IMMEDIATE EXPANSIONS BY VALUATION OF FIELDS

IMMEDIATE EXPANSIONS BY VALUATION OF FIELDS

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Abstract

The main subject of investigation is the so-called "immediate expansion" phenomenon in various first-order valued-field structures over the corresponding underlying field structures. In particular, certain "valued o-minimal fields", certain Henselian valued fields with non-divisible valued groups, and certain separably closed valued fields of finite imperfection degree, are shown to have this property.

Acknowledgements

I tend to think that this thesis is a collective effort of the whole human society, or of the entire universe, or of even more. In fact, it is not obvious to me whether things happen because we have free will. However, if I were forced to choose which to believe under certain threats, I would probably choose to believe that we do have free will (well, it might depend on the threats too). Be that as it may, the complexity of determining which factors are critical for a certain event to happen is still enormously huge. Who can guarantee that the flapping of some butterflies in Brazil many years ago does not have anything to do with my thesis? It thus makes sense to me to say that the current outcome exists because of the entire sequence of events happening in the past, up to the beginning of time, if there is a beginning of time and if causality really makes sense... In any case, here I am, without knowing whether things would turn out to be better or worse otherwise, feeling sincerely grateful to everything that things have turned out this way. But I shall be concise here.

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Contents

Chapter 1

Introduction and preliminaries

1.1 Introduction

Our main subject of investigation in this thesis is the *immediate expansion* phenomenon in various first-order valued-field structures over the corresponding underlying field structures. While there are examples of valued-field structures which do not have this phenomenon, we focus on those that do. In particular, we will prove that certain Henselian valued fields with non-divisible value groups, certain "valued o-minimal fields", separably closed valued fields of finite imperfection degree, have this phenomenon.

It is perhaps necessary to say a few words about how this investigation came about. It is said that model theory, to a very large extent, is the study of definable sets. Definable sets are to first-order structures as open sets are to topological spaces. Suppose that we have a first-order language $\mathscr{L}$, and a predicate P not in $\mathscr{L}$, it is a natural and basic question to ask how this new predicate P pertains to the language $\mathscr{L}$. One can in particular ask, assuming that we have a first-order $\mathscr{L} \cup \{P\}$-structure $\mathcal{M}$, what sort of new definable sets one will obtain in addition to those that are $\mathscr{L}$-definable over the universe of $\mathcal{M}$. More generally, suppose we have two first-order languages $\mathscr{L}_1$ and $\mathscr{L}_2$ and a set M, on which there is an $\mathscr{L}_1$-structure called $\mathcal{M}_1$ and an $\mathscr{L}_2$-structure called $\mathcal{M}_2$ such that every $\mathscr{L}_1$-definable set over M using parameters from M is $\mathscr{L}_2$-definable over M with parameters, then what sort of new definable sets are obtained with $\mathscr{L}_2$? Let us call $\mathcal{M}_2$ satisfying the assumption above an **expansion** of $\mathcal{M}_1$. More precisely, we approach the question through the notion of *intermediate structures*, in terms of definability; that is: suppose that $\mathcal{M}_2$ is an expansion of $\mathcal{M}_1$, then what are the *intermediate* first-order structures between $\mathcal{M}_1$ and $\mathcal{M}_2$? Of course, the basic situation is when there are no intermediate first-order structures between $\mathcal{M}_1$ and $\mathcal{M}_2$, in which case we say $\mathcal{M}_2$ is an **immediate expansion** of $\mathcal{M}_1$.

Investigations concerning expansions and reducts have been and are being carried out by many researchers, on various first-order structures from different perspectives. Most of the time these are not easy. There are results about reducts of ordered structures, graphs, fields, the arithmetics, etc.. In general, one has to restrict oneself into particular kinds of structures to give a "good" answer. In the hope of getting a good characterization of when it is true that $\mathcal{M}_2$ is an immediate expansion of $\mathcal{M}_1$, we want to restrict ourselves to the case where $\mathcal{M}_2$ is a valued field, $\mathscr{L}_1$ contains the language of rings $\{+, -, \times, 0, 1\}$ and $\mathscr{L}_2 = \mathscr{L}_1 \cup \{V\}$, where V is a predicate intended to be interpreted as a valuation on the underlying field. We made this choice first because the author is more familiar with valued fields than with other equally sensible choices. Second and heuristically, valuation theory has been observed to be useful in the algebraic investigation of various kinds of fields, in several branches of mathematics; valuations have been indispensable tools for algebraic number theorists; they are also tools for investigating resolution of singularities in algebraic geometry; the model theoretic investigation of valued fields has been a very active subject, providing applications to other subjects. All these seem to suggest that valuations are in some sense "very close" to the underlying field structure (using the addition and the multiplication) in the sense that they can provide relatively-easier-to-obtain and useful information about the field structure. This sort of "closeness" seems to occur in many situations on different kinds of fields, which suggests that perhaps one can hope to have a more organized and systematic way of investigation for the immediate expansion phenomenon in valued fields. One can of course, treat our investigation here as a beginning of an attempt to do this, and a beginning of an attempt to measure the "closeness" of valuations to addition and multiplication using first-order definability. This also to some extent explains our choice of investigating definable sets with parameters, rather than definable set without parameters; for example, when number theorists and algebraic geometers use valuations, they do not care whether they use any parameters from the underlying set or not (and in fact they use them freely).

In some practical situations, to study a field, one uses multiple valuations on this field, for example, in the study of global fields. But a good understanding of multiply valued fields should be built on a good understanding of singly valued fields. Under the current circumstance, we have not yet obtained a sufficiently good understanding of the latter, so we have not touched upon the former. One should view our investigation here as a beginning of a "local" investigation.

Now, we give a rough description of each chapter in this thesis.

Chapter 1 (Introduction and preliminaries) introduces the general scheme of the thesis, some preliminary knowledge that is going to be used throughout, and some convention that is going to be used subsequently. A particular section is also devoted to the discussion of expansions, reducts, and intermediate structures. The following chapters then focus on problems specifically related to valued fields.

Chapter 2 (Definable valuations) focuses on the case where $\mathscr{L}_1$ is the language of rings, $\mathcal{M}_1$ is a field, and $\mathscr{L}_2$ is $\mathscr{L}_1 \cup \{V\}$, with 'V' being the predicate for a valuation rings on M, and V is $\mathscr{L}_1$-definable (hence $\mathcal{M}_2$ is an immediate expansion of $\mathcal{M}_1$). A specific result proved in the chapter is that if V is a Henselian valuation ring on

M with a value group containing a convex non-divisible regular subgroup, then V is $\mathscr{L}_1$-definable over M.

Chapter 3 (T-convexly valued o-minimal fields) focuses on the case where $\mathscr{L}_1$ contains the language of rings, $\mathcal{M}_1$ is an o-minimal expansion of a field and $\mathscr{L}_2 = \mathscr{L} \cup \{V\}$, where V is a convex valuation ring on M with respect to the ordering on M. It is proved that in the case where V is so-called "T-convex", $\mathcal{M}_2$ is an immediate expansion of $\mathcal{M}_1$. This then generalizes a result known about RCVF (see [Haskell and Macpherson, 1998]).

Chapter 4 (Algebraically closed valued fields revisited) focuses on the case where $\mathcal{M}_2 \models$ ACVF and $\mathscr{L}_1$ is the language of rings. It has been proved in [Haskell and Macpherson, 1998] that in this case $\mathcal{M}_2$ is an immediate expansion of $\mathcal{M}_1$. We first isolate a critical result in their proof, and then give a new proof which does not rely on a strong result of Hrushovski (see [Haskell and Macpherson, 1998]) about strongly minimal expansions of algebraically closed fields.

Chapter 5 (Separably closed valued fields) focuses on the case where $\mathcal{M}_2$ is a separably closed valued field of positive characteristic and finite imperfection degree, $\mathscr{L}_1 = \mathscr{L}_{\mathrm{r}}$ and $\mathscr{L}_2 = \mathscr{L}_{\mathrm{r}} \cup \{V\}$. We first prove a quantifier elimination result for the theory of separably closed non-trivially valued fields of characteristic $p > 0$ and finite imperfection degree $e > 0$ (in a suitably nice language). This immediately gives us the result that this theory does not have the Independence Property. Eventually, we prove that $\mathcal{M}_2$ as mentioned above, is an immediate expansion of $\mathcal{M}_1$. We also give some remarks on the infinite imperfection degree case and on general valued stable fields.

Finally, in Appendix A, some well-known results in classical algebraic geometry has been collected for the convenience of the reader. These results are mostly used in Chapter 4 and Chapter 5.

1.2 Valued fields

A good reference for the basic valuation theory is [Engler and Prestel, 2005].

Recall that given a field K (with $0 \neq 1$), a subring V of K is said to be a **valuation ring** if it is true that for any $x \in K^\times := K \backslash \{0\}$, either $x \in V$ or $x^{-1} \in V$. For a valuation ring V, the set of all elements whose inverses are also in V, is called the set of **units**, denoted by $V^\times$. The group $V^\times$ is in fact a multiplicative subgroup of $K^\times$. The unique **maximal ideal** of V, denoted by $\mathfrak{m}_v$, is exactly the set of non-invertible elements in V, i.e. $\mathfrak{m}_v = V \backslash V^\times$.

Recall that an **ordered abelian group** is an abelian group with an ordering which is compatible with the group operation, that is if $(G, +)$ is an abelian group, then $(G, +, <)$ is an ordered abelian group if for all $a < b$ and $z \in G$, $a + z < b + z$. It follows that if G is finite, then $G = \{0\}$. An element of an ordered abelian group G is **positive** if it is larger than 0 (the identity element), **negative** if it is less than 0. An ordered abelian group is said to be **discrete** if there is a smallest positive element in this group; otherwise, one can prove that between any given two different elements of this group, there are always infinitely many elements, in which case the group is **dense**.

A subset Δ of an ordered abelian group Γ is said to be **convex**, if for all $a, b \in \Delta$ and $c \in \Gamma$ such that $a < c < b$, it is always true that $c \in \Delta$. The collection of all proper convex subgroups of an ordered abelian group is linearly ordered by set inclusion; the order type is called the **rank** of this ordered abelian group. An ordered abelian group of rank 1 is also called an **archimedean** ordered abelian group.

A **valuation** on K is a map $v : K \twoheadrightarrow \Gamma \cup \{\infty\}$, where Γ is an ordered abelian group and $\Gamma < \infty$, satisfying the following properties:

- for all $a \in K$, $v(a) = \infty$ if and only if $a = 0$;
- for all $a, b \in K$, $v(a + b) \geq \min\{v(a), v(b)\}$;
- for all $a, b \in K$, $v(ab) = v(a) + v(b)$.

It can be shown that these properties imply that for all $a, b \in K$, if $v(a) < v(b)$, then $v(a + b) = v(a)$. This latter property will be used frequently.

It is easy to verify that given a valuation map v, the set $\{x \in K \mid v(x) \geq 0\}$ is a valuation ring on K, and the set $\{x \in K \mid v(x) > 0\}$ is exactly the unique maximal ideal of that valuation ring.

We sometimes identify valuation maps with isomorphic ordered abelian groups, that is if $v_1 : K \twoheadrightarrow \Gamma_1 \cup \{\infty\}$ and $v_2 : K \twoheadrightarrow \Gamma_2 \cup \{\infty\}$ have the property that there is some isomorphism of ordered abelian groups $\varphi : \Gamma_1 \to \Gamma_2$ such that $\varphi \circ v_1 = v_2$, then we think of v_1 and v_2 as the same valuation map; in this case v_1 and v_2 give the same valuation ring. Given a valuation ring V, the quotient map $K^\times \twoheadrightarrow K^\times/V^\times$ induces a valuation map $v : K \to (K^\times/V^\times) \cup \{\infty\}$ which extends the quotient map and maps 0 to ∞. The ordering $xV^\times < yV^\times$ if $y/x \in \mathfrak{m}_v$, makes $K^\times/V^\times$ an ordered abelian group. So $K^\times/V^\times$ is also called the **value group** of the valuation ring V (or of the valuation v), usually denoted by vK.

Another quotient map $V \twoheadrightarrow V/\mathfrak{m}_v$ induces a map $\mathrm{res}_v : K \to (V/\mathfrak{m}_v) \cup \{\infty\}$, which extends the quotient map and maps $a \in K \backslash V$ onto ∞. This induced map is called the **place** map associated to the valuation ring V, and the field $V/\mathfrak{m}_v$ is usually called the **residue field** of V, denoted by Kv.

A valuation (or a valuation ring) is said to be **trivial** if $vK = \{0\}$, or equivalently $V = K$.

Notation 1.2.1. Given a field K with a valuation ring V (usually a capital letter), the corresponding valuation map (e.g. the one induced by the quotient map) is denoted by the corresponding lower-case letter v. The set $K \backslash \{0\}$ is denoted by $K^\times$. The value group is denoted by vK (occasionally, we use $vK^\times$ to denote the value group as well, to prevent potential ambiguity). We use $(vK) \cup \{\infty\}$ to denote the value group together with the infinity element, which is the full image set of the valuation map. The residue field is denoted by Kv, and the residue map is denoted by $\mathrm{res}_v : K \to (Kv) \cup \{\infty\}$. The maximal ideal of V is denoted by $\mathfrak{m}_v$ and the set of units in V is denoted by $V^\times$. If there is another valuation ring W on K, then we have the corresponding valuation map w, and $wK, Kw, \mathrm{res}_w, \mathfrak{m}_w$, etc..

A valuation ring V is said to be **Henselian** if it satisfies the so-called **Hensel's Lemma**: for any polynomial $f(X) = \sum a_i X^i \in V[X]$, if the corresponding polynomial $\mathrm{res}_v(f)(X) = \sum \mathrm{res}_v(a_i)X^i$ has a simple root $\alpha \in Kv$, then $f(X)$ has a root $a \in V$ and $\mathrm{res}_v(a) = \alpha$.

A **valued field** is a field K with a distinguished valuation ring V, usually denoted by (K, V). A valued field is Henselian if its valuation ring is Henselian. Sometimes we use (K, v) to denote a valued field, where v is the corresponding valuation map of V.

Suppose that L is a field extension of K, and W and V are valuation rings on L and K respectively. The valuation ring W is called an **extension** of V if $V = W \cap K$. If W is an extension of V, then vK can be naturally identified as an ordered subgroup of wL, and Kv can be naturally identified as a subfield of Lw.

Fact 1.2.2 (Chevalley; see [Engler and Prestel, 2005])**.** Suppose (K, V) is a valued field and L/K a field extension. Then there is some valuation ring W on L extending V.

It can be proved (see e.g. Chapter 4 of [Engler and Prestel, 2005]) that a valued field (K, V) is Henselian if and only if there is exactly one extension of V in the separable closure of K.

Fact 1.2.3 ([Engler and Prestel, 2005])**.** Suppose that L/K is a normal field extension, V is a valuation of K, W_1 and W_2 are valuation rings in L extending V. Then there exists $\sigma \in \mathrm{Aut}(L/K)$ with $\sigma(W_1) = \sigma(W_2)$.

Example 1.2.4. Let p be a fixed prime number. For any integer a, let $v_p(a)$ be the exponent m of p such that $p^m \mid a$ but $p^{m+1} \nmid a$. For any $a/b \in \mathbf{Q}$, define $v_p(a/b) = v_p(a) - v_p(b)$ (one checks that this is well defined on $\mathbf{Q}$). Then $(\mathbf{Q}, \mathbf{Z}_{(p)})$ is a valued field, where $\mathbf{Z}_{(p)}$ is the localization of $\mathbf{Z}$ at the prime ideal (p), i.e. it is the collection of elements of the form a/b with $p \nmid b$. The value group of this valued field is $\mathbf{Z}$ and the residue is $\mathbf{F}_p$ (the finite field of p-elements). This is not a Henselian valued field.

In fact, it is well-known that every non-trivial valuation on $\mathbf{Q}$ is of the form v_p for some prime number p.

Define $|x|_p := p^{-v_p(x)}$ for every $x \in \mathbf{Q}$, then one checks that $|\cdot|_p$ is a metric on $\mathbf{Q}$. The completion of $\mathbf{Q}$ with respect to this metric is called the p**-adic numbers**, denoted by $\mathbf{Q}_p$. Its corresponding valuation ring is called the ring of p**-adic integers**, denoted by $\mathbf{Z}_p$. The valued field $(\mathbf{Q}_p, \mathbf{Z}_p)$ is an "immediate extension" of $(\mathbf{Q}, \mathbf{Z}_{(p)})$, i.e. their value groups and residue fields are the same. Furthermore, $(\mathbf{Q}_p, \mathbf{Z}_p)$ is a Henselian valued field.

Example 1.2.5. One can do the same thing to $F[X]$, the ring of polynomials in one variable over some field F. Let $p(X) \in F[X]$ be some irreducible polynomial of degree larger than 0. Define $v_{p(X)}(f(X))$ to be the exponent m of $p(X)$ such that $p(X)^m \mid f(X)$ but $p(X)^{m+1} \nmid f(X)$. Extend the map $v_{p(X)}$ to $F(X)$ similarly. Then $(F(X), v_{p(X)})$ is a valued field which is not Henselian. The valuation ring is $F[X]_{(p(X))}$, the value group is $\mathbf{Z}$ and the residue field is $F[X]/(p(X))$ (a finite extension of F). The completion of $F(X)$ with respect to $v_{p(X)}$ is a **formal power series field** $F((T))$ (the field of Laurent series in T), where $T = p(X)$. This is also an immediate extension, and is Henselian.

Example 1.2.6. Given any field K and any ordered abelian group G, one can define the field of **formal power series** $K((T^G))$ to be the following set

$$\left\{ \sum_{g \in S \subseteq G} a_g T^g \,\middle|\, S \text{ is a well-ordered subset of } G, a_g \in K \right\}.$$

Then $K((T^G))$ with the natural addition and multiplication becomes a field. The natural valuation v on $K((T^G))$ is defined to be

$$v\left(\sum_{g \in S} a_g T^g\right) = \text{the smallest element } g \text{ of } S \text{ with } a_g \neq 0.$$

Then $(K((T^G)), v)$ is a Henselian valued field, with residue field K, and value group G. For more details, see for example Exercise 3.5.6 of [Engler and Prestel, 2005].

Given a valuation ring V on the field K, there is an induced topology, sometimes called the **valuation topology** associated to V, or the V**-topology**. A basis of this topology is the collection of all discs of the form $\{x \in K \mid v(x-a) > v(b)\}$, where $a, b \in K$. Under this V-topology, K becomes a topological field, that is the multiplication, addition and the additive and multiplicative inverses are continuous. This also means that polynomial functions and rational functions are continuous with respect to the V-topology. It follows that every Zariski-open set in K is also open in the V-topology (because if $p(\vec{a}) \neq 0$ and $\vec{b}$ is close enough to $\vec{a}$ with respect to the V-topology, then $p(\vec{b}) \neq 0$.)

Notation 1.2.7. In this thesis, when we write K^n, we mean the n-th Cartesian power of K. To denote the set of nth-powers of K, we use $P_n(K)$. An element of K^n when n is larger than or equal to 1, is usually denoted by $\vec{a} = (a_1, \ldots, a_n)$, $\vec{b} = (b_1, \ldots, b_n)$ etc.. When we say a point $\vec{a} \in K^n$ is **close enough** to a point $\vec{b} \in K^n$ we mean each entry of $v\left(\vec{a} - \vec{b}\right) := (v(a_1 - b_1), \ldots, v(a_n - b_n))$ is large enough (i.e. they are close enough with respect to the V-topology).

On a valued field (K, V), the set $\{x \mid v(x-a) > \gamma\}$ for some $a \in K$ and $\gamma \in vK$ is denoted by $D_v^>(a,\gamma)$, or simply $D^>(a,\gamma)$ if the valuation we refer to is clear from the context. The set $\{x \mid v(x-a) \geq \gamma\}$ will be denoted by $D_v^\geq(a,\gamma)$ or simply $D^\geq(a,\gamma)$.

On a field K, two valuation rings V and W are said to be **dependent** if they induce the same topology. One can prove (see e.g. Section 2.3 of [Engler and Prestel, 2005]) that if V and W are both non-trivial, then they are dependent if and only the smallest subring of K containing both V and W is not K.

It can be proved that

Fact 1.2.8 (F. K. Schmidt; see [Engler and Prestel, 2005])**.** If a field K admits two independent non-trivial Henselian valuations, then K is separably closed.

Now, suppose that we have a non-trivially valued field (K,V). Let W be a subring of K containing V. It follows easily that W is also a valuation ring, $\mathfrak{m}_W \subseteq \mathfrak{m}_V$, and $V^\times \subseteq W^\times$. Therefore, there is a natural quotient map $\gamma_{v,w} : vK \twoheadrightarrow wK$. The kernel of $\gamma_{v,w}$ is the set of elements $\Delta := \{v(x) | x \in K, w(x) = 0\}$, which is a convex subgroup of vK. Also, wK is isomorphic to $(vK)/\Delta$. Furthermore, $(Kw, V/\mathfrak{m}_w)$ is itself a valued field, with a value group Δ and residue field Kv.

In fact, we have

Fact 1.2.9 ([Engler and Prestel, 2005])**.** Suppose that (K,V) is non-trivially valued. Then there is a 1-1 correspondence between the set of convex subgroups Δ of vK and the set of the ideals $\mathfrak{p}$ of V (hence also with the valuation rings containing V). More explicitly, it is given by

$$\Delta \mapsto \{x \in K \mid v(x) > \Delta\};$$
$$\mathfrak{p} \mapsto \{\gamma \in vK \mid |\gamma| < v(\mathfrak{p})\}.$$

Fact 1.2.10 (see [Prestel and Ziegler, 1978])**.** Suppose that (K,V) is a valued field, $a_1,\ldots,a_n \in K$, $\epsilon \in vK$. Then there is some $\delta \in vK$ such that for all $b_1,\ldots,b_n \in K$ with

$$\prod_{i=1}^n (X-a_i) - \prod_{i=1}^n (X-b_i) \in D_v^>(0,\delta)[X],$$

there is a permutation σ of $1,\ldots,n$ such that $v(a_i - b_{\sigma(i)}) > \epsilon$ for all $i = 1,\ldots,n$.

Hensel's Lemma is in some sense (in terms of *t-Henselianity* loc.cit.) equivalent to the following "Implicit Function Theorem".

Fact 1.2.11 (see [Prestel and Ziegler, 1978])**.** Suppose that the valued field (K,V) is Henselian, $F(X_0,\ldots,X_n,Y)$ is a polynomial in $K[X_0,\ldots,X_n,Y]$, and $(\vec{a},b) := (a_0,\ldots,a_n,b) \in K^{n+2}$ is a zero of F which is not a zero of the formal partial derivative F_y. Then there are $\epsilon,\delta \in vK$ such that for all $\vec{a}\,'$ with $v(\vec{a}-\vec{a}\,') > \delta$, there is exactly one b' with $v(b-b') > \epsilon$ such that $(\vec{a}\,',b')$ is a zero of F. The map $\vec{a}\,' \mapsto b'$ is continuous.

1.3 Expansions and reducts; immediate and intermediate structures

In this thesis, the word “definable” always allows the use of parameters.

The usual definition of an expansion and a reduct of a first-order structure is a little bit too restricted for our purpose. Here we re-define these notions in a more general setting (by allowing parameters). We emphasize that this is different from the usual definitions of expansions and reducts. We refer the reader to Section 1.1 for a discussion of the heuristic reasons for allowing parameters.

Throughout this section, let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two first-order languages, and let $\mathcal{M}_1$ and $\mathcal{M}_2$ be an $\mathscr{L}_1$-structure and an $\mathscr{L}_2$-structure respectively. We further assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same universe (underlying set) M.

Definition 1.3.1. *The structure $\mathcal{M}_2$ is called an* **expansion** *of/over $\mathcal{M}_1$, and $\mathcal{M}_1$ is called a* **reduct** *of $\mathcal{M}_2$, if every $\mathscr{L}_1$-definable set over M is $\mathscr{L}_2$-definable over M. We use $\mathcal{M}_1 \sqsubseteq \mathcal{M}_2$ to denote that $\mathcal{M}_1$ is a reduct of $\mathcal{M}_2$.*

Definition 1.3.2. *The two structures $\mathcal{M}_1$ and $\mathcal{M}_2$ are said to have* **the same (first-order) structure** *if they are expansions (equivalently reducts) of each other. We use $\mathcal{M}_1 \approx \mathcal{M}_2$ to denote that $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same first-order structure, and $\mathcal{M}_1 \not\approx \mathcal{M}_2$ otherwise.*

If $\mathscr{L}$ is another language such that $\mathcal{M}_2$ is also an $\mathscr{L}$-structure, then we use $\mathcal{M}_2|\mathscr{L}$ to denote the $\mathscr{L}$-structure on M. Therefore $\mathcal{M}_1 \approx (\mathcal{M}_2|\mathscr{L}_1)$.

Definition 1.3.3. *Suppose $\mathcal{M}_1 \sqsubseteq \mathcal{M}_2$.*

We say that $\mathcal{M}_1$ is a **proper reduct** *of $\mathcal{M}_2$ if $\mathcal{M}_1 \not\approx \mathcal{M}_2$, in which case $\mathcal{M}_2$ is called a* **proper expansion** *of $\mathcal{M}_1$; this is denoted by $\mathcal{M}_1 \sqsubset \mathcal{M}_2$.*

We call $\mathcal{M}_1$ an **immediate reduct** *of $\mathcal{M}_2$, if for all $\mathcal{M}$ (in some language $\mathscr{L}$) with $\mathcal{M}_1 \sqsubseteq \mathcal{M} \sqsubseteq \mathcal{M}_2$, either $\mathcal{M}_1 \approx \mathcal{M}$ or $\mathcal{M} \approx \mathcal{M}_2$. This is denoted by $\mathcal{M}_1 ⧖ \mathcal{M}_2$ (or $\mathcal{M}_2 ⧖ \mathcal{M}_1$).* [1] *In this case we also say $\mathcal{M}_2$ is an* **immediate expansion** *of $\mathcal{M}_1$. We use $\mathcal{M}_1 ⧗ \mathcal{M}_2$ (or $\mathcal{M}_2 ⧗ \mathcal{M}_1$)*[2] *to mean that $\mathcal{M}_1$ is not an immediate reduct of $\mathcal{M}_2$. If $\mathcal{M}_1 ⧖ \mathcal{M}_2$, we also say the pair $(\mathcal{M}_1, \mathcal{M}_2)$ is immediate.*

Any $\mathcal{M}$ with $\mathcal{M}_1 \sqsubseteq \mathcal{M} \sqsubseteq \mathcal{M}_2$ is called an **intermediate structure** *of the pair $(\mathcal{M}_1, \mathcal{M}_2)$ (or $(\mathcal{M}_2, \mathcal{M}_1)$).*

Most of the time, we are going to be interested in the case that $\mathscr{L}_2 = \mathscr{L}_1 \cup \{P\}$, where P is a new symbol.

In the following, we let $\mathscr{L}_r$ be the language of rings, that is $\{+, -, \times, 0, 1\}$.

[1] The symbol ‘⧖’ is an empty hour glass, which could be thought of as meaning “there is no time” (hence “immediate”) or “there is nothing inside”. One can use $\mathcal{M}_1 ⧖\!\sqsubseteq \mathcal{M}_2$ or $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ (overlapping the two symbols) to emphasize that $\mathcal{M}_2$ is an expansion of $\mathcal{M}_1$. But most of the time, it will be obvious which one is the bigger structure.

[2] Obviously, “there is time” or “there is something inside”.

Example 1.3.4. Let $\mathscr{L}_1 = \mathscr{L}_{\mathrm{r}}$, and let $\mathscr{L}_2 = \mathscr{L}_{\mathrm{r}} \cup \{P\}$. Suppose that M is the set of rational numbers, and the predicate P is interpreted as the set of integers, then $\mathcal{M}_2$ ⧖ $\mathcal{M}_1$ just because $\mathbf{Z}$ is $\mathscr{L}_{\mathrm{r}}$-definable over $\mathbf{Q}$, by the Lagrange's Four-square Theorem.

Example 1.3.5. Let $\mathscr{L}_1 = \{+, \{\lambda_a\}_{a\in\mathbf{C}}\}$, and let $\mathscr{L}_2 = \mathscr{L}_1 \cup \{\times\}$. Let $M = \mathbf{C}$ and interpret λ_a as the multiplication by a. Then $\mathcal{M}_2$ ⧖ $\mathcal{M}_1$ by [Marker and Pillay, 1990].

Example 1.3.6. Let $\mathscr{L}_1 = \{+, <, \{\lambda_a\}_{a\in\mathbf{R}}\}$, and let $\mathscr{L}_2 = \mathscr{L}_1 \cup \{\times\}$. Let $M = \mathbf{R}$ and interpret λ_a as the multiplication by a. Then $\mathcal{M}_2$ ⧗ $\mathcal{M}_1$, which is a result of [Pillay et al., 1989] and [Marker et al., 1992].

Example 1.3.7. Let $\mathscr{L}_1 = \mathscr{L}_{\mathrm{r}}$ and $\mathscr{L}_2 = \mathscr{L}_{\mathrm{r}} \cup \{V\}$, with M being a field and V interpreted as a valuation ring on M. Then if V is $\mathscr{L}_{\mathrm{r}}$-definable over M, $\mathcal{M}_2$ ⧖ $\mathcal{M}_1$ is trivially true. This is the subject of Chapter 2.

Notice that the notion of immediate expansion is only defined for structures, not necessarily for theories, i.e. if $\mathcal{M}_2$ ⧖ $\mathcal{M}_1$ and $\mathcal{W}_2$ is an $\mathscr{L}_2$-structure which is $\mathscr{L}_2$-elementarily equivalent to $\mathcal{M}_2$, then it is not necessarily true that $\mathcal{W}_2$ ⧖ $(\mathcal{W}_2|\mathscr{L}_1)$. This phenomenon is similar to that of minimality (versus strong minimality).[3]

Example 1.3.8. Let M be a countable set with an equivalence relation E which has countably many equivalence classes, and whose equivalence classes are all finite and of different cardinalities. Then the first-order structure (M, E) is a typical example of a minimal but not strongly minimal structure. It is easy to see that (M, E) ⧖ $(M, =)$, but for any (W, E) which is $\{E\}$-elementarily equivalent to (M, E), as long as W is uncountable (in fact as long as W has an infinite equivalence class), (W, E) ⧗ $(W, =)$.

On the other hand, if we consider a different language, treating M as the structure $(M, E_1, E_2, \ldots)$, where E_i are predicates of the equivalence classes, then this time we do have the property that if $(W, E_1, E_2, \ldots)$ is $\{E_1, E_2, \ldots\}$-elementarily equivalent to $(M, E_1, E_2, \ldots)$, then $(W, E_1, E_2, \ldots)$ ⧖ $(W, =)$.

Notation 1.3.9. For two theories T_2 and T_1 in $\mathscr{L}_2$ and $\mathscr{L}_1$ respectively such that every model of T_2 is also a model of T_1, we write T_2 ⧖ T_1 to mean that for every model $\mathcal{M}_2$ of T_2, $\mathcal{M}_2$ ⧖ $(\mathcal{M}_2|\mathscr{L}_1)$.

It is well-known that every ω-saturated minimal structure is also strongly minimal. Similarly, we have

Theorem 1.3.10. *Assume that $\mathscr{L}_1 \subseteq \mathscr{L}_2$. Suppose that $\mathcal{W}_2$ is an ω-saturated $\mathscr{L}_2$-structure which is $\mathscr{L}_2$-elementarily equivalent to $\mathcal{M}_2$. Then $\mathcal{W}_2$ ⧖ $(\mathcal{W}_2|\mathscr{L}_1)$ implies that $\mathcal{M}_2$ ⧖ $\mathcal{M}_1$.*

[3] So one can define "strongly immediate expansions" similarly.

Proof. Suppose that $\mathcal{M}_2 ⧗ \mathcal{M}_1$. Then there are some $\mathscr{L}_2$-formulas $\phi_1(\vec{x},\vec{a})$, $\phi_2(\vec{x},\vec{a})$ with parameters $\vec{a}$ from M such that for every $\mathscr{L}_1$-formula $\theta(\vec{x},\vec{y})$, and all $\mathscr{L}_1 \cup \{\phi_1\}$-formula $\delta(\vec{x},\vec{y})$,

$$\forall \vec{y}\,[\theta(\vec{x},\vec{y}) \not\leftrightarrow \phi_1(\vec{x},\vec{a})]\,,$$
$$\forall \vec{y}\,[\delta(\vec{x},\vec{y}) \not\leftrightarrow \phi_2(\vec{x},\vec{a})]\,,$$

where by an $\mathscr{L}_1 \cup \{\phi_1\}$-formula, we mean an $\mathscr{L}_2$-formula that is obtained by first using the realization set of ϕ_1 as a predicate, and then replacing the predicate everywhere by the formula $\phi_1(\vec{x},\vec{a})$.

But this type (of $\vec{a}$ over $\emptyset$) is finitely realizable in $\mathcal{M}_2$, thus there is some element in the universe of $\mathcal{W}_2$ satisfying the same set of formulas, which means $\mathcal{W}_2 ⧗ (\mathcal{W}_2|\mathscr{L}_1)$. □

In the case where $\mathscr{L}_1 = \mathscr{L}_\mathrm{r}$, $\mathscr{L}_2 = \mathscr{L}_\mathrm{r} \cup \{V\}$, with M being a field and V interpreted as a valuation ring on M, there are examples known to be the case that $\mathcal{M}_2 ⧖ \mathcal{M}_1$. The main subject of the thesis will be exploring examples where $\mathcal{M}_2 ⧖ \mathcal{M}_1$. But there are also cases where $\mathcal{M}_2 ⧗ \mathcal{M}_1$. Here we give an example provided by F. Delon. First, recall that for an index set I, and a family of ordered abelian groups G_i indexed by $i \in I$, the **lexicographic product** of $\{G_i\}_{i\in I}$ is the subgroup

$$\prod_{i\in I}{}' G_i$$

of the product $\prod_{i\in I} G_i$, consisting of elements of well-ordered support (i.e. the set of indexes at which the coordinates are non-zero is well-ordered), ordered by $(a_i) < (b_i)$ if $a_{i_0} < b_{i_0}$, where i_0 is the minimal index i at which $a_i \neq b_i$. One checks that $\prod'_{i\in I} G_i$ is indeed an ordered abelian group.

Example 1.3.11 (Delon)**.** Let K be any field. Let I be the ordinal sum $\omega + \omega$. Consider a set of elements $\{x_i\}_{i\in I}$ which are algebraically independent over the field K. Let $\Gamma = \bigoplus_{i\in I} \mathbf{Z}$ be lexicographically ordered. Then one can make the field $L := K(x_i, i \in I)$ into a valued field such that $v(x_i) = 1_i$, where 1_i is the smallest positive element of the i-th factor $\mathbf{Z}$ of Γ. Call this valuation V or v. It follows from the construction that V is not $\mathscr{L}_\mathrm{r}$-definable over L (See Proposition 2.2.13 for a full proof). It is also true that the structure $(L,+,-,\times,0,1,V) ⧗ (L,+,-,\times,0,1)$. We will give a more detailed analysis of this example in Chapter 2 (Proposition 2.2.13).

In the following chapters, we focus on the situation where $\mathscr{L}_1$ is an extension of $\mathscr{L}_\mathrm{r}$ (in fact, most of time $\mathscr{L}_1 = \mathscr{L}_\mathrm{r}$), $\mathscr{L}_2 = \mathscr{L}_1 \cup \{V\}$, where V is a valuation ring on the field M; in this situation, we try to prove that many examples are immediate extensions.

Chapter 2

Definable valuations[†]

Given the language of rings $\mathscr{L}_{\mathrm{r}} = \{+, -, \times, 0, 1\}$, consider its extension by a single symbol $\mathscr{L}_{\mathrm{div}} := \mathscr{L}_{\mathrm{r}} \cup \{|\}$. Let (K, V) be a valued field. One interprets the symbol '$|$' in $\mathscr{L}_{\mathrm{div}}$ on K, as $x \mid y$ if and only if there is some $z \in V$ such that $xz = y$, or equivalently, $v(x) \leq v(y)$. Let $\mathcal{K}$ be the $\mathscr{L}_{\mathrm{div}}$-structure on K. We are interested in knowing when $\mathcal{K}$ ⧖ $(\mathcal{K}|\mathscr{L}_{\mathrm{r}})$.

As we have seen before, one trivial case where $\mathcal{K}$ ⧖ $(\mathcal{K}|\mathscr{L}_{\mathrm{r}})$ is true is when V is already $\mathscr{L}_{\mathrm{r}}$-definable over K. But to determine when V is $\mathscr{L}_{\mathrm{r}}$-definable over K, is in fact a very non-trivial task. In this chapter, we focus on addressing this problem. Classically, this is known as the problem of definable valuations.

