule with half the rank of $H_1(F;\mathbb{Z})$.

All the topologically slice knots are algebraically slice. As an example of a knot that is algebraically slice but not topologically slice, let $K = 7_7$, then $K\#K\#K\#K$ is the desired example.

Definition 1.10. The ribbon disk is the image $\alpha(D^2)$ of an immersion $\alpha : D^2 \longrightarrow \mathbb{R}^3$ whose only singularities are of the following form. Each component of the singular set is the image of a pair of closed intervals in D^2, one with endpoints on the boundary of D^2 and one entirely interior to D^2.

Definition 1.11. A knot K is called a ribbon knot if it bounds a ribbon disk.



We can locally resolve a ribbon singularity into 4-space to get back a slice disk. We can push a neighborhood of the singularity, into 4-space. Notice that at singularity, one part of the ribbon disk contains an arc, whose preimage lies entirely in the interior of D^2. So we push this part (shaded parts, in the above figure) into 4-space, and we can remove the self-intersection. As a result, we have:

Corollary 1.12. *Every ribbon knot is slice.*

But the reverse is a famous conjecture due to Fox, called the slice-ribbon conjecture (Problem 1.33 in Kirby's problem list [13]), and still is open.

Conjecture 1.13 (Slice-Ribbon Conjecture)**.** *Every slice knot is ribbon.*

Proposition 1.14. *The slice-ribbon conjecture is true for torus knots.*

Proof. The only slice torus knot, is the unknot, which is ribbon. □

As another obstruction for a knot to be slice, is the concordance genus.

Definition 1.15. Two knots K_1 and K_2 are called smoothly concordant if there is a smooth embedding $\phi : S^1 \times [0,1] \longrightarrow S^3 \times [0,1]$, whose boundary is $(K_1 \times \{0\}) \coprod (-K_2 \times \{1\})$.

Proposition 1.16. *A knot K is slice if and only if it is concordant to the unknot.*

Proof. Assume K is concordant to the unknot. The unknot bounds a disk. The interior of this disk, union $\phi(S^1 \times [0,1])$ is the slice disk.
Conversely, assume K is slice. Consider a trivial knot in the interior of the slice disk, it bounds a disk. Delete the interior of this disk, we can define an embedding ϕ, whose image is the rest of the slice disk. □

Definition 1.17. The concordance genus of a knot K, denoted by $g_c(K)$, is the minimum genus of a surface in S^3 whose boundary is a knot concordant to K.

Corollary 1.18. *A knot K is slice if and only if $g_c(K) = 0$.*

Proposition 1.19. *For a knot K, we have the following inequalities:*

$$\frac{1}{2}|\sigma(K)| \leq g_4(K) \leq g_c(K) \leq g_3(K)$$

And if K is slice, then $\sigma(K) = g_4(K) = g_c(K) = 0$.

These inequlities can be strict. For example, $0 = \frac{1}{2}|\sigma(4_1)| < g_4(4_1) = 1$, $1 = g_4(6_2) < g_c(6_2) = 2$ and $0 = g_c(6_1) < g_3(6_1) = 1$.

We have a binary operation on the set of all knots, called the connected sum. By this operation, the set of knots, forms a monoid. But existence of inverse element for each knot fails. To remove this problem, we define an equivalence relation on knots. Then the set of equivalence classes form a group, called the concordance group. The concordance group was introduced in 1966 by Fox and Milnor.

Definition 1.20. If K_1 and K_2 are two knots, we write $K_1 \sim K_2$ if K_1 is concordant to K_2.

This is an equivalence relation, and it is not hard to check that.

Now we denote the set of equivalence classes by $\mathcal{C}$, and consider the connected sum, as a binary operation on $\mathcal{C}$, i.e.

$$[K_1] + [K_2] = [K_1 \# K_2]$$

Theorem 1.21. *$\mathcal{C}$ is an abelian group.*

Proof. We know that the zero element in $\mathcal{C}$, is the set of all slice knots. Also we know that for a knot K, $K\#\overline{rK}$ is slice, so $-[K] = [\overline{rK}]$. The fact that $\mathcal{C}$ is abelian, is obvious. □

The group $\mathcal{C}$ is called the (smooth) concordance group. Fox and Milnor showed that this group is infinitely generated.

Proposition 1.22. *If K is reversible and amphicheiral knot which is not slice then $[K] \in \mathcal{C}$ is an order 2 element.*

Proof. By the assumption, $K = \overline{rK}$, so $[K] = -[K]$, and $2[K] = 0$. K is not slice, so $[K] \neq 0$, then the order of $[K]$ is 2. □

Example 1.23. 4_1 is not slice, because $\det(4_1) = 5$, which is not a perfect square, but it is reversible and amphicheiral, so the order of $[4_1]$ is 2.

Theorem 1.24. *Every knot K with $\sigma(K) \neq 0$ represents an element of infinite order in $\mathcal{C}$.*

Proof. Murasugi proved that $\sigma(K_1 \# K_2) = \sigma(K_1) + \sigma(K_2)$. If K^n is n times connected sum of K with itself, then $\sigma(K^n) = n\sigma(K) \neq 0$. So K^n is not slice, and $[K^n] = n[K] \neq 0$. So K is an element of infinite order. □

Example 1.25. $\sigma(3_1) = -2$, where 3_1 is the right-hand trefoil, so the right-hand trefoil represents an element of infinite order in $\mathcal{C}$.

Definition 1.26. A square matrix N is null-cobordant, if it is congruent to a matrix of the form $\begin{bmatrix} 0 & N_1 \\ N_2 & N_3 \end{bmatrix}$, where N_i are square matrices of the same size.

Definition 1.27. For matrices A_1 and A_2, we define the block sum $A_1 \oplus A_2 = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$. We say square matrices A_1 and A_2 are cobordant, if $A_1 \oplus (-A_2)$ is null-cobordant.

Levine defined a homomorphism $\phi : \mathcal{C} \longrightarrow \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$, by assigning to the concordance class of a knot, the cobordism class of any Seifert matrix of that knot. He proved that ϕ is surjective [16].

Recent work of Cochran, Orr and Teichner [4], has revealed a deeper structure to the knot concordance group. In that work a filtration of $\mathcal{C}_{\text{top}}$, the topological concordance group, by subgroups is defined (1997):

$$\ldots \subset \mathcal{F}_2 \subset \mathcal{F}_{1.5} \subset \mathcal{F}_1 \subset \mathcal{F}_{.5} \subset \mathcal{F}_0 \subset \mathcal{C}_{\text{top}}$$

It is shown that $\mathcal{F}_0$ corresponds to knots with trivial Arf invariant, $\mathcal{F}_{.5}$ corresponds to knots in the kernel of ϕ, and all knots in $\mathcal{F}_{1.5}$ have vanishing Casson-Gordon invariants. Using von Neumann η-invariants, it has been proved that each quotient is infinite. In fact, Cochran, Harvey and Leidy proved that

$$\forall n \ \mathrm{rank}(\mathcal{F}_n/\mathcal{F}_{n.5}) = \infty$$

This work places Levine's obstructions and those of Casson-Gordon in the context of an infinite sequence of obstructions, all of which reveal a finer structure to $\mathcal{C}_{\mathrm{top}}$.

If K is slice, then $\forall n \in \frac{1}{2}\mathbb{N} \quad K \in \mathcal{F}_n$. This filtration measures how far algebraically a knot is from being topologically slice.

Another filtration is as follows [11]:

$$\ldots \subset \mathcal{G}_{3.5} \subset \mathcal{G}_3 \subset \mathcal{G}_{2.5} \subset \mathcal{G}_2 \subset \mathcal{G}_{1.5} \subset \mathcal{C}$$

If K is slice, then $K \in \mathcal{G}_n$. This filtration measures how far geometrically a knot is from being slice. P. Horn proved that:

$$\forall n \geq 2 \ \mathrm{rank}(\mathcal{G}_n/\mathcal{G}_{n.5}) = \infty$$

These results show that, slice knots are fairly rare in the set of all knots.

For more details about concordance group, see [19].

1.1 Casson-Gordon Invariants

We begin by reviewing the linking form on $\mathrm{Tor}(H_1(M))$ for an oriented 3-manifold M. If x and y are curves representing torsion in the first homology, then $lk(x, y)$ is defined to be $(d \cap y)/n \in \mathbb{Q}/\mathbb{Z}$, where d is a 2-chain with boundary nx. Intersections are defined via transverse intersections of chains, and of course one must check that the value of the linking form is independent of the many choices in its definition. For a closed oriented 3-manifold the linking form is nonsingular in the sense that it induces an isomorphism from $\mathrm{Tor}(H_1(M))$ to $\mathrm{Hom}(\mathrm{Tor}(H_1(M)), \mathbb{Q}/\mathbb{Z})$.

Such a symmetric pairing on a finite abelian group, $l : H \times H \to \mathbb{Q}/\mathbb{Z}$, is called metabolic with metabolizer L if the linking form vanishes on $L \times L$ for some subgroup L with $|L|^2 = |H|$.

Let M_q denote the q-fold branched cover of S^3 branched over a given knot K, and let $\bar{M}_q$ denote 0-surgery on M_q along $\tilde{K}$, where $\tilde{K}$ is the lift of K to M_q. Here q will be a prime power.

Let x be an element of self-linking 0 in $H_1(M_q)$ and suppose that x is of prime power order, say p. Linking with x defines a homomorphism $\chi_x : H_1(M_q) \to \mathbb{Z}_p$. Furthermore, χ_x extends to give a $\mathbb{Z}_p$-valued character on $H_1(\bar{M}_q)$ which vanishes on the meridian of $\tilde{K}$. In turn, this character extends to give $\bar{\chi}_x : H_1(\bar{M}_q) \to \mathbb{Z}_p \oplus \mathbb{Z}$. Since x has self-linking 0, bordism theory implies that the pair $(\bar{M}_q, \bar{\chi}_x)$ bounds a 4-manifold, character, pair, (W, η).

