# DUALITIES

# DUALITIES BETWEEN FINITELY CLOSED SUBCATEGORIES OF MODULES

by

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# ABSTRACT

The thesis studies dualities between categories of modules which are finitely closed, i.e. closed under submodules, factor modules and finite direct sums. Omitting the requirement that the categories contain all finitely generated modules from the classical Morita situation provides a generalization which substantially increases the number of rings that possess such a duality.

In Chapter II we prove that a duality between two finitely closed categories A and B of modules is representable if and only if A and B consists of linearly compact modules. While a linearly compact finitely closed category of modules is always an $AB5^*$-category with no infinite direct sums, we demonstrate the converse in Chapter III for certain rings including all commutative ones, thus simplifying our characterization of representable dualities in these cases; we were however unable to prove this in general or to give a counterexample. In Chapter IV we show that a duality between two arbitrary finitely closed categories of modules over commutative rings may be decomposed into representable dualities between finitely closed categories of modules over local rings.

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## CHAPTER I

## HISTORICAL SURVEY, INTRODUCTION AND DEFINITIONS

### §1 Historical Survey and Introduction

The familiar duality between finite dimensional vector spaces has been generalized in various directions.

One fundamental contribution is due to Morita [19]. A Morita duality is a contravariant category equivalence between two subcategories of R-right and S-left-modules which are both finitely closed (i.e. closed under submodules, factor modules and finite direct sums) and contain all finitely generated modules. Matlis [18] extending earlier works of Grobner, Macaulay and others [see Gabriel [7]] showed that every complete commutative noetherian local ring R has a Morita duality. Morita [19] and Azumaya [2] demonstrated that the Morita dualities are exactly the functors hom(-,Q) induced by bimodules ${}_SQ_R$ which are injective cogenerators on both sides and satisfy $R = \text{End}\ {}_SQ$, $S = \text{End}\ Q_R$ and that the natural domain and range of such a duality are the Q-reflexive modules. The Q-reflexive modules are precisely the linearly compact modules (with the discrete topology) (see Mueller [21]). Osofsky [25] proved that if R has a Morita duality, then it is semi-perfect. Sandomierski's paper [28] is a good summary of the known results on Morita duality.

A second line of research started with Chevalley, Lefschetz [13, ChapterII] and Dieudonné [5] who developed the concept of linearly compact linearly topologized vector spaces and extended the ordinary duality for finite dimensional vector spaces to a duality between all abstract and all linearly compact linearly topologized vector spaces. Macdonald [16] generalized this for a commutative complete noetherian local ring A as follows. There is a duality theory between the category of all "linearly discrete" modules (which is canonically isomorphic to mod-A) and the category of all linearly compact linearly topologized module over A with the radical topology. If $A^*$ is the injective hull of the residue field of A then the dual of X is $X^* = \hom_A(X,A^*)$, but "hom" now denotes the set of all continuous homomorphisms and is topologized in a suitable way. Earlier Kaplansky [11] and Leptin [15] obtained similar results for the special case of complete discrete valuation rings. These results constitute an analogue of Pontryagin's duality theory. We note that each of these topological duality theories is an extension of a Morita duality.

A major problem of the above duality theory is the choice of the topology on the dual $X^*$. The reflexivity of any given module depends thereupon and there seems to be no optimal choice. Mueller [22] shows how every Morita duality may be enlarged to a duality for linearly topologized modules. The ambiguity of the choice of topology on the dual $X^*$ turns out to be an essential feature of his theory. He defines two topologies on a module $X_G$ to be equivalent if they have the same submodules closed, and proves that all linearly topological modules are reflexive up to equivalence.

A third group of results concerns Grothendieck categories. Gabriel [8, Chapter IV, §4] constructs the dual of a locally finite Grothendieck category as a subcategory of topological modules. Roos [27] works with locally noetherian categories and Oberst [24] extends their results to arbitrary Grothendieck categories.

The Roos-Oberst construction of the dual of a Grothendieck category U has three steps. First find a small finitely closed generating subcategory N of U, a ring S and an additive contravariant functor $F : N \to S\text{-mod}$ which is exact, fully faithful and transforms filtered unions into projective limits in S-mod. Second, setting $B = FN$, define a category $\tilde{B}$ of linearly topological S-modules. A B-topology on a S-module X is a Hausdorff linear topology with a basis of submodules V such that $X/V \in B$. A linear topology is strict if it is a complete B-topology and if each closed submodule Y is open whenever $X/Y \in B$ and the induced topology on Y is a B-topology. The objects of $\tilde{B}$ are the strict topological S-modules and the morphisms are the continuous S-homomorphisms. Third show that $\tilde{B}$ is the dual of U.

Oberst [24] has demonstrated that if N is any small finitely closed generating subcategory of U and E an injective cogenerator for which all $X \in N$ are isomorphic to submodules of E, then hom(-,E) : N → S-mod with S = hom(E,E) is an exact, fully faithful contravariant functor which transforms filtered unions into projective limits of S-mod. Clearly such an injective cogenerator always exists.

The subcategory B of S-mod in the Roos-Oberst duality construction is not necessarily a finitely closed subcategory of S-mod. If it is finitely closed, then it will consist of linearly compact modules. The special Grothendieck categories U which have a finitely closed generating subcategory N that is dual to a finitely closed linearly compact subcategory B of S-mod were studied by Goblot [10, Chapters V, VI]. He shows that such a U is equivalent to the category of discrete modules for some topological linearly compact ring R.

For us, the most valuable aspect of Goblot's work [10] is his study of dualities between subcategories of modules. Motivated by Morita duality he proves the following theorem as an application of the Roos-Oberst duality construction. If F : A → B is a duality between two finitely closed linearly compact subcategories of mod-R and S-mod with R (S) complete in the A-topology (B-topology), then there exists a bimodule Q which represents the duality.

Finally Lambek and Rattray [12] have used idempotent monads to study dualities. In this more general setting they have also studied localization.

This thesis studies dualities between two categories of R-right and S-left-modules which are finitely closed. Omitting the requirement that the categories contain all finitely generated modules from the Morita situation substantially increases the number of rings that possess such a duality. Although Goblot's thesis [10] must be considered our main reference, we emphasize that his categorical methods are not used; in fact, we believe that one of the accomplishments of this thesis is the ability to avoid his highly abstract categorical techniques (i.e. Roos-Oberst duality construction).

In Chapter II we prove a duality between two finitely closed categories A and B of modules to be representable if and only if A and B consist of linearly compact modules. While a linearly compact finitely closed category of modules is always an $AB5^*$-category with no infinite direct sums, we demonstrate the converse in Chapter III for certain rings including all commutative ones, thus simplifying our characterization of representable dualities in these cases; we were however unable to prove this in general or to give a counterexample. In Chapter IV we show that a duality between two arbitrary finitely closed categories of modules over commutative rings may be decomposed into representable dualities between finitely closed categories of modules over local rings.

We have also proved the following result (but as the proof is very similar to the one in Mueller [22] for the special case of a Morita duality, we will not write down the details). Let Q represent a duality between two finitely closed subcategories A and B of mod-R and S-mod, with $R = \mathrm{End}\ {}_SQ$ and $S = \mathrm{End}\ Q_R$. Then this duality may be extended to a duality between Top R and Top S (see Mueller [22]). Here Top R(Top S) is the category of all linearly topological modules over the ring R (S) with the A-topology (B-topology), where two topologies on a module X are identified if they have the same continuous homomorphisms into Q with the discrete topology. This enlarged duality then restricts to a duality which agrees with the Roos-Oberst construction between the Grothendieck category dis A and the category of strict linearly compact modules.

## §2 Finitely Closed Subcategories of Mod-R

Let R be a ring with unit and mod-R (R-mod) the category of all right (left) R-modules and all R-homomorphisms.

A topology $\tau$ on R is a (right) linear topology if $\tau$ is a ring topology on R having a fundamental system of neighbourhoods of zero consisting of right ideals. A module $M_R$ is $\tau$-discrete if the annihilator of each element of M is a right ideal open in $\tau$. Let dis $\tau$ be the full subcategory of mod-R whose objects are the $\tau$-discrete P-modules. The category dis $\tau$ is closed under submodules, factor modules and direct sums of mod-R. Also $\{R/I \mid I \in \tau\}$ is a set of generators for dis $\tau$. Hence dis $\tau$ is a Grothendieck category; it is a full coreflective subcategory of mod-R, hence limits are the coreflections of limits in mod-R, and colimits are the same as in mod R.

Definitions: (1) A full subcategory A of mod-R is finitely closed if and only if A is closed with respect to submodules, factor modules and finite direct sums of mod-R.

(2) An abelian category C is a AB5-category if for each object $X \in C$ the subobjects of X form a complete lattice and for all subobjects Y of X and all updirected families $(X_i)_{i \in I}$ of subobjects of X, $\bigcup_I (X_i \cap Y) = (\bigcup_I X_i) \cap Y$. The dual of an AB5-category is an $AB5^*$-category.

We remark that if A is a finitely closed subcategory of mod-R then A is an abelian category. The embedding $i : A \to$ mod-R is exact;

consequently, A-monomorphisms are one-one maps and A-epimorphisms are onto maps. Since mod-R is a Grothendieck category, clearly A is an AB5-category.

Let A be a finitely closed subcategory of mod-R, then there is a right linear topology on R defined as follows: a right ideal I of R is open if and only if $R/I \in A$. We shall refer to this topology as the A-topology. By dis A we mean the full subcategory of mod-R consisting of all discrete modules for the A-topology on the ring R. Note that $A \subseteq$ dis A and A generates dis A. Two finitely closed subcategories yield the same topology on R if and only if they have the same finitely generated R-modules.

Definition: Let A be a subcategory of mod-R. A is faithful if and only if $\text{ann}_R(A) = \{r \in R \mid Xr = 0 \text{ for all } X \in A\}$ is zero.

Proposition 2.1: Let A be a finitely closed subcategory of mod R. These are equivalent:

(1) The A-topology is Hausdorff.

(2) A is faithful.

Proof: $\text{ann}_R(A) = \cap I$ where the intersection is over the open right ideals I in the A-topology.

If A is a finitely closed subcategory of mod-R, then A is a finitely closed faithful subcategory of mod-$R/_{\text{ann}_R(A)}$.

## §3 Linearly Compact Modules

A Hausdorff linearly topologized module M is linearly compact if every finitely solvable system of congruences $x \equiv x_k \pmod{X_k}$ involving closed submodules $X_k$ is solvable. The phrase, "A is a linearly compact subcategory of mod R", means that each $X \in A$ is linearly compact in the discrete topology. The basic properties of linearly compact modules are developed in Lefschetz [13] and Zelinsky [30].

For us, the following four propositions are of particular interest:

Proposition 3.1: (Bourbaki [3], §2, Exercise 16). Let M be a Hausdorff linearly topologized module. These are equivalent:

(1) M is linearly compact.

(2) With every linear topology (Hausdorff or not) on M coarser than or equal to the given topology, M is complete.

Proposition 3.2: (Zelinsky [30] Proposition 9). If M is a linear topological module, N a closed submodule of M, then M is linearly compact if and only if N and M/N are linearly compact.

This implies that the subcategory of mod-R consisting of all discrete linearly compact modules is finitely closed.

Proposition 3.3: (Bourbaki [3], §2, Exercise 16). Let M be a linearly compact module, N a closed submodule of M, $\{P_i\}_{i \in I}$ a downwards directed family of closed submodules of M. Then $\bigcap_I (N+P_i) = N + \bigcap_I P_i$.

Consequently, if A is a finitely closed linearly compact subcategory of mod R, then A is an $AB5^*$-category.

<u>Proposition 3.4</u>: (Zelinsky [30], Proposition 6). Let M be linearly compact in the discrete topology. Then every independent collection of submodules in M is finite.

Thus if A is a linearly compact subcategory of mod-R, then A has no infinite direct sums of nonzero modules.

## §4 Duality

Let A and B be categories. A duality between A and B is a contravariant category equivalence; that is, a contravariant functor $F : A \to B$ for which there is a contravariant functor $H : B \to A$ and natural isomorphisms $HF \cong 1 : A \to A$ and $FH \cong 1 : B \to B$. We remark that a duality between A and B takes limits in A into colimits in B and vice versa.

Let $F : A \to B$ be a duality between finitely closed subcategories of mod-R and S-mod, respectively. A finitely closed subcategory of a module category is abelian, thus F is an exact additive contravariant functor. For any $X \in A$, the subobject lattices of X and FX are anti-isomorphic. Since A and B are both AB5-categories, they are also $AB5^*$-categories.

Let R and S be rings and ${}_SQ_R$ a bimodule. Q gives rise to a pair of contravariant functors $\hom_R(-,Q) : \text{mod-}R \to S\text{-mod}$ and $\hom_S(-,Q) : S\text{-mod} \to \text{mod-}R$. Both functors are left exact and take colimits to limits. For modules $X_R$ and ${}_SB$, we define ${}_SX^* = \hom_R(X,Q)$, $B_R^* = \hom_S(B,Q)$ and $X^{**} = (X^*)^*$. Similarly for homomorphisms, $f^* = \hom_R(f,Q)$. The map $i : X \to X^{**}$ given by $i(x) = \hat{x}$ where $\hat{x}f = f(x)$ for $f \in X^*$, yields a natural transformation from the identity functor to the double dual functor $(\ )^{**}$. A module X is called Q-reflexive if $i : X \to X^{**}$ is an isomorphism.

