SOLUTIONS OF A TWO-COMPONENT GINZBURG-LANDAU SYSTEM

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To my parents, my husband and my angel-Haoxiang

Abstract

We study Ginzburg–Landau equations for a complex vector order parameter $\Psi = (\psi_+, \psi_-) \in \mathbb{C}^2$. In particular, we consider entire solutions in all $\mathbb{R}^2$, which are obtained by blowing up around vortices, which occur in superconductivity and Bose-Einstein condensate. An important class of entire solutions are the symmetric vortex solutions in the plane $\mathbb{R}^2$, $\psi(x) = f_\pm(r)e^{in_\pm\theta}$, with given degrees $n_\pm \in \mathbb{Z}$. We prove existence, uniqueness, and asymptotic behavior of solutions as $r \to \infty$, and we also consider the monotonicity properties of solutions. Among the entire solutions we distinguish those which are local minimizers, and we show local minimizers must have degrees $n_\pm \in \{0, \pm 1\}$. For degrees $\deg(\Psi; \infty) = [\pm 1, 0]$ or $[0, \pm 1]$, we show stable solutions have coreless vortices, with $|\Psi(x)| \geq C > 0$. Finally, we consider the stability of the equivariant solutions with degree $[1, 1]$ of the Dirichlet problem in disks $\mathbb{D}_R$, as $R \to \infty$. Based on the discussion of monotonicity of symmetric vortex solutions, we find that the sign of parameter B still plays an important role in the stability of these solutions. When $B < 0$, the equivariant solution is stable in $\mathbb{D}_R$ for any R. On the other hand, there is an interval of values of $B > 0$ for which the equivariant solution is *unstable* for all sufficiently large disks $\mathbb{D}_R$.

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Chapter 1

Introduction

1.1 Two-component Ginzburg-Landau system

We study the structure of the vortices in two-component Ginzburg-Landau functional. Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded domain and $\Psi \in H^1(\Omega; \mathbb{C}^2)$. We define an energy functional

$$E_\epsilon(\Psi; \Omega) = \int_\Omega \frac{1}{2}|\nabla\Psi|^2 + \frac{1}{4\epsilon^2}[A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 + 2B(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2)] \,, \tag{1.1}$$

where $\Psi = [\psi_+(x), \psi_-(x)] \in \mathbb{C}^2$, $A_\pm > 0$, B and $\epsilon > 0$ are parameters. Energy functionals of a form similar to E_ϵ have been introduced in some physical models, and we will briefly describe it in the next section. Although the physical models are much more complex, we expect that the essential features of the singular limit $\epsilon \to 0$ in the physical systems will be well described by the simpler energy (1.1) above.

Throughout the thesis we make the following assumptions concerning the constants appearing in (1.1):

$$A_+, A_- > 0, \; B^2 < A_+A_-, \quad t_+, t_- > 0. \tag{H}$$

By hypothesis (H), the potential term in the energy

$$F(\Psi) = A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 + 2B(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2)$$

is nonnegative and coercive, and attains its minimum (of zero) when $|\psi_\pm| = t_\pm$. To pose a more concrete example, consider (1.1) in a disk $\Omega = \mathbb{D}_R$, with appropriate Dirichlet boundary conditions $\psi_\pm\big|_{\partial\mathbb{D}_R} = t_\pm e^{iN_\pm\theta}$. As $\epsilon \to 0$, minimizers Ψ should lie on the manifold in $\mathbb{C}^2$ on which the potential $F(\Psi)$ vanishes. That manifold is a 2-torus $\Sigma \subset \mathbb{S}^3 \subset \mathbb{C}^2$, parameterized by two real phases $\Psi = [\psi_+, \psi_-] = [t_+e^{i\alpha_+}, t_-e^{i\alpha_-}]$, and thus a Σ-valued map $\Psi(x)$ carries a pair of integer-valued degrees around any closed curve C,

$$\deg(\Psi; C) = [N_+, N_-], \quad N_+ = \deg(\psi_+; C), \quad N_- = \deg(\psi_-; C).$$

If the given Dirichlet boundary condition has nonzero degree in either component, then there is no finite energy map Ψ which takes values in Σ and satisfies those boundary conditions, and we expect that *vortices* of solutions will be created in the $\epsilon \to 0$, just as in the classical Ginzburg-Landau model [BBH94]. These solutions are the main subjects of this thesis. As is typical for Ginzburg-Landau equations, by blowing up around the core of a vortex at scale ϵ, we obtain an entire solution of the following system

$$\begin{cases} -\Delta\psi_+ + [A_+(|\psi_+|^2 - t_+^2) + B(|\psi_-|^2 - t_-^2)]\psi_+ = 0, \\ -\Delta\psi_- + [A_-(|\psi_-|^2 - t_-^2) + B(|\psi_+|^2 - t_+^2)]\psi_- = 0, \end{cases} \tag{1.2}$$

in all of $\mathbb{R}^2$. They describe the local structure of solutions of (1.7) near a vortex.

Many results in the thesis will concern *entire* solutions to (1.1), that is $\Omega = \mathbb{R}^2$ with $\epsilon = 1$. The entire solutions are obtained by blowing up at scale ϵ around a vortex. If the original Ψ^ϵ is energy minimizing in Ω, by blowing up, we obtain entire solutions for which the potential energy is integrable,

$$\int_{\mathbb{R}^2} F(\Psi)\, dx < \infty. \tag{1.3}$$

By the positive definiteness of F, this suggests that the solutions we seek should attain the asymptotic values

$$|\psi_\pm(x)| \to t_\pm \quad \text{as } |x| \to \infty.$$

That is, for $|x|$ large, the solutions may be written in polar form, $\psi_\pm(x) = \rho_\pm(x)e^{i\phi_\pm(x)}$ with real-valued $\rho_\pm(x), \phi_\pm(x)$, and $\rho_\pm(x) \simeq t_\pm$. The phases $\phi_\pm(x)$ may have nontrivial winding number around any large circle C_R enclosing the origin, and we will show that these solutions carry two integer *degrees* $n_\pm = \deg(\frac{\psi_\pm}{|\psi_\pm|}, C_R) \in \mathbb{Z}$. As there are no smooth Σ-valued functions with nontrivial degrees in a simply connected domain, solutions $\Psi(x)$ with nontrivial winding must vanish in one or more of its components $\psi_\pm$ to avoid singularities. These zeros are the *vortices* of the solution.

For the classical (single-component) Ginzburg-Landau equations in $\mathbb{R}^2$,

$$-\Delta u + (|u|^2 - 1)u = 0 \ , \tag{1.4}$$

the solutions are completely known. It is a complex-valued version of the Allen-Cahn for phase transitions (see [MM77]), leading to codimension 2, instead of codimension 1, singularities (the vortices). Bethuel, Brezis and Hélein [BBH94] considered the Dirichlet problem in a bounded domain $\Omega \subset \mathbb{R}^2$, with $u\big|_{\partial\Omega} = g$, a given function with $|g| = 1$ and nontrivial degree $N = \deg(g, \partial\Omega)$. They showed that, as $\epsilon \to 0$, energy minimizing solutions have exactly N vortices of degree one, and they described their location by minimizing a renormalized energy, which is derived by sharp estimate of the interaction energy between the vortices. Applying the technique of blowing up at scale ϵ around a vortex, it produces a locally minimizing solution to (1.4) with degree d at infinity. And the associate potential to (1.4) can be quantized with their degrees at infinity [BMR94]:

$$\int_{\mathbb{R}^2} (|u|^2 - 1)^2 = 2\pi d^2 \ .$$

With the radial form of solution $u(r, \theta) = f(r)e^{id\theta}$, Hervé-Hervé [HH94] pointed out that $f(r)$ is uniquely determined by the degree d. Combining results by Shafrir [Sha94], Sandier [San98] and Mironescu [Mir96], the unique nontrivial locally minimizing solutions is (up to symmetries) the degree-one equivariant solution $u = f(r)e^{i\theta}$. Bauman-Carlson-Phillips [BCP93] proved that the minimizer with degree 1 vortex vanishes at a unique point. Millot and Pisante [MP10] showed that any local minimizer for a 3D Ginzburg-Landau energy in (entire) $\mathbb{R}^3$ with $u : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying the energy growth condition must be $u(x) \sim \frac{x}{|x|}$ near infinity up to rotations and translations. Shafrir [Sha95] refined the Bethuel-Brezis-Hélein [BBH94] result by showing that the minimizer of the associated energy functional to (1.4) in Ω is close to the rescaled entire solution in the supremum norm. Mironescu [Mir95] discussed the stability of the radial solution to (1.4) in a disk with Dirichlet boundary condition, and mentioned that the equivariant solutions with degree one is stable while unstable when the degree is higher than 2 when the scale measurement ϵ is small.

For the case of Ginzburg-Landau model with magnetic field

$$F_\epsilon = \frac{1}{2}\int_\Omega |(\nabla - iA)u|^2 + |h - h_{\text{ex}}|^2 + \frac{1}{2\epsilon^2}(1 - |u|^2)^2 \ , \tag{1.5}$$

there are many phenomena which are observed due to the presence of the magnetic field, but the results on the local structure of the vortices are the same as for the simpler system (1.1). Given ϵ, the behavior of minimizers and critical points to (1.5) is determined by the value of the external field h_{ex}. There are three main critical values of h_{ex} or *critical fields* H_{c_1}, H_{c_2} and H_{c_3}, for which phase-transitions accour. At H_{c_1}, the first vortice(s) appear. Alama-Bronsard-Giorgi [ABG99] proved the uniqueness of N-vortex radially symmetric solution to (1.5) under the condition $\epsilon^{-2} \geq 2N^2$. Sandier and Serfaty [SS07] showed that normal state solution becomes minimizing in the full domain $\mathbb{R}^2$ at critical field H_{c_1} for (1.5). Giorgi and Phillips [GP99] presented that the normal states are the only solution to (1.5) when magnetic field of order ϵ^{-2}.

1.2 Physical motivation

In a model of two-component BEC from Eto *et al.* [EKN+11], we consider a pair of complex wave functions $\Phi = (\varphi_1, \varphi_2) \in \mathbb{C}^2$, defined in the sample domain $\Omega \subset \mathbb{R}^2$. The energy of the configuration is defined as

$$E(\varphi_1, \varphi_2) = \int_\Omega \left[\frac{\hbar^2}{2m_1} |\nabla \varphi_1|^2 + \frac{\hbar^2}{2m_2} |\nabla \varphi_2|^2 + \frac{1}{2} \left(g_1 |\varphi_1|^4 + g_2 |\varphi_2|^4 + 2 g_{12} |\varphi_1|^2 |\varphi_2|^2 \right) \right] dx,$$

where $m_1, m_2 > 0$ are the masses, and the coupling constants satisfy the positivity condition $g_1 g_2 - g_{12}^2 > 0$. The Gross-Pitaevskii equations govern the dynamics of the condensate,

$$\left. \begin{aligned} i\hbar \partial_t \varphi_1 &= -\frac{\hbar^2}{2m_1} \Delta \varphi_1 + g_1 |\varphi_1|^2 \varphi_1 + g_{12} |\varphi_2|^2 \varphi_1, \\ i\hbar \partial_t \varphi_2 &= -\frac{\hbar^2}{2m_2} \Delta \varphi_2 + g_{12} |\varphi_1|^2 \varphi_2 + g_2 |\varphi_2|^2 \varphi_2. \end{aligned} \right\} \tag{1.6}$$

A stationary equation of the desired form is obtained by considering standing wave solutions, $\varphi_i(x,t) = e^{-i\mu_i t/\hbar} u_i(x)$, $i = 1, 2$, where μ_i represent the chemical potentials:

$$\begin{aligned} -\frac{\hbar^2}{2m_1} \Delta \varphi_1 + g_1 |\varphi_1|^2 \varphi_1 + g_{12} |\varphi_2|^2 \varphi_1 &= \mu_1 \varphi_1, \\ -\frac{\hbar^2}{2m_2} \Delta \varphi_2 + g_{12} |\varphi_1|^2 \varphi_2 + g_2 |\varphi_2|^2 \varphi_2 &= \mu_2 \varphi_2. \end{aligned}$$

In the variational formulation of the stationary problem, the chemical potentials represent Lagrange multipliers, which arise because of the constraints on the masses of the two condensate species,

$$\int_\Omega |\varphi_i|^2 dx = \int_\Omega |u_i|^2 dx = N_i, \quad i = 1, 2.$$

By a rescaling of the dependent variables, $\psi_+ = \sqrt[4]{\frac{m_2}{m_1}}\, u_1$, $\psi_- = \sqrt[4]{\frac{m_1}{m_2}}\, u_2$, we may eliminate the masses m_i from the equations, and we obtain the system

$$\left\{ \begin{aligned} -\epsilon^2 \Delta \psi_+ + [A_+(|\psi_+|^2 - t_+^2) + B(|\psi_-|^2 - t_-^2)]\psi_+ = 0, \\ -\epsilon^2 \Delta \psi_- + [A_-(|\psi_-|^2 - t_-^2) + B(|\psi_+|^2 - t_+^2)]\psi_- = 0, \end{aligned} \right. \tag{1.7}$$

with $\epsilon^2 = \frac{\hbar^2}{\sqrt{m_1 m_2}}$, $A_+ = \frac{m_1}{m_2} g_1$, $A_- = \frac{m_2}{m_1} g_2$, $B = g_{12}$, and

$$t_+^2 = \frac{\mu_1 g_2 - \mu_2 g_{12}}{g_1 g_2 - g_{12}^2} \sqrt{\frac{m_2}{m_1}}, \qquad t_-^2 = \frac{\mu_2 g_1 - \mu_1 g_{12}}{g_1 g_2 - g_{12}^2} \sqrt{\frac{m_1}{m_2}}.$$

These equations are exactly the Euler-Lagrange equations corresponding to the energy (1.1). Following the procedure in Proposition 3.12 of [SS07], by blowing up around $\widetilde{\Psi}^\epsilon(x) = \Psi^\epsilon(p_\epsilon + \epsilon x)$, we obtain an entire solution of (1.2)

$$\begin{cases} -\Delta\psi_+ + [A_+(|\psi_+|^2 - t_+^2) + B(|\psi_-|^2 - t_-^2)]\psi_+ = 0, \\ -\Delta\psi_- + [A_-(|\psi_-|^2 - t_-^2) + B(|\psi_+|^2 - t_+^2)]\psi_- = 0. \end{cases}$$

In a more physically appropriate model for a two-component BEC [KTU03], the Laplacian in the Gross-Pitaevskii system (1.6) should be replaced by the Hamiltonians

$$H_i := -\frac{\hbar^2}{2m_i}\Delta + V_i^{trap} - \omega L_z, \qquad i = 1, 2,$$

with harmonic trapping potentials $V_i^{trap} = c_i^2|x|^2/2$, c_i constant $i = 1, 2$; angular momentum operator L_z; and (constant) angular speed Ω. While this is an essential step, both in modeling the confinement of the condensate and in describing the onset of vortices in the sample, these terms will not affect the general form (1.7) of the blow-up equations which describe the vortex profiles at length scale ϵ in the condensate. Indeed, the momentum operator plays much the same role as the magnetic vector potential in the GL model of superconductivity, and for rotations which are of moderate strength in ϵ, $\omega \ll \epsilon^{-1}$, the analysis of Proposition 3.12 of [SS07] may be used to derive (1.7) in limit $\epsilon \to 0$ after rescaling.

Some physics literature have introduced spin-coupled (or spinor) Ginzburg-Landau models for complex vector-valued order parameters in order to account for ferromagnetic (or antiferromagnetic) effects in high-temperature superconductors [KR98] and in optically confined Bose-Einstein condensates [IM02]. A change of variables in this models leads to (1.1) with balanced coefficients $A_+ = A_- = 1$, $t_\pm^2 = \frac{1}{2}$. A series of papers

[AB06], [ABM09] treat (1.1) and entire solutions in this special case. Aftalion-Mason-Wei [AMW] studied stable solutions of the two-component Bose-Eistein condensate due to the trapping potential and estimated the vortex-peak energy according to the parameters of the system. They derived an energy depending on the location of vortices and peaks and determined for which values of the experimental parameters, the lattice goes from triangular to square. These critical values agrees well with the ones found from the numerical computations of the full Gross-Pitaevskii equations of [MA11]. Berestycki-Lin-Wei-Zhao [BLWZ13] studied the bound state solutions of a class of two-component nonlinear elliptic systems with a large parameter tending to infinity. They obtained the separation of supports of ψ_+ and ψ_- in the case $B \gg \sqrt{A_+ A_-}$, which is a very different regime from the one treated in this thesis.

1.3 Main results

This thesis is primarily concerned with the existence, uniqueness, stability and other properties of solutions to (1.1). Our goal eventually is to characterize *all* solutions, or at least all *stable* solutions or we call it as *locally minimizing* solutions.

1.3.1 Symmetric vortex

In Chapter 2 we consider special entire solutions to (1.2). They are obtained by an equivariant ansatz, $\psi_\pm(x) = f_\pm(r)e^{in_\pm\theta}$ in polar coordinates (r, θ) in $\mathbb{R}^2$, with $f_\pm$ a pair of real-valued functions and given degree pair $[n_+, n_-] \in \mathbb{Z}^2$. By taking complex conjugates if necessary, we may assume that $n_\pm \geq 0$. By the equivariant ansatz, the associated system can be reduced to a system of ODEs. In the spirit of [ABM09], we show that for each fixed choice of degrees $n_\pm$ at infinity, there exist unique equivariant entire solutions satisfying (1.3). To determine the shape of the vortex profiles, we consider the asymptotic form of the solutions for $r \to \infty$ and prove that

$$f_\pm(r) = t_\pm + \frac{a_\pm}{r^2} + \frac{b_\pm}{r^4} + O(r^{-6}), \quad f'_\pm(r) = -\frac{2a_\pm}{r^3} + O(r^{-5}), \quad \text{as} \;\; r \to \infty, \tag{1.8}$$

with

$$a_\pm = \frac{1}{2}\frac{Bn_\mp^2 - A_\mp n_\pm^2}{(A_+A_- - B^2)t_\pm}, \tag{1.9}$$

and (rather complicated) constants $b_\pm$ given in (2.33). A formal asymptotic expansion of this form (in fact, an expansion to arbitrary order in $1/r$) may be obtained by simply substituting an ansatz into the system of equations and matching terms. Our results, presented in Theorem 2.4, provide rigorous confirmation of this expansion by means of sub-supersolution construction (Maple assisted) motivated by [CEQ94]. Moreover, we prove that the expansion is *uniform* in the coefficients $A_\pm, B, t_\pm$ lying in a compact set. The proof is completed using a new and original comparison principle (Lemma 2.3) for elliptic systems, which generalizes the one in [AB06].

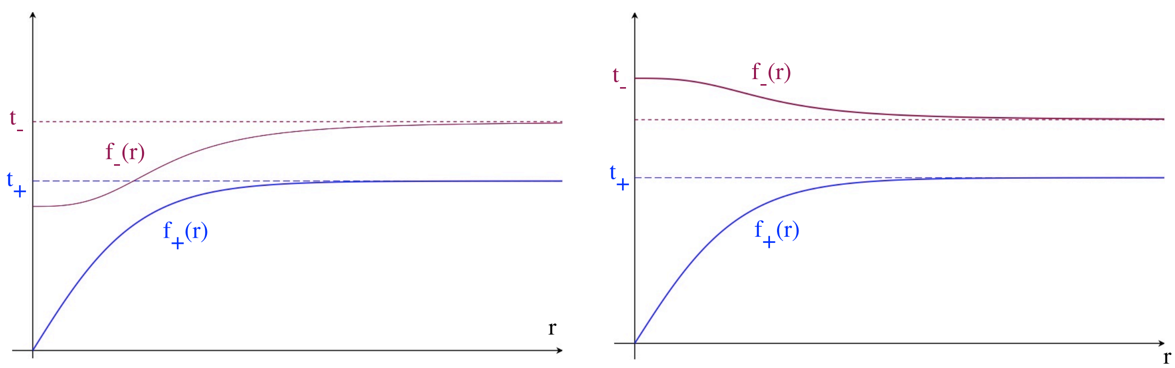
From the asymptotics (1.8), we see that the shape of the solutions depends strongly on the coefficients, in particular the sign of the interaction coefficient B. For $B < 0$, both components approach their limiting value $t_\pm$ from below, as is familiar from the classical GL vortices. However, for $B > 0$, this may no longer be the case, and for certain choices of $n_\pm$ and B one of the components will approach its limiting value from above. Such behavior was already noted in [ABM09] in the case $n_- = 0$, for a "balanced" system, $A_+ = A_-$, $t_+ = t_-$. Even in the case $n_\pm \neq 0$, our result implies that there are parameter regimes in which vortex profiles will be non-monotone. For the standard GL vortices, the vortex profile is known to be strictly monotone increasing in r. As suggested by the asymptotic expansion above, the validity of this property is strongly dependent on the value of B. We prove the following:

Theorem 1.1. *Let $A_+, A_- > 0$ be fixed, and B such that $B^2 < A_+A_-$. Assume $\Psi(x; B) = [f_+(r; B)e^{in_+\theta}, f_-(r; B)e^{in_-\theta}]$ is the equivariant solution for those parameters $A_\pm, B$.*

(i) *If* $B < 0$*, then* $f'_{\pm}(r; B) \geq 0$ *for all* $r > 0$ *for any degree* $[n_+, n_-]$*.*

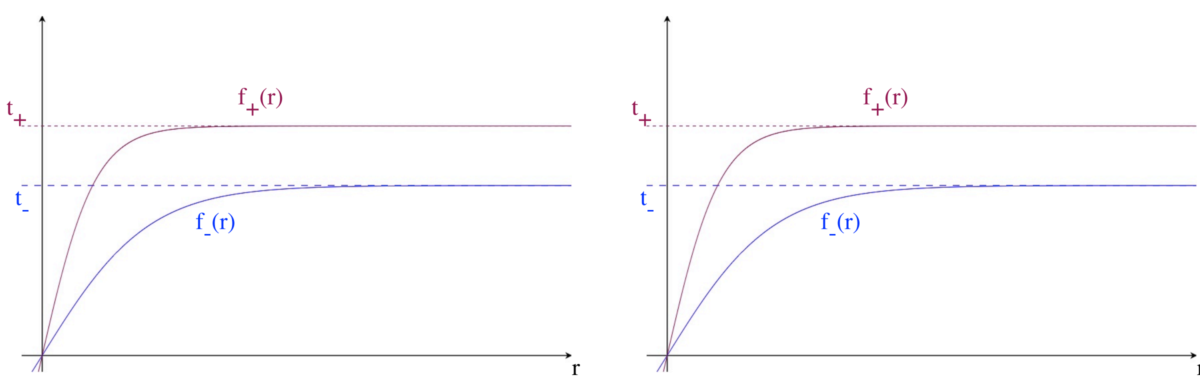
(ii) *If* $B > 0$*,* $n_+ \geq 1$ *and* $n_- = 0$*, then* $f'_+(r; B) \geq 0$ *and* $f'_-(r; B) \leq 0$ *for all* $r > 0$*.*

(iii) *For any pair* $[n_+, n_-]$ *with* $n_+ \neq 0 \neq n_-$*, there exists* $B_0 > 0$ *such that* $f'_{\pm}(r; B) \geq 0$ *for all* $r > 0$ *and all* B *with* $0 \leq B \leq B_0$*.*



$B < 0$ $\qquad$ $B > 0$

Figure 1.1: For the case $n_+ \geq 1$, $n_- = 0$



$B < 0$ $\qquad$ $0 < B < B_0$

Figure 1.2: For the case $n_+ \neq 0 \neq n_-$

We observe that the leading order term (1.9) in the asymptotic expansion suggests that the optimal value of B_0 in (iii) above is $\min\left\{A_-\frac{n_+^2}{n_-^2},\ A_+\frac{n_-^2}{n_+^2}\right\}$. Our proof of (iii) depends on perturbation of the case $B = 0$, by means of a compactness argument. The relationship between B_0 and the other parameters remains an open question.

1.3.2 Properties of entire solution

In Chapter 3 we consider the properties of general (not necessarily radial) entire solution to (1.2) satisfying (1.3). We prove that $\psi_\pm \sim t_\pm e^{i(n_\pm\theta+\beta_\pm)}$ uniformly outside of large ball B_R. With an improved sub-supersolution construction and comparison principle similar to the one in [BBH93] and [AB06], this uniform convergence is also true in $\mathcal{C}^k-$norm (Lemma 6.1). As was the case for the classical Ginzburg-Landau equations [BMR94], we also establish a phenomenon of quantization for the "mass" $\int_{\mathbb{R}^2} F(\Psi)$ (see Theorem 3.1), namely

Theorem 1.2. *Let (for any choice of* $[n_+, n_-]$*)* $\Psi = [\psi_+, \psi_-]$ *be a solution of* (1.2) *satisfying* (1.3)*. Then*

$$\int_{\mathbb{R}^2} \{A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 + 2B(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2)\}dx$$
$$= 2\pi(n_+^2 t_+^2 + n_-^2 t_-^2).$$

Measured on all of the $\mathbb{R}^2$, the energy defined in (1.1) with $\epsilon = 1$ of such a solution diverges. However, when properly renormalized, there is a well-defined core energy defined as a limit (see Lemma 3.2):

Proposition 1.3. *Let* Ψ *solve* (1.2) *in* $\mathbb{R}^2$ *satisfying* (1.3)*. Then, the following limit exists:*

$$\lim_{R\to\infty} [E(\Psi; \mathbb{D}_R) - \pi(t_+^2 n_+^2 + t_-^2 n_-^2) \ln R].$$

For entire solutions Ψ which are *stable* we expect more. We define locally minimizing solutions in $\mathbb{R}^2$ in the sense of De Giorgi: we say that Ψ is a *locally minimizing* *solution* of (1.2) if (1.3) holds and if for every bounded regular domain $\Omega \subset \mathbb{R}^2$,

$$E(\Psi; \Omega) \le E(\Phi; \Omega)$$

holds for every $\Phi = (\phi_+, \phi_-) \in H^1(\Omega; \mathbb{C}^2)$ with $\Phi\big|_{\partial\Omega} = \Psi\big|_{\partial\Omega}$. In the case of the single Gingzburg-Landau equation (1.4), it is known from Shafrir [Sha94] that the only nontrivial solutions which are locally minimizers for the associated energy to (1.4), have degree ± 1 at infinity. We show that the same is true for Ψ (Proposition 4.2):

Proposition 1.4. *A nontrivial local minimizer of* (1.2) *must have degrees* $n_\pm \in \{0, \pm 1\}$.

The 2-component system offers a larger variety of potentially local minimizing solutions (see [AMW]), we must consider degrees at infinity $\deg(\Psi; \infty) = [\pm 1, \pm 1], [\pm 1, 0]$ and $[0, \pm 1]$. The $[n_+, n_-] = [1, 0]$ or $[0, 1]$ vortices are very interesting, and different from usual GL vortices. Since one component has degree zero, that component is not required to vanish at any point. In Theorem 4.3 we show, for a class of systems including (1.2) that indeed $|\Psi(x)|$ is bounded away from zero in all $\mathbb{R}^2$. This is called a "coreless" vortex (see [AMW]) in physics. Another open question is to determine if the radial solution with $[n_+, n_-] = [1, 0]$ is in fact a local minimizer, and furthermore to show whether it is the *unique* local minimizing solution.

These results depend heavily of the strong $\mathcal{C}^k_{\text{loc}}$-convergence of solutions away from vortices. Energy bounds are not sufficient to obtain the sharp pointwise estimates needed to prove these results, and so we use elliptic regularity theory as in [BBH93] and some comparison principles for elliptic systems to improve the convergence of solutions. These methods are highly technical, and we include them in Chapter 5.

1.3.3 Stability & Instability in a disk

Chapter 4 concerns the stability of the equivariant solutions with degree $[1, 1]$ of the Dirichlet problem in the unit disk $\mathbb{D}_1$. Notice that the stability in dynamics depends on the type of evolution equation chosen for the model. For nonlinear Schrödinger evolutions, for example, there are many different definitions of stability (see [Pel11]). For convenience, we replace the usual parameter ϵ by $\lambda = \epsilon^{-2}$ in E_ϵ, write the energy functional as

$$E_\lambda(\Psi) = \int_{\mathbb{D}_1} \frac{1}{2}|\nabla\Psi|^2 + \frac{\lambda}{4}\left[A_+(|\psi_+|^2 - t_+^2)^2 \right.$$
$$\left. + A_-(|\psi_-|^2 - t_-^2)^2 + 2B(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2)\right] ,$$

and minima (or more generally, critical points) of E_λ over the space $\mathbf{H}$, consisting of all functions $\Psi \in H^1(\mathbb{D}_1; \mathbb{C}^2)$ with the symmetric boundary condition:

$$\Psi\Big|_{\partial\mathbb{D}_1} = [t_+e^{i\theta}, t_-e^{i\theta}]. \tag{1.10}$$

The associated Euler-Lagrange equations to E_λ with boundary condition (1.10) is as follows:

$$\begin{cases} -\Delta\psi_+ + \lambda[A_+(|\psi_+|^2 - t_+^2) + B(|\psi_-|^2 - t_-^2)]\psi_+ = 0, & \text{in } \mathbb{D}_1, \\ -\Delta\psi_+ + \lambda[A_-(|\psi_-|^2 - t_-^2) + B(|\psi_+|^2 - t_+^2)]\psi_- = 0, & \text{in } \mathbb{D}_1, \\ \psi_\pm = t_\pm e^{i\theta}, & \text{on } \partial\mathbb{D}_1. \end{cases} \tag{1.11}$$

We note that rescaling by $R = \sqrt{\lambda}$, this Dirichlet problem is equivalent to the Dirichlet problem in a very large disk $\mathbb{D}_R$, as $R \to \infty$ when $\epsilon \to 0$. Thus, this problem is also an approximation to the stability problem for entire solutions in all $\mathbb{R}^2$.

There are several previous papers on stability and bifurcation for classical Ginzburg-Landau equation. [Mir95] showed that the equivariant solutions lose their stability property when degree $d \geq 2$ and ϵ is small enough. As proved in [HH94], the radial solutions form a regular branch in $\mathcal{C}^k$ with respect to the parameter ϵ. Comte and

Mironescu [CM98] proved that the loss of stability (see [Mir95]) leads to the appearance of a bifurcation from this branch.

The two-component case was studied by [ABM09], in the "balanced" case, $A_\pm = 1$, $t_\pm^2 = \frac{1}{2}$. Analogous to the arguments in [ABM], we note that the stability of degree $[1,1]$ vortex to (1.11) depends on the parameter λ. Based on the discussion of monotonicity of radial solutions, we find that the sign of parameter B still plays an important role in the stability of degree one-one equivariant solution to (1.11). A detailed description of stability for equivariant solutions $\Psi = [f_+ e^{i\theta}, f_- e^{i\theta}]$ is given in Theorem 5.2 which follows the approaches in [Mir95]. On the other hand, motivated by the previous work for classical Ginzburg-Landau functional (see Mironescu [Mir95]), we use the analysis of the linearization of the energy functional E_λ to study the instability of equivariant solutions $\Psi = [f_+ e^{i\theta}, f_- e^{i\theta}]$ in a positive interval of parameter B (see Theorem 5.9). The main theorem is as follows:

Theorem 1.5. *Let $A_+, A_- > 0$ be fixed, and B such that $B^2 < A_+ A_-$. Assume $\Psi(x;R) = [f_+(r;R)e^{i\theta}, f_-(r;R)e^{i\theta}]$ is the equivariant solution for those parameters $A_\pm, B$ to* (1.11).

(i) *If $B < 0$, then $\Psi(x;R)$ is stable $\forall R > 0$, in the sense $E''_\lambda(\Psi)[\Phi] > 0$ $\forall \Psi \in H_0^1(\mathbb{D}_1; \mathbb{C}^2)$.*

(ii) *For any $B \in (0, B_0)$ with B_0 as in Theorem 1.1, there exists a unique constant $R_* = R_*(B) > 0$ such that $\Psi(x;R)$ is unstable for any $R > R_*$.*

Theorem 5.2 generalizes results of [ABM], which were restricted to the "balanced" case. When $A_\pm = 1$, $t_\pm^2 = \frac{1}{2}$ the system has additional symmetry properties, which enable the authors to reduce the problem of stability from a system to a single complex-valued equation. The results of [ABM] are therefore sharper, and they prove that when $B < 0$, the radial solution is *unique*. Note that when $B > 0$ it implies that a vortex of degree $[n_+, n_-] = [1, 1]$ is *not* radially symmetric, it must have non-coincident zeros in its two components, $\psi_\pm$.

Following [ABM], it is natural to suppose that bifurcation occurs at $R_*(B)$, when $B \in (0, B_0)$. As in [ABM], the unstable eigenfunctions correspond to separating the single $[1, 1]$ vortex at the origin in two antipodal vortices with degree $[n_+, n_-] = [1, 0]$ and $[0, 1]$, and so we conjecture that the same separation phenomenon holds in the more general case as well. This remains an interesting and challenging open question.

Chapter 2

Radial Solutions

In this chapter we consider radial solutions of the equations (1.2) of the form

$$\psi_+(x) = f_+(r)e^{in_+\theta}, \quad \psi_-(x) = f_-(r)e^{in_-\theta} \ ,$$

in polar coordinate (r, θ) with given degree pair $[n_+, n_-] \in \mathbb{Z}^2$. By taking complex conjugates if necessary, we may assume that $n_\pm \geq 0$. By the form of radial solution, we can reduce the Laplacian term into the sum of the following terms:

$$\begin{aligned} \triangle\psi_\pm &= \partial_{rr}\psi_\pm + \frac{1}{r}\partial_r\psi_\pm + \frac{1}{r^2}\partial_{\theta\theta}\psi_\pm \\ &= f''_\pm(r)e^{in_\pm\theta} + \frac{1}{r}f'_\pm(r)e^{in_\pm\theta} - \frac{n_\pm^2}{r^2}f_\pm e^{in_\pm\theta} \ . \end{aligned}$$

Therefore, when $f \geq 0$, the system (1.2) reduces to the following system of ODEs:

$$\left.\begin{aligned} -f''_+ - \frac{1}{r}f'_+ + \frac{n_+^2}{r^2}f_+ + \left[A_+(f_+^2 - t_+^2) + B(f_-^2 - t_-^2)\right] f_+ = 0, \quad \text{for } r \in (0, \ \infty), \\ -f''_- - \frac{1}{r}f'_- + \frac{n_-^2}{r^2}f_- + \left[A_-(f_-^2 - t_-^2) + B(f_+^2 - t_+^2)\right] f_- = 0, \quad \text{for } r \in (0, \ \infty), \\ f_\pm(r) \geq 0 \text{ for all } r \in [0, \ \infty), \\ f_\pm(r) \to t_\pm \ \text{ as } \ r \to \infty, \\ f_\pm(0) = 0 \ \text{ if } \ n_\pm \neq 0; \qquad f'_\pm(0) = 0 \ \text{ if } \ n_\pm = 0. \end{aligned}\right\} \tag{2.1}$$

2.1 Existence and uniqueness

We begin with their existence and uniqueness.

Lemma 2.1. *Let $n_\pm \in \mathbb{Z}$ be given and $A_+A_- - B^2 > 0$. Then there exists a unique solution $[f_+(r),\ f_-(r)]$ to* (2.1) *for $r \in [0,\ \infty)$ such that:*

$$f_\pm \in C^\infty\left((0,\ \infty)\right), \tag{2.2}$$

$$f_\pm(r) > 0 \quad \textit{for all } r > 0, \tag{2.3}$$

$$f_\pm(r) \sim r^{n_\pm} \ \textit{for } r \sim 0. \tag{2.4}$$

In particular, $\Psi(x) = [f_+(r)e^{in_+\theta},\ f_-(r)e^{in_-\theta}]$ is an entire solution of (1.2) *in $\mathbb{R}^2$ satisfying* (1.3).

Proof. To obtain the existence we consider the problem defined in the ball $B_R,\ R > 0$,

$$\begin{cases} -f_\pm'' - \dfrac{1}{r}f_\pm' + \dfrac{n_\pm^2}{r^2}f_\pm + \left[A_\pm(f_\pm^2 - t_\pm^2) + B(f_\mp^2 - t_\mp^2)\right]f_\pm = 0, \quad \text{for } 0 < r < R, \\ f_\pm(R) = t_\pm, \\ f_\pm(0) = 0 \ \text{ if } \ n_\pm \neq 0, \qquad f_\pm'(0) = 0 \ \text{ if } \ n_\pm = 0. \end{cases} \tag{2.5}$$

The existence of such a solution follows easily by minimization of the energy

$$\begin{aligned} &E^R_{n_+,\ n_-}(f_+,\ f_-) \\ &\quad = \frac{1}{2}\int_0^R \left\{ \sum_{i=\pm} \left[(f_i')^2 + \frac{n_\pm^2}{r^2}f_i^2 + \frac{1}{2}[A_i(f_i^2 - t_i^2)^2 + 2B(f_+^2 - t_+^2)(f_-^2 - t_-^2)] \right] \right\} rdr\ , \end{aligned} \tag{2.6}$$

over Sobolev functions satisfying the appropriate boundary conditions at $r = 0$ and $r = R$. Denote $[f_{R,+}(r),\ f_{R,-}(r)]$ as any solution of (2.5). By part (i) of Lemma 6.3, we have that $f_+^2(r) + f_-^2(r) \leq \Lambda$ for any solution to (2.1). Therefore, it follows that $f_{R,\pm} \in W^{2,p}_{\text{loc}}$ by L^p-estimate. Applying the Sobolev embedding, we have $f_{R,\pm} \in \mathcal{C}^1_{\text{loc}}$. By the standard elliptic estimates, $f_{R,\pm}$ is bounded in $\mathcal{C}^k_{\text{loc}}$ for $\forall k$ and there exists a subsequence $R_n \to \infty$ for which the solution $[f_{R,+}(r),\ f_{R,-}(r)] \to [f_{\infty,+}(r),\ f_{\infty,-}(r)]$ in $\mathcal{C}^{1,\alpha}_{\text{loc}}[0,\infty)$, and the limit functions $[f_{\infty,+}(r),\ f_{\infty,-}(r)]$ give (weak) solutions to the ODE on $(0,\infty)$ with the same boundary condition at $r=0$. And when r is near 0, the ODE maybe written in the form as:

$$-f''_{\pm} - \frac{f'_{\pm}}{r} + \frac{n_{\pm}^2}{r^2} f_{\pm} = g(r),$$

an ODE with singular point at $r=0$. By the Frobenius theory (see Appendix 3 of [BC89] or Lemma 5.9 of [ABM]), we deduce that the behavior $f_{\infty,\pm} \sim r^{n_{\pm}}$ near $r=0$. As a consequence (see Theorem 3.1 of [BC89]), we may conclude that $\psi_{\pm}(x) = f_{\infty,\pm}(r)e^{in_{\pm}\theta}$ is regular at $r=0$ and solves (1.2) in $\mathbb{R}^2$.

On the other hand, since $E^R_{n_+,\ n_-}(|f_+|,|f_-|) = E^R_{n_+,\ n_-}(f_+,\ f_-)$, it yields that $f_{\pm} \geq 0$ in $[0,R]$. By the local uniform convergence of $f_{\pm}$, we have that the limit $f_{\infty,\pm} \geq 0$ on $[0,\infty)$. Now suppose $f_{\pm}(r_0)=0$ be an interior minimum at some point $r_0 \in (0,\infty)$. Then $f_{\pm}(r_0) = 0 = f'_{\pm}(r_0)$, so $f_{\pm}(r_0) \equiv 0$ by uniqueness of solution to the initial-value problem, which is a contradiction. Hence, $f_{\pm} > 0$ in $(0,\infty)$. In order to obtain the existence, it suffices to establish (1.3). For this purpose, we derive a Pohozaev identity: we multiply the equation of $f_{R,\pm}$ by $r^2 f'_{R,\pm}(r)$ and integrate by parts with respect to $r \in (0,R)$. We get that

$$\begin{aligned} &-\frac{1}{2}[Rf'_{\pm}(R)]^2 + \frac{1}{2}n_{\pm}^2 t_{\pm}^2 - \frac{1}{2}A_{\pm}\int_0^R (f_{\pm}^2 - t_{\pm}^2)^2 r\,dr \\ &\qquad - B\int_0^R (f_{\pm}^2 - t_{\pm}^2) f_{\mp} f'_{\mp} r^2\,dr - B\int_0^R (f_{\pm}^2 - t_{\pm}^2)(f_{\mp}^2 - t_{\mp}^2) r\,dr = 0. \end{aligned} \tag{2.7}$$

Adding above $f_{\pm}$ identities together, we obtain the Pohozaev identity:

$$\begin{aligned} &\frac{1}{2}[Rf'_+(R)]^2 + \frac{1}{2}[Rf'_-(R)]^2 + \frac{1}{2}\int_0^R \left\{A_+(f_+^2 - t_+^2)^2 + A_-(f_-^2 - t_-^2)^2\right\} r\,dr \\ &\quad + 2B\int_0^R (f_+^2 - t_+^2)(f_-^2 - t_-^2) r\,dr + B\int_0^R \left[(f_+^2 - t_+^2) f_- f'_- + (f_-^2 - t_-^2) f_+ f'_+\right] r^2\,dr \\ &\quad = \frac{1}{2}(n_+^2 t_+^2 + n_-^2 t_-^2). \end{aligned} \tag{2.8}$$

Integrate by parts again and simplify the left side of above identity, we obtain that

$$\frac{1}{2}[Rf_+'(R)]^2 + \frac{1}{2}[Rf_-'(R)]^2 + \frac{1}{2}\int_0^R \left\{A_+(f_+^2 - t_+^2)^2 + A_-(f_-^2 - t_-^2)^2\right\} rdr \\ +2B\int_0^R (f_+^2 - t_+^2)(f_-^2 - t_-^2)rdr - B\int_0^R (f_+^2 - t_+^2)(f_-^2 - t_-^2)rdr = \frac{1}{2}(n_+^2 t_+^2 + n_-^2 t_-^2), \tag{2.9}$$

i.e.

$$[Rf_+'(R)]^2 + [Rf_-'(R)]^2 + \int_0^R \left\{A_+(f_+^2 - t_+^2)^2 + A_-(f_-^2 - t_-^2)^2\right\} rdr \\ +2B\int_0^R (f_+^2 - t_+^2)(f_-^2 - t_-^2)rdr = n_+^2 t_+^2 + n_-^2 t_-^2, \tag{2.10}$$

which is the Pohozaev identity. By uniform convergence on $[0, R_0]$ for any $R_0 > 0$ we have

$$\int_0^{R_0} \left\{A_+(f_+^2 - t_+^2)^2 + A_-(f_-^2 - t_-^2)^2 + 2B(f_+^2 - t_+^2)(f_-^2 - t_-^2)\right\} rdr \le n_+^2 t_+^2 + n_-^2 t_-^2,$$

and letting $R_0 \to \infty$ we establish the condition (1.3). This completes the existence of Lemma 2.1.

To show the uniqueness we use the idea of Brezis and Oswald [BO86]. Let $[n_+, n_-] \in \mathbb{Z}^2$ to be given, and suppose $[f_+,\ f_-]$ and $[g_+,\ g_-]$ are two solutions of (2.1). Denote by $\triangle_r f := \frac{1}{r}(rf'(r))'$ in the Laplacian for radial functions. Then we have:

$$-\frac{\triangle_r(f_\pm)}{f_\pm} + \frac{n_\pm^2}{r^2} f_\pm + A_\pm(f_\pm^2 - t_\pm^2) + B(f_\mp^2 - t_\mp^2) = 0,$$

$$-\frac{\triangle_r(g_\pm)}{g_\pm} + \frac{n_\pm^2}{r^2} g_\pm + A_\pm(g_\pm^2 - t_\pm^2) + B(g_\mp^2 - t_\mp^2) = 0,$$

therefore we get

$$-\frac{\triangle_r(f_+)}{f_+} + \frac{\triangle_r(g_+)}{g_+} = -A_+(f_+^2 - g_+^2) - B(f_-^2 - g_-^2), \tag{2.11}$$

$$-\frac{\triangle_r(f_-)}{f_-} + \frac{\triangle_r(g_-)}{g_-} = -A_-(f_-^2 - g_-^2) - B(f_+^2 - g_+^2). \tag{2.12}$$

We multiple (2.11) by $f_+^2 - g_+^2$ and integrate over $0 < r < \infty$. Since $\psi_\pm(x) = f_\pm(r)e^{in_\pm\theta}$ defines a solution of the system (1.2) satisfying (1.3), the estimates of Proposition 6.1 and Corollary 6.6 hold for $f_\pm, g_\pm$, we have

$$\begin{aligned}
&\int_0^\infty \left[-\frac{\triangle_r(f_+)}{f_+} + \frac{\triangle_r(g_+)}{g_+}\right](f_+^2 - g_+^2)rdr \\
&\quad= \int_0^\infty \left[-(rf_+')'f_+ - (rg_+')'g_+ + (rf_+')'\frac{g_+^2}{f_+} + (rg_+')'\frac{f_+^2}{g_+}\right]dr \\
&\quad= \int_0^\infty \left[(f_+')^2 + (g_+')^2\right]rdr - \int_0^\infty 2\left[\frac{g_+}{f_+}f_+'g_+' + \frac{f_+}{g_+}f_+'g_+'\right]rdr \\
&\qquad+ \int_0^\infty \left[(f_+')^2\frac{g_+^2}{f_+^2} + (g_+')^2\frac{f_+^2}{g_+^2}\right]rdr \\
&\quad= \int_0^\infty \left[(f_+')^2 - 2\frac{f_+}{g_+}f_+'g_+' + \frac{f_+^2}{g_+^2}(g_+')^2\right]rdr + \int_0^\infty \left[(g_+')^2 - 2\frac{g_+}{f_+}f_+'g_+' + \frac{g_+^2}{f_+^2}(f_+')^2\right]rdr \\
&\quad= \int_0^\infty \left|f_+' - \frac{f_+}{g_+}g_+'\right|rdr + \int_0^\infty \left|g_+' - \frac{g_+}{f_+}f_+'\right|rdr \\
&\quad= \int_0^\infty \left[A_+(f_+^2 - g_+^2)^2 + B(f_+^2 - g_+^2)(f_-^2 - g_-^2)\right]rdr,
\end{aligned} \tag{2.13}$$

i.e.

$$\begin{aligned}
&\int_0^\infty \left|f_+' - \frac{f_+}{g_+}g_+'\right|rdr + \int_0^\infty \left|g_+' - \frac{g_+}{f_+}f_+'\right|rdr \\
&\quad= -\int_0^\infty \left[A_+(f_+^2 - g_+^2)^2 + B(f_+^2 - g_+^2)(f_-^2 - g_-^2)\right]rdr.
\end{aligned} \tag{2.14}$$

Similarly, multiplying $f_-^2 - g_-^2$ to (2.12) and integrating over $0 < r < \infty$, we also have

$$\begin{aligned}
&\int_0^\infty \left|f_-' - \frac{f_-}{g_-}g_-'\right|rdr + \int_0^\infty \left|g_-' - \frac{g_-}{f_-}f_-'\right|rdr \\
&\quad= -\int_0^\infty \left[A_-(f_-^2 - g_-^2)^2 + B(f_+^2 - g_+^2)(f_-^2 - g_-^2)\right]rdr.
\end{aligned} \tag{2.15}$$

As in [BO86] adding above identities together we obtain that

$$\begin{aligned}
0 &\le \int_0^\infty \left|f_+' - \frac{f_+}{g_+}g_+'\right|rdr + \int_0^\infty \left|g_+' - \frac{g_+}{f_+}f_+'\right| + \left|f_-' - \frac{f_-}{g_-}g_-'\right|rdr \\
&\quad+ \int_0^\infty \left|g_-' - \frac{g_-}{f_-}f_-'\right|rdr \\
&= -\int_0^\infty \left\{A_+(f_+^2 - g_+^2)^2 + A_-(f_-^2 - g_-^2)^2 + 2B(f_+^2 - g_+^2)(f_-^2 - g_-^2)\right\}rdr.
\end{aligned} \tag{2.16}$$

Since we have the fact $A_+A_- - B^2 > 0$, it follows that

$$A_+u^2 + 2Buv + A_-v^2 \ge \lambda_s(u^2 + v^2),$$

where $\lambda_s > 0$ is the smallest eigenvalue of the matrix $\begin{bmatrix} A_+ & B \\ B & A_- \end{bmatrix}$. Hence,

$$A_+(f_+^2 - g_+^2)^2 + A_-(f_-^2 - g_-^2)^2 + 2B(f_+^2 - g_+^2)(f_-^2 - g_-^2) \\ \geq \lambda_s \left[(f_+^2 - g_+^2)^2 + (f_-^2 - g_-^2)^2\right] \geq 0.$$

Therefore,

$$\begin{aligned} 0 &\leq -\int_0^\infty \left\{A_+(f_+^2 - g_+^2)^2 + A_-(f_-^2 - g_-^2)^2 + 2B(f_+^2 - g_+^2)(f_-^2 - g_-^2)\right\} rdr \\ &\leq -\lambda_s \int_0^\infty \left[(f_+^2 - g_+^2)^2 + (f_-^2 - g_-^2)^2\right] rdr \leq 0, \end{aligned} \tag{2.17}$$

which implies that $f_\pm^2 - g_\pm^2 \equiv 0$, i.e. $f_\pm \equiv g_\pm$ for all $r \in (0, \infty)$, and we have proven the uniqueness. □

Remark 2.2. *If we consider the existence and uniqueness of radial solution to* (1.2) *for* $r \in (0, R]$, *the results are still held in the bounded disk. We just need to change the interval of the integration for* r *to* $(0, R]$, *together with the boundary conditions:* $f_\pm(R) = g_\pm(R) = t_\pm$.

2.2 Asymptotics

Before we introduce the asymptotics of radial solution, we establish the useful comparison lemma for radial solution at first.

Lemma 2.3. *Let* $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$ *be bounded functions in* $[R, \infty)$.

(A) *Assume* $\mathbb{A}$, $\mathbb{D} > 0$, $\mathbb{B}$, $\mathbb{C} \leq 0$ *and* $4\mathbb{A}\mathbb{D} - (\mathbb{B} + \mathbb{C})^2 > 0$ *in* $[R, \infty)$. *Then, if* u, v *are radial solutions of the following problem*

$$\begin{cases} -\Delta_r u + \dfrac{m^2}{r^2} u + \mathbb{A}u + \mathbb{B}v \leq 0 \quad in \quad [R, \infty), \\ -\Delta_r v + \dfrac{n^2}{r^2} v + \mathbb{C}u + \mathbb{D}v \leq 0 \quad in \quad [R, \infty), \end{cases}$$

with $u(R) \leq 0, v(R) \leq 0$, u, v *are bounded in* $[R, \infty)$ *with* $\int_0^\infty (u')^2 r dr < \infty$ *and* $\int_0^\infty (v')^2 r dr < \infty$, *we have that* $u \leq 0$ *and* $v \leq 0$ *in* $[R, \infty)$.

