AFFINE HJELMSLEV AND GENERALIZED AFFINE HJELMSLEV PLANES

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By

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ABSTRACT

The first chapter provides a discussion of "the" simplest affine Hjelmslev plane with non-trivial neighbour relation. In the second, we consider a geometry constructed over a locl ring and discuss the relationship between the A. H. ring properties and the A. H. plane axioms. In this way we introduce generalized A. H. planes - incidence structures with parallelism satisfying only the axioms induced by the local ring properties. In the remaining chapters, we coordinatize such a structure and give a proof of the Fundamental Theorem.

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TABLE OF CONTENTS

INTRODUCTION

An affine Hjelmslev plane (henceforth, called an A. H. plane) may be described as a geometry where more than one line may pass through two distinct points. Such a structure is usually defined by means of a neighbour relation and eight axioms. We begin this thesis with the construction of "the" simplest A. H. plane with non-trivial neighbour relation. (Simplest in the sense that it has the smallest number of points and lines.) We then discuss two non-isomorphic examples of simplest A. H. planes. The coordinate rings of these two geometries are special local rings called A. H. rings.

This leads, in a natural way, to the examination of an incidence structure with parallelism constructed over a local ring. Such a structure is shown to satisfy all but two of the axioms for an A. H. plane. We then discuss the relationship between the missing A. H. plane axioms and the missing A. H. ring properties.

The next section deals with an incidence structure with parallelism which satisfies only the A. H. plane axioms satisfied by an incidence structure constructed over a local ring. In the manner of Artin [1], we introduce two additional axioms and then coordinatize the structure by means of a local ring.

In the final section, we discuss automorphisms of the new structure and provide a proof of the fundamental theorem.

CHAPTER 1

Preliminary Definitions and Results

1.1. $<\mathbb{P},\mathbb{L},I,\|>$ is called an incidence structure with parallelism if and only if:

(a) $\mathbb{P}$ and $\mathbb{L}$ are sets.

(b) $I \subseteq \mathbb{P} \times \mathbb{L}$.

(c) $\| \subseteq \mathbb{L} \times \mathbb{L}$ is an equivalence relation (parallelism).

The elements of $\mathbb{P}$ are called points and are denoted by P, Q, R,... . The elements of $\mathbb{L}$ are lines and are denoted by l, m, n,... . $(P,l) \varepsilon I$ is written P I l and is read, "P is incident with l"; similarly, $(l,m) \varepsilon \|$ is written $l \| m$ and is read, "l is parallel to m". In addition, $l \wedge m = \{P \varepsilon \mathbb{P} \mid P\ I\ l, m\}$.

Two points, P and Q, are neighbours (written $P \sim Q$) if and only if there exist $l, m \varepsilon \mathbb{L}$, $l \neq m$ such that P, Q I l, m. Two lines, l and m, are neighbours (also written $l \sim m$) if and only if for any P I l, there exists a Q I m such that $P \sim Q$ and for any Q I m, there exists a P I l such that $Q \sim P$. The non-neighbouring relationship will be denoted by $\not\sim$.

An incidence structure with parallelism $\mathfrak{X} = <\mathbb{P},\mathbb{L},I,\|>$ is called an affine Hjelmslev plane (or an A. H. plane) if it satisfies the following axioms.

A1. For any $P, Q \in \mathbb{P}$, there exists $l \in \mathbb{L}$ such that $P, Q \; I \; l$. If $P \nsim Q$, we write $l = PQ$.

A2. There exist $P_1, P_2, P_3 \in \mathbb{P}$ such that $P_iP_j \nsim P_iP_k$ where $\{i, j, k\}$ is any permutation of $\{1, 2, 3\}$. $\{P_1, P_2, P_3\}$ is called a <u>triangle</u>.

A3. $\sim$ is transitive on $\mathbb{P}$.

A4. If $Q \; I \; l, m$, then $l \nsim m$ if and only if $\mathrm{card}\{P \; I \; l, m\}$ is one.

A5. If $l \nsim m$; $P, R \; I \; l$; $Q, R \; I \; m$ and $P \sim Q$, then $R \sim P, Q$.

A6. If $l \sim m$ and $n \nsim l$ with $P \; I \; l, n$ and $Q \; I \; m, n$, then $P \sim Q$.

A7. If $l \parallel m$; $P \; I \; l, n$ and $l \nsim n$, then $m \nsim n$ and there exists a point Q such that $Q \; I \; m, n$.

A8. For every $P \in \mathbb{P}$ and every $l \in \mathbb{L}$, there exists a unique line $L(P,l)$ such that $P \; I \; L(P,l)$ and $L(P,l) \parallel l$.

From A3 and the definition of the neighbour relation on $\mathbb{L}$, it is obvious that $\sim$ is transitive on $\mathbb{L}$ also.

<u>1.2. Lemma</u>. There exist three non-neighbouring lines through any point P.

<u>Proof</u>. By A2, there exists a triangle $\{P_1, P_2, P_3\}$. Then $L(P,P_1P_2)$, $L(P,P_2P_3)$, $L(P,P_1P_3)$ are three pairwise non-neighbouring lines through the point P.

<u>1.3. Lemma</u>. There are two non-neighbouring points on every line.

<u>Proof</u>. Consider any line 1. By A3 and A7, $L(P_i, 1) \not\sim P_i P_j$, $P_i P_k$, for some permutation (i, j, k) of (1, 2, 3). By A7, $1 \not\sim P_i P_j$, $P_i P_k$ and there exist points Q and R such that $Q = 1 \wedge P_i P_j$ and $R = 1 \wedge P_i P_k$.

If $Q \not\sim R$, then we are finished. Suppose that $Q \sim R$. By A5, $P_i \sim Q$, R. Clearly, Q, $R \not\sim P_j$, P_k. By A7, $L(P_k, 1) \not\sim P_i P_j$ and there exists a point $S = L(P_k, 1) \wedge P_i P_j$. By A5, $S \not\sim P_i$; hence, $S \not\sim Q$, R also. Again by A7, $L(S, P_i P_k) \not\sim 1$ and by A5, $Q \not\sim L(S, P_i P_k) \wedge 1$.

This implies for any point P on a line 1, there exists a point Q such that Q I 1 and $Q \not\sim P$; otherwise, all points on 1 would be neighbours.

<u>1.4. Lemma</u>. Let 1 and m be two parallel lines. If there exist P I 1 and Q I m such that $P \sim Q$, then $1 \sim m$.

<u>Proof</u>. Assume there exist points P and Q as defined in the lemma. By 1.3, there exists R I 1 such that $R \not\sim P$. Then $R \not\sim Q$ and $RQ \sim 1$ by A5. Then, by A7, $m \sim QR$ and by transitivity $1 \sim m$.

<u>1.5. Lemma</u>. Let $\mathfrak{X}$ be an A. H. plane with non-trivial neighbour relation. Then for any point P, there exists a point Q with $Q \sim P$, $Q \neq P$.

<u>Proof</u>. Choose any point P. If the neighbour relation is non-trivial,

then there exists a pair of neighbouring points R and S. If $P \sim R, S$, then we have the required point. Therefore, we may assume that $P \not\sim R, S$. There exists a line l such that $P \; I \; l$ and $l \not\sim PR$. By A7 and A4, $L(S,PR) \not\sim l$ and there exists a unique point $T = L(S,PR) \wedge l$. By 1.4, $L(S,PR) \sim PR$. By A6, $P \sim T$.

CHAPTER 2

The Simplest A. H. Plane with Non-trivial Neighbour Relation

2.1. In this chapter, we shall construct an A. H. plane with a non-trivial neighbour relation containing the minimum number of points and lines.

By definition, every A. H. plane must contain a triangle $\{A, B, C\}$. By A8, there exist L(A, BC), L(B, AC) and L(C, AB); A7 implies the existence of a fourth point $D = L(A, BC) \wedge L(C, AB)$; $D \not\sim A, B, C$ by A5. By A6 again, two possibilities exist: either $AC \wedge BD = \emptyset$ (Case 1) or $AC \wedge BD = E$, for some $E \in \mathbb{P}$ (Case 2). In either event, if the neighbour relation is non-trivial, it is possible to find a point F which is a neighbour of A and $F \neq A$ (by 1.5).

Obviously, the lines AB, AC, AD, BC, BD, CD are pairwise not neighbours; hence, F can be incident with at most one of these six lines. (We have already defined all the points of intersection of pairs of these lines. By definition, $F \neq A, B, C$; since $D \not\sim A$, $F \neq D$; by A5, $F \neq E$.) Further, if $X, Y \in \{B, C, D\}$ and $X \neq Y$ then $F \not I XY$; otherwise, $AX \not\sim XY$ implies, by A5, that $A \sim X$ which is a contradiction. Therefore, F is incident with at most one of AB, AC, AD and $F \not I BC, BD, CD$. Without loss of generality, we may assume $F \not I AC, AD$. There exists $L(F, AD) \not\sim AC, CD, BD$ and

$$G = L(F, AD) \wedge AC$$

$$H = L(F, AD) \wedge CD$$

$$J = L(F, AD) \wedge BD.$$

Clearly, these points are distinct from A, B, C, D, F, $L(F, AD) \wedge AB$. If H = G, then H = G = C and L(F, AD) = L(H, AD) = L(C, AD) = BC which implies F I BC; a contradiction. A similar argument may be used to show $J \neq H$.

<u>Case 1</u>: $AC \wedge BD = \emptyset$ (cf. Figures 2.1 and 2.2).

If G = J, G = $AC \wedge BD$ which would, of course, be a contradiction. Consider, now, the distinct lines L(J, AB), L(G, AB) which give rise to the points

$$K = L(J, AB) \wedge BC$$

$$M = L(J, AB) \wedge AD$$

$$N = L(G, AB) \wedge BC$$

$$P = L(G, AB) \wedge AD.$$

J $\not{I}$ AB, CD, L(G, AB) and G $\not{I}$ AB, CD, L(J, AB) imply M, K $\neq$ A, B, C, D, H, G, N, P, $L(F, AD) \wedge AB$ and N, P $\neq$ A, B, C, D, H, J, $L(F, AD) \wedge AB$. Also, F $\not{I}$ L(J, AB), L(G, AB) imply F $\neq$ M, K, N, P. Clearly, $M \neq K \neq J \neq M$ and $N \neq P \neq G \neq N$. Let Q = $L(H, BD) \wedge AB$. Since Q I AB, Q $\not{I}$ CD, L(J, AB), L(G, AB); hence, Q $\neq$ C, D, H, N, P, G, M, K, J.

If Q = F or Q = $L(F, AD) \wedge AB$, then Q = F = H or Q = $L(F, AD) \wedge AB$ = H, both of which are contradictions.

If Q = B, L(Q, BD) = BD which implies H I BD; however, H I CD. Hence, H = D; a contradiction.

Finally, if Q = A, then $Q \sim F$. By A7, $L(F, AD) \not\sim L(H, BD)$ and

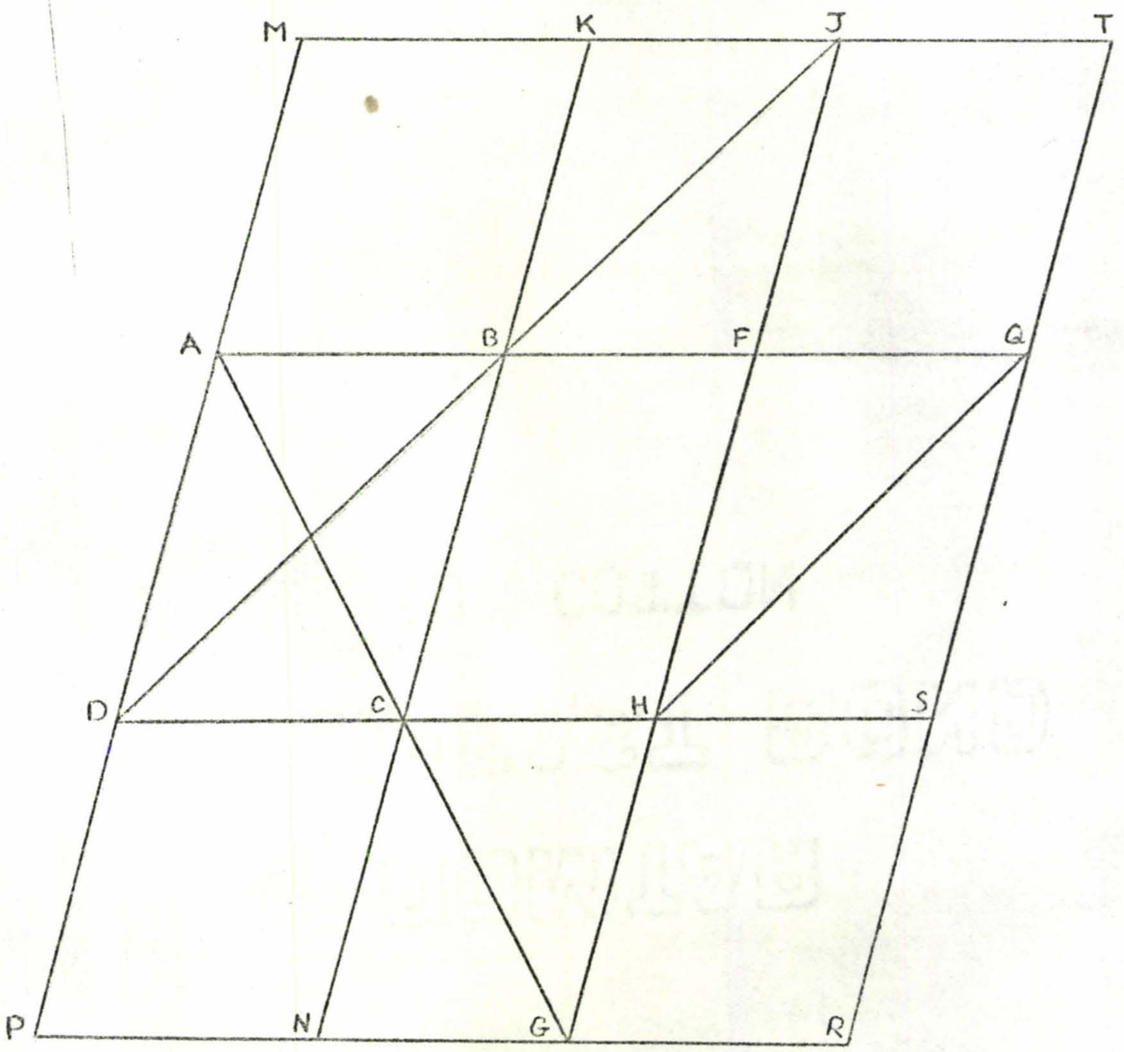

Figure 2.1.

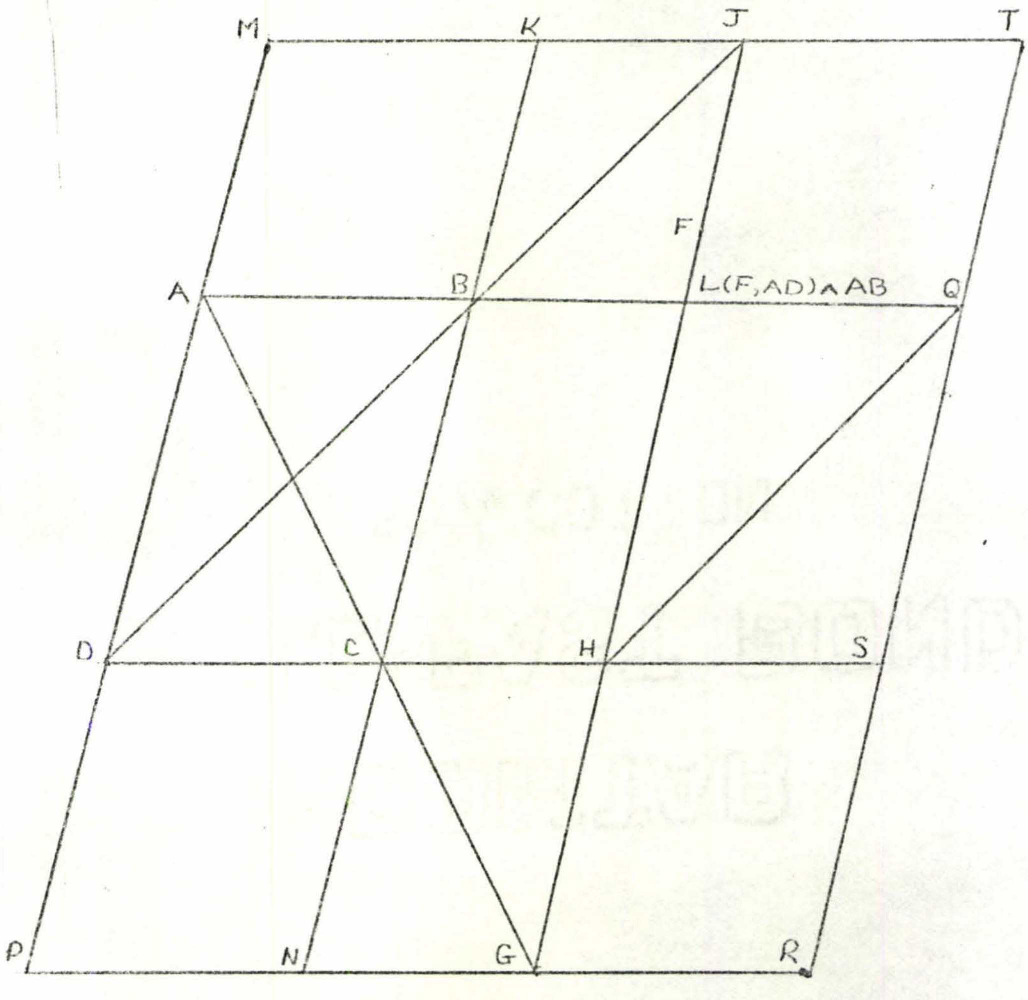

Figure 2.2.

so by A5 we would have $Q \sim H$. Since $L(H, BD) \not\sim CD$, this would imply $AB \sim CD$; a contradiction.

Now consider the line through Q parallel to AD. Since $Q \neq A, B, L(F, AD) \wedge AB$, $L(Q, AD) \neq AD, BC, L(F, AD)$ and we can define the points

$$R = L(Q, AD) \wedge L(G, AB)$$

$$S = L(Q, AD) \wedge CD$$

$$T = L(Q, AD) \wedge L(J, AB)$$

which are distinct from all the previously defined points. (All the previously defined points, with the exception of Q, are incident with one of the lines AD, BC, L(F, AD).) Further, they are mutually distinct as the lines AB, CD, L(J, AB) and L(G, AB) are distinct.

Therefore, the A. H. plane with F I AB must have at least sixteen points and if $F \not\mathrm{I}\ AB$ the A. H. plane so defined must have at least seventeen points.

<u>Case 2</u>: $AC \wedge BD = E$ (cf. Figure 2.3).

If $A \sim E$, $AD \not\sim BD$ would imply that $A \sim D$; a contradiction. Further, if $E \sim X$, for any X I AD, $AD \not\sim AC$ would imply $A \sim E$. Clearly, if $J = G$, then $J = G = E$ which gives $L(E, AD) = L(G, AD) = L(F, AD)$; however, $L(F, AD) \sim AD$ and so there exists X I AD such that $E \sim X$. Thus, $J \neq G$.