## §5 Examples

This section contains three examples of dualities between finitely closed subcategories of mod-R where R is a commutative ring. All these examples are well known, but we believe they illustrate the theory that follows.

Example 1: Let R be a commutative ring, E the minimal injective cogenerator (i.e. the injective hull of the direct sum of a set of representatives for the isomorphism types of simple modules in mod-R) and A the category of R-modules of finite length. A is a finitely closed subcategory of mod-R. By induction on the length of a module we see that each $X \in A$ is E-reflexive, hence $\hom_R(-,E) : A \to A$ is a duality. (See Anderson and Fuller [1], §23, Exercise 8).

Moreover, the category B of all E-reflexive modules is finitely closed. Clearly B is closed under finite direct sums. Let $0 \to Y \to X \to X/Y \to 0$ be an exact sequence with X E-reflexive. Now we have the following commutative diagram with exact rows, since E is injective:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & X/Y & \longrightarrow & 0 \\ & & \downarrow r & & \downarrow s & & \downarrow t & & \\ 0 & \longrightarrow & Y^{**} & \longrightarrow & X^{**} & \longrightarrow & (X/Y)^{**} & \longrightarrow & 0 \end{array}$$

As X is E-reflexive s is an isomorphism, and r and t are monomorphisms. Diagram chasing shows that r and t are also isomorphisms. Thus B is closed under submodules and factor modules. Therefore B is a finitely closed subcategory of mod-R and we have a duality $\hom_R(-,E) : B \to B$.

This example demonstrates that there exist dualities between finitely closed categories of modules over every commutative ring. Most of these dualities are not Morita dualities; in order for R to possess a Morita duality it must at least be linearly compact.

Example 2: Let $Z_p$ be the localization of the integers Z at the prime ideal pZ, and A the finitely closed subcategory consisting of all finite p-groups. The A-topology for $Z_p$ is the p-adic topology, and dis A equals the category of all p-groups. $C(p^\infty) = \varinjlim Z/p^n Z$ is an injective cogenerator of mod-$Z_p$ which is an object of dis A. Moreover, $\hom(-,C(p^\infty)) : A \to A$ is a duality.

The completion of $Z_p$ for the A-topology is $\hat{Z}_p$, the p-adic integers with the p-adic topology. The category of discrete $\hat{Z}_p$-modules is again the category of all p-groups. Since $Z_p$ is not linearly compact, mod-$Z_p$ does not have a Morita duality; however, our duality is part of a Morita duality in mod-$\hat{Z}_p$.

Example 3: Consider mod-Z the category of all abelian groups. For a group M, let tors $M = \{x \in M \mid \operatorname{ann}_Z x \neq 0\}$, the torsion subgroup of M. Each torsion group may be decomposed as $M = \bigoplus_{p \text{ prime}} M_p$ where $M_p$ is the p-component of M. We know that the torsion group $T = \bigoplus_{p \text{ prime}} C(p^\infty)$ is an injective cogenerator of mod-Z and $A = \{M \mid M \text{ is torsion and } M_p$ finite for all primes p$\}$ is a finitely closed subcategory of mod-Z.

Now dis A is the category of all torsion groups, and tors hom (–, T) : A → A is a duality. Since $\bigoplus_{p \text{ prime}} Z/pZ \in A$, A is not linearly compact. Thus by Chapter II Theorem 2.6, the duality is not representable.

An ideal of Z is open in the A-topology if and only if it is nonzero. The completion of Z in the A-topology is $\hat{Z} = \prod_{p \text{ prime}} \hat{Z}_p$ where $\hat{Z}_p$ is the p-adic integers with the p-adic topology. The category of discrete $\hat{Z}$-modules is again the torsion groups. For a torsion group X, the p-component $X_p$ is a $\hat{Z}_p$-module. Also for $X = \bigoplus_{p \text{ prime}} X_p$ we have tors hom $(X,T) = \bigoplus_{p \text{ prime}} \hom(X_p, C(p^\infty))$; that is, our duality while not representable as a whole is representable on the p-components and each p-component is a module over a local ring (See Chapter IV).

## CHAPTER II

## REPRESENTABLE DUALITIES

### §1 The Functor hom(-,Q)

In this chapter we will show that a duality between two finitely closed subcategories A and B of mod-R and S-mod is represented by a bimodule Q if and only if A and B consist of linearly compact modules. We begin by studying the functor hom(-,Q).

Definitions: (1) Let $M_X$ be the full subcategory of mod-R consisting of all modules isomorphic to submodules of factor modules of $X^n$, $n = 1,2,3,\ldots$, where $X^n$ is the product of n copies of the R-module X.

(2) Q $\varepsilon$ mod R is a self cogenerator if $Q_R$ cogenerates $M_Q$.

Proposition 1.1: (Sandomierski [28], Lemma 3.4) If $Q_R$ is a self cogenerator, $S = \text{End } Q_R$ and $X_R$ a R-module, then for $f_1,\ldots,f_n \in {}_XX^* = \hom_R(X,Q)$ and $g \in X^{**} = \hom_S(\hom_R(X,Q),Q)$ there exists $x \in X$ such that $\hat{x} f_i = g f_i$ for $i = 1,\ldots,n$.

Proof: Suppose the conclusion is false, then $p = (g f_1,\ldots,g f_n) \notin K = \{(f_1(x),\ldots,f_n(x)) \mid x \in X\}$ where $K_R$ is a submodule of ${Q^n}_R$. Since $Q_R$ is a self cogenerator, there is a R-homomorphism $h : {Q^n}_R \to Q_R$ such that $K \subseteq \ker(h)$ and $h(p) \neq 0$. Since $S = \text{End } Q_R$, $h = (S_1,\ldots,S_n)$, $S_i \in S$, with $h(g_1,\ldots,g_n) = \sum_{i=1}^{n} S_i g_i$.

Now $h(f_1 x, \ldots, f_n x) = \sum_{i=1}^{n} S_i f_i(x) = 0$ for each $x \in X$, so $\sum_{i=1}^{n} S_i g_i = 0$. Hence $h(p) = \sum_{i=1}^{n} S_i(g f_i) = \sum_{i=1}^{n} g(S_i f_i) = g(\sum_{i=1}^{n} S_i f_i) = 0$ a contradiction, and the proposition follows.

Let $Q_R$ be a self cogenerator and $S = \text{End } Q_R$. Consider the natural map $i : X \to X^{**} = \text{hom}_S(\text{hom}_R(X,Q),Q)$. Let $X^{**}$ be given the topology of simple convergence. This is a linear topology on $X^{**}$ which is complete. Moreover, it induces on X a linear topology with subbasis, $S = \{\text{Ker } f \mid f : X \to Q\}$. This topology on X is sometimes called the Q-topology. The Q-topology is Hausdorff if and only if Q cogenerates X. As $\text{Ker } i = \bigcap_{f \in \text{hom}_R(X,Q)} \text{Ker } f$, Lemma 1.1 says that $X^{**}$ is the Hausdorff completion of X with the Q-topology. Thus we have proved the following proposition:

<u>Proposition 1.2</u>: Let $Q_R$ be a self cogenerator, then $X^{**}$ is the Hausdorff completion of X in the Q-topology.

<u>Remarks</u>: (1) An submodule V of X is open in the Q-topology if and only if there exists an exact sequence $0 \to X/V \to Q_R^K$ for some integer K. Thus $X^{**} = \varprojlim_V X/V$ with V running over the open submodules for the Q-topology.

(2) If X is linearly compact, then X is Q-reflexive. This follows because a linearly compact module is complete for all linear topologies.

Definition: Let D be a Grothendieck category, Q an object of D and $S = hom_D(Q,Q)$. A module $_S X \in$ S-mod is Q-representable if there exists an object $M \in D$ such that $_S X$ is isomorphic to $hom_D(M,Q)$.

Proposition 1.3: (Goblot [10], Proposition 5.5). Let D be a Grothendieck category, Q an injective cogenerator of D and $S = hom_D(Q,Q)$. Then the following statements hold.

(a) For $M \in D$, $hom_D(M,Q)$ is a finitely generated S-module if and only if there exists an exact sequence $0 \to M \to Q^k$ for some integer k.

(b) If $_S B$ is a finitely generated S-submodule of $hom_D(M,Q)$, then $_S B$ is Q-representable by a quotient of M.

(c) Let $hom_D(M,Q)$ be a finitely generated S-module and $F: hom_D(M,Q) \to hom_D(N,Q)$, a S-homomorphism. Then there exists a $f : N \to M$ such that $hom_D(f,Q) = F$.

Proof: Let k be an integer and $(Q^k, (i_i))$ the direct sum of k copies of Q with its injections. Now $hom(Q,Q) = S$ and $hom_D(i_i,Q) = j_i^*$. thus $(S^k (j_i^*))$ is the product of k copies of S with its projections. Let $i_x$ $(1 \le x \le k)$ be the canonical injections of S into $S^k$: in particular, $((i_i), S^k, (j_i^*))$ is a biproduct.

Note that $hom_D(-,Q)$ is exact and faithful since Q is an injective cogenerator.

a) If $0 \to M \to Q^k$ is exact obviously $hom_D(M,Q)$ is a finitely generated S-module.

Assume that $\hom_D(M,Q)$ is finitely generated by $f_1,\ldots,f_k$. Consider the map $t = j_1f_1 + \ldots + j_kf_k : M \to Q^k$. Now $\hom_D(t,Q) = f_1^* j_1^* + \ldots + f_k^* j_k^* : S^k = \hom(Q^k,Q) \to \hom(M,Q)$. As $\hom_D(t,Q)\ i_x(1_S) = f_x$ where $1_S$ is the identity element of S, each $f_i$ is contained in the image of $\hom_D(t,D)$ and hence $\hom_D(t,Q)$ is onto. Since $\hom_D(-,Q)$ is faithful, t is an injection of M into $Q^k$.

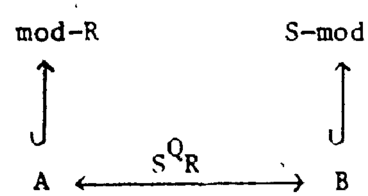
b) Let F be a S-homomorphism of $S^k$ into $\hom_D(M,Q)$. Define $f_x = F\ i_x(1_S) \in \hom_D(M,Q)$, thus $f_x^* = F\ i_x$. Define $f = j_1f_1 + \ldots + j_kf_k \in \hom_D(M,Q^k)$, thus $\hom_D(f,Q) = f^* = f_1^*\ i_1^* + \ldots + f_k^*\ j_k^* = F(i_1j_1^* + \ldots + i_kj_k^*) = F$ since $((i_j),\ S^k,\ (j_j^*))$ is a biproduct. As Im F = $\hom_D(\text{coim } f,Q)$, all finitely generated S-submodules of a Q-representable module are Q-representable. Clearly Coim f is a quotient of M.

c) By part (a) $\hom_D(M,Q)$ is a finitely generated S-module if and only if we have an exact sequence $0 \longrightarrow M \xrightarrow{r} Q^k$ for some integer k. Let $(S,M') =$ coker r, thus we have the exact sequence $0 \longrightarrow M \xrightarrow{r} Q^k \xrightarrow{s} M' \longrightarrow 0$. Let N be an object of D and F a S-homomorphism of $\hom_D(M,Q)$ into $\hom_D(N,Q)$. Consider $F\ r^* : S^k \to \hom_D(N,Q)$. As was shown in the proof of part (b), there exists a $g : N \to Q^k$ such that $g^* = F\ r^*$. As $s\ r = 0$ we have $g^*\ s^* = F\ r^*\ s^* = 0$; therefore, $s\ g = 0$ since $\hom_D(-,Q)$ is faithful. But r is the kernel of s, hence there exists a $f : N \to M$ such that $g = r\ f$. Therefore $g^* = f^*\ r^* = F\ r^*$ and $r^*$ is an epimorphism. Consequently $f^* = F$.

## §2 Representable Dualities

Let $R$ be a ring and $A$ a finitely closed subcategory of mod-$R$. From now on let us agree that unless otherwise stated the topology on the ring $R$ is the $A$-topology and dis $A$ refers to the subcategory of mod-$R$ consisting of all discrete modules for the $A$-topology.

In this section we are interested in the following situation:




where $A$, $B$ are finitely closed subcategories of mod-$R$, $S$-mod respectively. Furthermore, the duality between $A$ and $B$ is assumed to be representable by the bimodule ${}_SQ_R$.

Proposition 2.1: If the duality between $A$ and $B$ is represented by the bimodule ${}_SQ_R$, then $Q' = \{x \in Q \mid xR \in A\}$ and ${}'Q = \{x \in Q \mid Sx \in B\}$, the coreflections of $Q$ into dis $A$ and dis $B$ respectively, are $S$-$R$ bimodules with respect to the old multiplication. Moreover, $Q' = {}'Q$. In particular, $Q \in$ dis $A$ if and only if $Q \in$ dis $B$.

Proof: We will show that $Q'$ is a bimodule. First $\{x \in Q \mid xR \in A\} = \{x \in Q \mid \mathrm{ann}_R(x)$ is open in the $A$-topology$\}$. As $\mathrm{ann}_R(xr) = r^{-1}(\mathrm{ann}_R(x))$, we see that $xr \in Q'$ if $x \in Q'$. Also

$\mathrm{ann}_R(x) \subseteq \mathrm{ann}_R(sx)$, thus $sx \in Q'$ if $x \in Q'$.

