FUNCTORS ADJOINT ON THE RIGHT

REMARKS
ON
FUNCTORS ADJOINT ON THE RIGHT

By

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SCOPE AND CONTENTS: This thesis deals with functor categories of limit preserving functors. It provides certain information about such categories in general. It then proceeds to establish a dual equivalence between the category of sets and the category of zero-dimensional, compact, Hausdorff, Boolean lattices.

PREFACE

This thesis began as an attempt to clarify some of the notions discussed by J. Isbell in a paper delivered at the Kanpur Topological Conference, 1968. The first section of the second chapter does this.

Chapter one contains basic definitions and facts from category theory. Some of these facts are known, but are not easily found in the literature.

In Chapter two, after the expansion of some of Isbell's remarks, there is a discussion of the category of zero-dimensional, compact, Hausdorff, Boolean lattices $\mathcal{K}_0\mathcal{B}\ell$. A detailed proof is given to show that there is a dual equivalence between the category of sets and $\mathcal{K}_0\mathcal{B}\ell$. This latter fact is stated in [2] with a brief indication of proof. It must be pointed out that the proof given in this paper also demonstrates that every object in $\mathcal{K}_0\mathcal{B}\ell$ is a power of the two element object in $\mathcal{K}_0\mathcal{B}\ell$.

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CHAPTER 1

Preliminaries

This chapter is divided into two sections. The first section states some definitions and theorems from category theory that will help the reader understand this thesis. The second section provides motivation for the main body of this paper.

Section 1

In this section the notion of an adjunction is introduced and some interesting examples are explored. Afterwards, the Adjoint Functor Theorem is stated, the concept of representability is defined, and the two are then related.

First some general assumptions relating to the entire thesis must be stated. Ens will always denote the category of sets. Categorical duality will always be denoted by starring; for example, if $\mathcal{K}$ is a category, then its dual category will be written $\mathcal{K}^*$. Let $\mathcal{A}$ and $\mathcal{B}$ be categories. If a functor $F: \mathcal{A} \longrightarrow \mathcal{B}$ is contravariant, then F determines the two covariant functors $*F: \mathcal{A}^* \longrightarrow \mathcal{B}$ and $F^*: \mathcal{A} \longrightarrow \mathcal{B}^*$. Therefore all functors are assumed to be covariant except where stated otherwise.

In mathematics in general the notion of a function having an inverse is fundamentally important because it tells when various structures are essentially the same. In terms of category theory the notion of a functor having an adjoint plays a similar role.

<u>Definition</u>: An <u>adjunction</u> is an ordered triple $(F,G,\varphi): \mathcal{B} \longrightarrow \mathcal{A}$ where $F: \mathcal{B} \longrightarrow \mathcal{A}$ and $G: \mathcal{A} \longrightarrow \mathcal{B}$ are functors and φ is a natural equivalence

$$\varphi = \varphi_{b,a} : (Fb,a)_{\mathcal{A}} \cong (b,Ga)_{\mathcal{B}} \qquad (1.1)$$

The functor F is called a left adjoint for G, and G is called a right adjoint for F. The notation $F \dashv G$ will be used to indicate that there exists an adjunction $(F,G,\varphi): \mathcal{B} \longrightarrow \mathcal{A}$ and it is to be read as "F is the left adjoint of G".

In the context of this thesis this definition is sometimes unwieldly. The following definition and the two subsequent propositions provide a more relevant context for the notion of an adjunction. The proofs of the propositions are common in literature (see MacLane [10] p.116-121, or Mitchell [11] p.117-119).

<u>Definition</u>: Let $G:\mathcal{A} \longrightarrow \mathcal{B}$ be a functor and let b be an object of $\mathcal{B}$. A <u>universal morphism</u> from b to G is an ordered pair (a,u) where a is an object of $\mathcal{A}$ and $u: b \longrightarrow Ga$ is a morphism in $\mathcal{B}$ such that for any ordered pair (a',f) where a' is an object of $\mathcal{A}$ and $f: b \longrightarrow Ga'$ is a morphism of $\mathcal{B}$ there exists a unique morphism $f': a \longrightarrow a'$ in $\mathcal{A}$ with $Gf' \circ u = f$. This can be pictured in terms of the following diagrams:

$$\begin{tikzcd} a \arrow[d, "\exists! f'"] \\ a' \end{tikzcd} \qquad \begin{tikzcd} b \arrow[r, "u"] \arrow[dr, "f"'] & Ga \arrow[d, "Gf'"] \arrow[dl, phantom, "\circlearrowleft"] \\ & Ga' \end{tikzcd}$$

The dual of this notion is a universal morphism from G to b.

<u>Proposition 1.1</u> An adjunction $(F,G,\varphi): \mathcal{B} \longrightarrow \mathcal{A}$ determines

(i) A natural transformation $\eta: 1_{\mathcal{B}} \longrightarrow GF$, called the unit of the adjunction, which has the property that for any object b of $\mathcal{B}$ the ordered pair $(Fb,\ \eta_b: b \longrightarrow GFb)$ is universal from b to G, while for each $f: Fb \longrightarrow a$

$$\varphi f = Gf \circ \eta_b : b \longrightarrow Ga \tag{1.2}$$

(ii) A natural transformation $\varepsilon: FG \longrightarrow 1_{\mathcal{A}}$, called the counit of the adjunction, which has the property that each morphism ε_a is universal to a from F, while for each $g: b \longrightarrow Ga$

$$\varphi^{-1} g = \varepsilon_a \circ Fg : Fb \longrightarrow a \tag{1.3}$$

(iii) Both of the following composites are identities (of G, resp. F).

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G, \qquad F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F \qquad (1.4) \quad //$$

<u>Proposition 1.2</u> An adjunction is determined by functors $F: \mathcal{B} \longrightarrow \mathcal{A}$ and $G: \mathcal{A} \longrightarrow \mathcal{B}$ and natural transformations $\eta : 1_{\mathcal{B}} \longrightarrow GF$ and $\varepsilon: FG \longrightarrow 1_{\mathcal{A}}$ such that both composites (1.4) are identity transformations. Here φ is defined by (1.2) and φ^{-1} by (1.3). //

<u>Remarks</u>

(i) Because of this proposition, adjunctions are sometimes written as $(F, G, \eta, \varepsilon): \mathcal{B} \longrightarrow \mathcal{A}$.

(ii) Intuitively, Proposition 1.2 describes an adjunction by two identities on the unit and counit of an adjunction. This can be visualized by the following commutative diagrams.

$$\begin{tikzcd} G \arrow[r] \arrow[dr, equal] & GFG \arrow[d] \\ & G \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r] \arrow[dr, equal] & FGF \arrow[d] \\ & F \end{tikzcd}$$

The relationships expressed by the commutativity of these diagrams are sometimes called the triangle identities of an adjunction.

The notion of adjunction has been defined for covariant functors. Now suppose $S: \mathcal{A} \longrightarrow \mathcal{B}$ and $T: \mathcal{B} \longrightarrow \mathcal{A}$ are contravariant functors. S and T are called adjoint on the right if and only if there exists a natural equivalence $\varphi = \varphi_{a,b} : (a, Tb)_{\mathcal{A}} \cong (b, Sa)_{\mathcal{B}}$. To reduce this to the usual notion of an adjunction replace S and T by the covariant functors $*S: \mathcal{A}^* \longrightarrow \mathcal{B}$ and $T^*: \mathcal{B} \longrightarrow \mathcal{A}^*$. Then φ becomes $\bar{\varphi} = \bar{\varphi}_{b,a} : (T^*b, a)_{\mathcal{A}^*} \cong (b, {*Sa})_{\mathcal{B}}$. (i.e., $\varphi_{a,b} = \bar{\varphi}_{b,a}$), and hence there is an adjunction $(T^*, *S, \bar{\varphi}): \mathcal{B} \longrightarrow \mathcal{A}^*$. The unit of this adjunction is $\bar{\eta}: 1_{\mathcal{B}} \longrightarrow *ST^*$ and the counit is $\bar{\varepsilon}: T^{**}S \longrightarrow 1_{\mathcal{A}^*}$. In terms of φ, the unit and counit are $\eta: 1_{\mathcal{B}} \longrightarrow ST$ and $\varepsilon: 1_{\mathcal{A}} \longrightarrow TS$ respectively.

Utilizing the notion of an adjunction, the following definitions and proposition extend and clarify the notion of an isomorphism in terms of category theory.

Definitions

(i) A functor $S: \mathcal{A} \longrightarrow \mathcal{B}$ is called an <u>equivalence</u> of $\mathcal{A}$ with $\mathcal{B}$ if and only if there exists a functor $T: \mathcal{B} \longrightarrow \mathcal{A}$ such that $1_{\mathcal{A}} \cong TS$ and $ST \cong 1_{\mathcal{B}}$.

(ii) A <u>dual equivalence</u> is an equivalence of $\mathcal{A}$ with $\mathcal{B}^*$.

(iii) An <u>adjoint equivalence</u> of $\mathcal{A}$ with $\mathcal{B}$ is an adjunction $(T, S, \eta, \varepsilon): \mathcal{B} \longrightarrow \mathcal{A}$ in which both the unit $\eta: 1_{\mathcal{B}} \longrightarrow ST$ and the counit $\varepsilon: TS \longrightarrow 1_{\mathcal{A}}$ are natural equivalences.

(iv) A <u>dual adjoint equivalence</u> is an adjoint equivalence of $\mathcal{A}$ with $\mathcal{B}^*$.

Remark Let $(T,S,\eta,\varepsilon):\mathcal{B}\longrightarrow\mathcal{A}$ be an adjoint equivalence. Then since η and ε are natural equivalences, η^{-1} and ε^{-1} are also natural equivalences, and hence the triangle identities $\varepsilon T\circ T\eta = 1$ and $S\varepsilon\circ\eta S = 1$ can be written as $T\eta^{-1}\circ\varepsilon^{-1}T = 1$ and $\eta^{-1}S\circ S\varepsilon^{-1} = 1$ respectively. These identities then state that $(S,T,\eta^{-1},\varepsilon^{-1}):\mathcal{A}\longrightarrow\mathcal{B}$ is an adjunction with $\varepsilon^{-1}:1_{\mathcal{A}}\longrightarrow TS$ as unit and $\eta^{-1}:ST\longrightarrow 1_{\mathcal{B}}$ as counit. Therefore in an adjoint equivalence $(T,S,\eta,\varepsilon):\mathcal{B}\rightarrow\mathcal{A}$ the functor $T:\mathcal{B}\longrightarrow\mathcal{A}$ is the left adjoint of $S:\mathcal{A}\longrightarrow\mathcal{B}$ with the unit η and at the same time T is the right adjoint of S with unit ε^{-1}.

The following proposition clarifies the interconnection between the various kinds of equivalences. For the proof see MacLane [10], p.135-137.

Proposition 1.3 The following properties of a functor $S:\mathcal{A}\longrightarrow\mathcal{B}$ are equivalent.

(i) S is an equivalence of categories.

(ii) S is part of an adjoint equivalence $(T,S,\eta,\varepsilon):\mathcal{B}\longrightarrow\mathcal{A}$.

(iii) S is full, faithfull, and representative. //

Remark A functor $S:\mathcal{A}\longrightarrow\mathcal{B}$ is called full if and only if for any morphism $g\in(Sa,Sb)$ there exists a morphism $g'\in(a,b)$ such that $Sg' = g$. S is called faithfull if it preserves distinctness of morphisms. S is called representative if for every object b in $\mathcal{B}$ there exists an object a in $\mathcal{A}$ such that Sa is isomorphic to b.

In general in the literature there are many examples of adjunctions $(T,S,\varphi):\mathcal{B}\longrightarrow\mathcal{A}$. In order to make the foregoing material more meaningful some examples will now be given of adjunctions of the form $(T,S,\varphi):\mathcal{B}^*\longrightarrow\mathcal{A}$.

Example 1.1 Let $\mathcal{K}$ be a category. Suppose there are functors $\Box:\mathcal{K}\times\mathcal{K}\longrightarrow\mathcal{K}:(X,Y)\rightsquigarrow X\Box Y$ and $(\ )^{(\ )}:\mathcal{K}\times\mathcal{K}^*\longrightarrow\mathcal{K}:(X,Y)\rightsquigarrow X^Y$. Suppose further there are equivalences $\varphi_{X,Y,Z}:(X\Box Y,Z) = (X,Z^Y)$ which is natural in X, Y, and Z, and $\gamma_{X,Y}:X\Box Y = Y\Box X$ which is natural in X and Y. Fix A and put $\varphi_{X,Y}=\varphi_{Y,X,A}\circ(\gamma_{X,Y},i_A)\circ\varphi_{X,Y,A}$ where the composite is defined as the series of natural equivalences

$$(X,A^Y)\cong(X\Box Y,A)\cong(Y\Box X,A)\cong(Y,A^X)$$

Hence the composite is an equivalence $\varphi_{X,Y}:(X,A^Y)\cong(Y,A^X)$.