More generally, for any character $\chi : H_1(M_q) \to \mathbb{Z}_p$, there is a corresponding character $\bar{\chi} : H_1(\bar{M}_q) \to \mathbb{Z}_p \oplus \mathbb{Z}$. This character might not extend to a 4-manifold, but since the relevant bordism groups are finite, for some multiple $r\bar{M}_q$ the character given by $\bar{\chi}$ on each component does extend to a 4-manifold, character pair, (W, η).

Let Y denote the $\mathbb{Z}_p \times \mathbb{Z}$ cover of W corresponding to η. Using the action of $\mathbb{Z}_p \times \mathbb{Z}$ on $H_2(Y; \mathbb{C})$ one can form the twisted homology group $H_2^t(W; \mathbb{C}) = H_2(W; \mathbb{C}) \otimes_{\mathbb{C}[\mathbb{Z}_p \times \mathbb{Z}]} \mathbb{C}(t)$. (The action of $\mathbb{Z}_p$ on $\mathbb{C}(t)$ is given by multiplication by $e^{2\pi i/p}$.) There is a nonsingular hermitian form on $H_2^t(W; \mathbb{C})$ taking values in $\mathbb{C}(t)$. The Casson-Gordon invariant is defined to be the difference of this form and the intersection form of $H_2(W; \mathbb{C})$, both tensored with $\frac{1}{r}$, in $W(\mathbb{C}[t, t^{-1}]) \otimes \mathbb{Q}$. (In showing that this Witt class yields a well-defined obstruction to slicing a knot, the fact that $\Omega_4(\mathbb{Z}_p \oplus \mathbb{Z})$ is nonzero appears, and as a consequence one must tensor with $\mathbb{Q}$ to arrive at a well defined invariant, even in the case of χ_x in which it is possible to take $r = 1$.)

Definition 1.28. The Casson-Gordon invariant $\tau(M_q, \chi)$ is the class $(H_2^t(W;\mathbb{C}) - H_2(W;\mathbb{C})) \otimes \frac{1}{r} \in W(\mathbb{C}(t)) \otimes \mathbb{Q}$.

The main theorem of [2] states:

Theorem 1.29. *If K is slice, there is a metabolizer L for the linking form on $H_1(M_q)$ such that, for each prime power p and each element $x \in L$ of order p, $\tau(M_q, \chi_x) = 0$.*

The proof shows that if K is slice with slice disk D, then covers of $B^4 \setminus D$ can be used as the manifold W, and for this W the invariant vanishes.

Chapter 2

Slice-Ribbon Conjecture for 2-bridge Knots

In this section we summarize the paper by Paolo Lisca [18]. He applies Donaldson's theorem on the intersection forms of definite 4-manifolds to characterize the lens spaces which smoothly bound rational homology 4-dimensional balls.

Definition 2.1. Let $\mathbb{Q}_{>1}$ denote the set of rational numbers bigger than 1, and define maps $f, g : \mathbb{Q}_{>1} \longrightarrow \mathbb{Q}_{>1}$ by setting, for $\frac{p}{q} \in \mathbb{Q}_{>1}$, $p > q > 0$, $(p, q) = 1$,

$$f\left(\frac{p}{q}\right) = \frac{p}{p-q} \quad , \quad g\left(\frac{p}{q}\right) = \frac{p}{q'}$$

where $p > q' > 0$, $qq' \equiv 1 \pmod{p}$. Define $\mathcal{R} \subset \mathbb{Q}_{>1}$ to be the smallest subset of $\mathbb{Q}_{>1}$ such that $f(\mathcal{R}) \subset \mathcal{R}$, $g(\mathcal{R}) \subset \mathcal{R}$, and $\mathcal{R}$ contains the set of rational numbers $\frac{p}{q}$ such that $p > q > 0$, $(p, q) = 1$, $p = m^2$ for some $m \in \mathbb{N}$ and q is of one of the following types:

1. $mk \pm 1$ with $m > k > 0$ and $(m, k) = 1$.

2. $d(m \pm 1)$, where $d > 1$ divides $2m \mp 1$ and

3. $d(m \pm 1)$, where $d > 1$ is odd and divides $m \pm 1$.

So if $\frac{p}{q}$ is a positive number, in following way we can say whether it belongs to $\mathcal{R}$ or not. If p is not a perfect square, then $\frac{p}{q} \notin \mathcal{R}$. Otherwise, list all the numbers of the above three types. Then, first add all the numbers of the form $p - q$, then for each number q of the new list, find all q' such that $p > q' > 0$, $qq' \equiv 1 \pmod{p}$, and add them to the list. Do the same thing with these new numbers. After finitely many steps, this algorithm terminates. Now look at the denominator of $\frac{p}{q}$, if it is in the list, then $\frac{p}{q} \in \mathcal{R}$.

Example 2.2. We want to see, whether $\frac{25}{12} \in \mathcal{R}$ or not. First, $25 = m^2$ is a perfect square, so we continue the algorithm. Now list all the q's satisfying the above definition. We have $q = 4, 6, 9, 11, 14, 16, 18, 19, 21$, now apply f and g to all the fractions $\frac{25}{q}$. Notice that $f\left(\frac{25}{18}\right) = \frac{25}{7}$, so add 7 to the above list. If we apply f and g to the new list, we get nothing new. So the algorithm terminates. Since 12 does not belong to this list, so $\frac{25}{12} \notin \mathcal{R}$.

Theorem 2.3. *If K is a slice knot whose double branched cover is a lens space L, then $|H_1(L;\mathbb{Z})|$ is a perfect square.*

Proof. Corollary 3 on p. 213 of [22] implies that the 2-fold branched cover Σ_2 of any knot K has finite $H_1(\Sigma_2)$ with order given by $|\Delta_K(-1)| = |\det(K)|$. If K is slice, then $\Delta_K(t) = f(t)f(t^{-1})$ for some $f(t)\mathbb{Z}[t]$, hence $|H_1(\Sigma_2)| = \det(K)$ is a square integer. □

We know that the double branched cover of a 2-bridge knot $K(p, q)$ is the lens space $L(p, q)$. By the Theorem 2.3, if $K(p, q)$ is ribbon, then $|H_1(L(p, q);\mathbb{Z})| = p$ is a perfect square, say m^2.

Example 2.4. A twist knot with t full twist in it (a twist knot is a Whitehead double of the unknot), is a 2-bridge knot $K(4t + 1, 2)$. Since $4t + 1$ must be a perfect square, we conclude that $t = u(u - 1)$. In [2], by computing the signature, we see that the only twist knots that are slice, correspond to $t = 0$ (the unknot) and $t = 2$ (6_1).

If $K(p,q)$ is a 2-bridge knot, the following theorem, and the corollary after that, prove the slice-ribbon conjecture for 2-bridge knots.

Theorem 2.5. *Let $p > q > 0$ be coprime integers. Then, the following statements are equivalent.*

1. *The lens space $L(p,q)$ smoothly bounds a rational homology ball.*
2. *There exist:*
    - *A surface with boundary Σ, homeomorphic to a disk if p is odd and to the disjoint union of a disk and a Möbius band if p is even and*
    - *A ribbon immersion $i : \Sigma \longrightarrow S^3$ with $i(\partial\Sigma) = K(p,q)$.*
3. *$\frac{p}{q}$ belongs to $\mathcal{R}$.*

Corollary 2.6. *Let $p > q > 0$ be coprime integers with p odd. Then, the following statements are equivalent:*

1. *$\frac{p}{q}$ belongs to $\mathcal{R}$;*
2. *$K(p,q)$ is a ribbon knot;*
3. *$K(p,q)$ is a smoothly slice knot and*
4. *$L(p,q)$ smoothly bounds a rational homology ball.*

In particular, the slice-ribbon conjecture holds for 2-bridge knots.

The proof of Theorem 2.5 is based on the following idea. The 2-fold cover of B^4 branched along a slicing disk for $K(p,q)$ is a smooth rational homology ball with boundary the lens space $L(p,q)$ (see [23]). If a lens space $L(p,q)$ smoothly bounds a rational homology ball $W(p,q)$, one can form a smooth negative definite 4-manifold $X(p,q)$ by taking the union of $-W(p,q)$ with a canonical 4-dimensional plumbing $P(p,q)$ bounding $L(p,q)$.

Since $X(p,q)$ is negative definite, Donaldson's celebrated theorem [5] implies that the intersection lattice $Q_{X(p,q)}$ of $X(p,q)$ is isomorphic to the standard diagonal intersection lattice $\mathbb{D}^n$, where $n = b_2(X(p,q))$. Therefore there is an embedding of intersection lattices $Q_{P(p,q)} \hookrightarrow \mathbb{D}^n$, and since $-L(p,q) = L(p,p-q)$ smoothly bounds the rational homology ball $-W(p,q)$, for some n' there is an embedding $Q_{P(p,p-q)} \hookrightarrow \mathbb{D}^{n'}$ as well. The existence of both embeddings gives constraints on the pair (p,q) which eventually lead to the proof of Theorem 2.5.

We briefly mention the combinatorial machinery, which is used in the paper, to prove Theorem 2.5.

Let $\mathbb{D}$ denote the intersection lattice $(\mathbb{Z},(-1))$, and let $\mathbb{D}^n$ be the orthogonal direct sum of n copies of $\mathbb{D}$. Fix generators $e_1,\dots,e_n \in \mathbb{D}^n$ such that

$$e_i.e_j = -\delta_{ij}, \quad i,j = 1,\dots,n$$

Observe that the group of automorphisms $Aut(\mathbb{D}^n)$ contains the reflections across each hyperplane orthogonal to an e_i as well as all the transformations determined by the permutations of $\{e_1,\dots,e_n\}$. Given a subset $S = \{v_1,\dots,v_n\} \subseteq \mathbb{D}^n$, we define

$$\begin{aligned} E_i^S &:= \{j \in \{1,\dots,n\} | v_j.e_i \neq 0\}, \quad i = 1,\dots,n \\ V_i &:= \{j \in \{1,\dots,n\} | e_j.v_i \neq 0\}, \quad i = 1,\dots,n \\ p_i(S) &:= |\{j \in \{1,\dots,n\} | \ |E_j^S| = i\}|, \quad i = 1,\dots,n \end{aligned}$$

Let $v_1,\dots,v_n \in \mathbb{D}^n$ be elements such that, for $i,j \in \{1,\dots,n\}$,

$$v_i.v_j = \begin{cases} -a_i \leq -2 & \text{if } i = j \\ 0 \text{ or } 1 & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1 \end{cases} \tag{2.0.1}$$

for some integers a_i, $i = 1,\dots,n$.

