Optimal Precoder Design and Block-Equal QRS Decomposition for ML Based Successive Cancellation Detection

OPTIMAL PRECODER DESIGN AND BLOCK-EQUAL QRS DECOMPOSITION FOR ML BASED SUCCESSIVE CANCELLATION DETECTION

BY

DAN FANG, B.Eng.

A THESIS

SUBMITTED TO THE DEPARTMENT OF ELECTRICAL & COMPUTER ENGINEERING

AND THE SCHOOL OF GRADUATE STUDIES

OF MCMASTER UNIVERSITY

IN PARTIAL FULFILMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF APPLIED SCIENCE

Master of Applied Science (2013) McMaster University

(Electrical & Computer Engineering) Hamilton, Ontario, Canada

TITLE:	Optimal Precoder Design and Block-Equal QRS Decomposition for ML Based Successive Cancellation Detection
AUTHOR:	Dan Fang B.Eng., (Electrical Engineering) Southeast University, Nanjing, China
SUPERVISOR:	Dr. K. M. Wong
NUMBER OF PAGES:	xiii, 121

To my parents

Abstract

The Multiple-input and Multiple-output (MIMO) channel model is very useful for the presentation of a wide range of wireless communication systems. This thesis addresses the joint design of a precoder and a receiver for a MIMO channel model, in a scenario in which perfect channel state information (CSI) is available at both ends. We develop a novel framework for the transmitting-receiving procedure.

Under the proposed framework, the receiver decomposes the channel matrix by using a block QR decomposition, where $\mathbf{Q}$ is a unitary matrix and $\mathbf{R}$ is a block upper triangular matrix. The optimal maximum likelihood (ML) detection process is employed within each diagonal block of $\mathbf{R}$. Then, the detected block of symbols is substituted and subtracted sequentially according to the block QR decomposition based successive cancellation. On the transmitting end, the expression of probability of error based on ML detection is chosen as the design criterion to formulate the precoder design problem. This thesis presents a design of MIMO transceivers in the particular case of having 4 transmitting and 4 receiving antennas with full CSI knowledge on both sides. In addition, a closed-form expression for the optimal precoder matrix is obtained for channels satisfying certain conditions. For other channels not satisfying the specific condition, a numerical method is applied to obtain the optimal precoder matrix.

Acknowledgements

I feel very grateful to have spent two years studying and working with so many extraordinary people at McMaster University. Foremost, I thank my supervisor Professor Kon Max Wong for his invaluable guidance and thoughtful support throughout my M.A.Sc study. I have benefited tremendously from his keen observation, rigorous scientific attitude, and artistic temperament. His stimulating discussions and continuous encouragement inspired me throughout my pursuit of this degree. I also thank Professor Jian Kang Zhang. Without his expert guidance, insightful discussions and instructions with patience, this work could not been possible.

I would like to express my appreciation to my examination committee - Professor Kon Max Wong, Professor Jian Kang Zhang and Professor Timothy N. Davidson - for all the time they have taken to evaluate my thesis.

My appreciation also goes out to my friend and colleagues in the Department of Electrical and Computer Engineering. During these two years we worked and studied together, shared thoughts and had discussions with each other. I thank for all the help that I got from them.

Finally I would like to thank my family and Binbin Dai. Their endless love and support mean the world to me.

Acronyms

BER	Bit Error Rate
CSI	Channel State Information
DFT	Discrete Fourier Transform
EVD	Eigenvalue Decomposition
MIMO	Multiple-Input Multiple-Output
ML	Maximum Likelihood
MMSE	Minimum Mean Square Error
MMSE-DFE	Minimum Mean Square Error Decision Feedback Equalization
PSK	Phase-Shift Keying
QAM	Quadrature Amplitude Modulation
QoS	Quality of Service
SEP	Symbol Error Probability
SINR	Signal-to-Interference Plus Noise Ratio
SNR	Signal-to-Noise Ratio
SVD	Singular Value Decomposition
ZF	Zero-Forcing
ZF-DFE	Zero-Forcing Decision Feedback

Notation

$\mathbf{a}$	Column vector $\mathbf{a}$
$\mathbf{A}$	Matrix $\mathbf{A}$
a_{ij}	The ijth element of a matrix $\mathbf{A}$
$\mathbf{A}^{ij}$	A 2×2 matrix composed of a_{ii}, a_{ij}, a_{ji}, and a_{jj}, i.e., $\begin{pmatrix} m_{ii} & m_{ij} \\ m_{ji} & m_{jj} \end{pmatrix}$
$\mathbf{A}_{\ell_i \ell_j}$	The matrix that contains the ith and jth columns
$\overline{\mathbf{A}}_{k_1,k_2,\ldots,k_i}$	The remaining matrix after deleting columns $\mathbf{a}_{k_1}$, $\mathbf{a}_{k_2}$, $\ldots$, $\mathbf{a}_{k_i}$ from $\mathbf{A}$.
$\mathcal{P}_{\mathbf{A}}$	The projection matrix $\mathcal{P}_{\mathbf{A}} = \mathbf{I} - \mathbf{A}\mathbf{A}^{\dagger}$ that projects an arbitrary vector to the null space of $\mathbf{A}^H$.
$(\cdot)^{-1}$	The inverse of a matrix
$(\cdot)^T$	The transpose of a vector or matrix
$(\cdot)^H$	The Hermitian of a vector or matrix
$(\cdot)^{\dagger}$	The Pseudo inverse of a matrix
$\mathrm{tr}(\cdot)$	The trace operator
$\det(\cdot)$	The determinant operator
$\mathrm{diag}(a_1, a_2, \ldots, a_N)$	A diagonal matrix with diagonal entries $a_1, a_2, \ldots, a_N$

$\mathrm{P}(\cdot)$	The probability
$Q(\cdot)$	The Q function defined by $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}} dt$
$\mathrm{E}[\cdot]$	Expectation operator
$\mathrm{Q}[\cdot]$	Quantization operator
$\mathbf{I}_N$	$N \times N$ identity matrix
$\|\cdot\|$	The 2 norm of a vector or matrix
$\mathrm{Re}\{\cdot\}$	Real part of the variable in the curly bracket
$\mathbf{x} \prec_+ \mathbf{y}$	$\mathbf{x}$ is additively majorized by $\mathbf{y}$
$\mathbf{x} \prec_\times \mathbf{y}$	$\mathbf{x}$ is multiplicatively majorized by $\mathbf{y}$

Contents

List of Figures

Chapter 1

Introduction

1.1 Motivation

In recent years, wireless communication systems have played an important role in modern day human life. On the other hand, due to increasing demand of higher quality of service (QoS) and limited wireless communication resources, there are many technical challenges that a wireless system must confront. Multiple-input and multiple-output (MIMO) has received considerable attention in wireless communications. By employing multiple antennas at both the transmitter end and the receiver end, a wireless communication system with MIMO achieves higher data transmitting rate and reduces the fading effect within limited bandwidth and power resources [1] [2]. For point-to-point MIMO system, precoding is a critical part of data transmission, since the multiple subchannels are suitably weighted before being emitted from the transmitting antennas such that the communication quality is maximized in some sense at the receiver. Therefore the design of precoder is worth doing when the channel information is known to the transmitter and receiver.

Generally, the precoder design should match its corresponding receiving scheme. In the ideal case that the channel inputs are uniformly distributed, Maximum Likelihood (ML) detector achieves the optimal error performance [3]. Some optimal precoding policies for ML detector under different criteria have been proposed [4] [5]. In [4], the authors proposed to optimize the shape of the constellations such that the transmitted power is minimized. The linear precoding policy in [5], on the other hand, maximizes the input-output mutual information for MIMO Gaussian channel with arbitrary input distribution under a total transmitted power constraint. However, the complexity of ML detection increases dramatically fast with the number of antennas and the size of signal constellation [6], which makes its implementation extremely difficult. In practice, there are two streams of receiving schemes that are widely used. One is based on linear equalization, and the other is based on decision-feedback equalization. The most representative realizations of the linear receivers are the Zero-Forcing (ZF) receiver [7] [8] [9] [10] [11] [12] [13] [14] [15] and the Minimum Mean Squared Error (MMSE) receiver [16] [17] [18] [19] [20] [21] [22] [23] [24]. Linear receivers are very easy to be realized with low computation and implementation costs. However, in vertain channels they may lose a large amount of detection accuracy. An alternative nonlinear receiving scheme is decision-feedback detection [25] [26] [27] [28], which is also a relatively low complexity receiving structure compared to the ML detection. The typical examples are Zero-Forcing Decision Feedback Equalization (ZF-DFE) receiver [29] [30] and Minimum Mean Squared Error Decision Feedback Equalization (MMSE-DFE) receiver [31] [32] [33] [34] [35] [36]. The improved performance of such decision feedback equalizers over the linear equalizer has been proved both theoretically and experimentally in [37] [38] [39]. However, such schemes still sacrifice too much performance for the computational complexity.

For the independent Gaussian noise channel, the mutual information under a total power constraint is maximized if the channel inputs are also Gaussian and independent with waterfilling power allocation [40]. Although Gaussian inputs are optimal in the sense that the subchannels are diagonalized into a set of noninterfering channel vectors, it is such an ideal model that cannot be realized in practice. Practically, the channel inputs are chosen from discrete constellations, such as phase-shift keying (PSK) and quadrature amplitude modulation (QAM). For the non-Gaussian channel inputs, applying waterfilling policy leads to a performance quite different from the Gaussian ideal case. In [41], the results showed that the optimal waterfilling strategies for Gaussian inputs yields a significant loss of gain for discrete constellation inputs while linear precoding techniques may achieve higher transmission rate.

In this thesis, we assume that the receiver and transmitter have perfect knowledge of the channel state information (CSI). The channel inputs are from a specific discrete constellation. We presents a new framework aiming at achieving a better trade off between error performance and computation cost for the precoder and receiver design. The new receiver employs a detection strategy that combines the idea of ML detection and successive cancellation detection. The proposed receiver decomposes the channel matrix into a series of 2×2 blocks based on the proposed block-equal QRS decomposition and divides the received symbols into several 2×1 symbol blocks accordingly. For the detection of each block of 2 transmitted symbols, the receiver jointly detects the symbol block by using an ML detector and successively cancels the estimated signals block-by-block until all the transmitted symbols are successfully detected. Since that ML detector is capable to achieve the optimal error performance, intuitively, applying ML detector on each 2×1 symbol is superior to the traditional decision feedback equalization where the symbols are detected sequentially after each cancellation step. On the other hand, as we just repeatedly apply ML detection on the smallest possible size of problem, that is, detection of 2×1 symbol vector, the growth of complexity is relatively slow and acceptable. At the transmitter side, we design an optimal precoder compatible with the receiving scheme. The precoder is optimized such that it not only guarantees the existence of block-equal QRS decomposition, but also maximizes the minimum distance for a given finite signal constellation according to the criterion of ML detection. Specifically, we examine the case under which the MIMO channels have 4 transmitting antennas and 4 receiving antennas and transmitted symbols are 4-QAM signals. For some of the channels satisfying a certain condition, we obtain a closed-form solution for the optimal precoder. For other channels not satisfying the condition, a numerical solution method for the optimal precoder is given. A simulation example is given to illustrate the performance of our proposed precoder and detector.

1.2 Main Contributions

The main contributions of this work are as follows:

- We present a novel receiving strategy that is essentially different from any existing receiving schemes. It first combines the ML detection and successive cancellation detection successfully by taking the advantages of both of the methods and avoiding their shortcomings. The main idea is: Perform 2×2 block QRS decomposition on a given channel matrix $\mathbf{H}$, jointly detect the last two symbols together using ML detection, then every time successively cancel two newly detected symbols at one time and perform ML detection on next two symbols until all the symbols are detected. This can be looked upon as a generalized successive cancellation detection where each time we detect and cancel the data block by block instead of doing it symbol by symbol. The use of ML detector for joint detection of each 2×2 block enhances the detection accuracy significantly with just an affordable sacrifice in computational complexity because block size is small.

- The thesis explains the optimality of block-equal QRS decomposition in symbol detection. Also, it shows that the block-equal QRS decomposition of a full channel matrix can be equivalently transformed to that of a diagonal matrix $\mathbf{\Lambda}$. The decomposition can be written as $\mathbf{\Lambda S} = \mathbf{QR}$. We focus on the simplest 4×4 channel case where the channel matrix has the smallest size for block QRS decomposition. In this way, we mathematically prove that there is a necessary and sufficient condition of the existence of such decomposition $\mathbf{\Lambda S} = \mathbf{QR}$. Based on the necessary and sufficient condition, we also construct a specific closed form solution to $\mathbf{S}$ that makes $\mathbf{\Lambda S}$ a block-equal QRS decomposable matrix.

- We propose the corresponding optimal structure of the precoder matrix $\mathbf{F}$ that is compatible with the proposed receiving strategy. In particular, the optimal precoder has the following properties: (a) The resulting cascaded channel $\tilde{\mathbf{H}} = \mathbf{HF}$ has the block-equal QRS decomposition. (b) The precoder maximizes the minimum Euclidean distance of received signal constellation based on the criterion of ML detection. Therefore, it also minimizes the symbol error probability (SEP) for ML based block successive cancellation detection.

1.3 Structure of the Thesis

The thesis is organized as follows: In Chapter 2, we present the channel model and some related assumptions. Then we review some frequently used receiving schemes which are used for comparison analysis in computer simulation. In Chapter 3, we propose an optimal block QRS decomposition, named block-equal QRS decomposition. We show that for a channel matrix, the block QRS decomposition channel with equal diagonal block entries achieves optimal error performance. We study such a decomposition in a special case of 4×4 MIMO channel and find an equivalent condition to the existence of the unitary matrix $\mathbf{S}$ such that $\mathbf{\Lambda S} = \mathbf{QR}$. We further construct a particular closed form solution to $\mathbf{S}$ in special case. In Chapter 4, we apply the block-equal QRS decomposition on optimal precoder design in the 4×4 special case. An optimization problem is presented aiming to find the optimal precoder that maximizes the minimum Euclidean distance between two received symbol vectors under transmitting power constraint. For the case where the channel satisfies a certain condition, a closed-form solution for the block-equal QR decomposition given in Chapter 3 is used in constructing the optimal precoder matrix $\mathbf{F}$. On the other hand, when a channel does not satisfy that certain condition, a numerical search is used for obtaining the numerical result of the optimal precoder. In Chapter 5, we verify the performance of our proposed receiving scheme and corresponding precoder design by performming some computer simulations. The simulation results show that our method greatly improve the error performance compared with other commonly used detectors. In Chapter 6, we summarize our works and suggest directions for further research.

Chapter 2

Preliminaries

2.1 System Model

In this thesis, we consider a Multiple-Input and Multiple-Output (MIMO) system that is commonly used in various wireless communication systems. Specifically, the system has M transmitting antennas and N receiving antennas. At the transmitter end, multiple symbols are simultaneously transmitted through multiple transmitting antennas. Those transmitted symbols are assumed to be zero-mean, i.i.d. and uniformly distributed over a finite constellation alphabet $\mathcal{S}$. The transmitted symbols form a symbol vector $\mathbf{s} \in \mathcal{S}^M$. This symbol vector goes through a wireless communication channel the state information of which is available at both the transmitter and receiver. An additive noise component is applied on the channel. The received signal is modeled in the following matrix form:

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{n} \tag{2.1}$$

where $\mathbf{H} \in \mathbb{C}^{N\times M}$ is the channel matrix and $\mathbf{n} \in \mathbb{C}^{N}$ is a zero-mean, complex circularly-symmetric Gaussian noise vector with covariance matrix $2\tau^2\mathbf{I}$ and is assumed uncorrelated with transmitted signals. We assume that $M \leq N$, which means that $\mathbf{H}$ is a full column rank square matrix or tall matrix.

To enhance the performance of the detector at the receiving end, a precoder may be implemented at the transmitter. In such a case, the transmitted symbol vector is precoded first, then transmitted through the channel. In that case the received signal is modeled as:

$$\mathbf{y} = \mathbf{HFs} + \mathbf{n} \tag{2.2}$$

where $\mathbf{F} \in \mathbb{R}^{M\times M}$ is referred to as a precoder matrix added to the transmitter. We denote the product of channel matrix $\mathbf{H}$ and precoder matrix $\mathbf{F}$ as cascaded channel matrix $\tilde{\mathbf{H}} = \mathbf{HF}$, so that the channel model with precoder is

$$\mathbf{y} = \tilde{\mathbf{H}}\mathbf{s} + \mathbf{n} \tag{2.3}$$

which maintains consistency in form with the channel model without a precoder in Eq. (2.1).

2.2 Historical Overview of Receiver Designs

For a MIMO system with given channel state information (CSI), the receiver is designed to estimate the transmitted symbol vector $\mathbf{s}$ based on the knowledge of channel matrix $\mathbf{H}$ and the received signal $\mathbf{y}$. In addition to the additive channel noise, MIMO channel also faces interferences between different subchannels (figure 2.1). Therefore errors may occur and the estimated symbol vector $\hat{\mathbf{s}}$ may not be same as the transmitted symbol vector $\mathbf{s}$ because of the channel noise and interference. During the past years, several receiver design techniques were proposed to reduce the probability of errors. This section reviews some of the most representative techniques used for receiver design.

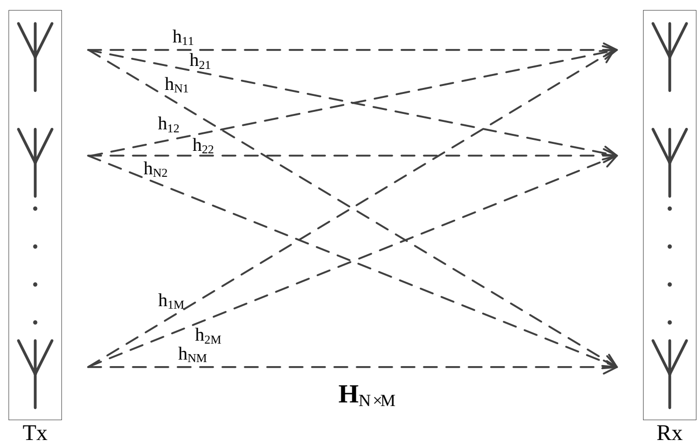

Figure 2.1: Interferences in MIMO System

2.2.1 Maximum Likelihood Detection

Under the channel model described in Eq. (2.1), for given channel matrix $\mathbf{H}$ and received signals $\mathbf{y}$, the probability of error defined as

$$P_e \triangleq P(\hat{\mathbf{s}} \neq \mathbf{s}|\mathbf{y}, \mathbf{H}) = 1 - P(\hat{\mathbf{s}} = \mathbf{s}|\mathbf{y}, \mathbf{H}) \tag{2.4}$$

where $P(\hat{\mathbf{s}} = \mathbf{s}|\mathbf{y}, \mathbf{H})$ is the probability of correct symbol estimation:

$$P(\hat{\mathbf{s}} = \mathbf{s}|\mathbf{y}, \mathbf{H}) = \frac{P(\hat{\mathbf{s}} = \mathbf{s}) f_{\mathbf{y}|\mathbf{s},\mathbf{H}}(\mathbf{y}|\hat{\mathbf{s}} = \mathbf{s}, \mathbf{H})}{f_{\mathbf{y}|\mathbf{H}}(\mathbf{y}|\mathbf{H})} \tag{2.5}$$

where $f_{\mathbf{y}|\mathbf{H}}(\mathbf{y}|\mathbf{H})$ is the conditional probability density function (p.d.f.) of $\mathbf{y}$ given channel $\mathbf{H}$. This conditional p.d.f. can be written as

$$f_{\mathbf{y}|\mathbf{s},\mathbf{H}}(\mathbf{y}|\hat{\mathbf{s}} = \mathbf{s}, \mathbf{H}) = f_{\mathbf{y}-\mathbf{Hs}}(\mathbf{y} - \mathbf{H}\hat{\mathbf{s}}) \tag{2.6}$$

$$= f_{\mathbf{n}}(\mathbf{y} - \mathbf{H}\hat{\mathbf{s}}) \tag{2.7}$$

where the joint probability density function of the white Gaussian noise vector $\mathbf{n}$ is given by

$$f_{\mathbf{n}}(\mathbf{n}) = \frac{1}{\pi^N \det(\mathbf{\Phi}_{\mathbf{nn}})} \exp\left(-\mathbf{n}^H {\mathbf{\Phi}_{\mathbf{nn}}}^{-1} \mathbf{n}\right) \tag{2.8}$$

Since the noise has a covariance matrix $\mathbf{\Phi}_{\mathbf{nn}} = 2\tau^2 \mathbf{I}$, the Eq. (2.6) can be further written as

$$f_{\mathbf{n}}(\mathbf{y} - \mathbf{H}\hat{\mathbf{s}}) = \frac{1}{\pi^N (2\tau^2)^N} \exp\left(-\frac{\|\mathbf{y} - \mathbf{H}\hat{\mathbf{s}}\|^2}{2\tau^2}\right) \tag{2.9}$$

Eq. (2.9), when treated as a function of $\mathbf{s}$, can be called likelihood function.

In Eq. (2.5), since both $P(\hat{\mathbf{s}} = \mathbf{s})$ and $f_{\mathbf{y}|\mathbf{H}}(\mathbf{y}|\mathbf{H})$ are independent of $\hat{\mathbf{s}}$, minimizing the probability of error is equivalent to maximizing the likelihood function. The ML detector, possesses the property that it minimizes the error probability by maximizing the likelihood function:

$$\hat{\mathbf{s}}_{\mathrm{ML}} = \arg\ \max_{\hat{\mathbf{s}}} \frac{1}{\pi^N (2\tau^2)^N} \exp\left(-\frac{\|\mathbf{y}-\mathbf{H}\hat{\mathbf{s}}\|^2}{2\tau^2}\right) \tag{2.10}$$

Note that the maximum of the likelihood function is achieved when $\|\mathbf{y}-\mathbf{H}\hat{\mathbf{s}}\|^2$ reaches the minimum value, therefore the estimation of ML detector is given by

$$\hat{\mathbf{s}}_{\mathrm{ML}} = \arg\ \min_{\hat{\mathbf{s}}\in\mathcal{S}^M} \|\mathbf{y}-\mathbf{H}\hat{\mathbf{s}}\|^2 \tag{2.11}$$

Similarly, when considering the channel model with precoder in Eq. (2.3), the estimation of ML detector is given by

$$\hat{\mathbf{s}}_{\mathrm{ML}} = \arg\ \min_{\hat{\mathbf{s}}\in\mathcal{S}^M} \|\mathbf{y}-\tilde{\mathbf{H}}\hat{\mathbf{s}}\|^2 \tag{2.12}$$

Eq. (2.12) also suggests an intuitive understanding of the criterion of ML detection: it chooses the $\hat{\mathbf{s}}$ that yields the minimum Euclidean distance between $\tilde{\mathbf{H}}\hat{\mathbf{s}}$ and the received signal $\mathbf{y}$.

Problem as shown in Eq. (2.12) are known to be NP-hard and not easy to solve in general [6]. However, when the problem size is small, e.g. $\mathbf{H} \in \mathbb{C}^{2\times 2}$, the problem is manageable for some available detection algorithms, such as exhaustive search, semidefinite relaxation and sphere decoding.

2.2.2 Linear Receivers

Despite the optimal performance of the ML detectors in terms of error rate, there is a variety of suboptimal receivers. Linear equalization is one category of such suboptimal receivers having as low as linear computational complexity. Using a linear receiver, the estimated transmitted symbol vector $\hat{\mathbf{s}}$ is obtained by applying a linear projection operation on received signals $\mathbf{y}$:

$$\hat{\mathbf{s}} = \mathbf{G}\mathbf{y} \tag{2.13}$$

There are two major linear strategies, Zero-forcing (ZF) Equalizer and Minimum Mean Square Error (MMSE) Equalizer.

Zero-Forcing Equalizer

A zero-Forcing Equalizer aims to null out the inter-stream interference completely. It is achieved by forcing $\mathbf{GH} = \mathbf{I}$ and $\mathbf{G}$ is given by the pseudoinverse of the matrix $\mathbf{H}$:

$$\mathbf{G}_{\mathrm{ZF}} = \mathbf{H}^{\dagger} = (\mathbf{H}^{H}\mathbf{H})^{-1}\mathbf{H}^{H} \tag{2.14}$$

The special case is when $\mathbf{H}$ is square and invertible, in this case the linear operation $\mathbf{G}$ is exactly the channel inverse:

$$\mathbf{G}_{\mathrm{ZF}} = \mathbf{H}^{-1} \tag{2.15}$$

Using ZF equalization, the estimated transmitted symbol vector is

$$\hat{\mathbf{s}} = \mathbf{G}_{ZF}\mathbf{y} = \mathbf{H}^{\dagger}\mathbf{H}\mathbf{s} + \mathbf{H}^{\dagger}\mathbf{n} = \mathbf{s} + \mathbf{H}^{\dagger}\mathbf{n} \tag{2.16}$$

Although the inter-stream interference is eliminated, the channel noise is still involved in the estimated signals. So, at high SNR, since that the inter-stream interference is the main concern, this equalizer performs well. However, when the channel is noisy and SNR is low, the channel noise becomes the dominant issue and the zero-forcing equalizer leads to very poor error performance.

Minimum Mean Square Error Equalizer

As the name implies, the Minimum Mean Square Error Equalizer minimizes the mean square error of the estimated signals.

$$\mathbf{G}_{\mathrm{MMSE}} = \arg \min_{\hat{\mathbf{s}} \in \mathcal{S}^M} \mathrm{E}[\|\mathbf{e}\|^2] \tag{2.17}$$

where the term error $\mathbf{e}$ is defined as the difference between transmitted signal and estimated signal:

$$\mathbf{e} = \hat{\mathbf{s}} - \mathbf{s} \tag{2.18}$$

$$= (\mathbf{GH} - \mathbf{I})\mathbf{s} + \mathbf{Gn} \tag{2.19}$$

The mean squared error is also equal to the trace of the covariance matrix of error:

$$\mathrm{E}[\|\mathbf{e}\|^2] = \mathrm{tr}(\mathrm{E}[\mathbf{e}\mathbf{e}^H]) = \mathrm{tr}\left(\mathbf{G}(\mathbf{H}\mathbf{H}^H + \mathbf{\Phi}_{\mathbf{nn}})\mathbf{G}^H - 2\mathrm{Re}\{\mathbf{GH}\} + \mathbf{I}\right) \tag{2.20}$$

where $\mathbf{\Phi_{nn}}$ denotes the covariance matrix of channel noise, i.e., $\mathbf{\Phi_{nn}} \triangleq \mathrm{E}[\mathbf{nn}^H]$. The MMSE solution of $\mathbf{G}$ that minimizes the mean squared error is given by

$$\mathbf{G}_{\mathrm{MMSE}} = (\mathbf{HH}^H + \mathbf{\Phi_{nn}})^{-1}\mathbf{H}^H \tag{2.21}$$

The MMSE Equalizer maximizes the signal-to-interference-plus-noise ratio (SINR) and finds an optimal trade off between the inter-stream interference and channel noise. This is the major advantage over ZF Equalizer especially at low-to-moderate SNR.

2.2.3 Successive Cancellation Detection

For the linear receiver, the data streams of different subchannels are detected separately. However for successive cancellation detector, the detection result of one subchannel is used for the detection of its following subchannels. Once a symbol of a subchannel is successfully detected, we can subtract it off from the received signals and reformulate our detection problem with one less subchannel. The process is continued until only one subchannel left and no more interference involved. This is the main philosophy behind successive cancellation detection.

Mathematically, successive cancellation detection can be described by the QR Decomposition. The procedure of QR decomposition based successive cancellation detection can be described as follows:

For simplicity, suppose that there is an $N \times N$ square channel matrix $\mathbf{H}$. Perform QR Decomposition on the channel matrix $\mathbf{H} = \mathbf{QR}$, where $\mathbf{Q}$ is a unitary matrix and $\mathbf{R}$ is an upper triangular matrix. The channel model in Eq. (2.1) can then be written in the following form:

$$\mathbf{y} = \mathbf{QRs} + \mathbf{n} \tag{2.22}$$

Left-multiply both sides by the matrix $\mathbf{Q}^H$,

$$\mathbf{Q}^H\mathbf{y} = \mathbf{Rs} + \mathbf{Q}^H\mathbf{n} \tag{2.23}$$

Define $\tilde{\mathbf{y}} \triangleq \mathbf{Q}^H\mathbf{y}$ and $\tilde{\mathbf{n}} \triangleq \mathbf{Q}^H\mathbf{n}$, Eq. (2.23) is rewritten as

$$\begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \vdots \\ \tilde{y}_N \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & \dots & r_{1N} \\ 0 & r_{22} & \dots & r_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r_{NN} \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_N \end{pmatrix} + \begin{pmatrix} \tilde{n}_1 \\ \tilde{n}_2 \\ \vdots \\ \tilde{n}_N \end{pmatrix} \tag{2.24}$$

Start the detection from the last symbol of Eq. (2.24). Since there is only one nonzero element in the last row of the upper triangular matrix $\mathbf{R}$. last symbol $\hat{s}_N$ can be easily estimated:

$$\hat{s}_N = \mathrm{Q}[\tilde{y}_1/r_{NN}] \tag{2.25}$$

where $\mathrm{Q}[\cdot]$ denote the operation of quantization. Assuming that the symbol $\hat{s}_N$ is detected correctly, we subtract the term s_N off from the second last row. Thus, the interference from the last symbol is eliminated and the second last symbol can be estimated:

$$\hat{s}_{N-1} = \mathrm{Q}[(\tilde{y}_{N-1} - r_{NN}\hat{s}_N)/r_{N-1\ N-1}] \tag{2.26}$$

Continue this subtract-and-estimate procedure until the first symbol is detected.

