Equivariant Gauge Theory and Four-Manifolds

by

Nima Anvari

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AUTHOR: Nima Anvari, B.Sc., M.Sc.

SUPERVISOR: Professor Ian Hambleton.

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Abstract

Let $p > 5$ be a prime and X_0 a simply-connected 4-manifold with boundary the Poincaré homology sphere $\Sigma(2,3,5)$ and even negative-definite intersection form $Q_{X_0} = \mathrm{E}_8$. We obtain restrictions on extending a free $\mathbb{Z}/p$-action on $\Sigma(2,3,5)$ to a smooth, homologically-trivial action on X_0 with isolated fixed points. It is shown that for $p = 7$ there is no such smooth extension. As a corollary, we obtain that there does not exist a smooth, homologically-trivial $\mathbb{Z}/7$-equivariant splitting of $\#^8 S^2 \times S^2 = E_8 \cup_{\Sigma(2,3,5)} \overline{E_8}$ with isolated fixed points. The approach is to study the equivariant version of Donaldson-Floer instanton-one moduli spaces for 4-manifolds with cylindrical ends. These are L^2-finite anti-self dual connections which asymptotically limit to the trivial product connection.

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Dedicated to my loving family....

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Introduction

1.1 Motivation

Group actions on four-manifolds present both a challenging and fascinating set of problems in low-dimensional topology. It is well-known that any finite group acts freely on some closed simply-connected four-manifold. On the other hand there are significant restrictions on which groups can act with fixed-sets on a particular four manifold as seen by the following

Theorem 1.1.1 ([Edm98]). *Let X be a closed simply-connected 4-manifold with $b_2 \geq 3$. If a finite group G acts on X locally linearly, pseudofreely, and homologically trivially, then G must be a cyclic group, acting semifreely, and the fixed point set consists of $b_2(X) + 2$ isolated points.*

The question about existence of $\mathbb{Z}/p$-actions in the topological category was answered more generally for any closed topological four-manifold by Edmonds:

Theorem 1.1.2 ([Edm87]). *Let X be a closed, simply-connected 4-manifold. For any prime $p > 3$ there exists a locally-linear homologically-trivial $\mathbb{Z}/p$-action on X with fixed sets consisting of isolated fixed points.*

So there are plenty of topological periodic symmetries on four-manifolds. When it comes to smooth symmetries, however, the story is different as with many facets of four dimensional topology. Our motivation comes from Kirby Problem 4.124 (see [Kir78]), also due to Allen Edmonds, which asks whether there are non-trivial periodic diffeomorphisms on the $K3 = \{z_1^4 + \cdots + z_4^4\} \subset \mathbb{C}P^3$(or more generally on a homotopy $K3$ surface) which are homologically-trivial. Early evidence that smooth finite symmetries on the other hand seem to be rigid come from considering complex surfaces, where it was known that homologically-trivial complex automorphisms are trivial. The case of involutions was considered by Ruberman [Rub95] and Matumoto [Mat92]. More recently however, Chen-Kwasik [CK07] have proved that there are no non-trivial symplectic symmetries which are homologically-trivial on the standard $K3$ surface with respect to any symplectic structure.

Since the fundamental work of Donaldson [Don83] on the instanton Yang-Mills theory there has been tremendous progress in extracting equivariant information from the moduli spaces in the works of Fintushel-Stern [FS85], Furuta [Fur89], Buchdahl-Kwasik-Schultz [BKS90], Braam-Matic [BM93] and Hambleton-Lee [HL95]. One of the strengths of applying Donaldson theory to the study of group actions is that in the negative-definite instanton one case, the compactification involves a copy of the original four manifold [Don83]. In the equivariant setup this gives a direct relationship between the fixed-set in X and the fixed-set in $\mathcal{M}_1(X)$ by the induced action on the moduli space. When X is indefinite as in the $K3$ example, we are unable to retrieve this particularly useful property. According to Freedman-Taylor [FT77] however, if the intersection form of smooth four manifold X is decomposed $Q_{X_0} \oplus Q_{X_1}$ then there exists an integral homology 3-sphere Σ which realizes the splitting $X = X_0 \cup_\Sigma X_1$. One strategy then would be to consider periodic diffeomorphisms which equivariantly split X into a $X_0 \cup_\Sigma X_1$ with X_0 definite and X_1 indefinite with a free action on some integral homology 3-sphere Σ. Our approach is to apply equivariant Donaldson-Floer theory to the definite side

X_0. For the $K3$ surface we can consider periodic homologically-trivial symmetries which equivariantly split a homotopy $K3 = X_0 \cup_{\Sigma(2,7,13)} X_1$ surface along a smoothly and freely embedded copy of $\Sigma(2,7,13)$ with intersection forms $Q_{X_0} = 2E_8$ and $Q_{X_1} = 3H$. The existence of such a smooth action

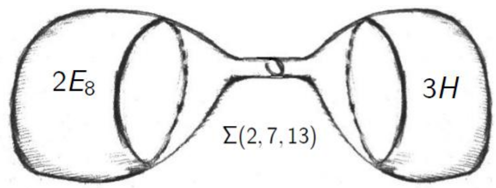

Figure 1.1: *Equivariant splitting of a homotopy $K3$ surface along a free action on $\Sigma(2,7,13)$.*

then fits into the framework of the following more general question:

Question. *Suppose $\Sigma(a,b,c)$ is the boundary of an even, negative definite, simply-connected four-manifold X_0. Does a free periodic action on $\Sigma(a,b,c)$ extend to a homologically-trivial smooth action on X_0 with isolated fixed points?*

Equivariant plumbing [Orl72, pg.23] always gives a smooth, homologically-trivial extension but requires at least one fixed 2-sphere, so we are asking for a smooth extension with just isolated fixed points. We first consider the Poincaré homology sphere $\Sigma(2,3,5)$ which splits $\#^8 S^2 \times S^2 = E_8 \cup_{\Sigma(2,3,5)} \overline{E_8}$.

Theorem (A). *Let X_0 be a smooth, simply-connected 4-manifold with boundary $\Sigma(2,3,5)$ and negative-definite intersection form $Q_{X_0} = E_8$. For any odd prime $p > 5$, if a free $\mathbb{Z}/p$-action on $\Sigma(2,3,5)$ extends to a smooth, homologically trivial action on X_0 with isolated fixed-points, then the rotation data of the fixed-points are (a,b) such that $a+b \equiv \pm 1 \pmod{p}$ or $a+b \equiv \pm 7 \pmod{p}$.*

The restriction on the primes is to ensure a free action on $\Sigma(2,3,5)$. When p is relatively prime to a,b,c, the free $\mathbb{Z}/p$ action on $\Sigma(a,b,c)$ is part of the circle action $t \cdot (x, y, z) = (t^{bc}x, t^{ac}y, t^{ab}z)$ of the Seifert fibered structure of $\Sigma(a, b, c)$. It can be shown (see [LS92] that this is the only free action on $\Sigma(a, b, c)$, called the *standard* action.

Similarly we can apply the same technique to get the following smooth constraints for a smooth homologically-trivial action extending $\Sigma(2, 7, 13)$ which embeds smoothly in a homotopy $K3$ surface.

Theorem (B). *Let X_0 be a smooth, simply-connected 4-manifold with boundary $\Sigma(2, 7, 13)$ and negative-definite intersection form $Q_{X_0} = 2E_8$. For any odd prime p not equal to $2, 7, 13$, if a free $\mathbb{Z}/p$-action on $\Sigma(2, 7, 13)$ extends to a smooth, homologically trivial action on X_0 with isolated fixed-points, then the rotation data of the fixed-points are (a, b) such that $a + b \equiv \pm 1 \pmod p$, $a + b \equiv \pm 15 \pmod p$, $a + b \equiv \pm 5 \pmod p$ or $a + b \equiv \pm 3 \pmod p$.*

For the Poincaré homology 3-sphere and $p = 7$ the necessary conditions for a smooth extension from Theorem A can be checked against the G-Signature formula for manifolds with boundary (see [APS75b]) and leads to the following

Theorem (C). *Let X_0 be as in Theorem A. Then for $p = 7$ the free action on $\Sigma(2, 3, 5)$ does not extend to a smooth, homologically-trivial action on X_0 with isolated fixed points.*

If t is the generator of the $\mathbb{Z}/p$-action, then this formula takes the following form:

$$\mathrm{Sign}(X, t) = \sum_i \frac{(t^{a_i} + 1)(t^{b_i} + 1)}{(t^{a_i} - 1)(t^{b_i} - 1)} - \eta_t(0) \tag{1.1}$$

where (a_i, b_i) are the rotation data at the fixed-points and $\eta_t(0)$ is the equivariant eta invariant [APS75b](see also Donnelly [Don78]). Since we are considering homologically-trivial actions the left-hand side is $\mathrm{Sign}(X) = -8$. The equivariant eta invariant is independent of the extended action and thus can be computed from the rotation data which arises from equivariant plumbing. To prove Theorem C, the rotation numbers satisfying the necessary conditions of Theorem A are checked against the G-signature formula using MAPLE.

Corollary. *There does not exist a smooth, homologically-trivial $\mathbb{Z}/7$-equivariant splitting of $\#^8 S^2 \times S^2 = E_8 \cup_{\Sigma(2,3,5)} \overline{E_8}$ with isolated fixed points inducing a free $\mathbb{Z}/7$-action on $\Sigma(2,3,5)$.*

1.2 Outline

In this section we will outline the arguments of the main results. Suppose X_0 is a smooth even negative-definite four manifold with boundary $\partial X_0 = \Sigma(2,3,5)$ and intersection form $Q_{X_0} = E_8$. Let $\pi = \mathbb{Z}/p$ be the cyclic group acting freely on $\Sigma(2,3,5)$ and suppose it extends to a smooth action on X_0 with isolated fixed points. Attach an infinite cylinder and consider the π-action on the non-compact Riemannian four manifold (X, g) with $X = X_0 \cup \Sigma(2,3,5) \times [0, \infty)$ with the π-action extending on the cylinder and a metric being both π-invariant and a product on the cylindrical-end. The moduli space of interest is $\mathcal{M}_1(X, \theta)$ consisting of anti-self dual connections on a trivial principal $SU(2)$-bundle over X with finite Yang-Mills action having one unit total Yang-Mills energy and asymptotic to the trivial product connection.

This moduli space has formal dimension five as in the closed negative-definite case and moreover as in the instanton-one theory of Donaldson, there is an end given by a collar $X \times (0, \lambda_0)$ due to Uhlenbeck bubbling of instantons. The moduli space inherits a π-action and this gives a relation between the fixed sets X^π and $\mathcal{M}^\pi(X, \theta)$. In particular the fixed points in X propagate one-parameter families of π-fixed ASD connections into the moduli space. Similarly, fixed 2-dimensional subsets propagate 3-parameter families of π-fixed ASD connections. Each of these strata is closed in the space of connections and has the structure of a closed submanifold in $\mathcal{B}^*$ so they cannot terminate in the irreducible component $\mathcal{M}^*(X, \theta)$ and since there are no reducibles it will be shown that they have to have another end which does not include the collar. Our goal is to study the compactification of these fixed sets that arise from the collar to obtain equivariant information about the group action on X. For cylindrical-end four manifolds this other end

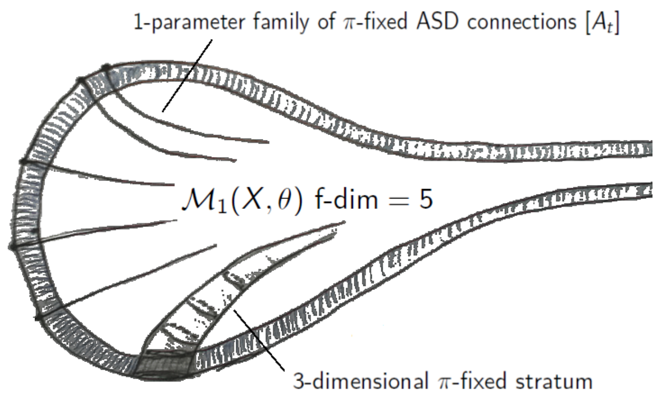

Figure 1.2: *The instanton one moduli space $\mathcal{M}_1(X,\theta)$ of ASD connections asymptotic to the trivial flat connection and total Yang-Mills energy one has a collar end as in the closed definite case. The fixed set propagates a one-parameter family of fixed ASD connections. Each of these fixed sets correspond to equivariant lifts of the π-action to the principal $SU(2)$-bundle which leave a one-parameter(respectively three-parameter) family of connections invariant. The equivariant lifts can be determined by pulling-back a equivariant bundle on the four sphere S^4 at the ideal boundary.*

comes from energy escaping down the cylinder leading to a charge-splitting. The version of Uhlenbeck compactness applicable here gives us a geometric limit consisting of finite energy ASD connections on our cylindrical-end four manifold and a sequence of ASD connections on a finite number of cylinders.

$$\overline{\mathcal{M}}_1(X,\theta) = \mathcal{M}_1(X,\theta) \cup X \cup \{\text{Floer Moduli spaces on } \Sigma \times \mathbb{R}\} \tag{1.2}$$

The amount of energy that can occur on each cylinder is determined by the Chern-Simons invariants of the limiting flat connections and has the value $CS(\beta) - CS(\alpha) \pmod{\mathbb{Z}}$.

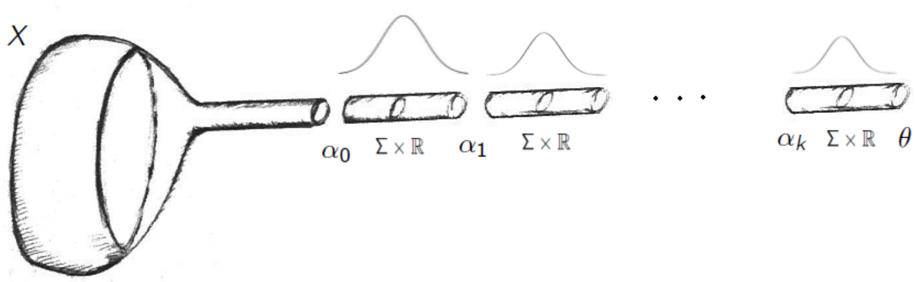

Figure 1.3: *When energy escapes down the cylindrical end, the limiting ASD connection has a curvature density that "ripples" finitely many times on cylinders before limiting to the trivial product connection.*

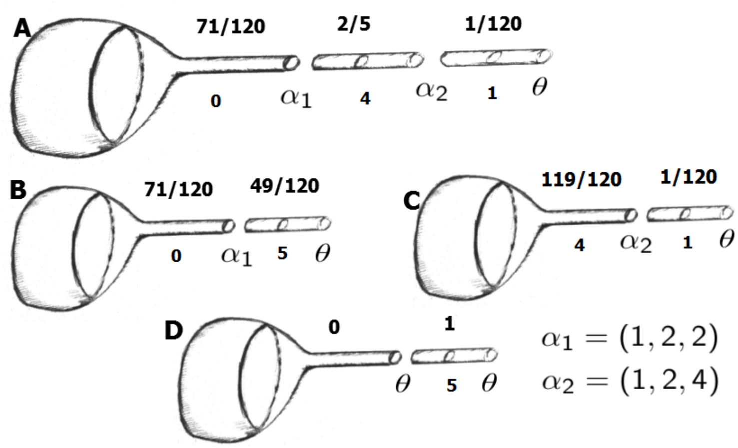

Figure 1.4: *The figure shows the charge splitting that can occur for* $\Sigma(2,3,5)$*. The number above is the Yang-Mills energy and is given by* $CS(\beta)-CS(\alpha) \pmod{\mathbb{Z}}$ *put in the range* $(0,1]$*. The value below is the formal dimension of ASD connections with the corresponding fixed energy. The total sum of the energy values is one and the formal sum of the dimensions is 5.*

If a charge-splitting contains a one-parameter family of invariant ASD connections under a equivariant lift of the π-action then it should descend to a one-parameter family of $SO(3)$-instantons on the quotient cylinder $Q \times \mathbb{R}$, where $Q = \Sigma(2,3,5)/\pi$. If this family of invariant connections was born at the Taubes collar from a fixed-point with rotation numbers (a,b) then the action on the trivial adjoint equivariant $SO(3)$-bundle over $\Sigma(2,3,5) \times \mathbb{R}$ is given by

$$t \cdot (x, s, U) = (tx, s, \phi(t)U) \tag{1.3}$$

where $x \in \Sigma(2,3,5)$, $s \in \mathbb{R}$, $U \in SO(3)$. In this formula ϕ is the isotropy representation $\phi : \pi \to SO(3)$ and $t \mapsto \begin{pmatrix} 1 & \\ & t^{a+b} \end{pmatrix}$ with $\mathbb{Z}/p = \langle t \rangle$. In the limit at $+\infty$ on $\Sigma(2,3,5) \times \mathbb{R}$ the trivial product connection descends to a flat reducible connection on Q whose $SO(3)$ holonomy representation is isomorphic to the isotropy representation ϕ.

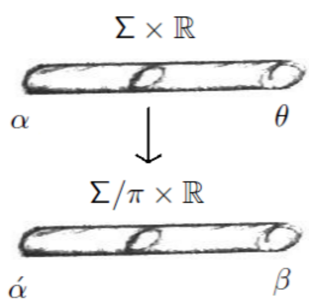

Figure 1.5: *An invariant ASD connection descends to an $SO(3)$ instanton on the quotient cylinder.*

The formal dimension of $SO(3)$-instantons in the quotient cylinder is 1 when the limiting flat connection has holonomy number ± 1 or $\pm 7 \pmod{p}$. This gives the required relation $a + b \equiv \pm 1$ or $\pm 7 \pmod{p}$.

Chapter 2

Background

Brieskorn homology spheres and more generally Seifert fibered 3-manifolds play a crucial role in low-dimensional topology. In the first section we will outline their definitions and properties that are relevant to our study of group actions and then look at some examples of group actions on four manifolds. See references [Orl72], [Fin77] for details on equivariant plumbing.