Definition 2.0.12. *A* **definable valuation** *is a valuation ring V on a field K which is $\mathscr{L}_{\mathrm{r}}$-definable over K.*

The main result of this chapter is the theorem stating that on a Henselian valued field (K, V), if vK contains a convex subgroup which is p-regular but not p-divisible, for some prime number p, then V is $\mathscr{L}_{\mathrm{r}}$-definable over K. This enables us to show further that if (K, V) is a Henselian valued field and vK is regular but not divisible, then V is in fact $\mathscr{L}_{\mathrm{r}}$-definable over K without parameters (i.e. 0-definable), generalizing the results of Fact 2.1.1 and Fact 2.1.10.

2.1 A brief overview

It seems that people became interested in definable valuation when they were considering Hilbert's Tenth Problem, its generalizations and problems related to defining the ring of integers for a number field, due to the strong connection between the valuation rings and the ring of integers.

The first and probably the simplest instance of a definable valuation (non-trivial, of course) is probably the fact that $\mathbf{Z}_p$ is $\mathscr{L}_{\mathrm{r}}$-definable over $\mathbf{Q}_p$. It is in fact, this

[†]The major results of this chapter have been re-written into a 4-page paper [Hong, 2013].

particular observation that inspired our main theorem of this chapter. For $\mathbf{Q}_p$, simply pick another prime $q \neq p$, then one argues that $\mathbf{Z}_p$ is exactly the set $\{x \in \mathbf{Q}_p \mid \exists y(y^q = px^q + 1)\}$. Note that this first-order definition uses a parameter. Ax proved a stronger statement, which could be rephrased as follows.

Fact 2.1.1 ([Ax, 1965])**.** Suppose that (K, V) is a Henselian valued field. If vK is isomorphic to $\mathbf{Z}$, then V is $\mathscr{L}_{\mathrm{r}}$-definable over K without parameters. In fact, as long as vK is a $\mathbf{Z}$-group (see the next section), then V is $\mathscr{L}_{\mathrm{r}}$-definable without parameters.

Definition 2.1.2 (see e.g. [Cassels and Fröhlich, 1967])**.** *A* **number field** *is a field which is a finite extension of* $\mathbf{Q}$*. A* **global field** *is a field which is either a number field or a finite extension of the function field* $F(T)$*, where* F *is a finite field. A* **non-archimedean local field** *is a completion of a global field with respect to a (non-trivial) valuation.*

The value group of a valuation on a global field or a non-archimedean local field is always isomorphic to $\mathbf{Z}$ (as an ordered abelian group). While the valuation on a non-archimedean local field is always Henselian, a non-trivial valuation on a global field is never Henselian. The valuation on a non-archimedean local field is $\mathscr{L}_{\mathrm{r}}$-definable for the simple reason mentioned above, but to prove the fact that a valuation on a global field is $\mathscr{L}_{\mathrm{r}}$-definable, one needs a certain amount of number theory.

Fact 2.1.3 (Rumely, [Rumely, 1980])**.** Every valuation ring on a global field is $\mathscr{L}_{\mathrm{r}}$-definable.

There are also some special cases about Henselian fields with real closed residue fields. See for example, [Jacob, 1979], [Jacob, 1981], [Delon and Farré, 1996] etc.. And some isolated cases on function fields, which are usually not very explicitly mentioned in the literature.

The next major advance was made by Koenigsmann in (mostly) two papers [Koenigsmann, 1995] and [Koenigsmann, 1994], using p-Henselian valuations and t-Henselian valuations. The notion of p-Henselian valuation was first introduced in [Wadsworth, 1983]. In [Koenigsmann, 1995], a result about definable p-Henselian valuations was proved, which has application in proving many Henselian valuations to be definable as well. We explain his main results here.

Recall that for a field K and a prime number p, the **maximal Galois p-extension**, or the **p-closure**, of K, usually denoted by $K(p)$, is the compositum (inside a fixed algebraic closure) of all finite p-extensions of K (namely, of all Galois extension of degree a p-power). A field K is said to be **p-closed** if $K = K(p)$.

Definition 2.1.4. *Suppose that* (K, V) *is a valued field. The valuation ring* V *is called a* **p-Henselian valuation** *ring if there is exactly one extension of* V *in the maximal Galois p-extension* $K(p)$ *of* K*.*

It follows easily that every Henselian valued field is also p-Henselian.

Fact 2.1.5 (p-Hensel's Lemma, [Koenigsmann, 1995])**.** Suppose that (K, V) is a valued field. Then (K, V) is p-Henselian if and only if for every polynomial $f(X) \in V[X]$ which splits in $K(p)$, and has a simple zero $\alpha \in Kv$, $f(X)$ has a zero $a \in V$ with $\mathrm{res}_v(a) = \alpha$.

Fact 2.1.6 ([Koenigsmann, 1995])**.** Suppose that K is not p-closed. Let

$$\mathcal{H}_{nc,p} := \{V \mid V \text{ is } p\text{-Henselian, } Kv \neq (Kv)(p)\ \},$$
$$\mathcal{H}_{c,p} := \{V \mid V \text{ is } p\text{-Henselian, } Kv = (Kv)(p)\ \}.$$

Then with respect to set inclusion, $\mathcal{H}_{nc,p}$ is linearly ordered, and $\mathcal{H}_{c,p}$ is upper directed and has a maximal element if non-empty. Furthermore, every element in $\mathcal{H}_{nc,p}$ is larger than or equal to (i.e. contains) every element in $\mathcal{H}_{c,p}$. If $\mathcal{H}_{c,p} = \emptyset$, then $\mathcal{H}_{nc,p}$ has a minimal element.

Definition 2.1.7 ([Koenigsmann, 1995])**.** *For any field K not p-closed, define the* **canonical p-Henselian valuation***, O_p, to be, the largest element in $\mathcal{H}_{c,p}$ if $\mathcal{H}_{c,p} \neq \emptyset$, or the smallest element in $\mathcal{H}_{nc,p}$ if $\mathcal{H}_{c,p} = \emptyset$. Naturally if K is p-closed, then we define O_p to be K.*

When we write $\zeta_p \in K$ for a field K and some prime number p, we mean that there is a primitive pth-root of unity in K, i.e. there is some ζ_p such that $\zeta_p^p = 1$ but $\zeta_p^k \neq 1$ for all $0 < k < p$.

A field is **Euclidean** if it is an ordered field whose every positive element is a square.

The main result of [Koenigsmann, 1995] is

Fact 2.1.8 ([Koenigsmann, 1995])**.** Suppose that $\zeta_p \in K$ or $\mathrm{char}(K) = p$. Then except in the case where $p = 2$ and Ko_p is Euclidean, O_p is always $\mathscr{L}_{\mathrm{r}}$-definable over K.

There are several other results obtained by Koenigsmann.

Fact 2.1.9 ([Koenigsmann, 1994])**.** Suppose that K is a field which is neither real closed nor separably closed, V is a non-trivial Henselian valuation ring on K. Then there is some valuation ring W on K which is $\mathscr{L}_{\mathrm{r}}$-definable such that V and W induce the same topology on K.

An ordered abelian group is **divisible** if for each element a, and each positive natural number n, there is another element b such that $nb = a$.

Fact 2.1.10 ([Koenigsmann, 2004])**.** Suppose that (K, V) is a Henselian valued field. Suppose that vK is archimedean, and not divisible. Then V is $\mathscr{L}_{\mathrm{r}}$-definable over K without parameters.

The following is an interesting case (which can be safely ignored within the context of this thesis):

Fact 2.1.11 ([Scanlon, 2008]). Every Rosenlicht differential valuation on a differentially closed field is trivial.

During the preparation of this thesis, there are two more papers about definable valuations being published; one is about Henselian valued fields with finite or pseudo-finite residue fields by Cluckers, Derakhshan, Leenknegt and Macintyre (see [Cluckers et al., 2013]); and the other is the work of Jahnke and Koenigsmann (see [Jahnke and Koenigsmann, 2012]) about Galois theoretic characterization of definable valuations. Both of these results adapt the viewpoint through the residue fields of the value fields, proving that under certain assumptions on the residue fields, the valuation rings are definable (most of the time without parameters) in the language of rings over the underlying field. The work of this thesis distinguishes itself as an approach through the value groups instead of residue fields.

2.2 Regular ordered abelian groups

Our main theorem concerns Henselian valuations with groups related to regular ordered abelian groups. Here we collect the results needed later for the next section. The notion of regular ordered groups originates from the work of [Robinson and Zakon, 1960] (in which they give a classification of these groups up to elementary equivalence), which turns out to be strongly related to the study of model theory of ordered abelian groups in general. Here we only include relevant results.

We refer the reader to the first chapter for some discussion on ordered abelian groups as well.

Let $\mathscr{L}_{\mathrm{OAG}}$ be the language of ordered abelian groups, i.e. $\{+, -, <, 0\}$.

Definition 2.2.1. *Let $n > 1$ be a positive integer. An element a in an ordered abelian group G is called* n**-divisible** *(in G) if there is some $b \in G$ such that $nb = b + b + \cdots + b = a$. Then G is* n**-divisible** *if every element of it is n-divisible. An ordered abelian group G is* **divisible***, if it is n-divisible for all positive integers n, or equivalently, p-divisible for all prime numbers p. Sometimes, we say G is* **non-divisible** *to mean that G is not divisible.*

Definition 2.2.2 ([Zakon, 1961]). *Let $n > 1$ be a positive integer. An ordered abelian group G is* n**-regular** *if every infinite convex subset of it has at least one element which is n-divisible in G.*

An ordered abelian group is called a **regular ordered abelian group**[1] *if it is p-regular for all primes p.*

[1]Originally, these groups were called "regularly ordered groups". The author has decided that "regular ordered abelian groups" is better, as the former seems to suggest that there is one ordering on G which is regular, but not necessarily the one in question.

Fact 2.2.3. It is well-known that every archimedean ordered abelian group is regular; see for example, [Robinson and Zakon, 1960] or [Zakon, 1961]. But the converse is not true, although it is true that every regular ordered abelian group is elementarily equivalent to an archimedean ordered abelian group in $\mathscr{L}_{\text{OAG}}$.

Definition 2.2.4 ([Ax and Kochen, 1965])**.** *A* **Z-group** *is an ordered abelian group G with a smallest positive element (i.e. G is discrete) with the property that $[G : nG] = n$ for all positive integers n.*

Fact 2.2.5. It can be proved that (see e.g. [Robinson and Zakon, 1960], [Zakon, 1961]) **Z**-groups are exactly the regular discrete ordered abelian groups. Equivalently, these are all the ordered abelian groups $\mathscr{L}_{\text{OAG}}$-elementarily equivalent to **Z**.

In particular, if G is p-regular, with a smallest positive element γ, then for any $g \in G$, exactly one of the p elements: $g, g + \gamma, g + 2\gamma, \ldots, g + (p-1)\gamma$ is p-divisible.

Fact 2.2.6 (cf. [Conrad, 1962])**.** Suppose that G is an ordered abelian group. Then G is p-regular if and only if for any H which is a non-zero convex subgroup of G, G/H is p-divisible.

Definition 2.2.7 ([Schmitt, 1984])**.** *For an ordered abelian group G, and an element $g \in G$, let $A(g)$ be the largest convex subgroup that does not contain g; let $B(g)$ be the smallest convex subgroup of G containing g. One can check that $A(g)$ is the collection of elements $h \in G$ such that for all natural numbers $n > 0$, $nh < g$; $B(g)$ is the collection of elements $h \in G$ such that there is some natural number n with $ng > h$.*

For any integer $n > 1$, let $A_n(g)$ be the smallest convex subgroup C of G such that $B(g)/C$ is n-regular; let $B_n(g)$ be the largest convex subgroup C of G such that $C/A(g)$ is n-regular.

$A_n(0)$ is defined to be the empty set and $B_n(0)$ is defined to be $\{0\}$.

Definition 2.2.8. *Given an ordered abelian group G, the collection of $A_n(g)$ for all non-negative elements $g \in G$, is clearly a linearly ordered set with respect to inclusion. We denote this ordered set by $A_n(G)$. The order type of $A_n(G)$ is called the* **principal n-regular rank** *of G.*

The following is a simple fact observed by Schmitt.

Fact 2.2.9 ([Schmitt, 1984])**.** $A_n(g)$ and $B_n(g)$ defined above are $\mathscr{L}_{\text{OAG}}$-definable in G using g as the only parameter.

We also have the following.

Fact 2.2.10 ([Delon and Farré, 1996])**.** Suppose that G is an ordered abelian group and H is a convex subgroup of G. If H is $\mathscr{L}_{\text{OAG}}$-definable over G, then there is some $n > 1$ such that

$$H = \bigcap_{g \notin H} A_n(g).$$

The first-order definability of sets in a valued field is related to the first-order definability of sets in the valued group. We have the following well-known proposition.

Proposition 2.2.11. *Suppose that (K, V) is a valued field. If $C \subseteq vK \cup \{\infty\}$ is definable in $vK \cup \{\infty\}$ in the language $\mathscr{L}_{\mathrm{OAG}} \cup \{\infty\}$, then the set $v^{-1}(C)$ is definable over K, in $\mathscr{L}_{\mathrm{div}}$.*

Proof. The proof is by induction on the complexity of formulas. □

Proposition 2.2.12. *Suppose that (K, V) is a Henselian valued field with* $\mathrm{char}(Kv) = 0$*, and W is a valuation ring containing V. Then W is $\mathscr{L}_{\mathrm{r}} \cup \{V\}$-definable over K if and only if the convex subgroup of vK corresponding to W is definable in the language of ordered abelian groups in vK.*

Proof. One direction is given by Proposition 2.2.11. The other direction mentioned in the proof of Theorem 4.4 in [Delon and Farré, 1996], which in fact comes from a result of Delon's thesis [Delon, 1982] about quantifier elimination of Henselian valued fields of residue characteristic 0. □

Now, we come back to Delon's example of a valued field with an intermediate structure between itself and the field structure (Example 1.3.11).

Proposition 2.2.13 (Delon)**.** *Let K be any field. Let I be the ordinal sum of two ω. Consider a set of elements $\{x_i\}_{i \in I}$ which are algebraically independent over the field K. Let $\Gamma = \bigoplus_{i \in I} \mathbf{Z}$ be lexicographically ordered. Notice that Γ is discrete, but not regular. Then the field $L := K(x_i, i \in I)$ has a valuation V defined by $v(x_i) = 1_i$, where 1_i is the smallest positive element of the i-th factor $\mathbf{Z}$ of Γ. It follows from the construction that the valuation ring V is not $\mathscr{L}_{\mathrm{r}}$-definable over L. It is also true that the structure $(L, +, -, \times, 0, 1, V)$ ⧗ $(L, +, -, \times, 0, 1)$.*

Proof. First, V is not $\mathscr{L}_{\mathrm{r}}$-definable because there are infinitely many x_i algebraically independent over K and one can permute infinitely many of them while fixing finitely many of them at the same time, to get the same field structure but different valuation rings.

Let W be the valuation ring corresponding to the convex subgroup Δ which consists of elements whose i-th coordinate is 0_i if i is in the first copy of ω.

Then by the same reason, W is not $\mathscr{L}_{\mathrm{r}}$-definable over K and V is not $\mathscr{L}_{\mathrm{r}} \cup \{W\}$-definable over K. It is enough to show that W is $\mathscr{L}_{\mathrm{r}} \cup \{V\}$-definable over K. This is because Δ is definable in vK, by using the fact that it is the union of $A_n(g)$, where $g \in vK$ satisfies that $A_n(g) = A_n(1_{i_0})$, with i_0 being the element 0 in the second copy of ω. □

It is not always true that a Henselian valued field with non-divisible value group will have its valuation ring $\mathscr{L}_{\mathrm{r}}$-definable over the field. Indeed, we give an example below. But under certain certain circumstances, as in the next section, we give some sufficient conditions.

Example 2.2.14. Let K be the field $\mathbf{Q}((t^G))$ with the natural valuation V, where G is the lexicographic product $\mathbf{Z}\times\mathbf{Q}\times\mathbf{Q}$. Let H be the convex subgroup $\{0\}\times\{0\}\times\mathbf{Q}$ of G. Then by Proposition 2.2.10, H is not $\mathscr{L}_{\mathrm{OAG}}$-definable in G. By Proposition 2.2.12, the valuation W containing V, corresponding to H is not $\mathscr{L}_{\mathrm{r}}$-definable over K.

2.3 The main theorem

Definition 2.3.1. *Fix a prime number p. An ordered abelian group is* **p-rendible** *if it is p-regular but not p-divisible. An ordered abelian group is* **rendible** *if it is regular but not divisible.*[2]

Notice that if an ordered abelian group G is not p-divisible, then $G \neq \{0\}$. Thus a p-rendible group is always not trivial. Every discrete ordered abelian group contains a convex subgroup which is rendible, e.g. the smallest convex subgroup containing the smallest positive element.

In the following, p always denotes a fixed prime number. The following is an elementary fact in field theory.

Fact 2.3.2 (see e.g. [Karpilovsky, 1989])**.** (1) Suppose that K is a field containing a primitive p-th root of unity. Then for any $a \in K\backslash P_p(K)$, $K[X]/(X^p - a)$ is a cyclic extension of degree p. In particular, $X^p - a$ splits in $K(p)$.

(2) Suppose that K is field of characteristic p. Then for any $a \in K$, if $X^p - X - a$ has a solution in K, then it splits in K; if $X^p - X - a$ does not have a solution in K, then $K[X]/(X^p - X - a)$ is a cyclic extension of degree p. In any case, $X^p - X - a$ splits in $K(p)$.

Lemma 2.3.3. *Suppose that the valued field (K, V) satisfies one of the following conditions:*

- *V is Henselian;*
- *V is p-Henselian, $\zeta_p \in K$, $\mathrm{char}(Kv) \neq p$;*
- *V is p-Henselian, $\mathrm{char}(K) = p$.*

If there is some $\epsilon \in K$ with $p \nmid v(\epsilon)$, then the set

$$\Phi_\epsilon := \{x \in K \mid v(\epsilon x^p) > 0\}$$

is $\mathscr{L}_{\mathrm{r}}$-definable over K.

[2] The word "rendible" is the adjective of "rend", which is short for "**re**gular **n**on-**d**ivisible".

Proof. First, suppose that V is Henselian. Consider the formula

$$\phi_\epsilon(x) := \text{“}\exists y(y^p - y^{p-1} = \epsilon x^p)\text{”}.$$

We show that ϕ_ϵ is a definition of Φ_ϵ.

On one hand, suppose $a \in K$ such that $\phi_\epsilon(a)$ is true. Then there is some $b \in K$ such that $b^p - b^{p-1} = \epsilon a^p$. Notice that because $p \nmid v(\epsilon)$, $v(\epsilon a^p)$ can not equal zero. If $v(\epsilon a^p) < 0$, then $v(b^p - b^{p-1}) < 0$, which implies that $v(b) < 0$ and $v(b^p - b^{p-1}) = v(b^p) = pv(b) = v(\epsilon a^p) = v(\epsilon) + pv(a)$. But this implies that $p \mid v(\epsilon)$, contradicting the assumption. Thus $v(\epsilon a^p) > 0$, and $a \in \Phi_\epsilon$.

On the other hand, suppose that $a \in K$ such that $a \in \Phi_\epsilon$, namely $v(\epsilon a^p) > 0$, then consider the polynomial $f(Y) = Y^p - Y^{p-1} - \epsilon a^p \in V[Y]$. The polynomial $\mathrm{res}_v(f)(Y) = Y^p - Y^{p-1} = Y^{p-1}(Y-1)$ has a simple zero $Y = 1$ in Kv. Thus by Hensel's Lemma, there is some $b \in K$ such that $b^p - b^{p-1} = \epsilon a^p$, which means $\phi_\epsilon(a)$ is true in K.

The proof for the other two cases of the hypothesis are more or less the same. The formula for the second case is "$\exists y(y^p - 1 = \epsilon x^p)$" and the formula for the third case is "$\exists y(y^p - y^{p-1} = \epsilon x^p)$". For the last two cases, instead of Hensel's Lemma, we use the p-Hensel's Lemma (see Fact 2.1.5) and Fact 2.3.2. □

Corollary 2.3.4. *Suppose that the valued field (K, V) satisfies one of the following conditions:*

- *V is Henselian;*
- *V is p-Henselian, $\zeta_p \in K$, $\mathrm{char}(Kv) \neq p$;*
- *V is p-Henselian, $\mathrm{char}(K) = p$.*

If vK is discrete, then V is $\mathscr{L}_\mathrm{r}$-definable over K.

Proof. Let $v(\epsilon)$, with $\epsilon \in K$, be the smallest positive element in vK. Then $p \nmid v(\epsilon)$ for some prime (in fact all primes) p. So by Lemma 2.3.3, $\Phi_\epsilon = \{x \in K \mid v(\epsilon x^p) > 0\}$ is $\mathscr{L}_\mathrm{r}$-definable over K.

Clearly $V \subseteq \Phi_\epsilon$. On the other hand, if $a \notin V$, then $v(\epsilon a^p) < 0$ since $v(\epsilon)$ is the smallest positive element in vK, which means $a \notin \Phi_\epsilon$.

Therefore $V = \Phi_\epsilon$ and we conclude that V is $\mathscr{L}_\mathrm{r}$-definable over K. □

Now we are ready to prove our main theorem of this chapter:

Theorem 2.3.5. *Suppose that the valued field (K, V) satisfies one of the following conditions:*

- *V is Henselian;*
- *V is p-Henselian, $\zeta_p \in K$, $\mathrm{char}(Kv) \neq p$;*

- *V is p-Henselian,* $\operatorname{char}(K) = p$.

If vK contains a convex p-rendible subgroup C, then V is $\mathscr{L}_{\mathrm{r}}$-definable over K.

Proof. The case where vK is discrete has already been proved in Corollary 2.3.4. Thus we may assume that vK is dense.

Let $\epsilon \in K$ be such an element that $v(\epsilon) \in C$ and $p \nmid v(\epsilon)$. By Lemma 2.3.3, the set $\Phi_\epsilon = \{x \in K \mid v(\epsilon x^p) > 0\}$ is $\mathscr{L}_{\mathrm{r}}$-definable over K.

Suppose first that V is Henselian. Let Ψ_ϵ be the set $\{\epsilon x^p \mid v(\epsilon x^p) > 0\}$. Define the set

$$\Omega_\epsilon := \left\{x^p - x^{p-1} \in K \,\middle|\, (\exists y \in K)\,(\exists z \in \Psi_\epsilon)\left[z\left(y^p - y^{p-1}\right) = x^p - x^{p-1}\right]\right\}.$$

It is clear that Ω_ϵ is $\mathscr{L}_{\mathrm{r}}$-definable over K. We show that $\Omega_\epsilon = \mathfrak{m}_v$. The valuation ring is then $\mathscr{L}_{\mathrm{r}}$-definable as $K\backslash(\mathfrak{m}_v\backslash\{0\})^{-1}$.

On one hand, if $a \in \Omega_\epsilon$, then there is some $x \in K$ such that $x^p - x^{p-1} = a$ and there are $y \in K$ and $z \in \Psi_\epsilon$ such that $z(y^p - y^{p-1}) = x^p - x^{p-1}$. Notice that $\Psi_\epsilon \subseteq \mathfrak{m}_v$. So if it were the case that $v(a) \leq 0$, then $v(z(y^p - y^{p-1})) = v(x^p - x^{p-1}) = v(x^p) \leq 0$, which means $v(z) + v(y^p - y^{p-1}) \leq 0$; because $v(z) > 0$, this implies that $v(y^p - y^{p-1}) < 0$; thus we get $v(z) + pv(y) = pv(x)$. But because $p \nmid v(z)$, the last equality is impossible, hence a contradiction.

On the other hand, if $a \in \mathfrak{m}_v$, then by Hensel's Lemma, there is some $x \in K$ such that $x^p - x^{p-1} = a$. Because C is p-rendible and vK is dense, there is some element $g \in K$ satisfying $-v(\epsilon) < v(g^p) < v(a) - v(\epsilon)$. Let z be the element ϵg^p, then $0 < v(z) < v(a)$. Thus, by Hensel's Lemma again, there is some $y \in K$ such that $y^p - y^{p-1} = a/z$, because $v(a/z) > 0$. Therefore, $a \in \Omega_\epsilon$.

For the case where V is p-Henselian, $\zeta_p \in K$ and $\operatorname{char}(Kv) \neq p$, one can prove similarly (using p-Hensel's Lemma) that $\mathfrak{m}_v$ is the set

$$\{x^p - 1 \in K \,|\, (\exists y \in K)\,(\exists z \in \Psi_\epsilon)\,[z\,(y^p - 1) = x^p - 1]\}.$$

Similarly, if V is p-Henselian and $\operatorname{char}(K) = p$, then one proves that $\mathfrak{m}_v$ is the set

$$\{x^p - x \in K \,|\, (\exists y \in K)\,(\exists z \in \Psi_\epsilon)\,[z\,(y^p - y) = x^p - x]\}.$$

In short, in all cases, V is indeed $\mathscr{L}_{\mathrm{r}}$-definable over K, using ϵ as the only parameter. □

The conclusion of the main theorem is that the valuation ring is $\mathscr{L}_{\mathrm{r}}$-definable with a single parameter. We can in fact improve the result to show that it is $\mathscr{L}_{\mathrm{r}}$-definable without parameters (0-definable), with stronger assumptions of course. To do this, we use Ax's trick from [Ax, 1965] and include the details of the argument.

Lemma 2.3.6. *Suppose that the valued field (K, V) satisfies the following conditions:*

- *V is Henselian,* $\operatorname{char}(Kv) \neq p$;

- *V is p-Henselian, $\zeta_p \in K$,* $\mathrm{char}(Kv) \neq p$.

If vK is p-rendible and discrete, then V is $\mathscr{L}_{\mathrm{r}}$-definable over K without parameters.

Proof. For any $\epsilon \in K$, we define the following sets:

$$\begin{aligned}\Phi'_\epsilon &:= \{x \in K \mid v(\epsilon x^p) > 0\},\\ \Phi_\epsilon &:= \{x \in K \mid \exists y (y^p - 1 = \epsilon x^p)\},\\ \Phi''_\epsilon &:= \{x \in K \mid v(\epsilon x^p) \geq 0\}.\end{aligned}$$

It follows from Hensel's Lemma or p-Hensel's Lemma that we have $\Phi'_\epsilon \subseteq \Phi_\epsilon$; if $p \nmid v(\epsilon)$, then $\Phi'_\epsilon = \Phi_\epsilon = \Phi''_\epsilon$.

We show that $\Phi_\epsilon \subseteq \Phi''_\epsilon$ provided that ϵ is not a pth-power. Suppose towards a contradiction that $x \in \Phi_\epsilon$ but $v(\epsilon x^p) < 0$. Then since there is some $y \in K$ such that $y^p - 1 = \epsilon x^p$, $0 > pv(y) = v(\epsilon) + pv(x)$. Therefore, $p \mid v(\epsilon)$. We have

$$1 - \left(\frac{1}{y}\right)^p = \epsilon \left(\frac{x}{y}\right)^p,$$

thus

$$\mathrm{res}_v(1) = \mathrm{res}_v\left(\epsilon\left(\frac{x}{y}\right)^p\right).$$

Therefore the polynomial $f(Y) = Y^p - \epsilon x^p/y^p \in V[Y]$ has the property that $\mathrm{res}_v(f)$ has a simple root in Kv, hence it has a root in K as well. But this contradicts the assumption that ϵ is not a pth-power.

To prove the lemma, we use Ax's trick in [Ax, 1965], taking the union of Φ_ϵ for suitable ϵ. Define the set

$$\begin{aligned}R := \{x \in K \mid x \in \Phi_\epsilon &\text{ for some } \epsilon, \text{ with the property that}\\ &\epsilon \text{ is not a } p\text{th-power and } \Phi_\epsilon \text{ is closed under multiplication }\}.\end{aligned}$$

Then $V = R$. To see this, on one hand, if $v(\epsilon)$ is the smallest positive element of the value group, then we have seen in Corollary 2.3.4 already that Φ_ϵ is actually V. It follows that $V \subseteq R$.

On the other hand, we are done if we can show that for any ϵ not a pth-power, $\Phi_\epsilon \subseteq V$ if Φ_ϵ is closed under multiplication. To see this, we notice that since $\Phi_\epsilon \subseteq \Phi''_\epsilon$, if $v(\epsilon) \leq 0$, then $\Phi_\epsilon \subseteq V$ is already true, because then for any $x \in \Phi_\epsilon$, $pv(x) \geq -v(\epsilon) \geq 0$. So we just to need to focus on the situation where $v(\epsilon) > 0$. Notice that if $v(\epsilon) > 0$, then for any $x \in \Phi_\epsilon$, $pv(x) \geq -v(\epsilon)$; in fact, for any $x \in \Phi_\epsilon$, $pv(x) > -v(\epsilon)$, because otherwise $x^2 \in \Phi_\epsilon$ as Φ_ϵ is closed under multiplication, but $v(\epsilon(x^2)^p) = -v(\epsilon) < 0$, contradicting the fact that $\Phi_\epsilon \subseteq \Phi''_\epsilon$.

Therefore, suppose towards a contradiction that there is some $w \in \Phi_\epsilon$ with $v(w) < 0$, then $v(\epsilon) > 0$ and $pv(w) > -v(\epsilon)$. Because $\Phi'_\epsilon \subseteq \Phi_\epsilon$, and vK is p-regular, we can then in particular find one p-divisible element in the interval $(-v(\epsilon), -v(\epsilon)+p\gamma]$; we can thus assume that w satisfies the condition that

$$0 > -v(\epsilon) + p\gamma \geq pv(w) > -v(\epsilon),$$

where $\gamma \in vK$ is the smallest positive element in vK. But then since Φ_ϵ is closed under multiplication, w^n is in Φ_ϵ as well, for all positive integer n. But since $-v(\epsilon) < p(-\gamma)$, $v(w^p) \leq -v(\epsilon) + p\gamma$, and γ is the smallest positive element in vK, we have

$$(p+1)v(w^p) \leq (p+1)(-v(\epsilon)+p\gamma) \leq -v(\epsilon) < 0.$$

This means $v(\epsilon(w^{p+1})^p) \leq 0$, contradicting the fact that $w^{p+1} \in \Phi_\epsilon$ as Φ_ϵ is closed under multiplication and $v(\epsilon x^p) \neq 0$ for all $x \in \Phi_\epsilon$.

We therefore conclude that R is indeed exactly V, which is $\mathscr{L}_{\mathrm{r}}$-definable over K without parameters. □

Lemma 2.3.7. *Suppose that (K, V) is a Henselian valued field with vK being p-rendible and dense. Then V is $\mathscr{L}_{\mathrm{r}}$-definable over K without parameters.*

Proof. For any $\epsilon \in K$, we define the following sets:

$$\begin{aligned}
\Phi'_\epsilon &= \{x \in K \mid v(\epsilon x^p) > 0\}, \\
\Psi'_\epsilon &= \{\epsilon x^p \mid x \in \Phi'_\epsilon\}, \\
\Phi_\epsilon &= \{x \in K \mid \exists y (y^p - y^{p-1} = \epsilon x^p)\}, \\
\Psi_\epsilon &= \{\epsilon x^p \mid x \in \Phi_\epsilon\}.
\end{aligned}$$

Then Ψ'_ϵ contains arbitrarily small positively valued elements in K, because vK is p-regular, as we have seen before. Also $\Psi'_\epsilon \subseteq \Psi_\epsilon$, and if $p \nmid v(\epsilon)$, then $\Phi_\epsilon = \Phi'_\epsilon$.

Define the set

$$\Omega_\epsilon = \left\{ x^p - x^{p-1} \,\middle|\, (\exists y \in K)\,(\exists z \in \Psi_\epsilon)\left[\left(y^p - y^{p-1}\right) z = x^p - x^{p-1}\right] \right\}.$$

We have seen that because of Hensel's Lemma, $\mathfrak{m}_v \subseteq \Omega_\epsilon$, and when it is the case that $p \nmid v(\epsilon)$, we also have $\Omega_\epsilon = \mathfrak{m}_v$.

Thus the set $\bigcap_{\epsilon \neq 0} \Omega_\epsilon$ is exactly $\mathfrak{m}_v$, which implies that V is $\mathscr{L}_{\mathrm{r}}$-definable over K without parameters. □

Remark 2.3.8. One can prove similarly, using the p-Hensel's Lemma, that if (K, V) is p-Henselian, $\zeta_p \in K$, $\mathrm{char}(Kv) \neq p$, vK is p-rendible, then V is $\mathscr{L}_{\mathrm{r}}$-definable over K without parameters.

Theorem 2.3.9. *Suppose that (K, V) is Henselian with vK being rendible. Then V is $\mathscr{L}_{\mathrm{r}}$-definable over K without parameters.*

Proof. If vK is dense, then by Lemma 2.3.7, V is $\mathscr{L}_{\mathrm{r}}$-definable over K without parameters.

If vK is discrete, then the value group is p-rendible for all primes p. Choose one p, such that $\mathrm{char}(Kv) \neq p$. By Lemma 2.3.6, V is also $\mathscr{L}_{\mathrm{r}}$-definable over K without parameters. □

Because every **Z**-group is rendible and every non-divisible archimedean ordered abelian group is also rendible, Theorem 2.3.9 is a generalization of both Fact 2.1.1 and Fact 2.1.10.

2.4 Some miscellaneous connections

The results in the previous section give sufficient conditions on the value group for a valuation ring to be $\mathscr{L}_{\mathrm{r}}$-definable. None of these conditions are necessary. Indeed, given any ordered abelian group, there is always a valued field whose value group is that group and whose valuation ring is definable without parameters.

Remark 2.4.1. Given an ordered abelian group G, consider the value field $K := \mathbf{Q}((t^G))$ with the natural valuation ring V. Then K is an ordered field with respect to the lexicographic ordering induced from K. By Lemma 4.3.6 of [Engler and Prestel, 2005], every Henselian valuation ring on K is convex, hence comparable (in terms of set inclusion) to the natural valuation ring V. But V is the smallest 2-Henselian valuation with a residue field which is not 2-closed. Clearly Kv is not Euclidean, and $\zeta_2 \in K$. We thus conclude that V is the canonical 2-Henselian valuation of K, which is $\mathscr{L}_{\mathrm{r}}$-definable over K without parameters, by Fact 2.1.8.

Under certain circumstances, Theorem 2.3.5 follows from Fact 2.1.8. Let us begin with a lemma.

Lemma 2.4.2. *If K is p-closed, then for any valuation ring V on K, Kv is p-closed; if furthermore $\zeta_p \in K$ or* $\mathrm{char}(K) = p$*, then vK is p-divisible.*

Proof. For the fact that Kv is p-closed, see Theorem 4.2.6 of [Engler and Prestel, 2005]. Here we prove that vK is p-divisible if $\zeta_p \in K$ or $\mathrm{char}(K) = p$.

If $\zeta_p \in K$ (and $\mathrm{char}(K) \neq p$), suppose towards a contradiction that there is some $\gamma \in vK$ which is not p-divisible. Then pick some $a \in K$ such that $v(a) = \gamma$; the polynomial $X^p - a$ is separable over K, so it splits over the separable closure of K. Let $L = K(\lambda)$ be the splitting field of $X^p - a$. By Theorem 2.4 of Chapter 7 in [Karpilovsky, 1989], L/K is a cyclic extension of degree diving p. But then $[L : K]$ has to be p, which in turn implies that $\lambda \in K(p) = K$, a contradiction.

If $\mathrm{char}(K) = p$, then again, suppose towards a contradiction that there is some $\gamma \in vK$ which is not p-divisible and negative. Then pick some $a \in K$ such that $v(a) = \gamma$; the polynomial $X^p - X - a$ can not have a solution in K (otherwise $p \mid v(a)$). But then Theorem 2.7 of Chapter 7 in [Karpilovsky, 1989] implies that $X^p - X - a$ gives rise to a p extension, which must be contained in $K(p) = K$, a contradiction. □

Example 2.4.3. It is possible for a p-closed field, not to have a primitive p-th root of unity, and to be of characteristic other than p, in which case, the value group could be not p-divisible. An example given by Koenigsmann to the author through personal communication is the following: consider a prime $l \neq p$ and $l, p > 2$ satisfying $l \not\equiv 1$ mod p, then $\mathbf{Q}_l(p)$ does not have a primitive p-th root of unity. It can be proved (see e.g. [Serre, 1979]) that $\mathbf{Q}_l(p)$ is unramified over $\mathbf{Q}_l$, which means that the value group is still $\mathbf{Z}$, not p-divisible.