Now consider $L(J, AB) \neq AB, CD$ and $L(G, AB) \neq AB, CD, L(J, AB)$. By A7, we have the following points:

$$K = L(J, AB) \wedge BC$$

$$M = L(J, AB) \wedge AD$$

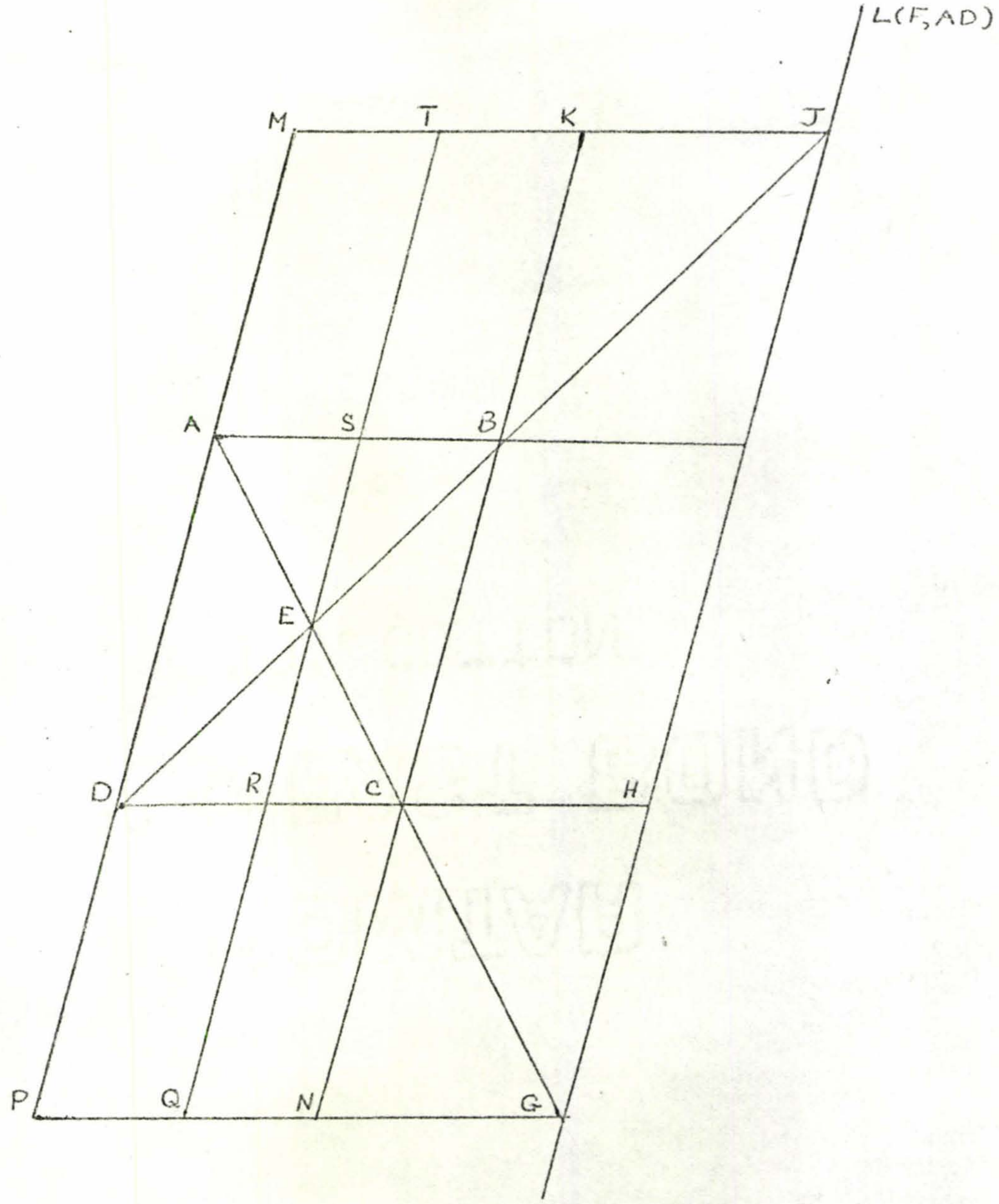

Figure 2.3.

$N = L(G, AB) \wedge BC$

$P = L(G, AB) \wedge AD.$

Clearly, these points are mutually distinct and differ from all the previously defined points including $L(F, AD) \wedge AB$ (as in Case 1).

Finally, let

$Q = L(E, AD) \wedge L(G, AB)$

$R = L(E, AD) \wedge CD$

$S = L(E, AD) \wedge AB$

$T = L(E, AD) \wedge L(J, AB).$

Obviously, $L(E, AD) \neq AD, BC, L(F, AD)$ and as we have already noted $L(J, AB)$, AB, CD, $L(G, AB)$ all differ. Therefore, these four points are different from any of the previously defined points and are mutually distinct.

Therefore, such a geometry contains at least seventeen points if F I AB and at least eighteen points if F $\not{I}$ AB.

We have determined that an A. H. plane with a non-trivial neighbour relation must contain at least sixteen points. If we examine the structure illustrated in Figure 2.1, we can obtain the following equivalence classes of points determined by the neighbour relation:

$\{P, A, F, G\}$

$\{D, M, J, H\}$

$\{C, K, S, T\}$

$\{N, R, B, Q\}.$

If this structure is an A. H. plane, then all lines must contain the same number of points (cf. [12]). Since the lines AD, BC, L(F, AD) and L(Q, AD) are distinct and parallel, they each contain four points and hence, all lines would contain four points. (cf. [12], 2.10).

2.2. Claim: Any three pairwise non-neighbouring points are not collinear, if the plane has only sixteen points.

Clearly, three pairwise non-neighbouring points are not collinear with any line parallel to AD or parallel to AB. Without loss of generality, we can consider the line AK. There must exist two more points incident with the line AK; however, since AK $\nsim$ AD, BC, AB, MK, these two points cannot be incident with these other lines; thus, the two points must be selected from the four points H, S, G, R. In this case, the four points are pairwise not neighbours and we can see immediately that AK $\neq$ HG, SR, HS, GR. If AK = HR, then by A5, AK $\nsim$ QS and hence K $\sim$ T implies T $\sim$ R; a contradiction. Therefore, AK = GS, but A $\sim$ G and K $\sim$ S; thus, we have only two non-neighbouring points on a line.

2.3. Claim: Three distinct neighbouring points are not collinear, if the plane has only sixteen points.

Assume there exist three neighbouring collinear points. Without loss of generality, consider A, F, G I l (some l ε L). Then A, F I l, AB imply by A4 that l $\sim$ AB. Similarly, A,G I l, AC imply l $\sim$ AC. Thus, by transitivity, AB $\sim$ AC; a contradiction.

2.4. These facts enable us to discover the number of lines in such a structure. Each line is uniquely determined by two non-neighbouring points and each of our lines contains four such pairs of points. Using combinatoric methods, we find that there are:

$$\binom{16}{2} = 120 \text{ pairs of points}$$

$$4 \cdot \binom{4}{2} = 24 \text{ pairs of neighbouring points.}$$

Hence, there are 96 pairs of non-neighbouring points. From these 96 pairs, we obtain $\frac{96}{4} = 24$ distinct lines.

Consequently, if there exists an A. H. plane with non-trivial neighbour relation which has exactly sixteen points, it must have twenty-four lines.

2.5. We shall now examine a particular incidence structure. Define $H = \mathbb{Z} \pmod 4$. Consider an incidence structure with parallelism $\mathfrak{X} = \langle \mathbb{P}, \mathbb{L}, I, \parallel \rangle$ where

$$\mathbb{P} = \{(x, y) \mid x, y \in H\},$$

$$\mathbb{L} = \mathbb{L}_1 \cup \mathbb{L}_2, \text{ with}$$

$$\mathbb{L}_1 = \{[a, b]_1 \mid [a, b]_1 = \{(ya + b, y) \quad y \in H\};\ b \in H,\ a = 0, 2\};$$

$$\mathbb{L}_2 = \{[a, b]_2 \mid [a, b]_2 = \{(x, xa + b) \quad x \in H\};\ a, b \in H\}.$$

I is set inclusion.

Elements of $\mathbb{L}_1$ ($\mathbb{L}_2$) are called <u>lines of the first</u> (<u>second</u>) <u>kind</u>. $[a, b]_i$ are called the <u>coordinates of the line</u>. For any $l = [a, b]_i$ and $m = [c, d]_j$, $l \parallel m$ if and only if $i = j$ and $a = c$.

In addition, we define an equivalence relation $\sim$ on P by

$(a, b) \sim (c, d)$ if and only if $a - c, b - d = 0, 2$. Two lines are defined to be in the relation $\sim$ if for each point P incident with either line, there exists a point Q incident with the other line such that $P \sim Q$.

We shall show that this structure is an A. H. plane with the above equivalence relation as the neighbour relation. Further, as this structure contains only sixteen points and twenty-four lines, it must be "the" simplest A. H. plane with a non-trivial neighbour relation.

From the definitions we obtain the following equivalence classes induced by the $\sim$ relation on $\mathbb{P}$:

$$\{(0, 0), (0, 2), (2, 0), (2, 2)\}$$
$$\{(0, 1), (0, 3), (2, 1), (2, 3)\}$$
$$\{(1, 0), (1, 2), (3, 0), (3, 2)\}$$
$$\{(1, 1), (1, 3), (3, 1), (3, 3)\}$$

and the following equivalence classes induced by the $\sim$ relation on $\mathbb{L}$:

$$\{[0, 0]_2, [0, 2]_2, [2, 0]_2, [2, 2]_2\}$$
$$\{[0, 1]_2, [0, 3]_2, [2, 1]_2, [2, 3]_2\}$$
$$\{[1, 0]_2, [1, 2]_2, [3, 0]_2, [3, 2]_2\}$$
$$\{[1, 1]_2, [1, 3]_2, [3, 1]_2, [3, 3]_2\}$$
$$\{[0, 0]_1, [0, 2]_1, [2, 0]_1, [2, 2]_1\}$$
$$\{[0, 1]_1, [0, 3]_1, [2, 1]_1, [2, 3]_1\}.$$

From the equivalence classes of the lines, we can see immediately that for any $[m, n]_i$, $[p, q]_j \varepsilon$ L, $[m, n]_i \sim [p, q]_j$ if and only if $i = j$ and $m - p, n - q = 0, 2$. We can make one additional observation: two lines of the first kind do not meet unless they are in the $\sim$ relation.

Next, we prove that the eight axioms of an A. H. plane hold in $\mathfrak{X}$.

<u>A1</u>. <u>For any</u> P, Q ε $\mathbb{P}$, <u>there exists</u> l ε $\mathbb{L}$ <u>such that</u> P, Q I l.

Consider any (a, b), (c, d) ε $\mathbb{P}$. Clearly, if $a = c$, then (a, b), (c, d) I $[0, a]_1$; if $b = d$, then (a, b), (c, d) I $[0, b]_2$. Therefore, assume $a \neq c$, $b \neq d$. If $(a - c) \neq 0, 2$, then $(a - c)^{-1}$ exists and $(a - c) = (a - c)^{-1}$. Hence, (a, b), (c, d) I $[(b - d)(a - c), b - a(b - d)(a - c)]_2$. Similarly, if $(a - c) = 2$ and $(b - d) \neq 0, 2$, then (a, b), (c, d) I $[(a - c)(b - d), a - (a - c)(b - d)b]_1$. Finally, if $(a - c), (b - d) = 2$, then (a, b), (c, d) I $[1, a - b]_2$.

<u>A2</u>. <u>There exists a triangle</u>.

Consider three points (0, 0), (1, 0), (0, 1). It is readily apparent that (0, 0), (1, 0) I $[0, 0]_2$; (0, 0), (0, 1) I $[0, 0]_1$; (1, 0), (0, 1) I $[3, 1]_2$. Further, since all points incident with $[0, 0]_1$ are of the form (0, b) for some b ε H and $(1, 0) \not\sim (0, b)$, we have $[0, 0]_1 \not\sim [0, 0]_2$, $[3, 1]_2$; similarly, all points incident with $[0, 0]_2$ are of the form (a, 0) and $(a, 0) \not\sim (0, 1)$; hence, $[0, 0]_2 \not\sim [3, 1]_2$.

<u>A3</u>. $\sim$ <u>is transitive on</u> $\mathbb{P}$.

This follows from the fact that $\sim$ is an equivalence relation.

<u>A4</u>. *If* P I l, m, *then* $l \nsim m$ *if and only if* card$\{P \in \mathbb{P} \mid P \text{ I } l, m\} = 1$.

We need only consider two cases: l and m are both of the second kind and l and m are of different kinds.

Let $l = [n, p]_2$, $m = [q, r]_2$, where $l \nsim m$ and (a, b) I l, m. Since $b = an + p$ and $b = aq + r$, we have $an + p = aq + r$ and $a(n - q) = r - p$. If $n - q = 0, 2$, then $r - p = 0, 2$ and $[n, p]_2 \sim [q, r]_2$. Thus, $n - q \neq 0, 2$. For any (c, d) I l, m, we have $cn + p = d = cq + r$; hence, $c(n - q) = r - p = a(n - q)$. Since $n - q \neq 0, 2$, $(n - q)^{-1}$ exists and is equal to $n - q$. Thus, (a, c) = (c, d).

Finally, if the two lines are of different kinds, say $[n, p]_1 = l$, $[q, r]_2 = m$ and (a, b), (c, d) I l, m, then $a - bn = c - dn$ and $b - aq = d - cq$. Hence, $b = d - cq + aq$ and $a - bn = c - dn$. This implies $a(1 - qn) = c(1 - qn)$. However, since $[n, p]_1 \in \mathbb{L}_1$, $n = 0, 2$; hence, $nq = 0, 2$ and $1 - nq \neq 0, 2$. Therefore, $(1 - nq)^{-1} = 1 - nq$ and so (a, b) = (c, d).

Conversely, consider any pair of neighbouring lines (without loss of generality, let the pair be $[m, n]_2$, $[p, q]_2$) which pass through a point (a, b). From earlier results, $m - p = 0, 2$ and $n - q = 0, 2$. Therefore, we have $b = am + n$ and $b = ap + q$ which imply $am + n = ap + q$ and $a(m - p) = q - n$.

If $m - p = 0$, then $q - n = 0$ and the two lines are equal. If $m - p = 2$ and $q - n = 0, 2$, then since

$$(a + 2)m + n = am + 2m + n = b + 2m;$$

$$(a + 2)p + q = ap + 2p + q = b + 2(p - m) + 2m = b + 2m,$$

we have $(a + 2, b + 2m)$ is incident with $[m, n]_2$ and $[p, q]_2$, also. Hence, if neighbouring lines meet then they do so in more than one point.

A5. If $l \not\sim m$; $P, R \ I \ l$; $Q, R \ I \ m$ and $P \sim Q$, then $R \sim P, Q$.

Consider, first $[m, n]_1 \not\sim [p, q]_2$ with $(a, b) \ I \ [m, n]_1, [p, q]_2$; $(c, d) \ I \ [m, n]_1$; $(e, f) \ I \ [p, q]_2$ and $(c, d) \sim (e, f)$. Therefore,

$$a - c = (b - d)m \quad \text{and} \quad b - f = (a - e)p.$$

However, $[m, n]_1 \in L_1$, so $m = 0, 2$ which implies $a - c = 0, 2$. Also, since $(c, d) \sim (e, f)$, $c - e, f - d = 0, 2$ and hence, $a - e = 0, 2$. This in turn implies that $b - f = (a - e)p = 0, 2$ and $b - d = 0, 2$. Thus, $(a, b) \sim (c, d), (e, f)$.

By an earlier remark, two non-neighbouring lines of the same kind that intersect must be lines of the second kind. Consider $[m, n]_2 \not\sim [p, q]_2$ with $(a, b), (c, d) \ I \ [m, n]_2$ and $(a, b), (e, f) \ I \ [p, q]_2$ where $(c, d) \sim (e, f)$. As before, $m - p \neq 0, 2$. Clearly, $(c, d) \sim (e, f)$ implies that $d - f = 0, 2$. However,

$$\begin{aligned} d - f &= (cm + n) - (ep + q) \\ &= cm + (b - am) - ep - (b - ap) \\ &= cm - am - ep + ap \\ &= (c - a)m + (a - e)p. \end{aligned}$$

Now, if $m = 0, 2$, $(c - a)m = 0, 2$ and $(a - e)p$ must equal 0 or 2. However, $m = 0, 2$ and $m - p \neq 0, 2$ imply $p \neq 0, 2$; hence,

$a - e = 0, 2$ and $a - c = a - e + e - c = 0, 2$. If $m \neq 0, 2$, then $p = 0, 2$ and by the same method as above we obtain $a - e, a - c = 0, 2$. In addition, $b - d = m(a - c)$ and $b - f = p(a - e)$ which imply $b - d, b - f = 0, 2$.

A6. *If* $l \sim m$ *and* $n \nsim l$ *with* $P \, I \, l, n$ *and* $Q \, I \, m, n$, *then* $P \sim Q$.

Again, we have two cases: all three lines are of the second kind or l and m are of one kind and n is of the other.

Let $[m, n]_2 \sim [p, q]_2$ and $[m, n]_2 \nsim [r, s]_2$. By the transitivity of the neighbour relation on the points, $[p, q]_2 \nsim [r, s]_2$. This implies that $m - p = 0, 2$; $n - q = 0, 2$; $m - r \neq 0, 2$ and $p - r \neq 0, 2$. Let $(a, b) \, I \, [m, n]_2, [r, s]_2$ and $(c, d) \, I \, [p, q]_2, [r, s]_2$. Clearly, since $(a, b), (c, d) \, I \, [r, s]_2$, we have $b - d = ar - cr$. If $r = 0, 2$, then $b - d = 0, 2$ which implies

$$\begin{aligned} 0, 2 &= b - d \\ &= (am + n) - (cp + q) \\ &= am + n - cp - q \\ &= am - cp + n - q. \end{aligned}$$

But $n - q = 0, 2$; hence, $am - cp = 0, 2$. This implies

$$\begin{aligned} 0, 2 &= am - ap + ap - cp \\ &= a(m - p) + (a - c)p. \end{aligned}$$

However, $m - p = 0, 2$ and so $a(m - p) = 0, 2$. Further, since we assumed $r = 0, 2$, $p \neq 0, 2$ and thus, $a - c = 0, 2$.

If $r \neq 0, 2$, then $r = r^{-1}$ and $m, p = 0, 2$. Therefore, $b - d = (a - c)r$ implies

$$a - c = (b - d)r$$
$$= (am + n - cp - q)r$$
$$= (n - q)r + (am - cp)r$$
$$= 0, 2.$$

Thus, $b - d = 0, 2$ also.

In the other case, we may, without loss of generality, let $[m, n]_2 \sim [p, q]_2$ and $[m, n]_2 \not\sim [r, s]_1$, where $(a, b)\ I\ [m, n]_2, [r, s]_1$ and $(c, d)\ I\ [p, q]_2, [r, s]_1$. Therefore, $a - c = (b - d)r$.

If $r = 0, 2$, then $a - c = 0, 2$ and

$$b - d = am + n - (cp + q)$$
$$= (br + s)m + n - (dr + s)p - q$$
$$= brm + sm + n - drp - sp - q$$
$$= r(bm - dp) + s(m - p) + n - q$$
$$= 0, 2.$$

If $r \neq 0, 2$, then $m, p = 0, 2$ and

$$b - d = (am + n) - (cp + q)$$
$$= am - cp + (n - q)$$
$$= 0, 2.$$

Hence, $a - c = (b - d)r = 0, 2$ also.

A7. <u>If</u> $l \parallel m$; $P\ I\ l, n$ <u>and</u> $l \not\sim n$, <u>then</u> $m \not\sim n$ <u>and there exists</u> $Q\ I\ m, n$.

Once again, we have two possibilities: all three lines may be of the second kind or the two parallel lines may be of one kind and the third line of the other kind.

First of all, consider $[m, n]_2 \parallel [m, p]_2$, $[m, n]_2 \nparallel [q, r]_2$ and (a, b) I $[m, n]_2$, $[q, r]_2$. By an earlier result, $m - q \neq 0, 2$ (since $[m, n]_2 \nparallel [q, r]_2$) and thus $[m, p]_2 \nparallel [q, r]_2$ also. Since (a, b) I $[m, n]_2$, $[q, r]_2$, we have $am + n = b = aq + r$. Hence,
$a = (r - n)(m - q)$ and

$$(r - n)(m - q)m + n = b = (r - n)(m - q)q + r.$$

Take the point $((r - p)(m - q), (r - p)(m - q)m + p)$. It is clearly incident with the line $[m, p]_2$. However,

$$\begin{aligned} &(r - p)(m - q)m + p \\ &= (r - n + n - p)(m - q)m + p + n - n \\ &= (r - n)(m - q)m + n + (n - p)(m - q)m + p - n \\ &= (r - n)(m - q)q + r + (n - p)(m - q)m + p - n \\ &= (r - n)(m - q)q + (n - p)(m - q)(m - q + q) + r + p - n \\ &= (r - p)(m - q)q + (n - p) + r + p - n \\ &= (r - p)(m - q)q + r. \end{aligned}$$

Thus, $((r - p)(m - q), (r - p)(m - q)m + p)$ I $[q, r]_2$.

Next, without loss of generality, consider $[m, n]_2 \parallel [m, p]_2$, $[m, n]_2 \nparallel [q, r]_1$ and (a, b) I $[m, n]_2$, $[q, r]_1$. Clearly, $[m, p]_2 \nparallel [q, r]_1$. Since (a, b) I $[m, n]_2$, $[q, r]_1$, we have $b - n = am = bqm + rm$ and $b(1 - qm) = rm + n$. However, $q = 0,2$ and so $(1 - qm) \neq 0, 2$. Thus, $b = (rm + n)(1 - qm)$ implies $a = (re + n)(1 - qm)q + r$ and

$$(rm + n)(1 - qm) = b = ((rm + n)(1 - qm)q + r)m + n.$$

Now, consider the point $((rm + p)(1 - qm)q + r, (rm + p)(1 - qm))$ I $[q, r]_1$. In addition,

$$((rm + p)(1 - qm)q + r)m + p$$
$$= ((rm + n)(1 - qm)q + r)m + n + (p - n)(1 - qm)qm + p - n$$
$$= (rm + n)(1 - qm) + (p - n)((1 - qm)qm + 1)$$
$$= (rm + n)(1 - qm) + (p - n)(qm - (qm)^2 + 1)$$
$$= (rm + n)(1 - qm) + (p - n)(1 - qm)$$
$$= (rm + p)(1 - qm).$$

Thus, $((rm + p)(1 - qm)q + r, (rm + p)(1 - qm))\ I\ [m, p]_2$.