2.1 Brieskorn Homology Spheres

Integral homology spheres are closed oriented 3-manifolds Σ with the homology of the standard 3-sphere, $H_1(\Sigma;\mathbb{Z}) = 0$. The particular class of homology spheres which we shall be interested in are the **Brieskorn homology spheres** which have the description as being the link of a complex singularity. More precisely, let $a_1, a_2, a_3 \geq 2$ be integers and $V(a_1, a_2, a_3) = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1^{a_1} + z_2^{a_2} + z_3^{a_3} = 0\}$ be a complex surface which has a singularity at the origin in $\mathbb{C}^3$, then the intersection $\Sigma(a_1, a_2, a_3) = V(a_1, a_2, a_3) \cap S^5$ has real codimension 3 in $\mathbb{C}^3$ so it defines a closed oriented 3-manifold with the orientation assigned as the boundary of a singular complex surface. It is an integral homology 3-sphere if and only if a_1, a_2, a_3 are relatively prime. All the $\Sigma(a_1, a_2, a_3)$ bound canonical negative-definite resolution 4-manifolds obtained by plumbing (see [Orl72], [NR78b]) and admit a natural circle action given by

$$t \cdot (z_1, z_2, z_3) = (t^{a/a_1} z_1, t^{a/a_2} z_2, t^{a/a_3} z_3) \tag{2.1}$$

where $a = a_1 a_2 a_3$. This action is fixed point free and with three singular orbits. The points $\{z_i = 0\}$ have isotropy $\mathbb{Z}/a_i$. This circle action which gives $\Sigma(a_1, a_2, a_3)$ the structure of a Seifert fibration which we discuss next.

It is useful to have the alternative description of Brieskorn homology spheres in terms of Seifert fibered spaces. Consider the circle bundle over the 2-sphere with Euler number b. Take three disjoint disks D_i^2 in S^2 and perform rational Dehn surgery on the three solid tori $D_i^2 \times S^1$. The result is an oriented closed 3-manifold $\Sigma(b; (a_1, b_1), (a_2, b_2), (a_3, b_3))$ with fundamental group

$$\pi_1(\Sigma) = \langle x_1, x_2, x_3, h \,|\, [h, x_i] = 1, x_i^{a_i} h^{b_i} = 1, x_1 x_2 x_3 = h^{-b} \rangle \tag{2.2}$$

This manifold is an integral homology 3-sphere if and only if

$$a_1 a_2 a_3 \sum_i^3 \frac{b_i}{a_i} = \pm 1 + b \cdot a_1 a_2 a_3 \tag{2.3}$$

The b_i are uniquely determined modulo a_i and this manifold is orientation preserving homeomorphic to $\Sigma(a_1, a_2, a_3)$. It is called the **Seifert fibered** homology 3-sphere with **Seifert invariants** $\Sigma(b; (a_i, b_i))$ [Orl72].

Let $\pi = \mathbb{Z}/p \subset S^1$ denote a free periodic action on $\Sigma(a_1, a_2, a_3)$ with Seifert invariants $\Sigma(b = 0; (a_i, b_i))$ then $Q = \Sigma(a_1, a_2, a_3)/\pi$ the quotient is also a Seifert fibered space but no longer a $\mathbb{Z}$-homology sphere. It has Seifert invariants $Q(b = 0; (a_1, p \cdot b_1), (a_2, p \cdot b_2), (a_3, p \cdot b_3))$ [NR78b] and fundamental group

$$\pi_1(Q) = \langle x_1, x_2, x_3, h \,|\, [h, x_i] = 1, x_i^{a_i} h^{p b_i} = 1, x_1 x_2 x_3 = 1 \rangle \tag{2.4}$$

with first homology $\mathbb{Z}/p$. We can thus think of Q as either a $\mathbb{Z}$-homology lens space or a $\mathbb{Q}$-homology sphere.

2.2 Representation Variety

The $SU(2)$ representation variety of $\pi_1(Y)$ is given by representations of the fundamental group into $SU(2)$:

$$Rep(\pi_1) = \text{Hom}(\pi_1(Y), SU(2)) \tag{2.5}$$

given the compact-open topology and where $\pi_1(Y)$ has the discrete topology. This space $\text{Rep}(\pi_1)$ is always compact, as it can be thought of as a real algebraic variety [Sav00]. The group $SU(2)$ acts on the space of representations by conjugation and the quotient is denoted by $\mathcal{R}(Y) = \text{Rep}(\pi_1)/SU(2)$ and called the **character variety**. It is well known that flat connections on Y up to gauge are in one-to-one correspondence with the character variety. A flat connection α is called **irreducible** if its stabilizer is ± 1 and **reducible** if its image is contained in $U(1)$ as a representation. For integral homology 3-spheres Σ there are only irreducible connections apart from the trivial product connection. A representation $\alpha : \pi_1(\Sigma) \to SU(2)$ thought of as a flat connection has an associated elliptic complex:

$$\Omega^0(\Sigma, \operatorname{ad} P) \to \Omega^1(\Sigma, \operatorname{ad} P) \to \Omega^2(\Sigma, \operatorname{ad} P) \to \Omega^3(\Sigma, \operatorname{ad} P) \tag{2.6}$$

we denote the cohomology groups by $H^*(Y, \operatorname{ad} \alpha)$. The group $H^0(Y, \operatorname{ad} \alpha)$ can be identified with the Lie algebra of the stabilizer of α in $SU(2)$ and so α is reducible if and only if $H^0(Y, \alpha)$ is non-zero. Following the terminology in [Don02] we call the flat connection α **acyclic** if both cohomology groups $H^0(Y, \alpha)$ and $H^1(Y, \alpha)$ vanish and **non-degenerate** if $H^1(Y, \alpha)$ vanishes but not necessarily $H^0(Y, \alpha)$.

The representation variety is called non-degenerate if every flat connection in $\mathcal{R}(Y)$ is non-degenerate. This is the case with Brieskorn homology 3-spheres $\Sigma(a_1, a_2, a_3)$ [FS90]. Thought of as a Seifert fibered space $\Sigma(b; (a_i, b_i))$ it is always possible to choose the invariants so that $b = 0$, in which case the fundamental group of $\Sigma(a_1, a_2, a_3)$ is given by

$$\pi_1(\Sigma) = \langle x_1, x_2, x_3, h \,|\, [h, x_i] = 1, x_i^{a_i} h^{b_i} = 1, x_1 x_2 x_3 = 1 \rangle \tag{2.7}$$

where $a_1 a_2 a_3 \sum_i^3 b_i/a_i = 1$. The irreducible flat connections can be determined in the following way (see [FS90],[Sav00]). Let $\alpha : \pi_1(\Sigma(a_1, a_2, a_3)) \to SU(2)$ be a irreducible representation. Since h is central $\alpha(h) = \pm 1$, then up to conjugation we have

$$\alpha(x_i) = \begin{pmatrix} e^{\pi i \ell_i / a_i} & 0 \\ 0 & e^{-\pi i \ell_i / a_i} \end{pmatrix} \tag{2.8}$$

for some rotation numbers $0 < \ell_i < a_i$. Moreover, ℓ_i is even if $\alpha(h)^{b_i} = 1$ and odd if $\alpha(h)^{b_i} = -1$. Conversely, a triple of rotation numbers (ℓ_1, ℓ_2, ℓ_3) will define a representation if they satisfy the following condition

$$\left| \frac{\ell_1}{a_1} - \frac{\ell_2}{b_2} \right| < \frac{\ell_3}{a_3} < 1 - \left| \frac{\ell_1}{a_1} + \frac{\ell_2}{a_2} \right| \tag{2.9}$$

Example 2.2.1. *In this example we list the rotation numbers for the representations of* $\Sigma(2, 3, 5)$*. The Seifert invariants are chosen as*

$$\Sigma(b = 0, (2, 3), (3, 4), (5, -14)).$$

There are only two irreducible representations and they have rotation numbers $(1, 2, 2)$ *and* $(1, 2, 4)$*.*

Similarly we can determine all the irreducible flat connections on $\Sigma(2, 7, 13)$.

Example 2.2.2. *We list the rotation numbers for the representations of*

$\Sigma(2,7,13)$. *The Seifert invariants are chosen as*

$$\Sigma(b=0,(2,3),(7,10),(13,-38)).$$

There are only 12 *irreducible representations and they have rotation numbers* $(1,2,4),(1,2,6),(1,2,8),(1,2,10)(1,4,2),(1,4,4)(1,4,6),(1,4,8),(1,4,10)$ $(1,4,12),(1,6,6),(1,6,8)$.

2.3 Group Actions on Four-Manifolds

We will look at some examples of finite cyclic group actions on four-manifolds with and without boundary.

2.3.1 G-Signature Theorem

Recall that if a finite group G acts **freely** on X if no non-trivial group element has a fixed point, and the action is **pseudofree** if the action is free on the complement of a discrete set in X. A group action is called **semifree** if it is free on the complement of the fixed point set of the whole group. An action is **locally linear** if each point has a neighbourhood invariant under the isotropy group at the point on which the action is equivalent to a linear action of the isotropy group on some euclidean space. In particular, smooth actions are always locally linear, but not every locally linear action is smooth. A orientation-preserving action is said to be **homologically trivial** if the induced action on integral homology $H_2(X,\mathbb{Z})$ groups is the identity.

The rotation numbers of a group actions are required to satisfy the G-signature theorem for any locally linear action.

$$\mathrm{Sign}(X,g)=\sum_i -\cot(\pi a_i/p)\cot(\pi b_i/p)+\sum_j F_j^2\csc^2(\pi c_j/p) \tag{2.10}$$

where the first sum is over isolated fixed points with rotation numbers (a_i,b_i)

and the second summation is over 2-dimensional fixed sets with rotation c_j on the normal fiber to the fixed surface F_j. It will be convenient to write this formula in the following form:

$$\mathrm{Sign}(X,t) = \sum_i \left(\frac{t^{a_i}+1}{t^{a_i}-1}\right)\left(\frac{t^{b_i}+1}{t^{b_i}-1}\right) - \sum_j \frac{4F_j t^{c_j}}{(t^{c_j}-1)^2} \tag{2.11}$$

and since we are only looking at homologically-trivial actions, the left hand side is just $\mathrm{Sign}(X)$.

2.3.2 The Four Sphere

Let $X = S^4$ be unit four-sphere in $\mathbb{R}^5 = \mathbb{C}^2 \times \mathbb{R}$ be described by $|z_1|^2 + |z_2|^2 + x^2 = 1$. Let $t = e^{\frac{2\pi i}{p}}$ denote the generator for $G = \mathbb{Z}/p$ for an odd prime p. Then the action defined by $t(z_1, z_2, x) \mapsto (t^a z_1, t^b z_2, x)$ has two isolated fixed points $(0, 0, \pm 1)$ when $a, b \not\equiv 0 \pmod p$ with rotation numbers (a, b) and $(a, -b)$. There is a fixed 2-sphere when one of a or b is zero, so for example if $a = 0$ then the fixed 2-sphere is given by $\{z_2 = 0\}$ and the rotation on the normal fiber is given by $(t^b, 1)$. The G-signature theorem in this case gives

$$0 = \left(\frac{t^a+1}{t^a-1}\right)\left(\frac{t^b+1}{t^b-1}\right) + \left(\frac{t^a+1}{t^a-1}\right)\left(\frac{t^{-b}+1}{t^{-b}-1}\right) \tag{2.12}$$

2.3.3 Complex Projective Space

Let $X = \mathbb{C}P^2$ denote complex projective space, we can define a linear action of $\mathbb{Z}/p \subset \mathrm{PGL}_3(\mathbb{C})$ by $t[z_1, z_2, z_3] = [t^a z_1, t^b z_2, z_3]$ with a and b integers modulo p. Note that $[t^a z_1, t^b z_2, z_3] = [z_1, t^{b-a} z_2, t^{-a} z_3] = [t^{a-b} z_1, z_2, t^{-b} z_3]$, hence when $a \not\equiv b \pmod p$, the action has three fixed points: $[0,0,1], [1,0,0], [0,1,0]$ with rotation numbers given respectively by $(a, b), (b-a, -a), (a-b, -b)$. When $a \equiv b \pmod p$, the action has one isolated fixed point $[0, 0, 1]$ with rotation number (a, a) and a fixed 2-sphere $\{z_3 = 0\}$ with action on the normal fiber given by the action of t^a. The fixed 2-sphere represents $\mathbb{C}P^1 \subset \mathbb{C}P^2$

and so has Euler number 1. We can obtain from these linear models the corresponding actions on $\overline{\mathbb{C}P}$ by reversing orientation, so that the rotation data are now $(a, b), (a + b, -a), (a + b, -b)$ and the fixed 2-sphere has Euler number -1. Let us record here the G-signature theorem in this case.

$$-1 = \left(\frac{t^a+1}{t^a-1}\right)\left(\frac{t^b+1}{t^b-1}\right) + \left(\frac{t^{a+b}+1}{t^{a+b}-1}\right)\left(\frac{t^{-a}+1}{t^{-a}-1}\right) + \left(\frac{t^{a+b}+1}{t^{a+b}-1}\right)\left(\frac{t^{-b}+1}{t^{-b}-1}\right)$$

similarly in the case when we have a fixed point and a fixed 2-sphere:

$$-1 = \left(\frac{t^a+1}{t^a-1}\right)\left(\frac{t^{-a}+1}{t^{-a}-1}\right) + \frac{4t^c}{(t^c-1)^2}$$

We can form more complex actions by equivariant connected sum of copies of $\overline{\mathbb{C}P^2}$.

2.3.4 Equivariant Plumbing

Write $S^2 = D_+ \cup D_-$ as the upper and lower hemispheres and consider the trivial D^2-bundle over each hemisphere. We would like to glue them together using the cocycle map $F : S^1 \mapsto \mathrm{Aut}(D^2)$ sending $z \mapsto z^k$ for some integer k. Then we obtain a D^2-bundle with Euler number k.

Let $E(e)$ denote the total space of a D^2-bundle over the 2-sphere S^2 with Euler number e. Given two such disk bundles $E(e_1)$ and $E(e_2)$, we can consider the operation of **plumbing** as follows. Choose disks in the base of each $E(e_i) \to S^2$, the bundles over these disks are trivial $D_i^2 \times D^2$. We use the gluing map μ which exchanges the fiber and base coordinates - the result is a smooth 4-manifold with boundary (after rounding corners) $P = E(e_1) \cup_\mu E(e_2)$. More generally, we can plumb according to a graph Γ whose vertices have integer weights indicating the Euler number of the

D^2-bundles. The intersection matrix is given by

$$a_{ij} = \begin{cases} e_i & \text{if } i = j \\ 1 & \text{if } i \text{ is connected to } j \\ 0 & \text{otherwise.} \end{cases}$$

The $\partial P(\Gamma)$ is an integral homology 3-sphere if and only if $\det(a_{ij}) = \pm 1$. Every Seifert fibered homology sphere can be obtained as the boundary of a plumbed four manifold. Now define a S^1-action on $D^2_+ \times D$ by $t \cdot (re^{i\theta}, se^{i\delta}) = (re^{i(\theta+at)}, se^{i(\delta+bt)})$ and similarly for the lower hemisphere with a and b replaced with c and d. We would like to use a map F to glue the trivial D^2-bundles together equivariantly to get a D^2-bundle with a S^1-action. The map $F : \partial D^2_+ \times D^2 \to \partial D^2_- \times D^2$ defined by $F(e^{i\theta}, se^{i\delta}) = (e^{-i\theta}, se^{i(-k\theta+\delta)})$ gives a D^2-bundle with Euler number k and is S^1-equivariant if

$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -k & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{2.13}$$

This gives a D^2-bundle over the 2-sphere with Euler number k and admits an effective S^1-action if $(a, b) = 1$ and $(c, d) = 1$. To equivariantly plumb with another D^2 bundle over a 2-sphere we just need to let $\tilde{a} = d$ and $\tilde{b} = c$.

Example 2.3.1. *Let $p > 5$. Equivariantly plumbing along the E_8 diagram gives one fixed 2-sphere (the centre node) and 7 isolated fixed points with rotation numbers* $(-4, 5), (-3, 4), (-2, 3) \times 2, (-1, 2) \times 3$.

2.4 Chern-Simons Theory

In this section we will review the basic definition and results about Chern-Simons theory that will be relevant for us later.

2.4.1 The Functional

Given a 3-manifold Y and a principal $SU(2)$ bundle $P \to Y$ (necessarily trivial), a one-parameter family of connections $\{A_t\}_{t\in I}$ on Y corresponds to a connection A on $Y \times I$. Given a path γ in $\mathcal{A}(Y)$ from the trivial product connection θ to some other connection α, the Chern-Simons functional $CS : \mathcal{A}(Y) \to \mathbb{R}$ is defined by

$$CS(\alpha) := \frac{1}{8\pi^2} \int_{Y\times I} Tr(F_A \wedge F_A) \tag{2.14}$$

The value of the this functional on flat-connections are called **Chern-Simon invariants**. Similarly, we can define a relative Chern-Simons functional by taking a path connecting any two connections say α and β and define

$$CS(\alpha, \beta) = \frac{1}{8\pi^2} \int_{Y\times I} Tr(F_A^2). \tag{2.15}$$

Since $\mathcal{A}(Y)$ is contractible this is independent of path and we can homotope this to one that passes through the trivial product connection. Splitting the integral into two gives $CS(\alpha, \beta) = CS(\beta) - CS(\alpha)$.

Theorem 2.4.1. *(Properties of Chern-Simons) Let α be a connection on Y.*
(1) (Multiplicative under finite covers) If $f : \tilde{Y} \mapsto Y$ is a finite cover of order n, then $CS(f^*\alpha) \equiv nCS(\alpha) \mod \mathbb{Z}$.
(2) (Invariance under flat cobordisms) If (Z, α) is a flat cobordism between (Y_1, α_1) and (Y_2, α_2) then $CS_{Y_1}(\alpha_1) \equiv CS_{Y_2}(\alpha_2) \mod \mathbb{Z}$.
(3) (Locally Constant) If α_t is a path of flat connections in $\mathcal{R}(Y)$ then $CS_Y(\alpha_0) = CS_Y(\alpha_1)$.

