CALCULATION OF TOLERANCES BASED ON MINIMUM COST

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by

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Scope and Contents: A study of gradient optimization techniques, in particular as applied to the cost-tolerances problem, is made. Three efficient techniques are used to obtain the tolerances based on minimum cost. The optimization techniques are the Fletcher-Powell method, a more recent method proposed by Fletcher and a method based on a more general objective function proposed by Jacobson and Oksman.

Firstly, the constrained cost-tolerance problem, under special constraints, is transformed to an unconstrained optimization problem using two methods.
Secondly, the three gradient methods are applied to get the optimum set of tolerances.

ABSTRACT

A possible mathematical formulation of the practical problem of computer - aided design for any system or any engineering design subject to tolerances on the n independent parameters is used to solve some special cases under certain restrictions. The sequential unconstrained minimization technique of Fiacco - McCormick is used to get the optimum solution. The scheme used is : starting from arbitrary initial acceptable or unacceptable designs and culminating in designs which under reasonable restrictions are acceptable.

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CHAPTER 1

INTRODUCTION

Today many important decisions are made by choosing a quantitative measure of effectiveness and then optimizing it. Deciding how to design, build, regulate, or operate a physical or economic system ideally involves three steps. First, one should know how the system variables interact. Second, one needs a single measure of system effectiveness expressible in terms of the system variables. Finally, one should choose those values of the system variables yielding optimum effectiveness. Thus optimization and choice are closely related. Optimization is decisive because it narrows down the possible choices to one - the best one.

In the design of any machine or mechanism it is necessary to assign tolerances to all components. This combination of these tolerances must be sufficiently tight to guarantee that the device will perform as intended. Another consideration that is equally as important for a mass produced item is that the manufacturing cost should be as low as is consistent with the quality and proper operation of the device.

The design engineer is usually anxious to assign tight tolerances to all components in order to guarantee proper operation and a quality product.

The manufacturing engineer is more interested in increasing the production and lowering the cost and hence encourages a loosening of tolerances.

The problem investigated in this work is the problem of optimal design cost subject to tolerances in some special cases. Recently published work (References 1 to 4) has yielded some practical insight into the nature of the problem.

Many types of objective functions (more appropriately, cost functions) can be formulated. In this work two possible objective functions have been investigated.

It was assumed that the parameter tolerances can be independently specified, and the design parameters and tolerances can be continuously varied. The tolerance regions are defined by a set of linear constraints imposed by the device specifications.

The present work is based on the above approach with the use of efficient minimization techniques. Firstly the constrained optimization problem is transformed to an unconstrained one using two methods described in chapter 3.

Then three gradient methods, because they utilize gradient information, are applied to solve the unconstrained optimization problem. These techniques show a rapid rate of convergence. A comparative study is made between the three most efficient gradient techniques which are described in chapter 2.

The results of applying these three techniques to some special cases of the cost-tolerances problem are shown in chapter 4.

CHAPTER 2

UNCONSTRAINED OPTIMIZATION METHODS

The existence of optimization problems is as old as mathematics. The first systematic techniques for the solution of these problems stem from the development of the calculus and are associated with the names Newton, Lagrange and Cauchy (who made the first application of the method of the steepest descent). However, little substantive progress was made until the middle of this century, when development was greatly accelerated by the availability of computers and by an increasing requirement for the solution of decision problems.

2.1 Concepts and definitions

Assuming that the unconstrained optimization problem is defined as

Minimize the function U where

$$U \triangleq U(\underline{\phi}) \tag{1}$$

where

$$\underline{\phi} \triangleq \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{bmatrix}$$

U is called the *objective function*

$\underline{\phi}$ represents a set of *independent parameters*

The problem, where the values of the components of $\underline{\phi}$ are not restricted by side conditions or constraints is called an *unconstrained* optimization problem.

The classic methods of calculus give necessary conditions

$$\frac{\partial U}{\partial \phi_1} = \frac{\partial U}{\partial \phi_2} = \ldots\ldots = \frac{\partial U}{\partial \phi_n} = 0 \tag{2}$$

which must be satisfied by a stationary **point.**

In this sense the problem of finding a stationary point of a function is equivalent with that of solving a system of nonlinear equations.