Let $S = \{v_1,\dots,v_n\} \subseteq \mathbb{D}^n$ be a subset which satisfies (2.0.1). We define the intersection graph of S as the graph having as vertices $v_1,\dots,v_n$, and an edge between v_i and v_j if and only if $v_i.v_j = 1$ for $i, j = 1, \dots, n$. The number of connected components of the intersection graph of S will be denoted by $c(S)$.

We shall say that an element $v_j \in S$ is isolated or final if it is, respectively, an isolated vertex or a leaf of the intersection graph, and it is internal otherwise.

Given elements $e, v \in \mathbb{D}^n$ with $e.e = -1$, we shall denote by $\pi_e(v)$ the projection of v in the direction orthogonal to e:

$$\pi_e(v) := v + (v.e)e \in \mathbb{D}^n$$

Two elements $v, w \in \mathbb{D}^n$ are linked if there exists $e \in \mathbb{D}^n$ with $e.e = -1$ such that

$$v.e \neq 0 \quad \text{and} \quad w.e \neq 0$$

A set $S \subset \mathbb{D}^n$ is irreducible if, given two elements $v, w \in S$, there exists a finite sequence $v_0 = v, \dots, v_k = w \in S$ such that v_i and v_{i+1} are linked for $i = 0, \dots, k-1$. A set which is not irreducible is reducible.

Definition 2.7. A subset $S = \{v_1, \dots, v_n\} \subseteq \mathbb{D}^n$ is good if it is irreducible and its elements satisfy (2.0.1).

Definition 2.8. Given a subset $S = \{v_1, \dots, v_n\} \subseteq \mathbb{D}^n$, define

$$I(S) := \sum_{i=1}^{n} (-v_i.v_i - 3) \in \mathbb{Z}$$

Lemma 2.9. *Let $S = \{v_1, v_2, v_3\} \subset \mathbb{D}^3 =< e_1, e_2, e_3 >$ be a good subset with $I(S) < 0$. Then, up to applying to S an element of $Aut(\mathbb{D}^3)$ and possibly replacing (v_1, v_2, v_3) with (v_3, v_2, v_1), one of the following holds:*

(1) $(v_1, v_2, v_3) = (e_1 - e_2, e_2 - e_3, -e_2 - e_1)$

(2) $(v_1, v_2, v_3) = (e_1 - e_2, e_2 - e_3, e_1 + e_2 + e_3)$

(3) $(v_1, v_2, v_3) = (e_1 + e_2 + e_3, -e_1 - e_2 + e_3, e_1 - e_2)$

Moreover,

$$(p_1(S), p_2(S), c(S), I(S)) = \begin{cases} (1,1,1,-3) & \textit{in case } (1) \\ (0,2,2,-2) & \textit{in case } (2) \\ (0,1,2,-1) & \textit{in case } (3) \end{cases}$$

In particular $(a_1, a_2, a_3) \in \{(2,2,2), (2,2,3), (3,3,2)\}$.

Proof. See [18, pages 435–436]. □

Given integers $a_1, \ldots, a_n \geq 2$, we shall use the notation

$$[a_1, \ldots, a_n]^- := a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_n}}}$$

and for any integer $t \geq 0$ we shall write

$$(\ldots, 2^{[t]}, \ldots) := (\ldots, \overbrace{2, \ldots, 2}^{t}, \ldots)$$

Lemma 2.10. *Let $p > q \geq 1$ be coprime integers, and suppose that $\frac{p}{q} = [a_1, \ldots, a_n]^-$ and $\frac{p}{p-q} = [b_1, \ldots, b_m]^-$, with $a_1, \ldots, a_n \geq 2$ and $b_1, \ldots, b_m \geq 2$. Then,*

$$\sum_{i=1}^{n}(a_i - 3) + \sum_{j=1}^{m}(b_j - 3) = -2$$

Proof. See [18, page 437]. □

Definition 2.11. A subset $S = \{v_1, \ldots, v_n\} \subseteq \mathbb{D}^n$ such that

$$v_i.v_j = \begin{cases} -a_i \leq -2 & \text{if } i = j \\ 1 & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1 \end{cases} \tag{2.0.2}$$

for $i, j = 1, \ldots, n$ will be called standard.

Lemma 2.12. *Suppose that $n > 3$, and let $S_n = \{v_1, \dots, v_n\} \subseteq \mathbb{D}^n$ be a good subset such that $E_i^{S_n} = \{s\}$ for some $i, s \in \{1, \dots, n\}$. Then,*

1. *v_s is internal,*

2. *for some $1 \leq j \leq n$ we have $V_s = \{i, j\}$, $E_j^{S_n} = \{s-1, s, s+1\}$ and $|v_{s-1}.e_j| = |v_s.e_j| = |v_{s+1}.e_j| = 1$ and*

3. *for some $t \in \{s-1, s+1\}$ the set*

$$S_{n-1} := S_n \setminus \{v_s, v_t\} \cup \{\pi_{e_j}(v_t)\} \subset < e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n > \cong \mathbb{D}^{n-1}$$

is good, $|E_j^{S_{n-1}}| = 1$ and $I(S_{n-1}) = I(S_n) + 2 + v_s.v_s$.

Moreover, if S_n is standard then so is S_{n-1}.

Proof. Since $E_i^{S_n} = \{s\}$, if $|V_s| = 1$, then v_s is linked with no other elements of S_n. And this is impossible, because S_n is irreducible. So $|V_s| \geq 2$, if $|V_s| > 2$, the set obtained from S_n by replacing v_s with $\pi_{e_i}(v_s)$ would still satisfy (2.0.1), but it would consist of n independent vectors contained in the span of the $n-1$ vectors $e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n$, giving a contradiction. Therefore $|V_s| = 2$, i.e. $V_s = \{i.j\}$ for some $j \neq i$. If $|v_s.e_j| > 1$, then we get a contradiction as before by replacing v_s with $\pi_{e_i}(v_s)$. Hence, we conclude $|v_s.e_j| = 1$. Since $E_i^{S_n} = \{s\}$, and $|V_s| = 2$, if v_k is linked with v_s, then $v_k.e_j \neq 0$. If v_s is isolated, $v_s.v_{s+1}$, or $v_s.v_{s-1}$ is zero. Say for example $v_s.v_{s+1} = 0$, then $v_{s+1}.e_j = 0$, so we cannot find any sequence to link v_s to v_{s+1}, contradiction to the assumption of irreducibility.

We need to show that v_s is not final. By contradiction, suppose e.g. that $v_s.v_{s-1} = 0$ and $v_s.v_{s+1} = 1$ (the discussion in the case $v_s.v_{s-1} = 1$, $v_s.v_{s+1} = 0$ is similar). Let $l \geq 1$ be the largest natural number such that the set $\{v_s, \dots, v_{s+l}\}$ has connected intersection graph. If $a_{s+1}, \dots, a_{s+l} = 2$, since $|v_s.e_j| = 1$, and $v_s.v_{s+1} = 1$, so $|v_{s+1}.e_j| = 1$, and because $a_{s+1} = 2$, some entry of v_{s+1}, other than i-th entry is ± 1. Similarly we can conclude that each of $v_{s+2}, \dots, v_{s+l}$, also has two nonzero entry equal to ± 1. Because of (2.0.1), two successive elements must have one nonzero entry in common, and one nonzero entry in a position, different from the others, so $|\cup_{i=1}^{l} V_{s+i}| = l+2$. Since S_n is irreducible and $E_i^{S_n} = \{s\}$, this gives a contradiction. Therefore $a_{s+h} > 2$ for some $1 \leq h \leq l$. Choose h to be as small as possible. Then, it is easy to verify that for some $k \in \{1, \dots, n\}$, $V_{s+h} \cap V_{s+h-1} = \{e_k\}$ and $|v_{s+h}.e_k| = 1$. Since $|\cup_{i=0}^{h-1} V_{s+i}| = h+1$, it follows that by eliminating the vectors $v_s, v_{s+1}, \dots, v_{s+h-1}$ and replacing v_{s+h} with $\pi_{e_k}(v_{s+h})$ one obtains a set of $n-h$ independent vectors contained in the span of $n-(h+1)$ vectors. This contradiction shows that v_s must be internal, i.e. $v_{s-1}.v_s = v_s.v_{s+1} = 1$.

Now observe that, since $E_i^{S_n} = \{s\}$, we must have $j \in V_{s-1} \cap V_{s+1}$. If $a_{s-1} = a_{s+1} = 2$ then $v_{s-1}.v_{s+1} = 0$ implies $V_{s-1} = V_{s+1}$, and it is easy to verify that either $n = 3$ or S is reducible. If $a_{s-1}, a_{s+1} > 2$ then, since clearly $|v_{s-1}.e_j| = |v_{s+1}.e_j| = 1$, one gets a contradiction by eliminating v_s and replacing v_{s-1} and v_{s+1}, respectively, with $\pi_{e_j}(v_{s-1})$ and $\pi_{e_j}(v_{s+1})$. We conclude that either (i) $a_{s-1} > 2$ and $a_{s+1} = 2$ or (ii) $a_{s+1} > 2$ and $a_{s-1} = 2$. By symmetry, it suffices to consider the case $a_{s+1} > 2$ and $a_{s-1} = 2$. Since $|v_{s-1}.e_j| = |v_{s+1}.e_j| = 1$, we have $v_{s-1}.\pi_{e_j}(v_{s+1}) = 1$. Therefore the elements of the set

$$S_{n-1} := \{v_1, \dots, v_n\} \setminus \{v_s, v_{s+1}\} \cup \{\pi_{e_j}(v_{s+1})\}$$

satisfy (2.0.1). Moreover, the formula $I(S_{n-1}) = I(S_n) + 2 + v_s.v_s$ is straightforward to check. Since $E_i^{S_n} = \{s\}$ we have $E_j^{S_n} = \{s-1, s, s+1\}$, therefore the only vectors linked to v_s are v_{s-1} and v_{s+1}. Since v_{s-1} and $\pi_{e_j}(v_{s+1})$ are linked to each other, it follows easily that S_{n-1} is irreducible. The fact that if S_n is standard then so is S_{n-1} is evident from the definition of S_{n-1}. $\square$