We will write down the well-known ASD equations on the cylinder and its relation to Chern-Simons flow. Let X denote $Y \times \mathbb{R}$ with orientation

$$dt \wedge dy_1 \wedge dy_2 \wedge dy_3.$$

We note the following identifications between forms on the cylinder and forms on the cross-section $\Omega^1(Y\times\mathbb{R}) = \Omega^0(Y)\oplus\Omega^1(Y)$ which maps $\alpha\, dt+\beta \mapsto (\alpha,\beta)$. Similarly, $\Omega^2(Y\times\mathbb{R}) = \Omega^1(Y)\oplus\Omega^2(Y)$ sending $dt\wedge\alpha+\beta\mapsto(\alpha,\beta)$. The four-dimensional Hodge star in the product metric dt^2+g_Y acts on 2-forms on the cylinder by

$$*_4(dt\wedge\alpha+\beta) = *_3\alpha + dt\wedge *_3\beta. \tag{2.16}$$

Theorem 2.4.2. *A 2-form $dt\wedge\alpha+\beta$ on $Y\times\mathbb{R}$ is ASD if and only if $\beta = - *_3\,\alpha$.*

Proof. The previous equation give $*_3(\alpha)+dt\wedge *_3(\beta) = -dt\wedge\alpha-\beta$, so the ASD equation becomes $\beta = - *_3\,\alpha$. □

As a result anti-self dual 2-forms are of the form $dt\wedge\alpha - *_3\alpha$. Similarly, self-dual forms are $dt\wedge\alpha + *_3\alpha$, in particular we see that self-dual 2-forms $\Omega^2_+(Y\times\mathbb{R})$ can be identified with $\Omega^1(Y)$ via the map $dt\wedge\alpha+*_3\alpha\mapsto\alpha$.

Let $\mathbf{A} = A(t)dt+\alpha(t)$ be a connection one-form on $Y\times\mathbb{R}$ with $A(t)$ a path of connections on Y. Then the curvature can be computed as $F_{\mathbf{A}} = d\mathbf{A}+\mathbf{A}\wedge\mathbf{A}$ which simplifies to

$$F_{\mathbf{A}} = F_{A(t)} + dt\wedge(\frac{\partial A(t)}{\partial t} - d_A(\alpha)).$$

using the covariant derivative $d_A(\alpha) = d\alpha+[A,\alpha]$. So the ASD equations on the cylinder are given by a gradient flow equations

$$\frac{\partial A_t}{\partial t} - d_A(\alpha) = - *_3\, F_{A_t} \tag{2.17}$$

this flow equation is gauge-equivalent to the following:

Theorem 2.4.3. *Let A_t be a one-parameter family of connections on Y.*

Then the curvature is F_{A_t} is ASD if and only if

$$\frac{\partial A_t}{\partial t} = - *_3 F_{A_t}.$$

2.4.2 Invariants of Flat Connections

In this section we record some Chern-Simons invariants for certain 3-manifolds which we will need later. We begin with lens spaces.

We consider flat connections on the lens space $L(p,q)$ as homomorphisms of the fundamental group $\alpha : \pi_1(L(p,q)) \mapsto SU(2)$ up to conjugation via holonomy representation. Since $\pi_1(L(p,q)) = \mathbb{Z}/p$ all flat connections are reducible and we label these flat connections by $\alpha(\ell)$ which send the generator of the fundamental group μ to ζ^ℓ where ζ is the primitive root of unity $e^{\frac{2\pi i \ell}{p}}$. As a representation into $SU(2)$ we get

$$\alpha(\ell)(\mu) = \begin{pmatrix} \zeta^\ell & \\ & \zeta^{-\ell} \end{pmatrix}$$

Note that $\alpha(\ell)$ is isomorphic to $\alpha(p-\ell)$ since these matrices are conjugate, so we only need to consider $0 < \ell < [p/2]$. With this, the Chern-Simons invariant is given by

Theorem 2.4.4 ([KK90, pg.354])**.** *Let $L(p,q)$ be a lens space then the Chern-Simons invariants are given by*

$$CS(\alpha(\ell)) \equiv \frac{q^* \ell^2}{p} \pmod{\mathbb{Z}} \tag{2.18}$$

where $q^* q \equiv 1 \pmod{p}$.

Theorem 2.4.5 ([FS90])**.** *Let $\Sigma(a_1, a_2, a_3)$ be a Brieskorn homology 3-sphere. Suppose the Seifert invariants are chosen so that b is even. If one of the a_i is even assume it is a_1 and further arrange the invariants b_i to be even for*

$i \neq 1$. Let α be an irreducible $SU(2)$ representation determined by rotation numbers (ℓ_1, ℓ_2, ℓ_3) then

$$CS(\alpha) \equiv \frac{-e^2}{4a_1a_2a_3} \pmod{\mathbb{Z}} \tag{2.19}$$

where $e = a_1a_2a_3 \sum_{i=1}^{3} \frac{\ell_i}{a_i}$.

Let $\pi = \mathbb{Z}/p$ act as the standard action on $\Sigma(a_1, a_2, a_3)$. Then the quotient $Q = \Sigma/\pi$ is a rational homology sphere(or a $\mathbb{Z}$-homology lens space) with Seifert invariants $Q(b = 0; (a_i, pb_i))$ and we will need a formula for the Chern-Simons invariant of reducible flat connections. Note that if we take the p-fold cover we get the trivial product connection on $\Sigma(a_1, a_2, a_3)$ and since Chern-Simons is multiplicative under finite covers we expect an expression of the form

$$CS(Q, \alpha(k)) \equiv \frac{n}{p} \mod \mathbb{Z}$$

for some integer n. This is indeed the case as can be verified using Auckly's technique [Auc94b]:

Theorem 2.4.6. *$CS(Q, \alpha(k)) \equiv \frac{n_0 k}{p} \pmod{\mathbb{Z}}$ where n_0 satisfies $n_0a_1a_2a_3 \equiv k \pmod{p}$.*

Proof. This is obtained by using a representation $\rho(n_0, n_1, n_2, n_3) : \pi_1(Q) \to U(1)$ where n_0 satisfies $a_1a_2a_3 \cdot n_0 \equiv k \pmod{p}$ and $n_i = b_i$ for $i \neq 0$. This representation sends $h \to e^{2\pi ik/p}$ and $x_i \mapsto 1$. The Seifert invariants satisfy

$$\sum_i \frac{b_i}{a_i} = \frac{p}{a_1a_2a_3} \tag{2.20}$$

and the formula for the Chern-Simons invariant of the corresponding flat connection is given in [Auc94b, pg.234] as

$$CS(Q,\rho) \equiv -\sum_{j=1}^{3} \frac{\rho_j n_j^2 + n_j(n_0 + c/2 + \sum_{i=1}^{3} n_i/a_i)/(b + \sum_i b_i/a_i)}{a_j} - \frac{(n_0 + c/2)(n_0 + c/2 + \sum_i n_i/a_i)}{b + \sum_i b_i/a_i} \pmod{\mathbb{Z}}$$

with $c = 0$ and ρ_j satisfies $a_j\sigma_j - b_j\rho_j = 1$ for some integers σ_j. This then simplifies to $\dfrac{n_0 k}{p}$ mod $\mathbb{Z}$. □

The Chern-Simons invariants for the irreducible flat connections on Q can be computed by using an $SO(3)$ flat cobordism as with $\Sigma(a_1, a_2, a_3)$ but also again from [Auc94b, pg.232].

Theorem 2.4.7 ([Auc94b])**.** *For an irreducible flat $SU(2)$ connection α on Q*

$$CS(Q,\alpha) \equiv -\sum_{i=1}^{3}\left(\frac{\rho_i {\ell_i}^2}{a_i} + \frac{\ell_i}{a_i}\right) + \frac{1}{4}\left(b + \sum_{i=1}^{3} \frac{pb_i}{a_i}\right) \pmod{\mathbb{Z}} \tag{2.21}$$

where ρ_i satisfies $a_i\sigma_i - (pb_i)\rho_i = 1$ for some integers σ_i, $i = 1..3$.

Chapter 3

Equivariant Yang-Mills Moduli Spaces

We give the setup for the equivariant moduli space of a cylindrical end 4-manifold.

3.1 The L^2-finite Moduli Space

Yang-Mills moduli spaces over non-compact four-manifolds were studied by Floer [Flo88], Taubes [Tau87], [Tau93], Morga-Mrowka-Ruberman [MMR94], Donaldson [Don02], with some of the analysis originating in [LM85].

3.1.1 Yang-Mills Functional

Let X_0 denote a smooth, simply-connected four-manifold with non-empty boundary $\partial X_0 = Y$. We consider (X, g) to be the associated Riemannian 4-manifolds obtained by adding cylindrical-end: $X = X_0 \cup \mathrm{End}(X)$, where $\mathrm{End}(X)$ is orientation-preserving isometric to $Y \times [0, \infty)$ with the product metric $g\,|_{\,\mathrm{End}(X)} = ds^2 + g_Y$. Consider a principal $SU(2)$ bundle P over X (necessarily trivial) and recall the definition of the **Yang-Mills energy** functional acting on the space of connections $\mathcal{YM} : \mathcal{A}(P) \to \mathbb{R}^+$ defined by

$$\mathcal{YM}(A) = \frac{-1}{8\pi^2}\int_X Tr(F_A \wedge *F_A) = \frac{1}{8\pi^2}\int_X |\, F_A \,|^2 = \|F_A\|^2_{L^2} \tag{3.1}$$

where F_A is ad(P)-valued curvature 2-form of the connection and $*$ is the Hodge star operator associated to the Riemannian metric. This functional is both conformally-invariant and gauge-invariant under the action of $\mathcal{G}(P) = \mathrm{Aut}(P)$ and so descends to a well-defined functional on the space of connections modulo gauge: $\mathcal{B}(P) = \mathcal{A}/\mathcal{G}$ depending only on the conformal structure of the metric g. The Hodge-star operator is an involution acting on 2-forms and gives the ± 1 - eigenspace decomposition:

$$\Omega^2(\mathrm{ad}(P)) = \Omega^2_+(\mathrm{ad}(P)) \oplus \Omega^2_-(\mathrm{ad}(P)) \tag{3.2}$$

into self-dual and anti-self dual(ASD) forms. The L^2-finite moduli-spaces are anti-self dual connections modulo gauge with finite Yang-Mills action:

$$\mathcal{M}(X, g) = \{[A] \in \mathcal{B}(P) \,|\, F_A^+ = 0, \|F_A\|^2_{L^2} < \infty\} \tag{3.3}$$

It is a fundamental result that g-ASD connections with finite Yang-Mills energy converge to flat connections down the cylindrical-end [MMR94],[Tau87]. Let (X, g) denote a smooth Riemannian four manifold with a cylindrical-end $Y \times [0, \infty)$.

Theorem 3.1.1 (Convergence to Flat Connections,[Don02])**.** *Suppose A is a L^2-finite g-ASD connection on X. Then the restrictions $A\,|_{Y\times\{t\}}$ over the* End(X) *converge modulo gauge in C^∞ over Y to a unique flat connection $\alpha \in \mathcal{R}(Y)$.*

Let $i_t : Y \times \{t\} \hookrightarrow Y \times [0, \infty)$ denote the inclusion map, then this convergence result tells us that $\lim_{t\to\infty}[i_t^* A\,|_{\,\mathrm{End}\,X}]$ exists in $\mathcal{R}(Y)$. This defines a boundary map ∂_∞ [MMR94]

$$\partial_\infty : \mathcal{M}(X, g) \to \mathcal{R}(Y) \tag{3.4}$$
$$[A] \mapsto \lim_{t\to\infty} [i_t^* A\,|_{\,\mathrm{End}\,X}] \tag{3.5}$$

If X has more than one end, then there exists a boundary map for each end. When X is the cylinder $Y \times \mathbb{R}$ with product metric g then there exist finite energy g-ASD connections on the cylinder if and only if there exists a gradient-flow line for the Chern-Simons functional connecting the flat-connections on the ends. It turns out that the energy of these moduli-spaces can only take on a discrete set of values determined by the Chern-Simons invariant.

Proposition 3.1.2 (Yang-Mills Energy and Chern-Simons)**.** *Let A be a finite energy g-ASD connection on $Y \times \mathbb{R}$ with energy $\mathcal{YM}(A) = k$ and limiting flat connections α and β. Then*

$$k \equiv CS(\beta) - CS(\alpha) \pmod{\mathbb{Z}}. \tag{3.6}$$

Let A_t denote the restriction of A to $Y \times \{t\}$, then we have

$$\begin{aligned} 8\pi^2 k &= \int_{Y\times\mathbb{R}} Tr(F_A^2) \\ &= \lim_{t\to\infty} \int_{Y\times(-t,0)} Tr(F_A^2) + \int_{Y\times(0,t)} Tr(F_A^2) \\ &= \lim_{t\to\infty} 8\pi^2\{CS(A_{-t}, A_0) + CS(A_0, A_t)\} \\ &= \lim_{t\to\infty} 8\pi^2\{CS(A_t) - CS(A_{-t})\} \\ &= 8\pi^2(CS(\beta) - CS(\alpha)). \end{aligned} \tag{3.7}$$

Similarly, for a cylindrical end 4-manifold $X = X_0 \cup \mathrm{End}(X)$ where $\mathrm{End}(X)$ is isometric to $Y \times \mathbb{R}$ and A is a finite energy g-ASD connection on X we have

$$\begin{aligned}
8\pi^2\mathcal{YM}(A) &= \int_X Tr(F_A^2) + \int_{\mathrm{End(X)}} Tr(F_A^2) \\
&= \lim_{t\to\infty} 8\pi^2(CS(A_0) + CS(A_0, A_t)) \\
&= 8\pi^2 CS(\alpha).
\end{aligned} \tag{3.8}$$

From this it follows that the energy is congruent modulo integers to the value of the Chern-Simons invariant on the limiting flat connection on the End(X) and so the Yang-Mills energy takes on a discrete set of values given by $CS(\alpha)$ (mod $\mathbb{Z}$). The convergence to flat connections of finite energy g-ASD connections on P partitions the moduli space:

$$\mathcal{M}(X, g) = \bigsqcup_{\alpha\in\mathcal{R}(Y)} \mathcal{M}(X, \alpha). \tag{3.9}$$

and since the energy of each ASD connection A is congruent to $CS(\alpha)$ (mod $\mathbb{Z}$) we get a further partition by energy $\mathcal{M}(X, \alpha) = \bigsqcup_\ell \mathcal{M}_\ell(X, \alpha)$.

3.1.2 Fredholm Analysis and Exponential Decay

Let us briefly review the description of the moduli space in the case of a closed four-manifold X, see [DK90] for details. The global description of the moduli space $\mathcal{M}(X, g)$ can be described by the $\mathcal{G}$-equivariant map $F^+\colon \mathcal{A}(P) \to \Omega^+_X(\mathrm{ad}\,P)$ and so defines a Fredholm section Ψ of the infinite dimensional vector bundle $\mathcal{A}^*(P) \times_{\mathcal{G}} \Omega^+_X(\mathrm{ad}\,P) \to \mathcal{B}^*(P)$ when these spaces are completed to suitable Sobolev spaces. The zero set is the irreducible component of the moduli space $\mathcal{M}^*(X, g)$, and when Ψ is in general position we can give the moduli space the structure of a smooth manifold.

We would like give a description of the neighbourhood of a g-ASD connection A in the moduli space, by considering the perturbed connection $A+a$ for some $a \in \Omega^1(\mathrm{ad}\,P)$. The condition that $A + a$ be anti-self dual is given by

$$F_A^+ = d_A^+ a + (a \wedge a)^+ = 0 \tag{3.10}$$

The solutions to this equation admit large symmetry, i.e. the gauge group acts on the set of solutions. A local transversal condition is imposed so that only perturbations a are considered which are transversal to the action of the gauge group. This is the Coulomb gauge condition $d_A^* a = 0$ so that $T_{[A]}\mathcal{B}^* = \{a \in \Omega_X^1(\operatorname{ad} P) \,|\, d_A^* a = 0\}$. Together we get a system of first order non-linear partial differential equations

$$d_A^+ a + (a \wedge a)^+ = 0 \tag{3.11}$$

$$d_A^* a = 0 \tag{3.12}$$

whose solutions describe a neighbourhood of $[A]$ in the moduli space $\mathcal{M}(X, g)$ up to the action of the stabilizer Γ_A. The linearisation gives the anti-self duality deformation operator $D_A = d_A^* + d_A^+$ whose kernel describes the local infinitesimal deformations of A in the moduli space, that is:

$$T_{[A]}\mathcal{M}^*(X, g) = \{a \in \Omega_X^1(\operatorname{ad} P) \,|\, d_A^+ a = d_A^* a = 0\} = \ker D_A \tag{3.13}$$

We can give a finite dimensional model by restricting to a slice $\psi : T_{A,\varepsilon} \to \Omega_X^+(\operatorname{ad} P)$ where $T_{A,\varepsilon} = \{A + a \,|\, d_A^* a = 0, \|a\| < \varepsilon\}$. Then the differential $D\psi_A : \ker d_A^* \to \Omega_X^+(\operatorname{ad} P)$ is Fredholm and the Kuranishi method [FU91] gives that locally we can write

$$\psi(a) = D\psi_A(a) + \phi(a) \tag{3.14}$$

where $\phi : H_A^1 \to H_A^2$, and $\phi^{-1}(0)/\Gamma_A \hookrightarrow \mathcal{M}(X, g)$ describes the neighbourhood of $[A]$ in the moduli space.