For a fixed object A in $\mathcal{K}$ define a contravariant functor $F:\mathcal{K}\longrightarrow\mathcal{K}:X\rightsquigarrow A^X$. This defines functors $F^*:\mathcal{K}\longrightarrow\mathcal{K}^*:X\rightsquigarrow A^X$ and $^*F:\mathcal{K}^*\longrightarrow\mathcal{K}:X\rightsquigarrow A^X$. Then by the previous assumption on $\mathcal{K}$,

$$(F^*X,Y)_{\mathcal{K}^*}=(A^X,Y)_{\mathcal{K}^*}=(Y,A^X)_{\mathcal{K}}\cong(X,A^Y)_{\mathcal{K}}=(X,{}^*FY)_{\mathcal{K}}.$$

Hence $(^*F,F^*,\varphi_{X,Y}):\mathcal{K}^*\longrightarrow\mathcal{K}$ is an adjunction and $^*F\dashv F^*$; i.e. F is adjoint on the right to itself.

The general setting is illustrated by the following special cases.

(i) Let $\mathcal{X}$ = Ens. Here $\Box = \times$, the usual cartesian product for sets, and $A^B = \{\text{all functions } B \longrightarrow A\}$. Let U, Y, and Z be sets. Any function $f : U \times Y \longrightarrow Z$ of two variables naturally determines a function $\varphi_f : Y \longrightarrow Z^U$ of one variable (in Y) whose values are functions of a second variable (in U). More precisely, $\varphi_f(y)(u) = f(u,y)$ for all $u \in U$ and for all $y \in Y$. It will now be shown that the assignment $f \rightsquigarrow \varphi_f$ defines an equivalence $\varphi : (U \times Y, Z) = (Y, Z^U)$ natural in X, Y, and Z.

To show that the assignment $f \rightsquigarrow \varphi_f$ is one-to-one, suppose $f \rightsquigarrow \varphi$ and $g \rightsquigarrow \varphi$. Then for all $y \in Y$ and for all $u \in U$, $f(u,y) = \varphi(y)(u) = g(u,y)$. This implies that the functions f and g are equal. To show that the assignment $f \rightsquigarrow \varphi_f$ is onto let $\psi \in (Y, Z^U)$. Hence $\psi : Y \longrightarrow Z^U$ and each $y \in Y$ determines $\psi(y) : U \longrightarrow Z$. Then each $u \in U$ determines $\psi(y)(u) \in Z$. Therefore a function $f : U \times Y \longrightarrow Z$ is defined by $f(u,y) = \psi(y)(u)$ and $f \rightsquigarrow \psi$.

If Y and U are sets, then there is a natural isomorphism from $Y \times U$ to $U \times Y$ which takes (y,u) to (u,y). So for a fixed set A, the general setting says that $(U, A^Y) \cong (Y, A^U)$.

In summary, what this discussion shows is that for a fixed A the functor $A^{(\ )}$ is adjoint on the right to itself.

(ii) Let $\mathcal{K}$ be a category of modules over a commutative ring R with unit. Let $\square = \otimes$, the usual tensor product, and let $A^B = \mathrm{Hom}(B,A)$ as an R-module.

Let M, N, and P be R-modules. Any R-bilinear function $f: M \times N \longrightarrow P$ determines functions $\bar{f}_m = f(m,-)$ from N to P. Since f is linear in n for a fixed $m \in M$, $\bar{f}_m = f(m,-): N \longrightarrow P$ is linear, and so $\bar{f}_m \in P^N$. Thus the assignment $m \rightsquigarrow \bar{f}_m$ is a function $\bar{f}: M \longrightarrow P^N$. By assumption f is also linear in m for a fixed $n \in N$ and hence $\bar{f}: M \longrightarrow P^N$ is linear and so $\bar{f} \in \mathrm{Hom}(M,P^N)$.

The following argument will show that $f \rightsquigarrow \bar{f}$ is an isomorphism by constructing an inverse for it. Given any $\bar{f}: M \longrightarrow P^N$ define f by $f(m,n) = \bar{f}_m(n)$. Then the linearity of $\bar{f}$ implies that $f(m,n)$ is linear in m, and the linearity of each $\bar{f}_m: N \longrightarrow P$ implies that $f(m,n)$ is linear in n. Therefore f is bilinear and $\bar{f} \rightsquigarrow f$ is the desired inverse.

Hence the R-module S of R-bilinear mappings $M \times N \longrightarrow P$ is in natural one-to-one correspondence with $\mathrm{Hom}(M,P^N)$. If M and N are R-modules then recall that the tensor product of M and N is defined to be the ordered pair (T,g), where T is an R-module (usually denoted by $T = M \otimes N$) and $g: M \times N \longrightarrow T$ is a bilinear map which has the following universality property: Given any R-module P and any R-bilinear mapping $f: M \times N \longrightarrow P$

there exists a unique R-linear map $f':T \longrightarrow P$ such that $f = f' \circ g$. Now this property implies that every bilinear function $f:M \times N \longrightarrow P$ in S has the form $f(m,n) = f(m \otimes n)$ for a unique $f':T \longrightarrow P$. Now the R-bilinear mappings from $M \times N$ to P are naturally equivalent firstly to $\mathrm{Hom}(M,P^N)$ and secondly to $\mathrm{Hom}(M \otimes N,P)$. Hence $\mathrm{Hom}(M \otimes N,P) \cong \mathrm{Hom}(M,P^N)$.

Now for any R-modules M and N there is always a natural isomorphism φ from $M \otimes N$ to $N \otimes M$ defined by $\varphi : m \otimes n \rightsquigarrow n \otimes m$. Hence a category of modules over a commutative ring R with unit is an example of the general setting. In this category the functor $P^{(\ )}$ is self adjoint.

Specializing the example of modules, let $\mathcal{K}$ be a category of finite dimensional vector spaces over a field K. For any object $X \in \mathcal{K}$ put $F(X) = K^X = X^*$, the usual vector space dual; i.e., the set of linear functions from X to K. Then F is a contravariant functor from $\mathcal{K}$ to $\mathcal{K}$. By the first part of this example F is adjoint on the right to itself. Since X is finite dimensional, the equality $X = X^{**}$ defines natural equivalence $X \longrightarrow FFX$, and $FFX \longrightarrow X$ (the unit and counit of the adjunction). Hence this is an example of a category which is dually adjointly equivalent with itself.

Before we proceed to our next example, a lemma and its corollary have to be proven.

Lemma 1.4 Every diagram in the full subcategory of Ens consisting of the empty set and the singletons commutes.

Proof:

Let

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \arrow[d] \\ c \arrow[r] & d \end{tikzcd}$$

be a diagram in this subcategory of Ens.

If $a = \emptyset$, then there is exactly one map $a \longrightarrow d$.

If $a \neq \emptyset$, then b, c, and d are also not empty.

Hence $a \longrightarrow b \longrightarrow d = a \longrightarrow c \longrightarrow d$. //

Corollary 1.5 Let $\mathcal{K}$ be any category and let $F, G: \mathcal{K} \longrightarrow$ Ens be functors such that

(1) Fk and Gk are either the empty set or a singleton for all objects $k \in \mathcal{K}$.

(2) $Fk \neq \emptyset$ if and only if $Gk \neq \emptyset$.

Then $F \cong G$.

Proof:

Define $\varphi_k : Fk \longrightarrow Gk$ as follows:

$\varphi_k = \emptyset$ if they are both empty.

φ_k is the only map otherwise.

For any $u: k \longrightarrow k'$ the lemma above implies that the following diagram is always commutatuive.

$$\begin{tikzcd} Fk \arrow[r, "\varphi_k"] \arrow[d, "Fu"'] & Gk \arrow[d, "Gu"] \\ Fk' \arrow[r, "\varphi_{k'}"'] & Gk' \end{tikzcd}$$

Hence $F \cong G$. //

(iii) Let $\mathcal{K}$ be a pseudocomplemented lattice. Fix an object $k \in \mathcal{K}$. Let a be an object in $\mathcal{K}$. By definition an object $b \in \mathcal{K}$ is the pseudocomplement of a relative to k if and only if $b = \{\sup x \in \mathcal{K} : x \wedge a \leq k\}$. This b is denoted as $b = k^a$. For any object $y \in \mathcal{K}$, $y \leq b = k^a = a \wedge k \Leftrightarrow y \wedge a \leq k$. In this example let $\square = \wedge$, the usual meet in a lattice and let k^x = the pseudocomplement of x relative to k. Notice that $(y \wedge x, k) \neq \emptyset$ if and only if $(y, k^x) \neq \emptyset$. Hence by Corollary 1.5 the two functors $\mathcal{K}^* \times \mathcal{K}^* \times \mathcal{K} \longrightarrow \mathrm{Ens}$ defined by $(y,x,k) \rightsquigarrow (y \wedge x, k)$ and $(y,x,k) \rightsquigarrow (y, k^x)$ are naturally equivalent. In any lattice $\mathcal{L}$ there is a functor $F: \mathcal{L} \longrightarrow \mathcal{L} : x \wedge y \rightsquigarrow y \wedge x$. Hence by the general theory there exists an equivalence $\varphi = \varphi_{y,x} : (y, k^x) \rightsquigarrow (x, k^y)$, which is natural in x and y; i.e., the functor $k^{(\ )}$ given by taking pseudocomplements with respect to k is adjoint to itself on the right.

Specializing the previous example of a pseudocomplemented lattice let $\mathcal{K}$ be a Boolean lattice. In this case let $k = 0$, the least element of the lattice. Then for any object $x \in \mathcal{K}$, $0^x = x'$ the complement of x. The functor $0^{(\ )} : \mathcal{K} \longrightarrow \mathcal{K}$ is then simply the order inverting map of taking complements. Since $(a')' = a$ in a Boolean lattice, $0^{(\ )}$ is actually a dual equivalence. This is the familiar self-duality of Boolean lattices.

Example 1.2 (Stone Duality) Let $\mathcal{B}_s$ be the category of Boolean spaces and their continuous maps. Let $\mathcal{B}\ell$ be the category of Boolean lattices and their homomorphisms. Let $B: \mathcal{B}_s \longrightarrow \mathcal{B}\ell^*$ be the functor which maps any Boolean space X to its Boolean lattice of open-closed sets BX. For any $f:X \longrightarrow Y$ in $\mathcal{B}_s$, let $Bf:BY \longrightarrow BX$ be the map $A \rightsquigarrow f^{-1}(A)$ where A is any open-closed subset of Y. Let $\Omega: \mathcal{B}\ell^* \longrightarrow \mathcal{B}_s$ be the functor which maps any Boolean lattice L to its Stone space ΩL. For any $h:X \longrightarrow Y$ in $\mathcal{B}\ell$, $\Omega h: \Omega Y \longrightarrow \Omega X$ is the map $U \rightsquigarrow h^{-1}(U)$ where U is any ultrafilter in Y. Then $B \dashv \Omega$ and B and Ω provide an equivalence of $\mathcal{B}_s$ with $\mathcal{B}\ell^*$.

Proof:

BX is a Boolean lattice since finite unions, finite intersections, and complements of open-closed sets are again such sets. Bf is a Boolean lattice homomorphism since $f^{-1}(A \wedge B) = f^{-1}(A) \wedge f^{-1}(B)$, $f^{-1}(A') = f^{-1}(A)'$, $f^{-1}(A \vee B) = f^{-1}(A) \vee f^{-1}(B)$, and $B(fg) = (Bf)(Bg)$.

Consider the functor $\Omega: \mathcal{B}\ell^* \longrightarrow \mathcal{B}_s$ defined above. Here ΩL is the ultrafilter space of L. Its topology is generated by the basis $\{U: a \in U \in \Omega\} = \Omega_a$ for $a \in L$: since firstly, $\Omega_a \cap \Omega_b = \Omega_{a \wedge b}$, and secondly, $a,b \notin U \Rightarrow a',b' \in U \Rightarrow a' \wedge b' \in U \Rightarrow \Omega_a \cup \Omega_b = \Omega_{a \vee b}$.

ΩL is a Boolean space; i.e., it is zero-dimensional, compact, and Hausdorff. To see the zero-dimensionality notice that $\Omega = \Omega_e = \Omega_a \cup \Omega_{a'}$, and that

$\Omega_a \cap \Omega_{a'} = \Omega_{a \wedge a'} = \Omega_0 = \emptyset$ implies that $(\Omega_a)' = \Omega_{a'}$. To prove the Hausdorff property let U and V be the ultrafilters and suppose $U \neq V$. Then there exists an $a \in U$ where $a \notin V$. Hence $a \in U$ and $a' \in V$, which implies that $U \in \Omega_a$ and $V \in \Omega_{a'}$. To prove the compactness property consider $X \subseteq L$ where $\Omega = \bigcup_{a \in X} \Omega_a$ and suppose $\Omega \supsetneqq \bigcup_{a \in F} \Omega_a$ for all finite $F \subseteq X$. Now $\bigcup_{\substack{a \in F \\ F \text{ finite}}} \Omega_a = \Omega_{VF}$. The assumption that $\Omega_{VF} \subsetneqq \Omega$ implies that there exists a $U \in \Omega$ such that $VF \notin U$. This implies that $(VF)' \in U$ which in turn implies that $0 \neq a'_1 \wedge \dots \wedge a'_n \in U$ for $F = \{a_1, \dots, a_n\} \subseteq X$. Hence X' generates a proper filter which is contained in some ultrafilter U. But there exists an $a \in X$ such that $a \in U$, which is a contradiction since a' is also in U. Therefore ΩL is compact.