This detection scheme can be used to improve the performance of linear receivers, like ZF receiver and MMSE receiver. The corresponding successive cancellation detection shcemes are respectively called Zero-Forcing Decision Feedback Equalization (ZF-DFE) and Minimum Mean Squared Error Decision Feedback Equalization (MMSE-DFE). It has been shown experimentally that the successive cancellation receivers invariably improve the error performance compared to their linear version of receivers.

The major concern of successive cancellation is the problem of error propagation. A lot of studies have been proposed to find the optimally ordered successive cancellation to minimize the probability of error propagation such that each time the transmitted symbol is successfully cancelled with very high probability. Specifically, an equal-diagonal QRS decomposition was proposed in [38]. This QRS decomposition has following form:

$$\mathbf{HS} = \mathbf{QR} \tag{2.27}$$

where $\mathbf{S}$ is a unitary matrix, $\mathbf{Q}$ and $\mathbf{R}$ are the factors in QR decomposition. This factorization possesses such a special property that $\mathbf{R}$ has equal diagonal entries, which naturally enable an optimally ordered column permutation for successive cancellation. Meanwhile, the unitary matrix $\mathbf{S}$ serves as the optimal precoder. Such $\mathbf{S}$ minimizes the block error probability and maximize the lower bound of the minimum

Euclidean distance between each pair of received symbol vectors. Applying this optimal QRS decomposition in ZF-DFE and MMSE-DFE further improves the detection performance. We also include the algorithm for designing the QRS decomposition in Appendix A.

Chapter 3

Block-Equal QRS decomposition

In this chapter, we introduce our main contribution in the mathematical derivation of a novel matrix factorization method called block-equal QRS decomposition. A new receiving scheme based on block-equal QRS decomposition is also proposed.

3.1 A New Transmitting and Receiving Scheme Based on Block-Equal QRS decomposition

We have acknowledged that ML detection achieves the optimal error performance but is hard to realize, and successive cancellation detection is much easier to implement but sacrifices too much performance. Our goal is to design a novel transmitting and receiving scheme having an error performance as close to ML detection as possible with an affordable computational cost. The new scheme is a combination of ML detection and successive cancellation detection. Instead of applying QR decomposition on the channel matrix, we introduce a block-wise QR decomposition so that the successive cancellation is proceeded in a block-by-block manner. The symbols in one block are jointly detected at the same time, where ML detection can be used within each of the blocks. We design such a strategy to take advantage of both ML detection and successive cancellation detection.

In the proposed scheme, the cascaded channel matrix $\tilde{\mathbf{H}}$ is decomposed into the product of a unitary matrix $\tilde{\mathbf{Q}}$ and a block upper triangular matrix $\tilde{\mathbf{R}}$. Applying the decomposition on the channel model (2.3), we have

$$\mathbf{y} = \tilde{\mathbf{Q}}\tilde{\mathbf{R}}\mathbf{s} + \mathbf{n} \tag{3.1}$$

Left-multiply both sides by the matrix $\tilde{\mathbf{Q}}^H$,

$$\tilde{\mathbf{Q}}^H\mathbf{y} = \tilde{\mathbf{R}}\mathbf{s} + \tilde{\mathbf{Q}}^H\mathbf{n} \tag{3.2}$$

Define $\tilde{\mathbf{y}} \triangleq \tilde{\mathbf{Q}}^H\mathbf{y}$ and $\tilde{\mathbf{n}} \triangleq \tilde{\mathbf{Q}}^H\mathbf{n}$. Eq. (3.2) is then rewritten as

$$\tilde{\mathbf{y}} = \tilde{\mathbf{R}}\mathbf{s} + \tilde{\mathbf{n}} \tag{3.3}$$

To depict the channel model clearly, we further write the Eq. (3.3) in the following matrix form:

$$\begin{pmatrix} \tilde{\mathbf{y}}_1 \\ \tilde{\mathbf{y}}_2 \\ \vdots \\ \tilde{\mathbf{y}}_N \end{pmatrix} = \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} & \dots & \mathbf{R}_{1N} \\ \mathbf{0} & \mathbf{R}_{22} & \dots & \mathbf{R}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{R}_{NN} \end{pmatrix} \begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \\ \vdots \\ \mathbf{s}_N \end{pmatrix} + \begin{pmatrix} \tilde{\mathbf{n}}_1 \\ \tilde{\mathbf{n}}_2 \\ \vdots \\ \tilde{\mathbf{n}}_N \end{pmatrix} \tag{3.4}$$

where the diagonal block entries $\{\mathbf{R}_{ii}\}$ are $n \times n$ square matrices, and both the transmitted and received symbol vectors are segmented into symbol blocks of dimension $n \times 1$ correspondingly.

Based on the channel model in Eq. (3.4), our proposed receiving scheme works as follows:

Algorithm 1. ***ML Based Successive Cancellation Detection***

1. *Start with the last row of Eq.* (3.4), *which is* $\tilde{\mathbf{y}}_N = \mathbf{R}_{NN}\mathbf{s}_N + \tilde{\mathbf{n}}_N$. *Estimate the last transmitted symbol block* $\mathbf{s}_N$ *by using ML detection:*

$$\hat{\mathbf{s}}_N = \arg \min_{\mathbf{s}_N \in \mathcal{S}} \|\tilde{\mathbf{y}}_N - \mathbf{R}_{NN}\mathbf{s}_N\|^2 \tag{3.5}$$

2. *Consider the second last row, which is* $\tilde{\mathbf{y}}_{N-1} = \mathbf{R}_{N-1,N-1}\mathbf{s}_{N-1} + \mathbf{R}_{N-1,N}\mathbf{s}_N + \tilde{\mathbf{n}}_{N-1}$. *Assuming that the transmitted symbol segment* $\hat{\mathbf{s}}_N$ *is detected correctly, we cancel it from the second last row, which leads to*

$$\tilde{\mathbf{y}}_{N-1} - \mathbf{R}_{N-1,N}\hat{\mathbf{s}}_N = \mathbf{R}_{N-1,N-1}\mathbf{s}_{N-1} + \tilde{\mathbf{n}}_{N-1} \tag{3.6}$$

Again, we employ ML detection for the estimation of the second last transmitted symbol block $\mathbf{s}_{N-1}$*:*

$$\hat{\mathbf{s}}_{N-1} = \arg \min_{\mathbf{s}_{N-1}} \|(\tilde{\mathbf{y}}_{N-1} - \mathbf{R}_{N-1,N}\hat{\mathbf{s}}_N) - \mathbf{R}_{N-1,N-1}\mathbf{s}_{N-1}\|^2 \tag{3.7}$$

3. *Repeat this cancellation process successively and apply ML detector on each block-based joint detection of transmitted symbols until all the symbols are detected successfully.*

Such a receiving process basically shares the idea with the successive cancellation detection. However, this new scheme makes a critical difference. The strategy of detecting the symbols block-by-block instead of doing it symbol-by-symbol creates a framework under which a joint detection of more than one symbol is allowed. A joint detection of a block of symbols often yields superior performance to the hard decision of a single symbol, which is used in conventional successive cancellation detection. When the block size is small, ML detection can be applied on the joint detection of each block with manageable computation complexity. In that sense, we may take advantage of the optimal error performance provided by ML detector at each step of block detection and enjoy an overall performance enhancement of the system as a whole.

3.2 The Optimality of Block-Equal QRS Decomposition

Our proposed receiving scheme is a combination of successive cancellation detection (for inter-block detection) and ML detection (within each block entry $\mathbf{R}_{ii}$ along the diagonal of the upper triangular matrix $\mathbf{R}$). Considering the block symbol error probability P_{ei} of the ith block $\mathbf{R}_{ii}$, it has a nearest neighbour approximation as shown below [42, p. 185]:

$$P_{\mathrm{e}i} \approx \bar{N}_{\mathrm{k}i} Q\left(\frac{d_{\min i}}{2\tau}\right) \tag{3.8}$$

where the Q function is defined as [42, p. 41]:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{t^2}{2}} dt \tag{3.9}$$

In this context, assume that the transmitted symbol constellation is $\mathcal{S}$, the ith transmitted symbol block of dimension n, denoted as $\mathbf{s}_i$, belongs to the symbol vector set $\mathcal{S}^n$. For each possible transmitted symbol block $\mathbf{s}_i$, we may represent $\mathbf{R}_{ii}\mathbf{s}_i$ at the receiver side as a point in the received signal space $\mathcal{X}^n$. $d_{\min i}$ is then defined as the minimum Euclidean distance between each pair of different points, $\mathbf{R}_{ii}\mathbf{s}_i$ and $\mathbf{R}_{ii}\bar{\mathbf{s}}_i$, in $\mathcal{X}^n$.

$$d_{\min i} = \min_{\mathbf{s}_i, \bar{\mathbf{s}}_i \in \mathcal{S}^n, \mathbf{s}_i \neq \bar{\mathbf{s}}_i} \|\mathbf{R}_{ii}(\mathbf{s}_i - \bar{\mathbf{s}}_i)\| \tag{3.10}$$

$\bar{N}_{\mathrm{k}i}$ denotes the average kissing number for the ith transmitted symbol block. The kissing number of each possible point $\mathbf{R}_{ii}\mathbf{s}_j$, denoted as $N_{\mathrm{k}ij}$, is defined as the number of the nearest neighbours, achieving the minimum distance $d_{\min i}$ from them to that point. The average kissing numbers is then defined as

$$\bar{N}_{\mathrm{k}i} = \frac{\sum_{\mathbf{s}_j \in \mathcal{S}^n} N_{\mathrm{k}ij}}{|\mathcal{X}^n|} \tag{3.11}$$

where $|\mathcal{X}^n|$ denotes the cardinality of transmitted symbol block set $\mathcal{X}^n$.

The nearest neighbour approximation in Eq. (3.8) shows that the term $d_{\min,i}$ dominates the error performance of the ith ML detector for that block. A good precoder should be designed such that $d_{\min i}$ is maximized.

From the whole channel matrix point of view, we successively perform the ML detection block-by-block until all the received symbols are detected. The probability of correctly detecting all the blocks is given by

$$P_{\mathrm{c}} = \prod_{i=1}^{n}(1 - P_{\mathrm{e}i}) \tag{3.12}$$

Therefore, the overall block symbol error probability is:

$$\begin{aligned} P_{\mathrm{e}} &= 1 - P_{\mathrm{c}} &(3.13)\\ &= \sum_{i=1}^{n} P_{\mathrm{e}i} - \sum_{i\neq j}^{n} P_{\mathrm{e}i}P_{\mathrm{e}j} + \ldots + (-1)^{n-1}\prod_{i=1}^{n} P_{\mathrm{e}i} &(3.14)\end{aligned}$$

The dominant term of P_e is the first term containing the sum of error probabilities for all the blocks. Thus, the overall block symbol error probability is upper bounded by its dominant term, the sum of each $P_{\mathrm{e}i}$:

$$P_{\mathrm{e}} \leq \sum_{i=1}^{n} P_{\mathrm{e}i} \tag{3.15}$$

The sum of all $P_{\mathrm{e}i}$s, however, is dominated by the maximum term over all the error probabilities, which is given by $\max\{P_{\mathrm{e}i}\}$. Eq. (3.8) reveals the fact that the dominant term of each $P_{\mathrm{e}i}$ is the corresponding Q function $Q(\frac{d_{\min i}}{2\tau})$. Thus, $\max\{P_{\mathrm{e}i}\}$ is largely determined by $\min\{d_{\min i}\}$, which suggests that the overall error performance is determined by every minimum of $d_{\mathrm{min}i}$.

An intuitive conjecture is, the optimal precoder should then be designed such that each block of the block QRS decomposition has the equal $d_{\mathrm{min}i}$, so that the overall error performance of the whole system is optimal.

$$\min_{1\leq i\leq n} d_{\min i} = d_{\min 1} = d_{\min 2} = \ldots = d_{\min n} \tag{3.16}$$

In Section 4.2, we will explain that if each diagonal block of $\tilde{\mathbf{R}}$ has equal minimum distance $d_{\min}$, then the blocks themselves also equal to each other. In order to guarantee the equal $d_{\min i}$ of each block, the block QRS decomposition is forced to have equal diagonal $\tilde{\mathbf{R}}$-block. We call such a decomposition as block-equal QRS decomposition.

For simplicity, let $d_{\min}$ be the equal minimum distance over all minimum distances:

$$d_{min} \triangleq \min_{1\leq i\leq n} d_{min,i} \tag{3.17}$$

In order to achieve the optimal error performance for the ML based intra-block detection scheme, such block QRS decomposition is supposed to have maximal possible $d_{\min}$. Note that $\mathbf{F}$ serves as the precoder matrix, such that the channel matrix $\mathbf{H}$ can be factorized as $\mathbf{HF} = \tilde{\mathbf{Q}}\tilde{\mathbf{R}}$. The precoder matrix $\mathbf{F}$ should be designed according to the following criteria:

$$\begin{aligned} \max_{\mathbf{F}} \quad & d_{\min} \\ \text{s.t.} \quad & \text{transmitting power constraint} \end{aligned} \tag{3.18}$$

3.3 Block-Equal QRS Decomposition of 4×4 Channel Matrix

In Section 3.2, we have explained that the optimal detection performance can be achieved by finding a unitary matrix $\mathbf{F}$, such that for a given channel matrix $\mathbf{H}$, the product $\mathbf{HF}$ has block-equal QRS decomposition $\tilde{\mathbf{H}} = \mathbf{HF} = \tilde{\mathbf{Q}}\tilde{\mathbf{R}}$. The output of each data block is then detected by a ML detector. Since ML detection is NP-hard, the computational complexity grows dramatically fast as the problem size increases. To control the computational overhead, the block size should be chosen as small as possible. The smallest possible size of a block is 2. Thus, we assume that the cascaded channel matrix $\tilde{\mathbf{H}}$ should be decomposed into $\tilde{\mathbf{Q}}\tilde{\mathbf{R}}$ where $\tilde{\mathbf{R}}$ has equal 2×2 diagonal blocks.

To facilitate the QR decomposition of $\tilde{\mathbf{H}}$ into equal 2×2 diagonal blocks in $\tilde{\mathbf{R}}$, we consider the simplest case in which the dimension of $\tilde{\mathbf{H}}$ is 4×4. Substituting the singular value decomposition of $\tilde{\mathbf{H}}$ into the equation of block-equal QRS decomposition,

$$\tilde{\mathbf{H}} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{V}^H = \tilde{\mathbf{Q}}\tilde{\mathbf{R}} \tag{3.19}$$

Left-multiply $\mathbf{U}^H$ on both sides of Eq. (3.19),

$$\boldsymbol{\Lambda}\mathbf{V}^H = \mathbf{U}^H\tilde{\mathbf{Q}}\tilde{\mathbf{R}} \tag{3.20}$$

We may express the 4×4 block-equal upper triangular matrix in the following form:

$$\tilde{\mathbf{R}} = \begin{pmatrix} \mathbf{R}_{eq} & \mathbf{R}_{12} \\ \mathbf{0} & \mathbf{R}_{eq} \end{pmatrix} \tag{3.21}$$

and further define the singular value decomposition of each diagonal block $\mathbf{R}_{eq}$ as

$$\mathbf{R}_{eq} = \mathbf{Q}_{eq}\boldsymbol{\Lambda}_{eq}\mathbf{U}_{eq} \tag{3.22}$$

Substituting the singular value decomposition of $\mathbf{R}_{eq}$ into Eq. (3.21), we have

$$\tilde{\mathbf{R}} = \begin{pmatrix} \mathbf{Q}_{eq} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{eq} \end{pmatrix} \begin{pmatrix} \boldsymbol{\Lambda}_{eq} & \breve{\mathbf{R}}_{12} \\ \mathbf{0} & \boldsymbol{\Lambda}_{eq} \end{pmatrix} \begin{pmatrix} \mathbf{U}_{eq} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_{eq} \end{pmatrix} \tag{3.23}$$

where $\mathbf{R}_{12}$ and $\breve{\mathbf{R}}_{12}$ has the following relationship:

$$\mathbf{R}_{12} = \mathbf{Q}_{eq}\breve{\mathbf{R}}_{12}\mathbf{U}_{eq} \tag{3.24}$$

We define the block matrices in Eq. (3.23) sequentially as $\mathbf{Q}_{\mathbf{R}}$, $\mathbf{R}$, and $\mathbf{U}_{\mathbf{R}}$:

$$\mathbf{Q}_{\mathbf{R}} = \begin{pmatrix} \mathbf{Q}_{eq} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{eq} \end{pmatrix} \quad \mathbf{R} = \begin{pmatrix} \boldsymbol{\Lambda}_{eq} & \breve{\mathbf{R}}_{12} \\ \mathbf{0} & \boldsymbol{\Lambda}_{eq} \end{pmatrix} \quad \mathbf{U}_{\mathbf{R}} = \begin{pmatrix} \mathbf{U}_{eq} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_{eq} \end{pmatrix} \tag{3.25}$$

Substituting $\tilde{\mathbf{R}} = \mathbf{Q}_{\mathbf{R}}\mathbf{R}\mathbf{U}_{\mathbf{R}}$ into Eq. (3.20), we further have

$$\boldsymbol{\Lambda}\mathbf{V}^H = \mathbf{U}^H\tilde{\mathbf{Q}}\mathbf{Q}_{\mathbf{R}}\mathbf{R}\mathbf{U}_{\mathbf{R}} \tag{3.26}$$

Rewrite Eq. (3.26),

$$\mathbf{\Lambda}\mathbf{V}^H\mathbf{U_R}^H = \mathbf{U}^H\tilde{\mathbf{Q}}\mathbf{Q_R}\mathbf{R} \tag{3.27}$$

Note that the matrices $\mathbf{U}$, $\mathbf{V}$, $\mathbf{U_R}$, and $\mathbf{Q_R}$ are unitary matrices, we can absorb the product $\mathbf{V}^H\mathbf{U_R}^H$ as one unitary matrix $\mathbf{S}$ and absorb the product $\mathbf{U}^H\tilde{\mathbf{Q}}\mathbf{Q_R}$ as another unitary matrix $\mathbf{Q}$,

$$\mathbf{S} = \mathbf{V}^H\mathbf{U_R}^H \tag{3.28}$$

$$\mathbf{Q} = \mathbf{U}^H\tilde{\mathbf{Q}}\mathbf{Q_R} \tag{3.29}$$

so that Eq. (3.27) has a compact form:

$$\mathbf{\Lambda}\mathbf{S} = \mathbf{Q}\mathbf{R} \tag{3.30}$$

Therefore, it is equivalent to considering an diagonal problem: finding the block-equal QRS decomposition for a given 4×4 diagonal matrix $\mathbf{\Lambda}$, whose diagonal elements are the singular values of a full 4×4 matrix $\tilde{\mathbf{H}}$.

Problem 1. *Given an diagonal matrix $\mathbf{\Lambda} = diag(\sqrt{\sigma_1}, \sqrt{\sigma_2}, \sqrt{\sigma_3}, \sqrt{\sigma_4})$, where $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \sigma_4 \geq 0$, find an unitary matrix $\mathbf{S} \in \mathbb{R}^{4\times4}$ such that $\mathbf{\Lambda}\mathbf{S} = \mathbf{Q}\mathbf{R}$, where $\mathbf{Q} \in \mathbb{R}^{4\times4}$ is an unitary matrix and $\mathbf{R} \in \mathbb{R}^{4\times4}$ has the following structure:*

$$\mathbf{R} = \begin{pmatrix} \mathbf{\Lambda}_{eq} & \breve{\mathbf{R}}_{12} \\ \mathbf{0} & \mathbf{\Lambda}_{eq} \end{pmatrix} \tag{3.31}$$

Specifically, the diagonal block entries $\mathbf{\Lambda}_{eq} = diag(\sqrt{r_1}, \sqrt{r_2})$ and $\breve{\mathbf{R}}_{12} \in \mathbb{R}^{2\times2}$ can be *any* 2×2 *matrix.*

Assume that the elements in $\mathbf{S}$ is written as s_{ij}, $i, j = 1, 2, 3, 4$. We have the following necessary and sufficient condition of the existence of such block-equal QRS decomposition.

Theorem 1. *There exists such a unitary matrix* $\mathbf{S}$ *that satisfies Problem 1 if and only if the following 7 equations have solution:*

$$\begin{aligned}
&s_{11}^2 + s_{21}^2 + s_{31}^2 + s_{41}^2 = 1 \\
&s_{12}^2 + s_{22}^2 + s_{32}^2 + s_{42}^2 = 1 \\
&s_{11}s_{12} + s_{21}s_{22} + s_{31}s_{32} + s_{41}s_{42} = 0 \\
&\sigma_1 s_{11}^2 + \sigma_2 s_{21}^2 + \sigma_3 s_{31}^2 + \sigma_4 s_{41}^2 = r_1 \\
&\sigma_1 s_{12}^2 + \sigma_2 s_{22}^2 + \sigma_3 s_{32}^2 + \sigma_4 s_{42}^2 = r_2 \\
&\sigma_1 s_{11}s_{12} + \sigma_2 s_{21}s_{22} + \sigma_3 s_{31}s_{32} + \sigma_4 s_{41}s_{42} = 0 \\
&\frac{\sigma_1^2 s_{11}^2 + \sigma_2^2 s_{21}^2 + \sigma_3^2 s_{31}^2 + \sigma_4^2 s_{41}^2}{r_1} + \frac{\sigma_1^2 s_{12}^2 + \sigma_2^2 s_{22}^2 + \sigma_3^2 s_{32}^2 + \sigma_4^2 s_{42}^2}{r_2} \\
&= \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 - r_1 - r_2
\end{aligned} \tag{3.32}$$

(Proof: See Appendix C.)

In Eq. (3.32), the 7 equations are in terms of 8 variables s_{ij}, $i = 1, 2$, $j = 1, 2, 3, 4$. The other variables σ_1, σ_2, σ_3, σ_4, r_1, and r_2 are the coefficients satisfying the following relation:

$$\sigma_1\sigma_2\sigma_3\sigma_4 = r_1^2 r_2^2 \tag{3.33}$$

The proof of Eq. (3.33) is as follows:

Proof. Write the equation $\mathbf{\Lambda S} = \mathbf{QR}$ in its matrix form:

$$\begin{pmatrix} \sqrt{\sigma_1} & & & \\ & \sqrt{\sigma_2} & & \\ & & \sqrt{\sigma_3} & \\ & & & \sqrt{\sigma_4} \end{pmatrix} S = Q \begin{pmatrix} \sqrt{r_1} & 0 & \multicolumn{2}{c}{\breve{\mathbf{R}}_{12}} \\ 0 & \sqrt{r_2} & & \\ \multicolumn{2}{c}{\mathbf{0}} & \sqrt{r_1} & 0 \\ & & 0 & \sqrt{r_2} \end{pmatrix} \tag{3.34}$$

The determinant of the left-hand side should be equal to that of the right-hand side, therefore,

$$\sqrt{\sigma_1\sigma_2\sigma_3\sigma_4} = r_1 r_2 \tag{3.35}$$

□

Moreover, when the 6 variables further satisfy a certain condition, the following theorem give an explicit solution to the 7 equations in Theorem 1.

Theorem 2. *When r_1 and r_2 satisfy that*

$$r_1 = \sqrt{\sigma_1\sigma_2} \tag{3.36}$$

$$r_2 = \sqrt{\sigma_3\sigma_4} \tag{3.37}$$

a specific solution of $\mathbf{S}$ *is given by*

$$\mathbf{S} = \begin{bmatrix} \sqrt{\dfrac{r_1-\sigma_2}{\sigma_1-\sigma_2}} & 0 & \sqrt{\dfrac{\sigma_1-r_1}{\sigma_1-\sigma_2}} & 0 \\ \sqrt{\dfrac{\sigma_1-r_1}{\sigma_1-\sigma_2}} & 0 & -\sqrt{\dfrac{r_1-\sigma_2}{\sigma_1-\sigma_2}} & 0 \\ 0 & \sqrt{\dfrac{r_2-\sigma_4}{\sigma_3-\sigma_4}} & 0 & \sqrt{\dfrac{\sigma_3-r_2}{\sigma_3-\sigma_4}} \\ 0 & \sqrt{\dfrac{\sigma_3-r_2}{\sigma_3-\sigma_4}} & 0 & -\sqrt{\dfrac{r_2-\sigma_4}{\sigma_3-\sigma_4}} \end{bmatrix} \tag{3.38}$$

(Proof: See Appendix E)

From Theorem 2, we know that the specific solution of $\mathbf{S}$ can be written as a function of $\mathbf{\Lambda}$:

$$\mathbf{S} = f_{d-qr}(\mathbf{\Lambda}) \tag{3.39}$$

where the notation $f_{d-qr}(\cdot)$ is the function mapping $\mathbf{\Lambda}$ to its corresponding $\mathbf{S}$.

Take the inverse on both sides of the diagonal block-equal QRS decomposition, we have

$$\mathbf{S}^H\mathbf{\Lambda}^{-1} = \mathbf{R}^{-1}\mathbf{Q}^H \tag{3.40}$$

$$\mathbf{\Lambda}^{-1}\mathbf{Q} = \mathbf{S}\mathbf{R}^{-1} \tag{3.41}$$

Note that the inverse of an upper triangular matrix is still an upper triangular matrix. Therefore, Eq. (3.41) is also a block-equal QRS decomposition. This observation implies that $\mathbf{Q}$ can also be obtained by using the same function $f_{d-qr}(\cdot)$ as long as we replace the argument of the function by its inverse $\mathbf{\Lambda}^{-1}$, i.e.,

$$\mathbf{Q} = f_{d-qr}(\mathbf{\Lambda}^{-1}) \tag{3.42}$$

Using the result of Theorem 2 twice, we are able to find both $\mathbf{S}$ and $\mathbf{Q}$. Then, $\mathbf{R}$ is given by

$$\mathbf{R} = \mathbf{Q}^H \mathbf{\Lambda} \mathbf{S} \tag{3.43}$$

Then, we develop the diagonal block-equal QRS decomposition to the block-equal QRS decomposition for any given 4×4 full channel matrix $\mathbf{H}$. From Eqs. (3.29) and (3.23), $\tilde{\mathbf{Q}}$ and $\tilde{\mathbf{R}}$ are expressed in the following equations:

$$\tilde{\mathbf{Q}} = \mathbf{U}\mathbf{Q}{\mathbf{Q}_{\mathbf{R}}}^H, \quad \tilde{\mathbf{R}} = \mathbf{Q}_{\mathbf{R}}\mathbf{R}\mathbf{U}_{\mathbf{R}} \tag{3.44}$$

Therefore the block equal QR decomposition of a 4×4 cascaded channel matrix is given by

$$\tilde{\mathbf{H}} = \tilde{\mathbf{Q}}\tilde{\mathbf{R}} = (\mathbf{U}\mathbf{Q}{\mathbf{Q}_{\mathbf{R}}}^H)(\mathbf{Q}_{\mathbf{R}}\mathbf{R}\mathbf{U}_{\mathbf{R}}) = \mathbf{U}\mathbf{Q}\mathbf{R}\mathbf{U}_{\mathbf{R}} \tag{3.45}$$

For notational simplicity, we can absorb the product of $\tilde{\mathbf{Q}}$ and $\mathbf{Q}_{\mathbf{R}}$ as $\tilde{\mathbf{Q}}_1$ and define a new matrix $\tilde{\mathbf{R}}_1$ as $\mathbf{R}\mathbf{U}_{\mathbf{R}}$:

$$\tilde{\mathbf{Q}}_1 = \tilde{\mathbf{Q}}\mathbf{Q}_{\mathbf{R}} = (\mathbf{U}\mathbf{Q}{\mathbf{Q}_{\mathbf{R}}}^H)\mathbf{Q}_{\mathbf{R}} = \mathbf{U}\mathbf{Q} \tag{3.46}$$

$$\tilde{\mathbf{R}}_1 = \mathbf{R}\mathbf{U}_{\mathbf{R}} \tag{3.47}$$

Using the new notations $\tilde{\mathbf{Q}}_1$ and $\tilde{\mathbf{R}}_1$, the block-equal QRS decomposition can also be written as

$$\tilde{\mathbf{H}} = \tilde{\mathbf{Q}}_1\tilde{\mathbf{R}}_1 \tag{3.48}$$

In practice, the block-equal QRS decomposition we apply is in the form of Eq. (3.48), where the unitary matrix $\tilde{\mathbf{Q}}_1$ actually absorbs the left-hand side unitary matrix $\mathbf{U}_{\mathbf{R}}$ and the block upper triangular matrix $\tilde{\mathbf{R}}_1$ only contains two parts, the diagonal block upper triangular matrix $\mathbf{R}$ and the unitary matrix $\mathbf{U}_{\mathbf{R}}$.