We now review the case when X is the cylinder $Y \times \mathbb{R}$, the anti-self duality deformation operator is $D_A : \Omega_X^1(\operatorname{ad} P) \to \Omega^0 \oplus \Omega_+^2(X, \operatorname{ad} P)$ sending $\omega \mapsto (-d_A^* \omega, d_A^+ \omega)$ using the following identifications $\Omega_X^1 = \Omega_Y^0 \oplus \Omega_Y^1$ via a map Φ sending $adt+b \mapsto (a, b)$. Similarly, we have seen that we can identify 1-forms on Y with self-dual 2-forms on the cylinder via a map $\Psi : \Omega^1_Y \mapsto \Omega^2_+(X)$ sending $\xi \mapsto dt \wedge \xi + *_3\xi$. Thus on the cylinder we may write the deformation operator $\Psi^{-1}D_A\Phi : \Omega^0_Y \oplus \Omega^1_Y \to \Omega^0_Y \oplus \Omega^1_Y$ as

$$\frac{\partial}{\partial t} + L_{A_t}.$$

That is, have

$$D_A\Phi = \Psi(\frac{\partial}{\partial t} + L_{A_t})$$

To see the form of L_{A_t} we use the identifications to see the form of the operators d^+_A and d^*_A.

$$d^+_A(a, b) = \frac{\partial b}{\partial t} + *_3 d_{A_t} b - d_{A_t} a$$

and

$$d^*_A(a, b) = -\frac{\partial a}{\partial t} + d^*_{A_t} b$$

Thus we can write

$$D_A = \frac{\partial}{\partial t} + L_{A_t} = \frac{\partial}{\partial t} + \begin{pmatrix} 0 & -d^*_{A_t} \\ -d_{A_t} & *_3 d_{A_t} \end{pmatrix}$$

over the cylindrical-end. Now if A is a connection over a four manifold X with a cylindrical end $Y \times [0, \infty)$ with acyclic flat limit α, then D_A is a Fredholm operator. This situation changes if we relax the acyclic condition to non-degenerate flat limits. The deformation operator D_A will no longer be Fredholm. To recover a Fredholm operator, weighted Sobolev spaces $L^2_{k,\delta}$ are used, see (3.17) below. Let λ^+ denote the first non-zero positive eigenvalue of L_α. Then we have $D_A : L^2_{2,\delta} \to L^2_{1,\delta}$ is a Fredholm operator if the weight $\delta < \lambda^+$.

With this, we can give a refinement of the decay to flat connections down the cylindrical end.

Theorem 3.1.3 (Exponential Decay,[Don02])**.** *Let A be an L^2-finite g-ASD connection with non-degenerate limiting flat connection α and λ^+ the first positive eigenvalue of L_α. Then over* $\mathrm{End}(X)$ *the curvature F_A decays exponentially:*

$$|\, F_A \,| \leq Ce^{-\lambda^+ t} \tag{3.15}$$

for some constant C. Moreover, if A converges to the flat connection α then it is gauge-equivalent to a connection $A_0 + a$ where A_0 is the pull-back of α on Y to $Y \times \mathbb{R}_+$ and a is a 1-form which satisfies

$$|\, a \,| \leq Ce^{-\lambda^+ t} \tag{3.16}$$

This motivates the following definitions. Let $X = X_0 \cup \mathrm{End}(X)$ denote the cylindrical-end 4-manifold and A_0 any connection on X that extends a given flat connection α over the $\mathrm{End}(X) = Y \times [0, \infty)$. Let $W : X \to [0, \infty)$ be a weight function that is zero on X_0 and equal to $e^{\delta t}$ over the $\mathrm{End}(X)$. Denote by $\Omega^i_{k,\delta}(X, \mathrm{ad}\, P)$ to be the completion of compactly supported forms $\Omega^i_X(\mathrm{ad}\, P)$ by the weighted norm $L^2_{k,\delta}$, that is

$$\|a\|^2_{k,\delta} = \int_X W\{\, |\, \nabla^k_{A_0} a \,|^2 + \cdots + |\, \nabla_{A_0} \,|^2 + |\, a \,|^2 \} \tag{3.17}$$

Denote by $\mathcal{A}(\alpha) = \{A_0 + a \,|\, a \in L^2_{2,\delta}(\Omega^1_X(\mathrm{ad}\, P))\}$ and $\mathcal{G}(\alpha) = \{u \in \mathcal{G} \,|\, \bigtriangledown_{A_0} u \in L^2_{2,\delta}\}$. These are respectively, the space of connections that limit exponentially to α on the cylindrical-end and gauge transformations that stabilize α on the end. The decay results tell us that the moduli space $\mathcal{M}(X, \alpha) = \mathcal{A}(\alpha)/\mathcal{G}(\alpha)$ captures all the finite-energy instantons that are asymptotic to α. The anti-self duality deformation operator is

$$D_{A,\delta} = d^*_{A,\delta} + d^+_A : L^2_{2,\delta}(\mathrm{ad}\, P \otimes T^*X) \to L^2_{1,\delta}(\mathrm{ad}\, P \otimes \Lambda^2_+ T^*X) \tag{3.18}$$

where $d^*_{A,\delta}$ is the adjoint operator with respect to the weighted norm:

$$\int_X W^2(d_A f, a) = \int_X W^2(f, d^*_{A,\delta} a) \tag{3.19}$$

where f and a have compact support. Then $d^*_{A,\delta} = W^{-2} d^*_A W^2$ and $D_{A,\delta}$ is Fredholm for δ which is smaller than the first non-zero eigenvalue of the self-adjoint elliptic operator L_α on $Y \times [0, \infty)$ (see [LM85],[Don02]). The index can be computed from [APS75a] which we consider next.

3.1.3 Formal Dimension and Index

In this section we give the formulas for the formal dimension of the moduli spaces $\mathcal{M}_\ell(X, \alpha)$.

$$\begin{aligned} \operatorname{Ind} D_A &= -\int_X \hat{A}(X) ch(S^+ \otimes \operatorname{ad} P) - \frac{1}{2}(\dim\ker(L_\alpha) - \eta_\alpha(0)) \\ &= -\int_X \hat{A}(X) ch(S^+) ch(\operatorname{ad} P) - \frac{1}{2}(\dim\ker(L_\alpha) - \eta_\alpha(0)) \\ &= -\frac{1}{2}\int_X (\mathcal{L} + \mathcal{E})(X)(3 - 8c_2(P)) - \frac{1}{2}(\dim\ker(L_\alpha) - \eta_\alpha(0)) \\ &= 8c_2(P) - \frac{3}{2}(\chi + \sigma + \eta_\theta(0))(X) - \frac{1}{2}h_\alpha + \frac{1}{2}\eta_\alpha(0). \end{aligned}$$

since by the Hirzebruch signature theorem

$$\int_X \frac{1}{3} p_1(\operatorname{ad} P) = \operatorname{Sign}(X) + \eta_\theta(0) \tag{3.20}$$

The relative second Chern class is the energy on X:

$$\ell = c_2(P) = \frac{1}{8\pi^2}\int_X Tr(F_A^2) \equiv CS(\alpha) \mod \mathbb{Z} \tag{3.21}$$

Together we get

$$8\ell - \frac{3}{2}(\chi+\sigma)(X) - \frac{1}{2}h_\alpha - \frac{3}{2}\eta_\theta(0) + \frac{1}{2}\eta_\alpha(0) \tag{3.22}$$

The Atiyah-Patodi-Singer rho invariant is $\rho(\alpha) = \eta_\alpha(0) - 3\eta_\theta(0)$ this gives the following formal dimension formula

$$\dim \mathcal{M}_\ell(X,\alpha) = 8\ell - \frac{3}{2}(\chi+\sigma)(X) - \frac{1}{2}(h^1_\alpha + h^0_\alpha) + \frac{1}{2}\rho(\alpha) \tag{3.23}$$

where $h^i_\alpha = dim_{\mathbb{R}} H^i(\Sigma, \operatorname{ad}\alpha))$ for $i = 0, 1$. The corresponding dimension formula of a Floer-type moduli space is

$$\dim \mathcal{M}_\ell(\Sigma\times\mathbb{R},\alpha,\beta) = 8\ell - \frac{1}{2}(h_\alpha + h_\beta) + \frac{1}{2}(\rho(\beta) - \rho(\alpha)) \tag{3.24}$$

with $h_\alpha = h^1_\alpha + h^0_\alpha$, and similarly for h_β with $\ell \equiv CS(\beta) - CS(\alpha) \mod \mathbb{Z}$.

3.2 Equivariant Gauge Theory

In this section we introduce the equivariant moduli space which will be our principal object of study: the instanton one moduli space. This equivariant moduli space is also studied in [BKS90] in a different context.

3.2.1 The Equivariant Moduli Space

Now in the equivariant setting we let $\pi = \langle t\rangle$ denote the finite cyclic group $\mathbb{Z}/p$ with p an odd prime acting smoothly and homologically trivially on X_0 extending a free π-action on its boundary $\partial X_0 = Y$. Let g be a π-invariant metric on $X = X_0 \cup \operatorname{End}(X)$ which is a product metric on the $\operatorname{End}(X) = Y \times \mathbb{R}$. Let A_0 denote a connection which extends the trivial product connection θ on the end $Y \times [0,\infty)$. The second relative Chern class of A_0 is an integer since

$$\ell = c_2(A_0) = \frac{1}{8\pi^2}\int_X Tr(F_{A_0}^2) \equiv CS(\theta) \in \mathbb{Z} \tag{3.25}$$

Consider the space of connections on P with fixed energy ℓ:

$$\mathcal{A}_\ell(\theta) = \{A_0 + a \,|\, a \in L^2_{2,\delta}(\Omega^1_X(\mathrm{ad}\, P))\} \tag{3.26}$$

and the corresponding gauge group $\mathcal{G}_\ell(\theta)$. The group π acts on the space of connections modulo gauge by pull-back, moreover since the action of $t \in \pi$ is orientation-preserving we have $c_2(t \cdot A_0) = c_2(A_0)$ so the charge is preserved. The quotient space $\mathcal{B}_\ell(\theta) = \mathcal{A}_\ell(\theta)/\mathcal{G}_\ell(\theta)$ inherits a π-action and with our choice of π-invariant metric the anti-self duality equations $F_A = -* F_A$ are π-invariant and so we get an induced action on $\mathcal{M}_\ell(X,\theta)$. We choose A_0 so that $c_2(A_0) = 1$, this is the equivariant instanton-one moduli space $\mathcal{M}_1(X,\theta)$ we study to extract information about the original π-action on X.

3.2.2 Equivariant Lifts

Equivariant gauge theory studies lifts of the π-action on (X,g) to a π-action on the principal $SU(2)$ bundle P over X and the induced π-action on the space of connections modulo gauge $\mathcal{B}(P)$. We give an outline below, for more details see [BM93] or [HL92]. Let $\mathrm{Diff}(X)$ denote the group of diffeomorphisms of X then there is an exact sequence:

$$1 \to \mathcal{G}(P) \to \mathrm{Aut}(P) \to \mathrm{Diff}(X) \tag{3.27}$$

where $\mathrm{Aut}(P)$ are bundle automorphisms of P which do not necessarily cover the identity. A lift of the π-action to P is a homomorphism $\pi \to \mathrm{Aut}(P)$ which projects back to π under the above map to $\mathrm{Diff}(X)$. Let $\mathcal{G}(\pi) \subset \mathrm{Aut}(P)$ denote the bundle automorphisms which cover the π-action on X, then we also have an exact sequence:

$$1 \to \mathcal{G}(P) \to \mathcal{G}(\pi) \to \pi \to 1 \tag{3.28}$$

The natural action of $\mathcal{G}(\pi)$ on $\mathcal{A}(P)$ is well-defined up to gauge transformations so we get an induced action of $\mathcal{G}(\pi)/\mathcal{G}(P) = \pi$ on $\mathcal{B}(P)$. Let $[A]$ denote a π-fixed connection in $\mathcal{B}(P)$ then the following sequence is exact:

$$1 \to \mathcal{G}_A \to \mathcal{G}_A(\pi) \to \pi \to 1 \tag{3.29}$$

where $\mathcal{G}_A$ denotes the stabilizer of A in the gauge group and $\mathcal{G}_A(\pi)$ denotes the stabilizer in $\mathrm{Aut}(P)$. There then exists a lift of the action on X to the principal bundle leaving the connection A invariant if and only if the above sequence splits. For irreducible $SO(3)$ connections the existence and uniqueness of a lift follows since $\mathcal{G}_A$ is trivial, but for irreducible π-fixed connections $[A]$ we have $\mathcal{G}_A = \{\pm 1\}$ and so we have $\mathcal{G}_A(\pi)$ is either $\mathbb{Z}/2p$ or $\mathbb{Z}/p \times \mathbb{Z}/2$. So in general there will be a lift when p is odd. Alternatively, we can always consider the double cover $\widetilde{\pi} = \mathbb{Z}/2p$ action on the bundle which covers the π-action on X and leaves the connection A π-invariant. Choosing a different representative in $[A]$ gives an equivalent lift and if I is a set parametrizing equivalence classes of lifts then we get disjoint union of the fixed-set

$$\mathrm{Fix}(\mathcal{B}^*, \pi) = \bigsqcup_{i \in I} \mathcal{A}_i^*/\mathcal{G}_i = \bigsqcup_{i \in I} \mathcal{B}_i^* \tag{3.30}$$

where $\mathcal{A}_i$ are i-invariant connections and $\mathcal{G}_i$ are i-invariant gauge transformations. The fixed sets may intersect at a reducible where $\mathcal{G}_A \neq 1$ there may be more than one lift of the action leaving A invariant.

Theorem 3.2.1 ([Fur89],[BM93]). *The image of $\mathcal{B}_i$ in $\mathcal{B}$ is closed, $\mathcal{B}_i^*$ is a closed smooth submanifold of $\mathcal{B}^*$ of infinite co-dimension.*

With a π-invariant metric, the anti-self duality equations are π-invariant and so we get an induced action on the moduli space $\mathcal{M}(X, g) \subset \mathcal{B}(P)$. If a flat connection $\alpha \in \mathcal{R}(Y)$ is π-fixed we further get a π-action on $\mathcal{M}(X, \alpha)$. The following is a decomposition of the fixed set in the moduli space according to distinct equivariant lifts.

$$\mathcal{M}^*(X, g)^\pi = \bigsqcup_i \mathcal{M}_i^*(X, g) \cap \mathcal{B}_i \tag{3.31}$$

3.3 Compactification of Moduli Spaces

In the case of a closed definite 4-manifold, Uhlenbeck compactness [DK90] provides a picture for the ends of the moduli space. Informally, the result states that given an infinite sequence of ASD connections there are a finite set of points on the manifold and a subsequence up to gauge that converges to an ASD connection, where the curvature accumulates in integral amounts of the total energy around those points. This is the bubbling phenomenon in the instanton theory. When the manifold has cylindrical-ends, there is the possibility that energy is lost at the end $Y \times [0, \infty)$ and leads to broken-trajectories of the Chern-Simons flow on the cylinder.

3.3.1 Uhlenbeck Compactness

Let us recall the compactness theorem of Uhlenbeck for the instanton-one moduli space over a closed, simply-connected Riemannian four manifold (X, g).

Theorem 3.3.1 (Uhlenbeck,[DK90]). *Let $\{A_n\}$ denote a sequence of ASD connection on a $SU(2)$-bundle with charge $c_2(P) = 1$. Then there exists a subsequence also denoted by $\{A_n\}$ such that one of the following holds:*

- *For each A_n there exists a gauge-equivalent connection $\widetilde{A_n}$ such that $\{\widetilde{A_n}\}$ converges in the C^∞-topology on X to a ASD connection $A \in \mathcal{M}_1(X)$.*

- *There exists a point $x \in X$ and trivializations $\rho_n : X - \{x\} \times SU(2) \to E\,|_{X-x}$ such that $\rho_n^*(A_n\,|_{X-x})$ converges in C^∞-topology on compact subsets to the trivial product connection and the curvature densities $|\,F_{A_n}\,|^2$ converge as measures to $8\pi^2\delta(x)$.*

Uhlenbeck compactness for cylindrical-end four manifolds still holds, however there is the possibility that energy is lost down the cylindrical-end and this leads to weak limits in the following way. Suppose $\{A_n\} \subset \mathcal{M}_1(X,\theta)$ is an infinite sequence of g-ASD connections on X. Then after passing to a subsequence again still denoted by $\{A_n\}$ one of the following holds.

- the subsequence converges in the C^∞-topology on compact subsets of X to a g-ASD connection $A \in \mathcal{M}_1(X,\theta)$.
- there exist adapted bundles $P(0) \to X, P(1), ..., P(k) \to Y \times \mathbb{R}$ with corresponding L^2-finite g-ASD connections $A(0), A(1), ..., A(k)$ such that $[A_n\,|_X]$ converges in C^∞-topology on compact subsets of X to $A(0)$ and the curvature densities $|\,Tr(F^2_{A_n})\,|^2$ converge as measures to $|\,Tr(F^2_{A(0)})\,|^2$. Similarly, appropriate translations $[c^*_{t_n(i)}A_n\,|_{Y\times\mathbb{R}}]$ converge in C^∞-topology on compact subsets of $Y \times \mathbb{R}$ to $A(i)$ for $i = 1...k$. These connections have compatible boundary values $\partial_\infty([A(0)]) = \partial^-_\infty([A(1)])$ and $\partial^-_\infty([A(i)]) = \partial^+_\infty([A(i+1)])$ and $[A_k] = \theta$.

We can then define weak limits as idealized ASD connections given as a tuple of gauge equivalence class of connections $[\mathbf{A}] := ([A_0], [A_1], \cdots, [A_k])$ where $[A_0] \in \mathcal{M}_{\ell_0}(X, \alpha_0)$ and $[A_i] \in \mathcal{M}_{\ell_i}(\Sigma \times \mathbb{R}, \alpha_{i-1}, \alpha_i)$, α_i are flat connections on Σ and have compatible boundary values $\partial_\infty(A_i) = \partial_\infty(A_{i+1})$. The following is convergence without loss of energy ([MMR94],6.3.3) stated in our setting for the moduli space $\mathcal{M}_1(X,\theta)$ which summarizes the compactness properties of a sequence of ASD connections.