(B) *Assume* $\mathbb{A}$, $\mathbb{D} > 0$, $\mathbb{B}$, $\mathbb{C} \geq 0$ *and* $4\mathbb{A}\mathbb{D} - (\mathbb{B} + \mathbb{C})^2 > 0$ *in* $[R, \infty)$. *Then, if* u, v *are radial solutions of the following problem*

$$\begin{cases} -\Delta_r u + \dfrac{m^2}{r^2} u + \mathbb{A} u + \mathbb{B} v \leq 0 \quad in \quad [R, \infty), \\ -\Delta_r v + \dfrac{n^2}{r^2} v + \mathbb{C} u + \mathbb{D} v \geq 0 \quad in \quad [R, \infty), \end{cases}$$

with $u(R) \leq 0 \leq v(R)$, u, v *are bounded in* $[R, \infty)$ *with* $\int_0^\infty (u')^2 r dr < \infty$ *and* $\int_0^\infty (v')^2 r dr < \infty$, *we have that* $u \leq 0 \leq v$ *in* $[R, \infty)$.

Proof. To verify (A), multiply the respective equations by $u^+ = \max(u, 0)$ and $v^+ = \max(v, 0)$ and integrate by parts in $[R, T]$ for $\forall T \geq R > 0$. Since u and v are radial solutions, we have

$$-\int_R^T \triangle u \cdot u^+ dr = -\int_R^T (ru')' u^+ dr = -\left(u' u^+ r\Big|_R^T\right) + \int_R^T [(u^+)']^2 r dr,$$

hence,

$$-\left(u' u^+ r\Big|_R^T\right) + \int_R^T \left\{[(u^+)']^2 + \frac{m^2}{r^2}(u^+)^2 + [\mathbb{A}(u^+)^2 + \mathbb{B}(v^+ + v^-)u^+]\right\} r dr \leq 0,$$

i.e.

$$\begin{aligned} &\int_R^T \left\{[(u^+)']^2 + \frac{m^2}{r^2}(u^+)^2 + [\mathbb{A}(u^+)^2 + \mathbb{B}(v^+ + v^-)u^+]\right\} r dr \\ &\quad \leq u'(T) u^+(T) T - u'(R) u^+(R) R \\ &\quad \leq u'(T) u^+(T) T \ , \end{aligned} \tag{2.18}$$

since $u'(R) u^+(R) R = 0$ by $u(R) \leq 0, u^+(R) = 0$.

Now we make a claim:

Claim A: There exists $T_n \to \infty$ such that $u'(T_n) T_n \to 0$.

Argument by contradiction. If not, we have $u'(r)r \geq c_0$ for all $r \geq T_0$. It follows that $u'(r) \geq \frac{c_0}{r}$, then we obtain that $[u'(r)]^2 r \geq \frac{c_0^2}{r} \notin L^1$, which is a contradiction.

Combine boundness of u in $[R, \infty)$ and the result of above claim, we take $T \to \infty$ on the both sides of (2.18), we get that

$$\int_R^\infty \left\{ [(u^+)']^2 + \frac{m^2}{r^2}(u^+)^2 + [\mathbb{A}(u^+)^2 + \mathbb{B}u^+v^+ + \mathbb{B}u^+v^-] \right\} rdr \leq 0 \ . \tag{2.19}$$

Similarly, we have the inequality for v^+:

$$\int_R^\infty \left\{ [(v^+)']^2 + \frac{n^2}{r^2}(v^+)^2 + [\mathbb{D}(v^+)^2 + \mathbb{C}u^+v^+ + \mathbb{C}u^-v^+] \right\} rdr \leq 0 \ . \tag{2.20}$$

Since $\mathbb{B}u^+v^- > 0$, $\mathbb{C}u^-v^+ > 0$ with $u^+ = \max(u, 0) > 0$, $v^+ = \max(v, 0) > 0$, $v^- = \min(v, 0) < 0$, we get that

$$\begin{aligned} &\int_R^\infty \left\{ [(u^+)']^2 + \frac{m^2}{r^2} \right\} rdr + \int_R^\infty [\mathbb{A}(u^+)^2 + \mathbb{B}u^+v^+] rdr \\ &\leq \int_R^\infty \left\{ [(u^+)']^2 + \frac{m^2}{r^2} \right\} rdr + \int_R^\infty [\mathbb{A}(u^+)^2 + \mathbb{B}u^+v^+ + \mathbb{B}u^+v^-] rdr \\ &\leq 0, \end{aligned}$$

and

$$\begin{aligned} &\int_R^\infty \left\{ [(v^+)']^2 + \frac{n^2}{r^2} \right\} rdr + \int_R^\infty [\mathbb{D}(v^+)^2 + \mathbb{C}u^+v^+] rdr \\ &\leq \int_R^\infty \left\{ [(v^+)']^2 + \frac{n^2}{r^2} \right\} rdr + \int_R^\infty [\mathbb{D}(v^+)^2 + \mathbb{C}u^+v^+ + \mathbb{C}u^-v^+] rdr \\ &\leq 0. \end{aligned}$$

Therefore, we deduce that

$$\begin{aligned} \int_R^\infty \Big\{ [(u^+)']^2 + [(v^+)']^2 + \frac{m^2(u^+)^2 + n^2(v^+)^2}{r^2} \\ + \mathbb{A}(u^+)^2 + (\mathbb{B} + \mathbb{C})u^+v^+ + \mathbb{D}(v^+)^2 \Big\} rdr \leq 0 \ . \end{aligned} \tag{2.21}$$

Note that the matrix associated to the quadratic form is positive definite in $[R, \infty)$ by hypothesis, so there exists a function $\lambda^+ > 0$ with

$$\mathbb{A}(u^+)^2 + (\mathbb{B} + \mathbb{C})u^+v^+ + \mathbb{D}(v^+)^2 \geq \lambda^+ \left[(u^+)^2 + (v^+)^2 \right] > 0 \ .$$

In consequence,

$$0 < \int_R^\infty \left\{ [(u^+)']^2 + [(v^+)']^2 + \frac{m^2(u^+)^2 + n^2(v^+)^2}{r^2} + \lambda^+ \left[(u^+)^2 + (v^+)^2\right] \right\} rdr \le 0 \ ,$$

which implies that $u^+ \equiv 0 \equiv v^+$ in $[R, \infty)$. Therefore $u(r) \le 0$ and $v(r) \le 0$ in $[R, \infty)$.

To prove (B) we multiply the first inequality by u^+, but multiply the second inequality by $v^- = \min(v, 0) \le 0$. Then, integrate by parts, we get that

$$\int_R^T \left\{ [(u^+)']^2 + \frac{m^2}{r^2}(u^+)^2 \right\} rdr + \int_R^T [\mathbb{A}(u^+)^2 + \mathbb{B}u^+v^+ + \mathbb{B}u^+v^-] rdr \le u'(T)u^+(T)T$$

and

$$\int_R^T \left\{ [(v^-)']^2 + \frac{n^2}{r^2}(v^-)^2 \right\} rdr + \int_R^T [\mathbb{C}u^+v^- + \mathbb{C}u^-v^- + \mathbb{D}(v^-)^2] rdr \le v'(T)v^-(T)T$$

by $v(R) \ge 0$ and $v^-(R) = 0$. In the following we will make a similar claim to Claim A as in the proof of part (A):

Claim B: There exists $T_n \to \infty$ such that $v'(T_n)T_n \to 0$.

If not, we have $v'(r)r \ge c_0$ for all $r \ge T_0$. It yields that $(v')^2 r \ge \frac{c_0}{r} \notin L^1$, which contradicts to our hypothesis.

This result together with the boundness of u and v in $[R, \infty)$, we obtain as $T \to \infty$

$$\int_R^\infty \left\{ [(v^-)']^2 + \frac{n^2}{r^2}(v^-)^2 \right\} rdr + \int_R^\infty [\mathbb{C}u^+v^- + \mathbb{C}u^-v^- + \mathbb{D}(v^-)^2] rdr \le 0. \qquad (2.22)$$

Since $\mathbb{B}u^+v^+ \ge 0$, $\mathbb{C}u^-v^- \ge 0$ with $u^+ = \max(u, 0) > 0$, $u^- = \min(u, 0) < 0$, $v^+ = \max(v, 0) > 0$, $v^- = \min(v, 0) < 0$, we can drop $\mathbb{B}u^+v^+$ and $\mathbb{C}u^-v^-$ two terms without affecting the inequalities (2.19) and (2.22). We add (2.19) and (2.22) together, and note that the matrix associated to the quadratic for is positive definite in $[R, \infty)$ by hypothesis, so there exists a function $\lambda^+ > 0$ with

$$\mathbb{A}(u^+)^2 + (\mathbb{B} + \mathbb{C})u^+v^- + \mathbb{D}(v^-)^2 \ge \lambda^+[(u^+)^2 + (v^-)^2] > 0 \ .$$

Therefore, we obtain that

$$\begin{aligned}
&\int_R^\infty \left\{[(u^+)']^2 + [(v^-)']^2 + \frac{m^2(u^+)^2 + n^2(v^-)^2}{r^2} + \lambda^+[(u^+)^2 + (v^-)^2]\right\} rdr \\
&\quad \le \int_R^\infty \left\{[(u^+)']^2 + [(v^-)']^2 + \frac{m^2(u^+)^2 + n^2(v^-)^2}{r^2} + \mathbb{A}(u^+)^2 + (\mathbb{B}+\mathbb{C})u^+v^- + \mathbb{D}(v^-)^2\right\} rdr \\
&\qquad \le 0\ .
\end{aligned}$$

Consequently,

$$\int_R^\infty \left\{[(u^+)']^2 + [(v^-)']^2 + \frac{m^2(u^+)^2 + n^2(v^-)^2}{r^2} + \lambda^+[(u^+)^2 + (v^-)^2]\right\} rdr \le 0\ , \quad (2.23)$$

which implies that $u^+ \equiv 0 \equiv v^-$ in $[R, \infty)$. Therefore $u(r) \le 0$ and $v(r) \ge 0$ in $[R, \infty)$. □

After this preliminary, we introduce the asymptotics of the radial solution at ∞.

Theorem 2.4. *Let $[f_+, f_-]$ be the solution of (2.1) with degree pair $[n_+, n_-]$ at ∞, then we have*

$$f_\pm = t_\pm + \frac{a_\pm}{r^2} + \frac{b_\pm}{r^4} + O(r^{-6}) \qquad as \ \ r \to \infty,$$

with

$$a_\pm = \frac{1}{2}\frac{Bn_\mp^2 - A_\mp n_\pm^2}{(A_+A_- - B^2)t_\pm},$$

and

$$b_\pm = -\frac{A_\mp^2(8n_\pm^2 + n_\pm^4)t_\mp^2 - BA_\mp(2n_\pm^2 + 8)n_\mp^2 t_\mp^2 - 8BA_\pm n_\mp^2 t_\pm^2 + B^2(8t_\pm^2 n_\pm^2 + n_\mp^4 t_\mp^2)t_\pm^2}{8(A_+A_- - B^2)^2 t_\pm^3 t_\mp^2}.$$

More specifically, let $A_\pm \ge 0$ and $B_0 > 0$ so that $A_+A_- - B_0^2 > 0$. Then, there exist positive constants C_1, C_2, R such that

$$\left| f_\pm(r) - \left(t_\pm + \frac{a_\pm}{r^2} + \frac{b_\pm}{r^4}\right)\right| \le \frac{C_1}{r^6}, \quad (2.24)$$

$$\left| f_\pm'(r) + \frac{2a_\pm}{r^3}\right| \le \frac{C_2}{r^5}, \quad (2.25)$$

hold for all $r \ge R$ and all B, $|B| \le B_0$.

Proof. We treat the cases $B > 0$ and $B < 0$ separately. We start with $B > 0$.

Step 1. Construction of a sub-supsolution pair. Let

$$\bar{w}_+ = t_+ + a_+ \frac{R^2}{r^2} + b_+ \frac{R^4}{r^4} + \bar{c}_+ \frac{R^6}{r^6}, \tag{2.26}$$

$$\underline{w}_- = t_- + a_- \frac{R^2}{r^2} + b_- \frac{R^4}{r^4} + \underline{c}_- \frac{R^6}{r^6}, \tag{2.27}$$

where $a_\pm, b_\pm, \bar{c}_+, \underline{c}_-$ and R are to be chosen so that

$$\text{LHS}(\bar{w}_+) = -\bar{w}_+'' - \frac{\bar{w}_+'}{r} + \frac{n_+^2}{r^2}\bar{w}_+ + [A_+(\bar{w}_+^2 - t_+^2) + B(\underline{w}_-^2 - t_-^2)]\bar{w}_+ \geq 0, \tag{2.28}$$

$$\text{LHS}(\underline{w}_-) = -\underline{w}_-'' - \frac{\underline{w}_-'}{r} + \frac{n_-^2}{r^2}\underline{w}_- + [A_-(\underline{w}_-^2 - t_-^2) + B(\bar{w}_+^2 - t_+^2)]\underline{w}_- \leq 0, \tag{2.29}$$

for all $r \geq R$, and $f_+(R) \leq \bar{w}_+(R)$, $f_-(R) \geq \underline{w}_-(R)$.

Using Maple software, we expand (2.28) and (2.29), which is a polynomial in even power of r^{-1},

$$\text{LHS}(\bar{w}_+) = \sum_{k=1}^{9} M_{2k}^+ \left(\frac{R}{r}\right)^{2k}, \tag{2.30}$$

$$\text{LHS}(\underline{w}_-) = \sum_{k=1}^{9} M_{2k}^- \left(\frac{R}{r}\right)^{2k}, \tag{2.31}$$

where $M_{2k}^\pm = M_{2k}^\pm(A_\pm, B, R, a_\pm, b_\pm, \bar{c}_+, \underline{c}_-)$ is a polynomial in its arguments. The coefficients

$$M_2^\pm = [n_\pm^2 + 2R^2(A_\pm a_\pm t_\pm + B a_\mp t_\mp)]t_\pm$$

may be set to zero by choosing

$$a_\pm = \frac{1}{2}\frac{Bn_\mp^2 - A_\mp n_\pm^2}{(A_+A_- - B^2)t_\pm}. \tag{2.32}$$

Similarly, the next term

$$M_4^\pm = [R^2 a_\pm(n_\pm^2 - 4) + R^4(3A_\pm t_\pm a_\pm^2 + Bt_\pm a_\mp^2 + 2Bt_\mp a_+ a_- t_\mp) + 2R^4(A_\pm t_\pm^2 b_\pm + Bt_+ t_- b_\mp)]t_\pm$$

vanish by the choice

$$b_\pm = -\frac{A_\mp^2(8n_\pm^2 + n_\pm^4)t_\mp^2 - BA_\mp(2n_\pm^2 + 8)n_\mp^2 t_\mp^2 - 8BA_\pm n_\mp^2 t_\pm^2 + B^2(8t_\pm^2 n_\pm^2 + n_\mp^4 t_\mp^2)t_\pm^2}{8(A_+A_- - B^2)^2 t_\pm^3 t_\mp^2}. \tag{2.33}$$

Using Maple software, the values of $a_\pm, b_\pm$ may then be substituted into the expansions of (2.28) and (2.29), and the expressions for $M_{2k}^\pm$ may be viewed as functions of R. The exact form of the coefficients $M_{2k}^\pm$ is very complex, but as we will choose R large, we are only interested in the leading order of each. We obtain:

$$
\begin{gathered}
M_6^+ = 2(B\underline{c}_- t_- + A_+ \bar{c}_+ t_+)t_+ + O(R^{-6}),\\
M_6^- = 2(B\bar{c}_+ t_+ + A_- \underline{c}_- t_-)t_- + O(R^{-6}),\\
M_8^\pm = O(R^{-2}), \qquad M_{10}^\pm = O(R^{-4}),\\
M_{12}^+ = (A_+\bar{c}_+^2 + B\underline{c}_-^2)t_+ + 2(B\underline{c}_- t_- + A_+\bar{c}_+ t_+)\bar{c}_+ + O(R^{-6}),\\
M_{12}^- = (A_-\underline{c}_-^2 + B\bar{c}_+^2)t_- + 2(B\bar{c}_+ t_+ + A_-\underline{c}_- t_-)\underline{c}_- + O(R^{-6}),\\
M_{14}^\pm = O(R^{-2}), \qquad M_{16}^\pm = O(R^{-4}),\\
M_{18}^+ = (A_+\bar{c}_+^2 + B\underline{c}_-^2)\bar{c}_+,\\
M_{18}^- = (A_-\underline{c}_-^2 + B\bar{c}_+^2)\underline{c}_-,
\end{gathered}
$$

here $O(R^{-n})$ denotes terms which are small as $R \to \infty$ uniformly for $|B| \le B_0$.

As $M_6^\pm$ is the leading order term, we choose $\bar{c}_+, \underline{c}_-$ in order that it gives the correct signs, and so that it dominates the other terms for $r \ge R$.

Without loss of generality, we consider $[\tilde{c}_+, \tilde{c}_-]$ satisfy the following system:

$$
\begin{cases}
A_+ t_+ \tilde{c}_+ + B t_- \tilde{c}_- = 1 \ , \\
B t_+ \tilde{c}_+ + A_- t_- \tilde{c}_- = -1 \ ,
\end{cases}
$$

it follows that

$$
\tilde{c}_+ = \frac{A_- + B}{(A_+A_- - B^2)t_+} > 0 \ , \qquad \tilde{c}_- = -\frac{A_+ + B}{(A_+A_- - B^2)t_-} < 0 \ . \tag{2.34}
$$

Let $\bar{c}_+ = \delta\tilde{c}_+, \underline{c}_- = \delta\tilde{c}_-$ with $0 < \delta < 1$ to be chosen later, hence $\bar{c}_+ > 0$, $\underline{c}_- < 0$, and

$$
M_6^+ = 2\delta t_+ > 0, \qquad M_6^- = -2\delta t_- < 0. \tag{2.35}
$$

By choosing R sufficiently large, we obtain that

$$
|M_8^\pm|, |M_{10}^\pm|, |M_{14}^\pm|, |M_{16}^\pm| \le \frac{1}{20}|M_6^\pm|, \tag{2.36}
$$

and so we have that

$$\left|M_8^{\pm}\left(\frac{R}{r}\right)^8 + M_{10}^{\pm}\left(\frac{R}{r}\right)^{10} + M_{14}^{\pm}\left(\frac{R}{r}\right)^{14} + M_{16}^{\pm}\left(\frac{R}{r}\right)^{16}\right|$$
$$\leq \frac{1}{5}|M_6^{\pm}|\left(\frac{R}{r}\right)^6, \quad \text{for } \forall r \geq R. \quad (2.37)$$

Also,

$$\begin{aligned}|M_{12}^{+}| &= |\delta^2(A_+\tilde{c}_+^2 + B\tilde{c}_-^2)t_+ + 2\delta^2\tilde{c}_+| \\ &= \delta^2|(A_+\tilde{c}_+^2 + B\tilde{c}_-^2)t_+ + 2\tilde{c}_+| \\ &\leq c_1(A_{\pm}, B, t_{\pm})\delta^2 \\ &\leq \frac{1}{5}|M_6^{+}| = \frac{2}{5}\delta t_+, \end{aligned} \quad (2.38)$$

for R sufficiently large, provided we choose $\delta \leq \frac{2t_+}{5c_1}$.

Meanwhile,

$$\begin{aligned}|M_{12}^{-}| &= |\delta^2(A_+\tilde{c}_+^2 + B\tilde{c}_-^2)t_+ + 2\delta^2\tilde{c}_+| \\ &= \delta^2|(A_-\tilde{c}_-^2 + B\tilde{c}_+^2)t_- + 2\tilde{c}_-| \\ &\leq c_2(A_{\pm}, B, t_{\pm})\delta^2 \\ &\leq \frac{1}{5}|M_6^{-}| = \frac{2}{5}\delta t_-, \end{aligned} \quad (2.39)$$

for R sufficiently large, provided we choose $\delta \leq \frac{2t_-}{5c_2}$. Therefore, if we choose $\delta \leq \min\{\frac{2t_+}{5c_1}, \frac{2t_-}{5c_2}\}$, it yields that $|M_{12}^{\pm}| \leq \frac{1}{5}|M_6^{\pm}|$. So we can deduce that

$$\left|M_{12}^{\pm}\left(\frac{R}{r}\right)^{12}\right| \leq \frac{1}{5}\left|M_6^{\pm}\left(\frac{R}{r}\right)^6\right|, \quad (2.40)$$

and $M_{18}^{\pm}$ has the appropriate sign to apply the comparison lemma.

By (2.30)-(2.31) and (2.35), (2.36), (2.37), (2.40), we deduce that

$$\sum_{k=1}^{9} M_{2k}^{+}\left(\frac{R}{r}\right)^{2k} \geq -\frac{1}{5}M_6^{+}\left(\frac{R}{r}\right)^6 - \frac{1}{5}M_6^{+}\left(\frac{R}{r}\right)^6 + M_6^{+}\left(\frac{R}{r}\right)^6$$

$$
\begin{aligned}
&= \frac{3}{5}M_6^+\left(\frac{R}{r}\right)^6 = \frac{6}{5}\delta t_+\left(\frac{R}{r}\right)^6 \\
&> \frac{3}{5}\delta t_+\left(\frac{R}{r}\right)^6 > 0,
\end{aligned} \tag{2.41}
$$

$$
\begin{aligned}
\sum_{k=1}^{9} M_{2k}^-\left(\frac{R}{r}\right)^{2k} &\leq \frac{1}{5}|M_6^-|\left(\frac{R}{r}\right)^6 + \frac{1}{5}|M_6^-|\left(\frac{R}{r}\right)^6 + M_6^-\left(\frac{R}{r}\right)^6 \\
&= -\frac{1}{5}M_6^-\left(\frac{R}{r}\right)^6 - \frac{1}{5}M_6^-\left(\frac{R}{r}\right)^6 + \frac{1}{5}M_6^-\left(\frac{R}{r}\right)^6 \\
&= \frac{3}{5}M_6^-\left(\frac{R}{r}\right)^6 = -\frac{6}{5}\delta t_-\left(\frac{R}{r}\right)^6 \\
&\leq -\frac{3}{5}\delta t_-\left(\frac{R}{r}\right)^6 < 0,
\end{aligned} \tag{2.42}
$$

for all $r \geq R$. Thus, $\bar{w}_+$, $\underline{w}_-$ satisfy (2.28) and (2.29), as desired.

Now we consider $\bar{w}_+$ and $\underline{w}_-$ at $r = R$. Combine (2.26), (2.27) and (2.32) we have $\bar{w}_+ = t_+ + \bar{c}_+ - O(R^{-2})$, $\underline{w}_- = t_- + \underline{c}_- - O(R^{-2})$. Since $f_\pm(r) \to t_\pm$ as $r \to \infty$ and $\bar{c}_+ > 0, \underline{c}_- < 0$, we deduce that $\bar{w}_+(R) > f_+(R)$ and $\underline{w}_-(R) < f_-(R)$ for $R \geq R_0$ sufficiently large. This completes Step 1.

Step 2. We will apply Lemma 2.3 to show that $\bar{w}_+(r)$ is a sup-solution to f_+-equation of (2.1) and $\underline{w}_-(r)$ is a sub-solution to f_--equation of (2.1). Let $u = f_+ - \bar{w}_+$, $v = f_- - \underline{w}_-$, together with equations (2.1), (2.28) and (2.29), and denote

$$
L_\pm u := -\Delta_r u + \frac{n_\pm^2}{r^2}u.
$$

We have that

$$
\begin{aligned}
L_+u &= L_+f_+ - L_+\bar{w}_+ \\
&= A_+(t_+^2 - f_+^2)f_+ + B(t_-^2 - f_-^2)f_+ + A_+(\bar{w}_+^2 - t_+^2)\bar{w}_+ + B(\underline{w}_-^2 - t_-^2)\bar{w}_+ \\
&= A_+(\bar{w}_+^3 - f_+^3) - A_+t_+^2(\bar{w}_+ - f_+) + Bt_-^2(f_+ - \bar{w}_+) + B(\underline{w}_-^2\bar{w}_+ - f_-^2 f_+) \\
&= [A_+t_+^2 + Bt_-^2 - A_+(\bar{w}_+^2 + \bar{w}_+f_+ + f_+^2)]u + B(\underline{w}_-^2\bar{w}_+ - f_-^2 f_+) \\
&= [A_+t_+^2 + B(t_-^2 - f_-^2) - A_+(\bar{w}_+^2 + \bar{w}_+f_+ + f_+^2)]u - B(\underline{w}_- + f_-)\bar{w}_+ v\ ,
\end{aligned}
$$

$$
\begin{aligned}
L_- v &= L_- f_- - L_- \underline{w}_- \\
&= A_-(t_-^2 - f_-^2)f_- + B(t_+^2 - f_+^2)f_- + A_-(\underline{w}_-^2 - t_-^2)\underline{w}_- + B(\bar{w}_+^2 - t_+^2)w_- \\
&= A_-(\underline{w}_-^3 - f_-^3) - A_- t_-^2(\underline{w}_- - f_-) + Bt_+^2(f_- - \underline{w}_-) + B(\bar{w}_+^2 \underline{w}_- - f_+^2 f_-) \\
&= [A_- t_-^2 + B(t_+^2 - f_+^2) - A_-(\underline{w}_-^2 + \underline{w}_- f_- + f_-^2)]v - B(\bar{w}_+ + f_+)\underline{w}_- u \ ,
\end{aligned}
$$

and thus (2.28) and (2.29) imply that

$$
L_+ u + [A_+(\bar{w}_+^2 + \bar{w}_+ f_+ + f_+^2) - A_+ t_+^2 + B(f_-^2 - t_-^2)]u + B(\underline{w}_- + f_-)\bar{w}_+ v \leq 0 \ , \tag{2.43}
$$

$$
L_- v + B(\bar{w}_+ + f_+)\underline{w}_- u + [A_-(\underline{w}_-^2 + \underline{w}_- f_- + f_-^2) - A_- t_-^2 + B(f_+^2 - t_+^2)]v \geq 0 \ . \tag{2.44}
$$

As in Lemma 2.3, we have a system on $[R, \infty)$ of the form:

$$
\begin{cases}
L_+ u + \mathbb{A}(r)u + \mathbb{B}(r)v \leq 0, & u(R) \leq 0, \\
L_- v + \mathbb{C}(r)u + \mathbb{D}(r)v \geq 0, & v(R) \geq 0,
\end{cases}
$$

with

$$
\begin{aligned}
\mathbb{A}(r) &= A_+(\bar{w}_+^2 + \bar{w}_+ f_+ + f_+^2) - A_+ t_+^2 + B(f_-^2 - t_-^2), \\
\mathbb{B}(r) &= B(\underline{w}_- + f_-)\bar{w}_+, \quad \mathbb{C}(r) = B(\bar{w}_+ + f_+)\underline{w}_-, \\
\mathbb{D}(r) &= A_-(\underline{w}_-^2 + \underline{w}_- f_- + f_-^2) - A_- t_-^2 + B(f_+^2 - t_+^2).
\end{aligned}
$$

Now, we check the hypothesis of Lemma 2.3 with the uniform convergence of $f_\pm(r)$ as $r \to \infty$:

$$
\begin{aligned}
\mathbb{A}(r) &\to 2A_+ t_+^2, & \mathbb{B}(r) &\to 2Bt_+ t_-, \\
\mathbb{C}(r) &\to 2Bt_+ t_-, & \mathbb{D}(r) &\to 2A_- t_-^2,
\end{aligned}
$$

and

$$
\begin{aligned}
4\mathbb{A}\mathbb{D} - (\mathbb{B} + \mathbb{C})^2 &\to 4 \cdot 2A_+ t_+^2 \cdot 2A_- t_-^2 - (4Bt_+ t_-)^2 \\
&= 16 t_+^2 t_-^2 (A_+ A_- - B^2) > 0 \qquad \text{for } R \text{ large enough.}
\end{aligned}
$$

All conditions of Lemma 2.3 are satisfied, it yields that $u(r) \leq 0, v(r) \geq 0$ in $[R, \infty)$, i.e. $f_+(r) \leq \bar{w}_+(r), f_-(r) \geq \underline{w}_-(r)$ in $[R, \infty)$.

Step 3. We repeat the above procedures with the roles $\pm$ of $\bar{w}_+$ and $\underline{w}_-$ under the condition $B > 0$. We may obtain $\underline{c}_+, \bar{c}_-, R$ so that $f_+(r) \geq \underline{w}_+(r), f_-(r) \leq \bar{w}_-(r)$ in $[R, \infty)$. Combine the results from above steps, we get that

$$\underline{w}_+(r) \leq f_+(r) \leq \bar{w}_+(r), \quad \underline{w}_-(r) \leq f_-(r) \leq \bar{w}_-(r), \quad \text{for } \forall r \geq R, \tag{2.45}$$

in the case $B > 0$.

Step 4. Now we consider the case $B < 0$. This is similar to the previous cases, but we must set up part (A) of the Comparison Lemma 2.3. Firstly we construct a supsolution pair $\bar{w}_+$ as in (2.26), $\bar{w}_-$ as following,

$$\bar{w}_- = t_- + a_- \frac{R^2}{r^2} + b_- \frac{R^4}{r^4} + \bar{c}_- \frac{R^6}{r^6}, \tag{2.46}$$

with $a_\pm, b_\pm, \bar{c}_\pm$ and R are to be chosen so that

$$\text{LHS}(\bar{w}_+) = -\bar{w}_+'' - \frac{\bar{w}_+'}{r} + \frac{n_+^2}{r^2}\bar{w}_+ + [A_+(\bar{w}_+^2 - t_+^2) + B(\bar{w}_-^2 - t_-^2)]\bar{w}_+ \geq 0, \tag{2.47}$$

$$\text{LHS}(\bar{w}_-) = -\bar{w}_-'' - \frac{\bar{w}_-'}{r} + \frac{n_-^2}{r^2}\bar{w}_- + [A_-(\bar{w}_-^2 - t_-^2) + B(\bar{w}_+^2 - t_+^2)]\bar{w}_- \geq 0, \tag{2.48}$$

for all $r \geq R$, and $f_+(R) \leq \bar{w}_+(R), f_-(R) \leq \bar{w}_-(R)$.

Using Maple software, we expand (2.47) and (2.48), which is a polynomial in even power of r^{-1}, as in (2.30) and

$$\text{LHS}(\bar{w}_-) = \sum_{k=1}^{9} M_{2k}^- \left(\frac{R}{r}\right)^{2k}, \tag{2.49}$$

respectively, where $M_{2k}^\pm = M_{2k}^\pm(A_\pm, B, R, a_\pm, b_\pm, \bar{c}_\pm)$ is a polynomial in its arguments. Similarly, we choose $a_\pm, b_\pm$ as in (2.32) and (2.33) separately so that $M_2^\pm$ may be zero as in the same formulas in Step 1. Then, repeat the same procedure as in Step 1, we get the exact forms for $M_6^\pm, M_{12}^\pm$ and $M_{18}^\pm$ as the same as in Step 1 but with $\underline{c}_-$ replaced by $\bar{c}_-$, and $M_8^\pm, M_{10}^\pm$ and $M_{14}^\pm$ as the same as in Step 1.

As $M_6^\pm$ is still the leading order term, we choose $\bar{c}_\pm$ in order that it gives the correct sign, and so that it dominates the other terms for all $r \geq R$.

Without loss of generality, we consider $[\hat{c}_+, \hat{c}_-]$ satisfy the following system:

$$\begin{cases} A_+ t_+ \hat{c}_+ + B t_- \hat{c}_- = 1, \\ B t_+ \hat{c}_+ + A_- t_- \hat{c}_- = 1, \end{cases}$$

we obtain that

$$\hat{c}_+ = \frac{A_- - B}{(A_+ A_- - B^2) t_+} > 0, \qquad \hat{c}_- = \frac{A_+ - B}{(A_+ A_- - B^2) t_-} > 0.$$

Let $\bar{c}_\pm = \delta \hat{c}_\pm$ with $0 < \delta < 1$ to be chosen later, hence $\bar{c}_\pm > 0$, and

$$M_6^\pm = 2\delta t_\pm > 0. \tag{2.50}$$

By choosing R large enough, we can also get (2.36), (2.37) and

$$\begin{aligned} |M_{12}^+| &= |\delta^2 (A_+ \hat{c}_+^2 + B \hat{c}_-^2) t_+ + 2\delta^2 \hat{c}_+| \\ &= \delta^2 |(A_+ \hat{c}_+^2 + B \hat{c}_-^2) t_+ + 2\hat{c}_+| \\ &\le c_5(A_\pm, B, t_\pm) \delta^2 \\ &\le \frac{1}{5} M_6^+ = \frac{2}{5} \delta t_+, \end{aligned} \tag{2.51}$$

which implies that $\delta \le \frac{2t_+}{5c_5}$.

Meanwhile,

$$\begin{aligned} |M_{12}^-| &= |\delta^2 (A_- \hat{c}_-^2 + B \hat{c}_+^2) t_- + 2\delta^2 \hat{c}_-| \\ &= \delta^2 |(A_- \hat{c}_-^2 + B \hat{c}_+^2) t_- + 2\hat{c}_-| \\ &\le c_6(A_\pm, B, t_\pm) \delta^2 \\ &\le \frac{1}{5} M_6^- = \frac{2}{5} \delta t_-, \end{aligned} \tag{2.52}$$

which implies that $\delta \le \frac{2t_-}{5c_6}$. Therefore, if we choose $\delta \le \min\{\frac{2t_+}{5c_5}, \frac{2t_-}{5c_6}\}$, it follows that $|M_{12}^\pm| \le \frac{1}{5}|M_6^\pm|$. And for $M_{18}^\pm$, we have

$$|M_{18}^\pm| = \delta^3 (A_\pm \hat{c}_\pm^2 + B \hat{c}_\mp^2) \hat{c}_\pm.$$

Since there is cubic term of δ in $M_{18}^{\pm}$, it can still be controlled by $M_6^{\pm}$.

By (2.30), (2.49) and (2.50), (2.36)-(2.37) and (2.40), we deduce that

$$\begin{aligned}\sum_{k=1}^{9} M_{2k}^{\pm}\left(\frac{R}{r}\right)^{2k} &\geq -\frac{1}{5}M_6^{\pm}\left(\frac{R}{r}\right)^{6} - \frac{1}{5}M_6^{\pm}\left(\frac{R}{r}\right)^{6} + M_6^{\pm}\left(\frac{R}{r}\right)^{6} \\ &= \frac{3}{5}M_6^{\pm}\left(\frac{R}{r}\right)^{6} = \frac{6}{5}\delta t_{\pm}\left(\frac{R}{r}\right)^{6} > 0,\end{aligned} \tag{2.53}$$

for all $r \geq R$.

Now we consider $\bar{w}_{\pm}$ at $r = R$. Combine (2.26), (2.46) and (2.32), we have $\bar{w}_{\pm} = t_{\pm} + \bar{c}_{\pm} - O(R^{-2})$. Since $f_{\pm} \to t_{\pm}$ as $r \to \infty$ and $\bar{c}_{\pm} > 0$, we obtain that $\bar{w}_{\pm}(R) > f_{\pm}(R)$ for $R \geq R_0$ sufficiently large. This finishes Step 4.

Step 5. We will apply Lemma 2.3 to show that $\bar{w}_{\pm}(r)$ is supsolution to (1.2). Let $u = f_+ - \bar{w}_+$, $v = f_- - \bar{w}_-$, we have that

$$\begin{aligned} L_+ u &= [A_+(t_+^2 - f_+^2) + B(t_-^2 - f_-^2)]f_+ + [A_+(\bar{w}_+^2 - t_+^2) + B(\bar{w}_-^2 - t_-^2)]\bar{w}_+ \\ &= -A_+(f_+^3 - \bar{w}_+^3) + A_+ t_+^2 (f_+ - \bar{w}_+) + B t_-^2 (f_+ - \bar{w}_+) + B(\bar{w}_-^2 \bar{w}_+ - f_-^2 f_+) \\ &= [-A_+(f_+^2 + f_+\bar{w}_+ + \bar{w}_+^2) + A_+ t_+^2 + B t_-^2 - B f_-^2]u - B(\bar{w}_- + f_-)\bar{w}_+ v \\ &= [A_+ t_+^2 + B(t_-^2 - f_-^2) - A_+(f_+^2 + f_+\bar{w}_+ + \bar{w}_+^2)]u - B(\bar{w}_- + f_-)\bar{w}_+ v, \end{aligned}$$

and

$$\begin{aligned} L_- v &= [A_-(t_-^2 - f_-^2) + B(t_+^2 - f_+^2)]f_- + [A_-(\bar{w}_-^2 - t_-^2) + B(\bar{w}_+^2 - t_+^2)]\bar{w}_- \\ &= -A_-(f_-^3 - \bar{w}_-^3) + A_- t_-^2 (f_- - \bar{w}_-) + B t_+^2 (f_- - \bar{w}_-) + B(\bar{w}_+^2 \bar{w}_- - f_+^2 f_-) \\ &= [-A_-(f_-^2 + f_-\bar{w}_- + \bar{w}_-^2) + A_- t_-^2 + B t_+^2 - B f_+^2]v - B(\bar{w}_+ + f_+)\bar{w}_- u \\ &= -B(\bar{w}_+ + f_+)\bar{w}_- u + [-A_-(f_-^2 + f_-\bar{w}_- + \bar{w}_-^2) + A_- t_-^2 + B(t_+^2 - f_+^2)]v, \end{aligned}$$

thus by (2.47)-(2.48) we have that

$$L_+ u + [A_+(f_+^2 + f_+\bar{w}_+ + \bar{w}_+^2) + B(f_-^2 - t_-^2) - A_+ t_+^2]u + B(\bar{w}_- + f_-)\bar{w}_+ v \leq 0,$$

$$L_- v + B(\bar{w}_+ + f_+)\bar{w}_- u + [A_-(f_-^2 + f_-\bar{w}_- + \bar{w}_-^2) + B(f_+^2 - t_+^2) - A_- t_-^2]v \leq 0,$$

and $u(R) = f_+(R) - \bar{w}_+(R) \le 0$, $v(R) = f_-(R) - \bar{w}_-(R) \le 0$. Therefore, we have the exact forms of the inequality system as in the part (A) of Lemma 2.3 with

$$\mathbb{A}(r) = A_+(f_+^2 + f_+\bar{w}_+ + \bar{w}_+^2) + B(f_-^2 - t_-^2) - A_+t_+^2, \tag{2.54}$$

$$\mathbb{B}(r) = -B(\bar{w}_- + f_-)\bar{w}_+, \qquad \mathbb{C}(r) = -B(\bar{w}_+ + f_+)\bar{w}_-, \tag{2.55}$$

$$\mathbb{D}(r) = A_-(f_-^2 + f_-\bar{w}_- + \bar{w}_-^2) + B(f_+^2 - t_+^2) - A_-t_-^2. \tag{2.56}$$

With the uniform convergence of $f_\pm(r)$ as $r \to \infty$, we obtain that

$$\mathbb{A}(r) \to 2A_+t_+^2, \qquad \mathbb{D}(r) \to 2A_-t_-^2,$$

$$\mathbb{B}(r) \to -2Bt_+t_-, \qquad \mathbb{C}(r) \to -2Bt_+t_-,$$

and

$$\begin{aligned}4\mathbb{A}\mathbb{D} - (\mathbb{B}+\mathbb{C})^2 &\to 4 \cdot 2A_+t_+^2 \cdot 2A_-t_-^2 - (4Bt_+t_-)^2 \\ &= 16t_+^2t_-^2(A_+A_- - B^2) > 0 \qquad \text{for} \quad R \;\text{ large enough.}\end{aligned}$$

Hence we can simply apply part (A) of Lemma 2.3 to get that $u(r) \le 0$, $v(r) \le 0$ in $[R, \infty)$, i.e. $f_\pm(r) \le \bar{w}_\pm(r)$ in $[R, \infty)$.

Step 6. We repeat the same procedure as in Step 4 and Step 5, but with a pair of sub-solutions $\underline{w}_\pm$ to (1.2) instead of sup-solutions $\bar{w}_\pm$. Let $\underline{w}_+$ be defined as in the following

$$\underline{w}_+ = t_+ + a_+\frac{R^2}{r^2} + b_+\frac{R^4}{r^4} + \underline{c}_+\frac{R^6}{r^6}, \tag{2.57}$$

and $\underline{w}_-$ defined as in (2.27), with $a_\pm, b_\pm, \underline{c}_\pm$ and R are to be chosen so that

$$\text{LHS}(\underline{w}_\pm) = -\underline{w}_\pm'' - \frac{\underline{w}_\pm'}{r} + \frac{n_\pm^2}{r^2}\underline{w}_\pm + [A_\pm(\underline{w}_\pm^2 - t_\pm^2) + B(\underline{w}_\mp^2 - t_\mp^2)]\underline{w}_\pm \le 0, \tag{2.58}$$

for all $r \ge R$, and $\underline{w}_\pm(R) \le f_\pm(R)$. And we choose $\underline{c}_\pm = -\delta\hat{c}_\pm$ with the same δ as in Step 4 so that $M_6^\pm$ is still the leading terms in (2.30) and (2.49) with which $\bar{w}_\pm$

replaced by $\underline{w}_\pm$. Therefore, we can simply get that

$$\begin{aligned}
\mathrm{LHS}(\underline{w}_\pm) &\leq M_6^\pm \left(\frac{R}{r}\right)^6 + \frac{1}{5}|M_6^\pm| \left(\frac{R}{r}\right)^6 + \frac{1}{5}|M_6^\pm| \left(\frac{R}{r}\right)^6 \\
&= \left(M_6^\pm - \frac{1}{5}M_6^\pm - \frac{1}{5}M_6^\pm\right)\left(\frac{R}{r}\right)^6 \\
&= \frac{3}{5}M_6^\pm \left(\frac{R}{r}\right)^6 = -\frac{6}{5}\delta t_\pm \left(\frac{R}{r}\right)^6 < 0. \qquad (2.59)
\end{aligned}$$

Next, with $u = f_+ - \underline{w}_+$, $v = f_- - \underline{w}_-$, we repeat the similar process as in Step 5 and apply part (A) of Lemma 2.3 again, we can obtain that $\underline{w}_\pm(r) \leq f_\pm(r)$ in $[R, \infty)$.

Step 7. Combing all the results from above steps, we have that

$$\underline{w}_\pm(r) \leq f_\pm(r) \leq \bar{w}_\pm(r) \qquad \text{in} \quad [R, \infty). \qquad (2.60)$$

for both cases of B.

From the sub-supersolution argument, we have that

$$\left| f_\pm - \left(t_\pm + \frac{\tilde{a}_\pm}{r^2} + \frac{\tilde{b}_\pm}{r^4}\right)\right| \leq \frac{C_1}{r^6},$$

with $\tilde{a}_\pm = a_\pm R^2, \tilde{b}_\pm = b_\pm R^4$, $a_\pm$ and $b_\pm$ defined as same as in (2.32) and (2.33) respectively, and $C_1 = \min\{c_\pm R^6\}$ with $c_+ = \min\{|\bar{c}_+|, |\underline{c}_+|\}$, $c_- = \min\{|\underline{c}_-|, |\bar{c}_-|\}$.

Next we want to show that $|f'_\pm(r) + \frac{2\tilde{a}_\pm}{r^3}| \leq \frac{C_2}{r^5}$ for some constant C_2. Following the idea in [CEQ94], let $W_\pm(r) = f_\pm - \left(t_\pm + \frac{\tilde{a}_\pm}{r^2} + \frac{\tilde{b}_\pm}{r^4}\right) = f_\pm - w_\pm + \frac{c_\pm}{r^6}$(for convenience, we drop bar and underline of $w_\pm$ and $c_\pm$ as in the formulas of $w_\pm$ shown in previous steps), hence $W_\pm(r) = O(r^{-6})$. Therefore, we deduce that

$$\begin{aligned}
&- W''_+ - \frac{1}{r}W'_+ + \frac{n_+^2}{r^2}W_+ \\
&= -\left(f''_+ - w''_+ + \frac{c_+}{r^8}\right) - \frac{1}{r}\left(f'_+ - w'_+ + \frac{c_+}{r^7}\right) + \frac{n_+^2}{r^2}\left(f_+ - w_+ + \frac{c_+}{r^6}\right) \\
&= -f''_+ - \frac{1}{r}f'_+ + \frac{n_+^2}{r^2}f_+ - \left(-w''_+ - \frac{1}{r}w'_+ + \frac{n_+^2}{r^2}w_+\right) + O(r^{-8}) \\
&= [A_+(t_+^2 - f_+^2) + B(t_-^2 - f_-^2)]f_+ + [A_+(w_+^2 - t_+^2) + B(w_-^2 - t_-^2)]w_+ + O(r^{-8})
\end{aligned}$$

$$= -\mathbb{A}W_+ - \mathbb{B}W_- + \mathbb{A}\frac{c_+}{r^6} + \mathbb{B}\frac{c_-}{r^6} + O(r^{-8}),$$

i.e.

$$-W_+'' - \frac{1}{r}W_+' + \frac{n_+^2}{r^2}W_+ + \mathbb{A}W_+ + \mathbb{B}W_- = O(r^{-6}) + O(r^{-8}) \approx O(r^{-6}) \quad \text{for} \quad r \geq R.$$

Similarly, we have

$$-W_-'' - \frac{1}{r}W_-' + \frac{n_-^2}{r^2}W_- + \mathbb{C}W_+ + \mathbb{D}W_- = O(r^{-6}) + O(r^{-8}) \approx O(r^{-6}) \quad \text{for} \quad r \geq R,$$

with $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$ defined as same as in the above process of sub-supersolution in different cases. Therefore, $\frac{1}{r}(rW_\pm')' = O(r^{-6})$. On the other hand, notice that for each $k > 1$ there exists a point $r_k \in (k, 2k)$ such that $W_\pm'(r_k) = (W_\pm(2k) - W_\pm(k))/k = O(k^{-7}) = O(r_k^{-7})$, then we have that

$$\begin{aligned} |-rW_\pm'(r) + r_k W_\pm'(r_k)| &= \left| \int_r^{r_k} \frac{1}{r}(rW_\pm')' r dr \right| \\ &\leq c_0 \int_r^{r_k} r^{-5} dr \to c_0 \int_r^{\infty} r^{-5} dr = \frac{C_2}{r^4} \quad \text{as } k \to \infty. \end{aligned}$$

Together with $r_k W_\pm'(r_k) = O(r_k^{-6}) \to 0$ as $k \to \infty$, we have that $|rW_\pm'(r)| \leq \frac{C_2}{r^4}$ for $r \geq R$ sufficiently large, i.e. $|W_\pm'(r)| \leq \frac{C_2}{r^5}$ for $r \geq R$ sufficiently large. Hence, we deduce that $|f_\pm'(r) + \frac{2\tilde{a}_\pm}{r^3}| \leq \frac{C_2}{r^5}$ for all $r \geq R$. □

2.3 Monotonicity

Next we will present the proof on the monotonicity of the radial solutions. First, we define the spaces as in [ABM09] :

$$X_0 := H^1((0,\infty); rdr),$$

$$X_n := \left\{ u \in X_0 : \int_0^\infty \frac{u^2}{r^2} rdr < \infty \right\}, \quad \|u\|_{X_n}^2 = \int_0^\infty \left[(u')^2 + u^2 + \frac{n^2}{r^2}u^2 \right] rdr.$$

Of course the spaces $X_n, n \neq 0$, are all equivalent, but we define them this way for notational convenience. It is not difficult to show (see [AB06]) that for $|n| \geq 1$, X_n is continuously embedded in the space of continuous functions on $(0, \infty)$ which vanish at $r = 0$ and $r \to \infty$, and that $\mathcal{C}_0^\infty((0,\infty))$ is dense in X_1. It is possible to define a global variational framework for the equivariant problems in affine spaces based on X_{n_+}, X_{n_-} to prove existence of solutions. The energy is the same as in (2.6), except it must be "renormalized" to prevent divergence of the $\frac{n_{n_\pm}^2}{r^2}$ term at infinity. Here we are only interested in the (formal) second variation of this renormalized energy,

$$\begin{aligned} D^2 E_{n_+,n_-}(f_+, f_-)[u_+, u_-] &:= \int_0^\infty \left[(u_+')^2 + (u_-')^2 + \frac{n_+^2}{r^2} u_+^2 + \frac{n_-^2}{r^2} u_-^2 \right] r dr \\ &+ \int_0^\infty \left\{ [A_+(f_+^2 - t_+^2) + B(f_-^2 - t_-^2)] u_+^2 + [A_-(f_-^2 - t_-^2) + B(f_+^2 - t_+^2)] u_-^2 \right\} r dr \\ &+ \int_0^\infty 2(A_+ f_+^2 u_+^2 + A_- f_-^2 u_-^2 + 2B f_+ f_- u_+ u_-) r dr, \end{aligned} \tag{2.61}$$

defined for $[u_+, u_-] \in X_{n_+} \times X_{n_-}$.

We have the following fact about radial solutions:

Lemma 2.5. *For any $n_\pm \in \mathbb{Z}$, if $[f_+,\ f_-]$ is the (unique) radial solution of* (2.1),

$$D^2 E_{n_+,n_-}(f_+, f_-)[u_+, u_-] > 0 \quad \text{for all} \quad [u_+, u_-] \in X_{n_+} \times X_{n_-} \setminus \{[0,0]\}.$$

In other words, the radial solutions are non-degenerate local minimizers of the renormalized energy. An analogous statement for the Ginzburg-Landau equation with magnetic field was derived in [ABG99], and this observation then became the main step in the proof of uniqueness of equivariant solutions proved there. The basic idea is that were there two admissible solutions to the equivariant vortex equations, each being a local minimizer of the energy there would be a third, non-minimizing solution via the Mountain-pass theorem. The argument was achieved by restriction to a convex constraint set (to eliminate the possibility of non-admissible solutions, which might not be local minimizers). The method works because the constraints play the role of a sub-supersolution pair for the Ginzburg-Landau equations, and hence the mountain pass solutions obtained would lie in the interior of the constraint set. Unfortunately, in our vector-valued case the sub-solution structure is nor apparent and the argument does not seem to carry over.