We now establish that $Q' = {}'Q$. By definition $x \in Q'$ implies $xR \in A$, hence $\mathrm{hom}_R(xR, Q_R) \in B$. Also $\hat{x} : \mathrm{hom}_R(xR, Q) \to {}_SQ$ defined by $f \rightsquigarrow f(x)$ is a S-homomorphism which contains x in its image since $xR \subseteq Q$. As B is closed under submodules and factor modules and $Sx \subseteq \mathrm{im}\, \hat{x}$, we have $Sx \in B$. Thus $x \in {}'Q$. Similarly, $x \in {}'Q$ implies $x \in Q'$ and we have $Q' = {}'Q$.

The last statement is trivial.

Consider $f \in \mathrm{hom}_R(X,Q)$ with $X \in A$. Since A is finitely closed, $f(X)$ is an object of A; consequently, it is contained in $Q'$. Thus for each $X \in A$, there is an isomorphism $t_X : \mathrm{hom}_R(X,Q) \to \mathrm{hom}_R(X,Q')$ where $t_X(f)$ is the factorization of $f \in \mathrm{hom}_R(X,Q)$ through $Q'$. These isomorphisms are natural, hence $\mathrm{hom}_R(-,Q) \mid A \cong \mathrm{hom}_R(-,Q') \mid A$. As $Q' = {}'Q$, we also have $\mathrm{hom}_S(-,Q) \mid B \cong \mathrm{hom}_S(-,Q') \mid B$. Hence we have the following corollary:

<u>Corollary 2.2</u>: A representable duality between A and B may always be represented by a bimodule ${}_SQ_R$ with $Q \in \mathrm{dis}\ A$ and $Q \in \mathrm{dis}\ B$.

<u>Proposition 2.3</u>: Let A, B be finitely closed subcategories of mod-R and S-mod respectively, and $F : A \to B$ a functor. If ${}_SQ_R$ and ${}_SW_R$ are two bimodules in dis A which represent F, then Q and W are isomorphic as bimodules

<u>Proof</u>: Since $\hom_R(-,Q) \mid A$ and $\hom_R(-,W) \mid A$ both represent F there is a natural isomorphism

$$t : \hom_R(-,Q) \mid A \to \hom(-,W) \mid A .$$

As Q and W are in dis A, we also have:

$$_SQ = \underset{\to}{\bigcup} \operatorname{ann}_Q(I) \cong \underset{\to}{\bigcup} \hom_R(R/I,\ Q)$$

$$_SQ = \underset{\to}{\bigcup} \operatorname{ann}_Q(I) \cong \underset{\to}{\bigcup} \hom_R(R/I,\ Q)$$

where the filtered unions are taken over the set of open right ideals for the A-topology. Define $f : {_SQ} \to {_SW}$ as the unique S-homomorphism making the following diagram commute for all open right ideals:

$$\begin{array}{ccc}
_SQ & \overset{f}{\dashrightarrow} & _SW \\
\uparrow & & \uparrow \\
\operatorname{ann}_Q(I) & & \operatorname{ann}_W(I) \\
\wr\| & & \wr\| \\
\hom_R(R/I,\ Q) & \xrightarrow[t_{R/I}]{} & \hom_R(R/I,\ W)
\end{array}$$

Clearly f is a S-isomorphism.

Let $r \in R$, $r^{-1}I = \{x \in R \mid rx \in I\}$ and $K_r : R/_{r^{-1}I} \to R/I$ be the R-homomorphism defined by $K_r(\bar{I}) = \bar{r}$. As $Q \in$ dis A, for each

$q \in Q$ there exists an open right ideal I and a $\rho \in \hom_R(R/I, Q)$ such that $\rho(\bar{1}) = q$. Now $\hom(K_r, Q)\,\rho \in \hom_R(R/_{r^{-1}I}, Q)$ and $\hom_R(K_r, Q)\,\rho\,(\bar{1}) = qr$. By the naturality of the transformation t and the definition of f, we have $f(qr) = t_{R/_{r^{-1}I}} \hom_R(K_r, Q)\,\rho\,(\bar{1}) = \hom_R(K_r, W)\, t_{R/I}\, \rho(\bar{1}) = f(q)\, r$. Thus f is also a R-homomorphism.

Consequently, ${}_SQ_R$ is isomorphic as a bimodule to ${}_SW_R$.

Remark: By Corollary 2.2 and Proposition 2.3 each representable duality between A and B is represented by an unique bimodule ${}_SQ_R$ with $Q \in$ dis A and $Q \in$ dis B.

Notation: The phrase, "the duality between A and B is represented by the bimodule ${}_SQ_R$", shall imply from now on that $Q \in$ dis A and $Q \in$ dis B.

Theorem 2.4: If the duality between A and B is represented by the bimodule ${}_SQ_R$, then $Q_R$ $({}_SQ)$ is an injective cogenerator of dis A (dis B).

Proof: We show that $Q_R$ is an injective cogenerator of dis A.

Claim 1: $Q_R$ is A-injective.

Let $0 \to Y \to X$ be an exact sequence in A. Since Q represents the duality, we have the exact sequence $\hom_R(X,Q) \to \hom_R(Y,Q) \to 0$.

Thus Q is A-injective.

<u>Claim 2</u>: $Q_R$ is dis A-injective.

Let

$$\begin{array}{ccccc} 0 & \longrightarrow & X_0 & \hookrightarrow & X' \\ & & \Big\downarrow{\scriptstyle \phi_{X_0}} & & \\ & & Q_R & & \end{array}$$

where $X'$ $\varepsilon$ dis A, $X_0$ is a submodule of $X'$ and $\phi_{X_0} : X \to Q_R$ is any R-homomorphism. Let $T = \{(X, \phi_X) \mid X_0 \subseteq X \subseteq X' \text{ and } \phi_X \mid X_0 = \phi_{X_0}\}$. Order T as follows: $(X, \phi_X) \leq (Y, \phi_Y)$ if and only if $X \hookrightarrow Y$ and $\phi_Y \mid X = \phi_X$. Clearly this ordering is inductive, hence by Zorn's lemma there exists a maximal element $(M, \phi_M)$. If $M = X'$ we have proved the claim.

Assume $M \neq X'$. Select $a \in X' - M$, then $M \subsetneq M + aR$. Consider the following diagram:

$$\begin{array}{ccc} M \cap aR & \hookrightarrow & aR \\ \Big\downarrow{\scriptstyle \phi_M \mid M \cap aR} & & \\ Q_R & & \end{array}$$

Note that $M \cap aR$ and $aR$ are elements of A. Thus there exists a map $f : aR \to Q_R$ such that $f \mid M \cap aR = \phi_M \mid M \cap aR$. Define $h : M + aR \to Q_R$ by $h(M + ar) = \phi_M(M) + f(ar)$. This is a well defined R-homomorphism. Also $(M,\phi_M) \lneq (M+aR, h)$, but $(M,\phi_M)$ was maximal, a contradiction.

Claim 3: $Q_R$ is a cogenerator of dis A.

First $Q_R$ cogenerates Y if and only if for each $0 \neq y \in Y$ there exists a $f : Y \to Q_R$ such that $f(y) \neq 0$. Secondly, for a bimodule ${}_SQ_R$, the natural map $X \to X^{**}$ is a monomorphism if and only if $Q_R$ cogenerates X. Thus $Q_R$ cogenerates the category A. Let $X \in \text{dis } A$ and $0 \neq x \in X$. As $X \in \text{dis } A$, $xR \in A$ thus there exists a map $\bar{f} : xR \to Q_R$ such that $\bar{f}(x) \neq 0$. By injectivity of $Q_R$, there is a map $f : X \to Q_R$ such that $f \mid xR = \bar{f}$; in particular, $f(x) \neq 0$. Thus $Q_R$ cogenerates X.

We wish to show that if we have a representable duality between A and B, then A and B consist of linearly compact modules. For future reference we will formulate this first as follows:

Proposition 2.5: Let A, B be finitely closed subcategories of mod-R and S-mod respectively, ${}_SQ_R$ a bimodule which is an injective object of dis A and $\hom_R(-,Q) \mid A$ a duality between A and B. Then B consists of linearly compact modules.

Proof: Let $X \in A$ and $(X_j)$ $j \in J$ an updirected family of subobjects of X. In mod-R we have an exact sequence:

$$0 \longrightarrow \varinjlim_{\text{mod-}R} X_j \longrightarrow X \longrightarrow \varinjlim_{\text{mod-}R} X/X_j \longrightarrow 0$$

As A is finitely closed, both $\varinjlim_{\text{mod-}R} X_j$ and $\varinjlim_{\text{mod-}R} X/X_j$ are elements of A. Consequently, $\varinjlim_{A} X_j = \varinjlim_{\text{mod-}R} X_j$ and $\varinjlim_{A} X/X_j = \varinjlim_{\text{mod-}R} X/X_j$.

Also we see that A is an AB5-category.

Let Y be an object of B and $(Y_j)$ $j \in J$ a downdirected family of subobjects of Y with intersection I. Note that $I = \varprojlim_{B} Y_j = \varprojlim_{S\text{-mod}} Y_j$. By duality Y, I, and the $Y_j$ are represented by X, V, and the $X_j$ which form an updirected family of quotients of X with $V = \varinjlim_{A} X_j$. For each $j \in J$, we have the exact sequence:

$$0 \longrightarrow K_j \longrightarrow X \longrightarrow X_j \longrightarrow 0$$

Thus we have the exact sequence:

$$0 \longrightarrow \varinjlim K_j \longrightarrow X \longrightarrow \varinjlim X_j = V \longrightarrow 0$$

with the colimits taken in mod-R [we showed above that colimits of this type in A coincide with the corresponding mod-R colimit]. As $\hom_R(-,Q)$ is exact on A (Q is A-injective) and takes colimits to limits in S-mod, we see that $Y/I = \varprojlim_{S\text{-mod}} Y/Y_j$. Thus for all downdirected families $(Y_j)$ $j \in J$ of submodules of Y the natural map $Y \to \varprojlim_{S\text{-mod}} Y/Y_j$ is onto, hence Y is linearly compact.

Theorem 2.6: If the duality between A and B is represented by a bimodule ${}_SQ_R$, then A and B are linearly compact.

Proof: By Theorem 2.4, the duality is represented by a bimodule ${}_SQ_R$ which is injective in dis A and injective in dis B. Now apply Proposition 2.5.

Notation: Let R, S be rings, ${}_SQ_R$ a bimodule, V a right ideal of R and Y a S-submodule of ${}_SQ$, then

(1) $\mathrm{ann}_Q(V) = \{q \in Q \mid qV = 0\} \cong \hom_R(R/V, Q)$

(2) $\mathrm{ann}_R(Y) = \{r \in R \mid Yr = 0\}$.

Proposition 2.7: Let the duality between A and B be represented by the bimodule ${}_SQ_R$. For each $f \in \mathrm{End}\ {}_SQ$ and each finite family $q_1,\dots,q_n$ of elements of Q, there exists a $r \in R$ such that $f(q_i) = q_i\, r$, for $i = 1,\dots,n$.

Proof: Let $f \in \mathrm{End}\ {}_SQ$ and $q_1,\dots,q_n$ elements of Q. Consider $\mathrm{ann}_Q\ \mathrm{ann}_R\ \{q_1,\dots,q_n\}$. Clearly $q_1,\dots,q_n$ are elements of $\mathrm{ann}_Q\ \mathrm{ann}_R\ \{q_1,\dots,q_n\}$. We note that $V = \mathrm{ann}_R\ \{q_1,\dots,q_n\}$ is the kernel of the homomorphism $t : R \to Q_R^n$ defined by $t(1) = (q_1,\dots,q_n)$. Thus R/V is an object of A [Q ε dis A and A contains all finitely generated modules of dis A]. Hence $R/V \cong \hom_S(\hom_R(R/V, Q), Q)$. As $\hom_R(R/V, Q) \cong \mathrm{ann}_Q V = \mathrm{ann}_Q \mathrm{ann}_R\{q_1, \ldots, q_n\}$, we see that $f \mid \mathrm{ann}_Q \mathrm{ann}_R\{q_1, \ldots, q_n\}$ is represented by some $\bar{r} \in R/V$. Let $r$ be an element of the coset $\bar{r}$, then $f(q_i) = q_i r$ for $i = 1, \ldots, n$.

Proposition 2.8: Let the duality between $A$ and $B$ be represented by the bimodule ${}_SQ_R$. If ${}_SY$ is a finitely generated S-submodule of Q, then $\mathrm{ann}_Q \mathrm{ann}_R Y = {}_SY$.

Proof: Clearly ${}_SY \subseteq \mathrm{ann}_Q \mathrm{ann}_R (Y)$.

Let $q_1, \ldots, q_n$ generate ${}_SY$ and $V = \mathrm{ann}_R Y$. Consider the monomorphism $f : R/V \to Q_R^n$ defined by $f(\bar{1}) = (q_1, \ldots, q_n)$. Let $p \in \mathrm{ann}_Q \mathrm{ann}_R Y$, then a map $g : R/V \to Q_R$ exists defined by $g(\bar{1}) = p$. As $Q_R$ is self-injective, a map $h : Q_R^n \to Q_R$ exists such that $hf = g$. Since $Q_R^n$ is a direct sum, there exists $f_1, \ldots, f_n : Q_R \to Q_R$ such that $h(X_1, \ldots, X_n) = \sum_{i=1}^{n} f_i(X_i)$ for all $(X_1, \ldots, X_n) \in Q_R^n$. By Proposition 2.7, expressed in terms of $S$, there exist $s_1, \ldots, s_n \in S$ such that $f_i(q_i) = s_i q_i$ for $i = 1, \ldots, n$. Hence $p = g(\bar{1}) = h\, f(\bar{1}) = h(q_1, \ldots, q_n) = \sum_{i=1}^{n} f_i(q_i) = \sum_{i=1}^{n} s_i\, q_i$. Hence $p \in {}_SY$ and $\mathrm{ann}_Q \mathrm{ann}_R (Y) \subseteq {}_SY$.