Chapter 4

Optimal Precoder Design Based on Block-Equal QRS Decomposition

This chapter applies the mathematical conclusions in Chapter 3 on the optimal precoder design for the proposed receiving shceme in 4×4 case.

4.1 The Structure of Precoder F

Based on the ML detection criterion, and the optimality of block-equal QRS decomposition, the optimal precoder is designed to maximize the minimum distance $d_{\min}$, which is mentioned in section 3.1, Eq. (3.18). We write the optimization problem of the precoder design in the following form:

Problem 2. The Main Optimization Problem for Precoder Design

$$\begin{aligned}
\max_{\mathbf{F}} \quad & d_{\min} \\
\text{s.t.} \quad & \tilde{\mathbf{H}} = \mathbf{H}\mathbf{F} \textit{ has a block-equal QRS decomposition} \\
& \operatorname{tr}(\mathbf{F}^H\mathbf{F}) = 1
\end{aligned} \tag{4.1}$$

Instead of optimize $\mathbf{F}$ as a whole, we may first investigate the structure of $\mathbf{F}$ and then determine each component of it. Define the singular value decompositions of $\mathbf{H}$ and $\mathbf{F}$ as follows:

$$\mathbf{H} = \mathbf{U_H}\mathbf{\Lambda_H}\mathbf{V_H}^H \tag{4.2}$$

$$\mathbf{F} = \mathbf{U_F}\mathbf{\Lambda_F}\mathbf{V_F}^H \tag{4.3}$$

where $\mathbf{U_F}$, $\mathbf{\Lambda_F}$ and $\mathbf{V_F}^H$ are to be determined.

A necessary condition for an optimal precoder is that it diagonalizes the channel matrix $\mathbf{H}$, i.e., $\mathbf{U_F} = \mathbf{V_H}$(See Appendix F), then $\tilde{\mathbf{H}}$ is equal to

$$\tilde{\mathbf{H}} = \mathbf{H}\mathbf{F} = \mathbf{U_H}(\mathbf{\Lambda_H}\mathbf{\Lambda_F})\mathbf{V_F}^H \tag{4.4}$$

Comparing the equation with the singular value decomposition of the cascaded channel matrix $\tilde{\mathbf{H}}$, that is

$$\tilde{\mathbf{H}} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^H \tag{4.5}$$

Thus, we obtain the following equalities:

$$\mathbf{U_H} = \mathbf{U} \quad \mathbf{\Lambda_H \Lambda_F} = \mathbf{\Lambda} \quad \mathbf{V_F}^H = \mathbf{V}^H \tag{4.6}$$

Also, from Eq. (3.45), we know that the block-equal QRS decomposition of $\tilde{\mathbf{H}}$ is $\tilde{\mathbf{H}} = \mathbf{UQRU_R}$. Substitute $\mathbf{\Lambda S} = \mathbf{QR}$ into the right-hand side, we have

$$\tilde{\mathbf{H}} = \mathbf{U\Lambda SU_R} \tag{4.7}$$

We also compare Eq. (4.7) with the SVD of $\tilde{\mathbf{H}}$ in Eq. (4.5), which leads to

$$\mathbf{V}^H = \mathbf{SU_R} \tag{4.8}$$

From Eqs. (4.3), (4.6), and (4.8), the structure of the optimal precoder is obtained by

$$\mathbf{F} = \mathbf{V_H \Lambda_F S U_R} \tag{4.9}$$

Under our assumption, the channel information $\mathbf{H}$ is given. Thus we can calculate $\mathbf{V_H}$ and $\mathbf{\Lambda_H}$ based on $\mathbf{H}$. Therefore, to optimize precoder matrix $\mathbf{F}$ is equivalent to optimize both of the matrices $\mathbf{\Lambda_F}$, $\mathbf{S}$, and $\mathbf{U_R}$. According to Theorem 2, $\mathbf{S}$ depends on $\mathbf{\Lambda}$, which is the product of $\mathbf{\Lambda_H}$ and $\mathbf{\Lambda_F}$. When $\mathbf{\Lambda_H}$ is known, $\mathbf{S}$ depends on $\mathbf{\Lambda_F}$. Thus, in order to design the optimal precoder, we are supposed to design the matrices $\mathbf{\Lambda_F}$ and $\mathbf{U_R}$ that maximize the minimum distance $d_{\min}$. The optimization problem 2 is equivalent to the following reformulation:

Problem 3. Reformulation of Problem 2 in Terms of $\mathbf{\Lambda_F}, \mathbf{S}, \mathbf{U_R}$

$$\begin{aligned}
\max_{\mathbf{\Lambda_F},\mathbf{S},\mathbf{U_R}} \quad & d_{\min} \\
\text{s.t.} \quad & \mathbf{\Lambda_F} \textit{ is a diagonal matrix with nonnegative diagonal elements} \\
& \mathbf{S} \textit{ is a unitary matrix} \\
& \mathbf{U_R} \textit{ is a block diagonal matrix, each diagonal block is a } 2\times 2 \textit{ unitary matrix} \\
& \mathbf{\Lambda S} = \mathbf{QR} \\
& \mathbf{\Lambda} = \mathbf{\Lambda_H \Lambda_F} \\
& \mathrm{tr}({\mathbf{\Lambda_F}}^H \mathbf{\Lambda_F}) = 1
\end{aligned} \tag{4.10}$$

4.2 Optimal Unitary Matrix $\mathbf{U_R}$ and the Corresponding $d_{\min}$

In this section, the choice of matrix $\mathbf{U_R}$ and its relation with $d_{\min}$ is discussed under case that all the transmitted symbols are chosen from 4-QAM constellation denoted by $\mathcal{S}$. Since $\mathbf{U_R} = \mathrm{diag}(\mathbf{U}_{eq}, \mathbf{U}_{eq})$, the essential issue is to determine $\mathbf{U}_{eq}$.

By introducing block-equal QRS decomposition, the cascaded channel $\tilde{\mathbf{H}}$ can be decomposed as a series of identical 2×2 block-channel. In the specific 4×4 case, each block-channel is represented by the matrix $\mathbf{R}_{eq}$.

$$\begin{pmatrix} \tilde{\mathbf{y}}_1 \\ \tilde{\mathbf{y}}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{R}_{eq} & \mathbf{R}_{12} \\ \mathbf{0} & \mathbf{R}_{eq} \end{pmatrix} \begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \end{pmatrix} + \begin{pmatrix} \tilde{\mathbf{n}}_1 \\ \tilde{\mathbf{n}}_2 \end{pmatrix} \tag{4.11}$$

Within each block-channel, ML detection deals with the following optimization problems for searching a 2×1 transmitted symbol vector $\mathbf{s}_i, i = 1, 2$:

$$\hat{\mathbf{s}}_2 = \arg \min_{\mathbf{s}_2 \in \mathcal{S}^2} \|\tilde{\mathbf{y}}_2 - \mathbf{R}_{eq}\mathbf{s}_2\|^2 \tag{4.12}$$

$$\hat{\mathbf{s}}_1 = \arg \min_{\mathbf{s}_1 \in \mathcal{S}^2} \|(\tilde{\mathbf{y}}_1 - \mathbf{R}_{12}\hat{\mathbf{s}}_2) - \mathbf{R}_{eq}\mathbf{s}_1\|^2 \tag{4.13}$$

When the signal to noise ratio (SNR) is high, the symbol error probability (SEP) is dominated by the minimum distance $d_{\min}$ (See Eq. (3.17)). Under the assumption that the transmitted symbols are uncorrelated and have unitary energy, $d_{\min}$ depends on the 2×2 block-channel matrix $\mathbf{R}_{eq}$, that is, $d_{\min}$ is a function of $\mathbf{R}_{eq}$:

$$d_{\min} = d_{\min}(\mathbf{R}_{eq}) = \sqrt{\min_{\mathbf{s}, \hat{\mathbf{s}} \in \mathcal{S}^2, \mathbf{s} \neq \hat{\mathbf{s}}} (\mathbf{s} - \hat{\mathbf{s}})^H \mathbf{R}_{eq}^H \mathbf{R}_{eq} (\mathbf{s} - \hat{\mathbf{s}})} \tag{4.14}$$

We employ Eq. (3.48) as the form of block-equal QRS decomposition, where $\tilde{\mathbf{R}}_1 = \mathbf{R}\mathbf{U}_{\mathbf{R}}$. Thus the equal diagonal block $\mathbf{R}_{eq}$ of $\tilde{\mathbf{R}}_1$ is given by

$$\mathbf{R}_{eq} = \mathbf{\Lambda}_{eq}\mathbf{U}_{eq} \tag{4.15}$$

where $\mathbf{\Lambda}_{eq} = \text{diag}(\sqrt{r_1}, \sqrt{r_2})$ and $\mathbf{U}_{eq} \in \mathbb{R}^{2\times 2}$ is a unitary matrix which can be characterized by several parameters. In particular, we examine the following parameterization of $\mathbf{U}_{eq}$ [43].

$$\mathbf{U}_{eq} = \begin{pmatrix} e^{j\alpha}\cos\theta & e^{j(\alpha+\gamma)}\sin\theta \\ -e^{j(-\gamma+\beta)}\sin\theta & e^{j\beta}\cos\theta \end{pmatrix} \tag{4.16}$$

In this parameterized form of $\mathbf{U}_{eq}$, $\alpha, \beta \in [0, 2\pi]$ is independent of the matrix

$\mathbf{R}_{eq}^H \mathbf{R}_{eq}^H$. Therefore, according to Eq. (4.14), α and β do not affect $d_{\min}$. However, the values of $\theta \in [0, 2\pi]$ and $\gamma \in [0, \pi]$ have an impact on $d_{\min}$. Thus, without loss of generality, we can set $\alpha = \beta = 0$. When the squared singular values, r_1 and r_2, of $\mathbf{R}_{eq}$ are fixed, i.e., $\mathbf{\Lambda}_{eq}$ is given, we can also say that $d_{\min}$ is a function of θ and γ:

$$d_{\min}(\mathbf{R}_{eq}) = d_{\min}(\theta, \gamma) \tag{4.17}$$

The parameters θ and γ should be carefully chosen such that the minimum distance is maximized, i.e.,

$$\begin{aligned} \max_{\theta,\gamma} \quad & d_{\min}(\theta, \gamma) \\ \text{s.t.} \quad & r_1 \text{ and } r_2 \text{ are fixed} \end{aligned} \tag{4.18}$$

From Eq. (4.14), it can be shown that $d_{\min}$ is symmetric about any multiple of $\pi/2$ for θ and γ. Hence, in the Problem (4.18) we can seek for closed-form solutions to θ and γ by only considering the case where $0 \leq \theta \leq \frac{\pi}{4}$ and $0 \leq \gamma \leq \frac{\pi}{4}$ without loss of generality.

To solve Problem (4.18), we firstly classify the 2×2 block-channel into four categories where r_1 and r_2 satisfy different conditions [43]:

- **Condition 1**: $(r_1 + r_2)/(r_1 - r_2) \geq 2$
- **Condition 2**: $\sqrt{2} \leq (r_1 + r_2)/(r_1 - r_2) \leq 2$
- **Condition 3**: $\frac{5\sqrt{74}}{37} \leq (r_1 + r_2)/(r_1 - r_2) \leq \sqrt{2}$
- **Condition 4**: $(r_1 + r_2)/(r_1 - r_2) \leq \frac{5\sqrt{74}}{37}$

The following Lemma 1 then provides the corresponding closed-form solutions to this optimization problem under different categories.

Lemma 1. *[43] For a 2×2 block-channel with given singular value matrix $\mathbf{\Lambda}_{eq} = diag(\sqrt{r_1}, \sqrt{r_2})$, the right-hand side unitary matrix is parameterized as*

$$\mathbf{U}_{eq} = \begin{pmatrix} \cos\theta & e^{j(\gamma)}\sin\theta \\ -e^{j(-\gamma)}\sin\theta & \cos\theta \end{pmatrix} \tag{4.19}$$

the optimal solution to the optimization problem (4.18) is given by

- *When $(r_1 + r_2)/(r_1 - r_2) \geq 2$,*

$$\theta = \gamma = \frac{\pi}{4} \tag{4.20}$$

and the maximum of the objective function is

$$\max d_{\min}(\theta, \gamma) = \sqrt{2(r_1 + r_2)} \tag{4.21}$$

- *When $\sqrt{2} \leq (r_1 + r_2)/(r_1 - r_2) \leq 2$,*

$$\theta = \frac{\pi}{4} \qquad \gamma = \arccos\left(\frac{r_1 + r_2}{2(r_1 - r_2)}\right) \tag{4.22}$$

and the maximum of the objective function is

$$\max d_{\min}(\theta, \gamma) = \sqrt{2(r_1 + r_2)} \tag{4.23}$$

- *When* $\frac{5\sqrt{74}}{37} \leq (r_1 + r_2)/(r_1 - r_2) \leq \sqrt{2}$,

$$\theta = \frac{\pi}{4} \qquad\qquad \gamma = \arcsin\left(\frac{r_1 + r_2}{2(r_1 - r_2)}\right) \tag{4.24}$$

and the maximum of the objective function is

$$\max d_{\min}(\theta, \gamma) = \sqrt{4(r_1 + r_2) - 2\sqrt{4(r_1 - r_2)^2 - (r_1 + r_2)^2}} \tag{4.25}$$

- *When* $(r_1 + r_2)/(r_1 - r_2) \leq \frac{5\sqrt{74}}{37}$,

$$\theta = \arctan\left(\sqrt{\frac{1-\nu}{1+\nu}}\right) \tag{4.26}$$

$$\gamma = \arcsin\left(\frac{r_1}{2(r_1 - r_2)}\sqrt{\frac{1-\nu}{1+\nu}} + \frac{r_2}{2(r_1 - r_2)}\sqrt{\frac{1-\nu}{1+\nu}}\right) \tag{4.27}$$

where $\nu = \frac{\sqrt{2(r_1-r_2)^2-(r_1+r_2)^2}}{\sqrt{3}(r_1-r_2)}$, *and the maximum of the objective function is*

$$\max d_{\min}(\theta, \gamma) = \sqrt{2(r_1 + r_2) - \frac{2}{\sqrt{3}}\sqrt{2(r_1 - r_2)^2 - (r_1 + r_2)^2}} \tag{4.28}$$

Lemma 1 shows that, when r_1 and r_2 are fixed, both the minimum distance $d_{\min}$ and the optimal design of the unitary matrix $\mathbf{U}_{eq}$ are determined by r_1 and r_2. According to Eq. (3.47), $\tilde{\mathbf{R}}_1$ is equal to the product of $\mathbf{R}$ and $\mathbf{U}_{\mathbf{R}}$:

$$\tilde{\mathbf{R}}_1 = \mathbf{R}\mathbf{U}_{\mathbf{R}} = \begin{pmatrix} \mathbf{R}_{eq} & \mathbf{R}_{12} \\ \mathbf{0} & \mathbf{R}_{eq} \end{pmatrix} = \begin{pmatrix} \mathbf{\Lambda}_{eq}\mathbf{U}_{eq} & \mathbf{R}_{12}\mathbf{U}_{eq} \\ \mathbf{0} & \mathbf{\Lambda}_{eq}\mathbf{U}_{eq} \end{pmatrix} \tag{4.29}$$

Here r_1 and r_2 are the diagonal elements of $\mathbf{\Lambda}_{eq}$. Therefore, when r_1 and r_2 are fixed,

$\mathbf{R}$ and $\mathbf{U_R}$ are also fixed. That is, as long as we find the optimal r_1 and r_2 that maximize $d_{\min}$, the optimal diagonal blocks of $\tilde{\mathbf{R}}_1$ have to be equal. In this sense, Lemma 1 shows the way to achieve our proposed block-equal QRS decomposition.

4.3 Optimization of Power Loading Matrix $\mathbf{\Lambda_F}$

In Section 4.1, we have claimed that the optimization problem of precoder design is equivalent to Problem 3 whose variables are $\mathbf{\Lambda_F}$, $\mathbf{S}$, and $\mathbf{U_R}$. Problem 3 states that:

$$\begin{aligned} \max_{\mathbf{\Lambda_F},\mathbf{S},\mathbf{U_R}} \quad & d_{\min} \\ \text{s.t.} \quad & \mathbf{\Lambda S} = \mathbf{QR},\ \mathbf{\Lambda} = \mathbf{\Lambda_H}\mathbf{\Lambda_F},\ \mathrm{tr}(\mathbf{\Lambda_F}^H\mathbf{\Lambda_F}) = 1 \end{aligned} \tag{4.30}$$

Section 4.2 further gives the closed-form expression of the objective function $d_{\min}$ in terms of the variables r_1 and r_2, that is, $d_{\min}(r_1, r_2)$. Moreover, the optimal $\mathbf{U_R}$ is also determined by r_1 and r_2. We then rewrite the Problem 3 as follows,

Problem 4. Reformulation of Problem 3 in Terms of $\mathbf{\Lambda_F}, \mathbf{S}, r_1, r_2$

$$\begin{aligned} \max_{\mathbf{\Lambda_F},\mathbf{S},r_1,r_2} \quad & d_{\min}(r_1, r_2) \\ \text{s.t.} \quad & \mathbf{\Lambda_F} \textit{ is a diagonal matrix with nonnegative diagonal elements} \\ & \mathbf{S} \textit{ is a unitary matrix} \\ & r_1 \geq r_2 \geq 0 \\ & \mathbf{\Lambda S} = \mathbf{QR} \\ & \mathbf{\Lambda} = \mathbf{\Lambda_H}\mathbf{\Lambda_F} \\ & \mathrm{tr}(\mathbf{\Lambda_F}^H\mathbf{\Lambda_F}) = 1 \end{aligned} \tag{4.31}$$

Specifically, we describe the diagonal matrices $\mathbf{\Lambda}$, $\mathbf{\Lambda_H}$, and $\mathbf{\Lambda_F}$ by using their diagonal elements, which are the singular values of their corresponding matrices:

$$\mathbf{\Lambda} = \mathrm{diag}(\sqrt{\sigma_1}, \sqrt{\sigma_2}, \sqrt{\sigma_3}, \sqrt{\sigma_4}) \tag{4.32}$$

$$\mathbf{\Lambda_H} = \mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}, \sqrt{\lambda_4}) \tag{4.33}$$

$$\mathbf{\Lambda_F} = \mathrm{diag}(\sqrt{\mu_1}, \sqrt{\mu_2}, \sqrt{\mu_3}, \sqrt{\mu_4}) \tag{4.34}$$

$$\mathbf{\Lambda}_{eq} = \mathrm{diag}(\sqrt{r_1}, \sqrt{r_2}) \tag{4.35}$$

The diagonal elements of $\mathbf{\Lambda_H}$ are given by the channel state information (CSI), while the diagonal elements of $\mathbf{\Lambda_F}$ need to be optimized. Essentially, these diagonal elements of $\mathbf{\Lambda_F}$ serve as power loading factors of the precoder. It should be worth noting that in this thesis the all the diagonal elements are assumed to be decreasingly ordered. That is,

$$\sqrt{\sigma_1} \geq \sqrt{\sigma_2} \geq \sqrt{\sigma_3} \geq \sqrt{\sigma_4} \tag{4.36}$$

$$\sqrt{\lambda_1} \geq \sqrt{\lambda_2} \geq \sqrt{\lambda_3} \geq \sqrt{\lambda_4} \tag{4.37}$$

$$\sqrt{\mu_1} \geq \sqrt{\mu_2} \geq \sqrt{\mu_3} \geq \sqrt{\mu_4} \tag{4.38}$$

$$\sqrt{r_1} \geq \sqrt{r_2} \tag{4.39}$$

Based on the notations, we transform the second and third constraints in Problem 4 from the matrix forms to their equivalent constraints in terms of the singular values of each matrix:

$$\sigma_i = \lambda_i \mu_i, i = 1, 2, 3, 4 \tag{4.40}$$

$$\mu_1 + \mu_2 + \mu_3 + \mu_4 = 1 \tag{4.41}$$

The first constraint in Problem 4 ensures that the cascaded channel matrix has block-equal QRS decomposition. From the first constraint $\mathbf{\Lambda S} = \mathbf{QR}$, we can get to the following equality:

$$\mathbf{S}^H \mathbf{\Lambda}^H \mathbf{\Lambda S} = \mathbf{R}^H \mathbf{R} \tag{4.42}$$

Note that σ_1, σ_2, σ_3, σ_4 are the eigenvalues of the matrix $\mathbf{S}^H \mathbf{\Lambda}^H \mathbf{\Lambda S}$ in descending order on the left-hand side. On the right hand side, the diagonal elements of $\mathbf{R}^H \mathbf{R}$ are the Cholesky values, which are given by r_1, r_1, r_2, and r_2 in descending order. The following lemma states the relationship between these values [44, p. 234].

Lemma 2. *If $\mathbf{A} \in \mathbb{R}^{n \times n}$ is a positive semi-definite matrix with eigenvalues λ_i and Cholesky values d_i, $i = 1, \ldots, n$, then*

$$\mathbf{d} \prec_\times \boldsymbol{\lambda} \tag{4.43}$$

where $\mathbf{d} = (d_{[1]}, \ldots, d_{[n]})$ and $\boldsymbol{\lambda} = (\lambda_{[1]}, \ldots, \lambda_{[n]})$ (The brackets in the subscripts indicates that the sequences are arranged in descending order). Conversely, if the two sequences $\mathbf{d}$ and λ satisfy the relationship described in Eq. (4.43), then there exists a positive semi-definite matrix $\mathbf{A}$ with eigenvalues λ_i and Cholesky values d_i, $i = 1, \ldots, n$.

Applying Lemma 2 to Eq. (4.42), a necessary but not sufficient condition of the first constraint $\mathbf{\Lambda S} = \mathbf{QR}$ is given by

$$(r_1, r_1, r_2, r_2) \prec_\times (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \tag{4.44}$$

Changing the first constraint of Problem 3 with its necessary but not sufficient condition (4.44), and replacing other two constraints with their equivalent versions, it leads to a new problem as follows:

Problem 5. Optimization Problem with Enlarged Feasible Set

$$\begin{aligned}
\max_{\substack{\mu_1,\mu_2,\mu_3,\mu_4\\ r_1,r_2}} \quad & d_{\min}(r_1, r_2) \\
\text{s.t.} \quad & (r_1, r_1, r_2, r_2) \prec_\times (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \\
& \sigma_i = \lambda_i \mu_i, i = 1, 2, 3, 4 \\
& \mu_1 + \mu_2 + \mu_3 + \mu_4 = 1
\end{aligned} \tag{4.45}$$

Since that the first constraint of Problem 5 is just a necessary condition of the constraint $\mathbf{\Lambda S} = \mathbf{QR}$, the solution to Problem 5 cannot guarantee that a specific block-equal QRS decomposition exists. In fact, the feasible set of Problem 4 is a subset of that of the Problem 5. In other words, Problem 5 is a relaxed problem.

To solve Problem 5, we decompose it into two sub-problems. The first sub-problem seeks the optimal r_1 and r_2 in terms of σ_1, σ_2, σ_3 and σ_4 that maximize $d_{\min}$ under the condition that the power loading factors μ_1, μ_2, μ_3 and μ_4 are fixed. Because σ_i depends on μ_i, so r_1 and r_2 can be expressed in terms of μ_1, μ_2, μ_3 and μ_4. The second sub-problem further finds optimal solutions of μ_1, μ_2, μ_3 and μ_4 that maximize $d_{\min}(r_1, r_2)$ under the precoder power constraint, which is $\mu_1 + \mu_2 + \mu_3 + \mu_4 = 1$.

4.4 Optimal Solution to the First Sub-Problem

As it was stated, the first step is to solve the following sub-problem. We remove the second and third constraints form Problem 5, because μ_1, μ_2, μ_3 and μ_4 are assumed to be fixed in the first sub-problem .

Problem 6. First Sub-problem of Problem 5

$$\begin{aligned} \max_{r_1,r_2} \quad & d_{\min} \\ \text{s.t.} \quad & (r_1, r_1, r_2, r_2) \prec_\times (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \end{aligned} \tag{4.46}$$

where $d_{\min}^2$ changes as r_1 and r_2 satisfy different 2×2 block-channel condition. According to the Lemma 1, the value of $d_{\min}^2$ can be concluded as:

$$d_{\min}^2 = \begin{cases} 2(r_1+r_2), & \dfrac{r_1+r_2}{(r_1-r_2)} \geq 2 \\ 2(r_1+r_2), & \sqrt{2} \leq \dfrac{r_1+r_2}{(r_1-r_2)} \leq 2 \\ 4(r_1+r_2) - 2\sqrt{3r_1^2+3r_2^2-10r_1r_2}, & \dfrac{5\sqrt{74}}{37} \leq \dfrac{r_1+r_2}{(r_1-r_2)} \leq \sqrt{2} \\ 2(r_1+r_2) - \dfrac{2}{\sqrt{3}}\sqrt{r_1^2+r_2^2-6r_1r_2}, & \dfrac{r_1+r_2}{(r_1-r_2)} \leq \dfrac{5\sqrt{74}}{37} \end{cases} \tag{4.47}$$

and the majorization constraint is equivalent to the following three conditions:

$$r_1 \geq \sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \tag{4.48}$$

$$r_1 \leq \sqrt{\sigma_1\sigma_2} \tag{4.49}$$

$$r_2 = \sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}/r_1 \tag{4.50}$$

(Proof: See the Appendix G.1)

4.4.1 Optimal r_1 and r_2 in the Condition 1

Under Condition 1, Problem 6 has following equivalent form:

$$\begin{aligned} \max_{r_1,r_2} \quad & 2(r_1+r_2) \\ \text{s.t.} \quad & (r_1+r_2)/(r_1-r_2) \geq 2 \\ & r_1 \geq \sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\ & r_1 \leq \sqrt{\sigma_1\sigma_2} \\ & r_2 = \sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}/r_1 \end{aligned} \tag{4.51}$$

For notational simplicity purpose, we introduce a new parameter $\rho_\sigma = \sqrt{(\sigma_1\sigma_2)/(\sigma_3\sigma_4)}$. There are several situations where ρ_σ falls in different ranges.

Situation A. When $\rho_\sigma \leq 3$, this problem can be reformulated as:

$$\begin{aligned} \max_{r_1} \quad & r_1 + \sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}/r_1 \\ \text{s.t.} \quad & r_1 \geq \sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\ & r_1 \leq \sqrt{\sigma_1\sigma_2} \end{aligned} \tag{4.52}$$

(Reformulation of Eq. (4.52): See Appendix G.2)

The optimal r_1 and r_2 is

$$r_1 = \sqrt{\sigma_1\sigma_2} \tag{4.53}$$

$$r_2 = \sqrt{\sigma_3\sigma_4} \tag{4.54}$$

The corresponding maximum of the squared minimum distance is

$$\max d_{\min}^2 = 2(\sqrt{\sigma_1\sigma_2} + \sqrt{\sigma_3\sigma_4}) \tag{4.55}$$

$$= 2(\rho_\sigma + 1)\sqrt{\sigma_3\sigma_4} \tag{4.56}$$

Situation B. When $\rho_\sigma > 3$, the problem can be formulated as:

$$\begin{aligned} \max_{r_1} \quad & r_1 + \sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}/r_1 \\ \text{s.t.} \quad & r_1 \geq \sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\ & r_1 \leq \sqrt{3}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \end{aligned} \tag{4.57}$$

(Reformulation of Eq. (4.57): See Appendix G.2)

The optimal r_1 and r_2 is

$$r_1 = \sqrt{3}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \tag{4.58}$$

$$r_2 = \frac{1}{\sqrt{3}}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \tag{4.59}$$

The corresponding maximum of the squared minimum distance is

$$\max d_{\min}^2 = 2(\sqrt{3} + \frac{1}{\sqrt{3}})\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \tag{4.60}$$

$$= \frac{8}{\sqrt{3}}\sqrt{\rho_\sigma}\sqrt{\sigma_3\sigma_4} \tag{4.61}$$

4.4.2 Optimal r_1 and r_2 in Condition 2

Under Condition 2, Problem 6 has following equivalent form:

$$\begin{aligned} \max_{r_1,r_2} \quad & 2(r_1+r_2) \\ \text{s.t.} \quad & \sqrt{2} \le (r_1+r_2)/(r_1-r_2) \le 2 \\ & r_1 \ge \sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\ & r_1 \le \sqrt{\sigma_1\sigma_2} \\ & r_2 = \sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}/r_1 \end{aligned} \tag{4.62}$$

There are several situations where ρ_σ falls in different ranges.