Proof. We follow [ABG99], and note that

$$f^2(r)\left[\left(\frac{u(r)}{f(r)}\right)'\right]^2 = (u')^2 - 2\frac{uu'f'}{f} + u^2\frac{(f')^2}{f^2} = (u')^2 - \left(\frac{u^2}{f}\right)' f' \ .$$

Let $u_\pm \in \mathcal{C}_0^\infty((0,\infty))$ (if $n_\pm = 0$, take $u_\pm \in \mathcal{C}_0^\infty([0,\infty))$ instead). Then $[\frac{u_+^2}{f_+}, \frac{u_-^2}{f_-}]$ gives an admissible test function in the weak form of the (2.1),

$$\begin{aligned}
0 &= DE_{n_+,n_-}(f_+,f_-)\left[\frac{u_+^2}{f_+}, \frac{u_-^2}{f_-}\right] \\
&= \int_0^\infty \left[f_+'\left(\frac{u_+^2}{f_+}\right)' + f_-'\left(\frac{u_-^2}{f_-}\right)' + \frac{n_+^2}{r^2}f_+\frac{u_+^2}{f_+} + \frac{n_-^2}{r^2}f_-\frac{u_-^2}{f_-}\right] rdr \\
&\quad + \int_0^\infty \Bigg\{A_+(f_+^2 - t_+^2)f_+\frac{u_+^2}{f_+} + A_-(f_-^2 - t_-^2)f_+\frac{u_-^2}{f_-} \\
&\quad + B\left[(f_+^2 - t_+^2)f_-\frac{u_-^2}{f_-} + (f_-^2 - t_-^2)f_+\frac{u_+^2}{f_+}\right]\Bigg\} rdr \\
&= \int_0^\infty \Bigg\{(u_+')^2 - f_+^2\left[\left(\frac{u_+^2}{f_+}\right)'\right]^2 + (u_-')^2 - f_-^2\left[\left(\frac{u_-^2}{f_-}\right)'\right]^2 + \frac{n_+^2}{r^2}u_+^2 + \frac{n_-^2}{r^2}u_-^2 \\
&\quad + A_+(f_+^2 - t_+^2)u_+^2 + A_-(f_-^2 - t_-^2)u_-^2 \\
&\quad + B\left[(f_+^2 - t_+^2)u_-^2 + (f_-^2 - t_-^2)u_+^2\right]\Bigg\} rdr \ .
\end{aligned}$$

After the regrouping, we get the following useful identity:

$$\begin{aligned}
&\int_0^\infty \Bigg\{(u_+')^2 + (u_-')^2 + \frac{n_+^2}{r^2}u_+^2 + \frac{n_-^2}{r^2}u_-^2 \\
&\quad + A_+(f_+^2 - t_+^2)u_+^2 + A_-(f_-^2 - t_-^2)u_-^2 + B\left[(f_+^2 - t_+^2)u_-^2 + (f_-^2 - t_-^2)u_+^2\right]\Bigg\} rdr \\
&\qquad = \int_0^\infty \left\{f_+^2\left[\left(\frac{u_+}{f_+}\right)'\right]^2 + f_-^2\left[\left(\frac{u_-}{f_-}\right)'\right]^2\right\} rdr \geq 0.
\end{aligned}$$

Compare the formula of the second variation $D^2E_{n_+,n_-}(f_+,f_-)$ and the above identity, simply use the fact that $\lambda_s>0$ is the smallest eigenvalue of the matrix $\begin{bmatrix} A_+ & B \\ B & A_- \end{bmatrix}$, we obtain that

$$\begin{aligned}
&D^2E_{n_+,n_-}(f_+,f_-)[u_+,u_-] \\
&\quad = \int_0^\infty \left\{ f_+^2\left[\left(\frac{u_+'}{f_+}\right)'\right]^2 + f_-^2\left[\left(\frac{u_-'}{f_-}\right)'\right]^2 + 2(A_+f_+^2u_+^2 + A_-f_-^2u_-^2 + 2Bf_+f_-u_+u_-)\right\} rdr \\
&\quad \geq 2\int_0^\infty (A_+f_+^2u_+^2 + A_-f_-^2u_-^2 + 2Bf_+f_-u_+u_-) rdr \\
&\quad \geq 2\lambda_s \int_0^\infty (f_+^2u_+^2 + f_-^2u_-^2) rdr \\
&\quad \geq 0, \qquad (2.62)
\end{aligned}$$

valid for all $u_\pm \in \mathcal{C}_0^\infty((0,\infty))$ (or, $u_\pm \in \mathcal{C}_0^\infty([0,\infty))$ if the respective $n_\pm = 0$). The case of general $u_\pm \in X_1$ (or X_0, in case one of $n_\pm = 0$) follows by density. It is clear that $D^2E_{n_+,n_-}(f_+,f_-) \geq 0$. If it were zero for some $[u_+,u_-]$, then we would have $f_+u_+ = -f_-u_- = 0$ almost everywhere. Since $f_\pm(r)>0$ for $r>0$, we conclude that $D^2E_{n_+,n_-}(f_+,f_-)>0$ as claimed. □

Based on the above preliminaries, we establish the following theorem:

Theorem 2.6. *Assume* $\Psi(x) = [f_+(r)e^{in_+\theta}, f_-(r)e^{in_-\theta}]$ *is an equivariant solution satisfying* (1.3), *and* $A_+A_- - B^2 > 0$.

(i) *If* $B<0$, *then* $f_\pm'(r) \geq 0$ *for all* $r>0$ *for any degree* $[n_+,n_-]$.

(ii) *If* $B>0$, $n_+\geq 1$ *and* $n_-=0$, *then* $f_+'(r)\geq 0$ *and* $f_-'(r)\leq 0$ *for all* $r>0$.

Proof. Let $u_\pm(r) := f_\pm'(r)$. Differentiating (2.1), we get

$$\begin{aligned}
-f_\pm''' - \frac{1}{r}f_\pm'' + \frac{n_\pm^2+1}{r^2}f_\pm' + [A_\pm(f_\pm^2 - t_\pm^2) + B(f_\mp^2 - t_\mp^2)]f_\pm' \\
+ 2A_\pm f_\pm^2 f_\pm' + 2Bf_\pm f_\mp f_\mp' - \frac{2n_\pm^2}{r^3}f_\pm = 0,
\end{aligned}$$

i.e.

$$
\begin{aligned}
-u_\pm'' - \frac{1}{r}u_\pm' + \frac{n_\pm^2+1}{r^2}u_\pm + [A_\pm(f_\pm^2 - t_\pm^2) + B(f_\mp^2 - t_\mp^2)]u_\pm \\
+ 2A_\pm f_\pm^2 u_\pm + 2Bf_\pm f_\mp u_\mp - \frac{2n_\pm^2}{r^3} f_\pm = 0. \quad (2.63)
\end{aligned}
$$

Now define

$$
v_\pm = \min\{0, u_\pm\} \le 0, \qquad w_\pm = \max\{0, u_\pm\} \ge 0,
$$

then $u_\pm = v_\pm + w_\pm$.

In the following, we divide our argument into two cases. Firstly, assume $B < 0$ and $A_+A_- - B^2 > 0$. We multiply the respective equation in (2.63) by $v_\pm$, use the facts $v_+w_+ = 0$ and $v_-w_- = 0$ and integrate by parts. By (2.3) and (2.4), if $n_\pm \ge 1$, $u_\pm(r) = f_\pm'(r) > 0$ in some neighborhood $r \in (0, \delta)$. Thus, in case $n_\pm \ge 1$, $v_\pm$ is supported away from $r = 0$. By the proof of Theorem 2.4 we may conclude that $v_\pm \in X_{n_\pm}$. Moreover,

$$
\int_0^\infty v_\pm \left(u_\pm'' + \frac{1}{r}u_\pm' \right) rdr = -\int_0^\infty (v_\pm')^2 rdr,
$$

with no boundary terms. In case $n_\pm = 0$, we have $u_\pm \in X_0$ by the regularity of solutions, and $u_\pm(0) = f_\pm'(0) = 0$. The integration by parts formula above again holds with no boundary terms in this case as well. Therefore, with all above facts, we have the following equations

$$
\begin{aligned}
0 = \int_0^\infty \Bigg\{ (v_\pm')^2 + \frac{n_\pm^2+1}{r^2}v_\pm^2 + [A_\pm(f_\pm^2 - t_\pm^2) + B(f_\mp^2 - t_\mp^2)]v_\pm^2 \\
+2A_\pm f_\pm^2 v_\pm^2 + 2Bf_\pm f_\mp u_\mp v_\pm - \frac{2n_\pm^2}{r^3} f_\pm v_\pm \Bigg\} rdr\ .
\end{aligned}
$$

Then, use the facts $u_\pm = v_\pm + w_\pm$ and add above two equations together

$$
\begin{aligned}
0 = \int_0^\infty \Bigg\{ (v_+')^2 + (v_-')^2 + \frac{n_+^2}{r^2}v_+^2 + \frac{n_-^2}{r^2}v_-^2 \\
+ [A_+(f_+^2 - t_+^2) + B(f_-^2 - t_-^2)]v_+^2 + [A_-(f_-^2 - t_-^2) + B(f_+^2 - t_+^2)]v_-^2
\end{aligned}
$$

$$+ 2(A_+f_+^2v_+^2 + A_-f_-^2v_-^2) + 4Bf_+f_-v_+v_- + 2Bf_+f_-(w_-v_+ + w_+v_-) \\ -\frac{2n_+^2}{r^3}f_+v_+ - \frac{2n_-^2}{r^3}f_-v_-\Big\} rdr \ .$$

Compare with the formula of the second variation $D^2E_{n_+,n_-}(f_+,f_-)$ in (2.62), we obtain

$$0 = D^2E_{n_+,n_-}(f_+,f_-)[v_+,v_-] + 2B\int_0^\infty f_+f_-(w_-v_+ + w_+v_-)rdr \\ + \int_0^\infty \frac{1}{r^2}(v_+^2+v_-^2)rdr - 2\int_0^\infty \frac{1}{r^3}(n_+^2f_+v_+ + n_-^2f_-v_-)rdr \ .$$

Since $n_+^2f_+v_+$, $n_-^2f_-v_-$, w_-v_+ and w_+v_- are all negative, together with $B<0$, we get that

$$\begin{aligned} 0 &\le D^2E_{n_+,n_-}(f_+,f_-)[v_+,v_-] \\ &= -\int_0^\infty \frac{1}{r^2}(v_+^2+v_-^2)rdr + 2\int_0^\infty \frac{1}{r^3}(n_+^2f_+v_+ + n_-^2f_-v_-)rdr \\ &\quad - 2B\int_0^\infty f_+f_-(w_-v_+ + w_+v_-)rdr \\ &\le -\int_0^\infty \frac{1}{r^2}(v_+^2+v_-^2)rdr \\ &< 0, \end{aligned}$$

which is a contradiction unless $v_\pm \equiv 0$, i.e.unless $f_\pm'(r) \ge 0$ for all $r>0$. This proves (i).

Now assume $B>0$ and $n_+\ge 1$ and $n_-=0$. This time we multiply the equation of u_+ by v_+ and the equation of u_- by v_-, and integrate by parts again. Just as in the previous case, $w_-\in X_0$, and the boundary term in the integration will all vanish. We obtain that:

$$0 = \int_0^\infty \Big\{(v_+')^2 + \frac{n_+^2+1}{r^2}v_+^2 + [A_+(f_+^2-t_+^2) + B(f_-^2-t_-^2)]v_+^2 \\ +2A_+f_+^2v_+^2 + 2Bf_+f_-u_-v_+ - 2\frac{n_+^2}{r^3}f_+v_+\Big\} rdr \ ,$$

and

$$0 = \int_0^\infty \left\{ (w_-')^2 + \frac{n_-^2+1}{r^2} w_-^2 + [A_-(f_-^2 - t_-^2) + B(f_+^2 - t_+^2)] w_-^2 \right.$$
$$\left. + 2A_- f_-^2 w_-^2 + 2B f_- f_+ u_+ w_- - 2\frac{n_-^2}{r^3} f_- w_- \right\} r dr \ .$$

As in the first case, we add the above two equations and compare to the formula of the second variation $D^2 E_{n_+,n_-}(f_+, f_-)$ in (2.62), we get

$$\begin{aligned}
0 &\le D^2 E_{n_+,n_-}(f_+, f_-)[v_+, w_-] \\
&= -\int_0^\infty \frac{1}{r^2}(v_+^2 + w_-^2) r dr - 2B \int_0^\infty f_+ f_- (v_- v_+ + w_+ w_-) r dr \\
&\quad + 2\int_0^\infty \left(\frac{n_+^2}{r^3} f_+ v_+ + \frac{n_-^2}{r^3} f_- w_- \right) r dr \\
&\le -\int_0^\infty \frac{1}{r^2}(v_+^2 + w_-^2) r dr \\
&< 0 \ ,
\end{aligned}$$

which is a contradiction unless $v_+ \equiv 0$, $w_- \equiv 0$, i.e. $f_+'(r) \ge 0$ and $f_-'(r) \le 0$ for all $r > 0$. Therefore, we complete our proof. □

Meanwhile, we observe that method of Theorem 2.6 cannot work, since asymptotics shows that we don't have same behaviour for every $B > 0$. Method based on compactness, uniform estimates in the asymptotics result of radial solutions, we have the following result.

Theorem 2.7. *Let* $[f_+, f_-]$ *be the solution of* (2.1) *with degree pair* $[n_+, n_-]$ *at* ∞*, and* $n_\pm \ne 0$*. If* $\forall A_\pm > 0$*, there exists* $B_0 > 0$ *such that* $f_\pm'(r) > 0$ $\forall r \in (0, \infty)$*,* $\forall B \in [0, B_0]$*.*

Proof. Denote $f_\pm(r; B)$ the solution of (2.5) with coefficient B. When $B = 0$ the system decouples, and each of $f_\pm(r)$ solves a rescaled equation for the standard Ginzburg-Landau vortices. For these solutions, it is well-known that $f_\pm'(r; 0) > 0$ for $\forall r > 0$

(see [HH94]). Now choose an interval $[0, B_0]$ for which $a_\pm < 0$ (which are stated in Theorem 2.4) for $\forall B \in [0, B_0]$. By asymptotics in Theorem 2.4, there exists $R > 0$ such that

$$f'_\pm(r; B) > 0 \quad \text{for} \quad \forall r \geq R, \forall B \in [0, B_0], \tag{2.64}$$

here $f'_\pm(r; B)$ depends on both R and parameter B.

Suppose that this theorem is false, then there exists a sequence $B_k \to 0$ and $r_k \in (0, \infty)$ such that $f'_\pm(r_k; B_k) \leq 0$. By (2.64), we have that $0 < r_k \leq R$, so there is a subsequence r_{k_j} with $r_{k_j} \to r_0$. By elliptic regularity, $f_\pm(\cdot\,; B) \to f_\pm(\cdot\,; 0)$ in $\mathcal{C}^k_{\text{loc}}$ for all k, and hence

$$f'_\pm(r_0; 0) = \lim_{j\to\infty} f'_\pm(r_{k_j}; B_{k_j}) \leq 0,$$

which is a contradiction. We complete the proof. □

Chapter 3

Properties of Entire Solution

In this chapter we will discuss the entire solutions $\Psi(x) = [\psi_+(x), \psi_-(x)]$ for (1.2) in all of $\mathbb{R}^2$. As stated in introduction, solutions to (1.2) obtained by blowing up will satisfy the integrable condition (1.3). For $\Psi \in H^1_{\text{loc}}(\mathbb{R}^2; \mathbb{C}^2)$ satisfying (1.3), we denote

$$
\begin{aligned}
&E(\Psi; \Omega) := E_1(\Psi; \Omega) \\
&= \int_\Omega \frac{1}{2}|\nabla \Psi|^2 + \frac{1}{4}[A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 + 2B(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2)] \ .
\end{aligned}
\tag{3.1}
$$

As was the case for the classical Ginzburg-Landau equation [BMR94], the integral in (1.3) can be quantized.

Theorem 3.1. *Let (for any choice of* $[n_+, n_-]$*)* $\Psi = [\psi_+, \psi_-]$ *be a solution of* (1.2) *satisfying* (1.3)*. Then*

$$
\int_{\mathbb{R}^2} \{A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 + 2B(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2)\} dx = 2\pi(n_+^2 t_+^2 + n_-^2 t_-^2).
$$

Proof. First, we need to construct the associated Pohozaev identity to (1.2). Multiplying $\sum_j x_j \partial_j \psi_+$ to ψ_+-equation of (1.2) and integrating over B_r. We obtain the left hand side as follows:

$$
\begin{aligned}
&-\int_{B_r} \Delta\psi_+ \sum_j x_j\partial_j\psi_+ \\
&\quad= -\int_{S_r} \frac{\partial\psi_+}{\partial\nu}\sum_j x_j\partial_j\psi_+ + \int_{B_r}\sum_{k,j}\partial_k\psi_+\partial_k(x_j\partial_j\psi_+) \\
&\quad= -\int_{S_r} r\cdot\frac{\partial\psi_+}{\partial\nu}\cdot\frac{\partial\psi_+}{\partial r} + \int_{B_r}\sum_{k,j}\partial_k\psi_+(\partial_k\psi_+ + x_j\partial_k\partial_j\psi_+) \\
&\quad= -\int_{S_r} r\left|\frac{\partial\psi_+}{\partial\nu}\right|^2 + \int_{B_r}|\nabla\psi_+|^2 + \int_{B_r}\sum_{k,j}\partial_j(\partial_k\psi_+)\cdot\partial_k\psi_+\cdot x_j \\
&\quad= -\int_{S_r} r\left|\frac{\partial\psi_+}{\partial\nu}\right|^2 + \int_{B_r}|\nabla\psi_+|^2 \\
&\qquad + \left[\int_{S_r}\frac{1}{2}|\nabla\psi_+|^2\cdot(x\cdot\nu)ds - \int_{B_r}\frac{1}{2}|\nabla\psi_+|^2\cdot\operatorname{div}(x_1,x_2)dx\right] \\
&\quad= -\int_{S_r} r\left|\frac{\partial\psi_+}{\partial\nu}\right|^2 + \int_{B_r}|\nabla\psi_+|^2 \\
&\qquad + \left[\int_{S_r}\frac{1}{2}|\nabla\psi_+|^2\cdot r ds - \int_{B_r}|\nabla\psi_+|^2 dx\right] \\
&\quad= -\int_{S_r} r\left|\frac{\partial\psi_+}{\partial\nu}\right|^2 + \int_{S_r}\frac{r}{2}|\nabla\psi_+|^2 \\
&\quad= -\int_{S_r} r\left|\frac{\partial\psi_+}{\partial\nu}\right|^2 + \frac{r}{2}\int_{S_r}\left|\frac{\partial\psi_+}{\partial\nu}\right|^2 + \left|\frac{\partial\psi_+}{\partial\tau}\right|^2 \\
&\quad= \frac{r}{2}\int_{S_r}\left|\frac{\partial\psi_+}{\partial\tau}\right|^2 - \left|\frac{\partial\psi_+}{\partial\nu}\right|^2, \qquad (3.2)
\end{aligned}
$$

and the right hand side is

$$
\begin{aligned}
&-\int_{B_r}[A_+(|\psi_+|^2-t_+^2)+B(|\psi_-|^2-t_-^2)]\psi_+\sum_j x_j\partial_j\psi_+ \\
&\quad= -\int_{B_r}\sum_j[A_+(|\psi_+|^2-t_+^2)+B(|\psi_-|^2-t_-^2)]\cdot x_j\cdot\frac{1}{2}\partial_j\psi_+^2 \\
&\quad= -\int_{B_r}\sum_j x_j[A_+(|\psi_+|^2-t_+^2)+B(|\psi_-|^2-t_-^2)]\frac{1}{2}\partial_j(|\psi_+|^2-t_+^2) \\
&\quad= -\frac{1}{4}\int_{B_r}\sum_j x_j\cdot A_+\partial_j[(|\psi_+|^2-t_+^2)^2] - \frac{1}{2}\int_{B_r}\sum_j B(|\psi_-|^2-t_-^2)\cdot x_j\cdot\partial_j(|\psi_+|^2-t_+^2)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{4}\int_{S_r} A_+(|\psi_+|^2 - t_+^2)^2 \cdot (x\cdot\nu) + \frac{1}{4}\int_{B_r} A_+(|\psi_+|^2 - t_+^2)^2 \cdot \mathrm{div}(x_1, x_2) \\
&\quad - \frac{1}{2}\int_{B_r}\sum_j B(|\psi_-|^2 - t_-^2)\cdot x_j \cdot \partial_j(|\psi_+|^2 - t_+^2) \\
&= -\frac{r}{4}\int_{S_r} A_+(|\psi_+|^2 - t_+^2)^2 + \frac{1}{2}\int_{B_r} A_+(|\psi_+|^2 - t_+^2)^2 \\
&\quad - \frac{1}{2}\int_{B_r}\sum_j B(|\psi_-|^2 - t_-^2)\cdot x_j \cdot \partial_j(|\psi_+|^2 - t_+^2), \qquad (3.3)
\end{aligned}
$$

i.e. we have an identity for ψ_+:

$$
\begin{aligned}
&\frac{r}{2}\int_{S_r}\left|\frac{\partial\psi_+}{\partial\tau}\right|^2 - \left|\frac{\partial\psi_+}{\partial\nu}\right|^2 = \\
&\quad -\frac{r}{4}\int_{S_r} A_+(|\psi_+|^2 - t_+^2)^2 + \frac{1}{2}\int_{B_r} A_+(|\psi_+|^2 - t_+^2)^2 \\
&\quad -\frac{1}{2}\int_{B_r}\sum_j B(|\psi_-|^2 - t_-^2)\cdot x_j\cdot\partial_j(|\psi_+|^2 - t_+^2). \qquad (3.4)
\end{aligned}
$$

Similarly, we have an identity for ψ_-:

$$
\begin{aligned}
&\frac{r}{2}\int_{S_r}\left|\frac{\partial\psi_-}{\partial\tau}\right|^2 - \left|\frac{\partial\psi_-}{\partial\nu}\right|^2 = \\
&\quad -\frac{r}{4}\int_{S_r} A_-(|\psi_-|^2 - t_-^2)^2 + \frac{1}{2}\int_{B_r} A_-(|\psi_-|^2 - t_-^2)^2 \\
&\quad -\frac{1}{2}\int_{B_r}\sum_j B(|\psi_+|^2 - t_+^2)\cdot x_j\cdot\partial_j(|\psi_-|^2 - t_-^2). \qquad (3.5)
\end{aligned}
$$

Adding (3.4) and (3.5) together, we can obtain that

$$
\begin{aligned}
&\frac{r}{2}\int_{S_r}\left|\frac{\partial\psi_+}{\partial\tau}\right|^2 + \left|\frac{\partial\psi_-}{\partial\tau}\right|^2 - \frac{r}{2}\int_{S_r}\left|\frac{\partial\psi_+}{\partial\nu}\right|^2 + \left|\frac{\partial\psi_-}{\partial\nu}\right|^2 \\
&= -\frac{r}{4}\int_{S_r} A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 + \frac{1}{2}\int_{B_r} A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 \\
&\quad - \frac{B}{2}\int_{B_r}\sum_j \left[(|\psi_-|^2 - t_-^2)\cdot x_j\cdot\partial_j(|\psi_+|^2 - t_+^2) + (|\psi_+|^2 - t_+^2)\cdot x_j\cdot\partial_j(|\psi_-|^2 - t_-^2)\right] \\
&= -\frac{r}{4}\int_{S_r} A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 + \frac{1}{2}\int_{B_r} A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 \\
&\quad - \frac{B}{2}\int_{B_r}\sum_j \partial_j[(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2)]\cdot x_j
\end{aligned}
$$

$$= -\frac{r}{4}\int_{S_r} A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 + \frac{1}{2}\int_{B_r} A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2$$
$$- \frac{B}{2}\int_{S_r} (|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2)\cdot(x\cdot\nu) + \frac{B}{2}\int_{B_r} (|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2)\cdot \mathrm{div}(x_1, x_2)$$
$$= -\frac{r}{4}\int_{S_r} A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 + \frac{1}{2}\int_{B_r} A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2$$
$$- \frac{r}{2}\int_{S_r} B(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2) + \int_{B_r} B(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2),$$

i.e.

$$\int_{S_r} \left|\frac{\partial \psi_+}{\partial \tau}\right|^2 + \left|\frac{\partial \psi_-}{\partial \tau}\right|^2 + \frac{1}{2}\int_{S_r} G(\psi_+, \psi_-)$$
$$= \int_{S_r} \left|\frac{\partial \psi_+}{\partial \nu}\right|^2 + \left|\frac{\partial \psi_-}{\partial \nu}\right|^2 + \frac{1}{r}\int_{B_r} G(\psi_+, \psi_-), \quad (3.6)$$

with $G(\psi_+, \psi_-) = A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 + 2B(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2)$.

Now define $E(R) := \int_{B_R} G(\psi_+, \psi_-)dx$, $E := \int_{\mathbb{R}^2} G(\psi_+, \psi_-)dx$, and note that $E(R) \to E$ as $R \to \infty$.

Claim: $\frac{1}{\ln R}\int_0^R \frac{E(r)}{r}dr \to E$ as $R \to \infty$.

Indeed, it needs to show that $\int_0^R \frac{E(r)}{r}dr \to \infty$. It is clear that $E(r)$ is increasing as $r \to \infty$, i.e. $E(r) \geq E(R_0) > 0$ if $r \geq R_0$. Then, we have

$$\int_{R_0}^R \frac{E(r)}{r}dr \geq E(R_0)\int_{R_0}^R \frac{1}{r}dr = E(R_0)\ln\frac{R}{R_0} \to \infty, \quad \text{as} \quad R \to \infty.$$

Hence,

$$\lim_{R\to\infty}\frac{1}{\ln R}\int_0^R \frac{E(r)}{r}dr = \lim_{R\to\infty}\frac{E(R)/R}{1/R} = \lim_{R\to\infty} E(R) = E.$$

Integrating (3.6) over $r \in (0, R)$:

$$\int_{B_R} \left|\frac{\partial \psi_+}{\partial \tau}\right|^2 + \left|\frac{\partial \psi_-}{\partial \tau}\right|^2 + \frac{1}{2}\int_{B_R} G(\psi_+, \psi_-)$$
$$= \int_{B_R} \left|\frac{\partial \psi_+}{\partial \nu}\right|^2 + \left|\frac{\partial \psi_-}{\partial \nu}\right|^2 + \int_0^R \frac{E(r)}{r}dr, \quad (3.7)$$

i.e.

$$\int_{B_R} \left|\frac{\partial\psi_+}{\partial\tau}\right|^2 + \left|\frac{\partial\psi_-}{\partial\tau}\right|^2 + \frac{1}{2}E(R) = \int_{B_R} \left|\frac{\partial\psi_+}{\partial\nu}\right|^2 + \left|\frac{\partial\psi_-}{\partial\nu}\right|^2 + \int_0^R \frac{E(r)}{r}dr. \tag{3.8}$$

Now we estimate each term of the above Pohozaev identity. By (6.15), Step 2 and Step 3 in the proof of Lemma 6.3, we have

$$\frac{\partial\psi_\pm}{\partial\nu} = \frac{\partial\rho_\pm}{\partial\nu}e^{i\varphi} + i\rho_\pm\frac{\partial\varphi_\pm}{\partial\nu}e^{i\varphi},$$

$$\begin{aligned}\left|\frac{\partial\psi_\pm}{\partial\nu}\right|^2 &= \left|\frac{\partial\rho_\pm}{\partial\nu}\right|^2 + \rho_\pm^2\left|\frac{\partial\varphi_\pm}{\partial\nu}\right|^2 \\ &\le |\nabla\rho_\pm|^2 + \Lambda\left|\nabla\varphi_\pm\right|^2 < \infty. \end{aligned}\tag{3.9}$$

On the other hand,

$$\begin{aligned}\frac{\partial\psi_\pm}{\partial\tau} &= \nabla\psi_\pm\cdot\tau = \nabla(\rho_\pm e^{i\varphi})\cdot\tau \\ &= (\nabla\rho_\pm e^{i\varphi} + i\rho_\pm\nabla\varphi_\pm e^{i\varphi})\cdot\tau \\ &= \frac{\partial\rho_\pm}{\partial\tau}e^{i\varphi} + i\rho_\pm e^{i\varphi}\left(n_\pm\frac{V}{r} + \nabla\phi_\pm\right)\cdot\tau \\ &= \frac{\partial\rho_\pm}{\partial\tau}e^{i\varphi} + i\rho_\pm e^{i\varphi}\left(n_\pm\frac{V\cdot\tau}{r} + \frac{\partial\phi_\pm}{\partial\tau}\right),\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\partial\psi_\pm}{\partial\tau}\right|^2 &= \left|\frac{\partial\rho_\pm}{\partial\tau}\right|^2 + \rho_\pm^2\left|n_\pm\frac{V\cdot\tau}{r} + \frac{\partial\phi_\pm}{\partial\tau}\right|^2 \\ &\le |\nabla\rho_\pm|^2 + \rho_\pm^2\frac{n_\pm^2}{r^2} + \rho_\pm^2\left|\frac{\partial\phi_\pm}{\partial\tau}\right|^2 + 2\rho_\pm^2\frac{n_\pm}{r}\frac{\partial\phi_\pm}{\partial\tau} \\ &\le |\nabla\rho_\pm|^2 + \rho_\pm^2\frac{n_\pm^2}{r^2} + \rho_\pm^2\left|\nabla\phi_\pm\right|^2 + 2\rho_\pm^2\frac{n_\pm}{r}\nabla\phi_\pm. \end{aligned}\tag{3.10}$$

Then, by (3.10) we have that

$$\left|\left|\frac{\partial\psi_\pm}{\partial\tau}\right|^2 - t_\pm^2\frac{n_\pm^2}{r^2}\right| \le |\nabla\rho_\pm|^2 + |\rho_\pm^2 - t_\pm^2|\frac{n_\pm^2}{r^2} + \Lambda|\nabla\phi_\pm|^2 + 2\Lambda\frac{n_\pm}{r}|\nabla\phi_\pm|,$$

which implies that

$$
\begin{aligned}
&\int_{B_R\setminus B_{R_0}} \left| \left|\frac{\partial \psi_\pm}{\partial \tau}\right|^2 - t_\pm^2 \frac{n_\pm^2}{r^2} \right| \\
&\quad \le \int_{B_R\setminus B_{R_0}} |\nabla \rho_\pm|^2 + \Lambda |\nabla \phi_\pm|^2 + n_\pm^2 \left[\int_{B_R\setminus B_{R_0}} |\rho_\pm^2 - t_\pm^2|^2\right]^{1/2} \left[\int_{B_R\setminus B_{R_0}} \frac{1}{r^4}\right]^{1/2} \\
&\qquad + 2\Lambda n_\pm \left[\int_{B_R\setminus B_{R_0}} \frac{1}{r^2}\right]^{1/2} \left[\int_{B_R\setminus B_{R_0}} |\nabla \phi_\pm|^2\right]^{1/2} \\
&\quad \le C_1 + C_2 n_\pm^2 \left[\int_{B_R\setminus B_{R_0}} |\rho_\pm^2 - t_\pm^2|^2\right]^{1/2} + 2\Lambda n_\pm C_3 \left[\int_{B_R\setminus B_{R_0}} \frac{1}{r^2}\right]^{1/2} \\
&\quad = C_4 + 2\Lambda \sqrt{2\pi} n_\pm C_3 \left(\ln \frac{R}{R_0}\right)^{1/2} \\
&\quad = C_4 + C_5 (\ln R)^{1/2}, \qquad (3.11)
\end{aligned}
$$

where C_4 and C_5 are some constants.

On the other hand,

$$
\int_{B_R\setminus B_{R_0}} t_\pm^2 \frac{n_\pm^2}{r^2} = \int_0^{2\pi} \int_{R_0}^{R} t_\pm^2 \frac{n_\pm^2}{r^2} r\,dr\,d\theta = 2\pi t_\pm^2 n_\pm^2 (\ln R - \ln R_0). \qquad (3.12)
$$

And we know that by (3.12)

$$
\begin{aligned}
\int_{B_R} \left|\frac{\partial \phi_\pm}{\partial \tau}\right|^2 &= \int_{B_R\setminus B_{R_0}} \left|\frac{\partial \phi_\pm}{\partial \tau}\right|^2 + \int_{B_{R_0}} \left|\frac{\partial \phi_\pm}{\partial \tau}\right|^2 \\
&= \int_{B_R\setminus B_{R_0}} \left(\left|\frac{\partial \phi_\pm}{\partial \tau}\right|^2 - \frac{n_\pm^2 t_\pm^2}{r^2} \right) + \int_{B_{R_0}} \left|\frac{\partial \phi_\pm}{\partial \tau}\right|^2 + \int_{B_R\setminus B_{R_0}} \frac{n_\pm^2 t_\pm^2}{r^2} \\
&= \int_{B_R\setminus B_{R_0}} \left(\left|\frac{\partial \phi_\pm}{\partial \tau}\right|^2 - \frac{n_\pm^2 t_\pm^2}{r^2} \right) + \int_{B_{R_0}} \left|\frac{\partial \phi_\pm}{\partial \tau}\right|^2 \\
&\qquad + 2\pi t_\pm^2 n_\pm^2 \ln R - 2\pi t_\pm^2 n_\pm^2 \ln R_0,
\end{aligned}
$$

it yields that

$$
\left| \int_{B_R} \left|\frac{\partial \phi_\pm}{\partial \tau}\right|^2 - 2\pi t_\pm^2 n_\pm^2 \ln R \right|
$$

$$
\begin{aligned}
&= \left| \int_{B_R \backslash B_{R_0}} \left(\left| \frac{\partial \phi_\pm}{\partial \tau} \right|^2 - \frac{n_\pm^2 t_\pm^2}{r^2} \right) + \int_{B_{R_0}} \left| \frac{\partial \phi_\pm}{\partial \tau} \right|^2 - 2\pi t_\pm^2 n_\pm^2 \ln R_0 \right| \\
&\leq \int_{B_R \backslash B_{R_0}} \left| \left| \frac{\partial \phi_\pm}{\partial \tau} \right|^2 - \frac{n_\pm^2 t_\pm^2}{r^2} \right| + C \\
&\leq C_5 (\ln R)^{1/2} + C_6,
\end{aligned}
$$

where C_5, C_6 are some constants. Therefore, we have

$$
\left| \frac{1}{\ln R} \int_{B_R} \left| \frac{\partial \phi_\pm}{\partial \tau} \right|^2 - 2\pi t_\pm^2 n_\pm^2 \right| \leq C_5 (\ln R)^{-1/2} + C_6 (\ln R)^{-1}. \tag{3.13}
$$

Together Claim, (3.9) and (3.13), dividing (3.8) by $\ln R$ and letting R go to infinity, we obtain that

$$
E = 2\pi (t_+^2 n_+^2 + t_-^2 n_-^2),
$$

i.e.

$$
\int_{\mathbb{R}^2} \{A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 + 2B(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2)\} dx = 2\pi (t_+^2 n_+^2 + t_-^2 n_-^2).
$$

We complete the proof of Theorem 3.1. □

Measured on all of $\mathbb{R}^2$, the energy (defined in (3.1)) of such a solution diverges. However, when properly renormalized, there is a well-defined core energy $W_0(\Psi)$, defined as the limit below:

Lemma 3.2. *Let Ψ solve* (1.2) *in $\mathbb{R}^2$ satisfying* (1.3)*. Then, the following limit exists:*

$$
W_0(\Psi) := \lim_{R \to \infty} [E(\Psi; \mathbb{D}_R) - \pi (t_+^2 n_+^2 + t_-^2 n_-^2) \ln R]. \tag{3.14}
$$

Proof. By the estimates in Lemma 6.3, there exists $R_0 > 0$ for which the solution $\Psi(x)$ admits a decomposition for $|x| \geq R_0$ in the following form:

$$
\left.
\begin{array}{c}
\psi_\pm(x) = \rho_\pm(x) e^{i\alpha_\pm(x)}, \quad \alpha_\pm(x) = n_\pm \theta + \chi_\pm(x), \\
\text{with } \chi_\pm(x) \to \phi_\pm(\text{constants}) \text{ uniformly as } |x| \to \infty.
\end{array}
\right\} \tag{3.15}
$$

Without loss of generality, we may take $\phi_\pm = 0$, so $\chi_\pm(x) \to 0$ uniformly as $|x| \to \infty$. Moreover, the estimates in Lemma 6.3 also imply that for large r,

$$|\rho_\pm - t_\pm| \le \frac{c}{r^2} \tag{3.16}$$

$$|\nabla \rho_\pm(x)| \le \frac{c}{r^3} \tag{3.17}$$

$$\int_{|x|\ge R_0} \left[|\nabla\rho_\pm|^2 + |\nabla\chi_\pm|^2\right] < \infty, \tag{3.18}$$

for R_0 sufficiently large that the decomposition (3.15) holds.

First, by the argument in [BMR94], we observe that for any $R > R_0$, by integration by parts we have

$$\int_{\mathbb{D}_R\setminus\mathbb{D}_{R_0}} \nabla\chi_\pm \cdot \nabla\theta = 0 \ . \tag{3.19}$$

In the following we denote $\mathbb{D}_R \setminus \mathbb{D}_{R_0}$ as $A_{R\setminus R_0}$ for convenience.

Then, with the similar computation in Lemma 6.3, we have

$$\begin{aligned} E(\Psi; A_{R\setminus R_0}) &= \frac{1}{2}\int_{A_{R\setminus R_0}} \sum_\pm \left[|\nabla\rho_\pm|^2 + \rho_\pm^2(n_\pm^2|\nabla\theta|^2 + |\nabla\chi_\pm|^2 + 2n_\pm\nabla\theta\cdot\nabla\chi_\pm)\right] \\ &\quad + \frac{1}{4}\int_{A_{R\setminus R_0}} [A_+(\rho_+^2 - t_+^2)^2 + A_-(\rho_-^2 - t_-^2)^2 + 2B(\rho_+^2 - t_+^2)(\rho_-^2 - t_-^2)] \\ &= \frac{1}{2}\int_{A_{R\setminus R_0}} \sum_\pm \left[|\nabla\rho_\pm|^2 + \rho_\pm^2\left(\frac{n_\pm^2}{r^2} + |\nabla\chi_\pm|^2 + 2n_\pm\nabla\theta\cdot\nabla\chi_\pm\right)\right] \\ &\quad + \frac{1}{4}\int_{A_{R\setminus R_0}} [A_+(\rho_+^2 - t_+^2)^2 + A_-(\rho_-^2 - t_-^2)^2 + 2B(\rho_+^2 - t_+^2)(\rho_-^2 - t_-^2)] \ . \end{aligned}$$

Therefore, by (3.19),

$$\begin{aligned} &E(\Psi; A_{R\setminus R_0}) - \pi(t_+^2n_+^2 + t_-^2n_-^2)\ln\left(\frac{R}{R_0}\right) \\ &= \frac{1}{2}\int_{A_{R\setminus R_0}} \sum_\pm \left[|\nabla\rho_\pm|^2 + \rho_\pm^2\left(\frac{n_\pm^2}{r^2} + |\nabla\chi_\pm|^2 + 2n_\pm\nabla\theta\cdot\nabla\chi_\pm\right)\right] \\ &\quad + \frac{1}{4}\int_{A_{R\setminus R_0}} [A_+(\rho_+^2 - t_+^2)^2 + A_-(\rho_-^2 - t_-^2)^2 + 2B(\rho_+^2 - t_+^2)(\rho_-^2 - t_-^2)] \\ &\quad - \pi(t_+^2n_+^2 + t_-^2n_-^2)\ln\left(\frac{R}{R_0}\right) \end{aligned}$$

$$= \frac{1}{2}\int_{A_{R\setminus R_0}} \sum_{\pm} \left\{ |\nabla\rho_\pm|^2 + (\rho_\pm^2 - t_\pm^2)\frac{n_\pm^2}{r^2} + \rho_\pm^2|\nabla\chi_\pm|^2 + 2n_\pm(\rho_\pm^2 - t_\pm^2)\nabla\theta\cdot\nabla\chi_\pm \right\}$$
$$+ \frac{1}{2}\int_{A_{R\setminus R_0}} \sum_{\pm} \left\{ t_\pm^2\frac{n_\pm^2}{r^2} + 2n_\pm t_\pm^2 \nabla\theta\cdot\nabla\chi_\pm \right\}$$
$$+ \frac{1}{4}\int_{A_{R\setminus R_0}} [A_+(\rho_+^2 - t_+^2)^2 + A_-(\rho_-^2 - t_-^2)^2 + 2B(\rho_+^2 - t_+^2)(\rho_-^2 - t_-^2)]$$
$$- \pi(t_+^2 n_+^2 + t_-^2 n_-^2)\ln\left(\frac{R}{R_0}\right), \tag{3.20}$$

and

$$\int_{A_{R\setminus R_0}} \sum_{\pm} t_\pm^2 \frac{n_\pm^2}{r^2} = \int_0^{2\pi}\int_{R_0}^{R} t_+^2 n_+^2 \frac{1}{r^2} r dr d\theta + \int_0^{2\pi}\int_{R_0}^{R} t_-^2 n_-^2 \frac{1}{r^2} r dr d\theta$$
$$= 2\pi(t_+^2 n_+^2 + t_-^2 n_-^2)\int_{R_0}^{R}\frac{1}{r^2} r dr$$
$$= 2\pi(t_+^2 n_+^2 + t_-^2 n_-^2)\ln\left(\frac{R}{R_0}\right). \tag{3.21}$$

Hence, by (3.19)-(3.21), we have that

$$E(\Psi; A_{R\setminus R_0}) - \pi(t_+^2 n_+^2 + t_-^2 n_-^2)\ln\left(\frac{R}{R_0}\right) = \int_{A_{R\setminus R_0}} f, \tag{3.22}$$

with

$$f = \frac{1}{2}\sum_{\pm} \left\{ |\nabla\rho_\pm|^2 + (\rho_\pm^2 - t_\pm^2)\frac{n_\pm^2}{r^2} + \rho_\pm^2|\nabla\chi_\pm|^2 + 2n_\pm(\rho_\pm^2 - t_\pm^2)\nabla\theta\cdot\nabla\chi_\pm \right\}$$
$$+ \frac{1}{4}[A_+(\rho_+^2 - t_+^2)^2 + A_-(\rho_-^2 - t_-^2)^2 + 2B(\rho_+^2 - t_+^2)(\rho_-^2 - t_-^2)]. \tag{3.23}$$

Now, using the estimates (3.16)-(3.18), it follows that

$$\int_{A_{R\setminus R_0}} |\nabla\rho_\pm|^2 \le \int_{A_{R\setminus R_0}} \frac{c}{r^3} = 2\pi c\int_{R_0}^{R}\frac{1}{r^3} r dr = 2\pi c\frac{R - R_0}{RR_0} < \infty, \tag{3.24}$$

$$\int_{A_{R\setminus R_0}} (\rho_\pm^2 - t_\pm^2)\frac{n_\pm^2}{r^2} \le \int_{A_{R\setminus R_0}} |\rho_\pm + t_\pm||\rho_\pm - t_\pm|\frac{n_\pm^2}{r^2}$$
$$\le M\int_{A_{R\setminus R_0}} \frac{c}{r^2}\frac{n_\pm^2}{r^2} = 2\pi c\int_{R_0}^{R}\frac{1}{r^4} r dr = \pi c\frac{R^2 - R_0^2}{R^2 R_0^2} < \infty. \tag{3.25}$$

From (3.18), suppose that $\lim_{R_0\to\infty}\int_{|x|\geq R_0} |\nabla\rho_\pm|^2 + |\nabla\chi_\pm|^2 = c$ for some finite constant c. Also we have $\lim_{R_0\to\infty}\int_{|x|\geq \frac{R_0}{2}} |\nabla\rho_\pm|^2 + |\nabla\chi_\pm|^2 = c$. Therefore, it implies that $\lim_{R_0\to\infty}\int_{\frac{R_0}{2}\leq|x|\leq R_0} |\nabla\chi_\pm|^2 = 0$ for R_0 sufficiently large. By the uniform boundness of $\rho_\pm$, we have

$$\int_{A_{R\setminus R_0}} \rho_\pm^2|\nabla\chi_\pm|^2 \leq c\int_{A_{R\setminus R_0}} |\nabla\chi_\pm|^2 < \infty. \tag{3.26}$$

By the uniform boundness of $\rho_\pm$ and (3.19), it follows that

$$\begin{aligned}\int_{A_{R\setminus R_0}} (\rho_\pm^2 - t_\pm^2)\nabla\theta\cdot\nabla\chi_\pm &\leq \int_{A_{R\setminus R_0}} (|\rho_\pm^2| + |t_\pm^2|)\nabla\theta\cdot\nabla\chi_\pm \\ &\leq c\int_{A_{R\setminus R_0}} \nabla\theta\cdot\nabla\chi_\pm = 0,\end{aligned} \tag{3.27}$$

$$\begin{aligned}\int_{A_{R\setminus R_0}} (\rho_\pm^2 - t_\pm^2)^2 &= \int_{A_{R\setminus R_0}} (\rho_\pm + t_\pm)^2(\rho_\pm - t_\pm)^2 \\ &\leq c\int_{A_{R\setminus R_0}} (\rho_\pm - t_\pm)^2 \leq c\int_{A_{R\setminus R_0}} \frac{1}{r^4} = c\frac{R^2 - R_0^2}{R^2R_0^2},\end{aligned} \tag{3.28}$$

$$\begin{aligned}\int_{A_{R\setminus R_0}} (\rho_+^2 - t_+^2)(\rho_-^2 - t_-^2) &= \int_{A_{R\setminus R_0}} (\rho_+ + t_+)(\rho_- + t_-)(\rho_+ - t_+)(\rho_- - t_-) \\ &\leq \int_{A_{R\setminus R_0}} \frac{1}{r^4} = c\frac{R^2 - R_0^2}{R^2R_0^2}.\end{aligned} \tag{3.29}$$

Combining (3.24)-(3.29), we obtain that f is integrable in $\mathbb{R}^2\setminus\mathbb{D}_{R_0}$; writing

$$E(\Psi; A_{R\setminus R_0}) - \pi(t_+^2n_+^2 + t_-^2n_-^2)\ln\left(\frac{R}{R_0}\right) = \int_{A_{R\setminus R_0}} f, \tag{3.30}$$

i.e.

$$\begin{aligned}&E(\Psi;\mathbb{D}_R) - \pi(t_+^2n_+^2 + t_-^2n_-^2)\ln R \\ &\qquad = E(\Psi;\mathbb{D}_{R_0}) - \pi(t_+^2n_+^2 + t_-^2n_-^2)\ln R_0 + \int_{A_{R\setminus R_0}} f\ ,\end{aligned}$$

we conclude that the limit $W_0(\Psi)$ exists as $R\to\infty$. $\square$

Next we will end this chapter with two patching arguments as in [Sha94]. These patching discussions will be useful tools in the proof of Proposition 4.2 in next chapter.

Lemma 3.3. *Let* Ψ *be an entire solution of* (1.2) *satisfying* (1.3)*. Then, there exists a family* $\tilde{\Psi}_R \in H^1(\mathbb{D}_R;\mathbb{C}^2)$ *of functions such that*

$$\tilde{\Psi}_R(x) = \Psi(x) \qquad \text{for } |x| \le \tfrac{R}{2},$$
$$\tilde{\Psi}_R(x) = \left[t_+ e^{i(n_+\theta+\phi_+)}, t_- e^{i(n_-\theta+\phi_-)}\right], \qquad \text{on } |x| = R \text{ for constants } \phi_\pm \in \mathbb{R},$$
$$\int_{\mathbb{D}_R} |\nabla\tilde{\Psi}_R|^2 = \int_{\mathbb{D}_R} |\nabla\Psi|^2 + o(1),$$
$$\int_{\mathbb{D}_R} A_+(|\tilde{\psi}_+|^2 - t_+^2)^2 + A_-(|\tilde{\psi}_-|^2 - t_-^2)^2 + 2B(|\tilde{\psi}_+|^2 - t_+^2)(|\tilde{\psi}_-|^2 - t_-^2)$$
$$= \int_{\mathbb{D}_R} A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 + 2B(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2) + o(1),$$

as $R\to\infty$*. In particular,*

$$E(\tilde{\Psi};\mathbb{D}_R) = E(\Psi;\mathbb{D}_R) + o(1), \quad \text{as } R\to\infty.$$

Proof. The proof follows [ABM], we provide details here for completeness. We apply the same decomposition (3.15) for $\psi_\pm(x)$, $|x|\ge R_0$, as in the proof of Lemma 3.2. Define the cutoff function

$$L(r) = L_R(r) = \begin{cases} 0, & \text{if } r \le \frac{R}{2}, \\ \frac{\ln(2r/R)}{\ln 2}, & \text{if } \frac{R}{2} \le r \le R, \\ 1, & \text{if } r \ge R, \end{cases}$$

and

$$L'(r) = L'_R(r) = \begin{cases} 0, & \text{if } r \le \frac{R}{2}, \\ \frac{1}{r\ln 2}, & \text{if } \frac{R}{2} \le r \le R, \\ 1, & \text{if } r \ge R, \end{cases}$$

We define our modification

$$\left.\begin{aligned} \tilde{\Psi}_R = \left(\tilde{\psi}_{R,+}, \tilde{\psi}_{R,-}\right), \qquad \tilde{\psi}_{R,\pm} = \tilde{\rho}_\pm(x)\exp[i\tilde{\alpha}_\pm], \\ \tilde{\rho}_\pm(x) = t_\pm L(r) + (1-L(r))\rho_\pm(x), \qquad \tilde{\alpha}_\pm(x) = n_\pm\theta + (1-L(r))\chi_\pm(x). \end{aligned}\right\} \tag{3.31}$$

It is easy to have that

(1) when $|x| \le \frac{R}{2}, L(r) = 0,$

$$\begin{aligned} \tilde{\psi}_{R,\pm}(x) &= [t_\pm L(r) + (1-L(r))\rho_\pm(x)]\exp[i(n_\pm\theta + (1-L(r))\chi_\pm(x))] \\ &= \rho_\pm(x)\exp[i(n_\pm\theta + \chi_\pm(x))] = \psi_{R,\pm}(x), \end{aligned}$$

i.e. $\tilde{\Psi}_R(x) = \Psi(x)$,

(2) when $|x| = R, L(r) = 1, \tilde{\psi}_{R,\pm}(x) = t_\pm\exp[in_\pm\theta]$,

$\tilde{\Psi}_R(x) = [t_+\exp(i(n_+\theta+\phi_+)), t_-\exp(i(n_-\theta+\phi_-))]$ for some constants $\phi_\pm \in \mathbb{R}$.

