On Complete Non-compact Ricci-flat Cohomogeneity One Manifolds

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Abstract

We present an alternative proof of the existence theorem of Böhm using ideas from the study of gradient Ricci solitons on the multiple warped product cohomogeneity one manifolds by Dancer and Wang. We conclude that the complete Ricci-flat metric converges to a Ricci-flat cone. Also, starting from a $4n$-dimensional $\mathbb{H}P^n$ base space, we construct numerical Ricci-flat metrics of cohomogeneity one in $(4n+3)$ dimensions whose level surfaces are $\mathbb{C}P^{2n+1}$. We show the local Ricci-flat solution is unique (up to homothety). The numerical results suggest that they all converge to Ricci-flat Ziller cone metrics even if $n = 2$.

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To my parents, grandparents and great grandparents

Introduction

A Riemannian metric $\hat{g}$ on $\widehat{M}$ is called Einstein if its Ricci tensor is a multiple of the metric, i.e. $\mathrm{Ric}(\hat{g}) = \lambda \cdot \hat{g}$. If $\lambda = 0$, the metric is called Ricci-flat. If we require $\hat{g}$ to be invariant under a Lie group G acting properly on $\widehat{M}$ with principal orbits of codimension one, then the PDE $\mathrm{Ric}(\hat{g}) = \lambda \cdot \hat{g}$ becomes an ODE on I together with appropriate boundary conditions to ensure that we have a smooth metric. The position of a singular orbit of the G-action gives an irregular singular point of this ODE.

Ricci-flat metrics are high dimensional Euclidean signature analogues of solutions of Einstein's equation in vacuo. Ricci-flat manifolds often have special holonomy group, which is the group of linear transformations arising from parallel transport along closed loops. Important cases are Calabi-Yau manifolds and hyperkähler manifolds, which play an important role in theoretical physics such as in superstring theory and the study of gravitational instantons.

In this article we consider only complete non-compact cohomogeneity one Ricci-flat manifolds. The first mathematicians to study these were Calabi, Page, and Bérard Bergery. In 1975 Calabi [Cal75] constructed cohomogeneity one non-positive Kähler Einstein metrics on certain complex line bundles over compact Kähler manifolds. Later, in [Cal79] he constructed a complete hyperkähler metric on $T^*\mathbb{C}P^n$ of cohomogeneity one. After Yau constructed the first examples of closed Ricci-flat Kähler manifold which are not flat, the physicist D. Page [Pag78] constructed the first compact example of an inhomogeneous non-Kähler positive Einstein metric. The first systematic study of cohomogeneity one Einstein metrics was carried out by Bérard Bergery [BB81], who provided a theoretical setting for Page's work. More recently, [WW98] and [DW98] have used the construction due to [Cal75] and [BB81] to obtain large families of Kähler as well as non-Kähler Einstein manifolds.

Here, we are interested in the existence of complete Ricci-flat metrics on two types of non-compact cohomogeneity one manifolds. The principal orbit in the first case is a multiple product manifold; in the second case the principal orbit is $\mathbb{C}P^{2n+1}$ and the singular orbit is $\mathbb{H}P^n$. The first family of complete, non-compact Ricci-flat metrics of cohomogeneity one was found by Böhm [Böh99] as a result of further study of the dynamic properies of the cohomogeneity one Einstein equations.

In this thesis, we will present an alternative proof of this existence theorem of Böhm using ideas from the study of steady gradient Ricci solitons on the same underlying manifolds by Dancer & Wang. The interest in the second family originated from the construction of an explicit complete G_2-holonomy metric on a certain vector bundle over S^4 by [BS89] and [GPP90]. In this thesis we examine this example and its high dimensional analogues.

Now we turn to our main results. Let G be a compact Lie group acting on $\widehat{M}$ with cohomogeneity one. Let $P = G/K$ be the multiple warped product principal orbit type and let $\hat{g}$ be a G-invariant metric on $\widehat{M}$. We can write

$$\hat{g} = dt^2 + g(t) \tag{0.1}$$

where $g(t)$ is a smooth curve of G-invariant metrics on P. The cohomogeneity one Ricci-flat equation for $\hat{g}$ is given by an ordinary differential equation for $g(t)$ [EW00]. We introduce a variable to change the ODE to a non-singular polynomial dynamical system. We were inspired by a Lyapunov function found in [DHW13]

$$\mathcal{F} := \frac{\sum_{i=1}^{r}(X_i^2 + Y_i^2) - \frac{1}{n}(\sum_{i=1}^{r}\sqrt{d_i}X_i)^2}{\prod_{i=1}^{r} Y_i^{\frac{2d_i}{n}}} \tag{0.2}$$

in our new variables. Using this function, the geometry of the Ricci-flat system is especially well understood. We then consider the Lie group triple (G, H, K)=$(Sp(n+1), Sp(n)Sp(1), Sp(n)U(1))$. We use two Einstein metrics on $\mathbb{C}P^{2n+1}$ to construct two Ricci-flat cones. For $n \geqslant 1$, our numerical solutions converge to the Ziller cone metric, not the other, which verifies the general Convergence Theorem 11.1 in [Böh99]. For $n = 1$, we conclude there is a unique (up to homothety) Ricci-flat 7-dimensional metric. So, the Ricci-flat 7-dimensional metric must have G_2 holonomy.

The content of this article is as follows. In Chapter 1 we derive the warped product cohomogeneity one Ricci-flat equation and state the smoothness condition for local solutions, following [EW00]. In Chapter 2 we give an alternative proof of Böhm's extistence theorem (Theorem A [Böh99]) using the new variables. In Chapter 3, we apply the new variables to the Ricci-flat equations for the cohomogeneity one manifold (G, H, K)=$(Sp(n+1), Sp(n)Sp(1), Sp(n)U(1))$ and describe the two homogeneous Einstein metrics on $\mathbb{C}P^{2n+1}$, following [Zil82]. In Chapter 4 we give numerical solutions for the above two Ricci-flat equations and discover that the geometry of the principal orbits converges to the geometry of the Ziller cone metric.

Chapter 1

Cohomogeneity one Riemannian manifold

In this dissertation, we investigate two families of Ricci-flat manifolds. In the first case, $\widehat{M}$ is a multiple warped product over an interval, which is the example discussed in [DW09]. In the second case, let $\widehat{M}$ be of cohomogeneity one with respect to an isometric group $Sp(n+1)$-action whose principal orbit P is $\mathbb{C}P^{2n+1}$ and whose singular orbit Q is $\mathbb{H}P^n$, For n=1 the explicit Ricci-flat solution was found in [GPP90] and when $n > 2$ the general convergence theorem for the Ricci-flat solution was found in [Böh99].

1.1 Homogeneous spaces

A connected Riemannian manifold (M, g) is said to be *homogeneous* if its full isometry group $I(M, g)$ acts transitively on M. $I(M, g)$ is a Lie group. Throughout this dissertation, let $G \subset I(M, g)$ be a connected compact subgroup which still acts transitively on M. Then the Riemannian manifold (M, g) is called G-homogeneous. Moreover, let $x \in M$ and K denote the isotropy group at x. Then M can be identified with the homogeneous manifold G/K. Furthermore, since G consists of isometries, it must acts effectively on G/K (and the corresponding linear isotropy representation of K in $Gl(T_xM)$ is faithful, i.e. injective) ([Bes87], 7.11, 7.12). The Riemannian metric g can be considered as a G-invariant metric on G/K.

Let us consider the Lie group triple (G, H, K) where G is a Lie group and H, K two compact subgroups of G with $K \subset H$. Following the notation of ([Bes87], 9.79), let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{h} \supset \mathfrak{k}$ the corresponding subalgebras for H and K. We choose once and for all an $\mathrm{Ad}_G(H)$-invariant complement $\mathfrak{p}_-$ to $\mathfrak{h}$ in $\mathfrak{g}$, and an $\mathrm{Ad}_G(K)$-invariant complement $\mathfrak{p}_+$ to $\mathfrak{k}$ in $\mathfrak{h}$. Then $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ is an $\mathrm{Ad}_G(K)$-invariant complement to $\mathfrak{k}$ in $\mathfrak{g}$. An $\mathrm{Ad}_G(H)$-invariant scalar product $(.,.)$ on $\mathfrak{p}_-$ defines a G-invariant Riemannian metric g_Q on $Q = G/H$, and an $\mathrm{Ad}_G(K)$-invariant scalar product on $\mathfrak{p}_+$ defines a H-invariant Riemannian metric $g_{H/K}$ on H/K. Finally, the orthogonal direct sum of these scalar products on $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ defines a G-invariant Riemannian metric g_P on $G/K = P$.

1.1.1 Adjoint representation and isotropy representation

Let G/K be a homogeneous space and the projection be $\pi : G \to G/K$, $\pi(g) = gK$. Let $X \in \mathfrak{g}$ and $\exp tX$ be the corresponding one-parameter subgroup. The differential $d\pi_e : \mathfrak{g} \to T_o(G/K)$, where $o = \pi(e) = K$ can be computed in the following way,

$$d\pi_e(X) = \frac{d}{dt}(\pi \circ \exp tX)\Big|_{t=0} = \frac{d}{dt}((\exp tX)K)\Big|_{t=0}.$$

From $\ker d\pi_e = \mathfrak{k}$ and $d\pi$ is onto, we obtain the canonical isomorphism

$$\mathfrak{g}/\mathfrak{k} \cong T_o(G/K).$$

Since G is a compact Lie group, then by the averaging procedure, there exists an $\mathrm{Ad}_G(K)$-invariant complement $\mathfrak{p}$ of $\mathfrak{k}$ in $\mathfrak{g}$. As an immediate consequence of the above isomorphism, we have the canonical isomorphism

$$\mathfrak{p} \cong T_o(G/K).$$

With this isomorphism, we can show that the isotropy representation of K in $T_o(G/K)$ is identified with the restriction of the adjoint representation of K on $\mathfrak{g}$ to $\mathfrak{p}$. In fact, it suffices to show that the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{p} & \xrightarrow{\mathrm{Ad}_G(k)} & \mathfrak{p} \\ {\scriptstyle d\pi_e|_{\mathfrak{p}}}\big\downarrow & & \big\downarrow{\scriptstyle d\pi_e|_{\mathfrak{p}}} \\ T_o(G/K) & \xrightarrow{(d\tau_k)_o} & T_o(G/K) \end{array}$$

For details see the proof of Proposition 4.5 in [Arv03].

1.2 Basic structure of a cohomogeneity one manifold

In this chapter we discuss the basic structure of cohomogeneity one isometric actions. For more detail we refer the reader to [Bre72], [Zil09], [DW11] [WZ86] and [EW00].

First of all, there exists a maximum orbit type G/K for G on M (i.e, K is conjugate to a subgroup of each isotropy group and is itself an isotropy group). The union of the orbits of type G/K is open and dense in M and its image in M/G is connected. We refer to [Bre72] page 179 for the proof. An orbit of maximum orbit type is called a *principal orbit*. If $P \approx G/K$ is a principal orbit and $Q \approx G/H$ is any orbit, K is conjugate to a subgroup of H and we may assume that $K \subset H$. Then there is an equivariant map $P \to Q$, which is a fiber bundle projection $G/K \to G/H$ with fiber H/K. If dim $P >$ dim Q, then Q is called a *singular orbit*.

Let G be a compact Lie group acting by isometries on an $(n+1)$-dimension connected Riemannian manifold $(\widehat{M}, \hat{g})$ with cohomogeneity one, i.e. the codimension of the principal orbit is one. Choosing a principal point $p \in \widehat{M}$, let $P = G \cdot p$ be a principal orbit (homogeneous space) with isotropy group $K = G_p$. Pick a singular point $q \in \widehat{M}$; then $Q = G/H$ is a singular orbit with isotropy group $H = G_q$. In this dissertation, we will only consider the case, $M/G = I = [0, +\infty)$. Denote the orbit projection by

$$\pi : \widehat{M} \to \widehat{M}/G = I.$$

Then the inverse image of $(0, +\infty)$ consists of the principal orbits, and $\pi^{-1}(0)$ is the singular orbit Q. Choose a geodesic $\gamma(t) : I \to \widehat{M}$, parametrized by arclength, intersecting all principal orbits orthogonally. Then, there is an equivariant diffeomorphism

$$\begin{aligned} \Phi : \mathring{I} \times (G/K) &\longrightarrow \quad M_0 \\ (t, g \cdot K) &\longmapsto \quad g \cdot \gamma(t) \end{aligned} \tag{1.1}$$

where $M_0 \subset \widehat{M}$ is the open and dense subset consisting of all points lying on principal orbits, K denotes the principal isotropy group of $\gamma(t)$ and $\mathring{I} = (0, \infty)$. Then, $P_t = \Phi(t, G/K) = G/G_{\gamma(t)}$ is the principal orbit passing through $\gamma(t)$. The connected homogeneous space $P = G/K$ is an abstract copy of the principal orbits and has dimension n.

1.2.1 Cohomogeneity one metric

We now discuss how to describe cohomogeneity one Riemannian metrics on $\widehat{M}$. Following from the map (1.1), for each $t \in (0, \infty)$, $\gamma(t)$ corresponds to constant isotropy group K and the induced metrics on G/K form a one-parameter family of G-invariant metrics $g(t)$. the pullback of the metric $\hat{g}|M_0$ is

$$\Phi^*(\hat{g}) = \mathrm{d}t^2 + g(t)$$

and $g(t)$ can be viewed as a family of G-invariant metrics on G/K along the geodesic $\gamma(t)$.

We identify $\mathfrak{p}$ with the tangent space G/K at $\gamma(t)$, $t \in \mathring{I}$ and $\mathrm{Ad}_G(K)|_{\mathfrak{p}}$ can be identified with the isotropy representation. As we discussed in section 1.1, if G is compact, we can firstly choose an appropriate $\mathrm{Ad}(K)$-invariant inner product h on $\mathfrak{p}$. Then, we decompose $\mathfrak{p}$ into its h-orthogonal real $\mathrm{Ad}_G(K)$-irreducible subspaces,

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \cdots \oplus \mathfrak{p}_r,$$

where $\mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \cdots \oplus \mathfrak{p}_k = \mathfrak{p}_+ \cong T(H/K)$ and $\mathfrak{p}_{k+1} \oplus \mathfrak{p}_{k+2} \oplus \cdots \oplus \mathfrak{p}_r = \mathfrak{p}_- \cong T(G/H)$. The set of G-invariant metrics $g(t)$ on G/K can be identified with the set of $\mathrm{Ad}_G(K)$-invariant inner products $<,>_t$ on $\mathfrak{p}$. If the $\mathfrak{p}_i$'s are pairwise inequivalent $\mathrm{Ad}_G(K)$-representations, by Schur's Lemma, $<,>_t|_{\mathfrak{p}_i} = g_i^2(t)h|_{\mathfrak{p}_i}$, for some functions $g_1^2, \ldots, g_r^2$. In this dissertation, we will assume the actions of $\mathrm{Ad}_G(K)$ on the $\mathfrak{p}_i$'s are inequivalent to each other, so that such a decomposition is unique up to ordering and the $\mathfrak{p}_i$'s are orthogonal to one another automatically, that is,

$$g(t)|_{\mathfrak{p}} \backsim <,>_t = g_1(t)^2 h\ |_{\mathfrak{p}_1} \perp g_2(t)^2 h\ |_{\mathfrak{p}_2} \perp \ldots \perp g_r(t)^2 h\ |_{\mathfrak{p}_r}\ . \tag{1.2}$$

Remark 1.1. Conversely, given an $\mathrm{Ad}_G(K)-$invariant metric $<,>_t$ along the unit speed geodesic $\gamma(t), t \in (0, +\infty)$, we can recover the metric on $\widehat{M_0}$ using the G-action.

1.3 Cohomogeneity one Ricci-flat equation

Following the notation of [EW00] and [DW11], we will denote by $\widehat{\nabla}$ and $\widehat{\mathrm{Ric}}$ respectively the Levi-Civita connection and the Ricci tensor of the Riemannian manifold $(\widehat{M}, \hat{g})$. Let ∇ and Ric denote the objects for (P_t, g_t), for a given time t. $N = d\Phi(\frac{\partial}{\partial t})$, is the unit ($G$-equivariant) normal field along P_t. We let $L(t)$ be the shape operator of the orbit $P_t = \Phi(\{t\} \times P)$, which is defined by

$$L(t)X = \widehat{\nabla}_X N$$

for any vector field $X \in TP_t$. We then can view $L(t)$ as a one-parameter family of (g_t-symmetric) endomorphisms on TP via Φ. We have, for $X, Y \in TP$ and $t \in (0, +\infty)$,

$$\dot{g}(t)(X, Y) = 2g(t)(L(t)X, Y),$$

where $\cdot$ denotes d/dt. Note that by G-invariance, the trace $\mathrm{tr}L_t$ (mean curvature) is constant along P_t for a fixed t, as shown in [EW00]. Using the Gauss and Codazzi equations together with the Riccati equation for L, one obtains for $X, Y \in TP$,

$$\begin{aligned}
\widehat{\mathrm{Ric}}(X,Y) &= \mathrm{Ric}(X,Y) - \mathrm{tr}(L)g(L(X),Y) - g(\dot{L}(X),Y),\\
\widehat{\mathrm{Ric}}(N,N) &= -\mathrm{tr}(\dot{L}) - \mathrm{tr}(L^2),\\
\widehat{\mathrm{Ric}}(X,N) &= -\mathrm{tr}(X \neg d^{\nabla} L),
\end{aligned}$$

where $d^{\nabla}L$ is the TP-valued 2-form on P which is the covariant exterior derivative of L regarded as TP-valued 1-form on P, and $\neg$ denotes the interior product. We denote by $r(t)$ the Ricci endomorphism of TP for metrics $g(t)$. Then $g(r(X),Y) = \mathrm{Ric}(X,Y)$. Similary, $\hat{r}$ denotes the Ricci endomorphism for metric $\hat{g}$.

For $\hat{g}$ to be an Ricci-flat metric (i.e. the Einstein constant is zero), we have

Proposition 1.2. *The Ricci-flat condition for the metric $\hat{g}$ on M_0 is given by*

$$\dot{g} = 2gL \tag{1.3a}$$

$$\dot{L} = -(\mathrm{tr}L)L + r \tag{1.3b}$$

$$\mathrm{tr}(\dot{L}) = -\mathrm{tr}(L^2) \tag{1.3c}$$

$$\mathrm{tr}(X \neg d^{\nabla} L) = 0 \tag{1.3d}$$

for all $X \in TP$.

Remark 1.3. If we take the trace of (1.3b) and use (1.3c), we obtain the equation for conservation law by

$$s - (\mathrm{tr}L)^2 + \mathrm{tr}(L^2) = 0, \tag{1.4}$$

where $s(t) = \mathrm{tr}(r(t))$ denotes the scalar curvature of $g(t)$.

1.3.1 Ricci-flat equations for multiple warped product

We next consider the case where $\widehat{M}$ is a multiple warped product over an interval, following the notation and approach of [DW09].

Let $\widehat{M}^{n+1}$ be multiple warped product manifold with compact Lie group G-acting by isometries which has a open dense set $I \times P$ foliated by diffeomorphic hypersurfaces P_t of real dimension n. In this dissertation, we set $G = SO(d_1+1) \times G_2 \times \cdots \times G_r$, $H = SO(d_1+1) \times H_2 \times \cdots \times H_r$ and $K = SO(d_1) \times H_2 \times \cdots \times H_r$. The hypersurface P_t along the unit speed geodesic $\gamma(t)$ is $S^{d_1} \times G_2/H_2 \times \cdots \times G_r/H_r$. Each homogeneous space $(G_i/H_i, h_i)$, with compact Lie group G_i acting by isometries, is an isotropy irreducible manifold with positive Einstein constants λ_i and dimension d_i. The singular orbit $Q = G/H = G_2/H_2 \times \cdots \times G_r/H_r$.

The metric $\hat{g}$ on the open dense set of $\widehat{M}$ can be described in the form

$$dt^2 + \sum_{i=1}^{r} g_i^2(t)h_i. \tag{1.5}$$

The Ricci endomorphism $r(t)$ of TP_t for metrics $g(t)$ is given by $g(r(X), Y) = \mathrm{Ric}(X, Y)$. So for a multiple warped product manifold,

$$r(t) = \mathrm{diag}\Big(\frac{\lambda_1}{g_1^2}\mathbb{I}_{d_1}, \ldots, \frac{\lambda_r}{g_r^2}\mathbb{I}_{d_r}\Big), \tag{1.6}$$

where $\mathbb{I}_{d_i}$ denotes the identity matrix of size d_i. d_i=dim$(\mathfrak{p}_i)$ for $i = 1, ..., r$.
The shape operator is given by

$$L(t) = \mathrm{diag}(\frac{\dot{g}_1}{g_1}\mathbb{I}_{d_1}, \cdots, \frac{\dot{g}_r}{g_r}\mathbb{I}_{d_r}).$$

So (1.3b) in the Ricci-flat equations becomes the system on $\mathring{I} \times P$

$$\begin{cases} -\frac{\ddot{g_1}}{g_1} + \frac{\dot{g_1}^2}{g_1^2} - (\frac{d_1\dot{g_1}}{g_1} + \ldots + \frac{d_r\dot{g_r}}{g_r})\frac{\dot{g_1}}{g_1} + \frac{\lambda_1}{g_1^2} = 0, \\ \qquad\qquad\vdots \\ -\frac{\ddot{g_r}}{g_r} + \frac{\dot{g_r}^2}{g_r^2} - (\frac{d_1\dot{g_1}}{g_1} + \ldots + \frac{d_r\dot{g_r}}{g_r})\frac{\dot{g_r}}{g_r} + \frac{\lambda_r}{g_r^2} = 0. \end{cases} \tag{1.7}$$

By Proposition 3.18 in [BB81], we know if the summands in the decomposition of $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \cdots \oplus \mathfrak{p}_r$ are pairwise distinct, then $\widehat{\mathrm{Ric}}(X, N) = 0$ is automatically true. In addition, by the remark of Lemma 2.4 and Corollary 2.6 in [EW00] and [Bac86], we know if (1.3d) is satisfied, then $(\widehat{\mathrm{Ric}}(N, N))v^2$ is constant in time. Here, $v(t)$ denotes the volume distortion of $g(t)$. Therefore, if we can prove that $\hat{g}$ is smooth, then $\lim_{t\to 0} v(t) = 0$, and so $\widehat{\mathrm{Ric}}(N, N) = 0$ i.e. (1.3c) is satisfied too.

Remark 1.4. In the multiple warped product manifold $I \times S^{d_1} \times M_2 \times \cdots \times M_r$, with metric $dt^2 + \sum_{i=1}^r g_i^2(t)h_i$, if (M_i, h_i) are inhomogeneous, then the resulting equations are equivalent to those coming from the above cohomogeneity one manifolds. This is because in the derivation of the Einstein equation (cf. Lemma 2.4 [EW00]), we only use the geometry of an equi-distant family of hypersurfaces. Therefore the paper [DW09], which discusses more general cases, was using the same equation and the same way of changing the variables.