Let

$$\underline{g}\ (\underline{\phi}) \triangleq \begin{bmatrix} \frac{\partial U}{\partial \phi_1} \\ \frac{\partial U}{\partial \phi_2} \\ \vdots \\ \frac{\partial U}{\partial \phi_n} \end{bmatrix} \tag{3}$$

called the *gradient vector* and

$$\underline{G} \triangleq \begin{bmatrix} \frac{\partial^2 U}{\partial \phi_1^2} & \frac{\partial^2 U}{\partial \phi_1 \partial \phi_2} & \cdots & \frac{\partial^2 U}{\partial \phi_1 \partial \phi_n} \\ \frac{\partial^2 U}{\partial \phi_2 \partial \phi_1} & & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial^2 U}{\partial \phi_n \partial \phi_1} & \cdots & \cdots & \frac{\partial^2 U}{\partial \phi_n^2} \end{bmatrix} \tag{4}$$

called the *Hessian matrix*.

The matrix $\underline{H}$ denotes the inverse Hessian, $\underline{G}^{-1}$, which will be approximated by different formulae in each of the gradient techniques as shown in chapter 3.

In recent years unconstrained problems have been attacked successfully by a number of direct search and descent methods.

2.2 Direct search

Methods which do not rely explicitly on evaluation or estimation of partial derivatives of the objective function at any point are usually called *direct search* methods. They rely on the sequential examination of trial solutions in which each solution is compared with the best obtained up to that time, with a strategy usually based on past experience for deciding where the next trial solution should be located.

Methods of this type are useful in the early stages of optimization and can provide efficient information about a region in which a minimum is located. In general they do not give a rapid rate of ultimate convergence and hence are inefficient for finding a minimum with high precision.

For more than one independent variable, a method was developed by Rosenbrock [5] . This method uses a set of n mutually orthogonal directions in each cycle of searches. This set of directions is then rotated so that it adopts itself to the directions of most rapid decrease of the objective function.

The method of Hooke and Jeeves [6] can be regarded as a further development of the Rosenbrock technique . This method changes the parameters one at a time starting from an initial point, but once the full series of perturbations has been completed it takes a step along the direction joining the last and the initial point. This method is called the Pattern Search method.

2.3 Methods using conjugate directions

There are efficient techniques which utilize additional information about the function to be minimized. In these methods the solution of a general optimization problem is found by solving a sequence of one-dimensional problems. These methods are called *descent methods*. The most important descent methods for solving general minimization problems are the *conjugate direction* methods. These assume that in a neighbourhood of a minimum the function can be closely approximated by a positive definite quadratic form.

There is the possibility of using $-\underline{\nabla}U$ as the descent vector at each stage. This resulting method is called the *steepest descent* method.

2.4 Methods used in this work

2.4.1 Definitions

Let

$$\underline{\delta} \triangleq \begin{bmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \vdots \\ \Delta\phi_n \end{bmatrix} \tag{5}$$

called the *increment vector*

In all minimization methods $\underline{\delta}$ is chosen so that

$$U(\underline{\phi}+\underline{\delta}) < U(\underline{\phi}) \tag{6}$$

Another n-dimensional vector $\underline{s}$ will denote the direction in which $\underline{\delta}$ is taken.

Algorithms terminate after one or more of the following criteria are satisfied

(a) if the change in the objective function becomes less than a small positive number.

(b) if the absolute values of the elements of the increment vector become smaller than another small positive number

(c) if the norm of the gradient vector becomes less than another small positive number.

As a safeguard the algorithms should go through n iterations, where n is the number of variables, after terminating criterion is satisfied, before the program terminates. A comparison between the results from each of the next three gradient method are given in tables 3, 6, 9.