Proposition 2.13. *Suppose that $n \geq 3$, and let $S = \{v_1, \dots, v_n\} \subseteq \mathbb{D}^n$ be a good subset such that $I(S) < 0$ and $p_1(S) > 0$. Then,*

1. *S is standard,*

2. *$|v_i.e_j| = 1$ for every $i, j = 1, \dots, n$ and*

3. If $n \geq 4$ there exist $h, t \in \{1, \ldots, n\}$ and $s \in \{1, n\}$ such that

$$E_h^S = \{s, t\}, a_s = 2 \text{ and } a_t > 2$$

Proof. See [18, page 440]. □

Definition 2.14. Let $S = \{v_1, \ldots, v_n\} \subseteq \mathbb{D}^n$ be a subset satisfying (2.0.1) and such that $|v_i.e_j| = 1$ for every $i, j = 1, \ldots, n$. Suppose that there exist $1 \leq h, s, t \leq n$ such that $E_h^S = \{s, t\}$ and $a_t > 2$. Then, we say that the subset $S' \subset < e_1, \ldots, e_{h-1}, e_{h+1}, \ldots, e_n > \cong \mathbb{D}^{n-1}$ defined by $S' = S \setminus \{v_s, v_t\} \cup \{\pi_{e_h}(v_t)\}$ is obtained from S by a contraction, and we write $S \searrow S'$. Moreover, we say that S is obtained from S' by an expansion, and we write $S' \nearrow S$.

Definition 2.15. Let $S' = \{v_1, \ldots, v_n\} \subseteq \mathbb{D}^n$, $n \geq 3$, be a good subset, and suppose there exists $1 < s < n$ such that $C' = \{v_{s-1}, v_s, v_{s+1}\} \subseteq S'$ gives a connected component of the intersection graph of S', with $v_{s-1}.v_{s-1} = v_{s+1}.v_{s+1} = -2$, $v_s.v_s < -2$ and $E_j^{S'} = \{s-1, s, s+1\}$ for some j. Let $S \subset \mathbb{D}^m$ be a subset of order $m \geq n$ obtained from S' by a sequence of expansions by final (-2)-vectors attached to C', so that $c(S) = c(S')$ and there is a natural $1-1$ correspondence between the sets of connected components of the intersection graphs of S and S'. Then, the connected component $C \subset S$ corresponding to $C' \subset S'$ is a bad component of S. The number of bad components of S will be denoted by $b(S)$.

Theorem 2.16. *Suppose that $n > 3$ and $S = \{v_1, \ldots, v_n\} \subseteq \mathbb{D}^n$ is a good subset with no bad components and such that $p_1(S) = 0$, $p_2(S) > 0$ and $I(S) < 0$. Then, there exist $i, s, t \in \{1, \ldots, n\}$ such that the set*

$$S' = S \setminus \{v_s, v_t\} \cup \{\pi_{e_i}(v_t)\} \subset < e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n > \cong \mathbb{D}^{n-1}$$

is good. Moreover, $I(S') \leq I(S)$, $b(S') \leq 1$ and if $b(S') = 1$ then $v_s.v_s < -2$ and $I(S') \leq I(S) - 1$.

Proof. See [18, pages 446–447]. □

Proposition 2.17. *Suppose that $n \geq 3$, and let $S = \{v_1, \ldots, v_n\} \subseteq \mathbb{D}^n$ be a good subset with no bad components such that $I(S) < 0$. Then, $|v_i.e_j| \leq 1$ for every $i, j = 1, \ldots, n$.*

Proof. See [18, pages 447–449]. □

Proposition 2.18. *Suppose that $n \geq 4$, and let $S = \{v_1, \ldots, v_n\} \subseteq \mathbb{D}^n$ be a good subset with no bad components such that $I(S) < 0$. Then, for some i, s, t the set*

$$S' = S \setminus \{v_s, v_t\} \cup \{\pi_{e_i}(v_t)\} \subset < e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n > \cong \mathbb{D}^{n-1}$$

is good and has no bad components. Moreover, either $(I(S'), c(S')) = (I(S), c(S))$ or

$$I(S') \leq I(S) - 1 \text{ and } c(S') \leq c(S) + 1$$

Proof. See [18, pages 449–451]. □

Corollary 2.19. *Suppose that $n \geq 3$, and let $S_n = \{v_1, \ldots, v_n\} \subseteq \mathbb{D}^n$ be a good subset with no bad components and such that $I(S_n) < 0$. Then $I(S_n) \in \{-1, -2, -3\}$, there exists a sequence of contractions $S_n \searrow S_{n-1} \searrow \cdots \searrow S_3$ such that, for each $k = 3, \ldots, n-1$ the set S_k is good, has no bad components and we have either $(I(S_k), c(S_k)) = (I(S_{k+1}), c(S_{k+1}))$ or*

$$I(S_k) \leq I(S_{k+1}) - 1 \text{ and } c(S_k) \leq c(S_{k+1}) + 1$$

Moreover:

1. *If $p_1(S_n) > 0$ then $I(S_n) = -3$, S_n is standard and one can choose the above sequence so that $I(S_k) = -3$ and S_k is standard for every $k = 1, \ldots, n-1$ and*

2. *If $I(S_n) + c(S_n) \leq 0$ then S_3 is given, up to applying an automorphism of $\mathbb{D}^3$, by either (1) or (2) in Lemma 2.9, if $I(S_n) + c(S_n) < 0$ then the former case occurs.*

Proof. See [18, pages 451–452]. □

Theorem 2.20. *Let $n \geq 3$, and let $S_n = \{v_1, \ldots, v_n\} \subseteq \mathbb{D}^n$ be a standard subset such that $I(S_n) < 0$. Then, $I(S_n) \in \{-1, -2, -3\}$ and there is a sequence of contractions $S_n \searrow \cdots \searrow S_3$ such that for every $k = 1, \ldots, n-1$ the set S_k is standard and $I(S_k) \leq I(S_{k+1})$.*

Proof. See [18, pages 454–456]. □

Now we identify the strings $(a_1, \ldots, a_n)$ corresponding to standard subsets $S \subset \mathbb{D}^n$ with $I(S) \in \{-1, -2, -3\}$. These results will be used to prove Theorem 2.5.

Lemma 2.21. *Let $n \geq 3$ and let $S_n = \{v_1, \ldots, v_n\} \subseteq \mathbb{D}^n$ be a standard subset such that $I(S_n) = -3$. Suppose $v_i.v_i = -a_i$ for $i = 1, \ldots, n$. Then, the string $(a_1, \ldots, a_n)$ is obtained from $(2, 2, 2)$ via a finite sequence of operations of the following types:*

$$\begin{array}{ll} (1) & (s_1, s_2, \ldots, s_{k-1}, s_k) \mapsto (s_1 + 1, s_2, \ldots, s_{k-1}, s_k, 2) \\ (2) & (s_1, s_2, \ldots, s_{k-1}, s_k) \mapsto (2, s_1, s_2, \ldots, s_{k-1}, s_k + 1) \end{array}$$

It follows that either $(a_1, \ldots, a_n)$ or $(a_n, \ldots, a_1)$ is of the form

$$\begin{gathered} (c_k + 1, 2^{[c_{k-1}-1]}, c_{k-2} + 2, \ldots, c_3 + 2, 2^{[c_2-1]}, \\ c_1 + 2, 2^{[c_1+1]}, c_2 + 2, \ldots, c_{k-1} + 2, 2^{[c_k-1]}) \\ (c_1 + 1, 2^{[c_1+1]}) \end{gathered}$$

for some integers $c_1, \ldots, c_k \geq 1$ and $k \geq 3$.

Proof. See [18, page 457]. □

Lemma 2.22. *Let $n \geq 4$, and let $S_n = \{v_1, \ldots, v_n\} \subseteq \mathbb{D}^n$ be a standard subset such that $I(S_n) = -2$. Suppose $v_i.v_i = -a_i$ for $i = 1, \ldots, n$. Then, either $(a_1, \ldots, a_n)$ or $(a_n, \ldots, a_1)$ is of one of the following types:*

$$\begin{array}{ll} (1) & (2^{[t]}, 3, 2 + s, 2 + t, 3, 2^{[s]}) \ , \ s, t \geq 0 \\ (2) & (2^{[t]}, 3 + s, 2, 2 + t, 3, 2^{[s]}) \ , \ s, t \geq 0 \end{array}$$

Proof. See [18, pages 457–459]. □

Lemma 2.23. *Let $n \geq 4$, and let $S_n = \{v_1, \ldots, v_n\} \subseteq \mathbb{D}^n$ be a standard subset such that $I(S_n) = -1$. Suppose $v_i.v_i = -a_i$ for $i = 1, \ldots, n$. Then, either $(a_1, \ldots, a_n)$ or $(a_n, \ldots, a_1)$ is of one of the following types:*

$$\begin{array}{ll} (1) & (t+2, s+2, 3, 2^{[t]}, 4, 2^{[s]}) \ , \ s,t \geq 0 \\ (2) & (t+2, 2, 3+s, 2^{[t]}, 4, 2^{[s]}) \ , \ s,t \geq 0 \\ (3) & (3+t, 2, 3+s, 3, 2^{[t]}, 3, 2^{[s]}) \ , \ s,t \geq 0 \end{array}$$

Proof. See [18, pages 459–460]. □

Lemma 2.24. *Let $p > q \geq 1$ be coprime integers, and suppose that $\frac{p}{q} = [a_1, \ldots, a_n]^-$, where either $(a_1, \ldots, a_n)$ or $(a_n, \ldots, a_1)$ is of the form*

$$\begin{gathered} (c_k + 1, 2^{[c_{k-1}-1]}, c_{k-2} + 2, \ldots, c_3 + 2, 2^{[c_2-1]}, \\ c_1 + 2, 2^{[c_1+1]}, c_2 + 2, \ldots, c_{k-1} + 2, 2^{[c_k-1]}) \\ (c_1 + 1, 2^{[c_1+1]}) \end{gathered}$$

for some integers $c_1, \ldots, c_k \geq 1$ and $k \geq 3$. Then, if p is odd $K(p,q)$ bounds an immersed ribbon disk, if p is even the 2-component link $K(p,q)$ bounds the image under a ribbon immersion of the disjoint union of a disk and a Möbius band.