Theorem 3.3.2 (Bubbling Phenomenon and Energy Splitting)**.** *Every infinite sequence of L^2-finite g-ASD connections in $\mathcal{M}_1(X,\theta)$ which does not have a convergent subsequence in $\mathcal{M}_1(X,\theta)$ will have a subsequence which bubbles* *to a standard instanton on the four-sphere or else has an ideal limit point consisting of an idealized g-ASD connection* $[\boldsymbol{A}] := ([A_0], [A_1], \cdots, [A_k = \theta])$ *which satisfies* $\sum_{i=0}^{k} \mathcal{YM}(A_i) = 1$.

3.3.2 The Taubes Construction

The Taubes map constructs nearly anti-self dual connections on X by grafting the standard instanton on the four sphere S^4 to the trivial product connection on X using a cut-off function. This process disturbs anti-self duality but the self-dual part has a small uniform bound on the curvature and this allows for a small perturbation so that the resulting connection is anti-self dual. We outline the construction below, see [Tau87] for details and also [BKS90] for the equivariant case.

Choose an oriented orthonormal frame at a point $x \in X$ to identify a neighbourhood of x with a small ball of radius λ_0. Since X has bounded geometry we can make this choice of λ_0 independent of the point by letting it be less than the injectivity radius of X. Define a degree one map $f : X \to S^4$ which maps the point x to the north pole and collapses everything outside the ball to the south-pole. Take the standard instanton I on a principal $SU(2)$ bundle Q with $c_2(Q) = 1$ with concentration $1/\lambda$ at the north-pole for $\lambda \in (0, \lambda_0)$. Pulling back I and extending it using a cut-off function to be trivially flat outside of the ball neighbourhood gives a connection $A(x, \lambda)$ which satisfies the following bounds:

$$\Big(\int_X W \,|\, F_A \,|^{\,p} \Big)^{1/p} \leq (\text{const}) W^{1/p} \lambda^{4/p-2} \tag{3.32}$$

and the self dual part

$$\Big(\int_X W \,|\, F_A^+ \,|^{\,p} \Big)^{1/p} \leq (\text{const}) W^{1/p} \lambda^{2/p} \tag{3.33}$$

a different choice of orthonormal frame gives a gauge-equivalent connection.

The Taubes map is given by $T : X \times (0, \lambda_0) \to \mathcal{B}(X)$ sending (x, λ) to $[A(x, \lambda)]$. Consider a perturbation term $A(x, \lambda) + a(x, \lambda)$ with the condition that $F^+_{A+a} = 0$, this leads to the following equation

$$0 = F^+_A + d^+_A a + (a \wedge a)^+ \tag{3.34}$$

and since the operator d^+_A is not elliptic, the condition $a = d^*_A u$ for some $u \in \Omega^+_X(\operatorname{ad} P)$ is imposed. Now the equation to be solved is:

$$d^+_A d^*_A u = -F^+_A - (d^*_A u \wedge d^*_A u)^+ \tag{3.35}$$

for a smooth solution u which then gives an ASD connection $\widetilde{A}(x, \lambda) = A(x, \lambda) + a(x, \lambda)$ where $a = d^*_A u$. Define a map $\widetilde{T} : X \times (0, \lambda_0) \to \mathcal{M}^\delta_1(X, \theta)$ sending $[A(x, \lambda)]$ to $[\widetilde{A}(x, \lambda)]$. Here δ denotes the δ-decay used in the Sobolev completion.

Proposition 3.3.3 ([Tau87]). *There exists $\delta_1 > 0$ such that for any $\delta \in (0, \delta_1)$, the moduli space $\mathcal{M}^\delta_1(X, \theta)$ is non-empty. There is an open set $\mathcal{K} \subset \mathcal{M}_1(X, \theta)$ such that for small λ_0, $\mathcal{K}$ is diffeomorphic to $X \times (0, \lambda_0)$ and isotopic in $\mathcal{B}$ to the image of the Taubes map $\widetilde{T}$.*

Since π acts by orientation preserving isometries and the metric is π-invariant we have by construction

$$t \cdot \widetilde{T}(x, \lambda) = \widetilde{T}(t \cdot x, \lambda) \tag{3.36}$$

This gives an equivariant collar in the moduli space and a partial compactification

$$\overline{\mathcal{M}}_1(X, \theta) = \mathcal{M}_1(X, \theta) \cup X \times (0, \lambda_0) \tag{3.37}$$

consisting of highly concentrated ASD connections. In particular, the equivariant moduli space $\mathcal{M}_1(X, \theta)$ is non-empty when X^π is non-empty. For connections $[A] \in X \times (0, \lambda_0)$ Taubes also gives that $H^2_A = 0$ [Law85, Theorem 3.38, pg. 81] so that a neighbourhood of the collar is smooth 5-manifold and these connections are irreducible. The fixed set X^π give rise a family of ASD connections which correspond to an equivariant lifts of the π-action on X to a $\widetilde{\pi} = \mathbb{Z}/2p$-action on the principal $SU(2)$-bundle.

We will study the compactification of the fixed-set $\mathcal{M}_1(X,\theta)^\pi$ to obtain information about the action on X.

3.4 Linear Actions on the Four Sphere

Consider the unit four-sphere S^4 lying in $\mathbb{R}^5 = \mathbb{C}^2 \times \mathbb{R}$ described by $|z_1|^2 + |z_2|^2 + x^2 = 1$. Let $\pi = \mathbb{Z}/p = \langle t \rangle$ with p odd prime and $t = e^{2\pi i/p}$. A π-action on S^4 is defined by the representation $\mathbb{C}^2(a,b) \oplus \mathbb{R}$, i.e. the map $t \cdot (z_1, z_2, x) \mapsto (t^a z_1, t^b z_2, x)$. Recall this action has two isolated fixed points $(0,0,\pm 1)$ when $a, b \neq 0$ with rotation numbers at the north pole (a,b) and south pole $(a,-b)$. Since t^k for k non-zero modulo p is also a suitable generator of $\mathbb{Z}/p$, it will simplify matters if we change the generator so that the action is conjugate to $t(z_1, z_2, x) = (tz_1, t^q z_2, x)$ where $ka \equiv 1 \pmod p$ and $q \equiv kb \pmod p$ and the rotation numbers at north and south poles are given by $(1,q)$ and $(1,-q)$.

Remove an invariant disk D around the south pole and consider the non-compact 4-manifold $X = (S^4 - D) \cup \mathrm{End}(X)$ where $\mathrm{End}(X)$ is orientation preserving isometric to a product $S^3 \times [0,\infty)$ with the product metric. Consider the equivariant instanton-one moduli space $(\mathcal{M}_1(X,\theta), \pi)$ of ASD connections with one unit of total Yang-Mills energy and asymptotic to the trivial product connection on the cylindrical-end. This moduli space is 5-dimensional and has a one-parameter family γ of π-fixed ASD connections emerging from the Taubes collar $X \times [0, \lambda_0)$ generated by the fixed point on the north pole. This arc γ corresponds to a lift of the action on X to a $\widetilde{\pi}$-bundle structure which admits one-parameter family of $\widetilde{\pi}$-invariant ASD connections where $\widetilde{\pi} = \langle \widetilde{t} \rangle$ with $\widetilde{t} = e^{\pi i/p}$.

We can determine the equivariant bundle structure from the map f used in the construction of the Taubes collar as follows. Let the fixed point p in X^π have rotation numbers (a, b). Take a small π-invariant neighbourhood of p and consider the map which sends the point p to the north pole and collapses everything outside this ball to the south pole. This is just the map $f\colon X \to S^4$ but now the four-sphere has a π-action which lifts to a principal bundle Q with $c_2(Q) = 1$ and has isotropy representation $\pm(b-a)$ at the north pole and $\pm(a+b)$ at the south pole [FL86]. Pulling-back this equivariant-bundle Q under the map f gives us an equivariant bundle structure on X. The isotropy representations over the fixed point $p = (a, b)$ has weights $\pm(b-a)$ and the action on the $P\,|_{\,\mathrm{End}(X)} = S^3 \times [0, \infty) \times SU(2)$ is given by

$$\widetilde{t} \cdot (x, s, U) = (tx, s, \phi(\widetilde{t})U) \tag{3.38}$$

where ϕ is the isotropy representation $\widetilde{\pi} \to SU(2)$ with weights $\pm(a+b)$, that is

$$\widetilde{t} \mapsto \begin{pmatrix} \widetilde{t}^{a+b} & \\ & \widetilde{t}^{-(a+b)} \end{pmatrix} \tag{3.39}$$

Since there are no reducibles, the closure of γ in the moduli space must give rise to the only possible energy splitting $\mathcal{M}_0(X, \theta) \times \mathcal{M}_1(\theta, \theta)$ where the latter moduli space is 5-dimensional and leaves behind the flat equivariant bundle $\mathcal{M}_0(X, \theta)$ containing the trivial product connection and formal dimension -3. Note the dimension count $-3 + 3 + 5 = 5$ since there is a gluing parameter $SO(3)$ for the trivial product connection:

$$\mathcal{M}_0^\pi(X, \theta) \times_\theta \mathcal{M}_1^\pi(X, \theta) \to \mathcal{M}_1^\pi(X, \theta) \tag{3.40}$$

There must then exist $\widetilde{\pi}$-invariant ASD connections on P over the cylinder $S^3 \times \mathbb{R}$. Modding out by the involution in $\mathbb{Z}/2p$ gives us the adjoint π-equivariant $SO(3)$ principal bundle and a one-parameter family of π-invariant $SO(3)$ ASD connections over $S^3 \times \mathbb{R}$, which then descend to a one-parameter family of $SO(3)$ instantons on the the quotient $L(p, q) \times \mathbb{R}$ with Pontryagin charge $4/p$. The action on the $SO(3)$-bundle is given by the adjoint of the isotropy representation $\operatorname{ad}\phi : \pi \to SO(3)$ $t \mapsto \begin{pmatrix} 1 & \\ & t^{a+b} \end{pmatrix}$ where $\mathbb{Z}/p = \langle t \rangle$. In the limit on $S^3 \times \mathbb{R}$ the trivial product connection descends to a flat reducible connection on $L(p, q)$ whose $SO(3)$ holonomy representation is isomorphic to the adjoint isotropy representation $\operatorname{ad}\phi$. In particular the holonomy number of the the flat connection in the quotient at $+\infty$ is $a + b \equiv 1 + q \pmod p$.

To check this we compute the formal dimension of $SO(3)$-instantons on $L(p, q) \times \mathbb{R}$. Let $m' = 1+q$ and $m = q-1$ be the $\mathbb{Z}/2p$ weights of the isotropy representations on the equivariant bundle and corresponding flat connections α and β. The $SO(3)$ energy in the quotient is given by the difference of the Chern-Simons invariants

$$CS(L(p; q), \beta) - CS(L(p; q), \alpha) \equiv q^*(\frac{(1 + q)^2 - (q - 1)^2}{p}) \equiv \frac{4q^*q}{p} \equiv \frac{4}{p}$$

modulo $4\mathbb{Z}$. Since the index of the anti-self duality deformation operator is given by

$$2\ell - \frac{1}{2}(h_\alpha + h_\beta) + \frac{1}{2}(\rho_\beta L(p; q) - \rho_\alpha L(p; q))$$

where $\ell \equiv CS(L(p; q), \beta) - CS(L(p; q), \alpha)$ we have

$$\dim \mathcal{M}_1^\pi(\theta, \theta) = \frac{8}{p} - 3 + n + \frac{2}{p} \sum_{k=1}^{p-1} \cot(\frac{\pi k}{p}) \cot(\frac{\pi q k}{p})(\sin^2(\frac{\pi k m'}{p}) - \sin^2(\frac{\pi k m}{p})) \tag{3.41}$$

where n is either 2, 1 or 0 depending on whether p divides both, one or none of m and m'. This formula is obtained by other methods in [Aus90] as the formal dimension of $\mathbb{Z}/p$-invariant ASD connections on the four sphere.

Chapter 4

Perturbations

In the closed definite case, the moduli space $\mathcal{M}_1(X)$ is the zero-section of an infinite dimensional vector bundle $\mathcal{A}(P) \times_{\mathcal{G}} \Omega^2_+(\operatorname{ad} P) \to \mathcal{B}(P)$. To get a manifold structure on $\mathcal{M}_1(X)$ there are usually two approaches. One approach due to Freed-Uhlenbeck [FU91] is to perturb a generic metric to get transversality. The other approach is to perturb the ASD equations ([DK90]4.3.6).

In the case of manifolds with cylindrical-ends, we are not able to perturb the metric since we require the metric to be a product on the cylinder and this may not be enough to get transversality. In the equivariant case, the situation is worse since we would have to perturb the metric equivariantly. In any case there are known obstructions to equivariant transversality, even in finite dimensions (see example 1.2 in [HL92]). It turns out however, that we can get an equivariant perturbation of the instanton equations that puts the moduli spaces into Bierstone general position (see [HL92]), an open and dense condition in the space of equivariant maps that gives the moduli space the structure of a Whitney stratifed space [Bie77].

4.1 Equivariant General Position

Let G denote a compact Lie group and let M, N be smooth G-manifolds with P a smooth G-invariant submanifold in N. We extend the usual definition of transversality by calling a smooth G-equivariant map $f : M \to N$ G-**equivariantly transverse** to P if $P \cap \mathrm{Im}(f) = \emptyset$ or

$$df_x(T_xM) \oplus T_yP = T_yN$$

for all $x \in M$ and $y = f(x) \in P$. However, there exist obstructions to perturbing a map f equivariantly into general position. The following example illustrates this in the simple case of a smooth G-map between finite dimensional vector spaces.

Example 4.1.1 (See also [HL92], example 1.2)**.** *Let V and W be finite dimensional G-vector spaces and $F : V \to W$ a smooth equivariant G-map that sends the origin to the origin. Suppose F is G-equivariantly transverse to $0 \in W$, then $dF_0 : V \to W$ is a surjective linear G-map and we get a G-isomorphism $V \cong \ker(dF_0) \oplus W$. In particular we see that W is a sub-representation of V, so*

$$[V] - [W] \in R^+(G). \tag{4.1}$$

Conversely, suppose W is a sub-representation of V, then there exists a surjective linear G-map $L : V \to W$ and a G-equivariant homotopy $F_t : V \to W$ defined by $F_t = F(1-t) + tL$ that sends the origin to the origin and perturbs $F_0 = F$ to equivariant general position. Thus we can realize an obstruction which is that $[V] - [W]$ be an actual representation.

Another approach is to consider the idea that a G-manifold M has a natural stratification consisting of points of the same orbit type. However, according to Bierstone, stratumwise transversality is not a generic condition in the sense that although the subspace of equivariant maps which are stratumwise transverse to P are dense in $\mathcal{C}^\infty_G(M, N)$ they are not generally open.

Bierstone's approach is to first consider defining the problem of equivariant transversality at the origin of W with respect to $0 \in V$ of a smooth, $f : V \to W$ two G-vector spaces V and W. The space $\mathcal{C}^\infty_G(V, W)$ of smooth G-equivariant maps is a module over the ring $\mathcal{C}^\infty_G(V)$ of smooth G-invariant functions on V. Then there exists a finite set of polynomial generators $F_1, \cdots F_k$ of $\mathcal{C}^\infty_G(V, W)$ so that

$$f(x) = \sum_{i=1}^{k} h_i(x) F_i(x) \tag{4.2}$$

where $h_i(x) \in \mathcal{C}^\infty_G(V)$. We can write this slightly differently as

$$f(x) = U \circ \text{graph}(h(x)) \tag{4.3}$$

where $U : V \times \mathbb{R}^k \to \mathbb{R}$ defined by $U(x, h) = \sum_{i=1}^{k} h_i F_i(x)$ and $\text{graph}(h) : V \to V \times \mathbb{R}^k$ defined by $\text{graph}(h(x)) = (x, h_1(x), \cdots, h_k(x))$. Then $f^{-1}(0) = U^{-1}(0) \cap \text{graph}(h)$.

Definition 4.1.1 ([Bie77])**.** Let $f : V \to W$ be a smooth map between G-vector spaces. Then f is in G**-equivariant general position** with respect to $0 \in W$ at $0 \in V$ if the graph of h is stratum-wise transverse to the affine algebraic variety $U^{-1}(0)$ at $0 \in V$.

This notion is well-defined in the sense that it does not depend on the choice of generators F_i and h_i.

Definition 4.1.2. Let $f : V \to W_1 \times W_2$ be a smooth G-equivariant map between G-vector spaces. Then f is in general position with respect to $W_1 \times \{0\}$ at $0 \in V$ if the projection $pr_2 \circ f : V \to W_2$ is in general position with respect to $0 \in W_2$ at $0 \in V$.

Definition 4.1.3 (Equivariant General Position)**.** Let $f : M \to N$ be a smooth equivariant map between G-manifolds and P a G-invariant submanifold of N. Then f is in equivariant general position with respect to P at x if for any slice of the orbit Gx, the G_x-equivariant map $df_x\,|_S : T_xS \to T_{f(x)}N$ is in general position with respect to $T_{f(x)}P$ at $0 \in T_xS$. A smooth equivariant map $f : M \to N$ is in general position with respect to a G-invariant submanifold P of N if it is in general position with respect to P at every point $x \in f^{-1}(P)$.