2.4.2 The Fletcher - Powell method

The main feature of the Fletcher - Powell method [7] is that the increment $\underline{\delta}$ is taken along the direction $\underline{s}$ where

$$\underline{s} = - \underline{H}\, \underline{g} \tag{7}$$

That is

$$\underline{\delta} = \alpha\, \underline{s} \tag{8}$$

where α is chosen to minimize $U(\underline{\phi}+\alpha\underline{s})$ along the direction of $\underline{s}$. The inverse Hessian $\underline{H}$ may initially be chosen to be any positive definite symmetric matrix. Then $\underline{H}$ is modified at each iteration from information presently available using the formula

$$\underline{H}_{i+1} = \underline{H}_i + \frac{\underline{\delta}\, \underline{\delta}^T}{\underline{\delta}^T\, \underline{\gamma}} - \frac{\underline{H}_i\, \underline{\gamma}\, \underline{\gamma}^T\, \underline{H}_i}{\underline{\gamma}^T\, \underline{H}_i\, \underline{\gamma}} \tag{9}$$

where

$$\underline{\delta} = - \alpha\, \underline{H}_i\, \underline{g}_i \tag{10}$$

and

$$\underline{\gamma} = \underline{g}_{i+1} - \underline{g}_i \tag{11}$$

It can be shown [7] that this method is stable, that is, formula (9) has the following property: if $\underline{H}_i$ is positive definite then $\underline{H}_{i+1}$ is also positive definite. Since we have chosen $\underline{H}_i$ from the beginning to be the identity matrix then all $\underline{H}_{i+1}$ will be positive definite.

It was shown also that if the objective function is quadratic the procedure will terminate after n iterations. This property of quadratic convergence depends on accurate location of the minimum along each direction of search, and this is the main disadvantage of the method.

2.4.3 The Fletcher method

The Fletcher method [8] is basically similar to the Fletcher - Powell method [7] in that both methods consider quadratic objective functions and the increment $\underline{\delta}$ depends on the gradient and the updating matrix $\underline{H}$.

In this method the property of quadratic convergence is replaced by another property which requires, for quadratic functions, that the eigenvalues of $\underline{H}$ must tend mononically to those of $\underline{G}^{-1}$.

The change ΔU in U on an iteration would be expected by Taylor's series to be approximately $\underline{g}^T\underline{\delta}$ for small $\underline{\delta}$, but much less than $\underline{g}^T\underline{\delta}$ in absolute value when the position of the minimum along a line is overestimated.

The change in $U(\underline{\phi})$ relative to $\underline{g}^T\underline{\delta}$ cannot become arbitrary small if

$$\frac{\Delta U}{\underline{g}^T\underline{\delta}} \gtrsim \mu \tag{12}$$

where $0<\mu<<1$, μ is assigned the value 0.0001.

In fact if the corrections are determined by

$$\underline{\delta} = -\lambda\,\underline{H}\,\underline{g} \tag{13}$$

then trying values of $\lambda = 1,\omega,\omega^2,\omega^3,\ldots$ $(0<\omega<1)$ for $\omega = 0.1$ will eventually produce a $\underline{\delta}$ which will satisfies equation (12). It is necessary now to find formula to possess the properties of positive definiteness and eigenvalue convergence. The new formula developed by Fletcher [8] is

$$\underline{H}_{i+1} = \underline{H}_i - \frac{\underline{\delta}\,\underline{\gamma}^T\,\underline{H}_i}{\underline{\delta}^T\,\underline{\gamma}} - \frac{\underline{H}_i\,\underline{\gamma}\,\underline{\delta}^T}{\underline{\delta}^T\,\underline{\gamma}} + \left(1 + \frac{\underline{\gamma}^T\,\underline{H}_i\,\underline{\gamma}}{\underline{\delta}^T\,\underline{\gamma}}\right)\frac{\underline{\delta}\;\underline{\delta}^T}{\underline{\delta}^T\,\underline{\gamma}} \tag{14}$$

where $\underline{\delta}$ and $\underline{\gamma}$ are defined in equations (10), (11). The use of formula (14) by itself might cause $\underline{H}$ to become unbounded. For this reason a condition is tested to choose between the two updating formulae (equations (9) and (14)).

This test condition is

if

$$\underline{\delta}^T\underline{\gamma} \gtrsim \underline{\gamma}^T\,\underline{H}_i\,\underline{\gamma} \tag{15}$$

then the formula (14) is used: otherwise formula (9) is used.