Situation A. When $\rho_\sigma < 3$, this problem is infeasible.

(Proof: See the Appendix G.3)

Situation B. When $3 \le \rho_\sigma \le (\sqrt{2}+1)^2$, the problem can be reformulated as

$$\begin{aligned} \max_{r_1} \quad & r_1 + \sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}/r_1 \\ \text{s.t.} \quad & r_1 \ge \sqrt{3}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\ & r_1 \le \sqrt{\sigma_1\sigma_2} \end{aligned} \tag{4.63}$$

The optimal r_1 and r_2 is

$$r_1 = \sqrt{\sigma_1\sigma_2} \tag{4.64}$$

$$r_2 = \sqrt{\sigma_3\sigma_4} \tag{4.65}$$

The corresponding maximum of the squared minimum distance is

$$\max d_{\min}^2 = 2(\sqrt{\sigma_1\sigma_2} + \sqrt{\sigma_3\sigma_4}) \tag{4.66}$$

$$= 2(\rho_\sigma + 1)\sqrt{\sigma_3\sigma_4} \tag{4.67}$$

Situation C. When $\rho_\sigma > (\sqrt{2}+1)^2$, the problem can be reformulated as

$$\begin{aligned} \max_{r_1} \quad & r_1 + \sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}/r_1 \\ \text{s.t.} \quad & r_1 \geq \sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\ & r_1 \leq (\sqrt{2}+1)\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \end{aligned} \tag{4.68}$$

The optimal r_1 and r_2 is

$$r_1 = (\sqrt{2}+1)\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \tag{4.69}$$

$$r_2 = (\sqrt{2}-1)\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \tag{4.70}$$

The corresponding maximum of the squared minimum distance is

$$\max d_{\min}^2 = 4\sqrt{2}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \tag{4.71}$$

$$= 4\sqrt{2}\sqrt{\rho_\sigma}\sqrt{\sigma_3\sigma_4} \tag{4.72}$$

4.4.3 Optimal r_1 and r_2 in Condition 3

Under Condition 3, Problem 6 has following equivalent form:

$$\begin{aligned}
\max_{r_1,r_2} \quad & 4(r_1+r_2) - 2\sqrt{3r_1^2+3r_2^2-10r_1r_2} \\
\text{s.t.} \quad & \tfrac{5\sqrt{74}}{37} \le (r_1+r_2)/(r_1-r_2) \le \sqrt{2} \\
& r_1 \ge \sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\
& r_1 \le \sqrt{\sigma_1\sigma_2} \\
& r_2 = \sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}/r_1
\end{aligned} \tag{4.73}$$

Rewriting the objective function as a function in terms of r_1+r_2, we have:

$$f(r_1+r_2) = 4(r_1+r_2) - 2\sqrt{3(r_1+r_2)^2 - 16r_1r_2} \tag{4.74}$$

$$= 4(r_1+r_2) - 2\sqrt{3(r_1+r_2)^2 - 16\sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}} \tag{4.75}$$

Setting the derivative of the minimum distance with respect to r_1+r_2 equal to zero, we obtain $r_1+r_2 = \frac{8}{\sqrt{3}}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}(\approx 4.6\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4})$. The upper bound of r_1+r_2, however, is given by

$$r_1+r_2 \le \left[\sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}} + \left(1\Big/\sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}}\right)\right]\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}(\approx 3.9\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}) \tag{4.76}$$

(Proof: See Appendix G.4.)

which is less than the value that makes the derivative of minimum distance to be zero. It means that the squared minimum distance is monotonically decreasing within the domain of r_1+r_2 (See Fig. 4.1). Thus, maximizing squared minimum distance is equivalent to minimizing r_1+r_2.

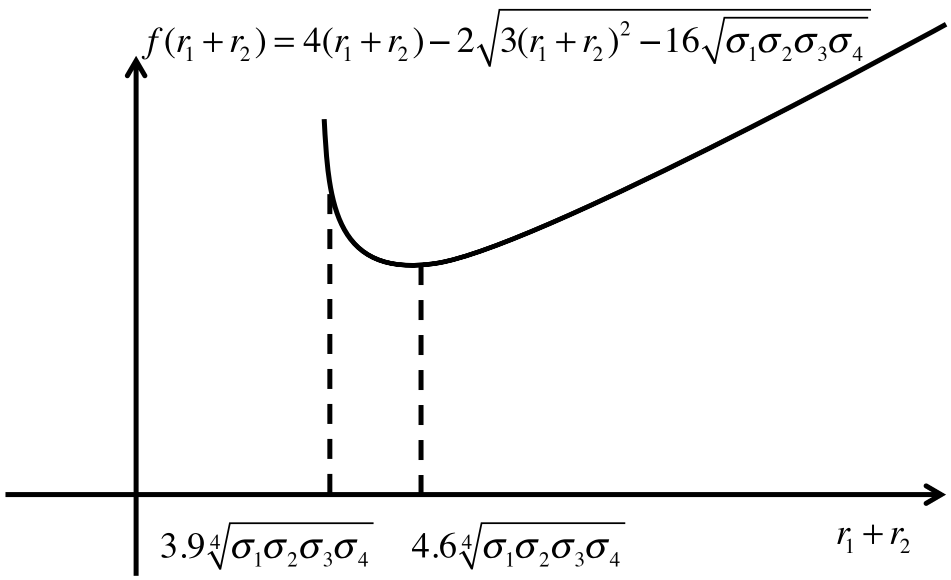

Figure 4.1: The Objective Function in Optimization Problem (4.73)

The problem is reformulated as:

$$\begin{aligned} \min_{r_1} \quad & r_1 + \sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}/r_1 \\ \text{s.t.} \quad & r_1 \le \sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\ & r_1 \ge (\sqrt{2}+1)\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\ & r_1 \le \sqrt{\sigma_1\sigma_2} \end{aligned} \tag{4.77}$$

To minimize $r_1 + r_2$, we have the following two situations with respect to the parameters ρ_σ:

Situation A. When $\rho_\sigma < (\sqrt{2}+1)^2$, this problem is infeasible.

Situation B. When $\rho_\sigma \ge (\sqrt{2}+1)^2$, the optimal r_1 and r_2 is

$$r_1 = (\sqrt{2}+1)\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \tag{4.78}$$

$$r_2 = (\sqrt{2}-1)\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \tag{4.79}$$

The corresponding maximum of the squared minimum distance is

$$\max d_{\min}^2 = 4\sqrt{2}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \tag{4.80}$$

$$= 4\sqrt{2}\sqrt{\rho_\sigma}\sqrt{\sigma_3\sigma_4} \tag{4.81}$$

4.4.4 Optimal r_1 and r_2 in Condition 4

Under Condition 4, Problem 6 has following equivalent form:

$$\begin{aligned} \max_{r_1,r_2} \quad & 2(r_1+r_2) - \tfrac{2}{\sqrt{3}}\sqrt{r_1^2+r_2^2-6r_1r_2} \\ \text{s.t.} \quad & (r_1+r_2)/(r_1-r_2) \le \tfrac{5\sqrt{74}}{37} \\ & r_1 \ge \sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\ & r_1 \le \sqrt{\sigma_1\sigma_2} \\ & r_2 = \sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}/r_1 \end{aligned} \tag{4.82}$$

Similar to Condition 3, we rewrite the objective function as a function in terms of r_1+r_2:

$$f(r_1+r_2) = 2(r_1+r_2) - \frac{2}{\sqrt{3}}\sqrt{(r_1+r_2)^2 - 8r_1r_2} \tag{4.83}$$

$$= 2(r_1+r_2) - \frac{2}{\sqrt{3}}\sqrt{(r_1+r_2)^2 - 8\sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}} \tag{4.84}$$

Setting the derivative of the minimum distance with respect to r_1+r_2 equal to zero, we obtain $r_1+r_2 = 2\sqrt{3}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}(\approx 3.46\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4})$. The lower bound of r_1+r_2, however, is given by

$$r_1 + r_2 \geq \left[\sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}} + \left(1/\sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}} \right) \right] \sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} (\approx 3.9\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}) \quad (4.85)$$

(Proof: See Appendix G.5.)

which is greater than the value that makes the derivative of minimum distance to be zero. It means that the squared minimum distance is monotonically increasing within the domain of $r_1 + r_2$. Thus, maximizing squared minimum distance is equivalent to maximizing $r_1 + r_2$.

The problem is reformulated as:

$$\begin{aligned} \max_{r_1} \quad & r_1 + \sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}/r_1 \\ \text{s.t.} \quad & r_1 \geq \sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}} \sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\ & r_1 \leq \sqrt{\sigma_1\sigma_2} \end{aligned} \quad (4.86)$$

Also, we have the following two situations with respect to the parameters ρ_σ:

Situation A. When $\rho_\sigma \leq \frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}$, this problem is infeasible.

Situation B. When $\rho_\sigma \geq \frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}$, the optimal r_1 and r_2 is

$$r_1 = \sqrt{\sigma_1\sigma_2} \quad (4.87)$$

$$r_2 = \sqrt{\sigma_3\sigma_4} \quad (4.88)$$

The corresponding maximum of the squared minimum distance is

$$\max d_{\min}^2 = 2(\sqrt{\sigma_1\sigma_2} + \sqrt{\sigma_3\sigma_4}) - \frac{2}{\sqrt{3}}\sqrt{\sigma_1\sigma_2 - 6\sqrt{\sigma_1\sigma_2\sigma_3\sigma_4} + \sigma_3\sigma_4} \tag{4.89}$$

$$= 2\left[\rho_\sigma + 1 - \frac{1}{\sqrt{3}}\sqrt{\rho_\sigma^2 - 6\rho_\sigma + 1}\right]\sqrt{\sigma_3\sigma_4} \tag{4.90}$$

4.4.5 Conclusion: Global Optimal r_1 and r_2

Since the optimal r_1 and r_2 are functions of σ_1, σ_2, σ_3, and σ_4, which are the squared singular values of $\tilde{\mathbf{H}} = \mathbf{HF}$, thus, given $\mathbf{H}$, we can adjust the the parameters in $\mathbf{F}$ so that r_1 and r_2 would assume values such that $d_{\min}^2(r_1, r_2)$ is of a maximum values.

Consider all the conditions from Condition 1 to Condition 4, we have obtain many different expressions of the maximum $d_{\min}^2$. There are overlaps between some cases within which two or more $\max d_{\min}^2$s are given. We compare the different $\max d_{\min}^2$s in the different regions of r_1 and r_2 and choose the one yielding the highest values of $\max d_{\min}^2$ within each region. Note that all the expressions are written as a function in terms of ρ_σ multiplied by a factor $\sqrt{\sigma_3\sigma_4}$. For the convenience of comparing the overlapped expressions of $\max d_{\min}^2$, we can remove the common factor $\sqrt{\sigma_3\sigma_4}$ and just compare those functions in the domain of ρ_σ. We then conclude all the expressions that divided by and their valid range in Table 4.1. Fig. 4.2 further depicts the relationship between those functions.

Therefore in summary, we have proposed the following choices of $\max d_{\min}^2$ under

Table 4.1: Conclusion of All Expressions of $\max d_{\min}^2$ in the First Sub-problem

	Condition 1	Condition 2	Condition 3	Condition 4
$\rho_\sigma \leq 3$	$f_1(\rho_\sigma)$[1]	-	-	-
$3 \leq \rho_\sigma \leq (\sqrt{2}+1)^2$	$f_2(\rho_\sigma)$[2]	$f_1(\rho_\sigma)$	-	-
$(\sqrt{2}+1)^2 \leq \rho_\sigma \leq \frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}$	$f_2(\rho_\sigma)$	$f_3(\rho_\sigma)$[3]	$f_3(\rho_\sigma)$	-
$\rho_\sigma \geq \frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}$	$f_2(\rho_\sigma)$	$f_3(\rho_\sigma)$	$f_3(\rho_\sigma)$	$f_4(\rho_\sigma)$[4]

[1] $f_1(\rho_\sigma) = 2(\rho_\sigma + 1)$
[2] $f_2(\rho_\sigma) = \frac{8}{\sqrt{3}}\sqrt{\rho_\sigma}$
[3] $f_3(\rho_\sigma) = 4\sqrt{2}\sqrt{\rho_\sigma}$
[4] $f_4(\rho_\sigma) = 2\left[\rho_\sigma + 1 - \frac{1}{\sqrt{3}}\sqrt{\rho_\sigma^2 - 6\rho_\sigma + 1}\right]$

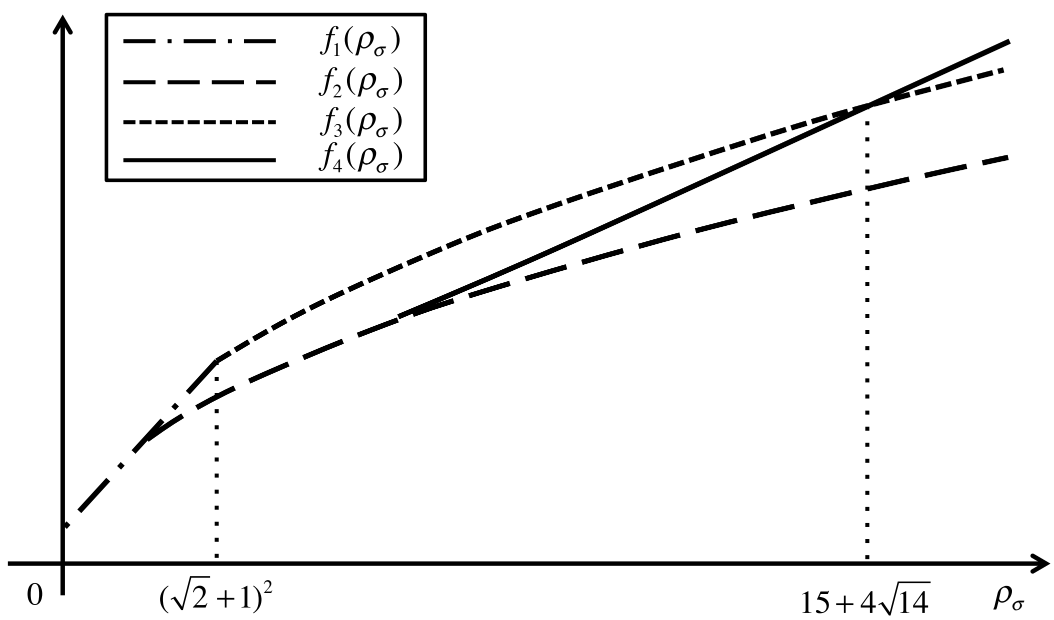

Figure 4.2: Comparsion of Multiple Expressions of $\max d_{\min}^2$ in the First Sub-problem

different conditions of ρ_σ:

$$d_{\min}^2 = \begin{cases} 2(\rho_\sigma + 1)\sqrt{\sigma_3\sigma_4}, & \rho_\sigma \le (\sqrt{2}+1)^2 \\ 4\sqrt{2}\sqrt{\rho_\sigma}\sqrt{\sigma_3\sigma_4}, & (\sqrt{2}+1)^2 \le \rho_\sigma \le 15 + 4\sqrt{14} \\ 2\left[\rho_\sigma + 1 - \frac{1}{\sqrt{3}}\sqrt{\rho_\sigma^2 - 6\rho_\sigma + 1}\right]\sqrt{\sigma_3\sigma_4}, & \rho_\sigma \ge 15 + 4\sqrt{14} \end{cases} \tag{4.91}$$

Corresponding to the values of global $\max d_{\min}^2$ are the values of optimal r_1 and r_2 which are shown as follows:

$$\begin{cases} r_1 = \sqrt{\sigma_1\sigma_2}, & r_2 = \sqrt{\sigma_3\sigma_4}, & \rho_\sigma \le (\sqrt{2}+1)^2 \\ r_1 = (\sqrt{2}+1)\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}, & r_2 = (\sqrt{2}-1)\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}, & (\sqrt{2}+1)^2 \le \rho_\sigma \le 15 + 4\sqrt{14} \\ r_1 = \sqrt{\sigma_1\sigma_2}, & r_2 = \sqrt{\sigma_3\sigma_4}, & \rho_\sigma \ge 15 + 4\sqrt{14} \end{cases} \tag{4.92}$$

4.5 Optimal Solution to the Second Sub-Problem

After solving the first sub-problem, r_1 and r_2 are expressed by optimal functions of $\{\sigma_i\}, i = 1, 2, 3, 4$ in the sense that $d_{\min}^2$ reaches the global maximum. Since we have obtained the optimal values of $d_{\min}$ in terms of r_1 and r_2, now, we tackle the second sub-problem by fixing r_1 and r_2 at their optimal functions in terms of $\{\sigma_i\}, i = 1, 2, 3, 4$ and then optimizing the objective function $d_{\min}$ with respect to $\{\sigma_i\}, i = 1, 2, 3, 4$.

Problem 7. Second Sub-problem of Problem 5

$$\begin{aligned} \max_{\mu_1,\mu_2,\mu_3,\mu_4} \quad & d_{\min}(\sigma_1,\sigma_2,\sigma_3,\sigma_4) \\ \text{s.t.} \quad & \sigma_i = \lambda_i \mu_i, i = 1,2,3,4 \\ & \mu_1+\mu_2+\mu_3+\mu_4 = 1 \end{aligned} \tag{4.93}$$

In Eq. (4.91), $d_{\min}^2$ is a piecewise function, of which the expression is depending on the channel parameter ρ_σ. We therefore solve the optimization problem under different cases of $\{\sigma_i\}, i = 1,2,3,4$ given by Eq. (4.91) respectively.

4.5.1 Case 1

In the first case, $\rho_\sigma \le (\sqrt{2}+1)^2$, $d_{\min}^2 = 2(\rho_\sigma+1)\sqrt{\sigma_3\sigma_4}$. Under this condition, Problem 7 has the following form:

$$\begin{aligned} \max_{\mu_1,\mu_2,\mu_3,\mu_4} \quad & 2(\rho_\sigma+1)\sqrt{\sigma_3\sigma_4} \\ \text{s.t.} \quad & \rho_\sigma \le (\sqrt{2}+1)^2 \\ & \sigma_i = \lambda_i \mu_i, i = 1,2,3,4 \\ & \mu_1+\mu_2+\mu_3+\mu_4 = 1 \end{aligned} \tag{4.94}$$

Denote $\rho_\lambda = \sqrt{\frac{\lambda_1\lambda_2}{\lambda_3\lambda_4}}$, $d_{\min}^2$ can be rewritten as $d_{\min}^2 = 2(\rho_\lambda\sqrt{\mu_1\mu_2} + \sqrt{\mu_3\mu_4})\sqrt{\lambda_3\lambda_4}$.

Lemma 3. *The optimal μ_1, μ_2, μ_3, and μ_4 that maximize the objective function in Problem (4.94) satisfy the following conditions:*

$$\mu_1 = \mu_2 \tag{4.95}$$

$$\mu_3 = \mu_4 \tag{4.96}$$

Proof. Since that

$$(\mu_1 - \mu_2)^2 \geq 0 \tag{4.97}$$

therefore

$$\mu_1\mu_2 \leq \frac{\mu_1^2 + \mu_2^2}{2} \tag{4.98}$$

The equality holds when μ_1 and μ_2 satisfy that $\mu_1 = \mu_2$.

Threrefore, $\sqrt{\mu_1\mu_2}$ is maximized when $\mu_1 = \mu_2$. Similarly, we can prove that the optimal μ_3 and μ_4 satisfy $\mu_3 = \mu_4$. □

Based on this result, we have $\mu_3 = 1/2 - \mu_1$. is reformulated as

$$\begin{aligned} \max_{\mu_1,\mu_3} \quad & \mu_1 \\ \text{s.t.} \quad & \mu_1 \leq \tfrac{(\sqrt{2}+1)^2}{2\left[\rho_\lambda + (\sqrt{2}+1)^2\right]} \\ & \mu_1 \geq 1/4 \end{aligned} \tag{4.99}$$

We discuss the solution to Problem (4.99) in the space spanned by λ_1, λ_2, λ_3 and λ_4. For notational simplicity, we express different regions of the whole space by using the parameter $\rho_\lambda = \sqrt{\frac{\lambda_1\lambda_2}{\lambda_3\lambda_4}}$.

Situation A. When $\rho_\lambda \leq (\sqrt{2}+1)^2$, we have

$$1/4 \leq \frac{(\sqrt{2}+1)^2}{2\left[\rho_\lambda + (\sqrt{2}+1)^2\right]} \tag{4.100}$$

the optimal $\{\mu_i\}, i = 1, 2, 3, 4$ and corresponding maximal minimum distance is:

$$\mu_1 = \mu_2 = \frac{(\sqrt{2}+1)^2}{2\left[\rho_\lambda + (\sqrt{2}+1)^2\right]} \tag{4.101}$$

$$\mu_3 = \mu_4 = \frac{\rho_\lambda}{2\left[\rho_\lambda + (\sqrt{2}+1)^2\right]} \tag{4.102}$$

$$\max d_{\min}^2 = \frac{(4+2\sqrt{2})\rho_\lambda}{\rho_\lambda + (\sqrt{2}+1)^2}\sqrt{\lambda_3\lambda_4} \tag{4.103}$$

Situation B. When $\rho_\lambda > (\sqrt{2}+1)^2$, however, $1/4 \geq \frac{(\sqrt{2}+1)^2}{2\left[\rho_\lambda+(\sqrt{2}+1)^2\right]}$, the problem is infeasible.

4.5.2 Case 2

In the second case, $(\sqrt{2}+1)^2 \leq \rho_\sigma \leq 15 + 4\sqrt{14}$, $d_{\min}^2 = 4\sqrt{2}\sqrt{\rho_\sigma}\sqrt{\sigma_3\sigma_4}$. Under this condition, Problem 7 has the following form:

$$\begin{aligned} \max_{\mu_1,\mu_2,\mu_3,\mu_4} \quad & 4\sqrt{2}\sqrt{\rho_\sigma}\sqrt{\sigma_3\sigma_4} \\ \text{s.t.} \quad & (\sqrt{2}+1)^2 \leq \rho_\sigma \leq 15 + 4\sqrt{14} \\ & \sigma_i = \lambda_i\mu_i, i = 1, 2, 3, 4 \\ & \mu_1 + \mu_2 + \mu_3 + \mu_4 = 1 \end{aligned} \tag{4.104}$$

Since that the parameters ρ_σ and ρ_μ have the following relation:

$$\rho_\sigma = \sqrt{\frac{\sigma_1\sigma_2}{\sigma_3\sigma_4}} = \sqrt{\frac{\mu_1\mu_2}{\mu_3\mu_4}}\rho_\lambda \tag{4.105}$$

Substitute Eq. (4.105) into the objective function of Problem (4.104), we have

$$4\sqrt{2}\sqrt{\rho_\sigma}\sqrt{\sigma_3\sigma_4} = 4\sqrt{2}\sqrt[4]{\mu_1\mu_2\mu_3\mu_4}\sqrt{\rho_\lambda}\sqrt{\lambda_3\lambda_4} \tag{4.106}$$

The inequality of arithmetic and geometric means says that:

$$\sqrt[4]{\mu_1\mu_2\mu_3\mu_4} \leq \frac{\mu_1 + \mu_2 + \mu_3 + \mu_4}{4} \tag{4.107}$$

with equality if and only if $\mu_1 = \mu_2 = \mu_3 = \mu_4$. Note that the third constraint of Problem (4.104) requires the sum of the four variables to be a constant. Therefore,

$$\sqrt[4]{\mu_1\mu_2\mu_3\mu_4} \leq \frac{1}{4} \tag{4.108}$$

Thus, the optimal values of μ_1, μ_2, μ_3, and μ_4 that maximize the $d_{\min}$ must satisfy that $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1/4$.

Substituting the optimal realtion $\mu_1 = \mu_2 = \mu_3 = \mu_4$ into Eq. (4.105) yields

$$\rho_\sigma = \rho_\lambda \tag{4.109}$$

Therefore the first constraint of Problem (4.104) suggests that ρ_λ should fall in the following region:

$$(\sqrt{2} + 1)^2 \leq \rho_\lambda \leq 15 + 4\sqrt{14} \tag{4.110}$$

Thus, we conclude our result in the following situations according to ρ_λ:

Situation A. When $(\sqrt{2} + 1)^2 \leq \rho_\lambda \leq 15 + 4\sqrt{14}$, the optimal $\{\mu_i\}, i = 1, 2, 3, 4$

and corresponding maximal minimum distance is:

$$\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1/4 \tag{4.111}$$

$$\max d_{\min}^2 = \sqrt{2\rho_\lambda}\sqrt{\lambda_3\lambda_4} \tag{4.112}$$

Situation B. When $\rho_\lambda \leq (\sqrt(2)+1)^2$ or $\rho_\lambda > 15+4\sqrt{14}$, the problem is infeasible.

4.5.3 Case 3

In the last case, $\rho_\sigma \geq 15 + 4\sqrt{14}$, $d_{\min}^2 = 2\left[\rho_\sigma + 1 - \frac{1}{\sqrt{3}}\sqrt{\rho_\sigma^2 - 6\rho_\sigma + 1}\right]\sqrt{\sigma_3\sigma_4}$. Under this condition, Problem 7 has the following form:

$$\begin{aligned}
\max_{\mu_1,\mu_2,\mu_3,\mu_4} \quad & 2\left[\rho_\sigma + 1 - \tfrac{1}{\sqrt{3}}\sqrt{\rho_\sigma^2 - 6\rho_\sigma + 1}\right]\sqrt{\sigma_3\sigma_4} \\
\text{s.t.} \quad & \rho_\sigma \geq 15 + 4\sqrt{14} \\
& \sigma_i = \lambda_i\mu_i, i = 1,2,3,4 \\
& \mu_1 + \mu_2 + \mu_3 + \mu_4 = 1
\end{aligned} \tag{4.113}$$

Substituting $\rho_\sigma = \sqrt{\frac{\mu_1\mu_2}{\mu_3\mu_4}}\rho_\lambda$ into the objective function of Problem (4.113),

$$2\left[\rho_\sigma + 1 - \frac{1}{\sqrt{3}}\sqrt{\rho_\sigma^2 - 6\rho_\sigma + 1}\right]\sqrt{\sigma_3\sigma_4} \tag{4.114}$$

$$= 2\left[\sqrt{\mu_1\mu_2}\rho_\lambda + \sqrt{\mu_3\mu_4} - \frac{1}{\sqrt{3}}\sqrt{\mu_1\mu_2\rho_\lambda^2 - 6\sqrt{\mu_1\mu_2\mu_3\mu_4}\rho_\lambda + \mu_3\mu_4}\right]\sqrt{\lambda\lambda_4} \tag{4.115}$$

Similar to Case 1, the optimal μ_1, μ_2, μ_3 and μ_4 should satisfy that $\mu_1 = \mu_2$ and $\mu_3 = \mu_4$. The $d^2_{\min}$ can then be rewritten as

$$d^2_{\min} = \sqrt{\lambda_3\lambda_4}2(\mu_1\rho_\lambda + \mu_3 - \frac{1}{\sqrt{3}}\sqrt{\mu_1^2\rho_\lambda^2 - 6\mu_1\mu_3\rho_\lambda + \mu_3^2}) \tag{4.116}$$

which can be treated as a function in terms of μ_1. By further simplify the Eq. (4.116) according to μ_1, we write the $d^2_{\min}$ as the following function:

$$f(\mu_1) = (\rho_\lambda - 1)\mu_1 - \frac{1}{\sqrt{3}}\sqrt{(\rho_\lambda^2 + 6\rho_\lambda + 1)\mu_1^2 - (3\rho_\lambda + 1)\mu_1 + \frac{1}{4}} \tag{4.117}$$

We calculate the second-order derivative of the objective function, which is given by:

$$f''(\mu_1) = \frac{2\rho_\lambda^2}{9\left[(\rho_\lambda^2 + 6\rho_\lambda + 1)\mu_1^2 - (3\rho_\lambda + 1)\mu_1 + \frac{1}{4}\right]^{3/2}} \tag{4.118}$$

Since $\rho_\lambda \geq 1$, therefore $f''(\mu_1) > 0$. So the objective function is a convex function regarding to μ_1. The maximum of a convex function would be one of the values on the boundaries of feasible domain.