With respect to the action of Ω on maps, notice that $f^{-1}(U)$ is an ultrafilter since if $a \notin f^{-1}(U)$ then $f(a) \notin U$, so $f(a)' \in U$ and hence $a' \in f^{-1}(U)$. The map f is continuous because $(\Omega f)^{-1}(\Omega_a) = \{U : (\Omega f)(U) \in \Omega_a\} = \{U : a \in f^{-1}(U)\} = \{U : f(a) \in U\} = \Omega_{f(a)}$. Notice also that $\Omega(fg) = (\Omega f)(\Omega g)$.

Now define $\eta : 1_{Bs} \longrightarrow \Omega B$ and $: 1_{Bl^*} \longrightarrow B\Omega$ as follows.

$\eta_X : X \longrightarrow \Omega BX : a \rightsquigarrow \{V : a \in V \in BX\}$

$\varepsilon_L : L \longrightarrow B\Omega L : c \rightsquigarrow \Omega_c = \{U : c \in U \in \Omega\}$.

The following argument will show that η as defined above is a natural isomorphism. For $U \in BX$, $\eta_X^{-1}(\Omega_U) = \{a : U \in \eta_X(a)\} = \{a : a \in U\} = U$. So η is continuous. Next, let $f: X \longrightarrow Y \in \mathcal{B}\ell$. To prove the naturality of η it must be shown that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\
\Omega BX \arrow[r, "\Omega Bf"'] & \Omega BY
\end{tikzcd}$$

Note that for any open-closed set V, $Bf(V) = f^{-1}(V)$. Then for $f: X \longrightarrow Y$, $Bf: BY \longrightarrow BX: V \rightsquigarrow f^{-1}(V)$. Hence for any ultrafilter $\mathcal{U}$, $\Omega Bf: \Omega BX \longrightarrow \Omega BY: \mathcal{U} \rightsquigarrow \{V: f^{-1}(V) \in \mathcal{U}\}$. Then $\eta_X(a)$ is mapped to $\{V: a \in f^{-1}(V)\} = \{V: f(a) \in V\} = \eta_X f(a)$. Therefore $\Omega Bf \circ \eta_X = \eta_Y \circ f$, and the diagram commutes.

To finish the argument it remains to show that η_X is one-to-one and onto for each X. This implies it is an isomorphism because the spaces are compact. Let $a \neq b$ where $a, b \in X$. Then there exist open-closed sets U, V in BX such that $a \in U$, $b \in V$, $a \notin V$, $b \notin U$. Hence $\eta_X(a) \neq \eta_X(b)$ and therefore η_X is one-to-one. Now for any $\mathcal{U} \in \Omega BX$ there exists an $a \in \bigcap_{V \in \mathcal{U}} V$ by compactness. So $\mathcal{U} \subseteq \eta_X(a)$ and therefore $\mathcal{U} = \eta_X(a)$ since $\mathcal{U}$ is an ultrafilter. Hence η_X is onto. This concludes the proof that η is a natural isomorphism.

The following argument will show that ε as previously defined is a natural isomorphism. That ε is a

Boolean lattice homomorphism comes from the facts that $\Omega_a \cap \Omega_b = \Omega_{a \wedge b}$, $\Omega_a \cup \Omega_b = \Omega_{a \vee b}$, $\Omega_0 = \emptyset$, and $\Omega_e = \Omega_L$. It will now be shown that ε_L is one-to-one and onto for each L. This will then imply that ε_L is an isomorphism for each L. First, $\varepsilon_L(c)$ is open-closed since $\Omega_c' = \Omega_{c'}$. Let $c \neq d$, where $c, d \in L$, then $c \wedge d < c \vee d$. To show that ε_L is one-to-one, it is sufficient to produce an ultrafilter containing one but not the other. Now $c \wedge d < c \vee d$ implies $(c \vee d) \wedge (c \wedge d)' \neq 0$. So there exists a $U \in \Omega L$ with $(c \vee d) \wedge (c \wedge d)' \in U$. Then $c \vee d \in U$; also, $(c \wedge d)' \in U$ and hence $c \wedge d \notin U$. Thus $U \in \Omega_{c \vee d}$ and $U \notin \Omega_{c \wedge d}$. Since $\Omega_c = \Omega_d$ implies $\Omega_{c \wedge d} = \Omega_{c \vee d}$ this shows $\Omega_c \neq \Omega_d$. ε_L is onto, since suppose $\Sigma \subsetneq \Omega L$ is an open-closed set. Then $\Sigma = \bigcup_{a \in L} \Omega_a$ compact implies $\Sigma = \Omega_{a_1} \cup \dots \cup \Omega_{a_k} = \Omega_{a_1 \vee \dots \vee a_k}$; i.e., the Ω_a are exactly the open-closed sets.

Let $f : K \longrightarrow L \in \mathcal{Bl}$. To prove the naturality of it must be shown that the following diagram commutes.

$$\begin{array}{ccc} K & \longrightarrow & L \\ \downarrow & & \downarrow \\ B\Omega K & \longrightarrow & B\Omega L \end{array}$$

Now $\Omega f : \Omega L \longrightarrow \Omega K : U \rightsquigarrow f^{-1}(U)$ and hence

$B\Omega f : B\Omega K \longrightarrow B\Omega L : \Sigma \rightsquigarrow \{U : f^{-1}(U) \in \Sigma\}$. So Ω_a is mapped to $\{U : f^{-1}(U) \in \Omega_a\} = \{U : f(a) \in U\} = \Omega_{f(a)}$. Therefore $B\Omega f \circ \varepsilon_K = \varepsilon_L \circ f$, and the diagram commutes. This concludes the proof that ε is a natural isomorphism.

What has been shown is that $\eta : 1_{\mathcal{B}s} \longrightarrow \Omega B$ and $\varepsilon : 1_{\mathcal{B}\ell^*} \longrightarrow B\Omega$ are natural equivalences. Hence by Proposition 1.2 there is adjoint equivalence between $\mathcal{B}s$ and $\mathcal{B}\ell^*$. //

<u>Example 1.3</u> (Galois Correspondences) Let $\mathcal{A}$ and $\mathcal{B}$ be quasi-ordered categories. Let $S : \mathcal{A} \longrightarrow \mathcal{B}$ and $T : \mathcal{B} \longrightarrow \mathcal{A}$ be order-inverting functors. Then S and T will be adjoint on the right if and only if for all objects $a \in \mathcal{A}$ and for all objects $b \in \mathcal{B}$,

$$b \leq Sa \text{ in } \mathcal{B} \quad \text{iff} \quad a \leq Tb \text{ in } \mathcal{A} \tag{1.5}$$

Proof:

Assume S and T are adjoint on the right. This implies that there is a natural equivalence $(b,Sa)_{\mathcal{B}} \cong (a,Tb)_{\mathcal{A}}$. Because $\mathcal{A}$ and $\mathcal{B}$ are quasi-ordered categories any hom set has at most one element. Therefore $b \leq Sa$ in $\mathcal{B}$ iff $a \leq Tb$ in $\mathcal{A}$.

Conversely assume $b \leq Sa$ in $\mathcal{B}$ iff $a \leq Tb$ in $\mathcal{A}$ and notice that this is equivalent to the assumption $b \leq Sa$ in $\mathcal{B}$ iff $Tb \leq a$ in $\mathcal{A}^*$. If $(b,Sa)_{\mathcal{B}} = \emptyset$ then also $(Tb,a)_{\mathcal{A}^*} = \emptyset$. If $(b,Sa)_{\mathcal{B}} \neq \emptyset$ then also $(Tb,a)_{\mathcal{A}^*} \neq \emptyset$. Therefore by corollary 1.5 there is a natural equivalence

$(-,S-)_{\mathcal{B}} = (T-,-)_{\mathcal{A}*}$ and hence T and S are adjoint on the right.

The unit and counit of this adjunction are defined by the inequalities $\eta : 1_{\mathcal{B}} \longrightarrow ST : b \leq STb$ and $\varepsilon : TS \longrightarrow 1_{\mathcal{A}*} : TSa \leq a$. These conditions come from (1.5) by letting a = Tb and b = Sa respectively. Similarly, the triangle identities of the adjunction are defined by the inequalities $Sa \leq STSa \leq Sa$ and $Tb \leq TSTb \leq Tb$.

Because the relations on $\mathcal{A}$ and $\mathcal{B}$ are only quasi-orders, it is not possible to replace the ordering symbols of the triangle identities by equality signs. If the orderings on $\mathcal{A}$ and $\mathcal{B}$ are partial orders, then the triangle identities become S = STS and T = TST respectively. If the orderings are only quasi-orders then all that can be said is that the unit of the adjunction is the inequality $b \leq STb$, the counit is the inequality $TSa \leq a$ and the inequalities $Sa \leq STSa \leq Sa$ and $Tb \leq TSTb \leq Tb$ are the triangle identities connecting the unit and counit.

An adjunction $(T,S,\varphi) : \mathcal{B} \longrightarrow \mathcal{A}*$ where $STS \cong S$ and $TST \cong T$ is called a Galois correspondence between $\mathcal{A}$ and $\mathcal{B}$. Note that for any Galois correspondence, φ can always be defined by the relation $b \leq Sa$ in $\mathcal{B}$ iff $Tb \leq a$ in $\mathcal{A}*$. //

The previous definitions and examples naturally lead to the question of when a functor has an adjoint. The following discussion gives necessary and sufficient conditions for a functor to have a left adjoint.

<u>Definition</u> Let $\mathcal{A}$ and $\mathcal{B}$ be categories and let $T:\mathcal{B} \longrightarrow \mathcal{A}$ be a functor. A set of objects $\{s_i\}_{i \in I}$ in $\mathcal{B}$ is called a solution set with respect to T for an object a in $\mathcal{A}$ if and only if for any object b in $\mathcal{B}$ and for any morphism $a \longrightarrow T(b)$ in $\mathcal{A}$, there exist morphisms $a \longrightarrow T(s_i)$ in $\mathcal{A}$ and $\alpha : s_i \longrightarrow b$ in $\mathcal{B}$ for some $i \in I$ such that the following diagram commutes.

$$\begin{tikzcd} a \arrow[r] \arrow[dr] & T(s_i) \arrow[d, "T(\alpha)"] \\ & T(b) \end{tikzcd}$$

For the proofs of the following theorem and its corollary see Mitchell [11] p. 124-126.

<u>Theorem 1.6</u> (Adjoint Functor Theorem)

Let $T: \mathcal{B} \longrightarrow \mathcal{A}$ be a functor where $\mathcal{B}$ is complete and locally small. Then T has a left adjoint if and only if it is a limit preserving functor which admits a solution set for every object in $\mathcal{A}$. //

Corollary 1.7 (Special Adjoint Functor Theorem)

If $\mathcal{B}$ is a complete and locally small category with a cogenerator, then $T:\mathcal{B} \longrightarrow \mathcal{A}$ has a left adjoint if and only if it is limit preserving. //

Remark The importance of Corollary 1.7 is that it points out circumstances under which the solution set requirement can be relaxed.

Another important categorical notion is that of representability. The definition will be followed by the Yoneda lemma which presents some basic information about representability. A proposition will then be stated which explains the connection between representable functors and those functors from an arbitrary category into Ens which have adjoints.

Definition Let $\mathcal{A}$ be a category. A functor $S:\mathcal{A} \longrightarrow \mathrm{Ens}$ is called representable if and only if it is naturally equivalent to a hom functor $H^a = (a,-)_{\mathcal{A}}$ for some object $a \in \mathcal{A}$.

For the proofs of the following three lemmas see MacLane [10] p.82-83.

Lemma 1.8 (Yoneda)

Let $S:\mathcal{A} \longrightarrow \mathrm{Ens}$ be a functor and let r be an object of $\mathcal{A}$. Then there is a bijection

$$Y: \left((r,-)_{\mathcal{A}} , S\right)_{\mathrm{Nat}} \cong S(r) \qquad (1.6)$$

which sends each natural transformation $\alpha:(r,-) \longrightarrow S$ to $\alpha_r 1_r$, the image of the identity $1_r : r \longrightarrow r$. //

Corollary 1.9

Let r and s be objects of $\mathcal{A}$. Each natural transformation $(r,-)_{\mathcal{A}} \longrightarrow (s,-)_{\mathcal{A}}$ has the form $(f,-)$ for a unique $f:r \longrightarrow s$. //

Now consider the functor $S:\mathcal{A} \longrightarrow Ens$ as an object in the functor category $Ens^{\mathcal{A}}$, and consider (S,r) as an object in the category $Ens^{\mathcal{A}} x \mathcal{A}$. Define an evaluation functor

$$E:Ens^{\mathcal{A}} x \mathcal{A} \longrightarrow Ens:(S,r) \rightsquigarrow Sr$$
$$:(F:S \rightarrow S', f:r \rightarrow r') \rightsquigarrow (F(f):Sf \rightarrow S'f)$$

Define a functor N as follows

$$N:Ens^{\mathcal{A}} x \mathcal{A} \longrightarrow Ens:(S,r) \rightsquigarrow ((r,-)_{\mathcal{A}} ,S)_{Nat}$$
$$:(F:S \rightarrow S', f:r \rightarrow r') \rightsquigarrow ((f,-)_{\mathcal{A}} ,F)_{Nat}$$

Lemma 1.10

The bijection (1.6) is a natural equivalence $y:N \longrightarrow E$ between functors $N,E:Ens^{\mathcal{A}} x \mathcal{A} \longrightarrow Ens$. For a morphism $f \in \mathcal{A}$, the correspondence $(f:s \rightarrow r) \rightsquigarrow ((f,-):(r,-) \rightarrow (s,-))$ is a faithful functor

$$Y:\mathcal{A}^* \longrightarrow Ens^{\mathcal{A}} \qquad (1.7)$$

called the Yoneda function. Its dual is the faithful functor

$$Y':\mathcal{A} \longrightarrow Ens^{\mathcal{A}^*} \qquad (1.8)$$

which sends $f:s$ —— r to the natural transformation $(-,f): \mathcal{A}^* \longrightarrow Ens:(-,s)_{\mathcal{A}} \longrightarrow (-,r)_{\mathcal{A}}$. //

There now remains one special case of an adjoint functor that has to be discussed. This section of the introduction will therefore conclude with the following proposition.