Corollary 2.4.4. *Let K be a field which is not p-closed. Suppose that $\zeta_p \in K$, or $\mathrm{char}(K) = p$, or when $p = 2$, the residue field of O_p is not Euclidean. If V is a p-Henselian valuation on K with vK containing a convex p-rendible subgroup C, then V is not finer than O_p (i.e. $O_p \subseteq V$) and is $\mathscr{L}_{\mathrm{r}}$-definable over K.*

Proof. We first show that if $\zeta_p \in K$, then for any valuation ring W with $\mathrm{char}(Kw) \neq p$, Kw has a primitive p-th root of unity. This is because, if $\zeta_p \in K$, then $\mathrm{res}_w(\zeta_p)^p = 1$ is also true, which means if $\mathrm{res}_w(\zeta_p) \neq 1$, then Kw has a primitive p-th root of unity. But it is impossible for $\mathrm{res}_w(\zeta_p)$ to be 1, otherwise we have $0 = \mathrm{res}_w(0) = \mathrm{res}_w(1 + \zeta_p + \zeta_p^2 + \cdots + \zeta_p^{p-1}) = p$ in Kw, which contradicts the assumption that $\mathrm{char}(Kw) \neq p$.

Since V is always comparable with O_p by Fact 2.1.6, there are two cases.

Suppose that $O_p \subseteq V$. Then there exists some convex subgroup Δ of the value group o_pK such that $(o_pK)/\Delta$ is isomorphic to vK. Pick any element $a \in K$ such that $0 < v(a) + \Delta \in C$ and $v(a) + \Delta$ is not p-divisible in C. Let $g = v(a)$. Then $A_n(g)$ in o_pK is exactly Δ, since $B(g)$ in o_pK has the property that $B(g)/\Delta$ is not p-divisible and p-regular in vK, which implies that $A_n(g)$ is at least Δ (Fact 2.2.6) and at most Δ (p-regularity). But then by Theorem 2.2.9, Δ is definable, and hence V is definable from O_p. So V is $\mathscr{L}_{\mathrm{r}}$-definable over K (using a as the only parameter).

Suppose now that $V \subsetneq O_p$. Then the residue field (Ko_p) is p-closed by the definition of O_p. Then O_p corresponds to a convex subgroup Δ of vK, which also contains a convex p-rendible subgroup. Also, V induces a valuation on Ko_p with the value group being Δ. But Δ is not p-divisible, therefore Ko_p is not p-closed; otherwise, since either $\zeta_p \in Ko_p$ or $\mathrm{char}(Ko_p) = p$, Δ has to be p-divisible. Therefore, we get a contradiction which means that this situation can not occur. □

Combining our main theorem and Corollary 2.4.4, we get the following.

Corollary 2.4.5. *Suppose that the valued field (K, V) satisfies one of the following conditions:*

- *V is Henselian;*
- *V is p-Henselian, $\zeta_p \in K$;*
- *V is p-Henselian, $\mathrm{char}(K) = p$.*

If vK contains a convex p-rendible subgroup C, then V is $\mathscr{L}_{\mathrm{r}}$-definable over K (using a single parameter).

In terms of defining a valuation ring up to the equivalence of topologies, we have the following results. Let us still begin with a lemma.

Lemma 2.4.6. *Suppose that G is an ordered abelian group with a non-dense principal p-regular rank. Suppose that G is not p-divisible. Then there exists some convex subgroup C of G such that G/C contains a convex p-rendible subgroup.*

Proof. It is enough to show that $B(g)/A_p(g)$ is not p-divisible for some $g > 0$. By assumption, there are some $g_1 < g_2$ such that $A_p(g_1) \subsetneq A_p(g_2)$ and for all $g_1 < g \in A_p(g_2)$, $A_p(g) = A_p(g_1)$. We may assume that $g_1 \neq 0$. Because $B(g_2)/A_p(g_1)$ is not p-regular, the equality

$$B(g_2)/A_p(g_2) = \frac{B(g_2)/A_p(g_1)}{A_p(g_2)/A_p(g_1)}$$

implies that either $B(g_2)/A_p(g_2)$ is not p-divisible, or for some $g_1 < g \in A_p(g_2)$, we have that $B(g)/A_p(g) = B(g)/A_p(g_1)$ is not p-divisible. □

Remark 2.4.7. It is actually possible for an ordered abelian group G to be not p-divisible and yet G/C p-divisible for all non-zero convex subgroup C. An example of such a group is given in [Conrad, 1955]. A similar example was also provided by Delon through personal communication.

Inspired by their examples, we give an example of an ordered abelian group G which is not p-divisible, but for any convex subgroup C, G/C does not contain a convex p-rendible subgroup. Such a group, of course, is going to have a dense principal p-regular rank.

Let $I = \mathbf{Q}$ be our index set with the usual ordering for $\mathbf{Q}$, and let $G_i \cong (\mathbf{Q}, +, <)$ be our ordered abelian group with index $i \in I$. Then we have the direct product of abelian groups $\prod_{i \in I} G_i$, which has an ordered subgroup

$$H := {\prod_{i \in I}}' G_i = \left\{ g = (g_i) \in \prod_{i \in I} G_i \,\middle|\, \text{supp}(g) \text{ is well-ordered} \right\},$$

where $(g_i) < (h_i)$ if $g_{i_0} < h_{i_0}$ with $i_0 = \min\{i \mid g_i \neq h_i\}$.

Given $a < b \in \mathbf{Q}$, let $s(a,b)$ be the sequence $\{b - (b-a)/n\}_{n=1}^{\infty}$. Define an element $e_{s(a,b)} = (e_i)$ of the direct product by

$$e_i = \begin{cases} 1, & i \in s(a,b), \\ 0, & i \notin s(a,b). \end{cases}$$

Now, let G be the ordered abelian subgroup of H generated by $\bigoplus_{i \in I} G_i$ and all the elements of the form $e_{s(a,b)}$.

If C_1 and C_2 are two different convex subgroups of G such that $C_1 \subsetneq C_2$ then C_2/C_1 is not p-divisible. It follows that, G has the property we want in the beginning of this remark. Indeed, $e_{s(a,b)}/2 + C_1$ is always not in C_2/C_1 for $a < b$ (as elements of I) with $C_1 < G_b^{>0} < G_a^{>0} \subseteq C_2$.

Theorem 2.4.8. *Suppose that the valued field (K, V), with K not p-closed, satisfies one of the following conditions:*

- *V is Henselian;*
- *V is p-Henselian, $\zeta_p \in K$;*
- *V is p-Henselian, $\mathrm{char}(K) = p$.*

If vK is not p-divisible with a non-dense principal p-regular rank, then there exists some non-trivial valuation ring containing V (hence inducing the same topology), which is $\mathscr{L}_{\mathrm{r}}$-definable over K.

Proof. This is a direct consequence of Corollary 2.4.5 and Lemma 2.4.6. □

There is also some connection the Independence Property. Recall that a complete theory T in some first-order language $\mathscr{L}$ is said to have the **Independence Property** if there is some $\mathscr{L}$-formula $\phi(\vec{x}, \vec{y})$ such that, for every model $\mathcal{M}$ of T, and for all n, there are $\vec{a}_1, \ldots, \vec{a}_n \in M$ and $\{b_J\}_{J \subseteq \{1,\ldots,n\}} \subseteq M$ such that $\mathcal{M} \models \phi(\vec{a}_i, \vec{b}_J)$ if and only if $i \in J$.

Corollary 2.4.9. *Suppose that the valued field (K, V) is Henselian; suppose furthermore that $\mathrm{char}(Kv) = 0$ and vK contains a convex p-rendible subgroup. Then K as a field in $\mathscr{L}_{\mathrm{r}}$ does not have the Independence Property if and only if K as a valued field in $\mathscr{L}_{\mathrm{div}}$ does not have the Independence Property if and only Kv as a field in $\mathscr{L}_{\mathrm{r}}$ does not have the Independence Property.*

Proof. The first equivalence is true because V is $\mathscr{L}_{\mathrm{r}}$-definable over K. The second equivalence is a direct consequence of a theorem due to Delon in [Delon, 1981] (Théorème 8), but see [Kaplan et al., 2011] Fact 5.2 for the version we need (that is, a Henselian valued field of residue characteristic 0 has the Independence Property if and only if the residue field has the Independence Property). □

Chapter 3

T-convexly valued o-minimal fields

Moving away from definable valuations, starting from this chapter, we will focus on valued fields whose valuations are not definable (in the smaller language). Typical examples of fields on which non-trivial valuations are not definable, are o-minimal fields and stable fields. On an o-minimal field, a non-trivial convex valuation ring is not definable in the smaller language because there is not any definable convex subring on the field; on a stable field, a non-trivial valuation ring is not definable because stable theories do not have the Order Property. In the following chapters, we focus on these two kinds of fields endowed with valuations.

Let $\mathscr{L}$ be a first-order language containing the language of ordered rings $\mathscr{L}_{\mathrm{or}} := \mathscr{L}_{\mathrm{r}} \cup \{<\}$. In terms of definability, because we are going to work with o-minimal fields (which are real closed fields) on which the orderings are always $\mathscr{L}_{\mathrm{r}}$-definable, it does not matter whether we consider languages containing $\mathscr{L}_{\mathrm{r}}$ or $\mathscr{L}_{\mathrm{or}}$.

3.1 Preliminaries

We refer the reader to [van den Dries, 1998] for basic knowledge about o-minimal structures, and to [Marker, 2002] for a summary about real closed fields.

Suppose R is a set, on which '$<$' is a linear ordering. A subset C of R is **convex** if for all $a, b, c \in R$ with $a < b < c$, $a, c \in C$, it always follows that $b \in C$. An **interval** is a convex subset of R quantifier-free definable in the language $\{<\}$. We will use the usual notation for intervals, e.g. (a, b), $[a, b)$, etc., where a, b could be $\pm\infty$; notice that an interval could be just a point, or even empty. An **ordered commutative ring** $(R, <)$ is a commutative ring R ($0 \neq 1$) with an ordering '$<$' which respects the ring operations (and satisfying that $0 < 1$). An **ordered field** is an ordered commutative ring which is also a field. An ordered field has a natural **order topology** associated to its ordering, which makes it a topological field.

A first-order structure $\mathscr{R}$, with the universe R, in a language containing the order relation '$<$' so that R is linearly ordered, is **o-minimal** if every definable subset of R is a finite union of intervals. A field R is **real closed** if it does not have any proper algebraic extension which is also an ordered field. A real closed field is o-minimal in the language of ordered rings. It is well-known that if R is real closed, then an element is non-negative if and only it is a square. In fact, an ordered field is **real closed** if and only if all of the non-negative elements are squares and every odd degree polynomial has a root. The class of all real closed fields is an elementary class in $\mathscr{L}_{\text{or}}$, whose theory is usually denoted by RCF. It is well-known that RCF has quantifier elimination in $\mathscr{L}_{\text{or}}$.

Fact 3.1.1 ([van den Dries, 1998]). Suppose that $\mathscr{R}$ is an o-minimal field in the language $\mathscr{L}$ containing $\mathscr{L}_{\text{or}}$. Then $\mathscr{R}$ has the **Intermediate Value Property**, that is, if $f : [a, b] \to R$ is definable and continuous, then for any value d between $f(a)$ and $f(b)$, there is some $c \in [a, b]$ with $f(c) = d$.

There is an important theorem which is the so-called Monotonicity Theorem for an o-minimal structure.

Fact 3.1.2 (see e.g. [van den Dries, 1998]). Suppose that $\mathscr{R}$ is an o-minimal structure with the universe R. Let $f : (a, b) \to R$ be a definable function. Then there are points $a_1, \dots, a_k \in (a, b)$ such that on each sub-interval (a_j, a_{j+1}), with $a_0 = a, a_{k+1} = b$, the function is either constant, or strictly monotone and continuous.

Different from minimality and immediate expansions, o-minimality is a first-order property, that is, if $\mathscr{R}$ is o-minimal in a language $\mathscr{L}'$ and $\mathscr{R}'$ is $\mathscr{L}'$-elementarily equivalent to $\mathscr{R}$, then $\mathscr{R}'$ is also o-minimal. This is a consequence of an important theorem called the Cell Decomposition Theorem.

Definition 3.1.3 ([van den Dries, 1998]). *Suppose that $\mathscr{R}$ is an o-minimal structure.*

Let $(i_1, \dots, i_m)$ be a sequence of zeros and ones. An $(i_1, \dots, i_m)$-cell is a definable subset of R^m obtained by induction on m as follows:

1. *a (0)-cell a singleton $\{r\}$; a (1)-cell is an open interval (a, b);*
2. *an $(i_1, \dots, i_m, 0)$-cell is the graph of a definable and continuous function on an $(i_1, \dots, i_m)$-cell; $(i_1, \dots, i_m, 1)$-cell is a subset of a set of the form $\{(\vec{x}, y) \in X \times R\}$ with X being an $(i_1, \dots, i_m)$-cell, satisfying the property that there are two functions $f < g$ being definable and continuous function, or the constant functions $-\infty, +\infty$, such that $f(\vec{x}) < y < g(\vec{x})$.*

Definition 3.1.4 ([van den Dries, 1998]). *Suppose that $\mathscr{R}$ is an o-minimal structure.*

A **decomposition** *of R^m is a special kind of partition of R^m into finitely many cells, defined by induction on m as follows:*

1. *a decomposition of R is a finite set of disjoint intervals whose union is R;*
2. *a decomposition of R^{m+1} is a finite partition of R^{m+1} into cells A such that the set of projections $\pi(A)$ (from R^{m+1} into R^m) is a decomposition of R^m.*

A decomposition of R^m **partitions** *a set $S \subset R^m$ if S is a union of cells in the decomposition.*

Fact 3.1.5 (Cell Decomposition Theorem, [van den Dries, 1998])**.** Suppose that $\mathscr{R}$ is an o-minimal structure.

(1) Given any definable sets $A_1, \ldots, A_k \subset R^m$ there is a decomposition of R^m partitioning each of $A_1, \ldots, A_k$;

(2) For each definable function $f : A \to R$, $A \subseteq R^m$, there is a decomposition $\mathcal{D}$ of R^m partitioning A such that the restriction $f|B : B \to R$ to each cell $B \in \mathcal{D}$ with $B \subseteq A$ is continuous.

Remark 3.1.6 ([van den Dries, 1998])**.** Suppose that $\mathscr{R}$ is an o-minimal field in the language $\mathscr{L}$ containing $\mathscr{L}_{\text{or}}$. Let $X \subseteq R$ be a definable nonempty set. We can "definably pick" an element $e(X)$, uniformly for all X. If X has a least element, then let $e(X)$ be that element; otherwise, let $a = \inf X$, $b = \sup\{x \in R \mid (a, x) \subseteq R\}$ and let

$$e(X) := \begin{cases} 0, & a = -\infty, b = +\infty, \\ b-1, & a = -\infty, b \in R, \\ a+1, & a \in R, b = +\infty, \\ (a+b)/2, & a, b \in R. \end{cases}$$

This procedure can be performed inductively to the higher Cartesian powers of R. In view of this, it follows that $\mathscr{R}$ in fact has definable Skolem functions.

Fact 3.1.7 (Definable Choice, [van den Dries, 1998])**.** Suppose that $\mathscr{R}$ is an o-minimal field in $\mathscr{L}$ containing $\mathscr{L}_{\text{or}}$.

(1) If $S \subseteq R^{m+n}$ is definable and $\pi : R^{m+n} \to R^m$ the projection on the first m coordinates, then there is a definable map $f : \pi S \to R^n$ such that the graph of f is contained in S.

(2) Each definable equivalence relation on a definable set X has a definable set of representatives.

Definition 3.1.8 ([Dickmann, 1987])**.** *Suppose that $\mathcal{M}$ is a first-order structure in a language containing a linear ordering. Then $\mathcal{M}$ is* **weakly o-minimal** *if every definable subset of M is a finite union of definable convex subsets of M. A theory is weakly o-minimal if all of its models are.*

Definition 3.1.9. *Suppose that $(R, <)$ is an ordered field. A valuation ring V on R is called a* **convex valuation** *(ring) if V is convex with respect to the ordering '$<$' on R. In this case R is called a* **convexly valued field***.*

Remark 3.1.10. If V is a convex valuation ring on R, then for all $x, y \in R$, with $|x| < |y|$, we know that $y \neq 0$ and $|x/y| < 1$; thus since $\pm 1 \in V$, we have $x/y \in V$, i.e. $v(x) \geq v(y)$. Therefore, we have

$$|x| \leq |y| \Longrightarrow v(x) \geq v(y).$$

The contra-positive of this statement is for all $x, y \in R$,

$$v(x) < v(y) \Longrightarrow |x| > |y|.$$

Proposition 3.1.11 (cf. [van den Dries and Lewenberg, 1995])**.** *Suppose that $\mathscr{R}$ is an o-minimal field in a language $\mathscr{L}$ containing $\mathscr{L}_{\text{or}}$ and V is a convex valuation ring on R. Suppose that the expansion $(\mathscr{R}, V)$ of $\mathscr{R}$ to the language $\mathscr{L} \cup \{V\}$ ('V' as a predicate for V) has quantifier elimination. Then $(\mathscr{R}, V)$ is weakly o-minimal.*

Proof. It is enough to show that every realization set of an atomic $\mathscr{L} \cup \{V\}$-formula in R is a finite union of definable convex sets. If the atomic $\mathscr{L} \cup \{V\}$-formula is an $\mathscr{L}$-formula, this follows from the o-minimality of $\mathscr{R}$. Otherwise, the atomic formula is of the form $V(f(x))$, where $f(x)$ is an $\mathscr{L}$-definable function on R. By the Monotonicity Theorem, the domain of f can be decomposed into a finite union of intervals, on each of which f is either constant or continuously monotone; then by Remark 3.1.10, the realization set of $V(f(x))$ has to be a finite union of definable convex subsets of R. $\square$

Let RCVF be the theory of real closed fields with non-trivial convex valuation rings, in the language $\mathscr{L}_{\text{or}} \cup \{|\}$.

Fact 3.1.12 ([Cherlin and Dickmann, 1983])**.** RCVF has quantifier elimination.

It follows that RCVF is weakly o-minimal.

Fact 3.1.13 ([Haskell and Macpherson, 1998])**.** RCVF ⧖ RCF.

It is natural to ask whether one can generalize Fact 3.1.13 to a more general context, for example, to **valued o-minimal fields** (i.e. o-minimal fields endowed with a valuation). The framework of T-convex valuation rings on o-minimal fields provides a good context for our investigation. It is perhaps not a surprise, since the original motivation of the work on T-convexity was to generalize the quantifier elimination for RCVF to more general valued o-minimal fields.

Definition 3.1.14. *Suppose that T is a complete o-minimal theory extending* RCF *(in the language $\mathscr{L}$ containing $\mathscr{L}_{\text{or}}$). Suppose that $\mathscr{R} \models T$. A convex subring*[1] *V of R is T-**convex** if every continuous function $f : R \to R$ which is $\mathscr{L}$-definable without parameters has the property that $f(V) \subseteq V$.*

Suppose that T is a complete o-minimal theory extending RCF in $\mathscr{L}$ (containing $\mathscr{L}_{\text{or}}$), $\mathscr{R} \models T$, and V is a non-trivial T-convex valuation ring on R. Let T_V be the theory of $(\mathscr{R}, V)$ in the language $\mathscr{L} \cup \{V\}$.

Fact 3.1.15 ([van den Dries and Lewenberg, 1995])**.** If T admits quantifier elimination and universal axiomatization, then T_V admits quantifier elimination.

[1] A convex subring is in fact the same as a convex valuation ring.

The following is usually called *expansion by definition*. Suppose that T is a complete theory in the language $\mathscr{L}$. For each $n \geq 1$ and for every $\phi(x_1, \ldots, x_n; y)$, an $\mathscr{L}$-formula, such that

$$T \vdash \forall x_1 \cdots \forall x_n \exists! y \phi(x_1, \ldots, x_n, y),$$

we add to the language $\mathscr{L}$ a function symbol f_ϕ and add to the theory T the defining axiom

$$\forall x_1 \cdots \forall x_n \phi(x_1, \ldots, x_n, f_\phi(x_1, \ldots, x_n)),$$

then, we get a new theory T^{df} in the new language $\mathscr{L}^{\mathrm{df}}$ extending T.

An immediate corollary of Fact 3.1.15 is the following:

Fact 3.1.16 ([van den Dries and Lewenberg, 1995])**.** Suppose that T is a complete o-minimal theory extending RCF in $\mathscr{L}$, $\mathscr{R} \models T$ and V is a non-trivial T-convex valuation ring on R. Then $(T^{\mathrm{df}})_V$ admits quantifier elimination.

3.2 The main theorem

In this section, we generalize Fact 3.1.13 to T-convexly[2] valued o-minimal fields. The original proof in [Haskell and Macpherson, 1998] of Fact 3.1.13 uses a result in [Holly, 1995], which is not used in our proof.

Throughout this section, we assume that T is a complete o-minimal theory extending RCF in the language $\mathscr{L}$ containing $\mathscr{L}_{\mathrm{or}}$, $\mathscr{R} \models T$, and V is non-trivial convex valuation in R.

3.2.1 The proof

Definition 3.2.1 ([Macpherson et al., 2000])**.** *A* **cut** *in R is a pair of subsets (C_1, C_2) of R such that $C_1 < C_2$ and $R = C_1 \cup C_2$. A cut (C_1, C_2) is called* **valuational** *if there is some positive element $\epsilon \in R$ such that*

$$(\forall x \in C_1)(\forall y \in C_2)(y - x > \epsilon).$$

A cut is **non-valuational** *if it is not valuational.*

Definition 3.2.2. *Let C be a non-empty convex subset of R. We say that C is* **right-valuational** *if C is bounded from above and the cut $(\{x \in R | \exists y \in C, x \leq y\}, \{x \in R | x > C\})$ is valuational. We say C is* **left-valuational** *if C is bounded from below and the cut $(\{x | x < C\}, \{x | \exists y \in C, x \geq y\})$ is valuational. And C is* **valuational** *if*

[2] The author uses 'T-convexly' instead of 'T-convex' here because we say a valuation is T-convex, not a valued o-minimal field is T-convex. This is different from the 'regular ordered abelian group' scenario.

C is right-valuational or left-valuational. We say C is **non-valuational** *if C is not valuational. We say that $\emptyset$ and R are non-valuational.*

Remark 3.2.3. Note that any non-trivial convex valuation ring V is a valuational convex subset of R. Indeed, let $x \in V$ and $y \in R \backslash V$, then $v(x-y) = v(y) < 0 = v(n)$ for any $n \in \mathbf{Z}$; and since V is convex, $|x-y| > n$ for any $n \in \mathbf{Z}$ by Remark 3.1.10; it then follows that V is both left-valuational and right-valuational.

Remark 3.2.4. If C is a convex subset of R which is right-valuational, then it clearly follows that C does not have a right-endpoint, i.e. a point that is smaller than any other point bigger than C and any other point in C.

Lemma 3.2.5. *If T_V has quantifier elimination, then for any given parameter set $X \subseteq R$, its $\mathscr{L} \cup \{V\}$-algebraic closure in R is the same as its $\mathscr{L}$-algebraic closure in R.*

Proof. It is obviously true that the $\mathscr{L}$-algebraic closure is always contained in the $\mathscr{L} \cup \{V\}$-algebraic closure. Since we have the ordering of the underlying field involved, the algebraic closure of X is the same as the definable closure of X regardless in $\mathscr{L}$ or $\mathscr{L} \cup \{V\}$. So it is enough to prove a stronger conclusion: if b is $\mathscr{L} \cup \{V\}$-definable using parameter set X, then b is $\mathscr{L}$-definable using parameter set X.

Since we have quantifier elimination for T_V, every formula can be written in disjunctive normal form. Therefore, we can assume that b is defined by the following formula:

$$\bigvee_{i=1}^{m} \left[\theta_i(x, \vec{a}) \wedge W_{i1}(g_{i1}(x, \vec{a})) \wedge \cdots \wedge W_{in_i}(g_{in_i}(x, \vec{a}))\right], \tag{3.2.1}$$

where the θ_i are $\mathscr{L}$-formulas and the g_{ij} are functions defined by $\mathscr{L}$-terms (they are actually $\mathscr{L}$-terms) and each W_{ij} is either V or $\neg V$; and $\vec{a}$ is a tuple of parameters in X.

By the assumption, we know that there exists some i_0 with $1 \leq i_0 \leq m$ such that b is defined by

$$\theta_{i_0}(x, \vec{a}) \wedge W_{i_0 1}(g_{i_0 1}(x, \vec{a})) \wedge \cdots \wedge W_{i_0 n_{i_0}}(g_{i_0 n_{i_0}}(x, \vec{a}))$$

Note that the realization sets of θ_i and W_{ij} are finite Boolean combinations of convex sets. Thus either b is one endpoint of the realization set of $\theta_{i_0}(x, \vec{a})$, or there exists some j_0 with $1 \leq j_0 \leq n_{i_0}$ such that b is one endpoint of the realization set of $W_{i_0 j_0}(g_{i_0 j_0}(x, \vec{a}))$. Note that in both cases, b is inside the realization set. We are done if it is the first case.

For the second case, we show that b is a point of discontinuity of $g_{i_0 j_0}$; and since by Fact 3.1.2 $g_{i_0 j_0}$ has only finitely many points of discontinuity and we have the ordering, b can be defined in the language $\mathscr{L}$ using $g_{i_0 j_0}$ without introducing additional parameters.

We write $f := g_{i_0j_0}$ for convenience. We can assume that $W_{i_0j_0}$ is actually V and b is a right endpoint of the realization set $B := \{x \in R | V(f(x, a))\}$, since the other cases could be proved similarly.

Since B is a finite Boolean combination of convex sets and b is one of the right-endpoints of B, we know that there exists some $y \notin B$ such that $(b, y) \cap B = \emptyset$, i.e. for any $z \in (b, y)$, $v(f(z)) < 0$. But then by Remark 3.2.3, $|f(b) - f(z)| > n$ for all $n \in \mathbf{Z}$. Thus f is not continuous at b. □

Lemma 3.2.6. *Assume that T_V has quantifier elimination. For any $\mathscr{R}_V$-definable set $A \subseteq R^l$ for some l, if the expansion of $\mathscr{R}$ to $\mathscr{L} \cup \{A\}$, denoted by $\mathscr{R}_A$, is o-minimal, then A is definable in $\mathscr{R}$.*

Proof. It suffices to prove that each $\mathscr{R}_A$-cell (i.e. an $\mathscr{L} \cup \{A\}$-definable set which is a cell in the o-minimal structure $\mathscr{R}_A$) is definable in $\mathscr{R}$. We prove the following assertions by induction on n:

$(1)_n$ Any $\mathscr{R}_A$-cell in R^n is $\mathscr{L}$-definable using the same parameters;

$(2)_n$ any $\mathscr{L} \cup \{A\}$-definable partial function $f : R^n \to R$ whose domain is an $\mathscr{R}_A$-cell is $\mathscr{L}$-definable using the same parameters.

$(1)_1$ is evidently true by the definition of o-minimality and the invariance of the algebraic closure; also $(1)_{n+1}$ is clearly true by the definition of cells and the induction hypothesis that $(1)_n$ and $(2)_n$ are true. So we just need to prove $(2)_{n+1}$ assuming $(1)_{n+1}$ and $(2)_n$ for any $n > 0$ (if $n = 0$, we just assume that $(1)_1$ is true).

Let $\mathscr{M}_A$ be an ω-saturated elementary extension of $\mathscr{R}_A$, and $\mathscr{M}$ be the corresponding reduct of $\mathscr{M}_A$ to the language $\mathscr{L}$. Then $\mathscr{M}$ is an elementary extension of $\mathscr{R}$. It is easy to see that if $(1)_n$ and $(2)_n$ are true for the pair $\mathscr{M}$ and $\mathscr{M}_A$, then they are also true for $\mathscr{R}$ and $\mathscr{R}_A$. (Note that we are paying attention to the parameters.) By this observation, there is no harm to assume that $\mathscr{R}_A$ is ω-saturated.

Now suppose that $f : C \to R$ is an $\mathscr{L} \cup \{A\}$-definable function defined on an $\mathscr{R}_A$-cell in R^{n+1}, and both f and C are $\mathscr{L} \cup \{A\}$-definable using $\vec{a}$ as parameters. By the invariance of the definable closure, we know that for each $\vec{x} \in C$, there exists a function $\theta_{\vec{x}}(\cdot)$ defined on C in the language $\mathscr{L}$, using parameters $\vec{a}$, such that $\theta_{\vec{x}}(\vec{x}) = f(\vec{x})$. Now, we consider

$$\Gamma(\vec{x}) := \{f(\vec{x}) \neq \theta_{\vec{y}}(\vec{x})\}_{\vec{y} \in C} \cup \{\vec{x} \in C\}.$$

We claim that $\Gamma(\vec{x})$ is not consistent with $\mathscr{R}_A$. Otherwise, suppose that $\Gamma(\vec{x})$ is satisfied by $\vec{b}$ in some model. Since $\mathscr{R}_A$ is ω-saturated, we can assume that there is some $\vec{t} \in R^{n+1}$ realizing $\Gamma(\vec{x})$. But then this implies that there is some $\vec{t} \in C$ such that for any $\vec{y} \in C$, $f(\vec{t}) \neq \theta_{\vec{y}}(\vec{t})$, which is a contradiction.

Thus, it follows from the Compactness Theorem that there exists a finite sequence $\vec{y}_1, \dots, \vec{y}_m$ such that

$$\mathscr{R}_A \models \forall \vec{x} \in C \bigvee_{i=1}^{m} (f(\vec{x}) = \theta_{\vec{y}_i}(\vec{x})).$$

Now, for each i, the set

$$X_i := \{\vec{x} \in C | f(\vec{x}) = \theta_{\vec{y}_i}(\vec{x})\}$$

is $\mathscr{L} \cup \{A\}$-definable over R, hence $\mathscr{L}$-definable in $\mathscr{R}$ by $(1)_{n+1}$. And therefore f is definable in $\mathscr{R}$ using the X_i and the $\theta_{\vec{y}_i}$. □

Lemma 3.2.7. *Assume that T_V has quantifier elimination, and A is an $\mathscr{L} \cup \{V\}$-definable subset of R^l. If there is a convex valuational subset B of R which is $\mathscr{L} \cup \{A\}$-definable, then V is $\mathscr{L} \cup \{A\}$-definable.*

Proof. Since T_V has quantifier elimination, using the notation in (3.2.1), we can assume that B is defined by

$$\bigvee_{i=1}^{m} \left[\theta_i(x, \vec{a}) \wedge W_{i1}(g_{i1}(x, \vec{a})) \wedge \cdots \wedge W_{in_i}(g_{in_i}(x, \vec{a}))\right].$$

Since only finite intersections and unions are involved, by intersecting with some appropriate interval I, we can further assume that B is defined by

$$\theta_i(x, \vec{a}) \cap I \quad \text{or} \quad W_{ij}(g_{ij}(x, \vec{a})) \cap I$$

for some i and j and B is right-valuational (the other cases can be proved similarly). It is impossible that we have the first case since any $\mathscr{L}$-definable convex subset of R is an interval, which is always non-valuational. Thus, we only need to consider the second case.

Let $\alpha \in B$ and $\beta > B$; we may assume that for any $x \in [\alpha, \beta] \cap B$, we have $W_{ij}(g_{ij}(x, \vec{a}))$ and for any $x \in [\alpha, \beta] \backslash B$, $\neg W_{ij}(g_{ij}(x, \vec{a}))$. By shrinking $[\alpha, \beta]$, we can assume that g_{ij} is continuous on $[\alpha, \beta]$, because it has only finitely many points of discontinuity, and B is right-valuational which means that B does not have a right-endpoint (in particular, B can not have a right-endpoint which is a point of discontinuity of g_{ij}). Furthermore, we can assume that $W_{ij} = V$. Then by the intermediate value property of o-minimal fields, the set

$$O = \{y \in R | |y| \leq |g_{ij}(x, \vec{a})|, \text{ for some } x \in [\alpha, \beta] \cap B\}.$$

is the valuation ring V. Indeed, on one hand, $O \subseteq V$ is obviously true because V is convex. On the other hand, for any $z \in V$, if z is not in O, we would get

$$|z| > |g_{ij}([\alpha, \beta] \cap B, \vec{a})|; \tag{3.2.2}$$

then since $\neg V(g_{ij}(\beta, \vec{a}))$ and $z \in V$, we know that $|z| < |g_{ij}(\beta, \vec{a})|$, by Remark 3.1.10. By the intermediate value property, we would have some $t \in [\alpha, \beta]$ such that $|g_{ij}(t, \vec{a})| = |z|$. But $t \notin [\alpha, \beta] \cap B$ by (3.2.2) and $t \notin [\alpha, \beta] \backslash B$ since $z \in V$. This contradiction proves that $z \in O$. Hence $V = O$ is $\mathscr{R}_A$-definable. □

Theorem 3.2.8. *Assume that T_V has quantifier elimination, and A is an $\mathscr{L} \cup \{V\}$-definable subset of R^l. Then either A is $\mathscr{R}$-definable or V is $\mathscr{R}_A$-definable.*

Proof. By the preceding lemma, it is enough to prove that an $\mathscr{R}_V$-definable convex non-valuational subset B of R is an interval. Again, we can assume that B is defined by $V(f(x, \vec{a})) \cap [\alpha, \beta]$ for some appropriate parameters α and $\beta > B$, and is not right-valuational. But for any $s \in B$ and any $t \in [\alpha, \beta] \backslash B$, we have $|f(s) - f(t)| > n$ for all $n \in \mathbf{Z}$. Since B is non-valuational, we can choose s and t to be arbitrarily closed, but this implies that f has a point of discontinuity in $[\alpha, \beta]$. But $[\alpha, \beta]$ could be shrunk, meaning that B has an endpoint as a point of discontinuity of f. Thus B is an interval. □

Theorem 3.2.9. *If V is T-convex, then $T_V ⧖ T$.*

Proof. Since V is T-convex, V is T^{df}-convex; and T_V^{df} admits quantifier elimination. Every model of T_V is of the form $(\mathscr{R}, V)$ where $\mathscr{R} \models T$ and V is T-convex. Thus the conclusion follows from Theorem 3.2.8. □

3.2.2 Examples

Definition 3.2.10 ([van den Dries and Lewenberg, 1995])**.** *An $\mathscr{L}$-definable function $f : R \to R$ is* **polynomially bounded** *if there is some n and a positive element $a \in R$ such that for all $r > a$, $|f(r)| < r^n$. The structure $\mathscr{R}$ is called* **polynomially bounded** *if every $\mathscr{L}$-definable function is polynomially bounded. The theory T is* **polynomially bounded** *if every model of T is.*

The relation between T-convexity and polynomially bounded o-minimal theories has been discussed quite thoroughly in [van den Dries and Lewenberg, 1995]. It is pointed out that for a complete polynomially bounded o-minimal field theory T, if T has an archimedean model, then every proper convex valuation is T-convex.

Fact 3.2.11 ([van den Dries and Lewenberg, 1995])**.** The following are equivalent:
(1) the theory T is polynomially bounded;
(2) for every $\mathscr{R} \models T$, every convex subring containing the prime model is T-convex;
(3) some model of T is polynomially bounded.

Corollary 3.2.12. *For every complete polynomially bounded o-minimal field theory T, if T has an archimedean model, then $T_V ⧖ T$ for every convex valuation ring on R.*

Example 3.2.13. We know that RCF is polynomially bounded and has an archimedean model, hence (for any proper convex valuation V) we have RCVF ⧖ RCF (note that V and '$|$' are $\mathscr{L}_{\mathrm{r}}$-interdefinable). This is the former result of Haskell-Macpherson, Fact 3.1.13.

We refer the reader to [van den Dries and Lewenberg, 1995] for related definitions and discussion of the following examples.

Example 3.2.14. Let T_{an} be the theory of the ordered field of reals with restricted analytic functions, in the language of $\mathscr{L}_{\mathrm{or}}$ augmented by a function symbol for each restricted analytic function. Let $\mathscr{R} = (R, <, \ldots) \models T_{\mathrm{an}}$. It can be shown that every one-variable function which is definable without parameters in this theory is polynomially bounded, thus every convex subring V of R is T_{an}-convex. Thus $T_{\mathrm{an},V}$ ⧖ T_{an}.