Proposition 2.9: If the duality between A and B is represented by the bimodule ${}_SQ_R$, then socle $(Q_R)$ = socle $({}_SQ)$. Socle $(Q_R)$ is essential in $Q_R$ and socle $({}_SQ)$ is essential in ${}_SQ$.

Proof: If for $0 \neq q \in Q$, $qR$ is simple, then $ann_R(q)$ is a maximal right ideal. If $0 \neq p \in Sq$, then $ann_R(Sp) = ann_R(Sq)$ by maximality. So $Sp = ann_Q\, ann_R(Sp) = ann_Q\, ann_R(Sq) = Sq$ (by Proposition 2.8), and it follows that $Sq$ is a simple left S-submodule of Q. By symmetry, socle $(Q_R)$ = socle $({}_SQ)$.

If $0 \neq q \in Q$, then since ${}_SQ$ is an injective cogenerator in dis B, there is an epimorphism $f : Sq \rightarrow Sx$ with $f(q) = x$ where $Sx$ is simple and $0 \neq x \in Q$. Since this homomorphism can be extended to a homomorphism in $hom_S(Q,Q)$, by Proposition 2.7 there is a $r \in R$ such that $qr = x$. It follows that socle $(Q_R)$ is essential in $Q_R$.

## §3 Completions

Let A and B be two dual finitely closed subcategories of mod-R and S-mod respectively. In this section, we will demonstrate that we may assume that R and S are complete and Hausdorff in the A-topology and B-topology respectively.

Proposition 3.1: Let R, S be rings, ${}_SQ_R$ a bimodule, $\tau$ a linear topology on R with R/V Q-reflexive for all right ideals V open in $\tau$, and $\underset{V \in \tau}{\underrightarrow{\cup}} \operatorname{ann}_Q(V) = Q$. Then End ${}_SQ = \varprojlim R/V$, the completion of R in the topology $\tau$.

Proof: Let V be an open right ideal in $\tau$. Consider $\operatorname{ann}_Q V = \hom_R(R/V, Q)$. Clearly $\{\widetilde{\operatorname{ann}_Q(V)}\}$ is an updirected family of S-submodules and by hypothesis its union is Q. As $R/V \cong \hom_S(\hom_R(R/V, Q), Q)$, we have $\hom_S(Q,Q) = \hom_S(\varinjlim \operatorname{ann}_Q(V), Q)$ $= \varprojlim \hom_S(\hom_R(R/V, Q),Q) \cong \varprojlim R/V$.

Corollary 3.2: If the duality between A and B is represented by the bimodule ${}_SQ_R$, then End ${}_SQ = \varprojlim R/V$, the completion of R in the A-topology.

Proof: Since a right ideal V of R is open in the A-topology if and only if $R/V \in A$; clearly, for each open right ideal V, R/V is Q-reflexive and $\underrightarrow{\cup} \operatorname{ann}_Q(V) = Q$ [as $Q \in$ dis A]. Thus by proposition 3.1, we have the result.

For linearly topologized ring R, dis R is the Grothendieck category of discrete modules. We recall that $X \in$ dis R if and only if for each $x \in X$ one has $\mathrm{ann}_R(x)$ open in R.

Proposition 3.3: If R is a linearly topologized ring and $\hat{R} = \varprojlim R/V$ its Hausdorff completion, then dis R is naturally isomorphic to dis $\hat{R}$. Moreover, $X_R \in$ dis R is a linearly compact R-module if and only if its image in dis $\hat{R}$ under the isomorphism is a linearly compact $\hat{R}$-module.

Proof: The homomorphism $i : R \to \hat{R}$ induced by the canonical epimorphisms $R \to R/V$ for V open is continuous. Hence trivially, each $\hat{R}$-module in dis $\hat{R}$ is a R-module in dis R.

Let $X_R \in$ dis R. As $\mathrm{ann}_R(x)$ is open in R for each $x \in X$, X can be made into a $\hat{R}$-module as follows. If $\hat{r} \in \hat{R}$ and $x \in X$, let $(r_i)$ be a generalized Cauchy sequence whose limit is $\hat{r}$, since $\mathrm{ann}_R(x)$ is open, it follows that $x\, r_i$ is constant for large i, and we define $x\, \hat{r}$ to be this constant value.

The two functors outlined on objects above, and mapping homomorphisms identically are the desired inverse isomorphisms between dis R and dis $\hat{R}$.

If $X_R$ and $X_{\hat{R}}$ correspond under the isomorphism between dis R and dis $\hat{R}$, then $X_R$ and $X_{\hat{R}}$ have the same underlying group and the same submodules. Thus $X_R$ is linearly compact if and only if $X_{\hat{R}}$ is.

Let A be a finitely closed subcategory of mod-R, and let $\hat{R}$ be the completion of R in the A-topology. By Proposition 3.3, we may identify dis A and dis $\hat{R}$ by means of the natural isomorphism. (Note that for the A-topology on R dis R = dis A). Under this identification A is a finitely closed subcategory of mod-$\hat{R}$ and the topology on $\hat{R}$ is the A-topology.

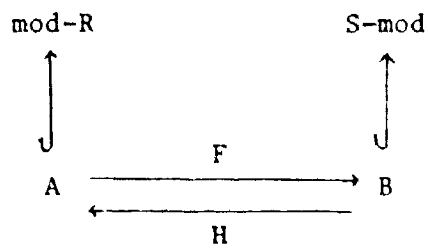
<u>Proposition 3.4</u>: If there is a duality between the finitely closed subcategories A and B of mod-R and S-mod (represented by the bimodule ${}_SQ_R$), then there is a duality between the finitely closed subcategories A and B of mod-$\hat{R}$ and $\hat{S}$-mod (represented by the bimodule ${}_{\hat{S}}Q_{\hat{R}}$) where $\hat{R}$ ($\hat{S}$) is the completion of R (S) in the A-topology (B-topology). Furthermore, if the duality is represented by Q, then $\hat{R}$ = End ${}_{\hat{S}}Q$, $\hat{S}$ = End $Q_{\hat{R}}$, $\hat{R}$ is linearly compact in the A-topology and $\hat{S}$ is linearly compact in the B-topology.

<u>Proof</u>: The first statement is immediate from the identification given by Proposition 3.3 of dis A with dis $\hat{R}$ and dis B with dis $\hat{S}$. We note that the representing bimodule Q is an object of dis A and an object of dis B.

If the duality is represented by a bimodule Q, then A and B consist of linearly compact modules (Theorem 2.6). Hence $\hat{R}$, the completion of R in the A-topology, is linearly compact. $\hat{R}$ = End ${}_{\hat{S}}Q$ by Corollary 3.2.

## §4 Construction of the Bimodule Q

In this section we consider the following situation:




Here A and B are finitely closed faithful subcategories of mod-R and mod-S. The functors F and G define a duality between A and B.

There is a linear topology on S, the B-topology which has, as a basis of zero, the left ideals V of S such that S/V is an object of B. With respect to the canonical epimorphisms, the S/V form a projective system whose limit is the completion of S in the B-topology. Applying the functor H to this projective system, we obtain a mono-inductive system in A. Let $Q_R = \varinjlim H(S/V)$ where the colimit is formed in mod-R. Note that $Q \in$ dis A.

<u>Proposition 4.1</u>: There exists a ring homomorphism $\Sigma : S \to \hom_R(Q,Q)$. Hence Q is a S-R bimodule.

<u>Proof</u>: For $s \in S$, define $\Sigma(s) : Q_R \to Q_R$ as follows: Let V be an open left ideal of S, then $s^{-1}V = \{x \in S \mid xs \in V\}$ is also an open left ideal. Now consider the family of S-homomorphisms,

$\rho_s : S/_{s^{-1}V} \to S/V$, defined by $\rho_s(\bar{1}) = \bar{s}$ for each open V. Applying H we get the following:

$$\begin{array}{ccc} Q_R & \overset{\Sigma(s)}{\dashrightarrow} & Q_R \\ \uparrow & & \uparrow \\ H(S/V) & \xrightarrow{H(\rho_S)} & H(S/s^{-1}V) \end{array}$$

where $\Sigma(s)$ is the unique R-homomorphism making the diagram commute for all V. Calculation now shows that $\Sigma$ is indeed a ring homomorphism.

The above proposition allows one to consider $hom_R(-,Q) : \text{mod-}R \to S\text{-mod}$ as a functor in the standard way.

<u>Proposition 4.2</u>: Let $Q = \varinjlim H(S/V)$. There exists a monic natural transformation $(\mu_X) : F \to hom(-,Q)$.

<u>Proof</u>: Since FX is an object of B, $FX = \varinjlim ann_{FX}(V)$. Secondly, let $J_V : ann_{FX}(V) \to hom_S(S/V, FX)$ be the standard isomorphism, then $J_X = \varinjlim J_V : \varinjlim ann_{FX}(V) \to \varinjlim hom_S(S/V, FX)$ is an isomorphism. Because F, H is a duality, we have an isomorphism $K_V : hom_S(S/V, FX) \to hom_R(X, H(S/V))$; thus $K_X = \varinjlim K_V : \varinjlim hom_S(S/V, FX) \to \varinjlim hom_R(X, H(S/V))$ is an isomorphism. Since $Q_R = \varinjlim H(S/V)$, we have a unique map

$t_X : \varinjlim \hom_R(X, H(S/V)) \to \hom_R(X, {}_SQ_R)$ such that $t_X b_V = \hom_R(X, \ell_V)$; $\ell_V : H(S/V) \to {}_SQ_R$ being the canonical inclusion and $b_V$ being the structure map of the filtered limit. Moreover, $t_X$ is a monomorphism.

For each $X \in A$, define $\mu_X : FX \to \hom_R(X,Q)$ by the sequence of homomorphisms:

$$FX = \varinjlim \operatorname{ann}_{FX} V \xrightarrow{J_X} \varinjlim \hom_S(S/V, FX) \xrightarrow{K_X} \varinjlim \hom_R(X, H(S/V)) \xrightarrow{t_X} \hom_R(X,Q).$$

That is, $\mu_X = t_X K_X J_X$. Moreover, $\mu_X$ is a monomorphism and is natural as $t_X$, $K_X$ and $J_X$ are.

Note that $\mu_X$ was really constructed as a group homomorphism, but simple checking shows that $\mu_X$ is actually an S-homomorphism.

Corollary 4.3: If $X$ is a finitely generated module, then $\mu_X$ is an isomorphism.

Proof: If $X$ is a finitely generated module $t_X$ is an isomorphism, hence so is $\mu_X$.

Proposition 4.4: If $B$ consists of linearly compact modules, then $\mu_X : FX \to \hom_R(X,Q)$ [for $Q = \varinjlim H(S/V)$] is an isomorphism for all $X \in A$.

Proof: Let $X$ be any object of $A$, then by Proposition 4.2 $\mu_X : FX \to \hom_R(X,Q)$ is a monomorphism. Let $\sigma_f : X_f \to X$ be the inclusion of a finitely generated submodule $X_f$ of $X$. Consider the following diagram:

$$\begin{array}{ccc} \hom_R(X,Q) & \xleftarrow{\ \mu_X\ } & FX \\ \downarrow{\scriptstyle \hom_R(\sigma_f,Q)} & & \downarrow{\scriptstyle F\,\sigma_f} \\ \hom_R(X_f,Q) & \xleftarrow[\ \mu_{X_f}\ ]{} & F\,X_f \end{array}$$

Now $F\,\sigma_f$, the dual of a monomorphism, is an onto map, and $\mu_{X_f}$ is an isomorphism since $X_f$ is a finitely generated module. Let $\phi \in \hom_R(X,Q)$. Then there exists an $b_f \in FX$ such that $\phi \mid X_f = \mu_{X_f}\ F\,\sigma_f(b_f)$ for each finitely generated submodule $X_f$ of X. Consider the congruence $b \equiv b_f$ (mod Ker $F\,\sigma_f$), f running over the finitely generated submodules $X_f$ of X. As the finitely generated submodules are an updirected family, this congruence is clearly finitely solvable. Since $FX \in B$ and B is linearly compact, the congruence is solvable. Let b be a solution, then $\mu_X(b) \mid X_f = \phi \mid X_f$ for all finitely generated submodules $X_f$ of X. Thus $\mu_X(b) = \phi$ and $\mu_X$ is an isomorphism.

<u>Theorem 4.5</u>: If A and B consist of linearly compact modules and ${}_SQ_R = \underrightarrow{U}\, H(S/V)$ is as constructed in Proposition 4.1, then

(1) F is naturally isomorphic to $\hom_R(-,Q) \mid A$

(2) H is naturally isomorphic to $\hom_S(-,Q) \mid B$.

Thus the duality between A and B is represented by the bimodule Q.

Proof: (1) This is Propositions 4.1, 4.2 and 4.4.