1.3.2 The Ricci endomorphism when principal orbit is G/K with 2 distinct irreducible summands

Suppose the principal orbit is $P = G/K$ and we choose a diagonal metric $g_1(t)^2 h \mid_{\mathfrak{p}_1} \perp g_2(t)^2 h \mid_{\mathfrak{p}_2} \perp \cdots \perp g_r(t)^2 h \mid_{\mathfrak{p}_r}$. If we denote $g_i(t)^2 = x_i(t)$, then we may use equation (1.3) in [WZ86] for the scalar curvature of P_t:

$$S = \frac{1}{2}\sum_i \frac{d_i b_i}{x_i} - \frac{1}{4}\sum_{ijk}\begin{bmatrix} k \\ ij \end{bmatrix}\frac{x_k}{x_i x_j}. \tag{1.8}$$

In this formula, for each i, b_i is defined by $B|_{\mathfrak{p}_i} = b_i h|_{\mathfrak{p}_i}$, where B is the negative of the Killing form of G, and d_i=dim$(\mathfrak{p}_i)$, and $\begin{bmatrix} k \\ ij \end{bmatrix} = \sum h([e_\alpha, e_\beta], e_\gamma)^2$, where the sum is taken over $\{e_\alpha\}, \{e_\beta\}, \{e_\gamma\}$, h -orthonormal bases for $\mathfrak{p}_i, \mathfrak{p}_j, \mathfrak{p}_k$ respectively.

Let $A_i = \frac{1}{2}\big(d_i b_i - \frac{1}{2}\begin{bmatrix} i \\ ii \end{bmatrix} - \sum_{j\neq i}\begin{bmatrix} j \\ ij \end{bmatrix}\big)$ as in [WZ86]. Then, when $r = 2$ and $\mathfrak{k} \oplus \mathfrak{p}_1$ is a subalgebra of $\mathfrak{g}$, we have

$$S = \frac{A_1}{x_1} + \frac{A_2}{x_2} - \frac{1}{4}\begin{bmatrix} 2 \\ 11 \end{bmatrix}\frac{x_2}{x_1^2} - \frac{1}{4}\begin{bmatrix} 1 \\ 22 \end{bmatrix}\frac{x_1}{x_2^2}. \tag{1.9}$$

Because of Schur's lemma, we let the Ricci tensor be given by

$$\mathrm{Ric} = \begin{pmatrix} r_1 \mathbb{I}_{d_1} & 0 \\ 0 & r_2 \mathbb{I}_{d_2} \end{pmatrix}.$$

Using the first variation formula for the Einstein-Hilbert action and personal discussion with Dr. Wang, we get

$$r_1 = -\frac{x_1^2}{d_1}\frac{\partial S}{\partial x_1} = \frac{A_1}{d_1} - \frac{1}{2d_1}\begin{bmatrix} 2 \\ 11 \end{bmatrix}\frac{x_2}{x_1} + \frac{1}{4}\frac{1}{d_1}\begin{bmatrix} 1 \\ 22 \end{bmatrix}\frac{x_1}{x_2^2}, \tag{1.10}$$

$$r_2 = -\frac{x_2^2}{d_2}\frac{\partial S}{\partial x_2} = \frac{A_2}{d_2} - \frac{1}{2d_2}\begin{bmatrix} 1 \\ 22 \end{bmatrix}\frac{x_1}{x_2} + \frac{1}{4}\frac{1}{d_2}\begin{bmatrix} 2 \\ 11 \end{bmatrix}\frac{x_2}{x_1^2}. \tag{1.11}$$

So, the Ricci endomorphism $r(t)$ is given by

$$r(t) = \begin{pmatrix} \frac{r_1}{x_1}\mathbb{I}_{d_1} & 0 \\ 0 & \frac{r_2}{x_2}\mathbb{I}_{d_2} \end{pmatrix}. \tag{1.12}$$

1.4 Smoothness condition around a singular orbit

It is natural to ask:"What conditions must the above metric $<,>_t$ satisfy in order for $\hat{g}$ on M_0 to extend to a smooth metric on $\widehat{M}$?" A necessary and sufficient condition was given by Eschenburg and Wang in [EW00] Lemma 1.1.

We need to look at the structure near the singular orbit of the cohomogeneity one manifold more closely to present the smoothness criterion. We follow the notation of ([EW00]).

The normal bundle of $Q = G/H$, $H = G_q$, is equivariantly diffeomorphic to

$$E = G \cdot V = G \times_H V.$$

where $V = T_q\widehat{M}/T_qQ = \mathbb{R}^{k+1}$ is the normal space at $q \in Q$. Then H acts on $G \times V$ by $(h, (g, v)) \to (gh^{-1}, hv)$. Here h act on V linearly via the slice representation, because h preserves $T_q(G/H)$, acts orthogonally. By the slice theorem (§ II.5 [Bre72]), the H-action is also cohomogeneity one, so H acts transitively on the unit sphere $S^k = H/K \subset V$, where $K \subset H \subset G$. Moreover, the principal orbit G/K can be identified with the unit sphere bundle $G \times_H S^k$.

Also by the slice theorem, a tubular neighbourhood of the singular orbit $Q = G/H$ can be described as

$$B_r(Q) = \pi^{-1}[0, r) = G \times_H D_r = E' \subset E$$

D_r is a disk of radius r in the slice V. Therefore, given a G-invariant Riemannian metric $\hat{g}_0$ on E, $\hat{g}_0|_{E'}$ can be transplanted to the metric $\hat{g}$ on a tubular neighbourhood $B_r(Q) \subset \widehat{M}$ and vice versa.

Therefore, we need to identify the smooth, G-invariant symmetric tensors on E, i.e. $a \in C^\infty(S^2TE)$. Since $E = G \cdot V$, we can restrict our attention to H-invariant $a \in C^\infty(S^2TE|V)$ (cf. [EW00]).
Let $\mathfrak{p}_-$ be an $\mathrm{Ad}(H)$-invariant complement of $\mathfrak{h}$ in $\mathfrak{g}$, which can be identified with the tangent space T_qQ. Let $\pi : E \to Q$ be the bundle map, then the tangent bundle TE splits into horizontal and vertical parts: $TE \cong \pi^*E \oplus \pi^*TQ$. These two pullback bundles are trivial on V and H-invariant, so we have $TE|V = V \times (V \oplus \mathfrak{p}_-)$, Therefore, the tensor field a is determined by some H-equivariant smooth mapping

$$a : V \to S^2(V \oplus \mathfrak{p}_-).$$

We can write a in "polar coordinates" $a_t : S^k \to S^2(V \oplus \mathfrak{p}_-)$, which is determined by the single value $a_t(v_0) \in S^2(V \oplus \mathfrak{p}_-)^K$, for a fixed $v_0 \in S^k$, since H acts transitively. Here we choose $K = H_{v_0} = G_{v_0}$.

Next, we introduce the vector space W of all smooth H-equivariant maps $L : S^k \to S^2(V \oplus \mathfrak{p}_-)$. For a fixed $v_0 \in S^k$, we define the evaluation map to be: $\epsilon : W \to S^2(V \oplus \mathfrak{p}_-)^K, \epsilon(L) = L(v_0)$. Let $W_p \subset W$ be the subspace consisting of restriction of all maps to unit sphere of H-equivariant homogeneous polynomials of degree p. (see page 113 of [EW00].)

Lemma 1.1 ([EW00]). *Let $t \to a_t(v_0) : \mathbb{R}_+ \to S^2(V \oplus \mathfrak{p}_-)^K$ be a smooth curve (i.e. at zero, the right-hand derivatives of all orders exists and are continuous from the right) with Taylor expansion (at zero) $a_t \sim \sum_p a_p t^p$. Then the map a defined by*

$$a : V\backslash\{0\} \to S^2(V \oplus \mathfrak{p}_-), a(v) = a_{|v|}\Big(\frac{v}{|v|}\Big)$$

has a smooth extension to 0 if and only if $a_p \in \epsilon(W_p)$ for all $p \geqslant 0$.

Remark 1.5. Following by [EW00], we will assume that the representations of K on $\mathfrak{p}_-$ and V have no equivalent irreducible factors. Then $S^2(V \oplus \mathfrak{p}_-)^K = S^2(V)^K \oplus S^2(\mathfrak{p}_-)^K$. Therefore,

$$a_t(v_0) = (\tilde{x}_+, x_-) : [0, \infty) \to S^2(V)^K \oplus S^2(\mathfrak{p}_-)^K.$$

Remark 1.6. $\mathfrak{p}_+$, the $\mathrm{Ad}(K)$-invariant complement of $\mathfrak{k}$ in $\mathfrak{h}$ can be identified with tangent space TS^k at v_0, (i.e. $v_0^\perp$). $T_{v_0}V = \mathbb{R}\{v_0\} \oplus \mathfrak{p}_+$. Finally, the tangent space of P (the unit normal bundle around Q) at point v_0 can be viewed as $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$. So the diagonal metric $dt^2 + <,>_t = dt^2 + g(t)|_{\mathfrak{p}}$ is really our $a_t(v_0) = (\tilde{x}_+, x_-)$.

Remark 1.7. For a non-compact manifold $\widehat{M}$ with a cohomogeneity one action, with orbit-space $[0, \infty)$, one can always just use G and subgroups K and H to recover its structure. It makes sense to denote $(\widehat{M}, \hat{g})$ as $(G, H, K, \hat{g})$ in the future.

1.4.1 The initial value problem in the multiply warped product case

Lemma 1.2. *Consider a multiple warped product G-manifold described in section 1.3.1. $H_1 = SO(d_1 + 1)$ acts on $V = \mathbb{R}^{d_1+1}$ orthogonally. $\mathfrak{p}_+$ corresponds to the tangent space of $S^{d_1} = \frac{SO(d_1+1)}{SO(d_1)}$ at v_0 and $\mathfrak{p}_-$ corresponds to tangent space at eH of the singular orbit $Q = G_2/H_2 \times \cdots \times G_r/H_r$. If we fix the background metric $h|_{\mathfrak{p}_+}$ which gives constant curvature 1 of the sphere S^{d_1}. Then, the metric $\hat{g}$ on $M_0 = \mathring{I} \times S^{d_1} \times G_2/H_2 \times \cdots \times G_r/H_r$ is smooth if and only if g_i are smooth in t, $g_1(t)$ is odd, $g_i(t)$ is even for $i > 1$ and $\dot{g}_1(0) = 1$.*

Proof. Choose "Cartesian coordinates" $(y_1, y_2, \dots y_{d_1+1})$ for V. Given a smooth curve of metrics along the geodesic

$$x = (\tilde{x}_+, x_-) : [0, r) \to S^2(V)^K \oplus S^2(\mathfrak{p}_-)^K,$$

the Taylor expansion expansion of x in a neighbourhood of zero is

$$x(t) \backsim \sum_{p=0}^{\infty} x_p t^p,$$

where $t = \sqrt{y_1^2 + y_2^2 + \cdots + y_{d_1+1}^2}$. By lemma 1.1 of [EW00], x is smooth if and only if $x_p \in \epsilon(W_p)$, which means that x_p is an H-equivariant homogeneous polynomial with values in $S^2(V \oplus \mathfrak{p}_-)^K$ of degree p on the sphere S_1^d. Denote by S^pV to be the vector space of degree p homogeneous polynomials on V. $\mathrm{Hom}(S^pV, S^2(V \oplus \mathfrak{p}_-))^H = \mathrm{Hom}(S^pV, S^2V)^H \oplus \mathrm{Hom}(S^pV, S^2\mathfrak{p}_-)^H \oplus \mathrm{Hom}(S^pV, V \otimes \mathfrak{p}_-)^H$ Since H is a product group, its representation on V is inequivalent to representation on $\mathfrak{p}_-$, so the last space is zero. So,

$$\mathrm{Hom}(S^pV, S^2(V \oplus \mathfrak{p}_-))^H = \mathrm{Hom}(S^pV, S^2V)^H \oplus \mathrm{Hom}(S^pV, S^2\mathfrak{p}_-)^H. \tag{1.13}$$

Since $H_1 = SO(d_1 + 1)$, the representation on S^2V of H decomposes to $\mathbb{1} \oplus U_2$, where U_2 is irreducible. As is well known (see, e.g., [FH91] Exercise 19.21, p.296) when p is odd,

$$S^pV = U_p \oplus S^{p-2}V = U_p \oplus U_{p-2} \oplus S^{p-4}V = \cdots = U_p \oplus U_{p-2} \oplus \cdots \oplus U_3 \oplus U_1 \tag{1.14}$$

where the U_i are irreducible and inequivalent to one another for $i = 2, 3, ..., p$. Those U_i are also inequivalent to U_2 and $\mathbb{1} = U_0$. By Schur's Lemma $\mathrm{Hom}(U_i, U_2)^H = 0$ $\mathrm{Hom}(U_i, \mathbb{1})^H = 0$ for all $i \geqslant 1$. This implies $\mathrm{Hom}(S^pV, S^2V)^H = 0$.

In addition, H acts on V by the representation $\rho_{d_1+1} \otimes \mathbb{1}$, where ρ_{d_1+1} is the standard representation of $SO(d_1 + 1)$. ρ_{d_1+1} is irreducible implies $\rho_{d_1+1} \otimes \mathbb{1}$ is irreducible. ($\mathbb{1}$ is trivial representation). Similarly, if we let η denote the representation of $H_2 \times H_3 \times \cdots \times H_r$ on $\mathfrak{p}_-$, then, H acts on $S^2(\mathfrak{p}_-)$ by the representation $\mathbb{1} \otimes S^2(\eta)$, which is inequivalent to the representation on V.
Suppose H decompose $S^2(\mathfrak{p}_-)$ into irreducible summands $N_1 \oplus \cdots \oplus N_k$. Then

$$\mathrm{Hom}(V, S^2(\mathfrak{p}_-)) = \mathrm{Hom}(V, N_1 \oplus \cdots \oplus N_k) \cong \oplus_{i=1}^k \mathrm{Hom}(V, N_i). \tag{1.15}$$

Here both V and N_i are inequivalent irreducible representation of H. Therefore, if $f_i \in \mathrm{Hom}(V, N_i)$ is a H-equivariant map, then $f_i = 0$ for all i by Schur's lemma. Hence,

$\mathrm{Hom}(V, S^2(\mathfrak{p}_-))^H = 0$. Similarly, U_j and N_i are inequivalent to one another too, for all i and j, so $\mathrm{Hom}(S^pV, S^2(\mathfrak{p}_-))^H = 0$.
Therefore, $\mathrm{Hom}(S^pV, S^2(V \oplus \mathfrak{p}_-))^H = 0$. Therefore $x_p = 0$ for all odd p.
Next since we know $S^{even}V = U_p \oplus U_{p-2} \oplus \cdots \oplus U_4 \oplus U_2 \oplus \mathbb{1}$, and $S^2V = U_2 \oplus \mathbb{1}$, then by Schur's Lemma, $\mathrm{Hom}(S^{\mathrm{even}}V, S^2V)^H \cong \mathbb{R} \oplus \mathbb{R}$.
So, x is a even function. This means $x_\pm$ are even.

One the other hand, $\hat{g}$ can be described in "cylindrical coordinates"about Q, by the map

$$\Phi : [0, r) \times P \to E', \Phi(t, v) = tv,$$

where P is viewed as the unit normal sphere bundle of Q. So the pullback of metric $\hat{g}$ onto $(0, r) \times P$ is

$$\Phi^*(\hat{g}) = dt^2 + g(t),$$

where

$$g(t) = \overbrace{t^2 x_+(t)}^{\tilde{x}_+(t)} \oplus x_-(t), \tag{1.16}$$

is a G-invariant metric on P. Compare this with the equation (1.2) So we have

$$\begin{cases} g_1(t)^2 h_1 = x_+(t) = \frac{\tilde{x}_+(t)}{t^2}, \\ g_2(t)^2 h_2 + \cdots + g_r(t)^2 h_r = x_-(t). \end{cases}$$

Then g_1 must be odd and g_i must be even for $i = 2, 3, ..., r$.

Notice that in (1.16), the t^2 comes from the fact, $\frac{\partial}{\partial\theta} = te_1$ where $e_1 = \frac{\partial}{\partial y_1}$ is a vector in "Cartesian coordinates" of Euclidean space V. Using e_1 we form a orthonormal basis for V respect to the Euclidean metric $\tilde{x}_+(0) = I$. Finally,

$$\tilde{x}_+(0) = I \Leftrightarrow \dot{g}_1(0) = 1.$$

□

1.5 Existences of local solutions

Theorem 1 ([EW00]). *Let G be a compact Lie group, H a closed subgroup with an orthogonal linear action on $V = \mathbb{R}^{k+1}$ which is transitive on the unit sphere S^k, and $E = G \times_H V$ be the vector bundle over $Q = G/H$ with fiber V. Denote by $\mathfrak{p}_-$ an ad-invariant complement of $\mathfrak{h}$ in $\mathfrak{g}$. Let $v_0 \in S^k$ have isotropy group K. Assume that as K-representations, V and $\mathfrak{p}_-$ have no irreducible sub-representations in common.*

Then, given any G-invariant metric g_Q on Q and any G-equivariant homomorphism $L_1 : E \to S^2(T^*Q)$, there exists a G-invariant Einstein metric on some open disk bundle E' of E with any prescribed sign (positive, zero, or negative) of the Einstein constant.

Remark 1.8. Since $E = G \cdot V$, a G-equivariant homomorphism L_1 is determined by a H-equivariant map $\alpha : V \to S^2(T_q^*Q)$. We will identify T_q^*Q and $T_qQ = \mathfrak{p}_-$ by g_Q. Then the H-equivariant map $\alpha : V \to S^2(\mathfrak{p}_-)$ vanishes, because $\mathrm{Hom}(V, S^2(\mathfrak{p}_-))^H = 0$ (cf. proof of Lemma 1.2). So the G-equivariant homomorphism $L_1 : E \to S^2(T^*Q)$ is trivial.

Let $q \in Q$ and $N_1, ..., N_{d_i+1}$ be an orthonormal basis of $V = (T_qQ)^\perp \cong \mathbb{R}^{d_i+1}$. The second fundamental form at q along the normal vector N_j is given by H-equivariant mapping

$$\begin{aligned} \alpha : V &\to S^2(\mathfrak{p}_-) \\ N_j &\to \alpha_{N_j}(X, Y) \end{aligned}$$

where $\alpha_{N_j}(X, Y) =< \widehat{\nabla}_X Y - \nabla_X Y, N_j >$.

Next, let L_{N_j} be the shape operator of Q along normal N_j given by $L_{N_j}X = \widehat{\nabla}_X N_j$. then we have following relation expressing the shape operator associated to the second fundamental form, that is:

$$< \widehat{\nabla}_X N_j, Y >= -\alpha_{N_j}(X, Y), \quad \text{where } X, Y \in T_qQ. \tag{1.17}$$

Lemma 1.9. *For the multiple warped product manifold in 1.3.1, the second order differential equations (1.7) have up to homothety an $r-2$ parameter family of solutions in a neighbourhood of the singular orbit.*

Proof. The singular orbit, $Q = M_2 \times \cdots \times M_r$, $M_i = G_i/H_i$ for $i = 2, \ldots r$ has G-invariant metric given by

$$g_2^2(0)h_2 + g_3^2(0)h_3 + \cdots + g_r^2(0)h_r, \tag{1.18}$$

where $g_i(0) \neq 0$ for $2 \leqslant i \leqslant r$. By Remark 1.8, all of the second fundamental forms of Q must vanished. By the above equation (1.17), the shape operator of the singular orbit Q at q must vanish too, i.e.

$$\dot{g}_i(0) = 0 \quad \text{for all } 2 \leqslant i \leqslant r \tag{1.19}$$

by $\dot{g} = 2gL$. By Theorem 1, there is an $r-1$ parameter family of local Ricci-flat metrics on $t \in [0, t_0)$ satisfying the system (1.7).

In addition, $\mathrm{Ric}(\lambda^2\hat{g})$=$\mathrm{Ric}(\hat{g}) = 0$. This implies if $\hat{g} = dt^2 + g(t)$ is a Ricci-flat metric then $\lambda^2\hat{g} = ds^2 + \lambda^2 g(\frac{s}{\lambda})$ is also a Ricci-flat metric where $s = \lambda t$. Therefore, by Theorem 1, given an initial value G-invariant metic $g_Q \sim g_2^2(0)h_2 + g_3^2(0)h_3 + \cdots + g_r^2(0)h_r$ we have a Ricci-flat metric on $t \in [0, t_0)$, which will be same as the Ricci-flat metric on $s \in [0, \frac{t_0}{g_2^2(0)})$ starting from the initial value $\frac{g_2^2(0)}{g_2^2(0)}h_2 + \frac{g_3^2(0)}{g_2^2(0)}h_3 + \cdots + \frac{g_r^2(0)}{g_2^2(0)}h_r$. Without loss of generality, we can fix $g_2^2(0)$ to be 1. Hence, we have up to homothety an $r-2$ family of local Ricci-flat metrics.

Let $h_1(s) = \lambda^2 g_1(\frac{s}{\lambda})$. Then $\frac{\mathrm{d}\,h_1}{\mathrm{d}\,s} = \frac{\mathrm{d}\,h_1}{\mathrm{d}\,t}\frac{\mathrm{d}\,t}{\mathrm{d}\,s} = \lambda\dot{g_1}\frac{1}{\lambda} = \dot{g_1}$. This implies that $\dot{g_1}(0) = 1 \Leftrightarrow \frac{\mathrm{d}\,h_1}{\mathrm{d}\,s}(0) = 1$. $\square$

Chapter 2

An alternative proof of Böhm's existence theorem

In this chapter, we shall give an alternative proof of Böhm's theorem that there exists an $r-2$ parameter family of complete Ricci-flat metrics on the multiple warped products.

Theorem 2 (Theorem 6.1 [Böh99])**.** *Let $l \geqslant 0$ and let G_1/K_1, G_2/K_2, ..., G_{l+1}/K_{l+1} be non-flat compact isotropy irreducible homogeneous spaces and let $k \geqslant 2$. Then*

$$\widehat{M} = \mathbb{R}^{k+1} \times G_1/K_1 \times G_2/K_2 \times \cdots \times G_{l+1}/K_{l+1} \tag{2.1}$$

carries an l-dimensional family of Ricci-flat metrics.

Notice that in our notation, $r = l+2$. In order to prove Theorem 2, first of all, inspired by [DW09], we transform equation 1.7 to a new first order system whose right hand side is given by polynomials. Then we find all of the stationary points of the new dynamical system. We are interested in two of the stationary points and trajectories which connect them. The first stationary point corresponds to the smooth initial values while the second stationary point represents the Ricci-flat cone solution. The question is how can we prove there exist complete Ricci-flat solutions such that they theoretically converge to this point. Since our manifolds fit into the more general framework of Eschenburg and Wang [EW00], there exist local smooth solutions of the Ricci-flat equation, which correspond to smooth G-invariant Einstein metrics on a tubular neighbourhood of Q. They give the new dynamical system local solutions. We extend the local solutions to global solutions. Finally, we find the corresponding cone solution of the cohomogeneity one Ricci-flat equation is a global attractor by modifying the Lyapunov function introduced by [DHW13].

2.1 A polynomial system

Since the Ricci-flat equation (1.7) is singular at $t = 0$, we introduce new variables to change the Ricci-flat equation to a polynomial system.

When $\mathfrak{p}_i$ are pairwise inequivalent, the shape operator is given by

$$L(t) = \mathrm{diag}(\frac{\dot{g}_1}{g_1}\mathbb{I}_{d_1}, \cdots, \frac{\dot{g}_n}{g_n}\mathbb{I}_{d_n}),$$

where $\mathbb{I}_m$ denotes the identity matrix of size m. So

$$\lim_{t\to 0} \mathrm{tr}L = +\infty,$$

due to the smoothness condition $g_1(0) = 0, \dot{g}_1(0) = 1$.

Applying the Cauchy–Schwartz inequality to (1.3c) we have

$$(\mathrm{tr}L)^{\cdot} \leqslant -\frac{1}{n}(\mathrm{tr}L)^2, \tag{2.2}$$

and so we know that $\mathrm{tr}L$ is non-increasing and the equality holds precisely at times where when $n(\mathrm{tr}L^2) = (\mathrm{tr}L)^2$.