A8. For every $l \varepsilon \mathbb{L}$ and every $P \varepsilon \mathbb{P}$, there exists $L(P,l) \varepsilon \mathbb{L}$ such that $L(P,l) \parallel l$ and $P\ I\ L(P,l)$.

Consider any $(a, b) \varepsilon \mathbb{P}$ and $[m, n]_1, [p, q]_2 \varepsilon \mathbb{L}$. Then $(a, b)\ I\ [m, a - bm]_1, [p, b - ap]_2$ and $[m, a - bm]_1 \quad [m, n]_1$; $[p, b - ap]_2 \quad [p, q]_2$.

2.6. By the properties of H, it is also clear that $\mathfrak{X}$ is a Desarguesian A. H. plane (cf. [7], 4.5). Since multiplication is commutative, $\mathfrak{X}$ is also Pappian.

2.7. It is interesting to note that not all A. H. planes with sixteen points and twenty-four lines are isomorphic. Consider the set $J = \mathbb{Z}_2[x] / (x^2)$, where $\mathbb{Z}_2[x]$ is the set of polynomials over the integers modulo 2. Thus, $J = \{0, 1, x, 1 + x\}$. Let $\mathfrak{X}_J = < \mathbb{P}', \mathbb{L}', I', \parallel >$, where

$$\mathbb{P}' = J \times J$$

$$\mathbb{L}' = \mathbb{L}'_1 \cup \mathbb{L}'_2, \text{ with}$$

$$\mathbb{L}'_1 = \{[a, b]_1 = \{(ya + b, y) \mid y \in J\} \mid a \in \{0, x\}, b \in J\};$$

$$\mathbb{L}'_2 = \{[a, b]_2 = \{(z, za + b) \mid z \in J\} \mid a, b \in J\}.$$

I' is set inclusion

$l \parallel m$ if and only if l and m are of the same kind with the same first coordinate.

Let $g : H \longrightarrow J$, where $g(0) = 0$, $g(1) = 1$, $g(2) = x$, $g(3) = 1 + x$. The map $f = (\varphi, \psi) : \mathcal{X} \longrightarrow \mathcal{X}_J$, where

$$\varphi : \mathbb{P} \longrightarrow \mathbb{P}'$$

$$(a, b) \rightsquigarrow (g(a), g(b)0$$

$$\psi : \mathbb{L} \longrightarrow \mathbb{L}'$$

$$[a, b]_i \rightsquigarrow [g(a + 2), g(b)]_i \text{ , if } a, b \in \{1, 3\}$$

$$[g(a), g(b)]_i \text{ , otherwise,}$$

is an I-isomorphism, but $\psi([1, 0]_2) = [1, 0]_2$ and $\psi([1, 1]_2) = [1 + x, 1]_2$ with $[1, 0]_2 \nparallel [1 + x, 1]_2$. Hence, f is not an isomorphism. In fact, this implies that no isomorphism between these two A. H. planes exists. (cf. [9], 3.1).

CHAPTER 3

Incidence Structures over Local Rings

3.1. In the last chapter, we constructed an A. H. plane. It is readily apparent that its coordinate ring, H, is an A. H. ring. It is not surprising that the incidence structure $\mathfrak{X}$ that we constructed over H is an A. H. plane, since in J. W. Lorimer and N. D. Lane's paper, "Desarguesian Affine Hjelmslev Planes", it is shown that all incidence structures constructed over A. H. rings are A. H. planes. We shall now examine the consequences of weakening the conditions on the coordinate ring by starting, instead, with just a local ring.

3.2. Theorem. If L is a ring with $0 \neq 1$, then the following are equivalent (cf. [5]):

1) L / Rad L is a division ring.
2) L has exactly one maximal ideal.
3) All non-units of L are contained in a proper ideal.
4) All non-units of L form a proper ideal η.
5) For all $a \in L$, either a or $1 - a$ is a unit.
6) For all $a \in L$, either a or $1 - a$ is right invertible.

3.3. A ring, L, with $0 \neq 1$ is called local if it satisfies one of the equivalent statements of Theorem 3.2.

An <u>A. H. ring</u> is a local ring, L, with two additional conditions:

1) $\eta = D_0$ (where D_0 is the set of divisors of zero plus 0 itself).

2) If a, b ε L, then a ε bL or b ε aL.

<u>3.4.</u> Let $\mathcal{A} = < \mathbb{P}, \mathbb{L}, I, \| >$ be an incidence structure with parallelism where

$$\mathbb{P} = L \times L, \text{ L a local ring;}$$

$$\mathbb{L} = \mathbb{L}_1 \cup \mathbb{L}_2 \text{ , with}$$

$$\mathbb{L}_1 = \{[m, n]_1 = \{(x, y) \in P \mid x = ym + n\} \quad m \in \eta,\ n \in L\};$$

$$\mathbb{L}_2 = \{[m, n]_2 = \{(x, y) \in P \mid y = xm + n\} \quad m, n \in L\};$$

I is set inclusion;

$[m, n]_i \| [p, q]_j$ if and only if $i = j$ and $m = p$.

We also define a <u>neighbour relation</u> on $\mathbb{P}$ by: $(a, b) \sim (c, d)$ if and only if $a - c,\ b - d \in \eta$. Two lines are defined to be neighbours if for any point on either line there exists a neighbouring point on the other line. We denote both of these neighbour relations by $\sim$.

<u>3.5. Lemma.</u> If $(a, b) \not\sim (c, d)$ and $(a, b), (c, d)$ I l, then $a - c \in \eta$ if and only if $l \in \mathbb{L}_1$.

<u>Proof.</u> Assume $(a, b), (c, d)$ I $[m, n]_1$. Then $a = bm + n$ and $c = dm + n$ imply $a - n = bm \in \eta$ and $c - n = dm \in \eta$. Hence, $a - c = a - n - c + n \in \eta$.

Next, consider two points $(a, b) \not\sim (c, d)$ with $a - c \in \eta$ and

$(a, b), (c, d)\ I\ l$, If $l = [m, n]_2$, for some $m, n \varepsilon L$, then $b - d = (a - c)m$. However, $b - d \not\varepsilon \eta$ and $(a - c)m \varepsilon \eta$; a contradiction. Since

$$b(b - d)^{-1}(a - c) - b(b - d)^{-1}(a - c) + a = a;$$

$$\begin{aligned} &d(b - d)^{-1}(a - c) - b(b - d)^{-1}(a - c) + a \\ &= -(b - d)(b - d)^{-1}(a - c) + a \\ &= -a + c + a \\ &= c, \end{aligned}$$

we have $(a, b), (c, d)\ I\ [(b - d)^{-1}(a - c), a - b(b - d)^{-1}(a - c)]_1$.

3.6. Lemma. For any point P incident with some line l, there exists a point Q also on l such that $Q \not\sim P$.

Proof. Since $1 \cdot 1 = 1$ and $a \cdot 0 = 0 \cdot a = 0$, for all $a \varepsilon L$, we have $1 \not\varepsilon \eta$ and $0 \varepsilon \eta$. Consider any $[m, n]_1$. Then $1 \cdot m + n = m + n$ and $0 \cdot m + n = n$ which imply $(m + n, 1), (n, 0)\ I\ [m, n]_1$. Now, for any $(a, b)\ I\ [m, n]_1$, either $b \varepsilon \eta$ or $b \not\varepsilon \eta$. In the first case, $1 - b \not\varepsilon \eta$, so $(a, b), (m + n, 1)\ I\ [m, n]_1$ and $(a, b) \not\sim (m + n, 1)$. If $b \not\varepsilon \eta$, then $b - 0 = b \not\varepsilon \eta$ and $(a, b), (n, 0)\ I\ [m, n]_1$ with $(a, b) \not\sim (n, 0)$.

Similarly, for any point P on a line of the second kind, there is a point which is not a neighbour of P but is incident with the same line.

3.7. Lemma. Two lines of different kinds are not neighbours.

Proof. Consider any two lines $[m, n]_1$, $[p, q]_2$, where $n, p, q \in L$ and $m \in \eta$. For any point $(a, b)\ I\ [p, q]_2$, there exists a point $(c, d)\ I\ [p, q]_2$ such that $(a, b) \not\sim (c, d)$. Hence, by 3.5, $a - c \notin \eta$.

Now assume $[m, n]_1 \sim [p, q]_2$. This implies that every point of $[p, q]_2$ is the neighbour of some point on $[m, n]_1$. In particular, there must exist points (e, f) and $(g, h)\ I\ [m, n]_1$ with $(e, f) \sim (a, b)$ and $(g, h) \sim (c, d)$. Clearly, $(e, f) \not\sim (g, h)$; however, by 3.5, $e - g \in \eta$ and by definition, $e - a \in \eta$ and $g - c \in \eta$. Thus, $a - c = (a - e) + (e - g) + (g - c) \in \eta$; a contradiction.

3.8. Lemma. Two lines of the same kind are neighbours if and only if their corresponding coordinates differ by a non-unit.

Proof. Consider two lines $[m, n]_2$ and $[p, q]_2$. Assume $m - p \in \eta$ and $n - q \in \eta$. If $(a, b)\ I\ [m, n]_2$, then $(a, ap + q)\ I\ [p, q]_2$ and

$$\begin{aligned} b - (ap + q) &= am + n - ap - q \\ &= a(m - p) + (n - q) \in \eta. \end{aligned}$$

Thus, $(a, b) \sim (a, ap + q)$. Similarly, we can find a neighbour of any point $(c, d)\ I\ [p, q]_2$, incident with $[m, n]_2$.

Next, consider any two non-neighbouring points (a, b) and (c, d) on $[m, n]_2$, where $[m, n]_2 \sim [p, q]_2$. Then there exist (e, f), $(g, h)\ I\ [p, q]_2$ with $(a, b) \sim (e, f)$ and $(c, d) \sim (g, h)$. Now, $(a, b) \sim (e, f)$ implies $b - f \in \eta$. Hence,

$$\begin{aligned} b - f &= am + n - ep - q \\ &= am + d - cm - ep - h + gp \end{aligned}$$

$$= (a - c)m + (g - e)p + d - h \ \varepsilon \ \eta.$$

Consequently, $(a - c)m + (g - e)p \ \varepsilon \ \eta$. Therefore,

$$(a - c)(m - p) + (a - e)p + (g - c)p$$

$$= (a - c)m + (a - c + g - e - a + c)p \ \varepsilon \ \eta.$$

Hence, $(a - c)(m - p) \ \varepsilon \ \eta$ which implies $m - p \ \varepsilon \ \eta$ since $a - c \not\varepsilon \ \eta$ by 3.5. In addition,

$$b - f = am + n - ep - q$$

$$= a(m - p) + ap - ep + n - q$$

$$= a(m - p) + (a - e)p + n - q \ \varepsilon \ \eta.$$

This implies that $n - q \ \varepsilon \ \eta$.

The result follows in a similar manner for two lines of the first kind. However, since the first coordinate of a line of the first kind is a non-unit, the first coordinates of the two lines of the first kind must necessarily differ by a non-unit. The rest follows as above.

<u>3.9. Remark.</u> If $(a, b) \ I \ [m, n]_2, [p, q]_2$, we have $am + n = b = ap + q$; hence, $a(m - p) = q - n$. Thus, if $m - p \ \varepsilon \ \eta$, then $q - n \ \varepsilon \ \eta$. Therefore, if $[m, n]_2 \nsim [p, q]_2$ and $[m, n]_2 \wedge [p, q]_2 \neq \emptyset$, then $m - p \not\varepsilon \ \eta$.

Several of the axioms of A. H. planes still hold in our new incidence structure $\mathcal{J}$. It is readily apparent that the neighbour relation is transitive on the set of points (A3). The points (0, 0), (0, 1) and (1, 0) form a triangle, where $(0, 0), (0, 1) \ I \ [0, 0]_1$; $(0, 0), (1, 0) \ I \ [0, 0]_2$; $(0, 1), (1, 0) \ I \ [-1, 1]_2$ (A2). The following system of lemmas give the additional axioms which hold in $\mathcal{J}$.

<u>3.10. Lemma</u>. If $(a, b)\ I\ g, h$ and $g \not\sim h$, then $\text{card}\{P\ I\ g, h\} = 1$.

<u>Proof</u>. Assume there exists $(c, d) \neq (a, b)$ such that $(c, d)\ I$ g, h also.

<u>Case 1</u>: Let $g = [m, n]_1$ and $h = [p, q]_2$.

Therefore,

$$b = ap + q = (bm + n)p + q = bmp + np + q;$$

$$d = cp + q = (dm + n)p + q = dmp + np + q.$$

Hence, $b(1 - mp) = np - q = d(1 - mp)$. However, $m \in \eta$ implies $1 - mp \notin \eta$ and so $(1 - mp)^{-1}$ exists. Thus, $b = (np + q)(1 - mp)^{-1}$ $= d$. Also, since $a - c = bm - dm = (b - d)m = 0$, we have $a = c$; a contradiction.

<u>Case 2</u>: Let $g = [m, n]_2$ and $h = [p, q]_2$.

By the above remark, $m - p \notin \eta$. Therefore, $am + n = b =$ $ap + q$ and $cm + n = d = cp + q$ imply $a = (q - n)(m - p)^{-1} = c$ and $b = am + n = cm + n = d$. Thus, $(a, b) = (c, d)$; a contradiction.

Finally, if two lines of the first kind meet, say $(a, b)\ I$ $[m, n]_1, [p, q]_1$, then $bm + n = a = bp + q$, which gives $b(m - p) =$ $q - n$. However, $m, p \in \eta$ implies $m - p \in$; hence, $q - p \in \eta$. Thus, the two lines are neighbours.

<u>3.11. Lemma</u>. If $g \not\sim h$; $P, R\ I\ g$; $Q, R\ I\ h$ and $P \sim Q$, then $R \sim P, Q$.

<u>Proof</u>. Let $P = (a, b)$; $Q = (c, d)$; $R = (e, f)$.

<u>Case 1</u>: Let $g = [m, n]_2$, $h = [p, q]_2$.

By an earlier remark, $m - p \notin \eta$. From the assumptions of the lemma,

$$
\begin{aligned}
b - d &= b - f - d + f \\
&= am + n - em - n - cp - q + cp + q \\
&= (a - e)m + (e - c)p \\
&= (a - e)(m - p) + (a - e)p + (e - c)p \\
&= (a - e)(m - p) + (a - c)p.
\end{aligned}
$$

However, $(a, b) \sim (c, d)$, Thus, $b - d$, $a - c \in \eta$. Therefore, $(a - e)(m - p) \in \eta$ and hence, $a - e \in \eta$. Finally, since $b - f = (a - e)m \in \eta$, $(a, b) \sim (e, f)$. Similarly, $(c, d) \sim (e, f)$.

Case 2: Let $g = [m, n]_1$, $h = [p, q]_2$.

Then $a - e = bm + n - fm - n = (b - f)m \in \eta$. However, $(a, b) \sim (c, d)$ implies $a - c \in \eta$ and $b - d \in \eta$ and so $c - e = c - a + a - e \in \eta$. In additon, $d - f = cp + q - ep - q = (c - e)p \in \eta$. Thus, (a, b), $(c, d) \sim (e, f)$.

As in the proof of the previous lemma, g and h cannot both be lines of the first kind, since if two lines of the first kind meet, they are neighbours.

3.12. Lemma. If $g \sim h$; $j \not\sim g$; $P \; I \; g, j$; $Q \; I \; h, j$, then $P \sim Q$.

Proof. Let $P = (a, b)$ and $Q = (c, d)$.

Case 1: Let $g = [m, n]_2$, $h = [p, q]_2$ and $j = [r, s]_1$. Clearly, $m - p \in \eta$, $n - q \in \eta$ and $r \in \eta$. Therefore,

$$a - c = br + s - dr - s = (b - d)r \in \eta$$

and

$$b - d = am + n - cp - q$$
$$= (br + s)m + n - (dr + s)p - q$$
$$= brm + sm + n - drp - sp - q$$
$$= brm + s(m - p) - drp + n - q \quad \varepsilon \ \eta.$$

Hence, $P \sim Q$.

<u>Case 2</u>: Let $g = [m, n]_2$, $h = [p, q]_2$ and $j = [r, s]_2$.

Then $m - p \ \varepsilon \ \eta$, $n - q \ \varepsilon \ \eta$ and $r - m \not\varepsilon \ \eta$. By the assumptions of the lemma, $am + n = b = ar + s$ and $cp + q = d = cr + s$. Hence, $b - ar = s = d - cr$. Therefore,

$$am + n - ar = cp + q - cr$$
$$a(m - r) - c(p - r) = q - n$$
$$a(m - r) - c(p - m + m - r) = q - n$$
$$(a - c)(m - r) = q - n + c(p - m) \quad \varepsilon \ \eta.$$

This implies $(a - c)(m - r) \ \varepsilon \ \eta$ and $a - c \ \varepsilon \ \eta$. Also, $b - d = (a - c)r \ \varepsilon \ \eta$. Thus, $P \sim Q$.

If g, h are lines of the first kind and j is a line of the second kind, the result follows from a proof similar to Case 1. Again, the three lines cannot all be of the first kind.

<u>3.13. Lemma</u>. If $g \parallel h$; $P \ I \ j, g$; $j \nparallel g$, then $j \nparallel h$ and there exists $Q \ I \ h, j$.

<u>Proof</u>. Take $P = (a, b)$, for some $a, b \ \varepsilon \ L$. $P \ I \ g, j$.

<u>Case 1</u>: Let $g = [m, n]_2$, then since $h \parallel g$, $h = [m, p]_2$, for some $p \ \varepsilon \ L$. Let $j = [q, r]_1$. Clearly, $j \nparallel h$ (by 3.8). Let

$c = a - (n - p)q(1 - mq)^{-1}$, which is well-defined since $q \in \eta$ and hence, $1 - mq \notin \eta$. Then $(c, cm + p)\ I\ [m, p]_2$. In addition,

$$
\begin{aligned}
&(cm + p)q + r \\
&= cmq + pq + r \\
&= (a - (n - p)q(1 - mq)^{-1})mq + pq + r \\
&= amq - (n - p)q(1 - mq)^{-1}mq + pq + r \\
&= amq - (n - p)q(1 - mq)^{-1}mq + pq + a - bq \\
&= amq - (n - p)q(1 - mq)^{-1}mq + pq + a - amq - nq \\
&= a - (n - p)q - (n - p)q(1 - mq)^{-1}mq \\
&= a - (n - p)q(1 + (1 - mq)^{-1}mq) \\
&= a - (n - p)q((1 - mq)^{-1}(1 - mq) + (1 - mq)^{-1}mq) \\
&= a - (n - p)q(1 - mq)^{-1} \\
&= c.
\end{aligned}
$$

Hence, $(c, cm + p)\ I\ [q, r]_1$ also.

Case 2: Let g and h be defined as in Case 1 and $j = [q, r]_2$. Since $g \; j$, $m - q \notin \eta$ and hence, $h \nmid j$ also. Take $c = a - (p - n)(m - q)^{-1}$, which is well-defined since $m - q \notin \eta$. Clearly, $(c, cm + p)\ I\ [m, p]_2$. Furthermore,

$$
\begin{aligned}
&cq + r \\
&= (a - (p - n)(m - q)^{-1})q + r \\
&= aq - (p - n)(m - q)^{-1}q + r \\
&= aq - (p - n)(m - q)^{-1}q + b - aq \\
&= am + n - (p - n)(m - q)^{-1}q \\
&= am - (p - n - p) - (p - n)(m - q)^{-1}q \\
&= am - (p - n)(1 + (m - q)^{-1}q) + p \\
&= am - (p - n)(m - q)^{-1}(m - q + q) + p
\end{aligned}
$$

$= (a - (p - n)(m - q)^{-1})m + p$

$= cm + p$

and so $(c, cm + p)\ I\ [q, r]_2$ also.

If g and h are lines of the first kind, j must be a line of the second kind. Clearly, $h \nparallel j$. The construction of the point Q is similar to the construction in Case 1.

As in the case of A. H. planes, we have a similar result which is even stronger.