Next, we want to show the other equalities in this lemma.

$$\begin{aligned} |\nabla\tilde{\rho}_\pm|^2 &= |t_\pm L'\nabla r + (1-L)\nabla\rho_\pm - L'\nabla r\rho_\pm|^2 \\ &= |(t_\pm - \rho_\pm)L'\nabla r + (1-L)\nabla\rho_\pm|^2 \\ &= |t_\pm - \rho_\pm|^2\frac{1}{r^2(\ln 2)^2}|\nabla r|^2 + (1-L)^2|\nabla\rho_\pm|^2 \\ &\quad + 2(1-L)(t_\pm - \rho_\pm)\frac{1}{r\ln 2}\nabla\rho_\pm\nabla r \\ &= |t_\pm - \rho_\pm|^2\frac{1}{r^2(\ln 2)^2} + (1-L)^2|\nabla\rho_\pm|^2 + \frac{2(1-L)}{r\ln 2}(t_\pm - \rho_\pm)\frac{\partial\rho_\pm}{\partial r}, \end{aligned}$$

then we have

$$\begin{aligned} |\nabla\tilde{\rho}_\pm|^2 - |\nabla\rho_\pm|^2 = (t_\pm - \rho_\pm)^2\frac{1}{r^2(\ln 2)^2} + (L^2 - 2L)|\nabla\rho_\pm|^2 \\ + \frac{2(1-L)}{r\ln 2}(t_\pm - \rho_\pm)\frac{\partial\rho_\pm}{\partial r}, \end{aligned}$$

Now let $\mathcal{A} = \{x\big|R/2 < |x| < R\}$ be an annular, we have that

$$\int_{\mathcal{A}}\big||\nabla\tilde{\rho}_\pm|^2 - |\nabla\rho_\pm|^2\big| \le \int_{\mathcal{A}}|t_\pm - \rho_\pm|^2\frac{1}{r^2(\ln 2)^2} + \int_{\mathcal{A}}|L^2 - 2L||\nabla\rho_\pm|^2$$

$$
\begin{aligned}
&+\int_{\mathcal{A}} \frac{2|1-L|}{r\ln 2}|t_\pm-\rho_\pm|\left|\frac{\partial\rho_\pm}{\partial r}\right| \\
&\le \int_{\mathcal{A}} \frac{c}{r^4}\cdot\frac{1}{r^2(\ln 2)^2}+\int_{\mathcal{A}} 3\cdot\frac{c}{r^6}+\int_{\mathcal{A}}\frac{2}{r\ln 2}\cdot\frac{c}{r^2}\cdot\frac{c}{r^3} \\
&= \frac{2\pi c}{(\ln 2)^2}\int_{R/2}^{R}\frac{1}{r^6}rdr+6\pi c\int_{R/2}^{R}\frac{1}{r^6}rdr+\frac{4\pi c}{\ln 2}\int_{R/2}^{R}\frac{1}{r^6}rdr \\
&= \frac{2\pi c}{(\ln 2)^2}\cdot\frac{2^4-1}{4R^4}+6\pi c\cdot\frac{2^4-1}{4R^4}+\frac{4\pi c}{\ln 2}\cdot\frac{2^4-1}{4R^4} \\
&\longrightarrow 0 \quad \text{as} \quad R\to\infty. \qquad (3.32)
\end{aligned}
$$

Now, we need to estimate $\tilde{\rho}_\pm^2|\nabla\tilde{\alpha}_\pm|^2-\rho_\pm^2|\nabla\alpha_\pm|^2$.

Firstly,

$$
\nabla\tilde{\alpha}_\pm = n_\pm\nabla\theta+(1-L)\nabla\chi_\pm - L'\cdot\nabla r\cdot\chi_\pm,
$$

then

$$
\begin{aligned}
|\nabla\tilde{\alpha}_\pm|^2 &= n_\pm^2|\nabla\theta|^2+(1-L)^2|\nabla\chi_\pm|^2+\frac{1}{r^2(\ln 2)^2}|\nabla r|^2\cdot|\chi_\pm|^2 \\
&\quad + 2n_\pm(1-L)\nabla\theta\cdot\nabla\chi_\pm-\frac{2n_\pm}{r\ln 2}\chi_\pm\nabla\theta\cdot\nabla r-\frac{2(1-L)}{r\ln 2}\chi_\pm\nabla\chi_\pm\cdot\nabla r \\
&= \frac{n_\pm^2}{r^2}+(1-L)^2|\nabla\chi_\pm|^2+\frac{1}{r^2(\ln 2)^2}|\chi_\pm|^2 \\
&\quad + 2n_\pm(1-L)\nabla\theta\cdot\nabla\chi_\pm-\frac{2(1-L)}{r\ln 2}\chi_\pm(\nabla\chi_\pm\cdot\hat{r}).
\end{aligned}
$$

Secondly, we have $\nabla\alpha_\pm = n_\pm\nabla\theta+\nabla\chi_\pm$, and

$$
\begin{aligned}
|\nabla\alpha_\pm|^2 &= n_\pm^2|\nabla\theta|^2+|\nabla\chi_\pm|^2+2n_\pm\nabla\theta\cdot\nabla\chi_\pm \\
&= \frac{n_\pm^2}{r^2}+|\nabla\chi_\pm|^2+2n_\pm\nabla\theta\cdot\nabla\chi_\pm.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|\nabla\tilde{\alpha}_\pm|^2-|\nabla\alpha_\pm|^2 &= (L^2-2L)|\nabla\chi_\pm|^2+\frac{1}{r^2(\ln 2)^2}|\chi_\pm|^2 \\
&\quad - 2n_\pm L\nabla\theta\cdot\nabla\chi_\pm-\frac{2(1-L)}{r\ln 2}\chi_\pm(\nabla\chi_\pm\cdot\hat{r}).
\end{aligned}
$$

By (3.18), we also have that $\int_{|x|\geq \frac{R_0}{2}} |\nabla\rho_\pm|^2 + |\nabla\chi_\pm|^2 < \infty$. Together with (3.18), it follows that

$$\lim_{R_0\to\infty}\int_{\frac{R_0}{2}\leq|x|\leq R_0} |\nabla\rho_\pm|^2 + |\nabla\chi_\pm|^2 = 0,$$

i.e.

$$\lim_{R_0\to\infty}\int_{\mathcal{A}} |\nabla\rho_\pm|^2 + |\nabla\chi_\pm|^2 = 0,$$

which implies that $\int_{\mathcal{A}} |\nabla\chi_\pm|^2 \to 0$ as $R\to\infty$. Meanwhile, we have that

$$\begin{aligned}\int_{\mathcal{A}} \frac{1}{r^2}|\chi_\pm|^2 &= \int_0^{2\pi}\int_{R/2}^{R} \frac{1}{r^2}|\chi_\pm|^2 r\,dr\,d\theta = 2\pi\int_{R/2}^{R}\frac{1}{r^2}|\chi_\pm|^2 r\,dr\\ &\leq 2\pi\int_{R/2}^{R}\frac{1}{r}dr\cdot\sup_{|x|\geq R/2}|\chi_\pm|^2 = 2\pi\ln 2\sup_{|x|\geq R/2}|\chi_\pm|^2\to 0,\end{aligned}$$

since $\chi_\pm\to 0$ uniformly as $R\to\infty$. Then, combining above estimates and applying Cauchy-Schwartz and Young's inequalities, we obtain that

$$\begin{aligned}\int_{\mathcal{A}} |\rho_\pm^2(|\nabla\tilde{\alpha}_\pm|^2 - |\nabla\alpha_\pm|^2)| &\leq \int_{\mathcal{A}} |\rho_\pm^2|\big||\nabla\tilde{\alpha}_\pm|^2 - |\nabla\alpha_\pm|^2\big| \leq c\int_{\mathcal{A}}\big||\nabla\tilde{\alpha}_\pm|^2 - |\nabla\alpha_\pm|^2\big|\\ &\leq c\Big\{\int_{\mathcal{A}}|L^2-2L||\nabla\chi_\pm|^2 + \int_{\mathcal{A}}\frac{1}{(\ln 2)^2}\frac{1}{r^2}|\chi_\pm|^2\\ &\quad+\int_{\mathcal{A}}2n_\pm L\nabla\chi_\pm\cdot\nabla\theta + \int_{\mathcal{A}}\frac{2|1-L|}{r\ln 2}\chi_\pm|\nabla\chi_\pm\cdot\hat{r}|\Big\}\\ &\leq c\Big\{\int_{\mathcal{A}}|\nabla\chi_\pm|^2 + \frac{1}{(\ln 2)^2}\int_{\mathcal{A}}\frac{1}{r^2}|\chi_\pm|^2 + \frac{2}{\ln 2}\int_{\mathcal{A}}\frac{\chi_\pm}{r}|\nabla\chi_\pm|\Big\}\\ &\leq c\Big\{\int_{\mathcal{A}}|\nabla\chi_\pm|^2 + \int_{\mathcal{A}}\frac{1}{r^2}|\chi_\pm|^2\Big\}\\ &\longrightarrow 0 \quad\text{as}\quad R\to\infty. \qquad (3.33)\end{aligned}$$

Next, by (3.16), uniform boundness of $\rho_\pm$ and Cauchy-Schwartz inequality, we deduce that

$$\begin{aligned}\int_{\mathcal{A}}\big||\nabla\alpha_\pm|^2(\tilde{\rho}_\pm^2-\rho_\pm^2)\big| &\leq \int_{\mathcal{A}}|\nabla\alpha_\pm|^2|\tilde{\rho}_\pm^2-\rho_\pm^2|\\ &= \int_{\mathcal{A}}|\nabla\alpha_\pm|^2|L(t_\pm-\rho_\pm)(\tilde{\rho}_\pm+\rho_\pm)|\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\mathcal{A}} \left| \frac{n_\pm^2}{r^2} + |\nabla\chi_\pm|^2 + 2n_\pm \nabla\theta \cdot \nabla\chi_\pm \right| \cdot |L| \cdot |t_\pm - \rho_\pm| \cdot |\tilde{\rho}_\pm + \rho_\pm| \\
&\leq \int_{\mathcal{A}} \frac{c}{r^2} \left| \frac{n_\pm^2}{r^2} + |\nabla\chi_\pm|^2 + 2\frac{n_\pm}{r}(\nabla\chi_\pm \cdot \hat{\theta}) \right| \\
&\leq \int_{\mathcal{A}} \frac{c}{r^2} \left(\frac{n_\pm^2}{r^2} + |\nabla\chi_\pm|^2 + 2\frac{n_\pm}{r}|\nabla\chi_\pm| \right) \\
&= c \int_0^{2\pi} \int_{R/2}^{R} \left(\frac{n_\pm^2}{r^4} + \frac{1}{r^2}|\nabla\chi_\pm|^2 + 2\frac{n_\pm}{r}|\nabla\chi_\pm| \right) r dr d\theta \\
&\leq c \left[\frac{3\pi n_\pm^2}{R^2} + \frac{4\pi}{R} \int_{R/2}^{R} |\nabla\chi_\pm|^2 dr + 2n_\pm \sqrt{2\pi \ln 2} \left(\int_{\mathcal{A}} |\nabla\chi_\pm|^2 \right)^{1/2} \right] \\
&\longrightarrow 0 \quad \text{as} \quad R \to \infty.
\end{aligned} \tag{3.34}
$$

Also, by Young's inequality, we get that

$$
\begin{aligned}
\int_{\mathcal{A}} |\tilde{\rho}_\pm^2 - \rho_\pm^2| |\nabla\alpha_\pm|^2 &= \int_{\mathcal{A}} |\tilde{\rho}_\pm + \rho_\pm| |\tilde{\rho}_\pm - \rho_\pm| |\nabla\alpha_\pm|^2 \\
&= \int_{\mathcal{A}} |\tilde{\rho}_\pm + \rho_\pm| |\tilde{\rho}_\pm - \rho_\pm| \left(\frac{n_\pm^2}{r^2} + |\nabla\chi_\pm|^2 + 2n_\pm \nabla\theta \cdot \nabla\chi_\pm \right) \\
&\leq \int_{\mathcal{A}} (|\tilde{\rho}_\pm| + |\rho_\pm|) |L(t_\pm - \rho_\pm)| \left(\frac{n_\pm^2}{r^2} + |\nabla\chi_\pm|^2 + 2n_\pm |\nabla\theta| \cdot |\nabla\chi_\pm| \right) \\
&\leq \int_{\mathcal{A}} \frac{c}{r^2} \left(\frac{n_\pm^2}{r^2} + |\nabla\chi_\pm|^2 + \frac{n_\pm}{r^4} + n_\pm |\nabla\chi_\pm|^2 \right) \\
&= \int_{\mathcal{A}} \frac{c}{r^2} \left(\frac{n_\pm^2}{r^2} + \frac{n_\pm}{r^4} + (n_\pm + 1)|\nabla\chi_\pm|^2 \right) \\
&\leq \int_{\mathcal{A}} \frac{c}{r^4} + \frac{c}{r^6} + \frac{c}{r^2}|\nabla\chi_\pm|^2 \\
&= \int_0^{2\pi} \int_{R/2}^{R} \frac{c}{r^4} r dr d\theta + \int_0^{2\pi} \int_{R/2}^{R} \frac{c}{r^6} r dr d\theta + \int_{\mathcal{A}} \frac{c}{r^2} |\nabla\chi_\pm|^2 \\
&= \frac{3\pi}{R^2} + \frac{15\pi}{2R^4} + \int_{\mathcal{A}} \frac{c}{r^2} |\nabla\chi_\pm|^2 \\
&\leq \frac{3\pi}{R^2} + \frac{15\pi}{2R^4} + \frac{4c}{R^2} \int_{\mathcal{A}} |\nabla\chi_\pm|^2 \\
&\longrightarrow 0 \quad \text{as} \quad R \to \infty.
\end{aligned} \tag{3.35}
$$

By Cauchy-Schwartz and Young's inequalities again, we get that

$$
\int_{\mathcal{A}} |\tilde{\rho}_\pm^2 - \rho_\pm^2| \big| |\nabla\tilde{\alpha}_\pm|^2 - |\nabla\alpha_\pm|^2 \big|
$$

$$
\begin{aligned}
&= \int_{\mathcal{A}} |\tilde{\rho}_{\pm} + \rho_{\pm}||\tilde{\rho}_{\pm} - \rho_{\pm}|\big||\nabla\tilde{\alpha}_{\pm}|^2 - |\nabla\alpha_{\pm}|^2\big| \\
&\leq \int_{\mathcal{A}} \frac{c}{r^2}\left|(L^2 - 2L)|\nabla\chi_{\pm}|^2 + \frac{1}{r^2(\ln 2)^2}|\chi_{\pm}|^2 - 2n_{\pm}L\nabla\chi_{\pm}\cdot\nabla\theta - \frac{2(1-L)}{r\ln 2}\chi_{\pm}(\chi_{\pm}\cdot\hat{r})\right| \\
&\leq \int_{\mathcal{A}} \left(3|\nabla\chi_{\pm}|^2 + \frac{1}{r^2(\ln 2)^2}|\chi_{\pm}|^2 + 2n_{\pm}|\nabla\chi_{\pm}|\cdot|\nabla\theta| + \frac{2}{r\ln 2}|\chi_{\pm}||\nabla\chi_{\pm}|\right) \\
&= \int_{\mathcal{A}} \frac{3c}{r^2}|\nabla\chi_{\pm}|^2 + \frac{c}{(\ln 2)^2}\frac{1}{r^4}|\chi_{\pm}|^2 + \frac{2cn_{\pm}}{r^4}|\nabla\chi_{\pm}| + \frac{2c}{\ln 2}\frac{1}{r^3}|\chi_{\pm}|\cdot|\nabla\chi_{\pm}| \\
&\leq \int_{\mathcal{A}} \frac{3c}{r^2}|\nabla\chi_{\pm}|^2 + 2c\pi\int_{R/2}^{R}\frac{1}{r^3}dr\cdot\sup_{|x|\geq R/2}|\chi_{\pm}|^2 + c\int_{\mathcal{A}}\frac{1}{r^6}|\chi_{\pm}|^2 + c\int_{\mathcal{A}}|\nabla\chi_{\pm}|^2 \\
&\leq \frac{4c}{R^2}\int_{\mathcal{A}}|\nabla\chi_{\pm}|^2 + \frac{3\pi}{R^2}\sup_{|x|\geq R/2}|\chi_{\pm}|^2 + \frac{15}{4R^4}\sup_{|x|\geq R/2}|\chi_{\pm}|^2 + c\int_{\mathcal{A}}|\nabla\chi_{\pm}|^2 \\
&\longrightarrow 0 \quad \text{as} \quad R\to\infty. \qquad (3.36)
\end{aligned}
$$

By (6.23)-(6.24) and (6.26), we obtain that

$$
\begin{aligned}
&\int_{\mathcal{A}} \big|\tilde{\rho}_{\pm}^2|\nabla\tilde{\alpha}_{\pm}|^2 - \rho_{\pm}^2|\nabla\alpha_{\pm}|^2\big| \\
&\quad \leq \int_{\mathcal{A}} \big|\rho_{\pm}^2(|\nabla\tilde{\alpha}_{\pm}|^2 - |\nabla\alpha_{\pm}|^2)\big| + \big||\nabla\alpha_{\pm}|^2(\tilde{\rho}_{\pm}^2 - \rho_{\pm}^2)\big| + \big|(\tilde{\rho}_{\pm}^2 - \rho_{\pm}^2)(|\nabla\tilde{\alpha}_{\pm}|^2 - |\nabla\alpha_{\pm}|^2)\big| \\
&\quad \longrightarrow 0 \quad \text{as} \quad R\to\infty. \qquad (3.37)
\end{aligned}
$$

From the previous estimates, we have that

$$
\int_{\mathcal{A}}\sum_{\pm}\left(|\nabla\tilde{\rho}_{\pm}|^2 - |\nabla\rho_{\pm}|^2 + \tilde{\rho}_{\pm}^2|\nabla\tilde{\alpha}_{\pm}|^2 - \rho_{\pm}^2|\nabla\alpha_{\pm}|^2\right) = o(1). \qquad (3.38)
$$

Therefore,

$$
\begin{aligned}
\int_{\mathbb{D}_R}|\nabla\tilde{\Psi}_R|^2 - |\nabla\Psi|^2 &= \int_{|x|\leq R/2}|\nabla\tilde{\Psi}_R|^2 - |\nabla\Psi|^2 + \int_{\mathcal{A}}|\nabla\tilde{\Psi}_R|^2 - |\nabla\Psi|^2 \\
&= \int_{|x|\leq R/2}|\nabla\Psi|^2 - |\nabla\Psi|^2 + \int_{\mathcal{A}}\sum_{\pm}\left(|\nabla\tilde{\rho}_{\pm}|^2 - |\nabla\rho_{\pm}|^2 + \tilde{\rho}_{\pm}^2|\nabla\tilde{\alpha}_{\pm}|^2 - \rho_{\pm}^2|\nabla\alpha_{\pm}|^2\right) \\
&= o(1) \quad \text{as} \quad R\to\infty.
\end{aligned}
$$

And, since $\tilde{\psi}_\pm = \psi_\pm$ on $|x| \le R/2$, we obtain that

$$\begin{aligned}
&\int_{\mathbb{D}_R} \Big\{ A_+[(|\tilde{\psi}_+|^2 - t_+^2)^2 - (|\psi_+|^2 - t_+^2)^2] + A_-[(|\tilde{\psi}_-|^2 - t_-^2)^2 - (|\psi_-|^2 - t_-^2)^2] \\
&\qquad +2B[(|\tilde{\psi}_+|^2 - t_+^2)(|\tilde{\psi}_-|^2 - t_-^2) - (|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2)] \Big\} \\
&= \left\{ \int_{|x|\le R/2} + \int_{\mathcal{A}} \right\} \Big\{ A_+[(|\tilde{\psi}_+|^2 - t_+^2)^2 - (|\psi_+|^2 - t_+^2)^2] + A_-[(|\tilde{\psi}_-|^2 - t_-^2)^2 - (|\psi_-|^2 - t_-^2)^2] \\
&\qquad +2B[(|\tilde{\psi}_+|^2 - t_+^2)(|\tilde{\psi}_-|^2 - t_-^2) - (|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2)] \Big\} \\
&= \int_{\mathcal{A}} \Big\{ A_+[(|\tilde{\psi}_+|^2 - t_+^2)^2 - (|\psi_+|^2 - t_+^2)^2] + A_-[(|\tilde{\psi}_-|^2 - t_-^2)^2 - (|\psi_-|^2 - t_-^2)^2] \\
&\qquad +2B[(|\tilde{\psi}_+|^2 - t_+^2)(|\tilde{\psi}_-|^2 - t_-^2) - (|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2)] \Big\}. \qquad (3.39)
\end{aligned}$$

Meanwhile, by (3.16) and the uniform bound of $\rho_\pm$ we have that

$$\begin{aligned}
&A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 + 2B(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2) \\
&\quad = A_+(\rho_+ + t_+)^2(\rho_+ - t_+)^2 + A_-(\rho_- + t_-)^2(\rho_- - t_-)^2 \\
&\qquad + 2B(\rho_+ + t_+)(\rho_- + t_-)(\rho_+ - t_+)(\rho_- - t_-) \\
&\quad \le \frac{c}{r^4},
\end{aligned}$$

i.e.

$$A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 + 2B(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2) = o\left(\frac{1}{r^4}\right). \qquad (3.40)$$

On the other hand, we have

$$\begin{aligned}
|\tilde{\psi}_\pm|^2 - t_\pm^2 = \tilde{\rho}_\pm^2 - t_\pm^2 &= (\tilde{\rho}_\pm^2 - t_\pm^2)(\tilde{\rho}_\pm^2 - t_\pm^2) \\
&= [t_\pm L + (1-L)\rho_\pm + t_\pm][t_\pm L + (1-L)\rho_\pm - t_\pm] \\
&= [(1+L)t_\pm + (1-L)\rho_\pm][(1-L)(\rho_\pm - t_\pm)],
\end{aligned}$$

hence,

$$A_+(|\tilde{\psi}_+|^2 - t_+^2)^2 + A_-(|\tilde{\psi}_-|^2 - t_-^2)^2 + 2B(|\tilde{\psi}_+|^2 - t_+^2)(|\tilde{\psi}_-|^2 - t_-^2) = o\left(\frac{1}{r^4}\right). \qquad (3.41)$$

Therefore, by (3.39)-(3.41), we obtain that

$$\int_{\mathcal{A}} \Big| [A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 + 2B(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2)]$$
$$-[A_+(|\tilde{\psi}_+|^2 - t_+^2)^2 + A_-(|\tilde{\psi}_-|^2 - t_-^2)^2 + 2B(|\tilde{\psi}_+|^2 - t_+^2)(|\tilde{\psi}_-|^2 - t_-^2)] \Big|$$
$$\longrightarrow 0 \quad \text{as} \quad R \to \infty,$$

which implies that

$$\int_{\mathbb{D}_R} \big\{ [A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 + 2B(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2)]$$
$$-[A_+(|\tilde{\psi}_+|^2 - t_+^2)^2 + A_-(|\tilde{\psi}_-|^2 - t_-^2)^2 + 2B(|\tilde{\psi}_+|^2 - t_+^2)(|\tilde{\psi}_-|^2 - t_-^2)] \Big\} = o(1). \quad (3.42)$$

Then it follows that

$$|E(\tilde{\Psi}_R; \mathbb{D}_R) - E(\Psi_R; \mathbb{D}_R)| = |E(\tilde{\Psi}_R; \mathcal{A}) - E(\Psi_R; \mathcal{A})| \longrightarrow 0,$$

as $R \to \infty$, which completes the proof. □

Repeating the same procedure as in the proof of Lemma 3.3, but with the choice

$$\hat{\rho}_\pm(x) = t_\pm(1 - L_R(r)) + L_R(r)\rho_\pm(x), \qquad \hat{\alpha}_\pm(x) = n_\pm\theta + L_R(r)\chi_\pm,$$

and L_R as the same as in the proof of Lemma 3.3, we obtain the opposite patching result, connecting a given solution outside a large ball $\mathbb{D}_R$ to the symmetric boundary condition on $\partial\mathbb{D}_{R/2}$:

Lemma 3.4. *Let Ψ be an entire solution of (1.2) satisfying (1.3). Then, there exists a family $\hat{\Psi}_R \in H^1(\mathbb{D}_R \setminus \mathbb{D}_{R/2}; \mathbb{C}^2)$ of functions so that*

$$\hat{\Psi}_R(x) = \Psi(x) \qquad \textit{for } |x| = R,$$

$$\hat{\Psi}_R(x) = \left[t_+ e^{i(n_+\theta + \phi_+)}, t_- e^{i(n_-\theta + \phi_-)}\right], \qquad \textit{on } |x| = \tfrac{R}{2} \textit{ for constants } \phi_\pm \in \mathbb{R},$$

$$\int_{\mathbb{D}_R \setminus \mathbb{D}_{R/2}} |\nabla \hat{\Psi}_R|^2 = \int_{\mathbb{D}_R \setminus \mathbb{D}_{R/2}} |\nabla \Psi|^2 + o(1),$$

$$\int_{\mathbb{D}_R \setminus \mathbb{D}_{R/2}} A_+(|\hat{\psi}_+|^2 - t_+^2)^2 + A_-(|\hat{\psi}_-|^2 - t_-^2)^2 + 2B(|\hat{\psi}_+|^2 - t_+^2)(|\hat{\psi}_-|^2 - t_-^2)$$
$$= \int_{\mathbb{D}_R \setminus \mathbb{D}_{R/2}} A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 + 2B(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2) + o(1),$$

as $R \to \infty$. In particular,

$$E(\hat{\Psi}_R; \mathbb{D}_R \setminus \mathbb{D}_{R/2}) = E(\Psi; \mathbb{D}_R \setminus \mathbb{D}_{R/2}) + o(1) \quad \textit{as } R \to \infty.$$

Proof. The proof follows [ABM], we provide details here for completeness. We apply the same decomposition (3.15) for $\psi_\pm(x)$, $|x| \geq R_0$, as in the proof of Lemma 3.2. Define the cutoff function

$$L(r) = L_R(r) = \begin{cases} 0, & \text{if } r \leq \frac{R}{2}, \\ \frac{\ln(2r/R)}{\ln 2}, & \text{if } \frac{R}{2} \leq r \leq R, \\ 1, & \text{if } r \geq R, \end{cases}$$

and

$$L'(r) = L'_R(r) = \begin{cases} 0, & \text{if } r \leq \frac{R}{2}, \\ \frac{1}{r \ln 2}, & \text{if } \frac{R}{2} \leq r \leq R, \\ 1, & \text{if } r \geq R, \end{cases}$$

We define our modification

$$\left.\begin{aligned} \hat{\Psi}_R = \left(\hat{\psi}_{R,+}, \hat{\psi}_{R,-}\right), \qquad \hat{\psi}_{R,\pm} = \hat{\rho}_\pm(x) \exp[i\hat{\alpha}_\pm], \\ \hat{\rho}_\pm(x) = t_\pm(1 - L_R(r)) + L_R(r)\rho_\pm(x), \qquad \hat{\alpha}_\pm(x) = n_\pm\theta + L_R(r)\chi_\pm(x). \end{aligned}\right\} \tag{3.43}$$

It is easy to have that

(1) when $|x| = R, L(r) = 1$,

$\hat{\psi}_{R,\pm} = \rho_\pm(x)e^{i(n_\pm\theta + \chi_\pm)}$, i.e. $\hat{\Psi}_R(x) = \Psi(x)$;

(2) when $|x| \leq \dfrac{R}{2}, L(r) = 0$,

$\hat{\psi}_{R,\pm}(x) = t_\pm e^{in_\pm\theta}$,

i.e. $\hat{\Psi}_R(x) = [t_+ e^{i(n_+\theta + \phi_+)}, t_- e^{i(n_-\theta + \phi_-)}]$ for some constants $\phi_\pm \in \mathbb{R}$.

Next, we want to show the other equalities in this lemma.

$$|\nabla\hat{\rho}_\pm|^2 = |-t_\pm L'\nabla r + L\nabla\rho_\pm + L'\nabla r\rho_\pm|^2$$

$$= |(\rho_\pm - t_\pm)L'\nabla r + L\nabla\rho_\pm|^2$$
$$= |\rho_\pm - t_\pm|^2 \frac{1}{r^2(\ln 2)^2} + L^2|\nabla\rho_\pm|^2 + \frac{2L}{r\ln 2}(\rho_\pm - t_\pm)\frac{\partial\rho_\pm}{\partial r},$$

then we have

$$|\nabla\hat{\rho}_\pm|^2 - |\nabla\rho_\pm|^2 = |\rho_\pm - t_\pm|^2 \frac{1}{r^2(\ln 2)^2} + (L^2-1)|\nabla\rho_\pm|^2 + \frac{2L}{r\ln 2}(\rho_\pm - t_\pm)\frac{\partial\rho_\pm}{\partial r},$$

Now let $\mathcal{A} = \{x \big| R/2 < |x| < R\}$ be an annulus, we have that

$$\begin{aligned}
\int_{\mathcal{A}} \big||\nabla\hat{\rho}_\pm|^2 - |\nabla\rho_\pm|^2\big| &\le \int_{\mathcal{A}} |\rho_\pm - t_\pm|^2 \frac{1}{r^2(\ln 2)^2} + \int_{\mathcal{A}} |L^2-1||\nabla\rho_\pm|^2 \\
&\quad + \int_{\mathcal{A}} \frac{2|L|}{r\ln 2}|\rho_\pm - t_\pm| \left|\frac{\partial\rho_\pm}{\partial r}\right| \\
&\le \int_{\mathcal{A}} \frac{C}{r^4}\cdot\frac{1}{r^2(\ln 2)^2} + \int_{\mathcal{A}} 2\cdot\frac{C}{r^6} + \int_{\mathcal{A}} \frac{2}{r\ln 2}\cdot\frac{C}{r^2}\cdot\frac{C}{r^3} \\
&= \frac{2\pi C}{(\ln 2)^2}\int_{R/2}^{R} \frac{1}{r^6} r\,dr + 4\pi C \int_{R/2}^{R} \frac{1}{r^6} r\,dr + \frac{4\pi C}{\ln 2}\int_{R/2}^{R} \frac{1}{r^6} r\,dr \\
&= \frac{2\pi C}{(\ln 2)^2}\cdot\frac{2^4-1}{4R^4} + 4\pi C\cdot\frac{2^4-1}{4R^4} + \frac{4\pi C}{\ln 2}\cdot\frac{2^4-1}{4R^4} \\
&\longrightarrow 0 \quad \text{as} \quad R\to\infty.
\end{aligned} \tag{3.44}$$

Now, we need to estimate $\hat{\rho}_\pm^2|\nabla\hat{\alpha}_\pm|^2 - \rho_\pm^2|\nabla\alpha_\pm|^2$. It is clear that $\nabla\hat{\alpha}_\pm = n_\pm\nabla\theta + L'\nabla r\cdot\chi_\pm + L\nabla\chi_\pm$, then

$$\begin{aligned}
|\nabla\hat{\alpha}_\pm|^2 &= n_\pm^2|\nabla\theta|^2 + L^2|\nabla\chi_\pm|^2 + \frac{1}{r^2(\ln 2)^2}|\nabla r|^2\cdot|\chi_\pm|^2 \\
&\quad + 2n_\pm L\nabla\theta\cdot\nabla\chi_\pm + \frac{2n_\pm}{r\ln 2}\chi_\pm\nabla\theta\cdot\nabla r + \frac{2L}{r\ln 2}\chi_\pm\nabla\chi_\pm\cdot\nabla r \\
&= \frac{n_\pm^2}{r^2} + L^2|\nabla\chi_\pm|^2 + \frac{1}{r^2(\ln 2)^2}|\chi_\pm|^2 \\
&\quad + 2n_\pm L\nabla\theta\cdot\nabla\chi_\pm + \frac{2n_\pm}{r\ln 2}\chi_\pm\nabla\theta\cdot\nabla r + \frac{2L}{r\ln 2}\chi_\pm\left[\nabla\chi_\pm\cdot\hat{r}\right],
\end{aligned}$$

$$\begin{aligned}
|\nabla\alpha_\pm|^2 &= n_\pm^2|\nabla\theta|^2 + |\nabla\chi_\pm|^2 + 2n_\pm\nabla\theta\cdot\nabla\chi_\pm \\
&= \frac{n_\pm^2}{r^2} + |\nabla\chi_\pm|^2 + 2n_\pm\nabla\theta\cdot\nabla\chi_\pm,
\end{aligned}$$

and

$$|\nabla\hat{\alpha}_{\pm}|^2 - |\nabla\alpha_{\pm}|^2 = (L^2-1)|\nabla\chi_{\pm}|^2 + \frac{1}{r^2(\ln 2)^2}|\chi_{\pm}|^2 \\ + 2n_{\pm}(L-1)\nabla\theta\cdot\nabla\chi_{\pm} + \frac{2L}{r\ln 2}\chi_{\pm}(\nabla\chi_{\pm}\cdot\hat{r}).$$

By (3.18), we also have that $\displaystyle\int_{|x|\geq\frac{R_0}{2}} |\nabla\rho_{\pm}|^2 + |\nabla\chi_{\pm}|^2 < \infty$. Together with (3.18), it follows that

$$\lim_{R_0\to\infty}\int_{\frac{R_0}{2}\leq|x|\leq R_0} |\nabla\rho_{\pm}|^2 + |\nabla\chi_{\pm}|^2 = 0,$$

i.e.

$$\lim_{R_0\to\infty}\int_{\mathcal{A}} |\nabla\rho_{\pm}|^2 + |\nabla\chi_{\pm}|^2 = 0,$$

which implies that $\displaystyle\int_{\mathcal{A}} |\nabla\chi_{\pm}|^2 \to 0$ as $R\to\infty$. Meanwhile, we have that

$$\int_{\mathcal{A}}\frac{1}{r^2}|\chi_{\pm}|^2 = \int_0^{2\pi}\int_{R/2}^{R}\frac{1}{r^2}|\chi_{\pm}|^2 r\,dr\,d\theta = 2\pi\int_{R/2}^{R}\frac{1}{r^2}|\chi_{\pm}|^2 r\,dr \\ \leq 2\pi\int_{R/2}^{R}\frac{1}{r}dr\cdot\sup_{|x|\geq R/2}|\chi_{\pm}|^2 = 2\pi\ln 2\sup_{|x|\geq R/2}|\chi_{\pm}|^2 \to 0,$$

since $\chi_{\pm}\to 0$ uniformly as $R\to\infty$.Then, combining above estimates and applying Cauchy-Schwartz and Young's inequalities, we obtain that

$$\begin{aligned}
&\int_{\mathcal{A}}\left|\hat{\rho}_{\pm}^2(|\nabla\hat{\alpha}_{\pm}|^2 - |\nabla\alpha_{\pm}|^2)\right| \\
&\leq \int_{\mathcal{A}}|\hat{\rho}_{\pm}|^2\cdot\left||\nabla\hat{\alpha}_{\pm}|^2 - |\nabla\alpha_{\pm}|^2\right| \\
&= \int_{\mathcal{A}}|t_{\pm}(1-L)+L\rho_{\pm}|^2\cdot\left||\nabla\hat{\alpha}_{\pm}|^2 - |\nabla\alpha_{\pm}|^2\right| \\
&\leq C\int_{\mathcal{A}}\left||\nabla\hat{\alpha}_{\pm}|^2 - |\nabla\alpha_{\pm}|^2\right| \\
&\leq C\int_{\mathcal{A}}\Big[|L^2-1||\nabla\chi_{\pm}|^2 + \frac{1}{r^2(\ln 2)^2}|\chi_{\pm}|^2 \\
&\qquad +2n_{\pm}|L-1|\nabla\theta\cdot\nabla\chi_{\pm} + \frac{2L}{r\ln 2}\chi_{\pm}(\nabla\chi_{\pm}\cdot\hat{r})\Big] \\
&\leq C\int_{\mathcal{A}}|\nabla\chi_{\pm}|^2 + C\int_{\mathcal{A}}\frac{1}{r^2}|\chi_{\pm}|^2
\end{aligned}$$

$$\longrightarrow 0 \quad \text{as} \quad R \to \infty. \tag{3.45}$$

Next, by (3.16), uniform boundness of $\rho_\pm$ and Cauchy-Schwartz inequality, we deduce that

$$\begin{aligned}
\int_{\mathcal{A}} \big| |\nabla \alpha_\pm|^2 (\hat{\rho}_\pm^2 - \rho_\pm^2) \big| &\leq \int_{\mathcal{A}} |\nabla \alpha_\pm|^2 |\hat{\rho}_\pm^2 - \rho_\pm^2| \\
&= \int_{\mathcal{A}} |\nabla \alpha_\pm|^2 |1-L|^2 |t_\pm^2 - \rho_\pm^2| \\
&\leq C \int_{\mathcal{A}} \frac{1}{r^4} \left| \frac{n_\pm^2}{r^2} + |\nabla \chi_\pm|^2 + 2n_\pm \nabla\theta \cdot \nabla \chi_\pm \right| \\
&\leq C \int_{\mathcal{A}} \frac{1}{r^6} + \frac{1}{r^4} |\nabla \chi_\pm|^2 \\
&\longrightarrow 0 \qquad \text{as} \quad R \to \infty.
\end{aligned} \tag{3.46}$$

By (3.45)-(3.46), we obtain that

$$\begin{aligned}
&\int_{\mathcal{A}} \left| \hat{\rho}_\pm^2 |\nabla \hat{\alpha}_\pm|^2 - \rho_\pm^2 |\nabla \alpha_\pm|^2 \right| \\
&\qquad \leq \int_{\mathcal{A}} |\hat{\rho}_\pm|^2 \left| |\nabla \hat{\alpha}_\pm|^2 - |\nabla \alpha_\pm|^2 \right| + \int_{\mathcal{A}} |\nabla \alpha_\pm|^2 \left| \hat{\rho}_\pm^2 - \rho_\pm^2 \right| \\
&\quad \longrightarrow 0 \quad \text{as} \quad R \to \infty.
\end{aligned}$$

Therefore,

$$\int_{\mathbb{D}_R \setminus \mathbb{D}_{R/2}} |\nabla \hat{\Psi}_R|^2 = \int_{\mathbb{D}_R \setminus \mathbb{D}_{R/2}} |\nabla \Psi|^2 + o(1).$$

Meanwhile, by (3.16) and the uniform bound of $\rho_\pm$ we have that

$$\begin{aligned}
&A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 + 2B(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2) \\
&\qquad = A_+(\rho_+ + t_+)^2(\rho_+ - t_+)^2 + A_-(\rho_- + t_-)^2(\rho_- - t_-)^2 \\
&\qquad\qquad + 2B(\rho_+ + t_+)(\rho_- + t_-)(\rho_+ - t_+)(\rho_- - t_-) \\
&\qquad \leq \frac{c}{r^4},
\end{aligned}$$

i.e.

$$A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 + 2B(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2) = o\left(\frac{1}{r^4}\right). \tag{3.47}$$

On the other hand, we have

$$|\hat{\psi}_\pm|^2 - t_\pm^2 = \hat{\rho}_\pm^2 - t_\pm^2 = L(\rho_\pm - t_\pm)\left(t_\pm(2-L) + L\rho_\pm\right),$$

hence

$$A_+(|\hat{\psi}_+|^2 - t_+^2)^2 + A_-(|\hat{\psi}_-|^2 - t_-^2)^2 + 2B(|\hat{\psi}_+|^2 - t_+^2)(|\hat{\psi}_-|^2 - t_-^2) = o\left(\frac{1}{r^4}\right). \quad (3.48)$$

Therefore, by (3.47)-(3.48), we obtain that

$$\int_{\mathbb{D}_R\setminus\mathbb{D}_{R/2}} \Big\{A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 + 2B(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2) \\ - \left[A_+(|\hat{\psi}_+|^2 - t_+^2)^2 + A_-(|\hat{\psi}_-|^2 - t_-^2)^2 + 2B(|\hat{\psi}_+|^2 - t_+^2)(|\hat{\psi}_-|^2 - t_-^2)\right]\Big\} = o(1).$$

Then, it follows that

$$\left|E(\hat{\Psi}_R;\mathcal{A}) - E(\Psi;\mathcal{A})\right| \longrightarrow 0,$$

as $R\to\infty$, i.e. $E(\hat{\Psi}_R;\mathbb{D}_R\setminus\mathbb{D}_{R/2}) = E(\Psi;\mathbb{D}_R\setminus\mathbb{D}_{R/2}) + o(1)$ as $R\to\infty$, which completes the proof. $\square$

Chapter 4

Local Minimizing Solution

In this chapter , we would like to relate solutions to (1.2) to energy minimization. If $\Omega \subset \mathbb{R}^2$ is a bounded domain, we may define an energy locally by

$$E(\Psi;\Omega) = \int_\Omega \frac{1}{2}|\nabla\Psi|^2 + \frac{1}{4}[A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 + 2B(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2)].$$

And we study solutions which are locally minimizing in the sense of De Giorgi:

Definition 4.1. *We say that Ψ is a locally minimizing solution of* (1.2) *if* (1.3) *holds and if for every bounded regular domain $\Omega \subset \mathbb{R}^2$,*

$$E(\Psi;\Omega) \leq E(\Phi;\Omega)$$

holds for every $\Phi = (\phi_+, \phi_-) \in H^1(\Omega;\mathbb{C}^2)$ with $\Phi\big|_{\partial\Omega} = \Psi\big|_{\partial\Omega}$.

Following Lemma 3.3 and Lemma 3.4 directly, we have the following proposition, which tells us that the local minimizers with degree $|n_\pm| \geq 2$ are unstable.

Proposition 4.2. *A nontrivial local minimizer of* (1.2) *must have degrees $n_\pm \in \{0, \pm 1\}$.*

Proof. The proof follows that of Theorem 2 in [Sha94] and Proposition 3.2 in [ABM]. Let Ψ be a local minimizer. If either $|n_+| > 2$ or $|n_-| > 2$, we must have $n_+^2 + n_-^2 > |n_+| + |n_-|$, and hence Lemma 3.2 implies that for all R sufficiently large,

$$\lim_{R\to\infty} \frac{E(\Psi; \mathbb{D}_R)}{\ln R} = \pi(t_+^2 n_+^2 + t_-^2 n_-^2) > \pi(t_+^2|n_+| + t_-^2|n_-|). \tag{4.1}$$

By Lemma 3.4, for R large we obtain $\widetilde{\Psi}$ with constants $\phi_\pm$ defined in $\mathbb{D}_R \setminus \mathbb{D}_{R/2}$. Denote by

$$G_*(U;\Omega) := G_\epsilon(U;\Omega) = \int_\Omega \frac{1}{2}|\nabla U|^2 + \frac{1}{4\epsilon^2}(|U|^2-1)^2,$$

the Ginzburg-Landau energy for $U \in H^1_{\mathrm{loc}}(\Omega;\mathbb{C})$.

For our positive definite condition $A_+A_- - B^2 > 0$, we have that there exist two positive eigenvalues λ_s and λ_M associated to the following matrix $\begin{bmatrix} A_+ & B \\ B & A_- \end{bmatrix}$, and λ_s denotes the smaller one, λ_M denotes the larger one. Then we have that

$$\lambda_s|\xi|^2 \le A_+\xi_+^2 + A_-\xi_-^2 + 2B\xi_+\xi_- \le \lambda_M|\xi|^2,$$

where $\xi = [\xi_+, \xi_-]$, $|\xi|^2 = \xi_+^2 + \xi_-^2$. Now let $\xi_+ = |\psi_+|^2 - t_+^2$, $\xi_- = |\psi_-|^2 - t_-^2$, we obtain that

$$\begin{aligned} A_+(|\psi_+|^2 - t_+^2)^2 &+ A_-(|\psi_-|^2 - t_-^2)^2 + 2B(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2) \\ &\le \lambda_M[(|\psi_+|^2 - t_+^2)^2 + (|\psi_-|^2 - t_-^2)^2] \\ &= \lambda_M t_+^4 \left(\left| \frac{\psi_+}{t_+} \right|^2 - 1 \right)^2 + \lambda_M t_-^4 \left(\left| \frac{\psi_-}{t_-} \right|^2 - 1 \right)^2 \\ &=: \lambda_M t_+^4(|U_R^+|^2 - 1)^2 + \lambda_M t_-^4(|U_R^-|^2 - 1)^2, \end{aligned} \tag{4.2}$$

with $U_R^\pm = \frac{\psi_\pm}{t_\pm}$ and $\nabla U_R^\pm = \frac{1}{t_\pm}\nabla\psi_\pm$. Taking $\Omega = \mathbb{D}_{R/2}$, let $U_R^\pm$ minimize the Ginzburg-Landua energy G_* with boundary condition $U_R^\pm\Big|_{\partial\mathbb{D}_{R/2}} = e^{i(n_\pm\theta + \phi_\pm)}$. Then,

$$G_{\epsilon_\pm}(U_R^\pm; \mathbb{D}_{R/2}) = \int_{\mathbb{D}_{R/2}} \left[\frac{1}{2}|\nabla U_R^\pm(x)|^2 + \frac{1}{4\epsilon_\pm^2}(|U_R^\pm(x)|^2 - 1)^2 \right] dx$$

$$
\begin{aligned}
&= \int_{\mathbb{D}_1} \frac{1}{2}|\nabla_y U_R^{\pm}(y)|^2 \cdot \frac{4}{R^2} \cdot \frac{R^2}{4} dy + \int_{\mathbb{D}_1} \frac{1}{4\epsilon_{\pm}^2}(|U_R^{\pm}(y)|^2 - 1)^2 \cdot \frac{R^2}{4} dy \\
&= \int_{\mathbb{D}_1} \left[\frac{1}{2}|\nabla U_R^{\pm}|^2 + \frac{R^2}{16\epsilon_{\pm}^2}(|U_R^{\pm}|^2 - 1)^2\right] dy \\
&= G_{\frac{2\epsilon_{\pm}}{R}}(U_R^{\pm}; \mathbb{D}_1), \quad \text{with} \;\; \epsilon_{\pm} = \sqrt{\frac{1}{\lambda_M t_{\pm}^2}}.
\end{aligned}
$$

Therefore, by the result of Brezis, Bethuel and Hélein (see [BBH94]), we have

$$
G_{\epsilon_{\pm}}(U_R^{\pm}; \mathbb{D}_{R/2}) = G_{\frac{2\epsilon_{\pm}}{R}}(U_R^{\pm}; \mathbb{D}_1) = \pi |n_{\pm}| \ln\left(\frac{R}{2\epsilon_{\pm}}\right) + O(1).
$$

Now let

$$
\Phi_R(x) = \begin{cases} \left(t_+ U_R^+, t_- U_R^-\right), & \text{in } \mathbb{D}_{R/2} \\ \hat{\Psi}_R(x), & \text{in } \mathbb{D}_R \setminus \mathbb{D}_{R/2} \end{cases}.
$$

Since Ψ is a local minimizer, we have that

$$
E(\Psi; \mathbb{D}_R) = E(\Psi; \mathbb{D}_{R/2}) + E(\Psi; \mathbb{D}_R \setminus \mathbb{D}_{R/2}),
$$

and

$$
\begin{aligned}
E(\Psi; \mathbb{D}_{R/2}) &\le \int_{\mathbb{D}_{R/2}} \frac{1}{2}(t_+^2 |\nabla U_R^+|^2 + t_-^2 |\nabla U_R^-|^2) + \frac{\lambda_M}{4}\left[t_+^4(|U_R^+|^2 - 1)^2 + t_-^4(|U_R^-|^2 - 1)^2\right] \\
&= t_+^2 G_{\epsilon_+}(U_R^+; \mathbb{D}_{R/2}) + t_-^2 G_{\epsilon_-}(U_R^-; \mathbb{D}_{R/2}) \\
&= t_+^2 G_{\frac{2\epsilon_+}{R}}(U_R^+; \mathbb{D}_1) + t_-^2 G_{\frac{2\epsilon_-}{R}}(U_R^-; \mathbb{D}_1) \qquad \text{with} \;\; \epsilon_{\pm} = \sqrt{\frac{1}{\lambda_M t_{\pm}^2}} \\
&= t_+^2 \pi |n_+| \ln\left(\frac{R}{2\epsilon_+}\right) + t_-^2 \pi |n_-| \ln\left(\frac{R}{2\epsilon_-}\right) + O(1).
\end{aligned}
$$

Hence, we obtain that

$$
\begin{aligned}
E(\Psi; \mathbb{D}_R) &= E(\Psi; \mathbb{D}_{R/2}) + E(\Psi; \mathbb{D}_R \setminus \mathbb{D}_{R/2}) \\
&\le t_+^2 \pi |n_+| \ln\left(\frac{R}{2\epsilon_+}\right) + t_-^2 \pi |n_-| \ln\left(\frac{R}{2\epsilon_-}\right) + E(\hat{\Psi}_R; \mathbb{D}_R \setminus \mathbb{D}_{R/2}) + O(1).
\end{aligned}
$$

From Lemma 3.2, it implies that

$$\begin{aligned}E(\Psi;\mathbb{D}_R\setminus\mathbb{D}_{R/2}) &= E(\Psi;\mathbb{D}_R)-E(\Psi;\mathbb{D}_{R/2})\\ &= \pi(t_+^2n_+^2+t_-^2n_-^2)\ln R-\pi(t_+^2n_+^2+t_-^2n_-^2)\ln(R/2)+O(1)\\ &= \pi(t_+^2n_+^2+t_-^2n_-^2)\ln 2+O(1),\end{aligned}$$

as $R\to\infty$, and

$$\begin{aligned}E(\Psi;\mathbb{D}_R) &\le t_+^2\pi|n_+|\ln(R/2\epsilon_+)+t_-^2\pi|n_-|\ln(R/2\epsilon_-)+E(\Psi_R;\mathbb{D}_R\setminus\mathbb{D}_{R/2})+O(1)\\ &= \pi\ln(R/2)(t_+^2|n_+|+t_-^2|n_-|)-t_+^2\pi|n_+|\ln(\epsilon_+)-t_-^2\pi|n_-|\ln(\epsilon_-)\\ &\quad+\pi(t_+^2n_+^2+t_-^2n_-^2)\ln 2+O(1)\\ &= \pi\ln R(t_+^2|n_+|+t_-^2|n_-|)-\pi\ln 2(t_+^2|n_+|+t_-^2|n_-|)+O(1)\\ &= \pi\ln R(t_+^2|n_+|+t_-^2|n_-|)+O(1),\end{aligned}$$

which contradicts to (4.1). □

After Proposition 4.2, we know that the local minimizers must have degrees $n_\pm\in\{0,\pm1\}$. For the solutions with degree $[n_+,n_-]=[1,1]$, we will discuss it later in Chapter 5. Now we want to obtain some pictures of $[1,0]$ type local minimizers. In general, for *any* system like (1.2), if Ψ is energy minimizer and $\deg(\psi_-;\infty)=0$, we don't expect ψ_- to vanish. And we show it in the following.