Example 3.2.15. Let $T_{\mathrm{an,powers}}$ be the theory of the ordered field of reals with restricted analytic functions and power functions, in the language of T_{an} augmented by function symbols for all power functions, where a power function is a function associated with a real number α such that it maps x to x^α if $x > 0$ and it maps x to 0 if $x \leq 0$. Let $\mathscr{R} = (R, <, \ldots) \models T_{\mathrm{an,powers}}$. It can be shown that every one-variable function definable without parameters in this theory is polynomially bounded. Hence for any proper convex valuation ring V, $T_{\mathrm{an,powers},V}$ ⧖ $T_{\mathrm{an,powers}}$.

Example 3.2.16. Let $T_{\mathrm{an,exp}}$ (resp. T_{exp}) be the theory of the ordered field of reals with restricted analytic functions (resp. just the theory of the ordered field of the reals) and exponential function e^x, in the language of T_{an} (resp. L_{or}) augmented by the unary function symbol exp. Let $\mathscr{R} = (R, <, \ldots) \models T_{\mathrm{an,exp}}$, (resp. $\mathscr{R} \models T_{\mathrm{exp}}$). It is shown in [van den Dries and Lewenberg, 1995] that a convex subring V of R is $T_{\mathrm{an,exp}}$-convex or T_{exp}-convex if and only if $\exp(V) \subseteq V$. Thus, if $\exp(V) \subseteq V$, then both $T_{\mathrm{an,exp},V}$ ⧖ T_{exp} and $T_{\mathrm{exp},V}$ ⧖ T_{exp}. In particular, the Log-exp series field (see [van den Dries et al., 2001]) as a model of $T_{\mathrm{an,exp}}$ is an immediate reduct of the valued o-minimal field structure.

Chapter 4

Algebraically closed valued fields

In this chapter, we focus on the situation where K is an algebraically closed field with a non-trivial valuation ring V. It has been proved in [Haskell and Macpherson, 1998] that if one considers $\mathcal{K} := (K, V)$ as an $\mathscr{L}_{\mathrm{r}} \cup \{V\}$-structure, then it is always true that $\mathcal{K} \text{⧖} (\mathcal{K}|\mathscr{L}_{\mathrm{r}})$. Their original proof used the notion of C-minimality, and depended on a result of Hrushovski (see [Haskell and Macpherson, 1998]) about strongly minimal expansions of algebraically closed fields. In this chapter, we show that these two things are not essential ingredients of the proof; we first give a review of their proof, without using the notion of C-minimality; then we give a new proof, which detaches the result from the theorem of Hrushovski about strongly minimal expansions of algebraically closed fields. The idea of the new proof will be used to prove the immediateness of the expansion in the context of separably closed valued fields of finite imperfection degree in the next chapter.

For related definitions and results in classical algebraic geometry, we refer the reader to the appendix. We want to emphasize that for an algebraic set Z in some K^n and a subfield $k \subseteq K$, when we say "Z is **definable** over k", it means that Z is model-theoretically definable over k, i.e. it is the realization set of a first-order formula with parameters coming from k; when we say "Z is **defined** over k", it means we are saying that the field of definition of Z is contained in k, i.e. the ideal associated to Z is generated by polynomials with coefficients coming from k.

Throughout this chapter, let $\mathscr{L}_{\mathrm{div}}$ be the language $\mathscr{L}_{\mathrm{r}} \cup \{|\}$, where '$|$' is the division predicate for the valuation (i.e. $x \mid y$ if and only $v(x) \leq v(y)$).

4.1 Preliminaries

Let ACF be the first-order theory of algebraically closed fields, axiomatized in the language $\mathscr{L}_{\mathrm{r}}$. Then

The following was first proved by Tarski. We refer the readers to, for example, [Marcja and Toffalori, 2003] and [Hodges, 1993], for more discussions about this fact and its history.

Fact 4.1.1 (See e.g. [Marcja and Toffalori, 2003])**.** ACF has quantifier elimination.

It follows immediately that ACF is in fact **strongly minimal**, that is for every model of ACF [1] the definable subsets of the universe are either finite or co-finite.

Let ACVF be the first-order theory of algebraically closed fields with a non-trivial valuation, axiomatized in the language $\mathscr{L}_{\text{div}}$ (i.e. a collection of $\mathscr{L}_{\text{div}}$-sentences expressing these properties).

Fact 4.1.2 ([Robinson, 1977])**.** ACVF has quantifier elimination.

Let $\mathcal{K}$ be a model of ACVF, with V as its valuation ring. The definable subsets of K (in $\mathscr{L}_{\text{div}}$) are well-understood—they are finite unions of "perforated discs". In fact, it is well-known that ACVF is *C-minimal* (but we are not going to discuss C-minimality in this thesis).

Definition 4.1.3 (cf. [Holly, 1995])**.** *A* **disc**[2] *in K is a set of the form $D_v^{>}(a,\gamma) = \{x \in K \mid v(x-a) > \gamma\}$ or $D_v^{\geq}(a,\gamma) = \{x \in K \mid v(x-a) \geq \gamma\}$, where $a \in K$ and $\gamma \in vK \cup \{-\infty\}$ with $-\infty < vK$.*

Notice that a disc could be empty, just one point or the whole field K. Every infinite disc is topologically both open and closed. Every point inside a disc is also a center of the disc. For two discs, either they are disjoint or one is contained in the other. Given finitely many discs which are not K, there is always a larger disc, which is also not K, containing them.

Definition 4.1.4 (cf. [Holly, 1995])**.** *A* **perforated disc**[3] *is a set of the form $D \backslash (E_1 \cup \cdots \cup E_n)$, where $D, E_1, \ldots, E_n$ are discs in K.*

It can be proved that

Fact 4.1.5 ([Holly, 1995])**.** Every definable subset of K is a finite Boolean combination of discs. In particular, every definable subset of K is a finite union of perforated discs.

The following theorem Theorem 4.1.9 about immediate expansions was originally proved using C-minimality. Here we provide a somewhat "simplified" proof without using C-minimality. The idea is essentially that of the original proof.

Lemma 4.1.6. *Suppose that $\mathcal{K} \models$ ACVF, and E is a finite subset of K which is $\mathscr{L}_{\text{div}}$-definable using parameters $\vec{a}$. Then E is contained in a finite $\mathscr{L}_{\text{r}}$-definable subset of K using parameters $\vec{a}$. In other words, the model theoretic algebraic closure operator is the same in $\mathcal{K}$ and in $\mathcal{K}|\mathscr{L}_{\text{r}}$.*

[1] Strictly speaking, every model of the complete theory ACF_p for some prime number p, where ACF_p is the theory of algebraically closed fields of characteristic p in the language $\mathscr{L}_{\text{r}}$, because strong minimality is only defined for complete theories.

[2] Note that one can define the same notion of a disc and a perforated disc on every valued field.

[3] In [Holly, 1995], a non-empty perforated disc is called a *Swiss cheese*. The author prefers the former name, as it is more acceptable to him to say "a perforated disc" and "a finite union of perforated discs" than to say "a Swiss cheese" and "a finite union of Swiss cheese(s)".

Proof. Because of the quantifier elimination, we can assume that E is defined by

$$(4.1.1) \qquad \bigvee_i \phi_i(x, \vec{a}) := \bigvee_i \left[\wedge_j (p_{i,j}(x) = 0) \wedge (g_i(x) \neq 0) \wedge \theta_i(x, \vec{a})\right],$$

where $p_{i,j}$ and g_i are polynomials using parameters in $\vec{a}$, and θ_i is a finite conjunction of formulas of the form $s(x) \mid t(x)$ or $s(x) \nmid t(x)$. After re-writing the first-order formula, we may assume that for each i, g_i contains all the polynomials $s(X), t(X)$ (as factors) occurring in θ_i. For example, one can write $s(x) \mid t(x)$ as

$$[t(x) = 0] \bigvee [s(x) \neq 0 \wedge t(x) \neq 0 \wedge (s(x) \mid t(x))],$$

and then re-arrange the boolean combination after doing the replacement.

It is enough to prove that every element in E is contained in a finite $\mathscr{L}_{\mathrm{r}}$-definable subset of K using parameters in $\vec{a}$. So assume that $b \in E$. Then $\phi_{i_0}(b, \vec{a})$ is true in K for some i_0.

Because of the continuity of the polynomial functions with respect to the valuation topology, and the assumption that all $s(X), t(X)$ occur as factors of $g_{i_0}(X)$, for every $c \in K$, if $v(b - c)$ is sufficiently large, then

$$\mathcal{K} \models (g_{i_0}(c) \neq 0) \wedge \theta_{i_0}(c, \vec{a}),$$

as long as there are g_{i_0} and θ_{i_0} occurring in the formula $\phi_{i_0}(b, \vec{a})$. Thus, for this i_0, the $p_{i_0,j}(x) = 0$ part must be non-empty, otherwise, E has to be infinite, because the realization set of $\phi_{i_0}(x, \vec{a})$ would contain all the c such that $v(c - b)$ is sufficiently large. Therefore, b is contained in the set $\{x \in K \mid p_{i_0,j}(x) = 0\}$ and we are done. □

Lemma 4.1.7 (cf. [Haskell and Macpherson, 1998])**.** *Suppose that F is an infinite field with a valuation ring V. Assume that E is an infinite and co-infinite subset of F which is a finite union of perforated discs (with respect to V), then V is definable over F in the language $\mathscr{L}_{\mathrm{r}} \cup \{E\}$.*

Proof. We may assume that V is a non-trivial valuation ring. Then the value group vF is infinite.

Because E is infinite and co-infinite, after adding finitely many points and removing finitely many points, we may assume that E is

$$\bigcup_i (D_i \backslash (G_{i1} \cup \cdots \cup G_{ij_i})),$$

where D_i, G_{ik} are all discs defined using radii in $vF \cup \{-\infty\}$, i.e. they are all of the form $\{x \in F \mid v(x - a) > \gamma\}$ or $\{x \in F \mid v(x - a) \geq \gamma\}$ for some $a \in F$ and $\gamma \in vF \cup \{-\infty\}$ (namely, γ can not be $+\infty$). We can also assume that this expression is the shortest of this kind. It then follows that all the D_i are different discs (as sets).

In particular at most one of the D_i is F; replace E by its complement $F \backslash E$ if one of the D_i is F. We may thus assume that all the discs D_i, G_{ij} are not F.

Let H be a disc $\{x \in F \mid v(x-a) > \gamma\}$ containing E with $a \in E$, $\gamma \neq -\infty$.

By performing a translation $x \mapsto (x-a)/b$ with $v(b) = \gamma$, and replacing E by its image, we can assume that H is $\mathfrak{m}_v$ and the image of E under v is an infinite subset of the value group.

It follows that the image of E under the valuation map is a proper non-empty subset of vF, say T (note that T is infinite). Let $L_t = \{x \in F \mid v(x) = t\}$ for each $t \in vF$.

We next prove that $\{x \in F \mid v(x) \in S\}$ is definable over F in the language $\mathscr{L}_{\mathrm{r}} \cup \{E\}$, for some proper non-empty subset S of vF. Essentially, the discs containing 0 determine what S is (up to finitely many elements in vF). There are two kinds of discs (which are not F), those containing 0 and those not containing 0. If a disc, say $\{x \in F \mid v(x-c) > \delta\}$ contains 0, then this disc is the same as $\{x \in F \mid v(x) > \delta\}$, because every point in the disc is a center of the disc; and in particular the disc contains L_t for all $t > \delta$. If $\{x \in F \mid v(x-c) > \delta\}$ does not contain 0, then $v(c) = v(0-c) \leq \delta$, which means for any point d inside the disc, $v(d) = v(c)$, namely the disc is totally contained in $L_{v(c)}$ (the same argument applies to discs of the form $\{x \in F \mid v(x-c) \geq \gamma\}$).

It is easy to see that because E is a finite union of perforated discs,

$$\text{there are only finitely many } t \in T \text{ such that } L_t \cap E \neq \emptyset \text{ and } L_t \cap E \neq L_t; \tag{4.1.2}$$

this ($L_t \cap E \neq \emptyset$ and $L_t \cap E \neq L_t$) occurs in the following two cases.

Suppose there is some $t \in T$ such that $L_t \cap E$, which is not empty and not the whole L_t, contains the complement (in L_t) of a finite union of discs of the form $\{x \in F \mid v(x-a) > \gamma\}$ with $\gamma \geq v(a) = t$ or $\{x \in F \mid v(x-a) \geq \gamma\}$ with $\gamma > v(a) = t$. Suppose the centers of the discs are $a_1, \ldots, a_n$. Then because there are infinitely many elements in L_t, there always exists some $u \in V \backslash \mathfrak{m}_v$ such that $\{ua_i\}_{i=1}^n \cap \{a_i\}_{i=1}^n = \emptyset$. Replacing E by $E \cup uE$ (T would stay the same), we have that $L_t \cap E = L_t$. After doing this, Condition (4.1.2) remains true.

Suppose there is some $t \in T$ such that $L_t \cap E$ is not empty and not the whole L_t, and $L_t \cap E$ is contained in a union of finitely many discs of the form $\{x \in F \mid v(x-a) > \gamma\}$ with $\gamma \geq v(a) = t$ or $\{x \in F \mid v(x-a) \geq \gamma\}$ with $\gamma > v(a) = t$. Suppose the centers of the discs are $a_1, \ldots, a_n$. Then because there are infinitely many elements in L_t, there always exists some $u \in V \backslash \mathfrak{m}_v$ such that $\{ua_i\}_{i=1}^n \cap \{a_i\}_{i=1}^n = \emptyset$. Then replacing E by $E \cap uE$ (T would stay the same, up to a finite subset), we have that $L_t \cap E = \emptyset$. After doing this, Condition (4.1.2) remains true.

After iterating this procedure, we can make sure that for each L_t, $L_t \cap E$ is either L_t or $\emptyset$; we get some proper non-empty subset S of vF such that $J := \{x \in F \mid v(x) \in S\}$ is definable from E. The set S is the image of some finite Boolean combination of discs containing 0, under the valuation map. It follows that the set of units in the valuation ring is

$$V^\times = V \backslash \mathfrak{m}_v = \{u \in F \mid uJ = J\}.$$

To see this, if $v(u) = 0$, then $v(uj) = v(j) = v(u^{-1}j)$ for any $j \in J$, thus $uJ \subset J$ and $u^{-1}J \subset J$; so $uJ = J$. On the other hand, suppose that $uJ = J$ but $v(u) > 0$ (otherwise, consider u^{-1}), then u translates those finitely many end-points of $v(J)$ in vF, which contradicts $uJ = J$.

It follows that the valuation ring is definable by $V = V^\times \cup (1 + V^\times)$ in $\mathscr{L}_{\mathrm{r}} \cup \{E\}$, because if $x \in (V \backslash V^\times) = \mathfrak{m}_v$, then $v(x-1) = v(1) = 0$, namely $x \in 1 + V^\times$. □

Fact 4.1.8 ([Hrushovski, 1992])**.** Suppose that the first-order structure $\mathcal{M}$ in a language $\mathscr{L}'$ containing $\mathscr{L}_{\mathrm{r}}$, is strongly minimal and $(\mathcal{M}|\mathscr{L}_{\mathrm{r}}) \models \mathrm{ACF}$. If $\mathrm{acl}^{\mathcal{M}}(\cdot) = \mathrm{acl}^{\mathcal{M}|\mathscr{L}_{\mathrm{r}}}(\cdot)$, then $\mathcal{M} \approx (\mathcal{M}|\mathscr{L}_{\mathrm{r}})$.

Theorem 4.1.9 ([Haskell and Macpherson, 1998])**.** ACVF $\between$ ACF*; that is, for any* $\mathcal{K} \models \mathrm{ACVF}$*,* $\mathcal{K} \between (\mathcal{K}|\mathscr{L}_{\mathrm{r}})$*.*

Proof. By Theorem 1.3.10, we may assume that $\mathcal{K}$ is ω-saturated (otherwise, take an ω-saturated $\mathscr{L}_{\mathrm{div}}$-elementary extension). Suppose that $D \subseteq K^n$ is an $\mathscr{L}_{\mathrm{div}}$-definable set.

First suppose that every definable subset of K in the language of $\mathscr{L}_{\mathrm{r}} \cup \{D\}$, with D being a new predicate, is either finite or co-finite, then $\mathcal{K}|(\mathscr{L}_{\mathrm{r}} \cup \{D\})$ is minimal. Because $\mathcal{K}$ is ω-saturated, $\mathcal{K}|(\mathscr{L}_{\mathrm{r}} \cup \{D\})$ is actually strongly minimal. But then one knows that the "acl$(\cdot)$" operator stays the same in $\mathcal{K}$ and $\mathcal{K}|\mathscr{L}_{\mathrm{r}}$, by Lemma 4.1.6. It then follows from Fact 4.1.8 that D is definable over K in $\mathscr{L}_{\mathrm{r}}$ already.

Now suppose that there is some definable subset of K in $\mathscr{L}_{\mathrm{r}} \cup \{D\}$ which is infinite and co-infinite. Then by Lemma 4.1.7 and Fact 4.1.5, V is definable over K in $\mathscr{L}_{\mathrm{r}} \cup \{D\}$. □

4.2 A new proof † ‡

As we have seen from the previous proof of the immediateness of the expansion, a non-trivial result Fact 4.1.8 was used. Here we give another proof, which does not depend on Fact 4.1.8. The technique of our new proof will also be "recycled" to be used in the context of separably closed valued fields in the next chapter. The main idea is that given an $\mathscr{L}_{\mathrm{div}}$-definable set E which is not $\mathscr{L}_{\mathrm{r}}$-definable, one should be able to explicitly define from E an infinite and co-infinite subset of K, which would then define the valuation ring, by Lemma 4.1.7.

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Through out this section, we again use $\mathcal{K} = (K, V, \ldots)$ to denote a model of ACVF. Let us start with a simple example below.

Example 4.2.1. Suppose that E is defined by

$$E := \{\vec{x} \in K^n \,|[f(\vec{x}) = 0] \wedge [v(g(\vec{x})) > 0]\},$$

where $f, g \in K[\vec{X}]$. If E is not $\mathscr{L}_{\mathrm{r}}$-definable, then V is $\mathscr{L}_{\mathrm{r}} \cup \{E\}$-definable over K.

Proof. Suppose that E is not $\mathscr{L}_{\mathrm{r}}$-definable over K.

It is easy to see that the image of E under the polynomial $g(\vec{X})$ is co-infinite, as any element of negative valuation is not in the image. The image is also infinite, for if not, name those finitely many elements $\epsilon_1, \ldots, \epsilon_n$. Then E is in fact defined by

$$[f(\vec{x}) = 0] \wedge [g(\vec{x}) \in \{\epsilon_1, \ldots, \epsilon_n\}],$$

so $\mathscr{L}_{\mathrm{r}}$-definable, contradicting the assumption.

Because $g(E)$ is $\mathscr{L}_{\mathrm{div}}$-definable over K, it is a finite union of perforated discs. Therefore, by Lemma 4.1.7, V is $\mathscr{L}_{\mathrm{r}} \cup \{E\}$-definable. □

Because in the language $\mathscr{L}_{\mathrm{r}}$, K is strongly minimal, for any $\mathscr{L}_{\mathrm{r}}$-definable set in K, the image of that set under a rational function is always finite or co-finite. If for the given $\mathscr{L}_{\mathrm{div}}$-definable set E, one can find a rational function which takes E onto an infinite co-infinite subset of K, then V is definable in $\mathscr{L}_{\mathrm{r}} \cup \{E\}$ over K. Most of the time, one can find such a function from the definition of E, like the one above. However, in the following example, it is not easy to see which rational function satisfies the desired property.

Example 4.2.2. Suppose that E is defined by

$$E := \Big\{(x, y) \mid [v(x) \geq 0 \wedge v(y) \geq 0] \bigvee [v(x) < 0 \wedge v(y) < 0]\Big\}.$$

Then V is $\mathscr{L}_{\mathrm{r}} \cup \{E\}$-definable over K.

Proof. The set $\{(a, y) \mid (a, y) \in E\}$ for any $a \in K$ is an infinite and co-infinite subset of K, which defines the valuation ring. □

Instead, in this example, the intersection with a line is the infinite and co-infinite subset of K we need. The method of intersecting the set with a line does not always work, although it works quite well when both E and its complement are Zariski-dense in the ambient affine space.

Example 4.2.3. Suppose that E is defined by

$$E := \Big\{(x, y) \mid (f(x, y) = 0) \bigwedge \\ \bigwedge \Big[(v(x) \geq 0 \wedge v(y) \geq 0) \bigvee (v(x) < 0 \wedge v(y) < 0)\Big]\Big\},$$

where $f(X,Y) \in K[X,Y]$. Suppose that E is not $\mathscr{L}_{\mathrm{r}}$-definable. Then V is $\mathscr{L}_{\mathrm{r}} \cup \{E\}$-definable over K.

Proof. We may assume that $\mathcal{K}$ is sufficiently saturated.

It is easy to see that simply intersecting E with a line would not work for this example, because for a "general" $f(X,Y)$, the intersection of $f(X,Y) = 0$ and a general line has only finitely many points. The idea of the proof here is to use a "straight family" of lines (or "a pencil of lines" through some point, where the point could be the infinity) to intersect with E, and then count the number of points in the intersection.

We may assume that $f(X,Y)$ is not the constant-zero polynomial. Otherwise, V is already $\mathscr{L}_{\mathrm{r}} \cup \{E\}$-definable by the Example 4.2.2. We may also assume that $f(X,Y)$ is irreducible; otherwise, take the Zariski-closure of E. There must be one irreducible component of the closure whose intersection with E is not $\mathscr{L}_{\mathrm{r}}$-definable; and since every irreducible Zariski-closed subset of K^2 which is not a point or the whole K^2, is the zero set of an ideal generated by one irreducible polynomial, we can then replace $f(X,Y)$ by a single irreducible polynomial. Suppose that the degree of $f(X,Y)$ is d, which is also the degree of the zero-set of $f(X,Y)$. Then the dimension of the zero-set of $f(X,Y)$ is one.

Observe that a line $Y = uX + b$ for some $u \in V \setminus \mathfrak{m}_v$ and $b \in V$, has the property that $v(X) \geq 0$ if and only if $v(Y) \geq 0$. It follows that all the intersection points of $Y = uX + b$ and $f(X,Y) = 0$ are inside E. Choose a general line $Y = uX + b$, which would then intersect with $f(X,Y) = 0$ at exactly d points.

As E is not $\mathscr{L}_{\mathrm{r}}$-definable, there must be some point (c,d) such that $f(c,d) = 0$ but $(c,d) \notin E$.

Now consider the family of lines $\{H_x := L + x(c, d - b) \mid x \in K\}$ where L is the general line we picked, $Y = uX + b$. For x very close to 0, H_x has an intersection with $Z_K(f(X,Y))$ whose points are very close to those of the intersection of H_0 and $Z_K(f(X,Y))$, which are in E. Thus there are infinitely many $x \in K$ such that the number of points in the intersection of H_x and $Z_K(f(X,Y))$ is exactly d. It follows from the strong minimality that for all but finitely many $x \in K$, $|H_x \cap Z_K(f(X,Y))| = d$. Notice that H_1 passes through (c,d). It follows that for all but finitely many x all of which are very close to 1, $|H_x \cap Z_K(f(X,Y))| = d$ and there is $\vec{w} \in H_x \cap Z_K(f(X,Y))$ with $v(\vec{w} - (c,d))$ very large; in particular $\vec{w} \notin E$. Thus, there are infinitely many $x \in K$ such that the number of points in the intersection of H_x and E is strictly less than d.

Therefore, the set

$$\{x \in K \mid |H_x \cap E| = d\}$$

is infinite and co-infinite in K, and is $\mathscr{L}_{\mathrm{r}} \cup \{E\}$-definable. We hence conclude that V is also $\mathscr{L}_{\mathrm{r}} \cup \{E\}$-definable. □

This strategy in Example 4.2.3 does not seem to work well in higher dimensions, especially when the Boolean combination of the expression involving valuation is more complicated. However, the idea of intersecting E with a family of lines and then counting the number of intersection points is a critical one.[4] We now head towards our proof.

Lemma 4.2.4. *Suppose that $\mathcal{K} \models$ ACVF is sufficiently saturated, P is an irreducible affine closed subset of K^n, with $\vec{c} \in P$, of dimension at least 1. Then there are infinitely many points $\vec{d} \in K^n$ with $v(\vec{c} - \vec{d})$ arbitrarily large and $\vec{d} \in P$.*

Proof. Suppose that P is defined over the subfield k of K. Viewing K as our "universal domain", suppose that $\vec{y} \in K^n$ is a generic point of P over k; then $k(\vec{y})$ is a regular field extension of k. We may assume that $y_1, \ldots, y_t$ is a separating transcendence basis (see Fact 5.1.1) of $k(\vec{y})/k$. By the saturatedness of $\mathcal{K}$, we can find infinitely many different sequences of $d_1, \ldots, d_t \in K$ algebraically independent over k, with $v(c_1 - d_1), v(c_2 - d_2), \ldots, v(c_t - d_t)$ arbitrarily large. Because $y_{t+1}, \ldots, y_n$ are (separably) algebraic over $y_1, \ldots, y_t$, we can find some primitive element α satisfying a (monic) separable polynomial over $k(y_1, \ldots, y_t)$ (in particular, α can be taken to be a linear combination of $y_{s+1}, \ldots, y_n$ over k, by Fact 5.1.2). We can find some $\beta \in K$ such that $k(y_1, \ldots, y_t, \alpha)$ and $k(d_1, \ldots, d_t, \beta)$ are isomorphic as fields. One can then find $d_{t+1}, \ldots, d_n \in K$ such that $k(y_1, \ldots, y_n)$ and $k(d_1, \ldots, d_n)$ are isomorphic as fields. But then $\vec{d} = (d_1, \ldots, d_n)$ also becomes a generic point of P over k. By the continuity of rational functions with respect to the valuation topology, as long as $v(c_1 - d_1), \ldots, v(c_t - d_t)$ are sufficiently large, $v(c_{t+1} - d_{t+1}), \ldots, v(c_n - d_n)$ will be sufficiently large too.

This construction of $\vec{d}$ clearly gives infinitely many different points. □

Lemma 4.2.5. *Let $\mathcal{K}$ be a model of* ACVF *which is sufficiently saturated, P an irreducible affine closed subset of K^n, which is of dimension 1 and non-degenerate. Let H be a hyperplane in K^n whose intersection with P contains a point $\vec{c}$.*

Suppose that L is an $(n-2)$-plane in H which does not contain $\vec{c}$, and suppose that $\vec{d}$ is another point outside of H. Then there are co-finitely many x with $v(x-0)$ sufficiently large, such that, the smallest hyperplane H_x, containing L and $x(\vec{d}-\vec{c})+\vec{c}$, has an intersection with P which contains a point arbitrarily close to $\vec{c}$.

Similarly, if $\vec{d}$ is a point outside of H, then $H'_x = H + x(\vec{d}-\vec{c})+\vec{c}$ has an intersection with P with contains a point arbitrarily close to $\vec{c}$, for all but finitely many x with $v(x-0)$ sufficiently large.

Proof. In fact, the union $\bigcup_{x \in K} H_x$ is K^n, and thus contains the whole P, which is an irreducible curve. Thus the affine closed set Q determined by the equations $\vec{y} \in P$ and $\vec{y} \in H_x$ in variable $(\vec{y}, x)$ is an irreducible curve in K^{n+1}. Because $\mathcal{K}$ is sufficiently saturated, and $(\vec{c}, 0) \in Q$, there are infinitely many generic points of Q which are arbitrarily closed to $(\vec{c}, 0)$. Because P is non-degenerate, the intersection of H_x with P must always be of finite cardinality. Therefore, if there are infinite many

[4] This idea was first suggested to the author by F. Delon.

generic points arbitrarily close to $(\vec{c}, 0)$, then there must be infinitely many x with $v(x-0)$ sufficiently large such that H_x has an intersection with P containing a point sufficiently closed to $\vec{c}$. But the set of x having this property is first-order definable, which is hence a finite union of perforated discs. Therefore, there can only be finitely many exceptions.

The other case can be proved similarly. □

Lemma 4.2.6. *Suppose that $E \subseteq K^n$ is an $\mathscr{L}_{\mathrm{div}}$-definable set. Suppose that the Zariski-closure of E in K^n is P, which is an irreducible affine closed set. Then there is a point $\vec{c} \in E$ such that for sufficiently large $\gamma \in vK$, $D_v^{>}(\vec{c}, \gamma) \cap P = D_v^{>}(\vec{c}, \gamma) \cap E$. We call such a point $\vec{c}$ a (P, E)-**interior point**.*

Proof. We may assume that E is defined by the formula,

$$\bigvee \phi_i(\vec{x}, \vec{a}) := \bigvee_i \left(\wedge_j (p_{i,j}(\vec{x}) = 0) \wedge (g_i(\vec{x}) \neq 0) \wedge \theta_i(\vec{x}, \vec{a})\right),$$

where $p_{i,j}$ and g_i are polynomials using parameters in $\vec{a}$, and θ_i is a finite conjunction of formulas of the form $s(x) \mid t(x)$ or $s(x) \nmid t(x)$. After re-writing the first-order formula, we may assume that for each i, g_i has all the polynomials $s(X), t(X)$ occurring in θ_i as factors.

For each i, the intersection $Z(\{p_{i,j}(\vec{x})\}_j) \cap P$ must be either all of P or be a lower dimensional sub-variety of P. Because the Zariski-closure of E is P, there must exist some i_0 such that the realization set of $\phi_{i_0}(\vec{x}, \vec{a})$ is not empty and $Z(\{p_{i_0,j}(\vec{x}\}_j) \supseteq P$. Any point $\vec{c}$ in P which is not in the lower dimensional sub-varieties of P obtained from $Z(\{p_{i,j}(\vec{x})\}_j) \cap P$ for $i \neq i_0$, will satisfy the property we want. □

Lemma 4.2.7. *Suppose that $E \subseteq K^n$ is an $\mathscr{L}_{\mathrm{div}}$-definable set which is not $\mathscr{L}_{\mathrm{r}}$-definable. Suppose that the Zariski-closure of E is P, which is an irreducible non-degenerate curve of degree d. Also suppose that $P \backslash E$ is Zariski-dense in P. If $d < 2n$, then V is $\mathscr{L}_{\mathrm{r}} \cup \{E\}$-definable.*

Proof. We may assume that $\mathcal{K}$ is sufficiently saturated.

By the assumption, every hyperplane intersects P at finitely many points. Pick $\vec{p}_1, \ldots, \vec{p}_n \in E$ with the property that any points close enough with respect to the valuation topology to them are in P if and only if they are in E (see Lemma 4.2.6). Let H be a hyperplane passing through these n points. Pick $\vec{q}_1, \ldots, \vec{q}_n \in (P \backslash E)$ with the property that any points close enough with respect to the valuation topology to them are in P if and only if they are not in E. Let H' be a hyperplane passing through these n points. Then $H \neq H'$, because $d < 2n$.

As the proofs are going to be similar, we may assume that H and H' are not parallel and let $L = H \cap H'$. Let $\vec{a} \in H$ and $\vec{b} \in H'$ be two points such that $\vec{a}, \vec{b} \notin L$. Then consider the family of hyperplanes $\{H_x \mid x \in K\}$, where H_x is the hyperplane spanned by L and $x(\vec{b} - \vec{a}) + \vec{a}$. Then for all but finitely many x which are very close to 0, H_x intersects E at at least n points; for all but finitely many x which are very close to 1, H_x intersects $P\backslash E$ at at least n points. Because any hyperplane intersects P at at most d points ($d < 2n$), there are infinitely and co-infinitely many $x \in K$ such that $|H_x \cap E| \geq n$. Thus V is $\mathscr{L}_{\mathrm{r}} \cup \{E\}$-definable. $\square$

Corollary 4.2.8. *Suppose that $E \subseteq K^2$ is an $\mathscr{L}_{\mathrm{div}}$-definable set which is not $\mathscr{L}_{\mathrm{r}}$-definable. Then V is $\mathscr{L}_{\mathrm{r}} \cup \{E\}$-definable.*

Proof. We may assume that $\mathcal{K}$ is sufficiently saturated.

The Zariski-closure of E must be of dimension either 1 or 2. If it is two, we may assume that the Zariski-closure of $K^2\backslash E$ is also K^2, otherwise, by replacing E with its complement, we are reduced to the 1-dimensional case. With this assumption about the complement, one can find $\vec{a} \in E$ and $\vec{b} \notin E$ such that any point $\vec{x}$ close enough to $\vec{a}$ is also in E, and any point close enough to $\vec{b}$ is also not in E. Then the set $\{x \in K \mid x(\vec{b} - \vec{a}) \in E\}$ is an infinite and co-infinite subset of K which is $\mathscr{L}_{\mathrm{r}} \cup \{E\}$-definable. So then V is also $\mathscr{L}_{\mathrm{r}} \cup \{E\}$-definable.

Now suppose that the Zariski-closure of E is an affine plane curve C. We may assume that C is irreducible (otherwise pick one irreducible component). Because E is not $\mathscr{L}_{\mathrm{r}}$-definable, $C\backslash E$ is also infinite (hence dense) in C.

Choose a large enough r with $p \nmid r$ such that the image of C under the Veronese map (see the appendix about this map) v_r, has a spanned linear space of dimension $N_1 = rd - d(d-3)/2 > D/2$, where $D = dr$ is the degree of $v_r(C)$. Then consider the "affinized" Veronese map $v_r^A : \mathbb{A}^2 \to \mathbb{A}^N$, which takes (x_1, x_2) to $(\ldots, x_1^{i_1} x_2^{i_2}, \ldots)$, with $0 \leq i_1 + i_2 \leq r$ (that is, the projective v_r with the last homogeneous coordinate equal to 1). Here $N = \binom{r+2}{2} + 1$.

Then $v_r^A(C)$ is irreducible and non-degenerate in $\mathbb{A}^{N_1}$ (a hyperplane in $\mathbb{A}^N$) of degree $D < 2N_1$. Notice that $v_r^A(E)$, whose Zariski-closure is $v_r^A(C)$, is also not $\mathscr{L}_{\mathrm{r}}$-definable and satisfies the conditions of Lemma 4.2.7. Therefore V is $\mathscr{L}_{\mathrm{r}} \cup \{v_r(E)\}$-definable, i.e. $\mathscr{L}_{\mathrm{r}} \cup \{E\}$-definable. $\square$

Proof of Theorem 4.1.9. We may assume that $\mathcal{K}$ is sufficiently saturated.

Assume that $E \subseteq K^n$ is $\mathscr{L}_{\mathrm{div}}$-definable but not $\mathscr{L}_{\mathrm{r}}$-definable. We may assume that the Zariski-closure of E is P, which is an irreducible affine closed subset of K^n. We may also assume that $P\backslash E$ is also Zariski-dense in P, and P is non-degenerate in the ambient space (otherwise, we are reduced to lower dimensional cases). Suppose that the dimension of P is m.

If $m = n$, then find $\vec{a} \in E$ and $\vec{b} \in P\backslash E$ such that all points close enough to $\vec{a}$ are in E and all points close enough to $\vec{b}$ are in $P\backslash E$. Then the set $\{x \in K \mid x(\vec{b} - \vec{a}) \in E\}$ is infinite and co-infinite in K, which then proves that V is $\mathscr{L}_{\mathrm{r}} \cup \{E\}$-definable, by Lemma 4.1.7.

If $m = 1$, then P is birationally equivalent (see Fact A.1.17) to a plane curve C, and up to a finite subset, E is $\mathscr{L}_{\mathrm{r}}$-definably in bijective correspondence with an $\mathscr{L}_{\mathrm{div}}$-definable but not $\mathscr{L}_{\mathrm{r}}$-definable subset of C. By Corollary 4.2.8, V is $\mathscr{L}_{\mathrm{r}}$-definable.

So we may assume that $2 \leq m \leq n-1$.

Iterating Theorem A.2.23 for $m-1$ times, we get that the intersection of P and a general $(n-m+1)$-plane is an irreducible non-degenerate curve. If we show that we can find such a curve containing an $\mathscr{L}_{\mathrm{r}} \cup \{E\}$-definable but not $\mathscr{L}_{\mathrm{r}}$-definable subset, then we are done (that is, we are back to the situation where $m=1$).

By Proposition A.2.37, there is always a general $(n-m+1)$-plane passing one (P,E)-interior point in E and one $(P,P\backslash E)$-interior point in $P\backslash E$. Such an $(n-m+1)$-plane would then intersect P at an irreducible non-degenerate curve C, whose intersection with E is not $\mathscr{L}_{\mathrm{r}}$-definable (because in C there are infinitely many points in E and infinitely many points in $P\backslash E$). Then we are reduced to the 1-dimensional case.