<u>3.14. Lemma.</u> Let $P_i = g_i \wedge j$, $j \nparallel g_i$ $(i = 1, 2)$ such that $g_1 \| g_2$. Then the following are equivalent:

1) $g_1 \sim g_2$

2) $P_1 \sim P_2$.

<u>Proof.</u> Assume 1). Let $P_1 = (a, b)$ and $P_2 = (c, d)$.

<u>Case 1</u>: Let $g_1 = [m, n]_2$; $g_2 = [m, p]_2$ and $j = [q, r]_1$.

This implies that $a - c = (b - d)q \in \eta$ (since $q \in \eta$). Also since $P_i\ I\ g_i$,

$$b - d = am + n - cm - p = (a - c)m + (n - p) \in \eta.$$

Thus, $P_1 \sim P_2$.

<u>Case 2</u>: Let g_1 and g_2 be defined as in Case 1. Let $j = [q, r]_2$.

Since $j \wedge g_i \neq \emptyset$ and $j \nparallel g_i$ $(i = 1, 2)$, we have $m - q \notin \eta$; cf. 3.8 and 3.9. If $g_1 \sim g_2$, then $n - p \in \eta$. Clearly,

$$am + n - cm - p = b - d = aq - cq$$

and

$$(a - c)(m - q) = p - n \quad \varepsilon \, \eta.$$

Since $m - q \not\varepsilon \, \eta$, we have $a - c \; \varepsilon \, \eta$ and $b - d \; \varepsilon \, \eta$ also. Thus, $P_1 \sim P_2$.

<u>Case 3</u>: Assume g_1 and g_2 are lines of the first kind. Then j must be a line of the second kind (since two lines of the first kind which meet are neighbours). Let $g_1 = [m, n]_1$; $g_2 = [m, p]_1$; $j = [q, r]_2$.

Then if $g_1 \sim g_2$,

$$a - c = bm + n - dm - p = (b - d)m + (n - p) \quad \varepsilon \, \eta$$

(since $m \; \varepsilon \, \eta$ and $n - p \; \varepsilon \, \eta$) and

$$b - d = aq + r - cq - r = (a - c)q \quad \varepsilon \, \eta.$$

Thus, $P_1 \sim P_2$.

Assume 2).

Let $g_1 = [m, n]_2$; $g_2 = [m, p]_2$; $P_1 = (a, b)$; $P_2 = (c, d)$.

Clearly, if $P_1 \sim P_2$,

$$n - p = b - am - d + cm = (b - d) + (c - a)m \quad \varepsilon \, \eta$$

(since $b - d \; \varepsilon \, \eta$ and $c - a \; \varepsilon \, \eta$). Thus, $g_1 \sim g_2$.

The proof is similar if g_1 and g_2 are lines of the first kind. The proof in this direction does not require the existence of j.

<u>3.15. Lemma</u>. For every point P and every line l, there exists a unique line L(P,l) such that $P \; I \; L(P,l)$ and $L(P,l) \parallel l$.

Proof. Take $P = (a, b)$, for some $a, b \in L$. If $P \, I \, l$, then l itself is the required line, so we need only consider the case where $P \not{I} \, l$. Take $l = [m, n]_2$ and consider the line $[m, b - am]_2$. Clearly, $[m, b - am]_2 \| [m, n]_2$ and since $am + (b - am) = b$, we have $(a, b) \, I \, [m, b - am]_2$. If $l = [m, n]_1$, then $(a, b) \, I \, [m, a - bm]_1$ and $[m, a - bm]_1 \| [m, n]_1$.

3.16. We have shown that the incidence structure $\mathcal{J}$ satisfies all the axioms of A. H. planes, with the exception of A1 and A4 in one direction. We shall examine these axioms next.

3.17. Lemma. Through any two non-neighbouring points, there exists exactly one line.

Proof. Consider the two non-neighbouring points (a, b) and (c, d). We discuss two cases: 1) $a - c \notin \eta$; 2) $a - c \in \eta$ and $b - d \notin \eta$.

Case 1: Since $a - c \notin \eta$, there exists $(a - c)^{-1} \notin \eta$. We have $(a, b), (c, d) \, I \, [(a - c)^{-1}(b - d), - a(a - c)^{-1}(b - d) + b]_2$ because

$$\begin{aligned} & c(a - c)^{-1}(b - d) - a(a - c)^{-1}(b - d) + b \\ = & - (a - c)(a - c)^{-1}(b - d) + b \\ = & - b + d + b \\ = & \, d. \end{aligned}$$

Therefore, there exists at least one line through the points (a, b) and (c, d).

If $(a, b), (c, d)$ are also incident with some line of the first kind, say $(a, b), (c, d)\ I\ [m, n]_1$, then $a - c = (b - d)m \in \eta$; a contradiction.

Further, if $(a, b), (c, d)$ are incident with some $[m, n]_2$, then $b - d = (a - c)m$ implies $m = (a - c)^{-1}(b - d)$; $b = a(a - c)^{-1}(b - d) + n$ implies $n = b - a(a - c)^{-1}(b - d)$. Thus, $[(a - c)^{-1}(b - d), b - a(a - c)^{-1}(b - d)]_2$ is the unique line between $(a, b), (c, d)$.

<u>Case 2</u>: If $a - c \in \eta$ and $b - d \notin \eta$, then $(b - d)^{-1}$ exists and $(a, b), (c, d)\ I\ [(b - d)^{-1}(a - c), a - b(b - d)^{-1}(a - c)]_1$ since

$$\begin{aligned} & d(b - d)^{-1}(a - c) + a - b(b - d)^{-1}(a - c) \\ &= - (b - d)(b - d)^{-1}(a - c) + a \\ &= - a + c + a \\ &= c. \end{aligned}$$

The uniqueness of this line is shown in the same manner as in Case 1.

<u>3.18</u>. At this point, we may note, in addition, that if the first coordinates of two points are the same and are equal to some $a \in L$, then both points are incident with the line $[0, a]_1$. Similarly, two points with second coordinate $b \in L$ are incident with $[0, b]_2$.

<u>3.19. Lemma</u>. There exists a point on each line.

<u>Proof</u>. Take any line l and let $m = L((0, 0), l)$. Since $[0, 0]_1 \nparallel$

$[0, 0]_2$, either $m \nsim [0, 0]_1$ or $m \nsim [0, 0]_2$. Without loss of generality, assume that $m \nsim [0, 0]_1$. Then by 3.13, $l \nsim [0, 0]_1$ and there exists a point P with $P\ I\ [0, 0]_1, l$.

<u>3.20. Lemma.</u> For any line l, there exist a point P such that $P \nsim X$, for all $X\ I\ l$.

<u>Proof.</u> Assume such a point does not exist. Then there exist three points Q, R and S on l which are neighbours of (0, 0), (1, 0) and (0, 1), respectively. Clearly, there exist lines R(0, 0) and R(0, 1). Therefore, by 3.11, $l \sim R(0, 0)$; $R(0, 0) \sim (0, 0)(0, 1)$; $l \sim R(1, 0)$ and $R(1, 0) \sim (0, 1)(1, 0)$. Thus, by transitivity, $(0, 0)(0, 1) \sim (0, 1)(1, 0)$; a contradiction.

<u>3.21. Lemma.</u> On any line l, there exist points P and Q such that $P \nsim Q$.

<u>Proof.</u> By 3.20, we can select R such that $R \nsim X$, for all $X\ I\ l$. At least two of the lines $[0, 0]_1$, $[0, 0]_2$, $[1, 0]_2$ are not neighbours of the line L((0, 0),l). Let these two lines be m and n. By 3.13, $L(R,m), L(R,n) \nsim L(R,l)$; hence, $L(R,m), L(R,n) \nsim l$ also and there exist unique points $P = L(R,m) \wedge l$ and $Q = L(R,n) \wedge l$. Since $m \nsim n$ and $R \nsim P, Q$, 3.11 implies $P \nsim Q$.

<u>3.22.</u> Using 3.21, it is easy to see that for any point P on

a line l, there exist a $Q \, I \, l$ such that $P \not\sim Q$. If it were otherwise, all points on the line would be neighbours; a contradiction.

<u>3.23. Lemma.</u> There exists a line through every pair of points if and only if for all $a, b \in L$ either $a \in bL$ or $b \in aL$.

<u>Proof.</u> Assume that through any two points of $\mathfrak{A}$, there exists a line. In particular, for any $a, b \in L$ there exists a line through $(0, 0)$ and (a, b). If $(0, 0), (a, b) \, I \, [m, n]_1$, for some $m \in \eta$ and $n \in L$, then $0 = 0 \cdot m + n = n$ and $a = b \cdot m + n = bm$. If $(a, b), (0, 0) \, I \, [p, q]_2$, for some $p, q \in L$, then $0 = 0 \cdot p + q = q$ and $b = a \cdot p + q = ap$. Hence, either $a \in bL$ or $b \in aL$.

Now assume that for any $a, b \in L$, either $a \in bL$ or $b \in aL$. Take any two points $(a, b), (c, d)$ and consider $a - c, b - d \in L$. By our assumption, either $(b - d) \in (a - c)L$ or $(a - c) \in (b - d)L$. If the first is true, then there exists $m \in L$ such that $b - d = (a - c)m$. Therefore, $b - am = d - cm$, which implies $(a, b), (c, d) \, I \, [m, b - am]_2$.

If $(a - c) \in (b - d)L$, but $(b - d) \notin (a - c)L$, then there exists $m \in \eta$ such that $a - c = (b - d)m$. If $m \notin \eta$, there would exist m^{-1} and $a - c = (b - d)m$ would imply $b - d = (a - c)m^{-1}$. Thus, $m \in \eta$. Since $a - bm = c - dm$, we have $(a, b), (c, d) \, I \, [m, a - bm]_1$.

<u>3.24. Remark.</u> If $(a, b) \not\sim (c, d)$, the lines defined above coincide

with the lines we constructed in 3.17. If $a - c \notin \eta$ and $b - d \in \eta$, then $a - c \neq (b - d)m$ for all $m \in L$, since $(b - d)m \in \eta$. Hence, under the assumption that $x \in yL$ or $y \in xL$ for all $x, y \in L$, $b - d = (a - c)m$ for some $m \in L$ and the lines $[m, b - am]_2$ and $[(a - c)^{-1}(b - d), b - a(a - c)^{-1}(b - d)]_2$ are the same. Further, if $b - d \notin \eta$ and $a - c \in \eta$, $a - c = (b - d)m$ for some $m \in L$, as above. However, $m \in \eta$ since otherwise $b - d = (a - c)m^{-1}$ and $a - c \notin \eta$; a contradiction. Thus, $[m, a - bm]_1 = [(b - d)^{-1}(a - c), a - b(b - d)^{-1}(a - c)]_1$. Finally, if $a - c$, $b - d \notin \eta$, then $b - d = (a - c)m$ for some $m \notin \eta$ and $a - c = (b - d)m^{-1}$. However, since $m^{-1} \notin \eta$, m^{-1} cannot be the first coordinate of a line of the first kind and as above, $b - d = (a - c)m$ implies (a, b), $(c, d)\ I\ [m, b - am]_2 = [(a - c)^{-1}(b - d), b - a(a - c)^{-1}(b - d)]_2$.

<u>3.25. Lemma.</u> The following are equivalent:

1) For $g, h \in \mathbb{L}$, $\operatorname{card}\{P\ I\ g, h\} = 1$ implies $g \not\sim h$.

2) $\eta = D_-$, where D_- is the set of left divisors of zero.

<u>Proof.</u> Assume 1). Consider any $r \in L \setminus D_-$ and choose some $m \in L$. Put $p = m - r$. Thus, $m - p \notin D_-$. Clearly, both $[m, 0]_2$ and $[p, 0]_2$ pass through $(0, 0)$. If $(a, b)\ I\ [m, 0]_2, [p, 0]_2$, then $0 = b - b = am - ap = a(m - p)$. However, $m - p \notin D_-$; therefore, $a = 0$ and $b = am = 0$. Thus, $\operatorname{card}\{[m, 0]_2 \wedge [p, 0]_2\} = 1$. By 1), $[m, 0]_2 \not\sim [p, 0]_2$ and so by 3.8, $r = m - p \notin \eta$. Hence, $\eta \subseteq D_- \subseteq \eta$ (cf. [6], 2.2).

Assume 2). Consider $[m, n]_2 \sim [p, q]_2$ such that there exists $(a, b)\ I\ [m, n]_2, [p, q]_2$. By 3.8, $m - p \in \eta = D_-$; thus, there exists $c \neq 0$ such that $c(m - p) = 0$. Hence,

$$\begin{aligned} q - n &= a(m - p) \\ &= c(m - p) + a(m - p) \\ &= (c + a)m - (c + a)p \end{aligned}$$

which implies $(c + a)m + n = (c + a)p + q$. Since $c \neq 0$, $c + a \neq a$ and we have two distinct points (a, b) and $(c + a, (c + a)m + n)$, incident with both $[m, n]_2$ and $[p, q]_2$.

If we have two lines of the first kind, the construction of a second point incident with both lines is similar to the above construction.

In addition, recall that two lines of different kinds cannot be neighbours.

3.26. Lemma. If (a, b) and (c, d) are two neighbouring points incident with a line of the second [first] kind, then they are incident with another line if and only if $a - c \in D_+$ [$b - d \in D_+$], where D_+ is the set of right divisors of zero.

Proof. Assume $(a, b), (c, d)\ I\ [m, n]_2, [p, q]_2$. Then $am + n = b = ap + q$ and $cm + n = d = cp + q$ imply $(a - c)(m - p) = 0$. However, $m - p \neq 0$; otherwise, $n - q = a(p - m) = 0$ and the lines are equal. Therefore, $a - c \in D_+$.

Next, assume $a - c \in D_+$. Thus, there exists $r \neq 0$ such that

$(a - c)r = 0$. Let $p = m + r$. Then

$$\begin{aligned} am + n &= a(p - r) + n \\ &= ap + (n - ar) \end{aligned}$$

and

$$\begin{aligned} cm + n &= c(p - r) + n \\ &= cp - cr + n \\ &= cp - ar + ar - cr + n \\ &= cp + (n - ar). \end{aligned}$$

Hence, $(a, b), (c, d) \; I \; [p, n - ar]_2$.

If (a, b) and (c, d) are two neighbouring points incident with a line of the first kind, then a similar argument shows that a second line through these two points exists if and only if $b - d \in D_+$.

CHAPTER 4

Generalized Affine Hjelmslev Planes

4.1. In the last chapter, we investigated the incidence structure constructed over a local ring and found it satisfies several of the axioms of an A. H. plane. We also showed that the missing axioms were equivalent to certain algebraic properties of the local ring. In this chapter, we shall consider the incidence structure satisfying the same axioms as the incidence structure over a local ring did. We show that under certain assumptions, such a structure may be coordinatized, in the manner of Artin [1], by a local ring.

Let $\mathfrak{A} = \langle \mathbb{P}, \mathbb{L}, I, \| \rangle$ be an incidence structure with parallelism. We define the neighbour relation on $\mathbb{P}$ to be an arbitrary equivalence relation on $\mathbb{P} \times \mathbb{P}$ which also satisfies the condition that if two points are not in the neighbour relation, there exists exactly one line between them. Two lines are defined to be neighbours if for any point on either line there exists a neighbouring point on the other line.

If $\mathfrak{A}$ also satisfies the following axioms, we call $\mathfrak{A}$ a generalized affine Hjelmslev plane (generalized A. H. plane).

G1. There exists a triangle.

G2. If $P \; I \; l, m$, then $l \nsim m$ implies $\text{card}\{P \; I \; l, m\} = 1$.

G3. If $P, Q \; I \; l$; $P, R \; I \; m$; $Q \sim R$ and $l \nsim m$, then $P \sim Q, R$.

G4. If $P \mathrel{I} l, m$; $Q \mathrel{I} l, n$; $l \not\sim m$ and $m \sim n$, then $P \sim Q$.

G5. If $l \parallel m$; $P \mathrel{I} l, n$ and $l \not\sim n$, then $m \not\sim n$ and there exists a $Q \mathrel{I} m, n$.

G6. For any $P \in \mathbb{P}$ and any $l \in \mathbb{L}$, there exists $L(P,l) \in \mathbb{L}$ such that $P \mathrel{I} L(P,l)$ and $L(P,l) \parallel l$.

It is clear from the definition that the neighbour relation on $\mathbb{L}$ is also an equivalence relation.

4.2. Remarks. Let $\mathfrak{A}$ be a generalized A. H. plane.

1) For any line l, there exists a point P such that $P \not\sim X$, for all $X \mathrel{I} l$. This is proved in the same manner as Lemma 3.20.

2) On any line l, there exist points P and Q such that $P \not\sim Q$; hence, for any $P \mathrel{I} l$, there exists $Q \mathrel{I} l$ with $Q \not\sim P$. The proof of this is similar to that of Lemma 3.21.

3) Through any point P, there exist three non-neighbouring lines. By G1, we may select a triangle with sides l_1, l_2, l_3. Since these lines are pairwise not neighbours, by G5, the three lines $L(P,l_1)$, $L(P,l_2)$, $L(P,l_3)$ are also pairwise not neighbours and all three pass through the point P.

This also implies that for any given line m and any point $P \mathrel{I} m$, there exist two lines n_1 and n_2 through P such that the three lines m, n_1, n_2 are pairwise not neighbours.

4.3. Lemma. If $g \parallel h$; $P \mathrel{I} g$; $Q \mathrel{I} h$ and $P \sim Q$, then $g \sim h$.

Proof. Choose R I h such that $R \not\sim Q$. Then $R \not\sim P$ and $RP \sim RQ$ by G3. By G5, $g \sim RP$ and by transitivity, $g \sim h$.

4.4. We now define some mappings on the set of points. We call a map, $\sigma : \mathbb{P} \to \mathbb{P}$, a dilatation of $\mathcal{A}$ if and only if P, Q I l implies σP I L(σQ,l), for points P and Q and a line l.

A dilatation $\hat{\tau}$ is a quasitranslation if $\hat{\tau}$ has no fixed points or $\hat{\tau}$ is the identity.

A line joining a point and its image under a dilatation is called a trace of the dilatation.

A quasitranslation τ is a translation if and only if every line parallel to a trace is also a trace.

It is clear that the identity map on P is a translation. Let D be the set of dilatations of a generalized A. H. plane $\mathcal{A}$ and let T be the set of translations.

4.5. Remark. If τ is a translation with a trace l and P is any point, then τP I L(P,l). By definition, L(P,l) = m is a trace of τ; hence, there exists a point Q such that Q, τQ I m. However, τ is a dilatation; therefore, τP I L(τQ,m) = m. Thus, P, τP I L(P,l).

4.6. Lemma. The images of two non-neighbouring points uniquely determine a dilatation.

Proof. Let P and Q be two non-neighbouring points with images σP and

σQ under some dilatation σ, respectively. By 4.2, there exists a point R with $R \not\sim X$, for all $X \ I \ PQ$. In particular, $R \not\sim P$, Q and so there exist lines RP and RQ. Clearly, the three lines are pairwise not neighbours. Therefore, $R = L(\sigma P, PR) \wedge L(\sigma Q, QR)$ is a uniquely determined point.

Now consider any point S. If $S \sim R$, then $S \not\sim X$, for all $X \ I \ PQ$ and as above $\sigma S = L(\sigma P, PS) \wedge L(\sigma Q, QS)$. If $S \not\sim R$, then there exists a line SR and $SR \not\sim RP$ or $SR \not\sim QR$. Without loss of generality, assume $SR \not\sim RP$. If $S \sim P$, then G3 would imply $P \sim R$; a contradiction. By G4, the lines SP and SR are not neighbours. Therefore, $\sigma S = L(\sigma P, SP) \wedge L(\sigma R, SR)$ defines a unique point.

Thus, σ is completely defined.

<u>4.7. Lemma</u>. The following are equivalent for a dilatation σ :

1) For any pair of non-neighbouring points, P and Q, we have $\sigma P \not\sim \sigma Q$.

2) σ is bijective.

<u>Proof</u>. Assume there exists a pair of non-neighbouring points, P and Q such that $\sigma P \not\sim \sigma Q$.

Let R' be a point such that $R' \not\sim X$, for all $X \ I \ \sigma P \sigma Q$. Then there exists a unique point $R = L(P, PR') \wedge L(Q, QR')$. Clearly, $\sigma R = R'$.