(2) From Propositions 4.1, 4.2 and 4.4 we know that H is represented by $_SW_R = \coprod \hom_R(R/I, Q)$. Also $_SQ = \coprod \mathrm{ann}_Q(I)$ since $Q \in \mathrm{dis}\ A$. [Each $H(S/V) \in \mathrm{dis}\ A$ and dis A is closed under colimits]. Let $f = \coprod f_I : {}_SW \to {}_SQ$ where $f_I : \hom(R/I, Q) \to \mathrm{ann}_Q(I)$ is defined by $f_I(\phi) = \phi(\bar{1})$. Thus f is an S-isomorphism since all the $f_I$ are S-isomorphism. We claim that f is also an R-homomorphism. Let $\phi$ be any element of W, then $\phi \in \hom_R(R/I, Q)$ for some I. By the definition of the R-action on W in Proposition 4.1, we know $\phi r \in \hom_R(R/_{r^{-1}I}, Q)$ and $\phi r(\bar{x}) = \phi(\overline{rx})$. Hence $f(\phi r) = \phi r(\bar{1}) = \phi(r\bar{1}) = \phi(\bar{r}) = \phi(\bar{1})\ r = f(\phi)\ r$. Thus $_SQ_R$ is isomorphic as a bimodule to $_SW_R$.

Thus $_SQ_R$ represents the duality, $Q \in \mathrm{dis}\ A$ and $Q \in \mathrm{dis}\ B$.

## CHAPTER III

## LINEARLY COMPACT SUBCATEGORIES OF MOD-R

### §1 Remarks on Limits

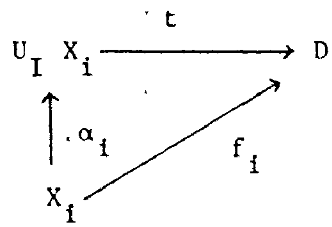
It is difficult to decide whether a given finitely closed subcategory A of mod-R is linearly compact or not. If A is linearly compact, then A is always an $AB5^*$-category and A has no infinite direct sums (of nonzero modules of mod-R). Goblot [10] shows that every finitely closed $AB5^*$-subcategory of mod-R with no infinite direct sums is dual to a finitely closed linearly compact subcategory of modules. In this chapter we show that for certain rings R a finitely closed $AB5^*$-subcategory A of mod-R with no infinite direct sums is linearly compact. This is done by showing certain limits in A are actually limits in mod-R.

I wish to thank Richard Squire for his help with the following proposition:

Proposition 1.1: Let C be an AB5-category and $\{u_i : X_i \to X\}_{i \in I}$ an updirected family of subobjects of X, then $\varinjlim_I X_i = U_I X_i$.

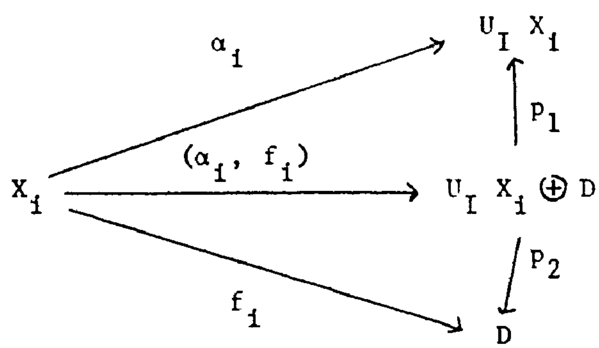
Proof: Let $\{f_i : X_i \to D\}_{i \in I}$ be a compatible family. For each $i \in I$, define $\alpha_i : X_i \to U_I X_i$ as the factorization of $u_i : X_i \to X$

through the union. To prove that $U_I\, X_i$ is the colimit we must show that there is an unique map $t : U_I\, X_i \to D$ making the diagram



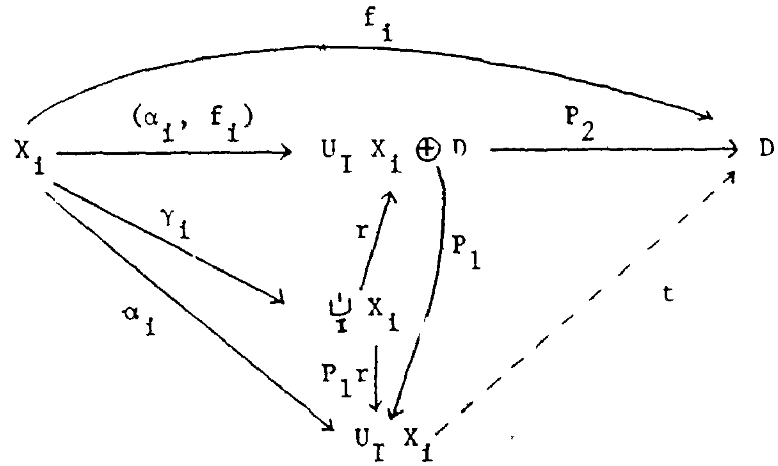

commute for all $i \in I$.

For existence of t we proceed as follows. Let $(\alpha_i, f_i)$ be the uniaue map defined by the diagram




where $p_1$ and $p_2$ are the projection maps. Note that $(\alpha_i, f_i)$ is a monomorphism as $\alpha_i$ is, and that the family $\{(\alpha_i, f_i) : X_i \to U_I\, X_i \oplus D\}_{i \in I}$ is updirected. Define $\dot{\bigsqcup}_I X_i \xrightarrow{r} U_I\, X_i \oplus D$ as the union of the family $\{(\alpha_i, f_i) : X_i \to U_I\, X_i \oplus D\}_{i \in I}$. For each $i \in I$ define $\gamma_i : X_i \to \dot{\bigsqcup}_I X_i$ as the factorization of $(\alpha_i, f_i) : X_i \to U_I\, X_i \oplus D$ through the union $\dot{\bigsqcup}_I X_i$.

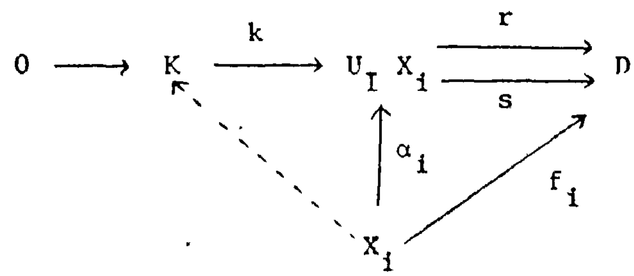
Consider the following commutative diagram for each $i \in I$.




We wish to show that $P_1 r$ is an isomorphism. Each $X_i$ factors through $\mathrm{im}(P_1 r)$, the image of $P_1 r$. As $\mathrm{im}(P_1 r)$ is a subobject of $\bigcup_I X$, it is also a subobject of X. Thus $\mathrm{im}(P_1 r) = \bigcup_I X_i$, as $\bigcup_I X_i$ is the least subobject of X through which each $X_i$ factors. Consequently, $P_1 r$ is an epimorphism. Let $K = \ker P_1 r$. Then by AB5, $K = K \cap \dot{\bigcup}_I X_i = \dot{\bigcup}_I (K \cap X_i)$. If $K \neq 0$ then $K \cap X_i \neq 0$ for some $i \in I$, but this is a contradiction as $\alpha_i$ is a monomorphism. As C is an abelian category, a map which is a monomorphism and a epimorphism is an isomorphism, thus $P_1 r$ is an isomorphism.

Define $t = P_2 r(P_1 r)^{-1}$. Diagram chasing shows that this is the desired map.

For uniqueness, assume that we have two maps r, s making the following diagram commute for all $i \in I$.




Let $(K,k)$ be the equalizer of $r$ and $s$. Hence $\alpha_i$ factors through $K$ for each $i \in I$. Note that $K$ is a subobject of $X$. Thus $K$ must equal $\cup_I X_i$ as $\cup_I X_i$ is the least subobject of $X$ through which each $X_i$ factors. Thus $r = s$ .

<u>Corollary 1.2</u>: If $C$ is an $AB5^*$-category and $\{u_i : X_i \to X\}_{i \in I}$ is a downdirected family of subobjects of $X$, then $\varprojlim_I X/X_i = X/\cap X_i$.

<u>Proof</u>: This is the dual of Proposition 1.1.

Let $A$ be a finitely closed subcategory of mod-R. We now consider the relationship between limits formed in A, dis A or mod-R.

<u>Proposition 1.3</u>: Let R be a ring, $A$ a finitely closed subcategory of mod-R and dis A the category of modules discrete for the A-topology, then the embedding $A \hookrightarrow$ dis A commutes with limits existing in A.

<u>Proof</u>: First, as dis A is a Grothendieck category, the limit in dis A of all diagrams exists. If $D: I \to A$ is a diagram in A with limit $(X, \Pi_i)$, then it has limit $(L, \phi_i)$ in dis A. Now the compatible family $(X, \Pi_i)$ factors over the compatible family $(L, \phi_i)$ by a unique homomorphism $\Pi : X \to L$. Also as $L$ is in dis A, $L$ is the

filtered union of subobjects $Y_k$ which lie in A. The restriction of the $\phi_i$ to one of these $Y_k$ yields a compatible family; hence, it factors over $(X, \Pi_i)$ by a homomorphism $f_k : Y_k \to X$. As $\sqcup = \bigcup\limits_{\to} Y_k$ and the $f_k$ are compatible with the order relations on the $Y_k$, we obtain a homomorphism $f : \sqcup \to X$. The homorphism $f$ is easily seen to be the inverse of $\Pi$.

Definition: A is a meager finitely closed subcategory of mod-P if for all $X \in A$ there exists a $Y \in A$ such that $X \subseteq Y$ and $Y$ is a finitely generated R-module.

Proposition 1.4: Let A be a meager finitely closed subcategory of mod-R. If the A-topology has a basis of two sided ideals, then the following are equivalent:

(1) A is an $AB5^*$-category.

(2) A is linearly compact.

Proof: 2) $\Rightarrow$ 1). Well known consequence of linearly compactness.

1) $\Rightarrow$ 2). Let $\{Y_i\}_{i \in I}$ be a downdirected family of submodules of $X \in A$. X is linearly compact if and only if $X/\cap Y_i = \varprojlim_{\text{mod-R}} X/Y_i$. Since A is an $AB5^*$-category, we know $\varprojlim_{A} X/Y_i = X/\cap Y_i$ (Corollary 1.2). By Proposition 1.3, $\varprojlim_{\text{dis } A} X/Y_i = \varprojlim_{A} X/Y_i = X/\cap Y_i$. We claim that $\varprojlim_{\text{dis } A} X/Y_i = \varprojlim_{\text{mod-R}} X/Y_i$. First as $X \in A$, it is a submodule of a finitely generated module. As the A-topology has a basis of two-sided ideals, there exists an ideal $V$ open in the A-topology such that $XV = 0$.

Secondly, $\varprojlim_{\text{mod-R}} X/Y_i$ is a submodule of $\prod_{\text{mod-R}} X/Y_i$. But $\left(\prod_{\text{mod-R}} X/Y_i\right) V = 0$; hence, $\varprojlim_{\text{mod-R}} X/Y_i$ is an object of dis A. Therefore, $\varprojlim_{\text{mod-R}} X/Y_i = \varprojlim_{\text{dis A}} X/Y_i = X/\cap Y_i$, and X is linearly compact.

Corollary 1.5: If R is commutative, a meager finitely closed $AB5^*$-subcategory of mod-R is linearly compact.

## §2 Duality Construction

Let A be a finitely closed subcategory of mod-R. In this section we demonstrate the following: A is dual to a finitely closed linearly compact subcategory of S-mod for some ring S if and only if A is an AB5$^*$-category and A has no infinite direct sums (of nonzero modules). The proof is adapted from R. Goblot [10].

Proposition 2.1: Let A be a finitely closed AB5$^*$-subcategory of mod-R with no infinite direct sums, and let $Q_R$ be an injective cogenerator of dis A which is essential over its socle. For any X ε A and any downdirected set $\{Y_i\}_{i \in I}$ of subobjects of X with intersection Y, we have

$$\varinjlim \hom_R(X/Y_i, Q) = \hom_R(X/Y, Q).$$

Secondly, all subobjects of Q contained in A are essential over finite socles.

Proof: We show the last statement first. Let T be a subobject of Q. As Q is essential over its socle, T is essential over its socle. If T is an object of A, then the socle of T must be finite as A contains no infinite direct sums.

Consider the unique homomorphism

$$t : \varinjlim \hom_R(X/Y_i, Q) \longrightarrow \hom_R(X/Y, Q)$$

such that $t\, u_i = \hom_R(P_i, Q)$, $P_i : X/Y \to X/Y_i$ being the canonical epimorphism and $u_i$ being the structure homomorphism of the colimit.

As $t$ is always a monomorphism, it suffices to show that $t$ is onto. Let $f \in \hom_R(X/Y, Q)$, let $g$ be the composition of $f$ and the surjection of $X$ onto $X/Y$, and let $K$ be the kernel of $g$. Note that $Y \subseteq K$. As $A$ satisfies AB5$^*$, we have $K = K + Y = K + \bigcap_I Y_i = \bigcap_I (K+Y_i)$. Since $X/K$ is an object of $A$ and isomorphic to a subobject of $Q$, it is essential over a finite socle. Thus $K = K + Y_i$ for some $i$. Hence $Y_i \subseteq K$ for some $i$, and $g$ factors over $X/Y_i$. Therefore, $t$ is onto.

Proposition 2.2: Let $A$ be a finitely closed AB5$^*$-subcategory of mod-R, $Q_R$ an injective cogenerator of dis $A$ and $S = \text{End}\ Q_R$. If for all $X \in A$ and for all downdirected sets $\{Y_i\}_{i \in I}$ of subobjects of $X$ with intersection $Y$, one has

$$\varinjlim \hom_R(X/Y_i, Q) = \hom_R(X/Y, Q),$$

then (1) $\hom_R(-,Q) \mid A$ is full.