2.4.4 The Jacobson - Oksman method

The Jacobson - Oksman method [9] differs from the previous two in that it is not based on quadratic functions, but on homogeneous functions.

Consider the homogeneous function

$$U(\underline{\phi}) = \frac{1}{\theta} (\underline{\phi} - \check{\underline{\phi}})^T \underline{g}(\underline{\phi}) + U(\check{\underline{\phi}}) \tag{16}$$

where θ is the degree of homogeneity, $\check{\underline{\phi}}$ is the location of the minimum and $U(\check{\underline{\phi}})$ is the minimum value.

The quadratic function considered earlier is

$$U(\underline{\phi}) = \frac{1}{2} (\underline{\phi} - \check{\underline{\phi}})^T \underline{Q}(\underline{\phi} - \check{\underline{\phi}}) + U(\check{\underline{\phi}}) \tag{17}$$

where $\underline{Q}$ is an n×n constant positive definite matrix (the second derivative matrix i.e. $Q_{ij} = \frac{\partial^2 U(\phi)}{\partial\phi_i \partial\phi_j}$).

By comparing equations (16) and (17) it can be seen that (17) is a special case of (16) with $\theta = 2$.

By rearranging equation (16) we have

$$\check{\underline{\phi}}^T g(\underline{\phi}) + \theta\ U(\underline{\phi}) - \theta\ U(\check{\underline{\phi}}) = \underline{\phi}^T \underline{g}(\underline{\phi}) \tag{18}$$

Let

$$\underline{v} \triangleq \underline{\phi}^T \underline{g}(\underline{\phi})$$

$$\underline{y} \triangleq [\underline{g}^T(\underline{\phi}) \qquad U(\underline{\phi}) \qquad -1]^T \tag{19}$$

$$\underline{\alpha} \triangleq [\underline{\check{\phi}}^T \qquad \theta \qquad x]^T$$

where

$$x = \theta\, U(\underline{\check{\phi}})$$

and $\underline{\alpha}$ and $\underline{y}$ are (n+2)-vecotrs with α containing the unknowns $(\underline{\check{\phi}}, \theta, U(\underline{\check{\phi}}))$.

For a point $\underline{\phi}_i$ equation (18) now becomes

$$\underline{y}_i^T\, \underline{\alpha} = v_i \tag{20}$$

If $\underline{v}$ and $\underline{y}$ are evaluated at (n+2) distinct points $\phi_1, \phi_2, \ldots, \phi_{n+2}$, so that the resultant $\underline{y}_i$'s are linearly independent, we can write

$$\begin{bmatrix} \underline{y}_1^T \\ \underline{y}_2^T \\ \vdots \\ \underline{y}_{n+2}^T \end{bmatrix} \begin{bmatrix} \\ \alpha \\ \\ \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n+2} \end{bmatrix} \tag{21}$$

or in matrix form

$$\underline{Y}\, \underline{\alpha} = \underline{v} \tag{22}$$

Since the $\underline{y}_i$'s are linearly independent, $\underline{Y}$ is invertible and we can solve for the unknown vector $\underline{\alpha}$.

$$\underline{\alpha} = \underline{Y}^{-1}\underline{v} \tag{23}$$

Matrix inversion is avoided by using a recursive formula as new $\underline{y}_i$ and $\underline{v}_i$ are evaluated. Starting with $\underline{P}_o = \underline{I}$, an $(n+2)\times(n+2)$ identity matrix and $\underline{v}_o = \underline{\alpha}_o$, an arbitrary initial guess, successive estimates of the vector $\underline{\alpha}$ are given by

$$\underline{\alpha}_{i+1} = \underline{\alpha}_i + \frac{\underline{P}_i \; \underline{e}_{i+1} \; (v_{i+1} - \underline{y}^T_{i+1} \; \underline{\alpha}_i)}{\underline{y}^T_{i+1} \; \underline{P}_i \; \underline{e}_{i+1}} \tag{24}$$

where $\underline{e}_{i+1}$ is a unit $(n+2)$-vector having unity as the $(i+1)^{th}$ element and zero elsewhere, and where $\underline{P}_i$ are obtained successively from the formula

$$\underline{P}_{i+1} = \underline{P}_i - \frac{\underline{P}_i \; \underline{e}_{i+1} \; (\underline{y}^T_{i+1} \; \underline{P}_i - \underline{e}^T_{i+1})}{\underline{y}^T_{i+1} \; \underline{P}_i \; \underline{e}_{i+1}} \tag{25}$$

It can be shown [9] that, for homogeneous functions, the algorithm finds the minimum $\check{\underline{\phi}}$, the degree of homogeneity θ, and the value of the minimum $U(\check{\underline{\phi}})$ after $(n+2)$ iterations.