Now let R' be a neighbour of some point on $\sigma P \sigma Q$. Clearly, R' cannot be a neighbour of both σP and σQ; say $R' \not\sim \sigma P$. Choose

$\sigma R\sigma Q \sim L(\sigma S, SR)$, $L(\sigma S, SQ)$; otherwise, $\sigma R \sim \sigma Q$ by G4. Similarly, $\sigma R\sigma Q \sim PR, PQ$ (since if $\sigma R\sigma Q$ is not a neighbour of PR or PQ then, using $PR \sim PQ$, G4 would imply $\sigma R \sim \sigma Q$). By the transitivity of the neighbour relation, $PR \sim \sigma R\sigma Q \sim L(\sigma S, SR)$; a contradiction. Thus, $\sigma Q \sim \sigma R$.

Case 2: Let $P \sim Q$.

Then there exists a point $S \not\sim P$ and a point $W \not\sim X$, for all $X \text{ I } PS$. By G3, we have $PW \sim WQ$ and $PS \sim SQ$; by G5, $L(\sigma W, PW) = PW \sim L(\sigma W, WQ)$ and $L(\sigma S, PS) = PS \sim L(\sigma S, SQ)$.

Now assume $P \not\sim \sigma Q$. By G4, $P\sigma Q \sim PW, PS$; therefore, $PW \sim PS$. However, since $W \not\sim X$, for all $X \text{ I } PS$, $PW \not\sim PS$; a contradiction. Thus, $P \sim \sigma Q$. Symmetrically, we obtain $P \sim \sigma R$. Then $\sigma R \sim P \sim \sigma Q$.

4.9. Lemma. One point and its image uniquely determine any translation that has a trace.

Proof. Let τ be a given translation with a trace m. Take any point P with image τP under the translation τ. By 4.5, $P, \tau P \text{ I } L(P,m) = l$. By 4.2, there exists a point Q with $Q \not\sim X$, for all $X \text{ I } l$. From the definition of dilatation, we obtain $\tau Q = L(\tau P, PQ) \wedge L(Q, l)$, a well-defined single point.

As we now have two non-neighbouring points and their images, 4.6 gives us the desired result.

4.10. Remark. A translation without traces is, of course, completely determined by the images of two non-neighbouring points.

4.11. Lemma. The composition of two dilatations is again a dilatation.

Proof. Let σ_1 and σ_2 be two dilatations. Consider the composition $\sigma_1 \circ \sigma_2$. If P, Q I l, then $\sigma_2 P$ I $L(\sigma_2 Q, l)$ and

$$\sigma_1(\sigma_2 P) \text{ I } L(\sigma_1(\sigma_2 Q), L(\sigma_2 Q, l)) = L(\sigma_1(\sigma_2 Q), l).$$

4.12. Lemma. For any translation τ with a trace l if $\tau P \sim P$, for some point P, then $\tau Q \sim Q$, for all points Q.

Proof. For a given translation τ, assume there exists a point P with $\tau P \sim P$. Consider any other point Q.

If $Q \not\sim X$, for all X I L(P,l), then by definition, $\tau Q = L(Q,l) \wedge L(\tau P, PQ)$. Since $P \sim \tau P$ and $Q \not\sim P, \tau P$, G3 implies that $PQ \sim \tau PQ$; hence, by G5, $\tau PQ \sim L(\tau P, PQ)$. However, $L(\tau P, PQ) \not\sim L(Q,l)$ by G5, so by G4, $Q \sim \tau Q$.

If $Q \sim Y$, for some Y I L(P,l), then there exists an R such that $R \not\sim X$, for all X I L(P,l) and so $\tau R \sim R$. There also exists a point S such that $S \not\sim X$, for all X I PR. Clearly, $S = L(\tau P, PS) \wedge L(\tau R, RS)$. Since $PS \not\sim RS$ by G4, either $PS \not\sim L(S,l)$ or $RS \not\sim L(S,l)$. Without loss of generality, assume $PS \not\sim L(S,l)$. Since $S \not\sim P$, $S \not\sim \tau P$ and by G3, $PS \sim \tau PS$. By G5, $\tau PS \sim L(\tau P, PS)$; therefore, by G4, $S \sim \tau S$.

There exists $m \in \{PR, RS, PS\}$ such that $Q \not\sim X$, for all X I m. By a similar discussion to the one above replacing PR by m and S by Q, we obtain $Q \sim \tau Q$.

We call any translation which maps a point P to a point Q, where $Q \sim P$, a *neighbour translation*. Let N be the set of neighbour translations.

4.13. We now introduce a new axiom.

G7. For any pair of points P and Q, there exists a translation taking P to Q.

A generalized A. H. plane in which G7 holds is called a *generalized A. H. translation plane* (or a generalized T. plane).

4.14. A *minor Desarguesian configuration*, C1, (cf. Figure 4.1) is a set of six points P_i, Q_i $(i = 1, 2, 3)$ and eight lines p_i, g_i $(i = 1, 2, 3)$; q_1, q_2 satisfying the following conditions:

1) $g_i \parallel g_j$; $i, j = 1, 2, 3$.

2) $P_i, Q_i \; I \; g_i$; $I = 1, 2, 3$.

3) $P_i, P_j \; I \; p_k$; (i, j, k) is a permutation of $(1, 2, 3)$.

4) $Q_2, Q_3 \; I \; q_1$; $Q_1, Q_3 \; I \; q_2$.

5) $p_1 \parallel q_1$; $p_2 \parallel q_2$.

6) $p_1, p_2 \nparallel g_3$.

We say that a generalized A. H. plane has property D1 if and only if for each minor Desarguesian configuration we have $Q_2 \; I \; L(Q_1, p_3)$.

4.15. Theorem. In a generalized A. H. plane, G7 implies D1.

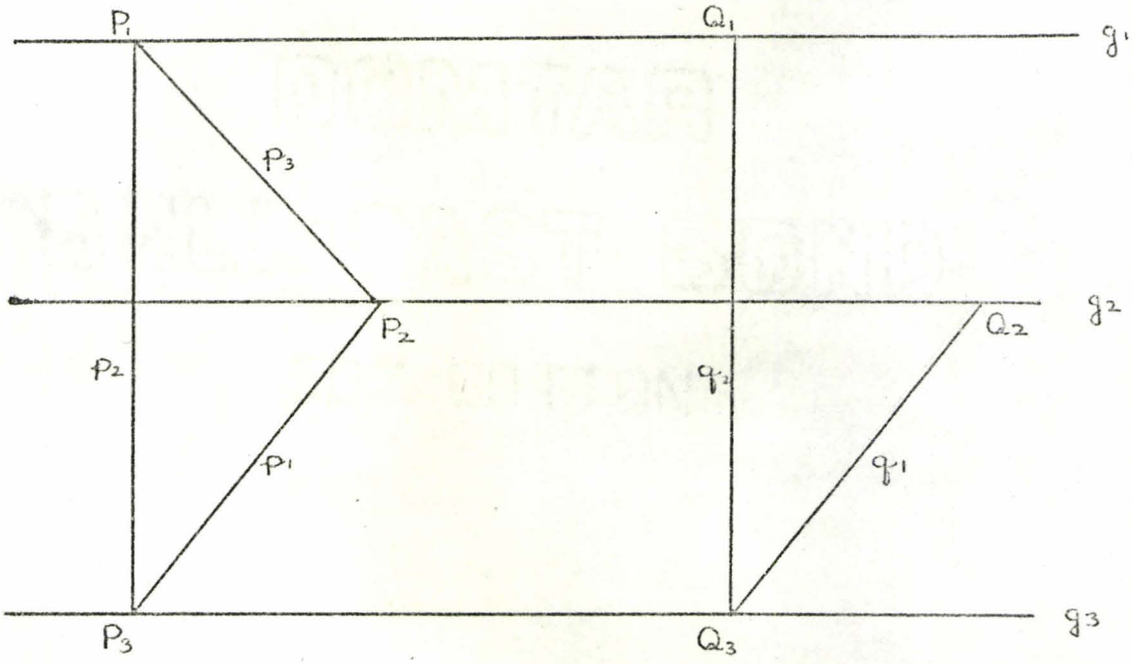

Figure 4.1.

Proof. Consider any minor Desarguesian configuration, Cl. By G7, there exists a translation, τ, which maps P_1 to Q_1. Clearly,

$$\tau P_3 = L(Q_1,p_2) \wedge L(P_3,g_1)$$
$$= q_2 \wedge g_3$$
$$= Q_3$$

and

$$\tau P_2 = L(Q_3,p_1) \wedge L(P_2,g_3)$$
$$= q_1 \wedge g_2$$
$$= Q_2.$$

Hence, by definition, $Q_2 \ I \ L(Q_1,p_3)$.

4.16. Lemma. Translations are bijective.

Proof. Case 1: Let τ be a translation with a line l as a trace.

Select a point P and take any point Q such that $Q \nsim X$, for all $X \ I \ L(P,l)$. Then $\tau Q = L(\tau P, PQ) \wedge L(Q,l)$. If $\tau P \sim \tau Q$, then by 4.3, $L(P,l) = L(\tau P,l) \sim L(\tau Q,l) = L(Q,l)$. Since $L(P,l) \nsim PQ$, G4 implies $P \sim Q$; a contradiction. Hence, $\tau P \nsim \tau Q$ and by 4.7, τ is bijective.

Case 2: Let τ be a translation without traces.

Then for any point P, $P \sim \tau P$. Consider a pair of non-neighbouring points P and Q. Clearly, $\tau P \sim P \nsim Q \sim \tau Q$. Thus, as before τ is bijective.

4.17. Remark. Let τ be any translation. Then for any pair of non-neighbouring points P and Q, we have $\tau P \nsim \tau Q$. If we were to assume the existence of a pair of non-neighbouring points P and Q with $\tau P \sim \tau Q$, then for a point R with $R \not\sim X$, all $X \, I \, PQ$, we have $\tau R = L(\tau P, PR) \wedge L(\tau Q, QR)$; cf. proof of Lemma 4.6. By G3, $\tau R \sim \tau P, \tau Q$. However, for any point S, there exists a line l in $\{PQ, QR, PR\}$ such that $S \not\sim X$, for all $X \, I \, l$. By the same reasoning as above, we obtain $\tau S \sim \tau P, \tau Q, \tau R$. Thus, τ cannot be bijective by 4.7 and so it cannot be a translation.

Henceforth, we assume that $\mathcal{A}$ is a generalized A. H. translation plane.

<u>4.18. Lemma</u>. The inverse map of a translation with traces is also a translation with the same lines as traces.

<u>Proof</u>. Take any translation τ with a line l as a trace. Since τ is bijective, the inverse map is defined on all points P. If there is a line m with τP and τQ on m and $L(\tau P, l) \neq L(\tau Q, l)$, then we may consider two cases.

<u>Case 1</u>: $L(\tau P, l) \not\sim L(\tau Q, l)$.

By 4.3, $\tau P \not\sim \tau Q$ and $P \not\sim Q$. Therefore, there exist unique lines $\tau P \tau Q$ and PQ; hence, $\tau P \, I \, L(\tau Q, PQ)$ implies $P \, I \, L(Q, \tau P \tau Q) = L(Q, m)$.

<u>Case 2</u>: $L(\tau P, l) \sim L(\tau Q, l)$.

There exists a point R such that $\tau R \not\sim X$, for all $X \, I \, L(\tau P, l)$. By definition, $\tau R \not\sim X$, for all $X \, I \, L(\tau Q, l)$ also. It is clear that $\tau R \tau P \not\sim L(\tau P, l)$, $\tau R \tau Q \not\sim L(\tau Q, l)$ and $L(R, l) \not\sim L(P, l), L(Q, l)$. By Case 1 and G5, this implies $RP \nparallel L(\tau P,1) = L(P,1)$ and $RQ \nparallel L(\tau Q,1) = L(Q,1)$. By G5, $RP, RQ \nparallel L(R,1)$. We now have $P \ I \ L(Q,m)$ by D1.

Finally, if $L(\tau P,1) = L(\tau Q,1)$ and $L(\tau P,1)$ is the only line through τP and τQ, then $L(\tau P,1)$ must be the only line through P and Q by the definition of dilatation. If there exists a line $m \neq L(\tau P,1)$ with $\tau P, \tau Q \ I \ m$, then for any point τR with $\tau R \nparallel X$, for all $X \ I \ L(\tau P,1)$, we have $\tau R \nparallel X$, for all $X \ I \ m$ also. By definition, $\tau R\tau P, \tau R\tau Q \nparallel L(\tau P,1), m$. Hence, there exists a point $S = L(Q,m) \wedge L(R,\tau R\tau P)$ and

$$\begin{aligned}\tau S &= L(\tau R, L(R,\tau R\tau P)) \wedge L(\tau Q, L(Q,m)) \\ &= \tau R\tau R \wedge m \\ &= \tau P.\end{aligned}$$

By the injectivity of τ, we have $P = S$. Therefore, $P \ I \ L(Q,m)$.

Thus, τ^{-1} is a dilatation.

It is clear that τ^{-1} has no fixed points unless τ is the identity, and that all the traces of τ are traces of τ^{-1}. Hence, τ^{-1} is a translation.

<u>4.19. Lemma</u>. The set of translations is closed under functional composition.

<u>Proof</u>. Consider two translations τ_1 and τ_2. By 4.11, $\tau_1 \circ \tau_2$ is a dilatation. Choose a point P and let $Q = (\tau_1 \circ \tau_2)P$, $R = \tau_2 P$.

Suppose, first, that τ_1 has a trace l_1.

We may assume that $R \ I \ l_1$. Thus, $Q \ I \ l_1$ also. If $\tau_1 \circ \tau_2$ has a fixed point S then $(\tau_1 \circ \tau_2)S = S$. Since τ_1 is bijective,

τ_1^{-1} exists and $\tau_2 S = \tau_1^{-1} S$. However, as τ_1 has a trace, τ_1^{-1} is also a translation. Hence, $\tau_2 = \tau_1^{-1}$. Therefore, $\tau_1 \circ \tau_2 = i$, the identity translation. On the other hand, if $\tau_1 \circ \tau_2$ has no fixed points then it is at least a quasitranslation. We now have four possibilities.

<u>Case 1</u>: There exist lines l_2 and m such that P, R I l_2 and P, Q I m.

a) First, suppose l_1, l_2, m are mutually distinct and two of them are neighbours. Then there exists a line n through P such that $n \not\sim l_2$, $L(P,l_1)$, m. Now take any point S I n such that $S \not\sim P$. Then

$$\tau_{PQ} S = L(S,m) \wedge L(Q,n)$$
$$\tau_2 S = L(S,l_2) \wedge L(R,n)$$

and

$$\tau_1(\tau_2 S) = L(\tau_2 S, l_1) \wedge L(Q,n).$$

We have three parallel lines n, L(R,n) and L(Q,n). In addition, $l_2 \not\sim n$ and $m \not\sim n$. By D1, $\tau_{PQ} S$ I $L(\tau_2 S, l_1)$; therefore,

$$\tau_{PQ} S = L(\tau_2 S, l_1) \wedge L(Q,n) = \tau_1(\tau_2 S).$$

Thus, $\tau_{PQ} = \tau_1 \circ \tau_2$ by 3.6.

b) Next, suppose l_1, l_2, m are mutually distinct and pairwise not neighbours and P, Q, R are also pairwise not neighbours. We have $(\tau_1 \circ \tau_2)P = Q = \tau_{PQ} P$. In addition,

$$\tau_2 Q = L(Q,l_2) \wedge L(R,m),$$
$$\tau_2 R = l_2 \wedge L(\tau_2 Q, l_1),$$
$$\tau_{PQ} R = L(R,m) \wedge L(Q,l_2) = \tau_2 Q$$

and

$$\tau_1(\tau_2 R) = L(\tau_2 R, l_1) \wedge L(Q,l_2)$$
$$= L(\tau_2 Q, l_1) \wedge L(Q,l_2).$$

However, $\tau_2 Q$ I $L(\tau_2 Q, l_1)$, $L(Q, l_2)$; therefore, by G2 and G5, $\tau_1(\tau_2 R) = \tau_2 Q = \tau_{PQ} R$. By 3.6, $\tau_1 \circ \tau_2 = \tau_{PQ}$ and hence, $\tau_1 \circ \tau_2$ is a translation.

c) Suppose all three points are incident with some line l. There exists S such that $S \not\sim X$, for all X I l. Thus, there exist lines SP, SQ and SR. Clearly, $SP \sim SQ$ if and only if $P \sim Q$; $SQ \sim SR$ if and only if $Q \sim R$ and $SP \sim SR$ if and only if $P \sim R$. If $SP \sim SQ$, then by a), we have $\tau_{PQ} = \tau_{SQ}\tau_{PS}$. If $SP \not\sim SQ$, then $P \not\sim Q$ and we have $\tau_{PQ} = \tau_{SQ}\tau_{PS}$ by b). Similarly, $\tau_2 = \tau_{SR}\tau_{PS}$ and $\tau_1 = \tau_{SQ}\tau_{RS}$. Therefore,

$$\begin{aligned} \tau_1\tau_2 &= (\tau_{SQ}\tau_{RS})(\tau_{SR}\tau_{PS}) \\ &= \tau_{SQ}\tau_{PS} \\ &= \tau_{PQ}. \end{aligned}$$

Hence, $\tau_1 \circ \tau_2$ is a translation.

d) Finally, suppose that l_1, l_2 and m are mutually distinct and pairwise non-neighbouring and P, Q, R are neighbours. Then there exists S I l_2 such that $S \not\sim P$. Clearly, $S \not\sim Q$, R and $SP \sim SQ$. Hence, by a), $\tau_{PQ} = \tau_{SQ}\tau_{PS}$ and $\tau_{SQ} = \tau_{RQ}\tau_{SR}$. By c), $\tau_{PR} = \tau_{SR}\tau_{PS}$. Therefore,

$$\tau_{PQ} = \tau_{SQ}\tau_{PS} = (\tau_{RQ}\tau_{SR})\tau_{PS} = \tau_{RQ}\tau_{PR} = \tau_1\tau_2.$$

<u>Case 2</u>: τ_1 and τ_2 are translations with traces, but there is no line through P and Q.

If there exists a point S such that S, $\tau_1\tau_2(S)$ I l for some line l, then $\tau_1\tau_2 = \tau_{S,\tau_1\tau_2(S)}$ by Case 1. However, this implies that P, $\tau_1\tau_2(P)$ I $L(P, l)$; a contradiction. Thus, $\tau_1\tau_2$ has no traces and the condition that all lines parallel to a trace be traces is satisfied vacuously.

Case 3: There exists a line m with P, Q I m, but no line through P and R.

Thus, l_1 and m are distinct and τ_2 has no traces. By Case 3, $\tau_2 = \tau_1^{-1}\tau_{PQ}$. Hence, $\tau_1\tau_2 = \tau_{PQ}$. Thus, $\tau_1 \circ \tau_2$ is a translation.

Case 4: Neither τ_2 nor τ_{PQ} have traces. The result follows in the same manner as Case 3, using Case 4 instead of Case 1.

Next, we assume that τ_1 is without traces. This implies that $R \sim Q$. Therefore, there exists a point S such that $S \not\sim P, R, Q$. By earlier cases, $\tau_1 = \tau_{SQ}\tau_{RS}$, $\tau_{PS} = \tau_{RS}\tau_2$ and $\tau_{PQ} = \tau_{SQ}\tau_{PS}$. Therefore,

$$\begin{aligned} \tau_1\tau_2 &= (\tau_{SQ}\tau_{RS})\tau_2 \\ &= \tau_{SQ}(\tau_{RS}\tau_2) \\ &= \tau_{PQ}. \end{aligned}$$

Thus, $\tau_1 \circ \tau_2$ is a translation.

4.20. Remark. The proof of Lemma 4.19 yields that if $\tau_1\tau_2(P) = Q$, then $\tau_1\tau_2 = \tau_{PQ}$. Hence, if τ_{PQ} is a translation without traces, we may select a point R such that $R \not\sim P, Q$ and we have $\tau_{PQ} = \tau_{RQ}\tau_{PR}$. Since τ_{RQ} and τ_{PR} are translations with traces RQ and PR, respectively, they are completely determined and hence, so is τ_{PQ}. Thus, there is only one translation taking P to Q. This implies that the translations without traces are also completely determined by the image of a single point.

4.21. Lemma. The inverse of any translation is also a translation.

Proof. Take any translation τ. If τ has a trace, then τ^{-1} is a translation by 4.18.

Assume τ has no traces and select any point P. Clearly, $P \sim \tau P$. There exists a point Q such that $Q \nsim P, \tau P$. By 4.19, $\tau = \tau_{Q,\tau P}\tau_{PQ}$. However, $\tau_{Q,\tau P}$ and τ_{PQ} have the lines $Q\tau P$ and PQ as traces respectively. By 4.18, $\tau_{Q,\tau P}^{-1}$ and τ_{PQ}^{-1} are translations; hence, $\tau_{PQ}^{-1}\tau_{Q,\tau P}^{-1} \varepsilon$ T. However,

$$\tau(\tau_{PQ}^{-1}\tau_{Q,\tau P}^{-1}) = (\tau_{Q,\tau P}\tau_{PQ})(\tau_{PQ}^{-1}\tau_{Q,\tau P}^{-1}) = i$$

(where i is the identity translation). Therefore, $\tau^{-1} = \tau_{PQ}^{-1}\tau_{Q,\tau P}^{-1} \varepsilon$ T.