(2) $B = \hom_R(A,Q)$ is a finitely closed linearly compact subcategory of S-mod.

Thus $\hom_R(-,Q) : A \to B$ is a duality.

Proof: The functor $\hom_R(-,Q) : A \to$ S-mod is exact and faithful. Let us show that it is full. Let $M$ and $N$ be two objects of $A$, and $F$ a S-homomorphism from $\hom_R(M,Q)$ to $\hom_R(N,Q)$. From Chapter II Proposition 1.3, the filtered family of finitely generated submodules of $\hom_R(M,Q)$ is represented by a filtered projective system $(M/M_i)$ of quotients of $M$. As $\hom_R(M/\cap M_i, Q) = \varinjlim \hom_R(M/M_i, Q) = \hom_R(M,Q)$

and Q is an injective cogenerator, we see that $\cap M_i$ must equal zero. [if $\cap M_i \neq 0$ then $\hom_R(M/\cap M_i, Q)$ is a proper submodule of $\hom_R(M,Q)$]. As A is an AB5*-category, we have $\varprojlim_A M/M_i = M/\cap M_i = M$. The homomorphism F induces a filtered system of homomorphism $(F_i)$ from the $\hom_R(M/M_i, Q)$ into $\hom_R(N,Q)$ which by Chapter II Proposition 1.3 are representable by a filtered system $(f_i)$ of maps from N to $M/M_i$. As $M = \varprojlim_A M/M_i$, we get $f : N \to M$, a map inducing the system $(f_i)$. As $\hom_R(f,Q)$ induces the $F_i = \hom_R(f_i, Q)$ on the $\hom_R(M/M_i, Q)$, by uniqueness, $\hom_R(f,Q) = F$. Thus $\hom_R(-,Q) \mid A$ is full; consequently, $\hom_R(A,Q)$ is the dual of A.

We will now show that $\hom_R(A,Q)$ is a finitely closed subcategory of S-mod. As $\hom_R(X,Q) \oplus \hom_R(Y,Q) = \hom_R(X\oplus Y, Q)$, it is clearly closed under finite direct sums. Let X be a S-submodule of $\hom_R(M,Q)$, $M \in A$. Now $X = \varinjlim X_i$ where $X_i$ is a finitely generated S-module. By Chapter II Proposition 1.3, each $X_i$ is representable by a quotient $M/M_i$ of M. Now $\hom_R(M/\cap M_i, Q) = \varinjlim \hom_R(M/M_i, Q) = \varinjlim X_i = X$. Thus X is representable. By exactness, $\hom_R(M,Q)/X$ is representable by the subobject $\cap M_i$ of M. Hence $\hom_R(A,Q)$ is a finitely closed subcategory.

The S-modules of $\hom_R(A,Q)$ are linearly compact for the discrete topology (Chapter II Proposition 2.5).

<u>Proposition 2.3</u>: Let R and S be rings. Assume A and B are finitely closed subcategories of mod-R and S-mod respectively which are dual. If B has no infinite direct sums, then A has no infinite direct sums.

<u>Proof</u>: In anv Grothendieck category the direct sum is a subobject of the product. Now consider the Grothendieck category dis B. Direct sums in dis B are direct sums in S-mod. Let dis B - $\prod_{i\in I} Y_i$ be the product in dis B, then $\bigoplus_{i\in I} Y_i \subseteq$ dis B - $\prod_{i\in I} Y_i$. By Proposition 1.3, limits in B are limits in dis B.

Let $X \in A$ correspond to $X^0 \in B$ under the duality. If $X \neq 0$, then $X^0 \neq 0$. Let I be an infinite set. Assume $X_i \neq 0$ for $i \in I$, and $\bigoplus_{i\in I} X_i \in A$. Then dis B - $\prod_{i\in I} X_i^0 \in B$; hence $\bigoplus_{i\in I} X_i^0 \in B$ as B is finitely closed. This contradicts the given hypothesis, thus A has no infinite direct sums.

For convience, we collect the last three propositions as the following theorem:

<u>Theorem 2.4</u>: Let A be a finitely closed subcategory of mod-R. These are equivalent:

(1) A is an $AB5^*$-category and has no infinite direct sums.

(2) Let $Q_R$ be an injective cogenerator of dis A which is essential over its socle, $S = \text{End } Q_R$ and $B = \hom_R(A,Q)$. Then B is a finitely closed linearly compact subcategory of S-mod, and $\hom_R(-,Q) : A \to B$ is a duality.

(3) There exists a ring S and a finitely closed linearly compact subcategory B of S-mod such that A is dual to B.

Proof: 1) $\Rightarrow$ 2). This is Proposition 2.1 and Proposition 2.2

2) $\Rightarrow$ 3) It is obvious.

3) $\Rightarrow$ 1) As B is a finitely closed subcategory of S-mod, it is an AB5-category, thus A is an AB5$^*$-category. Since B is linearly compact it has no infinite direct sums. Hence by Proposition 2.3, A has no infinite direct sums.

Proposition 2.5: Let A be a finitely closed linearly compact subcategory of mod-R. If $Q_R$ is an injective cogenerator of dis A with essential socle, $S = \text{End } Q_R$ and $B = \hom_R(A,Q)$, then B is a finitely closed linearly compact subcategory of S-mod and the duality between A and B is represented by the bimodule ${}_SQ_R$.

Proof: From Theorem 2.4 we know B is a linearly compact finitely closed subcategory of S-mod and A and B are dual.

Let $X \in A$. Since Q is an injective cogenerator of dis A, the natural map $i : X \to X^{**} = \hom_S(\hom_R(X,Q),Q)$ is a monomorphism. By Chapter II Proposition 1.2, $X^{**}$ is the completion of X with the Q-topology. As X is linearly compact it is complete for all linear topologies. Thus i is onto and Q represents the duality.

## §3 The Leptin Topology

Definition: Let X be a R-module. X is cofinite if for all downdirected families $\{X_i\}_{i \in I}$ of subobjects of X with $\cap X_i = 0$ there exists an $i \in I$ with $X_i = 0$.

Remark: Let A be a finitely closed subcategory of mod-R and $A^0$, its dual, a finitely closed subcategory of S-mod. $X \in A$ is cofinite if and only if $X^0 \in A^0$ is a finitely generated S-module.

Proposition 3.1: $X \in$ mod-R is cofinite if and only if X is an essential extension of a finite socle.

Proof: If X is cofinite, the set of nonzero submodules of X ordered by containment is inductive, then by Zorn's lemma, every nonzero submodule of X contains a simple submodule. Hence X is essential over its socle. This socle has finite length since an infinite direct sum of nonzero modules cannot be cofinite.

Assume X is essential over a finite socle. Since the socle is of finite length, it is cofinite (it is artinian). Let S be the socle of X and $(X_i)_{i \in I}$ a downdirected family of subobjects of X with $\cap X_i = 0$, then there exists an $i \in I$ such that $S \cap X_i = 0$. Hence $X_i = 0$ which proves that X is cofinite.

Let $\tau$ be a Hausdorff linear topology on a module X. Following Bourbaki [3, Chapter III, §2, Exercise 18], we define $\tau^*$ to be the linear topology on X with fundamental system of neighbourhoods of zero the filter basis generated by the submodules of X which are open under $\tau$ and completely-meet-irreducible. If $\tau$ is linearly compact so is $\tau^*$; moreover, $\tau^*$ is the coarsest Hausdorff linear topology on X coarser than $\tau$. $\tau^*$ is sometimes called the Leptin Topology.

Proposition 3.2: Let $\tau$ be a linearly compact Hausdorff topology on R. If $Q_R \in$ dis $\tau^*$ is an injective cogenerator of dis $\tau^*$ which is essential over its socle, then $Q_R$ is an injective cogenerator of dis $\tau$. Also if $M \in$ dis $\tau$ is essential over its socle, then $M \in$ dis $\tau^*$.

Proof: As $\tau^* \subseteq \tau$ clearly dis $\tau^* \subseteq$ dis $\tau$. Moreover, dis $\tau^*$ and dis $\tau$ have the same simple objects. Thus $Q_R$ is an injective cogenerator of dis $\tau$ if and only if $Q_R$ is dis $\tau$-injective. Let $E(Q_R)$ be the injective hull of $Q_R$ in dis $\tau$. If $x \in E(Q_R)$, then xR is essential over its socle. As xR is linearly compact socle (xR) is finite, and $\text{ann}_R(x)$ is the finite intersection of completely-meet-irreducible right ideal of R open for $\tau^*$. This implies that $E(Q_R)$ is an object of dis $\tau^*$. Since $Q_R$ is dis $\tau^*$-injective, $E(Q_R)$ must equal $Q_R$.

If $M \in$ dis $\tau$ is essential over its socle, then the same argument that shows $E(Q_R) \in$ dis $\tau^*$ shows $M \in$ dis $\tau^*$.

Definition: Let $\tau_1$ and $\tau_2$ be two Hausdorff linear topologies on X. $\tau_1$ and $\tau_2$ are Leptin equivalent if and only if $\tau_1^* = \tau_2^*$.

<u>Corollary 3.3</u>: Let $\tau_1$ and $\tau_2$ be two Leptin equivalent Hausdorff linearly compact topologies on R. These are equivalent:

(1) $Q_R$ is an injective cogenerator of dis $\tau_1$ which is essential over its socle.

(2) $Q_R$ is an injective cogenerator of dis $\tau_2$ which is essential over its socle.

<u>Proof</u>: 1)$\Rightarrow$2) Let $\tau^* = \tau_1^* = \tau_2^*$. Now dis $\tau^* \subseteq$ dis $\tau_1$ and Q is an object of dis $\tau^*$ by Proposition 3.2. Hence $Q_R$ is an injective cogenerator of dis $\tau^*$. Now apply Proposition 3.2 for $\tau = \tau_2$.

Let A and B be two finitely closed subcategories of mod-R. The A-topology equals the B-topology if and only if A and B contain the same finitely generated (or equivalently cyclic) modules. Also the Leptin A-topology equals the Leptin B-topology if and only if A and B have the same finitely generated cofinite modules.

<u>Proposition 3.4</u>: A completely-meet-irreducible submodule of a linearly topologized module is closed if and only if it is open. An arbitrary submodule is closed if and only if it is the intersection of (completely-meet-irreducible) open submodules.

<u>Proof</u>: The closure of any submodule A is the intersection of the submodules A + W for all open submodules W.

Now $\tau_1$ and $\tau_2$ are Leptin equivalent if and only if they have the same submodules closed. If $\tau_1$ and $\tau_2$ are Leptin equivalent and $\tau_1$ is topological linearly compact, then $\tau_2$ is also.

## §4 Results for Rings

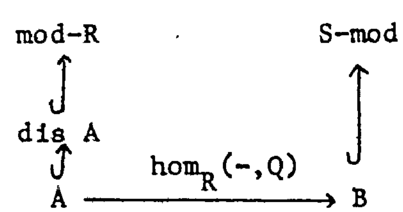
Let A be a finitely closed subcategory of mod-R, then A is a finitely closed generating subcategory of dis A.

Definition: For A, a finitely closed subcategory of mod-R, let $A_F = \{Y \mid$ there exists $X \in A$ finitely generated such that $Y \subseteq X$ (as modules)$\}$.

Theorem 4.1: Let A be a finitely closed $AB5^*$-subcategory of mod-R with no infinite direct sums. If $A_F$ is linearly compact, then A is linearly compact.

Proof: Let $\hat{R}$ be the Hausdorff completion of R in the A-topology. By Chapter II Proposition 3.3, we identify dis A and dis $\hat{R}$ by means of the isomorphism. Thus A is a finitely closed faithful $AB5^*$-subcategory of mod-$\hat{R}$ with no infinite direct sums, and the completion topology on $\hat{R}$ is the A-topology. Moreover, $X \in$ dis A is linearly compact if and only if $X \in$ dis $\hat{R}$ is linearly compact. Consequently, we may assume that R is complete and Hausdorff in the A-topology.

Let $Q_R$ be an injective cogenerator in dis A which is essential over its socle. By Theorem 2.4, we have the following situation:



where B is a finitely closed linearly compact subcategory of S-mod, $\hom_R(-,Q)$ is a duality between A and B and $S = \text{End } Q_R$ is a linearly compact ring for the B-topology. [Note that since Q is the union of submodules in A, the B-topology on S is Hausdorff].

Let $D = \hom_R(A_F, Q)$. As $A_F$ is finitely closed, D is finitely closed. Moreover, D is contained in B. Since $A_F$ is linearly compact, the duality between $A_F$ and D is represented by the bimodule ${}_SQ_R$ ( Proposition 2.5). As $R/V \in A$ implies $R/V \in A_F$, we see dis A = dis $A_F$. Since $Q \in$ dis $A_F$, Q is an injective cogenerator of dis D which is essential over its socle. (See Chapter II Proposition 2.1, Chapter II Theorem 2.4 and Chapter II Proposition 2.9). By Chapter II Corollary 3.1, $\text{End }_SQ = R$.

As $A_F$ contains all finitely generated modules of A, D contains all cofinite modules of B. Thus the B-topology is Leptin equivalent to the D-topology. [As the B-topology on S is Hausdorff linearly compact, so will the D-topology be Hausdorff linearly compact.] From Corollary 3.3, ${}_SQ$ is an injective cogenerator of dis B.