CHAPTER 3

METHODS FOR HANDLING CONSTRAINTS

An important class of applications for the techniques discussed in the previous chapter is to the solution of constrained optimization problems that have been suitably transformed.

There are two main types of transformation that can be used to transform a constrained optimization problem to an unconstrained one. First, by suitably transforming the independent variables it may be possible to introduce new variables which are unconstrained. Second, it is possible to transform the objective function by adding severe penalties to it whenever a constraint is violated in such a way that the unconstrained optimization techniques are forced to find minima in the feasible region. In this case the solution is found as the limit of a sequence of solutions to suitably transformed problems.

3.1 Transformation of the independent variables

Simple transformations, whose application is limited to certain forms of inequality constraints, have been summarized by Box [10]. These transform the independent variables and leaves the objective function unaltered. Assuming some or all independent variables are subject to constant lower and upper constraints such as,

$$\ell_i \leq \phi_i \leq u_i \tag{26}$$

If, for example, ϕ_i has to be positive then the following transformation can be used

$$\phi_i = \mathrm{abs}(\bar{\phi}_i) \ , \qquad \phi_i = \bar{\phi}_i^2 \qquad , \ \phi_i = e^{\bar{\phi}_i} \tag{27}$$

In a general case of constant lower and upper constraints as in equation (26), we can apply the transformation

$$\phi_i = \ell_i + (u_i - \ell_i) \sin^2 \bar{\phi}_i \tag{28}$$

After applying any of these transformations the unconstrained optimum of an objective function with respect to $\bar{\phi}_i$ variables is sought.

3.2 Transformation of the objective function

There are two methods used in this work to transform the constrained objective function to an unconstrained one.

Comparisons between the results from these two techniques are given in tables 4, 7, 10.

3.2.1 The Fiacco - McCormick Technique

This approach was first suggested by Caroll [11] and further developed by Fiacco - McCormick [12].

It was suggested that the solution to the constrained minimization problem

$$\text{minimize} \quad U(\underline{\phi}) \tag{29}$$

subject to the constraints

$$g_i(\underline{\phi}) \geq 0 \quad i = 1, 2, \ldots, m \tag{30}$$

might be found as the limit as $r_k \to 0$ to the unconstrained minimization problem.

$$\text{Minimize } P(\underline{\phi}, r_k) = U(\underline{\phi}) + r_k\, G(\underline{g}(\underline{\phi})) \tag{31}$$

where $r_k > 0$ and $G(\underline{g}(\underline{\phi}))$ has the following properties

(1) $G(\underline{g}(\underline{\phi}))$ is continuous for $\underline{g}\ (\underline{\phi}) > \underline{0}$

(2) $G(\underline{g}(\underline{\phi})) \to \infty$ for any $g_i(\underline{\phi}) \to 0$

One possible form of $G(\underline{g})$ is

$$G(\underline{g}) = \sum_{i=1}^{m} \frac{1}{g_i(\underline{\phi})} \tag{32}$$

This form was the one chosen to be used in this work. However, other alternatives are possible, e.g.,

$$G(\underline{g}) = -\sum_{i=1}^{m} \log\ (g_i(\underline{\phi})) \tag{33}$$

The definition of a feasible point is that point which belongs to a feasible region i.e.

$$R = \{\underline{\phi} \mid g_i(\underline{\phi}) \geq 0,\ i = 1,2,\ldots,m\} \tag{34}$$

An interior-feasible region is defined as

$$R^\circ = \{\underline{\phi} \mid g_i(\underline{\phi}) > 0,\ i = 1,2,\ldots,m\} \tag{35}$$

Let us assume that we know at least one point $\underline{\phi}^\circ \in R^\circ$ and that we can use one of the efficient unconstrained optimization techniques.