Situation A. When $\rho_\lambda \leq 15 + 4\sqrt{14}$, the problem is reformulated as

$$\begin{aligned} \max_{\mu_1} \quad & (\rho_\lambda - 1)\mu_1 - \tfrac{1}{\sqrt{3}}\sqrt{(\rho_\lambda^2 + 6\rho_\lambda + 1)\mu_1^2 - (3\rho_\lambda + 1)\mu_1 + \tfrac{1}{4}} \\ \text{s.t.} \quad & \mu_1 \geq \tfrac{(15+4\sqrt{14})}{2\left[\rho_\lambda + (15+4\sqrt{14})\right]} \\ & \mu_1 \leq 1/2 \end{aligned} \tag{4.119}$$

the feasible domain of the problem is $\frac{(15+4\sqrt{14})}{2\left[\rho_\lambda + (15+4\sqrt{14})\right]} \leq \mu_1 \leq 1/2$. The values of the objective function at two boundary points are given by:

$$d_{\min}^2\big|_{\mu_1=\frac{(15+4\sqrt{14})}{2[\rho_\lambda+(15+4\sqrt{14})]}} = \frac{2\sqrt{2}(4\sqrt{2}+2\sqrt{7}-\sqrt{15+4\sqrt{14}})}{\rho_\lambda+(15+4\sqrt{14})}\rho_\lambda\sqrt{\lambda_3\lambda_4} \tag{4.120}$$

$$d_{\min}^2|_{\mu_1=1/2} = (1-\frac{1}{\sqrt{3}})\sqrt{\lambda_3\lambda_4}\rho_\lambda \tag{4.121}$$

Compare the two boundary values, if $\rho_\lambda \leq \frac{3+15\sqrt{3}+4\sqrt{42}-6\sqrt{30+8\sqrt{14}}}{3-\sqrt{3}}(\approx 6.6673)$,

$$\mu_1 = \mu_2 = \frac{(15+4\sqrt{14})}{2\left[\rho_\lambda+(15+4\sqrt{14})\right]} \tag{4.122}$$

$$\mu_3 = \mu_4 = \frac{\rho_\lambda}{2\left[\rho_\lambda+(15+4\sqrt{14})\right]} \tag{4.123}$$

$$\max d_{\min}^2 = \frac{2\sqrt{2}(4\sqrt{2}+2\sqrt{7}-\sqrt{15+4\sqrt{14}})}{\rho_\lambda+(15+4\sqrt{14})}\rho_\lambda\sqrt{\lambda_3\lambda_4} \tag{4.124}$$

otherwise, if $\frac{3+15\sqrt{3}+4\sqrt{42}-6\sqrt{30+8\sqrt{14}}}{3-\sqrt{3}} \leq \rho_\lambda \leq 15+4\sqrt{14}$,

$$\mu_1 = \mu_2 = 1/2 \tag{4.125}$$

$$\mu_3 = \mu_4 = 0 \tag{4.126}$$

$$\max d_{\min}^2 = (1-\frac{1}{\sqrt{3}})\rho_\lambda\sqrt{\lambda_3\lambda_4} \tag{4.127}$$

Situation B. When $\rho_\lambda > 15+4\sqrt{14}$, the feasible domain of the problem is $1/4 \leq \mu_1 \leq 1/2$. The problem is reformulated as

$$\begin{array}{ll} \max\limits_{\mu_1} & (\rho_\lambda-1)\mu_1 - \frac{1}{\sqrt{3}}\sqrt{(\rho_\lambda^2+6\rho_\lambda+1)\mu_1^2-(3\rho_\lambda+1)\mu_1+\frac{1}{4}} \\ \text{s.t.} & 1/4 \leq \mu_1 \leq 1/2 \end{array} \tag{4.128}$$

The values of the objective function at two boundary points are given by:

$$d^2_{\min}|_{\mu_1=1/4} = \frac{1}{2}\left(\rho_\lambda + 1 - \sqrt{\frac{\rho_\lambda^2 - 6\rho_\lambda + 1}{3}}\right)\sqrt{\lambda_3\lambda_4} \tag{4.129}$$

$$d^2_{\min}|_{\mu_1=1/2} = (1 - \frac{1}{\sqrt{3}})\rho_\lambda\sqrt{\lambda_3\lambda_4} \tag{4.130}$$

For any possible $\rho_\lambda > 15 + 4\sqrt{14}$, it holds that

$$\mu_1 = \mu_2 = 1/2 \tag{4.131}$$

$$\mu_3 = \mu_4 = 0 \tag{4.132}$$

$$\max d^2_{\min} = (1 - \frac{1}{\sqrt{3}})\rho_\lambda\sqrt{\lambda_3\lambda_4} \tag{4.133}$$

4.5.4 Conclusion: Global Optimal $\mathbf{\Lambda_F}$

The previous sections discussed the optimal solution to the $\mathbf{\Lambda_F}$, which actually controls the power loading of the precoder. Similarly to what happens for the solutions to the first sub-problem, it turns out that the optimal solutions under different cases have some overlaps. We have also simplified the expressions of $\max d^2_{\min}$ in a united form composing a function of ρ_λ and a common factor $\lambda_3\lambda_4$. For the convenience of comparison purpose, we do not need to consider the common factor. Thus, we conclude all the expressions of $\max d^2_{\min}$ as functions in terms of ρ_λ in Table 4.2. The relationship between these functions are shown in Fig. 4.3.

Table 4.2: Conclusion of all expressions of $\max d_{\min}^2$ in the Second Sub-problem

	Condition 1	Condition 2	Condition 3
$\rho_\sigma \leq (\sqrt{2}+1)^2$	$f_1(\rho_\lambda)$[1]	-	$f_3(\rho_\lambda)$[3]
$(\sqrt{2}+1)^2 \leq \rho_\lambda \leq 6.67$	-	$f_2(\rho_\lambda)$[2]	$f_3(\rho_\lambda)$
$6.67 \leq \rho_\lambda \leq 15+4\sqrt{14}$	-	$f_2(\rho_\lambda)$	$f_4(\rho_\lambda)$[4]
$\rho_\lambda \geq 15+4\sqrt{14}$	-	-	$f_4(\rho_\lambda)$

[1] $f_1(\rho_\lambda) = \frac{(4+2\sqrt{2})\rho_\lambda}{\rho_\lambda+(\sqrt{2}+1)^2}$
[2] $f_2(\rho_\lambda) = \sqrt{2\rho_\lambda}$
[3] $f_3(\rho_\lambda) = \frac{2\sqrt{2}(4\sqrt{2}+2\sqrt{7}-\sqrt{15+4\sqrt{14}})}{\rho_\lambda+(15+4\sqrt{14})}\rho_\lambda$
[4] $f_4(\rho_\lambda) = (1-\frac{1}{\sqrt{3}})\rho_\lambda$

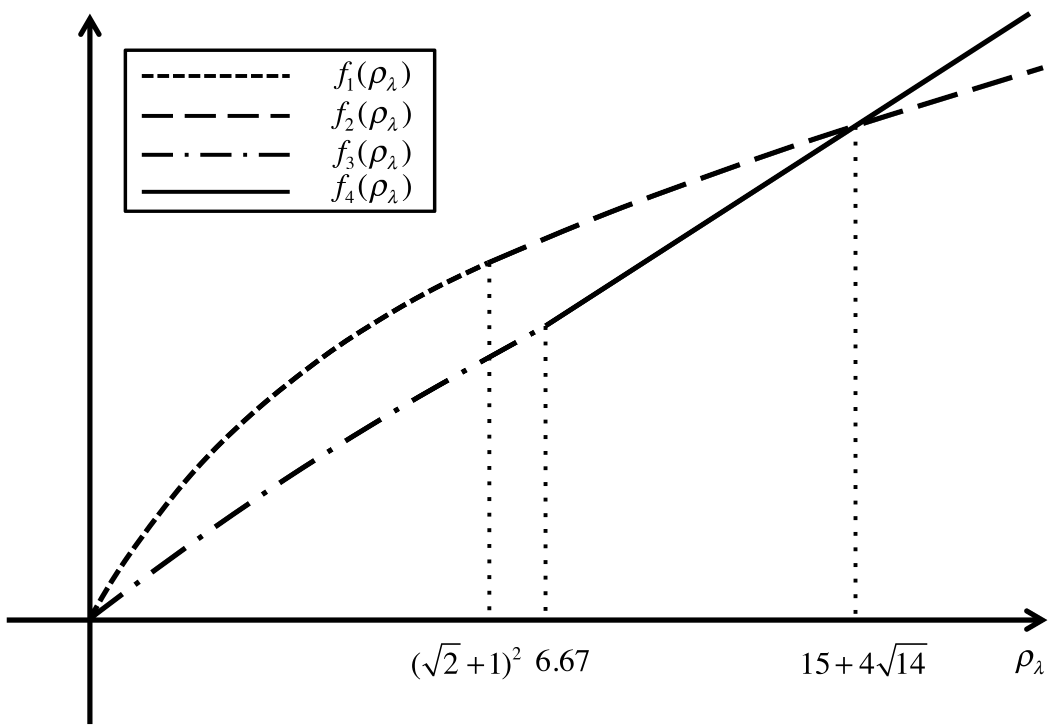

Figure 4.3: Comparsion of Multiple Expressions of $\max d_{\min}^2$ in the Second Sub-problem

Then, we summarize the results in choice of $\max d^2_{\min}$ as follows:

$$d^2_{\min} = \begin{cases} \dfrac{(4+2\sqrt{2})\rho_\lambda}{\rho_\lambda + (\sqrt{2}+1)^2}\sqrt{\lambda_3\lambda_4}, & \rho_\lambda \leq (\sqrt{2}+1)^2 \\ \sqrt{2\rho_\lambda}\sqrt{\lambda_3\lambda_4}, & (\sqrt{(2)}+1)^2 \leq \rho_\lambda \leq 2(2+\sqrt{3}) \\ (1-\dfrac{1}{\sqrt{3}})\rho_\lambda\sqrt{\lambda_3\lambda_4}, & \rho_\lambda \geq 2(2+\sqrt{3}) \end{cases} \tag{4.134}$$

Corresponding to the values of global $\max d^2_{\min}$ are the values of optimal μ_1, μ_2, μ_3 and μ_4 which are shown as follows:

$$\begin{cases} \mu_1 = \mu_2 = \dfrac{(\sqrt{2}+1)^2}{2\left[\rho_\lambda + (\sqrt{2}+1)^2\right]}, & \\ \mu_3 = \mu_4 = \dfrac{\rho_\lambda}{2\left[\rho_\lambda + (\sqrt{2}+1)^2\right]}, & \rho_\lambda \leq (\sqrt{2}+1)^2 \\ \mu_1 = \mu_2 = \mu_3 = \mu_4 = 1/4, & (\sqrt{2}+1)^2 \leq \rho_\lambda \leq 3(2+\sqrt{3}) \\ \mu_1 = \mu_2 = 1/2, \mu_3 = \mu_4 = 0 & \rho_\lambda \geq 3(2+\sqrt{3}) \end{cases} \tag{4.135}$$

The optimal values of r_1 and r_2 obtained in Eq. (4.92) are the choices for the $\mathbf{R}$ matrix in block-equal QR decomposition at the receiver. The optimal values of μ_1, μ_2, μ_3, and μ_4 obtained in Eq. (4.135), on the other hand, are the choices of singular values for the precoder matrix $\mathbf{F}$ at the transmitter. In Eq. (4.135), the optimal μ_1, μ_2, μ_3, and μ_4 have different optimal values under different channel conditions which is defined according to the value of ρ_λ as follows:

- Good Channels: $\rho_\lambda \leq (\sqrt{2}+1)^2$
- Moderate Channels: $(\sqrt{2}+1)^2 \leq \rho_\lambda \leq 3(2+\sqrt{3})$
- Bad Channels: $\rho_\lambda \geq 3(2+\sqrt{3})$

4.6 Optimal Solution to the Precoder Matrix F

In Section 3.3, we have shown that the block-equal QRS decomposition is possible if a solution exists for the set of 7 equations of (3.32). This set of quadratic equations consists of eight unknowns $\{s_{ij}\}, i = 1, 2$ and $j = 1, 2, 3, 4$ together with the coefficients σ_1, σ_2, σ_3, σ_4, r_1, and r_2. The above two optimization sub-problems of Problem 5 together yield the optimal values of those coefficients which can now be employed to solve the set of quadratic equations. After solving these equations, we may formulate the block-equal QRS decomposition described in Section 3.3. Problem 5, however, is a relaxed problem, which cannot guarantee the existence of block-equal QRS decomposition. It has a larger feasible set than that of the original Problems 2, 3, and 4. We can conclude that these optimal coefficients are within the feasible set of the original Problems 2, 3, and 4 if they lead to a solution of the set of 7 quadratic equations. Therefore, in this section, we examine the solution of the set of 7 equations using those optimal coefficients obtained in the previous two sections.

4.6.1 Good Channel Case

In Eq. (4.135), the first case says that, when $\rho_\lambda \leq (\sqrt{2}+1)^2$, the parameter ρ_σ is

$$\rho_\sigma = \sqrt{\frac{\sigma_1\sigma_2}{\sigma_3\sigma_4}} = \rho_\lambda\sqrt{\frac{\mu_1\mu_2}{\mu_3\mu_4}} = (\sqrt{2}+1)^2 \tag{4.136}$$

Combine this result with Eq. (4.92), saying that the optimal r_1 and r_2 satisfy that

$$r_1 = \sqrt{\sigma_2\sigma_2} \quad r_2 = \sqrt{\sigma_3\sigma_4} \quad \text{when } \rho_\sigma \leq (\sqrt{2}+1)^2 \tag{4.137}$$

It can be found that as long as the singular values of the channel satisfy the condition that $\rho_\lambda \leq (\sqrt{2}+1)^2$, the optimal r_1 and r_2 are $\sqrt{\sigma_1\sigma_2}$ and $\sqrt{\sigma_3\sigma_4}$. Theorem 2 says, when r_1 and r_2 has such relation with the singular values of the cascaded channel matrix $\tilde{\mathbf{H}}$ that $r_1 = \sqrt{\sigma_1\sigma_2}$ and $r_2 = \sqrt{\sigma_3\sigma_4}$, a specific closed form solution of $\mathbf{S}$ is given. This result shows that our optimal solution to Problem 5 perfectly matches the condition under which the cascaded channel has the proposed closed-from block-equal QRS decomposition. Therefore our specific solution to the matrix $\mathbf{S}$ shown in Eq. (3.38) can be employed to compose the optimal precoder matrix according to Eq. (4.9).

4.6.2 Moderate Channel Case

In the second case of Eq. (4.135), things are not as ideal as the first case. When $(\sqrt{2}+1)^2 \leq \rho_\lambda \leq 3(2+\sqrt{3})$, $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1/4$, which indicates that the optimal precoder uniformly allocates power over all the data streams. In such a case, the parameter ρ_σ is given by

$$\rho_\sigma = \sqrt{\frac{\mu_1\mu_2}{\mu_3\mu_4}}\rho_\lambda = \rho_\lambda \tag{4.138}$$

Therefore the range of ρ_σ is also $[(\sqrt{2}+1)^2, 3(2+\sqrt{3})]$. According to Eq. (4.92), under the case that ρ_σ falls in that range, the corresponding optimal r_1 and r_2 are

$$r_1 = (\sqrt{2}+1)\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \quad r_2 = (\sqrt{2}-1)\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \tag{4.139}$$
$$\text{when } (\sqrt{2}+1)^2 \leq \rho_\lambda \leq 3(2+\sqrt{3})$$

In this case, those optimal values of r_1 and r_2 do not satisfy the condition such that we can have the closed-form $\mathbf{S}$ in Theorem 2. However, it does not mean that such a matrix $\mathbf{S}$ based on this optimal solution does not exist. According to Theorem 1, the existence of a solution to the 7 equations is equivalent to the existence of a block-equal QRS decomposition of the cascaded channel. Thus, as long as we can get a set of solution to the 7 equations, that set of solution gives us one specific realization of $\mathbf{S}$. To find a solution to the equations, we employ a numerical optimization method by using computational tool MATLAB Optimization Toolbox. In such a case, an optimal $\mathbf{S}$ need to be found through numerical method. Thus, we consider the following optimization problem:

Problem 8. Optimization Problem for Numerical Result of S (Least Square Solution)

$$
\begin{aligned}
\min_{s_ij, i=1,2,3,4 j=1,2} \quad & |s_{11}^2 + s_{21}^2 + s_{31}^2 + s_{41}^2 - 1|^2 + |s_{12}^2 + s_{22}^2 + s_{32}^2 + s_{42}^2 - 1|^2 + \\
& |s_{11}s_{12} + s_{21}s_{22} + s_{31}s_{32} + s_{41}s_{42}|^2 + \\
& |\sigma_1 s_{11}^2 + \sigma_2 s_{21}^2 + \sigma_3 s_{31}^2 + \sigma_4 s_{41}^2 - r_1|^2 + \\
& |\sigma_1 s_{12}^2 + \sigma_2 s_{22}^2 + \sigma_3 s_{32}^2 + \sigma_4 s_{42}^2 - r_2|^2 + \\
& |\sigma_1 s_{11}s_{12} + \sigma_2 s_{21}s_{22} + \sigma_3 s_{31}s_{32} + \sigma_4 s_{41}s_{42}|^2 + \\
& |\tfrac{\sigma_1^2 s_{11}^2 + \sigma_2^2 s_{21}^2 + \sigma_3^2 s_{31}^2 + \sigma_4^2 s_{41}^2}{r_1} + \tfrac{\sigma_1^2 s_{12}^2 + \sigma_2^2 s_{22}^2 + \sigma_3^2 s_{32}^2 + \sigma_4^2 s_{42}^2}{r_2} \\
& \quad - (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 - r_1 - r_2)|^2 \\
\text{s.t.} \quad & r_1 = (\sqrt{2} + 1)\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\
& r_2 = (\sqrt{2} - 1)\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\
& \sigma_i = \lambda_i \mu_i, i = 1, 2, 3, 4 \\
& \mu_1 = \mu_2 = \mu_3 = \mu_4 = 1/4
\end{aligned}
\tag{4.140}
$$

From Eq. (3.32), we take the sum of the squares of all equations, and minimize it. We generated 200 random channels and each time evaluated the optimum parameters σ_1, σ_2, σ_3, σ_4, r_1, and r_2. Using these parameters in Problem 8, we minimized the sum of squares. We noted that, for each set of randomly generated channel, a solution for Problem 8 is arrived at. Thus, we can conclude that there exists a solution for the block-equal QRS decomposition. It can be shown in the simulation experiments that for each of the randomly generated channels, a local optimum can be found quickly and it can minimize the value of the objective function as small as a given tolerance (e.g., 1×10^{-12}). Therefore, for the second case, it can be concluded that based on the optimal solution $\mathbf{\Lambda_F}$ and $\mathbf{U_R}$ obtained from Problem 5, there exists numerical solution to the matrix $\mathbf{S}$ such that the resulting cascaded matrix $\tilde{\mathbf{H}}$ has block-equal QRS decomposition.

4.6.3 Bad Channel Case

The last case in Eq. (4.135) is that, when $\rho_\lambda \geq 3(2+\sqrt{3})$, the optimal power loading factor are $\mu_1 = \mu_2 = 1/2$ and $\mu_3 = \mu_4 = 0$. We may not use the parameter ρ_σ to indicate the region within the space expanded by $\{\sigma_i\}, i = 1, 2, 3, 4$. Instead, we consider the values of $\{\sigma_i\}$ directly. In this case, the optimal values of $\{\sigma_i\}$ are:

$$\sigma_1 = \frac{1}{2}\lambda_1 \quad \sigma_2 = \frac{1}{2}\lambda_2 \quad \sigma_3 = 0 \quad \sigma_4 = 0 \tag{4.141}$$

Therefore, the following inequality holds for any arbitrarily large positive number k:

$$\sigma_1\sigma_2 \geq k\sigma_3\sigma_4 = 0 \tag{4.142}$$

Let $k = (15 + 4\sqrt{14})^2$. The relationship between the four variables also satisfy that $\sigma_1\sigma_2 \geq (15 + 4\sqrt{14})^2\sigma_3\sigma_4$. Thus, according to the result shown in Eq. (4.92), the optimal r_1 and r_2 is

$$r_1 = \sqrt{\sigma_2\sigma_2} \quad r_2 = \sqrt{\sigma_3\sigma_4} \quad \text{when } \rho_\lambda \geq 3(2 + \sqrt{3}) \tag{4.143}$$

This result shows that the optimal choices of r_1 and r_2 are the same as those in the Good Channel Case. Therefore, the specific realization of matrix $\mathbf{S}$ given in in Eq. (3.38) is also suitable for constructing the optimal precoder matrix in the Bad Channel Case. Substituting the specific realization solution of $\mathbf{S}$, the optimal $\mathbf{\Lambda_F}$ and $\mathbf{U_R}$ into Eq. (4.9) then leads to the optimal $\mathbf{F}$.

4.6.4 Conclusion

In conclusion, a solution to the set of 7 quadratic equations always exists in good and bad channel conditions. In moderate channel condition, the simulation results show that numerical solutions usually exist. This indicates that the optimal solution to Problem 5 can always yield a precoder that guarantees the block-equal QRS decomposition. That is, although the relaxed Problem 5 enlarges the feasible set, its optimal solution of **F** actually locates in the original feasible set of Problems 2, 3, and 4. So, the optimal solution to Problem 5 is also the optimal solution to Problems 2, 3, and 4.

Chapter 5

Simulation and Discussion

In this chapter, we present a set of computer simulations on the proposed block-equal QR decomposition based optimal transmission and receiving scheme.

5.1 Simulation System Model

We only consider the 4×4 special case which has been theoretically discussed in the previous chapters: There are 4 transmitting antennas and 4 receiving antennas. Each transmitting antenna sends a data stream of intended symbols randomly picked from a 4-QAM constellation. At each time slot, the transmitted symbol vector will be precoded before being sent through a randomly generated channel. From the discussion about the existence of optimal precoder in Chapter 4, a closed form solution of the optimal precoder for our proposed receiving scheme has been obtained when the channel satisfies $\rho_\lambda \leq (\sqrt{2}+1)^2$. However, it is still an open question on finding the closed form solution while $(\sqrt{2}+1)^2 \leq \rho_\lambda \leq 3(2+\sqrt{3})$. In this situation without available closed form solution, we use numerical solution instead. At the output of the channel, the received signal is a composition of channel output signal and an additive white Gaussian noise. We assume that the noise is white, i.e., $\mathbf{\Phi_{nn}} = 2\tau^2\mathbf{I}$. The combined ML-BEQRS (Block-Equal QRS Decomposition) detector is applied at the receiver end to estimate the transmitted symbol vector. In the simulation, we change the level of SNR in a range from -5dB to 20dB and verify the error performance for each SNR level.

Depending on the value of channel condition parameter ρ_λ, the optimal precoder matrix is achieved by either closed-form solution or numerical solution. In Section 4.6, we have discussed the solutions of the optimal precoder under different cases, which are good channel case, moderate channel case and bad channel case. For the good channel case and bad channel case, we use the closed-form solution to construct the precoder. For the moderate channel case, we use the numerical solution instead. We evaluate our proposed method in three cases saparately .

5.2 Other Frameworks for Comparison

For the purpose of examine the error performance of our proposed design, we compare our precoder and detector with the following transmission and receiving strategies:

1. **Zero-Forcing (ZF) Equalization**

   The procoder for ZF Equalization is [8]

$$\mathbf{F}_{\mathrm{ZF}} = \sqrt{\frac{1}{M\mathrm{tr}(\mathbf{\Theta}^{-1/2})}}\mathbf{V}\mathbf{\Theta}^{-1/4}\mathbf{D}$$

where $\mathbf{\Theta}$ and $\mathbf{V}$ comes from the eigenvalue eigenvalue decomposition $\mathbf{H}^H\mathbf{\Phi_{nn}}^{-1}\mathbf{H} = \mathbf{V\Theta V}^H$ and $\mathbf{D}$ is a normalized discrete Fourier Transform (DFT) matrix. In 4×4 case $M = 4$.

2. **Minimum Mean Squared Error (MMSE) Equalization**

   The precoder for MMSE Equalization is [20]

$$\mathbf{F}_{\text{MMSE}} = \sqrt{\frac{1}{M}}\mathbf{V\Delta D}$$

   where $\mathbf{V}$, $\mathbf{D}$ and M has the same meaning as ZF equalization, and $\mathbf{\Delta}$ is a diagonal matrix with the diagonal elements determined by

$$|\delta_{ii}|^2 = \begin{cases} \frac{1}{r}\left(1 + \sum_{j=1}^{r} \theta_j^{-1}\right) - \theta_i^{-1}, & i \leq r \\ 0, & otherwise \end{cases} \tag{5.1}$$

   where θ_i are diagonal elements of $\mathbf{\Theta}$, coming from the eigenvalue decomposition $\mathbf{H}^H\mathbf{\Phi_{nn}}^{-1}\mathbf{H} = \mathbf{V\Theta V}^H$, and $r \leq 4$ is the largest integer satisfying

$$\frac{1}{\theta_i} < \frac{1}{r}\left(1 + \sum_{j=1}^{r} \theta_j^{-1}\right)$$

3. **Zero-Forcing Decision Feedback Equalization (ZF-DFE)**

   The structure of such decision feedback transmission and detection framework is depicted in Figure 5.1.

   The transceiver with ZF-DFE can be jointly designed using the following procedure [45]: (1) The precoder $\mathbf{F}_{\text{ZF-DFE}} = \sqrt{1/M}\mathbf{VK}$, where $\mathbf{V}$ is given by the eigenvalue decomposition of $\mathbf{H}^H\mathbf{\Phi_{nn}}^{-1}\mathbf{H} = \mathbf{V\Theta V}^H$, $\mathbf{K}$ is chosed such that

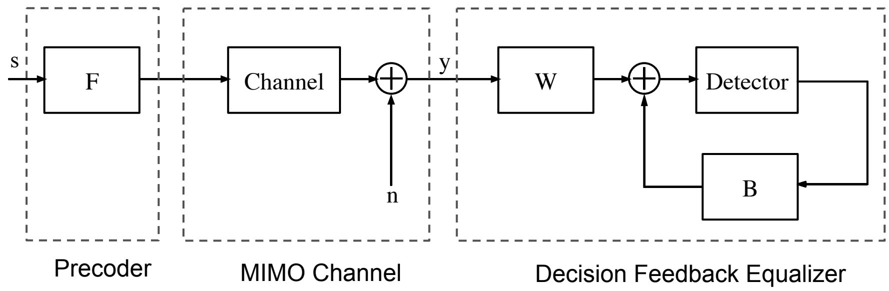

Figure 5.1: Model of Decision Feedback Transmission and Detection

the mean squared error achieves its lower bound. (2) The feedback matrix $\mathbf{B}_{\text{ZF-DFE}} = \mathbf{U}_{\text{ZF-DFE}} - \mathbf{I}$, where $\mathbf{U}_{\text{ZF-DFE}}$ is a unit diagonal upper triangular matrix given by the Cholesky factorization $(\mathbf{HF}_{\text{ZF-DFE}})^H \mathbf{\Phi_{nn}}^{-1}(\mathbf{HF}_{\text{ZF-DFE}}) = \mathbf{U}^H_{\text{ZF-DFE}}\mathbf{C}_{\text{ZF-DFE}}\mathbf{U}_{\text{ZF-DFE}}$. (3) The feedforward matrix $\mathbf{W}_{\text{ZF-DFE}} = (\mathbf{B}_{\text{ZF-DFE}} + \mathbf{I})(\mathbf{HF}_{\text{ZF-DFE}})^{\dagger}$.