Now Theorem 2.4 applied to B instead of A tells us that $\hom_S(B,Q)$ consists of linearly compact R-modules. [B has no infinite direct sums since B is linearly compact.] Now if $X \in A$, then $i : X \to X^{**} = \hom_S(\hom_R(X,Q),Q)$ is a monomorphism, but $X^{**} \in \hom_S(B,Q)$ is linearly compact, thus X is also.

Corollary 4.2: Let R be a topological linearly compact ring with topology $\tau$. If $A \subseteq \text{dis } \tau$ is a finitely closed AB5$^*$-subcategory of mod-R with no infinite direct sums, then A is linearly compact.

Proof: Let $0 \to Y \to X \to X/Y \to 0$ be an exact sequence, then X is linearly compact if and only if Y and X/Y are linearly compact.

$X \in \text{dis } \tau$ is finitely generated if and only if there exist $V_i \in \tau$, $i = 1,\ldots,n$, such that we have an exact sequence $R/V_1 \oplus \cdots \oplus R/V_n \to X \to 0$. Since each $R/V_i$ is linearly compact ($\tau$ is topological linearly compact), X is linearly compact. Thus $A_F$ is linearly compact. Now apply Theorem 4.1.

Corollary 4.3: Let R be a commutative ring. If A is a finitely closed AB5$^*$-subcategory of mod-R with no infinite direct sums, then A is linearly compact.

Proof: $A_F$ is a meager finitely closed subcategory, thus by Corollary 1.5, $A_F$ is linearly compact. Now apply Theorem 4.1.

Definition: A ring R is right fully bounded if for all prime ideals p, the ring R/p has the property that every essential right ideal contains a nonzero two sided ideal.

Theorem 4.4: (Gabriel [8, Lemma 2, page 423] and Cauchon [4, Corollaire 2, page 1156]) Let R be a right noetherian ring. These are equivalent:

(1) R is right fully bounded.

(2) R has condition (H) : For every right ideal I there exist $b_1, \ldots, b_n \in R$ such that $\mathrm{ann}_R(R/I) = b_1^{-1} I \cap \ldots \cap b_n^{-1} I$.

Proposition 4.5: Let R be a right noetherian and right fully bounded ring. If $\tau$ is a linear topology on R, then $\tau$ has a basis of two sided ideals.

Proof: By Theorem 4.4, R has condition (H). If $I \in \tau$, then $x^{-1} I \in \tau$ for all $x \in R$. Note that $\mathrm{ann}_R(R/I)$ is the largest two sided ideal contained in I. By condition (H), there exists $b_1, \ldots, b_n$ such that $\mathrm{ann}_R(R/I) = b_1^{-1} I \cap \ldots \cap b_n^{-1} I$. As finite intersections of open ideals are open, $\mathrm{ann}_R(R/I)$ is open.

Proposition 4.6: Let R be a right noetherian and right fully bounded ring. If A is a finitely closed $AB5^*$-subcategory of mod-R with no infinite direct sums, then A is linearly compact.

Proof: Now $A_F$ is a meager finitely closed $AB5^*$-subcategory of mod-R. As $R/V \in A$ implies $R/V \in A_F$, the A-topology on R is also the $A_F$-topology on R. By Proposition 4.5, the A-topology has a basis of two sided ideals. From Proposition 1.4, we see that $A_F$ is linearly compact. Now apply Theorem 4.1 to show that A is linearly compact.

Proposition 4.7: Let R be a ring such that every right R-module is essential over its socle. If A is a finitely closed $AB5^*$-subcategory of mod-R with no infinite direct sums, then A is linearly compact.

Proof: If $X \varepsilon A$, then the socle of X is finite (A has no infinite direct sums). Thus X is a cofinite module.

Let Q be an injective cogenerator of dis A which is essential over its socle, $S = \text{End } Q_R$ and $B = \text{hom}_R(A,Q)$. By Theorem 2.4, B is a finitely closed linearly compact subcategory of S-mod and $\text{hom}_R(-,Q) : A \to B$ is a duality. For $X \varepsilon A$, X is complete in the Q-topology since it is cofinite. Thus X is Q-reflexive (Chapter II Proposition 1.2). Hence ${}_SQ_R$ represents the duality between A and B. Thus A is linearly compact . ( Chapter II Theorem 2.6).

## CHAPTER IV

## DUALITY BETWEEN ARBITRARY FINITELY CLOSED SUBCATEGORIES OF MODULE CATEGORIES OVER COMMUTATIVE RINGS

### §1 Introduction

In this chapter all rings will be commutative.

Definitions: For A, a finitely closed subcategory of mod-R

(1) $A_F = \{Y \mid$ there exists $X \in A$ finitely generated such that $Y \subseteq X$ (as modules)$\}$.

(2) $A_\ell = \{X \mid X \in A$ and X is linearly compact$\}$.

$A_F$ and $A_\ell$ are both finitely closed subcategories of mod-R.

Let

$$\begin{array}{ccc} \text{mod-}R & & S\text{-mod} \\ \uparrow & & \uparrow \\ A & \underset{H}{\overset{F}{\rightleftarrows}} & B \end{array}$$

be a duality between the finitely closed faithful subcategories A and B of mod-R and S-mod respectively. Let $\hat{R}$ and $\hat{S}$ be the Hausdorff completions of R and S in the A-topology and B-topology respectively. By Chapter II Proposition 3.3, we identify dis A with dis $\hat{R}$ and dis B with dis $\hat{S}$. Thus A and B are finitely closed subcategories of mod-$\hat{R}$ and $\hat{S}$-mod, and the completion topologies on $\hat{R}$ and $\hat{S}$ are the A-topology and B-topology respectively. Consequently, we may assume that R and S are complete and Hausdorff in the A-topology and the B-topology respectively.

Since $A_F$ is dual to a finitely closed subcategory of S-mod, it is AB5$^*$; thus $A_F \subseteq A_\ell$ (ChapterIII Corollary 1.5). Similarly $B_F \subseteq B_\ell$ . As the linearly compact finitely closed subcategory $A_\ell$ has no infinite direct sums, the finitely closed subcategory $F(A_\ell)$ also has no infinite direct sums (ChapterIII Proposition 2.3). Thus $F(A_\ell)$ is linearly compact (Chapter III Corollary 4.3); consequently, $F(A_\ell) \subseteq B_\ell$. Similarly $H(B_\ell) \subseteq A_\ell$. Therefore, F and H restrict to a duality between $A_\ell$ and $B_\ell$.

By ChapterII Theorem 4.5, there exists a bimodule ${}_SQ_R$ such that the duality between $A_\ell$ and $B_\ell$ is represented by Q, $Q_R$ is an injective cogenerator essential over its socle in dis A and ${}_SQ$ is an injective cogenerator essential over its socle in dis B.

Since $R/V \in A_F \subseteq A_\ell$ for each open ideal V and R is complete in the A-topology, R is a topological linearly compact ring. Similarly S is a topological linearly compact ring. From Chapter II Proposition 3.1, we have $R = \text{End}\ {}_SQ$ and $S = \text{End}\ Q_R$.

<u>Proposition 1.1</u>: Let R and S be commutative rings, and ${}_SQ_R$ a bimodule. If $S = \text{End}\ Q_R$ and $R = \text{End}\ {}_SQ$, then $\rho : R \to S$ defined by $\rho(r)q = qr$ for all $q \in Q$ and $\sigma : S \to R$ defined by $q\sigma(s) = sq$ for

all $q \in Q$ are inverse ring isomorphisms.

Proof: Since R is commutative, $\rho(r) \in \text{End } Q_R = S$ for all $r \in R$. Also as S is commutative, $\rho(s) \in \text{End } {}_SQ = R$ for all $s \in S$. Both $\rho$ and $\sigma$ are ring homomorphisms.

Now $\rho\, \sigma(s)q = q\sigma(s) = sq$ for all $q \in Q$ and $s \in S$. Hence $\rho\, \sigma(s) = s$. Thus $\rho\sigma = 1_S$. Similarly $\sigma\rho = 1_R$.

Each S-module ${}_SX$ may be regarded as a R-module ${}_RX$ where $rx = \rho(r)x$ for all $x \in X$. Every S-homomorphism $f : {}_SX \to {}_SY$ becomes a R-homomorphism $f : {}_RX \to {}_RY$. Thus we have a functor $K$ : S-mod $\to$ R-mod, and similarly a functor $L$ : R-mod $\to$ S-mod. Clearly $KL = 1_{R\text{-mod}}$ and $LK = 1_{S\text{-mod}}$, hence S-mod is isomorphic to R-mod.

Consider the bimodule ${}_SQ_R$ : then ${}_RQ_R$ with $rq = \rho(r)\, q$ for all $q \in Q$ is also a bimodule, ${}_RQ$ is essential over its socle and $rq = qr$ for all $q \in Q$ by the definition of $\rho$. For all $X_R \in$ mod-R, $K(\hom_R(X_R, {}_SQ_R)) = \hom_R(X_R, {}_RQ_R)$.

Since B is a finitely closed faithful subcategory of S-mod, $K(B) = C$ is a finitely closed faithful subcategory of R-mod. As X and K(X) have the same underlying group and the same submodules, X is linearly compact if and only if K(X) is linearly compact. Hence $K(B_\ell) = C_\ell$ and $L(C_\ell) = B_\ell$. Composing the duality with the isomorphism between S-mod and R-mod, we obtain a new duality:

$$\begin{array}{ccc}
\text{mod-}R & & R\text{-mod} \\
\uparrow & & \uparrow \\
A & \underset{H\ L}{\overset{K\ F}{\rightleftarrows}} & C
\end{array}$$

with A and C finitely closed faithful subcategories of mod-R and R-mod. Moreover, the duality restricts to a duality between $A_\ell$ and $C_\ell$ which is represented by ${}_RQ_R$, and R operates the same on both sides of Q.

## §2 Topological Linearly Compact Commutative Rings

Let R be a commutative ring with ideal neighbourhoods of zero, $V_\alpha$, and let M be an open prime ideal of R. Define $W_\alpha = \{x \in R \mid sx \in V_\alpha \text{ for some } s \notin M\}$. If $sx \in V_\alpha$ and $ty \in V_\alpha$ for $s, t \notin M$, then $st(x-y) = t(sx) - s(ty) \in V_\alpha$ while $st \notin M$ because M is prime. Thus $x-y \in W_\alpha$. Also $xr \in W_\alpha$ when $r \in R$ because $sxr \in V_\alpha$. Hence $W_\alpha$ is an (open) ideal in R (for $V_\alpha \subseteq W_\alpha$). Following Zelinsky [29, page 435], we call the topology with ideal neighbourhood base of zero, $W_\alpha$, the M-adic topology on R. We remark that the construction of $W_\alpha$ depends only on $V_\alpha$ and M.

If R is a Dedekind domain (unique factorization of ideals) with ideal neighbourhoods of zero, and if M is a prime all of whose powers are open, then the M-adic topology on R is the ordinary one, because $W_\alpha$ is just the highest power of M dividing $V_\alpha$.

Theorem 2.1: (Zelinsky [30, Theorem 2, page 87] and [29, Theorem 6, page 438]) Let R be a commutative linearly compact ring with unit. Then R is algebraically and topologically isomorphic to a direct product of local rings. The local rings are exactly the M-adic completions of R, M ranging over all the open maximal ideals of R.

Corollary 2.2: (Zelinsky) If R is a commutative topological linearly compact ring and if M is an open maximal ideal of R, then the M-adic completion of R is a local ring.

Proposition 2.3: Let R be a commutative linearly compact ring with topology $\tau$, M an open maximal ideal and $\tau^*$ the Leptin topology of $\tau$. Then the underlying abstract rings of the M-adic completion with respect to $\tau$ and M-adic completion with respect to $\tau^*$ are equal (or canonical isomorphic).

Proof: For each $V \in \tau$, let $B_V$ be the corresponding open ideal of the M-adic topology with respect to $\tau$. Define $N_\tau = \bigcap_{V \in \tau} B_V$. Note that $N_\tau \subseteq M$, as $B_M = M$ because M is prime ($sx \in M$ and $s \notin M$ implies $x \in M$). Since $V \subseteq B_V$, the M-adic topology with respect to $\tau$ is coarser than $\tau$. As R is linearly compact, $R/N_\tau$ is the M-adic completion with respect to $\tau$. (R is complete for all topologies coarser than $\tau$). By Corollary 2.2 $R/N_\tau$ is a local ring. Clearly $\bar{M} = M/N_\tau$ is the unique maximal ideal.

Let $N_\tau \subseteq V \in \tau$. Assume $sx \in V$ for some $s \notin M$, then $\bar{s}\bar{x} = \overline{sx} \in \bar{V} = V/N_\tau$. Now $s \notin M$ implies $\bar{s} \notin \bar{M}$. Hence $\bar{s}$ is a unit in $R/N_\tau$. ($R/N_\tau$ is a local ring with maximal ideal $\bar{M}$). Consequently, $\bar{x} \in \bar{V}$ which implies $x \in V$. Thus $B_V = V$.