Hence, V is $\mathscr{L}_{\mathrm{r}} \cup \{E\}$-definable. □

Chapter 5

Separably closed valued fields † ‡

In this chapter, we focus on the situation where we have a separably closed field of positive characteristic and finite imperfection degree, K, with a non-trivial valuation ring V. Let $\mathcal{K}$ be the $\mathscr{L}_{\text{div}}$-structure on (K, V), then we prove that $\mathcal{K} \between (\mathcal{K}|\mathscr{L}_{\text{r}})$. The same question remains open in the infinite imperfection degree case. Almost all results depend on a quantifier elimination result we obtained.

We emphasize once more that for an algebraic set V in some $(K^{\text{alg}})^n$ and a subfield $k \subseteq K$, when we say "V is **definable** over k", it means that V is model-theoretically definable over k, i.e. it is the realization set of a first-order formula with parameters coming from k; when we say "V is **defined** over k", it means we are saying that the field of definition of V is contained in k, i.e. the ideal associated to V is generated by polynomials with coefficients coming from k.

5.1 Preliminaries

This section collects some preliminary knowledge about separably closed fields and separably closed valued fields.

5.1.1 Separably closed fields

A detailed reference for the p-independence relation and separable extensions of fields could be found in [Karpilovsky, 1989]. A good summary is for example [Delon, 1998]. In particular, some aspects of the model theory of separably closed field has been studied in [Eršov, 1967], [Wood, 1979], [Delon, 1988], [Delon, 1998], and many more

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‡The research leading to these results (except the one-sorted quantifier elimination) was done during the author's 5-month stay in Université Paris Diderot - Paris 7, under the supervision of Dr. Françoise Delon.

papers (including some from the point of view of differentially closed fields). Here we only give a fast, partial review.

Let K be a field (not necessarily separably closed). A polynomial over K is said to be **separable** if all of its roots are different. An algebraic field extension L over K is said to be **separable** if all the minimal polynomials of elements of L over K are separable. For any field K, there is always a maximal separable algebraic extension, called the **separable closure** (or the **separably algebraic closure**) of K, denoted by K^{sep}.

Recall that our notation for the set of nth-powers in a field K, is $P_n(K)$, not K^n (which is the n-th Cartesian power of K as a set).

A field K is **perfect** if K is of characteristic 0 or $P_p(K) = K$ where $0 < p = \text{char}(K)$; equivalently, K is perfect if the separable closure of K, is the same as the algebraic closure of K, denoted by K^{alg}. Hence a separably closed field of characteristic 0 is always algebraically closed.

In the following, we assume that K is of characteristic $p > 0$. A useful identity for a field of characteristic p is that for all $a, b \in K$, $(a + b)^p = a^p + b^p$. Recall that over a field of characteristic $p > 0$, a polynomial f has a multiple root if and only if f and its formal derivative f' has a common factor. Thus if f is irreducible and has a multiple root, then f' has to be zero; it follows that in this case, $f(X) = g(X^p)$ for some separable polynomial $g \in K[X]$. Thus, if K is separably closed, then because every separable polynomial splits into linear factors over K, every polynomial over K is a product of constants in K and polynomials of the form $X^{p^n} - a$ with $a \in K$.

In general, K is a field extension of $P_p(K)$. For $x \in K$ and $A \subseteq K$, x is said to be p**-independent** over A in K if $x \notin (P_p(K))(A)$, i.e. x not in the field generated by $P_p(K)$ and A. A set A is said to be a p**-independent** subset of K if for all $x \in A$, we have $x \notin P_p(K)(A \backslash \{x\})$. A p**-basis** of K is a maximal p-independent subset of K. A field extension L over K is said to be **separable** if the p-independence relation in K is preserved in L. Any purely transcendental extension is separable. One can check that when L/K is algebraic, this agrees with the definition above, although in the general case, it is no longer always true that if L/K is separable, then L is separable over any subfield containing K.

Suppose that $B = \{b_i\}_{i \in I}$ is a p-basis of K, then the set of monomials $\{\prod_{i \in I} b_i^{j(i)}\}_j$, where j runs through all the maps from I to the set $\{0, \dots, p-1\}$ with finite supports, is a vector-space basis of K viewed as a vector space over $P_p(K)$; this is in fact an equivalent definition of p-basis. The cardinality of different p-bases are the same, denoted by $p^{e(K)}$, with $e(K)$ a natural number or ∞ (one can prove that it is always of this form); the exponent $e(K)$ is sometimes simply denoted by e if the field we are referring to is clear from the context, and is called the **Eršov invariant**, or the **imperfection degree** of K. One can see that $e(K) = 0$ if and only if K is perfect.

Suppose that $B = \{b_i\}_{i \in I}$ is a p-basis of K, then the coordinates of an element in K with respect to the basis of monomials $\{\prod_i b_i^{j(i)}\}_j$ are called p**-coordinates**. In particular, we define functions $f_j : K \to K$ such that for all $x \in K$,

$$x = \sum_j f_j(x)^p \prod_{i \in I} b_i^{j(i)},$$

with j running through all the functions from I to $\{0, \dots, p-1\}$ with finite supports.

Following the notation of [Hrushovski, 1996], we define the function $\lambda_1 : K \to K^{p^e}$ to be $x \mapsto (f_j(x))_j$, and $\lambda_n = \lambda_1^{\circ n} = \lambda_1 \circ \cdots \circ \lambda_1$ (the composition of λ_1 with itself for n-times).[1]

We will make use of the following important theorem about separable field extensions.

Fact 5.1.1 (Separating transcendence basis theorem, e.g. [Karpilovsky, 1989] [2])**.** Suppose that $L := K(y_1, \dots, y_n)$ is a separable field extension of K. Then there exists a subset $\{y_{i_1}, \dots, y_{i_t}\}$, of $\{y_i\}_{i=1}^n$, which is algebraically independent over K, and such that L is separably algebraic over $K(y_{i_1}, \dots, y_{i_t})$.

The $y_{i_1}, \dots, y_{i_t}$ in the above theorem is called a **separating transcendence basis** of L/K.

The following is the well-known Primitive Element Theorem:

Fact 5.1.2 (see e.g. [Cox, 2012])**.** Suppose that $k(y_1, \dots, y_n)/k$ is a finite separable field extension, then there exists a **primitive element** $\alpha \in k(y_1, \dots, y_n)$ such that $k(y_1, \dots, y_n) = k(\alpha)$. Furthermore, if k is infinite, then α can be chosen to be of the form

$$\alpha = \mu_1 y_1 + \mu_2 y_2 + \cdots + \mu_n y_n,$$

where $\mu_1, \dots, \mu_n \in k$; in fact, this is true for all points in $\mathbf{P}_k^{n-1}$ with only finitely many exceptions.

Turning to the model theory aspect, we have

Fact 5.1.3 ([Eršov, 1967])**.** In the language of rings $\mathscr{L}_\mathrm{r}$, suppose that K and L are two separably closed field of characteristic $p > 0$. Then K and L are elementarily equivalent if and only if $e(K) = e(L) < \aleph_0$, or $e(K), e(L) = \infty$.

Fact 5.1.4 ([Wood, 1979])**.** In the language of rings $\mathscr{L}_\mathrm{r}$, the theory of separably closed fields is stable, but not superstable.

It is in fact a conjecture that all infinite stable fields are separably closed. We will see in the following a way of proving the stability without counting types directly.

Suppose that $0 < e < \aleph_0$. Let $\mathscr{L}_{p,e}$ be the union of $\mathscr{L}_\mathrm{r}$ and constant symbols $\{b_i\}_{i \in e}$ (a p-basis), and unary function symbols $\{f_j\}_{j \in p^e}$ (p-coordinate functions). Let

[1] Perhaps λ^n would be a better notation than λ_n.

[2] A separating transcendence basis is called a *separating transcendency basis* in [Karpilovsky, 1989]. This theorem is also a corollary of a theorem called *MacLane's Criterion* loc.cit..

$\mathrm{SCF}_{p,e}$ be the theory of separably closed fields of characteristic p, imperfection degree e, having $\{b_i\}_i$ as a p-basis with $\{f_j\}$ the corresponding p-coordinate functions. Then we have

Fact 5.1.5 ([Delon, 1988], [Delon, 1998]). $\mathrm{SCF}_{p,e}$ ($p > 0$, $0 < e < \aleph_0$) has quantifier elimination.

For the case where e is infinite, we refer the reader to Section 5.5; in particular, there is still a quantifier elimination result, in a slightly different language. Except in Section 5.5, we will always work with the finite imperfection degree case.

F. Point pointed out to the author that it was in [Srour, 1986] that Srour first showed that the theory of separably closed fields of a fixed characteristic and a fixed imperfection degree has quantifier elimination in a language slightly different from $\mathscr{L}_{p,e}$ (note that his language contains the language of fields which has a symbol for the inverse function $(\cdot)^{-1}$). F. Point also pointed out that Haran in [Haran, 1988] gave an explicit primitive recursive quantifier elimination algorithm for the finite imperfection degree case in the language given by Srour. The author would like to thank F. Point for pointing these out.

5.1.2 Separably closed valued fields

Suppose that K is a separably closed field of characteristic $p > 0$, with a valuation ring V. If V is trivial (i.e. $V = K$), then clearly the residue field is K itself and the value group is $\{0\}$. If V is not trivial, then because a separably closed field in the $\mathscr{L}_{\mathrm{r}}$ is always stable, V is not definable over K in $\mathscr{L}_{\mathrm{r}}$. But one can ask about the "defining power" of V compared to $\mathscr{L}_{\mathrm{r}}$ over the field K, in terms of "intermediate structures", which is the main subject of investigation of this chapter.

Recall that a valued field is Henselian if and only if there is exactly one extension of the valuation onto its separable closure. Due to this fact, a separably closed valued field is always Henselian.

Fact 5.1.6 ([Engler and Prestel, 2005]). If (K, V) is a separably closed non-trivially valued field (of any characteristic and any imperfection degree), then Kv is always algebraically closed and vK is always divisible.

It follows that a separably closed non-trivially valued field of positive characteristic is always NOT **algebraically maximal** (a valued field is algebraically maximal if there are no proper immediate algebraic extensions, that is, there is no proper algebraic extension which has the same residue field and value group). A valued field is said to be **separably algebraically maximal** if there is no proper immediate extension which is also a separable algebraic extension. It follows trivially that every separably closed valued field is separably algebraically maximal.

A valued field (K, V) of positive characteristic p, is called **Kaplansky**, if vK is p-divisible and for each $a_0, \ldots, a_n, b \in Kv$, the equation $x^{p^n} + a_{n-1}x^{p^{n-1}} + \cdots + a_1 x^p +$

$a_0x + b = 0$ has a solution in Kv. Thus, a separably closed non-trivially valued field of characteristic p is a Kaplansky valued field.

In [Delon, 1982], certain aspects of the model theory of algebraically maximal valued fields and separably algebraically maximal valued fields have been studied. In particular, it was shown that being algebraically maximal and being separably algebraically maximal are both first-order, in the language $\mathscr{L}_{\text{div}}$.

Fact 5.1.7 ([Delon, 1982])**.** Let k be a field of characteristic $p > 0$ satisfying that for all elements $a_0, \ldots, a_n, b \in k$ the equation $x^{p^n} + a_{n-1}x^{p^{n-1}} + \ldots a_1x^p + a_0x + b = 0$ has a solution in k, and G a non-trivial p-divisible ordered abelian group. In the language $\mathscr{L}_{\text{div}}$, let $T_e(k, G)$ be the first-order theory of Kaplansky, separably algebraically maximal valued fields of characteristic $p > 0$ and imperfection degree $e \in \{1, 2, \ldots, \infty\}$, with residue field elementarily equivalent to k in the language of rings, and with value group elementarily equivalent to G in the language of ordered abelian groups $\{+, -, <, 0\}$. Then $T_e(k, G)$ is complete.

Let $T_{eQ}(k, G)$ be the expansion of $T_e(k, G)$, in the language obtained by adding all the n-ary p-independence predicates (i.e. $Q_n(x_1, x_2, \ldots, x_n)$ if and only if $x_1, x_2, \ldots, x_n$ are p-independent) to $\mathscr{L}_{\text{div}}$. Then $T_{eQ}(k, G)$ is model-complete.

In particular, if k is an algebraically closed field, and G is a divisible ordered abelian group, then $T_e(k, G)$ is exactly[3] the first-order theory of separably closed non-trivially valued fields, of characteristic p and imperfection degree e, and this theory is complete by the theorem.

From this theorem (and its proof), it is possible to obtain a quantifier elimination result in a suitable one-sorted language (clearly one needs to consider about p-coordinate functions), as we obtain below. But our proof of the result about quantifier elimination here, in the next section, is self-contained and, most importantly, independent.

5.2 Quantifier elimination

In this section, we show that in the natural two-sorted language consisting of the valued-field sort and the value-group sort, the theory of non-trivially valued separably closed fields with characteristic $p > 0$ and finite imperfection degree $e > 0$, has quantifier elimination. It then follows that the induced structure on the value group is exactly the ordered abelian group structure; in particular, the value group is stably embedded (i.e. every definable set in the value group using parameters outside of the group is already definable using parameters from the group).

Fix e a finite positive integer, and $p > 0$ a prime number.

[3] It is enough to see that every separably algebraically maximal valued field with an algebraically closed residue field and a divisible value group is separably closed. This is true because the separable closure has the same residue field and value group.

Let $\mathscr{L}_{\mathrm{OAG}}$ be the language of ordered abelian groups $\{+, -, <, 0\}$. Let $\mathscr{L}_{p,e}$ be the union of $\mathscr{L}_{\mathrm{r}}$ and constant symbols $\{b_i\}_{i\in e}$ (a p-basis), and unary function symbols $\{f_i\}_{i\in p^e}$ (p-coordinate functions). Let $\mathscr{L}^2_{\mathrm{r}}$ be the two sorted language for valued fields, where the language for the field sort is $\mathscr{L}_{\mathrm{r}}$ and the language for the value group sort is $\mathscr{L}_{\mathrm{OAG}} \cup \{\infty\}$, and the function connecting these two sorts is the valuation function $\{v\}$. Let $\mathscr{L}^2_{p,e}$ be the two sorted language with $\mathscr{L}_{p,e}$ for the field sort, $\mathscr{L}_{\mathrm{OAG}} \cup \{\infty\}$ for the value group sort, and $\{v\}$ for the valuation function going from the field sort to the value-group sort.

To obtain a quantifier elimination result for some one-sorted language, one can instead prove a quantifier elimination result for a suitable two-sorted language, as we will do below. This version of quantifier elimination was proved in order to answer a question regarding the induced structure on the value group, which as we will see is exactly the ordered abelian group structure itself (hence o-minimal).

Let $\mathrm{SCVF}^2_{p,e}$ be the theory of non-trivially valued separably closed fields with characteristic $p > 0$ and finite imperfection degree $e > 0$, with $\{b_i\}_{i\in p}$ being a p-basis and $\{f_i\}_{i\in p^e}$ being the p-coordinate functions relative to the p-basis $\{b_i\}_i$, in the language $\mathscr{L}^2_{p,e}$, adjoined with the axiom "$v(b_0) > 0$".[4]

We denote the one-sorted reduct of $\mathrm{SCVF}^2_{p,e}$ to $\mathscr{L}_{p,e} \cup \{|\}$ by $\mathrm{SCVF}_{p,e}$.

Theorem 5.2.1. $\mathrm{SCVF}^2_{p,e}$ *has quantifier elimination.*

We first introduce a critical notion which we will use often, and then prove some lemmas before coming back to the proof of the theorem above. The phenomenon below has been observed first by Delon, but has never been explicitly defined.

Definition-Proposition 5.2.2. *Suppose that $\mathcal{K} \models \mathrm{SCVF}_{p,e}$. Suppose that we are given a quantifier-free $\mathscr{L}_{p,e} \cup \{|\}$-formula $\phi(\vec{x}, \vec{a})$ with $\vec{a} \in K^m$. Then for sufficiently large n, $\lambda_n(\{\vec{x} \in K^m \mid \phi(\vec{x}, \vec{a})\})$ is definable using a quantifier-free $\mathscr{L}_{\mathrm{div}}$-formula over K; any such a quantifier-free $\mathscr{L}_{\mathrm{div}}$-formula is called a* **$\lambda$-resolution** *of ϕ, denoted by $\lambda_{\mathrm{res}}(\phi)(\lambda_{\mathrm{res}}(\vec{x}), \lambda_{\mathrm{res}}(\vec{a}))$, or simply $\tilde{\phi}(\vec{y}, \tilde{\vec{a}})$, where $\vec{y}$ is the image of $\vec{x}$ under the corresponding λ_n (we simply write $\vec{y} = \lambda_{\mathrm{res}}(\vec{x})$).*

Proof. We give a quick proof of the fact that for sufficiently large n, $\lambda_n(\{\vec{x} \in K^m \mid \phi(\vec{x}, \vec{a})\})$ is definable using a quantifier-free $\mathscr{L}_{\mathrm{div}}$-formula over K.

It is easy to see that every f_j is additive, i.e. $f_j(x + y) = f_j(x) + f_j(y)$ using the uniqueness of the p-coordinate functions. It is also easy to see that $f_j(xy)$ could be express as a polynomial in the variables $\{f_k(x)\}_k \cup \{f_l(y)\}_l$. For example, $x = \sum_j f_j(x)^p \prod_i b_i^{j(i)}$, $y = \sum_j f_j(y)^p \prod_i b_i^{j(i)}$; so if we multiply them together, we get

$$xy = \sum_k h_k(x, y)^p \prod_i b_i^{k(i)},$$

[4] The condition "$v(b_0) > 0$" is to make sure that we have a constant symbol for an element with a non-zero value, which seems to be necessary in our proof of quantifier elimination. See Lemma 5.2.9.

where $h_k(x, y)$ is a polynomial in variables $\{f_k(x)\}_k \cup \{f_l(y)\}_l$ obtained from the product. But on the other hand by the definition of the p-coordinate functions, we have $xy = \sum_j f_j(xy)^p \prod_i b_i^{j(i)}$. This implies that $f_j(xy) = h_k(x, y)$. One can than do this inductively to see that the p-coordinate function of a polynomial is a polynomial using the p-coordinate functions of the original variables as variables. We then get an equivalent formula in which every term is a composition of a polynomial and λ_n for some n, one can then replace the terms $\lambda_n(x)$ with x being a variable symbol by new variables $\vec{z}$. This gives a λ-resolution. □

Remark 5.2.3. The λ-resolution of a formula ϕ is never unique. But all the λ-resolutions are $\mathscr{L}_{\mathrm{r}}$-interdefinable over K. In fact, for example, for any subset A of K, $\lambda_1(A) = \{(f_j(x)) \mid x \in A\}$, and $A = \{\sum_j x_j^p \prod_i b_i^{j(i)} \mid (x_j) \in \lambda_1(A)\}$ shows that A and $\lambda_1(A)$ are $\mathscr{L}_{\mathrm{r}}$-interdefinable over K.

Definition 5.2.4. *Suppose that $\mathcal{K} \models \mathrm{SCVF}^2_{p,e}$. Given a quantifier-free $\mathscr{L}^2_{p,e}$-formula whose free variables are all in the field-sort $\phi(\vec{x}, \vec{a}, \vec{\delta})$, with $\vec{a} \in K^m$ and $\vec{\delta} \in (vK)^l$, a* **λ-resolution** *of ϕ is a quantifier-free $\mathscr{L}^2_{\mathrm{r}}$-formula defining the image of the realization set of ϕ under λ_n, for sufficiently large n. It is denoted by $\lambda_{\mathrm{res}}(\phi)(\lambda_{\mathrm{res}}(\vec{x}), \lambda_{\mathrm{res}}(\vec{a}), \lambda_{\mathrm{res}}(\vec{\delta}))$, or simply $\tilde{\phi}(\vec{y}, \tilde{\vec{a}}, \tilde{\vec{\delta}})$.*

Lemma 5.2.5. *Suppose K is a separably closed field of characteristic $p > 0$ with a valuation v, then for any natural number n, $P_{p^n}(K^\times)$ is dense in K with respect to the valuation topology.*

Proof. We may assume that v is not trivial and $n > 0$. Then the value group is a divisible group without upper bound. It is obvious that $P_{p^n}(K^\times)$ is dense in a neighborhood of 0.

Suppose that z is an element of $K^\times$, and $\gamma > 0$ is an element of the value group of K; it is enough to show that there exists some $z_\gamma \in K^\times$ such that $v(z - z_\gamma^{p^n}) > \gamma$.

Case 1. $v(z) \geq 0$.

Pick $a \in K^\times$ such that $v(a) > \gamma$, then consider the polynomial $f(X) = X^{p^n} - aX - z$. Notice that the formal derivative of f, $f'(X) = -a \neq 0$. Thus f splits over K. In particular since $\infty \neq v(z) \geq 0$, there exists some root $z_\gamma \neq 0$ with $v(z_\gamma) \geq 0$. Then $v(z - z_\gamma^{p^n}) = v(az_\gamma) > \gamma$.

Case 2. $v(z) < 0$.

Pick $b \in K^\times$ such that $v(b^{p^n} z) \geq 0$ (it exists!). Let $\epsilon = v(b^{p^n}) + \gamma$. Then since $\epsilon > 0$, applying Case 1 (to $b^{p^n} z$ and ϵ), one has a $z' \neq 0$ such that $v(b^{p^n} z - z'^{p^n}) > \epsilon$. Therefore, letting $z_\gamma = z' b^{-1}$, we get $v(z - z_\gamma^{p^n}) > \gamma$. □

Remark 5.2.6. A corollary of Lemma 5.2.5 is that if K is separably closed, then K is dense (with respect to the valuation topology) in its algebraic closure K^{alg}, because for each $a \in K^{\mathrm{alg}}$ there is some n such that $a^{p^n} \in K$. This proves that separably closed field is "valuationally algebraically closed", a notion introduced in Definition 5.6.6.

In fact, Delon has proved (independently, and many years ago) that Lemma 5.2.5 is true for all separably algebraically maximal valued fields (using pseudo-convergent sequences). See Proposition 1.22 of [Delon, 1982].

For the ease of reading, the following is Lemma 3.27 (with its original proof) from [Van den Dries, 2004].

Lemma 5.2.7 (van den Dries). *Let (L, V) be a valued field extension of (K, W) (so $v|_K = w$) such that $Kw = Lv$. Let $a_1, \ldots, a_n \in K$, $n \geq 1$, and let $x \in L \backslash K$ be such that $v(x - a_i) \in vK$ for $i = 1, \ldots, n$. Then there exists $a \in K$ such that $v(x - a_i) = v(a - a_i)$ for $i = 1, \ldots, n$.*

Proof. Any $a \in K$ with $v(a - x) > v(a - a_i)$ for all i has the desired property. We may assume that $v(x - a_1) \geq v(x - a_i)$ for $i = 2, \ldots, n$. Since $v(x - a_1) \in vK$, we can take $b \in K$ such that $v(x - a_1) = v(b)$. So $v((x - a_1)/b) = 0$ and since $Kw = Lv$, $(x - a_1)/b = c + \epsilon$ with $c \in K$, $v(c) = 0$ and $v(\epsilon) > 0$. Then $a = a_1 + bc$ works because $x - a = b\epsilon$ and $v(b\epsilon) > v(x - a_i)$. □

Lemma 5.2.8. *Suppose that $\mathscr{L}^2$ is a two-sorted language with $\mathscr{L}_1$ being the language for the first sort, and the $\mathscr{L}_2$ being the language for the second sort. The only function between sorts is just a function v from the first sort to the second sort.*

Suppose that $\mathcal{M} := (M_1, M_2, v_M)$ and $\mathcal{N} := (N_1, N_2, v_N)$ are two $\mathscr{L}^2$-structures, and $\mathcal{F} := (F_1, F_2, v_M)$ is an $\mathscr{L}^2$-substructure of $\mathcal{M}$, and $f : \mathcal{F} \to \mathcal{N}$ is an $\mathscr{L}^2$-embedding. Suppose K_1 is an $\mathscr{L}_1$-substructure of M_1 containing F_1 and K_2 is an $\mathscr{L}_2$-substructure of M_2 containing F_2. Then

1. *If $f' : K_2 \to N_2$ is an $\mathscr{L}_2$-embedding extending $f|_{F_2}$, then $g := f \cup f' : (F_1, K_2, v_M) \to \mathcal{N}$ is an $\mathscr{L}^2$-embedding.*
2. *Suppose that $F_2 = M_2$. If $f' : (K_1, v_M(K_1), v_M) \to \mathcal{N}$ is an $\mathscr{L}^2$-embedding extending $f|_{(F_1, v_M(K_1), v_M)}$, then $g := f \cup f' : (K_1, M_2, v_M) \to \mathcal{N}$ is an $\mathscr{L}^2$-embedding.*

Proof. Any quantifier-free formula of $\mathscr{L}^2$ in free variables $\vec{x}_1$ in the first sort and free variables $\vec{x}_2$ in the second sort is a boolean combination of formulas of the form $\phi_1(\vec{x}_1)$ or $\phi_2(v(\vec{t}(\vec{x}_1)), \vec{x}_2)$, where ϕ_1 is a quantifier-free $\mathscr{L}_1$-formula, and ϕ_2 is a quantifier-free $\mathscr{L}_2$-formula with $\vec{t}$ being a tuple of $\mathscr{L}_1$-terms (so $v(\vec{t})$ means applying v to all the coordinates of $\vec{t}$).

In Case (1), since the field sort of the domain of g stays to be F_1, for any $\phi_1(\vec{x}_1)$ and $\vec{w} \in F_1$, $\phi_1(\vec{w}_1)$ is true in F_1 if and only if $\phi_1(f(\vec{w}_1)) = \phi_1(g(\vec{w}_1))$ is true in $g(F_1)$. For any $\vec{w} \in F_1$ and $\vec{k}_2 \in K_2$, since $v_M(\vec{t}(\vec{w})) \in F_2 \subseteq K_2$, $\phi_2(v_M(\vec{t}(\vec{w})), \vec{k}_2)$ is true in (F_1, K_2, v_M) if and only if $\phi_2(v_N(\vec{t}(f(\vec{w}))), g(\vec{k}_2)) = \phi_2(v_N(\vec{t}(g(\vec{w}))), g(\vec{k}_2))$ is true in $(g(F_1), g(K_2), v_N)$. Therefore, g is an $\mathscr{L}^2$-embedding.

In case (2), again it is clear that for all $\vec{w} \in K_1$, $\phi_1(\vec{w})$ is true in (K_1, M_2, v_M) if and only if $\phi_1(g(\vec{w}))$ is true in the image of g.

For any $\vec{k}_1 \in K_1$ and $\vec{m}_2 \in M_2$, for any formula of the form ϕ_2 mentioned above, $\phi_2(v_M(\vec{t}(\vec{k}_1)), \vec{m}_2)$ is true in $\mathcal{M}$, if and only if $\phi_2(f(v_M(\vec{t}(\vec{k}_1))), f(\vec{m}_2))$ is true in $\mathcal{N}$, if and only if $\phi_2(f'(v_M(\vec{t}(\vec{k}_1))), f(\vec{m}_2))$ is true in $\mathcal{N}$, if and only if $\phi_2(v_N(\vec{t}(f'(\vec{k}_1))), f(\vec{m}_2))$ is true in $\mathcal{N}$, if and only if $\phi_2(v_N(\vec{t}(g(\vec{k}_1))), g(\vec{m}_2))$ is true in $\mathcal{N}$. Therefore, g is indeed an $\mathscr{L}^2$-embedding. □

Lemma 5.2.9. *Suppose that $\mathcal{K}_1 := (K_1, \Gamma_1, v_1)$ and $\mathcal{K}_2 := (K_2, \Gamma_2, v_2)$ are two models of* $\mathrm{SCVF}^2_{p,e}$ *and $\mathcal{K}_2$ is $|\mathcal{K}_1|^+$-saturated. Suppose that $\mathcal{F} := (F, \Gamma_1, v_1)$ is an $\mathscr{L}^2_{\mathrm{r}}$-substructure of $\mathcal{K}_1$ and $f : \mathcal{F} \to \mathcal{K}_2$ is an $\mathscr{L}^2_{\mathrm{r}}$-embedding, and F is a separably closed subfield of K_1. Suppose that F is non-trivially valued.*

Then for any $a \in K_1$ which is transcendental over F, there is an $\mathscr{L}^2_{\mathrm{r}}$-embedding $f' : (F(a)^s, \Gamma, v_1) \to \mathcal{K}_2$ extending f.

Proof. We may assume that $v_1(a) \geq 0$ (otherwise do the argument for a^{-1}). We remark that the residue field Fv_1 is algebraically closed, because F is not trivially valued by the assumption.

There are three cases.

Case 1. $F(a)v_1 \neq Fv_1$ and $v_1(F(a)^\times) = v_1F$.

In this case $\mathrm{res}_{v_1}(a) \in F(a)v_1$ is transcendental over Fv_1; otherwise $F(a)v_1$ is going to be algebraic over Fv_1, contradicting the fact that Fv_1 is algebraically closed. Since $\mathcal{K}_2$ is $|\mathcal{K}_1|^+$-saturated, there exists some $b \in K_2$ with $\mathrm{res}_{v_2}(b) \in (K_2)v_2$ such that $\mathrm{res}_{v_2}(b)$ is transcendental over $f(F)v_2$. So b is transcendental over $f(F)$ as well. It is also clear that $v_1(a) = 0$ and $v_2(b) = 0$. Then by Corollary 2.2.2 of [Engler and Prestel, 2005], f extends to a valued field embedding which is also an $\mathscr{L}^2_{\mathrm{r}}$-embedding $f_1 : (F(a), \Gamma_1, v_1) \to \mathcal{K}_2$ sending a to b. But by Fact 1.2.3, f_1 clearly has an extension to the separable closure $f' : (F(a)^s, \Gamma_1, v_1) \to \mathcal{K}_2$.

Case 2. $F(a)v_1 = Fv_1$ and $v_1(F(a)^\times) \neq v_1F$.

In this case $v_1(a)$ might still be in v_1F. Since $F(a)$ is the quotient field of $F[a]$ and $v_1(F(a)^\times) \neq v_1F$, there is some $g(a) \in F[a]$ such that $v(g(a)) \notin v_1F$. Because F is separably closed, there are $c', c_i \in F$ and natural numbers n_i such that $g(a) = c' \prod_i (a^{p^{n_i}} - c_i)$. This implies that there is some factor $a^{p^{n_i}} - c_i$ with $v(a^{p^{n_i}} - c_i) \notin v_1F$. We assume that n is the minimal integer m such that there is some $w \in F$ with $v_1(a^{p^m} - w) \notin v_1F$; let c be a corresponding w.

Subcase 1. There is some $\gamma \in v_1F$ such that $v(a^{p^n} - c) < \gamma$.

In this case, by the denseness of $(F)^{p^n}$, there is some $d \in F$ such that $v(a^{p^n} - d^{p^n}) = v(a^{p^n} - c) \notin v_1F$, i.e. $v(a - d) \notin v_1F$. Then $f(v_1(a-d))$ is not in $f(v_1F)$ either. So for any $b \in K_2$ such that $v_2(b) = f(v_1(a-d))$, b is transcendental over $f(F)$. By Corollary 2.2.3 of [Engler and Prestel, 2005], the valued field isomorphism extending $f|_A$ by sending $a - d$ to b also extends f to an $\mathscr{L}^2_{\mathrm{r}}$-embedding $f_1 : (F(a-d), \Gamma_1, v_1) \to \mathcal{K}_2$.

Because $F(a-b)^s = F(a)^s$, it follows that f can be extended to an $\mathscr{L}_{\mathrm{r}}^2$-embedding $f' : (F(a)^s, \Gamma_1, v_1) \to \mathcal{K}_2$, by Fact 1.2.3.

Subcase 2. $v_1(a^{p^n} - c) > v_1 F$.

In this case, we may assume that $v_1(a-w) \in v_1 F$ for all $w \in F$ (i.e. $n \geq 1$), otherwise we can extend f by the same method as that in Subcase 1. It also follows from the minimality of n that c does not have a p-th root in F.

Then for any $e \in F$ such that $e \neq c$, we have $v_1(a^{p^n} - e) = v_1(c-e) \in v_1 F$. It follows that for any $m \geq n$ and w, if $v_1(a^{p^m} - w) > v_1 F$, then $w = c^{p^{(m-n)}}$.

Let $\delta = v_1(a^{p^n} - c)$. Then there is some $k \in K_2$ such that $v_2(k) = f(\delta)$. Then $v_2(k + f(c) - f(c)) = f(\delta)$. By the denseness of $(K_2^\times)^{p^n}$, there is some $b \in K_2^\times$ such that $v_2(b^{p^n} - f(c)) = f(\delta)$. It then follows that b is also transcendental over $f(F)$. For any $m \geq n$ and $w \in F$ with $w \neq c^{p^{m-n}}$, we have

$$\begin{aligned}
v_2(b^{p^m} - f(w)) &= v_2(b^{p^m} - f(c)^{p^{m-n}} + f(c)^{p^{m-n}} - f(w)) \\
&= v_2(f(c)^{p^{m-n}} - f(w)) \\
&= f(v_1(c^{p^{m-n}} - w)) \\
&= f(v_1(a^{p^m} - w));
\end{aligned}$$

for any $m \geq n$ and $w = c^{p^{m-n}}$, we have

$$\begin{aligned}
v_2(b^{p^m} - f(w)) &= v_2(b^{p^m} - f(c)^{p^{m-n}}) \\
&= p^{m-n} v_2(b^{p^n} - f(c)) \\
&= f(p^{m-n} v_1(a^{p^n} - c)) \\
&= f(v_1(a^{p^m} - w));
\end{aligned}$$

for any $m \leq n$ and $w \in F$, we have

$$\begin{aligned}
p^{n-m} v_2(b^{p^m} - f(w)) &= v_2(b^{p^n} - f(w)^{p^{n-m}}) \\
&= v_2(b^{p^n} - f(c) + f(c) - f(w^{p^{n-m}})) \\
&= v_2(f(c) - f(w^{p^{n-m}})) \\
&= f(v_1(c - w^{p^{n-m}})) \\
&= f(v_1(a^{p^n} - w^{p^{n-m}})) \\
&= f(p^{n-m} v_1(a^{p^m} - w)).
\end{aligned}$$

Therefore, extending $f|_F$ to $F(a)$ by sending a to b, we get a valued field isomorphism, because for all $g(a) \in F(a)$, we have that $f(v_1(g(a))) = v_2(g(b))$. Hence we also get an extension f' accordingly by Fact 1.2.3.

Case 3. $F(a)v_1 = Fv_1$ and $v_1(F(a)^\times) = v_1 F$.

For any $g(X) \in F[X]$, one can factorize g into $c\prod_j(X^{p^{n_j}} - c_j)$ with $c, c_j \in F$, since F is separably closed. It is enough to find some $b \in K_2 \backslash f(F)$ with b being transcendental over $f(F)$ such that for all i and $c_j \in F$, $f(v_1(a^{p^i} - c_j)) = v_2(b^{p^i} - f(c_j))$, Because then the field isomorphism $F(a) \to (f(F))(b)$ sending a to b is also a valued field isomorphism.

But by the saturation of $\mathcal{K}_2$, it is enough to show that for any finitely many elements $c_1, \ldots, c_n \in F$, and any $p^{\alpha_1}, \ldots, p^{\alpha_n}$, there exist infinitely many $z \in f(F)$ (to make sure that one can say that z is transcendental) such that for all i,

$$\infty \neq f(v_1(a^{p^{\alpha_i}} - c_i)) = v_2(z^{p_i^{\alpha}} - f(c_i)).$$

We may assume that $\alpha_1 \geq \alpha_i$ for all i. By Lemma 5.2.7 we can find some $w \in F$ such that for all i,

$$\infty \neq v_1(a^{p^{\alpha_1}} - c_i^{p^{\alpha_1 - \alpha_i}}) = v_1(w - c_i^{p^{\alpha_1 - \alpha_i}}). \tag{5.2.1}$$

By Lemma 5.2.5, by varying the accuracy of the approximation, there are infinitely many $t \in F$ such that $v_1(w - t^{p^{\alpha_1}})$ are sufficiently large and hence, for all i

$$v_1(a^{p^{\alpha_1}} - c_i^{p^{\alpha_1 - \alpha_i}}) = v_1(t^{p^{\alpha_1}} - c_i^{p^{\alpha_1 - \alpha_i}}),$$

namely,

$$v_1(a^{p^{\alpha_i}} - c_i) = v_1(t^{p^{\alpha_i}} - c_i).$$

Now this carries over to $\mathcal{K}_2$ because $t \in F$, that are infinitely many $z \in f(F)$ satisfying the conditions we wanted. Eventually, by Fact 1.2.3, we get an $\mathscr{L}_{\mathrm{r}}^2$-embedding f' : $(F(a)^s, \Gamma_1, v_1) \to \mathcal{K}_2$ extending f to include the separable closure of $F(a)$. □

Fact 5.2.10 (see e.g. [Marker, 2002])**.** Let ODAG be the theory of non-trivial ordered divisible abelian groups in the language $\mathscr{L}_{\mathrm{OAG}}$. Then ODAG is complete and has quantifier elimination. ODAG is o-minimal.