Proof. See [18, pages 461–462]. □

Lemma 2.25. *Let $p > q > 0$ be coprime integers, and suppose that $\frac{p}{q}$ is equal to one of the following:*

$$\begin{array}{ll} (1) & [2^{[t]}, 3, 2+s, 2+t, 3, 2^{[s]}]^- \ , \ s,t \geq 0 \\ (2) & [2^{[t]}, 3+s, 2, 2+t, 3, 2^{[s]}]^- \ , \ s,t \geq 0 \end{array}$$

Then, if p is odd $K(p,q)$ bounds a ribbon disk, if p is even the 2-component link $K(p,q)$ bounds the image under a ribbon immersion the disjoint union of a disk and a Möbius band.

Proof. See [18, pages 463–464]. □

Lemma 2.26. *Let $p > q > 0$ be coprime integers, and suppose that $\frac{p}{q}$ is equal to one of the following:*

$$\begin{array}{ll} (1) & [t+2, s+2, 3, 2^{[t]}, 4, 2^{[s]}]^- \ , \ s,t \geq 0 \\ (2) & [t+2, 2, 3+s, 2^{[t]}, 4, 2^{[s]}]^- \ , \ s,t \geq 0 \\ (3) & [3+t, 2, 3+s, 3, 2^{[t]}, 3, 2^{[s]}]^- \ , \ s,t \geq 0 \end{array}$$

Then, if p is odd $K(p,q)$ bounds a ribbon disk, if p is even the 2-component link $K(p,q)$ bounds the image under a ribbon immersion of the disjoint union of a disk and a Möbius band.

Proof. See [18, pages 465–466]. □

Lemma 2.27. *Suppose that $a_i \geq 2$ for $i = 1, \ldots, n$, are integers and*

$$[a_1, \ldots, a_n]^- = \frac{m^2}{mk \pm 1}, \ (m,k) = 1, \ 0 < k < m$$

Then,

$$[2, a_1, \ldots, a_n, a_n{+}1]^- = \frac{(2m-k)^2}{(2m-k)m \pm 1}, \ [a_1{+}1, a_2, \ldots, a_n, 2]^- = \frac{(m+k)^2}{(m+k)k \pm 1}$$

Proof. See [18, page 467]. □

Lemma 2.28. *Let $n \geq 3$ and let $S_n = \{v_1, \ldots, v_n\} \subseteq \mathbb{D}^n$ be a standard subset such that $I(S_n) = -3$. Suppose $v_i.v_i = -a_i$ for $i = 1, \ldots, n$. Then, $[a_1, \ldots, a_n]^- = \frac{m^2}{mk+1}$, for some integers m, k with $0 < m < k$ and $(m,k) = 1$.*

Proof. See [18, page 467]. □

Lemma 2.29. *Let $n \geq 4$, and let $S_n = \{v_1, \ldots, v_n\} \subseteq \mathbb{D}^n$ be a standard subset such that $I(S_n) = -2$. Suppose $v_i.v_i = -a_i$ for $i = 1, \ldots, n$. Then, either $[a_1, \ldots, a_n]^-$ or $[a_n, \ldots, a_1]^-$ is of one of the following forms:*

1. $\frac{m^2}{m^2 - d(m-1)}$, *where d divides $2m+1$ or*

2. $\frac{m^2}{m^2-d(m-1)}$, *where d is odd and divides $m-1$.*

Proof. See [18, page 468]. □

Lemma 2.30. *Let $n \geq 4$, and let $S_n = \{v_1, \ldots, v_n\} \subseteq \mathbb{D}^n$ be a standard subset such that $I(S_n) = -1$. Suppose $v_i.v_i = -a_i$ for $i = 1, \ldots, n$. Then, either $[a_1, \ldots, a_n]^-$ or $[a_n, \ldots, a_1]^-$ is of one of the following forms:*

1. $\frac{m^2}{d(m+1)}$, *where d is odd and divides $m+1$,*
2. $\frac{m^2}{d(m+1)}$, *where d divides $2m-1$ or*
3. $\frac{m^2}{m^2-d(m+1)}$, *where d is odd and divides $m+1$.*

Proof. See [18, pages 468–469]. □

2.1 Proof of the Main Theorem

Now we can prove Theorem 2.5. We first show that (2) implies (1). Let us assume that (2) holds. Let $\tilde{\Sigma} \subset B^4$ be a smoothly embedded surface obtained by pushing the interior of Σ inside the 4-ball. It is easy to check that (regardless of the parity of p) the inclusion $S^3 \setminus \partial\tilde{\Sigma} \subset B^4 \setminus \tilde{\Sigma}$ induces a surjective homomorphism

$$\phi : H_1(S^3 \setminus \partial\tilde{\Sigma}; \mathbb{Z}) \longrightarrow H_1(B^4 \setminus \tilde{\Sigma}; \mathbb{Z})$$

such that the homomorphism $H_1(S^3 \setminus \partial\tilde{\Sigma}; \mathbb{Z}) \longrightarrow \mathbb{Z}/2\mathbb{Z}$ defining the 2-fold cover $L(p, q) \longrightarrow S^3$ branched along $\partial\tilde{\Sigma} = K(p, q)$ factors through $H_1(B^4 \setminus \tilde{\Sigma}; \mathbb{Z})$ via ϕ. Therefore, the cover $L(p, q) \longrightarrow S^3$ extends to a 2-fold cover $W \longrightarrow B^4$ branched along $\tilde{\Sigma}$. We may assume that the distance function from the origin $B^4 \longrightarrow [0, 1]$ restricted to $\tilde{\Sigma}$ is a proper Morse function with only index-0 and index-1 critical points. This implies that W has a handlebody decomposition with only 0-, 1- and 2-handles (see e.g. [3, pages 30–31]). Therefore, from

$$b_0(W) - b_1(W) + b_2(W) = \chi(W) = 2\chi(B^4) - \chi(\tilde{\Sigma}) = 1$$

we deduce $b_1(W) = b_2(W)$. On the other hand, since $b_1(\partial W) = 0$ and $H_1(W, \partial W; \mathbb{Q}) \cong H^3(W; \mathbb{Q}) = 0$ the homology exact sequence of the pair $(W, \partial W)$ gives $b_1(W) = 0$, so it follows that $H_*(W; \mathbb{Q}) \cong H_*(B^4; \mathbb{Q})$, and (1) holds.

Now we show that (1) implies (3). Assume that Part (1) of the statement holds. It is a well-known fact that if $\frac{p}{q} = [a_1, \ldots, a_n]^-$ the lens space $L(p, q)$ smoothly bounds the 4-dimensional plumbing $P(p, q)$. The intersection form of $P(p, q)$ is negative definite. Hence, since $L(p, q) \cong L(p, p - q)$, if $L(p, q)$ smoothly bounds a rational homology 4-ball $W(p, q)$ we can construct the smooth, negative definite 4-manifolds

$$X(p, q) = P(p, q) \cup_\partial (-W(p, q)), \ X(p, p - q) = P(p, p - q) \cup_\partial W(p, q)$$

By Donaldson's theorem on the intersection form of definite 4-manifolds [5], the intersection forms of $X(p, q)$ and $X(p, p - q)$ are both standard diagonal. Hence, suppose that the intersection lattice of $X(p, q)$ is isomorphic to $\mathbb{D}^n$ and the intersection lattice of $X(p, p - q)$ is isomorphic to $\mathbb{D}^{n'}$. Clearly, the intersection lattices $H_2(P(p, q); \mathbb{Z}) \cong \mathbb{Z}^n$ and $H_2(P(p, p - q); \mathbb{Z}) \cong \mathbb{Z}^{n'}$ have bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_{n'}\}$ which satisfy (2.0.2). Therefore, via the embeddings $P(p, q) \subset X(p, q)$ and $P(p, p - q) \subset X(p, p - q)$ we can view the above bases as standard subsets $S \subset \mathbb{D}^n$ and $S' \subset \mathbb{D}^{n'}$ with associated strings $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_{n'})$, where $[b_1, \ldots, b_{n'}]^- = \frac{p}{p-q}$. We may assume without loss of generality that $I(S) < 0$. Then, by Theorem 2.20 and Lemma 2.9, Lemma 2.28, Lemma 2.29 and Lemma 2.30 it follows that (3) holds.

Finally, we show that (3) implies (2). Suppose that (3) holds, i.e. $\frac{p}{q} \in \mathcal{R}$. Then, since applying finitely many times the functions f and g of Definition 2.1 amounts to changing $K(p,q)$ by an isotopy or a reflection, we may assume that $p = m^2$ and q is of one of the three types given in Definition 2.1. We consider various cases separately.

First Case: ($q = mk \pm 1$, with $m > k > 0$ and $(m,k) = 1$) In view of Lemma 2.21 and Lemma 2.24, it suffices to show that the string of coefficients of the continued fraction expansion of $\frac{p}{q}$ is obtained from $(2,2,2)$ via a finite sequence of operations as in Lemma 2.21. Since $m^2 - (mk \mp 1) = m(m-k) \pm 1$ and either $m \geq 2k$ or $m \geq 2(m-k)$, up to replacing k with $m-k$ (and $K(p,q)$ with its mirror image $K(p,p-q)$) we may assume $m \geq 2k$. If $m = 2k$, since $(m,k) = 1$ we must have $m = 2$, $k = 1$ and $\frac{p}{q} = [2,2,2]^-$. If $m > 2k$, arguing by induction on m we may assume

$$\frac{(m-k)^2}{(m-k)k \pm 1} = [a_1, a_2, \ldots, a_n]^-$$

where $(a_1, a_2, \ldots, a_n)$ is obtained from $(2,2,2)$ as described above. But in view of Lemma 2.27 we have

$$\frac{m^2}{mk \pm 1} = [a_1 + 1, a_2, \ldots, a_n, 2]^-$$

so we are done.