For each $I \in \tau^*$, let $B_I^*$ be the corresponding open ideal of the M-adic topology with respect to $\tau^*$. Define $N_\tau^* = \bigcap_{I \in \tau^*} B_I^*$. Since $\tau^* \subseteq \tau$, I is open for $\tau$, and $B_I^* = B_I$. Thus $N_\tau \subseteq N_\tau^*$ (intersecting over smaller set). Now $N_\tau$ is closed in $\tau$ (for $B_V \in \tau$ as $V \subseteq B_V$ and $N_\tau = \bigcap_{V \in \tau} B_V$) hence $N_\tau = \cap V_\alpha$ where $V_\alpha \in \tau$ is completely-meet-irreducible. But $B^*_{V_\alpha} = B_{V_\alpha} = V_\alpha$ since $N_\tau \subseteq V_\alpha$.

Hence $N_\tau^* = \bigcap_{I \in \tau^*} B_I \subseteq \cap B_{V_\alpha} = \cap V_\alpha = N_\tau$ as $V_\alpha = B_{V_\alpha} \in \tau^*$ since $V_\alpha$ is completely-meet-irreducible. Thus $N_\tau^* = N_\tau$.

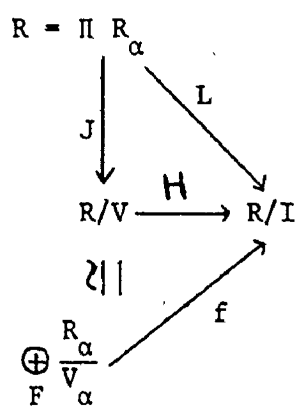
Again as R is linearly compact, $R/N_\tau = R/N_\tau^*$ is the M-adic completion with respect to $\tau^*$.

If $\tau_1$ and $\tau_2$ are two Leptin equivalent ring topologies on R, they have the same maximal ideals open.

<u>Corollary 2.4</u>: Let $\tau_1$ and $\tau_2$ be two linearly compact topologies on the commutative ring R which are Leptin equivalent. Then, for any open maximal ideal M, the underlying abstract rings of the M-adic completion with respect to $\tau_1$ and the M-adic completion with respect to $\tau_2$ are equal (or canonical isomorphic).

<u>Proposition 2.5</u>: Let $R_\alpha$ be a linear topological commutative ring and $R = \Pi R_\alpha$. If I is an open ideal of R, then $I = \Pi I_\alpha$ where $I_\alpha$ is an open ideal of $R_\alpha$ and $I_\alpha = R_\alpha$ for all but finite many $\alpha$.

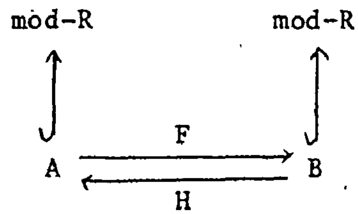
<u>Proof</u>: Let I be an open ideal of R. Then I contains an open ideal $V = \Pi V_\alpha$ where $V_\alpha$ is an open ideal of $R_\alpha$ and $V_\alpha = R_\alpha$ for all but finitely many $\alpha$. Let $F = \{\alpha \mid V_\alpha \neq R_\alpha\}$, then $R/V \cong \bigoplus_F R_\alpha/V_\alpha$, a finite direct sum of rings. Consider the following commutative diagram:




where J, H, and L are the natural epimorphisms and f is the composition of H with the isomorphism. Clearly f is onto. Let $K = \text{Ker } f$, then $K = \bigoplus_F K_\alpha$ where $K_\alpha$ is an ideal of $R_\alpha/V_\alpha$. (Let $k = (k_\alpha) \in K$, then $(0,\ldots,k_\alpha,\ldots,0) = k\, C_\alpha \in K$ where $C_\alpha = (0 \ldots 1 \ldots 0)$ is the identity of $R/V_\alpha$. It follows immediately that $K = \bigoplus_F K_\alpha$ where $K_\alpha = \{(0 \ldots k_\alpha \ldots 0) \mid$ there exists $k \in K$ with $\alpha$-component $k_\alpha\}$.) Diagram chasing now shows that $I = \Pi\, I_\alpha$ where $I_\alpha = R_\alpha$ if $\alpha \notin F$ and $I_\alpha = p_\alpha^{-1}(K_\alpha)$ where $p_\alpha : R_\alpha \to R_\alpha/V_\alpha$ is the canonical epimorphism if $\alpha \in F$.

## §3 Decomposition of Dualities

Let R be a commutative ring with unit. For the rest of this chapter we will assume a duality:




where A and B are finitely closed faithful subcategories of mod-R, and R is linearly compact in both the A-topology and the B-topology. The restriction of the duality between $A_\ell$ and $B_\ell$ is represented by $Q_R$, where $Q_R$ is an injective cogenerator in dis A and $Q_R$ is an injective cogenerator in dis A. $Q_R$ is essential over its socle. We note that both F and H are subfunctors of $\hom_R(-, Q_R)$. (See Chapter II Proposition 4.2).

Proposition 3.1: The A-topology on R and the B-topology on R are Leptin equivalent.

Proof: Let V be an open, completely-meet-irreducible ideal for the A-topology. Thus $R/V \in A_F \subseteq A$ is a finitely generated cofinite module. Note that $F \mid A_F = \hom_R(-,Q)$. Hence $\hom_R(R/V, Q) \cong \text{ann}_Q V$ is a finitely generated cofinite submodule of Q. As $Q \in$ dis A, we have $\text{ann}_Q V \in A_F$ since $A_F$ contains all finitely generated modules of dis A. Consequently, $\hom_R(\mathrm{ann}_Q V, Q) = \hom_R(\hom_R(R/V, Q), Q) \cong R/V$ is an element of B. Thus V is open for the B-topology.

Since R is linearly compact, we know by Theorem 2.1 that $R = \prod_{i \in I} R_i$ a product of local rings.

Proposition 3.2: If R has either the A-topology or the B-topology, then the same underlying abstract local rings arise in the decomposition of $R = \prod_{i \in I} R_i$ given by Theorem 2.1.

Proof: Since the A-topology and the B-topology are Leptin equivalent, they have the same maximal ideals open. Now apply Corollary 2.4.

We wish to decompose the duality $F : A \longrightarrow B$ into dualities between finitely closed subcategories of the mod-$R_i$.

Definitions: (1) For $i \in I$, the ideal $N_i = \{(x_i) \in \prod_I R_i \mid x_i = 0\}$.

(2) For $X \in$ mod-R, $X_i = \{x \in X \mid xN_i = 0\}$.

(3) For a class C of modules, $C_i = \{X \in C \mid X N_i = 0\}$.

(4) For categories C and $\{C_i\} i \in I$, $C = \prod_I C_i$ if each object of C is a product of objects of the $C_i$ and each morphism of C is a product of morphisms of the $C_i$.

Proposition 3.3: For any R-module X in dis A or dis B, $X = \bigoplus_I X_i$

Proof: Now $X \cap \sum_{j \neq i} X_j = 0$. For if $x \in X_i \cap \sum_{j \neq i} X_j$, then $N_i + K \subseteq \mathrm{ann}_R(x)$ where $K = \bigcap_{\text{finite}} N_j$ and $N_j \neq N_i$. Thus $\mathrm{ann}_R(x) = R$, hence $x = 0$. Also $\sum_I X_j = X$. For if $x \in X$ then $V = \mathrm{ann}_R(x)$ is an open ideal ($X \in$ dis A or $X \in$ dis B). Thus $V = \prod_{i \in I} V_i$ where $V_i$ is an ideal of $R_i$ and $V_i = R_i$ for all but finitely many $i \in I$. (Proposition 2.5). Consequently, $xR \cong R/V \cong \bigoplus_{\text{finite}} R_i/V_i$ and we see that $x \in \Sigma X_i$.

For $X,Y \in$ dis A, we have $\hom_R(X,Y) = \prod_{i \in I} \hom_R(X_i,Y_i)$ since $f(X_i)\, N_i = f(X_i\, N_i) = f(0) = 0$ for all $f \in \hom_R(X,Y)$. In particular, if W is an R-module such that $W N_i = 0$, then $\hom_R(W,Q) = \hom_R(W,Q_i)$. Note that $\hom_R(W,Q_i)\, N_i = 0$.

Let $A_i = \{X \in A \mid X N_i = 0\}$ and $B_i = \{Y \in B \mid Y N_i = 0\}$. Clearly both $A_i$ and $B_i$ are finitely closed subcategories of mod-R. Also we may consider $A = \prod_{i \in I} A_i$ and $B = \prod_{i \in I} B_i$. Since F and H are subfunctors of $\hom_R(-,Q)$, we see that the duality between A and B restricts to a duality between $A_i$ and $B_i$. Moreover, as $N_i$ annihilates both $A_i$ and $B_i$, they may be considered as finitely closed subcategories of mod-$R_i$ (Note that $R/N_i = R_i$).

<u>Proposition 3.4</u>: $A_i$ and $B_i$ are linearly compact subcategories of mod-R.

<u>Proof</u>: Now $A_i$ is a finitely closed AB5$^*$-subcategory of mod-R, as its dual $B_i$ is a finitely closed subcategory of mod-R. By Chapter III Corollary 4.3, it is enough to show that $A_i$ has no infinite direct sums.

Assume $A_i$ contains an infinite direct sum of nonzero modules. Then $A_i$ contains an infinite direct sum of nonzero cyclic modules because $A_i$ is closed under submodules. Consequently, as $A_i$ is closed under factor modules it contains an infinite direct sum of simple modules. But as $A_i$ is a finitely closed subcategory of mod-$R_i$ and $R_i$ is a local ring, all simple modules are isomorphic. Thus for an infinite set J, $\bigoplus_J M \in A_i$.

As $A_i \subseteq A$, we have $\bigoplus_J M \in A$. Because F is a duality between A and B, $F(\bigoplus_J M) = B - \prod_J FM$ where $B - \prod$ is the product in the category B. But $B - \prod_J FM = \text{dis } B - \prod_J FM$ (Chapter III Proposition 1.3) where dis $B - \prod$ is the product in the category dis B. Now FM is a simple module of B, thus $V = \text{ann}_S FM$ is an open left ideal of S for the B-topology. The product in mod-R, $\prod_J FM$, of J copies of FM is an object of dis B for if $x = (x_j) \in \prod FM$ then $Vx = 0$. Thus $F(\bigoplus_J M) = \prod_J FM \in B$. But $\prod_J FM$ does not satisfy the $AB5^*$ property. This is a contradiction since B is an $AB5^*$-category (dual to a finitely closed subcategory of mod-R). Consequently, $A_i$ has no infinite direct sums. Thus by Chapter III Corollary 4.3, $A_i$ consists of linearly compact modules.

Now $B_i$ is also a finitely closed $AB5^*$-subcategory of mod-R. Since $A_i$ its dual has no infinite direct sums, $B_i$ has no infinite direct sums (Chapter III Proposition 2.3). Thus by Chapter III Corollary 4.3, $B_i$ consists of linearly compact modules.

We recall that $Q = \bigoplus_{i \in I} Q_i$ where $Q_i = \{q \in Q \mid q N_i = 0\}$. Note that $X \in \text{dis } A_i$ if and only if $X \in \text{dis } A$ and $X N_i = 0$.

(for V is open in the $A_i$-topology if and only if $N_i \subseteq V$ and V is open in the A-topology). Clearly $Q_i$ is an injective cogenerator in dis $A_i$ and $Q_i$ is an injective cogenerator in dis $B_i$.

Corollary 3.5: The module $Q_i$ represent the duality between $A_i$ and $B_i$.

Proof: As $A_i$ and $B_i$ are linearly compact, we know that the duality between $A_i$ and $B_i$ is represented by Q. Moreover, $\hom_R(X,Q) = \hom_R(X,Q_i)$ for all $X \in A_i$, and $\hom_R(Y,Q) = \hom_R(Y, Q_i)$ for all $Y \in B_i$.

Remark: As dis $A_i$ and dis $B_i$ are both annihilated by $N_i$, these dualities may be considered to be in mod-$R_i$.

Theorem 3.6: If $F : A \to B$ is a duality, then $F(X) = \bigoplus_I \hom(X_i,Q_i)$ for all $X \in A$.

Proof: For $X \in A$, $X = \bigoplus_I X_i$ with $X_i \in A_i$ by Proposition 3.3. F is a subfunctor of $\hom_R(-,Q)$. Therefore, $F(X) \subseteq \hom_R(X,Q) = \hom_R(\bigoplus_I X_i,Q) = \prod_I \hom_R(X_i,Q_i)$.

Let $p_k : \bigoplus_I X_i \to X_k$ be the projection map, then $F(p_k) : F(X_k) \to F(\bigoplus_I X_k) = F(X)$ is an inclusion. Since $X_k \in A_k$ $F(X_k) = \hom_R(X_k,Q_k)$ (Corollary 3.5). Thus $\hom_R(X_k,Q_k) \subseteq F(X)$. Consequently, $\bigoplus_I \hom_R(X_i,Q_i) \subseteq F(X)$.

Now $F(X) \in B$. Thus by Proposition 3.3, $F(X) = \bigoplus_I [F(X)]_i$. Since $F(X) \subseteq \prod_I \hom_R(X_i,Q_i)$, clearly $[F(X)]_i \subseteq [\prod_I \hom_R(X_i,Q_i)]_i$.

But $f \in [\prod_I \hom_R(X_i,Q_i)]_i$ implies $f(X) \subseteq Q_i$ (as $N_i f = 0$) and thus $f \in \hom_R(X_i,Q_i)$. Therefore $[F(X)]_i \subseteq \hom_R(X_i,Q_i)$ and $F(X) \subseteq \bigoplus_I \hom_R(X_i,Q_i)$.

Thus we have shown $F(X) = \bigoplus_I \hom_R(X_i,Q_i)$.

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