Proof of Theorem 5.2.1. Suppose that we are given two models $\mathcal{K}_1 := (K_1, \Gamma_1, v_1)$ and $\mathcal{K}_2 := (K_2, \Gamma_2, v_2)$ of $\mathrm{SCVF}_{p,e}^2$, where $\mathcal{K}_2$ is $|K_1|^+$-saturated; suppose also that (A, Δ, v_1) is an $\mathscr{L}_{p,e}^2$-substructure of $\mathcal{K}_1$, which has an $\mathscr{L}_{p,e}^2$-embedding f into $\mathcal{K}_2$. We need to show that f can be extended to an $\mathscr{L}_{p,e}^2$-embedding of $\mathcal{K}_1$ into $\mathcal{K}_2$.

First, because the theory ODAG has quantifier elimination, one can extend the $\mathscr{L}_{\mathrm{OAG}} \cup \{\infty\}$-embedding of Δ onto the whole Γ_1, which would still be an $\mathscr{L}_{p,e}^2$-embedding of (A, Γ_1, v_1) into $\mathcal{K}_2$ extending f.

So we may assume that $\Delta = \Gamma_1$. It is also clear that we can assume that A is a field already, since the coordinate functions and the valuation of an element in the quotient field of A are uniquely determined by those of elements of A and the ring structure of A. Then because A is closed under the p-coordinate functions, we know that K_1 is separable over A, and since K_1 is separably closed, the separable closure of A, A^s, is contained in K_1. The same things are true for $f(A)$ in K_2.

For any $y \in A^s$, we have $A \subseteq A(y^p) \subseteq A(y)$ with the second field extension being purely inseparable, which implies that $A(y) = A(y^p) = A[y^p]$. Thus the p-coordinate functions of elements in $A(y)$ are uniquely determined by its field structure. Since A^s and $f(A)^s$ are isomorphic as fields, and all valuations on A^s are conjugated by elements in the Galois group of A^s/A, one can extend the embedding to an $\mathscr{L}^2_{p,e}$-embedding of (A^s, Γ_1, v_1). Therefore, we may furthermore assume that $A = A^s$.

Now for any $c \in K_1 \backslash A$, c is transcendental over A (because K_1 is a separable extension of A). We would like to extend the embedding to cover c. To do this, it is enough to realize, in $\mathcal{K}_2$ the image under f of the quantifier-free type of c over (A, Γ_1). Let $\phi(x, \vec{a}, \vec{\delta})$ be a quantifier-free $\mathscr{L}^2_{p,e}$-formula realized by c in $\mathcal{K}_1$. It is enough to show that $\phi(x, f(\vec{a}), f(\vec{\delta}))$ is realizable in $\mathcal{K}_2$.

After taking a λ-resolution of ϕ, one can assume that we want to realize $\tilde{\phi}(\vec{y}, \tilde{\vec{a}}, \tilde{\vec{\delta}})$ in $\mathcal{K}_2$, where $\tilde{\phi}$ is a quantifier-free $\mathscr{L}^2_{\mathrm{r}}$-formula, and $\vec{y} = \lambda_{\mathrm{res}}(x)$. Let $\vec{t} = \lambda_{\mathrm{res}}(c)$. It is enough to realize the $\mathscr{L}^2_{\mathrm{r}}$-diagram of $\vec{t}$ in $\mathcal{K}_2$.

Suppose that $d = \mathrm{tr.deg}(A(\vec{t})/A)$, and $t_{i_1}, \ldots, t_{i_d}$ a separating transcendence basis of $A(\vec{t})/A$ (note that K_1 is a separable extension of A).

By the previous lemmas, we can extend the $\mathscr{L}^2_{p,e}$-embedding f to an $\mathscr{L}^2_{\mathrm{r}}$-embedding of $(A(t_{i_1}), \Gamma_1, v_1)$. The embedding can then be extended to its separable closure. Then we extend the embedding to an $\mathscr{L}^2_{\mathrm{r}}$-embedding of $(A(t_{i_1})^s(t_{i_2}), \Gamma_1, v_1)$, etc. Eventually we will get an $\mathscr{L}^2_{\mathrm{r}}$-embedding covering $A(\vec{t})$. Therefore $\phi(x, f(\vec{a}), f(\vec{\delta}))$ can be realized in $\mathcal{K}_2$.

It then follows that the $\mathscr{L}^2_{p,e}$-substructure generated by (A, Γ_1, v_1) and c has an $\mathscr{L}^2_{p,e}$-embedding, extending f, into the $\mathcal{K}_2$. Eventually, we get an $\mathscr{L}^2_{p,e}$-embedding of $\mathcal{K}_1$ into $\mathcal{K}_2$ which finishes our proof. □

Corollary 5.2.11. $\mathrm{SCVF}_{p,e}$ *has quantifier elimination in* $\mathscr{L}_{p,e} \cup \{|\}$.

Proof. Any quantifier-free formula involving the valuation function v in the two sorted language whose free variables are all of the field sort, can be re-written as a quantifier-free formula in the one-sorted language $\mathscr{L}_{p,e} \cup \{|\}$. □

Corollary 5.2.12. *Suppose that* $\mathcal{K}$ *is a model of* $\mathrm{SCVF}_{p,e}$. *Then the induced structure on* vK *is exactly the ordered abelian group structure.*

Proof. Expand $\mathcal{K}$ to be a model of $\mathrm{SCVF}^2_{p,e}$. Then a set of the form $v(A)$, where A is an $\mathscr{L}_{p,e} \cup \{|\}$-definable subset of K^n, can (by the quantifier elimination) be defined by a quantifier-free $\mathscr{L}^2_{p,e}$-formula whose free variables are all in the value group sort. But any of such a quantifier-free $\mathscr{L}^2_{p,e}$-formula is equivalent to a quantifier-free $\mathscr{L}_{\mathrm{OAG}}$-formula. □

Recall that a complete theory T in some first-order language $\mathscr{L}$ is said to have the **Independence Property** if there is some $\mathscr{L}$-formula $\phi(\vec{x}, \vec{y})$ such that, for every model $\mathcal{M}$ of T, and for all n, there are $\vec{a}_1, \ldots, \vec{a}_n \in M$ and $\{b_J\}_{J \subseteq \{1,\ldots,n\}} \subseteq M$ such that $\mathcal{M} \models \phi(\vec{a}_i, \vec{b}_J)$ if and only if $i \in J$. The following was pointed out by Delon to the author.

Corollary 5.2.13. $\mathrm{SCVF}_{p,e}$ *does not have the Independence Property, that is, if* $\mathcal{K} \models \mathrm{SCVF}_{p,e}$*, then* $\mathrm{Th}(\mathcal{K})$ *does not have the Independence Property. In particular, the theory* $T_e(k, G)$ *in Theorem 5.1.7 does not have the Independence Property.*

Proof. Suppose that $\mathcal{K} \models \mathrm{SCVF}_{p,e}$. We want to show that for any $\mathscr{L}_{p,e} \cup \{|\}$-formula $\phi(\vec{x}, \vec{y})$, there is some n such that for all $\vec{a}_1, \ldots, \vec{a}_n \in K$ and $\{b_J\}_{J \subseteq \{1,\ldots,n\}} \subseteq K$, it is not true that $\mathcal{K} \models \phi(\vec{a}_i, \vec{b}_J)$ if and only if $i \in J$.

By the quantifier elimination, we may assume that $\phi(\vec{x}, \vec{y})$ is quantifier free. Taking a λ-resolution of ϕ. Then ϕ has the property we want if and only if $\lambda_{\mathrm{res}}(\phi)$ has the property we want. Because $\lambda_{\mathrm{res}}(\phi)$ is a quantifier-free $\mathscr{L}_{\mathrm{div}}$-formula, one can interpret it as a quantifier-free $\mathscr{L}_{\mathrm{div}}$-formula with parameters inside K^{alg}. It follows from the fact that ACVF_p does not have the Independence Property, that $\lambda_{\mathrm{res}}(\phi)$ must have the property we want. □

Remark 5.2.14. The argument above, using quantifier elimination and λ-resolution, can also be used to show that the theory of separably closed fields of characteristic p and imperfection degree e is stable, by showing that it does not have the Order Property (the idea of the proof is the same). B. Hart observed the similarity between the method we use for proving results related to the Independence Property and Order Property on separably closed (valued) fields and the techniques used in Hrushovsi's paper [Hrushovski, 2002], where he studied various model-theoretic properties of PAC-structures with their algebraic closures (which are usually strongly minimal structures with special properties). The author would like to thank B. Hart for bringing that paper to his attention.

5.3 Denseness

As definable sets in a model of ACVF are usually relatively easier to study than those in a separably closed valued subfield, we would like to know whether there is a good relation between these two collections of definable sets.

In the previous section, through obtaining a quantifier elimination result for the two sorted language $\mathscr{L}^2_{p,e}$, we deduced that the induced structure on the value group is exactly the ordered abelian group structure. One can naturally ask whether the same thing is true for the residue field. In order to answer this question, one can try to prove a quantifier elimination result for a suitable 3-sorted language. Instead of doing that, Delon suggested to the author to prove something easier which probably also gives more insight—to prove that any definable set in the separably closed valued field is dense (in the valuation topology) in some definable set in the algebraic closure, which has the same image in the value group and in the residue field.

Let $K \models \mathrm{SCVF}_{p,e}$; let K^{alg} be the (field-theoretic) algebraic closure of K. Then $K^{\mathrm{alg}} \models \mathrm{ACVF}_p$. Because ACVF_p has quantifier elimination in the language $\mathscr{L}_{\mathrm{div}}$, and for any $f(\vec{X}) \in K^{\mathrm{alg}}[\vec{X}]$, there is some natural number n such that $(f(\vec{X}))^{p^n} \in K[\vec{X}]$, we know that K is stably embedded (i.e. for every definable set in K^{alg}, the intersection of that set and K is the same as the intersection of K and a definable set in K^{alg} defined using parameters from K) into K^{alg} (as a field structure or as a valued-field structure). It follows that the pure valued-field structure on K is the same as the induced (first-order) structure on K in K^{alg}.

In this section, we assume that $\mathcal{K}$, as a model of $\mathrm{SCVF}_{p,e}$, is $\aleph_1$-saturated.

We show that for a quantifier-free $\mathscr{L}_{\mathrm{div}}$-definable set over K, there is a corresponding quantifier-free $\mathscr{L}_{\mathrm{div}}$-definable set in K^{alg} containing the original set in such a manner that the original set is dense with respect to the valuation topology. This in turn gives us information about the induced structure on the value group and the residue field, of K.

The idea in this subsection mostly comes from Delon.

Suppose that $D \subseteq K^l$. By quantifier elimination of $\mathrm{SCVF}_{p,e}$, we can suppose that D is defined by a quantifier-free formula in $\mathscr{L}_{p,e} \cup \{|\}$. Take a λ-resolution of the defining formula of D, one gets a set $\tilde{D}$ which is in some Cartesian power of K (not smaller than l), and is defined by a quantifier-free formula $\phi(\vec{x}, \vec{a})$ in $\mathscr{L}_{\mathrm{div}}$. We can assume that ϕ is of the following form

$$\bigvee_i \phi_i(\vec{x}, \vec{a}) := \bigvee_i \left[\wedge_j (f_{i,j}(\vec{x}) = 0) \wedge g_i(\vec{x}) \neq 0 \wedge \theta_i(\vec{x}, \vec{a})\right], \tag{5.3.1}$$

where the $f_{i,j}, g_i$ are polynomials in $K[\vec{X}]$ with parameters $\vec{a}$, and θ_i is a finite conjunction of formulas of the form $p(\vec{x}) \mid q(\vec{x})$ or $p(\vec{x}) \nmid q(\vec{x})$, with $p, q \in K[\vec{X}]$ using $\vec{a}$ as parameters. By using more disjunctions, we can in particular assume that for each i, the polynomials p, q occurring in θ_i are factors of g_i, that is to say that for each realization $\vec{v}$ of ϕ_i, $p(\vec{v}) \neq 0$ and $q(\vec{v}) \neq 0$ if p, q occurs inside θ_i (see also the discussion after Equation (4.1.1)).

Now for each i, since $\wedge_j (f_{i,j}(\vec{x}) = 0)$ defines an algebraic set in K, say it is V_i, we can write V_i into a finite union of irreducible algebraic sets in K, say $V_i = \cup_m P_{i,m}$. Then by Proposition A.1.16, $I_K(P_{i,m}) \otimes K^{\mathrm{alg}} = I_{K^{\mathrm{alg}}}(\overline{P_{i,m}}^{\mathrm{alg}})$. It follows that $\overline{P_{i,m}}^{\mathrm{alg}}$ is defined over K for each m. By Fact A.1.14, $\overline{P_{i,m}}^{\mathrm{alg}}$ is already absolutely irreducible, i.e. $I_{K^{\mathrm{alg}}}(\overline{P_{i,m}}^{\mathrm{alg}})$ is a prime ideal in $K^{\mathrm{alg}}[\vec{X}]$.

At this point, we remark that we can rewrite the definition of $\tilde{D}$ as

$$\bigvee_i \left(\vec{x} \in Z_K\left(\prod_m (I_K(P_{i,m}))\right) \wedge g_i(\vec{x}) \neq 0 \wedge \theta_i(\vec{x}, \vec{a})\right), \tag{5.3.2}$$

which is first order as the ideals $I_K(P_{i,m})$ are finitely generated. The above procedure defines an operator R on the quantifier-free $\mathscr{L}_{\mathrm{div}}$-formulas with parameters in K, so we call Formula (5.3.2) ("**the right definition**" of ϕ) $R(\phi)$. Notice that the parameters may not be $\vec{a}$ anymore.

Now, because $R(\phi)$ is a quantifier-free formula in $\mathscr{L}_{\text{div}}$ with parameters in K, we can interpret it as a quantifier-free $\mathscr{L}_{\text{div}}$-formula with parameters in K^{alg}. Let $R(\tilde{D})^{\text{alg}}$ be the realization set of $R(\phi)$ in K^{alg}.

We remark that in general, the realization set of ϕ in K^{alg} and the realization set of $R(\phi)$ in K^{alg} are different. This is related to the issue of field of definitions. For example, suppose that $a \in K$ is not a pth-power, then the realization set of $X^p - a = 0$ in K is empty, while it has a point in K^{alg}; the right definition of this formula is $1 = 0$.

Theorem 5.3.1. *Using the notation above, $\tilde{D}$ is a dense subset of $R(\tilde{D})^{\text{alg}}$, with respect to the valuation topology, inside K^{alg}.*

Proof. Suppose that all the $P_{i,m}$ are defined over $k \subseteq K$. As K is $\aleph_1$-saturated, we can treat K^{alg} as our "universal domain" for the classical algebraic geometry (see the appendix of this thesis about the classical algebraic geometry). The proof is very similar to that of Lemma 4.2.4.

Assume that $\vec{c} \in R(\tilde{D})^{\text{alg}}$. Then there is some fixed i and m such that $\vec{c}$ is a realization of

$$\vec{x} \in Z_K\left(I_K(P_{i,m})\right) \wedge g_i(\vec{x}) \neq 0 \wedge \theta_i(\vec{x}, \vec{a}).$$

Let $\vec{y}$ in K^{alg} be a generic point of $\overline{P_{i,m}}^{\text{alg}}$; we may assume that $\dim(P_{i,m}) \geq 1$; it follows that $k(\vec{y})$ is a regular extension of k, and $k(\vec{y})$ has a separating transcendence basis over k, say $y_1, y_2, \ldots, y_s$. It follows from Fact 5.1.2 that $y_{s+1}, \ldots, y_{|\vec{y}|}$ are rational functions of $y_1, \ldots, y_s$ over k.

Write $\vec{c} = (c_1, \ldots, c_s, \ldots)$. For each c_j, $1 \leq j \leq s$, by the saturatedness of K, we can find some $d_j \in K$, $1 \leq j \leq s$, such that $v(c_j - d_j)$ are very large and $d_1, \ldots, d_s$ are algebraically independent. By using the rational expressions of y_j, $j \geq s+1$ in terms of $y_1, \ldots, y_s$, we can find $d_j \in K$, $j \geq s+1$, such that $k(\vec{y})$ is isomorphic to $k(\vec{d})$ as a field. By the continuity of rational functions with respect to the valuation topology, it follows that as long as $v(c_j - d_j)$ are large enough, for $1 \leq j \leq s$, then $v(c_j - d_j)$ are large enough, for all j. Then $\vec{d} = (d_1, \ldots, d_s, d_{s+1}, \ldots, d_{|\vec{y}|})$, is also a generic point of $\overline{P_{i,m}}^{\text{alg}}$, with all $d_j \in K$. But by the continuity of polynomials again, as long as $v(d_j - c_j)$ are large enough, $g_i(\vec{d}) \neq 0$ and $\theta_i(\vec{d}, \vec{a})$ remain true.

Therefore, we proved the denseness. □

Corollary 5.3.2. *Using the notation above, suppose that D is $\mathscr{L}_{p,e} \cup \{|\}$-definable over K. Then there is some D^{alg} in K^{alg} which is $\mathscr{L}_{\text{div}}$-definable (using parameters in K) such that $D \subseteq D^{\text{alg}}$ and D is dense in D^{alg} (as subsets of K^{alg}) with respect to the valuation topology.*

Proof. Suppose that $\tilde{D} = \lambda_n(D)$ for some n. Define D^{alg} to be the set

$$\left\{ \vec{x} = (x_1, \ldots, x_l) \in \left(K^{\mathrm{alg}}\right)^l \;\middle|\; \left[\exists \vec{y} = (\vec{y}_1, \ldots, \vec{y}_l) \in R(\tilde{D})^{\mathrm{alg}}\right] \bigwedge_{k=1}^{l} \left(x_k = \sum_{j \in p^{ne}} y_{k,j}^{p^{ne}} \prod_i b_i^{j(i)} \right) \right\}.$$

It is clear that D^{alg} is $\mathscr{L}_{\mathrm{div}}$-definable using parameters from K. Because $\tilde{D}$ is dense in $R(\tilde{D})^{\mathrm{alg}}$, it follows by the continuity of polynomials that D is also dense in D^{alg}. □

Corollary 5.3.3. *Suppose that $\mathcal{K} = (K, V, \ldots) \models \mathrm{SCVF}_{p,e}$. Then the induced structure on vK is exactly the ordered abelian group structure and the induced structure on Kv is exactly the pure field structure.*

Proof. For any D which is $\mathscr{L}_{p,e} \cup \{|\}$-definable over K, we have seen that D is dense as a subset of $D^{\mathrm{alg}} \subseteq (K^{\mathrm{alg}})^n$. It follows that D and D^{alg} have the same image (up finitely many points; in fact they differ from each other by at most one element, ∞) under the valuation map, in the value group, and they have the same image (up to finitely many points) under the residue map in the residue field as well. The conclusion then follows from the fact that the induced structure on $v(K^{\mathrm{alg}})^{\times}$ and $K^{\mathrm{alg}}v$ from K^{alg} as a model of ACVF are respectively the pure ordered abelian group structure and the pure field structure. □

Remark 5.3.4. Related to this result, after this investigation, the author learned that in [Kollár, 2007], J. Kollár proved that every pseudo-algebraically closed field F has the following property: for any non-trivial valuation w on F^{alg}, for every affine variety V defined over F, $V(F)$ is dense in $V(F^{\mathrm{alg}})$ with respect to the topology induced by w. It is known that a separably closed field is pseudo-algebraically closed. One may be able to the generalize our result in this section to definable sets (in $\mathscr{L}_{\mathrm{div}}$) over a pseudo-algebraically closed field.

5.4 Immediate expansions

In this section, assume that $\mathcal{K} \models \mathrm{SCVF}_{p,e}$, with $e > 0$ being finite. Let V be the valuation ring on K. We want to show that $\mathcal{K}$ ⧖ $(\mathcal{K}|\mathscr{L}_{\mathrm{r}})$.

The main idea is that the proof for ACVF ⧖ ACF works for this case, as long as we make sure that everything can be done over K instead of K^{alg}.

Recall that the Grassmannian $\mathbf{G}(k,n)(K)$ is Zariski-dense in $\mathbf{G}(k,n)(K^{\mathrm{alg}})$ (Proposition A.2.29). Note that if two irreducible affine varieties $P_1 \subseteq (K^{\mathrm{alg}})^n$ and $P_2 \subseteq (K^{\mathrm{alg}})^n$ are defined over K, then $P_1 \cap P_2$ is NOT necessarily **defined** over K, even though it is **definable** over K; but we can use results in Appendix A.3 to ensure that this is the case generically.

It is easy to see that if $D \subseteq K^{\text{alg}}$ is a finite union of perforated discs, then $D \cap K$ is also a finite union of perforated discs—one can choose the centers and radii of the discs occurring in the expression of D (which have non-empty intersection with K) using parameters from K, as K is dense in K^{alg} in the valuation topology and $vK = v(K^{\text{alg}})^{\times}$. To define the valuation ring, by Lemma 4.1.7, it is enough to find an infinite and co-infinite subset of K, which is also a finite union of perforated discs, definable from the structure given; we do this by "ascending" to K^{alg} to make sure that one can indeed get a finite union of perforated discs, and then "descending" back to K.

We always assume that $\mathcal{K}$ is sufficiently saturated (in fact, $\mathcal{K}$ being ω-saturated is enough).

Theorem 5.4.1. *Suppose that D is a quantifier-free $\mathscr{L}_{p,e} \cup \{|\}$-definable set over K which is not $\mathscr{L}_r$-definable. Then none of the $\tilde{D}^{\text{alg}}$ is $\mathscr{L}_r$-definable over K^{alg}.*

Proof. Suppose that some $\tilde{D}^{\text{alg}}$ is $\mathscr{L}_r$-definable. Then there is some quantifier-free $\mathscr{L}_r$-formula ψ such that $\vec{x} \in \tilde{D}^{\text{alg}}$ if and only $\psi(\vec{x})$ is true in K^{alg}, for all $\vec{x}$ from K^{alg}. By the stable embeddedness of K as a field into K^{alg}, this equivalence carries downwards to K (that is $\vec{x} \in K$ is in $\tilde{D}$ if and only $\vec{x}$ satisfies a quantifier-free $\mathscr{L}_r$-formula). Because D is $\mathscr{L}_r$-definable over K if and only $\tilde{D}$ is $\mathscr{L}_r$-definable over K, the conclusion then follows. □

Lemma 5.4.2. *Suppose that $E \subseteq K^2$ is a quantifier-free $\mathscr{L}_{\text{div}}$-definable set which is not $\mathscr{L}_{\text{r}}$-definable. Then V is $\mathscr{L}_{\text{r}} \cup \{E\}$-definable.*

Proof. The Zariski-closure of E must be of dimension either 1 or 2. If it is two, we may assume that the Zariski-closure of $K^2 \backslash E$ in K^2 is also K^2, otherwise, by replacing E with its complement, we are reduced to the 1-dimensional case. With this assumption about the complement, one can find $\vec{a} \in E$ and $\vec{b} \notin E$ such that any point $\vec{x}$ close enough to $\vec{a}$ is also in E, and any point close enough to $\vec{b}$ is also not in E. Then the set $N := \{x \in K^{\text{alg}} \mid x(\vec{b} - \vec{a}) + \vec{a} \in E^{\text{alg}}\}$ is an infinite and co-infinite subset of K^{alg} which is $\mathscr{L}_{\text{r}} \cup \{E\}$-definable and hence is a finite union of perforated discs. Then $\{x \in K \mid x(\vec{b} - \vec{a}) + \vec{a} \in E\} = K \cap N$ is also a finite union of perforated discs which is infinite and co-infinite in K. So then V is also $\mathscr{L}_{\text{r}} \cup \{E\}$-definable.

Now suppose that the Zariski-closure of E is an affine plane curve C. We may assume that C is irreducible (otherwise pick one irreducible component). Because E is not $\mathscr{L}_{\text{r}}$-definable, $C \backslash E$ is also infinite (hence Zariski-dense) in C.

Choose a large enough r (with $p \nmid r$) such that the image of C under the Veronese map (see the appendix about this map) v_r has a spanned linear space of dimension $N_1 = rd - d(d-3)/2 > D/2$, where $D = dr$ is the degree of $v_r(C)$. Then consider the "affinized" Veronese map $v_r^A : \mathbb{A}^2(K) \to \mathbb{A}^N(K)$, which takes (x_1, x_2) to $(\ldots, x_1^{i_1} x_2^{i_2}, \ldots)$, with $0 \leq i_1 + i_2 \leq r$ (that is, the projective v_r with the last homogeneous coordinate equal to 1). Here $N = \binom{r+2}{2} + 1$.

Then $v_r^A(C)$ is irreducible and non-degenerate in $\mathbb{A}^{N_1}(K)$ (a hyperplane in $\mathbb{A}^N(K)$) of degree (i.e. the degree of $v_r^A(C)(K^{\mathrm{alg}})$) $D < 2N_1$. Notice that $v_r^A(E)$, whose Zariski-closure is $v_r^A(C)$, is also not $\mathscr{L}_{\mathrm{r}}$-definable. Furthermore, $v_r^A(E)$ is a quantifier-free $\mathscr{L}_{\mathrm{div}}$-definable set. This is because when one applies v_r^A to C^{alg}, which is defined over K, a point in C^{alg} is a K-point (i.e. with all its coordinates in K) if and only if its image under v_r^A is a K-point. And since $v_r^A(E^{\mathrm{alg}})$ is a quantifier-free $\mathscr{L}_{\mathrm{div}}$-definable set in K^{alg}, $v_r^A(E)$ as the set of K-realization points of that quantifier-free $\mathscr{L}_{\mathrm{div}}$-formula is also quantifier-free $\mathscr{L}_{\mathrm{div}}$-definable in K.

It follows that the set of points in $v_r^A(E)$ which are $(v_r^A(C)^{\mathrm{alg}}, v_r^A(E)^{\mathrm{alg}})$-interior points (and K-points at the same time) is Zariski-dense in $v_r^A(C)^{\mathrm{alg}}$; denote this set by G. Meanwhile the set of points in $v_r^A(C)\backslash v_r^A(E)$, which are also $(v_r^A(C)^{\mathrm{alg}}$, $v_r^A(C)^{\mathrm{alg}}\backslash v_r^A(E)^{\mathrm{alg}})$-interior points (and K-points) is also Zariski-dense in $v_r^A(C)^{\mathrm{alg}}$; denote this set by G'. Let H be a general hyperplane passing through N_1 points in G, and let H' be a general hyperplane passing through N_1 points in G'. By Proposition A.2.29, we may assume that H and H' are defined over K. Then by Proposition A.3.10, $H \cap v_r^A(C)^{\mathrm{alg}}$ and $H' \cap v_r^A(C)^{\mathrm{alg}}$ are both defined over K, as the pre-image of H and H' under v_r^A are general plane curves of degree r in $(K^{\mathrm{alg}})^2$. We may assume that $H \cap H' = L$, which is an $(N_1 - 2)$-plane defined over K. Note that because the degree of $v_r^A(C)^{\mathrm{alg}}$ is $D < 2N_1$, $H \cap v_r^A(E)^{\mathrm{alg}}$ has at least N_1 points and $H' \cap v_r^A(E)^{\mathrm{alg}}$ has at most $N_1 - 1$ points. Pick some K-point $\vec{a} \in H$ and some K-point $\vec{b} \in H'$ such that L and $\vec{a}$ span H, L and $\vec{b}$ span H'. Consider the family of hyperplanes

$$\mathcal{F} := \left\{ H_x \subseteq (k^{\mathrm{alg}})^{N_1} \mid H_x = \left\langle L, \vec{a} + x(\vec{b} - \vec{a}) \right\rangle, x \in K^{\mathrm{alg}} \right\}.$$

Notice that F is an irreducible family of hyperplanes (it is the line joining H in H' in the Grassmannian of hyperplanes) with both $H \cap v_r^A(C)^{\mathrm{alg}}$ and $H' \cap v_r^A(C)^{\mathrm{alg}}$ being defined over K, by Proposition A.3.10 again, except finitely many $x \in K$, $H_x \cap v_r^A(C)^{\mathrm{alg}}$ is defined over K. Therefore, except finitely many $x \in K$, $K^{N_1} \cap H_x \cap v_r^A(C)^{\mathrm{alg}} = (K^{\mathrm{alg}})^{N_1} \cap H_x \cap v^A(C)^{\mathrm{alg}}$.

But the set $\{x \in K^{\mathrm{alg}} \mid |H_x \cap v_r^A(E)^{\mathrm{alg}}| \geq N_1\}$ is an infinite co-infinite subset of K^{alg} which is also a finite union of perforated discs, up to a finite subset; therefore, up to finitely many points in K, $\{x \in K \mid |H_x \cap v_r^A(E) \cap K^{N_1}| \geq N_1\}$ is also an infinite co-infinite subset of K, which is a finite union of perforated discs.

Therefore V is $\mathscr{L}_{\mathrm{r}} \cup \{v_r(E)\}$-definable, i.e. $\mathscr{L}_{\mathrm{r}} \cup \{E\}$-definable. □

Theorem 5.4.3. $\mathrm{SCVF}_{p,e}$ ⧖ $\mathrm{SCF}_{p,e}$, *if* $e > 0$ *is finite.*

Proof. Suppose that $\mathcal{K} \models \mathrm{SCVF}_{p,e}$. We want to show that $\mathcal{K}$ ⧖ $(\mathcal{K}|\mathscr{L}_{\mathrm{r}})$. We may assume that $\mathcal{K}$ is sufficiently saturated.

Let E be an $\mathscr{L}_{\mathrm{div}}$-definable set over K which is not $\mathscr{L}_{\mathrm{r}}$-definable. We may assume that E is quantifier-free $\mathscr{L}_{\mathrm{div}}$-definable, after picking a λ-resolution. We may assume that E is obtained by its right definition already.

We may assume that the Zariski-closure of E is P, which is an irreducible affine closed subset of K^n. We may also assume that $P \backslash E$ is also dense in P, and P is non-degenerate in the ambient space (otherwise, we are reduced to lower dimensional cases). Suppose that the dimension of P is m.

If $m = n$, then find $\vec{a} \in E$ and $\vec{b} \in P \backslash E$ such that all points close enough to $\vec{a}$ are in E and all points close enough to $\vec{b}$ are in $P \backslash E$. Then the set $\{x \in K \mid x(\vec{b} - \vec{a}) \in E\}$ is infinite and co-infinite in K and is also a finite union of perforated discs, which then proves that V is $\mathscr{L}_{\mathrm{r}} \cup \{E\}$-definable.

If $m = 1$, then P is K-birationally equivalent (see Fact A.1.17) to a plane curve C, and up to a finite subset, E is $\mathscr{L}_{\mathrm{r}}$-definably in bijective correspondence to an $\mathscr{L}_{\mathrm{div}}$-definable but not $\mathscr{L}_{\mathrm{r}}$-definable subset of C. Because the K-birational equivalence map has the property that a point in the image is a K-point if and only if its pre-image is a K-rational point; Up to finitely many points, the image of E under this K-birational equivalence map is quantifier-free $\mathscr{L}_{\mathrm{div}}$-definable in K, because the same holds in the algebraic closure (which descends to K). By Lemma 5.4.2, V is $\mathscr{L}_{\mathrm{r}}$-definable.

So we may assume that $2 \leq m \leq n - 1$.

We use Fact A.3.6 and its proof. Suppose that P^{alg} is defined over a subfield k of K. One can choose an $(P^{\mathrm{alg}}, E^{\mathrm{alg}})$-interior point $\vec{a}$; we can find a generic point of P^{alg} over k, say $\vec{x} \in K^n$. We may assume that $k(\vec{x})$ has a separating transcendence basis containing x_1, x_2. Then there are co-infinitely many $c \in K$ such that $k(\vec{x})/k(x_1 + cx_2)$ is a regular field extension. As, $P \backslash E$ is Zariski-dense in P, one can in particular choose one $(P^{\mathrm{alg}}, P^{\mathrm{alg}} \backslash E^{\mathrm{alg}})$-interior point which is also in K^n, say $\vec{y}$ such that $x_1 + cx_2 = y_1 + cy_2$ (i.e. $c = (x_1 - y_1)/(y_2 - x_2)$) and $k(\vec{x})/k(x_1 + cx_2)$ is a regular field extension. The hyperplane $H := X_1 + cX_2 - (x_1 + cx_2)$ has the property that H is defined over $k(x_1 + cx_2) \subseteq K$, passing through $\vec{x}$ and $\vec{y}$, and $H \cap P^{\mathrm{alg}}$ is an irreducible variety of dimension $m - 1$ defined over K. One can iterate this process until one gets an irreducible affine curve C defined over K, passing through some $(P^{\mathrm{alg}}, E^{\mathrm{alg}})$-interior point in E and one $(P^{\mathrm{alg}}, P^{\mathrm{alg}} \backslash E^{\mathrm{alg}})$-interior point in $P \backslash E$. Such an irreducible curve $C(K)$ would then contain (C, E^{alg})-interior points and $(C, C \backslash E^{\mathrm{alg}})$-interior points. Then we are reduced to the 1-dimensional case.

Hence, V is $\mathscr{L}_{\mathrm{r}} \cup \{E\}$-definable. □

5.5 The infinite imperfection degree case

In this section, we make some remarks on separably closed valued fields with an infinite imperfection degree.

On a separably closed field K of characteristic $p > 0$ and imperfection degree $e = \infty$ (i.e. $e \geq \aleph_0$; the following functions can in fact be defined for the finite case too), as in [Delon, 1988] or [Srour, 1986], one can define the $(n+1)$-ary functions $f_{n,i}$ for all $n \in \omega$ and $i \in p^n$ such that $z = f_{n,i}(x; y_1, \ldots, y_n)$ if and only if one of the following is true:

- $y_1, \ldots, y_n$ are p-independent, $x \in K^p(y_1, \ldots, y_n)$ and z is the i-th p-coordinate of x with respect to $y_1, \ldots, y_n$;

- $y_1, \ldots, y_n$ are p-independent, $x \notin K^p(y_1, \ldots, y_n)$ and $z = 0$;
- $y_1, \ldots, y_n$ are not p-independent, $z = 0$.

Let $\mathscr{L}_{p,\infty}$ be the language obtained by taking the union of $\mathscr{L}_r$ and all the function symbols $\{f_{n,i}\}_{n\in\omega, i\in p^n}$. Let $\mathscr{L}^2_{p,\infty}$ be the corresponding 2-sorted language for separably closed valued fields, i.e. the field-sort language is $\mathscr{L}_{p,\infty}$, the value-group-sort language is $\mathscr{L}_{\mathrm{OAG}} \cup \{\infty\}$, and the only function symbol between sorts is the valuation map v. Let $\mathscr{L}^2_r$ be the sub-language of $\mathscr{L}^2_{p,\infty}$ with the field-sort language replaced by $\mathscr{L}_r$. Let $\mathscr{L}^{2,*}_{p,\infty}$ be $\mathscr{L}^2_{p,\infty} \cup \{*\}$ where '$*$' is a constant symbol (for an element with $v(*) \neq 0$). Naturally, one defines $\mathscr{L}^{2,*}_r$, $\mathscr{L}^*_{p,\infty}$, $\mathscr{L}^*_r$ in the same way.

Define $\mathrm{SCF}_{p,\infty}$ to be the theory of separably closed fields of characteristic $p > 0$ and imperfection degree $e = \infty$, in the language $\mathscr{L}_{p,\infty}$. Define $\mathrm{SCVF}^*_{p,\infty}$ to be the theory of separably closed non-trivially valued fields of characteristic $p > 0$ and imperfection degree $e = \infty$, with $v(*) \neq 0$, in the language $\mathscr{L}^*_{p,\infty}$. Define $\mathrm{SCVF}^{2,*}_{p,\infty}$ to be the expansion of $\mathrm{SCVF}^*_{p,\infty}$ to the language $\mathscr{L}^{2,*}_{p,\infty}$.

Theorem 5.5.1 ([Delon, 1988])**.** $\mathrm{SCF}_{p,\infty}$ *has quantifier elimination.*

Working with $\mathrm{SCVF}^{2,*}_{p,\infty}$, we do not really have a version of λ-resolution. As a consequence, the proofs in the finite case can not be re-used in the infinite case immediately. However, we do still have quantifier elimination. One can easily see that given a $\mathcal{K} \models \mathrm{SCVF}^{2,*}_{p,\infty}$, if we fix a p-basis, say $\{b_i\}_{i\in I}$, then for every finite subset $J \subseteq I$, we have a corresponding λ-resolution with respect to the set of elements $\{b_i\}_{i\in J}$, although in general it does not provide a bijection.