Lemma 2.1 (Proposition 3.2 [Böh99]). *Let $\widehat{M}^{n+1}$ be a complete Ricci-flat cohomogeneity one Einstein manifold whose principal orbit is not a torus. Then there exists no principal orbit P which is minimal, that is,*

$$\mathrm{tr}L > 0 \qquad \forall t > 0.$$

Proof. This proof is different from Böhm's proof.
By smoothness, we know that $\mathrm{tr}L > 0$ for all small t. In order to derive a contradiction, assume that $\mathrm{tr}L(t_0) = 0$ at some $t_0 > 0$. As $\mathrm{tr}L$ is non-increasing by(2.2), then there are only two cases.
Case one: Mean curvature of the principal orbit P remains a constant 0, i.e.

$$\mathrm{tr}L \equiv 0, \text{ for } \forall t \geqslant t_0.$$

Case two: Mean curvature becomes negative after t_0. In another words, there exists $t_1 > t_0$ and $a > 0$, such that

$$\mathrm{tr}L \leqslant -a, \text{ for } \forall t > t_1. \tag{2.3}$$

If case one is true, then

$$(\mathrm{tr}\dot{L}) = 0, \text{ for } \forall t > t_0. \tag{2.4}$$

On the other hand by (1.3c), it follows that $\mathrm{tr}(L^2) = 0$ for $\forall t > t_0$. Therefore, $L \equiv 0$ for $\forall t > t_0$. By (1.3b), $r_t = 0$ for $\forall t > t_0$. So G/H has a G-invariant Ricci-flat metric. By [AK75], the Ricci-flat metric is actually flat. Since G is compact, $(G/H, g)$ has to be a torus. This contradicts our assumption that G/H is not a torus. So, case one cannot be true.

Now, suppose case two is true. Then

$$\frac{\dot{v}}{v} = (\log v)^{\cdot} = \mathrm{tr}L \leqslant -a, \text{ for } \forall t > t_1,$$

by (2.3). Integrating the above, one gets,

$$\log v \,|_{t_1}^{t} = \int_{t_1}^{t} (\mathrm{tr}L)\mathrm{d}t \leqslant -a(t - t_1).$$

So,

$$\frac{v(t)}{v(t_1)} \leqslant e^{-a(t-t_1)}.$$

i.e.

$$v(t) \leqslant v(t_1)e^{-a(t-t_1)}.$$

Therefore the volume of $\widehat{M}$, given by $\int_0^\infty v\mathrm{d}t = \int_0^{t_1} v\mathrm{d}t + \int_{t_1}^\infty v\mathrm{d}t < \infty$. However, this contradicts Yau's Theorem (Theorem 7 [Yau76]) i.e. the volume of $(\widehat{M}, \widehat{g})$ must be infinity. Thus case two cannot be true either.

Therefore, $\mathrm{tr}L > 0$ for $\forall t > 0$. $\square$

Remark 2.2. Since $\mathrm{tr}L$ is positive for $\forall t > 0$, $\mathrm{tr}L$ is also a strictly decreasing map via inequality (2.2). We can then introduction a new independent variable by

$$\frac{\mathrm{d}}{\mathrm{d}s} := \frac{1}{\mathrm{tr}L}\frac{\mathrm{d}}{\mathrm{d}t}. \tag{2.5}$$

We use a prime to denote differentiation with respect to s.

2.1.1 For multiple warped product

Following [DW09] we introduce the new dependent variables,

$$X_i = \frac{\sqrt{d_i}}{\mathrm{tr}L}\frac{\dot{g}_i}{g_i}, \tag{2.6}$$

$$Y_i = \frac{\sqrt{d_i \lambda_i}}{g_i}\frac{1}{\mathrm{tr}L}. \tag{2.7}$$

for $i = 1, ..., r$. Notice that

$$\sum_{j=1}^{r} X_j^2 = \frac{-\mathrm{tr}(\dot{L})}{(\mathrm{tr}L)^2} = \frac{\mathrm{tr}(L^2)}{(\mathrm{tr}L)^2}, \tag{2.8}$$

$$\sum_{j=1}^{r} Y_j^2 = \frac{\mathrm{tr}(r_t)}{(\mathrm{tr}L)^2}. \tag{2.9}$$

We obtain from the Ricci-flat system (1.3) the following system in our new variables.

$$X_i' = X_i(\sum_{j=1}^{r} X_j^2 - 1) + \frac{Y_i^2}{\sqrt{d_i}}, \tag{2.10a}$$

$$Y_i' = Y_i(\sum_{j=1}^{r} X_j^2 - \frac{X_i}{\sqrt{d_i}}). \tag{2.10b}$$

From above system we get $X_i'X_i + Y_i'Y_i = 0$, so

$$\sum_{j=1}^{r}(X_j^2 + Y_j^2) = C, \tag{2.11}$$

where C is a constant. From Remark 1.3, we know the conservation law of the Ricci-flat system is $s - (\mathrm{tr}L)^2 + \mathrm{tr}(L^2) = 0$. Applying (2.8) and (2.9) to this equation we get

$$\sum_{j=1}^{r}(X_j^2 + Y_j^2) = 1. \tag{2.12}$$

In the following, we let $\mathcal{L} = \sum_{j=1}^{r}(X_j^2 + Y_j^2)$. From [DW09] (Remark 2.14) if we take the derivative with respect to s, we have

$$u' = \sum_{j=1}^{r} \sqrt{d_i} X_i - 1, \tag{2.13}$$

In the Ricci-flat case, $u' = 0$. We may define the quantity

$$\mathcal{H} := \sum_{j=1}^{r} \sqrt{d_i} X_i, \tag{2.14}$$

so that in the Ricci-flat case, $\mathcal{H} \equiv 1$.

2.1.2 Invariant of conservation law and $\mathcal{H}$

Lemma 2.3. *The sphere $\mathcal{L} = 1$ is invariant under the flow of the vector field (2.10).*

Proof. Since $\nabla(\mathcal{L}-1) = (2X_1, 2Y_1, 2X_2, 2Y_2, ..., 2X_r, 2Y_r)$, and by (2.10)

$$X_iX_i' = X_i^2(\sum_{j=1}^{r} X_j^2 - 1) + \frac{X_iY_i^2}{\sqrt{d_i}} = -X_i^2(\sum_{j=1}^{r} Y_j^2) + \frac{X_iY_i^2}{\sqrt{d_i}}, \tag{2.15}$$

$$Y_iY_i' = Y_i^2(\sum_{j=1}^{r} X_j^2 - \frac{X_i}{\sqrt{d_i}}). \tag{2.16}$$

It follows that

$$\begin{aligned} &\nabla(\mathcal{L}-1)\cdot(X_1', Y_1', X_2', Y_2', ..., X_r', Y_r') \\ &= 2\sum_{i=1}^{r}(X_iX_i' + Y_iY_i') \\ &= -\sum_{i=1}^{r} X_i^2 \sum_{j=1}^{r} Y_j^2 + \sum_{i=1}^{r}\frac{X_iY_i^2}{\sqrt{d_i}} + \sum_{i=1}^{r} Y_i^2 \sum_{j=1}^{r} X_j^2 - \sum_{i=1}^{r}\frac{X_iY_i^2}{\sqrt{d_i}} \\ &= 0. \end{aligned} \tag{2.17}$$

Therefore, $\{\mathcal{L} = 1\}$ is invariant under the flow (2.10). $\square$

Lemma 2.4. *The hypersurface $\mathcal{H} = \sum_{j=1}^{r}\sqrt{d_j}X_j = 1$ is invariant under the flow of the vector field (2.10).*

Proof. $\mathcal{H}$ is a constant. Hence $\mathcal{H}' = \sum_{j=1}^{r}\sqrt{d_j}X_j' = 0$.
Since $\nabla(\mathcal{H}-1) = (\sqrt{d_1}, 0, \sqrt{d_2}, 0, ..., \sqrt{d_r}, 0)$, so

$$\begin{aligned} &\nabla(\mathcal{H}-1)\cdot(X_1', Y_1', X_2', Y_2', ..., X_r', Y_r') \\ &= \sum_{j=1}^{r}\sqrt{d_j}X_j' \\ &= 0. \end{aligned} \tag{2.18}$$

Therefore, $\mathcal{H}-1$ is invariant under the flow of (2.10), and so is $\mathcal{H}$. $\square$

Remark 2.5. In the paper [DW09], the authors have studied the the existence of Ricci soliton structures on multiple warped products. The equation are the same ones here. In [DW09], the conservation law for a non-trival steady soliton is $\sum_{j=1}^{r}(X_j^2+Y_j^2) = C$ where $C < 1$, and the Lyapunov function $\sum_{j=1}^{r}(X_j^2+Y_j^2) < 1$. But as considering the Ricci-flat (in steady soliton) case, all the solution curves lie in the corresponding conservation law sphere $\sum_{j=1}^{r}(X_j^2+Y_j^2) = 1$ where $C = 1$. So, first of all, Ricci-flat solutions will not be the Ricci soliton solution. Secondly we cannot use the same Lyapunov function and similar arguments Proposition 3.7 [DW09] to derive the Ricci-flat trajectories's long time behaviour.

2.2 Stationary points

Proposition 2.6. *Let $A := \{i|Y_i \neq 0\}$. For an arbitrary (possibly empty) subset $A = \{m_1, ...m_l\} \subset \{1, ..., r\}$, all the equilibrium points of the first order ode system (2.10a) and (2.10b) lying in $\{\mathcal{L} = 1\}$ are given by*

(i) When $A = \varnothing$, $\{Y_i = 0, \text{ for all } \forall i, \sum_{j=1}^{r} X_j^2 - 1 = 0, \mathcal{H} \equiv 1\}$,

(ii) if $A \neq \varnothing$, $\{X_i = 0, Y_i = 0, \text{for} i \notin A, X_k = \frac{\sqrt{d_k}}{\sum_{i=1}^{l} d_{m_i}}, Y_k = \frac{\sqrt{(\sum_{i=1}^{l} d_{m_i} - 1)d_k}}{\sum_{i=1}^{l} d_{m_i}}, \text{ for } k \in A\}$.

Proof. A stationary point corresponds to zeros of the vector field given by (2.10). We have

$$X_i(\sum_{j=1}^{r} X_j^2 - 1) + \frac{Y_i^2}{\sqrt{d_i}} = 0, \tag{2.19a}$$

$$Y_i(\sum_{j=1}^{r} X_j^2 - \frac{X_i}{\sqrt{d_i}}) = 0. \tag{2.19b}$$

If $Y_i = 0$ for all $i = 1, ...r$ and $X_i \neq 0$ for some i, then (2.19) implies $\sum_{j=1}^{r} X_j^2 = 1 \Leftrightarrow A = \varnothing$, otherwise $X_1 = \cdots = X_r = 0$ holds.
If $Y_k \neq 0$ for $k \in A$ implies that $\sum_{j=1}^{r} X_j^2 - 1 \neq 0$. And by (2.19b) $\sum_{j=1}^{r} X_j^2 - \frac{X_k}{\sqrt{d_k}} = 0$ and $X_k \neq 0$ for $k \in A$. Solving these equations we have $X_k = \frac{\sqrt{d_k}}{\sum_{i=1}^{l} d_{m_i}}$. By (2.19a), we know that $\frac{\sqrt{d_k}}{\sum_{i=1}^{l} d_{m_i}}(\frac{1}{\sum_{i=1}^{l} d_{m_i}} - 1) + \frac{Y_k^2}{\sqrt{d_k}} = 0$. So $Y_k = \frac{\sqrt{(\sum_{i=1}^{l} d_{m_i} - 1)d_k}}{\sum_{i=1}^{l} d_{m_i}}$.

□

Remark 2.7. Consider the manifold $I \times M_1 \times ... \times M_r$ where (M_1, h_1) is the sphere with constant curvature 1 metric. Recall the smoothness conditions

$$g_1(0) = 0, \; g_i(0) > 0.(i > 1), \tag{2.20}$$

$$\dot{g}_1(0) = 1, \; \dot{g}_i(0) = 0.(i > 1). \tag{2.21}$$

They correspond in the new variable (X_i, Y_i) (2.6-2.7) to the conditions

$$X_1 = \frac{1}{\sqrt{d_1}}, \; X_i = 0.(i > 1), \tag{2.22}$$

$$Y_1 = \frac{\sqrt{d_1 - 1}}{\sqrt{d_1}}, \; Y_i = 0.(i > 1). \tag{2.23}$$

The most interesting stationary point is when $|A| = r$,

$$\{X_i = \frac{\sqrt{d_i}}{n}, Y_i = \frac{\sqrt{d_i(n-1)}}{n} \text{ for}, \forall i\}. \tag{2.24}$$

This critical point corresponds to the Ricci-flat cone (see example in Propostion 2.11). In this chapter, we will prove that this point is a global attractor by showing it possesses a Lyapunov functional.

2.3 The Ricci-flat cone of a positive Einstein metric

Proposition 2.8. *A manifold (N^n, g_N) is Einstein with constant $\lambda = n-1$ iff the corresponding cone (C_+N^n, g_c) is Ricci-flat, where $C_+N^n = \mathbb{R}_+ \times N^n$ is equipped with the metric $g_c = dt^2 + t^2 g_N$.*

Proof. We can view C_+N^n as a family of equidistant hypersurfaces. $g_c = dt^2 + t^2 g_N$ implies that $g_1(t) = t$. Assume (C_+N^n, g_c) is Ricci-flat. Then we can apply the Ricci-flat equation (1.3) on it.

$$L_t = \frac{\dot{g_1}}{g_1}\mathbb{I}_n = \frac{1}{t}\mathbb{I}_n. \tag{2.25}$$

Now,

$$\widehat{\mathrm{Ric}}(N, N) = -\mathrm{tr}(\dot{L}) - \mathrm{tr}(L^2) = -(-\frac{n}{t^2}) - \frac{n}{t^2} = 0. \tag{2.26}$$

The Ricci tensor of the metric g_c on (C_+N^n, g_c) will be

$$\begin{aligned}\widehat{\mathrm{Ric}}(X, Y) &= r_t - (\mathrm{tr}L)L - \dot{L} \\ &= r_t - \frac{n}{t}\frac{1}{t}\mathbb{I}_n - (-\frac{1}{t^2})\mathbb{I}_n \\ &= r_t + \frac{1-n}{t^2}\mathbb{I}_n.\end{aligned} \tag{2.27}$$

If $\widehat{Ric} \equiv 0$, then $r_t = \frac{n-1}{t^2}\mathbb{I}_n$ for all t. Set $t = 1 \Rightarrow r_1 = (n-1)\mathbb{I}_n$ and $t^2 g_N = g_N$. Therefore (C_+N^n, g_c) is Ricci-flat $\Leftrightarrow$ (N^n, g_N) is Einstein with constant $\lambda = n-1$. □

2.3.1 Cone metric corresponding to the stationary point - the hypersurface $S^{d_1} \times N^{d_2}$

Lemma 2.9. *Suppose that (S^{d_1}, h_1) and (N^{d_2}, h_2) are two Einstein manifolds, with Einstein constant $\lambda_1 = d_1 - 1$ and $\lambda_2 = d_2 - 1$, respectly. Then on the hyper-surface $S^{d_1} \times S^{d_2}$, there exists a Einstein metric h, such that $Ric\ (h) = (d_1 + d_2 - 1)h$.*

Proof. If there exist constants a and b such that $\mathrm{Ric}(ah_1) = (d_1+d_2-1)(ah_1)$, $\mathrm{Ric}(bh_2) = (d_1 + d_2 - 1)(bh_2)$, then

$$\mathrm{Ric}(ah_1) = \mathrm{Ric}(h_1) = (d_1 - 1)h_1 = \frac{d_1 - 1}{a}ah_1 = (d_1 + d_2 - 1)(ah_1),$$

$$\mathrm{Ric}(bh_2) = \mathrm{Ric}(h_2) = (d_2-1)h_1 = \frac{d_2-1}{b}bh_2 = (d_1+d_2-1)(bh_2).$$

This implies that $a = \frac{d_1-1}{d_1+d_2-1}$ and $b = \frac{d_2-1}{d_1+d_2-1}$. Define a metric h on $S^{d_1}\times S^{d_2}$

$$h = ah_1 \oplus bh_2 = \frac{d_1-1}{d_1+d_2-1}h_1 \oplus \frac{d_2-1}{d_1+d_2-1}h_2. \tag{2.28}$$

Therefore, $\mathrm{Ric}(h) = \mathrm{Ric}\ (ah_1\oplus bh_2) = (d_1+d_2-1)(ah_1\oplus bh_2) = (d_1+d_2-1)h.$ □

Corollary 2.10. *A cone $(\mathbb{R}_+\times S^{d_1}\times S^{d_2}, dt^2+t^2h)$ is Ricci-flat if $h = \frac{d_1-1}{d_1+d_2-1}h_1 \oplus \frac{d_2-1}{d_1+d_2-1}h_2$.*

Proof. By Lemma 2.9. □

Proposition 2.11. *If we create a cone whose metric is $dt^2+t^2h = dt^2+t^2\frac{d_1-1}{d_1+d_2-1}h_1 + t^2\frac{d_2-1}{d_1+d_2-1}h_2$, then this metric corresponds to the point*

$$(X_1,Y_1,X_2,Y_2) \quad = (\tfrac{\sqrt{d_1}}{d_1+d_2}, \tfrac{\sqrt{d_2}}{d_1+d_2}, \tfrac{\sqrt{d_1(d_1+d_2-1)}}{d_1+d_2}, \tfrac{\sqrt{d_2(d_1+d_2-1)}}{d_1+d_2}) \tag{2.29}$$

on the conservation law $\mathcal{L}=1$.

Proof. As before if we also denote the cone metric as $dt^2+g_1(t)^2h_1+g_2(t)^2h_2$, then

$$\begin{cases} g_1(t) &= t\sqrt{\frac{d_1-1}{d_1+d_2-1}},\\ g_2(t) &= t\sqrt{\frac{d_2-1}{d_1+d_2-1}},\end{cases}$$

$$\begin{cases} \dot g_1(t) &= \sqrt{\frac{d_1-1}{d_1+d_2-1}},\\ \dot g_2(t) &= \sqrt{\frac{d_2-1}{d_1+d_2-1}}.\end{cases}$$

Therefore,

$$\mathrm{tr}L = d_1\frac{\dot g_1}{g_1} + d_2\frac{\dot g_2}{g_2} = d_1\frac{1}{t}+d_2\frac{1}{t} = \frac{d_1+d_2}{t}. \tag{2.30}$$

By the change of variable (2.7)

$$X_1 = \frac{\sqrt{d_1}}{\mathrm{tr}L}\frac{\dot g_1}{g_1} = \frac{\sqrt{d_1}}{d_1+d_2}, \tag{2.31}$$

$$X_2 = \frac{\sqrt{d_2}}{\mathrm{tr}L}\frac{\dot g_2}{g_2} = \frac{\sqrt{d_2}}{d_1+d_2}, \tag{2.32}$$

$$Y_1 = \frac{\sqrt{d_1(d_1-1)}}{\mathrm{tr}L}\frac{1}{g_1} = \frac{\sqrt{d_1(d_1-1)}}{t\sqrt{\frac{d_1-1}{d_1+d_2-1}}}\frac{1}{\frac{d_1+d_2}{t}} = \frac{\sqrt{d_1(d_1+d_2-1)}}{d_1+d_2}, \tag{2.33}$$

$$Y_2 = \frac{\sqrt{d_2(d_2-1)}}{\mathrm{tr}L}\frac{1}{g_2} = \frac{\sqrt{d_2(d_2-1)}}{t\sqrt{\frac{d_2-1}{d_1+d_2-1}}}\frac{1}{\frac{d_1+d_2}{t}} = \frac{\sqrt{d_2(d_1+d_2-1)}}{d_1+d_2}. \tag{2.34}$$

□

Remark 2.12. For above proposition, when we take the multiple warped product manifold to be $\mathbb{R}^3 \times N^4$, where $d_1 = 2$ and $d_2 = 4$, then

$$(X_1, Y_1, X_2, Y_2) \quad = (\tfrac{\sqrt{d_1}}{d_1+d_2}, \tfrac{\sqrt{d_2}}{d_1+d_2}, \tfrac{\sqrt{d_1(d_1+d_2-1)}}{d_1+d_2}, \tfrac{\sqrt{d_2(d_1+d_2-1)}}{d_1+d_2}) = (\tfrac{\sqrt{2}}{6}, \tfrac{1}{3}, \tfrac{\sqrt{10}}{6}, \tfrac{\sqrt{5}}{3}).$$

In the next chapter (Figure 4.2), we will see that the numerical results actually converge to above point as $s \to +\infty$.

2.4 Local solution for X_i, Y_i

Recall the change of variables

$$dt = \frac{1}{\mathrm{tr}L} ds = \frac{1}{\sum_{i=1}^r d_i \frac{\dot{g}_i}{g_i}} ds. \tag{2.35}$$

By lemma 1.9, there exists smooth local solutions $(g_i, \dot{g}_i)$ on $[0, t_1)$ for some small $t_1 > 0$. Hence, for a small number $b > 0$, $g_i \neq 0$ for all i on $(0, b)$ and $\mathrm{tr}L = \sum_{i=1}^r d_i \frac{\dot{g}_i}{g_i} > 0$ and is finite. We denote the function $s - s_0 = \int_{t_0}^t \frac{d\tau}{\mathrm{tr}L})$ as

$$s = \phi(t), \qquad \text{on} \quad (0, b), \quad \text{where} \quad s_0 = \phi(t_0).$$

t_0 cannot be taken to zero, corresponding s_0 is finite and $\mathrm{tr}L(s_0)$=$\mathrm{tr}L(s(t_0))$=$\mathrm{tr}L(t_0)$ is finite too. Since $\frac{ds}{dt} > 0$, ϕ is invertible. Thus, $t = \phi^{-1}(s)$.

Lemma 2.13. *When $t \to 0$, then $s \to -\infty$.*

Proof. On the interval $(0, b)$ we can integrate $ds = \mathrm{tr}L dt$ because of the finiteness of $\mathrm{tr}L$.

$$\begin{aligned} s(t) - s_0 &= \int_{t_0}^t ds \\ &= \int_{t_0}^t \sum_{i=1}^r d_i \frac{\dot{g}_i}{g_i} d\tau \\ &= \sum_{i=1}^r d_i \log g_i(\tau)\Big|_{t_0}^t \\ &= \sum_{i=1}^r d_i \log g_i(t) - \log g_i(t_0) \end{aligned} \tag{2.36}$$

where $g_i(t_0) > 0$ is finite for all i, and by choosing an arbitrary $t_0 \in (0, b)$, s_0 is finite too.

Since $g_1(0) = 0$ and $g_i(0) > 0$ are finite for $i \geqslant 2$, $s(t) \to -\infty$ when $t \to 0$. $\square$

Corollary 2.14. *The ode system(2.10) has an $r-2$ parameter family of solutions $\gamma(t)$ s.t.* $\lim_{s\to-\infty} \gamma(s) = (\frac{1}{\sqrt{d_1}}, 0, \ldots, 0, \sqrt{\frac{d_1-1}{d_1}}, 0, \ldots, 0)$.