Second Case: ($q = d(m-1)$, where $d > 1$ divides $2m+1$) It suffices to show that (2) holds for $K(p, p-q)$. Since $d(m-1) < m^2$, we have $2m+1 > d > 1$, and d must be odd because it divides $2m+1$. Therefore we can write $d = 2s+3$ for some $s \geq 0$ and $2m+1 = d(2t+3)$ for some $t \geq 0$. Then $m = 2st + 3s + 3t + 4$, and as in the Lemma 2.29

$$\frac{m^2}{m^2 - d(m-1)} = [2^{[t]}, 3, s+2, t+2, 3, 2^{[s]}]^-$$

Therefore (2) holds by Lemma 2.25 (1).

Third Case: ($q = d(m+1)$, where $d > 1$ divides $2m-1$) Arguing as in the previous case, we can write $d = 2s+3$ and $2m-1 = d(2t+3)$ for some $s, t \geq 0$. Then, $m = 2st + 3s + 3t + 5$ and

$$\frac{m^2}{d(m+1)} = [t+2, 2, s+3, 2^{[t]}, 4, 2^{[s]}]^-$$

which implies (2) by Lemma 2.26 (2).

Fourth Case: ($q = d(m+1)$, where $d > 1$ is odd and divides $m+1$) Since $d(m+1) < m^2$ we have $m+1 > d > 1$, therefore we can write $d = 2s+3$ and $m+1 = d(t+2)$ for some $s, t \geq 0$. Then

$$\frac{m^2}{d(m+1)} = [t+2, s+2, 3, 2^{[t]}, 4, 2^{[s]}]^-$$

and (2) holds by Lemma 2.26 (1).

Fifth Case: ($q = d(m-1)$, where $d > 1$ is odd and divides $m-1$) As before, it suffices to prove that (2) holds for $K(p, p-q)$. We can write $d = 2s+3$ and $m-1 = d(t+1)$ for some $s, t \geq 0$. Then

$$\frac{m^2}{m^2 - d(m-1)} = [2^{[t]}, s+3, 2, t+2, 3, 2^{[s]}]^-$$

and (2) holds by Lemma 2.25 (2). This concludes the proof.

Chapter 3

Slice-Ribbon Conjecture for 3-stranded Pretzel Knots

In this section we summarize the paper by Joshua Greene and Stanislav Jabuka [9]. They use Lisca's approach [18] on 3-stranded pretzel knots. Thus, let $P(p,q,r)$ denote the 3-stranded pretzel knot with p, q and r half-twists in its strands. They further assume that p, q, r are odd and that $|p|,|q|,|r| \geq 3$. In the case when any of p, q or r equals ± 1, the corresponding pretzel knot $P(p,q,r)$ is a 2-bridge knot, and so Lisca's results [18] apply. The main result of their article is the next theorem:

Theorem 3.1. *Consider the pretzel knot $P(p,q,r)$ with p,q,r odd and with $|p|,|q|,|r| \geq 3$. Then $P(p,q,r)$ is slice if and only if either*

$$p+q=0 \quad \text{or} \quad p+r=0 \quad \text{or} \quad q+r=0$$

and in each of these cases $P(p,q,r)$ is a ribbon knot. All other pretzel knots $P(p,q,r)$ are of infinite order in the smooth knot concordance group.

Corollary 3.2. *The slice-ribbon conjecture is true for 3-stranded pretzel knots $P(p,q,r)$ with p,q,r odd.*

Proof. If $min\{|p|, |q|, |r|\} \geq 3$, the corollary follows from theorem 3.1. If instead $min\{|p|, |q|, |r|\} = 1$, the resulting $P(p, q, r)$ is a 2-bridge knot, in which case the corollary follows from the work of Lisca [18]. $\square$

Remark 3.3. When $min\{|p|, |q|, |r|\} = 1$ one direction of Theorem 3.1 still holds. Namely, if either of $p + q$, $q + r$ or $p + r$ is zero, then $P(p, q, r)$ is still slice. However the other direction of Theorem 3.1 no longer holds: for example, the knot $P(23, -3, 1)$ is ribbon according to [18].

Corollary 3.4. *Let $P(p, q, r)$ be a pretzel knot with p, q, r odd and with Alexander polynomial $\Delta_{P(p,q,r)}(t) = 1$. Then $P(p, q, r)$ is slice if and only if $P(p, q, r)$ is the unknot.*

Proof. By [17, pages 56–57], we know that

$$\Delta_{P(p,q,r)}(t) \doteq \frac{1}{4}\big((pq + qr + rp)(t^2 - 2t + 1) + t^2 + 2t + 1\big)$$

So the Alexander polynomial of $P(p, q, r)$ is trivial precisely when $pq + qr + rp = -1$. If $\min\{|p|, |q|, |r|\} = 1$, say $r = 1$ for concreteness, this equation implies that $p = -1$ or $q = -1$, and thus $P(p, q, r)$ is the unknot. If $\min\{|p|, |q|, |r|\} \geq 3$ we can use theorem 3.1. Without loss of generality, suppose that $q + r = 0$. Then the equation $pq + qr + pr = -1$ reduces to $qr = -1$, which is impossible for $|q|, |r| \geq 3$. $\square$

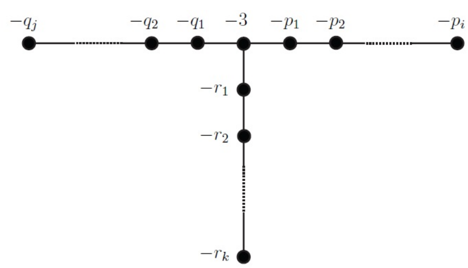
Example 3.5. If $K = P(3, -7, -5)$, then $\Delta_K(t) \doteq 1$, so by Freedman's result, K is topologically slice. This knot is not the unknot, so by Corollary 3.4 is not smoothly slice.

Let G be a finite weighted graph. Let $w(v)$ denote the weight of a vertex v of G. Order the vertices of G in an arbitrary manner and let v_i denote the i-th vertex. We denote the incidence matrix $A = A_G$ associated to G with respect to such an ordering. Assume from now on that G is a tree or a forest. To such a weighted graph G we shall associate a smooth 4-manifold with boundary $W(G)$ by plumbing together 2-disk bundles over S^2 according to instructions read off from G. Namely, for each vertex v of G we pick a disk bundle $D(v) \to S^2$ with Chern class $c_1(D(v)) = w(v)$. Given two such disk-bundles $D(v_1)$ and $D(v_2)$, we plumb them together if and only if the vertices v_1 and v_2 are connected by an edge of G. The intersection form of the resulting 4-manifold $W(G)$, expressed in terms of the basis of the spheres S^2 used in its construction, is the incidence matrix of G.

Given a pretzel knot $P(p,q,r) \subset S^3$ with $\min\{|p|,|q|,|r|\} \geq 3$, let $Y(p,q,r)$ denote the 3-manifold obtained as the 2-fold cover of S^3 branched along $P(p,q,r)$. These 3-manifolds are Seifert fibered spaces with three singular fibers and their plumbing descriptions are obtained according to the following recipe. Find continued fraction expansions of $\frac{p}{p-1}$, $\frac{q}{q-1}$ and $\frac{r}{r-1}$:

$$\frac{p}{p-1} = [p_1, \ldots, p_i]^- \ , \ \frac{q}{q-1} = [q_1, \ldots, q_j]^- \ , \ \frac{r}{r-1} = [r_1, \ldots, r_k]^-$$

Let $G = G(p,q,r)$ be the weighted graph as figure below, then
$Y(p,q,r) = \partial W(G(p,q,r))$.



Let X be a closed, smooth, oriented 4-manifold. $H_2(X;\mathbb{Z})/Tors$ is isomorphic to the free Abelian group $\mathbb{Z}^{b_2(X)}$. By Donaldson's Theorem [5], if the intersection form

$$Q_X : (H_2(X;\mathbb{Z})/Tors) \otimes (H_2(X;\mathbb{Z})/Tors) \longrightarrow \mathbb{Z}$$

is negative definite, then Q_X is diagonalizable. We refer to the pair $(\mathbb{Z}^{b_2(X)}, Q_X)$ as a lattice. By saying that Q_X is diagonalizable, we mean that the lattice

$(\mathbb{Z}^{b_2(X)}, Q_X)$ is isomorphic to the standard negative definite lattice $(\mathbb{Z}^{b_2(X)}, -Id)$ of the same dimension.

Sliceness obstruction: Let $K \subset S^3$ be a knot and let Y be the 2-fold cover of S^3 branched along K. Let W be any smooth negative definite 4-manifold with $\partial W = Y$. If K is slice, then the lattice $(\mathbb{Z}^{b_2(W)}, Q_W)$ must embed in the standard negative definite intersection lattice of equal rank, that is, there must exist a monomorphism $\phi : \mathbb{Z}^{b_2(W)} \to \mathbb{Z}^{b_2(W)}$ such that $Q_W(\alpha, \beta) = -Id(\phi(\alpha), \phi(\beta))$ for any $\alpha, \beta \in \mathbb{Z}^{b_2(W)} \cong H_2(W; \mathbb{Z})/Tors$.

Obstruction from Heegaard Floer homology: Assume that Y is a rational homology three-sphere, equipped with a spin^c structure $\mathfrak{t}$. We define a numerical invariant $d(Y, \mathfrak{t})$ for Y, which is the minimal degree of any non-torsion class in $HF^+(Y, \mathfrak{t})$ coming from $HF^\infty(Y, \mathfrak{t})$, see [20].

Let $K \subset S^3$ be a knot and let Y_K denote the 2-fold cover of S^3 branched over K. If K is slice, then the order of $H^2(Y_K; \mathbb{Z})$ is a square and there exists a subgroup $V \subset H^2(Y_K; \mathbb{Z})$ of square root order such that $d(Y_K, \mathfrak{s}) = 0$ for each $\mathfrak{s} \in V$. The subgroup V is the image of the restriction induced map $H^2(W_K; \mathbb{Z}) \to H^2(Y_K; \mathbb{Z})$, where W_K is the rational homology ball obtained by a 2-fold cover of D^4 branched over the slicing disk for K.