4. **Minimum Mean Squared Error Decision Feedback (MMSE-DFE) Equalization**

   The transceiver with MMSE-DFE can be jointly designed using the following procedure [45]: (1) The precoder $\mathbf{F}_{\text{MMSE-DFE}} = \mathbf{V}\mathbf{\Delta}\mathbf{K}^H$, where the definition of $\mathbf{V}$ is same as ZF-DFE and $\mathbf{\Delta}$ is a diagonal matrix with diagonal elements defined in equation (5.1). The unitary matrix $\mathbf{K}$ is chosed such that the mean squared error achieves its minimized lower bound. (2) The feedback matrix $\mathbf{B}_{\text{MMSE-DFE}} = \mathbf{U}_{\text{MMSE-DFE}} - \mathbf{I}$, where $\mathbf{U}_{\text{MMSE-DFE}}$ is a unit diagonal upper triangular matrix given by the Cholesky factorization $\mathbf{I} + (\mathbf{HF}_{\text{ZF-DFE}})^H \mathbf{\Phi_{nn}}^{-1}(\mathbf{HF}_{\text{ZF-DFE}}) = \mathbf{U}^H_{\text{MMSE-DFE}}\mathbf{C}_{\text{MMSE-DFE}}\mathbf{U}_{\text{MMSE-DFE}}$. (3) The feedforward matrix $\mathbf{W}_{\text{MMSE-DFE}} = (\mathbf{B}_{\text{MMSE-DFE}} + \mathbf{I})(\mathbf{HF}_{\text{MMSE-DFE}})^H[(\mathbf{HF}_{\text{MMSE-DFE}})(\mathbf{HF}_{\text{MMSE-DFE}})^H + \mathbf{\Phi_{nn}}]^{-1}$.

5. **Block QR Decomposition Detector without Proposed Optimal Precoder**

   The precoder matrix in this case is set to be $\mathbf{F} = \mathbf{I}$. Then use the block-by-block successive cancellation strategy.

6. **Maximum Likelihood Detector with Proposed Optimal Precoder**

   Employ our proposed optimal precoder matrix, then use ML detector, jointly detecting all 4 symbols.

7. **Maximum Likelihood Detector without Proposed Optimal Precoder**

   The precoder matrix in this case is set to be $\mathbf{F} = \mathbf{I}$. Then use ML detector to jointly detect all 4 symbols.

In all the simulation two set of 200 channels that satisfy the corresponding channel condition are randomly generated. In each scenario, our proposed method and other 7 methods that have been mentioned for comparison are tested over the fixed 200 channels. Their average bit error rates (BER) over all 200 fixed channel realizations are evaluated and compared with each other. As for each specific channel, the transmitter continuously send 10^6 symbol vectors, each symbol of which is independent and randomly picked from the 4-QAM constellation symbol set. The receiver detect those symbols by using the 8 methods separately.

5.3 Simulation Results

We illustrate the performance of our proposed block-equal QRS decomposition and the other 7 methods mentioned with a computer simulation that is shown in Fig. 5.2, Fig. 5.3 and Fig. 5.4. In the first plot Fig. 5.2, all the 8 methods are compared

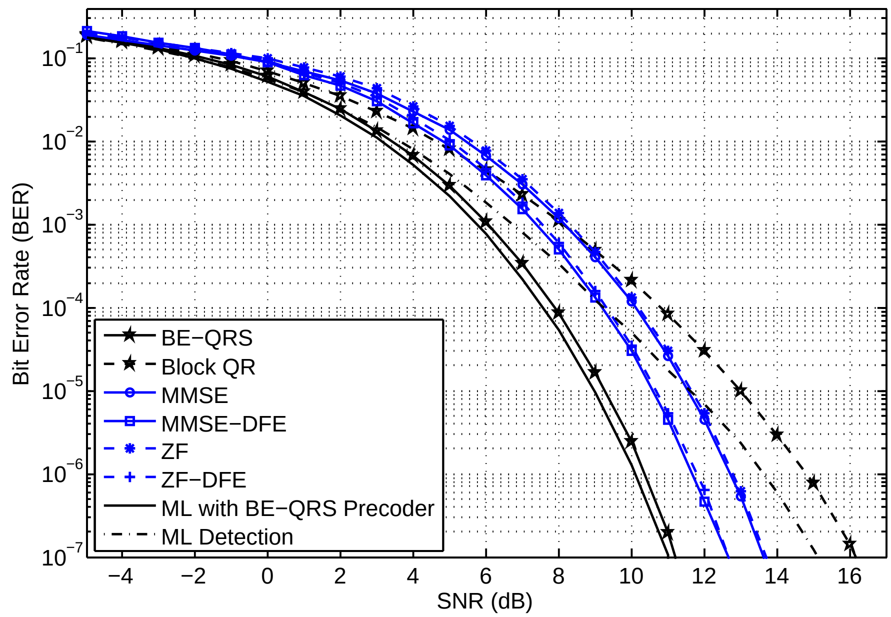

Figure 5.2: Error Performance Comparison in Good Channel Case

in the good channel case, in which our proposed method (the solid line with five-pointed star named BE-QRS) can get a closed-form precoder matrix **F**. The graph shows that when applying our proposed optimal precoder matrix, the ML detector that jointly detects all the 4 symbols together gets the optimal error performance (the solid line). Our proposed detection strategy achieves a performance curve that is very close to the optimal performance line. To be precise, the difference of SNR in dB when the bit error rate (BER) at the level of 10^{-7} is no greater than 0.2dB. Meanwhile, the performance of our proposed precoding and receiving scheme is far beyond the linear receivers (ZF and MMSE) and its decision feedback counterparts (ZF-DFE and MMSE-DFE). Among these four strategies, the MMSE-DFE (the solid line with square) possesses a better error performance curve. Specifically our methods outperforms the linear receivers (ZF and MMSE) by 4dB and outperforms the decision feedback receivers (ZF-DFE and MMSE-DFE) by 1.5dB at the BER level of 10^{-7}.

On the other hand, the performance of ML detector that jointly detects 4 symbols but has no optimal precoder (the dash-dot line) is not very good, and neither is the case of using block successive cancellation detection strategy without optimal precoding (the dashed line with five-pointed star). These two lines indicate that the implementation of a good precoder is a significant contribution to the improvement of error performance when the channel state is good.

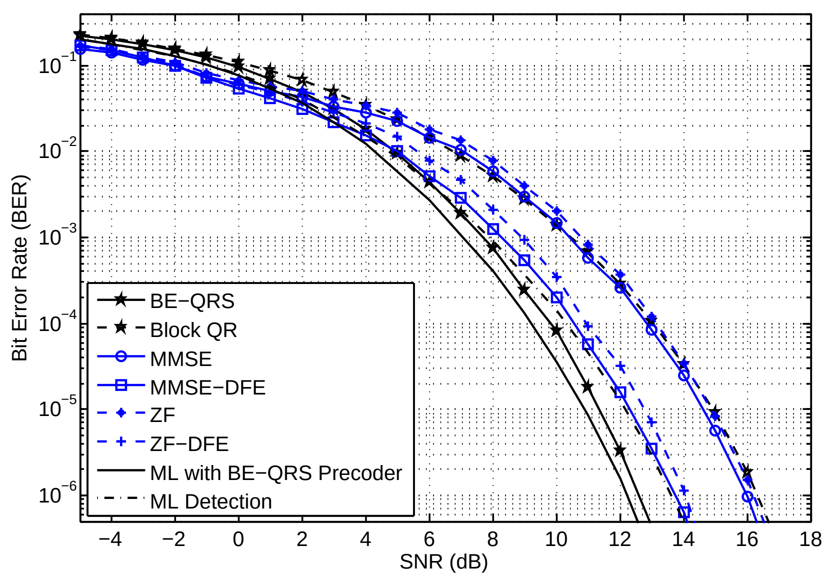

Figure 5.3: Error Performance Comparison in Moderate Channel Case

The second plot in Fig. 5.3 compare the same methods under the Scenario 2, where $(\sqrt{2}+1)^2 \leq \rho_\lambda \leq 3(2+\sqrt{3})$. Under this case, the channel condition is worse than the previous case and the closed-form solution is undetermined. However we can use numerical method to obtain the solution. The experiments shows that, by using the MATLAB Optimization Toolbox, we can easily solve the Problem 8 with a local optimal solution to a randomly given channel. Although the solution is just local optimum, every time it can minimize the objective function to a level less than 10^{-12}.

The performance comparison in Fig. 5.3 shows that our proposed method, the BE-QRS in the graph, still yields a good performance very close to the optimal error performance of jointly ML detector with the proposed optimal precoder for BE-QRS. Compared to the other linear and non-linear methods at the BER level of 10^{-7}, our method is superior to the method of MMSE-DFE, ZF-DFE, MMSE, ZF by 1.2dB, 1.4dB, 3.35dB and 3.6dB respectively. This result is very similar to the good channel case. However, there is also a difference between the results of two cases. In the moderate channel case, the jointly ML detector without optimal precoder achieve a relatively better performance compared to the previous case. The difference is caused by the change of channel condition. When channel condition gets worse, the use of jointly ML detection plays a more critical role than the use of precoder.

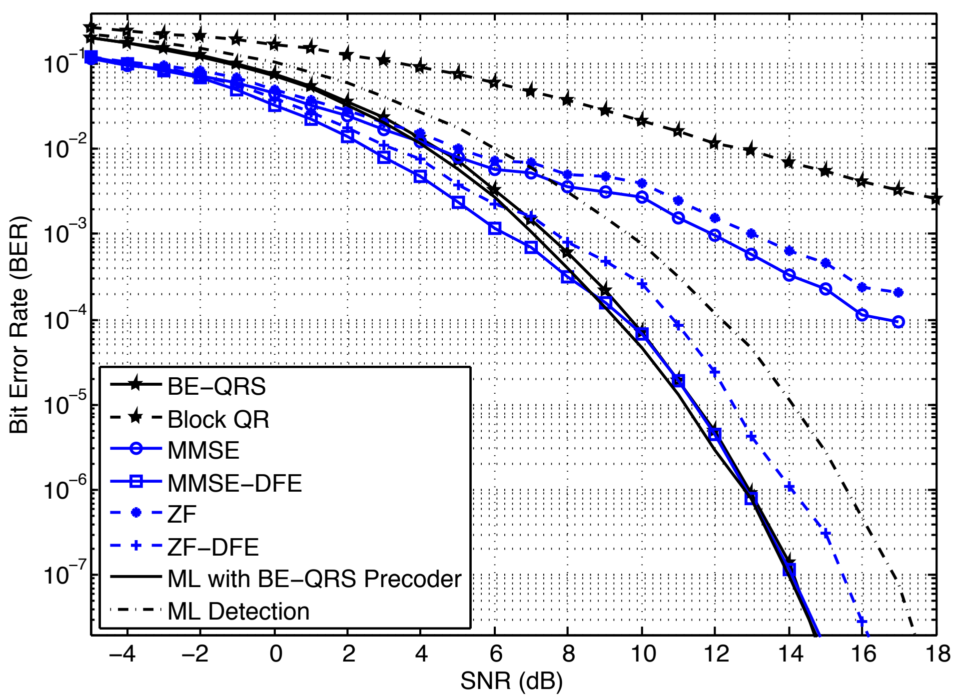

Figure 5.4: Error Performance Comparison in Bad Channel Case

The last plot in Fig. 5.4 shows the performance comparison under bad channel condition where $\rho_\lambda \geq 3(2+\sqrt{3})$. The performance of linear receivers (ZF and MMSE), and receiver without using precoder (Blcok-QR and ML Detection) remains quite poor. In this case the closed-form solution is available for constructing the optimal precoder. Unlike the results in the previous two cases, the methods of ZF-DFE and MMSE-DFE seem to achieve better performances than our proposed method and 4×4 ML detection when SNR is low (less than 9dB) and a competitive performance in moderate and high SNR range (from 10dB to 15dB). However, the comparison is unfair. Because of the poor channel condition, the methods of ZF-DFE and MMSE-DFE shut down some of the subchannels in low SNR cases. And shutting down subchannels results in transmission rate less than 4.

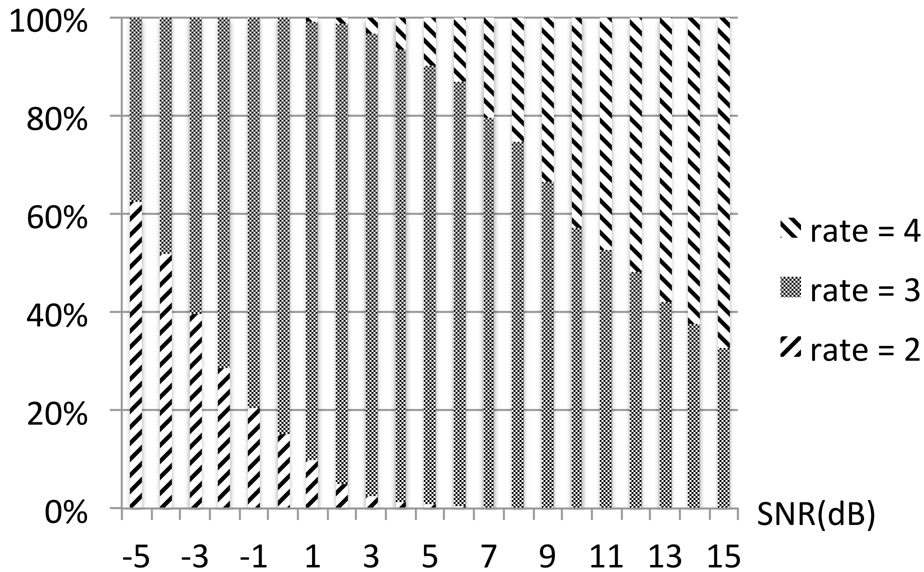

Figure 5.5: The Percentages of Channels with Different Transmission Rates in MMSE-DFE scheme

Figure 5.5 shows the transmission rates of the 200 channel realizations over all SNR conditions in MMSE-DFE method. We can find that for a significant portion of channel realizations, the transmission rate is less than 4 even when SNR is high. Specifically, when the SNR is in the range from 10dB to 15dB, our method achieve the same performance with MMSE-DFE and ZF-DFE. According to Figure 5.5, the transmission rate is 3 symbols/use for over 50% percent of the channel realizations as SNR = 10dB. When SNR = 15dB, the percentage of channels with transmission rate equal to 3 symbols/use is still larger than 30%. On the other hand, our proposed method and 4×4 ML detection with proposed optimal precoder always transmit symbols at the rate $r = 4$. In this sense, our method actually is superior to those nonlinear receivers.

Chapter 6

Conclusion and Future Work

6.1 Final Remarks

In this thesis, we have proposed a block-equal QR decomposition of a given 4×4 matrix. The decomposition has the following form

$$\tilde{\mathbf{H}}\mathbf{F} = \tilde{\mathbf{Q}}\tilde{\mathbf{R}} \tag{6.1}$$

where $\mathbf{F}$ is a unitary matrix to be determined, $\tilde{\mathbf{Q}}$ is a unitary matrix and $\tilde{\mathbf{R}}$ is a block-equal upper triangular matrix. This block-equal upper triangular matrix has equal 2×2 block entries along the diagonal. We proved that as long as there is a solution to a set of 7 equations with 8 variables, there exists a block-equal QR decomposition. We further construct the explicit solution for the block-equal QR decomposition under a special condition.

We applied the block-equal QR decomposition to design the optimal precoder for the transmitted signals composed of 4-QAM signals. The precoder has following good properties: (1) It lets the cascaded channel $\tilde{\mathbf{H}} = \mathbf{HF}$ to have the block-equal QR decomposition. (2) After performing the block-equal QR decomposition on the cascaded channel $\tilde{\mathbf{H}}$, each 2×2 block channel along the diagonal of $\tilde{R}$ applies same rotation on the 2×2 transmitted symbol vector, which leads to the same minimum Euclidean distance between two received signal vectors at the channel output. (3) It maximizes the minimum Euclidean distance between two received signal vectors at the channel output and minimizes the symbol error probability at the same time. The presented simulation results demonstrated the superior error performance of our optimal precoder-receiver pair.

6.2 Future Works

- In this thesis, we have proved that the equivalent condition of finding a block-equal QR decomposition is solving a set of 7 equations with 8 variables. When a certain condition is satisfied, we found a specific closed form factorization for any given channel matrix $\mathbf{H}$. However, when the condition is not satisfied, finding the closed form solution of a unitary matrix that leads to a block-equal QR decomposition is still an open question. We used the MATLAB Optimization Toolbox to achieve a numerical solution in this situation. However, a closed form solution is preferred since it greatly improves time efficiency and makes our proposed strategy easy to implemented for practical use.

- Our precoder and receiver design is based on a minimum sized 4×4 channel model where there are only 4 transmitting and receiving antennas in the MIMO system. Future investigations should consider the block-equal decomposition scheme for the channel with generalized dimension of $N \times M$, where $N \geq M$ and M is a multiple of 4. To generalize the idea of block-equal decomposition would be of significant meaning in practice. The results we have got can definitely contribute to this future work.

- We designed the optimal precoder for the case where transmitted signal is from 4-QAM constellation. Another direction of further study may be the extension of the symbol set to any square QAM constellations.

Appendix A

Equal-diagonal QRS decomposition and the Construction of the S-Factor

Theorem 3. *[38] For an arbitrary $M \times N$ matrix $\mathbf{H}$ with rank r, there exists a unitary matrix $\mathbf{S}$ such that $\mathbf{HS}$ has an equal-diagonal $\mathbf{R}$-factor, i.e.,*

$$\mathbf{HS} = \mathbf{QR} \tag{A.1}$$

where $\mathbf{Q}$ is an $M \times r$ column-wise orthonormal matrix and $\mathbf{R} = [\mathbf{R}_{r\times r} \quad \mathbf{0}_{r\times(N-r)}]$ with $\mathbf{R}_{r\times r}$ being the equal-diagonal $\mathbf{R}$-factor.

Algorithm 2 (Construction of the $\mathbf{S}$-factor)**.** *Following is the recursive algorithm to find the $\mathbf{S}$-factor of the QRS decomposition $\mathbf{HS} = \mathbf{QR}$.*

1) SVD *Perform the SVD of* $\mathbf{H} = \mathbf{H}\mathbf{\Lambda}\mathbf{V}$ *and let* $\check{\mathbf{H}} = \mathbf{H}\dot{(}\mathbf{V}^H)$, *i.e.,*

$$\check{\mathbf{H}} = \mathbf{U}\begin{pmatrix} \mathbf{\Lambda_{r\times r}} \\ \mathbf{0}_{(M-r)\times r} \end{pmatrix} \tag{A.2}$$

2) Initialization *Determine the first column of* $\check{\mathbf{S}}$, *i.e.,* $\check{\mathbf{s}}_1 = \begin{pmatrix} \check{S}_{11}, \ldots, \check{S}_{r1} \end{pmatrix}^T$, *such that constraints*

$$\check{\mathbf{s}}_1^H \check{\mathbf{H}}^H \check{\mathbf{H}} \check{\mathbf{s}}_1 = det\left(\check{\mathbf{H}}^H \check{\mathbf{H}}\right)^{1/r} \tag{A.3}$$

$$\check{\mathbf{s}}_1^H \check{\mathbf{s}}_1 = 1 \tag{A.4}$$

are satisfied.

3) Recursion *Reduce the dimension and decouple constraints. Set* $\check{\mathbf{s}}_{k+1} = \check{\mathbf{S}}_k^{\perp} \mathbf{z}_{k+1}$, *where* $\mathbf{z}_{k+1}$ *is any vector that satisfies*

$$\mathbf{z}_{k+1}^H \mathbf{C}^{(k)} \mathbf{z}_{k+1} = det\left(\check{\mathbf{H}}^H \check{\mathbf{H}}\right)^{1/r} \tag{A.5}$$

$$\mathbf{z}_{k+1}^H \mathbf{z}_{k+1} = 1 \tag{A.6}$$

with $\mathbf{C}^{(k)} = (\check{\mathbf{H}}\check{\mathbf{S}}_k^{\perp})^H \mathcal{P}_{\check{\mathbf{H}}\check{\mathbf{S}}_k} (\check{\mathbf{H}}\check{\mathbf{S}}_k^{\perp})$. *The notation* $\mathcal{P}_{\mathbf{A}} = \mathbf{I} - \mathbf{A}\mathbf{A}^{\dagger}$ *denots the projection matrix that projects an arbitrary vector to the null space of* $\mathbf{A}^H$.

4) Complete the S-factor

$$\mathbf{S} = [\mathbf{V}_r \check{\mathbf{S}} \quad \overline{(\mathbf{V}^H)}_{1,\ldots,r}] \tag{A.7}$$

where the overline $\overline{\mathbf{A}}_{k_1,k_2,\ldots,k_i}$ *denotes the remaining matrix after deleting columns*

$\mathbf{a}_{k_1}$, $\mathbf{a}_{k_2}$, ..., $\mathbf{a}_{k_i}$ *from* $\mathbf{A}$.

Appendix B

Majorization theory

Majorization theory [44] helps us simplify the matrices conditions into conditions of some single variables. This section introduces some basic relation symbols as well as some useful theorems that are used in the thesis .

Definition 1. *([44, p. xx]) For any* $\mathbf{x} \in \mathbb{R}^n$*, let*

$$x_{[1]} \geq \ldots \geq x_{[n]} \tag{B.1}$$

denote the the components of $\mathbf{x}$ *in decreasing order.*

Definition 2. *([44, 1.A.1]) For* $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$,

$$\mathbf{x} \prec_{+} \mathbf{y} \quad \textit{if} \begin{cases} \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, & k = 1, \ldots, n-1, \\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}. \end{cases} \tag{B.2}$$

When $\mathbf{x} \prec_{+} \mathbf{y}$, $\mathbf{x}$ *is said to be additively majorized by* $\mathbf{y}$*. (*$\mathbf{y}$ *additively majorizes* $\mathbf{x}$*).*

Parallel to the additive majorization, here is the definition of multiplicative majorizaton.

Definition 3. *([46, 2.3]) For* $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}_+^n$,

$$\mathbf{x} \prec_\times \mathbf{y} \quad \textit{if} \begin{cases} \prod_{i=1}^{k} x_{[i]} \leq \prod_{i=1}^{k} y_{[i]}, & k = 1, \ldots, n-1, \\ \prod_{i=1}^{n} x_{[i]} = \prod_{i=1}^{n} y_{[i]}. \end{cases} \tag{B.3}$$

When $\mathbf{x} \prec_\times \mathbf{y}$, $\mathbf{x}$ *is said to be multiplicatively majorized by* $\mathbf{y}$. *(*$\mathbf{y}$ *multiplicatively majorizes* $\mathbf{x}$*).*

Lemma 4. *([44, p. 234]) If* $\mathbf{A} \in \mathbb{R}^{n \times n}$ *is a positive semi-definite matrix with eigenvalues* x_i *and Cholesky values* y_i, $i = 1, \ldots, n$, *then*

$$\mathbf{x} \prec_\times \boldsymbol{y} \tag{B.4}$$

where $\mathbf{x} = (x_{[1]}, \ldots, x_{[n]})$ *and* $\boldsymbol{y} = (y_{[1]}, \ldots, y_{[n]})$. *Conversely, if the two sequences* $\mathbf{x}$ *and* $\mathbf{y}$ *satisfy the relationship described in Eq.* (B.4), *then there exists a positive semi-definite matrix* $\mathbf{A}$ *with eigenvalues* x_i *and Cholesky values* y_i, $i = 1, \ldots, n$.

Lemma 5. *([44, 9.H.1]) If* $\mathbf{U}$ *and* $\mathbf{V}$ *are* $n \times n$ *positive semi-definite Hermitian matrices with eigenvalues* $\lambda_i(\mathbf{U})$ *and* $\lambda_i(\mathbf{V})$, $i = 1, \ldots, n$, *then*

$$\prod_{i=1}^{k} \lambda_i(\mathbf{UV}) \leq \prod_{i=1}^{k} \lambda_i(\mathbf{U})\lambda_i(\mathbf{V}), \quad k = 1, \ldots, n, \tag{B.5}$$

$$\prod_{i=1}^{n} \lambda_i(\mathbf{UV}) = \prod_{i=1}^{n} \lambda_i(\mathbf{U})\lambda_i(\mathbf{V}). \tag{B.6}$$

Appendix C

Proof of Theorem 1

Separate the matrix $\mathbf{S}$ into two parts, each of which contain two columns of $\mathbf{S}$:

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_1 & \mathbf{S}_2 \end{pmatrix} \tag{C.1}$$

where $\mathbf{S}_i \in \mathbb{R}^{4\times 2}$, i = 1, 2.

If there exists such decomposition $\mathbf{\Lambda S} = \mathbf{QR}$, it can then be written in such a block matrices form:

$$\mathbf{\Lambda} \begin{pmatrix} \mathbf{S}_1 & \mathbf{S}_2 \end{pmatrix} = \mathbf{Q} \begin{pmatrix} \mathbf{\Lambda}_{eq} & \breve{\mathbf{R}}_{12} \\ \mathbf{0} & \mathbf{\Lambda}_{eq} \end{pmatrix} \tag{C.2}$$

Left-multiply both sides of Eq. (C.2) by its Hermitian,

$$\begin{pmatrix} \mathbf{S}_1^H \\ \mathbf{S}_2^H \end{pmatrix} \mathbf{\Lambda}^H \mathbf{\Lambda} \begin{pmatrix} \mathbf{S}_1 & \mathbf{S}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{\Lambda}_{eq}^H & \mathbf{0} \\ \breve{\mathbf{R}}_{12}^H & \mathbf{\Lambda}_{eq}^H \end{pmatrix} \mathbf{Q}^H \mathbf{Q} \begin{pmatrix} \mathbf{\Lambda}_{eq} & \breve{\mathbf{R}}_{12} \\ \mathbf{0} & \mathbf{\Lambda}_{eq} \end{pmatrix} \tag{C.3}$$

Let $\mathbf{\Lambda}^H \mathbf{\Lambda} = \text{diag}(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \triangleq \mathbf{\Sigma}$ and $\mathbf{\Lambda}_{eq}^H \mathbf{\Lambda}_{eq} = \text{diag}(r_1, r_2) \triangleq \mathbf{\Upsilon}$

The Eq. (C.3) is equivalent to

$$\begin{pmatrix} \mathbf{S}_1^H \mathbf{\Sigma} \mathbf{S}_1 & \mathbf{S}_1^H \mathbf{\Sigma} \mathbf{S}_2 \\ \mathbf{S}_2^H \mathbf{\Sigma} \mathbf{S}_1 & \mathbf{S}_2^H \mathbf{\Sigma} \mathbf{S}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{\Upsilon} & \mathbf{\Lambda}_{eq} \breve{\mathbf{R}}_{12} \\ \breve{\mathbf{R}}_{12}^H \mathbf{\Lambda}_{eq} & \breve{\mathbf{R}}_{12}^H \breve{\mathbf{R}}_{12} + \mathbf{\Upsilon} \end{pmatrix} \tag{C.4}$$

There actually are three equalities in Eq. (C.4), which are

$$\mathbf{S}_1^H \mathbf{\Sigma} \mathbf{S}_1 = \mathbf{\Upsilon} \tag{C.5}$$

$$\mathbf{S}_1^H \mathbf{\Sigma} \mathbf{S}_2 = \mathbf{\Lambda}_{eq} \breve{\mathbf{R}}_{12} \tag{C.6}$$

$$\mathbf{S}_2^H \mathbf{\Sigma} \mathbf{S}_2 = \breve{\mathbf{R}}_{12}^H \breve{\mathbf{R}}_{12} + \mathbf{\Upsilon} \tag{C.7}$$

Substitute $\breve{\mathbf{R}}_{12}$ in Eq. (C.6) into Eq. (C.7),

$$\mathbf{S}_2^H \left(\mathbf{\Sigma} - \mathbf{\Sigma} \mathbf{S}_1 \mathbf{\Upsilon}^{-1} \mathbf{S}_1^H \mathbf{\Sigma} \right) \mathbf{S}_2 = \mathbf{\Upsilon} \tag{C.8}$$

For notational simplicity, define that

$$\mathbf{A} \triangleq \mathbf{\Sigma} - \mathbf{\Sigma} \mathbf{S}_1 \mathbf{\Upsilon}^{-1} \mathbf{S}_1^H \mathbf{\Sigma} \tag{C.9}$$

Thus, the Eq. (C.8) can be simplified as

$$\mathbf{S}_2^H \mathbf{A} \mathbf{S}_2 = \mathbf{\Upsilon} \tag{C.10}$$

Note that both of $\mathbf{S}_i, i = 1, 2$ consist of two columns,

$$\mathbf{S}_1 = \begin{pmatrix} \mathbf{s}_1 & \mathbf{s}_2 \end{pmatrix}, \ \mathbf{S}_2 = \begin{pmatrix} \mathbf{s}_3 & \mathbf{s}_4 \end{pmatrix} \tag{C.11}$$