Theorem 4.1.2 ([Bie77]). *The subspace of smooth equivariant maps in general position with respect to P is open and dense in $\mathcal{C}^\infty_G(M, N)$ with respect to the C^∞-topology.*

4.2 Wilson Loop Perturbations

It is often the case that for general integral homology 3-spheres, perturbations of the Chern-Simons functional are used so that the critical points are non-degenerate flat connections. In the case of Brieskorn homology spheres the representation variety $\mathcal{R}(\Sigma(a, b, c))$ is already non-degenerate, however further perturbations of the ASD equations are still needed so that all the Floer moduli spaces are regular. In the non-equivariant setting, this was done by Floer [Flo88] or by Donaldson:

Proposition 4.2.1 ([Don02, pg 145, Prop. 5.17]). *For arbitrary small perturbations η, the critical points of $CS + \eta$ are non-degenerate and all the perturbed Floer instanton moduli spaces are regular.*

In our case we adapt Floer's method [Flo88, 2c] to make equivariant perturbations of the ASD equations over a cylindrical end four manifold X to obtain moduli spaces in Bierstone general position (see [HL92] for the case of a closed 4-manifold X and chart-by-chart perturbations).

For these we use Wilson loop perturbations in free π-orbits of embedded circles in X. The non-equivariant case is described in [Don87, pg.400-401]. We will review the construction of these equivariant perturbations below with notation from [Sav00, pg.129-130]. Let $\gamma : S^1 \times D^3 \to X$ be an embedded loop in X which is slightly thickened. Given a connection A and $x \in \gamma(S^2 \times D^3)$ let $\mathrm{Hol}_A(\gamma, x)$ denote the holonomy around the loop γ_x parallel to γ. If we denote by $\Pi : SU(2) \to \underline{su}(2)$ the map $u \mapsto u - \frac{1}{2} tr(u)\,\mathrm{Id}$ then $\Pi\,\mathrm{Hol}_A(\gamma)$ defines a section on $\mathrm{ad}(P)$ over $\gamma(S^1 \times D^3)$. If ω is 2-form compactly supported in $\gamma(S^1 \times D^3)$ then

$$\eta(\omega, \gamma, A) = \omega \otimes \Pi\,\mathrm{Hol}_A(\gamma) \in \Omega^2(X, \mathrm{ad}\,P) \tag{4.4}$$

Now given a finite set of embedded loops γ_i and 2-forms ω_i for $i = 1 \ldots m$ then define a linear combination

$$\sigma(A) = \sum_i^m \varepsilon_i \eta(\omega_i, \gamma_i, A) \tag{4.5}$$

and consider the perturbed ASD equations

$$F_A = - * F_A + \sigma_+(A). \tag{4.6}$$

where $\sigma_+(A)$ is the orthogonal projection onto $\Omega^2_+(X, \mathrm{ad}\,P)$. In the equivariant setting we use π-orbits of m freely embedded loops γ_i and consider

$$\sigma(A) = \sum_{i=1}^m \sum_{s \in \pi} \varepsilon_i \eta((s^{-1})^* \omega_i, s(\gamma_i), A) \tag{4.7}$$

and define

$$\widehat{\sigma}(A) = \sum_{t \in \pi} (t^{-1})^* \sigma(t^* A). \tag{4.8}$$

The perturbed section $F_A^+ + \widehat{\sigma}_+(A)$ is now $\mathcal{G}(\pi)$-equivariant and so the perturbed moduli space inherit a π-action as before.

Since Bierstone general position is an open-dense condition, a generic equivariant perturbation of the ASD equations give the moduli spaces the structure of a Whitney stratified space (see [Bie77] or [HL92] for details).

Chapter 5

Proof of Main Results

We are now in a position to prove the main results stated in the introduction.

5.1 Preliminaries

Let (X_0, π) denote a smooth, simply-connected four-manifold with even-negative definite intersection form Q_{X_0} and a smooth group action of $\pi = \mathbb{Z}/p$ which is homologically-trivial and free on the boundary $\partial X_0 = \Sigma$ that is an integral homology three sphere. Denote by (X, g) the non-compact Riemannian manifold $X = X_0 \cup \text{End}(X)$ where $\text{End}(X)$ is orientation-preserving isometric to the product $\Sigma \times \mathbb{R}$ with product metric on the end $g\,|_{\text{End}(X)} = ds^2 + g_\Sigma$ and extend the π-action in the obvious way and let $(\mathcal{M}_1(X, \theta), \pi)$ denote the equivariant moduli space of g-ASD connections asymptotic to the trivial product connection on the end with one unit of total Yang-Mills energy.

In general the fixed set X^π will consist of isolated fixed-points and disjoint spheres. The Taubes construction gives an equivariant smooth map $T : X \times (0, \lambda_0) \to \mathcal{M}_1(X, \theta)$ that is a diffeomorphism on its image, which consists of connections with highly concentrated curvature at a point in X with scale λ. In particular, at an isolated fixed point, we get a one-parameter family γ of π-fixed irreducible ASD connections propagating into the moduli space, according to section 3.2 they correspond to a unique equivariant bundle structure which leaves these connections invariant. Similarly for the fixed 2-spheres. The following lemma determines the equivariant bundle structures explicitly.

Lemma 5.1.1. *If a fixed point has rotation numbers (a, b) then the the equivariant lift of γ has isotropy representations over the fiber of this point given by $\mathbb{Z}/2p$-weights $\pm(b-a)$ and over the other fixed points $\pm(a+b)$. Similarly for the fixed 2-spheres.*

Proof. This comes from a map $f : X \to S^4$ and by pulling-back an equivariant π-bundle Q over the four-sphere with $c_2(Q) = 1$ [FL86] exactly as in the section on linear actions on the four-sphere. □

Lemma 5.1.2 ([HL95]). *If the π-action has at least 3 fixed points then the γ_i represent distinct equivariant bundle structures and are therefore disjoint in $\mathcal{M}_1^*(X, \theta)$.*

Proof. Suppose there are at least three fixed points of the π-action, p_i, say with rotation numbers $(a_1, b_1), (a_2, b_2), (a_3, b_3)$. Each of these fixed points lies at the collar end of the moduli space and is part of a π-fixed arc γ_i.We would like to show that none of these arcs can connect with each other in the moduli space. Suppose γ connects p_1 and p_2, this creates a cancelling pair so that $(a_2, b_2) = (a_1, -b_1)$. We will use the presence of the third distinct fixed point p_3 to show a contradiction. Because the point p_1 is fixed, there is an π-invariant ball $B(p_1)$ with a linear action so this allows us to construct an equivariant degree one map $f_1 : X \mapsto S^4$, now we can pull-back the equivariant bundle structure $Q \mapsto S^4$ via f_1 and get an equivariant bundle (X, f_1^*Q). Similarly, we can do this with a map f_2 about the point p_2, this gives an equivariant bundle structure (X, f_2^*Q'). Since these bundle structures are equivalent, the isotropy at p_3 has to agree and a comparison shows that either $2a \equiv 0$ modulo p or $2b \equiv 0$ modulo p, in either case we get a contradiction. □

A similar argument also gives the following

Lemma 5.1.3. *If the π-action has at least 3 fixed points, then one-dimensional fixed set generated by the fixed points in the Taubes boundary $X \times (0, \lambda_0)$ cannot split energy in the equivariant compactification of $\mathcal{M}_1(X, \theta)$ by $\mathcal{M}_0(X, \theta) \times \mathcal{M}_1(\theta, \theta)$.*

Proof. The idea is that $\mathcal{M}_0(X, \theta)$ has zero energy, so it leaves behind a flat equivariant bundle which identifies the isotropy over the fibres of each fixed point. Suppose γ is a one-parameter family of π-fixed ASD connections generated at the Taubes boundary from the fixed point with rotation numbers (a, b). Then the corresponding equivariant lift has isotropy over the fiber of this fixed point and is given by $(b - a, -(b - a))$ and $(a + b, -(a + b))$ over the other fixed points. In such a energy splitting a flat equivariant bundle identifies the isotropy over all the points, so $a + b = \pm(b - a)$ and this forces either $2a \equiv 0 \pmod{p}$ or $2b \equiv 0 \pmod{p}$. Since p is odd and (a, b) are rotation numbers for a fixed-point we get a contradiction. □

Lemma 5.1.4. *If a fixed 2-sphere in X_0 represents a non-trivial homology class with non-zero self-intersection, then the 3-dimensional π-fixed stratum generated at the Taubes boundary cannot bound off in $\mathcal{M}_1^*(X, \theta)$ and therefore must give way to a energy-splitting.*

Proof. A fixed 2-sphere S in X represents a non-trivial homology class $[S] \in H_2(X, \mathbb{Z})$. Let F denote the 3-dimensional stratum in the moduli space which arises from the Taubes boundary and suppose $\partial F = S$. Let $[c] = \mu([S]) \in H^2(\mathcal{B}^*, \mathbb{Z})$ and $i : X \to \mathcal{M}_1(X, \theta)$ denote the inclusion map, then $i^*\mu([S])$ is the Poincaré dual $PD([S]) \in H^2(X, \mathbb{Z})$ ([DK90] 5.3)and so $\langle i^*c, \partial[F] \rangle$ evaluates non-trivially. On the other hand, we have $\langle i^*c, \partial[F] \rangle = \langle i^*\delta(c), [F] \rangle = 0$ giving a contradiction. Thus F cannot bound off in the irreducible component of the moduli space and so must give rise to an energy splitting. □

5.2 Proof of Theorem A

Now suppose that π acts on $X = X_0 \cup \mathrm{End}(X)$ with just isolated fixed points in X_0. Consider the equivariant moduli space $(\mathcal{M}_1(X,\theta),\pi)$. According to the lemma of section 5.1, the equivariant compactification of the fixed-set $\mathcal{M}_1(X,\theta)^\pi$ that arises from the Taubes boundary must limit to a connection that takes energy down the cylindrical-end $\Sigma(2,3,5) \times [0,\infty)$ and gives an charge-splitting. This will involve the flat connections of $\Sigma(2,3,5)$, we record here a table that gives the necessary values for index calculations.

α	(ℓ_1,ℓ_2,ℓ_3)	$\mu(\alpha)$	$\rho(\alpha)/2$	$-8CS(\alpha) \mod 8$
1	(1,2,2)	5	-97/30=-3.23333...	49/15=3.266666...
2	(1,2,4)	1	-73/30=-2.43333...	1/15=0.066666...

Table 5.1: *For each flat connection α of $\Sigma(2,3,5)$ are listed values for the Floer μ-index modulo 8, one-half the Atiyah-Patodi-Singer ρ-invariant and -8 times the Chern-Simons invariant of the given flat connection. The values for the ρ-invariant can be computed using a flat $SO(3)$-cobordism to a disjoint union of lens spaces (see [Sav00, pg. 144], also the MAPLE program supplied in Appendix 6.2.3.*

The energy in $\mathcal{M}(\alpha_i,\theta)$ is given by $-CS(\Sigma(2,3,5),\alpha_i) \mod \mathbb{Z} \in (0,1]$. In an energy splitting, the moduli space has an end given by a local diffeomorphism

$$\mathcal{M}_{\ell_0}(X,\alpha_0) \times_{\alpha_0} \mathcal{M}_{\ell_1}(\alpha_0,\alpha_1) \times_{\alpha_1} \cdots \times_{\alpha_{k-1}} \mathcal{M}_{\ell_k}(\alpha_{k-1},\theta) \to \mathcal{M}_1(X,\theta)$$

where $\{\alpha_i\}_{i=1}^{k-1}$ are irreducible flat connections on $\Sigma(2,3,5)$ – this then leads to a dimension count

$$5 = \dim \mathcal{M}_{\ell_0}(X,\alpha_0) + \sum_{i=1}^{k} \dim \mathcal{M}_{\ell_i}(\alpha_{i-1},\alpha_i)$$

with $\alpha_k = \theta$ and as the convergence is without loss of energy we get the condition $\sum_{i=0}^{k} \ell_i = 1$. The dimensions can be determined modulo 8 by the formulas

$$\dim \mathcal{M}(\alpha, \beta) \equiv \mu(\alpha) - \mu(\beta) - \dim \mathrm{Stab}(\beta) \pmod 8 \tag{5.1}$$

and $\dim \mathcal{M}(X, \alpha) \equiv -\mu(\alpha) - 3 \pmod 8$ [Flo88] where μ is the Floer index and $\mu(\theta) = -3$. Imposing the energy condition allows one to determine the exact geometric dimensions and since there are only 2 irreducible flat connections on $\Sigma(2,3,5)$ denoted by $\alpha_1 = (1,2,2)$ and $\alpha_2 = (1,2,4)$ we have only the possibilities in Table 5.2.

	Charge-Splitting	Dimension	Energy
A	$\mathcal{M}(X, \alpha_1) \times \mathcal{M}(\alpha_1, \theta)$	0+5	71/120+49/120=1
B	$\mathcal{M}(X, \alpha_1) \times \mathcal{M}(\alpha_1, \alpha_2) \times \mathcal{M}(\alpha_2, \theta)$	0+4+1	71/120+2/5+1/120=1
C	$\mathcal{M}(X, \alpha_2) \times \mathcal{M}(\alpha_2, \theta)$	4+1	119/120+1/120=1
D	$\mathcal{M}(X, \theta) \times \mathcal{M}(\theta, \theta)$	0+5	0+1

Table 5.2: *All possible energy splitting in the compactification of* $\mathcal{M}_1(X, \theta)$.

We now investigate whether any of the charge-splittings given in Table 5.2 contain π-invariant ASD connections:

$$\mathcal{M}^{\pi}_{\ell_0}(X, \alpha_0) \times_{\alpha_0} \mathcal{M}^{\pi}_{\ell_1}(\alpha_0, \alpha_1) \times_{\alpha_1} \cdots \times_{\alpha_{k-1}} \mathcal{M}^{\pi}_{\ell_k}(\alpha_{k-1}, \theta) \to \mathcal{M}^{\pi}_1(X, \theta) \tag{5.2}$$

It follows from immediately from Lemma 5.1.3 that case D is ruled out. We now rule out the possibility of a 1-dimensional fixed set in the equivariant moduli space $(\mathcal{M}_1(X, \theta), \pi)$ splitting in case B of Table 5.2 in the following way. Suppose there was such a splitting, then there would exist a local diffeomorphism $\mathcal{M}^{\pi}_{71/120}(\alpha_1) \times_{\alpha_1} \mathcal{M}^{\pi}_{2/5}(\alpha_1, \alpha_2) \times_{\alpha_2} \mathcal{M}^{\pi}_{1/120}(\alpha_2, \theta) \to \mathcal{M}^{\pi}_1(X, \theta)$, but since fixed-sets in the moduli-space occur in even-codimension; the only possibility is 0+0+1. But a non-empty Floer moduli space in general-position has a translation action and thus must be at least one-dimensional. However we will prove a slightly stronger statement in the following:

Lemma 5.2.1. *The moduli space* $\mathcal{M}_{\ell}(\alpha_1, \alpha_2)$ *does not support* π*-fixed ASD* *connections with energy* $\ell = 2/5$ *for any odd prime* $p \geq 7$.

Proof. If there exists a π-fixed ASD connection with energy $\ell = 2/5$ in $\mathcal{M}^{\pi}_{\ell}(\alpha_1, \alpha_2)$ then it corresponds to an equivariant lift of the π-action to the principal bundle which leaves that connection invariant. Since a π-invariant connection descends to an $SO(3)$ connection on the cylinder $Q \times \mathbb{R}$ where $Q = \Sigma(2, 3, 5)/\pi$ is a rational homology 3-sphere, the moduli space in the quotient must be non-empty. Let α'_1 and α'_2 denote the irreducible limiting flat connections on $Q \times \mathbb{R}$. The connection in the quotient has energy or Pontryagin charge $4\ell/p = 8/5p$, however, a non-empty moduli space must have energy that is congruent modulo $4\mathbb{Z}$ (cf. [Sav00, Remark 5.6, pg. 102]) to the difference of the $SO(3)$ Chern-Simons invariants $CS(Q, \alpha'_2) - CS(Q, \alpha'_1)$. It follows from Auckly's formula (Theorem 2.4.7) that this difference has the form $n/30$ for some integer n. But now $\dfrac{n}{30} \not\equiv \dfrac{8}{5p} \mod 4\mathbb{Z}$ since if two rational numbers are congruent modulo integers then they must have the same denominator, but the former has denominator at most 30 and for the latter $p \geq 7$. It must be then that $\mathcal{M}^{\pi}_{\ell}(\alpha_1, \alpha_2)$ is empty.

□

It remains then to investigate the remaining cases $\mathcal{M}_{\ell}(\alpha_i, \theta)$; a similar argument as above provides the following more general proposition which will give us a necessary condition for the existence of π-invariant ASD connections:

Proposition 5.2.2. *Suppose a principal* $SU(2)$ *bundle over* $\Sigma(a_1, a_2, a_3) \times \mathbb{R}$ *admits* π*-invariant ASD connections with energy* $\ell \equiv \dfrac{e^2}{4a_1a_2a_3} \in (0, 1]$ *asymptotic to an irreducible flat connection* α *at* $-\infty$ *and the trivial product at* $+\infty$. *Then this connection descends to an* $SO(3)$ *ASD connection on the quotient* $Q \times \mathbb{R}$ *with energy* $4\ell/p$ *which limits to an irreducible connection still denoted by* α *at* $-\infty$ *and a flat* $U(1)$*-reducible connection* β *at* $+\infty$ *which has* $SO(3)$*-holonomy number* $\pm e \pmod p$.

Proof. Since an invariant connection descends to a $SO(3)$ ASD connection, the moduli space in the quotient is non-empty, this again gives the relation between the $SO(3)$ Chern-Simons invariants

$$CS(Q,\beta) - CS(Q,\alpha) \equiv \frac{4\ell}{p} \equiv \frac{e^2}{pa_1a_2a_3} \mod 4\mathbb{Z} \tag{5.3}$$

But the Chern-Simons invariant of the reducible connection is given by $CS(Q,\beta(k)) \equiv \dfrac{n_0 k}{p}$ for some integer n_0 such that $n_0(a_1a_2a_3) \equiv k \pmod p$ and where k is the $SO(3)$ holonomy number of the representation $\beta(k)$. On the other hand we have $CS(Q,\alpha) \equiv \dfrac{m}{a_1a_2a_3}$ for some integer m. Taking the difference gives

$$\frac{n_0k(a_1a_2a_3) - mp}{p(a_1a_2a_3)} \equiv \frac{e^2}{p(a_1a_2a_3)} \mod 4\mathbb{Z} \tag{5.4}$$

This implies that the numerators are congruent modulo $4p(a_1a_2a_3)\mathbb{Z}$ and so gives

$$k^2 \equiv e^2 \pmod p \tag{5.5}$$

since $\mathbb{Z}/p$ has no zero divisors completes the proof. □

This proposition gives a necessary condition for $\Sigma(a_1,a_2,a_3)\times\mathbb{R}$ with flat limits an irreducible at $-\infty$ and θ at $+\infty$ to admit invariant ASD connections: the energy in the numerator must be a square.