Consider the following slightly more general energy functional

$$G(\Psi;\Omega)=\int_\Omega\frac{1}{2}|\nabla\Psi|^2+\mathcal{F}(|\psi_+|^2,|\psi_-|^2), \tag{4.3}$$

with $\mathcal{F}(s_+,s_-)\ge 0$ for any $s_\pm$ and $\mathcal{F}(t_+^2,t_-^2)=0$. We consider entire solutions $\psi_\pm(x)$ to

$$-\Delta\psi_\pm+\mathcal{R}_\pm(|\psi_+|^2,|\psi_-|^2)\psi_\pm=0, \tag{4.4}$$

where $\mathcal{R}_\pm(|\psi_+|^2,|\psi_-|^2)=\frac{\partial\mathcal{F}}{\partial s_\pm}(|\psi_+|^2,|\psi_-|^2)$. We prove a generalization of Theorem 1.1 in [ABM09] as follows:

Theorem 4.3. *Suppose $\Psi = [\psi_+, \psi_-]$ is a locally minimizing solution with degree pair $[n_+, n_-] = [1, 0]$ of (4.4) with $\psi_\pm \to t_\pm e^{i(n_\pm\theta+\beta_\pm)}$ uniformly as $|x| \to \infty$. Then, there exists a constant $\phi_- \in [-\frac{\pi}{2}, \frac{\pi}{2})$ such that $\psi_-(x)e^{i\phi_-} > 0$ is real and positive in $\mathbb{R}^2$.*

Proof. We may assume without loss of generality that $\psi_-(x) \to t_-$ uniformly as $|x| \to \infty$. In particular, if we fix any $\delta < \frac{1}{2}t_-$, there exists a radius $R = R(\delta)$ such that $|\psi_-(x) - t_-| < \delta$ for all $|x| \geq R$. Let $\Omega = B_R(0)$, and for $x \in \Omega$ define

$$\widetilde{\psi}_+ = \psi_+(x), \qquad \widetilde{\psi}_-(x) = |\mathrm{Re}\psi_-(x)| + i\mathrm{Im}\psi_-(x).$$

Note that $\widetilde{\Psi} := [\widetilde{\psi}_+, \widetilde{\psi}_-] \in H^1(\Omega; \mathbb{C}^2)$, $G(\widetilde{\Psi}; \Omega) = G(\Psi; \Omega)$, and (by the choice of R) $\widetilde{\Psi}\Big|_{\partial\Omega} = \Psi\Big|_{\partial\Omega}$. Therefore, $\widetilde{\Psi}$ is also a local minimizer of G, in the sense described above. This yields that $\widetilde{\Psi}$ also satisfies the same Euler-Lagrange equation (4.4) as Ψ does in Ω. Since $\widetilde{\Psi}$ solves the Euler-Lagrange equation, we separate the real part and the imaginary part of the associated equation. Then, we have that $u = \mathrm{Re}\ \tilde{\psi}_-$ is a non-negative solution of the following problem

$$\begin{cases} -\triangle u + \mathcal{R}_-(|\tilde{\psi}_+|^2, |\tilde{\psi}_-|^2)u = 0, \\ u\Big|_{\partial\Omega = S_R} > 0 \quad \text{(by the choice of } R) \ . \end{cases}$$

Now we need to show $u > 0$ in Ω. Since $u = \mathrm{Re}\ \tilde{\psi}_- = |\mathrm{Re}\ \psi_-| \geq 0$ in Ω, suppose there exists x_0 inside Ω such that $u(x_0) = 0$ in Ω. Then $u(x_0) = 0$ is the minimum of u inside Ω. By the strong maximum principle, it implies that $u \equiv$constant inside Ω, which yields that $u \equiv 0$ inside Ω. Thus $u \equiv 0$ on $\partial\Omega$, which is a contradiction to $u > 0$ on $\partial\Omega$. Therefore, $u = \mathrm{Re}\ \tilde{\psi}_- > 0$ in Ω. Compare with $\Psi = [\psi_+, \psi_-] = [\psi_+, \mathrm{Re}\ \psi_- + i\mathrm{Im}\ \psi_-]$ and $\widetilde{\Psi} = [\tilde{\psi}_+, \tilde{\psi}_-] = [\psi_+, |\mathrm{Re}\ \psi_-| + i\mathrm{Im}\ \psi_-] = [\psi_+, \mathrm{Re}\ \psi_- + i\mathrm{Im}\ \psi_-]$, it follows that $\Psi = \widetilde{\Psi}$ and $\mathrm{Re}\ \psi_- > 0$ in $\Omega = B_R(0)$. Let R be sufficiently large, we have that
$\mathrm{Re}\ \psi_- > 0$ in $\mathbb{R}^2$.

Now let α be a constant with $|\alpha| < \frac{\pi}{2}$, and consider $\hat{\psi}_-(x) := \psi_-(x)e^{i\alpha}$. Note that $\hat{\Psi} := [\psi_+, \hat{\psi}_-]$ is also a solution to (4.4) with the same energy in any domain Ω.

By the uniform convergence of $\psi_\pm$ and our definition of $\hat{\psi}_-$, we have $\hat{\psi}_-(x) \to t_- e^{i\alpha}$ uniformly as $|x| \to \infty$. Choosing $\delta = \delta(\alpha) > 0$ such that $B_\delta(t_- e^{i\alpha})$ is fully contained inside the right half-plane $\{\text{Re } z > 0\}$, there exists a radius $R = R(\alpha)$ such that $|\hat{\psi}_-(x) - t_- e^{i\alpha}| < \delta$ whenever $|x| \geq R$. Repeating the same argument as above, we can get that Re $\hat{\psi}_-(x) > 0$ in $\mathbb{R}^2$. Equivalently,

$$\text{Re}\psi_- \cos\alpha - \text{Im}\psi_- \sin\alpha > 0,$$

i.e.

$$\text{Im}\psi_- < \text{Re}\psi_- \cot\alpha, \quad \text{when } 0 < \alpha < \frac{\pi}{2},$$

$$\text{Im}\psi_- > \text{Re}\psi_- \cot\alpha, \quad \text{when } -\frac{\pi}{2} < \alpha < 0.$$

Letting $\alpha \to \pm\frac{\pi}{2}$ we conclude that Im $\psi_-(x) \equiv 0$. $\square$

Applying Theorem 4.3 to

$$\mathcal{F}(s_+, s_-) = A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 + 2B(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2),$$

we can get the following corollary

Corollary 4.4. *Suppose Ψ is a locally minimizing solution of* (1.2) *with degree pair* $[n_+, n_-] = [1, 0]$. *Then, there exists a constant $\phi_- \in [-\frac{\pi}{2}, \frac{\pi}{2})$ such that $\psi_-(x)e^{i\phi_-} > 0$ is real and positive in $\mathbb{R}^2$.*

Chapter 5

Stability/Instability in $\mathbb{D}_1$

In this chapter we discuss the stability of the degree-one radial solutions of the Dirichlet problem on the unit disk $\mathbb{D}_1$:

$$\begin{cases} -\Delta\psi_+ + \lambda[A_+(|\psi_+|^2 - t_+^2) + B(|\psi_-|^2 - t_-^2)]\psi_+ = 0, & \text{in } \mathbb{D}_1, \\ -\Delta\psi_+ + \lambda[A_-(|\psi_-|^2 - t_-^2) + B(|\psi_+|^2 - t_+^2)]\psi_- = 0, & \text{in } \mathbb{D}_1, \\ \psi_\pm = t_\pm e^{i\theta}, & \text{on } \partial\mathbb{D}_1. \end{cases} \tag{5.1}$$

We consider critical points $\Psi \in \mathbf{H} := \{\Psi \in H^1(\mathbb{D}_1; \mathbb{C}^2) \ : \ \Psi\big|_{\partial\mathbb{D}_1} = (t_+e^{i\theta}, t_-e^{i\theta})\}$, which solves the Dirichlet problem above in the unit disk, for fixed B, as λ ranges in the half-line $\lambda \in (0, \infty)$.

In Lemma 2.1 , we already proved the existence of unique radial solution of (2.5) having the form $\Psi = [\psi_+, \psi_-] = [f_+(r)e^{in_+\theta}, f_-(r)e^{in_-\theta}]$ with $f_\pm > 0$ and $[n_+, n_-]$ is the degree pair for the solution. $f_\pm(r)$ is easily seen to satisfy (2.5). Moreover, the monotonicity of $f_\pm(r)$ has been established in various cases in Theorem 2.6. For convenience, we replace the usual parameter ϵ by $\lambda = \epsilon^{-2}$ in our energy E_ϵ, and write

$$E_\lambda(\Psi) = \int_{\mathbb{D}_1} \frac{1}{2}|\nabla\Psi|^2 + \frac{\lambda}{4}\left[A_+(|\psi_+|^2 - t_+^2)^2 \right.$$
$$\left. + A_-(|\psi_-|^2 - t_-^2)^2 + 2B(|\psi_+|^2 - t_+^2)(|\psi_-|^2 - t_-^2)\right] .$$

The associated second variation of energy E_λ around $\Psi = (\psi_+, \psi_-)$ in direction $\Phi = (\phi_+, \phi_-) \in H^1_0(\mathbb{D}_1; \mathbb{C}^2)$ is

$$E''_\lambda(\Psi)[\Phi] = \int_{\mathbb{D}_1} |\nabla\Phi|^2 + \lambda[A_+(|\psi_+|^2 - t_+^2) + B(|\psi_-|^2 - t_-^2)]|\phi_+|^2$$
$$+ \lambda[A_-(|\psi_-|^2 - t_-^2) + B(|\psi_+|^2 - t_+^2)]|\phi_-|^2$$
$$+ 2\lambda[A_+\langle\psi_+, \phi_+\rangle^2 + A_-\langle\psi_-, \phi_-\rangle^2 + 2B\langle\psi_+, \phi_+\rangle\langle\psi_-, \phi_-\rangle] \ , \quad (5.2)$$

with $\langle u, v\rangle = \mathrm{Re}(\bar{u}v) = \frac{\bar{u}v + u\bar{v}}{2}$.

First, we note that for λ small enough, there are no other solution to (1.11):

Proposition 5.1. *There exists λ^* so that for every $\lambda < \lambda^*$ the unique solution to (1.11) is $\Psi = [\psi_+, \psi_-] = [f_+(r)e^{i\theta}, f_-(r)e^{i\theta}]$.*

Proof. We firstly define the convex set $\mathcal{B} = \{\Psi \in \mathbf{H} : |\Psi| \le \Lambda \text{ in } \mathbb{D}_1\}$. By the result shown in previous chapter, any solution of (1.11) lies in $\mathcal{B}$. For any $\Psi \in \mathcal{B}$ and together with the positive definite condition, we have

$$E''_\lambda(\Psi)[\Phi] \ge \int_{\mathbb{D}_1} |\nabla\Phi|^2 + \lambda[A_+(|\psi_+|^2 - t_+^2) + B(|\psi_-|^2 - t_-^2)]|\phi_+|^2$$
$$+ \lambda[A_-(|\psi_-|^2 - t_-^2) + B(|\psi_+|^2 - t_+^2)]|\phi_-|^2$$
$$\ge \int_{\mathbb{D}_1} |\nabla\Phi|^2 - \lambda[(A_+t_+^2 + |B|t_-^2)|\phi_+|^2 + (A_-t_-^2 + |B|t_+^2)|\phi_-|^2]$$
$$\ge \int_{\mathbb{D}_1} |\nabla\Phi|^2 - \lambda[A_+t_+^2 + A_-t_-^2 + |B|(t_+^2 + t_-^2)]|\Phi|^2$$
$$= \int_{\mathbb{D}_1} |\nabla\Phi|^2 - \lambda C|\Phi|^2,$$

with constant $C = C(A_\pm, |B|, t_\pm) \ge 0$ independent of λ, Φ. By choosing λ^* sufficiently small that $C\lambda$ is smaller than the first Dirichlet eigenvalue of the Laplacian in $\mathbb{D}_1$ we may conclude that $E''_\lambda(\Psi)[\Phi]$ is a strictly positive definite quadratic form on $H^1_0(\mathbb{D}_1; \mathbb{C})$, for any $\Psi \in \mathcal{B}$. Thus, E_λ is strictly convex on $\mathcal{B}$, and hence it has a unique critical point. □

Motivated by [Mir95], we will use the Fourier decomposition to reconstruct our E''_λ, apply the variational method to show the following theorem:

Theorem 5.2. *Let* $\Psi = (\psi_+, \psi_-) = [f_+(r)e^{i\theta}, f_-(r)e^{i\theta}]$ *be a solution of* (1.2) *in* $\mathbb{D}_1$, *such that* $f'_\pm \geq 0$ *with* $B < 0$. *Then,* Ψ *is stable, in the sense* $E''_\lambda(\Psi)[\Phi] > 0 \ \forall\ \Psi \in H^1_0(\mathbb{D}_1; \mathbb{C}^2)$.

We note that by Theorem 2.6, the radial solution $f_\pm(r)$ to (1.2) is monotone increasing when $B < 0$. Now we divide our proof of Theorem 5.2 into the following steps.

Firstly, reformulation. Each $\phi_\pm \in H^1_0(\mathbb{D}_1; \mathbb{C})$ can be written in its Fourier modes in θ

$$\phi_\pm = \sum_{n\in\mathbb{Z}} b_n^\pm(r) e^{in\theta},$$

where $b_n^\pm(r) \in H^1_{\mathrm{loc}}((0,1];\mathbb{C})$. Using above Fourier decomposition, we have the following calculations for each term of the second variation E''_λ:

$$\begin{aligned}
&\int_{\mathbb{D}_1} |\nabla\Phi|^2 \\
&\quad = \int_0^{2\pi}\int_0^1 \sum_{n\in\mathbb{Z}} \left\{ |(b_n^+)'|^2 + |(b_n^-)'|^2 + |b_n^+|^2\frac{n^2}{r^2} + |b_n^-|^2\frac{n^2}{r^2} \right\} rdr \\
&\quad = \int_0^{2\pi}\int_0^1 \sum_{n=1}^{\infty} \left\{ |(b_{n+1}^+)'|^2 + |(b_{n+1}^-)'|^2 + \frac{(n+1)^2}{r^2}|b_{n+1}^+|^2 + \frac{(n+1)^2}{r^2}|b_{n+1}^-|^2 \right\} rdr \\
&\qquad + \int_0^{2\pi}\int_0^1 \sum_{n=0}^{\infty} \left\{ |(b_{1-n}^+)'|^2 + |(b_{1-n}^-)'|^2 + \frac{(1-n)^2}{r^2}|b_{1-n}^+|^2 + \frac{(1-n)^2}{r^2}|b_{1-n}^-|^2 \right\} rdr.
\end{aligned}$$

And

$$\begin{aligned}
\int_{\mathbb{D}_1} \langle \psi_\pm, \phi_\pm\rangle^2 &= \sum_{n\in\mathbb{Z}} \int_0^{2\pi}\int_0^1 \langle f_\pm e^{i\theta}, b_n^\pm e^{in\theta}\rangle^2 rdrd\theta \\
&= \frac{\pi}{2}\int_0^1 f_\pm^2 \sum_{n\in\mathbb{Z}} |b_{n+1}^\pm + \overline{b_{1-n}^\pm}|^2 rdr \\
&= \frac{\pi}{2}\int_0^1 f_\pm^2 \left\{ \sum_{n=0}^{\infty} |b_{n+1}^\pm + \overline{b_{1-n}^\pm}|^2 + \sum_{n=-\infty}^{-1} |b_{n+1}^\pm + \overline{b_{1-n}^\pm}|^2 \right\} rdr
\end{aligned}$$

$$= \frac{\pi}{2}\int_0^1 f_\pm^2 \left\{ \sum_{n=0}^{\infty} |b_{n+1}^\pm + \overline{b_{1-n}^\pm}|^2 + \sum_{n=1}^{\infty} |b_{n+1}^\pm + \overline{b_{1-n}^\pm}|^2 \right\} rdr,$$

where by changing the index we obtain the last identity.

Using the orthogonality, we get

$$\begin{aligned}
&\int_{\mathbb{D}_1} \langle \psi_+, \phi_+\rangle \langle \psi_-, \phi_-\rangle \\
&\quad = \int_0^{2\pi}\int_0^1 \sum_{n,m} \langle f_+ e^{i\theta}, b_n^+ e^{in\theta}\rangle \langle f_- e^{i\theta}, b_m^- e^{im\theta}\rangle rdrd\theta \\
&\quad = \sum_{n,m} \frac{1}{4}\int_0^{2\pi}\int_0^1 f_+ f_- [b_n^+ b_m^- e^{i(m+n-2)\theta} + b_n^+ \overline{b_m^-} e^{i(n-m)\theta} \\
&\qquad\qquad + \overline{b_n^+} b_m^- e^{i(m-n)\theta} + \overline{b_n^+ b_m^-} e^{i(2-n-m)\theta}] rdrd\theta \\
&\quad = \sum_{n+m=2} \frac{\pi}{2}\int_0^1 f_+ f_- (b_n^+ b_m^- + \overline{b_n^+ b_m^-}) rdr + \sum_{n=m} \frac{\pi}{2}\int_0^1 f_+ f_- (b_n^+ \overline{b_m^-} + \overline{b_n^+} b_m^-) rdr \\
&\quad = \sum_{n\in\mathbb{Z}} \frac{\pi}{2}\int_0^1 f_+ f_- (b_n^+ b_{2-n}^- + \overline{b_n^+ b_{2-n}^-} + b_n^+ \overline{b_n^-} + \overline{b_n^+} b_n^-) rdr \\
&\quad = \sum_{n\in\mathbb{Z}} \frac{\pi}{2}\int_0^1 f_+ f_- (b_{1+n}^+ b_{1-n}^- + \overline{b_{1-n}^+ b_{1+n}^-} + b_{1+n}^+ \overline{b_{1+n}^-} + \overline{b_{1-n}^+} b_{1-n}^-) rdr \\
&\quad = \left(\sum_{n=0}^{\infty} + \sum_{n=-\infty}^{-1}\right) \left\{ \frac{\pi}{2}\int_0^1 f_+ f_- (b_{1+n}^+ b_{1-n}^- + \overline{b_{1-n}^+ b_{1+n}^-} + b_{1+n}^+ \overline{b_{1+n}^-} + \overline{b_{1-n}^+} b_{1-n}^-) rdr \right\} \\
&\quad = \sum_{n=0}^{\infty} \frac{\pi}{2}\int_0^1 f_+ f_- (b_{1+n}^+ b_{1-n}^- + \overline{b_{1-n}^+ b_{1+n}^-} + b_{1+n}^+ \overline{b_{1+n}^-} + \overline{b_{1-n}^+} b_{1-n}^-) rdr \\
&\qquad + \sum_{n=1}^{\infty} \frac{\pi}{2}\int_0^1 f_+ f_- (b_{1-n}^+ b_{1+n}^- + \overline{b_{1+n}^+ b_{1-n}^-} + b_{1-n}^+ \overline{b_{1-n}^-} + \overline{b_{1+n}^+} b_{1+n}^-) rdr,
\end{aligned}$$

$$\begin{aligned}
\int_{\mathbb{D}_1} |\phi_\pm|^2 &= \int_0^{2\pi}\int_0^1 \sum_{n\in\mathbb{Z}} |b_n^\pm e^{in\theta}|^2 rdrd\theta \\
&= \int_0^{2\pi}\int_0^1 \sum_{n\in\mathbb{Z}} |b_n^\pm|^2 rdrd\theta \\
&= 2\pi \int_0^1 \sum_{n\in\mathbb{Z}} |b_n^\pm|^2 rdr
\end{aligned}$$

$$= 2\pi \int_0^1 \left\{ \sum_{n=1}^{\infty} |b_{n+1}^{\pm}|^2 + \sum_{n=0}^{\infty} |b_{1-n}^{\pm}|^2 \right\} rdr.$$

Therefore, we have the following quadratic forms associated to (5.2):

$$\begin{aligned}
&\mathcal{Q}_{\lambda}^{(n)}(b_{1+n}^{\pm}, b_{1-n}^{\pm}) \\
&\quad = 2\pi \int_0^1 \left[|(b_{1+n}^{+})'|^2 + |(b_{1-n}^{+})'|^2 + \frac{(1+n)^2}{r^2}|b_{1+n}^{+}|^2 + \frac{(1-n)^2}{r^2}|b_{1-n}^{+}|^2 \right] rdr \\
&\quad + 2\pi \int_0^1 \left[|(b_{1+n}^{-})'|^2 + |(b_{1-n}^{-})'|^2 + \frac{(1+n)^2}{r^2}|b_{1+n}^{-}|^2 + \frac{(1-n)^2}{r^2}|b_{1-n}^{-}|^2 \right] rdr \\
&\quad + 2\lambda\pi \int_0^1 \left[A_+ f_+^2 |b_{1+n}^{+} + \overline{b_{1-n}^{+}}|^2 + A_- f_-^2 |b_{1+n}^{-} + \overline{b_{1-n}^{-}}|^2 \right] rdr \\
&\quad + 4\lambda\pi B \int_0^1 f_+ f_- [\langle b_{1+n}^{+} + \overline{b_{1-n}^{+}}, b_{1+n}^{-}\rangle + \langle b_{1-n}^{+} + \overline{b_{1+n}^{+}}, b_{1-n}^{-}\rangle] rdr \\
&\quad + 2\lambda\pi \int_0^1 [A_+(f_+^2 - t_+^2) + B(f_-^2 - t_-^2)](|b_{1+n}^{+}|^2 + |b_{1-n}^{+}|^2) rdr \\
&\quad + 2\lambda\pi \int_0^1 [A_-(f_-^2 - t_-^2) + B(f_+^2 - t_+^2)](|b_{1+n}^{-}|^2 + |b_{1-n}^{-}|^2) rdr
\end{aligned}$$

for $n \neq 0$, and

$$\begin{aligned}
&\mathcal{Q}_{\lambda}^{(0)}(b_1^{\pm}) \\
&= 2\pi \int_0^1 \left\{ |(b_1^{+})'|^2 + |(b_1^{-})'|^2 + \frac{1}{r^2}(|b_1^{+}|^2 + |b_1^{-}|^2) \right\} rdr \\
&\quad + \lambda\pi \int_0^1 \left\{ A_+ f_+^2 |b_1^{+} + \overline{b_1^{+}}|^2 + A_- f_-^2 |b_1^{-} + \overline{b_1^{-}}|^2 + 4B f_+ f_- \langle b_1^{+} + \overline{b_1^{+}}, b_1^{-}\rangle \right\} rdr \\
&\quad + 2\lambda\pi \int_0^1 \left\{ [A_+(f_+^2 - t_+^2) + B(f_-^2 - t_-^2)]|b_1^{+}|^2 + [A_-(f_-^2 - t_-^2) + B(f_+^2 - t_+^2)]|b_1^{-}|^2 \right\} rdr \\
&= 2\pi \int_0^1 \left\{ |(b_1^{+})'|^2 + |(b_1^{-})'|^2 + \frac{1}{r^2}(|b_1^{+}|^2 + |b_1^{-}|^2) \right\} rdr \\
&\quad + 2\lambda\pi \int_0^1 \left\{ [A_+(f_+^2 - t_+^2) + B(f_-^2 - t_-^2)]|b_1^{+}|^2 + [A_-(f_-^2 - t_-^2) + B(f_+^2 - t_+^2)]|b_1^{-}|^2 \right\} rdr \\
&\quad + 4\lambda\pi \int_0^1 \left\{ A_+ f_+^2 (\mathrm{Re} b_1^{+})^2 + A_- f_-^2 (\mathrm{Re} b_1^{-})^2 + 2B f_+ f_- \mathrm{Re}(b_1^{+}) \mathrm{Re}(b_1^{-}) \right\} rdr.
\end{aligned}$$

Therefore, we can write (5.2) into the following

$$E''_\lambda(\Psi)[\Phi] = \mathcal{Q}^{(0)}_\lambda(b_1^\pm) + \mathcal{Q}^{(1)}_\lambda(b_2^\pm, b_0^\pm) + \sum_{n=2}^{\infty} \mathcal{Q}^{(n)}_\lambda(b_{1+n}^\pm, b_{1-n}^\pm). \tag{5.3}$$

Consquently, the operator $\mathcal{L}_\lambda$ associated to $E''_\lambda(\Psi)[\Phi]$ can be identified to a direct sum in Fourier modes,

$$\mathcal{L}_\lambda(\Phi) \cong \bigoplus_{n=0}^{\infty} \mathcal{L}^{(n)}_\lambda(b_{n+1}^\pm, b_{1-n}^\pm),$$

where the operators $\mathcal{L}^{(n)}_\lambda$ are associated to the quadratic forms $\mathcal{Q}^{(n)}_\lambda$. Define (in Fourier Space)

$$\widetilde{\mathcal{L}}_\lambda \widetilde{\Phi} := \bigoplus_{n\neq 1} \mathcal{L}^{(n)}_\lambda(b_{n+1}^\pm, b_{1-n}^\pm),$$

where $\widetilde{\Phi} = [\widetilde{\phi}_+, \widetilde{\phi}_-] = \left[\sum_{n\neq 0,2} b_n^+ e^{in\theta}, \sum_{n\neq 0,2} b_n^- e^{in\theta}\right]$, and so $\mathcal{L}_\lambda(\Phi) \cong \mathcal{L}^{(1)}_\lambda(b_2^\pm, b_0^\pm) \oplus \widetilde{\mathcal{L}}_\lambda \widetilde{\Phi}$, with $\tilde{\mathcal{Q}}_\lambda$ denote the quadratic form associated to $\tilde{\mathcal{L}}_\lambda$.

We have thus proven (from above) the following proposition:

Proposition 5.3. *We have $\mathcal{L}_\lambda \Phi \cong \mathcal{L}^{(1)}_\lambda(b_2^\pm, b_0^\pm) \oplus \widetilde{\mathcal{L}}_\lambda \widetilde{\Phi}$, where the operators $\mathcal{L}^{(n)}_\lambda$ are associated to the quadratic forms $\mathcal{Q}^{(n)}_\lambda$, the operator $\widetilde{\mathcal{L}}_\lambda$ is associated to the quadratic form $\widetilde{\mathcal{Q}}_\lambda$ and $\widetilde{\Phi} = [\widetilde{\phi}_+, \widetilde{\phi}_-] = \left[\sum_{n\neq 0,2} b_n^+ e^{in\theta}, \sum_{n\neq 0,2} b_n^- e^{in\theta}\right]$.*

As a consequence of this proposition, we define $a_1^\pm := i \left(\sum_{\substack{n\neq 0,2 \\ n\geq 0}} |b_n^\pm|^2 \right)^{1/2}$. Then, $a_1(r)$ is purely imaginary and we have

$$|a_1^\pm|^2 = \left| i \left(\sum_{\substack{n\neq 0,2 \\ n\geq 0}} |b_n^\pm|^2 \right)^{\frac{1}{2}} \right|^2 = \sum_{\substack{n\neq 0,2 \\ n\geq 0}} |b_n^\pm|^2.$$

By the facts

$$(a_1^\pm)' = i \left(\sum_{\substack{n\neq 0,2 \\ n\geq 0}} |b_n^\pm|^2 \right)^{-\frac{1}{2}} \sum_{\substack{n\neq 0,2 \\ n\geq 0}} \langle b_n^\pm, (b_n^\pm)' \rangle,$$

and

$$|(a_1^\pm)'|^2 = \frac{|\sum_{\substack{n\neq 0,2\\ n\geq 0}} \langle b_n^\pm, (b_n^\pm)'\rangle|^2}{\sum_{\substack{n\neq 0,2\\ n\geq 0}} |b_n^\pm|^2} \leq \frac{\sum_{\substack{n\neq 0,2\\ n\geq 0}} |b_n^\pm|^2 \sum_{\substack{n\neq 0,2\\ n\geq 0}} |(b_n^\pm)'|^2}{\sum_{\substack{n\neq 0,2\\ n\geq 0}} |b_n^\pm|^2} = \sum_{\substack{n\neq 0,2\\ n\geq 0}} |(b_n^\pm)'|^2,$$

we obtain that

$$|(a_1^\pm)'|^2 \leq \sum_{\substack{n\neq 0,2\\ n\geq 0}} |(b_n^\pm)'|^2. \tag{5.4}$$

On the other hand, by the positive definite condition $A_+A_- - B^2 > 0$, we can simply get that

$$\begin{aligned}
A_+f_+^2|b_{1+n}^+ + \overline{b_{1-n}^+}|^2 + A_-f_-^2|b_{1+n}^- + \overline{b_{1-n}^-}|^2 \\
+ 2Bf_+f_-[\langle b_{1+n}^+ + \overline{b_{1-n}^+}, b_{1+n}^-\rangle + \langle b_{1-n}^+ + \overline{b_{1+n}^+}, b_{1-n}^-\rangle] \\
= A_+f_+^2|b_{1+n}^+ + \overline{b_{1-n}^+}|^2 + A_-f_-^2|b_{1+n}^- + \overline{b_{1-n}^-}|^2 \\
+ 2Bf_+f_-\mathrm{Re}[(b_{1+n}^+ + \overline{b_{1-n}^+})(\overline{b_{1+n}^-} + b_{1-n}^-)] \\
\geq \lambda_s(f_+^2|b_{1+n}^+ + \overline{b_{1-n}^+}|^2 + f_-^2|b_{1+n}^- + \overline{b_{1-n}^-}|^2) > 0
\end{aligned}$$

for each $n > 0$ and $n \neq 1$, where $\lambda_s > 0$ is the smallest eigenvalue of the matrix $\begin{bmatrix} A_+ & B \\ B & A_- \end{bmatrix}$.

Similarly, under the same positive definite condition, we also have

$$\begin{aligned}
A_+f_+^2(\mathrm{Re}b_1^+)^2 + A_-f_-^2(\mathrm{Re}b_1^-)^2 + 2Bf_+f_-(\mathrm{Re}b_1^+)(\mathrm{Re}b_1^-) \\
\geq \lambda_s[f_+^2(\mathrm{Re}b_1^+)^2 + f_-^2(\mathrm{Re}b_1^-)^2] > 0,
\end{aligned}$$

with the same eigenvalue λ_s as above.

Therefore, for $n \neq 1$, we have that

$$\begin{aligned}
&\mathcal{Q}_\lambda^{(n)}(b_{1+n}^\pm, b_{1-n}^\pm) \\
&\quad\geq 2\pi \int_0^1 \left\{ |(b_{1+n}^+)'|^2 + |(b_{1-n}^+)'|^2 + \frac{(1+n)^2}{r^2}|b_{1+n}^+|^2 + \frac{(1-n)^2}{r^2}|b_{1-n}^+|^2 \right\} rdr
\end{aligned}$$

$$+ 2\pi \int_0^1 \left\{ |(b_{1+n}^-)'|^2 + |(b_{1-n}^-)'|^2 + \frac{(1+n)^2}{r^2}|b_{1+n}^-|^2 + \frac{(1-n)^2}{r^2}|b_{1-n}^-|^2 \right\} rdr$$
$$+ 2\lambda\pi \int_0^1 [A_+(f_+^2 - t_+^2) + B(f_-^2 - t_-^2)](|b_{1+n}^+|^2 + |b_{1-n}^+|^2) rdr$$
$$+ 2\lambda\pi \int_0^1 [A_-(f_-^2 - t_-^2) + B(f_+^2 - t_+^2)](|b_{1+n}^-|^2 + |b_{1-n}^-|^2) rdr. \tag{5.5}$$

By (5.4) and (5.5), it follows that

$$\begin{aligned}
\widetilde{\mathcal{Q}}_\lambda(\tilde{\Phi}) &= \sum_{\substack{n\neq 1\\ n\geq 0}} \mathcal{Q}_\lambda^{(n)}(b_{1+n}^\pm, b_{1-n}^\pm) \\
&\geq 2\pi \int_0^1 \left\{ |(a_1^+)'|^2 + |(a_1^-)'|^2 + \frac{1}{r^2}(|a_1^+|^2 + |a_1^-|^2) \right\} rdr \\
&\quad + 2\lambda\pi \int_0^1 [A_+(f_+^2 - t_+^2) + B(f_-^2 - t_-^2)]|a_1^+|^2 rdr \\
&\quad + 2\lambda\pi \int_0^1 [A_-(f_-^2 - t_-^2) + B(f_+^2 - t_+^2)]|a_1^-|^2 rdr \\
&=: Q_\lambda^{(0)}(a_1^\pm).
\end{aligned} \tag{5.6}$$

Meanwhile, we have that

$$\begin{aligned}
&\mathcal{Q}_\lambda^{(1)}(b_2^\pm, b_0^\pm) \\
&\quad = 2\pi \int_0^1 \left[|(b_2^+)'|^2 + |(b_0^+)'|^2 + \frac{4}{r^2}|b_2^+|^2 \right] rdr \\
&\qquad + 2\pi \int_0^1 \left[|(b_2^-)'|^2 + |(b_0^-)'|^2 + \frac{4}{r^2}|b_2^-|^2 \right] rdr \\
&\qquad + 2\lambda\pi \int_0^1 \left[A_+ f_+^2 |b_2^+ + \overline{b_0^+}|^2 + A_- f_-^2 |b_2^- + \overline{b_0^-}|^2 \right] rdr \\
&\qquad + 4\lambda\pi B \int_0^1 f_+ f_- [\langle b_2^+ + \overline{b_0^+}, b_2^- \rangle + \langle b_0^+ + \overline{b_2^+}, b_0^- \rangle] rdr \\
&\qquad + 2\lambda\pi \int_0^1 [A_+(f_+^2 - t_+^2) + B(f_-^2 - t_-^2)](|b_2^+|^2 + |b_0^+|^2) rdr \\
&\qquad + 2\lambda\pi \int_0^1 [A_-(f_-^2 - t_-^2) + B(f_+^2 - t_+^2)](|b_2^-|^2 + |b_0^-|^2) rdr.
\end{aligned}$$

The self-adjoint operator associated to $\mathcal{Q}_\lambda^{(1)}(b_2^\pm, b_0^\pm)$ is

$$\mathcal{L}_\lambda^{(1)}\begin{bmatrix} b_0^\pm \\ b_2^\pm \end{bmatrix} = \begin{bmatrix} -(b_0^\pm)'' - \frac{1}{r}(b_0^\pm)' + \lambda[A_\pm(f_\pm^2 - t_\pm^2) + B(f_\mp^2 - t_\mp^2)]b_0^\pm \\ \qquad +\lambda A_\pm f_\pm^2(\overline{b_2^\pm} + b_0^\pm) + \lambda B f_+ f_-(\overline{b_2^\mp} + b_0^\mp) \\ -(b_2^\pm)'' - \frac{1}{r}(b_2^\pm)' + \frac{4}{r^2}b_2^\pm + \lambda[A_\pm(f_\pm^2 - t_\pm^2) + B(f_\mp^2 - t_\mp^2)]b_2^\pm \\ \qquad +\lambda A_\pm f_\pm^2(b_2^\pm + \overline{b_0^\pm}) + \lambda B f_+ f_-(b_2^\mp + \overline{b_0^\mp}) \end{bmatrix}. \quad (5.7)$$

We perform a reduction of the operator $\mathcal{L}_\lambda^{(1)}$: define a quadratic form $Q_\lambda^{(1)}$ on real-valued radial functions $(a_2^\pm, a_0^\pm)$ by

$$\begin{aligned} Q_\lambda^{(1)}(a_2^\pm, a_0^\pm) &= 2\pi\int_0^1 \sum_{i=\pm}\left[|(a_2^i)'|^2 + |(a_0^i)'|^2 + \frac{4}{r^2}|a_2^i|^2\right] r\,dr \\ &+ 2\lambda\pi\int_0^1 [A_+(f_+^2 - t_+^2) + B(f_-^2 - t_-^2)](|a_2^+|^2 + |a_0^+|^2) r\,dr \\ &+ 2\lambda\pi\int_0^1 [A_-(f_-^2 - t_-^2) + B(f_+^2 - t_+^2)](|a_2^-|^2 + |a_0^-|^2) r\,dr \\ &+ 2\lambda\pi\int_0^1 [A_+f_+^2(a_0^+ - a_2^+)^2 + A_-f_-^2(a_0^- - a_2^-)^2 + 2Bf_+f_-(a_0^+ - a_2^+)(a_0^- - a_2^-)] r\,dr. \end{aligned}$$

The associated self-adjoint operator to $Q_\lambda^{(1)}$ is

$$\mathcal{M}_\lambda^{(1)}\begin{bmatrix} a_0^\pm \\ a_2^\pm \end{bmatrix} = \begin{bmatrix} -(a_0^\pm)'' - \frac{1}{r}(a_0^\pm)' + \lambda[A_\pm(f_\pm^2 - t_\pm^2) + B(f_\mp^2 - t_\mp^2)]a_0^\pm \\ \qquad +\lambda A_\pm f_\pm^2(a_0^\pm - a_2^\pm) + \lambda B f_+ f_-(a_0^\mp - a_2^\mp) \\ -(a_2^\pm)'' - \frac{1}{r}(a_2^\pm)' + \frac{4}{r^2}a_2^\pm + \lambda[A_\pm(f_\pm^2 - t_\pm^2) + B(f_\mp^2 - t_\mp^2)]a_2^\pm \\ \qquad +\lambda A_\pm f_\pm^2(a_2^\pm - a_0^\pm) + \lambda B f_+ f_-(a_2^\mp - a_0^\mp) \end{bmatrix}. \quad (5.8)$$

Denote

$$\tilde{\mu}_\lambda = \min_{\|\Phi\|_{L^2}^2=1} E_\lambda''(\Psi)[\Phi], \qquad \mu_\lambda = \min_{\|(a_0^\pm, a_1^\pm, a_2^\pm)\|_{L^2}^2=1} Q_\lambda(a_0^\pm, a_1^\pm, a_2^\pm),$$

with $Q_\lambda(a_0^\pm, a_1^\pm, a_2^\pm) = Q_\lambda^{(0)}(a_1^\pm) + Q_\lambda^{(1)}(a_2^\pm, a_0^\pm)$, $\mathcal{L}_\lambda$, $\mathcal{M}_\lambda^{(n)}(n = 0, 1)$ are the associated self-adjoint operators respectively, and we write
$\|(a_0^\pm, a_1^\pm, a_2^\pm)\|_{L^2}^2 := \sum_{n=\pm}\left[\|a_0^n\|_{L^2}^2 + \|a_1^n\|_{L^2}^2 + \|a_2^n\|_{L^2}^2\right].$

Proposition 5.4. *We have* $\tilde{\mu}_\lambda = \mu_\lambda, \forall\ \lambda > 0$.

Proof. Assume that Φ attains the minimum $\tilde{\mu}_\lambda$ under the constraint $\|\Phi\|^2_{L^2} = 1$. Then, by the Fourier decomposition of Φ and (5.3),

$$\begin{aligned}\tilde{\mu}_\lambda = E''_\lambda(\Psi)[\Phi] &= \mathcal{Q}^{(0)}_\lambda(b_1^\pm) + \mathcal{Q}^{(1)}_\lambda(b_2^\pm, b_0^\pm) + \sum_{n=2}^{\infty} \mathcal{Q}^{(n)}_\lambda(b^\pm_{1+n}, b^\pm_{1-n}) \\ &\geq Q^{(0)}_\lambda(a_1^\pm) + Q^{(1)}_\lambda(a_2^\pm, a_0^\pm),\end{aligned}$$

where $\|(a_0^\pm, a_1^\pm, a_2^\pm)\|^2_{L^2} = 1$ by the choice of $a_0^\pm, a_1^\pm, a_2^\pm$ as above. Therefore, $\tilde{\mu}_\lambda \geq \mu_\lambda$.

Conversely, if $(a_0^\pm, a_1^\pm, a_2^\pm)$ attain the minimum for μ_λ, we have $\Phi = [\phi_+, \phi_-]$ with $\phi_\pm = a_0^\pm + i a_1^\pm e^{i\theta} - a_2^\pm e^{2i\theta}$ and $\|\Phi\|^2_{L^2} = \|(a_0^\pm, a_1^\pm, a_2^\pm)\|^2_{L^2} = 1$. Hence,

$$\tilde{\mu}_\lambda \leq E''_\lambda(\Psi)[\Phi] = Q^{(0)}_\lambda(a_1^\pm) + Q^{(1)}_\lambda(a_2^\pm, a_0^\pm) = \mu_\lambda,$$

i.e. $\tilde{\mu}_\lambda \leq \mu_\lambda$. Therefore, we complete our proof. □

Denote

$$\mu^{(0)}_\lambda = \min_{\|a_1^\pm\|^2_{L^2}=1} Q^{(0)}_\lambda(a_1^\pm), \qquad \mu^{(1)}_\lambda = \min_{\|(a_0^\pm, a_2^\pm)\|^2_{L^2}=1} Q^{(1)}_\lambda(a_2^\pm, a_0^\pm).$$

By the definitions of μ_λ, $\mu^{(0)}_\lambda$ and $\mu^{(1)}_\lambda$, we have that $\mu_\lambda = \min\{\mu^{(0)}_\lambda, \mu^{(1)}_\lambda\}$. If we want to show μ_λ is positive, it is sufficient to show that both of $\mu^{(0)}_\lambda$ and $\mu^{(1)}_\lambda$ are positive.

Follow the approaches in [Mir95], we establish the propositions below.

Proposition 5.5. $\mu^{(0)}_\lambda > 0$ *for* $\forall\ B$.

Before we prove it, we need some preliminaries.

Lemma 5.6. Ψ_{rad} *is the only minimizer of* E_λ *in the class*

$$\mathcal{E} = \{V = (g_+(r)e^{i\theta}, g_-(r)e^{i\theta}) \big| V \in H^1(B_1; \mathbb{C}), g_\pm(1) = f_\pm(1) = t_\pm\}.$$

Proof. We have that

$$E_\lambda(V) = \pi \int_0^1 [|g_+'|^2 + |g_-'|^2 + \frac{1}{r^2}(|g_+|^2 + |g_-|^2)]rdr$$
$$+ \frac{\lambda\pi}{2}\int_0^1 [A_+(|g_+|^2 - t_+^2)^2 + A_-(|g_-|^2 - t_-^2)^2 + 2B(|g_+|^2 - t_+^2)(|g_-|^2 - t_-^2)]rdr.$$

If $\widetilde{V} = (|g_+|e^{i\theta}, |g_-|e^{i\theta})$, by the fact that $|\nabla|g_\pm|| \le |\nabla g_\pm|$ for $\forall g_\pm \in \mathbb{C}$, we have that $E_\lambda(\widetilde{V}) \le E_\lambda(V)$.

If V is a minimizer, so is $\widetilde{V}$. Then $\widetilde{V}$ is smooth, which implies $g_\pm(r) \neq 0$ for $r \in (0,1)$. And the equality occurs if $g_\pm \in \mathbb{R}$ (if $g_\pm$ is *not* real, we have the boundary condition: $g_\pm(1) = f_\pm(1) = t_\pm \in \mathbb{R}$, which implies that $g_\pm \in \mathbb{R}$).

From above analysis, the minimum of E_λ in $\mathcal{E}$ is obtained by a function $g_\pm(r)e^{i\theta}$ with $g_\pm(r) \ge 0$. But from the uniqueness result in previous chapter, $f_\pm$ are the only nonnegative solutions of

$$\begin{cases} -g_\pm'' - \dfrac{g_\pm'}{r} + \dfrac{g_\pm}{r^2} = \lambda[A_\pm(t_\pm^2 - f_\pm^2) + B(t_\mp^2 - f_\mp^2)]g_\pm, \\ g_\pm(1) = f_\pm(1). \end{cases}$$

Then by the uniqueness of above ODEs, we have $g_\pm \equiv f_\pm$. Therefore, $\Psi_{\mathrm{rad}} = (f_+(r)e^{i\theta}, f_-(r)e^{i\theta})$ is the *only* minimizer of E_λ in the class $\mathcal{E}$. □

As a consequence of this lemma, we find that $\mu_\lambda^{(0)} \ge 0$. In fact, $E_\lambda''(\Psi_{\mathrm{rad}})[w_+, w_-] \ge 0$, if $w = (w_+, w_-) \in \mathcal{F} = \{v = (g_+(r)e^{i\theta}, g_-(r)e^{i\theta}) \big| v \in H_0^1(B_1;\mathbb{C}), g_\pm(1) = 0\}$.

We have

$$\mu_\lambda^{(0)} = \min_{\substack{a_1^\pm \in H^1_{\mathrm{loc}}((0,1];[0,\infty)),\\ \int_0^1 (|a_1^+|^2 + |a_1^-|^2) rdr = 1}} E_\lambda''(\Psi_{\mathrm{rad}})(ia_1^\pm) \ge \min_{w=(w_+,w_-)\in\mathcal{F}} E_\lambda''(\Psi_{\mathrm{rad}})(w_\pm) \ge 0.$$

We claim that $\mu_\lambda^{(0)} > 0$. Suppose not. We obtain the existence of $w_\pm = ia_1^\pm(r)e^{i\theta}$, with $a_1^\pm \ge 0$, $\displaystyle\int_0^1 (|a_1^+|^2 + |a_1^-|^2)rdr = 1$ and $E_\lambda''(\Psi_{\mathrm{rad}})(w_\pm) = 0$. Then $w = (w_+, w_-)$ is a global minimizer of E''_λ, and hence verify the following equations:

$$-\Delta w_\pm + \lambda[A_\pm(f_\pm^2 - t_\pm^2) + B(f_\mp^2 - t_\mp^2)]w_\pm + 2\lambda[A_\pm\langle\psi_\pm, w_\pm\rangle + B\langle\psi_\mp, w_\mp\rangle]\psi_\pm = 0.$$

By the fact that $\langle f_\pm, ia_1^\pm\rangle = 0$ with $a_1^\pm \in H^1(B_1;\mathbb{R})$, we have $E''_\lambda(\Psi_{\text{rad}})(ia_1^\pm) = Q_\lambda^{(0)}(a_1^\pm)$. The Euler-Lagrange equations associated to $Q_\lambda^{(0)}(a_1^\pm)$ are

$$\begin{cases} -(a_1^\pm)'' - \dfrac{(a_1^\pm)'}{r} + \dfrac{a_1^\pm}{r^2} = -\lambda[A_\pm(f_\pm^2 - t_\pm^2) + B(f_\mp^2 - t_\mp^2)]a_1^\pm, \quad \text{in } [0,1], \\ a_1^\pm(0) = a_1^\pm(1) = 0. \end{cases} \tag{5.9}$$

Multiplying $rf_\pm$ to $a_1^\pm$-equation of (5.9) respectively, and integrating by parts, we obtain that

$$\begin{aligned} \int_0^1 \left[-(a_1^\pm)'' - \frac{(a_1^\pm)'}{r} + \frac{a_1^\pm}{r^2}\right] f_\pm r dr &= -(a_1^\pm)'(1)t_\pm + \int_0^1 \left[(a_1^\pm)' f_\pm' + \frac{a_1^\pm f_\pm}{r^2}\right] r dr \\ &= \lambda \int_0^1 [A_\pm(f_\pm^2 - t_\pm^2) + B(f_\mp^2 - t_\mp^2)]a_1^\pm f_\pm r dr. \end{aligned} \tag{5.10}$$

Also, we have that $f_\pm(r)$ satisfy the following equations

$$\begin{cases} -f_\pm'' - \dfrac{f_\pm'}{r} + \dfrac{f_\pm}{r^2} = -\lambda[A_\pm(f_\pm^2 - t_\pm^2) + B(f_\mp^2 - t_\mp^2)]f_\pm, \quad \text{in } [0,1], \\ f_\pm(1) = t_\pm. \end{cases} \tag{5.11}$$

After multiply $ra_1^\pm$ to $f_\pm$-equation of (5.11) respectively, and integrate by parts, we similarly get that

$$-f_\pm'(1)a_1^\pm(1) + \int_0^1 \left[(a_1^\pm)' f_\pm' + \frac{a_1^\pm f_\pm}{r^2}\right] r dr = \lambda \int_0^1 [A_\pm(f_\pm^2 - t_\pm^2) + B(f_\mp^2 - t_\mp^2)]a_1^\pm f_\pm r dr. \tag{5.12}$$

Therefore, by (5.10) and (5.12), we obtain that

$$-f_\pm'(1)a_1^\pm(1) = -(a_1^\pm)'(1)t_\pm,$$

which implies that $(a_1^\pm)'(1) = 0$. Together with $a_1^\pm(1) = 0$, by the uniqueness of ODEs, it yields that $a_1^\pm(r) \equiv 0$, which is a contradiction. We conclude that $\mu_\lambda^{(0)} > 0$, which completes the proof of Proposition 5.5.

Lemma 5.7. *If $B < 0$, then $\mu_\lambda^{(1)} > 0$.*

Proof. Let $F_\pm = \frac{1}{2}(a_0^\pm + a_2^\pm)$, $K_\pm = \frac{1}{2}(a_0^\pm - a_2^\pm)$, we can rewrite $Q_\lambda^{(1)}(a_2^\pm, a_0^\pm)$ in terms of $F_\pm$ and $K_\pm$:

$$\begin{aligned}
\widehat{Q}_\lambda^{(1)}(F_\pm, K_\pm) &= Q_\lambda^{(1)}(a_2^\pm, a_0^\pm) \\
&= 4\pi \int_0^1 \left[|F_\pm'|^2 + |K_\pm'|^2 + \frac{2}{r^2}|F_\pm - K_\pm|^2 \right] rdr \\
&\quad + 8\lambda\pi \int_0^1 [A_+ f_+^2 K_+^2 + A_- f_-^2 K_-^2 + 2B f_+ f_- K_+ K_-] rdr \\
&\quad + 4\lambda\pi \int_0^1 [A_+(f_+^2 - t_+^2) + B(f_-^2 - t_-^2)](F_+^2 + K_+^2) rdr \\
&\quad + 4\lambda\pi \int_0^1 [A_-(f_-^2 - t_-^2) + B(f_+^2 - t_+^2)](F_-^2 + K_-^2) rdr.
\end{aligned}$$

The quantity $\widehat{Q}_\lambda^{(1)}(F_\pm, K_\pm)$ decreases if we replace $F_\pm, K_\pm$ by $|F_\pm|, |K_\pm|$, i.e. we have

$$\widehat{Q}_\lambda^{(1)}(|F_\pm|, |K_\pm|) \leq \widehat{Q}_\lambda^{(1)}(F_\pm, K_\pm) = Q_\lambda^{(1)}(a_2^\pm, a_0^\pm),$$

which implies that there exists a minimizer pair $(F_\pm, K_\pm)$ with $F_\pm \geq 0$ and $K_\pm \geq 0$.