Proposition 1.11 Let $T: \mathcal{A} \longrightarrow \mathrm{Ens}$ be a functor. If T has a left adjoint, then T is representable.

Proof:

Let S be the left adjoint of T. Then there is a natural equivalence $(S-,-)_{\mathcal{A}} \cong (-,T-)_{\mathrm{Ens}}$. Let 1 be a one element set in Ens. Then the equivalence becomes $(S1,-) \cong (1,T-)$. Also $1_{\mathrm{Ens}} = (1,-)$. Composing with T, this becomes $T \cong (1,-) \circ T = (1,T-)$.

Then the equivalence $(S1,-) \cong (1,T-) \cong T$ implies that T is representable by S1. //

Section 2

By way of introduction, let $\mathcal{A}$ be an equational class of finitary algebras and their homomorphisms. Let Θ be the full subcategory of $\mathcal{A}$ generated by the free algebras with finite basis. This section will show how to extend a product preserving functor $A:\Theta^* \longrightarrow$ Ens to a limit preserving functor $\overline{A}: \mathcal{A}^* \longrightarrow$ Ens. After the introduction of functor categories, the Yoneda Lemma and Proposition 1.11 then imply that there is an equivalence between $\mathcal{A}$ and the functor category of limit preserving functors from $\mathcal{A}^*$ to Ens. For any categories $\mathcal{B}$ and $\mathcal{D}$, the Special Adjoint Functor Theorem (Corollary 1.7) will be applied to the category of limit preserving functors from $\mathcal{D}^*$ to $\mathcal{B}$, which is denoted $\mathcal{B}^{(\mathcal{D}^*)}$. This will lead to a proposition about a categorical equivalence between $\mathcal{B}^{(\mathcal{D}^*)}$ and a category $\mathcal{K}$ of certain pairs of functors. It is the functor category $\mathcal{B}^{(\mathcal{D}^*)}$ and the latter equivalence that will be studied in Chapter 2 of this thesis.

Proposition 1.12 Let $\mathcal{A}$ be an equational class of finitary algebras and let Θ be the full subcategory of all finitely generated free $\mathcal{A}$- algebras. Let $S:\Theta \longrightarrow \mathrm{Ens}$ be a contravariant functor which takes coproducts to products. Then S can be extended to a limit preserving functor T from $\mathcal{A}^*$ to Ens.

Proof:

Let $F_{\mathcal{A}}:\mathrm{Ens} \longrightarrow \mathcal{A}$ be the free $\mathcal{A}$- algebra functor. Let Θ be the full subcategory of $\mathcal{A}$ whose objects are all the $F_{\mathcal{A}}k$, $k \in N$. Now $SF_{\mathcal{A}}k = (SF_{\mathcal{A}}1)^k$. For convenience put $X = SF_{\mathcal{A}}1$. Let $(f_\alpha)_{\alpha \in I}$ be the operations and let $(n_\alpha)_{\alpha \in I}$ be the type of algebras in $\mathcal{A}$.

Define $h_\alpha : F_{\mathcal{A}}1 \longrightarrow F_{\mathcal{A}}n$ by $h_\alpha(x_1) = f_\alpha(x_1,\ldots,x_n)$, so $Sh_\alpha : X^n \longrightarrow X$, and there is an algebra $A = (X,(Sh_\alpha)_{\alpha \in I})$.

Claim: $A \in \mathcal{A}$.

Let p be an element of the absolutely free algebra F with n basis elements $z_1,\ldots,z_n$. In any algebra B of $\mathcal{A}$ write $p(b_1,\ldots,b_n)$ $(b_i \in B)$ for the image of p with respect to the map $F \longrightarrow B$ defined by $z_k \rightsquigarrow b_k$, and write p_B for this function. To prove that $A \in \mathcal{A}$ it must be shown that $p_A = q_A$ for all p and q where $p_{F_{\mathcal{A}}n} = q_{F_{\mathcal{A}}n}$. Take any $p \in F$ and define $\overline{p}:F_{\mathcal{A}}1 \longrightarrow F_{\mathcal{A}}n$ by $\overline{p}(x_1) = p(x_1,\ldots,x_n)$. If it can be proven that $S\overline{p} = p_A$ then $p_{F_{\mathcal{A}}n} = q_{F_{\mathcal{A}}n}$ will imply $\overline{p} = \overline{q}$ which in turn implies $p_A = q_A$ which establishes the claim.

It will now be shown that $S\overline{p} = p_A$. The coproduct in $\mathcal{A}$ is defined by $i_k:F_{\mathcal{A}}1 \longrightarrow F_{\mathcal{A}}n$ where $i_k(x_1) = x_k$,

$k = 1,\ldots,n$ are the injections. Now the facts that $S(F_{\mathcal{A}}\, n) = X^n$ and that S takes coproducts to products imply that $Si_k = pr_k$. Also, $pr_k = (z_k)_A$ and $i_k = \overline{z}_k$. Therefore $S\overline{z}_k = (z_k)_A$ and hence $S\overline{p} = p_A$ holds for z_k. Next, if $S\overline{p} = p_A$ holds for $p_1,\ldots,p_{n_\alpha}$ consider $p = f_\alpha(p_1,\ldots,p_{n_\alpha})$ where f_α is an operation in F. Then

$$\begin{aligned}\overline{p}(x_1) &= f_\alpha(p_1,\ldots,p_{n_\alpha})(x_1,\ldots,x_n)\\ &= f_\alpha(p_1(x_1,\ldots,x_n),\ldots,p_{n_\alpha}(x_1,\ldots,x_n))\\ &= f_\alpha(\overline{p}_1(x_1),\ldots,\overline{p}_{n_\alpha}(x_1)).\end{aligned}$$

Now $\overline{p}_k : F_{\mathcal{A}}\,1 \longrightarrow F_{\mathcal{A}}\,n$, so $S\overline{p}_k : SF_{\mathcal{A}}\,n \longrightarrow SF_{\mathcal{A}}\,1$ and hence $S\overline{p}_1 \sqcap \ldots \sqcap S\overline{p}_{n_\alpha} : SF_{\mathcal{A}}n \longrightarrow SF_{\mathcal{A}}n_\alpha$. So for $\overline{p} = (\overline{p}_1 \sqcup \ldots \sqcup \overline{p}_{n_\alpha}) \circ h_\alpha$ where

$$\begin{array}{ccccc} F_{\mathcal{A}}1 & \xrightarrow{h_\alpha} & F_{\mathcal{A}}n_\alpha & \xrightarrow{\overline{p}_1 \sqcup \cdots \sqcup \overline{p}_{n_\alpha}} & F_{\mathcal{A}}\,n \\ x_1 & \rightsquigarrow & f_\alpha(x_1,\ldots,x_n\ ) & \rightsquigarrow & f_{n_\alpha}(\overline{p}_1(x_1),\ldots,\overline{p}_{n_\alpha}(x_1)) \end{array}$$

we have $S\overline{p} = Sh_\alpha \circ (S\overline{p}_1 \sqcap \ldots \sqcap S\overline{p}_{n_\alpha}) = Sh_\alpha(p_{1A} \sqcap \ldots \sqcap p_{n_\alpha A})$. (The last equality comes from the fact that $S\overline{p}_A = p_A$ for $p_1,\ldots,p_{n_\alpha}$.) For $a_1,\ldots,a_n\ X^n$,

$$\begin{aligned}S\overline{p}(a_1,\ldots,a_n) &= Sh_\alpha(p_1(a_1,\ldots,a_n),\ldots,p_{n_\alpha}(a_1,\ldots,a_n))\\ &= f_\alpha(p_1,\ldots,p_{n_\alpha})(a_1,\ldots,a_n) = p(a_1,\ldots,a_n),\end{aligned}$$

since Sh_α is the αth operation of $\mathcal{A}$. This implies $S\overline{p} = p_A$ and therefore $A \in \mathcal{A}$.

Define $T: \mathcal{A}^* \longrightarrow \mathrm{Ens}$ by putting $TB = (B,A)$ for the algebra A defined above.

Claim: T and S agree on Θ.

Any object in Θ can be written as $F_{\mathcal{A}}k$ for some

$k \in N$. Then $TF_A k = (TF_A 1)^k = (F_A 1, A)^k = X^k = (SF_A 1)^k = SF_A k$ implies that T acts the same on objects as S. To show that T and S are the same on morphisms take a morphism $h: F_A 1 \longrightarrow F_A m$, and consider the following commutative diagram (where u is any morphism in $(F_A m, A)$).

$$\begin{tikzcd}
(F_A m, A) \arrow[r] \arrow[d, equal] & (F_A 1, A) \arrow[dd] \\
(F_A 1, A)^m \arrow[d, equal] & \\
X^m \arrow[r] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
u \arrow[r, squiggly] \arrow[dd, squiggly] & uh \arrow[d, squiggly] \\
 & uh(x_1) \arrow[d, equal] \\
(u(x_1), \ldots, u(x_m)) \arrow[r, squiggly] & Sh(u(x_1), \ldots, u(x_m))
\end{tikzcd}$$

Now, if $h = \bar{p}$ with $p \in Fm$ then $Sh = S\bar{p} = p_A$; and, $p_A(u(x_1), \ldots, u(x_m)) = u(p(x_1, \ldots, x_m)) = u(h(x_1))$. Hence T acts the same on the morphisms of Θ as S does.

Therefore the contravariant functor $S: \Theta \longrightarrow Ens$ which takes coproducts to products has been extended to a functor $T: \mathcal{A}^* \longrightarrow Ens$ which preserves limits. //

In order to allow the discussion to proceed in a smooth way, some definitions have to be formulated.

<u>Definitions</u>

(i) Let $\mathfrak{D}$ and $\mathfrak{D}'$ be categories. A functor $J: \mathfrak{D}' \longrightarrow \mathfrak{D}$ is called codense if and only if each object in $\mathfrak{D}$ is a colimit of objects JD' where D' is an object in $\mathfrak{D}'$. If $\mathfrak{D}'$ is a subcategory of $\mathfrak{D}$ and $J: \mathfrak{D}' \longrightarrow \mathfrak{D}$ is the embedding functor, then $\mathfrak{D}'$ is called a codense subcategory of $\mathfrak{D}$.

(ii) A <u>functor category</u> is a category whose objects are functors between two given categories and whose morphisms are the natural transformations between the functors. It is usually assumed that the natural transformations between any two given functors form a set and not a proper class. If $\mathfrak{D}$ and $\mathfrak{B}$ are categories, then the functor category whose objects are the functors between $\mathfrak{D}$ and $\mathfrak{B}$ is denoted $\mathfrak{B}^{\mathfrak{D}}$.

<u>Remark</u> Since limit preserving functors play an important role in much of the following, let $\mathfrak{B}^{(\mathfrak{D})}$ denote the functor category whose objects are the limit preserving functors from $\mathfrak{D}$ to $\mathfrak{B}$, and whose morphisms are the natural transformations between these functors. In this case it is possible to replace the restriction that the natural transformations between any two functors form a set by the weaker restriction that the category $\mathfrak{D}$ has a codense subcategory $\mathfrak{D}'$, where the objects of $\mathfrak{D}'$ form a set and not a proper class. The point involved is that the functors involved are limit preserving and hence their values on the objects of $\mathfrak{D}$ are determined by their values on the objects of $\mathfrak{D}'$.

<u>Proposition 1.13</u> Let $\mathcal{B}$ be any category. Every functor in $\mathcal{B}^{(\mathcal{A}^*)}$ has a left adjoint.

Proof:

Since every functor in $\mathcal{B}^{(\mathcal{A}^*)}$ is limit preserving, it is enough to show that $\mathcal{A}^*$ satisfies the hypothesis of the Special Adjoint Functor Theorem (Corollary 1.7).

$\mathcal{A}$ has a generator, namely the free $\mathcal{A}$- algebra with one basis element. Hence $\mathcal{A}^*$ has a cogenerator.