The irreducible flat connections α_1 and α_2 on $\Sigma(2,3,5)$ descend to irreducible flat connections on the quotient $\Sigma(2,3,5)/\pi$ which we still denote by α_i.

Theorem 5.2.3. *Let Q denote the rational homology sphere quotient $\Sigma(2,3,5)/\pi$ and $\ell = 49/120$. Then the formal dimension of the moduli space $\mathcal{M}_{4\ell/p}(Q\times\mathbb{R},\alpha_1,\beta)$ of $SO(3)$-ASD connections on the cylinder $Q\times\mathbb{R}$ with energy $4\ell/p$ that limit to α_1 at $-\infty$ and to a reducible connection β at $+\infty$ is 1 when the holonomy representation of the flat connection β is $\pm 7 \pmod p$. Similarly,* *when $\ell = 1/120$ the formal dimension of $\mathcal{M}_{4\ell/p}(Q \times \mathbb{R}, \alpha_2, \beta)$ is 1 when the holonomy representation of the flat connection β is $\pm 1 \pmod p$.*

Proof. This follows from the proposition. □

We can now finish the proof of Theorem A. Suppose there exists a smooth extension to X_0 with only isolated fixed points. If a fixed-point of the π-action on X_0 has rotation numbers (a, b) where a, b are non-zero integers well-defined modulo p then there is an equivariant lift corresponding to the 1-parameter family of π-fixed ASD connections in $\mathcal{M}_1^{\pi}(X, \theta)$ that it generates at the Taubes boundary. This is a $\widetilde{\pi}$-action on the principal $SU(2)$ bundle and has isotropy representation over the fixed point $p = (a, b)$ with weights $\pm(b - a)$ and the action on the $P|_{\text{End}(X)} = \Sigma(2, 3, 5) \times [0, \infty) \times SU(2)$ is given by

$$\widetilde{t} \cdot (x, s, U) = (tx, s, \phi(\widetilde{t})U) \tag{5.6}$$

where ϕ is the isotropy representation $\widetilde{\pi} \to SU(2)$ at ∞ with weights $\pm(a+b)$:

$$\widetilde{t} \mapsto \begin{pmatrix} \widetilde{t}^{a+b} & \\ & \widetilde{t}^{-(a+b)} \end{pmatrix} \tag{5.7}$$

we can mod out by the involution to get the π-equivariant adjoint $SO(3)$-bundle over $\Sigma(2, 3, 5) \times \mathbb{R}$ with action given by the adjoint representation :$t \mapsto \begin{pmatrix} 1 & \\ & t^{a+b} \end{pmatrix}$ with $\mathbb{Z}/p = \langle t \rangle$. In the limit at $+\infty$ on $\Sigma(2, 3, 5) \times \mathbb{R}$ the trivial product connection descends to a flat reducible connection on Q whose $SO(3)$ holonomy representation is isomorphic to the adjoint isotropy representation $\operatorname{ad}\phi$. Since this holonomy is ± 1 and $\pm 7 \pmod p$ this completes the proof.

We finish with an example that allows us to understand the equivariant compactification of the fixed set in the moduli space when the π-action on X is given by equivariant plumbing.

Example 5.2.4. *Let $p > 5$, we examine the fixed set in the moduli space for the equivariant plumbing actions (*see [Orl72] also [Fin77]*) which has one* *fixed* 2*-sphere and* 7 *isolated fixed points with rotation numbers*

$$(-4,5),(-3,4),(-2,3)\times 2,(-1,2)\times 3.$$

The equivariant lifts coincide with the holonomy representation $a+b\equiv\pm 1$ (mod p) *and so in particular, the* 1*-dimensional fixed sets originating in the Taubes boundary split energy in the equivariant compactification of* $\mathcal{M}_1(X,\theta)$ *according to case C of Table 5.2. The fixed* 2*-sphere is the boundary of a* 3*-dimensional stratum which splits according to case C of Table 5.2 with dimension 2+1.*

5.3 Index Computations

We can verify these claims by an index calculation using [APS75a], see the appendix for the MAPLE program. The formal dimension is given by

$$\dim \mathcal{M}_{4\ell/p}(Q\times\mathbb{R},\alpha,\beta)=\frac{8\ell}{p}-\frac{1}{2}(h_\alpha+h_\beta)+\frac{1}{2}(\rho_\beta(Q)-\rho_\alpha(Q)). \qquad (5.8)$$

Since α is irreducible and β is reducible we have $h_\alpha=0$ and $h_\beta=1$. The rho invariants for reducible flat connections are determined by Kwasik-Lawson ([KL93], pg.40) and is given by

$$\begin{aligned}\rho_\beta(Q)(l) = &-\frac{2}{p}\sum_{k=1}^{p-1}\sin^2(\frac{\pi kl}{p})+\frac{2}{30p}\sum_{k=1}^{p-1}\csc^2(\frac{\pi k}{p})\sin^2(\frac{\pi kl}{p})\\ &+\sum_{i=1}^{3}\frac{2}{pa_i}\sum_{m_1=0}^{p-1}\sum_{m_2=1}^{a_{i-1}}\cot(\frac{\pi m_2}{a_i})\cot(\frac{\pi m_1}{p}-\frac{\pi m_2 b_i}{a_i})\sin^2(\frac{\pi m_1 l}{p}) \qquad (5.9)\end{aligned}$$

where l is the rotation number for the holonomy representation of β in $SO(3)$. For irreducible flat connections α, the rho invariants can be calculated by an $SO(3)$-flat cobordism to a union of lens spaces $L(a_i,pb_i)$ using the mapping cylinder for the Seifert fibration of Q [Yu91] as in the case of $\Sigma(a_1,a_2,a_3)$

[Sav00, pg. 144]. In this way, the linear equivariant plumbing actions give us that the moduli space $\mathcal{M}_\ell(Q \times \mathbb{R}, \alpha_2, \beta)$ for $\ell = 1/120$ is non-empty and we get the following dimension:

$$\dim \mathcal{M}_\ell(Q \times \mathbb{R}, \alpha_2, \beta) = \frac{8}{p}(\frac{1}{120}) - \frac{1}{2} + \frac{1}{2}(\rho_\beta(Q)(1) - \rho_{\alpha_2}(Q)) = 1. \quad (5.10)$$

If we now turn-on a non-linear smooth π extension to X_0, we don't know if $\mathcal{M}_\ell(Q \times \mathbb{R}, \alpha_1, \beta)$ for $\ell = 49/120$ is non-empty but we have the following formal dimension:

$$\dim \mathcal{M}_\ell(Q \times \mathbb{R}, \alpha_1, \beta) = \frac{8}{p}(\frac{49}{120}) - \frac{1}{2} + \frac{1}{2}(\rho_\beta(Q)(7) - \rho_{\alpha_1}(Q)) = 1. \quad (5.11)$$

5.4 Proof of Theorem B

The argument for Theorem B is similar, but now there are more irreducible flat connections that contribute to the compactification but as we will see, not all the flat connections will contain the right amount of energy to realize a dimension splitting. To determine which ones contribute requires an extra computation. First we list the flat connections on $\Sigma(2, 7, 13)$ and its associated invariants that we will need later in Table 5.3.

α	(ℓ_1, ℓ_2, ℓ_3)	$\mu(\alpha)$	$\rho(\alpha)/2$	$-8CS(\alpha)$ mod 8
1	(1,2,4)	7	-969/182=-5.324175824...	289/91=3.17582417...
2	(1,2,6)	3	-1137/182=-6.24725274...	569/91=6.25274725...
3	(1,2,8)	7	-1081/182=-5.93956044...	233/91=2.56043956...
4	(1,2,10)	3	-801/182=-4.401098901...	9/91=0.09890109...
5	(1,4,2)	5	-733/182=-4.027472527...	225/91=2.47252747...
6	(1,4,4)	1	-1125/182=-6.181318681...	393/91=4.31868131...
7	(1,4,6)	5	-1293/182=-7.104395604...	673/91=7.39560439...
8	(1,4,8)	1	-1237/182=-6.796703297...	337/91=3.70329670...
9	(1,4,10)	5	-957/182=-5.258241758...	113/91=1.24175824...
10	(1,4,12)	1	-453/182=-2.489010989...	1/91=0.01098901...
11	(1,6,6)	1	-1189/182=-6.532967033...	361/91=3.96703296...
12	(1,6,8)	5	-1133/182=-6.225274725...	25/91=0.27472527...

Table 5.3: *For each flat connection α of $\Sigma(2,7,13)$, we have listed the the Floer μ-index modulo 8, 1/2 times the ρ-invariant (see MAPLE program in Appendix 6.2.3) and -8 times the Chern-Simons invariant.*

Table 5.4 lists all the possible dimension-splitting that contribute to a 5-dimensional moduli space. We seek to impose the energy conditions, that is we compute the dimensions by putting the Chern-Simon invariants of the flat connections in range of the energy $(0, 1]$. To determine the dimension of a splitting using the energy we will take case C as an example.

Example 5.4.1. *In case C the splitting is $\mathcal{M}_{\ell_0}(X, \alpha_0)\times\mathcal{M}_{\ell_1}(\alpha_0, \alpha_1)\times\mathcal{M}_{\ell_2}(\alpha_1, \theta)$, with $\mu(\alpha_0) = 3$ and $\mu(\alpha_1) = 1$, so let us take $\alpha_0 = (1, 2, 10)$ and $\alpha_1 = (1, 4, 12)$ as an example. The Chern-Simons can be computed from the table to be $CS(\alpha_0) \equiv -9/728$ modulo $\mathbb{Z}$ and $CS(\alpha_1) \equiv -1/728$. modulo $\mathbb{Z}$. Using that $\ell_0 \equiv CS(\alpha_0)$, $\ell_1 \equiv CS(\alpha_1) - CS(\alpha_0)$, and $\ell_2 \equiv -CS(\alpha_1)$, putting these values in the energy range $(0, 1)$ gives $\ell_0 = \frac{719}{728}$, $\ell_1 = \frac{1}{91}$ and $\ell_2 = \frac{1}{728}$. These satisfy $\sum_i \ell_i = 1$ and $\dim\mathcal{M}_{\ell_0}(X, \alpha_0) = 8\ell_0 + \rho(\alpha_0)/2 - 3/2 = 2$, $\dim\mathcal{M}_{\ell_1}(\alpha_0, \alpha_1) = 8\ell_1 - 1/2(h_{\alpha_0} + h_{\alpha_1}) + 1/2(\rho(\alpha_1) - \rho(\alpha_0)) = 2$ and $\dim\mathcal{M}_{\ell_2}(\alpha_1, \theta) = 8\ell_2 - \rho(\alpha_1)/2 - 3/2 = 1$.*

We would then need to do the same calculation with all α_0 and α_1 with

Case	Dimension Splitting	Flat Connections
A	0+4+1	$\mu(\alpha_0) = 5$, $\mu(\alpha_1) = 1$
B	0+2+2+1	$\mu(\alpha_0) = 5$, $\mu(\alpha_1) = 3$,$\mu(\alpha_2) = 1$
C	2+2+1	$\mu(\alpha_0) = 3$, $\mu(\alpha_1) = 1$
D	4+1	$\mu(\alpha_0) = 1$
E	0+5	$\mu(\alpha_0) = 5$
F	2+3	$\mu(\alpha_0) = 3$
G	0+5	$\alpha_0 = \theta$

Table 5.4: *All the possible dimension splitting without imposing energy conditions. This means that not all flat connections with the indicated Floer index will actually give the correct geometric dimension although they all give the same dimension modulo* 8*.*

$\mu(\alpha_0) = 3$ and $\mu(\alpha_1) = 1$, even though in the end we get the same dimensions modulo 8 we want to determine only the flat connections which give the positive geometric dimension splitting $2 + 2 + 1$. Ultimately we will want to rule fixed-set in charge splitting and since a similar argument as in 5.2 rules out cases A,B,C and G we will only give the computations for the remaining cases.

D	$\dim \mathcal{M}_{\ell_0}(X, \alpha)$	$\dim \mathcal{M}_{\ell_1}(\alpha, \theta)$
$\mu(\alpha) \equiv 1$	$8\ell_0 - 3/2 + \rho(\alpha)/2$	$8\ell_1 - \rho(\alpha)/2 - 3/2$
(1,4,4)	-4	9
(1,4,8)	-4	9
(1,4,12)	4	1
(1,6,6)	-4	9

Table 5.5: *In Case D among the irreducible flat connections α with $\mu(\alpha) \equiv 1$, only the flat connection with rotation numbers $(1, 4, 12)$ provides the right amount of energy that realizes the dimension splitting $4 + 1$.*

F	$\dim \mathcal{M}_{\ell_0}(X, \alpha)$	$\dim \mathcal{M}_{\ell_1}(\alpha, \theta)$
$\mu(\alpha) = 3$	$8\ell_0 + \rho(\alpha)/2 - 3/2$	$8\ell_1 - \rho(\alpha)/2 - 3/2$
(1,2,6)	-6	11
(1,2,10)	2	3

Table 5.6: *In case F among the irreducible flat connections α with $\mu(\alpha) \equiv 3$, only the flat connection with rotation numbers $(1, 2, 10)$ provides the right amount of energy that realizes the dimension splitting $2 + 3$.*

E	$\dim \mathcal{M}_{\ell_0}(X, \alpha)$	$\dim \mathcal{M}_{\ell_1}(\alpha, \theta)$
$\mu(\alpha) = 5$	$8\ell_0 + \rho(\alpha)/2 - 3/2$	$8\ell_1 - \rho(\alpha)/2 - 3/2$
(1,4,2)	0	5
(1,4,6)	-8	13
(1,4,10)	0	5
(1,6,8)	0	5

Table 5.7: *In case E among the irreducible flat connections α with $\mu(\alpha) \equiv 5$, the flat connections with rotation numbers $(1, 4, 2), (1, 4, 10), (1, 6, 8)$ provide the right amount of energy that realizes the dimension splitting $0 + 5$.*

Case	Charge Splitting	Flat Connections
D	$\mathcal{M}_{\ell_0}(X, \alpha_0) \times \mathcal{M}_{\ell_1}(\alpha_1, \theta)$	$\alpha_1 = (1, 4, 12)$
E	$\mathcal{M}_{\ell_0}(X, \alpha_0) \times \mathcal{M}_{\ell_1}(\alpha_0, \theta)$	$\alpha_0 = (1, 4, 2), (1, 4, 10), (1, 6, 8)$
F	$\mathcal{M}_{\ell_0}(X, \alpha_0) \times \mathcal{M}_{\ell_1}(\alpha_0, \theta)$	$\alpha_0 = (1, 2, 10)$

Table 5.8: *For case D we have $\ell_0 = 727/728$ and $\ell_1 = 1/728$. For case F, $\ell_0 = 719/728$ and $\ell_1 = 9/728$. In case E, there are three cases; if $\alpha_0 = (1, 4, 2)$ we have $\ell_0 = 503/728$ and $\ell_1 = 225/728$. If $\alpha_0 = (1, 4, 10)$ we have $\ell_0 = 615/728$ and $\ell_1 = 113/728$. And if $\alpha_0 = (1, 6, 8)$ we have $\ell_0 = 703/728$ and $\ell_1 = 25/728$.*

In Case E above, we can now rule out $\alpha_0 = (1, 4, 10)$ as the numerator in the energy $\ell_1 = 113/728$ is not a square, as a result the formal dimension in the quotient is not integral. The remaining cases give $a + b \equiv \pm 1, \pm 15, \pm 5$ and $\pm 3 \pmod p$.

Chapter 6

Appendix

In the first section we list the G-signature solutions to the E_8-problem for $p = 7$. Maple programs that were used for index computations are also given with examples.

6.1 G-Signature Solutions for $p = 7$

Let X_0 denote a four-manifold with boundary $\partial X_0 = \Sigma$ an integral homology 3-sphere. If a free $\mathbb{Z}/p = \langle t \rangle$ action on Σ extends to a locally linear, homologically-trivial action on X_0 (not necessarily free) then the G-signature theorem for manifolds with boundary is given in Atiyah-Patodi-Singer [APS75b]:

$$\mathrm{Sign}(X, t) = L(X, t) - \eta_t(0) \tag{6.1}$$

where $\eta_t(0)$ is the equivariant eta invariant or G-signature defect and $L(X, t)$ is the expression in the G-signature theorem for a closed 4-manifold. This invariant depends only on the 3-manifold Σ and not on how the action extends to the bounding four manifold X_0 nor does it depend on which four manifold the action is extending. To see this, suppose the action on Σ extends to another four manifold X_1, then consider the G-signature theorem on the closed four manifold double $X_0 \cup -X_1$.