Proof. By the local existence Lemma 1.9, if $g_i, \dot{g}_i$ satisfies the smoothness condition (2.20) then there exists a local smooth solution $(g_i, \dot{g}_i)$ of the ode system 1.7 on $(0, t_0]$. The change of coordinates

$$X_i = \frac{\sqrt{d_i}}{\sum_{i=1}^r d_i \frac{\dot{g}_i}{g_i}} \frac{\dot{g}_i}{g_i}, \tag{2.37}$$

$$Y_i = \frac{\sqrt{d_i \lambda_i}}{\sum_{i=1}^r d_i \frac{\dot{g}_i}{g_i}} \frac{1}{g_i}. \tag{2.38}$$

transforms the local smooth solutions of 1.7 to smooth solution of 2.10 on $(-\infty, s_0]$. $\square$

2.4.1 Property at Initial Point $\{X_1 = \frac{1}{\sqrt{d_1}}, Y_1 = \frac{\sqrt{d_1-1}}{\sqrt{d_1}}, X_i = Y_i = 0, (i \neq 1)\}$

Linearising the Ricci-flat equation (2.10) at the stationary point $\{X_1 = \frac{1}{\sqrt{d_1}}, Y_1 = \frac{\sqrt{d_1-1}}{\sqrt{d_1}}, X_i = Y_i = 0, (i \neq 1)\}$, we obtain a system whose matrix has a 2×2 block

$$\mathbf{A_1} = \begin{pmatrix} \frac{3}{d_1} - 1 & \frac{2\sqrt{d_1-1}}{d_1} \\ \frac{\sqrt{d_1-1}}{d_1} & 0 \end{pmatrix}$$

corresponding to X_1, Y_1. The remaining entries are diagonal

$$\mathbf{A_2} = \begin{pmatrix} \frac{1}{d_i} - 1 & 0 \\ 0 & \frac{1}{d_i} \end{pmatrix}$$

corresponding to $X_i, Y_i, (i > 1)$ [DW09]. The eigenvalues of A_1 are $\frac{1}{d_1} - 1$ and $\frac{2}{d_1}$, the corresponding eigenvectors being $v_1 = (1, \frac{\sqrt{d_1-1}}{1-d_1}, 0, ..., 0)^\top$ and $v_2 = (\frac{2}{\sqrt{d_1-1}}, 1, 0, ..., 0)^\top$.

The eigenvalues of A_i are $\frac{1}{d_i} - 1$ and $\frac{1}{d_i}$ and the corresponding eigenvalues are $v_i = (0, ..., 0, 1, 0, ..., 0)^\top$, $i \geq 3$. At the point p_0, 2.22, a normal vector of the sphere is

$N = (X_1, Y_1, X_2, Y_2, ...) = (\frac{1}{\sqrt{d_1}}, \frac{d_1-1}{\sqrt{d_1}}, 0, ..., 0)$. So we have

$$\begin{aligned} < v_1, N > &= 0, \\ < v_2, N > &\neq 0, \\ < v_i, N > &= 0 \ (\forall i \geqslant 3). \end{aligned}$$

Thus, there are $r-1$ eigendirections corresponding to positive eigenvalue $\frac{1}{d_1}$, and r eigendirections corresponding to negative eigenvalue $\frac{1}{d_1} - 1$. By the Hartman-Grobman Theorem, there is an $r-2$ parameter family of solutions starting from the point p_0 and lying in the sphere $\mathcal{L} = 1$.

2.5 Long term existence

Our next Theorem 3 shows that the local solution X_i, Y_i can be extended to $[s_0, +\infty)$.

Theorem 3. *Each $\gamma(s) = (X_i(s), Y_i(s))$ extends to a trajectory defined on $(-\infty, \infty)$ and $X_i(s), Y_i(s)$ are smooth in s.*

Proof. Assume that the solution of the initial problem only exists for finite time $s \in (-\infty, s_*)$, where $s_* = \sup\ \{s | \exists \text{solution curve on } (-\infty, s)\}$. Notice that the trajectory lies on the unit sphere, which is a compact set. Therefore, the right hand side of equations (2.10) are bounded and Lipschitz continuous on $[\gamma(s_*) - \delta, \gamma(s_*) + \delta]$ for some $\delta > 0$. So by the local existence theorem [Har02], there exists a solution of the initial value problem on $(s_* - \epsilon, s_* + \epsilon)$, where $\epsilon > 0$. This contradicts the definition of s_*. Therefore, the solution exists globally. □

Remark 2.15. Notice that our system (2.10) and the soliton system ([DW09] (2.7) (2.8)) are exactly the same in the steady case. This is because the equations (1.7) for g_i involve g_i and u, but in the steady case the X_i, Y_i equations (2.10) do not explicitly involve u.

One might want to ask whether the trajectories we found in $\mathcal{L} = 1$ are steady soliton trajectories with $\dot{u} \neq 0$.

Luckily, this is not the case. From the Remark 2.14 of [DW09], we know the complete Ricci-flat trajectories lie in $\mathcal{L} = 1$. And if $\dot{u} \neq 0$ and $\varepsilon = 0$, then C is non-zero, so $\mathcal{L} \neq 1$, which is a contradiction.

Therefore, the $r-2$ parameter family of trajectories X_i, Y_i contained in $\mathcal{L} = 1$ and starting from the stationary point $(\frac{1}{\sqrt{d_1}}, \frac{\sqrt{d_1-1}}{\sqrt{d_1}}, 0, \ldots, 0)$ are Ricci-flat solutions which correspond to the $r-2$ parameter family of local smooth Ricci-flat solutions $g_i, \dot{g}_i$ by Lemma 1.9.

2.5.1 Recovering g_i from X_i, Y_i

Let $\gamma(s)$, $-\infty < s < \infty$ be one of the solution curves of (2.10).

Theorem 4. *On the trajectory, $Y_i > 0$ on $(-\infty, +\infty)$ for all i.*

Proof. Since we have a smooth local solution under the condition (2.20) on $(0, t_1)$, $g_i(t) = l_i$(finite, non zero) for all i on $(0, b], b < t_1$. Then the definition of Y_i (2.7) implies that $Y_i > 0$ for $-\infty < s < \phi(b)$.
Now from the equation (2.10b) and $X_i \leqslant 1$ we know

$$\frac{\mathrm{d}}{\mathrm{d}\, s} Y_i^2 \geqslant -\frac{2}{\sqrt{d_i}} Y_i^2. \tag{2.39}$$

By a standard comparison argument it follows if $Y_i(s_*) > 0$ then $Y_i > 0$ on $[s_*, +\infty)$ [DW09]. □

Definition 2.16. *On $[s_0, +\infty)$, define*

$$\eta(s) = Ce^{-\int_{s_0}^{s} \sum X_i^2 \,\mathrm{d}\, x},$$

where $C > 0$ is a constant and

$$t(s) = \int_{s_0}^{s} \frac{\mathrm{d}\, \sigma}{\eta(\sigma)}. \tag{2.40}$$

Then $\mathrm{d}\, t = \frac{\mathrm{d}\, s}{\eta(s)}$.
Define

$$g_i(t) \quad = \quad \frac{\sqrt{d_i \lambda_i}}{\eta(s) Y_i} \tag{2.41}$$

for $s \in [s_0, +\infty)$.

Lemma 2.17. *$\eta(s) = \mathrm{tr} L$ and the positive constant is $C = \mathrm{tr} L(s_0)$.*

Proof.

$$\begin{aligned}
\frac{\mathrm{d}\, g_i}{\mathrm{d}\, t} &= \frac{\mathrm{d}\, g_i}{\mathrm{d}\, s}\eta(s) = \sqrt{d_i\lambda_i}\eta(s)\big(-\frac{\eta'(s)}{\eta(s)^2Y_i} - \frac{Y_i'}{\eta(s)Y_i^2}\big) \\
&= \sqrt{d_i\lambda_i}\big(\frac{\eta'}{\eta Y_i} - \frac{Y_i'}{Y_i^2}\big) \\
&= -\frac{\sqrt{d_i\lambda_i}}{Y_i}\big(-\sum_i X_i^2 + (\sum_i X_i^2 - \frac{X_i}{\sqrt{d_i}})\big) \\
&= \frac{\sqrt{d_i\lambda_i}}{Y_i}\frac{X_i}{\sqrt{d_i}} = \sqrt{\lambda_i}\frac{X_i}{Y_i}.
\end{aligned}$$

This implies that

$$\frac{\dot{g}_i}{g_i} = \frac{\sqrt{\lambda_i}X_i}{Y_i}\frac{\eta Y_i}{\sqrt{d_i\lambda_i}} = \eta\frac{X_i}{\sqrt{d_i}}.$$

By $\mathcal{H} = \sum_i \sqrt{d_i}X_i = 1$, we have

$$\sum_i d_i\frac{\dot{g}_i}{g_i} = \eta(\sum_i \sqrt{d_i}X_i) = \eta.$$

Therefore, $\eta = \mathrm{tr}L$ and $C = \mathrm{tr}L(s_0)$. □

Lemma 2.18. *On the trajectory* $\sum_{i=1}^r X_i^2 \geqslant \frac{1}{n}$ *and equality holds if and only if* $X_i = \frac{\sqrt{d_i}}{n}$*, for all* $1 \leqslant i \leqslant r$.

Proof. By the Cauchy-Schwartz inequality

$$\sum_{i=1}^r X_i^2 \sum_{i=1}^r \sqrt{d_i}^2 \geqslant (\sum_{i=1}^r \sqrt{d_i}X_i)^2 = 1. \tag{2.42}$$

So $\sum_{i=1}^r X_i^2 \geqslant \frac{1}{n}$. The equality holds if and only if $X_i = \lambda\sqrt{d_i}$ for some real constant λ. With the condition $\sum_{i=1}^r \sqrt{d_i}X_i = 1$, we get $\lambda = \frac{1}{n}$ so $X_i = \frac{\sqrt{d_i}}{n}$ for all i. □

Lemma 2.19. *When* $s \to +\infty$*, then* $t \to +\infty$.

Proof. By Lemma 2.18, $\sum_{i=1}^r X_i^2 \geqslant \frac{1}{n}$. Hence,

$$\eta(s) \leqslant Ce^{-\int_{s_0}^s \frac{1}{n}\,\mathrm{d}\, x} = Ce^{-\frac{s-s_0}{n}} = Ce^{\frac{s_0}{n}}e^{-\frac{s}{n}}.$$

Integrating

$$\mathrm{d}\, t = \frac{\mathrm{d}\, s}{\eta(s)} = \frac{1}{C}e^{\int_{s_0}^s \sum X_i^2\,\mathrm{d}\, x}\,\mathrm{d}\, s \geqslant \frac{1}{Ce^{\frac{s_0}{n}}}e^{\frac{s}{n}}\,\mathrm{d}\, s, \tag{2.43}$$

we obtain

$$t(s) - t_0 = \int_{s_0}^s \mathrm{d}\, t \geqslant \int_{s_0}^s \frac{1}{Ce^{\frac{s_0}{n}}}e^{\frac{x}{n}}\,\mathrm{d}\, x = \frac{1}{nCe^{\frac{s_0}{n}}}(e^{\frac{s}{n}} - e^{\frac{s_0}{n}}).$$

Since $nC > 0$ and $e^{\frac{s_0}{n}} > 0$ are constants, $t \to +\infty$ when $s \to +\infty$. □

Since s_0 must still correspond to t_0, for the above integration, we still denote on $[s_0, +\infty)$

$$t = \phi^{-1}(s), \quad \text{and} \quad s = \phi(t).$$

2.6 Convergence of the solutions

Definition 2.20. *We define the sets*

$$\mathcal{D} := \{(X_1, Y_1, \dots X_r, Y_r) | Y_1 > 0, Y_2 > 0, ..., Y_r > 0\}, \tag{2.44}$$

$$\mathcal{E} := \{(X_1, Y_1, \dots X_r, Y_r) | \sum_{i=1}^{r} X_i^2 + Y_i^2 = 1, \sum_{i=1}^{r} \sqrt{d_i} X_i = 1\}. \tag{2.45}$$

By invariant flow Lemma 2.3 and 2.4 and Theorem 4, at certain small time $t_0 > 0$ i.e. $s \neq -\infty$ the $r-2$ parameter local smooth solution must all lie in $\mathcal{E}$ for all t because of the Einstein condition.

2.6.1 Lyapunov function

We now state Lyapunov's stability theorem:

Theorem 5 ([HS74] Page 193 Theorem 1)**.** *Let $\bar{x} \in W \subset \mathbb{R}^n$ be an equilibrium for $x' = f(x)$, where $f : W \to \mathbb{R}^n$ is a C^1 map. Let $V = U \to \mathbb{R}$ be a continuous function defined on a neighborhood $U \subset W$ of x, differentiable on $U - \bar{x}$, where $\dot{V} : U \to \mathbb{R}$ defined by $\dot{V}(x) = DV(x)(f(x))$, such that*
(a) *$V(\bar{x}) = 0$ and $V(x) > 0$ if $x \neq \bar{x}$;*
(b) *$V' \leqslant 0$ in $U - \bar{x}$.*
Then, $\bar{x}$ is stable. Furthermore, if also
(c) *$V' < 0$ in $U - \bar{x}$.*
then $\bar{x}$ is asymptotically stable.

Inspired by [DHW13] (page 49) and C.Bohm's work (cf. [Böh99] page 142 function (8)), on the set $\mathcal{D} \cap \mathcal{E}$, where $Y_i > 0$ for $\forall i$, we define

$$\mathcal{F} := \prod_{i=1}^{r} Y_i^{-\frac{2d_i}{n}} \cdot \prod_{i=1}^{r} d_i^{\frac{d_i}{n}} = \prod_{i=1}^{r} (\frac{Y_i}{\sqrt{d_i}})^{-\frac{2d_i}{n}}. \tag{2.46}$$

$\mathcal{F}$ is finite. Let $\mathcal{V} = \mathcal{F} - \frac{n^2}{n-1}$. We define the directional derivative of $\mathcal{V}$ along the solution curve by

$$\begin{aligned} \mathcal{V}' : \mathcal{D} \cap \mathcal{E} \to & \quad \mathbb{R} \\ x \longmapsto & \quad \mathcal{V}'(x) \quad = \frac{d}{ds}\Big|_{s=s_0} \mathcal{V}(\varphi_s x) \end{aligned}$$

where $\varphi_s x$ is the trajectory of (2.10) passing though $x \in \mathcal{D} \cap \mathcal{E}$ when $s = s_0$.

In the following, we will apply Lyapunov stability theorem to the function $\mathcal{V}$.

Theorem 6. *Let* $\bar{x} = \{(X_1, Y_1, ..., X_r, Y_r) | \frac{Y_i}{\sqrt{d_i}} = \frac{\sqrt{n-1}}{n}$ *and* $X_i = \frac{\sqrt{d_i}}{n}$ *for all* $i\} \in \mathcal{D} \cap \mathcal{E}$ *be the equilibrium for the Ricci-flat equation (2.10).*
Then,
(a) $\mathcal{V}(\bar{x}) = 0$ *and* $\mathcal{V}(x) > 0$ *if* $x \neq \bar{x}$*;*

(b) $\mathcal{V}' \leqslant 0$ *in* $\mathcal{D} \cap \mathcal{E} - \bar{x}$.

(c) $\bar{x}$ *is stable.*

Proof. Part (a): It is equivalent to show that the function $\mathcal{F}$ on $\mathcal{D} \cap \mathcal{E}$ has a unique minimum $\frac{n^2}{n-1}$ at the point $\frac{Y_i}{\sqrt{d_i}} = \frac{\sqrt{n-1}}{n}$ and $X_i = \frac{\sqrt{d_i}}{n}$ for all i.

First of all, we introduce the inequality, $\prod_{k=1}^n a_k^{q_k} \leqslant \sum_{k=1}^n q_k a_k$, where $a_k, q_k > 0$ for all k, and $\sum_{k=1}^n q_k = 1$. The equality holds iff a_k are identical with each other for all k, (see [HLP88] page 17). Apply the this inequality to $\frac{1}{\mathcal{F}}$. Then

$$\begin{aligned} \prod_{i=1}^r (\frac{Y_i^2}{d_i})^{\frac{d_i}{n}} &\leqslant \sum_{i=1}^r \frac{d_i}{n} \frac{Y_i^2}{d_i} \\ &= \frac{1}{n} \sum_{i=1}^r Y_i^2 \\ &= \frac{1}{n}(1 - \sum_{i=1}^r X_i^2) \quad \text{(by the conservation law)} \\ &\leqslant \frac{1}{n}(1 - \frac{1}{n}). \end{aligned} \tag{2.47}$$

So

$$\mathcal{F} \geqslant \frac{1}{\prod_{i=1}^r (\frac{Y_i^2}{d_i})^{\frac{d_i}{n}}} \geqslant \frac{1}{\frac{1}{n}(1 - \frac{1}{n})} = \frac{n^2}{n-1} \tag{2.48}$$

and the equality holds iff $\{\frac{Y_i^2}{d_i} = \frac{Y_1^2}{d_1}$ for all i and $\sum_{i=1}^r X_i^2 = \frac{1}{n}\}$. In this case $\sum_{i=1}^r Y_i^2 = 1 - \frac{1}{n}$ implies $\frac{Y_i}{\sqrt{d_i}} = \frac{\sqrt{n-1}}{n}$ for all i. By lemma (2.18), we know $X_i = \frac{\sqrt{d_i}}{n}$ for all i.

Part(b): Notice that $\mathcal{V}' = \mathcal{F}'$ is defined as the directional derivative of $\mathcal{V}$ along the trajectory (2.10).
Since $0 < Y_i < 1$ on $\mathcal{D} \cap \mathcal{E}$, we have $\mathcal{F} > 0$ by definition.
In addition,

$$\frac{\mathcal{F}'}{\mathcal{F}} = (\log\mathcal{F})' = -\sum_{i=1}^{r} \frac{2d_i}{n}(\log Y_i/\sqrt{d_i})' = -\sum_{i=1}^{r} \frac{2d_i}{n}\frac{Y_i'}{Y_i}. \tag{2.49}$$

From the Ricci-flat equation (2.10) we have $\frac{Y_i'}{Y_i} = \left(\sum_{j=1}^r X_j^2\right) - \frac{X_i}{\sqrt{d_i}}$. so $\frac{d_i Y_i'}{Y_i} = d_i \sum_{j=1}^r X_j^2 - \sqrt{d_i} X_i$. Since $\mathcal{H} = \sqrt{d_i} X_i = 1$, we obtain

$$\sum_{i=1}^{r} d_i \frac{Y_i'}{Y_i} = \sum_{i=1}^{r} d_i \Big(\sum_{j=1}^{r} X_j^2\Big) - \sum_{i=1}^{r} \sqrt{d_i} X_i = n \sum_{i=1}^{r} X_i^2 - 1. \tag{2.50}$$

So by Lemma 2.18,

$$\frac{\mathcal{F}'}{\mathcal{F}} = -2\Big(\sum_{i=1}^{r} X_i^2 - \frac{1}{n}\Big) \leqslant 0. \tag{2.51}$$

Therefore $\mathcal{V}' = \mathcal{F}' \leqslant 0$. $\mathcal{F}' = 0$ holds iff and only if for all i, $X_i = \frac{\sqrt{d_i}}{n}$.

Part(c): Using (a) and (b), by Lyapunov stability theorem 5, the equilibrium point $\bar{x}$ is stable. □

Proposition 2.21. *The trajectory converges to $\bar{x}$ as s tends to $+\infty$.*

Proof. Recall that the omega limit set of the trajectory is the set

$$\Omega = \{(X^*, Y^*) : \exists s_k \to +\infty \quad \text{with} \quad (X(s_k), Y(s_k)) \to (X^*, Y^*)\}.$$

As our trajectories lie in $\sum_i X_i^2 + Y_i^2 = 1$, a compact set, we know from the theory ([Per01] § 3.2) that Ω is a non-empty, connected, compact set that is invariant under the flow of our equations. Moreover

$$\lim_{s \to +\infty} \mathcal{F}(\gamma(s)) = \mu \geqslant 0.$$

This implies $\Omega \subset \{\mathcal{F} = \mu\}$, where μ is a constant.
Now if Ω contains a point (X^*, Y^*) with $X_i \neq \frac{\sqrt{d_i}}{n}$, we see from (2.51) that $\mathcal{F}' < 0$ at this point, so $\mathcal{F} < \mu$. However, this contradicts the flow-invariance of Ω. Hence Ω is contained in the set $X_i = \frac{\sqrt{d_i}}{n}, (i = 1, ..., r)$. Furthermore, if Ω contains a point with $Y_i \neq \frac{\sqrt{n-1}\sqrt{d_i}}{n}$ we see from (2.10a) that some $X_i' \neq 0$, so after the flow $X_i \neq \frac{\sqrt{d_i}}{n}$ which implies $\mathcal{F}' < 0$ and $\mathcal{F} < \mu$ again contradicting flow-invariance.
Hence $\Omega = \{(X^*, Y^*) | X_i = \frac{\sqrt{d_i}}{n}, Y_i = \frac{\sqrt{d_i(n-1)}}{n}, i = (1, \dots, r)\}$, and the limiting value

$\mu = \frac{n^2}{n-1}$, showing that our $r-2$ parameter family of smooth solution curves do indeed converge to the $\bar{x}$. □

2.6.2 The geometric meaning of asymptotic solutions on multiple warped product

For a product manifold $N^n = M_1 \times \cdots \times M_r$, the cone metric of C_+N^n is $dt^2 + t^2 g_N$ where $g_N = \frac{\lambda_1}{n-1} h \mid_{\mathfrak{p}_1} + \cdots + \frac{\lambda_r}{n-1} h \mid_{\mathfrak{p}_r}$. If we denote the cone metric as $dt^2 + g_1(t)^2 h \mid_{\mathfrak{p}_1} + \cdots + g_r(t)^2 h \mid_{\mathfrak{p}_r}$, then

$$\begin{cases} g_1(t) & = & \frac{\lambda_1}{n-1} t, \\ & \vdots & \\ g_r(t) & = & \frac{\lambda_r}{n-1} t. \end{cases}$$

Therefore,

$$\mathrm{tr}L = \sum_{i=1}^{r} d_i \frac{\dot{g}_i}{g_i} = \frac{\sum_{i=1}^{r} d_i}{t}. \tag{2.52}$$

By an argument similar to Lemma 2.9 and Proposition 2.8, we know that $\mathbb{R}_+ \times M_1 \times \cdots \times M_r$ with metric $\mathrm{d}t^2 + t^2 \mathrm{g}_N$ is a Ricci-flat cone.

Lemma 2.22. *When $t \gg 0$, $trL \backsim \frac{C}{t}$, where C is a positive constant.*

Proof. By (2.8),

$$\left(\frac{1}{\mathrm{tr}L}\right)^{\cdot} = -\frac{\mathrm{tr}\dot{L}}{(\mathrm{tr}L)^2} = \sum_{i=1}^{r} X_i^2.$$

When $t \to +\infty$, $\sum_i X_i^2 = \frac{1}{n}$ a constant, therefore

$$\frac{1}{\mathrm{tr}L} \backsim \frac{1}{n} t.$$

□

Theorem 7. *Our Ricci-flat metrics are asymptotically conical. More precisely, the metric corresponding to our trajectory is,*

$$\hat{g} \backsim \mathrm{d}t^2 + t^2 \mathrm{g}_\infty$$

as $t \to +\infty$, where $\mathrm{g}_\infty = \frac{\lambda_1}{n-1} h \mid_{\mathfrak{p}_1} + \cdots + \frac{\lambda_r}{n-1} h \mid_{\mathfrak{p}_r}$ is the product Einstein metric on $M_1 \times \cdots \times M_r$.