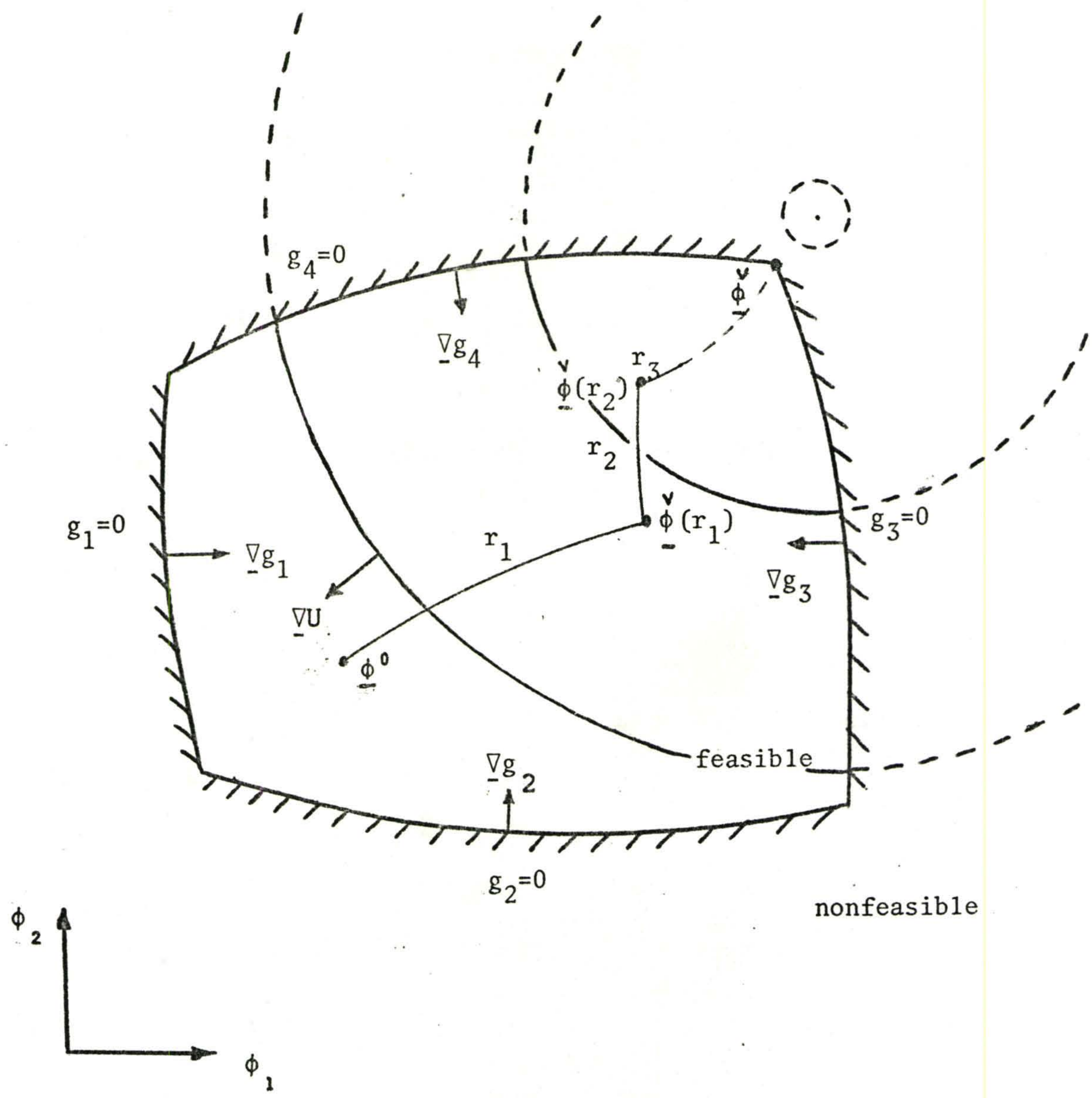

Figure (1)