4.22. Lemma. Composition of translations is commutative.

Proof. Consider any two translations τ_1 and τ_2. By 4.19, both $\tau_1\tau_2$ and $\tau_2\tau_1$ are translations. Choose any point P and let $Q = \tau_1\tau_2 P$ and $R = \tau_2 R$.

Case 1: P, Q I l_2; R, Q I l_1 and $l_1 \nsim l_2$.

Clearly, $Q = \tau_1(\tau_2 P) = l_1 \wedge L(\tau_1 P, l_2)$ and $\tau_2(\tau_1 P) = L(\tau_1 P, l_2) \wedge L(R, l_1)$. Thus, $\tau_1\tau_2 = \tau_2\tau_1$.

Case 2: P, R I l_2; R, Q I l_1 and $l_1 \sim l_2$.

There exists a line m through P with $m \nsim l_2$. Hence, $m \nsim L(P, l_1)$ and by G5, $m \nsim l_1$ and they meet. Take S I m such that $S \nsim P$. We have $S \nsim X$, for all X I l_2, $L(P, l_1)$ by G3. By G5, $L(P, l_1) \sim l_1$. Therefore, $S \nsim X$, for all X I l_1 also. Hence, SP, SR, $SQ \nsim l_1, l_2$. By 4.19, we have $\tau_1 = \tau_{SQ}\tau_{RS}$. By Case 1, $\tau_2\tau_{SQ} = \tau_{SQ}\tau_2$ and

$\tau_2\tau_{RS} = \tau_{RS}\tau_2$. Hence,

$$\begin{aligned}\tau_2\tau_1 &= \tau_2(\tau_{SQ}\tau_{RS}) \\ &= (\tau_2\tau_{SQ})\tau_{RS} \\ &= (\tau_{SQ}\tau_2)\tau_{RS} \\ &= \tau_{SQ}(\tau_2\tau_{RS}) \\ &= \tau_{SQ}(\tau_{RS}\tau_2) \\ &= (\tau_{SQ}\tau_{RS})\tau_2 \\ &= \tau_1\tau_2.\end{aligned}$$

Case 3: One or both of the translations τ_1 and τ_2 has no traces. Thus, at least two of the points P, Q, R are neighbours.

There exists a point S with $S \not\sim P, Q, R$. By 4.19, $\tau_2 = \tau_{SR}\tau_{PS}$; $\tau_1 = \tau_{SQ}\tau_{RS}$ and $\tau_{PQ} = \tau_{SQ}\tau_{PS}$. By Cases 1 and 2,

$$\begin{aligned}\tau_2\tau_1 &= (\tau_{SR}\tau_{PS})(\tau_{SQ}\tau_{RS}) \\ &= (\tau_{PS}\tau_{SR})(\tau_{RS}\tau_{SQ}) \\ &= \tau_{PS}\tau_{SQ} \\ &= \tau_{PQ} \\ &= \tau_1\tau_2.\end{aligned}$$

Thus, $\tau_1\tau_2 = \tau_2\tau_1$.

4.23. In 4.18 through 4.22, we have shown that the set of translations of a generalized A. H. translation plane is closed under functional composition and the taking of inverses. Composition was also shown to be commutative. The set of translations is, therefore, an abelian group.

We call a map, $a : T \to T$ ($\tau \rightsquigarrow \tau^a$), a trace-preserving

<u>endomorphism</u> of T if the traces of τ are among the traces of τ^a and for any two translations τ_1 and τ_2, we have $(\tau_1\tau_2)^a = \tau_1^a\tau_2^a$. Let L be the set of trace-preserving endomorphisms of T.

Consider the map which takes all translations to the identity translation. Clearly, it satisfies both the conditions of a trace-preserving endomorphism. We denote this special endomorphism by 0. We denote the identity endomorphism by 1. To remain consistent we shall denote the map taking each translation to its inverse by -1.

We now introduce two more maps on T. For $a, b \in L$, let $a + b$ be the map defined by $\tau^{a+b} = \tau^a\tau^b$ and ab the map defined by $\tau^{ab} = (\tau^b)^a$, for any $\tau \in T$.

<u>4.24. Lemma</u>. If $a, b \in L$, then $a + b, ab \in L$.

<u>Proof</u>. Take any translation τ. Since the traces of τ^a and τ^b include the traces of τ, the traces of τ are also traces of $\tau^{a+b} = \tau^a\tau^b$, by Case 1 of 4.19. Clearly, the traces of τ^{ab} include those of τ.

Now choose $\tau_1, \tau_2 \in T$. Then

$$\begin{aligned}(\tau_1\tau_2)^{a+b} &= (\tau_1\tau_2)^a(\tau_1\tau_2)^b\\ &= \tau_1^a\tau_2^a\tau_1^b\tau_2^b\\ &= \tau_1^a\tau_1^b\tau_2^a\tau_2^b\\ &= \tau_1^{a+b}\tau_2^{a+b},\end{aligned}$$

and

$$\begin{aligned}(\tau_1\tau_2)^{ab} &= ((\tau_1\tau_2)^b)^a\\ &= (\tau_1^b\tau_2^b)^a\end{aligned}$$

$$= (\tau_1{}^b)^a(\tau_2{}^b)^a$$
$$= \tau_1{}^{ab}\tau_2{}^{ab}.$$

4.25. Lemma. L is a ring.

Proof. By 4.24, L is closed under both addition and multiplication. Let $\tau \varepsilon$ T; a, b, c ε L. We have

$$\tau^{(a+b)+c} = \tau^{a+b}\tau^c$$
$$= (\tau^a\tau^b)\tau^c$$
$$= \tau^a(\tau^b\tau^c)$$
$$= \tau^a\tau^{b+c}$$
$$= \tau^{a+(b+c)};$$
$$\tau^{a+b} = \tau^a\tau^b = \tau^b\tau^a = \tau^{b+a};$$
$$\tau^{0+a} = \tau^0\tau^a = \tau^a = \tau^a\tau^0 = \tau^{a+0};$$
$$\tau^{a+(-1)a} = \tau^a\tau^{(-1)a} = \tau^a(\tau^a)^{-1} = i = \tau^0.$$

Thus, L is a commutative group under addition. We also have

$$\tau^{(ab)c} = (\tau^c)^{ab} = ((\tau^c)^b)^a = (\tau^{bc})^a = \tau^{a(bc)};$$
$$\tau^{1a} = (\tau^a)^1 = \tau^a = (\tau^1)^a = \tau^{a1};$$
$$\tau^{a(b+c)} = (\tau^{b+c})^a = (\tau^b\tau^c)^a = (\tau^b)^a(\tau^c)^a = \tau^{ab}\tau^{ac} = \tau^{ab+ac};$$
$$\tau^{(b+c)a} = (\tau^a)^{b+c} = (\tau^a)^b(\tau^a)^c = \tau^{ba}\tau^{ca} = \tau^{ba+ca}.$$

Thus, L is a ring.

Let L* be the multiplicative monoid consisting of the non-zero elements of L.

<u>4.26. Theorem.</u> Let P be a given point. For any $a \in L$, there exists a unique dilatation $\sigma = \sigma(a)$ which leaves P fixed such that $\tau_{PS}{}^a = \tau_{P,\sigma S}$, for all points S. If $a = 0$, then σ is the degenerate dilatation which maps every point into P.

<u>Proof.</u> Suppose such a dilatation exists. Then for any point S, we have $\sigma S = \tau_{P,\sigma S}P = \tau_{PS}{}^a P$. Thus, σ is completely determined. Therefore, if such a dilatation exists it is unique.

Now define a map σ by $\sigma S = \tau_{PS}{}^a P$, for all points S. Consider any pair of points Q and R which are incident with some line l. Clearly, $\tau_{PQ} = \tau_{RQ}\tau_{PR}$. Then

$$\tau_{PQ}{}^a = \tau_{RQ}{}^a \tau_{PR}{}^a$$

$$\tau_{PQ}{}^a P = \tau_{RQ}{}^a \tau_{PR}{}^a P$$

and so $\sigma Q = \tau_{RQ}{}^a(\sigma R)$. However, a preserves traces so all lines parallel to l are traces of $\tau_{RQ}{}^a$. Therefore, $\sigma Q \; I \; L(\sigma R, l)$ (cf. 4.5). Thus, σ is a dilatation. In addition, $\sigma P = \tau_{PP}{}^a P = P$. It is clear that σ is not the dilatation taking all points to the single point P unless $\tau_{PQ}{}^a = i$, for all Q. This is the case if and only if $a = 0$.

Finally, we have $\tau_{P,\sigma S} = \tau_{P,\tau_{PS}{}^a P} = \tau_{PS}{}^a$, for all points S.

<u>4.27. Lemma.</u> Let $\sigma \in D_P$ and $\tau \in T$. Then $\sigma\tau = \tau_{P,\sigma\tau P}$.

<u>Proof.</u> Let $\tau P = S$.

<u>Case 1</u>: Assume the translation τ has a trace l. Clearly,

$\sigma\tau P = \sigma S = \tau_{P,\sigma S}P = \tau_{P,\sigma S}(\sigma P) = \tau_{P,\sigma\tau P}\sigma P.$

Now take $Q \not\sim X$, for all $X \mathrel{I} L(P,1)$. This implies that $PQ \not\sim L(P,1)$. Let $\sigma Q = R$. Then

$$\tau_{P,\sigma S}\sigma Q = \tau_{P,\sigma S}R = L(R,1) \wedge L(\sigma S, PQ).$$

However, $\sigma(\tau Q) \mathrel{I} L(\sigma S, PQ)$, $L(\sigma Q,1) = L(R,1)$. Therefore, by 4.6, $\sigma\tau = \tau_{P,\sigma\tau P}$.

<u>Case 2</u>: Now assume the translation τ has no traces. There exist two non-neighbouring lines m and n through P. By G5, $L(S,n) \not\sim m$ and there exists a point $U = m \wedge L(S,n)$. Hence, $\tau = \tau_{US}\tau_{PU}$ and $\tau_{US}P = L(P,n) \wedge L(S,m)$. Let $\tau_{US}P = W$. By Case 1,

$$\sigma\tau = \sigma(\tau_{PW}\tau_{PU}) = \tau_{P,\sigma W}(\sigma\tau_{PU}) = \tau_{P,\sigma W}\tau_{P,\sigma U}\sigma.$$

We shall show that $\tau_{P,\sigma W}\tau_{P,\sigma U} = \tau_{P,\sigma S}$. We have

$$\tau_{P,\sigma W}\tau_{P,\sigma U}P = \tau_{P,\sigma W}(\sigma U) = L(\sigma U,n) \wedge L(\sigma W,m).$$

However, σ is a dilatation and so $\sigma S \mathrel{I} L(\sigma U,n), L(\sigma W,m)$. Thus, $\tau_{P,\sigma W}\tau_{P,\sigma U} = \tau_{P,\sigma S}$ and $\sigma\tau = \tau_{P,\sigma\tau P}\sigma$.

<u>4.28. Theorem</u>. Let P be a given point. The mapping $h_P : L^* \longrightarrow D_P$ defined by $h_P(a) = \sigma(a)$ is a monoid isomorphism.

<u>Proof</u>. To show that h_P is a monoid homomorphism, we prove that $\sigma(a)\sigma(b) = \sigma(ab)$ and $\sigma(1) = i$, for all $a, b \in L$. Take any point S and any $a, b \in L$. Then

$$\begin{aligned}\tau_{P,\sigma(a)\sigma(b)(S)} &= (\tau_{P,\sigma(b)(S)})^a \\ &= ((\tau_{PS})^b)^a \\ &= (\tau_{PS})^{ab} \\ &= \tau_{P,\sigma(ab)(S)}.\end{aligned}$$

and

$$\tau_{P,\sigma(1)(S)} = (\tau_{PS})^1 = \tau_{P,i(S)}.$$

Therefore, h_P is a monoid homomorphism.

Now let $\sigma(a) = \sigma(b)$. Then for all points S,

$$\tau_{PS}{}^a = \tau_{P,\sigma(a)(S)} = \tau_{P,\sigma(b)(S)} = \tau_{PS}{}^b.$$

Thus, a = b and h_P is injective.

It remains to show that h_P is surjective. Let $\sigma \in D_P$. Define a map, $a : T \rightarrow T$, by $\tau_{PS}{}^a = \tau_{P,\sigma S}$, for each S. We shall show that a is a trace-preserving endomorphism. Consider any two translations τ_1 and τ_2. Let $\tau_i P = Q_i$, for $i = 1, 2$. Then if $\tau_1\tau_2 P = \tau_1 Q_2 = S$, we have

$$(\tau_1\tau_2)^a P = \tau_{PS}{}^a P = \tau_{P,\sigma S} P = \tau_{P,\sigma(\tau_1 Q_2)} P = \sigma(\tau_1 Q_2)$$

and

$$\tau_1{}^a\tau_2{}^a P = \tau_1{}^a(\tau_{P,\sigma Q_2})P = \tau_{P,\sigma Q_1}\tau_{P,\sigma Q_2} P = \tau_{P,\sigma Q_1}(Q_2).$$

By 4.27, $\sigma\tau_1 = \tau_{P,\sigma\tau_1 P}\sigma$. Therefore, $(\tau_1\tau_2)^a = \tau_1{}^a\tau_2{}^a$. Take any translation $\tau = \tau_{PS}$. If there is no line through P and S, then the traces of τ are obviously among the traces of τ^a. Assume that τ has a trace l. We may assume that P I l and so S I l. Since σ is a dilatation, $\tau^a P = \sigma S$ I l. Thus, the traces of τ are traces of τ^a also.

4.29. Remark. An element of a monoid A is a non-unit if it is not invertible. Let η be the set of non-units of A. If η is an ideal, then A is local and η is the unique maximal ideal of A (cf. 3.2).

Let $\mathfrak{J}$ be a generalized T. plane. Consider D_P, the set of dilatations with fixed point P, and M_P, the set of degenerate dilatations with fixed point P. By 4.7, if $\sigma \in M_P$, then all points are mapped by σ to a set of neighbouring points. By 4.8, there can be no dilatation in D_P which takes neighbouring points to non-neighbouring points. Thus, M_P is the set of non-units of D_P. By 4.6, D_P is a monoid under functional composition. Take any $\sigma_1 \in M_P$ and $\sigma_2 \in D_P$. For two non-neighbouring points P and Q, 4.7 implies $\sigma_1 P \sim \sigma_1 Q$ and 4.8 yields $\sigma_2(\sigma_1 P) \sim \sigma_2(\sigma_1 Q)$. Also by 4.7, $\sigma_1(\sigma_2 P) \sim \sigma_1(\sigma_2 Q)$. Thus, M_P is an ideal and D_P is a local monoid.

In Theorem 4.28, we showed that h_P is a monoid isomorphism between L^* and D_P. Thus, L^* is also a local monoid and $a \in \eta$ (where η is the set of non-units of L^*) if and only if $\sigma(a) \in M_P$. We have the result that L is a local ring.

<u>4.30. Lemma</u>. Let $a \in L$. Then for any $\tau \in T \setminus N$, $\tau^a \in N$ if and only if $a \in \eta$.

<u>Proof</u>. Suppose that for some $\tau \in T \setminus N$, we have $\tau^a \in N$. Let $\tau P = Q$. Then there exists a dilatation $\sigma = \sigma(a)$ such that $\tau_{PS}{}^a = \tau_{P,\sigma S}$, for all translations τ_{PS}. In particular, $\tau^a = \tau_{P,\sigma Q}$. Consequently, $P \sim \sigma Q$; however, $P \not\sim Q$ implies, by 4.7, that $\sigma \in M_P$. Thus, $a \in \eta$.

Now assume $a \in \eta$. Then $\sigma = \sigma(a) \in M_P$ and for any non-neighbouring translation τ_{PQ}, we have $\tau_{PQ}{}^a = \tau_{P,\sigma Q} \in N$.

<u>4.31</u>. Before we can coordinatize our generalized T. plane $\mathfrak{P}$, we must introduce an additional axiom.

G8. If $\tau_1 \in T \setminus N$, $\tau_2 \in T$, and the traces of τ_1 are among the traces of τ_2, then there exists a trace-preserving endomorphism a such that $\tau_1{}^a = \tau_2$.

It is clear that τ_1 must be a translation with traces and therefore, τ_2 must also have traces.

We also introduce a second condition.

G8(P). For each collinear triple (PQR) of mutually distinct points with $P \neq Q$, there exists a dilatation σ which leaves P fixed and takes Q to R.

<u>4.32. Theorem.</u> In a generalized T. plane G8 is equivalent to G8(P).

<u>Proof.</u> Assume that G8 holds in a generalized T. plane $\mathfrak{A}$. Select any collinear triple (PQR) where $P \neq Q$. By G7, there exist translations τ_{PQ} and τ_{PR}. Since $P \neq Q$, $\tau_{PQ} \notin N$. By G8, there exists a trace-preserving endomorphism a of T such that $\tau_{PQ}{}^a = \tau_{PR}$. By 4.26, there exists a dilatation σ with fixed point P such that $\tau_{PS}{}^a = \tau_{P,\sigma S}$, for all points S. In particular, $\tau_{PQ}{}^a = \tau_{P,\sigma Q}$. Thus, $\tau_{PR} = \tau_{P,\sigma Q}$. If we now apply this to the point P, we obtain,

$$R = \tau_{PR}P = \tau_{P,\sigma Q}P = \sigma Q.$$

Thus, σ is the required dilatation.

Now assume G8(P) holds for all points P. Consider two translations τ_1 and τ_2, such that $\tau_1 \in T \setminus N$ and the traces of τ_1 are among the traces of τ_2. Take any point P and let $Q = \tau_1 P$ and $R = \tau_2 P$. Since $\tau_1 \notin N$, we have $P \neq Q$. Clearly, the three points are collinear. If $\tau_1 = \tau_2$, then the endomorphism 1 takes τ_1 to τ_2, so we may assume τ_1 and τ_2 are not equal. Hence, Q and R are distinct. The endomorphism -1 takes τ_1 to its inverse so we may also assume that P and R are distinct. By G8 (P), there exists a dilatation σ such that $\sigma P = P$ and $\sigma Q = R$. By 4.28, there exists $a \in L$ such that $\tau_{PS}{}^a = \tau_{P,\sigma S}$, for all points S. In particular, $\tau_1{}^a = \tau_{PQ}{}^a = \tau_{P,\sigma Q}$. Therefore, $\tau_1{}^a = \tau_{P,\sigma Q} = \tau_{PR} = \tau_2$. Thus, $\tau_2 = \tau_1{}^a$.

4.33. We now define a second configuration. A Desarguesian configuration, C2, consists of eight lines p_i, g_i $(i = 1, 2, 3)$; q_1, q_2 and seven points P, P_i, Q_i $(i = 1, 2, 3)$ with the following properties (cf. Figure 4.2).

1) $P \mathrel{I} g_i$ and $P_i, Q_i \mathrel{I} g_i$; $i = 1, 2, 3$.

2) $P_i, P_j \mathrel{I} p_k$; (i, j, k) is a permutation of $(1, 2, 3)$.

3) $p_1 \parallel q_1$; $p_2 \parallel q_2$.

4) $P \not\sim P_3$.

5) $p_1, g_1 \not\sim g_3$ or $p_2, g_2 \not\sim g_3$.

We say that a generalized A. H. plane has property D2 if and only if for any Desarguesian configuration we have $Q_2 \mathrel{I} L(Q_1, p_3)$.