Proof. For all i

$$g_i \dot{g}_i = \frac{X_i}{Y_i^2} \frac{\sqrt{d_i}\lambda_i}{\mathrm{tr}L}.$$

As $s \to +\infty$, by Theorem 6, $X_i \to \frac{\sqrt{d_i}}{n}$ and $Y_i \to \frac{\sqrt{d_i(n-1)}}{n}$ for all i and by Lemma 2.22 $\mathrm{tr}L \backsim \frac{C}{t}$, hence,

$$\frac{1}{2}(g_i^2)^{\cdot} = g_i\dot{g}_i \backsim \frac{\lambda_i}{n-1}t. \tag{2.53}$$

So,

$$g_i(t) \backsim \sqrt{\frac{\lambda_i}{n-1}}t, \tag{2.54}$$

as $s \to +\infty$. □

Chapter 3

Ricci-flat system for the triple $(Sp(n+1), Sp(n)Sp(1), Sp(n)U(1))$

Let $G = Sp(n+1)$, $H = Sp(n) \times Sp(1)$, and $K = Sp(n) \times U(1)$, then $\mathbb{C}P^{2n+1} \cong Sp(n+1)/(Sp(n) \times U(1))$ admits two $Sp(n+1)$-invariant Einstein metrics [Zil82], and $Sp(n+1)/(Sp(n) \times Sp(1)) = \mathbb{H}P^n$. We have the natural fibration

$$\mathbb{C}P^{2n+1} \longrightarrow \mathbb{H}P^n$$

with fibre $H/K = Sp(1)/U(1) = S^2$. $H = Sp(n) \times Sp(1)$ has a non-effective orthogonal representation on the slice $V = \mathbb{R}^3 \cong \mathrm{Im}\mathbb{H}$ with cohomogeneity one, i.e., it acts transitively on the unit sphere $S^2 \subset V$. Thus the $Sp(n+1)$-invariant metrics can be obtained by changing the standard metric on $\mathbb{C}P^{2n+1}$ in the direction tangent to the fibre S^2 and scaling the metric on the base. Similar to Chapter 2, we can derive its Ricci-flat system. Then by using the two homogeneous Einstein metrics of $\mathbb{C}P^{2n+1}$ [Zil82], we can create two Ricci-flat cones with cross section $\mathbb{C}P^{2n+1}$. Our numerical results show that there is a Ricci-flat solution that converges asympotically to for all to the Ziller metric.

3.1 Computing the isotropy representation

We refer to [BtD95] for the definition of symplectic groups. Sp(n) is identified with the subgroup of U($2n$) consisting of the matrices of the form

$$A = \left(\begin{array}{cc|cc|cc} a_{11} & -\overline{b_{11}} & \ldots & \ldots & a_{1n} & -\overline{b_{1n}} \\ b_{11} & \overline{a_{11}} & \ldots & \ldots & b_{1n} & \overline{a_{1n}} \\ \hline \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ \hline a_{n1} & -\overline{b_{n1}} & \ldots & \ldots & a_{nn} & -\overline{b_{nn}} \\ b_{n1} & \overline{a_{n1}} & \ldots & \ldots & b_{nn} & \overline{a_{nn}} \end{array}\right) \in \mathrm{U}(2n).$$

By [BtD95] page 9, a unitary matrix $A \in \mathrm{U}(2n)$ is symplectic if and only if $A^T J A = J$ where $J \in \mathrm{U}(2n)$ is given by

$$J = \mathrm{diag}(E, E, ..., E), \quad E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The group Sp($n;\mathbb{C}$) consists of all $2n \times 2n$ matrices A with complex entries that satisfy the condition $A^T J A = J$, and the compact symplectic group Sp(n)=Sp($n;\mathbb{C}$) $\cap$ U($2n$). The Lie algebra $\mathfrak{sp}(n)$ of the symplectic group Sp(n) is $\{X \in \mathfrak{gl}(2n, \mathbb{C}) : X + X^* = 0 \,\text{and}\, X^T J + JX = 0\}$. The conditions $X + X^* = 0$ and $X^T J + JX = 0$ imply that X has the form

$$X = \begin{pmatrix} h_{11} & A_{12} & \ldots & A_{1n} \\ -A_{12}^* & h_{22} & \ldots & A_{2n} \\ \vdots & & \vdots & \\ -A_{1n}^* & -A_{2n}^* & \ldots & h_{nn} \end{pmatrix},$$

where

$$h_{ii} = \begin{pmatrix} ix_i & -\overline{\gamma_i} \\ \gamma_i & -ix_i \end{pmatrix} A_{ij} = \begin{pmatrix} a_{ij} & -\overline{b_{ij}} \\ b_{ij} & \overline{a_{ij}} \end{pmatrix}, x_i \in \mathbb{R}, a_{ij},\, b_{ij},\, \gamma_i \in \mathbb{C}.$$

If we choose the background bi-invariant metric of $\mathfrak{sp}(n+1)$ to be $h(X, Y) = -2\mathrm{trace}(XY)$, then $\mathfrak{sp}(n+1)$ is spanned by the orthonormal basis

$$\{\lambda_k\}_{k=1}^{2n^2+5n+3} = \{\tfrac{1}{2\sqrt{2}} \begin{pmatrix} 0 & & & \\ & & F_{jk} & \\ & -F_{jk}^* & \ddots & \\ & & & 0 \end{pmatrix}, \tfrac{1}{2} \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & E_{jj} & \\ & & & 0 \end{pmatrix}_{2n+2\times 2n+2} : E_{jj} =$$

$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ or $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ $F_{jk} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$}. Consider the Ad(K)-invariant decomposition $\mathfrak{sp}(n+1) = \mathfrak{k} \oplus \mathfrak{p}$ with $h(\mathfrak{k}, \mathfrak{p}) = 0$. Embed $U(1) \hookrightarrow Sp(1) \simeq SU(2)$ by sending $e^{i\theta} \to \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$. Then $\mathfrak{u}(1) \subset \mathfrak{sp}(1)$ by mapping $ix \to \begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix}$. The Lie algebra of the subgroup K is

$$\mathfrak{k} = \mathfrak{sp}(n) \times \mathfrak{u}(1) = \left\{ \begin{pmatrix} \mathfrak{sp}(n) & & \\ & ix & 0 \\ & 0 & -ix \end{pmatrix} \Big| x \in \mathbb{R} \right\}.$$

Then

$$\mathfrak{p} = \mathfrak{g}/\mathfrak{k} = \frac{\mathfrak{sp}(n+1)}{\mathfrak{sp}(n) \times \mathfrak{u}(1)} = \left(\begin{array}{c|cc} 0 & \multicolumn{2}{c}{C} \\ \hline -C^* & 0 & -\bar{\gamma} \\ & \gamma & 0 \end{array} \right).$$

where C is a $2 \times 2n$ complex matrix with quaternionic structure. The $\mathrm{Ad}(K)$ representation on $\mathfrak{p}$ has an h-orthogonal $\mathrm{Ad}(K)$ invariant decomposition $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ such that $\mathrm{Ad}(K)|_{\mathfrak{p}_i}$ are irreducible. To see this we first recall.

Lemma 3.1. *Let G be a compact or semisimple Lie group. If we have a complex irreducible representation W and its dual W^* of G, Then $W \oplus W^* \cong V \otimes \mathbb{C}$, where V is a real irreducible representation of G.*

Lemma 3.2 ([Sam90] page 105)**.** *Let $\mathfrak{g}$ be the direct sum of two semisimple algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Then any finite-dim, complex irreducible representation Φ of $\mathfrak{g}$ is equivalent to a tensor product of complex irreducible representations Φ_1 and Φ_2 of $\mathfrak{g}_1$ and $\mathfrak{g}_2$.*

Lemma 3.3. *Consider the cohomogeneity one space given by the triple $G = Sp(n+1)$, $H = Sp(n)Sp(1)$, $K = Sp(n)U(1)$. If we choose an $\mathrm{Ad}_G|_H$-invariant complement $\mathfrak{p}_2$ to $\mathfrak{h}$ in $\mathfrak{g}$ and an $\mathrm{Ad}_G|_K$-invariant complement $\mathfrak{p}_1$ to $\mathfrak{k}$ in $\mathfrak{h}$, then the $\mathrm{Ad}_G|_K$ invariant decomposition $\mathfrak{p}_1 \oplus \mathfrak{p}_2$ is a sum of inequivalent irreducible real representations.*

Proof. Recall the standard representation ν_n of $\mathrm{Sp}(n)$ on $\mathbb{C}^{2n}$, let ϕ denote the standardnon-trivial 1-dim complex representation of $U(1)$. ϕ^* is the representation of $U(1)$ on the dual space. Recall also that $\mathrm{Ad}_{Sp(n)} \otimes \mathbb{C} = S^2\nu_n$. We compute

$$Ad_{Sp(n+1)} \otimes \mathbb{C}|_H = S^2(\nu_n \oplus \nu_1) \cong S^2(\nu_n) \oplus S^2(\nu_1) \oplus (\nu_n \otimes \nu_1), \tag{3.1}$$

$$(\nu_n \otimes \nu_1)|_K = \nu_n \otimes (\phi \oplus \phi^*) \cong (\nu_n \otimes \phi) \oplus (\nu_n \otimes \phi^*) \cong (\nu_n \otimes \phi) \oplus (\nu_n \otimes \phi)^*. \tag{3.2}$$

Recall H is the product of two simple groups $Sp(m)$ and $Sp(1)$, ν_n is complex irreducible and ϕ is one dimensional hence irreducible. So $\nu_n \otimes \phi$ is complex irreducible by Lemma 3.2. Therefore there exists a real irreducible representation V_1 such that the $(\nu_n \otimes \phi) \oplus (\nu_n \otimes \phi)^* \cong V_1 \otimes \mathbb{C}$ by Lemma 3.1. Similarly,

$$\begin{aligned}(S^2\nu_n \oplus S^2\nu_1)|_K &= S^2\nu_n \oplus S^2(\phi \oplus \phi^*) \\ &\cong S^2\nu_n \oplus S^2\phi \oplus S^2(\phi^*) \oplus (\phi \otimes \phi^*) \\ &\cong S^2\nu_n \oplus (S^2\phi \oplus (S^2\phi)^*) \oplus (\phi \otimes \phi^*) \\ &\cong S^2\nu_n \oplus (\phi \otimes \phi^*) \oplus V_2 \otimes \mathbb{C}.\end{aligned} \tag{3.3}$$

We have $\phi \otimes \phi^* \cong \mathbb{1}$ and $S^2\phi = \phi \otimes \phi$. Since $\mathbb{C} \otimes \mathbb{C} = \mathbb{C}$ is one dimensional complex vector space, $\phi \otimes \phi$ must be complex irreducible. Then V_2 is real irreducible in the last line of (3.3) by Lemma 3.1. The first two summands form the complexified adjoint representation of K, simply because

$$Ad_K \otimes \mathbb{C} = Ad_{Sp(n)} \otimes \mathbb{C} \oplus Ad_{U(1)} \otimes \mathbb{C} = S^2\nu_n \oplus \phi \otimes \phi^*. \tag{3.4}$$

Therefore, the $\mathfrak{p}_i$ in the $\mathrm{Ad}_G|_K$-invariant decomposition $\mathfrak{p}_1 \oplus \mathfrak{p}_2$ (complement to $\mathfrak{k}$ in $\mathfrak{g}$) are individually irreducible. □

More explicitly,

$$\mathfrak{p}_1 = \left(\begin{array}{c|cc} & & \\ \hline & 0 & -\bar{\gamma} \\ & \gamma & 0 \end{array}\right), \mathfrak{p}_2 = \left(\begin{array}{c|cc} 0 & \multicolumn{2}{c}{C} \\ \hline -C^* & 0 & 0 \\ & 0 & 0 \end{array}\right).$$

The real dimension d_1=dim$(\mathfrak{p}_1) = 2$, and d_2=dim$(\mathfrak{p}_2) = 4n$. By the section 1.1.1. This is the isotropy representation of G/K.

3.2 Scalar curvature function of Sp(n+1)/Sp(n)U(1)

Lemma 3.4. *There is a unique bi-invariant metric on $Sp(n+1)$ up to scalar multiplication. $S^2\nu_n$ is irreducible.*

Lemma 3.5. *The scalar curvature formula of an invariant metric $x_1 h\,|_{\mathfrak{p}_1} \perp x_2 h\,|_{\mathfrak{p}_2}$ on the homogeneous space $\frac{Sp(n+1)}{Sp(n)U(1)}$ is given by*

$$S = \frac{2}{x_1} + \frac{2n(n+2)}{x_2} - \frac{n x_1}{2x_2^2}. \tag{3.5}$$

Proof. Let B be the negative of the Killing form of $\mathfrak{sp}(n+1)$. Then $B(X, X) > 0$, except when $X = 0$. Let basis element λ_1 be

$$\lambda_1 = \frac{1}{2}\left(\begin{array}{cc|c} i & 0 & \\ 0 & -i & \\ \hline & & \end{array}\right)_{(2n+2)\times(2n+2)}$$

$$\begin{aligned} B(\lambda_1, \lambda_1) &= -\text{trace}(ad_{\lambda_1} \circ ad_{\lambda_1}) \\ &= -\sum_{i=1}^{2n^2+5n+3} h(ad_{\lambda_1} \circ ad_{\lambda_1}(\lambda_i), \lambda_i) \\ &= -\sum_{i=1}^{2n^2+5n+3} h([\lambda_1, [\lambda_1, \lambda_i]], \lambda_i) \\ &= n + 2. \end{aligned}$$

Since $h(\lambda_1, \lambda_1) = -2\text{trace}(\lambda_1 \cdot \lambda_1) = 1$, by Lemma 3.4, $B|_{\mathfrak{p}_i} = b_i h|_{\mathfrak{p}_i}$, with

$$b_i = n + 2, \quad \forall i = 1, 2.$$

We know by the definition given in [WZ86], $\begin{bmatrix} k \\ ij \end{bmatrix} = \sum (A_{\alpha\beta}^{\gamma})^2$, $A_{\alpha\beta}^{\gamma} = h([\mathfrak{p}_\alpha, \mathfrak{p}_\beta], \mathfrak{p}_\gamma)$. $[\mathfrak{p}_1, \mathfrak{p}_1] \in \mathfrak{k}$ implies that $\begin{bmatrix} 2 \\ 11 \end{bmatrix} = 0$ and $\begin{bmatrix} 1 \\ 11 \end{bmatrix} = 0$. So by the symmetric property of $\begin{bmatrix} k \\ ij \end{bmatrix}$ in all 3 indices, we only need to compute $\begin{bmatrix} 1 \\ 22 \end{bmatrix}$.

Choose the h-orthonormal basis of $\mathfrak{p}_1$ to be

$$\{\lambda_2, \lambda_3\} = \{\frac{1}{2}\left(\begin{array}{c|cc} & & \\ \hline & 0 & -1 \\ & 1 & 0 \end{array}\right), \frac{1}{2}\left(\begin{array}{c|cc} & & \\ \hline & 0 & i \\ & i & 0 \end{array}\right)\},$$

the h-orthonormal basis of $\mathfrak{p}_2$ to be

$$\{e_k\} = \{\frac{1}{2\sqrt{2}}\left(\begin{array}{c|c} & \vdots \\ & C_i \\ & \vdots \\ \hline \dots \quad -C_i^* \quad \dots & \end{array}\right), C_i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\}$$

It is easy to compute that $\sum h([e_k, e_l], \lambda_2)^2 + h([e_k, e_l], \lambda_3)^2 = 2n$. So $\begin{bmatrix} 1 \\ 22 \end{bmatrix} = 2n$. Therefore, by the formula (1.8), the scalar curvature of the invariant metric is

$$\begin{aligned} S &= \frac{1}{2}\sum_{i=1}^{2} \frac{d_i b_i}{x_i} - \frac{1}{4}\sum_{ijk} \begin{bmatrix} k \\ ij \end{bmatrix} \frac{x_k}{x_i x_j} \\ &= \frac{1}{2}(\frac{2(n+2)}{x_1} + \frac{4n(n+2)}{x_2}) - \frac{1}{4}(2n\frac{x_1}{x_2^2} + 2n\frac{x_2}{x_1 x_2} + 2n\frac{x_2}{x_2 x_1}) \\ &= (\frac{n+2}{x_1} + \frac{2n(n+2)}{x_2} - (\frac{n x_1}{2x_2^2} + \frac{n}{x_1})) \\ &= \frac{2}{x_1} + \frac{2n(n+2)}{x_2} - \frac{n x_1}{2x_2^2}. \end{aligned}$$

□

Lemma 3.6. *For the cohomogeneity one manifold $\widehat{M}$, where the principal orbit is the homogeneous space $\frac{Sp(n+1)}{Sp(n)U(1)}$ with an invariant metric $x_1 h \mid_{\mathfrak{p}_1} \perp x_2 h \mid_{\mathfrak{p}_2}$, its Ricci-flat equation restrict to on the principal part $M_0 \cong \mathring{I} \times P$ is given by*

$$\begin{cases} 2\frac{\ddot{g}_1}{g_1} + 4n\frac{\ddot{g}_2}{g_2} = 0, \\ -\frac{\ddot{g}_1}{g_1} - \frac{\dot{g}_1^2}{g_1^2} - \frac{4n\dot{g}_1\dot{g}_2}{g_1 g_2} + \frac{1}{g_1^2} + \frac{n}{4}\frac{g_1^2}{g_2^4} = 0, \\ -\frac{\ddot{g}_2}{g_2} - \frac{(4n-1)\dot{g}_2^2}{g_2^2} - \frac{2\dot{g}_1\dot{g}_2}{g_1 g_2} + \frac{n+2}{2g_2^2} - \frac{1}{4}\frac{g_1^2}{g_2^4} = 0. \end{cases} \tag{3.6}$$

Proof. Let Ricci tensor of the homogeneous metric be given by

$$\mathrm{Ric} = \begin{pmatrix} r_1 \mathrm{I}_2 & 0 \\ 0 & r_2 \mathrm{I}_{4\mathrm{n}} \end{pmatrix}.$$

Then by (1.10) and (1.11),

$$\begin{aligned} r_1 &= -\frac{x_1^2}{d_1}\frac{\partial S}{\partial x_1} = 1 + \frac{n}{4}\frac{x_1^2}{x_2^2}, \\ r_2 &= -\frac{x_2^2}{d_2}\frac{\partial S}{\partial x_2} = \frac{n+2}{2} - \frac{1}{4}\frac{x_1}{x_2}. \end{aligned}$$

The Ricci endomporphism $r(t)$ is given $g(r(X), Y) = \mathrm{Ric}(X, Y)$, where g is the invariant metric $x_1 h \mid_{\mathfrak{p}_1} \perp x_2 h \mid_{\mathfrak{p}_2}$, so

$$r(t) = \begin{pmatrix} \frac{r_1}{x_1}\mathrm{I}_2 & 0 \\ 0 & \frac{r_2}{x_2}\mathrm{I}_{4\mathrm{n}} \end{pmatrix} = \begin{pmatrix} (\frac{1}{x_1} + \frac{n}{4}\frac{x_1}{x_2^2})\mathrm{I}_2 & 0 \\ 0 & (\frac{n+2}{2x_2} - \frac{1}{4}\frac{x_1}{x_2^2})\mathrm{I}_{4\mathrm{n}} \end{pmatrix}. \tag{3.7}$$

If we denote $g_1^2 = x_1$ and $g_2^2 = x_2$ then by $\dot{g} = 2gL$, (1.3a), (1.3b) and (1.3c) , we know the shape operator is

$$L = \begin{pmatrix} \frac{\dot{g}_1}{g_1}\mathrm{I}_2 & 0 \\ 0 & \frac{\dot{g}_2}{g_2}\mathrm{I}_{4\mathrm{n}} \end{pmatrix} \tag{3.8}$$

and the Ricci-flat equation is the system (3.6), which $\mathrm{Ric}(\frac{\partial}{\partial t}, X)$ is missing. $\square$

3.3 Initial value and local existence

By Theorem 1, given any $Sp(n+1)$-invariant metric g_Q on $Q = \mathbb{H}P^n$ and any $Sp(n+1)$-invariant homomorphism $L_1 : E \to S^2(T^*\mathbb{H}P^n)$, there exists an $Sp(n+1)$-invariant Ricci-flat metric on some open disk bundle E' of E. Since $E = Sp(n+1)\cdot V$, $Sp(n+1)$-equivariant homomorphism L_1 is really determined by a linear $Sp(n)Sp(1)$-equivariant homomorphism $\alpha : V \to S^2(T_q^*\mathbb{H}P^n)$. We will identify $T_q^*\mathbb{H}P^n$ and $T_q\mathbb{H}P^n$ by g_Q. Therefore $T_q^*\mathbb{H}P^n = T_q\mathbb{H}P^n = \mathfrak{p}_-$. So if we fix an $Sp(n+1)$-invariant metric g_Q on $\mathbb{H}P^n$, then the question of how many $Sp(n+1)$-invariant Einstein metric on E' becomes the enumeration of $Sp(n)Sp(1)$-equivariant linear maps α.

Lemma 3.7. *Consider the Lie group triple $(Sp(n+1), Sp(n)Sp(1), Sp(n)U(1))$ with cohomogeneity one structure in Theorem 1. If the actions of $H = Sp(n)Sp(1)$ on $V = \mathbb{R}^3$ by $SO(3)$ is inequivalent to the irreducible H-irreducible H-representations in $S^2(\mathfrak{p}_-)$. Then, $Sp(n+1)$-equivariant homomorphism $L_1 : E \to S^2(T^*\mathbb{H}P^n)$ must be trivial.*

Proof. We only need to show that, the action of H on V is inequivalent to any irreducible H sub-representations in $S^2(\mathfrak{p}_-)$.

$\mathbb{1} \otimes Ad_{Sp(1)}$ is the representation of $H = Sp(m) \times Sp(1)$ on $V = \mathbb{R}^3$. Here, action of $Sp(m)$ on V is trivial and action of $Sp(1)$ on $\mathbb{R}^3 \cong \mathrm{Im}\mathbb{H}$ can be viewed as $SO(3)$ acting on $\mathbb{R}^3$, which is transitive on the unit sphere $S^2 = H/K = Sp(1)/U(1) \subset \mathbb{R}^3$.

$\mathbb{1} \otimes Ad_{Sp(1)}$ is irreducible because H is a product of two simple groups $Sp(m)$ and $Sp(1)$, and both $\mathbb{1}$ and $Ad_{Sp(1)}$ are irreducible so their tensor product is an irreducible representation of H by Lemma 3.2.