Since the associated system satisfied by $(a_2^\pm, a_0^\pm)$ with $\mu_\lambda^{(1)}$ playing the role of a Lagrange multiplier is as follows

$$\begin{cases}
-(a_0^\pm)'' - \frac{1}{r}(a_0^\pm)' + \lambda[A_\pm(f_\pm^2 - t_\pm^2) + B(f_\mp^2 - t_\mp^2)]a_0^\pm \\
\qquad\qquad + \lambda A_\pm f_\pm^2(a_0^\pm - a_2^\pm) + \lambda B f_+ f_-(a_0^\mp - a_2^\mp) = \mu_\lambda^{(1)} a_0^\pm, \quad \text{in } [0,1], \\
-(a_2^\pm)'' - \frac{1}{r}(a_2^\pm)' + \frac{4}{r^2}a_2^\pm + \lambda[A_\pm(f_\pm^2 - t_\pm^2) + B(f_\mp^2 - t_\mp^2)]a_2^\pm \\
\qquad\qquad + \lambda A_\pm f_\pm^2(a_2^\pm - a_0^\pm) + \lambda B f_+ f_-(a_2^\mp - a_0^\mp) = \mu_\lambda^{(1)} a_2^\pm, \quad \text{in } [0,1], \\
a_0^\pm(1) = a_2^\pm(1) = 0,
\end{cases} \tag{5.13}$$

we can change (5.13) in terms of $K_\pm$ and $F_\pm$ as following

$$\begin{cases} -F''_\pm - \dfrac{1}{r}F'_\pm + \dfrac{2}{r^2}(F_\pm - K_\pm) + \lambda[A_\pm(f_\pm^2 - t_\pm^2) + B(f_\mp^2 - t_\mp^2)]F_\pm = \mu_\lambda^{(1)}F_\pm, \quad \text{in} \ \ [0,1], \\ -K''_\pm - \dfrac{1}{r}K'_\pm + \dfrac{2}{r^2}(K_\pm - F_\pm) + \lambda[A_\pm(f_\pm^2 - t_\pm^2) + B(f_\mp^2 - t_\mp^2)]K_\pm \\ \qquad\qquad\qquad +2\lambda A_\pm f_\pm^2 K_\pm + 2\lambda B f_+ f_- K_\mp = \mu_\lambda^{(1)} K_\pm, \quad \text{in} \ \ [0,1], \\ K_\pm(1) = 0 = F_\pm(1), \end{cases} \tag{5.14}$$

with $F_\pm \geq 0$, $K_\pm \geq 0$. On the other hand, we have the fact that $\widetilde{F}_\pm = \frac{f_\pm}{r}$ and $\widetilde{K}_\pm = f'_\pm$ satisfy (5.14) with $\mu_\lambda^{(1)} = 0$ when $B < 0$, i.e.

$$\begin{cases} -\widetilde{F}''_\pm - \dfrac{1}{r}\widetilde{F}'_\pm + \dfrac{2}{r^2}(\widetilde{F}_\pm - \widetilde{K}_\pm) + \lambda[A_\pm(f_\pm^2 - t_\pm^2) + B(f_\mp^2 - t_\mp^2)]\widetilde{F}_\pm = 0, \quad \text{in} \ \ [0,1], \\ -\widetilde{K}''_\pm - \dfrac{1}{r}\widetilde{K}'_\pm + \dfrac{2}{r^2}(\widetilde{K}_\pm - \widetilde{F}_\pm) + \lambda[A_\pm(f_\pm^2 - t_\pm^2) + B(f_\mp^2 - t_\mp^2)]\widetilde{K}_\pm \\ \qquad\qquad\qquad +2\lambda A_\pm f_\pm^2 \widetilde{K}_\pm + 2\lambda B f_+ f_- \widetilde{K}_\mp = 0, \quad \text{in} \ \ [0,1], \\ \widetilde{K}_\pm(1) = f'_\pm(1) > 0, \qquad \widetilde{F}_\pm(1) = t_\pm. \end{cases}$$

We now multiply the $K_\pm$-equations of (5.14) by $\widetilde{K}_\pm r$ separately and integrate by parts, we obtain that

$$\begin{aligned} &- K'_\pm(1)\widetilde{K}_\pm(1) + \int_0^1 \widetilde{K}'_\pm K'_\pm r dr + \int_0^1 \frac{2}{r^2}(K_\pm - F_\pm)\widetilde{K}_\pm r dr \\ &\qquad + \int_0^1 \lambda[A_\pm(f_\pm^2 - t_\pm^2) + B(f_\mp^2 - t_\mp^2)]K_\pm \widetilde{K}_\pm r dr \\ &\qquad + 2\lambda \int_0^1 [A_\pm f_\pm^2 + B f_+ f_-]K_\pm \widetilde{K}_\pm r dr = \mu_\lambda^{(1)} \int_0^1 \widetilde{K}_\mp K_\pm r dr. \end{aligned} \tag{5.15}$$

Similarly, we multiply the $\widetilde{K}_\pm$-equations of (5.14) by $K_\pm r$ and integrate by parts to get that

$$\begin{aligned} &\int_0^1 \widetilde{K}'_\pm K'_\pm r dr + \int_0^1 \frac{2}{r^2}(\widetilde{K}_\pm - \widetilde{F}_\pm)K_\pm r dr \\ &\qquad + \int_0^1 \lambda[A_\pm(f_\pm^2 - t_\pm^2) + B(f_\mp^2 - t_\mp^2)]K_\pm \widetilde{K}_\pm r dr \\ &\qquad\qquad + 2\lambda \int_0^1 [A_\pm f_\pm^2 + B f_+ f_-]K_\pm \widetilde{K}_\pm r dr = 0. \end{aligned} \tag{5.16}$$

Therefore, by (5.15), (5.16) and the boundary condition $K_{\pm}(1) = 0$, we can get that

$$-K'_{\pm}(1)\widetilde{K}_{\pm}(1) + \int_0^1 \frac{2}{r^2}(\widetilde{F}_{\pm}K_{\pm} - \widetilde{K}_{\pm}F_{\pm})rdr = \mu_\lambda^{(1)} \int_0^1 K_{\pm}\widetilde{K}_{\pm}rdr. \tag{5.17}$$

Similarly, we have

$$-F'_{\pm}(1)\widetilde{F}_{\pm}(1) + \int_0^1 \frac{2}{r^2}(\widetilde{K}_{\pm}F_{\pm} - \widetilde{F}_{\pm}K_{\pm})rdr = \mu_\lambda^{(1)} \int_0^1 F_{\pm}\widetilde{F}_{\pm}rdr. \tag{5.18}$$

Combine (5.17) and (5.18), it follows that

$$\mu_\lambda^{(1)} \int_0^1 (\widetilde{K}_{\pm}K_{\pm} + \widetilde{F}_{\pm}F_{\pm})rdr = -K'_{\pm}(1)\widetilde{K}_{\pm}(1) - F'_{\pm}(1)\widetilde{F}_{\pm}(1). \tag{5.19}$$

Since $K_{\pm} \geq 0$, $F_{\pm} \geq 0$ in $[0, 1]$ and $K_{\pm}(1) = F_{\pm}(1) = 0$, we have $K'_{\pm}(1) \leq 0$, $F'_{\pm}(1) \leq 0$. We claim that $K'_{\pm}(1) < 0$, $F'_{\pm}(1) < 0$. Actually, if $K'_{\pm}(1) = 0 = F'_{\pm}(1)$, and with the boundary conditions $K_{\pm}(1) = 0 = F_{\pm}(1)$, it implies that zero is the only solution of (5.14) by uniqueness, i.e. $K_{\pm}(r) \equiv 0 \equiv F_{\pm}(r)$. It yields that $a_0^{\pm}(r) \equiv 0 \equiv a_2^{\pm}(r)$, which is impossible. Therefore, $K'_{\pm}(1) < 0$, $F'_{\pm}(1) < 0$. Since the right side of (5.19) is positive, and the each term on the left side of (5.19) is also positive except $\mu_\lambda^{(1)}$. Hence, we can obtain that $\mu_\lambda^{(1)} > 0$. This completes the proof. □

Proof of Theorem 5.2. By the fact $\mu_\lambda = \min\{\mu_\lambda^{(0)}, \mu_\lambda^{(1)}\}$, and together with Proposition 5.4-5.5 and Lemma 5.7, we have $\tilde{\mu}_\lambda > 0$, which gives us that $E''_\lambda(\Psi)[\Phi] > 0$. This completes the proof of Theorem 5.2. □

After we establish the stability of the radial solution with $[1, 1]$ degree pair, we will discuss the relations of the eigenvalues between the operators $\mathcal{L}_\lambda^{(1)}$ and $\mathcal{M}_\lambda^{(1)}$. Once we get the property of the eigenvalues between the operators $\mathcal{L}_\lambda^{(1)}$ and $\mathcal{M}_\lambda^{(1)}$, we can obtain the instability result in Theorem 5.9 when $B > 0$.

Lemma 5.8. *$\mu \in \mathbb{R}$ is an eigenvalue of $\mathcal{L}_\lambda^{(1)}$ over $L^2(([0, 1); rdr); \mathbb{C}^4)$ if only if it is an eigenvalue of $\mathcal{M}_\lambda^{(1)}$ over $L^2(([0, 1); rdr); \mathbb{R}^4)$. Moreover, if μ is a simple eigenvalue of*

$\mathcal{M}_\lambda^{(1)}$ *with eigenspace spanned by* $(a_0^\pm, a_2^\pm)$*, then*

$$\ker(\mathcal{L}_\lambda^{(1)} - \mu I) = \left\{ \ t(\xi a_0^\pm, -\overline{\xi} a_2^\pm) : \ \xi \in \mathbb{S}^1, \quad t \in \mathbb{R} \ \right\}.$$

Proof. Let $\mu \in \sigma(\mathcal{L}_\lambda^{(1)})$ with complex-valued eigenvectors $(b_0^\pm, b_2^\pm)$, that is

$$\mathcal{L}_\lambda^{(1)} \begin{bmatrix} b_0^\pm \\ b_2^\pm \end{bmatrix} = \mu \begin{bmatrix} b_0^\pm \\ b_2^\pm \end{bmatrix},$$

with $\mathcal{L}_\lambda^{(1)}$ defined in (5.7). We observe that $a_0^\pm = \text{Im} b_0^\pm$, $a_2^\pm = \text{Im} b_2^\pm$ will be eigenvectors of $\mathcal{M}_\lambda^{(1)}$ with μ. On the other hand, if $(a_0^\pm, a_2^\pm)$ are real-valued eigenvectors of $\mathcal{M}_\lambda^{(1)}$ with μ, then $(b_0^\pm, b_2^\pm) = (ia_0^\pm, ia_2^\pm)$ will be eigenvectors of $\mathcal{L}_\lambda^{(1)}$ with the same eigenvalue. Then, $\sigma(\mathcal{L}_\lambda^{(1)}) = \sigma(\mathcal{M}_\lambda^{(1)})$.

Now suppose μ is simple eigenvalue of $\mathcal{M}_\lambda^{(1)}$ with eigenspace spanned by $(a_0^\pm, a_2^\pm)$. If $(b_0^\pm, b_2^\pm)$ is an eigenfunction of $\mathcal{L}_\lambda^{(1)}$, then (by the observation above) $(\text{Im} b_0^\pm, \text{Im} b_2^\pm) = l(a_0^\pm, a_2^\pm)$ for $l \in \mathbb{R}$. Similarly, $(\text{Re} b_0^\pm, -\text{Re} b_2^\pm)$ is an eigenfunction of $\mathcal{M}_\lambda^{(1)}$, and so $(\text{Re} b_0^\pm, -\text{Re} b_2^\pm) = k(a_0^\pm, a_2^\pm)$ for $k \in \mathbb{R}$. Setting $t = \sqrt{k^2 + l^2}$ and $\xi = \frac{k+il}{t} \in \mathbb{S}^1$, we have $(b_0^\pm, b_2^\pm) = t(\xi a_0^\pm, -\overline{\xi} a_2^\pm)$ as desired. □

In order to study the dependence on λ of the eigenvalues of the linearized operator $\mathcal{M}_\lambda^{(1)}(a_0^\pm, a_2^\pm)$ (defined in (5.8)), we replace the dependence on λ by a dependence on the domain $(0, R)$, via a change of variables. Therefore, we obtain a new quadratic form:

$$\begin{aligned} Q_\lambda^{(1)}(a_0^\pm, a_2^\pm) &= 2\pi \int_0^R \sum_{i=\pm} \left[|(\hat{a}_2^i)'|^2 + |(\hat{a}_0^i)'|^2 + \frac{4}{r^2} |\hat{a}_2^i|^2 \right] r dr \\ &+ 2\pi \int_0^R \left[A_+ f_+^2 (\hat{a}_0^+ - \hat{a}_2^+)^2 + A_- f_-^2 (\hat{a}_0^- - \hat{a}_2^-)^2 + 2B f_+ f_- (\hat{a}_0^+ - \hat{a}_2^+)(\hat{a}_0^- - \hat{a}_2^-) \right] r dr \\ &+ 2\pi \int_0^R \left[A_+ (f_+^2 - t_+^2) + B(f_-^2 - t_-^2) \right] (|\hat{a}_0^+|^2 + |\hat{a}_2^+|^2) r dr \\ &+ 2\pi \int_0^R \left[A_- (f_-^2 - t_-^2) + B(f_+^2 - t_+^2) \right] (|\hat{a}_0^-|^2 + |\hat{a}_2^-|^2) r dr \\ &=: \widehat{Q}_R(\hat{a}_0^\pm, \hat{a}_2^\pm), \end{aligned}$$

where $\hat{a}_0^\pm(r) = a_0^\pm(rR)$, $\hat{a}_2^\pm(r) = a_2^\pm(rR)$, $R = \sqrt{\lambda}$.

The associated operator to $\widehat{Q}_R(\hat{a}_0^\pm, \hat{a}_2^\pm)$ is defined as $\widehat{\mathcal{M}}_R^{(1)}$. We observe that the first eigenvalue of $\widehat{\mathcal{M}}_R^{(1)}$ denoted by $\hat{\mu}_R$, is related to the first eigenvalue $\mu_\lambda^{(1)}$ via $\mu_\lambda^{(1)} = R^2\hat{\mu}_R$.

Finally, we obtain the following instability result for radial solution when $B > 0$:

Theorem 5.9. *For any $B \in (0, B_0)$, where B_0 is as in Theorem 2.7, there exists a unique constant $R_* = R_*(B) > 0$ such that $\psi_\pm = f_\pm(r; R)e^{i\theta}$ of* (1.2) *is unstable for any $R > R_*$.*

Proof. To do this we argue as in Theorem 2 in [Mir95]. Let $\hat{a}_0^\pm(r)$, $\hat{a}_2^\pm(r)$ be the ground-state eigenfunctions. Define $L_\pm = \hat{a}_0^\pm(r) + \hat{a}_2^\pm(r)$, $P_\pm = \hat{a}_0^\pm(r) - \hat{a}_2^\pm(r)$. Therefore, we can rewrite $\widehat{Q}_R(\hat{a}_0^\pm, \hat{a}_2^\pm)$ in terms of $L_\pm, P_\pm$ as follows:

$$\begin{aligned}\widehat{Q}_R(\hat{a}_0^\pm, \hat{a}_2^\pm) = \pi \int_0^R \sum_{i=\pm} &\left[|P_i'|^2 + |L_i'|^2 + \frac{2}{r^2}|L_i - P_i|^2\right] rdr \\ &+ 2\pi \int_0^R \left[A_+ f_+^2 P_+^2 + A_- f_-^2 P_-^2 + 2B f_+ f_- P_+ P_-\right] rdr \\ &+ \pi \int_0^R \left[A_+(f_+^2 - t_+^2) + B(f_-^2 - t_-^2)\right] (|P_+|^2 + |L_+|^2) rdr \\ &+ \pi \int_0^R \left[A_-(f_-^2 - t_-^2) + B(f_+^2 - t_+^2)\right] (|P_-|^2 + |L_-|^2) rdr \\ &=: \breve{Q}_R(L_\pm, P_\pm).\end{aligned}$$

As $R \to \infty$, the radial profile $f_\pm(\cdot, R) \to f_\pm^\infty(\cdot)$ in $\mathcal{C}^k([0, R])$ for all $R > 0$ and $k \in \mathbb{N}$, with $f_\pm^\infty$ the modulus of the unique entire equivariant solution of the form $\psi_\pm^\infty = f_\pm^\infty(r)e^{i\theta}$. Let $L_+ = \frac{f_+^\infty}{r}$, $L_- = -\frac{f_-^\infty}{r}$, $P_+ = (f_+^\infty)'$, $P_- = -(f_-^\infty)'$. Since $f_\pm^\infty$ vanish linearly at $r = 0$, $L_\pm(r)$ and $P_\pm(r)$ are regular near $r = 0$, $L_+ - P_+ = -r\left[\frac{f_+^\infty}{r}\right]'$, $L_- - P_- = r\left[\frac{f_-^\infty}{r}\right]'$ are well-defined in $\breve{Q}_R(L_\pm, P_\pm)$. Meanwhile, $P_\pm$ and $L_\pm$ satisfy the following equations:

$$\begin{cases} -(L_\pm)'' - \frac{1}{r}(L_\pm)' + \frac{2}{r}(L_\pm - P_\pm) + [A_\pm((f_\pm^\infty)^2 - t_\pm^2) + B((f_\mp^\infty)^2 - t_\mp^2)]L_\pm = 0, \\ -(P_\pm)'' - \frac{1}{r}(P_\pm)' + \frac{2}{r}(P_\pm - L_\pm) + [A_\pm((f_\pm^\infty)^2 - t_\pm^2) + B((f_\mp^\infty)^2 - t_\mp^2)]P_\pm \\ \qquad\qquad\qquad\qquad\qquad +2A_\pm(f_\pm^\infty)^2 P_\pm - 2Bf_+^\infty f_-^\infty P_\mp = 0. \end{cases}$$

Using above equations and integrating by parts, together with the asymptotic properties of radial solutions at infinity in Theorem 2.4, we can obtain that

$$\begin{aligned}
&\lim_{R\to\infty} \breve{Q}_R(L_\pm, P_\pm) \\
&\quad = \lim_{R\to\infty} \pi \left[P_\pm' P_\pm r\Big|_0^R + L_\pm' L_\pm r\Big|_0^R \right] \\
&\qquad + \pi \int_0^\infty \Big\{ -\frac{2}{r}(P_\pm - L_\pm)P_\pm - [A_\pm((f_\pm^\infty)^2 - t_\pm^2) + B((f_\mp^\infty)^2 - t_\mp^2)]P_\pm^2 \\
&\qquad\qquad -2A_\pm(f_\pm^\infty)^2 P_\pm^2 + 4Bf_+^\infty f_-^\infty P_+ P_- \Big\} rdr \\
&\qquad + \pi \int_0^\infty \Big\{ -\frac{2}{r}(L_\pm - P_\pm)L_\pm - [A_\pm((f_\pm^\infty)^2 - t_\pm^2) + B((f_\mp^\infty)^2 - t_\mp^2)]L_\pm^2 \Big\} rdr \\
&\qquad + \pi \int_0^\infty \frac{2}{r^2} |L_\pm - P_\pm|^2 rdr \\
&\qquad + 2\pi \int_0^\infty \left[A_+(f_+^\infty)^2 P_+^2 + A_-(f_-^\infty)^2 P_-^2 + 2Bf_+^\infty f_-^\infty P_+ P_- \right] rdr \\
&\qquad + \pi \int_0^\infty \left[A_+((f_+^\infty)^2 - t_+^2) + B((f_-^\infty)^2 - t_-^2) \right] (|P_+|^2 + |L_+|^2) rdr \\
&\qquad + \pi \int_0^\infty \left[A_-((f_-^\infty)^2 - t_-^2) + B((f_+^\infty)^2 - t_+^2) \right] (|P_-|^2 + |L_-|^2) rdr \\
&\quad = 8\pi B \int_0^\infty f_+^\infty f_-^\infty P_+ P_- rdr \\
&\quad = -8\pi B \int_0^\infty f_+^\infty f_-^\infty (f_+^\infty)'(f_-^\infty)' rdr < 0,
\end{aligned}$$

since by Theorem 2.7 we have $(f_\pm^\infty)'(r) > 0$ when $0 < B < B_0$. Denote $\breve{Q}_\infty(\breve{L}_\pm, \breve{P}_\pm) := \lim_{R\to\infty} \breve{Q}_R(L_\pm, P_\pm)$. From above computation, we have $\breve{Q}_\infty(\breve{L}_\pm, \breve{P}_\pm) < 0$. Set a cut-off function $\eta_R(x) = \eta_1(x/R)$ with $\eta_1(x) = 1$ for $0 \le x \le 1/2$, $\eta_1(x) = 0$ for $x > 1$ and $0 < \eta_1(x) \le 1$ for $1/2 < x < 1$. Define $L_\pm^R := L_\pm \eta_R$, $P_\pm^R := P_\pm \eta_R \in H_0^1([0, R))$. Then we have $L_+ = \frac{f_+^\infty}{r} = o(r^{-3})$, $L_- = -\frac{f_+^\infty}{r} = o(r^{-3})$, $L_+' = \left[\frac{f_+^\infty}{r}\right]' = o(r^{-2})$,

$L'_- = -\left[\frac{f_-^\infty}{r}\right]' = o(r^{-2})$, $P'_+ = \left[f_+^\infty\right]' = o(r^{-3})$ and $P'_- = -\left[f_+^\infty\right]' = o(r^{-3})$ by the asymptotic behaviours as in Theorem 2.4. Therefore, choose $\delta > 0$ (to be determined later) such that $\breve{Q}_\infty(\breve{L}_\pm, \breve{P}_\pm) < -\delta < 0$, we have that

$$\begin{aligned}
&\breve{Q}_R(L^R_\pm, P^R_\pm) \\
&\quad = \pi \int_0^{R/2} \sum_{i=\pm} \left[|P'_i|^2 + |L'_i|^2 + \frac{2}{r^2}|L_i - P_i|^2 \right] rdr \\
&\qquad + 2\pi \int_0^{R/2} \left[A_+ f_+^2 P_+^2 + A_- f_-^2 P_-^2 + 2B f_+ f_- P_+ P_- \right] rdr \\
&\qquad + \pi \int_0^{R/2} \left[A_+(f_+^2 - t_+^2) + B(f_-^2 - t_-^2) \right] (|P_+|^2 + |L_+|^2) rdr \\
&\qquad + \pi \int_0^{R/2} \left[A_-(f_-^2 - t_-^2) + B(f_+^2 - t_+^2) \right] (|P_-|^2 + |L_-|^2) rdr \\
&\qquad + \pi \int_{R/2}^{R} \sum_{i=\pm} \left[|P'_i|^2 + |L'_i|^2 + \frac{2}{r^2}|L_i - P_i|^2 \right] rdr \\
&\qquad + 2\pi \int_{R/2}^{R} \left[A_+ f_+^2 P_+^2 + A_- f_-^2 P_-^2 + 2B f_+ f_- P_+ P_- \right] rdr \\
&\qquad + \pi \int_{R/2}^{R} \left[A_+(f_+^2 - t_+^2) + B(f_-^2 - t_-^2) \right] (|P_+|^2 + |L_+|^2) rdr \\
&\qquad + \pi \int_{R/2}^{R} \left[A_-(f_-^2 - t_-^2) + B(f_+^2 - t_+^2) \right] (|P_-|^2 + |L_-|^2) rdr \\
&\quad \le \pi \int_0^{\infty} \sum_{i=\pm} \left[|P'_i|^2 + |L'_i|^2 + \frac{2}{r^2}|L_i - P_i|^2 \right] rdr \\
&\qquad + 2\pi \int_0^{\infty} \left[A_+ f_+^2 P_+^2 + A_- f_-^2 P_-^2 + 2B f_+ f_- P_+ P_- \right] rdr \\
&\qquad + \pi \int_0^{\infty} \left[A_+(f_+^2 - t_+^2) + B(f_-^2 - t_-^2) \right] (|P_+|^2 + |L_+|^2) rdr \\
&\qquad + \pi \int_0^{\infty} \left[A_-(f_-^2 - t_-^2) + B(f_+^2 - t_+^2) \right] (|P_-|^2 + |L_-|^2) rdr + \delta \\
&\quad = \breve{Q}_\infty(\breve{L}_\pm, \breve{P}_\pm) + \delta < 0
\end{aligned}$$

for R sufficiently large. Hence, we have $\breve{Q}_R(L^R_\pm, P^R_\pm) < \breve{Q}_\infty(\breve{L}_\pm, \breve{P}_\pm) + \delta < 0$ for $\forall\ R \ge R_0$ if $\delta < \frac{1}{2}|\breve{Q}_\infty(\breve{L}_\pm, \breve{P}_\pm)|$, which implies that the minimu of $\breve{Q}_R(L^R_\pm, P^R_\pm) < 0$. Thus, for R sufficiently large and $0 < B < B_0$, $\mu_1^{(1)}(R^2) = R^2 \hat{\mu}_1(R) < 0$. Since

$\mu_\lambda = \min\{\mu_\lambda^{(0)}, \mu_\lambda^{(1)}\} < \mu_\lambda^{(1)}$, we have $\mu_\lambda < 0$, which completes our proof. $\square$

Chapter 6

Strong Convergence Away From Vortices

In this chapter, we provide some asymptotic estimates of the behaviour of solutions at infinity. First let us state the asymptotic description of solutions:

Proposition 6.1. *Let* Ψ *be a solution of* (1.2) *in* $\mathbb{R}^2$ *satisfying* (1.3). *Then there exist constants* β_+, β_- *such that*

$$\psi_\pm \to t_\pm e^{i(n_\pm\theta+\beta_\pm)} \quad \textit{uniformly as } |x| \to \infty \ . \tag{6.1}$$

Moreover, for any degree pair $[n_+, n_-]$,

$$|\psi_\pm(x)|^2 = t_\pm^2 - \frac{A_\mp n_\pm^2 - Bn_\mp^2}{A_+A_- - B^2}\frac{1}{r^2} + o\left(\frac{1}{r^2}\right) \ , \tag{6.2}$$

as $r = |x| \to \infty$.

Remark 6.2. (6.1) *implies that the solution is asymptotically equivariant, and the asymptotic expansion* (6.2) *of* $|\psi_\pm|^2$ *agrees with the one in Chapter 2 which was derived for equivariant solutions.*

To prove Proposition 6.1 we use the following modification of a similar result in [BMR94]:

Lemma 6.3. *Let* Ψ *be an entire solution of* (1.2) *satisfying* (1.3).

(i) $|\Psi(x)|^2 \leq \min\{\frac{2M}{\lambda_s}, t_+^2 + t_-^2\}$, *where* $M = \max\{A_+t_+^2 + Bt_-^2, A_-t_-^2 + Bt_+^2\}$ *and* $A_+A_- - B^2 > 0$, $\lambda_s > 0$ *is the smallest eigenvalue of the matrix* $\begin{bmatrix} A_+ & B \\ B & A_- \end{bmatrix}$.

(ii) $|\psi_\pm|^2 \to t_\pm^2$ *uniformly as* $|x| \to \infty$.

(iii) There exist constants $R_0 > 0$, $n_\pm \in \mathbb{Z}$, *and smooth functions* $\rho_\pm(x)$, $\phi_\pm(x)$ *such that for all* $|x| \geq R_0$

$$\Psi(x) = [\psi_+(x), \psi_-(x)] = [\rho_+(x)e^{i(n_+\theta+\phi_+(x))}, \rho_-(x)e^{i(n_-\theta+\phi_-(x))}],$$

with .

$$\int_{|x|>R_0} (|\nabla\rho_\pm|^2 + |\nabla\phi_\pm|^2) < \infty. \tag{6.3}$$

Proof. We show the proof of statement (i). Let $V(x) = |\Psi|^2$, then $\frac{1}{2}\nabla V = \psi_+\nabla\psi_+ + \psi_-\nabla\psi_-$. Therefore,

$$\begin{aligned}
\frac{1}{2}\Delta V &= \psi_+\nabla\psi_+ + \psi_-\nabla\psi_- + |\nabla\Psi|^2 \\
&\geq \psi_+\nabla\psi_+ + \psi_-\nabla\psi_- \\
&= \frac{1}{\epsilon^2}[A_+(|\psi_+|^2 - t_+^2) + B(|\psi_-|^2 - t_-^2)] \\
&\qquad + \frac{1}{\epsilon^2}[A_-(|\psi_-|^2 - t_-^2) + B(|\psi_+|^2 - t_+^2)] \\
&= \frac{1}{\epsilon^2}\left[A_+|\psi_+|^4 + B|\psi_+|^2|\psi_-|^2 - (A_+t_+^2 + Bt_-^2)|\psi_+|^2\right. \\
&\qquad \left. + A_-|\psi_-|^4 + B|\psi_+|^2|\psi_-|^2 - (A_-t_-^2 + Bt_+^2)|\psi_-|^2\right] \\
&= \frac{1}{\epsilon^2}\left[A_+|\psi_+|^4 + 2B|\psi_+|^2|\psi_-|^2 + A_-|\psi_-|^4\right. \\
&\qquad \left. - (A_+t_+^2 + Bt_-^2)|\psi_+|^2 - (A_-t_-^2 + Bt_+^2)|\psi_-|^2\right] \\
&\geq \frac{1}{\epsilon^2}\left[\lambda_s(|\psi_+|^4 + |\psi_-|^4) - (A_+t_+^2 + Bt_-^2)|\psi_+|^2 - (A_-t_-^2 + Bt_-^2)|\psi_+|^2\right]
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{\epsilon^2}\left(\frac{\lambda_s}{2}(|\psi_+|^2+|\psi_-|^2)^2 - \max\{A_+t_+^2+Bt_-^2, A_-t_-^2+Bt_+^2\}(|\psi_+|^2+|\psi_-|^2)\right) \\
&= \frac{1}{\epsilon^2}\left(\frac{\lambda_s}{2}|\Psi|^4 - M|\Psi|^2\right) \qquad \text{where } M := \max\{A_+t_+^2+Bt_-^2, A_-t_-^2+Bt_+^2\} \\
&= \frac{1}{\epsilon^2}\left(\frac{\lambda_s}{2}V^2 - MV\right) \\
&= \frac{\lambda_s}{2\epsilon^2}V\left(V-\frac{2M}{\lambda_s}\right),
\end{aligned}
$$

i.e.

$$
\Delta V \geq \frac{\lambda_s}{\epsilon^2}V\left(V-\frac{2M}{\lambda_s}\right), \tag{6.4}
$$

where $\lambda_s > 0$ is the smallest eigenvalue of the matrix $\begin{bmatrix} A_+ & B \\ B & A_- \end{bmatrix}$ since $A_+A_- - B^2 > 0$.

Now let $v = V - \frac{2M}{\lambda_s}$, we can rewrite (6.4) into the following inequality:

$$
\begin{cases}
-\Delta v + a(x)v \leq 0 \qquad \text{in} \quad \Omega, \\
\quad v\Big|_{\partial\Omega} = t_+^2 + t_-^2 - \dfrac{2M}{\lambda_s},
\end{cases}
$$

with $a(x) = \frac{\lambda_s}{\epsilon^2}V > 0$.

If $t_+^2 + t_-^2 - \frac{2M}{\lambda_s} \leq 0$, by maximum principle $v \leq 0$ on Ω, i.e. $V \leq \frac{2M}{\lambda_s}$. On the other hand, if $t_+^2 + t_-^2 - \frac{2M}{\lambda_s} > 0$, by maximum principle again $\max_{\overline{\Omega}} v$ can be attained on $\partial\Omega$, i.e. $\max_{\overline{\Omega}} v = t_+^2 + t_-^2 - \dfrac{2M}{\lambda_s}$ and $v \leq t_+^2 + t_-^2 - \dfrac{2M}{\lambda_s}$, i.e. $V \leq t_+^2 + t_-^2$. From above analysis, we obtain that $V \leq \min\left\{\dfrac{2M}{\lambda_s}, t_+^2 + t_-^2\right\}$, i.e. $|\Psi|^2 \leq \min\left\{\dfrac{2M}{\lambda_s}, t_+^2 + t_-^2\right\}$.

Now we are going to show statements (ii) and (iii). First, we need to show that $\nabla\Psi \in L^\infty(\mathbb{R}^2)$, i.e. $\nabla\psi_\pm \in L^\infty(\mathbb{R}^2)$. Indeed, from (1.2) it follows that

$$
-\triangle\psi_\pm = [A_\pm(t_\pm^2 - |\psi_\pm|^2) + B(t_\mp^2 - |\psi_\mp|^2)]\psi_\pm.
$$

Then, by part (i), we have

$$\begin{aligned}|\triangle\psi_\pm| &\le (|A_\pm(t_\pm^2 - |\psi_\pm|^2)| + |B(t_\mp^2 - |\psi_\mp|^2)|)|\psi_\pm| \\ &\le (A_\pm t_\pm^2 + |B|t_\mp^2)|\psi_\pm| < C, \quad \forall\ x \in \mathbb{R}^2.\end{aligned}$$

By the Sobolev embedding, it follows that for any $p \in (2,\infty)$, $\forall\ x_0 \in \mathbb{R}^2$,

$$\begin{aligned}\|\nabla\psi_\pm\|_{L^\infty(B_1(x_0))} &\le \|\psi_\pm\|_{W^{2,p}(B_1(x_0))} \\ &\le C\left(\|\psi_\pm\|_{L^p(B_2(x_0))} + \|\triangle\psi_\pm\|_{L^p(B_2(x_0))}\right) \\ &\le C\|\psi_\pm\|_{L^\infty(B_2(x_0))} \le C.\end{aligned}$$

Hence, we have that $\nabla\psi_\pm \in L^\infty(\mathbb{R}^2)$, i.e. $\nabla\Psi \in L^\infty(\mathbb{R}^2)$.

Now it is time to show the uniform convergence of $|\psi_\pm|^2$ as $|x| \to \infty$. In fact, we use the argument by contradiction. Suppose that there is a sequence $|x_n| \to \infty$ such that $||\psi_+(x_n)| - t_+| \ge \delta$ for some $\delta > 0$. First, we consider the case for $|\psi_+(x_n)| - t_+ \ge \delta$. Then,

$$\begin{aligned}|\psi_+(x)| &\le |\psi_+(x) - \psi_+(x_n)| + |\psi_+(x_n)| \\ &\le \|\nabla\psi_+\|_{L^\infty}|x - x_n| + |\psi_+(x_n)| \\ &\le M_+ \cdot \frac{\delta}{2M_+} + t_+ - \delta \qquad \text{for} \quad x \in B(x_n, \frac{\delta}{2M_+}) \\ &= t_+ - \frac{\delta}{2},\end{aligned}$$

with $M_+ = \|\nabla\psi_+\|_{L^\infty}$. It follows then

$$(|\psi_+|^2 - t_+^2)^2 = (|\psi_+| + t_+)^2 \cdot (t_+ - |\psi_+|)^2 \ge t_+^2 \cdot \frac{\delta^2}{4}, \tag{6.5}$$

i.e. $(|\psi_+|^2 - t_+^2)^2 \ge t_+^2 \cdot \frac{\delta^2}{4}$. Therefore,

$$\int_{B(x_n, \frac{\delta}{2M_+})} (|\psi_+|^2 - t_+^2)^2 \ge \int_{B(x_n, \frac{\delta}{2M_+})} t_+^2 \cdot \frac{\delta^2}{4} dx = \frac{t_+^2\delta^2}{4} \cdot \frac{\pi\delta^2}{4M_+^2}. \tag{6.6}$$

On the other hand, by the positive definiteness of $F(\Psi)$, we have that $(|\psi_+|^2 - t_+^2)^2 + (|\psi_-|^2 - t_-^2)^2 \leq \frac{1}{\lambda_s} F(\Psi)$. Hence, by (1.3),

$$\int_{\mathbb{R}^2} A_+(|\psi_+|^2 - t_+^2)^2 + A_-(|\psi_-|^2 - t_-^2)^2 \leq \frac{A_+ + A_-}{\lambda_s} \int_{\mathbb{R}^2} F(\Psi) < \infty. \tag{6.7}$$

It is easy to obtain that

$$\int_{\mathbb{R}^2} (|\psi_\pm|^2 - t_\pm^2)^2 < \infty \tag{6.8}$$

for $A_\pm > 0$.

Since (6.8) holds, there is some R_0 such that

$$\int_{|x|>R_0} (|\psi_+|^2 - t_+^2)^2 < \frac{t_+^2 \delta^2}{4} \cdot \frac{\pi \delta^2}{4M_+^2}. \tag{6.9}$$

Since (6.6) and $|x_n| \to \infty$, this yields a contradiction.

Second, we discuss the case for $|\psi_+(x_n)| - t_+ \geq \delta$. Then, we have that

$$(|\psi_+|^2 - t_+^2)^2 = (|\psi_+| + t_+)^2 \cdot (t_+ - |\psi_+|)^2 \geq t_+^2 \cdot \delta^2, \tag{6.10}$$

which implies that

$$\int_{B(x_n, \frac{\delta}{2M_+})} (|\psi_+|^2 - t_+^2)^2 \geq \int_{B(x_n, \frac{\delta}{2M_+})} t_+^2 \cdot \delta^2 dx = \frac{\pi \delta^4 t^2}{4M_+^2}. \tag{6.11}$$

Since (6.8) holds, there is some R_0 such that

$$\int_{|x|>R_0} (|\psi_+|^2 - t_+^2)^2 < \frac{\pi \delta^4 t^2}{4M_+^2}. \tag{6.12}$$

Since (6.6) and $|x_n| \to \infty$, this yields a contradiction. Together with above two cases, we obtain the uniform convergence of $|\psi_+|^2$ as $|x| \to \infty$. Similarly we can also get $|\psi_-|^2 \to t_-^2$ uniformly as $|x| \to \infty$. Immediately by part (i), there exist smooth functions $\rho_\pm$, $\phi_\pm$ and contants $n_\pm \in \mathbb{Z}$ for $|x| \geq R_0$ such that

$$\Psi(x) = [\psi_+(x), \psi_-(x)] = [\rho_+(x) e^{i(n_+\theta + \phi_+(x))}, \rho_-(x) e^{i(n_-\theta + \phi_-(x))}].$$

To prove (6.3), we write the equations for $\varphi_\pm = n_\pm\theta + \phi_\pm$ and $\rho_\pm$ by (1.2):

$$-\triangle\rho_\pm + \rho_\pm|\nabla\varphi_\pm|^2 - i(2\nabla\rho_\pm\nabla\varphi_\pm - \rho_\pm\triangle\varphi_\pm) = [A_\pm(t_\pm^2 - \rho_\pm^2) + B(t_\mp^2 - \rho_\mp^2)]\rho_\pm.$$

After separating the real and imaginary parts of above equations, we can obtain that

$$\mathrm{div}(\rho_\pm^2\nabla\varphi_\pm) = 0, \tag{6.13}$$

$$-\triangle\rho_\pm + \rho_\pm|\nabla\varphi_\pm|^2 = [A_\pm(t_\pm^2 - \rho_\pm^2) + B(t_\mp^2 - \rho_\mp^2)]\rho_\pm. \tag{6.14}$$

Note that $\varphi_\pm = n_\pm\theta + \phi_\pm$, we have

$$\nabla\varphi_\pm = n_\pm\nabla\theta + \nabla\phi_\pm = n_\pm\frac{V}{r} + \nabla\phi_\pm, \tag{6.15}$$

where $V(x)$ is the vector-field in $\mathbb{R}^2\setminus\{0\}$ defined by $V(r\cos\theta, r\sin\theta) = (-\sin\theta, \cos\theta)$. Combining (6.13) and (6.15), we get that

$$\mathrm{div}(\rho_\pm^2\nabla\varphi_\pm) = \mathrm{div}[\rho_\pm^2(\frac{n_\pm}{r}\cdot V + \nabla\phi_\pm)] = 0 \quad \text{for} \quad |x| \geq R_0. \tag{6.16}$$

We will split our proof in the following several steps.

Step 1. We have that. for every $R > R_0$, $\displaystyle\int_{S_R}\rho_\pm^2\frac{\partial\phi_\pm}{\partial\nu} = 0$.

We show the proof of ψ_+ firstly. Consider the vector-field $D_+ = (\psi_+ \times \psi_{+,x_1}, \psi_+ \times \psi_{+,x_2})$ (which is well-defined and smooth on all of $\mathbb{R}^2$). Therefore,

$$\begin{aligned}
\mathrm{div}D_+ &= \frac{\partial}{\partial x_1}(\psi_+ \times \psi_{+,x_1}) + \frac{\partial}{\partial x_2}(\psi_+ \times \psi_{+,x_2})\\
&= \frac{\partial}{\partial x_1}[\mathrm{Im}(\overline{\psi}_+\psi_{+,x_1})] + \frac{\partial}{\partial x_2}[\mathrm{Im}(\overline{\psi}_+\psi_{+,x_2})]\\
&= \mathrm{Im}(\overline{\psi}_{+,x_1}\psi_{+,x_1}) + \mathrm{Im}(\overline{\psi}_+\psi_{+,x_1x_1}) + \mathrm{Im}(\overline{\psi}_{+,x_2}\psi_{+,x_2}) + \mathrm{Im}(\overline{\psi}_+\psi_{+,x_2x_2})\\
&= \mathrm{Im}(|\psi_{+,x_1}|^2) + \mathrm{Im}(\overline{\psi}_+\psi_{+,x_1x_1}) + \mathrm{Im}(|\psi_{+,x_2}|^2) + \mathrm{Im}(\overline{\psi}_+\psi_{+,x_2x_2})\\
&= \psi_+ \times \psi_{+,x_1x_1} + \psi_+ \times \psi_{+,x_2x_2}\\
&= \psi_+ \times \triangle\psi_+\\
&= \psi_+ \times \{[A_+(|\psi_+|^2 - t_+^2) + B(|\psi_-|^2 - t_-^2)]\psi_+\}
\end{aligned}$$

$$= 0,$$

i.e.

$$\text{div} D_+ = 0. \tag{6.17}$$

integrating (6.17) on B_R we have

$$\int_{B_R} \text{div} D_+ = \int_{S_R} D_+ \cdot \nu = 0 \qquad \text{for } \ \forall\ R > 0.$$

On the other hand, since $\psi_+ = \rho_+ e^{i\varphi_+}$, we obtain that

$$\begin{aligned} \psi_+ \times \psi_{+,x_1} &= \text{Im}(\overline{\psi}_+ \psi_{+,x_1}) = \text{Im}[\rho_+ e^{-i\varphi_+}(\rho_{+,x_1} e^{i\varphi_+} + i\rho_+ e^{i\varphi_+} \varphi_{+,x_1})] \\ &= \text{Im}(\rho_+ \rho_{+,x_1} + i\rho_+^2 \varphi_{+,x_1}) \\ &= \rho_+^2 \varphi_{+,x_1}. \end{aligned}$$

Similarly we also have $\psi_+ \times \psi_{+,x_2} = \rho_+^2 \varphi_{+,x_2}$. Therefore, $D_+ = (\rho_+^2 \varphi_{+,x_1}, \rho_+^2 \varphi_{+,x_2}) = \rho_+^2 \nabla \varphi_+$. By the fact $V \cdot \nu = 0$ on S_R, it follows that

$$\begin{aligned} 0 = \int_{S_R} D_+ \cdot \nu = \int_{S_R} \rho_+^2 \cdot \nabla \varphi_+ \cdot \nu &= \int_{S_R} \rho_+^2 (\frac{n_+}{r} V + \nabla \phi_+) \cdot \nu \\ = \int_{S_R} \rho_+^2 \cdot \frac{n_+}{r} V \cdot \nu + \rho_+^2 \nabla \phi_+ \cdot \nu \\ = \int_{S_R} \rho_+^2 \nabla \phi_+ \cdot \nu \ , \end{aligned}$$

i.e. $\int_{S_R} \rho_+^2 \frac{\partial \phi_+}{\partial \nu} = 0$. Similarly we can get $\int_{S_R} \rho_-^2 \frac{\partial \phi_-}{\partial \nu} = 0$ as well. Hence,

$$\int_{S_R} \rho_\pm^2 \frac{\partial \phi_\pm}{\partial \nu} = 0 \ . \tag{6.18}$$

Step 2. We have $\int_{\mathbb{R}^2 \setminus B_{R_0}} |\nabla \phi_\pm|^2 < \infty$.

We still consider the proof for the case of ψ_+ firstly. Set $\phi_{+,R} = \frac{1}{2\pi R} \int_{S_R} \phi_+$. Multiplying $\phi_+ - \phi_{+,R}$ to ϕ_+-equation of (6.13) and integrating over $A_R = B_R \setminus B_{R_0}$ we obtain

$$\int_{A_R} \text{div}(\rho_+^2 \nabla \varphi_+) \cdot (\phi_+ - \phi_{+,R}) = 0.$$

By the fact $V \cdot \nu = 0$, it then follows that

$$\begin{aligned}
&\int_{A_R} \rho_+^2 \nabla\varphi_+ (\nabla\phi_+ - \nabla\phi_{+,R}) \\
&\quad = \int_{\partial A_R} \rho_+^2 \nabla\varphi_+ \cdot \nu(\phi_+ - \phi_{+,R}) \\
&\quad = \int_{S_R} \rho_+^2 \nabla\varphi_+ \cdot \nu(\phi_+ - \phi_{+,R}) - \int_{S_{R_0}} \rho_+^2 \nabla\varphi_+ \cdot \nu(\phi_+ - \phi_{+,R}) \\
&\quad = \int_{S_R} \rho_+^2 (\frac{n_+}{r} V + \nabla\phi_+) \cdot \nu(\phi_+ - \phi_{+,R}) - \int_{S_{R_0}} \rho_+^2 (\frac{n_+}{r} V + \nabla\phi_+) \cdot \nu(\phi_+ - \phi_{+,R}) \\
&\quad = \int_{S_R} \rho_+^2 \nabla\phi_+ \cdot \nu(\phi_+ - \phi_{+,R}) - \int_{S_{R_0}} \rho_+^2 \nabla\phi_+ \cdot \nu(\phi_+ - \phi_{+,R}).
\end{aligned}$$

i.e.

$$\int_{A_R} \rho_+^2 \nabla\varphi_+ \nabla\phi_+ = \int_{S_R} \rho_+^2 \nabla\phi_+ \cdot \nu(\phi_+ - \phi_{+,R}) - \int_{S_{R_0}} \rho_+^2 \nabla\phi_+ \cdot \nu(\phi_+ - \phi_{+,R}). \quad (6.19)$$

It is clear that $V(r\cos\theta, r\sin\theta) = (-\sin\theta, \cos\theta) = (-\frac{x_2}{r}, \frac{x_1}{r}) = \frac{1}{r}(\frac{\partial x_1}{\partial\theta}, \frac{\partial x_2}{\partial\theta})$, and $V \cdot \nabla\phi_+ = \frac{1}{r}(\frac{\partial x_1}{\partial\theta}, \frac{\partial x_2}{\partial\theta}) \cdot \nabla\phi_+ = \frac{1}{r}\frac{\partial\phi_+}{\partial\theta}$, we have that

$$\begin{aligned}
\int_{A_R} \frac{n_+}{r} \cdot V \cdot \nabla\phi_+ &= \int_{A_R} \frac{n_+}{r} \cdot \frac{1}{r}\frac{\partial\phi_+}{\partial\theta} \\
&= \int_0^{2\pi} \int_{R_0}^R \frac{n_+}{r^2} \frac{\partial\phi_+}{\partial\theta} r dr d\theta \\
&= \int_{R_0}^R \frac{n_+}{r} \left(\int_0^{2\pi} \frac{\partial\phi_+}{\partial\theta} d\theta \right) dr \\
&= \int_{R_0}^R \frac{n_+}{r} [\phi_+(r, 2\pi) - \phi_+(r, 0)] dr \\
&= 0. \qquad (6.20)
\end{aligned}$$

By (6.18)-(6.19), we have

$$\begin{aligned}
\int_{A_R} \rho_+^2 \nabla\varphi_+ \nabla\phi_+ &= \int_{A_R} \rho_+^2 (\frac{n_+}{r} \cdot V + \nabla\phi_+) \nabla\phi_+ \\
&= \int_{A_R} \rho_+^2 \frac{n_+}{r} \cdot V \cdot \nabla\phi_+ + \rho_+^2 |\nabla\phi_+|^2
\end{aligned}$$

$$
\begin{aligned}
&= \int_{S_R} \rho_+^2 \nabla\phi_+ \cdot \nu(\phi_+ - \phi_{+,R}) - \int_{S_{R_0}} \rho_+^2 \nabla\phi_+ \cdot \nu(\phi_+ - \phi_{+,R}) \\
&= \int_{S_R} \rho_+^2 \frac{\partial\phi_+}{\partial\nu}(\phi_+ - \phi_{+,R}) - \int_{S_{R_0}} \rho_+^2 \frac{\partial\phi_+}{\partial\nu}\phi_+ + \int_{S_{R_0}} \rho_+^2 \frac{\partial\phi_+}{\partial\nu}\phi_{+,R} \\
&= \int_{S_R} \rho_+^2 \frac{\partial\phi_+}{\partial\nu}(\phi_+ - \phi_{+,R}) - \int_{S_{R_0}} \rho_+^2 \frac{\partial\phi_+}{\partial\nu}\phi_+ \\
&= \int_{S_R} \rho_+^2 \frac{\partial\phi_+}{\partial\nu}(\phi_+ - \phi_{+,R}) - C,
\end{aligned}
$$

i.e.

$$
\int_{A_R} \rho_+^2 \frac{n_+}{r} \cdot V \cdot \nabla\phi_+ + \rho_+^2 |\nabla\phi_+|^2 = \int_{S_R} \rho_+^2 \frac{\partial\phi_+}{\partial\nu}(\phi_+ - \phi_{+,R}) - C, \tag{6.21}
$$

where C is some constant independent of R.