Now we can sketch the proof of main theorem of this section, i.e. Theorem 3.1. By the following symmetries of pretzel knots

$$P(p,q,r) = P(r,p,q), \quad P(p,q,r) = P(r,q,p) \quad \overline{P(p,q,r)} = P(-p,-q,-r)$$

we can assume that p and r are positive. When $q > 0$ the signature of $P(p,q,r)$ is nonzero, and so $P(p,q,r)$ cannot be slice. Thus we turn to the case of $p, r \geq 3$ and $q \leq -3$.

Proposition 3.6. *Consider the pretzel knot $K = P(p,q,r)$ with $p, r \geq 3$ and $q \leq -3$ and all three of p, q, r odd. If K is slice then there exists an integer $\lambda \in \mathbb{Z}$ such that*

$$-q = p\lambda^2 + r(\lambda+1)^2$$

Proof. See [9, pages 9–12]. □

For now let $K \subset S^3$ be any slice knot and let Y_K be its 2-fold branched cover. Assume that Y_K bounds a negative definite plumbing X associated to a weighted graph G (which we assume is a forest) with n vertices. Let W_K be the rational homology 4-ball obtained by a 2-fold cover of D^4 branched over the slicing disk for K.

Let $\tilde{f}_1, \ldots, \tilde{f}_n \in H_2(X; \mathbb{Z})$ be the basis represented by the n 2-handles of X and let $f_1, \ldots, f_n \in H^2(X, Y_K; \mathbb{Z})$ be the basis of their Poincaré duals. Furthermore, let $e_1, \ldots, e_n \in H^2(X; \mathbb{Z})$ be the basis of the Hom-duals of $\tilde{f}_i : e_i(\tilde{f}_j) = \delta_{ij}$. With respect to these choices of bases, the restriction induced map $\gamma : H^2(X, Y_K; \mathbb{Z}) \to H^2(X; \mathbb{Z})$ is represented by the matrix G. The long exact sequence of the pair (X, Y_K)

$$0 \to H^2(X, Y_K; \mathbb{Z}) \xrightarrow{G} H^2(X; \mathbb{Z}) \xrightarrow{\delta} H^2(Y_K; \mathbb{Z}) \to 0$$

allows us to identify $H^2(Y_K; \mathbb{Z})$ with the cokernel of G (via δ).

Theorem 3.7. *With $K, Y_K, W_K, X, G, \{\tilde{f}_1, \ldots, \tilde{f}_n\}, \{f_1, \ldots, f_n\}$, and $\{e_1, \ldots, e_n\}$ as above, there exists a map $H^2(X \cup_{Y_K} W_K; \mathbb{Z}) \to H^2(X; \mathbb{Z})$ whose matrix representative A (with the given choices of bases) leads to a factorization $G = -AA^t$ with the additional property that (after identifying $H^2(Y_K; \mathbb{Z})$ with coker G) the image $H^2(W_K; \mathbb{Z}) \to H^2(Y_K; \mathbb{Z})$ is isomorphic to $(imA)/(imG)$ (via δ).*

Proof. See [9, pages 12–13]. □

We now return to the case of $K = P(p, q, r)$ and G the weighted graph. In this case we denote Y_K by $Y(p, q, r)$. The matrix A whose existence is asserted by theorem 3.7 is determined.

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -1 \\ \lambda & \lambda & \lambda & \lambda & \dots & \lambda+1 & \lambda+1 & \lambda+1 \end{bmatrix}$$

The i-th row of A, for $i = 1, \dots, p+r-1$ has a 1 in its i-th column, a -1 in its $(i+1)$-st column and zeros elsewhere. The $(p+r)$-th row of A has λ's in its first p columns and $(\lambda+1)$'s in its remaining r columns. An explicit calculation shows that indeed the factorization $G = -AA^t$ holds. Theorem 3.7 also tells us that the subgroup $V \subset H^2(Y(p,q,r);\mathbb{Z})$ is isomorphic to $(\mathrm{im}A)/(\mathrm{im}G)$ via the map $\delta : H^2(X;\mathbb{Z}) \to H^2(Y(p,q,r);\mathbb{Z})$. This makes it easy to find an upper bound on the number of vanishing correction terms $d(Y(p,q,r),\mathfrak{s})$ for $\mathfrak{s} \in V$. Towards this goal, pick $v = Ax \in \mathrm{im}A$. The term $v^tG^{-1}v$ simplifies to

$$v^tG^{-1}v = -x^tA^t(AA^t)^{-1}Ax = -x^tA^t(A^t)^{-1}A^{-1}Ax = -|x|^2$$

showing that

$$d(Y(p,q,r),\mathfrak{s}) = \max_{Ax \in Char_{\mathfrak{s}}(G)} \frac{p+r-|x|^2}{4} \tag{3.0.1}$$

The requirement that $v = Ax$ be characteristic translates into a condition on x itself:

$$v = Ax \text{ is characteristic } \Leftrightarrow v_i \equiv G_{ii} \pmod 2, \forall\, i$$

$$\begin{aligned} &\Leftrightarrow \textstyle\sum_j A_{ij}x_j \equiv \sum_j A_{ij}^2 \pmod 2, \forall\, i \\ &\Leftrightarrow \textstyle\sum_j A_{ij}x_j \equiv \sum_j A_{ij} \pmod 2, \forall\, i \end{aligned} \tag{3.0.2}$$

Since $\det(A)$ is odd (up to sign, it is the square-root of the knot determinant), the matrix A is invertible $\pmod 2$, and so the vector $x \pmod 2$ is uniquely determined by this condition. On the other hand, taking $x_i \equiv 1 \pmod 2 \ \forall\, i$, clearly satisfies this equation, so it must be the unique solution. Combining (3.0.1) with (3.0.2) we see that the only way for $d(Y(p,q,r),\mathfrak{s})$ to be zero for a given $\mathfrak{s} \in V$ is that the corresponding $x = (x_1, \dots, x_{p+r})$ have coordinates $x_i \in \{\pm 1\}$ for all $i = 1, \dots, p+r$. While there are 2^{p+r} such vectors $v = Ax$ in im$A$, there are significantly fewer equivalence classes of these vectors in $V = (\mathrm{im}A)/(\mathrm{im}G)$. To see this, define $l : \mathbb{Z}^{p+r} \to \mathbb{Z}$ by $l(x) = x_1 + \cdots + x_{p+r}$. Observe that the first $p+r-1$ columns of A^t generate the kernel of l, showing that any two vectors $v = Ax$ and $v' = Ax'$ with $x' = x + y$ and $y \in \mathrm{Ker}l \subset \mathrm{im}A^t$ belong to the same equivalence class in V. Finally, the functional l, when restricted to the set $\{x \in \mathbb{Z}^{p+r} | x_i \in \{\pm 1\}\}$, only takes on $p+r+1$ distinct values, showing that there can be at most that many characteristic covectors in $v \in V$ with vanishing correction terms. But according to obstruction from Heegaard Floer homology and the result of proposition 3.6, there need to be at least $|\mathrm{Coker}(A^t)| = |\det(A^t)| = |p\lambda + r(\lambda+1)|$ vanishing correction terms. Now we can establish that

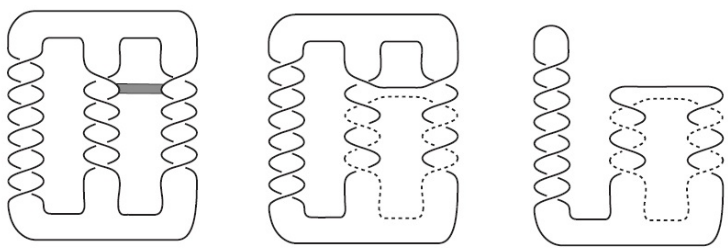
$$|p\lambda + r(\lambda+1)| > p + r + 1$$

when $p, r \geq 3$, unless $\lambda = 0, -1$. We have then proved

Proposition 3.8. *Let $P(p,q,r)$ be a pretzel knot with $|p|, |q|, |r| \geq 3$ and all three p, q, r odd. If $P(p,q,r)$ is slice then either $p+q=0$ or $q+r=0$ or $p+r=0$.*

Proposition 3.9. *Any pretzel knot $P(p,q,r)$ with either $p+q=0$ or $q+r=0$ or $p+r=0$ is ribbon.*

Proof. Without loss of generality (by the symmetries of pretzel knots) we can assume that $q+r=0$. If a knot K can be turned into an $(m+1)$-component unlink by attaching m bands to it for some $m \geq 1$ then K is ribbon. A pretzel knot $P(p, q, r)$ with $q + r = 0$ can easily be isotoped to a 2-component unlink after attaching a single band. See [9, pages 14–15]. $\square$



These two propositions prove the first part of Theorem 3.1. The rest of the theorem, is proved at the end of the paper [9].

Chapter 4

Further Results

4.1 Lecuona Results

Ana G. Lecuona has proved the slice-ribbon conjecture for a large family of Montesinos knots [14], and has found a necessary, and in some cases sufficient, condition for sliceness inside the family of pretzel knots $P(p_1, \ldots, p_n)$, with one p_i even [15]. In this section, we briefly discuss about her results.

Montesinos links are defined as the boundary of 2-dimensional plumbings with star-shaped plumbing graphs. A star-shaped graph is a connected tree with a distinguished vertex v_0 (called the central vertex) such that the degree of any vertex other than the central one is ≤ 2. In a weighted star-shaped graph Γ each vertex represents a twisted band, that is a D^1-bundle over S^1, embedded in S^3, with the number of half-twists given by the weight of the vertex. Bands are plumbed together precisely when the corresponding vertices are adjacent. The result of this plumbing construction is a surface $B_\Gamma \subset S^3$ whose boundary ML_Γ is, by definition, a Montesinos link. For another definition see [1, chapter 12].