By the computation (3.1) we did in Lemma 3.3, $\nu_m \otimes \nu_1$ is the isotropy representation of H on $\mathfrak{p}_-$. So $S^2(\nu_m \otimes \nu_1)$ is the representation of H on $S^2(\mathfrak{p}_-)$.

$$\begin{aligned} S^2(\nu_m \otimes \nu_1) &= (S^2\nu_m \otimes S^2\nu_1) \oplus (\Lambda^2\nu_m \otimes \Lambda^2\nu_1) \\ &= [Ad_{Sp(m)} \otimes \mathbb{C}] \otimes [Ad_{Sp(1)} \otimes \mathbb{C}] \oplus [(\mathbb{1} \oplus \Lambda_0) \otimes \mathbb{1}] \\ &= [Ad_{Sp(m)} \otimes \mathbb{C}] \otimes [Ad_{Sp(1)} \otimes \mathbb{C}] \oplus [\mathbb{1} \otimes \mathbb{1}] \oplus [\Lambda_0 \otimes \mathbb{1}]. \end{aligned}$$

Since $Sp(1)$ acts on $\Lambda^2\mathbb{C}$ by multiplication by $\det(q) = 1$, $\Lambda^2\nu_1$ must be trivial. Also, we used the fact

$$\Lambda^2\nu_m = \begin{cases} \mathbb{1} \oplus \Lambda_0 & m > 1 \\ \mathbb{1} & m = 1 \end{cases}$$

where Λ_0 is an irreducible representation. This implies $\Lambda_0 \otimes \mathbb{1}$ is irreducible. Moreover, a tensor product of trivial representations is still irreducible. Finally, notice that $Ad_{Sp(m)} \otimes \mathbb{C}$ is irreducible for all m, so $[Ad_{Sp(m)} \otimes \mathbb{C}] \otimes [Ad_{Sp(1)} \otimes \mathbb{C}]$ is irreducible by Lemma 3.2. Therefore, $S^2(\nu_m \otimes \nu_1)$ decomposes into three irreducible parts but none of them is equivalent to $\mathbb{1} \otimes Ad_{Sp(1)}$. Using Schur's lemma and an argument similar to the proof of Lemma 1.2, we see that $\mathrm{Hom}(V, S^2(\mathfrak{p}_-))^H = 0$. □

Lemma 3.8. *For the Lie group triple $(Sp(n+1), Sp(n)Sp(1), Sp(n)U(1))$ with cohomogeneity one structure in Theorem 1, the second order differential equations (3.6) have up to homothety a unique local Ricci-flat metric.*

Proof. For the Lie group triple $(G, H, K) = (Sp(m+1), Sp(m) \times Sp(1), Sp(m) \times U(1))$, the metric on M_0 has the form $dt^2 + g_1(t)^2|h_{\mathfrak{p}_+} + g_2(t)^2|h_{\mathfrak{p}_-}$ where we assume that, the metric on Q is given by $g_2(0)^2|h_{\mathfrak{p}_-}$. Here, $\mathfrak{p}_+ = \mathfrak{p}_1$, $\mathfrak{p}_- = \mathfrak{p}_2$ and $r = 2$. By Lemma 3.7, all of second fundamental forms of Q must vanish. So the shape operator of the singular orbit Q at q must vanish too, i.e.

$$\dot{g}_2(0) = 0$$

by $\dot{g} = 2gL$. By Theorem 1, there is an $r - 1 = 1$ parameter family of local Ricci-flat metrics. In addition, $\mathrm{Ric}(\lambda^2\hat{g})$=$\mathrm{Ric}(\hat{g}) = 0$. This implies if $\hat{g} = dt^2 + g(t)$ is a Ricci-flat metric then $\lambda^2\hat{g} = ds^2 + \lambda^2 g(\frac{s}{\lambda})$ is also a Ricci-flat metric where $s = \lambda t$. Moveover, let $h_1(s) = \lambda^2 g_1(\frac{s}{\lambda})$. Then $\frac{\mathrm{d}\,h_1}{\mathrm{d}\,s} = \frac{\mathrm{d}\,h_1}{\mathrm{d}\,t}\frac{\mathrm{d}\,t}{\mathrm{d}\,s} = \lambda\dot{g}_1\frac{1}{\lambda} = \dot{g}_1$. This implies that $\dot{g}_1(0) = 1 \Leftrightarrow \frac{\mathrm{d}\,h_1}{\mathrm{d}\,s}(0) = 1$. Therefore, the Ricci-flat metric on $t \in [0, t_0)$ given by $g_2^2(0)h|_{\mathfrak{p}_-}$ is same with the Ricci-flat metric on $s \in [0, \frac{t_0}{g_2^2(0)})$ given by $h|_{\mathfrak{p}_-}$. Without loss of generality, we can fix $g_2^2(0)$ to be 1. Hence, we have up to homothety a unique local Ricci-flat metric.

□

Remark 3.9. In particular, if we choose initial values given by $\tilde{x}_+(0) = I_+, x_-(0) = I_-$, then

$$\begin{cases} g_1(0) = 0 & g_2(0) \neq 0 \\ \dot{g}_1(0) = 1 & \dot{g}_2(0) = 0 \end{cases} \tag{3.9}$$

which is our necessary smoothness condition.

Lemma 3.10. *If we choose the background metric for $\mathfrak{sp}(n+1)$ to be*

$$< X, Y >= -2\, tr(X, Y), \tag{3.10}$$

then $\frac{H}{K} = \frac{Sp(1)}{U(1)} = S^2$ has constant sectional curvature 1.

Proof. For the chosen background metric, we can choose the orthonormal basis for $\mathfrak{p}_+ = \mathfrak{sp}(1)/\mathfrak{u}(1)$ to be

$$\{e_1 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, e_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\},$$

the orthonormal basis for $\mathfrak{sp}(1)$ to be

$$\{e_0 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, e_1, e_2\}. \tag{3.11}$$

Let B be the negative Killing form for $\mathfrak{sp}(1)$. By the scalar curvature formula in [Bes87] (7.39),

$$\begin{aligned}
S &= \frac{1}{2}\sum_{\alpha} B(e_\alpha, e_\alpha) - \frac{1}{4}\sum_{\alpha,\beta} < [e_\alpha, e_\beta]_{\mathfrak{p}_+}, [e_\alpha, e_\beta]_{\mathfrak{p}_+} > \\
&= \frac{1}{2}(B(e_1, e_1) + B(e_2, e_2)) - \frac{1}{4}(< [e_1, e_2]_{\mathfrak{p}_+}, [e_1, e_2]_{\mathfrak{p}_+} > + < [e_2, e_1]_{\mathfrak{p}_+}, [e_2, e_1]_{\mathfrak{p}_+} >) \\
&= \frac{1}{2}\Big(- \sum_{i=0}^{2} < [e_1, [e_1, e_i]], e_i > - \sum_{i=0}^{2} < [e_2, [e_2, e_i]], e_i > \Big) \\
&\quad - \frac{1}{4}\Big(< \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}_{\mathfrak{p}_+}, \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}_{\mathfrak{p}_+} > + < \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}_{\mathfrak{p}_+}, \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}_{\mathfrak{p}_+} > \Big) \\
&= \frac{1}{2}\Big(- < [e_1, [e_1, e_0]], e_0 > - < [e_1, [e_1, e_2]], e_2 > - < [e_2, [e_2, e_0]], e_0 > \\
&\quad - < [e_2, [e_2, e_1]], e_1 > \Big) - 0 \\
&= \frac{1}{2}(1 + 1 + 1 + 1) - 0 \\
&= 2,
\end{aligned}$$

which is the scalar curvature of the constant sectional curvature 1 metric on S^2. □

3.4 The $n = 1$ case

By starting from $\mathbb{H}P^n$, we can construct metrics of cohomogeneity one in $(4n + 3)$ dimensions with principal orbit $\mathbb{C}P^{2n+1}$. When $n = 1$, the cohomogeneity one manifold has dimension 7. The authors [BS89] and [GPP90] independently found an explicit solution corresponding to a metric with G_2 holonomy. In our notation, for an invariant metric $dt^2+g_1(t)^2h\mid_{\mathfrak{p}_1}\perp g_2(t)^2h\mid_{\mathfrak{p}_2}$ to have holomomy means that $g_1(t)$ and $g_2(t)$ satisfy the first order system

$$\begin{cases} \frac{\dot{g_1}}{g_1}+\frac{\dot{g_2}}{g_2}=-\sqrt{2}\frac{k}{g_1}, \\ \frac{\dot{g_2}}{g_2}=-\frac{kg_1}{\sqrt{2}g_2^2}. \end{cases} \tag{3.12}$$

It is easy to check that when $k^2=\frac{1}{2}$, the first order ode system (3.12) is a subsystem of Ricci-flat condition (3.6) with $n=1$.

Lemma 3.11 ([BS89] also [GPP90])**.** *The Lie group triple $G=Sp(2), H=Sp(1)Sp(1), K=Sp(1)U(1)$ has a unique (up to homothety) complete smooth G_2 holonomy metric.*

Proof. The system (3.12) becomes

$$\begin{cases} \dot{g_1} &= -\sqrt{2}k+\frac{k{g_1}^2}{\sqrt{2}{g_2}^2}, \\ \dot{g_2} &= -\frac{kg_1}{\sqrt{2}g_2}. \end{cases} \tag{3.13}$$

Plugging (3.13) into (3.6), then, we have $k^2=\frac{1}{2}$. When $k=-\frac{1}{\sqrt{2}}$, $\dot{g_1}(0)=-\sqrt{2}(-\frac{1}{\sqrt{2}})-\frac{g_1(0)^2}{2g_2(0)^2}=1$. But if $k=\frac{1}{\sqrt{2}}$, $\dot{g_1}(0)=-\sqrt{2}(\frac{1}{\sqrt{2}})+\frac{g_1(0)^2}{2g_2(0)^2}=-1$. So, k has to be $-\frac{1}{\sqrt{2}}$ in order satisfy our smoothness condition.

So,

$$\begin{cases} \dot{g_1} &= 1-\frac{{g_1}^2}{2{g_2}^2}, \\ \dot{g_2} &= \frac{g_1}{2g_2}. \end{cases} \tag{3.14}$$

Then, $\frac{\mathrm{d}g_1}{\mathrm{d}g_2}=\frac{2g_2}{g_1}-\frac{g_1}{g_2}\Rightarrow g_1g_2\mathrm{d}g_1=(2{g_2}^2-{g_1}^2)\mathrm{d}g_2\Rightarrow$

$$g_1g_2\mathrm{d}g_1+({g_1}^2-2{g_2}^2)\mathrm{d}g_2=0. \tag{3.15}$$

Call $P=g_1g_2, R={g_1}^2-2{g_2}^2$. Then, $\frac{\partial P}{\partial g_2}=g_1\neq\frac{\partial R}{\partial g_1}=2g_1$. We can use integrating factor $\mu=e^{\int\varphi(y)\mathrm{d}y}$, where $\varphi(y)=\frac{P_{g_2}-R_{g_1}}{-P}=\frac{\frac{\partial g_1g_2}{\partial g_2}-\frac{\partial({g_1}^2-2{g_2}^2)}{2g_1}}{-g_1g_2}=\frac{g_1-2g_1}{-g_1g_2}=\frac{1}{g_2}$. So, $\mu=e^{\int\frac{1}{g_2}\mathrm{d}g_2}=C_1e^{\ln g_2}=C_1g_2$. Multipling (3.15) by μ, we obtain $C_1g_1{g_2}^2\mathrm{d}g_1+C_1({g_1}^2g_2-2{g_2}^3)\mathrm{d}g_2=0$. So, there is a smooth function $f(g_1,g_2)$ such that $\mathrm{d}f=C_1g_1{g_2}^2\mathrm{d}g_1+C_1({g_1}^2g_2-2{g_2}^3)\mathrm{d}g_2=0$. Integrating, $f(g_1,g_2)=\int\mathrm{d}f=C_1\frac{1}{2}{g_1}^2{g_2}^2-C_1\frac{1}{2}{g_2}^4+C_2$. Since $\mathrm{d}f=0$, f is a constant function, i.e. ${g_1}^2{g_2}^2-{g_2}^4=C.\Rightarrow$

$$g_1=\pm\sqrt{\frac{C+{g_2}^4}{{g_2}^2}},\ \dot{g_2}=\pm\frac{1}{2}\sqrt{\frac{C+{g_2}^4}{{g_2}^4}}. \tag{3.16}$$

By the smoothness condition $g_1(0) = 0$, we have $g_1(0) = \pm g_2\sqrt{1+Cg_2{}^{-4}} \Rightarrow C = -g_2(0)^4 \Rightarrow$

$$g_1(g_2) = \pm g_2\sqrt{1-\left(\frac{g_2(0)}{g_2}\right)^4}, \tag{3.17}$$

In order that g_1 is real, we must have

$$g_2 \geqslant g_2(0) > 0. \tag{3.18}$$

So, $\frac{\mathrm{d}t}{\mathrm{d}g_2} = 2\sqrt{\frac{1}{1+Cg_2{}^{-4}}} = 2\sqrt{\frac{1}{1-(g_2(0)/g_2)^4}} > 0$, i.e. $t = T(g_2) = \int 2\sqrt{\frac{1}{1-(g_2(0)/g_2)^4}}\mathrm{d}g_2$ is monotone increasing. By the inverse function theorem. $\frac{\mathrm{d}g_2}{\mathrm{d}t} = \frac{1}{2}\sqrt{1-(g_2(0)/g_2)^4} > 0$, so the inverse function $g_2 = B(t)$ is monotone increasing too. Therefore, $g_2 \to g_2(0)$ as $t \to 0$ implies that $t \to 0$ as $g_2 \to g_2(0)$.

On the other hand, when $g_2 \to \infty$, $\frac{\mathrm{d}t}{\mathrm{d}g_2} = 2\sqrt{\frac{1}{1-(g_2(0)/g_2)^4}} \to 1$, so $t \to \infty$.

We still need to verify that $\dot{g}_2(0) = 0$. This is indeed true, as $\dot{g}_2(0) = \frac{g_1(0)}{2g_2(0)} = 0$, when $g_1(0) = 0, g_2(0) > 0$.

Hence, we have the following one-parameter global solutions of (3.14)

$$g_1(g_2) = \pm g_2\sqrt{1-(\frac{g_2(0)}{g_2})^4}, \tag{3.19}$$

$$t(g_2) = \int_{g_2(0)}^{g_2} 2\sqrt{\frac{1}{1-(\frac{g_2(0)}{r})^4}}dr. \tag{3.20}$$

where $g_2 \geqslant g_2(0) > 0$. The solution satisfes our smoothness conditions for 2-dimensional second order Ricci-flat system(3.6).

□

Proposition 3.12. *The Lie group triple $G = Sp(2), H = Sp(1)Sp(1), K = Sp(1)U(1)$ has unique (up to homothety) global Ricci-flat metric of cohomogeneity one and this metric has G_2 holonomy.*

Proof. By Lemma 3.8, there is a unique (up to homothety) Ricci-flat metric on $t \in [0, t_1)$ satisfing the smoothness condition

$$g_1(0) = 0, \quad \dot{g}_1(0) = 1, \quad g_2(0) > 0, \quad \dot{g}_2(0) = 0.$$

Since a G_2 holonomy metric is Ricci-flat, by uniqueness, this local solution must have G_2 holonomy. By Lemma 3.11, this G_2 holonomy metric exists for all $t \in [0, +\infty)$. Hence, the Ricci-flat metric is defined for all time. □

3.5 Einstein metrics on the principal orbit

From ([Bes87] 4.23) and [Zil82], we know there are only two $Sp(n+1)$ invariant Einstein metrics on the hypersurface $\mathbb{C}P^{2n+1} = \frac{Sp(n+1)}{Sp(n)U(1)}$. If the metric on the principal orbit is $x_1 h\mid_{\mathfrak{p}_1} + x_2 h\mid_{\mathfrak{p}_2}$, then the two homogeneous Einstein metrics are given by

$$\begin{cases} x_1 &= 2^{\frac{2n}{2n+1}}, \\ x_2 &= (\frac{1}{2})^{\frac{1}{2n+1}}, \end{cases} \tag{3.21}$$

and,

$$\begin{cases} x_1 &= (\frac{2}{n+1})^{\frac{2n}{2n+1}}, \\ x_2 &= (\frac{n+1}{2})^{\frac{1}{2n+1}}. \end{cases} \tag{3.22}$$

They are neither isometric nor homothetic.

Definition 3.13. *Let (M, g) and $(N, \breve{g})$ be two Riemannian manifolds. Let $\pi : M \to N$ be a Riemannian submersion. The canonical variation g_t of the metric g on N in defined by*

$$g_t \restriction \mathfrak{p}_+ \quad = tg \restriction \mathfrak{p}_+, \tag{3.23}$$

$$g_t \restriction \mathfrak{p}_- \quad = g \restriction \mathfrak{p}_-, \tag{3.24}$$

$$g_t(\mathfrak{p}_+, \mathfrak{p}_-) \quad = 0. \tag{3.25}$$

where $\mathfrak{p}_+$ is the vertical distribution (the tangent spaces to the fibres), and $\mathfrak{p}_-$ is the horizontal distribution. In the following we let M be $\mathbb{C}P^{2n+1}$ with the Fubini-Study metric and N be $\mathbb{H}P^n$. π is the map that takes a complex subspace to the corresponding quaternion line.

Following the notation by the Main Technical Lemma [Bes87] (9.74), when $t = 1$, it gives the standard metric on $\mathbb{C}P^{2n+1}$. When $t = 1/(n+1)$ gives a second Einstein metric on $\mathbb{C}P^{2n+1}$, called the Ziller metric [Zil82]. For the first pair of solution (3.21) $\frac{x_1}{x_2} = \frac{2^{2n/2n+1}}{2^{-1/2n+1}} = 2$, and for the second pair of solutions (3.22) $\frac{x_1'}{x_2'} = \frac{\frac{2}{n+1}^{2n/2n+1}}{\frac{2}{n+1}^{-1/2n+1}} = \frac{2}{n+1}$. So

$$\frac{x_1'}{x_2'} = \frac{1}{n+1}\frac{x_1}{x_2}. \tag{3.26}$$

Therefore, the second pair of solution (x_1', x_2') (3.22) is the Ziller metric and the first pair of solution (x_1, x_2) (3.21) is the Fubini-Study metric.

3.6 Change variables

In order to obtain numerical solutions and understand the asymptotic behavior, we change the variables to simplify the Ricci-flat system as in the multiple warped product situation.

Let

$$X_i = \frac{\sqrt{d_i}}{\mathrm{tr}L}\frac{\dot{g}_i}{g_i}, \tag{3.27}$$

$$Y_i = \frac{\sqrt{d_i}}{g_i}\frac{1}{\mathrm{tr}L}. \tag{3.28}$$

for $i = 1, 2$ and $d_1 = 2, d_2 = 4n$. Notice that

$$\sum_{j=1}^{r} X_j^2 = \frac{-\mathrm{tr}(\dot{L})}{(\mathrm{tr}L)^2} = \frac{\mathrm{tr}(L^2)}{(\mathrm{tr}L)^2}, \tag{3.29}$$

$$\sum_{j=1}^{r} Y_j^2 = \frac{\mathrm{tr}(r_t)}{(\mathrm{tr}L)^2}. \tag{3.30}$$

We obtain from the Ricci-flat system (3.6) the following equations in our new variables.

$$X_1' = X_1(\sum_{j=1}^{r} X_j^2 - 1) + \frac{Y_1^2}{\sqrt{2}} + \frac{\sqrt{2}}{64n}\frac{Y_2^4}{Y_1^2}, \tag{3.31a}$$

$$X_2' = X_2(\sum_{j=1}^{r} X_j^2 - 1) + \frac{(n+2)Y_2^2}{4\sqrt{n}} - \frac{\sqrt{n}}{16n^2}\frac{Y_2^4}{Y_1^2}, \tag{3.31b}$$

$$Y_1' = Y_1(\sum_{j=1}^{r} X_j^2 - \frac{X_1}{\sqrt{2}}), \tag{3.31c}$$

$$Y_2' = Y_2(\sum_{j=1}^{r} X_j^2 - \frac{X_2}{\sqrt{4n}}). \tag{3.31d}$$

The conservation law is

$$Y_1^2 + \frac{(n+2)Y_2^2}{2} - \frac{1}{16n}\frac{Y_2^4}{Y_1^2} + X_1^2 + X_2^2 - 1 = 0. \tag{3.32}$$

Notice that the quadratic form is no longer positive definite. So,the conservation law no longer gives a compact hypersurface in phase space.

Proposition 3.14. *The equilibrium points (X_1, X_2, Y_1, Y_2) of the first order ode system (3.31) lying in (3.32) are given by* $\left(\frac{\sqrt{2}}{2+4n}, \frac{\sqrt{4n}}{2+4n}, \frac{1}{\sqrt{2}}\sqrt{\frac{4n+1}{(2n+1)^2(n+1)}}, \sqrt{\frac{2n(4n+1)}{(2n+1)^2(n+1)}}\right)$, $\left(\frac{\sqrt{2}}{2+4n}, \frac{\sqrt{4n}}{2+4n}, \frac{1}{2}\sqrt{\frac{2(n+1)^2(4n+1)}{(2n+1)(2n^3+7n^2+5n+1)}}, \sqrt{\frac{2n(4n+1)(n+1)}{(2n+1)(2n^3+7n^2+5n+1)}}\right)$*, and* $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right)$.

Proof. A stationary point corresponds to zeros of vector field given by (3.31). Assume $B = Y_1^2 + \frac{(n+2)Y_2^2}{2} - \frac{1}{16n}\frac{Y_2^4}{Y_1^2}$, so $B + X_1^2 + X_2^2 = 1$.
If $X_2 = 0$ and $Y_2 = 0$, then by solving (3.31) we get

$$X_1 = \frac{1}{\sqrt{2}}, Y_1 = \frac{1}{\sqrt{2}}.$$

If $X_2 = 0$ and $Y_2 \neq 0$, we can derive a contradiction. If $X_2 \neq 0$ and $B = 0$, we also get a contradiction. Therefore, if $X_2 \neq 0$, B must be non-zero, i.e. $Y_2 \neq 0$ and $Y_1 \neq 0$. Then,

$$\begin{cases} X_1^2 + X_2^2 - \frac{X_1}{\sqrt{2}} = 0, \\ X_1^2 + X_2^2 - \frac{X_2}{\sqrt{4n}} = 0, \end{cases} \tag{3.33}$$

which implies $X_1 = \frac{X_2}{\sqrt{2n}}$. Through (3.31), we obtain $Y_1^2 = \frac{1}{4n}Y_2^2$ or $Y_1^2 = \frac{n+1}{4n}Y_2^2$. Plug these into (3.32), so

$$\begin{cases} Y_1 = \frac{1}{\sqrt{2}}\sqrt{\frac{4n+1}{(2n+1)^2(n+1)}}, \\ Y_2 = \sqrt{\frac{2n(4n+1)}{(2n+1)^2(n+1)}}, \end{cases} \tag{3.34}$$

or,

$$\begin{cases} Y_1 = \frac{1}{2}\sqrt{\frac{2(n+1)^2(4n+1)}{(2n+1)(2n^3+7n^2+5n+1)}}, \\ Y_2 = \sqrt{\frac{2n(4n+1)(n+1)}{(2n+1)(2n^3+7n^2+5n+1)}}. \end{cases} \tag{3.35}$$

□

Remark 3.15. The smoothness conditions are

$$g_1(0) = 0, \; g_2(0) > 0. \tag{3.36}$$

$$\dot{g}_1(0) = 1, \; \dot{g}_2(0) = 0. \tag{3.37}$$

In the new variable (X_i, Y_i) the above initial values determine the equilibrium point

$$(X_1, X_2, Y_1, Y_2) = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0). \tag{3.38}$$

3.7 Ricci-flat cone solutions

In Chapter 2 Lemma 2.9, we formed a Ricci-flat cone using the product Einstein metric and showed that it corresponds to one of the equilibrium points of the X_i, Y_i-system. Now we turn our attention to the cohomogeneity one manifold whose hypersurfaces are $\mathbb{C}P^{2n+1}$. By discussions in section 3.5 and Proposition 2.8, we can use the two homogeneous Einstein metrics on $\mathbb{C}P^{2n+1}$ to form two Ricci-flat cones $C_+\mathbb{C}P^{2n+1} = \mathbb{R}_+ \times \mathbb{C}P^{2n+1}$. Since $(\mathbb{C}P^{2n+1}, g_E)$ is Einstein, $\mathrm{Ric}(g_E) = \Lambda g_E$,

$$Ric(\lambda g_E) = Ric(g_E) = \Lambda g_E = \frac{\Lambda}{\lambda}(\lambda g_E). \tag{3.39}$$

By Proposition 2.8, the cone $(\mathbb{R}_+ \times \mathbb{C}P^{2n+1}, g_c)$ $g_c = dt^2 + t^2\lambda g_E$ is Ricci-flat if and only if $\frac{\Lambda}{\lambda} = m-1$, where $m = 4n+2$.