Then by (6.20) we have

$$
\begin{aligned}
\int_{A_R} \rho_+^2 |\nabla\phi_+|^2 &= \int_{S_R} \rho_+^2 \frac{\partial\phi_+}{\partial\nu}(\phi_+ - \phi_{+,R}) - \int_{A_R} \rho_+^2 \frac{n_+}{r} \cdot V \cdot \nabla\phi_+ - C \\
&= \int_{S_R} \rho_+^2 \frac{\partial\phi_+}{\partial\nu}(\phi_+ - \phi_{+,R}) + \int_{A_R} (t_+^2 - \rho_+^2) \frac{n_+}{r} \cdot V \cdot \nabla\phi_+ - C,
\end{aligned}
$$

it yields that

$$
\begin{aligned}
\int_{A_R} \rho_+^2 |\nabla\phi_+|^2 &\leq \int_{S_R} \rho_+^2 \left|\frac{\partial\phi_+}{\partial\nu}\right| \cdot |\phi_+ - \phi_{+,R}| + \int_{A_R} |t_+^2 - \rho_+^2| \cdot \frac{n_+}{r} \cdot |V| \cdot |\nabla\phi_+| + C \\
&\leq \int_{S_R} C_1 \left|\frac{\partial\phi_+}{\partial\nu}\right| \cdot |\phi_+ - \phi_{+,R}| + \int_{A_R} |t_+^2 - \rho_+^2| \cdot \frac{n_+}{r} \cdot |\nabla\phi_+| + C. \qquad (6.22)
\end{aligned}
$$

By Cauchy-Schwartz inequality and the Poincaré's inequality, we obtain that

$$
\left|\int_{S_R} \frac{\partial\phi_+}{\partial\nu}(\phi_+ - \phi_{+,R})\right| \leq \left[\int_{S_R} \left|\frac{\partial\phi_+}{\partial\nu}\right|^2\right]^{1/2} \left[\int_{S_R} |\phi_+ - \phi_{+,R}|^2\right]^{1/2}, \tag{6.23}
$$

$$
\int_{S_R} |\phi_+ - \phi_{+,R}|^2 \leq R^2 \int_{S_R} |\nabla_T \phi_+|^2 . \tag{6.24}
$$

Combining (6.23) and (6.24) and Young's inequality, we get that

$$\begin{aligned}
\left|\int_{S_R} \frac{\partial \phi_+}{\partial \nu}(\phi_+ - \phi_{+,R})\right| &\le \left[\int_{S_R} \left|\frac{\partial \phi_+}{\partial \nu}\right|^2\right]^{1/2} \left[\int_{S_R} |\phi_+ - \phi_{+,R}|^2\right]^{1/2} \\
&\le \left[\int_{S_R} \left|\frac{\partial \phi_+}{\partial \nu}\right|^2\right]^{1/2} \left[R^2 \int_{S_R} |\nabla_T \phi_+|^2\right]^{1/2} \\
&= R \left[\int_{S_R} \left|\frac{\partial \phi_+}{\partial \nu}\right|^2\right]^{1/2} \left[\int_{S_R} |\nabla_T \phi_+|^2\right]^{1/2} \\
&\le \frac{R}{2} \left[\int_{S_R} \left|\frac{\partial \phi_+}{\partial \nu}\right|^2 + \int_{S_R} |\nabla_T \phi_+|^2\right] \\
&= \frac{R}{2} \int_{S_R} |\nabla \phi_+|^2. \qquad (6.25)
\end{aligned}$$

On the other hand, by (6.8) we obtain that

$$\begin{aligned}
n_+ \int_{A_R} |t_+^2 - \rho_+^2| \frac{|\nabla \phi_+|}{r} &\le \frac{n_+}{R_0} \int_{A_R} |t_+^2 - \rho_+^2||\nabla \phi_+| \\
&\le \frac{n_+}{R_0} \left[\int_{A_R} (t_+^2 - \rho_+^2)^2\right]^{1/2} \left[\int_{A_R} |\nabla \phi_+|^2\right]^{1/2} \\
&\le C \left[\int_{A_R} |\nabla \phi_+|^2\right]^{1/2}, \qquad (6.26)
\end{aligned}$$

where C is some constant.

Denote $\Lambda = \min\{\frac{2M}{\lambda_s}, t_+^2 + t_-^2\}$ with M denoted as in part (i). Since $|\psi_+|^2 \to t_+^2$ uniformly as $|x| \to \infty$. For fixed $\delta = t_+^2 - \frac{3\Lambda}{4}$, there exists a radius $R = R(\delta)$ such that $\big||\psi_+|^2 - t_+^2\big| < \delta$, i.e. $-\delta < |\psi_+|^2 - t_+^2 < \delta$. Therefore, $|\psi_+|^2 > t_+^2 - \delta = \frac{3\Lambda}{4}$, i.e.

$$|\psi_+|^2 > \alpha := \frac{3\Lambda}{4} > \frac{\Lambda}{2}\ . \qquad (6.27)$$

Hence, by (6.22)-(6.27), we have that

$$\begin{aligned}
\alpha \int_{A_R} |\nabla \phi_+|^2 &< \int_{A_R} \rho_+^2 |\nabla \phi_+|^2 \\
&\le \int_{S_R} \Lambda \cdot \frac{\partial \phi_+}{\partial \nu}(\phi_+ - \phi_{+,R}) + \int_{A_R} |t_+^2 - \rho_+^2| \cdot \frac{n_+}{r} \cdot |\nabla \phi_+| + C
\end{aligned}$$

$$\leq \frac{R\Lambda}{2}\int_{S_R}|\nabla\phi_+|^2 + C_1\left[\int_{A_R}|\nabla\phi_+|^2\right]^{1/2} + C_2\ ,$$

which implies that

$$\int_{A_R}|\nabla\phi_+|^2 < \frac{R\Lambda}{2\alpha}\int_{S_R}|\nabla\phi_+|^2 + C_1\left[\int_{A_R}|\nabla\phi_+|^2\right]^{1/2} + C_2, \tag{6.28}$$

where C_1, C_2 are some constants.

Set, $R \geq R_0$,

$$f(R) = \int_{A_R}|\nabla\phi_+|^2 = \int_0^{2\pi}\int_{R_0}^{R}|\nabla\phi_+|^2 r dr d\theta\ , \tag{6.29}$$

then

$$f'(R) = R\int_0^{2\pi}|\nabla\phi_+|^2 d\theta = \int_{S_R}|\nabla\phi_+|^2 dS_R. \tag{6.30}$$

Hence, by (6.28)-(6.30),

$$f(R) \leq \frac{R\Lambda}{2\alpha}f'(R) + C + Cf(R)^{1/2}\ . \tag{6.31}$$

The desired conclusion of Step 2 now follows from the Claim below:

Claim: Any function satisfying (6.31) and $f(R) \leq CR$ for $\forall R \geq R_0$ is bounded on $(R_0, +\infty)$.

In fact, from (6.31), it follows that

$$\begin{aligned} f(R) &\leq \frac{R\Lambda}{2\alpha}f'(R) + C + Cf(R)^{1/2} \\ &= Rf'(R)\left(\frac{\Lambda}{2\alpha} + \frac{Cf(R)^{1/2}}{Rf'(R)}\right) + C \\ &=: \frac{1}{\beta}Rf'(R) + C\ , \end{aligned} \tag{6.32}$$

where $\frac{1}{\beta} = \frac{\Lambda}{2\alpha} + \frac{Cf(R)^{1/2}}{Rf'(R)}$.

In the following, we need to show that $\beta > 1$, i.e.

$$\frac{1}{\beta} = \frac{\Lambda}{2\alpha} + \frac{Cf(R)^{1/2}}{Rf'(R)} = \frac{\Lambda Rf'(R) + 2\alpha Cf(R)^{1/2}}{2\alpha Rf'(R)} < 1\ .$$

It is equivalent to show that $\Lambda R f'(R) + 2\alpha C f(R)^{1/2} < 2\alpha R f'(R)$, i.e. $R\left[f'(R)(\Lambda - 2\alpha) + \frac{2\alpha C}{R} f(R)^{1/2}\right] < 0$. If we can prove that $f'(R)(\Lambda - 2\alpha) + \frac{2\alpha C}{R} f(R)^{1/2}$ is negative, we've done.

Since $f(R) \leq CR$ for $\forall R \geq R_0$, we have that

$$\begin{aligned} f'(R)(\Lambda - 2\alpha) + \frac{2\alpha C}{R} f(R)^{1/2} &\leq f'(R)(\Lambda - 2\alpha) + \frac{2\alpha C}{R}(CR)^{1/2} \\ &= f'(R)(\Lambda - 2\alpha) + \frac{2\alpha C^{3/2}}{R^{1/2}}, \end{aligned}$$

i.e.

$$f'(R)(\Lambda - 2\alpha) + \frac{2\alpha C}{R} f(R)^{1/2} \leq f'(R)(\Lambda - 2\alpha) + \frac{2\alpha C^{3/2}}{R^{1/2}} \ . \tag{6.33}$$

The second term of the right hand side of (6.33) is small enough if R is sufficiently large, and the first term of the right hand side of the above inequality is negative since $f'(R)$ is positive and (6.27). Therefore, $f'(R)(\Lambda - 2\alpha) + \frac{2\alpha C}{R} f(R)^{1/2}$ is negative, which implies $\Lambda R f'(R) + 2\alpha C f(R)^{1/2} < 2\alpha R f'(R)$. Hnece, $\frac{1}{\beta} < 1$, which yields that

$$f(R) \leq \frac{R}{\beta} f'(R) + C \tag{6.34}$$

with $\beta > 1$. Set $g(R) = f(R) - C$, so that $g(R) \leq \frac{R}{\beta} g'(R)$ by (6.34). Then, multiply $\frac{\beta}{R^{\beta+1}}$ to this inequality to obtain that

$$\frac{\beta}{R^{\beta+1}} g(R) \leq \frac{1}{R^{\beta}} g'(R),$$

i.e. $(R^{-\beta} g(R))' \geq 0$.

We claim that $g(R) \leq 0$ for $\forall R \geq R_0$. Argument by contradiction. Suppose not, then $g(R_1) > 0$ for some $R_1 \geq R_0$. Since $(R^{-\beta} g(R))' \geq 0$, i.e. $R^{-\beta} g(R)$ is increasing in R. Therefore, $R^{-\beta} g(R) \geq R_1^{-\beta} g(R_1)$ for $\forall R \geq R_1$, i.e. $g(R) \geq \left(\frac{R}{R_1}\right)^{\beta} g(R_1)$ for $\forall R \geq R_1$. Since $g(R) = f(R) - C$, then we obtain that

$$f(R) - C \geq \left(\frac{R}{R_1}\right)^{\beta} g(R_1) \ , \tag{6.35}$$

it follows that

$$\begin{aligned} f(R) &\geq \left(\frac{R}{R_1}\right)^{\beta} g(R_1) + C \\ &= \left(\frac{R}{R_1}\right)^{\beta-1} \frac{g(R_1)}{R_1} R + C \\ &\geq \frac{g(R_1)}{R_1} R + C \\ &=: CR + C > CR \quad \text{for} \quad R \geq R_0, \end{aligned}$$

which is a contradiction. Hence, we get the result for ϕ_+. Similarly we can get the result for ϕ_-. Therefore, we finish the proof of Step 2.

Step 3. We have $\displaystyle\int_{\mathbb{R}^2 \setminus B_{R_0}} |\nabla \rho_\pm|^2 < \infty$.

Indeed, we firstly claim that

$$\int_{\mathbb{R}^2 \setminus B_{R_0}} (t_\pm - \rho_\pm) |\nabla \varphi_\pm|^2 < \infty \ . \tag{6.36}$$

Since

$$|\nabla \varphi_\pm| = \left| n_\pm \frac{V}{r} + \nabla \phi_\pm \right| \leq \frac{n_\pm}{r} + |\nabla \phi_\pm|,$$

$$\begin{aligned} |\nabla \varphi_\pm|^2 &\leq \left(\frac{n_\pm}{r} + |\nabla \phi_\pm|\right)^2 \\ &= \frac{n_\pm^2}{r^2} + |\nabla \phi_\pm|^2 + 2\frac{n_\pm}{r}|\nabla \phi_\pm| \\ &\leq 2\left(\frac{n_\pm^2}{r^2} + |\nabla \phi_\pm|^2\right) \ . \end{aligned}$$

Then, by Step 2 and (6.8), we have

$$\begin{aligned} \int_{\mathbb{R}^2 \setminus B_{R_0}} (t_\pm - \rho_\pm) |\nabla \varphi_\pm|^2 &\leq 2 \int_{\mathbb{R}^2 \setminus B_{R_0}} (t_\pm - \rho_\pm) \left(\frac{n_\pm^2}{r^2} + |\nabla \phi_\pm|^2\right) \\ &= 2 \int_{\mathbb{R}^2 \setminus B_{R_0}} (t_\pm - \rho_\pm) \frac{n_\pm^2}{r^2} + 2 \int_{\mathbb{R}^2 \setminus B_{R_0}} (t_\pm - \rho_\pm) |\phi_\pm|^2 \\ &\leq 2n_\pm^2 \int_{\mathbb{R}^2 \setminus B_{R_0}} (t_\pm - \rho_\pm) \frac{1}{r^2} + 2 \int_{\mathbb{R}^2 \setminus B_{R_0}} t_\pm |\phi_\pm|^2 \end{aligned}$$

$$
\begin{aligned}
&\leq 2n_\pm^2 \int_{\mathbb{R}^2\setminus B_{R_0}} (t_\pm - \rho_\pm)\frac{1}{r^2} + C_1 \\
&\leq 2n_\pm^2 \left[\int_{\mathbb{R}^2\setminus B_{R_0}} (t_\pm - \rho_\pm)^2\right]^{1/2} \left[\int_{\mathbb{R}^2\setminus B_{R_0}} \frac{1}{r^4}\right]^{1/2} + C_1 \\
&= 2C_2 n_\pm^2 \left(\int_{\mathbb{R}^2\setminus B_{R_0}} \frac{(t_\pm^2 - \rho_\pm^2)^2}{(t_\pm + \rho_\pm)^2}\right)^{1/2} + C_1 \\
&\leq \frac{2C_2 n_\pm^2}{t_\pm^2} \left(\int_{\mathbb{R}^2\setminus B_{R_0}} (t_\pm^2 - \rho_\pm^2)^2\right)^{1/2} + C_1 \\
&< \infty,
\end{aligned}
$$

i.e.

$$
\int_{\mathbb{R}^2\setminus B_{R_0}} (t_\pm - \rho_\pm)|\nabla\varphi_\pm|^2 < \infty. \tag{6.37}
$$

Fix some smooth function η such that

$$
\eta(x) = \begin{cases} 1, & \text{if } |x| \leq 1, \\ 0 < \eta < 1, & \text{if } 1 < |x| < 2, \\ 0, & \text{if } |x| \geq 2. \end{cases} \tag{6.38}
$$

Set $\eta_R(x) = \eta(x/R)$. Since $\rho_\pm$ satisfy the following two equations respectively:

$$
-\triangle\rho_+ + \rho_+|\nabla\varphi_+|^2 = A_+(t_+^2 - \rho_+^2)\rho_+ + B(t_-^2 - \rho_-^2)\rho_+, \tag{6.39}
$$

$$
-\triangle\rho_- + \rho_-|\nabla\varphi_-|^2 = A_-(t_-^2 - \rho_-^2)\rho_- + B(t_+^2 - \rho_+^2)\rho_-. \tag{6.40}
$$

In the following we consider the case of ρ_+ at first, the case for ρ_- is similar to the proof of ρ_+. Multiplying (6.39) by $(\rho_+ - t_+)\eta_R$, integrating over $\mathbb{R}^2 \setminus B_{R_0}$ and integrating by parts:

$$
\begin{aligned}
&\int_{\mathbb{R}^2\setminus B_{R_0}} (-\triangle\rho_+ + \rho_+|\nabla\varphi_+|^2)(\rho_+ - t_+)\eta_R \\
&\quad = \int_{S_{R_0}} \frac{\partial\rho_+}{\partial\nu}(\rho_+ - t_+)\eta_R + \int_{\mathbb{R}^2\setminus B_{R_0}} \nabla\rho_+\nabla[(\rho_+ - t_+)\eta_R]
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \int_{\mathbb{R}^2\setminus B_{R_0}} \rho_+|\nabla\varphi_+|^2(\rho_+ - t_+)\eta_R \\
&= \int_{S_{R_0}} \frac{\partial\rho_+}{\partial\nu}(\rho_+ - t_+)\eta_R + \int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla\rho_+|^2\eta_R \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2\setminus B_{R_0}} \nabla(\rho_+ - t_+)^2\nabla\eta_R + \int_{\mathbb{R}^2\setminus B_{R_0}} \rho_+|\nabla\varphi_+|^2(\rho_+ - t_+)\eta_R \\
&= -\int_{\mathbb{R}^2\setminus B_{R_0}} [A_+(t_+^2 - \rho_+^2)(t_+ - \rho_+) + B(t_-^2 - \rho_-^2)(t_+ - \rho_+)]\rho_+\eta_R,
\end{aligned}
$$

i.e. we have the following equation

$$
\begin{aligned}
&\int_{S_{R_0}} \frac{\partial\rho_+}{\partial\nu}(\rho_+ - t_+)\eta_R + \int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla\rho_+|^2\eta_R \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2\setminus B_{R_0}} \nabla(\rho_+ - t_+)^2\nabla\eta_R + \int_{\mathbb{R}^2\setminus B_{R_0}} \rho_+|\nabla\varphi_+|^2(\rho_+ - t_+)\eta_R \\
&= -\int_{\mathbb{R}^2\setminus B_{R_0}} [A_+(t_+ - \rho_+)^2(t_+ + \rho_+) + B(t_-^2 - \rho_-^2)(t_+ + \rho_+)]\rho_+\eta_R,
\end{aligned}
\tag{6.41}
$$

Similarly, we can get the equation for ρ_-:

$$
\begin{aligned}
&\int_{S_{R_0}} \frac{\partial\rho_-}{\partial\nu}(\rho_- - t_-)\eta_R + \int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla\rho_-|^2\eta_R \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2\setminus B_{R_0}} \nabla(\rho_- - t_-)^2\nabla\eta_R + \int_{\mathbb{R}^2\setminus B_{R_0}} \rho_-|\nabla\varphi_-|^2(\rho_- - t_-)\eta_R \\
&= -\int_{\mathbb{R}^2\setminus B_{R_0}} [A_-(t_- - \rho_-)^2(t_- + \rho_-) + B(t_+^2 - \rho_+^2)(t_- - \rho_-)]\rho_-\eta_R,
\end{aligned}
\tag{6.42}
$$

Adding (6.41) and (6.42) together, we have that

$$
\begin{aligned}
&\int_{S_{R_0}} \frac{\partial\rho_+}{\partial\nu}(\rho_+ - t_+)\eta_R + \frac{\partial\rho_-}{\partial\nu}(\rho_- - t_-)\eta_R + \int_{\mathbb{R}^2\setminus B_{R_0}} (|\nabla\rho_+|^2 + |\nabla\rho_-|^2)\eta_R \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2\setminus B_{R_0}} \nabla(\rho_+ - t_+)^2\nabla\eta_R + \nabla(\rho_- - t_-)^2\nabla\eta_R \\
&\quad + \int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla\varphi_+|^2(\rho_+ - t_+)\rho_+\eta_R + \int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla\varphi_-|^2(\rho_- - t_-)\rho_+\eta_R \\
&= -\int_{\mathbb{R}^2\setminus B_{R_0}} [A_+(t_+ - \rho_+)^2(t_+ + \rho_+) + B(t_-^2 - \rho_-^2)(t_+ - \rho_+)]\rho_+\eta_R \\
&\quad - \int_{\mathbb{R}^2\setminus B_{R_0}} [A_-(t_- - \rho_-)^2(t_- + \rho_-) + B(t_+^2 - \rho_+^2)(t_- - \rho_-)]\rho_-\eta_R.
\end{aligned}
\tag{6.43}
$$

Define $C_+ := (t_+ + \rho_+)\rho_+$, $C_- := (t_- + \rho_-)\rho_-$, $D := (t_- + \rho_-)\rho_+ + (t_+ + \rho_+)\rho_-$, where C_+ and C_- are positive, and we can get the upper bound for D:

$$\begin{aligned}
|D| &= |(t_- + \rho_-)\rho_+ + (t_+ + \rho_+)\rho_-| \\
&\le (|t_-| + |\rho_-|)|\rho_+| + (|t_+| + |\rho_+|)|\rho_-| \\
&\le (t_- + \Lambda)\Lambda + (t_+ + \Lambda)\Lambda \\
&= \Lambda(t_+ + t_- + 2\Lambda) \\
&=: C_0.
\end{aligned}$$

Using above information, we can rewrite the RHS of (6.43) and get an estimate:

$$\begin{aligned}
&-\int_{\mathbb{R}^2\setminus B_{R_0}} [A_+(t_+ - \rho_+)^2(t_+ + \rho_+) + B(t_-^2 - \rho_-^2)(t_+ - \rho_+)]\rho_+\eta_R \\
&\qquad - \int_{\mathbb{R}^2\setminus B_{R_0}} [A_-(t_- - \rho_-)^2(t_- + \rho_-) + B(t_+^2 - \rho_+^2)(t_- - \rho_-)]\rho_-\eta_R \\
&= -\int_{\mathbb{R}^2\setminus B_{R_0}} [A_+(t_+ - \rho_+)^2 C_+ + A_-(t_- - \rho_-)^2 C_-]\eta_R \\
&\qquad - \int_{\mathbb{R}^2\setminus B_{R_0}} B(t_+ - \rho_+)(t_- - \rho_-)D\eta_R \\
&\le -\int_{\mathbb{R}^2\setminus B_{R_0}} [A_+(t_+ - \rho_+)^2 C_+ + A_-(t_- - \rho_-)^2 C_-]\eta_R \\
&\qquad + \int_{\mathbb{R}^2\setminus B_{R_0}} |B||t_+ - \rho_+||t_- - \rho_-||D|\eta_R \\
&\le \int_{\mathbb{R}^2\setminus B_{R_0}} |B||t_+ - \rho_+||t_- - \rho_-||D|\eta_R \\
&\le C_0 \int_{\mathbb{R}^2\setminus B_{R_0}} |B||t_+ - \rho_+||t_- - \rho_-| \\
&\le C_3 \int_{\mathbb{R}^2\setminus B_{R_0}} (t_+ - \rho_+)^2 + (t_- - \rho_-)^2, \qquad (6.44)
\end{aligned}$$

Combining (6.43) and (6.44), we obtain that

$$
\begin{aligned}
&\int_{S_{R_0}} \frac{\partial \rho_+}{\partial \nu}(\rho_+ - t_+)\eta_R + \frac{\partial \rho_-}{\partial \nu}(\rho_- - t_-)\eta_R + \int_{\mathbb{R}^2\setminus B_{R_0}} (|\nabla \rho_+|^2 + |\nabla \rho_-|^2)\eta_R \\
&\qquad + \frac{1}{2}\int_{\mathbb{R}^2\setminus B_{R_0}} \nabla(\rho_+ - t_+)^2 \nabla \eta_R + \nabla(\rho_- - t_-)^2 \nabla \eta_R \\
&\qquad + \int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla \varphi_+|^2 (\rho_+ - t_+)\rho_+ \eta_R + \int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla \varphi_-|^2 (\rho_- - t_-)\rho_+ \eta_R \\
&\quad \le C_3 \int_{\mathbb{R}^2\setminus B_{R_0}} (t_+ - \rho_+)^2 + (t_- - \rho_-)^2,
\end{aligned}
$$

then it yields that

$$
\begin{aligned}
\int_{\mathbb{R}^2\setminus B_{R_0}} (|\nabla \rho_+|^2 + |\nabla \rho_-|^2)\eta_R &\le \int_{S_{R_0}} \frac{\partial \rho_+}{\partial \nu}(t_+ - \rho_+)\eta_R + \frac{\partial \rho_-}{\partial \nu}(t_- - \rho_-)\eta_R \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^2\setminus B_{R_0}} \nabla(\rho_+ - t_+)^2 \nabla \eta_R + \nabla(\rho_- - t_-)^2 \nabla \eta_R \\
&\quad + \int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla \varphi_+|^2 (t_+ - \rho_+)\rho_+ \eta_R + \int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla \varphi_-|^2 (t_- - \rho_-)\rho_+ \eta_R \\
&\quad + C_3 \int_{\mathbb{R}^2\setminus B_{R_0}} (t_+ - \rho_+)^2 + (t_- - \rho_-)^2 \\
&\le \max\{t_+, t_-\} \int_{S_{R_0}} \left|\frac{\partial \rho_+}{\partial \nu}\right| + \left|\frac{\partial \rho_-}{\partial \nu}\right| + \frac{1}{2}\int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla(\rho_+ - t_+)^2 \nabla \eta_R| + |\nabla(\rho_- - t_-)^2 \nabla \eta_R| \\
&\quad + \Lambda \int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla \varphi_+|^2 (t_+ - \rho_+) + \Lambda \int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla \varphi_-|^2 (t_- - \rho_-) \\
&\quad + C_3 \int_{\mathbb{R}^2\setminus B_{R_0}} (t_+ - \rho_+)^2 + (t_- - \rho_-)^2. \qquad (6.45)
\end{aligned}
$$

On the other hand, by (6.8), we have

$$
\begin{aligned}
\int_{\mathbb{R}^2\setminus B_{R_0}} (t_+ - \rho_+)^2 + (t_- - \rho_-)^2 &= \int_{\mathbb{R}^2\setminus B_{R_0}} \left[\frac{t_+^2 - \rho_+^2}{t_+ + \rho_+}\right]^2 + \left[\frac{t_-^2 - \rho_-^2}{t_- + \rho_-}\right]^2 \\
&\le \int_{\mathbb{R}^2\setminus B_{R_0}} \left[\frac{t_+^2 - \rho_+^2}{t_+}\right]^2 + \left[\frac{t_-^2 - \rho_-^2}{t_-}\right]^2 \\
&= \frac{1}{t_+^2}\int_{\mathbb{R}^2\setminus B_{R_0}} (t_+^2 - \rho_+^2)^2 + \frac{1}{t_-^2}\int_{\mathbb{R}^2\setminus B_{R_0}} (t_-^2 - \rho_-^2)^2 \\
&< \infty. \qquad (6.46)
\end{aligned}
$$

Also, note that

$$\begin{aligned}\left|\int_{\mathbb{R}^2\setminus B_{R_0}} \nabla(\rho_\pm - t_\pm)^2 \nabla\eta_R\right| &= \left|\int_{\mathbb{R}^2\setminus B_{R_0}} (\rho_\pm - t_\pm)^2 \Delta\eta_R\right| \\ &\leq \int_{\mathbb{R}^2\setminus B_{R_0}} [(\rho_+ - t_+)^2 + (\rho_- - t_-)^2]\Delta\eta_R \\ &\leq \frac{1}{R^2}\int_{\mathbb{R}^2\setminus B_{R_0}} (\rho_+ - t_+)^2 + (\rho_- - t_-)^2 \\ &\leq \frac{C}{R^2}, \qquad (6.47)\end{aligned}$$

where C is some positive constant.

Hence, from (6.37), (6.45) to (6.47), we obtain that

$$\begin{aligned}&\int_{\mathbb{R}^2\setminus B_{R_0}} (|\nabla\rho_+|^2 + |\nabla\rho_-|^2)\eta_R \\ &\quad\leq \frac{C}{R^2} + C + \max\{t_+, t_-\}\int_{S_{R_0}} \left|\frac{\partial\rho_+}{\partial\nu}\right| + \left|\frac{\partial\rho_-}{\partial\nu}\right| \\ &\quad\leq \frac{C}{R^2} + C, \qquad (6.48)\end{aligned}$$

where C is some constant.

Therefore, the right hand side of (6.48) remains bounded as $R \to \infty$. Letting $R \to \infty$ in (6.48) we obtain that

$$\int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla\rho_+|^2 + |\nabla\rho_-|^2 < \infty.$$

If either of $\int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla\rho_\pm|^2 = \infty$, we get the contradiction. Therefore, $\int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla\rho_\pm|^2 < \infty$. We complete the proof of Step 3. After the above three steps, we finish the proof of (6.3). □

In the following, we will give the details of the proof of Proposition 6.1, which follows the papers [BBH93] and [AB06], although many details must be modified in the case of our system of equations.

Proof of Proposition 6.1. The proof follows the lines of [BBH93] and [Sha94] (see also [ABM09]). Let $R_m \to \infty$ be any increasing divergent sequence, $\epsilon_m = 1/R_m$, and let $0 < a < 1 < b$ be fixed. Denote by $\Omega = B_b(0) \setminus \overline{B_a(0)}$ and $\Omega_m = B_{bR_m}(0) \setminus \overline{B_{aR_m}(0)}$. Consider the rescaled solutions $\Psi^{\epsilon_m}(x) = [\psi_{m+}(x), \psi_{m-}(x)] = \Psi(R_m x) = [\psi_+(R_m x), \psi(R_m x)]$. Then Ψ^{ϵ_m} satisfies

$$-\Delta\psi_{m\pm} + \frac{1}{\epsilon_m^2}[A_\pm(|\psi_{m\pm}|^2 - t_\pm^2) + B(|\psi_{m\mp}|^2 - t_\mp^2)]\psi_{m\pm} = 0 \quad \text{in } \Omega. \tag{6.49}$$

We apply Lemma 6.3 to obtain $R_0 > 0$ and $\rho_\pm, \phi_\pm$ defined for $|x| \geq R_0$. Since large $|x|$ is equivalent to large m we may write, for large m, $\psi_{m\pm} = \rho_{m\pm}\exp(i(n_\pm\theta + \phi_{m\pm}(x)))$.

Step 1. $\Psi^{\epsilon_m} \to \Psi^*$ strongly in $H^1(\Omega)$.

As in [ABM09], by (6.3) and the scaling $y = R_m x$ we calculate that

$$\begin{aligned}
\int_\Omega |\nabla\Psi^{\epsilon_m}|^2 &= \int_\Omega |\nabla\Psi(R_m x)|^2 dx \\
&= \int_{\Omega_m} R_m^2|\nabla\Psi(y)|^2\frac{1}{R_m^2}dy = \int_{\Omega_m} |\nabla\Psi(y)|^2 dy \\
&= \int_{\Omega_m} \sum_{i=\pm}[|\nabla\rho_i|^2 + \rho_i^2|n_i\nabla\theta + \nabla\phi_i|^2] \\
&= \int_{\Omega_m} \sum_{i=\pm}[|\nabla\rho_i|^2 + \rho_i^2(n_i^2|\nabla\theta|^2 + |\nabla\phi_i|^2 + 2n_i\nabla\theta\cdot\nabla\phi_i)] \\
&= \int_{\Omega_m} \sum_{i=\pm}[|\nabla\rho_i|^2 + \rho_i^2\frac{n_i^2}{r^2} + \rho_i^2|\nabla\phi_i|^2 + 2\rho_i^2\frac{n_i^2}{r^2}\nabla\phi_i\cdot(-x_2, x_1)] \\
&= \int_{\Omega_m} \sum_{i=\pm} t_i^2\frac{n_i^2}{r^2} + O(1) \\
&= \int_0^{2\pi}\int_{aR_m}^{bR_m} \sum_{i=\pm} t_i^2\frac{n_i^2}{r^2} r\,dr\,d\theta + O(1) \\
&= 2\pi(t_+^2 n_+^2 + t_-^2 n_-^2)\ln\left(\frac{b}{a}\right) + O(1).
\end{aligned} \tag{6.50}$$

Up to a subsequence, we get that $\Psi^{\epsilon_m} \rightharpoonup \Psi^*$ in $H^1(\Omega;\Sigma)$, and

$$\Psi^*(x) = [t_+e^{i(n_+\theta+\beta_+)}, t_-e^{i(n_-\theta+\beta_-)}] \tag{6.51}$$

with $\beta_\pm$ real constants. In fact, since Ψ^* takes values in Σ we may write it locally as $\Psi^* = [t_+\exp(i\varphi_+(x)), t_-\exp(i\varphi_-(x))]$, where $\varphi_\pm$ are possibly multivalued, real-valued functions. Then, by Cauchy-Schwarz inequality, we derive a lower bound which matches (6.50):

$$\begin{aligned}
\int_\Omega |\nabla\psi_\pm^*|^2 = \int_\Omega |\nabla(t_\pm e^{i\varphi_\pm})|^2 &= \int_\Omega t_\pm^2 |\nabla\varphi_\pm|^2 \\
&\geq t_\pm^2 \int_\Omega (\nabla\varphi_\pm\cdot\tau)^2 \\
&= t_\pm^2 \int_a^b \int_0^{2\pi} (\nabla\varphi_\pm\cdot\tau)^2 r\,d\theta\,dr \\
&= t_\pm^2 \int_a^b \left(\int_{S_r} (\nabla\varphi_\pm\cdot\tau)^2 dS_r\right) dr \\
&= t_\pm^2 \int_a^b \left(\int_{S_r} \left(\frac{\partial\varphi_\pm}{\partial\tau}\right)^2 dS_r\right) dr \\
&\geq t_\pm^2 \int_a^b \frac{\left(\int_{S_r} \partial\varphi_\pm/\partial\tau\, dS_r\right)^2}{2\pi r} dr \\
&= t_\pm^2 \int_a^b \frac{(2\pi n_\pm)^2}{2\pi r} dr \\
&= 2\pi t_\pm^2 n_\pm^2 \int_a^b \frac{1}{r} dr \\
&= 2\pi t_\pm^2 n_\pm^2 \ln(b/a)\,.
\end{aligned}$$

By lower semi-continuity, we have

$$\int_\Omega |\nabla\Psi_\pm^*|^2 \leq 2\pi \ln(b/a)\,(t_+^2 n_+^2 + t_-^2 n_-^2).$$

Combining above two inequalities, we obtain $\int_\Omega |\nabla\Psi_\pm^*|^2 = 2\pi\ln(b/a)\,(t_+^2 n_+^2 + t_-^2 n_-^2)$.

Step 2. Let $A_\epsilon := \frac{1}{2}|\nabla\Psi^{\epsilon_m}|^2$. Then, whenever $|\psi_\pm^{\epsilon_m}| \geq \frac{1}{T}$ with $\frac{1}{T} := \min\left\{\frac{1}{2}t_+^2, \frac{1}{2}t_-^2\right\}$,

$$-\Delta A_\epsilon + \frac{1}{2}|D^2\Psi^{\epsilon_m}|^2 \leq 4TA_\epsilon^2, \qquad \text{if } \; A_+A_- - B^2 > 0,\; A_\pm > 0. \tag{6.52}$$

In the following, we write $\epsilon = \epsilon_m$, dropping the subscript, and often write $\Psi^{\epsilon_m} =$

$\Psi = (\psi_+, \psi_-)$ for simplicity if notation. By a simple calculation, we have that

$$\Delta\left(\frac{1}{2}|\nabla\psi_\pm|^2\right) = |D^2\psi_\pm|^2 + \sum_{i=1}^{2}\psi_{\pm,x_i}\cdot\Delta(\psi_{\pm,x_i}).$$

From (1.2), we have that

$$\begin{aligned}\Delta(\psi_{+,x_i}) &= \frac{1}{\epsilon^2}[A_+(|\psi_+|^2 - t_+^2) + B(|\psi_-|^2 - t_-^2)]\psi_{+,x_i} \\ &\quad + \frac{1}{\epsilon^2}[2A_+\langle\psi_+, \psi_{+,x_i}\rangle + 2B\langle\psi_-, \psi_{-,x_i}\rangle]\psi_+ \\ &= \frac{1}{\epsilon^2}[A_+(|\psi_+|^2 - t_+^2) + B(|\psi_-|^2 - t_-^2)]\psi_{+,x_i} \\ &\quad + \frac{2}{\epsilon^2}[A_+\langle\psi_+, \psi_{+,x_i}\rangle + B\langle\psi_-, \psi_{-,x_i}\rangle]\psi_+.\end{aligned}$$

Similarly,

$$\begin{aligned}\Delta(\psi_{+,x_i}) &= \frac{1}{\epsilon^2}[A_-(|\psi_-|^2 - t_-^2) + B(|\psi_+|^2 - t_+^2)]\psi_{-,x_i} \\ &\quad + \frac{2}{\epsilon^2}[A_-\langle\psi_-, \psi_{-,x_i}\rangle + B\langle\psi_+, \psi_{+,x_i}\rangle]\psi_-.\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta\left(\frac{1}{2}|\nabla\psi_+|^2\right) &= |D^2\psi_+|^2 + \sum_{i=1}^{2}\psi_{+,x_i}\cdot\Delta(\psi_{+,x_i}) \\ &= |D^2\psi_+|^2 + \frac{1}{\epsilon^2}[A_+(|\psi_+|^2 - t_+^2) + B(|\psi_-|^2 - t_-^2)]|\nabla\psi_+|^2 \\ &\quad + \frac{2}{\epsilon^2}[A_+\langle\psi_+, \nabla\psi_+\rangle^2 + B\langle\psi_+, \nabla\psi_+\rangle\langle\psi_-, \nabla\psi_-\rangle].\end{aligned}$$

Similarly,

$$\begin{aligned}\Delta\left(\frac{1}{2}|\nabla\psi_-|^2\right) &= |D^2\psi_-|^2 + \frac{1}{\epsilon^2}[A_-(|\psi_-|^2 - t_+^2) + B(|\psi_+|^2 - t_+^2)]|\nabla\psi_-|^2 \\ &\quad + \frac{2}{\epsilon^2}[A_-\langle\psi_-, \nabla\psi_-\rangle^2 + B\langle\psi_+, \nabla\psi_+\rangle\langle\psi_-, \nabla\psi_-\rangle].\end{aligned}$$

By the positive definite condition,

$$-\Delta A_\epsilon + |D^2\Psi|^2 = -\frac{1}{\epsilon^2}[A_+(|\psi_+|^2 - t_+^2) + B(|\psi_-|^2 - t_-^2)]|\nabla\psi_+|^2$$

$$
\begin{aligned}
&- \frac{1}{\epsilon^2}[A_-(|\psi_-|^2 - t_-^2) + B(|\psi_+|^2 - t_+^2)]|\nabla\psi_-|^2 \\
&- \frac{2}{\epsilon^2}[A_+\langle\psi_+, \nabla\psi_+\rangle^2 + A_-\langle\psi_-, \nabla\psi_-\rangle^2 + B\langle\psi_+, \nabla\psi_+\rangle\langle\psi_-, \nabla\psi_-\rangle] \\
\leq & -\frac{\Delta\psi_+}{\psi_+}|\nabla\psi_+|^2 - \frac{\Delta\psi_-}{\psi_-}|\nabla\psi_-|^2 - \frac{2}{\epsilon^2}\lambda_s[\langle\psi_+, \nabla\psi_+\rangle^2 + \langle\psi_-, \nabla\psi_-\rangle^2] \\
\leq & -\frac{\Delta\psi_+}{\psi_+}|\nabla\psi_+|^2 - \frac{\Delta\psi_-}{\psi_-}|\nabla\psi_-|^2.
\end{aligned}
$$

By the fact $|\Delta\psi_\pm| \leq \sqrt{2}|D^2\psi_\pm|$ and Cauchy's inequality, we also have

$$
\begin{aligned}
&- \Delta A_\epsilon + |D^2\Psi|^2 \\
&\leq \frac{|\Delta\psi_+|}{|\psi_+|}|\nabla\psi_+|^2 + \frac{|\Delta\psi_-|}{|\psi_-|}|\nabla\psi_-|^2 \\
&\leq \frac{\sqrt{2}|D^2\psi_+|}{|\psi_+|}|\nabla\psi_+|^2 + \frac{\sqrt{2}|D^2\psi_-|}{|\psi_-|}|\nabla\psi_-|^2 \\
&= \sqrt{2}\left(\frac{|D^2\psi_+|}{|\psi_+|}|\nabla\psi_+|^2 + \frac{|D^2\psi_-|}{|\psi_-|}|\nabla\psi_-|^2\right) \\
&\leq \sqrt{2}\left(\frac{1}{2\sqrt{2}}|D^2\psi_+|^2 + \frac{\sqrt{2}}{2}\frac{|\nabla\psi_+|^4}{|\psi_+|^2}\right) + \sqrt{2}\left(\frac{1}{2\sqrt{2}}|D^2\psi_-|^2 + \frac{\sqrt{2}}{2}\frac{|\nabla\psi_-|^4}{|\psi_-|^2}\right) \\
&= \frac{1}{2}\left(|D^2\psi_+|^2 + |D^2\psi_-|^2\right) + \left(\frac{|\nabla\psi_+|^4}{|\psi_+|^2} + \frac{|\nabla\psi_-|^4}{|\psi_-|^2}\right) \\
&\leq \frac{1}{2}|D^2\Psi|^2 + T(|\nabla\psi_+|^4 + |\nabla\psi_-|^4) \\
&\leq \frac{1}{2}|D^2\Psi|^2 + T(|\nabla\psi_+|^2 + |\nabla\psi_-|^2)^2 \\
&= \frac{1}{2}|D^2\Psi|^2 + T|\nabla\Psi|^4 \\
&= \frac{1}{2}|D^2\Psi|^2 + 4TA_\epsilon^2,
\end{aligned}
$$

i.e. $-\Delta A_\epsilon + \frac{1}{2}|D^2\Psi|^2 \leq 4TA_\epsilon^2$, as desired.

Step 3. Ψ^{ϵ_m} is bounded in H^2_{loc} and $\nabla\Psi^{\epsilon_m}$ is bounded in L^∞_{loc} .

Since $\Psi \to \Psi^*$ strongly in $H^1(\Omega)$, for given $\delta > 0$ (to be determined later), we may choose R sufficiently small so that

$$
\int_{B(x_0,R)} |\nabla\Psi|^2 < \delta, \qquad \forall\ x_0 \in \Omega,\ \forall\epsilon. \tag{6.53}
$$

Fix a point $x_0 \in \Omega$ and set $d = \mathrm{dist}(x_0, \partial\Omega)$. Let ζ be a smooth function with support in $B(x_0, r)$ with $r = \min(d/2, R)$ such that $\zeta = 1$ on $B(x_0, r/2)$. Multiplying ζ^2 to (6.52) and integrating by parts we have

$$\int_\Omega -A_\epsilon \Delta(\zeta^2) + \frac{1}{2}\int_\Omega |D^2\psi_\pm|^2\zeta^2 \leq 4T\int_\Omega A_\epsilon^2\zeta^2,$$

then by Cauchy-Schwartz inequality and Cauchy's inequality,

$$\begin{aligned}
\frac{1}{2}\int_\Omega |D^2\psi_\pm|^2\zeta^2 &\leq 4T\int_\Omega A_\epsilon^2\zeta^2 + \int_\Omega A_\epsilon\Delta(\zeta^2) \\
&\leq 4T\int_\Omega A_\epsilon^2\zeta^2 + \left[\int_\Omega (A_\epsilon\zeta)^2\right]^{1/2}\left[\int_\Omega \left(\frac{\Delta(\zeta^2)}{\zeta}\right)^2\right]^{1/2} \\
&\leq 4T\int_\Omega A_\epsilon^2\zeta^2 + \epsilon\int_\Omega (A_\epsilon\zeta)^2 + \frac{1}{4\epsilon}\int_\Omega \left(\frac{\Delta(\zeta^2)}{\zeta}\right)^2 \\
&= C_1\int_\Omega A_\epsilon^2\zeta^2 + C_2,
\end{aligned}$$

with C_1, C_2 are some constants. This implies that

$$\frac{1}{2}\int_\Omega |D^2\psi_\pm|^2\zeta^2 \leq C_1\int_\Omega A_\epsilon^2\zeta^2 + C_2 = C_3\int_\Omega |\nabla\Psi|^4\zeta^2 + C_2,$$

i.e.

$$\begin{aligned}
\int_\Omega |D^2\psi_\pm|^2\zeta^2 &\leq C_3\int_\Omega |\nabla\Psi|^4\zeta^2 + C_2 \\
&\leq C_3\int_\Omega (|\nabla\psi_+|^2 + |\nabla\psi_-|^2)^2\zeta^2 + C_2 \\
&\leq 2C_3\int_\Omega (|\nabla\psi_+|^4 + |\nabla\psi_-|^4)\zeta^2 + C_2.
\end{aligned}$$

Recall that $W^{1,1}(\Omega) \subset L^2(\Omega)$ (since $\Omega \subset \mathbb{R}^2$) and

$$\left(\int_\Omega \varphi^2\right)^{1/2} \leq C\int_\Omega |\nabla\varphi| + |\varphi|, \qquad \forall\varphi \in W^{1,1}(\Omega)$$

(see (28) in [BBH93]), we obtain that

$$\left(\int_\Omega \zeta^2|\nabla\psi_\pm|^4\right)^{1/2} \leq C\int_\Omega \left|\nabla(\zeta|\nabla\psi_\pm|^2)\right| + \zeta|\nabla\psi_\pm|^2$$

$$
\begin{aligned}
&= C\int_\Omega \nabla\zeta\cdot|\nabla\psi_\pm|^2 + 2\zeta|\nabla\psi_\pm|\cdot|D^2\psi_\pm| + \zeta\cdot|\nabla\psi_\pm|^2\\
&\le C\int_\Omega |\nabla\psi_\pm|^2 + C\int_\Omega \zeta|\nabla\psi_\pm|\cdot|D^2\psi_\pm|\\
&\le C + C\int_\Omega \zeta|\nabla\psi_\pm|\cdot|D^2\psi_\pm|,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\int_\Omega \zeta^2|\nabla\psi_\pm|^4 &\le C + C\left(\int_\Omega \zeta|\nabla\psi_\pm|\cdot|D^2\psi_\pm|\right)^2\\
&= C + 2C\int_\Omega \zeta|\nabla\psi_\pm|\cdot|D^2\psi_\pm| + C\left(\int_\Omega \zeta|\nabla\psi_\pm|\cdot|D^2\psi_\pm|\right)^2\\
&\le C + C\left(\int_\Omega \zeta|\nabla\psi_\pm|\cdot|D^2\psi_\pm|\right)^2\\
&\le C + C\left[\int_\Omega \zeta^2|D^2\psi_\pm|^2\right]\left[\int_\Omega |\nabla\psi_\pm|^2\right]\\
&\le C + C\delta\int_\Omega \zeta^2|D^2\psi_\pm|^2,
\end{aligned}
$$

i.e. $\int_\Omega \zeta^2|\nabla\psi_\pm|^4 \le C + C\delta\int_\Omega \zeta^2|D^2\psi_\pm|^2$.

Add the two inequalities for ψ_+, ψ_- together, we get that

$$
\begin{aligned}
\int_\Omega \zeta^2\left[|\nabla\psi_+|^4 + |\nabla\psi_-|^4\right] &\le C\delta\int_\Omega \zeta^2\left(|D^2\psi_+|^2 + |D^2\psi_-|^2\right) + C\\
&= C\delta\int_\Omega \zeta^2|D^2\Psi|^2 + C,
\end{aligned}
$$

i.e. we obtain that

$$
\begin{aligned}
\int_\Omega \zeta^2|D^2\psi_\pm|^2 &\le C\int_\Omega \zeta^2\left[|\nabla\psi_+|^4 + |\nabla\psi_-|^4\right] + C\\
&\le C\delta\int_\Omega \zeta^2|D^2\Psi|^2 + C. \qquad (6.54)
\end{aligned}
$$

Then, we add the two inequalities of $|D^2\psi_+|^2$ and $|D^2\psi_-|^2$ together, we get that

$$
\int_\Omega \zeta^2|D^2\Psi|^2 \le C\delta\int_\Omega \zeta^2|D^2\Psi|^2 + C.
$$

If we choose δ sufficiently small, we may absorb $\int_\Omega \zeta^2|D^2\Psi|^2$ into the LHS of (6.54). Hence, it yields that

$$\int_\Omega \zeta^2|D^2\Psi|^2 \leq C,$$

which implies that Ψ is bounded in $H^2_{\text{loc}}(\Omega)$.

Now we will show $\nabla\Psi$ is bounded in L^∞_{loc}. Since Ψ is bounded in H^2_{loc}, we have that $\nabla\Psi$ is bounded in H^1_{loc}. By the Sobolev embedding $H^1 \hookrightarrow L^q$ for $\forall\, q < \infty$, we have that $\nabla\Psi$ is bounded in L^q_{loc} for $\forall\, q < \infty$. Then, we obtain that

$$\begin{aligned}
\int_K |\nabla\Psi|^{2q} &\leq \left[\int_K \left(|\nabla\Psi|^{2q}\right)^{q/2q} dx\right]^{2q/q} \left[\int_K 1^{1-\frac{q}{2q}} dx\right]^2 \\
&\leq \left[\int_K |\nabla\Psi|^q dx\right]^2 |K|^2 \\
&\leq \|\nabla\Psi\|^{2q}_{L^q(K)}|\Omega|^2 < C, \qquad \text{where} \quad K \subset\subset \Omega,
\end{aligned}$$

i.e. $|\nabla\Psi|^2 \in L^q_{\text{loc}}$ for $\forall\, q < \infty$, which implies that $A_\epsilon \in L^q_{\text{loc}}$ for $\forall\, q < \infty$. From (6.52), we deduce that

$$-\Delta A_\epsilon \leq 4TA_\epsilon^2 - \frac{1}{2}|D^2\Psi|^2 \leq 4TA_\epsilon^2 =: f_\epsilon.$$

By using the similar argument as above, we can get that $f_\epsilon \in L^q_{\text{loc}}$ for $\forall\, q < \infty$. Therefore, we have that $-\Delta A_\epsilon \leq f_\epsilon$ with $f_\epsilon \in L^q_{\text{loc}}$ for $\forall\, q < \infty$.

Now assume that

$$\begin{cases} -\Delta\varphi_\epsilon = f_\epsilon & \text{in } \ 2K, \\ \varphi_\epsilon = 0 & \text{on } \ \partial(2K), \end{cases}$$

where $K = \overline{B_R(x_0)}$, $2K = \overline{B_{2R}(x_0)}$ with f_ϵ defined as above. Applying L^q-interior estimate, it yields that $\|\varphi_\epsilon\|_{W^{2,q}(K)} \leq C\|f_\epsilon\|_{L^q(2K)} \leq C$. It follows that φ_ϵ is bounded in $L^q(K)$. By the Sobolev embedding $W^{2,q} \hookrightarrow \mathcal{C}^{1,\alpha}$, we deduce that φ_ϵ is bounded in $\mathcal{C}^{1,\alpha}_{\text{loc}}$, which implies that φ_ϵ is bounded in L^∞_{loc}. Let $\psi = A_\epsilon - \varphi_\epsilon$, then we have $-\Delta\psi = -\Delta A_\epsilon + \Delta\varphi_\epsilon \leq 0$ and $\|\psi\|_{L^q(K)} \leq \|A_\epsilon\|_{L^q(K)} + \|\varphi_\epsilon\|_{L^q(K)} \leq C$. On the other hand, by the mean-value inequality and Hölder's inequality, we obtain that

$$|\psi(x)| \leq \frac{1}{|B_R(x)|}\int_{B_R(x)} |\psi(y)|dy$$

$$\leq \frac{1}{|B_R(x)|}\|\psi\|_{L^q}\,|B_R(x)|^{1-\frac{1}{q}}$$
$$\leq C(R),$$

i.e. $\|\psi\|_{L^\infty_{\text{loc}}} \leq C$. Therefore, $\|A_\epsilon\|_{L^\infty_{\text{loc}}} \leq \|\psi\|_{L^\infty_{\text{loc}}} + \|\varphi_\epsilon\|_{L^\infty_{\text{loc}}} \leq C$, which yields that $A_\epsilon \in L^\infty_{\text{loc}}$, i.e. $\|\nabla\Psi\|_{L^\infty_{\text{loc}}} \leq C$.

Step 4. $\|\Delta\Psi^{\epsilon_m}\|_{L^\infty_{\text{loc}}} \leq C$ uniformly in ϵ.