4.34. Theorem. In a generalized T. plane G8 implies D2.

Proof. Consider any Desarguesian configuration, C2. Without loss of generality, we may assume $p_1, g_1 \not\sim g_3$. By G4, we have $p_1 \not\sim g_2$ and $p_2 \not\sim g_1$. By 4.32, G8 implies G8(P). Therefore, there exists a

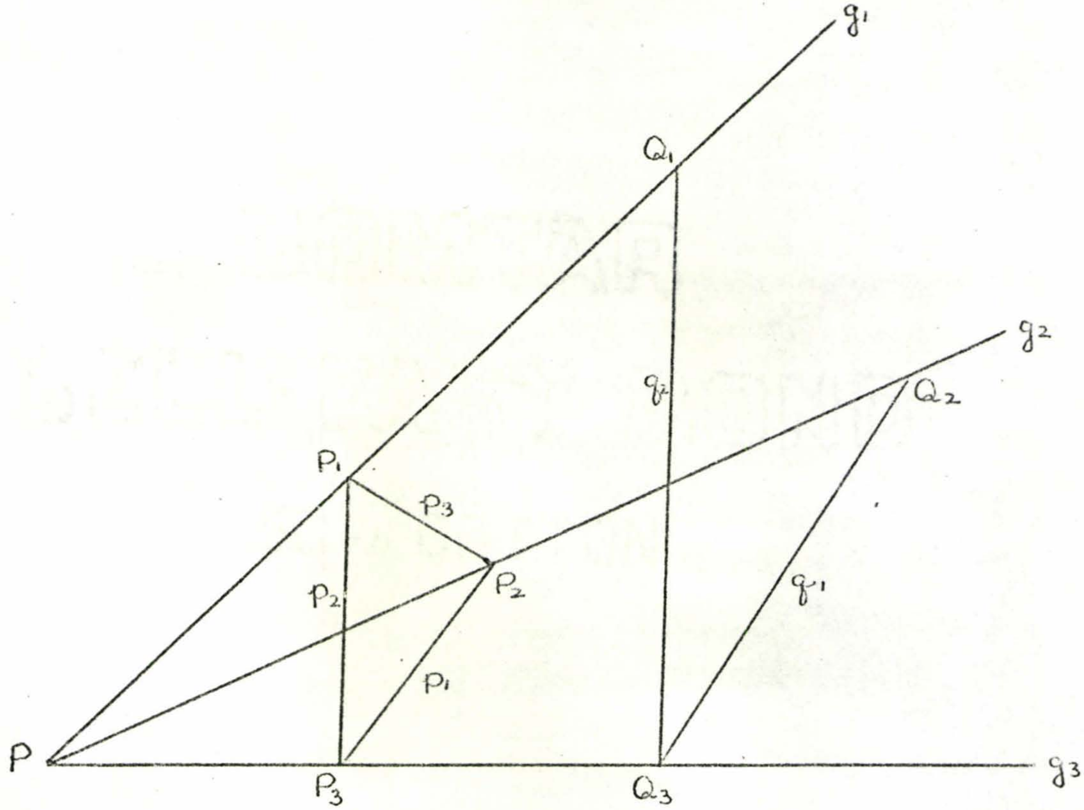

Figure 4.2.

dilatation, $\sigma \varepsilon D_P$, such that $\sigma P_3 = Q_3$. By definition, $\sigma P_2 = L(P_2,p_1) \wedge g_2 = Q_2$ and $\sigma P_1 = L(P_1,p_2) \wedge g_1 = Q_1$. Since σ is a dilatation, we have $Q_2 = \sigma P_2 \ I \ L(Q_1,p_3)$. Thus, we have property D2.

Henceforth, we assume that both G7 and G8 hold in a generalized A. H. plane $\mathfrak{A}$.

<u>4.35. Theorem</u>. Let τ_1 and τ_2 be non-neighbouring translations such that for some point P, $P\tau_1P \not\sim P\tau_2P$. Then for any translation τ, there exist unique endomorphisms a, b ε L such that $\tau = \tau_1^a\tau_2^b$.

<u>Proof</u>. Take τ_1, τ_2 and P as defined in the theorem. Select any translation τ and let $\tau P = Q$. Since $P\tau_1P \not\sim P\tau_2P$, $L(Q,P\tau_1P) \not\sim P\tau_2P$ and there exist translations τ_{PR} and τ_{RQ}. Clearly, $P\tau_2P$ is a trace of τ_{PR} and $P\tau_1P$ a trace of τ_{RQ}. By G8, there exist a, b ε L such that $\tau_{RQ} = \tau_1^a$ and $\tau_{PR} = \tau_2^b$. Thus, $\tau = \tau_{RQ}\tau_{PR} = \tau_1^a\tau_2^b$.

Now suppose there also exist c, d ε L such that $\tau = \tau_1^c\tau_2^d$. Then $\tau_1^a\tau_2^b = \tau_1^c\tau_2^d$ and $\tau_1^{a-c} = \tau_2^{d-b}$. However, $P\tau_1P$ and $P\tau_2P$ are traces of τ_1^{a-c} and τ_2^{d-b}, respectively and $P\tau_1P \not\sim P\tau_2P$. Therefore, by G2, the two lines can meet in only one point. Hence, $P, \tau_1^{a-c}P = \tau_2^{d-b}P \ I \ P\tau_1P, P\tau_2P$ if and only if $\tau_1^{a-c}P = \tau_2^{d-b}P = P$. Thus, $\tau_1^{a-c} = \tau_2^{d-b} = i$. Consequently, $a - c = 0 = d - b$ and $a = c$, $b = d$. Thus, the representation, $\tau = \tau_1^a\tau_2^b$, is unique.

4.36. We are now ready to introduce coordinates. We do this in the same manner as Klingenberg did in [3]. Choose any triangle $\{O, X, Y\}$ as a coordinate frame for $\mathfrak{A}$. Let $l_1 = OX$, $l_2 = OY$. Define $\tau_1 = \tau_{OX}$ and $\tau_2 = \tau_{OY}$. Take any point P. By G7, there exists a translation τ_{OP} and by 3.35, there exist $a, b \in L$ such that $\tau_{OP} = \tau_1^a \tau_2^b$. We shall call (a, b) the coordinates of the point P with respect to the coordinate frame $\{O, X, Y\}$. Let $P_1 = l_1 \wedge L(P, l_2) = (a, 0)$ and $P_2 = l_2 \wedge L(P, l_1) = (0, b)$.

Since $\tau_{OO} = \tau_1^0 \tau_2^0$, O has the coordinates (0, 0). Similarly, X has the coordinates (1, 0) and Y has (0, 1).

Throughout the rest of this chapter, we remain in a fixed coordinate frame $\{O, X, Y\}$ of $\mathfrak{A}$.

4.37. Lemma. Let P and Q be points of $\mathfrak{A}$ with coordinates (a, b) and (c, d) respectively. Then $P \sim Q$ if and only if $a - c \in \eta$ and $b - d \in \eta$.

Proof. We first show $P \sim Q$ if and only if $P_1 \sim Q_1$ and $P_2 \sim Q_2$. Assume $P \sim Q$. By 4.3, $L(P, l_1) \sim L(Q, l_1)$ and $L(P, l_2) \sim L(Q, l_2)$. By G4, $P_1 \sim Q_1$ and $P_2 \sim Q_2$. Now suppose that $P_1 \sim Q_1$ and $P_2 \sim Q_2$. Then $L(P_1, l_2) \sim L(Q_1, l_2)$ and $L(P_2, l_1) \sim L(Q_2, l_1)$. By G4, $P \sim L(P, l_1) \wedge L(Q, l_2)$ and $Q \sim L(P, l_1) \wedge L(Q, l_2)$. Thus, $P \sim Q$.

Clearly, $\tau_1^a O = P_1$, $\tau_1^c O = Q_1$, $\tau_2^b O = P_2$ and $\tau_2^d O = Q_2$. We have $\tau_1^{a-c} = \tau_{Q_1 P_1}$ and $\tau_2^{b-d} = \tau_{Q_2 P_2}$. By 4.30, $a - c \in \eta$

if and only if $Q_1 \sim P_1$ and $b - d \in \eta$ if and only if $Q_2 \sim P_2$.

Therefore, $P \sim Q$ if and only if $a - c \in \eta$ and $b - d \in \eta$.

4.38. Remark. If $\tau = \tau_1^a \tau_2^b$ and P is any point with coordinates (x, y), then $P = (a + x, b + y)$. Since P has the coordinates (x, y), $P = \tau_1^x \tau_2^y O$. Hence, $\tau P = \tau_1^a \tau_2^b P = \tau_1^a \tau_2^b (\tau_1^x \tau_2^y O) = \tau_1^{a+x} \tau_2^{b+y} O$ $= (a + x, b + y)$.

4.39. Now consider any line l and any point $P = (a, b)$ with $P \, I \, l$. There exists a point Q incident with l such that $Q \not\sim P$ and $c, d \in L$ such that $\tau_{PQ} = \tau_1^c \tau_2^d$. Any point R incident with l may be expressed as the image of τ_{PQ}^t, for some $t \in L$. Also, if $t \in L$, then $\tau_{PQ}^t P \, I \, l$. Therefore, $\{R \mid R \, I \, l\} = \{\tau_{PQ}^t P \mid t \in L\}$. In addition, $\tau_{PQ}^t P = (\tau_1^c \tau_2^d)^t P = \tau_1^{tc} \tau_2^{td} P = (tc + a, td + b)$. Hence, $\{R \mid R \, I \, l\} = \{(tc + a, td + b) \mid t \in L\}$.

We showed in 4.37, that $P = (a, b)$ and $Q = (c, d)$ are neighbours if and only if $a - c, b - d \in \eta$. Therefore, if $P \not\sim Q$, then either $a - c$ or $b - d \notin \eta$.

We now define two kinds of lines. A line l is of the first kind if and only if l has a representation of the form

$$l = \{P \mid P \, I \, l\} = \{(tm + n, t) \mid t \in L\},$$

where $m \in \eta$ and $n \in L$. We write $l = [m, n]_1$. A line l is of the second kind if and only if l has a representation of the form

$$l = \{P \mid P \, I \, l\} = \{(t, tm + n) \mid t \in L\},$$

where $m, n \in L$. We write $l = [m, n]_2$.

4.40. Theorem. For each line, there exist $m, n \varepsilon L$ such that $1 = [m, n]_1$ or $1 = [m, n]_2$. Conversely, given $m, n \varepsilon L$, there exists a line $1 = [m, n]_2$. If $m \varepsilon \eta$, then there exists $1 = [m, n]_1$.

Proof. By 4.39, each line 1 has a representation of the form $\{(tc + a, td + b) \mid t \varepsilon L\}$, where (a, b) I 1 and the non-neighbouring translation $\tau_1^c \tau_2^d$ has 1 as a trace. Since $\tau_1^c \tau_2^d \not\varepsilon N$, either $c \not\varepsilon \eta$ or $d \not\varepsilon \eta$. If $c \not\varepsilon \eta$, then let $s = tc + a$. Clearly, $t = (s - a)c^{-1}$ and $td + b = (s - a)c^{-1}d + b = sc^{-1}d - ac^{-1}d + b$. Thus, $1 = [c^{-1}d, b - ac^{-1}d]_2$. If $c \varepsilon \eta$, then $d \not\varepsilon \eta$. Let $s = td + b$. Clearly, $t = (s - b)d^{-1}$ and $tc + a = (s - b)d^{-1}c + a = sd^{-1}c - bd^{-1}c + a$. Thus, $1 = [d^{-1}c, a - bd^{-1}c]_1$.

Now take any $m, n \varepsilon L$. Let $\tau = \tau_1^1 \tau_2^m$ and P be the point with coordinates $(0, n)$. Then $P = \tau_1^1 \tau_2^m P = (1, m + n)$ and $(0, n) \not\sim (1, m + n)$. However, $n = 0m + n$ and $m + n = 1m + n$. Thus, $(0, n)(1, m + n) = [m, n]_2$.

If $m \varepsilon \eta$, let $\tau = \tau_1^m \tau_2^1$. Let P be the point with coordinates $(n, 0)$. Then $\tau P = \tau_1^m \tau_2^1 P = (m + n, 1)$ and $(n, 0) \not\sim (m + n, 1)$. However, $n = 0m + n$ and $m + n = 1m + n$. Thus, $(n, 0)(m + n, 1) = [m, n]_1$.

4.41. Lemma. A line of the first kind and a line of the second kind intersect in a unique point.

Proof. Consider any two lines $[m, n]_1$ and $[u, v]_2$ which are of different kinds. By definition, $m \varepsilon \eta$; hence, um, $mu \varepsilon \eta$. Since L is a local ring, $(1 - um)$, $(1 - mu) \not\varepsilon \eta$ (cf. 3.2). First, assume the two lines intersect in some point (x, y). Then $x = ym + n$ and $y = xu + v$. Hence, $x = (vm + n)(1 - um)^{-1}$ and $y = (nu + v)(1 - mu)^{-1}$. Thus, the point of intersection is unique if it exists.

Now consider the point $((vm + n)(1 - um)^{-1}, (nu + v)(1 - mu)^{-1})$. Then

$$
\begin{aligned}
&(vm + n)(1 - um)^{-1}u + v \\
&= ((vm + n)(1 - um)^{-1}u(1 - mu) + v(1 - mu))(1 - mu)^{-1} \\
&= (vm(1 - um)^{-1}u - (1 - um)^{-1}umu - u) \\
&\qquad + n((1 - um)^{-1}u - (1 - um)^{-1}umu) + v)(1 - mu)^{-1} \\
&= (vm((1 - um)^{-1}(1 - um)u - u) \\
&\qquad + n(1 - um)^{-1}(1 - um)u + v)(1 - mu)^{-1} \\
&= (nu + v)(1 - mu)^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
&(nu + v)(1 - mu)^{-1}m + n \\
&= ((nu + v)(1 - mu)^{-1}m(1 - um) + n(1 - um)(1 - um)^{-1} \\
&= (vm + n)(1 - um)^{-1}.
\end{aligned}
$$

Thus, $((vm + n)(1 - um)^{-1}, (nu + v)(1 - mu)^{-1})$ is always incident with $[m, n]_1$ and $[u, v]_2$.

4.42. Lemma. Two lines $[m, n]_i$ and $[u, v]_j$ are parallel if and only if they are of the same kind and $m = u$.

Proof. G6 and 4.41 imply that two parallel lines must be of the same kind.

Suppose both lines are of the first kind; ie., we have $[m, n]_1$ and $[u, v]_1$. Then there exists a translation τ with $[m, n]_1$ as a trace such that $\tau O = (m, 1)$; hence, $\tau = \tau_1^{\,m}\tau_2^{\,1}$. Similarly, the translation $\tau_1^{\,u}\tau_2^{\,v}$ has $[u, v]_1$ as a trace. Clearly, $\tau_1^{\,m}\tau_2^{\,1}$, $\tau_1^{\,u}\tau_2^{\,v} \notin N$; therefore, each has only a single parallel pencil of traces. Hence, $[m, n]_1 \parallel [u, v]_1$ if and only if there exists a trace-preserving endomorphism a such that $\tau_1^{\,u}\tau_2^{\,1} = (\tau_1^{\,m}\tau_2^{\,1})^a = \tau_1^{\,am}\tau_2^{\,a}$; ie., such that $a = 1$ and $u = am = m$. Thus, $u = m$.

A similar discussion gives the same result if the lines are of the second kind.

4.43. Thus, if a generalized A. H. plane $\mathfrak{A}$ satisfies G7 and G8, it may be coordinatized by a local ring L. As in the previous chapter, we have the additional results that $a \in bL$ or $b \in aL$ for all $a, b \in L$ if and only if there is a line through every pair of points and that $\eta = D_-$ if and only if for any two lines g and h, $\mathrm{card}\{P \; I \; g, h\} = 1$ implies $g \nparallel h$. All the other results regarding the behavior of lines and points proved in the last chapter also hold in $\mathfrak{A}$.

CHAPTER 5

The Fundamental Theorem of Generalized A. H. Planes

5.1. In this chapter, we discuss the fundamental theorem of a generalized A. H. plane which is coordinatized by a local ring.

Henceforth, we let $\mathcal{A}$ be such a generalized A. H. plane with L as its coordinate ring. Let O be the point with coordinates (O, O).

5.2. Lemma. If $P \neq O$, then every point on the line OP is of the form tP, for some $t \in L$.

Proof. Let $P = (a, b)$.

If OP is a line of the first kind, then $OP = [m, O]_1$, for some $m \in \eta$ since $O = Om + n = n$. By 3.5, $a = a - O \in \eta$ and $b = b - O \notin \eta$. Now consider any point $Q = (c, d)$ on $[m, O]_1$. Clearly,

$$d = d(b^{-1}b) = (db^{-1})b$$

and

$$(db^{-1})a = db^{-1}bm = dm = c.$$

Thus, $Q = (db^{-1})P$.

If OP is a line of the second kind, then it must have second coordinate O as above. Then $OP = [m, O]_2$, for some $m \in L$ and by 3.5, $a \notin \eta$. Hence, for any $Q = (c, d)$ on $[m, O]_2$, $Q = (ca^{-1})P$.

In addition, it is clear that for any $t \in L$, tP I l whenever

O, P I 1.

5.3. Lemma. If g and h are non-neighbouring lines such that O, P I g and O, Q I h, then $P + Q = L(P,h) \wedge L(Q,g)$ (where the addition is componentwise).

Proof. Let $P = (a, b)$ and $Q = (c, d)$.

Case 1: The lines are of different kinds, say $g = [m, 0]_1$ and $h = [n, 0]_2$, for some $m \varepsilon L$, $n \varepsilon \eta$.

Clearly, $L(P,h) = [n, b - an]_2$ and $L(Q,g) = [m, c - dm]_1$. Since

$$(a + c)n + b - an = cn + b = b + d$$

and

$$(b + d)m + c - dm = bm + c = a + c,$$

we have $(a + c, b + d)$ I $L(P,h)$, $L(Q,g)$. By 3.10 and 3.13, the two lines intersect in only one point. Thus, $P + Q = L(P,h) \wedge L(Q,g)$.

Case 2: Both lines are of the second kind, say $g = [m, 0]_2$ and $h = [n, 0]_2$, for some $m, n \varepsilon L$. It is readily apparent that $L(P,h) = [n, b - an]_2$ and $L(Q,g) = [m, d - cm]_2$. Therefore,

$$(a + c)n + b - an = cn + b = d + b = am + d = (a + c)m + d - cm;$$

hence, $(a + c, b + d)$ I $L(P,h)$, $L(Q,g)$. Again by 3.10 and 3.13, $P + Q = L(P,h) \wedge L(Q,g)$.

5.4. Lemma. If $O \not\sim P$ and $Q \not\sim X$, for all X I OP, then P and Q are (left) linearly independent.

Proof. Let $P = (a, b)$ and $Q = (c, d)$ and consider $mP + nQ = 0$, for some $m, n \in L$. Thus, $ma + nc = 0$ and $mb + nd = 0$. Since $0 \not\sim P$, we have $a \notin \eta$ or $b \notin \eta$ and since $Q \not\sim (ta, tb)$ for any $t \in L$, we have $c - ta \notin \eta$ or $d - tb \notin \eta$.

If $a \notin \eta$, then $m = -nca^{-1}$ and so $-nca^{-1}b + nd = 0$. If $n \neq 0$, then $-ca^{-1}b + d \in D\_ \subseteq \eta$ which implies that $0 = c - (ca^{-1})a \notin \eta$; a contradiction. Thus, $n = 0$ and hence,

$$m = m(aa^{-1}) = (ma)a^{-1} = 0a^{-1} = 0.$$

A similar proof gives the result when $b \notin \eta$.

5.5. An automorphism of a generalized A. H. plane is a pair of functions, $f = (\varphi, \psi) : \mathfrak{H} \to \mathfrak{H}$ such that

1) $\varphi : \mathbb{P} \to \mathbb{P}$ and $\psi : \mathbb{L} \to \mathbb{L}$ are bijections;

2) $P \mathrel{I} l$ if and only if $\varphi P \mathrel{I} \psi l$;

3) $l \parallel m$ if and only if $\psi l \parallel \psi m$.

Although it is not immediately apparent that the neighbour relation remains unchanged under such a map f, this result will follow from later work.

Clearly, the automorphisms of $\mathfrak{H}$ form a group, Aut $\mathfrak{H}$. Moreover, if $f \in \text{Aut}\,\mathfrak{H}$, then $\{Q \mid Q \mathrel{I} \psi l\} = \{\varphi P \mid P \mathrel{I} l\}$; hence, we may write $f = (f,f)$. From the definitions, it is obvious that $\text{card}\,\{P \mathrel{I} l, m\} = 1$ implies $f(l \wedge m) = fl \wedge fm$ and $f(L(P,l)) = L(fP,fl)$.

Let $(\text{Aut}\,\mathfrak{H})_0$ be the subgroup of Aut $\mathfrak{H}$ which map 0 to 0.

5.6. Lemma. The number of lines between P and Q is equal to the number of lines between fP and fQ.

Proof. Assume there is no line between points P and Q. If there exists a line between fP and fQ, then by the bijectivity of f on L, there would exist a line m, the preimage of the line through fP and fQ. Clearly, $P, Q \; I \; m$; a contradiction.

The other cases are proved in a similar fashion.

5.7. Theorem. If $f \in (\mathrm{Aut}\, \mathfrak{A})_0$, then $f(P + Q) = fP + fQ$, for all P and Q in $\mathfrak{A}$.

Proof. We may assume that $P, Q \neq 0$.