Proposition 3.16. *For the two homogeneous Einstein metrics g_{FS} (3.21) and g_Z (3.22) of $\mathbb{C}P^{2n+1}$, the corresponding cone metrics $dt^2+t^2\lambda g_E$ correspond to the following equilibrium points in the X_i, Y_i space:*

$$(X_1, X_2, Y_1, Y_2) = (\tfrac{\sqrt{d_1}}{d_1+d_2}, \tfrac{\sqrt{d_2}}{d_1+d_2}, \tfrac{1}{\sqrt{2}}\sqrt{\tfrac{4n+1}{(2n+1)^2(n+1)}}, \sqrt{\tfrac{2n(4n+1)}{(2n+1)^2(n+1)}}),$$
$$(X_1, X_2, Y_1, Y_2) = (\tfrac{\sqrt{d_1}}{d_1+d_2}, \tfrac{\sqrt{d_2}}{d_1+d_2}, \tfrac{1}{2}\sqrt{\tfrac{2(n+1)^2(4n+1)}{(2n+1)(2n^3+7n^2+5n+1)}}, \sqrt{\tfrac{2n(4n+1)(n+1)}{(2n+1)(2n^3+7n^2+5n+1)}}),$$

respectively. Both of these two points lie on the conservation law hypersurface $\mathcal{L} = 1$.

Proof.

$$\Lambda = \frac{S_g}{4n+2} = \frac{\frac{2}{x_1} + \frac{n(2n+4)}{x_2} - \frac{n}{2}\frac{x_1}{x_2^2}}{4n+2}. \tag{3.40}$$

plugging the first pair of solution (Fubini-Study metric) (3.21) into the equality (3.40), we get

$$\begin{aligned}\lambda &= \frac{\Lambda}{4n+2-1}\\ &= \frac{S_g}{(4n+2)(4n+1)}\\ &= (\frac{2}{2^{\frac{2n}{2n+1}}} + \frac{n(2n+4)}{(1/2)^{\frac{1}{2n+1}}} - \frac{n}{2}\frac{2^{\frac{2n}{2n+1}}}{(1/2)^{\frac{2}{2n+1}}})\frac{1}{(4n+2)(4n+1)}\\ &= \frac{(2n+1)(n+1)}{(4n+2)(4n+1)}2^{\frac{1}{2n+1}}\end{aligned} \tag{3.41}$$

So,

$$\lambda g_E = \begin{pmatrix} \lambda x_1 h\mid_{\mathfrak{p}_1} & 0 \\ 0 & \lambda x_2 h\mid_{\mathfrak{p}_2} \end{pmatrix}$$

If we denote the cone metric as $dt^2 + t^2\lambda g_E$, then by section 2.3.1, $g_1 = t\sqrt{\lambda x_1}, g_2 = t\sqrt{\lambda x_2}, \mathrm{tr}L = \frac{2+4n}{t}$, since $d_1 = 2, d_2 = 4n$.
By the change of variable (2.7)

$$X_1 = \frac{\sqrt{d_1}}{\mathrm{tr}L}\frac{\dot{g}_1}{g_1} = \frac{\sqrt{d_1}}{d_1+d_2}, \tag{3.42}$$

$$X_2 = \frac{\sqrt{d_2}}{\mathrm{tr}L}\frac{\dot{g}_2}{g_2} = \frac{\sqrt{d_2}}{d_1+d_2}, \tag{3.43}$$

$$Y_1 = \frac{\sqrt{d_1}}{trL}\frac{1}{g_1} = \frac{\sqrt{2}}{\frac{2+4n}{t}}\frac{1}{t\sqrt{\lambda x_1}} = \frac{1}{\sqrt{2}}\sqrt{\frac{(4n+1)}{(2n+1)^2(n+1)}}, \tag{3.44}$$

$$Y_2 = \frac{\sqrt{d_2}}{trL}\frac{1}{g_2} = \frac{\sqrt{4n}}{\frac{2+4n}{t}}\frac{1}{t\sqrt{\lambda x_2}} = \sqrt{\frac{2n(4n+1)}{(2n+1)^2(n+1)}}. \tag{3.45}$$

Similarly, plugging the second pair of solutions (Ziller metric)(3.22) into the equality (3.40), we get

$$\begin{aligned}\lambda &= \frac{S_g}{(4n+2)(4n+1)} \\ &= \Big(\frac{2}{(2/n+1)^{\frac{2n}{2n+1}}} + \frac{n(2n+4)}{(n+1/2)^{\frac{1}{2n+1}}} - \frac{n}{2}\frac{(2/n+1)^{\frac{2n}{2n+1}}}{((n+1/2)^{\frac{2}{2n+1}}}\Big)\frac{1}{(4n+2)(4n+1)} \\ &= \frac{2n^3+7n^2+5n+1}{(4n+2)(4n+1)}2^{\frac{1}{2n+1}},\end{aligned} \tag{3.46}$$

and,

$$X_1 = \frac{\sqrt{d_1}}{\mathrm{tr}L}\frac{\dot{g}_1}{g_1} = \frac{\sqrt{d_1}}{d_1+d_2} = \frac{\sqrt{2}}{2+4n}, \tag{3.47}$$

$$X_2 = \frac{\sqrt{d_2}}{\mathrm{tr}L}\frac{\dot{g}_2}{g_2} = \frac{\sqrt{d_2}}{d_1+d_2} = \frac{2\sqrt{n}}{2+4n}, \tag{3.48}$$

$$Y_1 = \frac{\sqrt{d_1}}{trL}\frac{1}{g_1} = \frac{\sqrt{2}}{\frac{2+4n}{t}}\frac{1}{t\sqrt{\lambda x_1}} = \frac{1}{2}\sqrt{\frac{2(n+1)(4n^2+5n+1)}{4n^4+16n^3+17n^2+7n+1}}, \tag{3.49}$$

$$Y_2 = \frac{\sqrt{d_2}}{trL}\frac{1}{g_2} = \frac{\sqrt{4n}}{\frac{2+4n}{t}}\frac{1}{t\sqrt{\lambda x_2}} = \sqrt{\frac{2n(4n^2+5n+1)}{4n^4+16n^3+17n^2+7n+1}}. \tag{3.50}$$

□

Remark 3.17. When $n = 1$, the second solution corresponds to $Y_1 = \frac{\sqrt{2}}{3}$ $Y_2 = \frac{2}{3}$. $X_1 = \frac{\sqrt{2}}{6}, X_2 = \frac{1}{3}$, which is the point to which our solution converges (see Figure(4.7)).

Chapter 4

Numerical solutions

4.1 Warped product $\mathbb{R}^3 \times N^4$, N^4 is an Einstein manifold

By 2.10, when $r = 2$, $d_1 = 2$ and $d_2 = 4$, the Ricci-flat system becomes

$$\begin{cases} X_1' = X_1(X_1^2 + X_2^2 - 1) + \frac{Y_1^2}{\sqrt{2}} \\ X_2' = X_2(X_1^2 + X_2^2 - 1) + \frac{Y_2^2}{\sqrt{4}} \\ Y_1' = Y_1(X_1^2 + X_2^2 - \frac{X_1}{\sqrt{2}}) \\ Y_2' = Y_2(X_1^2 + X_2^2 - \frac{X_2}{\sqrt{4}}) \end{cases}$$

The conservation law is given by

$$X_1^2 + X_2^2 + Y_1^2 + Y_2^2 = 1. \tag{4.1}$$

The solution lies in the conservation law hypersurface and the vector field is tangent to the conservation law's hypersurface. We use (4.1) to solve for X_1 and plug into above dynamic system. We obtain

$$\begin{cases} X_2' = X_2(-Y_1^2 - Y_2^2) + \frac{Y_2^2}{\sqrt{4}} \\ Y_1' = Y_1(1 - Y_1^2 - Y_2^2 - \frac{\sqrt{1 - X_2^2 - Y_1^2 - Y_2^2}}{\sqrt{2}}) \\ Y_2' = Y_2(1 - Y_1^2 - Y_2^2 - \frac{X_2}{\sqrt{4}}) \end{cases} \tag{4.2}$$

We will use the `ode45` solver in Mathlab to solve the above ode system (4.2). The `ode45` solver uses Dormand-Prince pairs (DOPRI5). The local truncation error is $\mathcal{O}(h^5)$ [DP80], which implies that its global truncation error is $\mathcal{O}(h^4)$.

We take the initial values nearby the point $(X_1, X_2, Y_1, Y_2) = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0)$, for example at $s_0 = 0$,

$$(X_2, Y_1, Y_2) = (0, \frac{1}{\sqrt{2}}, 0.001). \tag{4.3}$$

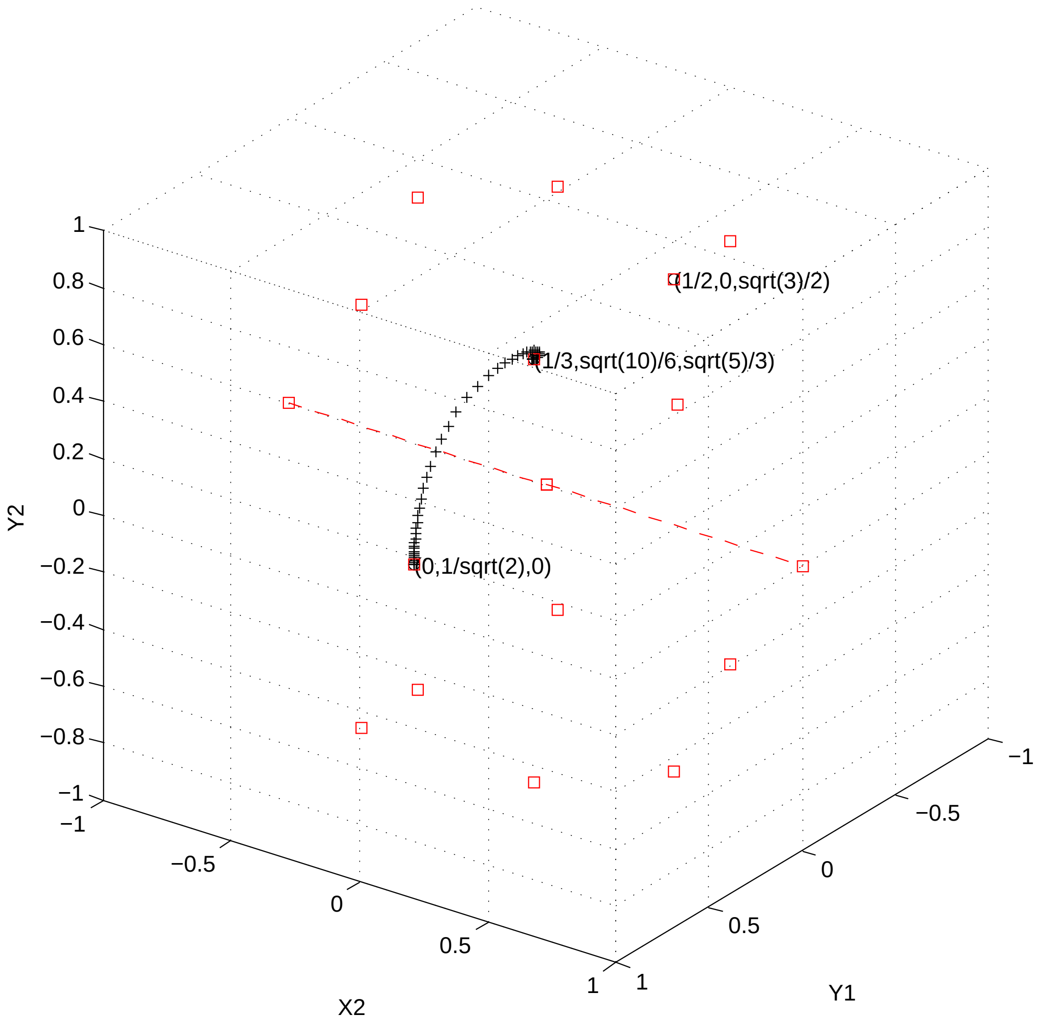

FIGURE 4.1: Starting from $(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0)$

TABLE 4.1: Asymptotical point for $\mathbb{R}^3 \times N^4$ - initial $(0, \frac{1}{\sqrt{2}}, 0.001), s_0 = 0$

	X_1	X_2	Y_1	Y_2
$s \to -\infty$	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	0
$s \to +\infty$	$\frac{\sqrt{2}}{6}$	$\frac{1}{3}$	$\frac{\sqrt{10}}{6}$	$\frac{\sqrt{5}}{3}$
$s_* = 100 >> 0$		0.3334	0.5270	0.7453
e		3.8779×10^{-5}	1.3264×10^{-5}	1.6187×10^{-5}

The global truncation error is $e = \left|(X_i, Y_i) - \text{numercial results}(X_i(s_*), Y_i(s_*))\right|$.

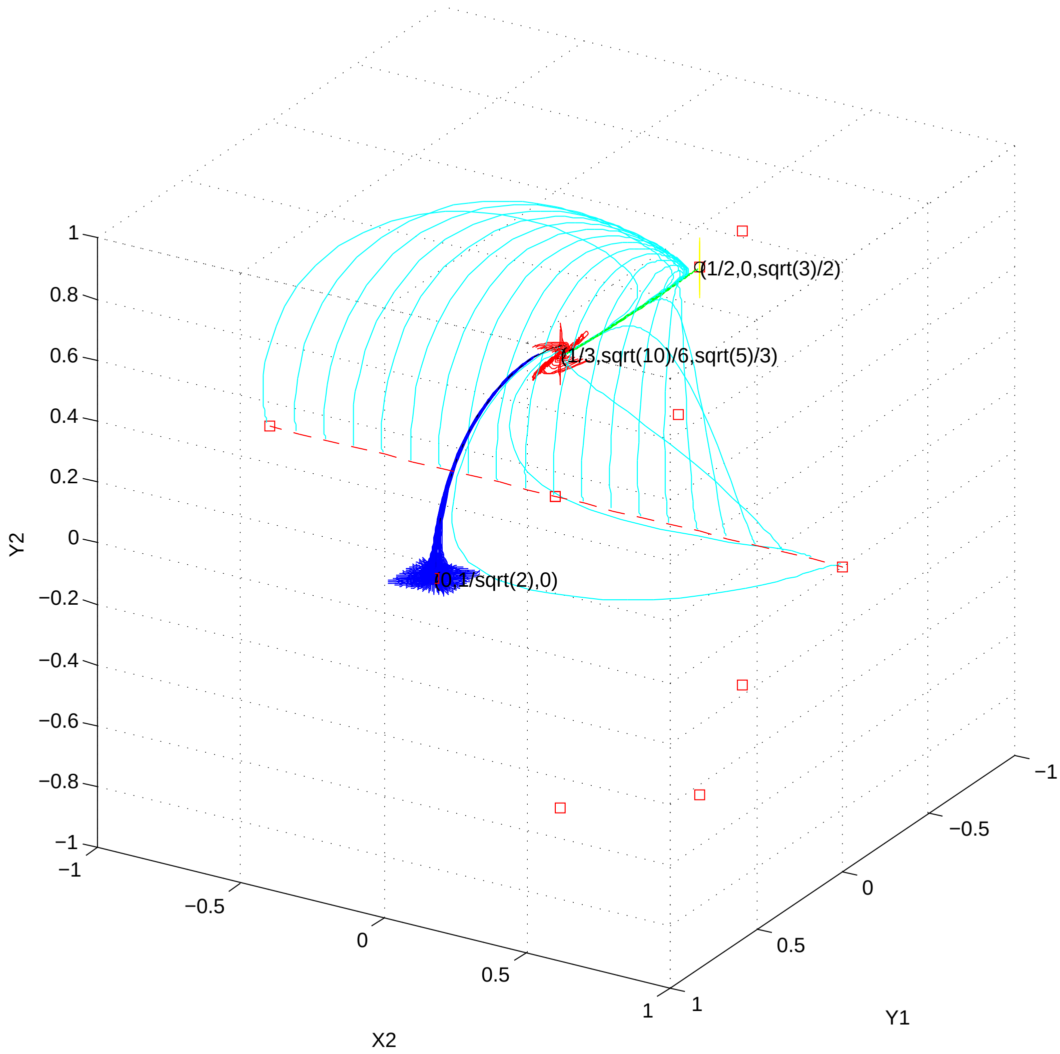

FIGURE 4.2: $\mathbb{R}^3 \times N^4$

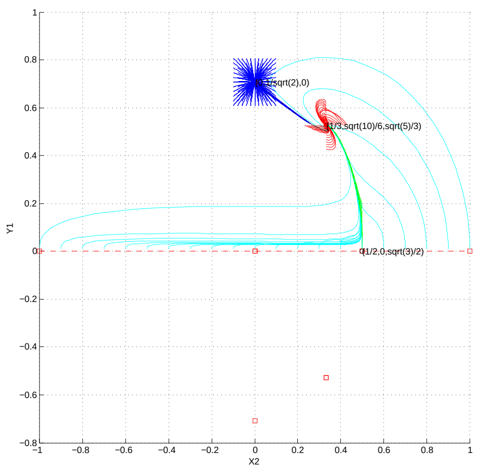

FIGURE 4.3: $\mathbb{R}^3 \times N^4, X_2Y_1$ plane

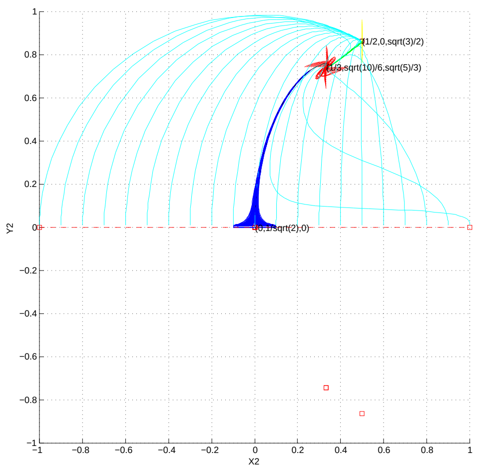

FIGURE 4.4: $\mathbb{R}^3 \times N^4, X_2Y_2$ plane

Remark 4.1. From the picture, the solutions starting near the initial value $(X_2, Y_1, Y_2) = (0, \frac{1}{\sqrt{2}}, 0)$ appear to converge to $(X_2, Y_1, Y_2) = (\frac{1}{3}, \frac{\sqrt{10}}{6}, \frac{\sqrt{5}}{3})$. Moreover, by examing the picture of trajectories near $(\frac{1}{3}, \frac{\sqrt{10}}{6}, \frac{\sqrt{5}}{3})$, this point looks asymptotically stable. This obstervation is compatible with Theorem 6.

4.2 Example $Sp(n+1), Sp(n)Sp(1), Sp(n)U(1)$

As $n = 1$, we solve (3.13) with initial values $g_1(0) = 0, \dot{g}_1(0) = 1, g_2(0) > 0, \dot{g}_2(0) = 0$ by ode 45 solver

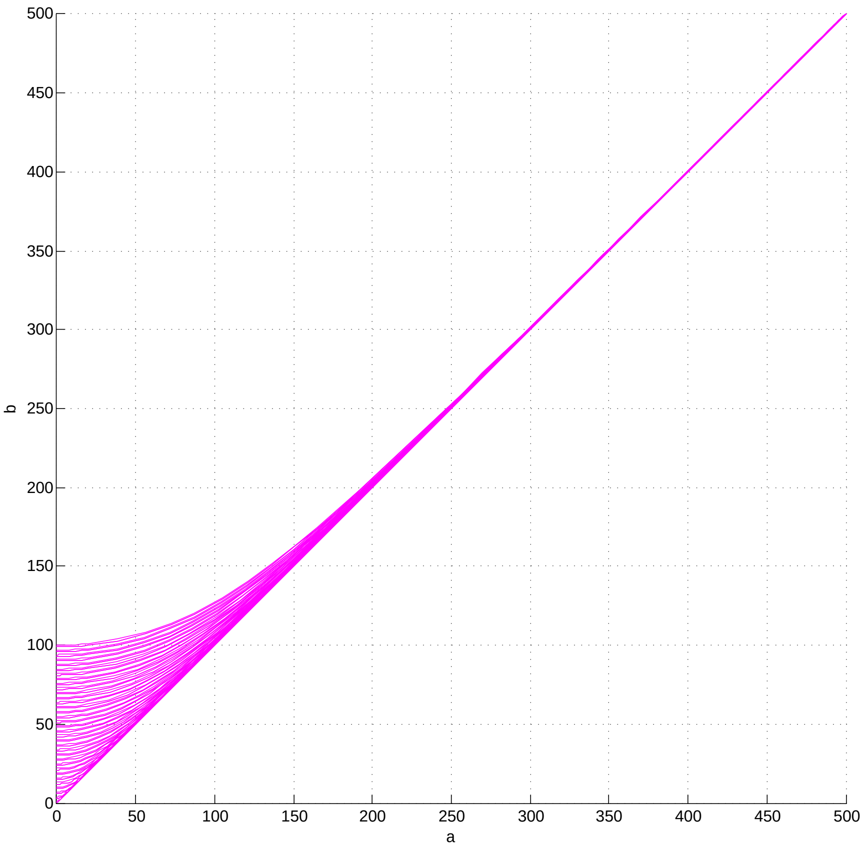

FIGURE 4.5: Sp(2)/Sp(1)U(1)

TABLE 4.2: The slope at $t >> 0$, if $n = 1$

Initial value $b(0)$	70	60	50	40	30	20	10
Slope $\frac{dg_1}{dg_2}$ at $t = 76$	0.9996	0.9998	0.9999	0.9999	1	1	1

Remark 4.2. While $n = 1$, the Fubini-Study metric in (3.21) is $x_1 = 2^{\frac{2}{3}}$ and $x_2 = 2^{-\frac{1}{3}}$ and Ziller metric in (3.22) is $x_1 = x_2 = 1$. So by section 3.7 the Ziller cone

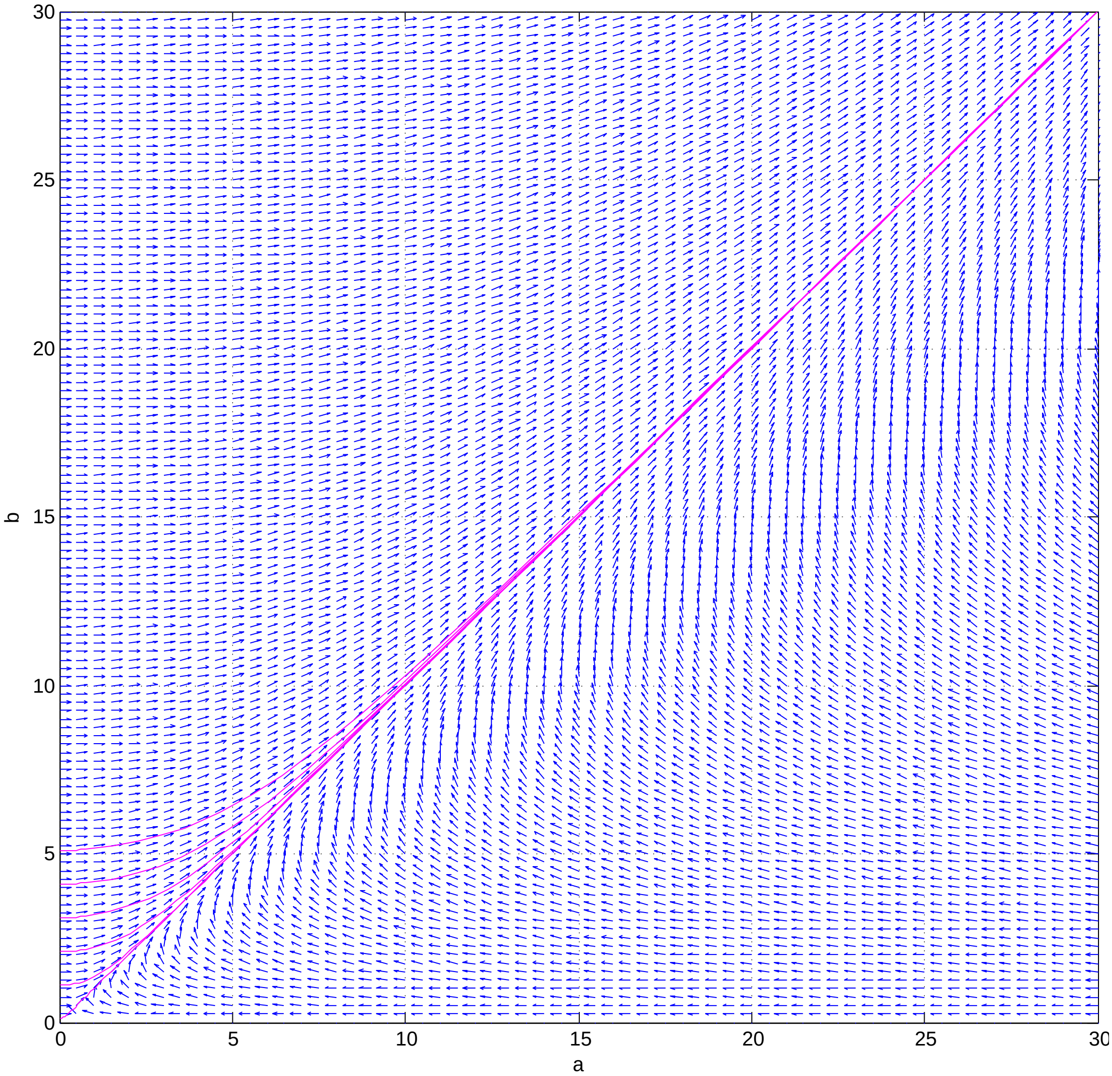

FIGURE 4.6: Vector field a, b axis for $n = 1$ case

$(\mathbb{R}_+ \times \mathbb{C}P^{2n+1})$'s metric $g_c = dt^2 + t^2\lambda h|_{\mathfrak{p}_1} + t^2\lambda h|_{\mathfrak{p}_2}$, where $g_1(t) = g_2(t)$. On the other hand, in the Figure 4.5, $a = g_1$ and $b = g_2$, and the numerical solutions of subsystem (3.14) have the slope $\frac{\mathrm{d}g_1}{\mathrm{d}g_2} \backsim 1$ for $t >> 0$ (see Table 4.2). So we conclude the one parameter family (3.19) of solutions converges to the cone over the Ziller metric.