Claim: $\mathcal{A}^*$ is complete.

This will be divided into two parts. It will be show that $\mathcal{A}$ has coequalizers, then it will be shown that $\mathcal{A}$ has coproducts.

Let $\mathfrak{A} = (A,(f_\alpha)_{\alpha \in I})$, $\mathfrak{B} = (B,(f_\alpha)_{\alpha \in I}) \in \mathcal{A}$ and let $u,v\colon \mathfrak{B} \longrightarrow \mathfrak{A} \in \mathcal{A}$. The coequalizer of u and v will now be constructed. Put

$\bar{A} = \{(u(b),v(b)) : b \in \mathfrak{B}\}$. Let θ be the smallest congruence on $\mathfrak{A}$ for which $\bar{A}$ is contained in the kernel of the natural map $\varkappa_\theta\colon \mathfrak{A} \longrightarrow \mathfrak{A}/\theta$. It will now be shown that $\varkappa_\theta$ is the coequalizer of u and v. Assume that there exists an $f\colon \mathfrak{A} \longrightarrow \mathfrak{X} \in \mathcal{A}$ for which $fu = fv$. Now $fu = fv \Rightarrow \bar{A} \leq \ker(f) \Longrightarrow \theta \leq \ker(f)$. Hence by the second isomorphism theorem for algebras, there exists a unique $w\colon \mathfrak{A}/\theta \longrightarrow \mathfrak{X}$ such that $w \circ \varkappa_\theta = f$.

Now for coproducts. Let $\{A_\alpha : \alpha \in \Lambda\}$ be a set of algebras in $\mathcal{A}$. It is required to prove that they have a coproduct. Let $\{X_\alpha : \alpha \in \Lambda\}$ be a set of disjoint sets. Let $X = \bigcup_{\alpha \in \Lambda} X_\alpha$. Let FX be the free algebra with basis X_α . Suppose there are morphisms $p_\alpha : FX_\alpha \longrightarrow A_\alpha$, which are onto, and with kernel θ_α .

Define $j_\alpha : FX_\alpha \longrightarrow FX$ by natural injection where FX_α is taken as a subalgebra of FX. Let θ be the congruence generated by the θ_α . Take $A = FX/\theta$ and let $v : FX \longrightarrow A = FX/\theta$ be the natural homomorphism.

Now the kernel of each vj_α contains θ_α . Hence there exist $i_\alpha : A_\alpha \longrightarrow A$ such that the following diagram commutes.

$$\begin{tikzcd} FX_\alpha \arrow[r, "j_\alpha"] \arrow[d, "p_\alpha"'] & FX \arrow[d, "v"] \\ A_\alpha \arrow[r, "i_\alpha"'] & A = FX/\theta \end{tikzcd}$$

It will now be demonstrated that A is the coproduct of $\{A_\alpha : \alpha \in \Lambda\}$. Suppose there are morphisms $\{f_\alpha : A_\alpha \longrightarrow B : \alpha \in \Lambda\}$. The sets X_α are disjoint and there are morphisms $\{f_\alpha \circ p_\alpha : FX_\alpha \longrightarrow B : \alpha \in \Lambda\}$. Now, since FX is free, these extend to $h : FX \longrightarrow B$ defined by $h|_{X_\alpha} = f_\alpha \cdot p_\alpha|_{X_\alpha}$. Now $\ker(h) \geq \theta_\alpha$ for all α ; hence, there exists a morphism $f : A \longrightarrow B$ such that $h = fv$. Then $f_\alpha = fi_\alpha$ and hence f is unique. Hence $\mathcal{A}^*$ is complete.

It remains to discuss the hypothesis of co-local smallness for $\mathcal{A}$. In [7] Isbell proves that $\mathcal{A}$ is colocally small. //

<u>Remark</u> Propositions 1.11 and 1.13 together say that in particular each functor in $\mathrm{Ens}^{(\mathcal{A}^*)}$ is representable; hence, there is an equivalence of $\mathcal{A}$ with $\mathrm{Ens}^{(\mathcal{A}^*)}$.

In the previous proposition it was shown that any limit preserving functor $A: \mathcal{A}^* \longrightarrow \mathcal{B}$ has a left adjoint. Conversely, by corollary 1.7, if a functor $A: \mathcal{A}^* \longrightarrow \mathcal{B}$ has a left adjoint then A is limit preserving. Define a category $\mathcal{K}$ as follows. The objects of $\mathcal{K}$ are ordered pairs (T,S) of functors, where $S: \mathcal{A}^* \longrightarrow \mathcal{B}$ is limit preserving, and where $T: \mathcal{B} \longrightarrow \mathcal{A}^*$ is a left adjoint of S. The morphisms of $\mathcal{K}$ are ordered pairs of natural transformations (τ,σ) where $\tau: T \longrightarrow T'$ and $\sigma: S \longrightarrow S'$ are natural transformations which are defined for objects $(T,S),(T',S') \in \mathcal{K}$. Composition is the usual composition of natural transformations.

<u>Remark</u> If $T \longrightarrow\!\!| \, S$ and also $T' \longrightarrow\!\!| \, S$, then there exists a natural equivalence $\alpha: T \cong T'$ (Mitchell [11] p.124).

<u>Proposition 1.14</u> There is an equivalence of $\mathcal{B}^{(\mathcal{A}^*)}$ and $\mathcal{K}$.

Proof:

Define $\varphi: \mathcal{K} \longrightarrow \mathcal{B}^{(\mathcal{A}^*)}: (T,S) \rightsquigarrow S$ where $T \longrightarrow\!\!| \, S$

$: ((\tau,\sigma):(T,S) \longrightarrow (T',S')) \rightsquigarrow (\sigma: S \longrightarrow S')$

where τ and σ are natural transformations.

Now φ is clearly well defined. It will now be proven that φ is an equivalence of $\mathcal{K}$ with $\mathcal{D}^{(\mathcal{A}^*)}$ by showing that φ is full faithful, and representative.

Let $\sigma: S \longrightarrow S' \in \mathcal{D}^{(\mathcal{A}^*)}$. By Proposition 1.13, S and S' have left adjoints, say T and T' respectively. From Mitchell [11] p.122 it is known that there exists a unique natural transformation $\tau: T \longrightarrow T'$. Then $(\tau,\sigma):(T,S) \longrightarrow (T',S')$ is natural in $\mathcal{K}$ and $\varphi(\tau,\sigma) = \sigma$. Therefore φ is full.

Now suppose $(\tau,\sigma),(\tau',\sigma') \in \mathcal{K}$ where we can assume $\sigma \not\cong \sigma'$ without loss of generality. Then $\varphi(\tau,\sigma) = \sigma \not\cong \sigma' = \varphi(\tau',\sigma')$. As an aside, notice that if $\tau \cong \tau'$ this forces $\sigma \cong \sigma'$ since (in the above notation) $T \longrightarrow\!\!| S$, $T' \longrightarrow\!\!| S'$, and $\tau \cong \tau'$ implies $\sigma \cong \sigma'$ by Mitchell [10] p.122. Hence φ is faithfull.

By Proposition 1.13 each object $S \in \mathcal{D}^{(\mathcal{A}^*)}$ has a left adjoint say T.. Therefore $(T,S) \in \mathcal{K}$ and $\varphi(T,S) = S$. Therefore φ is representative. //

<u>Corollary 1.15</u> If $\mathcal{D}$ is replaced by Ens in the proposition then there is an equivalence $\mathcal{A} \cong \mathrm{Ens}^{(\mathcal{A}^*)} \cong \mathcal{K}$.

Proof:

The first equivalence is clear from the remark preceeding the definition of the category $\mathcal{K}$. The second equivalence is clear from the proposition. //

Intuitively, the corollary points out that the equivalence between $\mathcal{A}$ and $\mathcal{K}$ assigns to each object $a \in \mathcal{A}$ a pair of functors (T,S) where $T \longrightarrow\!\!| \, S$ and where $a \in \mathcal{A}$ is a representing object for the limit preserving functor $S: \mathcal{A}^* \longrightarrow \mathrm{Ens}$. Conversely, each such pair (T,S) has connected to it a representing object $a \in \mathcal{A}$. When $\mathcal{D}$ is not the category of sets, it makes no sense to talk about representing objects. The closest analogy is to discuss those pairs of adjoint functors which are members of the category $\mathcal{K}$. In order to proceed, some definitions must first be made.

<u>Definitions</u>

(i) The functor category $\mathcal{D}^{(\mathcal{A}^*)}$ is called the category of <u>$\mathcal{A}$ - algebras in $\mathcal{D}$</u>.

(ii) Next, given categories $\mathcal{B}$ and $\mathcal{D}$, proposition 1.14 and the foregoing definition motivate the idea of defining a <u>$\mathcal{D}$ - object in $\mathcal{B}$</u> as a functor $F: \mathcal{D}^* \longrightarrow \mathcal{B}$ which preserves limits.

It is important to notice that each limit preserving functor $F: \mathcal{D}^* \longrightarrow \mathcal{B}$ will have an adjoint because suitable assumptions will be made on $\mathcal{D}$. Therefore, whenever it is convenient, the category $\mathcal{B}^{(\mathcal{D}^*)}$ may be replaced by the equivalent category $\mathcal{K}$. Chapter two will discuss some of the properties of a functor category $\mathcal{B}^{(\mathcal{D}^*)}$.

CHAPTER 2

Some Properties of Categories of the Form $\mathcal{B}^{(\mathcal{D}^*)}$

This chapter is divided into two sections. The first section will continue the discussion of the functor category $\mathcal{B}^{(\mathcal{D}^*)}$ which began in Chapter one. All the categories involved will be assumed complete, locally small, and to have cogenerators. The generator property involved will later be stated explicitly. The second section will specialize the discussion to the functor category $\mathcal{B}_s^{(\mathcal{B}l^*)}$ where $\mathcal{B}l$ and $\mathcal{B}_s$ are the categories of Boolean lattices and Boolean spaces respectively.

Section 1

The purpose of this section is to generalize a situation for topological groups. An object in the category of topological groups may be thought of as an underlying set TG together with two structures making TG a topological space and a group in a compatible way. Functors $F_{\mathcal{J}}$ and $F_{\mathcal{G}}$ can be defined from the category of topological groups $\mathcal{J}\mathcal{G}$ to the category of topological spaces $\mathcal{J}$, or to the category of groups $\mathcal{G}$, by mapping any topological group to its underlying space or to its underlying group respectively. Each of these functors may be followed by the underlying set functor U, which maps any space or group to its underlying set. The point about all this is that the process is commutative; i.e., the following diagram is commutative.

$$\begin{tikzcd} \mathcal{T}\mathcal{G} \arrow[r, "F_{\mathcal{T}}"] \arrow[d, "F_{\mathcal{G}}"'] & \mathcal{T} \arrow[d, "U"] \\ \mathcal{G} \arrow[r, "U"'] & \mathcal{E}ns \end{tikzcd}$$

It is this commutative situation that will be generalized. Let $\mathcal{B}$ and $\mathcal{D}$ be categories with distinguished objects B and D respectively. Define hom- functors as follows:

$$H^{B}: \mathcal{B} \longrightarrow \mathrm{Ens}: B' \rightsquigarrow (B,B')_{\mathcal{B}} = H^{B}(B')$$

$$H^{D}: \mathcal{D} \longrightarrow \mathrm{Ens}: D' \rightsquigarrow (D,D')_{\mathcal{D}} = H^{D}(D')$$

In the next few paragraphs functors $V_{\mathcal{B}} : \mathcal{B}^{(\mathcal{D}^*)} \longrightarrow \mathcal{B}$ and $V_{\mathcal{D}} : \mathcal{B}^{(\mathcal{D}^*)} \longrightarrow \mathcal{D}$ are defined so that the following diagram commutes in some sense.

$$\begin{tikzcd} \mathcal{B}^{(\mathcal{D}^*)} \arrow[r, "V_{\mathcal{B}}"] \arrow[d, "V_{\mathcal{D}}"'] & \mathcal{B} \arrow[d, "H^{B}"] \\ \mathcal{D} \arrow[r, "H^{D}"'] & \mathcal{E}ns \end{tikzcd} \qquad (2.1)$$

<u>Proposition 2.1</u> For any object A in $\mathfrak{D}^*$, evaluation at A is a functor $E_A : \mathfrak{B}^{(\mathfrak{D}^*)} \longrightarrow \mathfrak{B}$.

Proof:

For any object X in $\mathfrak{B}^{(\mathfrak{D}^*)}$ put $E_A(X) = X(A)$.

For any morphism $\alpha : X \longrightarrow Y$ in $\mathfrak{B}^{(\mathfrak{D}^*)}$ put $E_A(\alpha) = \alpha(A) : X(A) \longrightarrow Y(A)$.

This makes sense because α is a natural transformation from X to Y. By the definition of a natural transformation, E_A preserves identities. By the definition of composition of natural transformations, E_A preserves composition.

Hence E_A is a functor. //

Define $V_{\mathfrak{B}}$ as the evaluation at the distinguished object D of $\mathfrak{D}$;i.e. $V_{\mathfrak{B}} : \mathfrak{B}^{(\mathfrak{D}^*)} \longrightarrow \mathfrak{B} : X \rightsquigarrow X(D)$.