Case 1: There exist non-neighbouring lines l and m such that $0, P \; I \; l$ and $0, Q \; I \; m$.

By 5.3,

$$\begin{aligned} f(P + Q) &= f(L(P,m) \wedge L(Q,l)) \\ &= f(L(P,m)) \wedge f(L(Q,l)) \\ &= L(fP, fm) \wedge L(fQ, fl). \end{aligned}$$

Clearly, $0, fP \; I \; fl$ and $0, fQ \; I \; fm$. Using the method employed in the proof of 5.3, it is readily apparent that $fP, fQ \; I \; L(fP,fm)$, $L(fQ,fl)$; however, we have just shown that these two lines intersect in the single point $f(P + Q)$. Thus, $f(P + Q) = fP + fQ$.

Case 2: Lines between the pairs of points $0, P$ and $0, Q$ do not necessarily exist.

Let $P = (a, b)$ and $Q = (c, d)$. Consider the points

$(a, 0), (c, 0), (a + c, 0)\ I\ [0, 0]_2$. Clearly, $(1, 1) \not\sim (x, 0)$, for all $x \in L$ and $(0, 0), (1, 1)\ I\ [1, 0]_2$. Therefore, by Case 1,

$$f((c, 0) + (1, 1)) = f(c, 0) + f(1, 1)$$

and

$$f((a + c, 0) + (1, 1)) = f(a + c, 0) + f(1, 1).$$

Since $(c + 1, 1) \not\sim (0, 0)$, there exists a unique line through them, say l, and since $(c + 1, 1) \not\sim (x, 0)$, all $x \in L$, we have $l \not\sim [0, 0]_2$. Therefore, by Case 1,

$$\begin{aligned} f(a + c, 0) + f(1, 1) &= f(a + c + 1, 1) \\ &= f((a, 0) + (c + 1, 1)) \\ &= f(a, 0) + f(c + 1, 1) \\ &= f(a, 0) + f(c, 0) + f(1, 1). \end{aligned}$$

Hence, $f(a + c, 0) = f(a, 0) + f(c, 0)$. Similarly, $f(0, b + d) = f(0, b) + f(0, d)$.

Moreover, since $(a, 0), (c, 0), (a + c, 0)\ I\ [0, 0]_2$ and $(0, b), (0, d), (0, b + d)\ I\ [0, 0]_1$, Case 1 implies

$$f(a, b) = f(a, 0) + f(0, b),$$

$$f(c, d) = f(c, 0) + f(0, d)$$

and

$$f(a + c, b + d) = f(a + c, 0) + f(0, b + d).$$

Therefore,

$$\begin{aligned} f(a + c, b + d) &= f(a + c, 0) + f(0, b + d) \\ &= f(a, 0) + f(c, 0) + f(0, b) + f(0, d) \\ &= f(a, b) + f(c, d); \end{aligned}$$

ie., $f(P + Q) = fP + fQ$.

<u>5.8. Theorem.</u> If $f \varepsilon (\text{Aut } \mathfrak{J})_0$, then there exists a unique ring isomorphism $\lambda = \lambda_f \varepsilon \text{ Aut } L$ such that $f(tP) = \lambda(t)fP$.

<u>Proof.</u> We first show that if such a λ exists, it is unique. Assume there exists $\lambda' \varepsilon \text{ Aut } L$ such that $\lambda(t)fP = f(tP) = \lambda'(t)fP$, for each P. By the surjectivity of f on P, we may choose Q such that $fQ = (1, 0)$. Then

$$f(tQ) = \lambda'(t)(1, 0) = (\lambda'(t), 0) = (\lambda(t), 0).$$

Hence, $\lambda'(t) = \lambda(t)$, for all t.

We next establish the existence of λ. Take any point P such that $fP \not I O$. Clearly, for any $t \varepsilon L$, the point tP is incident with the unique line l through O and P (cf. 5.6). Hence, $f(tP) \; I \; fl = OfP$. By 5.2, f(tP) may be expressed as a multiple of fP, say $f(tP) = sfP$, for some $s \varepsilon L$. For each $t \varepsilon L$, let $\lambda(t,P) = s$. We now show that $\lambda(t,P)$ is independent of the choice of P, where $fP \sim O$.

Choose any other point Q such that $fQ \not I O$. Let the unique line through the points O and Q be h (cf. 5.6).

<u>Case 1</u>: $fl \not I fh$.

By 3.11, $O \not I fP$ implies that $fP \not I X$, for all $X \; I \; fh$; hence, fP and fQ are linearly independent by 5.4. Also by 3.11, $fQ \not I Y$, for all $Y \; I \; fl$. Thus, $fl \not I L(fQ,fl)$. By 3.13, $L(fP,fh) \not I l$ and since $f(P + Q) = fP + fQ \; I \; L(fP,fh), L(fQ,fl)$, 3.11 implies that $O \not I f(P + Q)$. Using 5.7, we obtain,

$$\lambda(t,P)fP + \lambda(t,Q)fQ = f(tP) + f(tQ)$$

$$= f(t(P + Q))$$

$$= \lambda(t,P + Q)f(P + Q)$$

$$= \lambda(t,P + Q)fP + \lambda(t,P + Q)fQ.$$

However, fP and fQ are linearly independent. Thus, $\lambda(t,P) = \lambda(t,P + Q) = \lambda(t,Q)$.

<u>Case 2</u>: $fl \sim fh$.

Since f is an automorphism, there exists a line j such that $O \; I \; fj$ and $fj \not\sim fl, fh$. There also exists a point $R \; I \; j$ such that $fR \not\sim O$. Applying Case 1 to the pairs of lines fl and fj; fh and fj in turn, we obtain $\lambda(t,P) = \lambda(t,R) = \lambda(t,Q)$.

Since $\lambda(t,P)$ is independent of the choice of P as long as $fP \not\sim O$, we may replace $\lambda(t,P)$ by $\lambda(t)$. We have $f(tP) = \lambda(t)fP$, for all P with $fP \not\sim O$.

In addition, we shall show that $f(tP) = \lambda(t)fP$ even if $fP \sim O$. Choose any Q such that $fQ \not\sim O$, then if $fP \sim O$, we have $f(P - Q) = fP + f(-Q) \not\sim O$. Hence,

$$\begin{aligned} f(tP) &= f(t(P - Q) + tQ) \\ &= f(t(P - Q)) + f(tQ) \\ &= \lambda(t)f(P - Q) + \lambda(t)fQ \\ &= \lambda(t)(f(P - Q) + fQ) \\ &= \lambda(t)f(P - Q + Q) \\ &= \lambda(t)fP. \end{aligned}$$

Finally, we show that the map λ is a ring isomorphism. Using the same method as above, we see that there exists a map μ such that $f^{-1}(tP) = \mu(t)f^{-1}P$, for each point P and each $t \in L$. For any $t \in L$,

$$\lambda(\mu(t))(1, 0) = \lambda(\mu(t))f(f^{-1}(1, 0))$$

$$= f(\mu(t)f^{-1}(1, 0))$$

$$= f(f^{-1}(t)(1, 0))$$

$$= t(1, 0).$$

Thus, $\lambda(\mu(t)) = t$. Similarly, $\mu(\lambda(t)) = t$, for any $t \in L$.

Next, consider any two points P and Q different from O such that $fP = Q$. Then for any $s, t \in L$,

$$\lambda(s + t)Q = \lambda(s + t)fP$$

$$= f((s + t)P)$$

$$= f(sP + tP)$$

$$= f(sP) + f(tP)$$

$$= \lambda(s)fP + \lambda(t)fP$$

$$= (\lambda(s) + \lambda(t))Q.$$

Hence, $\lambda(s + t) = \lambda(s) + \lambda(t)$. In addition, for $s, t \in L$,

$$\lambda(st)Q = \lambda(st)fP$$

$$= f((st)P)$$

$$= f(s(tP))$$

$$= \lambda(s)f(tP)$$

$$= \lambda(s)\lambda(t)Q.$$

Thus, $\lambda(st) = \lambda(s)\lambda(t)$. Therefore, λ is an automorphism of L.

5.9. Remark. We have just shown that for any point P and any $t \in L$, $f(tP) = \lambda(t)fP$. Therefore, in particular, if $P = (1, 0)$ or $P = (0, 1)$, $f(t(1, 0)) = \lambda(t)f(1, 0)$ or $f(t(0, 1)) = \lambda(t)f(0, 1)$, for any $t \in L$. However, any point (a, b) can be expressed as a linear combination of (1, 0) and (0, 1); ie., $(a, b) = a(1, 0) + b(0, 1)$.

Hence,

$$f(a, b) = f(a(1, 0) + b(0, 1))$$
$$= f(a(1, 0)) + f(b(0, 1))$$
$$= \lambda(a)f(1, 0) + \lambda(b)f(0, 1).$$

Since f is surjective on P, this implies that any point in $\mathfrak{A}$ may be expressed as a linear combination of f(1, 0) and f(0, 1); thus, $\{f(1, 0), f(0, 1)\}$ is a basis for the set of points and f is a non-singular semi-linear transformation with respect to the module structure on L X L.

<u>5.10. Theorem</u>. The map $\alpha : (\text{Aut } \mathfrak{A})_o \longrightarrow \text{Aut } L$ defined by $\alpha(f) = \lambda_f$, is a group epimorphism whose kernel is the general linear group, $(G. L. \mathfrak{A})_o$, of the module structure on L X L. Thus,

$$(\text{Aut } \mathfrak{A})_o / (G. L. \mathfrak{A})_o \cong \text{Aut } L.$$

<u>Proof</u>. From 5.8, it is clear that α defines a mapping. Since

$$(f_1 \circ f_2)(tP) = f_1(\lambda_{f_2}(t) f_2 P)$$
$$= \lambda_{f_1}(\lambda_{f_2}(t)) f_1(f_2 P)$$
$$= \lambda_{f_1}(\lambda_{f_2}(t))(f_1 \circ f_2)P,$$

for $f_1, f_2 \in (\text{Aut } \mathfrak{A})_o$, we have $\lambda_{f_1 \circ f_2} = \lambda_{f_1} \circ \lambda_{f_2}$ by the uniqueness of λ in 5.8.

We next show that α is surjective. Take any $\lambda \in \text{Aut } L$. Define f by:

$$f(x, y) = \lambda(x)(1, 0) + \lambda(y)(0, 1) = (\lambda(x), \lambda(y))$$

and

$$f([m, n]_i) = [\lambda(m), \lambda(n)]_i,$$

where $x, y, m, n \in L$ and $[m, n]_i$ is a line of the i^{th} kind in 当.
Since λ is bijective, f must be bijective on both the points and the lines; $\lambda(0) = 0$ implies $f(0, 0) = (0, 0)$; $\lambda(1) = 1 = \lambda(aa^{-1}) = \lambda(a)\lambda(a^{-1})$ implies $a \in \eta$ if and only if $\lambda(a) \in \eta$.

Consider any $(a, b)\ I\ [m, n]_1$. Then

$$\begin{aligned}\lambda(b)\lambda(m) + \lambda(n) &= \lambda(bm) + \lambda(n)\\ &= \lambda(bm + n)\\ &= \lambda(a)\end{aligned}$$

which implies $f(a, b)\ I\ f([m, n]_1)$ and conversely. Similarly, $f(a, b)\ I\ f([m, n]_2)$ if and only if $(a, b)\ I\ [m, n]_2$. Thus, it is clear that $f \in (\mathrm{Aut}\ 当)_o$. Since $f(aP) = \lambda(a)fP$, $\alpha(f) = \lambda$.

Finally, we show that $\ker \alpha = (G.\ L.\ 当)_o$. Let $f \in \ker \alpha$; ie., $\lambda_f = i$ and so $f(aP) = \lambda_f(a)fP = afP$, for all P. Therefore, $f \in (G.\ L.\ 当)_o$. Conversely, if $f \in (G.\ L.\ 当)_o$, then the uniqueness of λ shown in 5.8 implies that $\lambda_f = i$.

<u>5.11</u>. Now that we have shown that $f(tP) = \lambda(t)fP$, for all points P and all $t \in L$, we can show that the neighbour relation is preserved by all $f \in (\mathrm{Aut}\ 当)_o$.

<u>5.12. Lemma</u>. If $f \in (\mathrm{Aut}\ 当)_o$, then $P \sim Q$ if and only if $fP \sim fQ$.

<u>Proof</u>. Let λ be the automorphism of L associated with f. Since

$\lambda \in \text{Aut } L$, if $a \notin \eta$, then $1 = \lambda(aa^{-1}) = \lambda(a)\lambda(a^{-1})$; ie., $\lambda(a) \notin \eta$. Let $c \in L$. If $\lambda(c) \notin \eta$, put $b = \lambda(c)$. Then there exists $d \in L$ such that $\lambda(d) = b^{-1}$. Therefore, $1 = bb^{-1} = \lambda(c)\lambda(d) = \lambda(cd) = \lambda(1)$ and since $\lambda \in \text{Aut } L$, $cd = 1$. Hence, $a \notin \eta$ if and only if $\lambda(a) \notin \eta$.

Now consider any P such that $P \sim O$ (such a P exists if and only if the neighbour relation is non-trivial; cf. 1.5). Let $P = (a, b)$; $f(1, 0) = (c, d)$ and $f(0, 1) = (g, h)$. Then

$$f(a, b) = \lambda(a)f(1, 0) + \lambda(b)f(0, 1)$$
$$= (\lambda(a)c + \lambda(b)g, \lambda(a)d + \lambda(b)h).$$

Since $a, b \in \eta$, we have $\lambda(a), \lambda(b) \in \eta$; hence, $\lambda(a)c + \lambda(b)g$, $\lambda(a)d + \lambda(b)h \in$. Thus, $fP \sim O$.

Finally, for any pair of neighbouring points P and Q, $P - Q \sim O$. Therefore, $O \sim f(P - Q) = fP - fQ$. Thus, $fP \sim fQ$.

Since f^{-1} is also an automorphism which leaves O fixed, $fP \sim fQ$ also implies $P \sim Q$.

<u>5.13. Lemma</u>. If $f \in (\text{Aut } \mathfrak{H})_O$, then $l \sim m$ if and only if $fl \sim fm$.

<u>Proof</u>. Assume $l \sim m$. If P I fl, then there exists Q I l such that $fQ = P$; however, there exists R I m such that $Q \sim R$. Clearly, fR I fm and by the last lemma, $fQ \sim fR$. Similarly, if we consider any Q I fm, there exists an R I fl such that $Q \sim R$.

Since $f^{-1} \in (\text{Aut } \mathfrak{H})_O$, $fl \sim fm$ also implies $l \sim m$.

5.14. <u>5.14</u>. Now consider any pair of points $P = (a, b)$, $Q = (c, d)$. Clearly, the map $\varphi : \mathbb{P} \to \mathbb{P}$ defined by $\varphi(x, y) =$

$(x + c - a, y + d - b)$, takes P into Q. We can use φ to define a map $\psi : \mathbb{L} \to \mathbb{L}$ in the following manner. Take any line l, then by 3.21, there exist two points R and S on l such that $R \not\sim S$. Clearly, $\varphi R \not\sim \varphi S$ and so there exists a unique line m with $\varphi R, \varphi S$ I m. Let $\psi(l) = m$.

<u>5.15. Lemma</u>. If $f = (\varphi, \psi)$, where φ, ψ are defined as above, then $f \in \text{Aut}\ \mathfrak{A}$.

<u>Proof</u>. It is clear that φ is bijective and by definition, if P I l, then φP I ψl and conversely.

Let $g = [m, n]_1$ and $h = [m, p]_1$ be a pair of parallel lines then $(n, 0), (m + n, 1)$ I $[m, n]_1$ and $(p, 0), (m + p, 1)$ I $[m, p]_1$. Therefore,

$$(n + c - a, d - b), (m + n + c - a, 1 + d - b) \text{ I } \psi([m, n]_1)$$

and

$$(p + c - a, d - b), (m + p + c - a, 1 + d - b) \text{ I } \psi([m, p]_1);$$

hence,

$$\psi([m, n]_1) = [m, n + c - a - dm + bm]_1$$

and

$$\psi([m, p]_1) = [m, p + c - a - dm + bm]_1$$

using the methods of 3.17. Therefore, parallelism is preserved for lines of the first kind. If the two lines are of the second kind, the result follows in a similar fashion. In addition, the same argument may be used to show that if $\psi l \parallel \psi m$, then $l \parallel m$.

The surjectivity of ψ is clear since for any $[m, n]_1$ and $[p, q]_2$,

$$\psi([m, n - c + a + dm - bm]_1) = [m, n]_1$$

and

$$\psi([p, q - d + b + cp - ap]_2) = [p, q]_2.$$

We show finally that ψ is injective. If two lines have the same image, then they must be of the same kind and have the same first coordinate. Without loss of generality, consider two lines of the first kind, $[m, n]_1$ and $[m, p]_1$. If they have the same image, then

$$n + c - a - dm + bm = p + c - a - dm + bm;$$

hence, $n = p$.

Thus, $f \varepsilon$ Aut 当.

5.16. We shall call the automorphism which takes P to Q, as defined above, f_{PQ}.

5.7 and 5.8 together with 5.15 completely determine the structure of all automorphisms of 当. If we take any $f \varepsilon$ Aut 当 such that $f(0) = P$, for some point P and let $f' = f_{PO} \circ f \;\; \varepsilon \; (\text{Aut 当})_0$, then we have $f = f_{OP} \circ f'$ and therefore, f has the form

$$f(a, b) = \lambda(a)f'(1, 0) + \lambda(b)f'(0, 1) + P$$

$$= \lambda(a)f(1, 0) + \lambda(b)f(0, 1) + (1 - \lambda(a) - \lambda(b))P.$$

This, then, is what one calls the Fundamental Theorem.

Finally, we show that any automorphism of 当 preserves the neighbour relation.

<u>5.17. Lemma.</u> If $f \in \mathrm{Aut}\,\mathfrak{H}$, then $P \sim Q$ if and only if $fP \sim fQ$.

<u>Proof.</u> Consider any pair of neighbouring points P and Q. Let $f(O) = R$. Recall that f can be put in the form, $f = f_{OR} \circ f'$, where $f' \in (\mathrm{Aut}\,\mathfrak{H})_o$. By 5.12, f' preserves the neighbour relation. Therefore,

$$fP - fQ = (f'P + R) - (f'Q + R)$$
$$= f'P - f'Q \in \eta \times \eta.$$

Since $f^{-1} \in \mathrm{Aut}\,\mathfrak{H}$, the same argument implies $P \sim Q$ if $fP \sim fQ$.

<u>5.18.</u> An argument similar to the proof of 5.13 implies that $l \sim m$ if and only if $fl \sim fm$.

APPENDIX

We give two examples of local rings which are not A. H. rings. We then construct generalized A. H. planes over these rings and show that they are not A. H. planes.

Consider the set Q of rationals with denominator not divisible by a fixed prime p. It is clear that Q is a local ring, with the non-units being those elements of Q with numerator divisible by p; however, $D_o = \{0\} \neq \eta$. Therefore, Q is not an A. H. ring. If we take any pair of points $a = \frac{a}{a'}\,p^i$, $b = \frac{b}{b'}\,p^j \in Q$, where a, a', b, b' are not divisible by p and $i \geq j$, then

$$\frac{a}{a'}p^i = \frac{b}{b'}p^j \frac{b'}{b} \frac{a}{a'}p^{i-j} \in Q.$$

Therefore, if we construct a generalized A. H. plane over Q in the manner of Chapter 3, it is clear that for distinct $\frac{a}{b}, \frac{c}{d} \in \eta$, we have $[\frac{a}{b}, 0]_2$, $[\frac{c}{d}, 0]_2$ are neighbouring lines through (0, 0). However, for any point (x, y) to be incident with both these lines $0 = x(\frac{a}{b} - \frac{c}{d})$. Since $\frac{a}{b} - \frac{c}{d} \not\in \{0\} = D_o$, $x = 0$ and $y = x\,\frac{a}{b} = 0$. Therefore, such a structure is a generalized A. H. plane which is not an A. H. plane.

Consider now any field F. Let $R = \left\{\frac{p(x,y)}{q(x,y)} \;\middle|\; p(x,y),\ q(x,y) \in F[x,y];\ q(0,0) \neq 0\right\}$. Clearly, R is a local ring with elements of the form $\frac{p(x,y)}{q(x,y)}$, where $p(0,0) = 0$, as non-units. Once again, $D_o = \{0\} \neq \eta$. In addition, it is clear for the polynomials x and y that $x \not\in yR$ and $y \not\in xR$. If we construct a generalized A. H. plane over R in the manner of Chapter 3, it is clear that there is no line through the points (0, 0) and (x, y).

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