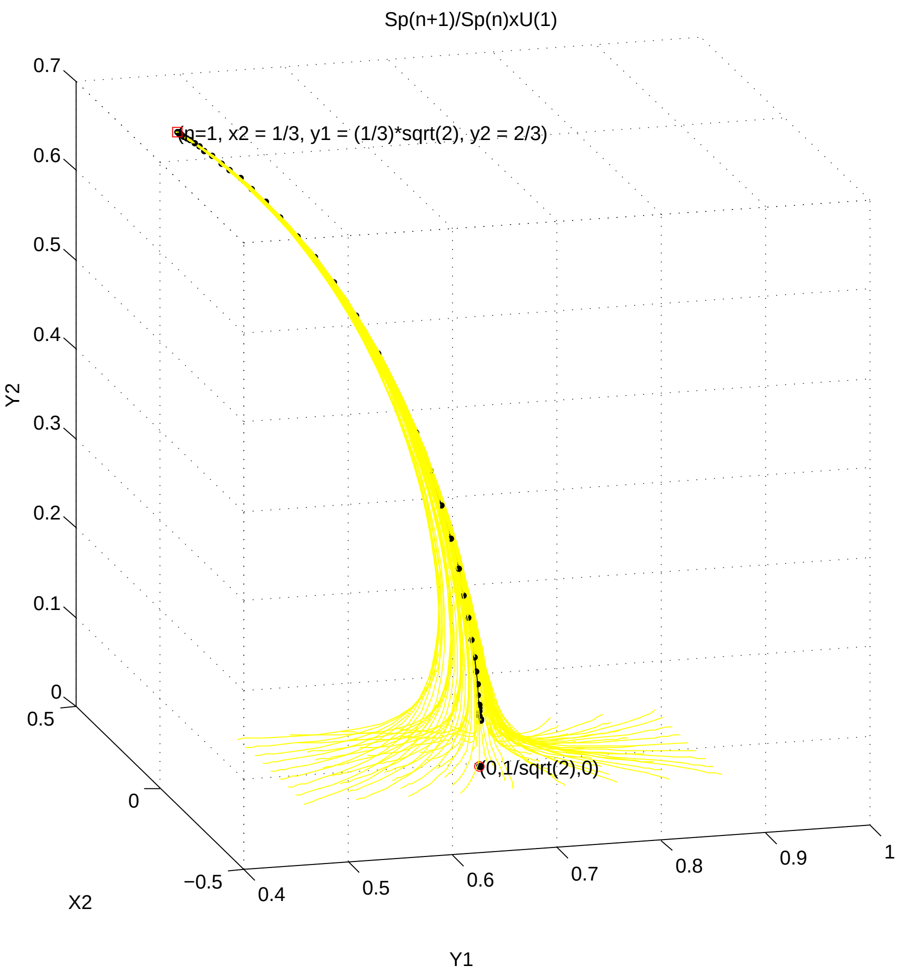

FIGURE 4.7: n=1

For general case, the trajectory lies in the hypersurface given by the conservation law. Using this surface to solve for X_1, we have

$$X_1 = \sqrt{1 - X_2^2 - A} \tag{4.4}$$

where $A = Y_1^2 + \frac{(n+2)Y_2^2}{2} - \frac{1}{16n}\frac{Y_2^4}{Y_1^2}$. Without loss of generality, we take $X_1 > 0$. Then the Ricci-flat system restricted to X_2, Y_1, Y_2 becomes

$$\begin{cases} X_2' = X_2(-A) + \frac{(n+2)Y_2^2}{4\sqrt{n}} - \frac{\sqrt{n}}{16n^2}\frac{Y_2^4}{Y_1^2} \\ Y_1' = Y_1(1 - A - \frac{\sqrt{1-X_2^2-A}}{\sqrt{2}}) \\ Y_2' = Y_2(1 - A - \frac{X_2}{\sqrt{4n}}) \end{cases} \tag{4.5}$$

The initial value for this system which satisfies the smoothness condition is given by $(X_1, X_2, Y_1, Y_2) = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0)$ as $s \to -\infty$.

Taking the above system (4.5) with initial values near the above point, say $(X_2, Y_1, Y_2) = (0, \frac{1}{\sqrt{2}} + 0.001, 0.01) \approx (0, 0.7081, 0.01)$ and $s_0 = 0$, the `ode45` solver gives the following asymptotical numerical solution.

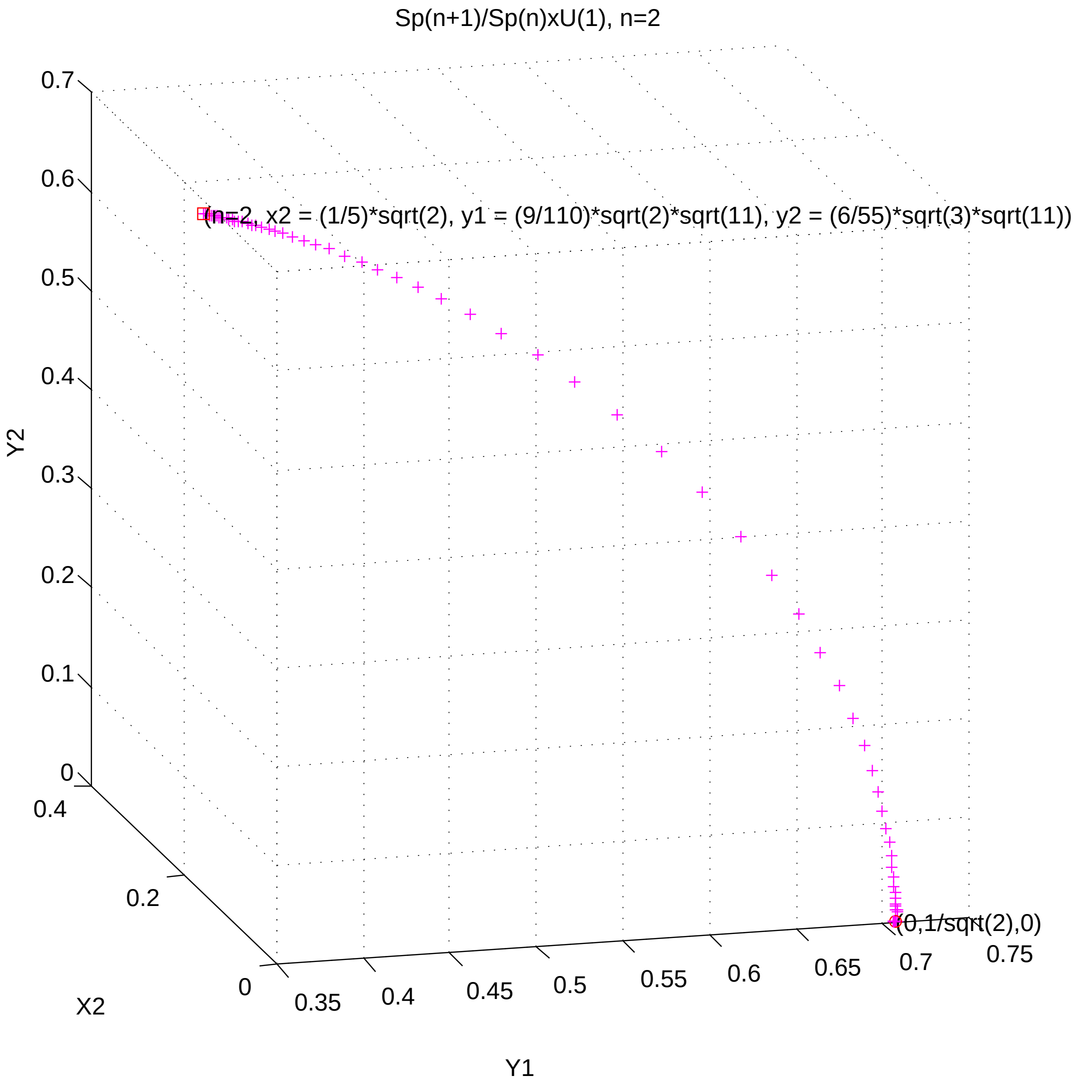

FIGURE 4.8: n=2, Initial$(0, 0.7081, 0.01)$

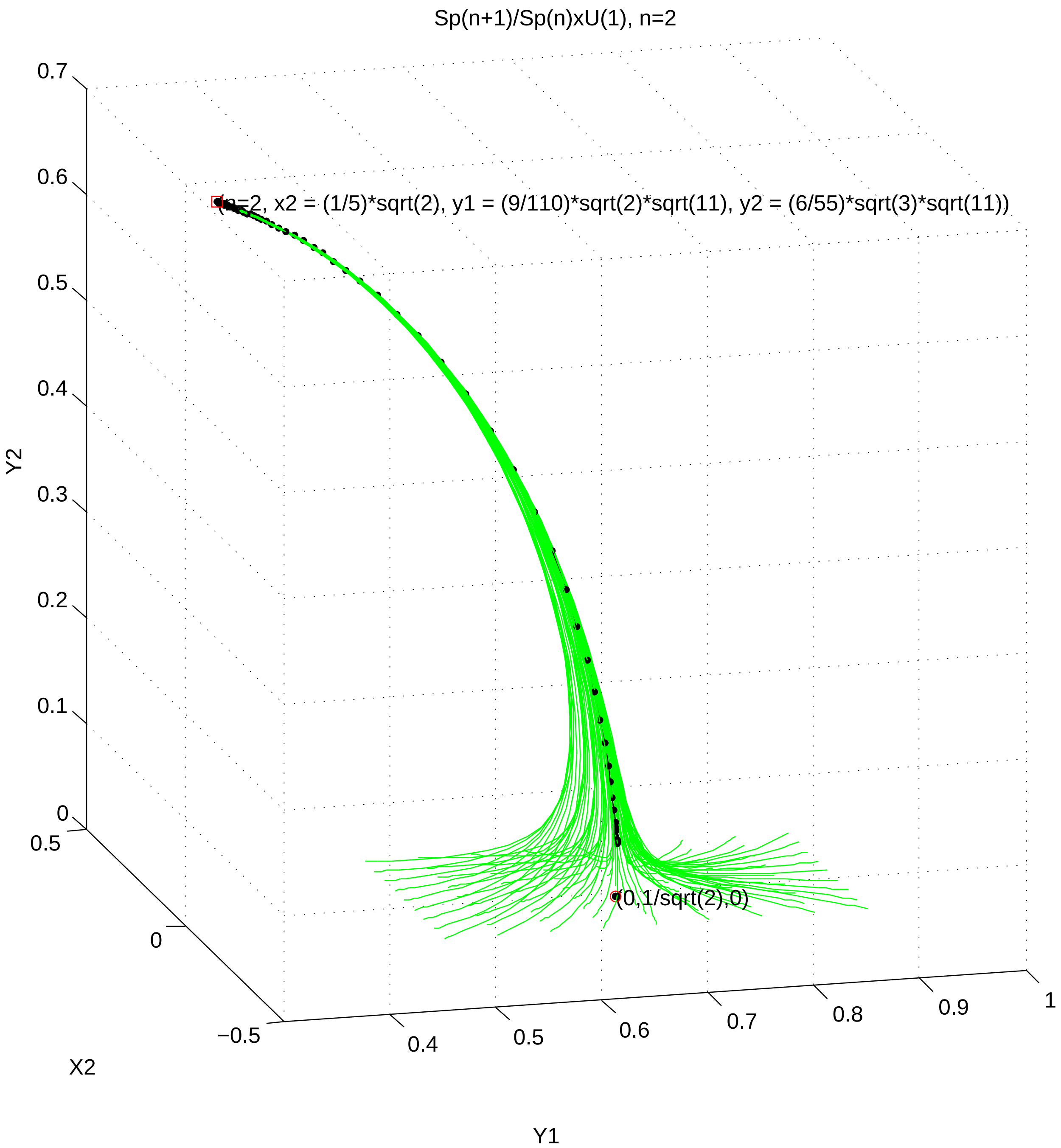
Sp(n+1)/Sp(n)xU(1), n=2
(n=2, x2 = (1/5)*sqrt(2), y1 = (9/110)*sqrt(2)*sqrt(11), y2 = (6/55)*sqrt(3)*sqrt(11))
(0,1/sqrt(2),0)
Y2
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
X2
0.5
0
−0.5
Y1
0.4
0.5
0.6
0.7
0.8
0.9
1

Figure 4.9: n=2

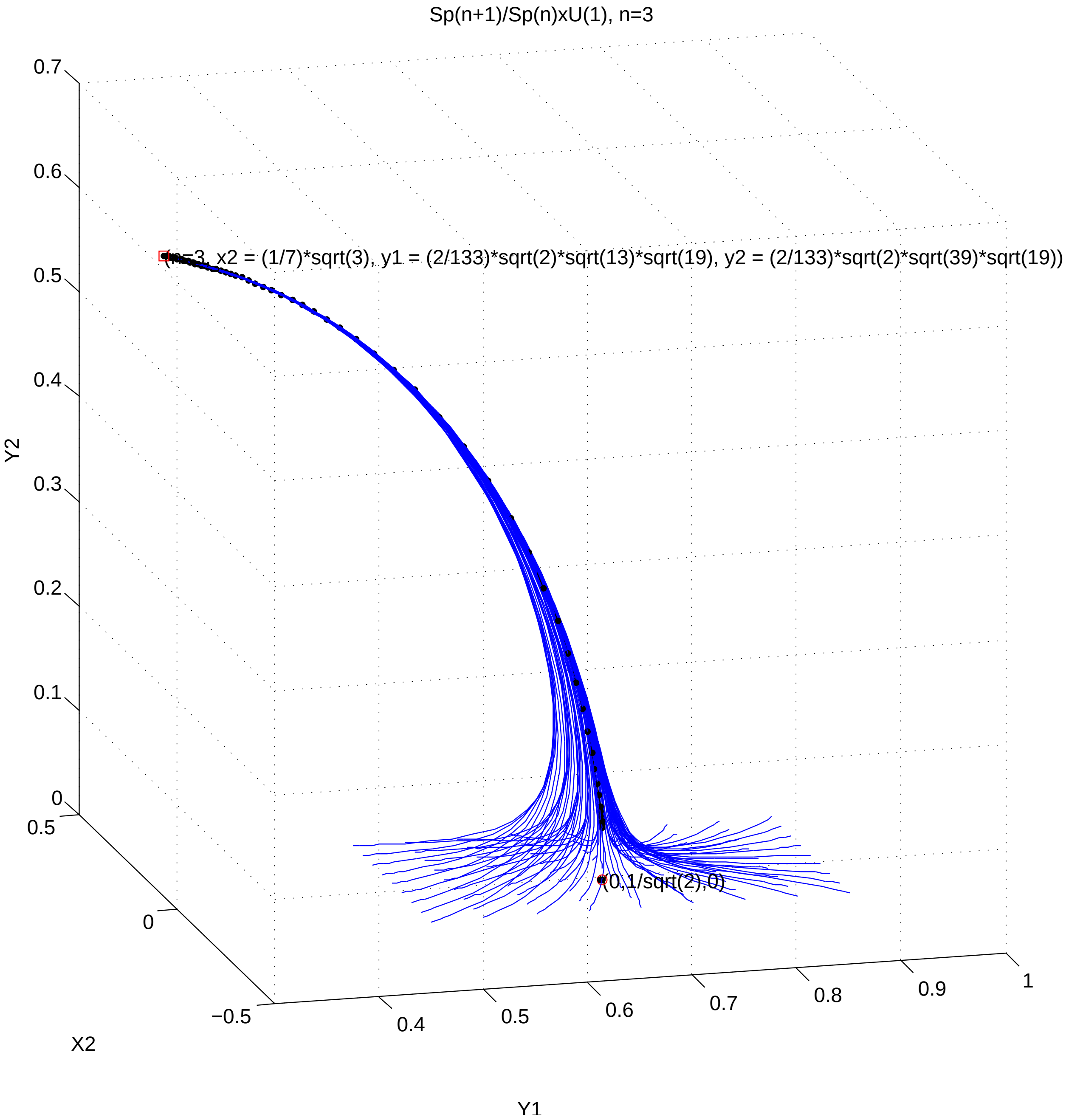

FIGURE 4.10: n=3

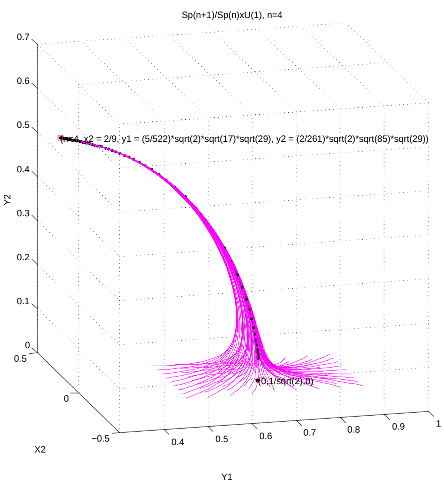

FIGURE 4.11: n=4

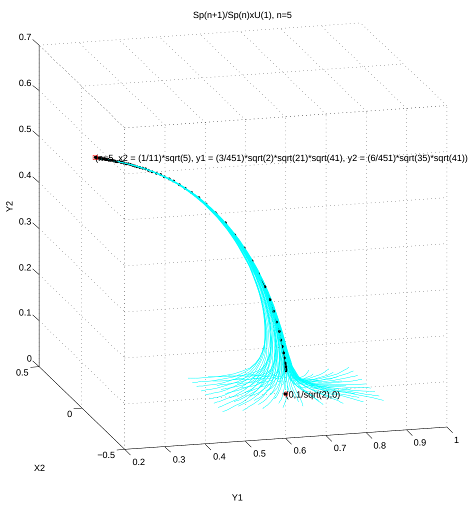

FIGURE 4.12: n=5

TABLE 4.3: Asymptotical point for $(Sp(n+1), Sp(n)Sp(1), Sp(n)U(1))$

		X_1	X_2	Y_1	Y_2
Ziller metric	$n = 1$	$\frac{\sqrt{2}}{6}$	$\frac{1}{3}$	$\frac{\sqrt{2}}{3}$	$\frac{2}{3}$
numerical			0.3333	0.4714	0.6667
error			1.5709×10^{-5}	5.1886×10^{-6}	5.5572×10^{-6}
Ziller metric	$n = 2$	$\frac{\sqrt{2}}{10}$	$\frac{\sqrt{2}}{5}$	$\frac{9\sqrt{22}}{110}$	$\frac{6\sqrt{33}}{55}$
numerical			0.2829	0.3838	0.6267
error			2.8775×10^{-5}	5.1894×10^{-6}	6.6544×10^{-6}
Ziller metric	$n = 3$	$\frac{\sqrt{2}}{14}$	$\frac{\sqrt{3}}{7}$	$\frac{2\sqrt{2\times13\times19}}{133}$	$\frac{2\sqrt{2\times39\times19}}{133}$
numerical			0.2475	0.3342	0.5789
error			3.0038×10^{-5}	5.2377×10^{-6}	4.6476×10^{-6}
Ziller metric	$n = 4$	$\frac{\sqrt{2}}{18}$	$\frac{2}{9}$	$\frac{5\sqrt{2\times17\times29}}{522}$	$\frac{2\sqrt{2\times85\times29}}{261}$
numerical			0.2223	0.3008	0.5380
error			4.4283×10^{-5}	6.5752×10^{-6}	5.4050×10^{-6}
Ziller metric	$n = 5$	$\frac{\sqrt{2}}{22}$	$\frac{\sqrt{5}}{11}$	$\frac{\sqrt{2\times21\times41}}{451}$	$\frac{6\sqrt{35\times41}}{451}$
numerical			0.2033	0.2760	0.5040
error			5.6049×10^{-5}	7.8158×10^{-6}	5.5697×10^{-6}

where the global truncation error $e = \left|(X_i, Y_i) -\right.$ numercial solution $\left.(X_i(s_*), Y_i(s_*))\right|$ and we take $s_* = 400 >> 0$ to get the above numerical solution.

Remark 4.3. We compare the points in the numerical solutions of system (4.5) to the Ziller metric for s large, and $n = 1, 2, 3, 4, 5$. We find that the Ricci-flat solution converges to the cone over the Ziller metric rather than the cone over the Fubini-Study metric. For $n \geqslant 3$, this corroborates with Theorem 6.4 and the Convergence Theorem 11.1 of [Böh99]. The case $n = 1$ and $n = 2$ are not covered in Theorem 6.4 of [Böh99]. But the case $n = 1$ can be solved explicitly; according to [BS89] and [GPP90], this manifold has holonomy type G_2.

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