To define $V_{\mathfrak{D}} : \mathfrak{B}^{(\mathfrak{D}^*)} \longrightarrow \mathfrak{D}$ proceed as follows. Form the composite $H^B X : \mathfrak{D}^* \longrightarrow \text{Ens}$. The functor $H^B X$ is a composite of limit preserving functors and therefore (by corollary 1.7) it has a left adjoint. Hence by Proposition 1.11 it is representable. For each $X \in \mathfrak{B}^{(\mathfrak{D}^*)}$ choose a representing object $V_{\mathfrak{D}}(X)$ in $\mathfrak{D}$ for the composite functor $H^B X$.

<u>Proposition 2.2</u> $V_{\mathfrak{D}}$ as defined above is (the object function of) a functor $\mathfrak{B}^{(\mathfrak{D}^*)} \longrightarrow \mathfrak{D}$.

Proof:

To define $V_{\mathcal{D}}$ on morphisms proceed as follows. Let $\alpha : X \longrightarrow Y$ be a morphism in $\mathcal{B}^{(\mathcal{D}^*)}$. Form the composites $H^B X$ and $H^B Y$ and choose as representing objects $V_{\mathcal{D}}(X)$ and $V_{\mathcal{D}}(Y)$ respectively. Since α is a natural transformation, $H^B\alpha$ is a natural transformation from $H^B X$ to $H^B Y$. By corollary 1.9 $H^B\alpha$ has the form $(f,-)$ for a unique morphism $f : V_{\mathcal{D}}(X) \longrightarrow V_{\mathcal{D}}(Y)$. Put $V_{\mathcal{D}}(\alpha) = f$. By the uniqueness of f, $V_{\mathcal{D}}(\alpha)$ is well defined.

The fact that $V_{\mathcal{D}}$ satisfies the axioms for a functor comes from the facts that $H^B\alpha$ and α are natural transformations. //

Each $X \in \mathcal{B}^{(\mathcal{D}^*)}$ has a left adjoint, say $X^{\#}$. Then, by adjointness, $H^B X$ is represented by $X^{\#}B$. Hence,

$$\begin{aligned} H^D V_{\mathcal{D}}(X) &= (D, V_{\mathcal{D}}(X))_{\mathcal{D}} = (V_{\mathcal{D}}(X), D)_{\mathcal{D}^*} \\ &\cong (X^{\#}B, D)_{\mathcal{D}^*} \cong (B, X(D))_{\mathcal{B}} \\ &= (B, V_{\mathcal{B}}(X))_{\mathcal{B}} = H^B V_{\mathcal{B}}(X). \end{aligned}$$

The natural equivalence of $(V_{\mathcal{D}}(X), D)_{\mathcal{D}^*}$ and $(X^{\#}B, D)_{\mathcal{D}^*}$ comes from the uniqueness of the representing object. The natural equivalence of $(X^{\#}B, D)_{\mathcal{D}^*}$ and $(B, X(D))_{\mathcal{B}}$ comes from the fact that $X^{\#} \dashv X$. Therefore $H^D V_{\mathcal{D}}(X)$ and $H^B V_{\mathcal{B}}(X)$ are naturally equivalent for each $X \in \mathcal{B}^{(\mathcal{D}^*)}$. Essentially what this argument says is that the functor

$$\mathbf{H} : \mathcal{B}^{(\mathcal{D}^*)} \longrightarrow \mathrm{Ens} : X \rightsquigarrow (B, X(D))_{\mathcal{B}} \cong (D, V_{\mathcal{D}}X)_{\mathcal{D}}$$

has been defined so that the composites $H^B V_{\mathcal{B}}$ and $H^D V_{\mathcal{D}}$ are naturally equivalent functors. Intuitively, for each limit preserving functor $X \in \mathcal{B}^{(\mathcal{D}^*)}$, $H^B X$ is an object in $\mathcal{D}$ on the set $S = H^B X(D)$ and $X(D)$ is an object in $\mathcal{B}$ on the same set S.

Consider diagram (2.1). There are two topics that must be discussed with respect to this diagram. These are existence and uniqueness. For existence it will be shown that if $W: \mathcal{D}^* \longrightarrow \mathrm{Ens}$ is a representable functor and if V is an object of $\mathcal{B}$ such that $H^B(V) = W(D)$ then W can be lifted to a limit preserving functor $X: \mathcal{D}^* \longrightarrow \mathcal{B}$ with $H^B X = W$ and $X(D) = V$. For uniqueness it is necessary to show that if $V_{\mathcal{B}}(X) = V_{\mathcal{B}}(X')$ and $V_{\mathcal{D}}(X) = V_{\mathcal{D}}(X')$ then $X \cong X'$.

Uniqueness will be considered first. In order to proceed, some definitions have to be made.

Definitions

(i) An object D in a category $\mathcal{D}$ is called a (Freyd) generator for $\mathcal{D}$ if and only if the functor $H^D: \mathcal{D} \longrightarrow \mathrm{Ens}: D' \rightsquigarrow (D,D')_{\mathcal{D}}$ is faithful; i.e., for any distinct morphisms $f,g: E \longrightarrow F$ in $\mathcal{D}$ there exists a morphism $e: D \longrightarrow E$ such that $fe \neq ge$.

(ii) Let F and E be objects in a category $\mathcal{D}$. F is called a proper subobject of E if and only if there exists a monomorphism $F \longrightarrow E$ which is not an isomorphism.

(iii) An object D in a category $\mathcal{D}$ is called a <u>Grothendieck generator</u> for $\mathcal{D}$ if and only if for any object E in $\mathcal{D}$ and for any proper subobject F of E, there exists a morphism $e: D \longrightarrow E$ which does not factor through F.

<u>Remark</u> In Isbell [5] it is proved that in a locally small and co-complete category $\mathcal{D}$, an object D is a Grothendieck generator for $\mathcal{D}$ if and only if no proper subclass of the class of objects of $\mathcal{D}$ includes D and is closed under the formation of colimits. Thus every object of $\mathcal{D}$ can be constructed from D by transfinitely iterated formation of colimits.

Now the theorem about uniqueness can be stated and proven.

<u>Theorem 2.3</u> If $B \in \mathcal{B}$ is a generator and $D \in \mathcal{D}$ is a Grothendieck generator, then $V_{\mathcal{D}}(X)$ and $V_{\mathcal{B}}(X)$ determine $X \in \mathcal{B}^{(\mathcal{D}^*)}$ up to natural equivalence; i.e., if $H^B X \cong H^B X'$ and $X(D) \cong X'(D)$ then there exists a natural equivalence $\alpha : X \cong X'$.

Proof:

Let $X, X' \in \mathcal{B}^{(\mathcal{D}^*)}$ and assume $X(D) = X'(D)$. Then $\alpha_D = 1: X(D) \longrightarrow X'(D)$ is a natural equivalence between X and X' on the full subcategory of $\mathcal{D}^*$ consisting of the one object D.

Next suppose $\alpha: X|_{\mathcal{R}} \longrightarrow X'|_{\mathcal{R}}$ is defined and is a natural equivalence between X and X' for a full subcategory $\mathcal{R}$ of $\mathcal{D}$, where $D \in \mathcal{R}$. Since D is a Grothendieck generator, every object in $\mathcal{D}$ can be written as an iterated colimit of D. Hence, to prove the theorem, it is sufficient to extend α over one more object Q which is a colimit of a diagram E in $\mathcal{R}$.

Let $Q = \varinjlim E$ where $E:J \longrightarrow \mathcal{R}$ with $h_i:E(i) \longrightarrow Q$ the colimit maps. By the assumptions on X, the functor X takes this to $XQ = \varprojlim XE$ with $Xh_i:XE(i) \longrightarrow XQ$ the limit maps. By assumption α is a natural equivalence; hence, $XQ = \varprojlim X'E$ with the limit maps $\beta_i \circ Xh_i$ where $\beta_i = \alpha_{E(i)}$. Since X' takes colimits to limits, it is also true that $X'Q = \varprojlim X'E$ with limit maps $X'h_i$. By the uniqueness of limits this induces a unique isomorphism, $\beta:XQ \longrightarrow X'Q$ where $\beta_i Xh_i = X'h_i \beta$.

By assumption $H^B X \cong H^B X'$. Then, since H^B preserves limits $H^B(\beta)$ must be of the form $H^B XQ \longrightarrow H^B X'Q$. Extend α to α' by putting $\alpha_Q = \beta$. Now the transformation $\alpha':X|_{\mathcal{R}'} \longrightarrow X'|_{\mathcal{R}'}$ formed from α and α_Q is natural because $H^B\alpha'$ is natural and H^B is faithful. (Note: H^B is faithful because B is a generator by assumption.) //

Now for existence. In addition to the previous assumptions, assume further that in $\mathcal{B}$ the following condition holds. For any object B' in $\mathcal{B}$ and for any bijection $f:H^B(B') \longrightarrow S$ in Ens there exists an object B'' in $\mathcal{B}$ and an isomorphism $\varphi:B \longrightarrow B''$ in $\mathcal{B}$ such that $H^B(\varphi) = f$. This is called the transportability assumption.

<u>Theorem 2.4</u> Let $W: \mathcal{A}^* \longrightarrow$ Ens be a representable functor

and let V be an object of $\mathcal{B}$ such that $H^B(V) = W(D)$. Assume further that the following conditions hold.

(a) D is a generator.

(b) Every morphism from D to a copower of D is a coordinate injection.

(c) For every object Y of $\mathcal{B}$ and for every subset S of $H^B(Y)$ the subfunctor hS of h_Y whose values $hS(Q)$ consist of all $f:Q \longrightarrow Y$ such that $H^B(f)$ factors through S, is representable.

Then, W can be lifted to a limit preserving functor $X: \mathcal{D}^* \longrightarrow \mathcal{B}$ with $H^B X = W$ and $X(D) = V$.

Proof:

Let $I.D$ be the I-th copower of D. Define $X(I.D) = V.I$. By the transportability assumption there is an I-th power object on the underlying set $W(I.D)$. By assumption (b) this affects the lifting from Ens to $\mathcal{B}$ for the full subcategory of copowers of D.

Since D is a generator any other object D' in is an epimorphic image of the copower $H(D').D = D'$. Hence $W(D')$ is a subset S of $W(|D'|)$. Now $X(D')$ can be constructed in $X(|D'|)$ by (c). This takes care of all the morphisms from D to D'; i.e., the coordinates of $|D'| \longrightarrow D'$ are mapped to coordinates of $X(D') \longrightarrow X(|D'|)$. Morphisms from a copower $I.D$ to D' are described by their coordinates $D \longrightarrow D'$ and are mapped accordingly.

For any $D' \longrightarrow D''$, X is defined on the composite $|D'| \longrightarrow D' \longrightarrow D''$ which maps to $X(D'') \longrightarrow V.|D'|$. Since H^B takes this to a map factoring through $W(D')$, it factors uniquely through $H^D(D')$.

Claim: X is a functor.

X has been properly defined for objects and morphisms. It remains to show that if

$\beta\alpha : D'' \longrightarrow D''' \longrightarrow D'$ is a morphism in $\mathfrak{D}$,

then $X(\beta\alpha) = X(\alpha)X(\beta)$ in $\mathfrak{B}$.

For $X(\beta\alpha)$, X is defined on the composite $|D''| \longrightarrow D'' \longrightarrow D'$ which maps to $X(D') \longrightarrow V.|D''|$.

For $X(\beta)$ and $X(\alpha)$, X is defined on the respective composites $D''' \longrightarrow D'$ and $D'' \longrightarrow D'''$ which map to $X(D') \longrightarrow V.|D'''|$ and $X(D''') \longrightarrow V.|D''|$.

Now $V.|D'''| \longrightarrow X(D''')$ is an identity. Hence,

$X(\alpha)X(\beta) = X(D') \longrightarrow V.|D''| = X(\beta\alpha)$.

Claim: X is limit preserving.

Now $H^BX = W$ by construction. Then $H^BX = W$ takes colimits of $\mathfrak{D}$ to limits in Ens. Since H^B is a hom functor, X preserves the limits in $\mathfrak{D}^*$.

Hence W has been lifted to a limit preserving functor $X : \mathfrak{D}^* \longrightarrow \mathfrak{B}$. //

Theorem 2.3 makes use of the facts that D is a Grothendieck generator and that B is a generator. The following example shows that it is not possible to weaken this hypothesis so that B and D are both generators.

<u>Example 2.1</u> Let $\mathcal{B}$ and $\mathcal{D}$ be complete lattices with more than one atom. Let D and B be the least elements of $\mathcal{D}$ and $\mathcal{B}$ respectively. Then X(D) (the greatest element) and $H^B X$ are essentially the same for all X but not all X are isomorphic.