The Eq. (C.5) and Eq. (C.10) can be written as

$$\mathbf{s}_1^H \mathbf{\Sigma} \mathbf{s}_1 = r_1 \tag{C.12}$$

$$\mathbf{s}_2^H \mathbf{\Sigma} \mathbf{s}_2 = r_2 \tag{C.13}$$

$$\mathbf{s}_1^H \mathbf{\Sigma} \mathbf{s}_2 = 0 \tag{C.14}$$

$$\mathbf{s}_3^H \mathbf{A} \mathbf{s}_3 = r_1 \tag{C.15}$$

$$\mathbf{s}_4^H \mathbf{A} \mathbf{s}_4 = r_2 \tag{C.16}$$

$$\mathbf{s}_3^H \mathbf{A} \mathbf{s}_4 = 0 \tag{C.17}$$

Since that $\mathbf{S}$ is a unitary matrix, the vectors $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_3$ and $\mathbf{s}_4$ are orthonormal to each other. Combining the Eq. (C.17) and the orthonormality of $\mathbf{S}$, there exists a homogeneous linear system with respect to $\mathbf{s}_4$:

$$\begin{pmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \mathbf{s}_3 & \mathbf{A}\mathbf{s}_3 \end{pmatrix}^H \mathbf{s}_4 = 0 \tag{C.18}$$

which indicates that $\mathbf{A}\mathbf{s}_3$ is a linear combination of $\mathbf{s}_1$, $\mathbf{s}_2$, and $\mathbf{s}_3$:

$$\mathbf{A}\mathbf{s}_3 = a\ \mathbf{s}_1 + b\ \mathbf{s}_2 + c\ \mathbf{s}_3 \tag{C.19}$$

According to the definition of $\mathbf{A}$ in Eq. (C.9),

$$\mathbf{A} = \mathbf{\Sigma} - \left(\frac{\mathbf{\Sigma s}_1\mathbf{s}_1^H\mathbf{\Sigma}}{r_1} + \frac{\mathbf{\Sigma s}_2\mathbf{s}_2^H\mathbf{\Sigma}}{r_2}\right) \tag{C.20}$$

Substitute the expression of $\mathbf{A}$ into the Eq. (C.19),

$$\left[\mathbf{\Sigma} - \left(\frac{\mathbf{\Sigma s}_1\mathbf{s}_1^H\mathbf{\Sigma}}{r_1} + \frac{\mathbf{\Sigma s}_2\mathbf{s}_2^H\mathbf{\Sigma}}{r_2}\right)\right]\mathbf{s}_3 = a\ \mathbf{s}_1 + b\ \mathbf{s}_2 + c\ \mathbf{s}_3 \tag{C.21}$$

Left-multiply both sides of Eq. (C.21) by the factor $\mathbf{s}_1^H$:

$$\text{Left-hand Side} = \mathbf{s}_1^H\left[\mathbf{\Sigma} - \left(\frac{\mathbf{\Sigma s}_1\mathbf{s}_1^H\mathbf{\Sigma}}{r_1} + \frac{\mathbf{\Sigma s}_2\mathbf{s}_2^H\mathbf{\Sigma}}{r_2}\right)\right]\mathbf{s}_3 \tag{C.22}$$

$$= \mathbf{s}_1^H\mathbf{\Sigma s}_3 - \left(\frac{\mathbf{s}_1^H\mathbf{\Sigma s}_1\mathbf{s}_1^H\mathbf{\Sigma s}_3}{r_1} + \frac{\mathbf{s}_1^H\mathbf{\Sigma s}_2\mathbf{s}_2^H\mathbf{\Sigma s}_3}{r_2}\right) \tag{C.23}$$

$$\text{Because of equation (C.12) and (C.14),} \tag{C.24}$$

$$= \mathbf{s}_1^H\mathbf{\Sigma s}_3 - \mathbf{s}_1^H\mathbf{\Sigma s}_3 = 0 \tag{C.25}$$

$$\text{Right-hand Side} = a\ \mathbf{s}_1^H\mathbf{s}_1 + b\ \mathbf{s}_1^H\mathbf{s}_2 + c\ \mathbf{s}_1^H\mathbf{s}_3 = a \tag{C.26}$$

$$\text{Left-hand Side} = \text{Right-hand Side} \tag{C.27}$$

$$\Rightarrow a = 0 \tag{C.28}$$

Left-multiply both sides of Eq. (C.21) by the factor $\mathbf{s}_2^H$:

$$\text{Left-hand Side} = \mathbf{s}_2^H \left[\mathbf{\Sigma} - \left(\frac{\mathbf{\Sigma}\mathbf{s}_1\mathbf{s}_1^H\mathbf{\Sigma}}{r_1} + \frac{\mathbf{\Sigma}\mathbf{s}_2\mathbf{s}_2^H\mathbf{\Sigma}}{r_2} \right) \right] \mathbf{s}_3 \tag{C.29}$$

$$= \mathbf{s}_2^H\mathbf{\Sigma}\mathbf{s}_3 - \left(\frac{\mathbf{s}_2^H\mathbf{\Sigma}\mathbf{s}_1\mathbf{s}_1^H\mathbf{\Sigma}\mathbf{s}_3}{r_1} + \frac{\mathbf{s}_2^H\mathbf{\Sigma}\mathbf{s}_2\mathbf{s}_2^H\mathbf{\Sigma}\mathbf{s}_3}{r_2} \right) \tag{C.30}$$

$$\text{Because of equation (C.13) and (C.14),} \tag{C.31}$$

$$= \mathbf{s}_2^H\mathbf{\Sigma}\mathbf{s}_3 - \mathbf{s}_2^H\mathbf{\Sigma}\mathbf{s}_3 = 0 \tag{C.32}$$

$$\text{Right-hand Side} = a\ \mathbf{s}_2^H\mathbf{s}_1 + b\ \mathbf{s}_2^H\mathbf{s}_2 + c\ \mathbf{s}_2^H\mathbf{s}_3 = b \tag{C.33}$$

$$\text{Left-hand Side} = \text{Right-hand Side} \tag{C.34}$$

$$\Rightarrow b = 0 \tag{C.35}$$

Left-multiply both sides of Eq. (C.21) by the factor $\mathbf{s}_3^H$:

$$\text{Left-hand Side} = \mathbf{s}_3^H \left[\mathbf{\Sigma} - \left(\frac{\mathbf{\Sigma}\mathbf{s}_1\mathbf{s}_1^H\mathbf{\Sigma}}{r_1} + \frac{\mathbf{\Sigma}\mathbf{s}_2\mathbf{s}_2^H\mathbf{\Sigma}}{r_2} \right) \right] \mathbf{s}_3 \tag{C.36}$$

$$= \mathbf{s}_3^H\mathbf{\Sigma}\mathbf{s}_3 - \left(\frac{\mathbf{s}_3^H\mathbf{\Sigma}\mathbf{s}_1\mathbf{s}_1^H\mathbf{\Sigma}\mathbf{s}_3}{r_1} + \frac{\mathbf{s}_3^H\mathbf{\Sigma}\mathbf{s}_2\mathbf{s}_2^H\mathbf{\Sigma}\mathbf{s}_3}{r_2} \right) \tag{C.37}$$

$$= \mathbf{s}_3^H\mathbf{S}\mathbf{s}_3 = r_1 \quad \text{(equation C.15)} \tag{C.38}$$

$$\text{Right-hand Side} = a\ \mathbf{s}_3^H\mathbf{s}_1 + b\ \mathbf{s}_3^H\mathbf{s}_2 + c\ \mathbf{s}_3^H\mathbf{s}_3 = c \tag{C.39}$$

$$\text{Left-hand Side} = \text{Right-hand Side} \tag{C.40}$$

$$\Rightarrow c = r_1 \tag{C.41}$$

Therefore, substituting that $a = b = 0,\ c = r_1$, the Eq. (C.19) can then be rewritten as

$$\mathbf{A}\mathbf{s}_3 = r_1\mathbf{s}_3 \tag{C.42}$$

Similarly, there also exists the following homogeneous linear system with respect of $\mathbf{s}_3$:

$$\begin{pmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \mathbf{s}_3 & \mathbf{A}\mathbf{s}_4 \end{pmatrix}^H \mathbf{s}_3 = 0 \tag{C.43}$$

A similar result can be obtained by using the same trick:

$$\mathbf{A}\mathbf{s}_4 = r_2\mathbf{s}_4 \tag{C.44}$$

Let A be right-multiplied by $\mathbf{s}_1$ and $\mathbf{s}_2$, we have

$$\mathbf{A}\mathbf{s}_1 = \left[\mathbf{\Sigma} - \left(\frac{\mathbf{\Sigma}\mathbf{s}_1\mathbf{s}_1^H\mathbf{\Sigma}}{r_1} + \frac{\mathbf{\Sigma}\mathbf{s}_2\mathbf{s}_2^H\mathbf{\Sigma}}{r_2}\right)\right]\mathbf{s}_1 \tag{C.45}$$

$$= \mathbf{\Sigma}\mathbf{s}_1 - \frac{\mathbf{\Sigma}\mathbf{s}_1\mathbf{s}_1^H\mathbf{\Sigma}\mathbf{s}_1}{r_1} - \frac{\mathbf{\Sigma}\mathbf{s}_2\mathbf{s}_2^H\mathbf{\Sigma}\mathbf{s}_1}{r_2} \tag{C.46}$$

$$= \mathbf{\Sigma}\mathbf{s}_1 - \mathbf{\Sigma}\mathbf{s}_1 - \mathbf{0} \tag{C.47}$$

$$= \mathbf{0} \tag{C.48}$$

$$\mathbf{A}\mathbf{s}_2 = \left[\boldsymbol{\Sigma} - \left(\frac{\boldsymbol{\Sigma}\mathbf{s}_1\mathbf{s}_1^H\boldsymbol{\Sigma}}{r_1} + \frac{\boldsymbol{\Sigma}\mathbf{s}_2\mathbf{s}_2^H\boldsymbol{\Sigma}}{r_2}\right)\right]\mathbf{s}_2 \tag{C.49}$$

$$=\boldsymbol{\Sigma}\mathbf{s}_1 - \frac{\boldsymbol{\Sigma}\mathbf{s}_1\mathbf{s}_1^H\boldsymbol{\Sigma}\mathbf{s}_2}{r_1} - \frac{\boldsymbol{\Sigma}\mathbf{s}_2\mathbf{s}_2^H\boldsymbol{\Sigma}\mathbf{s}_2}{r_2} \tag{C.50}$$

$$=\boldsymbol{\Sigma}\mathbf{s}_2 - \mathbf{0} - \boldsymbol{\Sigma}\mathbf{s}_2 \tag{C.51}$$

$$=\mathbf{0} \tag{C.52}$$

Therefore, A can be factorized as

$$\mathbf{A} = \begin{pmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \mathbf{s}_3 & \mathbf{s}_4 \end{pmatrix} \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & r_1 & \\ & & & r_2 \end{pmatrix} \begin{pmatrix} \mathbf{s}_1^H \\ \mathbf{s}_2^H \\ \mathbf{s}_3^H \\ \mathbf{s}_4^H \end{pmatrix} = \mathbf{S} \begin{pmatrix} \mathbf{0} & \\ & \boldsymbol{\Upsilon} \end{pmatrix} \mathbf{S}^H \tag{C.53}$$

which indicates that each columns of $\mathbf{S}$ is an eigenvector of $\mathbf{A}$ and the eigenvalues are 0, 0, r_1, and r_2.

Definition 4. *Let matrix* $\mathbf{M} = \{m_{pq}\}$ *be an n-by-n matrix. The notation* $\mathbf{M}^{ij}$ *is defined as*

$$\mathbf{M}^{ij} \triangleq \begin{pmatrix} m_{ii} & m_{ij} \\ m_{ji} & m_{jj} \end{pmatrix}, 1 \leq i < j \leq n \tag{C.54}$$

Theorem 4. *Let* $\mathbf{M}$ *be an n-by-n symmetric matrix with eigenvalues* κ_1, κ_2, ..., κ_n.

The following equalities holds

$$\sum_{i=1}^{n} \kappa_i = tr(\mathbf{M}) \tag{C.55}$$

$$\sum_{i \neq j}^{n} \kappa_i \kappa_j = \sum_{i \neq j}^{n} \det(\mathbf{M}^{ij}) \tag{C.56}$$

Apply the Theorem 4 on the matrix $\mathbf{A}$, we get the following two equations:

$$\mathrm{tr}(\mathbf{A}) = r_1 + r_2 \tag{C.57}$$

$$\sum_{i \neq j}^{4} \det(\mathbf{A}^{ij}) = r_1 r_2 \tag{C.58}$$

For the Eq. (C.57),

$$\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} \sigma_i - \left(\frac{\mathbf{s}_1^H \boldsymbol{\Sigma}^2 \mathbf{s}_1}{r_1} + \frac{\mathbf{s}_2^H \boldsymbol{\Sigma}^2 \mathbf{s}_2}{r_2} \right) = r_1 + r_2 \tag{C.59}$$

For each term of the Eq. (C.58), $\det(\mathbf{A}^{ij})$ is given by

$$\det(\mathbf{A}^{ij}) = \det(\boldsymbol{\Sigma}^{ij} - \boldsymbol{\Sigma}_{\ell_i \ell_j}^H \mathbf{S}_1 \boldsymbol{\Upsilon}^{-1} \mathbf{S}_1^H \boldsymbol{\Sigma}_{\ell_i \ell_j}) \tag{C.60}$$

where $\boldsymbol{\Sigma}_{\ell_i \ell_j}$ is the matrix that contains the ith and jth columns

$$= \det(\Sigma^{ij}) \det(\mathbf{I} - \boldsymbol{\Upsilon}^{-1} \mathbf{S}_1^H \boldsymbol{\Sigma}_{\ell_i \ell_j} \boldsymbol{\Sigma}^{ij-1} \boldsymbol{\Sigma}_{\ell_i \ell_j}^H \mathbf{S}_1) \tag{C.61}$$

$$= \sigma_i \sigma_j \det(\boldsymbol{\Upsilon}^{-1}) \det(\boldsymbol{\Upsilon} - \mathbf{S}_1^H \boldsymbol{\Sigma}_{\ell_i \ell_j} \boldsymbol{\Sigma}^{ij-1} \boldsymbol{\Sigma}_{\ell_i \ell_j}^H \mathbf{S}_1) \tag{C.62}$$

$$= \frac{\sigma_i \sigma_j}{r_1 r_2} \det \begin{pmatrix} r_1 - (\sigma_i s_{i1}^2 + \sigma_j s_{j1}^2) & \sigma_i s_{i1} s_{i2} + \sigma_j s_{j1} s_{j2} \\ \sigma_i s_{i1} s_{i2} + \sigma_j s_{j1} s_{j2} & r_2 - (\sigma_i s_{i2}^2 + \sigma_j s_{j2}^2) \end{pmatrix}$$

Therefore Eq. (C.58) can be rewritten as

$$\sum_{i\neq j}^{4} \det\left(\mathbf{A}^{ij}\right) \tag{C.63}$$

$$= \frac{\sigma_1\sigma_2}{r_1r_2} \det \begin{pmatrix} r_1 - (\sigma_1 s_{11}^2 + \sigma_2 s_{21}^2) & \sigma_1 s_{11}s_{12} + \sigma_2 s_{21}s_{22} \\ \sigma_1 s_{11}s_{12} + \sigma_2 s_{21}s_{22} & r_2 - (\sigma_1 s_{12}^2 + \sigma_2 s_{22}^2) \end{pmatrix} \tag{C.64}$$

$$+ \frac{\sigma_3\sigma_4}{r_1r_2} \det \begin{pmatrix} r_1 - (\sigma_3 s_{31}^2 + \sigma_4 s_{41}^2) & \sigma_3 s_{31}s_{32} + \sigma_4 s_{41}s_{42} \\ \sigma_3 s_{31}s_{32} + \sigma_4 s_{41}s_{42} & r_2 - (\sigma_3 s_{32}^2 + \sigma_4 s_{42}^2) \end{pmatrix} \tag{C.65}$$

$$+ \frac{\sigma_1\sigma_3}{r_1r_2} \det \begin{pmatrix} r_1 - (\sigma_1 s_{11}^2 + \sigma_3 s_{31}^2) & \sigma_1 s_{11}s_{12} + \sigma_3 s_{31}s_{32} \\ \sigma_1 s_{11}s_{12} + \sigma_3 s_{31}s_{32} & r_2 - (\sigma_1 s_{12}^2 + \sigma_3 s_{32}^2) \end{pmatrix} \tag{C.66}$$

$$+ \frac{\sigma_2\sigma_4}{r_1r_2} \det \begin{pmatrix} r_1 - (\sigma_2 s_{21}^2 + \sigma_4 s_{41}^2) & \sigma_2 s_{21}s_{22} + \sigma_4 s_{41}s_{42} \\ \sigma_2 s_{21}s_{22} + \sigma_4 s_{41}s_{42} & r_2 - (\sigma_2 s_{22}^2 + \sigma_4 s_{42}^2) \end{pmatrix} \tag{C.67}$$

$$+ \frac{\sigma_1\sigma_4}{r_1r_2} \det \begin{pmatrix} r_1 - (\sigma_1 s_{11}^2 + \sigma_4 s_{41}^2) & \sigma_1 s_{11}s_{12} + \sigma_4 s_{41}s_{42} \\ \sigma_1 s_{11}s_{12} + \sigma_4 s_{41}s_{42} & r_2 - (\sigma_1 s_{12}^2 + \sigma_4 s_{42}^2) \end{pmatrix} \tag{C.68}$$

$$+ \frac{\sigma_2\sigma_3}{r_1r_2} \det \begin{pmatrix} r_1 - (\sigma_2 s_{21}^2 + \sigma_3 s_{31}^2) & \sigma_2 s_{21}s_{22} + \sigma_3 s_{31}s_{32} \\ \sigma_2 s_{21}s_{22} + \sigma_3 s_{31}s_{32} & r_2 - (\sigma_2 s_{22}^2 + \sigma_3 s_{32}^2) \end{pmatrix} \tag{C.69}$$

$$= \frac{\sigma_1\sigma_2\sigma_3\sigma_4}{r_1r_2} \left[(s_{31}s_{42} - s_{32}s_{41})^2 + (s_{11}s_{22} - s_{12}s_{21})^2 + (s_{21}s_{42} - s_{22}s_{41})^2 \right. \tag{C.70}$$

$$\left. + (s_{11}s_{32} - s_{12}s_{s31})^2 + (s_{21}s_{32} - s_{22}s_{31})^2 + (s_{11}s_{42} - s_{12}s_{41})^2\right] = r_1r_2$$

Therefore,

$$(s_{31}s_{42} - s_{32}s_{41})^2 + (s_{11}s_{22} - s_{12}s_{21})^2 + (s_{21}s_{42} - s_{22}s_{41})^2 \tag{C.71}$$

$$+ (s_{11}s_{32} - s_{12}s_{s31})^2 + (s_{21}s_{32} - s_{22}s_{31})^2 + (s_{11}s_{42} - s_{12}s_{41})^2 = \frac{r_1^2 r_2^2}{\sigma_1\sigma_2\sigma_3\sigma_4}$$

Revisiting the Eq. (C.3), we calculate the determinant of each side, which yields

$$r_1^2 r_2^2 = \sigma_1\sigma_2\sigma_3\sigma_4 \tag{C.72}$$

Substitute it into the Eq. (C.71), it becomes

$$\begin{aligned}&(s_{31}s_{42} - s_{32}s_{41})^2 + (s_{11}s_{22} - s_{12}s_{21})^2 + (s_{21}s_{42} - s_{22}s_{41})^2 \\ &+ (s_{11}s_{32} - s_{12}s_{s31})^2 + (s_{21}s_{32} - s_{22}s_{31})^2 + (s_{11}s_{42} - s_{12}s_{41})^2 = 1\end{aligned} \tag{C.73}$$

Actually, it can be shown that when the vector $\mathbf{s}_1$ and $\mathbf{s}_2$ are orthonormal, the Eq. (C.73) is always true. (Proof: See Appendix D). In other words, this equation is redundant under our conditions. So far, despite of the redundant Eq. (C.73), seven equations with respect to the eight elements have been found. The existence of the solution to the 7 equations are equivalent to the existence of such a matrix $\mathbf{S}$ that satisfies the conditions in Challenge 1:

$$s_{11}^2 + s_{21}^2 + s_{31}^2 + s_{41}^2 = 1 \tag{C.74}$$

$$s_{12}^2 + s_{22}^2 + s_{32}^2 + s_{42}^2 = 1 \tag{C.75}$$

$$s_{11}s_{12} + s_{21}s_{22} + s_{31}s_{32} + s_{41}s_{42} = 0 \tag{C.76}$$

$$\sigma_1 s_{11}^2 + \sigma_2 s_{21}^2 + \sigma_3 s_{31}^2 + \sigma_4 s_{41}^2 = r_1 \tag{C.77}$$

$$\sigma_1 s_{12}^2 + \sigma_2 s_{22}^2 + \sigma_3 s_{32}^2 + \sigma_4 s_{42}^2 = r_2 \tag{C.78}$$

$$\sigma_1 s_{11}s_{12} + \sigma_2 s_{21}s_{22} + \sigma_3 s_{31}s_{32} + \sigma_4 s_{41}s_{42} = 0 \tag{C.79}$$

$$\frac{\sigma_1^2 s_{11}^2 + \sigma_2^2 s_{21}^2 + \sigma_3^2 s_{31}^2 + \sigma_4^2 s_{41}^2}{r_1} + \frac{\sigma_1^2 s_{12}^2 + \sigma_2^2 s_{22}^2 + \sigma_3^2 s_{32}^2 + \sigma_4^2 s_{42}^2}{r_2} = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 - r_1 - r_2 \tag{C.80}$$

Appendix D

The Proof of the Redundancy of Equation (C.71)

$$
\begin{aligned}
&2 \times \text{Left-hand Side}\\
=&2\left[(s_{31}^2 s_{42}^2 + s_{32}^2 s_{41}^2 - 2s_{31}s_{32}s_{41}s_{42}) + (s_{11}^2 s_{22}^2 + s_{12}^2 s_{21}^2 - 2s_{11}s_{12}s_{21}s_{22})\right.\\
&+ (s_{21}^2 s_{42}^2 + s_{22}^2 s_{41}^2 - 2s_{21}s_{22}s_{41}s_{42}) + (s_{11}^2 s_{32}^2 + s_{12}^2 s_{31}^2 - 2s_{11}s_{12}s_{31}s_{32})\\
&\left.+ (s_{21}^2 s_{32}^2 + s_{22}^2 s_{31}^2 - 2s_{21}s_{22}s_{31}s_{32}) + (s_{11}^2 s_{42}^2 + s_{12}^2 s_{41}^2 - 2s_{11}s_{12}s_{41}s_{42})\right]\\
=&(s_{11}^2 s_{22}^2 + s_{12}^2 s_{21}^2 - 2s_{11}s_{12}s_{21}s_{22}) + (s_{11}^2 s_{32}^2 + s_{12}^2 s_{31}^2 - 2s_{11}s_{12}s_{31}s_{32}) +\\
&(s_{11}^2 s_{42}^2 + s_{12}^2 s_{41}^2 - 2s_{11}s_{12}s_{41}s_{42}) + (s_{11}^2 s_{22}^2 + s_{12}^2 s_{21}^2 - 2s_{11}s_{12}s_{21}s_{22}) +\\
&(s_{21}^2 s_{42}^2 + s_{22}^2 s_{41}^2 - 2s_{21}s_{22}s_{41}s_{42}) + (s_{21}^2 s_{32}^2 + s_{22}^2 s_{31}^2 - 2s_{21}s_{22}s_{31}s_{32}) +\\
&(s_{31}^2 s_{42}^2 + s_{32}^2 s_{41}^2 - 2s_{31}s_{32}s_{41}s_{42}) + (s_{11}^2 s_{32}^2 + s_{12}^2 s_{31}^2 - 2s_{11}s_{12}s_{31}s_{32}) +\\
&(s_{21}^2 s_{32}^2 + s_{22}^2 s_{31}^2 - 2s_{21}s_{22}s_{31}s_{32}) + (s_{31}^2 s_{42}^2 + s_{32}^2 s_{41}^2 - 2s_{31}s_{32}s_{41}s_{42}) +\\
&(s_{21}^2 s_{42}^2 + s_{22}^2 s_{41}^2 - 2s_{21}s_{22}s_{41}s_{42}) + (s_{11}^2 s_{42}^2 + s_{12}^2 s_{41}^2 - 2s_{11}s_{12}s_{41}s_{42})
\end{aligned}
$$

$$
\begin{aligned}
=&s_{11}^2(s_{22}^2+s_{32}^2+s_{42}^2)+s_{12}^2(s_{21}^2+s_{31}^2+s_{41}^2)-2s_{11}s_{12}(s_{21}s_{22}+s_{31}s_{32}+s_{41}s_{42})\\
&+s_{21}^2(s_{12}^2+s_{42}^2+s_{32}^2)+s_{22}^2(s_{11}^2+s_{41}^2+s_{31}^2)-2s_{21}s_{22}(s_{11}s_{12}+s_{41}s_{42}+s_{31}s_{32})\\
&+s_{31}^2(s_{42}^2+s_{12}^2+s_{22}^2)+s_{32}^2(s_{41}^2+s_{11}^2+s_{21}^2)-2s_{31}s_{32}(s_{41}s_{42}+s_{11}s_{12}+s_{21}s_{22})\\
&+s_{41}^2(s_{32}^2+s_{22}^2+s_{12}^2)+s_{42}^2(s_{31}^2+s_{21}^2+s_{11}^2)-2s_{41}s_{42}(s_{31}s_{32}+s_{21}s_{22}+s_{11}s_{12})
\end{aligned}
$$

Since that $s_{11}^2+s_{21}^2+s_{31}^2+s_{41}^2=1$, $s_{12}^2+s_{22}^2+s_{32}^2+s_{42}^2=1$

and $s_{11}s_{12}+s_{21}s_{22}+s_{31}s_{32}+s_{41}s_{42}=0$, therefore,

$$
\begin{aligned}
=&s_{11}^2(1-s_{12}^2)+s_{12}^2(1-s_{11}^2)+2s_{11}^2s_{12}^2+s_{21}^2(1-s_{22}^2)+s_{22}^2(1-s_{21}^2)+2s_{21}^2s_{22}^2\\
&+s_{31}^2(1-s_{32}^2)+s_{32}^2(1-s_{31}^2)+2s_{31}^2s_{32}^2+s_{41}^2(1-s_{42}^2)+s_{42}^2(1-s_{41}^2)+2s_{41}^2s_{42}^2\\
=&s_{11}^2+s_{12}^2+s_{21}^2+s_{22}^2+s_{31}^2+s_{32}^2+s_{41}^2+s_{42}^2=2=2\times\text{Right-hand Side}
\end{aligned}
$$

Appendix E

Proof of Theorem 2

Under this particular case that $r_1 = \sqrt{\sigma_1\sigma_2}$ and $r_2 = \sqrt{\sigma_3\sigma_4}$, we assume that the matrix $\mathbf{S}_1$ has the following special form:

$$\mathbf{S}_1 = \begin{bmatrix} s_{11} \neq 0 & s_{12} = 0 \\ s_{21} \neq 0 & s_{22} = 0 \\ s_{31} = 0 & s_{32} \neq 0 \\ s_{41} = 0 & s_{42} \neq 0 \end{bmatrix} \tag{E.1}$$

With the assumption above, the 7 equations in Theorem 1 can be simplified into the following 5 equations:

$$s_{11}^2 + s_{21}^2 = 1 \tag{E.2}$$

$$s_{32}^2 + s_{42}^2 = 1 \tag{E.3}$$

$$\sigma_1 s_{11}^2 + \sigma_2 s_{21}^2 = r_1 \tag{E.4}$$

$$\sigma_3 s_{32}^2 + \sigma_4 s_{42}^2 = r_2 \tag{E.5}$$

$$\frac{\sigma_1^2 s_{11}^2 + \sigma_2^2 s_{21}^2}{r_1} + \frac{\sigma_3^2 s_{32}^2 + \sigma_4^2 s_{42}^2}{r_2} = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 - r_1 - r_2 \tag{E.6}$$

Combine Eq. (E.2) and Eq. (E.4), we have:

$$\begin{cases} s_{11}^2 + s_{21}^2 = 1 \\ \sigma_1 s_{11}^2 + \sigma_2 s_{21}^2 = r_1 \end{cases} \Rightarrow \begin{cases} s_{11}^2 = (r_1 - \sigma_2)/(\sigma_1 - \sigma_2) \\ s_{21}^2 = (\sigma_1 - r_1)/(\sigma_1 - \sigma_2) \end{cases} \tag{E.7}$$

Similarly, combine Eq. (E.3) and Eq. (E.5), we have:

$$\begin{cases} 0s_{32}^2 + s_{42}^2 = 1 \\ \sigma_3 s_{32}^2 + \sigma_4 s_{42}^2 = r_2 \end{cases} \Rightarrow \begin{cases} s_{32}^2 = (r_2 - \sigma_4)/(\sigma_3 - \sigma_4) \\ s_{42}^2 = (\sigma_3 - r_2)/(\sigma_3 - \sigma_4) \end{cases} \tag{E.8}$$

Substituting the expressions of s_{11}^2, s_{21}^2, s_{32}^2 and s_{42}^2 into the Eq. (E.6):

$$\frac{\sigma_1^2 \frac{r_1-\sigma_2}{\sigma_1-\sigma_2} + \sigma_2^2 \frac{\sigma_1-r_1}{\sigma_1-\sigma_2}}{r_1} + \frac{\sigma_3^2 \frac{r_2-\sigma_4}{\sigma_3-\sigma_4} + \sigma_4^2 \frac{\sigma_3-r_2}{\sigma_3-\sigma_4}}{r_2} = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 - r_1 - r_2 \tag{E.9}$$

$$\frac{r_1(\sigma_1+\sigma_2) - \sigma_1\sigma_2}{r_1} + \frac{r_2(\sigma_3+\sigma_4) - \sigma_3\sigma_4}{r_2} = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 - r_1 - r_2 \tag{E.10}$$

$$\frac{\sigma_1\sigma_2}{r_1} + \frac{\sigma_3\sigma_4}{r_2} = r_1 + r_2 \tag{E.11}$$

We notice that when we set r_1 and r_2 as their optimal values $r_1 = \sqrt{\sigma_1\sigma_2}$ and

$r_2 = \sqrt{\sigma_3\sigma_4}$, the Eq. (E.6) holds. Thus, we can conclude that our assumption is correct, i.e.,

$$\mathbf{S}_1 = \begin{bmatrix} \sqrt{(r_1-\sigma_2)/(\sigma_1-\sigma_2)} & 0 \\ \sqrt{(\sigma_1-r_1)/(\sigma_1-\sigma_2)} & 0 \\ 0 & \sqrt{(r_2-\sigma_4)/(\sigma_3-\sigma_4)} \\ 0 & \sqrt{(\sigma_3-r_2)/(\sigma_3-\sigma_4)} \end{bmatrix} \tag{E.12}$$

is a set of solution to the 7 equations in this case. And **S** is given by:

$$\mathbf{S} = \begin{bmatrix} \sqrt{\frac{r_1-\sigma_2}{\sigma_1-\sigma_2}} & 0 & \sqrt{\frac{\sigma_1-r_1}{\sigma_1-\sigma_2}} & 0 \\ \sqrt{\frac{\sigma_1-r_1}{\sigma_1-\sigma_2}} & 0 & -\sqrt{\frac{r_1-\sigma_2}{\sigma_1-\sigma_2}} & 0 \\ 0 & \sqrt{\frac{r_2-\sigma_4}{\sigma_3-\sigma_4}} & 0 & \sqrt{\frac{\sigma_3-r_2}{\sigma_3-\sigma_4}} \\ 0 & \sqrt{\frac{\sigma_3-r_2}{\sigma_3-\sigma_4}} & 0 & -\sqrt{\frac{r_2-\sigma_4}{\sigma_3-\sigma_4}} \end{bmatrix} \tag{E.13}$$

Appendix F

The Optimality of a Precoder that Diagonalizes the Channel

We Assume that:

1. The eigenvalues of matrix $\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}$ in descending order are given by sequence $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$. The Cholesky values of $\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}$ in descending order, on the other hand, are given by (r_1, r_2, r_3, r_4)

2. The eigenvalues of matrix $\mathbf{H}^H\mathbf{H}$ in descending order are given by sequence $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

3. The eigenvalues of matrix $\mathbf{F}^H\mathbf{F}$ in descending order are given by sequence $(\mu_1, \mu_2, \mu_3, \mu_4)$.