Theorem 5.5.2. $\mathrm{SCVF}^{2,*}_{p,\infty}$ *has quantifier elimination.*

Proof. The proof is similar to the finite case; each time a new element is to be added to the embedding, only a finite subset of the p-basis would be used. Thus the method for the finite case still works. □

Corollary 5.5.3. $\mathrm{SCVF}^*_{p,\infty}$ *has quantifier elimination.*

Due to time constraint, this investigation on the infinite imperfection degree case can not be carried on further. Delon has provided a counting-coheir argument to show that $\mathrm{SCVF}_{p,\infty}$ does not have the Independence Property. But we shall not record it here. We pose several questions, hoping that a slight alteration, of the proof for the finite imperfection case for the non-Independence Property and immediate expansions, would still work for the infinite imperfection degree case.

Question 5.5.4. Suppose that $\mathcal{K} \models \mathrm{SCVF}^*_{p,\infty}$ with $\{b_i\}_{i\in I}$ being a p-basis. Suppose that D is $\mathscr{L}_{\mathrm{div}}$-definable but not $\mathscr{L}_{\mathrm{r}}$-definable over K. Then is it true that $D \cap (P_p(K)(b_0, \ldots, b_n))^n$ is still not $\mathscr{L}_{\mathrm{r}}$-definable, as long as n is sufficiently large?

Question 5.5.5. Suppose that $\mathcal{K} \models \mathrm{SCVF}^*_{p,\infty}$ with $\{b_i\}_{i\in I}$ being a p-basis. Suppose that an $\mathscr{L}^*_{p,\infty} \cup \{|\}$-formula $\phi(\vec{x}, \vec{y})$ witnesses the Independence Property. Then is it true that it also witnesses it in $P_p(K)(b_0, \ldots, b_n)$ as long as n is sufficiently large?

5.6 Some remarks on valued stable fields

Related to the conjecture that every infinite stable field is separably closed, one can ask what we know about a stable field with a non-trivial valuation.

Definition 5.6.1. *A* **valued stable field** *is an infinite stable field (in the sense of model theory) endowed with a valuation.*

A valued stable field can be treated as an $\mathscr{L} \cup \{|\}$-structure, where $\mathscr{L}$ is an extension of the language of rings in which the underlying field is stable and '$|$' is the 'division' predicate for the valuation as usual.

It is known that (see e.g. [Delon, 1998]) $\mathrm{SCF}_{p,e}$ has elimination of imaginaries when e is finite. It is not known whether $\mathrm{SCF}_{p,\infty}$ has elimination of imaginaries or not. One can naturally ask whether $\mathrm{SCVF}_{p,e}$ has elimination of imaginaries or not. As it usually happens, valued fields rarely have elimination of imaginaries in a one-sorted language.

Definition 5.6.2 ([Poizat, 2000])**.** *A theory T* **(strongly) eliminates imaginaries** *if for every formula $f(\vec{x}, \vec{a})$ with parameters $\vec{a}$ in a model $\mathcal{M}$ of T there is a tuple $\vec{b}$ with the following property: If σ is an automorphism of an elementary extension of $\mathcal{M}$, it preserves the formula $f(\vec{x}, \vec{a})$ (namely, for every $\vec{x}$, $f(\vec{x}, \vec{a}) \leftrightarrow f(\sigma(\vec{x}), \vec{a})$) if and only if it fixes every element of $\vec{b}$. A theory T is said to* **weakly eliminate imaginaries** *if for every formula $f(\vec{x}, \vec{a})$ with parameters in $\mathcal{M}$, there is a smallest algebraically closed set $A \subseteq M$ such that $f(\vec{x}, \vec{a})$ is definable over A.*

Remark 5.6.3 ([Poizat, 2000])**.** It can be shown that if T eliminates imaginaries then T weakly eliminates imaginaries, because the former implies that the algebraic closure of $\vec{b}$ is the smallest algebraically closed set over which A is definable. But in general weak elimination of imaginaries does not imply the strong one.

We have the following[5]

Theorem 5.6.4. *A valued stable field, with a non-trivial valuation, does not have (weak) elimination of imaginaries in the language $\mathscr{L} \cup \{|\}$ as long as $\mathrm{acl}_{\mathscr{L}\cup\{|\}} = \mathrm{acl}_{\mathscr{L}}$.*

Proof. Suppose that $\mathcal{K}$ is a valued stable field; let $\mathcal{M}$ be a sufficiently $\mathscr{L} \cup \{|\}$-saturated elementary extension of $\mathcal{K}$. Then $\mathcal{M}$ is also a valued stable field. In the following, we work inside $\mathcal{M}$.

[5]The author would like to thank Dr. Martin Bays and Dr. Deirdre Haskell for their discussion with him.

For any element $m \in M$, define

$$\Phi_m = \{x \in M \mid v(x - m) > 0\},$$

which is clearly definable in the language $\mathscr{L} \cup \{|\}$.

To show that $\mathcal{M}$ does not weakly eliminate imaginaries, it is enough to show that Φ_m does not have a minimal algebraically closed defining set, for some $m \in M$. Because $\mathrm{acl}_{\mathscr{L}\cup\{|\}} = \mathrm{acl}_{\mathscr{L}}$, we denote them just by acl.

Since the $\mathrm{acl}(\emptyset)$ is of cardinality $|\mathscr{L}|$, the number of sets $\mathscr{L} \cup \{|\}$-definable over $\mathrm{acl}(\emptyset)$ is at most of cardinality $|\mathscr{L}|$. Because $\mathcal{M}$ is sufficiently large, we can find some b such that Φ_b is not $\mathrm{acl}(\emptyset)$-definable (in $\mathscr{L} \cup \{|\}$). We may also assume that b is of positive valuation.

In the following, unless specified, we work in the language $\mathscr{L}$.

Since $\mathcal{M}|\mathscr{L}$ is stable, by the theory of stable groups, working in the language $\mathscr{L}$, there is a unique generic type over M, which is generic both for the additive group and the multiplicative group (see e.g. Theorem 5.10 of [Poizat, 2001], or Lemma 2.3.1 of [Wagner, 1997]). Denote this type by $\mathrm{tp}(e/M)$ where e is some element in an elementary extension of $\mathcal{M}$. By Proposition 5.3 of [Poizat, 2001], $\mathrm{tp}(e/M)$ does not fork over the empty set. Therefore, $\mathrm{tp}(e/b)$ does not fork over the empty set either. But this type is realizable in $\mathcal{M}$ as $\mathcal{M}$ is sufficiently saturated. So suppose $a \in M$ realizes $\mathrm{tp}(e/b)$. Then $\mathrm{tp}(a/b) = \mathrm{tp}(e/b)$ does not fork over the empty set, i.e. $a \mathop{\smile\!\!\!\!\!\!|}_{\emptyset} b$.

In the following, we use that fact that in a stable group G, if g is generic over G, $\mathrm{tp}(h/G)$ is finitely realizable in G, g and h are independent over G, then gh is generic over G and $gh \mathop{\smile\!\!\!\!\!\!|}_G h$ (so $gh \mathop{\smile\!\!\!\!\!\!|}_{\emptyset} h$ if $h \in G$). For a proof, see the 'pivotal' Proposition 2.1.11 in [Wagner, 1997].

Case 1. $v(a) > 0$. Then since $(e + b) \mathop{\smile\!\!\!\!\!\!|}_{\emptyset} b$, we have $(a + b) \mathop{\smile\!\!\!\!\!\!|}_{\emptyset} b$. But then we have $\Phi_{(a+b)} = \Phi_b$ while $\mathrm{acl}(a + b) \cap \mathrm{acl}(b) = \mathrm{acl}(\emptyset)$, which implies that $\mathcal{M}$ does not have weak elimination of imaginaries in $\mathscr{L} \cup \{|\}$.

Case 2. $v(a) \leq 0$. Then since e^{-1} is also generic over M, we know that $e^{-1}b$ is also generic over M and $e^{-1}b \mathop{\smile\!\!\!\!\!\!|}_{\emptyset} b$. Furthermore, $(e^{-1}b + b)$ is again generic over M, and thus $(a^{-1}b + b) \mathop{\smile\!\!\!\!\!\!|}_{\emptyset} b$. But then $\Phi_{a^{-1}b+b} = \Phi_b$ and $\mathrm{acl}(a^{-1}b + b) \cap \mathrm{acl}(b) = \mathrm{acl}(\emptyset)$, and $\mathcal{M}$ does not have weak elimination of imaginaries in $\mathscr{L} \cup \{|\}$ again. □

Corollary 5.6.5. $\mathrm{SCVF}_{p,e}$ *does not have elimination of imaginaries.*

Proof. It is enough to see that the algebraic closure operator in the valued-field structure is the same as that of the field structure. (cf. Lemma 4.1.6.) □

Given an infinite stable field, the conjecture says that it should be separably closed. This conjecture is in general considered hard among the model theorists. Guided by the philosophy that valuations are in general useful in the algebraic study of fields, one might try to approach it from the valuation theoretic point of view.

Definition 5.6.6. *A field K is* **valuationally algebraically closed***, if for any non-trivial valuation V on K^{alg}, K is dense in K^{alg} with respect to the topology induced by V.*

A field K is **immediately algebraically closed***, if for any non-trivial valuation V on K, and for any extension W of V to K^{alg}, K^{alg} is an immediate valued-field extension of K.*

It follows easily that if K is valuationally algebraically closed, then K is always immediately algebraically closed. Naturally if K is algebraically closed, then it is valuationally algebraically closed. Any non-trivial valuation on an algebraically closed field has an algebraically closed residue field and a divisible value group; it follows that if K is valuationally algebraically closed or immediately algebraically closed, then for any non-trivial valuation ring V on K, Kv is algebraically closed and vK is divisible (the last two conditions combined is in fact equivalent to saying that K is immediately algebraically closed).

We have seen that if K is separably closed, then K is also valuationally algebraically closed. In the following, we give an example, showing that $\mathbf{R}$ is in fact valuationally algebraically closed too.

Example 5.6.7. Not every valuationally algebraically closed field is separably closed. Every pseudo-algebraically closed field is valuationally algebraically closed by Proposition 11.5.3 of [Fried and Jarden, 2008]. But not every valuationally algebraically closed field is pseudo-algebraically closed; for example, $\mathbf{R}$ is valuationally algebraically closed[6], but it is not pseudo-algebraically closed as an ordered field is not pseudo-algebraically closed.

Proof. Denote again by v a valuation on $\mathbf{R}$ which is not trivial. The algebraic closure of $\mathbf{R}$ is $\mathbf{C} = \mathbf{R}[i]$; we denote an extension of v on $\mathbf{C}$ by v again. For any $a + bi$ with $a, b \in \mathbf{R}$, we want to show that we can find some $x \in \mathbf{R}$ such that $v(a + bi - x)$ is arbitrarily large. We may assume that $b \neq 0$.

Notice that $[(a - x) + bi] + [(a - x) - bi] = 2(a - x)$. If $v(a - x + bi)$ is very large, in particular larger than $v(bi)$, then $v(a - x) = v(bi) = v(b)$ as $i^2 = -1$. Thus, $v(a - x - bi) = v(2(a - x)) = v(2) + v(b)$. So we want

$$\begin{aligned} v(a + bi - x) &= v((a - x)^2 + b^2) - v(a - x - bi) \\ &= v((a - x)^2 + b^2) - v(2b) \end{aligned}$$

to be very large. It is enough to make sure that $v((a - x)^2 + b^2)$ can be very large.

Because v is not trivial, there must be some element $\epsilon \in \mathbf{R}^{\times}$ with $v(\epsilon) > 0$ and $\epsilon > 1$. To see this, assuming the opposite, i.e. for all $z > 1$, $v(z) \leq 0$; this implies that v is convex, hence trivial on $\mathbf{R}$. For example, if we have $0 < a < b$ with $b \in V$, then because $b/a > 1$, $v(b/a) \leq 0$, i.e. $v(b) \leq v(a)$ which implies that $a \in V$ too.

[6] In fact, one can prove that a real closed field is archimedean if and only if it is valuationally algebraically closed.

Let $y \in \mathbf{R}$ be such that $y > 0$ and $v(y)$ is very large, then there is some n, such that $v(\epsilon^n y)$ is also very large with the property that $\epsilon^n y > b^2$ (because $\mathbf{R}$ is archimedean). This means that there is an $x \in \mathbf{R}$ such that $(a-x)^2 = \epsilon^n y - b^2 > 0$.

We conclude that $\mathbf{R}$ is dense in $\mathbf{C}$ with respect to the valuation induced by v. Therefore, $\mathbf{R}$ is valuationally algebraically closed. □

Question 5.6.8. Is every immediately algebraically closed field also valuationally algebraically closed?

Question 5.6.9. Suppose that (K, V) is a valued stable field, then is the (model-theoretic) algebraic closure in the valued-field structure the same as the algebraic closure in the smaller language (in which K is stable)?

Question 5.6.10. Is it true that every valued stable field is valuationally algebraically closed?

Question 5.6.11. Is it true that every valued stable field, which is also valuationally algebraically closed, is separably closed?

Question 5.6.12. Suppose that $(K, +, -, \times, 0, 1)$ is valuationally algebraically closed, is it true that $(K, +, -, \times, 0, 1, V) \mathrel{⧖} (K, +, -, \times, 0, 1)$ for any valuation ring V on K? If this is not true, what about if K is pseudo-alegbraically closed, or pseudo-finite?

Appendix A

Classical algebraic geometry

The material in this chapter is "classical algebraic geometry" in the sense that we work with zero-sets of polynomials over some fields. Nothing scheme theoretic is involved. The main references for this chapter are [Lang, 1958], [Hartshorne, 1977], [Harris, 1992] and [Fried and Jarden, 2008].

A variety in this thesis is always NOT assumed to be irreducible.

A.1 Affine varieties, generic points, etc.

Suppose that K is a field.

A **zero-set** of a polynomial $f(\vec{X}) \in K[X_1, \ldots, X_n]$ is a set of the form $\{\vec{a} \in K^n \mid f(\vec{a}) = 0\}$. A zero-set of a set of polynomials A is usually denoted by $Z_K(A)$, or simply $Z(A)$, if the K we refer to is clear from the context. An **affine variety** or an **affine algebraic set** in/over K, is a zero-set of a collection of polynomials over K. It is well known that every ideal in $K[X_1, \ldots, X_n]$ for some n is always finitely generated, thus an affine variety is a zero-set of finitely many polynomials. One can prove that K^n has a topology with all affine varieties as the closed sets; this topology is called the **Zariski topology** on K^n. The set K^n with the Zariski topology is usually denoted by $\mathbf{A}^n(K)$, or simply $\mathbf{A}^n$, called the **affine space** $\mathbf{A}^n$. A non-empty affine variety is **irreducible** if it is not a union of two proper Zariski-closed subsets.

Given a subset Y of $\mathbb{A}^n$, there is an **ideal of** Y in $K[\vec{X}]$ over K, which is

$$I_K(Y) := \{f \in K[\vec{X}] \mid (\forall \vec{a} \in Y) f(\vec{a}) = 0\}.$$

Again we omit the subscript K if it clear from the context what K is.

Fact A.1.1 (see [Hartshorne, 1977])**.**

(a) If $T_1 \subseteq T_2$ are subsets of $K[\vec{X}]$, then $Z(T_1) \supseteq Z(T_2)$;

(b) if $Y_1 \subseteq Y_2$ are subsets of $\mathbb{A}^n$, then $I(Y_1) \supseteq I(Y_2)$;

(c) for any two subsets Y_1, Y_2 of $\mathbb{A}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$;

(d) if K is algebraically closed, then for any ideal $\mathfrak{a}$ of $K[\vec{X}]$, $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ (the **radical** of $\mathfrak{a}$);[1]

(e) for any subset Y of $\mathbb{A}^n$, $Z(I(Y)) = \bar{Y}$, the **Zariski-closure** of Y, i.e. the smallest Zariski-closed set containing Y.

Clause (d) above is also called the *Hilbert's Nullstellensatz.*

Fact A.1.2 (see [Hartshorne, 1977])**.** Suppose that K is algebraically closed. Then there is a one-to-one inclusion reversing correspondence between affine algebraic sets in $\mathbb{A}^n$ and radical ideals (i.e. ideals which are equal to their radicals) in $K[\vec{X}]$, given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$.

Furthermore, an affine variety is irreducible if and only if its ideal is prime.

Fact A.1.3 (see [Hartshorne, 1977])**.** Every non-empty affine variety Y in $\mathbb{A}^n$ is a finite union of irreducible affine varieties $Y_1 \cup \cdots \cup Y_m$, for some m. If we require that $Y_i \nsubseteq Y_j$ if $i \neq j$, then the Y_i are unique (up to permutation), in which case they are called the **irreducible components** of Y.

Definition A.1.4 (see [Hartshorne, 1977])**.** *Suppose that Y is an affine variety. A function $f : Y \to K$ is* **regular** *at a point $\vec{p}$, if there is a Zariski-open neighborhood U of $\vec{p}$ such that $\vec{p} \in U \subseteq Y$ and polynomials $g(\vec{X}), h(\vec{X}) \in K[\vec{X}]$ with $h(\vec{X})$ nowhere zero on U, such that $f(\vec{X}) = g(\vec{X})/h(\vec{X})$ on U. A function f is regular on Y (resp. U) if it is regular at every point on Y (resp. U).*

Fact A.1.5 (see [Hartshorne, 1977])**.** A regular function is continuous if we identify K with $\mathbb{A}^1$.

A **quasi-affine variety** is an affine variety or a Zariski-open subset of an affine variety.

Definition A.1.6 (see [Hartshorne, 1977])**.** *Suppose that $X \subseteq \mathbb{A}^n, Y \subseteq \mathbb{A}^m$ are two quasi-affine varieties. Then a* **morphism** *$\varphi : X \to Y$ is a Zariski-continuous map such that for every open set $U \subseteq Y$ and every regular function $f : V \to K$, the function $f \circ \varphi : \varphi^{-1}(U) \to K$ is regular.*

We want to study the behaviour of varieties as we change the underlying fields. It turns out the early Weil-style analysis is suitable for our purpose, although it is often criticised as not being functorial enough by modern algebraic geometers, who prefer to use schemes.

Definition A.1.7 (see [Lang, 1958])**.** *A field K is called a* **universal domain** *if it is algebraically closed and of infinite transcendence degree over its prime sub-field.*

[1] $\sqrt{\mathfrak{a}} := \{f \in K[\vec{X}] \mid (\exists r \in \mathbb{N}_{>0}) f^r \in \mathfrak{a}\}$.

Let k be a subfield of K with K being of infinite transcendence degree over k. Let $\mathfrak{p}$ be a prime ideal of $k[X_1, \ldots, X_n]$, the quotient map

$$\pi : k[X_1, \ldots, X_n] \to K[X_1, \ldots, X_n]/\mathfrak{p},$$

induces an isomorphism on k (we identify k with its image). Let $\xi_i = \pi(X_i)$. Then $f(\vec{X}) \in \mathfrak{p}$ if and only $f(\vec{\xi}) = 0$.

Fact A.1.8 ([Lang, 1958])**.** Let $k(\vec{\xi})$ be a finitely generated extension of k. There exists an isomorphism of $k(\xi)$ into K which is the identity on k.

Proof. We may assume that $\xi_1, \ldots, \xi_r$ is a transcendence basis of $k(\vec{\xi})/k$. Let the elements $x_1, \ldots, x_r \in K$ be algebraically independent over k. Then there is an isomorphism $k(\xi_1, \ldots, \xi_r) \to k(x_1, \ldots, x_r)$. This isomorphism can be extended to $k(\vec{\xi}) \to k(\vec{x})$ for some $\vec{x} \in K^n$, because K is algebraically closed. □

Definition A.1.9 ([Lang, 1958])**.** *A point* $(x_1, \ldots, x_n) \in K^n$ *satisfying the condition in the proof of Lemma A.1.8, i.e. satisfying an isomorphism*

$$k(X_1, \ldots, X_n)/\mathfrak{p} \to k(x_1, \ldots, x_n)$$

which restricts to identity on k*, is called a* **generic zero** *or a* **generic point** *of* $\mathfrak{p}$ *(with respect to* k *and* K*).*

If $V \subseteq k^n$ is the zero set of its associated ideal $\mathfrak{p} := I_k(V)$ which is also a prime ideal in $k[\vec{X}]$, we also called a generic point of $\mathfrak{p}$ (with respect to K and k) a **generic point** of V (with respect to k and K).

Fact A.1.10 ([Lang, 1958])**.** The dimension (see Definition A.2.14) of $Z_K(\mathfrak{p})$ is the transcendence degree of $k(\vec{x})/k$ for some (equivalent any) generic point $\vec{x} \in K$.

Definition A.1.11 ([Lang, 1958])**.** *An ideal* $\mathfrak{A} \subseteq K[\vec{X}]$ *has a* **basis** *in* k*, if there exists a set of generators of* $\mathfrak{A}$ *in* $K[\vec{X}]$ *whose elements are polynomials with coefficients in* k*. If* $\mathfrak{A}$ *has a basis in* k*, then we say that* k *is a* **field of definition** *for* $\mathfrak{A}$*, or that* $\mathfrak{A}$ *is* **defined** *over* k*.* [2]

Fact A.1.12 ([Lang, 1958])**.** Let $\mathfrak{A}$ be an ideal of $K[\vec{X}]$. There exists a minimal field of definition for $\mathfrak{A}$, i.e. there is a field $k_0 \subseteq K$ such that $\mathfrak{A}$ has a basis in k_0, and for any field k, if $\mathfrak{A}$ has a basis in k, then $k \supseteq k_0$. This k_0 is always finitely generated over the prime field.

Furthermore, if σ is an automorphism of K then $\mathfrak{A}^\sigma = \mathfrak{A}$ if and only if σ leaves every element of k_0 fixed.

Because K is algebraically closed, A is the zero-set of $I_K(A)$; the **field of definition** of A is defined to be the field of definition of $I_K(A)$.

[2] This should not be confused with "being *definable* over k".

Fact A.1.13 ([Lang, 1958])**.** Suppose that V is $Z_K(\mathfrak{P})$ with $\mathfrak{P}$ being a prime ideal in $K[\vec{X}]$. Suppose that k is a field of definition for V. Let $\vec{x} \in K^n$ be a generic point of $\mathfrak{p} := \mathfrak{P} \cap k[\vec{X}]$ (with respect to k). Then $k(\vec{x})$ is separable over k and k is relatively algebraically closed in $k(\vec{x})$.[3]

Fact A.1.14 ([Lang, 1958])**.** Suppose that we have a prime ideal $\mathfrak{p}$ in $k[\vec{X}]$, and V as an affine algebraic set of the ideal $\mathfrak{p} \otimes k^{\text{alg}}$ in $(k^{\text{alg}})^n$. Suppose that V is defined over k (that is $I_{k^{\text{alg}}}(V) = \mathfrak{p} \otimes k^{\text{alg}}$). Suppose that V is decomposed into $V = \cup_j V_j$ where the V_j are the irreducible components (of V over k^{alg}). Then the irreducible components $\{V_j\}_j$ are mutually conjugated by automorphisms of k^{alg} fixing k.

Fact A.1.15 ([Lang, 1958])**.** Let $\mathfrak{a}$ be an ideal in $k[\vec{X}]$ and let L be an extension of k. Let f be an element in $\mathfrak{a}L[\vec{X}]$. Then f can be written as a finite sum $f = \sum_i c_j f_j(\vec{X})$ where $\{c_j\}_j$ is a set of elements in L linearly independent over k, and $f_j(\vec{X}) \in k[\vec{X}]$ for all j. When f is written in such a way $f_j(\vec{X})$ are all in $\mathfrak{a}$ and they are uniquely determined.

The following proposition was pointed out by Delon:

Proposition A.1.16. *Suppose that k is separably closed, and k^{alg} is the algebraic closure of k. If V is an algebraic set in k, that is there are finitely many polynomials $f_1, \dots, f_r \in k[\vec{X}]$ such that*

$$V = \{\vec{x} \in k \mid f_1(\vec{x}) = 0, \dots, f_r(\vec{x}) = 0\},$$

then $I_k(V) \otimes k^{\text{alg}} = I_{k^{\text{alg}}}(\bar{V}^{\text{alg}})$, where $\bar{V}^{\text{alg}}$ is the **Zariski-closure** *of V in k^{alg}.*

Proof. We first prove that the zero set of $I_k(V)$ in k^{alg}, which is denoted by $Z_{k^{\text{alg}}}(I_k(V))$ is $\bar{V}^{\text{alg}}$.

It is clear that V is contained in $Z_{k^{\text{alg}}}(I_k(V))$. For any $g(\vec{X}) \in k^{\text{alg}}[\vec{X}]$, with the property that $V \subseteq Z_{k^{\text{alg}}}(g)$, we need to show that $Z_{k^{\text{alg}}}(I_k(V)) \subseteq Z_{k^{\text{alg}}}(g)$. By the assumption on g, for any $\vec{v} \in V$, $g(\vec{v}) = 0$. Because k is separably closed, there is some natural number m such that $g^{p^m} \in k[\vec{X}]$. Therefore, $g^{p^m} \in I_k(V)$. Then by the definition of the zero set, for any $\vec{v} \in Z_{k^{\text{alg}}}(I_k(V))$, $g^{p^m}(\vec{v}) = 0$, which in turn implies that $g(\vec{v}) = 0$. Therefore, we get that $Z_{k^{\text{alg}}}(I_k(V)) \subseteq Z_{k^{\text{alg}}}(g)$.

Back to the proof of the proposition, we have that

$$I_k(V) \subseteq I_{k^{\text{alg}}}(Z_{k^{\text{alg}}}(I_k(V))) = I_{k^{\text{alg}}}(\bar{V}^{\text{alg}}).$$

So $I_k(V) \otimes k^{\text{alg}} \subseteq I_{k^{\text{alg}}}(\bar{V}^{\text{alg}})$. On the other hand, suppose that we are given one $g \in I_{k^{\text{alg}}}(\bar{V}^{\text{alg}})$; then we can write (for some n)

$$g(\vec{X}) = \sum_{i=1}^{n} c_i g_i(\vec{X}),$$

[3] This kind of field extensions are also known as *regular extensions.*

where $g_i \in k[\vec{X}]$ and the c_i are elements of k^{alg} which are linearly independent over k. If we show that $g_i \in I_k(V)$, then $g \in I_k(V) \otimes k^{\mathrm{alg}}$.

For any $\vec{v} \in V$, because $g \in I_{k^{\mathrm{alg}}}(\bar{V}^{\mathrm{alg}})$, we have

$$0 = g(\vec{v}) = \sum_{i=1}^{n} c_i g_i(\vec{v}),$$

with $g_i(\vec{v}) \in k$. But because the c_i are linearly independent over k, all the $g_i(\vec{v})$ are zero, which is exactly what we want. □

Fact A.1.17 ([Fried and Jarden, 2008]). Every irreducible affine curve C in K which is defined over k is k-birationally equivalent to a plane curve C' (in K^2); that is up to finitely many points, there is a isomorphism from C to C' (given by rational functions over k).

A.2 Projective varieties, dimensions, degrees, etc.

Suppose that K is again a field.

Definition A.2.1 ([Hartshorne, 1977]). *A* **graded ring** *is a ring S together with a decomposition $S = \bigoplus_{d \geq 0} S_d$ of S into a direct sum of abelian groups S_d, such that for any $d, e \geq 0$, $S_d \cdot S_e \subseteq S_{d+e}$. An element of S_d is called a* **homogeneous element of degree** *d. An ideal $\mathfrak{a} \subseteq S$ is a* **homogeneous ideal** *if $\mathfrak{a} = \bigoplus_{d \geq 0}(\mathfrak{a} \cap S_d)$.*

Fact A.2.2 ([Hartshorne, 1977]). An ideal of a graded ring is homogeneous if and only if it can be generated by homogeneous elements. The sum, product, intersection, and radical of homogeneous ideals are homogeneous. A homogeneous ideal $\mathfrak{a}$ is prime if and only if for any two homogeneous elements f, g such that $fg \in \mathfrak{a}$, it follows that $f \in \mathfrak{a}$ or $g \in \mathfrak{a}$.

We can regard the ring $K[X_0, \ldots, X_n]$ as a graded ring, where a homogeneous element of degree d (now called a **homogeneous polynomial of degree** d) is a linear combination of monomials of degree d.

Consider the set-quotient obtained by $K^{n+1} \backslash \{0\}$ modulo the equivalence relation '$\sim$', where for any $\vec{a}, \vec{b} \in K^{n+1}$, $\vec{a} \sim \vec{b}$ if and only if there is some $\lambda \in K^{\times}$ such that $\vec{a} = \lambda \vec{b}$. That is, we identify a 1-dimensional linear subspace of K^{n+1} with one point. An element of this set-quotient is usually denoted by $[a_0 : a_1 : \ldots : a_n]$, which is the equivalence class of $(a_0, \ldots, a_n)$, in which case the a_i are called the **homogeneous coordinates** of the point $[a_0 : \ldots : a_n]$. For any homogeneous polynomial $f \in K[X_0, \ldots, X_n]$, it makes sense to talk about the 'zero-set' of f in $(K^{n+1} \backslash \{0\}) / \sim$; that is, a **zero** of f is an element $[a_0 : \ldots : a_n]$ such that $f(a_0, \ldots, a_n) = 0$; because f is homogeneous, this is well-defined—it does not depend on the choice of representative for the equivalence class '$\sim$'. A zero set of a set of homogeneous polynomials $A \subseteq$

$K[X_0, \ldots, X_n]$ is denoted by $Z_K(A)$. One can show that the zero-sets generate a topology on $(K^{n+1}\backslash\{0\})/\sim$ in which they are exactly the closed sets.

Definition A.2.3. *The* **projective n-space** *over K is $(K^{n+1}\backslash\{0\})/\sim$ endowed with the Zariski topology generated by the zero-sets of sets of homogeneous polynomials. It is denoted by $\mathbf{P}^n(K)$, or simply $\mathbf{P}^n$.*

Sometimes, given a vector space V over K, we use $\mathbf{P}(V)$ to denote the set of all 1-dimensional subspace of V.

Definition A.2.4. *A zero-set of a set of homogeneous polynomials in $\mathbf{P}^n$ is called a* **projective variety**. *An open subset of a projective variety is called a* **quasi-projective variety**.

Let $U_i \subseteq \mathbf{P}^n$ be the set of points whose i-th homogeneous coordinate is not zero. Then the map

$$[a_0 : \ldots : a_{i-1} : a_i : a_{i+1} : \ldots a_n] \mapsto (a_0/a_i, \ldots, a_{i-1}/a_i, a_{i+1}/a_i, \ldots, a_n)$$

from U_i to $\mathbf{A}^n$ is well-defined and gives a bijection of sets. Thus we identify $\mathbf{A}^n$ as an open subset U_i of $\mathbf{P}^n$. If $X \subseteq \mathbf{P}^n$ is a projective variety, say $X = Z(\{F_j\}_i)$, where F_i is a homogeneous polynomial of degree d_j in $K[X_0, \ldots, X_n]$, then clearly $X \cap \mathbf{A}^n$, under that identification, is the zero set of the polynomials $\{f_j(Y_1, \ldots, Y_n)\}_j$, where

$$\begin{aligned} f_j(Y_1, \ldots, Y_n) &:= F_j(X_0, \ldots, X_n)/X_i^{d_j} \\ &= F_j(X_0/X_j, \ldots, X_{i-1}/X_j, 1, X_{i+1}/X_i, \ldots, X_n/X_i). \end{aligned}$$

This process is invertible. One can see that if we have an affine sub-variety of $\mathbf{A}^n$ (for the ease of notation, we identify it with U_0), being the zero-set of $\{f_j(X_1, \ldots, X_n)\}_j$, then assuming that

$$f_j(X_1, \ldots, X_n) := \sum a_{i_1,\ldots,i_n} X_1^{i_1} \cdots X_n^{i_n},$$

we can get

$$\begin{aligned} F_j(X_0, \ldots, X_n) &:= X_0^{d_j} f_j(X_1/X_0, \ldots, X_n/X_0) \\ &= \sum a_{i_1,\ldots,i_n} X^{(d_j - \sum_k i_k)} X_1^{i_1} \cdots X_n^{i_n}. \end{aligned}$$

We refer to these two operations as the **affinization** and the **projectivization**.

Fact A.2.5 ([Hartshorne, 1977]). The **projective closure**, i.e. the Zariski-closure of an affine variety X (being identified as a subset of $\mathbf{A}^n \cong U_i$) in $\mathbf{P}^n$, is the zero-set of the ideal of the projectivization of the ideal of X.

Definition A.2.6 ([Harris, 1992])**.** *An inclusion of vector spaces over the field K, $W \cong K^{k+1} \hookrightarrow V \cong K^{n+1}$ induces a map $\mathbf{P}W \hookrightarrow \mathbf{P}V$; the image Λ of such a map is called a* **linear subspace** *of dimension k in $\mathbf{P}V$, or a k-***plane** *in $\mathbf{P}V$. In case $n = k - 1$, we called Λ a* **hyperplane***. In case $k = 1$, we call Λ a* **line***. A linearly subspace is a projective sub-variety of $\mathbf{P}^n$*

Suppose that $\Gamma \subseteq \mathbf{P}V$ is a subset, then the **span** *of Γ, is the smallest linear subspace of $\mathbf{P}V$ containing Γ.*

Fact A.2.7 ([Harris, 1992])**.** If $\Lambda = \mathbf{P}W$ of dimension k in $\mathbf{P}V$, then the space of $(k+1)$-planes containing Λ is the projective space $\mathbf{P}(V/W)$.

Definition A.2.8 ([Harris, 1992])**.** *A* **hypersurface** *is a sub-variety of $\mathbf{P}^n$ which is the zero-set of a single homogeneous polynomial.*

Definition A.2.9 ([Hartshorne, 1977])**.** *Suppose that Y is a quasi-projective variety in $\mathbf{P}^n$.*

A function $f : Y \to K$ is **regular** *at a point $\vec{p} \in Y$, if there is an open neighborhood U with $\vec{p} \in U \subseteq Y$ and homogeneous polynomials $g, h \in K[X_0, \ldots, X_n]$ of the same degree, such that h is nowhere zero on U and $f = g/h$ on U. We say that f is* **regular** *on Y if it is regular at every point in Y.*

Definition A.2.10 ([Hartshorne, 1977])**.** *Suppose that X and Y are two quasi-projective varieties, and $\varphi : X \to Y$ is a map. Then φ is said to be a* **morphism** *if φ is continuous and for every open set $V \subseteq Y$ and every regular function $f : V \to K$, $f \circ \varphi : \varphi^{-1}(V) \to K$ is regular. An* **isomorphism** *is a morphism with a two sided inverse.*

Fact A.2.11 ([Hartshorne, 1977])**.** If $X \subseteq \mathbf{P}^n$ is a projective variety and $\varphi : X \to \mathbf{P}^n$ is a morphism, then the image of $\varphi(X)$ is also a projective variety.

Let $G(k,n)$ be the set of k-dimensional linear subspaces of the vector space K^n; or in general let $G(k,V)$ be the set of k-dimensional linear subspaces of the vector space V. Then $G(k,n)$ is the same as the set of $(k-1)$-planes in $\mathbf{P}^{n-1}$, so sometimes it is also written as $\mathbf{G}(k-1, n-1)$. This kind of sets are called the **Grassmannians**. One can see that they are generalizations of the projective spaces.

We want to embed $G(k,n)$ into some projective space, so that it becomes a projective variety. The embedding is done as follows (following [Harris, 1992] almost verbatim). Suppose that V is an n-dimensional vector space over K. If $W \subseteq V$ is the k-dimensional linear subspace spanned by vectors $v_1, \ldots, v_k$, then we consider the element $v_1 \wedge \ldots \wedge v_k \in \bigwedge^k(V)$ in the k-th exterior power of V. If we have a different basis for W, then the corresponding element in $\bigwedge^k(V)$ is obtained by multiplying the original one by the determinant of the matrix of basis change. Thus we can define a map $\psi : G(k,V) \to \mathbf{P}\left(\bigwedge^k V\right)$ as

$$W \mapsto [v_1 \wedge \ldots \wedge w_k].$$

One can recover the original vector space W as the set of vectors $v \in V$ satisfiying $v \wedge (v_1 \wedge \ldots \wedge v_k) = 0 \in \bigwedge^{k+1} V$. The map φ is called the **Plücker embedding** of $G(k, V)$. The homogeneous coordinates on $\mathbf{P}(\wedge^k V)$ are called the **Plücker coordinates** of $G(k, V)$.