Follow the process in [BBH93], we establish the following lemma:

Lemma 6.4. *Let $w_0(r)$ be the solution of*

$$\begin{cases} -\epsilon^2\Delta w_0 + \mu w_0 = 0, & \textit{in} \quad B(0,R), \\ w_0 = 1, & \textit{on} \quad \partial B(0,R). \end{cases} \tag{6.55}$$

(A) When $B<0, A_+A_- - B^2 > 0$, we have, for $\epsilon < \frac{3\sqrt{\mu}}{4}R$,

$$0 < w_0(r) \leq \exp\left\{\frac{\sqrt{\mu}}{4\epsilon R}(r^2 - R^2)\right\} \quad \textit{on} \;\; B(0,R),$$

with $\mu = \min\{t_+^2(A_+ + Br), t_-^2(A_- + \frac{B}{r})\}$.

(B) When $B>0, A_+A_- - B^2 > 0$, we have, for $\epsilon < \frac{3\sqrt{\mu}}{4}R$,

$$0 < w_0(r) \leq \exp\left\{\frac{\sqrt{\mu}}{4\epsilon R}(r^2 - R^2)\right\} \quad \textit{on} \;\; B(0,R),$$

with $\mu = \min\{t_+^2(A_+ - Br), t_-^2(A_- - \frac{B}{r})\}$.

Proof. We show the proof in a general way. Let $w(x) = \bar{w}(cx)$ with $\bar{x} = cx$, $x \in B(0,R)$ and $\bar{w}$ is the solution to (6.55), w is the solution to (6.55) when $\mu = 1$. Then, we have $\Delta_x w = c^2\Delta_{\bar{x}}\bar{w}$, and $-\epsilon^2\Delta w + w = -\epsilon^2c^2\Delta_{\bar{x}}\bar{w} + \bar{w} = c^2\left[-\epsilon^2\Delta\bar{w} + \frac{1}{c^2}\bar{w}\right] = 0$. Therefore, after the scaling, we have

$$\begin{cases} -\epsilon^2\Delta\bar{w} + \frac{1}{c^2}\bar{w} = 0, & \text{in} \quad B(0,\bar{R}), \\ \bar{w} = 1, & \text{on} \quad \partial B(0,\bar{R}), \end{cases} \tag{6.56}$$

with $c = \frac{1}{\sqrt{\mu}}$ and $\bar{R} = cR$.

According to the result in [BBH93], it is clear that $w(r) \le \exp\{\frac{1}{4\epsilon R}(r^2 - R^2)\}$ on $B(0,R)$. After the scaling above, we have that

$$\bar{w}(\bar{r}) \le \exp\left\{\frac{1}{4\epsilon\sqrt{\mu}\bar{R}}(\mu\bar{r}^2 - \mu\bar{R}^2)\right\} = \exp\left\{\frac{\sqrt{\mu}}{4\epsilon\bar{R}}(\bar{r}^2 - \bar{R}^2)\right\} \quad \text{on} \quad B(0,\bar{R}).$$

Then, for $\epsilon < \frac{3}{4}R = \frac{3}{4}\sqrt{\mu}R$, i.e. for $\epsilon < \frac{3\sqrt{\mu}}{4}\bar{R}$, we have

$$0 < \bar{w}(\bar{r}) \le \exp\left\{\frac{\sqrt{\mu}}{4\epsilon\bar{R}}(\bar{r}^2 - \bar{R}^2)\right\} \quad \text{on} \quad B(0,\bar{R}).$$

Therefore, we finish our proof. □

Proof of Step 4. We will use sup-sub solution argument for the following proof and we divide the proof into two cases. Assume $B(x_0,R) \subset \Omega$. By translating axes, we may assume $x_0 = 0$.

Case 1. When $B < 0$, $B^2 < A_+A_-$, choose r with $\frac{-B}{A_-} < r < \frac{A_+}{-B}$, then let $a_\pm > 0$ with $a_+^2 + a_-^2 = 1$ and $r = \frac{a_-}{a_+}$ (i.e. $r = \tan\alpha$, $\alpha \in (0,\frac{\pi}{2})$, $a_+ = \cos\alpha$, $a_- = \sin\alpha$). We define $w_\pm = \frac{a_\pm}{t_\pm^2}w_0$, where w_0 is a solution to (6.55). Denote the limit operators for the associated equation as follows:

$$L_0^+(w_+,w_-) = -\frac{\epsilon^2}{2}\Delta w_+ + A_+t_+^2w_+ + Bt_-^2w_-, \tag{6.57}$$

$$L_0^-(w_-,w_+) = -\frac{\epsilon^2}{2}\Delta w_- + A_-t_-^2w_- + Bt_+^2w_+. \tag{6.58}$$

Make a substitution to (6.57)-(6.58), we have

$$\begin{aligned} L_0^+(w_+,w_-) &= \frac{a_+}{t_+^2}\left[-\frac{\epsilon^2}{2}\Delta w_0 + t_+^2\left(A_+ + B\frac{a_-}{a_+}\right)w_0\right] \\ &= \frac{a_+}{t_+^2}\left[-\frac{\epsilon^2}{2}\Delta w_0 + t_+^2\left(A_+ + Br\right)w_0\right] \\ &\ge \frac{a_+}{2t_+^2}\left[-\epsilon^2\Delta w_0 + 2\mu w_0\right] \\ &= \frac{a_+}{2t_+^2}\mu w_0, \end{aligned}$$

i.e. $L_0^+(w_+, w_-) \geq \frac{a_+\mu}{2t_+^2}w_0$. Similarly, we have $L_0^-(w_-, w_+) \geq \frac{a_-\mu}{2t_-^2}w_0$.

Let

$$L_-^+(w_+, w_-) = -\frac{\epsilon^2}{2}\Delta w_+ + A_+|\psi_+|^2 w_+ + B|\psi_-|^2 w_-,$$

and

$$L_-^-(w_-, w_+) = -\frac{\epsilon^2}{2}\Delta w_- + A_-|\psi_-|^2 w_- + B|\psi_+|^2 w_+.$$

Therefore, by the uniform convergence of $|\psi_\pm|^2$ and the lower bound for the limit operators $L_0^\pm$, we obtain that

$$\begin{aligned}
L_-^+(w_+, w_-) &= -\frac{\epsilon^2}{2}\Delta w_+ + A_+t_+^2 w_+ + Bt_-^2 w_- + [A_+(|\psi_+|^2 - t_+^2)w_+ + B(|\psi_-|^2 - t_-^2)w_-] \\
&\geq \frac{a_+\mu}{2t_+^2}w_0 + \left[A_+(|\psi_+|^2 - t_+^2)\frac{A_+a_+}{t_+^2} + B(|\psi_-|^2 - t_-^2)\frac{Ba_-}{t_-^2}\right]w_0 \\
&\geq \frac{a_+\mu}{4t_+^2}w_0 > 0, \qquad \text{for } \epsilon \text{ small enough.}
\end{aligned}$$

Similarly, we have that $L_-^-(w_-, w_+) \geq \frac{a_+\mu}{4t_+^2}w_0 > 0$. Therefore, we have $L_-^+(w_+, w_-) > 0$ and $L_-^-(w_-, w_+) > 0$, which implies that $[w_+, w_-]$ is sup-solution to $L_-^\pm$. Then, by the upper bound of w_0 in part (A) of Lemma 6.4, we get that

$$0 < w_\pm \leq \frac{a_\pm}{t_\pm^2}\exp\left\{\frac{\sqrt{\mu}}{4\epsilon R}(r^2 - R^2)\right\} \quad \text{on} \quad B(0, R) \quad \text{for} \quad \epsilon < \frac{3\sqrt{\mu}}{4}R. \tag{6.59}$$

Define $\varphi^\pm = \epsilon^2 X^\pm$, with $X^\pm := \frac{1}{\epsilon^2}\left[A_\pm(|\psi_\pm|^2 - t_\pm^2) + B(|\psi_\mp|^2 - t_\mp^2)\right]$ and $X^\pm\big|_{\partial\Omega} = 0$, so $\psi_\pm$ solves

$$-\Delta\psi_\pm + X^\pm\psi_\pm = 0.$$

We then calculate the equations satisfied by $\varphi_\pm$

$$-\frac{\epsilon^2}{2}\Delta\varphi^+ + A_+|\psi_+|^2\varphi^+ + B|\psi_-|^2\varphi^- = \epsilon^2 E_+, \tag{6.60}$$

$$-\frac{\epsilon^2}{2}\Delta\varphi^- + A_-|\psi_-|^2\varphi^- + B|\psi_+|^2\varphi^+ = \epsilon^2 E_-, \tag{6.61}$$

with $E_\pm = -(A_\pm|\nabla\psi_\pm|^2 + B|\nabla\psi_\mp|^2)$, $\|E_\pm\|_{L^\infty} \leq E_\pm^0$ and $E_\pm^0$ are constants, and denote the left side of (6.60)-(6.61) by $L^+(\varphi^+, \varphi^-)$, $L^-(\varphi^-, \varphi^+)$ separately.

Now consider $\overline{\varphi^+} = \epsilon^2\bar{u} + c_+w_+$ and $\overline{\varphi^-} = \epsilon^2\bar{v} + c_-w_-$ with $\bar{u}, \bar{v}$ are some constants to be chosen later. Also consider $\overline{\varphi^+} = \epsilon^2\bar{u} + c_1w_+$, $\overline{\varphi^-} = \epsilon^2\bar{v} + c_1w_-$ with c_1 to be determined later. Indeed, $\|\varphi^+\|_{L^\infty(B_R)} \leq A_+(\Lambda + t_+^2) + B(\Lambda + t_-^2)$, we choose c_+ such that $c_+w_+\big|_{\partial B_R} > A_+(\Lambda + t_+^2) + B(\Lambda + t_-^2) \geq \|\varphi^+\|_{L^\infty(B_R)}$. Similarly, choose c_- so that $c_-w_-\big|_{\partial B_R} > A_-(\Lambda + t_-^2) + B(\Lambda + t_+^2) \geq \|\varphi^-\|_{L^\infty(B_R)}$. Define $c_1 = \max\{c_+, c_-\}$, we obtain that $c_1w_\pm > c_\pm w_\pm \geq \|\varphi^\pm\|_{L^\infty(B_R)}$, which implies that $c_1w_\pm - \varphi^\pm\big|_{\partial B_R} < 0$. Hence, $w_\pm$ is sup-solution to $L^\pm(w_+, w_-) \geq 0$ with $w_\pm\big|_{\partial B_R} = \frac{a_\pm}{t_\pm^2}$.

Let $u = \varphi^+ - \overline{\varphi^+}$, $v = \varphi^- - \overline{\varphi^-}$, then by easy computations, we have

$$\begin{aligned}
L^+u &= L^+(\varphi^+, \varphi^-) - L^+(\overline{\varphi^+}, \overline{\varphi^-}) \\
&= \epsilon^2E_+ - \left[-\frac{\epsilon^2}{2}\Delta(\epsilon^2\bar{u} + c_1w_+) + A_+|\psi_+|^2(\epsilon^2\bar{u} + c_1w_+) + B|\psi_-|^2(\epsilon^2\bar{v} + c_1w_-)\right] \\
&= \epsilon^2E_+ - \left[-\frac{\epsilon^2}{2}\Delta(c_1w_+) + A_+|\psi_+|^2(c_1w_+) + B|\psi_-|^2(c_1w_+)\right] - \epsilon^2(A_+|\psi_+|^2\bar{u} + B|\psi_-|^2\bar{v}) \\
&= \epsilon^2E_+ - c_1L^+(w_+, w_-) - \epsilon^2(A_+|\psi_+|^2\bar{u} + B|\psi_-|^2\bar{v}) \\
&\leq \epsilon^2E_+ - \epsilon^2(A_+|\psi_+|^2\bar{u} + B|\psi_-|^2\bar{v}),
\end{aligned}$$

i.e.

$$L^+u \leq \epsilon^2(E_+ - A_+|\psi_+|^2\bar{u} - B|\psi_-|^2\bar{v}). \tag{6.62}$$

By substituting $v = \varphi^- - \overline{\varphi^-}$ to (6.61), we similarly have that

$$L^+u \leq \epsilon^2(E_- - A_-|\psi_-|^2\bar{v} - B|\psi_+|^2\bar{u}). \tag{6.63}$$

We now introduce a pair of modified comparison principles which will use to continue the arguments. These are suitably modified from analogous comparison principles in [AB06].

Lemma 6.5. *Let $A_\pm$ and B be constants, Ω be a bounded domain.*

(A) Assume $A_\pm > 0$, $B < 0$ and $A_+A_- - B^2 > 0$. Then, if u, v solve

$$\begin{cases} -\Delta u + A_+|\psi_+|^2u + B|\psi_-|^2v \leq 0, & u\big|_{\partial\Omega} \leq 0, \\ -\Delta v + A_-|\psi_-|^2v + B|\psi_+|^2u \leq 0, & v\big|_{\partial\Omega} \leq 0, \end{cases}$$

we have $u \leq 0$ and $v \leq 0$ in Ω.

(B) Assume $A_\pm > 0$, $B > 0$ and $A_+A_- - B^2 > 0$. Then, if u, v solve

$$\begin{cases} -\Delta u + A_+|\psi_+|^2 u + B|\psi_-|^2 v \leq 0, & u\big|_{\partial\Omega} \leq 0, \\ -\Delta v + A_-|\psi_-|^2 v + B|\psi_+|^2 u \geq 0, & v\big|_{\partial\Omega} \geq 0, \end{cases}$$

we have $u \leq 0$ and $v \geq 0$ in Ω.

Proof. The proof is similar to Lemma 2.3. To verify (A), multiply the respective equation by $u_+ = \max(u, 0)$ and $v_+ = \max(v, 0)$ and integrate by parts. There will be no boundary terms, and we obtain that

$$\int_\Omega \left[|\nabla u_+|^2 + A_+|\psi_+|^2 u_+^2 + B|\psi_-|^2 u_+ v_+ + B|\psi_-|^2 u_+ v_-\right] \leq 0, \tag{6.64}$$

$$\int_\Omega \left[|\nabla v_+|^2 + A_+|\psi_-|^2 v_+^2 + B|\psi_+|^2 u_+ v_+ + B|\psi_+|^2 u_- v_+\right] \leq 0, \tag{6.65}$$

where $u_- = \min(u, 0) \leq 0$, $v_- = \min(v, 0) \leq 0$. Multiply (6.64), (6.65) by t_+^2 and t_-^2 respectively and drop positive terms $Bt_+^2|\psi_-|^2 u_+ v_-$, $Bt_-^2|\psi_+|^2 u_- v_+$, we get that

$$\int_\Omega t_+^2|\nabla u_+|^2 + A_+ t_+^2|\psi_+|^2 u_+^2 + Bt_-^2|\psi_-|^2 u_+ v_+ \leq 0,$$

$$\int_\Omega t_-^2|\nabla v_+|^2 + A_- t_-^2|\psi_-|^2 v_+^2 + Bt_-^2|\psi_+|^2 u_+ v_+ \leq 0.$$

We add the two inequalities above, and note that the matrix associated to the quadratic form is positive definite in $\bar{\Omega}$ by hypothesis and the uniform convergence of $\psi_\pm$, so there exists a function $\alpha > 0$ with

$$A_+ t_+^2|\psi_+|^2 u_+^2 + B(t_+^2|\psi_-|^2 + t_-^2|\psi_+|^2) u_+ v_+ + A_- t_-^2|\psi_-|^2 v_+^2 \geq \alpha(u_+^2 + v_+^2) > 0.$$

In consquence,

$$\begin{aligned} \int_\Omega t_+^2|\nabla u_+|^2 &+ t_-^2|\nabla v_+|^2 + \alpha(u_+^2 + v_+^2) \\ &\leq \int_\Omega \left[t_+^2|\nabla u_+|^2 + t_-^2|\nabla v_+|^2 + A_+ t_+^2|\psi_+|^2 u_+^2\right. \\ &\qquad \left. + B(t_+^2|\psi_-|^2 + t_-^2|\psi_+|^2) u_+ v_+ + A_- t_-^2|\psi_-|^2 v_+^2\right] \leq 0, \end{aligned}$$

which implies that $\int_\Omega t_+^2|\nabla u_+|^2 + t_-^2|\nabla v_+|^2 < 0$, and we conclude that $u_+ \equiv 0, v_+ \equiv 0$ in Ω. Therefore, $u \le 0$ and $v \le 0$ in Ω.

To prove (B) we again multiply the first equation by u_+ to get (6.64), but multiply the second by $v_- = \min(v, 0) \le 0$, to obtain

$$\int_\Omega \left[|\nabla v_-|^2 + A_+|\psi_-|^2 v_-^2 + B|\psi_+|^2 u_+ v_- + B|\psi_+|^2 u_- v_-\right] \le 0. \tag{6.66}$$

Then, multiply (6.64), (6.66) by t_+^2, t_-^2 respectively and drop off the non-affecting terms $Bt_+^2|\psi_-|^2 u_+ v_+ > 0$, $Bt_+^2|\psi_+|^2 u_- v_- > 0$, we get that

$$\int_\Omega t_+^2|\nabla u_+|^2 + A_+ t_+^2|\psi_+|^2 u_+^2 + Bt_-^2|\psi_-|^2 u_+ v_- \le 0,$$

$$\int_\Omega t_-^2|\nabla v_-|^2 + A_- t_-^2|\psi_-|^2 v_-^2 + Bt_-^2|\psi_+|^2 u_+ v_- \le 0.$$

We add the two inequalities above, and note that the matrix associated to the quadratic form is positive definite in $\bar\Omega$ by hypothesis and the uniform convergence of $\psi_\pm$, so there exists a function $\beta > 0$ with

$$A_+ t_+^2|\psi_+|^2 u_+^2 + B(t_+^2|\psi_-|^2 + t_-^2|\psi_+|^2)u_+ v_- + A_- t_-^2|\psi_-|^2 v_-^2 \ge \beta(u_+^2 + v_-^2) > 0.$$

Similarly, as the proof in part (A), we conclude $u_+ \equiv 0, v_- \equiv 0$ in Ω. Therefore, $u \le 0$ and $v \ge 0$ in Ω. □

Back to the proof of Step 4, according to the result of Lemma 6.5, we treat each case separately. For any $\delta > 0$, there exists $\epsilon_0 > 0$ such that whenever $\epsilon < \epsilon_0$, we have

$$\begin{aligned} A_+|\psi_+|^2 &= A_+ t_+^2 + o(1) \ge (A_+ - \delta)t_+^2, \\ B|\psi_-|^2 &= Bt_-^2 + o(1) \ge (B - \delta)t_-^2, \\ A_-|\psi_-|^2 &= A_- t_-^2 + o(1) \ge (A_- - \delta)t_-^2, \\ B|\psi_+|^2 &= Bt_+^2 + o(1) \ge (B - \delta)t_+^2, \end{aligned}$$

for all $x \in \bar\Omega$. We fix δ sufficiently small so that the same sign conditions hold for these constants: $(A_+ - \delta)t_+^2 > 0, (A_- - \delta)t_-^2 > 0, (B - \delta)t_+^2 < 0, (B - \delta)t_-^2 < 0$ and

$[(A_+-\delta)(A_--\delta)-(B-\delta)^2]t_+^2t_-^2>0$. We now choose $\bar{u},\bar{v}$ to be the unique solution to the following system

$$\begin{cases} (A_+-\delta)t_+^2\bar{u}+(B-\delta)t_-^2\bar{v}=E_+^0, \\ (B-\delta)t_+^2\bar{u}+(A_--\delta)t_-^2\bar{v}=E_-^0, \end{cases}$$

that is,

$$\bar{u}=\frac{E_+^0(A_--\delta)-E_-^0(B-\delta)}{[(A_+-\delta)(A_--\delta)-(B-\delta)^2]t_+^2}>0, \qquad \bar{v}=\frac{E_-^0(A_+-\delta)-E_+^0(B-\delta)}{[(A_+-\delta)(A_--\delta)-(B-\delta)^2]t_-^2}>0.$$

Therefore, we have our system for u and v as defined before

$$\begin{cases} L^+u\le\epsilon^2[E_+^0-(A_+-\delta)t_+^2\bar{u}-(B-\delta)t_-^2\bar{v}]\le 0, \ \ u\big|_{\partial B_R}<0, \\ L^+v\le\epsilon^2[E_-^0-(A_--\delta)t_-^2\bar{v}-(B-\delta)t_+^2\bar{u}]\le 0, \ \ v\big|_{\partial B_R}<0. \end{cases}$$

Applying part (A) of Lemma 6.5, we conlude $u\le 0, v\le 0$, i.e. $\varphi^+\le\epsilon^2\bar{u}+c_+w_+$, $\varphi^-\le\epsilon^2\bar{v}+c_-w_-$, which yields

$$X^+\le\bar{u}+c_+\frac{w_+}{\epsilon^2}, \qquad X^-\le\bar{v}+c_-\frac{w_-}{\epsilon^2}, \qquad \text{in} \ \ B(x_0,R).$$

Applying the L'Hospital's rule to the upper bound of $w_\pm$ as ϵ goes to zero in (6.59), we obtain $X^+\le\bar{u}$ and $X^-\le\bar{v}$ as $\epsilon\to 0$. For a complementary lower bound we note that $-X^+$, $-X^-$ satisfy a system of the exact same form, but with $-E_+^0$, $-E_-^0$ on the right side. We conclude that

$$\|X^\pm\|_{L^\infty(B(x_0,R/2))}\le\max\{\bar{u},\bar{v}\}.$$

Since $B(x_0,R)\subset\Omega$ for $\forall R$, by covering argument $\overline{\Omega}$ is compact. Connect back to $\Delta\psi_\pm$, $\|\Delta\psi_\pm\|_{L^\infty_{\mathrm{loc}}}\le C$ in case $B<0$.

Case 2. When $B>0, A_+A_->B^2$, choose r with $\frac{B}{A_-}<r<\frac{A_+}{B}$, then let $a_\pm>0$ with $a_+^2+a_-^2=1$ and $r=\frac{a_-}{a_+}$ (i.e. $r=\tan\alpha$, $\alpha\in(0,\frac{\pi}{2})$, $a_+=\cos\alpha$, $a_-=\sin\alpha$). We define $w_\pm=\frac{a_\pm}{t_\pm^2}w_0$, where w_0 is a solution to (6.55). As in the proof of case 1, we define the limit operators to the associated equations as follows:

$$L_0^+(w_+,w_-)=-\frac{\epsilon^2}{2}\Delta w_++A_+t_+^2w_+-Bt_-^2w_-, \tag{6.67}$$

$$L_0^-(w_-, w_+) = -\frac{\epsilon^2}{2}\Delta w_- + A_- t_-^2 w_- - B t_+^2 w_+. \tag{6.68}$$

Substitueting the formula of $w_\pm$ in case 2 to (6.67) and (6.68), by similar calculation we obtain that

$$L_0^+(w_+, w_-) \geq \frac{a_+\mu}{2t_+^2} w_0, \qquad L_0^-(w_-, w_+) \geq \frac{a_-\mu}{2t_-^2} w_0. \tag{6.69}$$

Therefore by the uniform convergence of $|\psi_\pm|^2$ and (6.69), we have

$$\begin{cases} L_+^+(w_+, w_-) = -\frac{\epsilon^2}{2}\Delta w_+ + A_+|\psi_+|^2 w_+ - B|\psi_-|^2 w_- \geq \frac{a_+\mu}{4t_+^2} w_0 > 0, \\ L_+^-(w_-, w_+) = -\frac{\epsilon^2}{2}\Delta w_- + A_-|\psi_-|^2 w_- - B|\psi_+|^2 w_+ \geq \frac{a_-\mu}{4t_-^2} w_0 > 0, \end{cases}$$

as $\epsilon \to 0$. Let $u = \varphi^+ - \epsilon^2 \bar{u} - c_1 w_+$, $z = \varphi^- + \epsilon^2 \underline{z} + c_1 w_-$, where $\varphi^\pm$, c_1 are defined as in the case 1 and $\bar{u}$, $\underline{z}$ are constants to be chosen later. With the similar calculation, we have

$$L^+ u \leq \epsilon^2 (B|\psi_-|^2 \underline{z} - A_+|\psi_+|^2 \bar{u}), \qquad L^- z \geq \epsilon^2(-E_-^0 + A_-|\psi_-|^2 \underline{z} - B|\psi_+|^2 \bar{u}),$$

For the following proof, we do essentially the same thing, but apply part (B) of Lemma 6.5. Again we fix a $\delta > 0$ so that for ϵ small enough, we have

$$A_\pm|\psi_\pm|^2 = A_\pm t_\pm^2 + o(1) \geq (A_\pm - \delta)t_\pm^2, \qquad B|\psi_\pm|^2 = B t_\pm^2 + o(1) \leq (B + \delta)t_\pm^2,$$

for all $x \in \bar{\Omega}$, so that $(A_\pm - \delta)t_\pm^2 > 0$, $(B + \delta)t_\pm^2 > 0$ and $[(A_+ - \delta)(A_- - \delta) - (B + \delta)^2]t_+^2 t_-^2 > 0$. We now choose $\bar{u}$ and $\underline{z}$ to be the unique solution of the system

$$\begin{cases} (B + \delta)t_-^2 \underline{z} - (A_+ - \delta)t_+^2 \bar{u} = 0, \\ (A_- - \delta)t_-^2 \underline{z} - (B + \delta)t_+^2 \bar{u} = E_-^0, \end{cases}$$

with E_-^0 as before. In fact,

$$\bar{u} = \frac{(B + \delta)E_-^0}{[(A_+ - \delta)(A_- - \delta) - (B + \delta)^2]t_+^2} > 0, \quad \underline{z} = \frac{(A_+ - \delta)E_-^0}{[(A_+ - \delta)(A_- - \delta) - (B + \delta)^2]t_-^2} > 0.$$

Then, from above argument we have

$$\begin{cases} L^+u \leq 0, & u\big|_{\partial B_R} < 0, \\ L^-z \geq 0, & z\big|_{\partial B_R} > 0. \end{cases}$$

Applying Lemma 6.5 in case (B), we have $u \leq 0$ and $z \geq 0$. From part (B) in Lemma 6.4 and L'Hospital's rule to $w_\pm$ as $\epsilon \to 0$, we get

$$X^+ \leq \bar{u}, \qquad X^- \geq -\underline{z}$$

in $\bar{\Omega}$. For a complementary lower bound and upper bound, we note that $-X^+$, $-X^-$ satisfy a system of the exact same form, but with $-E_\pm$ on the right side of the equations. We conlude that $\|X^\pm\|_{L^\infty_{\text{loc}}} \leq \max\{\bar{u}, \underline{z}\}$, i.e. $\|\Delta\psi_\pm\|_{L^\infty_{\text{loc}}} \leq C$ uniformly for $x \in \bar{\Omega}$. This completes Step 3.

Step 5. $\|\Psi^{\epsilon_m} - \Psi^*\|_{L^\infty_{\text{loc}}(\Omega)} \leq C\epsilon^2$.

Fix R, let $B(x_0, 2R) \subset \Omega$. Write $\Psi = [\rho_+ e^{i\varphi_+}, \rho_- e^{i\varphi_-}]$ and $\Psi^* = [t_+ e^{i\phi^*_+}, t_- e^{i\phi^*_-}]$. Then, from (1.2), we have

$$-t_\pm^2 \Delta(\varphi_\pm - \phi^*_\pm) = \operatorname{div}[(\rho_\pm^2 - t_\pm^2)\nabla\varphi_\pm]. \tag{6.70}$$

By elliptic regularity in [GT01], we may conclude

$$\sup_{B(x_0,R)} (\varphi_\pm - \phi^*_\pm) \leq C\|(\rho_\pm^2 - t_\pm^2)\nabla\varphi_\pm\|_{L^\infty(B(x_0,2R))} + C\|\varphi_\pm - \phi^*_\pm\|_{L^\infty(B(x_0,2R))},$$

which implies

$$\|\varphi_\pm - \phi^*_\pm\|_{L^\infty(B(x_0,R))} \leq C\|(\rho_\pm^2 - t_\pm^2)\nabla\varphi_\pm\|_{L^\infty(B(x_0,2R))}.$$

From $\rho_\pm \to t_\pm$ uniformly in ϵ, we obtain $\rho_\pm \geq \frac{t_\pm}{2}$ for ϵ sufficiently small. Since $|\nabla\psi_\pm|^2 = |\nabla\rho_\pm|^2 + \rho_\pm^2|\nabla\varphi_\pm|^2$, it yields $\frac{t_\pm^2}{4}|\nabla\varphi_\pm|^2 \leq |\nabla\psi_\pm|^2$, i.e. $\|\nabla\varphi_\pm\|_{L^\infty(\Omega)} \leq \frac{2}{t_\pm}|\nabla\psi_\pm|_{L^\infty(\Omega)} \leq C$ is uniformly bounded. By Step 4, we also have $|X^\pm| \leq C$, and hence

$$A_+\rho_+^2 + B\rho_-^2 = A_+t_+^2 + Bt_-^2 + O(\epsilon^2),$$

$$B\rho_+^2 + A_-\rho_-^2 = Bt_+^2 + A_-t_-^2 + O(\epsilon^2).$$

Solving the above for $\rho_\pm$ individually, we have

$$\rho_\pm^2 - t_\pm^2 = O(\epsilon^2).$$

Therefore, together with the lower bound of $\rho_\pm$ as above, we have

$$\|\rho_\pm - t_\pm\|_{L^\infty(\Omega)} \le \frac{C\epsilon^2}{\|\rho_\pm + t_\pm\|_{L^\infty(\Omega)}} \le \frac{2C}{3t_\pm^2}\epsilon^2 := C\epsilon^2,$$

hence

$$\begin{aligned}
\|\rho_\pm e^{i\varphi_\pm} - t_\pm e^{i\phi_\pm^*}\|_{L^\infty(\Omega)} &\le \|\rho_\pm - t_\pm\|_{L^\infty(\Omega)}\|e^{i\varphi_\pm}\|_{L^\infty(\Omega)} + \|t_\pm(e^{i\varphi_\pm} - e^{i\phi_\pm^*})\|_{L^\infty(\Omega)} \\
&\le C\epsilon^2 + t_\pm\|e^{i\varphi_\pm} - e^{i\phi_\pm^*}\|_{L^\infty(\Omega)} \\
&\le C\epsilon^2 + t_\pm\|ie^{i\xi}\nabla\xi\|_{L^\infty(\Omega)}\|\varphi_\pm - \phi_\pm^*\|_{L^\infty(\Omega)} \\
&\le C\epsilon^2 + C\|(\rho_\pm^2 - t_\pm^2)\nabla\varphi_\pm\|_{L^\infty(\Omega)} \\
&\le C\epsilon^2 + C\|\rho_\pm^2 - t_\pm^2\|_{L^\infty(\Omega)}\|\nabla\varphi_\pm\|_{L^\infty(\Omega)} \\
&\le C\epsilon^2,
\end{aligned}$$

which implies the desired result.

Step 6. $\|\nabla\varphi_\pm\|_{\mathcal{C}^k_{\mathrm{loc}}} \le C$, $\|X^\pm\|_{\mathcal{C}^k_{\mathrm{loc}}} \le C$.

The proof follows as in [BBH93], via induction on k. Fix any ball $B_R \Subset \Omega$. Following the steps in [BBH93] (with our definition of $X^\pm$) the argument is identical through their estimate (67): assuming the statement of Step 6 is true for k, we have

$$\|\nabla\rho_\pm\|_{\mathcal{C}^k(B_R)} \le C, \qquad \|\nabla\varphi_\pm\|_{\mathcal{C}^{k+1}(B_R)} \le C, \qquad \|\epsilon X^\pm\|_{\mathcal{C}^{k+1}(B_R)} \le C. \tag{6.71}$$

From the formula of $\psi_\pm = [\rho_+e^{i\varphi_+}, \rho_-e^{i\varphi_-}]$, we know that $\rho_\pm$ and $\varphi_\pm$ satisfy the following systems

$$-\Delta\rho_\pm = -\rho_\pm|\nabla\varphi_\pm|^2 + X^\pm\rho_\pm, \tag{6.72}$$

$$-\Delta\varphi_\pm = 2\frac{\nabla\rho_\pm}{\rho_\pm}\nabla\varphi_\pm. \tag{6.73}$$

By the results in Step 4 and Step 5, it is clear that (6.71) is true for $k=0$. Assume that, for $\forall\ k$, the induction assumptions are held. Then the right hand side of (6.72) is $\mathcal{C}^k_{\text{loc}}$ bounded, which implies the right hand side of (6.72) is in $W^{k,p}$ for $\forall\ p<\infty$. By L^p-estimate, we have $\|\rho_\pm\|_{W^{k+2,p}_{\text{loc}}} \leq C$ for $\forall\ p<\infty$. From Sobolev embedding $W^{k+2,p} \hookrightarrow \mathcal{C}^{k+1,p}$, $\|\rho_\pm\|_{\mathcal{C}^{k+1}_{\text{loc}}}$ is bounded. Therefore, $\|\nabla\rho_\pm\|_{\mathcal{C}^k_{\text{loc}}} \leq C$.

On the other hand, the right hand side of (6.73) is in $\mathcal{C}^k_{\text{loc}}$, which is also in $W^{k,p}$ for $\forall\ p<\infty$. Apply L^p-estimate again, we obtain $\|\varphi_\pm\|_{W^{k+2,p}_{\text{loc}}} \leq C$ for $\forall\ p<\infty$. By the facts that $\|\nabla\rho_\pm\|_{W^{k+1,p}_{\text{loc}}} \leq C$ and $\|\nabla\varphi_\pm\|_{W^{k+1,p}_{\text{loc}}} \leq C$, we have $\frac{1}{\rho_\pm} \in \mathcal{C}^{k+1}_{\text{loc}}$, which implies $D^{k+1}(\rho_\pm^{-1}) \in L^\infty_{\text{loc}}$. Compute the L^p-norm on the right hand side of (6.73) after differentiating it, we obtain the right hand side of (6.73) is in $W^{k+1,p}$ for $\forall\ p<\infty$. Applying L^p-estimate again, it yields $\|\varphi_\pm\|_{W^{k+3,p}_{\text{loc}}} \leq C$ for $\forall\ p<\infty$. Then from Sobolev embedding, we get $\|\varphi_\pm\|_{\mathcal{C}^{k+2,p}_{\text{loc}}} \leq C$, which implies that $\|\nabla\varphi_\pm\|_{\mathcal{C}^{k+1}_{\text{loc}}} \leq C$. By induction $\|\nabla\varphi_\pm\|_{\mathcal{C}^k_{\text{loc}}} \leq C$ holds with $k+1$ instead of k.

Now define $Y^\pm := D^{k+1}X^\pm$, where D^{k+1} denotes any partial derivative of order $k+1$, and $X^\pm$ satisfy the following equations:

$$\begin{cases} -\frac{\epsilon^2}{2}\Delta X^+ + A_+|\psi_+|^2X^+ + B|\psi_-|^2X^- = E_+, \\ -\frac{\epsilon^2}{2}\Delta X^- + A_-|\psi_-|^2X^- + B|\psi_+|^2X^+ = E_-, \end{cases} \tag{6.74}$$

where $E_\pm$ defined as in Step 4. By differentiating the system (6.74), we obtain a system of the same form for $Y^\pm$,

$$\begin{cases} -\frac{\epsilon^2}{2}\Delta Y^+ + A_+|\psi_+|^2Y^+ + B|\psi_-|^2Y^- = \tilde{E}_+, \\ -\frac{\epsilon^2}{2}\Delta Y^- + A_-|\psi_-|^2Y^- + B|\psi_+|^2Y^+ = \tilde{E}_-, \end{cases} \tag{6.75}$$

with $A_\pm$, B as before, $\tilde{E}_\pm$ depending on the derivatives of $\rho_\pm$, $\varphi_\pm$ of order at most $k+1$, and on derivatives of $X^\pm$ of order at most k. Especially, $\tilde{E}_\pm$ are uniformly bounded in B_R by (6.71),

$$\|\tilde{E}_\pm\|_{L^\infty(B_R)} \leq C_0.$$

Also by (6.71), we obtain the following bound on $Y^\pm$ on ∂B_R, for any fixed ball

$B_R \Subset \Omega$,

$$\|Y^{\pm}\|_{L^\infty(\partial B_R)} \le C_1/\epsilon.$$

In the following, we separate our proof into two cases.

Case 1. When $A_+A_- - B^2 > 0$, $B < 0$, let $V^{\pm} = \epsilon^2 Y^{\pm}$, then $Y^{\pm}$ satisfy the system as follows

$$\begin{cases} -\frac{\epsilon^2}{2}\Delta V^+ + A_+|\psi_+|^2 V^+ + B|\psi_-|^2 V^- = \epsilon^2 \tilde{E}_+, \\ -\frac{\epsilon^2}{2}\Delta V^- + A_-|\psi_-|^2 V^- + B|\psi_+|^2 V^+ = \epsilon^2 \tilde{E}_-, \end{cases} \tag{6.76}$$

with $\|\tilde{E}_\pm\|_{L^\infty(B_R)} \le \tilde{E}^0_\pm$ and $\|V^\pm\|_{L^\infty(\partial B_R)} \le C_1$. We do essentially the same thing as in Step 5. Let $\overline{V^+} = \epsilon^2\bar{u} + c_1 w_+$ and $\overline{V^-} = \epsilon^2\bar{v} + c_1 w_-$ with $\bar{u}, \bar{v}, w_\pm$ and c_1 as defined in Step 4. Then by similar computations as before and by the definition of $u = V^+ - \overline{V^+}$ and $v = V^- - \overline{V^-}$, we choose the constants $\bar{u}$, $\bar{v}$ such that

$$\begin{cases} L^+u \le 0, \quad u\big|_{\partial B_R} < 0, \\ L^+v \le 0, \quad v\big|_{\partial B_R} < 0, \end{cases} \tag{6.77}$$

with the operators $L^\pm$ as defined before. We repeat the similar process as in Step 4, use the same definitions of $w_\pm$ as in case 1 of Step 4, apply part (A) of Lemma 6.5, we obtain

$$Y^+ \le \bar{u} + \frac{c_1}{\epsilon^2}w_+ = \bar{u} + \frac{a_+c_1}{t_+^2\epsilon^2}w_0, \quad Y^- \le \bar{v} + \frac{c_1}{\epsilon^2}w_- = \bar{v} + \frac{a_-c_1}{t_-^2\epsilon^2}w_0.$$

Then from the exponential bound of w_0 in Lemma 6.4, we have $Y^\pm = D^{k+1}X^\pm$ uniformly bounded above in B'_R for any radius $R' < R$. Applying the same argument to $-Y^\pm$, we obtain a matching lower bound, and conclude that $X^\pm$ is bounded in $\mathcal{C}^{k+1}_{\text{loc}}$.

Case 2. When $A_+A_- - B^2 > 0$, $B > 0$, we choose instead $u = \epsilon^2 Y^+ - \epsilon^2\bar{u} - c_1 w_+$, $z = \epsilon^2 Y^- + \epsilon^2 \underline{z} + c_1 w_-$ with $w_\pm$ denoted as in case 2 of Step 4. Now, the conclusions follow from part (B) of Lemma 6.4 and Lemma 6.5.

Step 7. $\|X^\pm + \frac{1}{t_\pm^2}|\nabla\psi^*_\pm|^2\|_{\mathcal{C}^k_{\text{loc}}} \le C\epsilon^2$.

From the definition of $X^{\pm}$ and Step 6, if we fix any compact subset K of Ω, we have

$$
\begin{aligned}
\||\psi_\pm|^2 - t_\pm^2\|_{\mathcal{C}^k(K)} &\le \epsilon^2 \left\| \frac{A_\mp X^\pm}{A_+A_- - B^2} \right\|_{\mathcal{C}^k(K)} + \epsilon^2 \left\| \frac{B X^\mp}{A_+A_- - B^2} \right\|_{\mathcal{C}^k(K)} \\
&\le C\epsilon^2 \|X^\pm\|_{\mathcal{C}^k(K)} + C\epsilon^2 \|X^\mp\|_{\mathcal{C}^k(K)} \\
&\le C(K,k)\epsilon^2,
\end{aligned}
$$

which implies

$$
\|\rho_\pm^2 - t_\pm^2\|_{\mathcal{C}^k(K)} \le C(K,k)\epsilon^2, \tag{6.78}
$$

and

$$
\|2\rho_\pm \nabla \rho_\pm\|_{\mathcal{C}^{k-1}_{\text{loc}}} \le C(K,k-1)\epsilon^2.
$$

From the uniform convergence of $\rho_\pm^2$ in ϵ, we choose $\epsilon = \min\{\frac{3}{4}t_+^2, \frac{3}{4}t_-^2\}$ such that $\rho_\pm^2 \ge \frac{1}{4}t_\pm^2$, i.e. $\rho_\pm \ge \frac{1}{2}t_\pm$ when ϵ samll enough. Then we have

$$
t_\pm \|\nabla \rho_\pm\|_{\mathcal{C}^{k-1}_{\text{loc}}} \le \|2\rho_\pm \nabla \rho_\pm\|_{\mathcal{C}^{k-1}_{\text{loc}}} \le C(K,k-1)\epsilon^2,
$$

i.e. $\|\nabla \rho_\pm\|_{\mathcal{C}^{k-1}_{\text{loc}}} \le C(K,k-1)\epsilon^2$. Differentiating the right hand side of (6.73), we get

$$
-\Delta[D^k(\varphi_\pm - \phi_\pm^*)] = \frac{1}{t_\pm^2} \text{div}[D^k(\rho_\pm^2 - t_\pm^2)\nabla \varphi_\pm],
$$

then

$$
\begin{aligned}
\|D^k(\varphi_\pm - \phi_\pm^*)\|_{L^\infty_{\text{loc}}} &\le C\|D^k(\rho_\pm^2 - t_\pm^2)\nabla\varphi_\pm\|_{L^\infty_{\text{loc}}} \\
&\le C \sum_{|i|+|j|=k} \|\partial^i(\rho_\pm^2 - t_\pm^2)\|_{L^\infty_{\text{loc}}} \|\partial^j(\nabla\varphi_\pm)\|_{L^\infty_{\text{loc}}} \\
&\le C\|\partial^k(\rho_\pm^2 - t_\pm^2)\|_{L^\infty_{\text{loc}}} \|\partial^k(\nabla\varphi_\pm)\|_{L^\infty_{\text{loc}}} \\
&= C\|\rho_\pm^2 - t_\pm^2\|_{\mathcal{C}^k_{\text{loc}}} \|\nabla\varphi_\pm|\|_{\mathcal{C}^k_{\text{loc}}} \\
&\le C\epsilon^2,
\end{aligned}
$$

i.e. $\|D^k(\varphi_\pm - \phi_\pm^*)\|_{L^\infty_{\text{loc}}} \le C\epsilon^2$, which implies $\|\varphi_\pm - \phi_\pm^*\|_{\mathcal{C}^k_{\text{loc}}} \le C\epsilon^2$. Since k is any integer, it is as same as

$$
\|\varphi_\pm - \phi_\pm^*\|_{\mathcal{C}^{k+1}_{\text{loc}}} \le C\epsilon^2. \tag{6.79}
$$

Therefore, by Step 6 and (6.79)

$$\begin{aligned}\||\nabla\varphi_\pm|^2-|\nabla\phi_\pm^*|^2\|_{\mathcal{C}^k_{\rm loc}} &\le \|\nabla\varphi_\pm+\nabla\phi_\pm^*\|_{\mathcal{C}^k_{\rm loc}}\|\nabla\varphi_\pm-\nabla\phi_\pm^*\|_{\mathcal{C}^k_{\rm loc}}\\ &\le \left(\|\nabla\varphi_\pm\|_{\mathcal{C}^k_{\rm loc}}+\|\nabla\phi_\pm^*\|_{\mathcal{C}^k_{\rm loc}}\right)\|\nabla\varphi_\pm-\nabla\phi_\pm^*\|_{\mathcal{C}^k_{\rm loc}}\\ &\le C\epsilon^2. \end{aligned}\tag{6.80}$$

Now, let $U^\pm=X^\pm+\frac{1}{t_\pm^2}|\nabla\psi_\pm^*|^2$, we obtain a system of equations for $U^\pm$ of the form,

$$\begin{cases} -\dfrac{\epsilon^2}{2}\Delta U^+ + A_+|\psi_+|^2U^+ + B|\psi_-|^2U^- = F_+,\\ -\dfrac{\epsilon^2}{2}\Delta U^- + A_-|\psi_-|^2U^- + B|\psi_+|^2U^+ = F_-, \end{cases}\tag{6.81}$$

with

$$F_+=-\frac{A_+}{t_+^2}(t_+^2|\nabla\psi_+|^2-|\psi_+|^2|\nabla\psi_+^*|^2)-\frac{B}{t_-^2}(t_-^2|\nabla\psi_-|^2-|\psi_-|^2|\nabla\psi_-^*|^2)-\frac{\epsilon^2}{2t_+^2}\Delta(|\nabla\psi_+^*|^2).$$

And by (6.78) and (6.80), we have

$$\begin{aligned}&\left\|t_\pm^2|\nabla\psi_\pm|^2-|\psi_\pm|^2|\nabla\psi_\pm^*|^2\right\|_{\mathcal{C}^k_{\rm loc}}\\ &\quad\le t_\pm^2\left\||\nabla\psi_\pm|^2-|\nabla\psi_\pm^*|^2\right\|_{\mathcal{C}^k_{\rm loc}}+\left\||\psi_\pm|^2-t_\pm^2\right\|_{\mathcal{C}^k_{\rm loc}}\left\|\nabla\psi_\pm^*\right\|^2_{\mathcal{C}^k_{\rm loc}}\\ &\quad= t_\pm^2\left\||\nabla\psi_\pm|^2-|\nabla\psi_\pm^*|^2\right\|_{\mathcal{C}^k_{\rm loc}}+\|\rho_\pm^2-t_\pm^2\|_{\mathcal{C}^k_{\rm loc}}\left\|\nabla\psi_\pm^*\right\|^2_{\mathcal{C}^k_{\rm loc}}\\ &\quad\le C\epsilon^2,\end{aligned}$$

i.e. $\|F_+\|_{\mathcal{C}^k_{\rm loc}}\le C\epsilon^2$, and similarly for F_-. Following the proofs in Step 6, dividing the proof into two cases and applying Lemma 6.4-Lemma 6.5, we can obtain the desired results.

Step 7. By the first step, we have $\Psi^{\epsilon_m}\to\Psi^*$ strongly in $H^1(\Omega;\Sigma)$. In addition, we have

$$\Psi^*(x)=[t_+e^{i(n_+\theta+\beta_+)},t_-e^{i(n_-\theta+\beta_-)}]$$

with $\beta_\pm$ real constants as desired. From the assumption $\psi_\pm^*=t_\pm e^{i\phi_\pm^*}=t_\pm e^{i(n_\pm\theta+\beta_\pm)}$, we have

$$|\nabla\phi_\pm^*|^2=|n_\pm\nabla\theta+\nabla\beta_\pm|^2=n_\pm^2|\nabla\theta_\pm|^2=\frac{n_\pm^2}{r^2},$$

and

$$|\nabla\psi^*_\pm|^2 = t^2_\pm|ie^{i\phi^*_\pm}\nabla\phi^*_\pm|^2 = t^2_\pm|\nabla\phi^*_\pm|^2 = t^2_\pm\frac{n^2_\pm}{r^2}.$$

Applying the local convergence results away from vortices for the singularly perturbed problem (6.49) from above steps, we have $\Psi^{\epsilon_m} \to \Psi^*$ in $\mathcal{C}^k_{\rm loc}(\Omega)$ for any $k \geq 0$, and

$$\left\|\frac{1}{\epsilon_m^2}[A_\pm(|\psi_{m\pm}|^2 - t^2_\pm) + B(|\psi_{m\mp}|^2 - t^2_\mp)] + \frac{1}{t^2_\pm}|\nabla\psi^*_\pm|^2\right\|_{\mathcal{C}^k_{\rm loc}(\Omega)} \longrightarrow 0, \quad \text{for all } k \geq 0.$$

Note that the $\mathcal{C}^k_{\rm loc}$ convergence of Ψ_m to Ψ^* and computations above imply that we may replace $\frac{1}{t^2_\pm}|\nabla\psi^*_\pm|^2$ by $\frac{n^2_\pm}{r^2}$ in the above estimate. Recall that $\Omega = B_b(0) \setminus \overline{B_a(0)}$ with $0 < a < 1 < b$ be fixed, so $\partial B_1(0) \subset \Omega$. Evaluating along $\partial B_1(0) \subset \Omega$,

$$\left\|R^2_m[A_\pm(|\psi_{m\pm}|^2 - t^2_\pm) + B(|\psi_{m\mp}|^2 - t^2_\mp)] + n^2_\pm\right\|_{L^\infty(\partial B_1(0))} \longrightarrow 0.$$

Since R_m is an arbitrary divergent sequence, we may conclude that the above holds for general $r \to 0$, that is,

$$[A_\pm(|\psi_{m\pm}|^2 - t^2_\pm) + B(|\psi_{m\mp}|^2 - t^2_\mp)] + \frac{n^2_\pm}{r^2} = o\left(\frac{1}{r^2}\right)$$

uniformly as $|x| = r \to \infty$. This then yields that

$$|\psi_+|^2 = t^2_+ - \frac{A_-n^2_+ - Bn^2_-}{A_+A_- - B^2}\frac{1}{r^2} + o\left(\frac{1}{r^2}\right),$$

$$|\psi_-|^2 = t^2_- - \frac{A_+n^2_- - Bn^2_+}{A_+A_- - B^2}\frac{1}{r^2} + o\left(\frac{1}{r^2}\right),$$

as $r \to \infty$. The conclusion (6.2) then follows immediately.

To obtain the uniform limit of $\phi_\pm(x)$ (as defined in Lemma 6.3), we note that by taking the imaginary part of (1.2) in polar coordinate, we obtain the same equation (for the conservation of current) as in the classical Ginzburg-Landau equation

$$\mathrm{div}(\rho^2_\pm(n_\pm\nabla\theta + \nabla\phi_\pm)) = 0.$$

Therefore, the uniform convergence of $\phi_{\pm}(x)$ to $\beta_{\pm}$ as $|x|$ goes to ∞ follows exactly as in [Sha94]. □

We note the following estimates which will be useful in our study of equivariant solutions in Chapter 2:

Corollary 6.6. *Under the hypothesis as in Proposition 6.1, with* $\rho_{\pm} = |\psi_{\pm}|$*, we have*

$$\frac{\partial \rho_{\pm}}{\partial r} = \frac{A_{\mp} n_{\pm}^2 - B n_{\mp}^2}{(A_+ A_- - B^2) t_{\pm}} \frac{1}{r^3} + o\left(\frac{1}{r^3}\right),$$

$$\frac{\partial^2 \rho_{\pm}}{\partial \rho^2} = -\frac{3(A_{\mp} n_{\pm}^2 - B n_{\mp}^2)}{(A_+ A_- - B^2) t_{\pm}} \frac{1}{r^4} + o\left(\frac{1}{r^4}\right).$$

Proof. The proof easily follows by differentiation in the $\mathcal{C}^k_{\mathrm{loc}}$ estimates above. □

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