Since $S^3 = \partial D^4$, we can push the interior of B_Γ into the interior of D^4. It follows that the double covering of D^4 branched over B_Γ is the 4-dimensional plumbing M_Γ, obtained by plumbing D^2-bundles over S^2 according to the graph Γ, which defined the Montesinos link. The boundary $Y_\Gamma := \partial M_\Gamma$ is a Seifert space (see [21] for a proof) with as many singular fibers as legs of the graph Γ. A leg of a star-shaped graph is any connected component of the graph obtained by removing the central vertex. The involution u that defines the covering $M_\Gamma \to D^4 \simeq M_\Gamma/u$, turns the Seifert space Y_Γ into the double covering of S^3 branched along the Montesinos link ML_Γ. Restricting our attention to three-legged star-shaped graphs Γ, it is well known [1, chapter 12] that the Seifert space Y_Γ is the double covering of S^3 branched along exactly one Montesinos link (up to link isotopy). In [14], she studies the family $\mathcal{P}$ of all three-legged connected plumbing graphs Γ such that:

1. $I(\Gamma) := \sum_{i=0}^{n}(a_i - 3) < -1$, where by $-a_1, \ldots, -a_n$ she denotes the weights of the vertices of Γ, and

2. the central vertex has weight less or equal to -3 and every non central vertex has weight less or equal to -2.

Theorem 4.1. *Consider $\Gamma \in \mathcal{P}$. The Seifert space Y_Γ is the boundary of a rational homology ball W if and only if there exist a surface Σ and a ribbon immersion $\Sigma \looparrowright S^3$ such that $\partial\Sigma = ML_\Gamma$ and $\chi(\Sigma) = 1$.*

Corollary 4.2. *The slice-ribbon conjecture holds true for all Montesinos knots ML_Γ with $\Gamma \in \mathcal{P}$.*

Proof. Let $\Gamma \in \mathcal{P}$ be such that the knot $ML_\Gamma \subset S^3$ is slice. Let $D^2 \hookrightarrow D^4$ be a smooth slicing disc for ML_Γ and W the 2-fold cover of D^4 branched along D^2. It is well known [12, Lemma 17.2] that W is a rational homology ball and that $\partial W = Y_\Gamma$. It follows immediately from Theorem 4.1 that the knot ML_Γ is ribbon. $\square$

Given nonzero integers $p_1, \ldots, p_n$ the pretzel link $P(p_1, \ldots, p_n)$ is obtained by taking n pairs of parallel strands, introducing p_i half twists on the i-th pair, with the convention $p_i > 0$ for right-hand crossings and $p_i < 0$ for left-hand crossings, and connecting the strands with n pairs of bridges. If more than one of the p_i is even or if n is even and none of the p_i is even then $P(p_1, \ldots, p_n)$ is a link. In all other cases it is a knot. Inside the family of pretzel knots $P(p_1, \ldots, p_n)$ we limit our considerations to those with one even parameter and moreover we fix $n \geq 3$ and $|p_i| > 1$ for all i. Note that if $n \leq 2$ or if $n = 3$ and one of the p_i satisfies $p_i = \pm 1$, then the pretzel knot is a 2-bridge knot.

Consider the following set:

$$\mathcal{E} = \{a, -a-2, -\frac{(a+1)^2}{2}, q_1, -q_1, \ldots, q_m, -q_m\}$$

where $m \geq 0$, $a, |q_i| \geq 3$ odd and $a \equiv 1, 11, 37, 47, 49, 59 \pmod{60}$. The main result of [15] is the following.

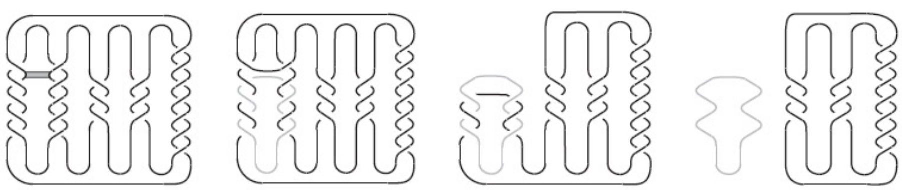
Theorem 4.3. *Let $K = P(p_1, \ldots, p_n)$ be a slice pretzel knot with one even parameter and such that $\{p_1, \ldots, p_n\} \nsubseteq \mathcal{E}$. Then, the n-tuple of integers $(p_1, \ldots, p_n)$ can be reordered so that it has the form*

1. *$(q_1, -q_1 \pm 1, q_2, -q_2, \ldots, q_{\frac{n}{2}}, -q_{\frac{n}{2}})$ if n is even,*
2. *$(q_0, q_1, -q_1, \ldots, q_{\frac{n-1}{2}}, -q_{\frac{n-1}{2}})$ if n is odd.*

Proposition 4.4. *The slice-ribbon conjecture holds true for pretzel knots of the form $P(p_1, p_2, p_3)$ where $p_1, p_2, p_3 \in \mathbb{Z}$ and $\{p_1, p_2, p_3\} \nsubseteq \mathcal{E}$.*

Proposition 4.5 (Ribbon Algorithm)**.** *Let $K = P(p_1, \ldots, p_n)$ be a pretzel knot and let $p_{n+1} := p_1$. While for some $j \in \{1, \ldots, n\}$ it holds $p_j = -p_{j+1}$, we reduce the number of parameters to $n-2$ and repeat with the the knot $P(p_1, \ldots, p_{j-1}, p_{j+2}, \ldots, p_n)$. If at the end of the sequence of reductions we are left with a pretzel knot with exactly one parameter or with two parameters a and b satisfying $a = -b - 1$, then K is ribbon.*

Proof. On a pretzel knot K, whenever there are two adjacent strands p_1 and p_2 with the same number of crossings but of opposite signs, we can perform the ribbon move shown in figure below, which simplifies the pretzel knot yielding the disjoint union of an unknot and a new pretzel knot K'. The knot K' is equal to K without p_1 and p_2. Therefore, if n is odd and after the sequence of reductions the set of parameters defining K consists of only one integer, we have that after performing $\frac{n-1}{2}$ ribbon moves on K we obtain the disjoint union of $\frac{n+1}{2}$ unknots. Thus, K is ribbon. On the other hand, if n is even and after the sequence of reductions the set of parameters defining K consists of exactly two integers a and b satisfying $b = -a - 1$, then after performing $\frac{n}{2} - 1$ ribbon moves on K, we obtain, since $P(a, -a-1)$ is the unknot, the disjoint union of $\frac{n}{2}$ unknots. Thus again, K is ribbon. □



Corollary 4.6. *Let $K = P(p_1, \ldots, p_n)$ be a pretzel knot satisfying the assumptions of Theorem 4.3. Then the above Ribbon Algorithm shows that for certain orderings of the parameters, K is slice.*

4.2 Potential Counterexamples for the Slice-Ribbon Conjecture

We briefly discuss about a source of generating potential counterexamples for the slice-ribbon conjecture. For details, see [8].

Theorem 4.7 (Property R). *If 0-framed surgery on a knot $K \subset S^3$ yields $S^1 \times S^2$ then K is the unknot.*

This theorem was proved by David Gabai [7].

Problem 1.82 in Kirby's problem list [13] conjectures a generalization to links: If surgery on an n-component link L yields the connected sum $\#_n S^1 \times S^2$, then L becomes the unlink after suitable handle slides.

In [8] by studying a family of knots that might be counterexamples to the generalized property R conjecture, another family of links denoted by $L_{n,k}$ has been introduced, which are a generalization of the previous family. And they show that $L_{n,k}$ is slice [8, pages 23–25]. The authors do not know whether $L_{n,k}$ is ribbon except in the special cases $n = 0, 1$ or $k = 0$ or $(n, k) = (2, 1)$. This method appears to be the only currently known source of potential counterexamples to the slice-ribbon conjecture (for knots or links).

4.3 Some Open Questions

Casson and Gordon [2] observed that if $K(p, q)$ is a smoothly slice knot then p is a perfect square. Moreover, they proved that if the 2-bridge knot $K(m^2, q)$ is ribbon then

$$\frac{2}{m^2}\sum_{s=1}^{m^2-1}\cot\left(\frac{\pi s}{m^2}\right)\cot\left(\frac{\pi q s}{m^2}\right)\sin^2\left(\frac{\pi r s}{m}\right) = \pm 1, \quad r = 1, \ldots, m-1 \quad (4.3.1)$$

Casson and Gordon [2, page 188] used (4.3.1) to show that if a 2-bridge knot $K(m^2, q)$ is ribbon and $m \leq 105$ then $\frac{m^2}{q}$ belongs to $\mathcal{R}$.

Question 4.8. If (4.3.1) implies that the knot $K(m^2, q)$ is smoothly slice?

Question 4.9. Is there any element of order 4 in $\mathcal{C}$?

In [10], the following theorem has been proved:

Theorem 4.10. *All 2-bridge knots of 12 or fewer crossings have smooth concordance order $1, 2$ or ∞.*

The results of [18] identify all the slice 2-bridge knots, that is all 2-bridge knots of concordance order 1. We know that the signature of a knot is additive under the connected sum, so every knot with nonzero signature has infinite order in the smooth concordance group (cf. Theorem 1.24). All the 2-bridge knots are invertible. So if a 2-bridge knot $K(p,q)$ is equivalent to its mirror image, which happens for instance if $p|(q^2+1)$, and if it is not slice, then it must have concordance order 2.

Question 4.11. What can we say about the concordance order of 2-bridge knots with more than 12 crossings?

Conjecture 4.12. *If* $\{p_1,\ldots,p_n\} \subset \mathcal{E}$ *(see [15]), then the pretzel knot* $P(p_1,\ldots,p_n)$ *is not slice.*

Conjecture 4.13. *The pretzel knots* $P(p_1,\ldots,p_n)$ *with* $|p_i| > 1$ *for all* i*, that are ribbon, are precisely those detected by the algorithm in Proposition 4.5.*

Question 4.14. For $n \geq 2$, $k \neq 0$ and $(n,k) \neq (2,1)$, is $L_{n,k}$ (see [8]) a ribbon link? Are the slice knots made by band-summing its components always ribbon? Are they ever ribbon?

And finally, the old and famous slice-ribbon conjecture (Conjecture 1.13), is it true in general, or we can find a counterexample for it?

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