If we choose an identification $V \cong K^n$, we can represent the plane W by the $k \times n$ matrix M_W whose rows are the vectors v_i; the matrix M_W is determined up to multiplication on the left by an invertible $k \times k$ matrix. It can be shown that the Plücker coordinates are then just the maximal minors of the matrix M_W.

In order to see that $G(k, V)$ is embedded as a projective variety, it is enough to characterize the totally decomposable vectors $\omega \in \wedge^k V$. A vector $\omega \in \wedge^k V$ is **totally decomposable** if it is a product of vectors in V, i.e. $\omega = v_1 \wedge \ldots \wedge v_k$ for some set of vectors. Given $\omega \in \wedge^k V$ and $v \in V$, ω can be expressed as $v \wedge \varphi$ for some $\varphi \in \wedge^{k-1} V$ if and only if $\omega \wedge v = 0 \in \wedge^{k+1} V$, in which case we say v **divides** ω. Thus ω is totally decomposable if and only if the space of vectors v dividing ω is k-dimensional. For each $\omega \in \wedge^k V$, consider the map

$$\begin{aligned} \varphi(\omega) : & V \to \wedge^{k+1} V \\ & v \mapsto \omega \wedge v. \end{aligned}$$

One can see that $[\omega] \in \psi(G(k, n))$ if and only if $\operatorname{rank}(\varphi(\omega))$ is $n - k$. But $\operatorname{rank}(\varphi(\omega))$ is always not less than $n - k$, thus

$$[\omega] \in \psi(G(k, n)) \Longleftrightarrow \operatorname{rank}(\varphi(\omega)) \leq n - k.$$

Because the map $\theta := \omega \mapsto \varphi(\omega)$ from $\wedge^k V$ to $\operatorname{Hom}(V, \wedge^{k+1} V)$ is linear, $G(k, V)$ is the zero-set of the $(n - k + 1) \times (n - k + 1)$ minors of the matrix of θ.

Suppose that a_i is one Plücker coordinate of $\mathbf{P}(\wedge^k V)$ after identifying V with K^n with the standard basis $\{\mathbf{e}_i\}_i$, where $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, whose only non-zero coordinate is the i-th coordinate which is 1. Then the set U_i, which is the set of points with $a_i \neq 0$ in $\mathbf{P}(\wedge^k V)$, is isomorphic to an affine space. For instance, we examine the U_0, where the a_0 is the maximal minor obtained from the first k columns. After a change of basis or performing elementary row operations, we can assume that the $k \times n$ matrix associated to a k-dimensional linear subspace W is in fact of the form

$$\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & x_{1,1} & x_{1,2} & \ldots & x_{1,n-k} \\ 0 & 1 & 0 & \ldots & 0 & x_{2,1} & x_{2,2} & \ldots & x_{2,n-k} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & x_{k,1} & x_{k,2} & \ldots & x_{k,n-k} \end{pmatrix}. \tag{A.2.1}$$

One checks that this gives an isomorphism $U_0 \cap G(k, n) \cong \mathbf{A}^{k(n-k)}$.

Fact A.2.12 ([Harris, 1992])**.** $G(k, n)$ is an irreducible projective variety (of dimension $k(n - k)$).

Definition A.2.13 ([Harris, 1992]). *By a* **general k-plane** *in $\mathbf{P}^n$, we mean a point inside some (previously fixed) Zariski-open dense subset of $\mathbf{G}(k, n)$.*

For example, if we say "given a projective variety X in $\mathbf{P}^n$, a general k-plane intersects X at finitely many points", then what we mean is: "the subset, of points in $\mathbf{G}(k, n)$ which correspond to k-planes intersecting X at an infinite subset, is the complement of a Zariski-open dense subset of $\mathbf{G}(k, n)$." It can be proved that for any quasi-projective variety X in $\mathbf{P}^n$, there is always a smallest integer k such that a general $(n-k)$-plane in $\mathbf{P}^n$ intersects X at finitely many points (and furthermore the number of the points is the same for all general $(n-k)$-planes.

Definition A.2.14 ([Harris, 1992]). *The* **dimension** *of an irreducible quasi-projective variety $X \subseteq \mathbf{P}^n$ is the smallest integer k such that a general $(n-k-1)$-plane $\Lambda \subseteq \mathbf{P}^n$ is disjoint from X. Equivalently, the dimension of an irreducible quasi-projective $X \subseteq \mathbf{P}^n$ is the integer k such that a general $(n-k)$-plane in $\mathbf{P}^n$ intersects X at finitely many points. The dimension of X is denoted by $\dim(X) := k$.*

Fact A.2.15 ([Harris, 1992]). If X is an irreducible variety and $Y \subseteq X$ is a proper closed sub-variety, then $\dim(Y) < \dim(X)$.

Fact A.2.16 ([Harris, 1992]). If $X \subseteq \mathbf{P}^n$ is a k-dimensional projective variety and Λ is any linear subspace of dimension $l \geq n-k$, then Λ must intersect X.

Fact A.2.17 ([Harris, 1992]). $\dim(G(k, n)) = \dim\left(\mathbf{A}^{k(n-k)}\right) = k(n-k)$.

Definition A.2.18 ([Harris, 1992]). *Let $X \subseteq \mathbf{P}^n$ be an irreducible quasi-projective variety of dimension k. The degree of X, denoted by $\deg(X)$, is the number of points of the intersection of a general $(n-k)$-plane with X.*

The following two items are not critical to the understanding of the results in this thesis.

Definition A.2.19 ([Harris, 1992]). *Suppose that X and Y are two projective varieties in $\mathbf{P}^n$ and that their intersection has irreducible components Z_i. Then X and Y are said to intersect* **generically transversely** *if, for each i, X and Y intersect tranversely at a general point $p_j \in Z_i$, i.e. for each p_j a general point in Z_i the tangent spaces of X and Y at p_j span the whole tangent space of $\mathbf{P}^n$ at p_j.*

Fact A.2.20 (Bézout's Theorem, [Harris, 1992]). Let X and Y be two irreducible projective varieties in $\mathbf{P}^n$ of dimension k and l with $k + l \geq n$, and suppose that they intersect generically transversely. Then

$$\deg(X \cap Y) = \deg(X)\deg(Y).$$

In particular, if $k + l = n$, then $X \cap Y$ consists of $\deg(X)\deg(Y)$ points.

Definition A.2.21 ([Harris, 1992]). *A quasi-projective variety is said to be* **non-degenerate** *if it does not lie in any hyperplane.*

Fact A.2.22 ([Harris, 1992]). Let $C \subseteq \mathbf{P}^d$ be any irreducible non-degenerate curve (i.e. projective variety of dimension 1). Then $\deg(C) \geq d$.

Fact A.2.23 ([Harris, 1992]). Let X be an irreducible non-degenerate quasi-projective variety of dimension $k \geq 1$. Let H be a general hyperplane and $Y = X \cap H$. Then Y is non-degenerate in H; and if furthermore $k \geq 2$ then Y is irreducible as well.

Definition A.2.24 ([Harris, 1992]). *We define a family of maps called* **Segre maps**. *For given numbers m, n, denote by $\mathbf{P}^n \times \mathbf{P}^m$ the set-theoretic Cartesian product of the set $\mathbf{P}^n$ and the set $\mathbf{P}^m$. Then $\sigma : \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^{(n+1)(m+1)-1}$, is the map*

$$\sigma([X_0, \ldots, X_n], [Y_0, \ldots, Y_m]) = [\ldots, X_i Y_j, \ldots].$$

Fact A.2.25 ([Harris, 1992]). The image of this map, denoted by $\Sigma_{n,m}$, is a projective variety, which is the zero-set of polynomials $Z_{i,j}Z_{k,l} - Z_{i,l}Z_{k,j}$. If $X \subseteq \mathbf{P}^n$ and $Y \subseteq \mathbf{P}^m$ are two projective (resp. quasi-projective) varieties, then treating $X \times Y$ set-theoretically, we have that $\sigma(X \times Y) \subseteq \Sigma_{n,m}$ is a projective (resp. quasi-projective) variety.

Definition A.2.26. *Given two quasi-projective varieties $X \subseteq \mathbf{P}^n$ and $Y \subseteq \mathbf{P}^m$, the set-theoretic product $X \times Y$, endowed with the structure of $\sigma(X \times Y)$, is called the* **product variety** *of X and Y, also denoted by $X \times Y$.*[4]

Note that the topology on the product variety $X \times Y$ is not the product topology of the topology on X and the topology on Y.

Fact A.2.27 ([Harris, 1992]). If $X \subseteq \mathbf{P}^n$ is projective variety, $\varphi : X \to \mathbf{P}^m$ is a morphism, then the graph of φ viewed as a subset of $\mathbf{P}^n \times \mathbf{P}^m$ is a sub-variety.

Proposition A.2.28. *Suppose that E is a Zariski-dense subset of the projective variety X and F is a Zariski-dense subset of the projective variety Y. Then $E \times F$ as a subset of $X \times Y$ is also Zariski-dense.*

Proof. For any point $\vec{e} \in E$, $\{\vec{e}\} \times Y$ is isomorphic to Y as a projective variety. Thus the Zariski-closure of $\{\vec{e}\} \times F$ is $\{\vec{e}\} \times Y$. It follows that for any point $\vec{y} \in Y$, $E \times \{\vec{y}\}$ is in the Zariski-closure of $E \times F$. But the Zariski-closure of $E \times \{\vec{y}\}$ is $X \times \{\vec{y}\}$. Therefore, $X \times Y$ is contained in the Zariski-closure of $E \times F$, which implies that $E \times F$ is Zariski-dense in $X \times Y$. □

Proposition A.2.29. *Suppose that K is an infinite subfield of L. Then the Grassmannian $G(k, n)(K)$ of k-planes in K is Zariski-dense in $G(k, n)(L)$.*

[4]One can check that it is indeed the product in the categorical sense.

Proof. By the identification $U_0 \cap G(k,n) \cong \mathbf{A}^{k(n-k)}$ via Equation (A.2.1), it is enough to verify that $\mathbf{A}^m(K)$ is Zariski-dense in $\mathbf{A}^m(K)$ for all m. But this is true when K is an infinite subfield of L, which can be proved by induction on the dimension and by Proposition A.2.28. $\square$

Fact A.2.30 ([Harris, 1992])**.** Let X be a irreducible quasi-projective variety and $\pi : X \to \mathbf{P}^n$ a regular map; let Y be the closure of the image. For any $\vec{p} \in X$, let $X_{\vec{p}} = \pi^{-1}(\pi(p))$ be the fiber of π through $\vec{p}$. Let $\mu(\vec{p}) = \dim(X_p)$. Then $\mu(\vec{p})$ is an upper-semicontinuous function of $\vec{p}$; that is for all m, the set of all $\vec{p}$ such that $\mu(\vec{p}) \geq m$ is Zariski-closed in X. Moreover, if μ is the minimal value of $\mu(\vec{p})$ on X, then

$$\dim(X) = \dim(Y) + \mu.$$

Fact A.2.31 ([Harris, 1992])**.** Let X be an irreducible projective variety and $\pi : X \to \mathbf{P}^n$ any regular map; let $Y = \pi(X)$ be its image. For any $\vec{q} \in Y$, let $\lambda(\vec{q}) = \dim(\pi^{-1}(\vec{q}))$. Then $\lambda(\vec{q})$ is an upper-semicontinuous function of $\vec{q}$. Moreover, if λ is the minimal value of $\lambda(\vec{q})$ on Y, then

$$\dim(X) = \dim(Y) + \lambda.$$

Fact A.2.32 ([Harris, 1992])**.** Let $\pi : X \to Y$ be a morphism of projective varieties with Y being irreducible. Suppose that all fibers $\pi^{-1}(\vec{p})$ of π are irreducible of the same dimension. Then X is irreducible.

Fact A.2.33 ([Harris, 1992])**.** Projective varieties of the following form are called **flag manifolds**:

$$\mathbf{F}(k,l,n) = \{(\Gamma, \Lambda) \mid \Gamma \subseteq \Lambda\} \subseteq \mathbf{G}(k,n) \times \mathbf{G}(l,n).$$

They are irreducible of dimension

$$\begin{aligned}\dim(\mathbf{F}(k,l,n)) &= \dim(\mathbf{G}(k,n)) + \dim(\mathbf{G}(l-k-1, n-k-1)) \\ &= (k+1)(n-k) + (l-k)(n-l).\end{aligned}$$

Fact A.2.34 ([Harris, 1992])**.** Let $\Phi \subseteq \mathbf{G}(k,n)$ be any sub-variety. Then the union

$$\Psi = \bigcup_{\Lambda \in \Phi} \Lambda$$

inside $\mathbf{P}^n$ is also a projective variety.

Fact A.2.35 ([Harris, 1992])**.** Let $X \subseteq \mathbf{P}^n$ be a projective variety. Then the set of k-planes meeting X is a sub-variety of the Grassmannian $\mathbf{G}(k,n)$.

Fact A.2.36 ([Harris, 1992])**.** Similar to Fact A.2.7, the variety of k-planes containing a linear subspace of dimension $l < k$, is isomorphic to $\mathbf{G}(k-l-1, n-l-1)$.

Proposition A.2.37. *Suppose that X is an irreducible non-degenerate*[5] *projective variety of dimension $m < n$ in $\mathbf{P}^n$, E and F two subsets of X which are both Zariski-dense in X and disjoint. Then for any $0 \leq k \leq m-1$, there is some general $(n-m+k)$-plane passing through some point in E and some point in F.*

Proof. Consider the morphism:

$$s : (X \times X) \setminus \Delta \to \mathbf{G}(1, n)$$
$$(\vec{p}, \vec{q}) \mapsto \overline{\vec{p}\vec{q}},$$

where Δ is the diagonal in $X \times X$, and $\overline{\vec{p}\vec{q}}$ is the line determined by $\vec{p}$ and $\vec{q}$. Denoting the image of s by S, the Zariski-closure of S is called the **variety of secant lines**, denoted by $\mathscr{S}(X)$. It is known ([Harris, 1992]) that $\mathscr{S}(X)$ is an irreducible projective variety with $\dim(\mathscr{S}(X)) = 2m$. Because $E \times F$ is Zariski-dense in $X \times X$, its image, $s(E \times F)$, under the continuous map s, has to be Zariski-dense in S, hence in $\mathscr{S}(X)$ as well.

Consider the two projections π_1 and π_2 from the flag manifold $\mathbf{F}(1, n-m+k, n)$ to $\mathbf{G}(1, n)$ and $\mathbf{G}(n-m+k, n)$ respectively. $\pi_2\pi_1^{-1}(\mathscr{S}(X))$ must contain a general $(n-m+k)$-plane, as a general $(n-m+k)$-plane passes through two distinct points on X. Because each point $(\vec{p}, \vec{q}) \in E \times F$ has the property that $\pi_1^{-1}(s(\vec{p}, \vec{q})) = \mathbf{G}(n-m+k-2, n-2)$, the dimension of the Zariski-closure of $\pi_1^{-1}(s(E \times F))$ is $\dim(\mathscr{S}(X)) + \dim(\mathbf{G}(n-m+k-2, n-2))$. Furthermore, since each fiber of π_1 on $\mathscr{S}(X)$ has the same dimension, $\pi_1^{-1}(\mathscr{S}(X))$ is irreducible, with $\dim(\pi_1^{-1}(\mathscr{S}(X))) = \dim(\mathscr{S}(X)) + \dim(\mathbf{G}(n-m+k-2, n-2))$ as well. It follows that $\pi_1^{-1}(s(E \times F))$ is Zariski-dense in $\pi_1^{-1}(\mathscr{S}(X))$. But this implies that there must be a general $(n-m+k)$-plane inside $\pi_2\pi_1^{-1}(s(E \times F))$ (π_2^{-1} takes an open subset into an open subset), which is exactly the conclusion we wanted to prove. □

Definition A.2.38 ([Harris, 1992]). *For each $n \geq 0$ and $r \geq 0$, the* **Veronese map** *of degree r is defined to be the map $v_r : \mathbb{P}^n \to \mathbb{P}^N$, which takes $[x_0, \ldots, x_n]$ to $[\ldots, x^I, \ldots]$, where x^I ranges over all monomials of degree r in the variables $x_0, x_1, \ldots, x_n$. One can show that $N = \binom{n+r}{r} - 1$.*

It is easy to see that when $r \geq 1$, v_r is a one-to-one function. The image $v_r(\mathbb{P}^n)$ is in fact a projective sub-variety of $\mathbb{P}^N$. It can be also proved that if X is a projective sub-variety of $\mathbb{P}^n$, then $v_r(X)$ is also a projective sub-variety of $v_r(\mathbb{P}^N)$. In fact, X and $v_r(X)$ are isomorphic. And the corresponding "affinized" $v_r^A(X)$ is isomorphic to X.

Suppose that X is a projective hypersurface in $\mathbb{P}^n$ defined by a homogeneous polynomial of degree d (which generates the ideal of the zero set). Consider the r-th Veronese map v_r on X, where r is at least d and $p \nmid r$. Clearly $v_r(X)$ should be of

[5] As we can see from the proof, being of degree larger than 1 is enough for the conclusion to hold.

dimension m, inside $\mathbb{P}^N$ where $N+1 = \binom{n+r}{r}$. One can show that the span of $v_r(X)$ is of dimension, [6]

$$\binom{r+n}{r} - \binom{r-d+n}{r-d}.$$

Furthermore, the degree of $v_r(X)$ is $r^m d$.

In the case where $n = 2$ and $m = 1$, i.e. when X is a plane curve in $\mathbb{P}^2$ (in particular, X must be defined by a single equation of degree d), the dimension of the span of $v_r(X)$ is

$$\binom{r+2}{r} - \binom{r-d+2}{r-d} = rd - d(d-3)/2.$$

The degree of $v_r(X)$ is rd. It follows that if the degree of the Veronese map, r, is sufficiently large, then $v_r(X)$ is a non-degenerate curve in $\mathbb{P}^{N_1}$ of degree rd, where $N_1 = rd - d(d-3)/2$ and $2N_1 > dr$.

A.3 General intersections

In order to prove that $\mathrm{SCVF}_{p,e}$ ⧖ $\mathrm{SCF}_{p,e}$ (see Chapter 5), we will need to use some results about intersecting a given irreducible variety defined over a separably closed field by linear varieties defined over the the same field. This is a critical step to make the descend from the algebraic closure to the separably closed sub-field that we are working in possible.

Throughout this section, let K be a separably closed field of characteristic $p > 0$, which has an infinite transcendence degree over its prime field. As usual, we use K^{alg} to denote the (field-theoretic) algebraic closure of K. The letter k will alway denote a (most of the time countable) subfield of K^{alg}.

The reader can find most of the contents (except the part about the resultants) of this section in the book [Fried and Jarden, 2008].

From now on, we are going to call an affine algebraic set $V \subseteq (K^{\mathrm{alg}})^n$ which is the zero set of a set of polynomials in $k[\vec{X}]$ an **k-algebraic set**, and if V is not a finite union of proper k-algebraic subsets, then it will be called an **irreducible k-variety**. It is easy to see that V is an irreducible k-variety if and only if its associated ideal in $k[\vec{X}]$ is prime.

Definition A.3.1 ([Lang, 1958]). *Given two points $\vec{x}, \vec{x}' \in (K^{\mathrm{alg}})^n$, $\vec{x}'$ is a **k-specialization** of $\vec{x}$ if every polynomial in $k[\vec{X}]$ vanishes at $\vec{x}$ also vanishes at $\vec{x}'$.*

It follows from the definition that we have the following: if $V \subseteq (K^{\mathrm{alg}})^n$ is an algebraic set whose associated ideal in $k[\vec{X}]$ is a prime ideal $\mathfrak{p}$, and $\vec{x}$ is a generic point

[6] This is because the dimension of the vector space of homogeneous polynomials of degree r in variables $X_0, \ldots, X_n$ has dimension $\binom{r+n}{r}$; the dimension of the subspace of homogeneous polynomials of degree r in $X_0, \ldots, X_n$ vanishing on C is $\binom{r-d+n}{r-d}$. The dimension of the span of the image is the difference of these two. This is an argument from Page 264 of [Harris, 1992].

of $\mathfrak{p}$ over k, then a point $\vec{x}' \in (K^{\text{alg}})^n$ is in V if and only if it is a k-specialization of $\vec{x}$. Also, suppose $V \subseteq (K^{\text{alg}})^n$ is a k-algebraic set such that there is a point $\vec{x} \in V$ such that all points in V are k-specializations of $\vec{x}$, then $I_k(V)$ has to be $\{f(\vec{X}) \in k[\vec{X}] \mid f(\vec{x}) = 0\}$, which is thus a prime ideal in $k[\vec{X}]$; so in this case V is an irreducible k-variety and $\vec{x}$ is a generic point of V over k.

In order to discuss about the results regarding general intersections of an irreducible variety with a linear variety, we need to make a detour into the notion of derivations of fields.

Definition A.3.2 ([Fried and Jarden, 2008]). *Suppose that E is a field. A map $D : E \to E$ is called a* **derivation** *of E if $D(x + y) = D(x) + D(y)$ and $D(xy) = D(x)y + xD(y)$ for all $x, y \in E$. If D vanishes on a subfield F of E then we say that D is a* **derivation** *of the field extension E/F.*

Fact A.3.3 ([Fried and Jarden, 2008]). Let E/F be a finitely generated field extension. Then E/F is separably algebraic if and only if 0 is the only derivation of E/F.

Fact A.3.4 ([Fried and Jarden, 2008]). Let E/F be a finitely generated separable extension of positive characteristic p and $t \in F$. Then

(a). If there exists a derivation D of E/F such that $D(t) \neq 0$, than E is a separable extension of $F(t)$.

(b). If t is transcendental over F and $E/F(t)$ is separable, then there exists a derivation D of E/F such that $D(t) \neq 0$.

Fact A.3.5 (Matsusaka-Zariski, [Fried and Jarden, 2008]). Let E/F be a finitely generated regular field extension, $y, z \in E$ algebraically independent over F. Suppose that there exists a derivation D of E/F such that $D(z) \neq 0$. Then there exists a finite subset C of F such that E is a regular extension of $F(y + cz)$ for all $c \in F \backslash C$.

Fact A.3.6 ([Fried and Jarden, 2008]). Every affine irreducible variety V (i.e. in K^{alg}) of dimension $r \geq 1$ defined over k contains an affine irreducible curve C (i.e. an irreducible K^{alg}-variety of dimension 1) defined over k.

Proof. Here we only provide the proof to the fact that there exist (infinitely many different choices of) algebraically independent elements $t_1, \ldots, t_{r-1}$ over k such that V contains an affine irreducible curve C defined over $k(t_1, \ldots, t_{r-1})$. For the full proof of the original assertion, see [Fried and Jarden, 2008].

Note that in this section, we always assume that K is of positive characteristic.

If $r = 1$, then the conclusion is trivial. Thus we may assume that $r \geq 2$. Let $\vec{x} = (x_1, \ldots, x_n)$ be a generic point of V over k. Thus $k(\vec{x})/k$ is a regular field extension and $k(\vec{x})$ has a separating transcendence basis amongst the coordinates of $\vec{x}$, say $x_1, x_2, \ldots, x_s$, where $n \geq s \geq 2$. Then $k(\vec{x})/k(x_2)$ is a separable field extension. Thus by Fact A.3.4 there exists a derivation D of $k(\vec{x})/k$ such that $D(x_2) \neq 0$. By

Fact A.3.5, there exists co-finitely many $c \in k$, with $t = x_1 + cx_2$, such that $k(\vec{x})/k(t)$ is a regular field extension.

Let H_t be the hyperplane in $(K^{\mathrm{alg}})^n$ defined over $K(t)$ by $X_1 + cX_2 - t = 0$. Then $V \cap H_t$ is defined over $k(t)$. To see this, observe that $\vec{x} \in V \cap H_t$; if another points $\vec{y}$ is in $V \cap H_t$, then $y_1 + cy_2 = t$, thus $\vec{y}$ is a $k(t)$-specialization of $\vec{x}$. Therefore, $V \cap H_t$ is a irreducible variety defined over $k(t)$ and $\vec{x}$ is a generic point of $V \cap H_t$. The dimension of $V \cap H_t$ is the transcendental degree of $k(\vec{x})/k(t)$, which is $r - 1$. One can then iterate this argument until we get a set of algebraically independent elements $t_1, \dots, t_{r-1}$ such that $V \cap H_{t_1} \cap \cdots \cap H_{t_{r-1}}$ is an irreducible curve defined over $k(t_1, \dots, t_{r-1})$. □

Fact A.3.6 does not say anything about the case where V is an affine curve, i.e. the 1-dimensional case where the intersection is a set of finitely many points. In the following, we use resultants to prove some results regarding affine curves, which will be used to prove that $\mathrm{SCVF}_{p,e}$ ⧖ $\mathrm{SCF}_{p,e}$.

Recall (see e.g. [Fischer, 2001]) that for a commutative ring A and two polynomials in $A[X]$,

$$f(X) = a_m X^m + a_{m-1} X^{m-1} + \cdots + a_1 X + a_0,$$
$$g(X) = b_n X^n + b_{n-1} X^{n-1} + \cdots + b_1 X + b_0,$$

where $a_m \neq 0 \neq b_n$, the **resultant** $\mathrm{Res(f, g)}$ of f and g is the determinant of the $(n+m) \times (m+n)$ **Sylvester matrix** of f and g, as shown in Equation (A.3.1).

$$\text{(A.3.1)} \qquad \mathrm{Res}(f, g) = \begin{vmatrix} a_m & a_{m-1} & a_{m-2} & \cdots & a_0 & & & \\ & a_m & a_{m-1} & a_{m-2} & \cdots & a_0 & & \\ & & \ddots & \ddots & \ddots & \cdots & \ddots & \\ & & & a_m & a_{m-1} & a_{m-2} & \cdots & a_0 \\ b_n & b_{n-1} & b_{n-2} & \cdots & b_0 & & & \\ & b_n & b_{n-1} & b_{n-2} & \cdots & b_0 & & \\ & & \ddots & \ddots & \ddots & \cdots & \ddots & \\ & & & b_n & b_{n-1} & b_{n-2} & \cdots & b_0 \end{vmatrix}$$

Fact A.3.7 ([Fischer, 2001])**.** Suppose that A is an unique factorization domain, $f, g \in A[X]$ with non-zero leading coefficients. Then f and g have a common factor of degree at least 1 in $A[X]$, if and only if, $\mathrm{Res}(f, g) = 0$ in A.

Remark A.3.8. If there is exactly one of f and g being of degree 0, say f. Then $\mathrm{Res}(f, g) = f^n$, and one can check that the equivalence asserted in Fact A.3.7 is still valid. Similarly, one can easily check that as long as one of the leading coefficients of f and g is not zero (so the other one could be zero or not), the equivalence still holds.

Remark A.3.9. In the expansion of $\mathrm{Res}(f,g)$ as a polynomial of the coefficients, there is a term $a_m^n b_0^m$ and a term $\pm a_0^n b_n^m$. See for example the circled items below to get $a_m^n b_0^m$:

$$\begin{vmatrix} \textcircled{a_m} & a_{m-1} & a_{m-2} & \cdots & a_0 & \cdots & \cdots & \cdots \\ & \textcircled{a_m} & a_{m-1} & a_{m-2} & \cdots & a_0 & \cdots & \cdots \\ & & \textcircled{\ddots} & \ddots & \ddots & \cdots & \ddots & \cdots \\ & & & \textcircled{a_m} & a_{m-1} & a_{m-2} & \cdots & a_0 \\ b_n & b_{n-1} & b_{n-2} & \cdots & \textcircled{b_0} & \cdots & \cdots & \cdots \\ & b_n & b_{n-1} & b_{n-2} & \cdots & \textcircled{b_0} & \cdots & \cdots \\ & & \ddots & \ddots & \ddots & \cdots & \textcircled{\ddots} & \cdots \\ & & & b_n & b_{n-1} & b_{n-2} & \cdots & \textcircled{b_0} \end{vmatrix}.$$

We are going to use this fact below.

Now, let C be an irreducible affine plane curve defined over $k \subseteq K$. It follows that the vanishing ideal of C in $k[X,Y]$ or $(K^{\mathrm{alg}})[X,Y]$ is generated by a single irreducible polynomial in $k[X,Y]$, assume that it is $f(X,Y)$. We want to show that

Proposition A.3.10. *Let C be an irreducible affine curve in $(K^{\mathrm{alg}})^2$ defined over $k \subseteq K$. A general plane curve of degree d over K, which is the zero locus of $g(X,Y) = \sum_{i+j\leq d} A_{ij}X^iY^j \in K[X,Y]$, has an intersection with C (of finitely many points) which is defined over K.*

Proof. To do this, let $g(X,Y)$ be a polynomial of degree d with undetermined coefficients A_{ij} as follows:

$$g(X,Y) = \sum_{i+j\leq d} A_{ij}X^iY^j.$$

These kind of polynomials are clearly parametrized by their coefficients $(A_{ij}) \in \mathbf{P}^{\binom{2+d}{2}-1}$; one can think of it as the parametrization for the hyperplane sections of the Veronese embedding of $\mathbf{P}^2$. Notice that g is not necessarily irreducible. We may also assume that $I_{K^{\mathrm{alg}}}(C)$ is generated by a single (absolutely) irreducible polynomial $f(X,Y) \in k[X,Y]$.

Note that by results in the previous section, a general $Z_{K^{\mathrm{alg}}}(g(X,Y))$ would only intersect C at finite many points in $(K^{\mathrm{alg}})^2$. Therefore, we just need to figure out when an intersection with finite many points is defined over K.

One can consider both f and g as elements in $(K[X])[Y]$; then the resultant of f and g as elements in $(K[X])[Y]$, is $\mathrm{Res}(f(X,-),g(X,-)) \in K[X]$. If $f(X,-) \in (K[X])[Y]$ is of degree 0, then $f(X,Y)$ is in fact in $K[X]$; but then by the assumption of C being irreducible and defined over k, contained inside K, $f(X,Y)$ is a linear polynomial in $K[X]$, which we may assume to be monic, say $f(X,Y) = X-a$, with $a \in K$. Then $g(a,Y) = \sum_{i+j\leq d} A_{ij}a^iY^j$. As long as $g(a,Y) \in K[Y]$ is separable (i.e. does not have multiple roots), all the solutions to the equations $(f(X,Y)=0) \wedge (g(X,Y) = 0)$ are in K, therefore the intersection is defined over K. But to say that a polynomial

$h(Y) \in K[Y]$ is does not multiple roots, it is the same as saying that $h(Y)$ and $h'(Y)$ does not have common roots in $K[Y]$, which by Fact A.3.7, is the same as saying that $\mathrm{Res}(h, h') \neq 0$. This means that any (A_{ij}) satisfying[7] $\mathrm{Res}(g(a, -), \frac{\partial g}{\partial Y}(a, -)) \neq 0$ would satisfy our requirement that the corresponding intersection is defined over K.

We may therefore assume that $f(X, -)$ and $f(-, Y)$ both have degrees at least 1 in Y and in X respectively. We proceed by using the same trick—if one makes sure that $\mathrm{Res}(f(X, -), g(X, -))$ and $\mathrm{Res}(f(-, Y), g(-, Y))$ are both polynomials without multiple roots, then they will split into linear factor in $K[Y]$ and $K[X]$ respectively, then the solutions of $(f(X, Y) = 0) \wedge (g(X, Y) = 0)$ will all have to be in K^2. Here we have to be more careful about the leading coefficients. We dictate that $A_{d0} \neq 0 \neq A_{0d}$, so that at least the leading coefficients of $g(X, Y)$ considered as polynomials in X and in Y respectively are both non-zero. Thus, the following conditions make sure that $Z_{K^{\mathrm{alg}}}(f, g)$ is in K^2:

$$\begin{cases} A_{d0} \neq 0, \\ A_{0d} \neq 0, \\ \mathrm{Res}(f(X, -), g(X, -)) \text{ does not have multiple roots,} \\ \mathrm{Res}(f(-, Y), g(-, Y)) \text{ does not have multiple roots.} \end{cases} \tag{A.3.2}$$

We just need to make sure that the set of all $(A_{ij})_{ij}$ satisfying these conditions is not empty.

By the assumption that the degree of f with respect to X and Y are both larger than or equal to 1, we may assume that

$$f(X, Y) = \sum_{i=0}^{m_Y} F_i^X(X) Y^i \in (K[X])[Y]$$

and

$$f(X, Y) = \sum_{i=0}^{m_X} F_i^Y(Y) X^i \in (K[Y])[X],$$

with $m_Y, m_X > 0$ and $F_{m_Y}^X(X), F_{m_X}^Y(Y) \neq 0$. It follows that if one writes $\mathrm{Res}(f(X, -), g(X, -))$ as a polynomial in $(K[X])[\{A_{ij}\}_{ij}]$ there is exactly one term which is a multiple of $A_{d0}^{m_Y}$, i.e. $\pm a_{m_Y}(X)^d (A_{d0} X^d)^{m_Y}$; this implies that $\mathrm{Res}(f(X, -), g(X, -))$ is a non-trivial polynomial in the variables $\{A_{ij}\}_{ij}$, and that it is also a non-constant polynomial in X. By the same reason, $\mathrm{Res}(f(-, Y), g(-, Y))$ is a non-trivial polynomial in the variables $\{A_{ij}\}_{ij}$ because of the term involving $A_{0d}^{m_X}$ is non-trivial; and it is also a non-constant polynomial in Y. We remark that this shows that for a general $\{A_{ij}\}_{ij}$, the intersection $Z_{K^{\mathrm{alg}}}(f) \cap Z_{K^{\mathrm{alg}}}(g)$ is finite, because both resultants are non-trivial polynomials in single variables X and Y respectively (thus f and g can only have finitely many common zeros).

[7] Note that this condition gives a Zariski-open set which is not empty.

$$\text{(A.3.3)} \qquad \begin{cases} A_{d0} \neq 0, \\ A_{0d} \neq 0, \\ \operatorname{Res}(P(Y), P'(Y)) \neq 0, \\ \operatorname{Res}(Q(Y), Q'(Y)) \neq 0. \end{cases}$$

Denote $\operatorname{Res}(f(X,-), g(X,-))$ by $P(Y)$ and $\operatorname{Res}(f(-,Y), g(-,Y))$ by $Q(X)$. Let $P'(Y)$ and $Q'(X)$ be their formal derivatives with respect to Y and X respectively. Then because P and Q are non-trivial polynomials, Condition (A.3.2) is equivalent to Condition (A.3.3).

This gives a Zariski-open condition for the parameters $\{A_{ij}\}_{ij}$. We still have to show that this Zariski-open set is non-empty. In order to do that, we show that there is some particular element inside this set. We are going to use a union of d lines to construct such an element. [8]

Suppose that the degree of $f(X,Y)$ is m, because $Z_{K^{\text{alg}}}(f)$ is defined over k, the degree of a general line in $(K^{\text{alg}})^2$ will intersect $Z_{K^{\text{alg}}}(f)$ at m points, which are not necessarily all in K^2. By Proposition A.2.29, a general line in K^2 will intersect $Z_{K^{\text{alg}}}(f)$ at m points too and one can make sure that all these points have distinct X-coordinates and distinct Y-coordinates respectively, because $m > 1$ by our assumption and two points on $Z_{K^{\text{alg}}}(f)$ have the same X-coordinates or the same Y-coordinates if and only if they are on the same horizontal or vertical line (so one can achieve this by picking slant lines). After picking the first line, avoiding all the finitely many points in $Z_{K^{\text{alg}}}(f)$ with the same X- or Y-coordinates (there are only finitely many of them, because $m > 1$ and f is absolutely irreducible), one can find a second general line intersecting $Z_{K^{\text{alg}}}(f)$ at m points which have distinct X- and Y-coordinates. One iterates this process until we get d general lines. The equations of these lines gives a curve of degree d, defined by the product of the equations of these lines, which will be our sample $g(X,Y)$. Note that this sample g will satisfy that $A_{d0} \neq 0$ and $A_{0d} \neq 0$. Because all the common zeros of f and g have distinct X- and Y-coordinates respectively, the corresponding $P(Y)$ and $Q(Y)$ would be polynomials with all roots being simple roots in K^{alg}. Because the degree of P and Q are at most md (this is a simple exercise about resultants), the leading coefficients of P and Q are not zero (because each of them has md distinct roots already). This means that $\operatorname{Res}(P(Y), P'(Y)) \neq 0$ and $\operatorname{Res}(Q(X), Q'(X)) \neq 0$.

We thus conclude our proof. □

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