Proof:

Every object of $\mathcal{D}$ and $\mathcal{B}$ is a generator since any hom functor $H^{D'}$ (for $D' \in \mathcal{D}$) or $H^{B'}$ (for $B' \in \mathcal{B}$) is trivially faithful. Trivially, because any hom set in $\mathcal{D}$ or $\mathcal{B}$ has at most one element. There is no Grothendieck generator in a complete lattice. To see this let C be an atom of the lattice and notice that the zero of the lattice is a proper subobject of C. An object E (different from the zero) cannot be a Grothendieck generator for the lattice because there does not exist a map $E \longrightarrow C$. The zero of the lattice cannot be a Grothendieck generator because any map from the zero can be factored through the subobjects of any object.

Let $X, X' \in \mathcal{B}^{(\mathcal{D}^*)}$ and let D be the least element of $\mathcal{D}$. Then $X(D) = X'(D)$ since a complete lattice has a unique greatest element. Now $H^B X(D')$ has exactly one element for any object $D' \in \mathcal{D}$ because any hom set in a lattice contains at most one element and

$B \leq X(D')$ for all $B \in \mathcal{B}$, $D' \in \mathcal{D}$. Then $H^B X(D')$ and $H^B X'(D')$ are isomorphic for each object $D' \in \mathcal{D}$. The fact that not all $X \in \mathcal{B}^{(\mathcal{D}^*)}$ are naturally equivalent comes from the fact that if this were so it would imply that any two lattice homomorphisms between two given complete lattices would be naturally equivalent and therefore equal. //

In ending this section, it must be pointed out that Isbell in [6] shows by counter-example that these theorems do not apply in the case where $\mathcal{B} = \mathcal{D} = \mathcal{T}$, the category of topological spaces.

Section 2

Considering the remarks made in introducing the first section of this chapter, this section will show that the ideas developed in Section 1 provide a true generalization of what is intuitively expected in the case of topological groups. The procedure will be to show that this situation holds in the general case of topological algebras. This section will then proceed to study in detail some aspects of one such category of topological algebras, namely the category $\mathcal{K}_0\mathcal{B}\ell$ whose objects are the zero-dimensional, compact, Hausdorff, Boolean lattices.

To begin with, let $\mathcal{A}$ be an equational class of algebras of type τ, let $\mathcal{T}$ be the category of topological spaces, and let $\mathcal{H}$ be the category of topological $\mathcal{A}$ - algebras.

<u>Proposition 2.5</u> There is an equivalence $\mathcal{H} \cong \mathcal{T}^{(\mathcal{A}^*)}$.

Proof:

We prove in detail part of this assertion.

For $A = (Y,(f_\alpha)_{\alpha\in I},O) \in \mathcal{H}$ let the underlying algebra of A be denoted by $A_a = (Y,(f_\alpha)_{\alpha\in I})$ and let the underlying topological space of A be denoted by $A_S = (Y,O)$.

For $B \in \mathcal{A}$, let $[B,A]$ be the space with the underlying set (B,A_a) and with the subspace topology from $A_s^{|B|}$. For $h:B \longrightarrow B' \in \mathcal{A}$, let $[h,A]$ be the map $[B',A] \longrightarrow [B,A] : u \rightsquigarrow u \circ h$. The map $u \rightsquigarrow u \circ h$ is continuous since $pr_b(u \circ h) = u(h(b)) = pr_{h(b)}(u)$ is continuous. Then it is clear that there is a functor $[-,A] : \mathcal{A}^* \longrightarrow \mathcal{T}$.

Take a space X. Define an algebra $\langle X,A \rangle$ as follows. The underlying set of the algebra is the set of all continuous functions $f:X \longrightarrow A_s$. The operations of the algebra are functionally defined; i.e., for the functions $u_1, \ldots, u_{n_\alpha}$ the operations $\varphi_\alpha(u_1, \ldots, u_{n_\alpha}) : X \longrightarrow A_s$ is defined by the composite mapping

$$X \xrightarrow{u_1 \sqcap \cdots \sqcap u_{n_\alpha}} A_s^{n_\alpha} \xrightarrow{f_\alpha} A_s$$

$$x \rightsquigarrow (u_1(x), \ldots, u_{n_\alpha}(x)) \rightsquigarrow f_\alpha(u_1(x), \ldots, u_{n_\alpha}(x)).$$

This map is continuous since A is a topological algebra. For a continuous $w:X \longrightarrow X'$ let $\langle w,A \rangle$ be the map $\langle X',A \rangle \longrightarrow \langle X,A \rangle : u \rightsquigarrow u \circ w$. The map $u \rightsquigarrow u \circ w$ is continuous since

$$\begin{aligned} \varphi_\alpha(u_1, \ldots, u_{n_\alpha}) \circ w &= f_\alpha(u_1 \sqcap \ldots \sqcap u_{n_\alpha}) \circ w \\ &= f_\alpha(u_1 \circ w \sqcap \ldots \sqcap u_{n_\alpha} \circ w) \\ &= \varphi_\alpha(u_1 \circ w, \ldots, u_{n_\alpha} \circ w). \end{aligned}$$

Hence there is a functor $\langle -,A \rangle : \mathcal{T} \longrightarrow \mathcal{A}^*$.

It is clear that $[-,A]$ and $\langle -,A \rangle$ are adjoint on the right where the correspondences $(X,[B,A]) \rightleftarrows (B,\langle X,A \rangle)$ are given by $h \rightsquigarrow \bar{h}$ where $\bar{h}(b)(x) = h(x)(b)$ and $\tilde{g} \leftarrow g$

where $\tilde{g}(x)(b) = g(b)(x)$. Hence $[-,A] \in \mathcal{T}^{(\mathcal{A}^*)}$.

Next, for $h: A \longrightarrow C \in \mathcal{TA}$ the maps $[B,A] \longrightarrow [B,C]$ defined by $u \rightsquigarrow h \circ u$ are continuous since $pr_b(h \circ u) = h(u(b)) = (h \circ pr_b) \circ u$ for each $b \in B$. Thus there exists a natural transformation $[-,h] : [-,A] \longrightarrow [-,B]$. So in all, we have defined a functor $\beta : \mathcal{TA} \longrightarrow \mathcal{T}^{(\mathcal{A}^*)}$ where $\beta A = [-,A]$ and $\beta h = [-,h]$.

This β is faithful. To show this take $h,k: A \longrightarrow C \in \mathcal{TA}$ with $h \neq k$. Then there exists an $a \in A$ where $h(a) \neq k(a)$. Take B free on $\{x_1\}$ in $\mathcal{A}$ and $u: B \longrightarrow A_a$ with $u(x_1) = a$. Then $h \circ u \neq k \circ u$. Thus $(\beta h)_B \neq (\beta k)_B$; hence, $\beta h \neq \beta k$.

It will now be proven that β is full. For any natural transformation $t: [-,A] \longrightarrow [-,B]$, we have the natural transformation $s: (-,A_a) \longrightarrow (-,B_a)$ on the level of underlying sets, and hence by the dual of Proposition 1.9, s has the form $(-,f)$ where $f: A_a \longrightarrow B_a \in \mathcal{A}$. In particular, $s_{F1} : (F1,A_a) \longrightarrow (F1,B_a) : u \rightsquigarrow f \circ u$ is the underlying set map of $t_{F1} : [F1,A] \longrightarrow [F1,B]$. Since we can show (which is omitted here) that $A \rightsquigarrow [F1,A]$ and $A \rightsquigarrow A_s$ are naturally equivalent functors, this shows that f is continuous; i.e., $f: A \longrightarrow B \in \mathcal{TA}$, and $t = [-,f]$.

It remains to show that β is representative. Given any $G \in \mathcal{T}^{(\mathcal{A}^*)}$, let $X = GF1$. Operations $\varphi_\alpha : X^{n_\alpha} \longrightarrow X$ are defined by $\varphi_\alpha = Gh_\alpha$ where $h_\alpha : F1 \longrightarrow Fn : h(x_1) = f_\alpha(x_1, \ldots, x_n)$. Then $A = (X, (\varphi_\alpha)_{\alpha \in I})$ is a topological algebra of the required type. That in fact $A \in \mathcal{TA}$ follows, by considering underlying sets, from the proof of Proposition 1.12. Then, G and [-,A] have isomorphic values for F1 since $X = GF1$ and $[F1,A] \cong A_s = X$. Moreover, for the underlying set functor $U: \mathcal{T} \longrightarrow Ens$ we have $UG \cong (-,A_a)$ by the proof of Proposition 1.12, and hence the hypothesis of Theorem 2.3 are satisfied with $\mathcal{B} = \mathcal{T}$, $\mathcal{D} = \mathcal{A}$, $D = F1$, and $H^B = U$. It follows that $G \cong [-,A]$. //

In order to link this discussion with the earlier discussion, notice that the distinguished objects in the categories $\mathcal{T}$ and $\mathcal{A}$ are the one point space and the free algebra on one generator respectively. With the natural forgetful functors $F_{\mathcal{T}}$ and $F_{\mathcal{A}}$ from $\mathcal{TA}$ to $\mathcal{T}$ and $\mathcal{A}$ respectively, and with the added category equivalence β, we now have from diagram (2.1) on p.36 the following diagram.

$$\begin{tikzcd}
\mathcal{TA} \arrow[dr, "\beta"] \arrow[drr, "F_{\mathcal{T}}"] \arrow[ddr, "F_{\mathcal{A}}"'] & & \\
& \mathcal{T}^{(A^*)} \arrow[r, "V_{\mathcal{T}}"] \arrow[d, "V_{\mathcal{A}}"] & \mathcal{T} \arrow[d, "U = H^1"] \\
& \mathcal{A} \arrow[r, "W = H^{F1}"'] & \mathcal{E}ns
\end{tikzcd}$$

In order to show that the new definitions generalize the usual procedure for topological algebras, it is necessary to show that $F_{\mathcal{A}} \cong V_{\mathcal{A}} \circ \beta$ and $F_{\mathcal{T}} \cong V_{\mathcal{T}} \circ \beta$.

<u>Proposition 2.6</u> $F_{\mathcal{A}} \cong V_{\mathcal{A}} \circ \beta$

Proof:

To begin with it must be remarked that $V_{\mathcal{A}}$ is defined so that $(B, V_{\mathcal{A}} G) \cong UGB$ is natural in G and B. For $A \in \mathcal{H}$, $\beta A = [-, A]$. Hence $(V_{\mathcal{A}} \circ \beta)A = V_{\mathcal{A}}([-, A])$. Thus, by the above remark, $(B, V_{\mathcal{A}} \beta A) = U[B, A] = (B, A_a)$ is natural in B and A. Hence $V_{\mathcal{A}} \circ \beta A \cong A_a = F_{\mathcal{A}} A$; i.e., $V_{\mathcal{A}} \circ \beta = F_{\mathcal{A}}$. //

<u>Proposition 2.7</u> $F_{\mathcal{T}} = V_{\mathcal{T}} \circ \beta$

Proof:

For $A \in \mathcal{H}$, $V_{\mathcal{T}} \circ \beta(A) = [F1, A] \cong A_s = F_{\mathcal{T}}(A)$. Hence $V_{\mathcal{T}} \circ \beta = F_{\mathcal{T}}$. //

These two propositions have shown that the new definitions provide a true generalization of the usual definitions associated with a topological algebra. As mentioned in the introduction to this section, attention will now be focused on the category $\mathcal{K}_0\mathcal{B}\ell$ of zero-dimensional, compact, Hausdorff, Boolean lattices. Let $\mathcal{B}_s$ be the category of Boolean spaces and let $\mathcal{B}\ell$ be the category of Boolean lattices respectively. As a general notational convenience, if $L \in \mathcal{K}_0\mathcal{B}\ell$, then let L_s and L_a denote the underlying space or lattice respectively.

Proposition 2.8 $\mathcal{K}_0\mathcal{B}\ell \cong \mathcal{B}_s^{(\mathcal{B}\ell^*)}$

Proof:

In the context of topological algebras $\mathcal{K}_0\mathcal{B}\ell$ can be considered as a full subcategory of all topological Boolean lattices. By theorem 2.5 there exists an equivalence $\mathcal{T}\mathcal{B}\ell \cong \mathcal{T}^{(\mathcal{B}\ell^*)}$. For $A \in \mathcal{K}_0\mathcal{B}\ell$, $[B,A]$ is a closed subspace of $A_s^{|B|}$, hence Boolean. Thus $[-,A] \in \mathcal{B}_s^{(\mathcal{B}\ell^*)}$. Next, given the limit preserving functor $T: \mathcal{B}\ell^* \to \mathcal{B}_s$, then the construction of $A \in \mathcal{T}\mathcal{B}\ell$ such that $[-,A] \cong T$ (this is a special case of the previous general discussion) shows that $A_s = T(F1)$. Thus $A_s \in \mathcal{B}_s$ and so $A \in \mathcal{K}_0\mathcal{B}\ell$. //

Among the limit preserving functors from $\mathcal{B}\ell^*$ to $\mathcal{B}_s$ is the Stone Duality functor. The following proposition proves this fact in a way that points out that $2 \in \mathcal{K}_0\mathcal{B}\ell$ corresponds to this functor under the equivalence described in the previous proposition.

<u>Proposition 2.9</u> The Stone Duality functor $H: \mathcal{B}\ell^* \longrightarrow \mathcal{B}s$ may be defined as that functor which maps any Boolean lattice L to Hom $(L, 2_a) \leq 2_s^{|L|}$.