What follows will be a list which gives solutions to the G-signature theorem for a $\mathbb{Z}/7$ locally-linear, homologically-trivial extension to X_0 with $\partial X_0 = \Sigma(2,3,5)$ and consisting of only isolated fixed-points:

$$-8 = \sum_{i=1}^{9} \left(\frac{t^{a_i}+1}{t^{a_i}-1} \right) \left(\frac{t^{b_i}+1}{t^{b_i}-1} \right) - \eta_t(0) \tag{6.2}$$

with rotation numbers (a_i, b_i). Since the action is homologically trivial, there are $b_2(X)+1 = 9$ fixed points and the value for the equivariant eta invariant is determined from the equivariant plumbing action to be $\eta_t(0) = 2.83798866$. There are 12 possible rotation numbers (when $p = 7$) for any given fixed point and as it may be possible for more than one fixed point to have the same rotation number, the total number of possible G-signature checks is given by counting combinations with repetition:

$$\frac{(n+k-1)!}{k!(n-1)!} \tag{6.3}$$

where we are choosing out of $n = 12$ rotation numbers $k = 9$ at a time. So the total number of checks is 167960 for $p = 7$. MAPLE found 45 solutions, where the G-signature came out to be -8 (up to rounding). The table below is given in two columns, the first is the number location in the array out of the total 167960, the second is the rotation numbers. Note that none of the solutions satisfy the condition that for all i, $a_i + b_i \equiv \pm 1 \pmod{p}$ or $a_i + b_i \equiv 0 \pmod{p}$ so these solutions cannot be realized as rotation numbers for a smooth, homologically-trivial extension by a $\mathbb{Z}/7$ action on $\Sigma(2,3,5)$.

10231	(1,1) (1,1) (1,1) (1,-3) (1,-1) (1,-1) (2,2) (2,2) (3,3)
18029	(1,1) (1,1) (1,2) (1,-3) (1,-2) (1,-1), (2,2) (2,2) (3,3)
22704	(1,1) (1,1) (1,3) (1,-3) (1,-3) (1,-1) (2,2) (2,2) (3,3)
24774	(1,1) (1,1) (1,3) (1,-1) (2,2) (2,-3) (2,-3) (3,3) (3,3)
25570	(1,1) (1,1) (1,-3) (1,-3) (1,-3) (1,-2) (1,-2) (2,2) (2,2)
26919	(1,1) (1,1) (1,-3) (1,-2) (1,-2) (2,2) (2,-3) (2,-3) (3,3)
27695	(1,1) (1,1) (1,-3) (1,-1) (2,2) (2,2) (2,2) (2,-2) (3,3)
27707	(1,1) (1,1) (1,-3) (1,-1) (2,2) (2,2) (2,3) (2,-3) (3,3)
27732	(1, 1) (1, 1) (1, -3) (1, -1) (2, 2) (2, 2) (3, 3) (3, 3) (3, -3)
28606	(1, 1) (1, 1) (1, -2) (1, -2) (1, -2) (2, 2) (2, 2) (2, 3) (3, 3)
36470	(1, 1) (1, 2) (1, 2) (1, 3) (1, -1) (2, 2) (2, 2) (3, 3) (3, 3)
37393	(1, 1) (1, 2) (1, 2) (1, -3) (1, -2) (1, -2) (2, 2) (2, 2) (3, 3)
42068	(1, 1) (1, 2) (1, 3) (1, -3) (1, -3) (1, -2) (2, 2) (2, 2) (3, 3)
43760	(1, 1) (1, 2) (1, 3) (1, -2) (2, 2) (2, -3) (2, -3) (3, 3) (3, 3)
44949	(1, 1) (1, 2) (1, -3) (1, -3) (1, -3) (1, -3) (2, 2) (2, 2) (2, -3)
45966	(1, 1) (1, 2) (1, -3) (1, -3) (2, 2) (2, -3) (2, -3) (2, -3) (3, 3)
46681	(1, 1) (1, 2) (1, -3) (1, -2) (2, 2) (2, 2) (2, 2) (2, -2) (3, 3)
46693	(1, 1) (1, 2) (1, -3) (1, -2) (2, 2) (2, 2) (2, 3) (2, -3) (3, 3)
46718	(1, 1) (1, 2) (1, -3) (1, -2) (2, 2) (2, 2) (3, 3) (3, 3) (3, -3)
51160	(1, 1) (1, 2) (2, -3) (2, -3) (2, -3) (2, -3) (2, -3) (3, 3) (3, 3)
53388	(1, 1) (1, 3) (1, 3) (1, -3) (1, -3) (1, -3) (2, 2) (2, 2) (3, 3)
54408	(1, 1) (1, 3) (1, 3) (1, -3) (2, 2) (2, -3) (2, -3) (3, 3) (3, 3)
55135	(1, 1) (1, 3) (1, 3) (1, -2) (2, 2) (2, 2) (2, 3) (3, 3) (3, 3)
57329	(1, 1) (1, 3) (1, -3) (1, -3) (2, 2) (2, 2) (2, 2) (2, -2) (3, 3)
57341	(1, 1) (1, 3) (1, -3) (1, -3) (2, 2) (2, 2) (2, 3) (2, -3) (3, 3)
57366	(1, 1) (1, 3) (1, -3) (1, -3) (2, 2) (2, 2) (3, 3) (3, 3) (3, -3)
62130	(1, 1) (1, 3) (2, 2) (2, 2) (2, -3) (2, -3) (2, -2) (3, 3) (3, 3)
62250	(1, 1) (1, 3) (2, 2) (2, 3) (2, -3) (2, -3) (2, -3) (3, 3) (3, 3)

62330	(1, 1) (1, 3) (2, 2) (2, -3) (2, -3) (3, 3) (3, 3) (3, 3) (3, -3)
63754	(1, 1) (1, -3) (1, -3) (1, -3) (2, 2) (2, 2) (2, 2) (2, 3) (2, 3)
68403	(1, 1) (1, -3) (2, 2) (2, 2) (2, 2) (2, 2) (2, -2) (2, -2) (3, 3)
68432	(1, 1) (1, -3) (2, 2) (2, 2) (2, 2) (2, 3) (2, -3) (2, -2) (3, 3)
68474	(1, 1) (1, -3) (2, 2) (2, 2) (2, 2) (2, -2) (3, 3) (3, 3) (3, -3)
68499	(1, 1) (1, -3) (2, 2) (2, 2) (2, 3) (2, 3) (2, -3) (2, -3) (3, 3)
68534	(1, 1) (1, -3) (2, 2) (2, 2) (2, 3) (2, -3) (3, 3) (3, 3) (3, -3)
68604	(1, 1) (1, -3) (2, 2) (2, 2) (3, 3) (3, 3) (3, 3) (3, -3) (3, -3)
71847	(1, 1) (1, -2) (2, 2) (2, 2) (2, 2) (2, 3) (2, 3) (2, 3) (3, 3)
80018	(1, 2) (1, 2) (1, 2) (1, 3) (1, -2) (2, 2) (2, 2) (3, 3) (3, 3)
80970	(1, 2) (1, 2) (1, 2) (1, -3) (1, -3) (2, 2) (2, 2) (2, -3) (3, 3)
83332	(1, 2) (1, 2) (1, 2) (2, 2) (2, -3) (2, -3) (2, -3) (3, 3) (3, 3)
84693	(1, 2) (1, 2) (1, 3) (1, 3) (1, -3) (2, 2) (2, 2) (3, 3) (3, 3)
88183	(1, 2) (1, 2) (1, 3) (2, 2) (2, 2) (2, 2) (2, -2) (3, 3) (3, 3)
88212	(1, 2) (1, 2) (1, 3) (2, 2) (2, 2) (2, 3) (2, -3) (3, 3) (3, 3)
88256	(1, 2) (1, 2) (1, 3) (2, 2) (2, 2) (3, 3) (3, 3) (3, 3) (3, -3)
91161	(1, 2) (1, 2) (1, -3) (2, 2) (2, 2) (2, 2) (2, 3) (2, 3) (3, 3)

Table 6.1: *This table lists rotation numbers for the 45 solutions to the G-signature theorem for manifolds with boundary* $\Sigma(2,3,5)$ *and* $p = 7$*. See MAPLE program 6.2.2.*

6.2 MAPLE Programs

We include a collection of MAPLE programs that where used during the investigation of the Thesis.

6.2.1 Four Sphere

This MAPLE program computes the dimension of the invariant stratum in the moduli space for linear $\mathbb{Z}/p$ actions on the four sphere. If this action has rotation numbers (a, b) and $(a, -b)$ at the poles then equivalently, this is the same dimension of $SO(3)$-instantons on the cylinder $L(p, r, s) \times \mathbb{R}$ where $L(p, a, b)$ is a lens space and the instanton has energy $8/p$ limiting to reducible $U(1)$ flat connections whose $SO(3)$ holonomy numbers are $b - a \mod p$ at

$-\infty$ and $a+b \pmod p$ at $+\infty$. This gives for any odd prime p not dividing a and b:

$$\frac{8}{p} - 3 + m + \rho(L(p,a,b);a+b) - \rho(L(p,a,b);b-a) = 1. \qquad (6.4)$$

where m is the number of non-trivial holonomy representations.

```
> restart:
  p := 7:

  Digits := 15:

  a := 2:
  b := 3:

  m := 2:
```

$$\mathrm{Rho}(l) := \frac{2}{p} \cdot sum\left(\cot\left(\frac{\mathrm{Pi}\cdot k\cdot a}{p}\right)\cdot\cot\left(\frac{\mathrm{Pi}\cdot k\cdot b}{p}\right)\cdot\sin^2\left(\frac{\mathrm{Pi}\cdot k\cdot l}{p}\right), k = 1..p-1\right):$$

$$Index := \frac{8}{p} - 3 + m + \mathrm{Rho}(a+b) - \mathrm{Rho}(b-a):$$

```
evalf(Index);
```

$$1.00000000000000 \tag{1}$$

```
>
```

6.2.2 G-Signature Solutions

This MAPLE program looks for solutions (rotation numbers) of the G-signature theorem for manifolds X_0 with boundary $\Sigma(2,3,5)$, where $G = \mathbb{Z}/7$ acts freely on $\Sigma(2,3,5)$ and extends to a locally linear, homologically-trivial action on X_0 with intersection form $Q_{X_0} = -E_8$.

```
> restart;
with(ArrayTools) :
with(combinat) :
with(linalg) :

 # accuracy control.
 Digits := 12 :

 p := 7 :

 # rotation numbers which arise from equivariant plumbing on the E8 diagram.
R := Array([[-4,-3,-2,-2,-1,-1,-1], [5, 4, 3, 3, 2, 2, 2]]) :

 A := sum(-cot('R[1, i]'·Pi/p)·cot('R[2, i]'·Pi/p), i=1..7) :

 B := -2·csc^2(1·Pi/p) :

 # equivariant eta invariant
 ETAg := 8 + (A + B) :
#evalf(ETAg);
 evalf(A + B − ETAg);

 k := choose([a, b, c, d, e, f, g, h, m, n, q, r, a, b, c, d, e, f, g, h, m, n, q, r, a, b, c, d, e, f, g, h, m,
    n, q, r, a, b, c, d, e, f, g, h, m, n, q, r, a, b, c, d, e, f, g, h, m, n, q, r, a, b, c, d, e, f, g, h, m, n,
    q, r, a, b, c, d, e, f, g, h, m, n, q, r, a, b, c, d, e, f, g, h, m, n, q, r, a, b, c, d, e, f, g, h, m, n, q, r]
    , 9) :

 # counts the  total number of  G−signature checks.
vectdim(k);

 # list of rotation numbers for p=7.
a := (1, 1) :
 b := (1, 2) :
 c := (1, 3) :
 d := (1,-3) :
 e := (1,-2) :
 f := (1,-1) :
 g := (2, 2) :
 h := (2, 3) :
 m := (2,-3) :
 n := (2,-2) :
q := (3, 3) :
 r := (3,-3) :


    # will store the value of the g−signature  for the range of cases in the vector k.    :   MAX
    range is 1..167960=vectdim.
UU := Array(5000..15000) :
```

```
# can change these values of i depending on the cases needed to be verified but also adjust
  the array UU above accordingly.
for i from 5000 to 15000 do
  U := 0 :

  for l from 1 to 9 do

  # for case i in UU, adds the rotation numbers of the fixed points.
  U := -cot(k[i, l][1]·Pi / p)·cot(k[i, l][2]·Pi / p) + U :

end do:
# the g-signature for case i and records it in UU.
UU[i] := evalf(U - ETAg) :

# if the g-signature value UU[i] is close -8 print the rotation numbers and the g - signature.
if -8.01 < UU[i] < -7.99 then
        print(i, k[i], UU[i]); end if;
end do:


# plot(UU, x = 0 ..5, y = -7 .. -9);
```

-8.

167960

5897, [1, 1, 1, 1, 1, 1, 1, 2, 1, 3, 1, -1, 1, -1, 2, -2, 2, -2], -8.0079251641

7904, [1, 1, 1, 1, 1, 1, 1, 3, 1, 3, 1, -1, 1, -1, 3, -3, 3, -3], -7.9936445118

10065, [1, 1, 1, 1, 1, 1, 1, -3, 1, -2, 1, -1, 2, -2, 2, -2, 3, -3], -8.0079251641

10231, [1, 1, 1, 1, 1, 1, 1, -3, 1, -1, 1, -1, 2, 2, 2, 2, 3, 3], -8.0000000000

10864, [1, 1, 1, 1, 1, 1, 1, -2, 1, -2, 1, -1, 3, -3, 3, -3, 3, -3], -7.9936445118

13821, [1, 1, 1, 1, 1, 2, 1, 2, 1, 3, 1, -2, 1, -1, 2, -2, 2, -2], -8.0079251641 **(1)**

>

6.2.3 Index Computations

This MAPLE program computes the dimension of invariant ASD connections on $\Sigma(2,3,5) \times \mathbb{R}$ which limits to a irreducible at $-\infty$ and the trivial product connection θ at $+\infty$. This dimension coincides with the dimension of $SO(3)$ instantons in the quotient cylinder $Q \times \mathbb{R}$ where $Q = \Sigma(2,3,5)/\pi$. The limiting flat connections are again a irreducible at $-\infty$ and a reducible $U(1)$ flat connection at $+\infty$.

```
> restart;
  with(ArrayTools) :
  Digits := 20 :

  p := 19 :

  #Seifert invariants
  a[1] := 2 : b[1] := 3 :
  a[2] := 3 : b[2] := 4 :
  a[3] := 5 : b[3] := -14 :



A := a[1]·a[2]·a[3] :
  S := Array([[0, 0, 0], [0, 0, 0]]) :
  S := Array([[a[1], a[2], a[3]], [b[1], b[2], b[3]]]) :


  # Seifert invariants of the quotient rational homology 3-sphere
NewS := proc(p :: integer) :: Array

local P :
  P := Array([[0, 0, 0], [0, 0, 0]]) :
  P := Array([[a[1], a[2], a[3]], [p·b[1], p·b[2], p·b[3]]]) :

return P;

 end proc:

  # rho invariant of lens space
  GetRhoLens := proc(a :: integer, b, c, k :: integer) :: rational

  local Rho :
  Rho := 2/a ·sum(cot(Pi·i·b/a)·cot(Pi·i·c/a)·sin^2(Pi·i·k/a), i = 1 ..a − 1) :


  return Rho;

  end proc:



    GetRhoReducible := proc(k :: integer) :: rational

  local Rho1, Rho2 :
  local Rho :
   Rho1 := -2/p ·sum(sin^2(Pi·n·k/p), n = 1 ..p − 1) + 2/(p·A) ·sum(csc^2(Pi·n/p)
```

$\cdot\sin^2\left(\frac{\mathrm{Pi}\cdot n\cdot k}{p}\right), n=1..p-1\Big)$:

$Rho2 := sum\left(\frac{2}{p\cdot 'S[1,l]'}\cdot sum\left(sum\left(\cot\left(\frac{\mathrm{Pi}\cdot j}{'S[1,l]'}\right)\cdot\cot\left(\frac{\mathrm{Pi}\cdot i}{p}-\frac{\mathrm{Pi}\cdot j\cdot 'S[2,l]'}{'S[1,l]'}\right)\cdot\sin^2\left(\frac{\mathrm{Pi}\cdot i\cdot k}{p}\right), j=1..'S[1,l]'-1\right), i=0..p-1\right), l=1..3\right)$:

Rho := $evalf(Rho1 + Rho2)$:

return Rho;

end proc:

 # computes Atiyah-Patodi-Singer rho invariant of rational homology 3-sphere using a flat cobordism to a disjoint union of lens spaces.

$GetRho :=$ **proc**$(T :: Array, U :: Array) :: rational$

local R :
local Rho :

$R := \frac{-4}{a_1}\cdot sum\left(\cot\left(\frac{\mathrm{Pi}\cdot k}{a_1}\right)\cdot\cot\left(\frac{\mathrm{Pi}\cdot k\cdot T[2,1]}{a_1}\right)\cdot\sin^2\left(\frac{\mathrm{Pi}\cdot k\cdot U[1,1]}{a_1}\right), k=1..a_1-1\right) + \frac{-4}{a_2}\cdot sum\left(\cot\left(\frac{\mathrm{Pi}\cdot k}{a_2}\right)\cdot\cot\left(\frac{\mathrm{Pi}\cdot k\cdot T[2,2]}{a_2}\right)\cdot\left(\sin\left(\frac{\mathrm{Pi}\cdot k\cdot U[1,2]}{a_2}\right)\right)^2, k=1..a_2-1\right) + \frac{-4}{a_3}\cdot sum\left(\cot\left(\frac{\mathrm{Pi}\cdot k}{a_3}\right)\cdot\cot\left(\frac{\mathrm{Pi}\cdot k\cdot T[2,3]}{a_3}\right)\cdot\left(\sin\left(\frac{\mathrm{Pi}\cdot k\cdot U[1,3]}{a_3}\right)\right)^2, k=1..a_3-1\right) - 3$;

Rho := $evalf\left(\frac{R}{2}\right)$;
return Rho;

end proc:

$T := Array([[0,0,0],[0,0,0]])$:
$T := NewS(p)$:

$L1 := Array([[1,2,4]])$:
$L2 := Array([[1,2,2]])$:

$evalf\left(\frac{8}{p}\cdot\left(\frac{1}{120}\right) + GetRhoReducible(1) - GetRho(T, L1) - \frac{1}{2}\right)$;
$evalf\left(\frac{8}{p}\cdot\left(\frac{49}{120}\right) + GetRhoReducible(7) - GetRho(T, L2) - \frac{1}{2}\right)$;

$$1.0000000000000000001$$
$$1.0000000000000000001 \tag{1}$$

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