Based on Lemma 4 (See Appendix B), the sequence of Cholesky values of one matrix is multiplicatively mahorized by its sequence of eigenvalues:

$$(r_1, r_2, r_3, r_4) \prec_\times (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \tag{F.1}$$

Since $\tilde{\mathbf{H}} = \mathbf{HF}$, the sequences of eigenvalues satisfy the following relationship according to Lemma 5 (See Appendix B):

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \prec_\times (\lambda_1\mu_1, \lambda_2\mu_2, \lambda_3\mu_3, \lambda_4\mu_4) \tag{F.2}$$

Combining Eqs. (F.1) and (F.2), we have the following new majorization relationship:

$$(r_1, r_2, r_3, r_4) \prec_\times (\lambda_1\mu_1, \lambda_2\mu_2, \lambda_3\mu_3, \lambda_4\mu_4) \tag{F.3}$$

According to Lemma 4, if the two sequences satisfy the condition, then there exists a matrix, whose Cholesky values are given by the left-hand side sequence and whose eigenvalues are given by the right-hand side sequence. If the eigenvalues of $\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}$ are $(\lambda_1\mu_1, \lambda_2\mu_2, \lambda_3\mu_3, \lambda_4\mu_4)$, Then the matrix $\tilde{\mathbf{H}}^H\tilde{\mathbf{H}}$ satisfies the condition in Lemma 4. Therefore the diagonalization of the channel $\mathbf{H}$ by the precoder $\mathbf{F}$ is a sufficient condition for the existence of an optimal precoder.

Appendix G

Mathematical Derivations

G.1 Equivalent Conditions

$$(r_1, r_1, r_2, r_2) \prec_\times (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \tag{G.1}$$

According to the definition of multiplicative majorizaton stated in Def. 3, Eq. (G.1) is equivalent to the following equality and inequalities:

$$r_1 \leq \sigma_1 \tag{G.2}$$

$$r_1^2 \leq \sigma_1\sigma_2 \tag{G.3}$$

$$r_1^2 r_2 \leq \sigma_1\sigma_2\sigma_3 \tag{G.4}$$

$$r_1^2 r_2^2 \leq \sigma_1\sigma_2\sigma_3\sigma_4 \tag{G.5}$$

Note that the sequences in Eq. (G.1) are decreasingly ordered. Substituting $r_1 \geq r_2$ and $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \sigma_4$ into the equality and inequalities above and removing the redundant ones, we have the following 3 conditions that are equivalent to Eq. (G.1):

$$r_1 \geq \sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \tag{G.6}$$

$$r_1 \leq \sqrt{\sigma_1\sigma_2} \tag{G.7}$$

$$r_2 = \sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}/r_1 \tag{G.8}$$

G.2 Reformulation in Eqs. (4.52),(4.57)

Substitutting $r_2 = \sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}/r_1$ into problem (4.51), the objective function and the constrants can be simplified as

$$\begin{aligned} \max_{r_1} \quad & r_1 + \sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}/r_1 \\ \text{s.t.} \quad & r_1 \leq \sqrt{3}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\ & r_1 \geq \sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\ & r_1 \leq \sqrt{\sigma_1\sigma_2} \end{aligned} \tag{G.9}$$

Situation A. When $\rho_\sigma \leq 3$,

$$\sqrt{\frac{\sigma_1\sigma_2}{\sigma_3\sigma_4}} \leq 3 \tag{G.10}$$

$$\sqrt[4]{\frac{\sigma_1\sigma_2}{\sigma_3\sigma_4}} \leq \sqrt{3} \tag{G.11}$$

$$\sqrt[4]{\sigma_1\sigma_2} \leq \sqrt{3}\sqrt[4]{\sigma_3\sigma_4} \tag{G.12}$$

$$\sqrt{\sigma_1\sigma_2} \leq \sqrt{3}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \tag{G.13}$$

Since that $\sqrt{\sigma_1\sigma_2} \leq \sqrt{3}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}$, the feasible set of r_1 is upper bounded by $\sqrt{\sigma_1\sigma_2}$. And the condition $r_1 \leq \sqrt{3}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}$ is redundant. Thus, problem (4.51) can be reformulated as

$$\begin{aligned} \max_{r_1} \quad & r_1 + \sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}/r_1 \\ \text{s.t.} \quad & r_1 \geq \sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\ & r_1 \leq \sqrt{\sigma_1\sigma_2} \end{aligned} \tag{G.14}$$

Situation B. On the contrary, when $\rho_\sigma > 3$, $\sqrt{\sigma_1\sigma_2} \geq \sqrt{3}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}$, the feasible set of r_1 is upper bounded by $\sqrt{3}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}$. And the condition $r_1 \leq \sqrt{\sigma_1\sigma_2}$ is redundant. Thus, problem (4.51) can be reformulated as

$$\begin{aligned} \max_{r_1} \quad & r_1 + \sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}/r_1 \\ \text{s.t.} \quad & r_1 \geq \sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\ & r_1 \leq \sqrt{3}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \end{aligned} \tag{G.15}$$

G.3 Reformulation in Eqs. (4.63),(4.68)

Substitutting $r_2 = \sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}/r_1$ into problem (4.62), the objective function and the constrants can be simplified as

$$\begin{aligned} \max_{r_1} \quad & r_1 + \sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}/r_1 \\ \text{s.t.} \quad & r_1 \geq \sqrt{3}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\ & r_1 \leq (\sqrt{2}+1)^2\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\ & r_1 \leq \sqrt{\sigma_1\sigma_2} \end{aligned} \tag{G.16}$$

Situation A. When $\rho_\sigma < 3$, $\sqrt{\sigma_1\sigma_2} \leq \sqrt{3}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}$, which means one of the upper bounds of r_1 is greater than its lower bound. There does not exist such a r_2 satisfying all the three constraints at the same time. So this problem is infeasible when $\rho_\sigma < 3$.

Situation B. When $3 \leq \rho_\sigma \leq (\sqrt{2}+1)^2$, $\sqrt{3}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \leq \sqrt{\sigma_1\sigma_2} \leq (\sqrt{2}+1)^2\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}$. The feasible set of r_1 is bounded by $\sqrt{3}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}$ and $\sqrt{\sigma_1\sigma_2}$. Thus, problem (4.62) can be reformulated as

$$\begin{aligned} \max_{r_1} \quad & r_1 + \sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}/r_1 \\ \text{s.t.} \quad & r_1 \geq \sqrt{3}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\ & r_1 \leq \sqrt{\sigma_1\sigma_2} \end{aligned} \tag{G.17}$$

Situation C. When $\rho_\sigma > (\sqrt{2}+1)^2$, $\sqrt{3}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \leq (\sqrt{2}+1)^2\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \leq \sqrt{\sigma_1\sigma_2}$. The feasible set of r_1 is bounded by $\sqrt{3}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}$ and $(\sqrt{2}+1)^2\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}$. Thus, problem (4.62) can be reformulated as

$$\begin{aligned} \max_{r_1} \quad & r_1 + \sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}/r_1 \\ \text{s.t.} \quad & r_1 \geq \sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\ & r_1 \leq (\sqrt{2}+1)\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \end{aligned} \tag{G.18}$$

G.4 Upper Bound of $r_1 + r_2$ under Condition 3

Too find the upper bound of $r_1 + r_2$, we solve the following optimization problem:

$$\begin{aligned} \max_{r_1} \quad & r_1 + \sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}/r_1 \\ \text{s.t.} \quad & r_1 \leq \sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\ & r_1 \geq (\sqrt{2}+1)\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\ & r_1 \leq \sqrt{\sigma_1\sigma_2} \end{aligned} \tag{G.19}$$

To solve the problem, we need to discuss the relationship between $\sqrt{\sigma_1\sigma_2}$ and the other two boundaries $(\sqrt{2}+1)\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}$ and $\sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}$ (See Fig. G.1):

a) $\sqrt{\sigma_1\sigma_2} \leq (\sqrt{2}+1)\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \leq \sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}$

The problem is infeasible.

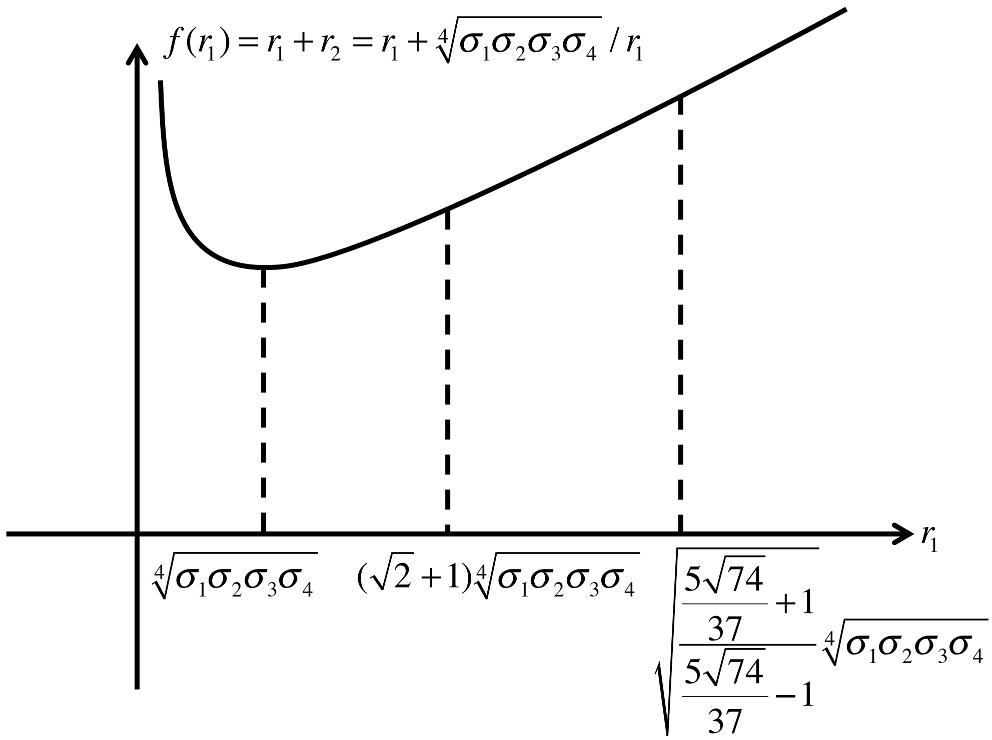

Figure G.1: The Relationship of Different Boundaries in Problem (G.19)

b) $(\sqrt{2}+1)\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \leq \sqrt{\sigma_1\sigma_2} \leq \sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}$

The feasible set of r_1 should be $(\sqrt{2}+1)\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \leq r_1 \leq \sqrt{\sigma_1\sigma_2}$. The corresponding maximum value of $r_1 + r_2$ is

$$(r_1 + r_2)_{\max} = \sqrt{\sigma_1\sigma_2} + \sqrt{\sigma_3\sigma_4} \tag{G.20}$$

Note that the following inequality also holds under this situation:

$$(r_1+r_2)_{\max} \leq \left[\sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}} + \left(1/\sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}}\right)\right]\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \tag{G.21}$$

c) $(\sqrt{2}+1)\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \leq \sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \leq \sqrt{\sigma_1\sigma_2}$

The feasible set of r_1 should be $(\sqrt{2}+1)\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \leq r_1 \leq \sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}$. The corresponding maximum value of r_1+r_2 is

$$(r_1+r_2)_{\max} = \left[\sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}} + \left(1/\sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}}\right)\right]\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \tag{G.22}$$

Combine the results in Eqs. (G.21) and (G.22), we can draw the conclusion that $\left[\sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}} + \left(1/\sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}}\right)\right]\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}$ is always a upper bound of r_1+r_2.

G.5 Lower Bound of r_1+r_2 under Condition 4

Too find the lower bound of r_1+r_2, we solve the following optimization problem:

$$\begin{aligned} \min_{r_1} \quad & r_1 + \sqrt{\sigma_1\sigma_2\sigma_3\sigma_4}/r_1 \\ \text{s.t.} \quad & r_1 \geq \sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \\ & r_1 \leq \sqrt{\sigma_1\sigma_2} \end{aligned} \tag{G.23}$$

To solve the problem, we need to discuss the relationship between two boundaries $\sqrt{\sigma_1\sigma_2}$ and $\sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}$ (See Fig. G.2):

a) $\sqrt{\sigma_1\sigma_2} \leq \sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}$

The problem is infeasible.

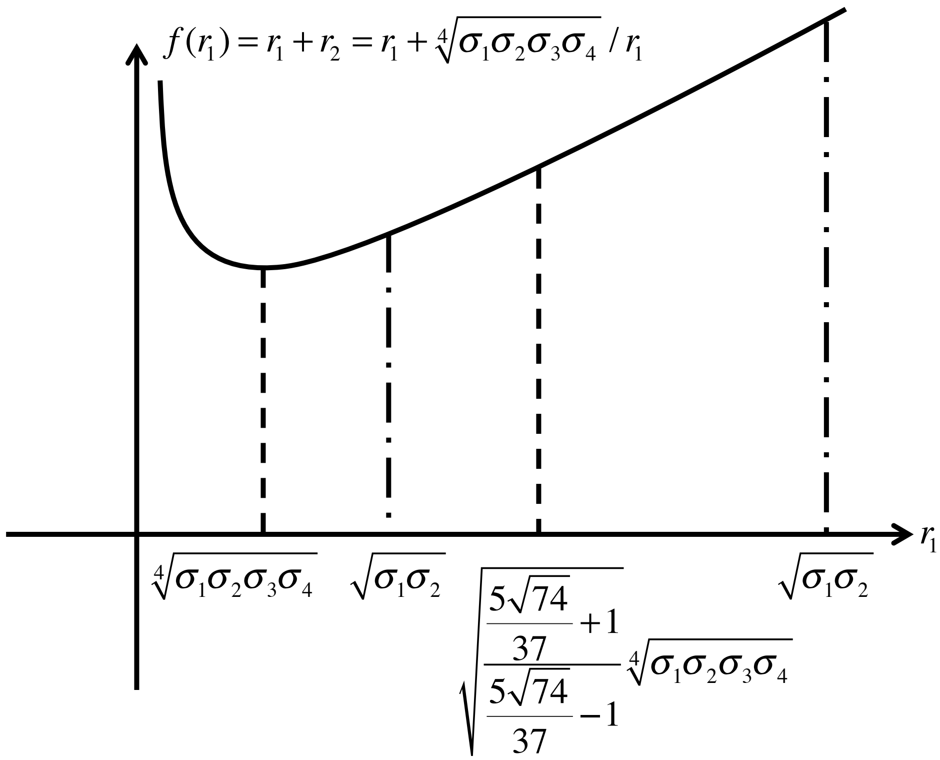

Figure G.2: The Relationship of Different Boundaries in Problem (G.23)

b) $\sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \leq \sqrt{\sigma_1\sigma_2}$

The feasible set of r_1 should be $\sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}}\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \leq r_1 \leq \sqrt{\sigma_1\sigma_2}$. The corresponding minimum value of $r_1 + r_2$ is

$$(r_1 + r_2)_{\min} = \left[\sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}} + \left(1/\sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}}\right)\right]\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4} \tag{G.24}$$

Therefore, $r_1 + r_2$ is lower bounded by $\left[\sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}} + \left(1/\sqrt{\frac{\frac{5\sqrt{74}}{37}+1}{\frac{5\sqrt{74}}{37}-1}}\right)\right]\sqrt[4]{\sigma_1\sigma_2\sigma_3\sigma_4}$.

Bibliography

[1] E. Telatar, "Capacity of multi-antenna Gaussian channels," *European Transactions on Telecommunications*, vol. 10, no. 6, pp. 585–595, 1999.

[2] G. J. Foschini and M. J. Gans, "On limits of wireless communications in a fading environment when using multiple antennas," *Wireless Personal Communications*, vol. 6, no. 3, pp. 311–335, 1998.

[3] S. Verdú, "Minimum probability of error for asynchronous Gaussian multiple-access channels," *IEEE Transactions on Information Theory*, vol. 32, no. 1, pp. 85–96, 1986.

[4] S. Bergman and B. Ottersten, "Lattice-based linear precoding for MIMO channels with transmitter CSI," *IEEE Transactions on Signal Processing*, vol. 56, no. 7, pp. 2902–2914, 2008.

[5] F. Pérez-Cruz, M. R. Rodrigues, and S. Verdú, "MIMO Gaussian channels with arbitrary inputs: Optimal precoding and power allocation," *IEEE Transactions on Information Theory*, vol. 56, no. 3, pp. 1070–1084, 2010.

[6] S. Verdú, "Computational complexity of optimum multiuser detection," *Algorithmica*, vol. 4, no. 1-4, pp. 303–312, 1989.

[7] D. Gore, R. W. Heath Jr, and A. Paulraj, "On performance of the zero forcing receiver in presence of transmit correlation," in *Proceedings 2002 IEEE International Symposium on Information Theory*. IEEE, 2002, p. 159.

[8] Y. Ding, T. N. Davidson, Z.-Q. Luo, and K. M. Wong, "Minimum BER block precoders for zero-forcing equalization," *IEEE Transactions on Signal Processing*, vol. 51, no. 9, pp. 2410–2423, 2003.

[9] Q. H. Spencer, A. L. Swindlehurst, and M. Haardt, "Zero-forcing methods for downlink spatial multiplexing in multiuser MIMO channels," *IEEE Transactions on Signal Processing*, vol. 52, no. 2, pp. 461–471, 2004.

[10] L.-U. Choi and R. D. Murch, "A transmit preprocessing technique for multiuser MIMO systems using a decomposition approach," *IEEE Transactions on Wireless Communications*, vol. 3, no. 1, pp. 20–24, 2004.

[11] Z. Pan, K.-K. Wong, and T.-S. Ng, "Generalized multiuser orthogonal space-division multiplexing," *IEEE Transactions on Wireless Communications*, vol. 3, no. 6, pp. 1969–1973, 2004.

[12] K.-K. Wong, "Maximizing the sum-rate and minimizing the sum-power of a broadcast 2-user 2-input multiple-output antenna system using a generalized zeroforcing approach," *IEEE Transactions on Wireless Communications*, vol. 5, no. 12, pp. 3406–3412, 2006.

[13] Z. Shen, R. Chen, J. G. Andrews, R. W. Heath, and B. L. Evans, "Low complexity user selection algorithms for multiuser MIMO systems with block diagonalization," *IEEE Transactions on Signal Processing*, vol. 54, no. 9, pp. 3658–3663,

2006.

[14] P. S. Udupa and J. S. Lehnert, "Optimizing zero-forcing precoders for MIMO broadcast systems," *IEEE Transactions on Communications*, vol. 55, no. 8, pp. 1516–1524, 2007.

[15] A. Wiesel, Y. C. Eldar, and S. Shamai, "Zero-forcing precoding and generalized inverses," *IEEE Transactions on Signal Processing*, vol. 56, no. 9, pp. 4409–4418, 2008.

[16] K.-H. Lee and D. Petersen, "Optimal linear coding for vector channels," *IEEE Transactions on Communications*, vol. 24, no. 12, pp. 1283–1290, 1976.

[17] J. Yang and S. Roy, "Joint transmitter-receiver optimization for multi-input multi-output systems with decision feedback," *IEEE Transactions on Information Theory*, vol. 40, no. 5, pp. 1334–1347, 1994.

[18] ——, "On joint transmitter and receiver optimization for multiple-input-multiple-output (MIMO) transmission systems," *IEEE Transactions on Communications*, vol. 42, no. 12, pp. 3221–3231, 1994.

[19] A. Scaglione, P. Stoica, S. Barbarossa, G. B. Giannakis, and H. Sampath, "Optimal designs for space-time linear precoders and decoders," *IEEE Transactions on Signal Processing*, vol. 50, no. 5, pp. 1051–1064, 2002.

[20] S. S. Chan, T. N. Davidson, and K. M. Wong, "Asymptotically minimum BER linear block precoders for MMSE equalisation," in *IEE Proceedings- Communications*, vol. 151, no. 4. IET, 2004, pp. 297–304.

[21] D. N. C. Tse and O. Zeitouni, "Linear multiuser receivers in random environments," *IEEE Transactions on Information Theory*, vol. 46, no. 1, pp. 171–188, 2000.

[22] M. Joham, W. Utschick, and J. A. Nossek, "Linear transmit processing in MIMO communications systems," *IEEE Transactions on Signal Processing*, vol. 53, no. 8, pp. 2700–2712, 2005.

[23] D. A. Schmidt, M. Joham, and W. Utschick, "Minimum mean square error vector precoding," in *IEEE 16th International Symposium on Personal, Indoor and Mobile Radio Communications, 2005*, vol. 1. IEEE, 2005, pp. 107–111.

[24] Y.-C. Liang, G. Pan, and Z. Bai, "Asymptotic performance of MMSE receivers for large systems using random matrix theory," *IEEE Transactions on Information Theory*, vol. 53, no. 11, pp. 4173–4190, 2007.

[25] C. A. Belfiore and J. H. Park Jr, "Decision feedback equalization," *Proceedings of the IEEE*, vol. 67, no. 8, pp. 1143–1156, 1979.

[26] G. K. Kaleh, "Channel equalization for block transmission systems," *IEEE Journal on Selected Areas in Communications*, vol. 13, no. 1, pp. 110–121, 1995.

[27] D. Williamson, R. A. Kennedy, and G. W. Pulford, "Block decision feedback equalization," *IEEE Transactions on Communications*, vol. 40, no. 2, pp. 255–264, 1992.

[28] N. Zervos, S. Pasupathy, and A. N. Venetsanopoulos, "Low Complexity Maximum Likelihood Equalizers," in *IEEE International Conference on Communications*, 1984, pp. 1259–1262.

[29] R. Price, "Nonlinearly feedback equalized PAM versus capacity for noisy filter channels," in *IEEE International Conference on Communications*, 1972.

[30] W. Lee and F. Hill, "A maximum-likelihood sequence estimator with decision-feedback equalization," *IEEE Transactions on Communications*, vol. 25, no. 9, pp. 971–979, 1977.

[31] J. Salz, "Optimum mean-square decision feedback equalization," *Bell System Technical Journal*, vol. 52, no. 8, pp. 1341–1373, 1973.

[32] D. D. Falconer and G. J. Foschini, "Theory of minimum mean-square-error QAM systems employing decision feedback equalization," *Bell System Technical Journal*, vol. 52, no. 10, p. 18211849, 1973.

[33] N. Al-Dhahir and J. M. Cioffi, "MMSE decision-feedback equalizers: finite-length results," *IEEE Transactions on Information Theory*, vol. 41, no. 4, pp. 961–975, 1995.

[34] ——, "Block transmission over dispersive channels: Transmit filter optimization and realization, and MMSE-DFE receiver performance," *IEEE Transactions on Information Theory*, vol. 42, no. 1, pp. 137–160, 1996.

[35] N. Al-Dhahir and A. H. Sayed, "The finite-length multi-input multi-output MMSE-DFE," *IEEE Transactions on Signal Processing*, vol. 48, no. 10, pp. 2921–2936, 2000.

[36] F. Xu, T. N. Davidson, and K. Wong, "Design of block-by-block transceivers with successive detection," in *4th IEEE Workshop on Signal Processing Advances in Wireless Communications, 2003.* IEEE, 2003, pp. 502–506.

[37] F. Xu, T. N. Davidson, J.-K. Zhang, and K. M. Wong, "Design of block transceivers with decision feedback detection," *IEEE Transactions on Signal Processing*, vol. 54, no. 3, pp. 965–978, 2006.

[38] J.-K. Zhang, A. Kavcic, and K. M. Wong, "Equal-diagonal QR decomposition and its application to precoder design for successive-cancellation detection," *IEEE Transactions on Information Theory*, vol. 51, no. 1, pp. 154–172, 2005.

[39] G. Golden, C. Foschini, R. A. Valenzuela, and P. Wolniansky, "Detection algorithm and initial laboratory results using V-BLAST space-time communication architecture," *Electronics Letters*, vol. 35, no. 1, pp. 14–16, 1999.

[40] T. M. Cover and J. A. Thomas, *Elements of Information Theory.* Wiley-Interscience, 1991.

[41] A. Lozano, A. M. Tulino, and S. Verdú, "Optimum power allocation for parallel Gaussian channels with arbitrary input distributions," *IEEE Transactions on Information Theory*, vol. 52, no. 7, pp. 3033–3051, 2006.

[42] J. Proakis and M. Salehi, *Digital Communications, 5th Edition.* McGraw-Hill Science/Engineering/Math, 2007.

[43] J.-K. Zhang and A.-Z. Wong, "Complete characterization of ML-optimal precoders for two-by-two MIMO channels," McMaster University, Tech. Rep., 2013.

[44] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications.* Academic Press, 1980.

[45] F. Xu, "Design of block-by-block transceivers with decision feedback equalization," Master's thesis, McMaster University, 2003.

[46] D. P. Palomar and Y. Jiang, “MIMO transceiver design via majorization theory,” *Foundations and Trends in Communications and Information Theory*, vol. 3, no. 4, pp. 331–551, 2006.