Proof:

Define $G: \mathcal{B}s \longrightarrow \mathcal{B}\ell : X \rightsquigarrow (X, 2_s)$ where $(X, 2_s)$ is taken as a Boolean sublattice of $2_a^{|X|}$

$: (f: X \text{—} Y) \rightsquigarrow ((Y, 2_s) \longrightarrow (X, 2_s) : g \rightsquigarrow g \circ f)$

Now $Gf(g \vee h) = (g \vee h) \circ f = (g \circ f) \vee (h \circ f)$. Also $Gf(g \wedge h) = (g \wedge h) \circ f = (g \circ f) \wedge (h \circ f)$, and $Gf(a') = Gf(a)'$. Therefore, as defined, Gf is a lattice homomorphism.

Recall from example 1.2 that the functor $S: \mathcal{B}s \longrightarrow \mathcal{B}\ell$ was defined to be the functor that mapped any Boolean space X to the Boolean lattice of its open-closed sets.

Define $\alpha: S \longrightarrow G : SX \rightsquigarrow GX : A \rightsquigarrow \chi_A$ where χ_A is the characteristic function for A (as a subset of X).

Claim: α_X is a natural equivalence between S and G.

It is clear that α_X is an isomorphism for each $X \in \mathcal{B}s$. It remains to check the naturality. For $f: X \longrightarrow Y \in \mathcal{B}s$ consider the following diagram.

$$\begin{tikzcd}
SY \arrow[r, "\alpha_Y"] \arrow[d, "Sf"'] & GY \arrow[d, "Gf"] \\
SX \arrow[r, "\alpha_X"'] & GX
\end{tikzcd}$$

Then $Gf\circ\alpha_Y(A) = Gf\circ\chi_A = \chi_A\circ f$

Also $\alpha_X\circ Sf(A) = \alpha_X(f^{-1}(A)) = \chi_{f^{-1}(A)}$

Note that $\forall x\in X$, $\chi_A\circ f(x) = \chi_A(f(x)) = \begin{cases} 1 \text{ if } f(x) \in A \\ 0 \text{ if } f(x) \notin A \end{cases}$

Hence $\chi_A\circ f = \chi_{f^{-1}(A)}$ and therefore $Gf\circ\alpha_Y = \alpha_X\circ Sf$.

Define $H:\mathcal{B}\ell \longrightarrow \mathcal{B}s : L \rightsquigarrow (L,2_a) \subseteq 2_s^{|L|}$.

This H is just a specific example of the functor $[-,A]$ of the previous discussion in Proposition 2.5.

Recall from example 1.2 that the functor $T:\mathcal{B}\ell \longrightarrow \mathcal{B}s$ was defined to be the functor which maps any lattice L to its ultrafilter space ΩL.

Define $\gamma:T \longrightarrow H: TL \rightsquigarrow HL: U \rightsquigarrow \chi_U$, where χ_U is the characteristic function for U.

Claim: γ is a natural equivalence between T and H.

First, it will be shown that γ_L is a homeomorphism for each $L \in \mathcal{B}\ell$. In order to prove that γ_L is continuous, it suffices to show that all $pr_a\circ\gamma_L$ for a L are continuous. Now $pr_a\circ\gamma_L(U) = pr_a(\chi_U) = \chi_U(a)$ $= \varphi_a(U)$, so that $pr_a\circ\gamma_L = \varphi_a$. Then $\varphi_a^{-1}\{1\}$ $=\{U:\varphi_a(U) = 1\} = \{U:\chi_U(a) = 1\} = \{U: a\in U\in\Omega\} = \Omega_a$. Also $\varphi_a^{-1}\{0\} = (\Omega_a)' = \Omega_{a'}$. Thus φ_a is continuous and therefore all $pr_a\circ\gamma_L$ are continuous. Therefore γ_L is continuous. Note that if $\chi \in (L, 2_a)$ then $U_\chi = \chi^{-1}\{1\}$ is an ultrafilter. Now, $U_{\chi_U} = U$ and $\chi_{U_\chi} = \chi$. Hence the mappings $U \rightsquigarrow \chi_U$ and $\chi \rightsquigarrow U_\chi$ are inverses. Therefore γ_L is one-to-one and onto. Hence, because TH and HL are compact Hausdorff spaces, γ_L is a homeomorphism.

It remains to check the naturality of γ. Let $f: L \longrightarrow K$ and consider the following diagram.

$$\begin{tikzcd} TK \arrow[r, "\gamma_K"] \arrow[d, "Tf"'] & HK \arrow[d, "Hf"] \\ TL \arrow[r, "\gamma_L"'] & HL \end{tikzcd}$$

Let $U \in TK$. Then $Hf \circ \gamma_K(U) = Hf\,\chi_U = \chi_U f$. Also $\gamma_L \circ Tf(U) = \gamma_L(f^{-1}(U)) = \chi_{f^{-1}(U)}$. But $\chi_U f = \chi_{f^{-1}(U)}$. Therefore $\gamma_L \circ Tf = Hf \circ \gamma_K$ and hence the diagram commutes. Therefore γ is natural.

This completes the proof of this proposition. //

The rest of this thesis will concern itself with comparing the categories Ens and $\mathcal{K}_o\mathcal{B}\ell$.

Define $P:\mathrm{Ens} \longrightarrow \mathcal{K}_0\mathcal{B}\ell : X \rightsquigarrow 2^X$, where 2 is the two element object in $\mathcal{K}_0\mathcal{B}\ell$.

$:(f:X \longrightarrow Y) \rightsquigarrow (Pf:2^Y \longrightarrow 2^X : u \rightsquigarrow u\circ f)$.

Then P is a contravariant functor.

<u>Lemma 2.10</u> P is faithful.

Proof:

Let $f,g:X \longrightarrow Y$ where $f \neq g$. Then there exists an $x \in X$ such that $f(x) \neq g(x)$. Now $Pf,Pg:2^Y \longrightarrow 2^X$, hence for $u \in 2^Y$ and $x \in X$, $pr_x \circ Pf(u) = pr_x(u\circ f) = u(f(x))$ and $pr_x \circ Pg(u) = pr_x(u \circ g) = u(g(x))$. Since $f(x) \neq g(x)$ there exists a $u \in 2^Y$ such that $u(f(x)) \neq u(g(x))$, (for example $u = \chi_{\{f(x)\}}$).

Thus $pr_x \circ Pf \neq pr_x \circ Pg$ and hence $Pf \neq Pg$. //

<u>Lemma 2.11</u> Any morphism $h:2^X \longrightarrow 2$ in $\mathcal{K}_0\mathcal{B}\ell$ is a projection.

Proof:

This proof will first show that h factors through a projection $2^X \longrightarrow 2^Y$ for $Y \subseteq X$, Y finite. Then X may be assumed finite and the proof of the proposition will then be given.

Note that $h^{-1}\{1\}$ is an ultrafilter U_h. Since h is continuous, U_h is open-closed and hence it is a neighbourhood of $1 \in 2^X$.

For $x \in X$, put $U_x = p_x^{-1}\{1\}$. Now p_x is a projection; hence, it is continuous and therefore U_x is open-closed.

Note that $\bigcap_{x\in X} U_x = \{1\}$. In a compact Hausdorff space X any filter basis $\mathfrak{B}$ of closed neighbourhoods of $a \in X$ with $\bigcap_{V\in\mathfrak{B}} V = \{a\}$ is a basis for the neighbourhood filter of $a \in X$. This implies that there exist $x_1,\ldots,x_n \in X$ such that $U_h \supseteq U_{x_1} \cap \ldots \cap U_{x_n}$.

It must now be shown for the restriction map $r:2^X \longrightarrow 2^{\{x_1,\ldots,x_n\}}$ that if $u,v \in 2^X$ and if $u(x_i) = v(x_i)$ for $i = 1,\ldots,n$ then $h(u) = h(v)$. If this holds then $g:2^{\{x_1,\ldots,x_n\}} \longrightarrow 2$ can be defined by $g\circ r = h$, where g will then be continuous trivially (since its domain is finite).

Now $u(x_i) = v(x_i)$ implies $u\vee v' \in \bigcap U_x$ so $h(u\vee v') = 1$ and therefore $h(u)\vee h(v)' = 1$. Hence $h(u) \geqslant h(v)$. By symmetry, also $h(v) \geqslant h(u)$. Hence $h(u) = h(v)$.

Hence h factors as $2^X \xrightarrow[r]{} 2^{\{x_1,\ldots,x_n\}} \xrightarrow[g]{} 2$.

Now to prove the proposition. Suppose for each $x \in X$, $h \neq p_x$. Hence there exists a $u_x \in 2^X$ with $h(u_x) \neq u_x(x)$. Now, if $u_x(x) = 1$, consider u_x' instead. So it may be assumed that $u_x(x) = 0$ and $h(u_x) = 1$, for each x. Using the assumption that X is finite put $u = \bigwedge u_x$. Then $u(z) = \bigwedge u_x(z) = 0$ implies that $u = 0$ for all $z \in X$ and $h(u) = 0$. Also $h(u) = \bigwedge h(u_x) = 1$ which is contradictory. Hence $h = p_x$ and therefore any $h:2^X \longrightarrow 2$ in $\mathcal{K}_0\mathcal{B}\ell$ is a projection. //

<u>Lemma 2.12</u> $P:\mathrm{Ens} \longrightarrow \mathcal{K}_0\mathcal{B}\ell : X \rightsquigarrow 2^X$ is full.

Proof:

Let $h: 2^X \longrightarrow 2^Y$ be a morphism in $\mathcal{K}_0\mathcal{B}\ell$ and consider any projection $p_y: 2^Y \longrightarrow 2$. (Note that if $p_x = p_{z'}$, then $x = z'$.) Then $p_y h = p_x$.

Define $u: Y \longrightarrow X : u(y) = x$ where $p_y h = p_x$; i.e., $p_{u(y)} = p_y h$.

Then $Pu: 2^X \longrightarrow 2^Y : \xi \rightsquigarrow Pu(\xi) = \xi u$.

So $p_y Pu = p_{u(y)} = p_y h$ implies that $Pu = h$. //

The following lemma points out the importance played by the two element object in $\mathcal{K}_0\mathcal{B}\ell$.

<u>Lemma 2.13</u> Any object $A \in \mathcal{K}_0\mathcal{B}\ell$ is isomorphic to $2^{(A,2)}$.

Proof:

Define $\eta: A \longrightarrow 2^{(A,2)}$ by $\eta(x)(\varphi) = \varphi(x)$ for $x \in A$ and $\varphi \in (A,2)$. Now A is compact and η is continuous; hence, the image is closed.

Claim: η is dense. This will then imply that η is onto.

The standard basic open sets of $2^{(A,2)}$ are given by distinct $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_k \in (A,2)$ consisting of all $f \in 2^{(A,2)}$ such that $f(\varphi_i) = 1$ and $f(\psi_i) = 0$. To show that there is a $\eta(x)$ among these, an $x \in A$ must be found such that $\varphi_i(x) = 1$ and $\psi_i(x) = 0$. Putting $U_i = \varphi_i^{-1}\{1\}$ and $V_j = \psi_j^{-1}\{1\}$ this says that an $x \in A$ must be found such that $x \in \bigcap U_i$ and $x' \in \bigcap V_j$. Now, since $U_i \neq V_j$ for all i and j, $\bigcap U_i \nsubseteq V_j$ for each j (fix j; pick

$z_i \in U_i$ with $z_i' \in V_j$; then $\vee z_i$ belongs to $\cap U_i$ and $(\vee z_i)' = \wedge z_i' \in V_j)$. Hence $x_i \in \cap U_i$ with $x_j' \in V_j$, and for $x = \wedge x_j$, $x \in \cap U_i$, $x' = \vee x_j' \in \cap V_j$.

Claim: η is one-to-one.

Let $a,b \in A$ where $a,b \neq 0$, and $a \neq b$. Then to show $\eta(a) \neq \eta(b)$ notice that $\eta(x)(\varphi) = \varphi(x)$, and hence this amounts to finding a $\varphi \in (A,2)$ with $\varphi(a) \neq \varphi(b)$. It is a known fact (see for intance [1]) that any $A \in \mathcal{K}_0\mathcal{B}\ell$ is pro-finite. Hence there exists an $h: A \longrightarrow B$ in $\mathcal{K}_0\mathcal{B}\ell$ where B is finite and $h(a) \neq h(b)$. Then there exists $g: B \longrightarrow 2$ such that $g(h(a)) \neq g(h(b))$. Then $g \circ h \in (A,2)$ and it maps a and b differently. //

The following theorem summarizes and finishes the present discussion of $\mathcal{K}_0\mathcal{B}\ell$.

<u>Theorem 2.14</u> The functor P provides a dual equivalence between Ens and $\mathcal{K}_0\mathcal{B}\ell$.

Proof:

$P: \text{Ens} \longrightarrow \mathcal{K}_0\mathcal{B}\ell : X \longrightarrow 2^X$ is a full, faithful and representative contravariant functor by lemmas 2.12, 2.10, and 2.13 respectively. Therefore it defines dual equivalence between Ens and $\mathcal{K}_0\mathcal{B}\ell$. //

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