Sequence of solutions of a 2-dimensional problem

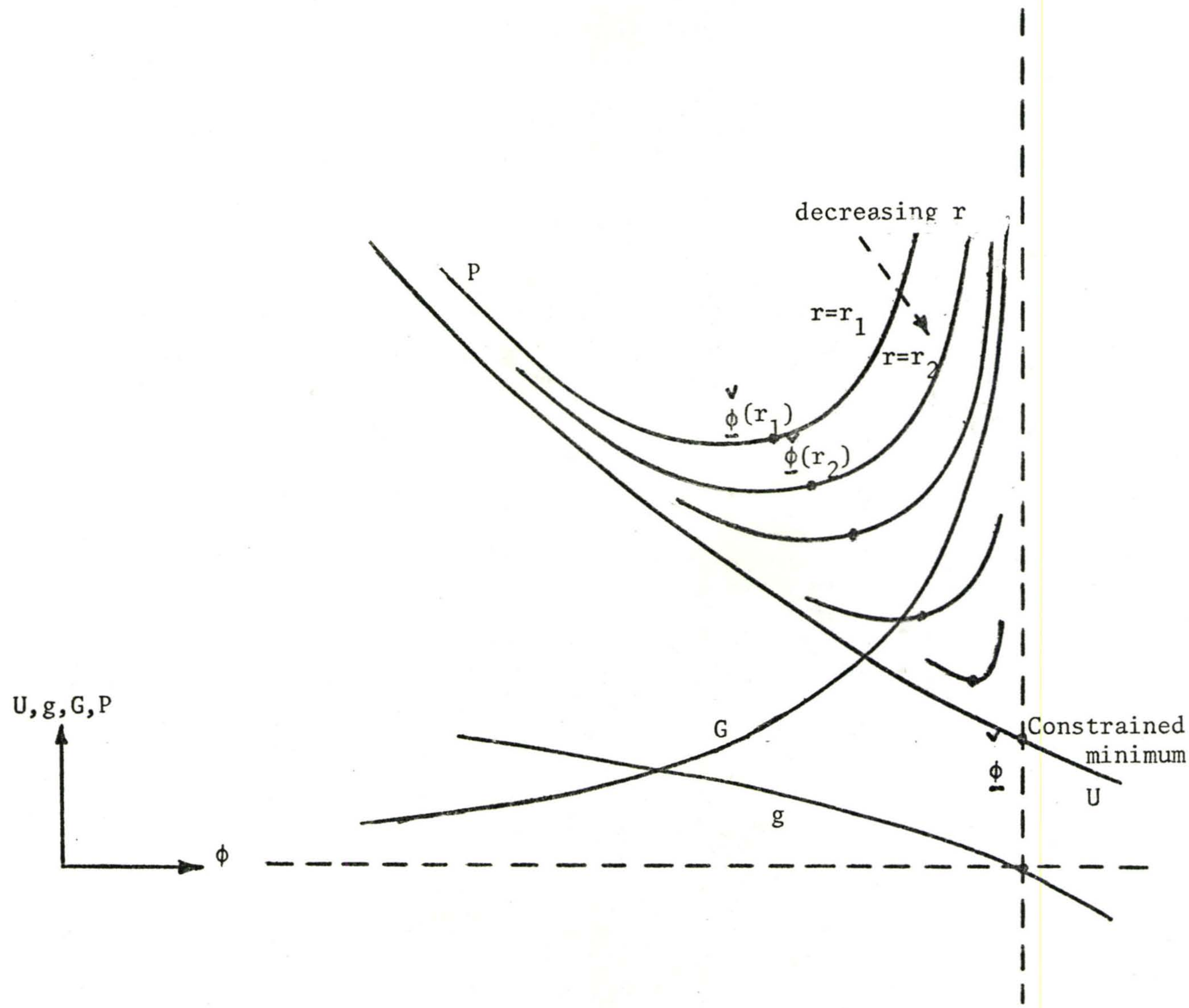

Figure (2)

The effect of r on the unconstrained function P.

The following procedure is then possible:

Starting from $\underline{\phi}^{\circ}$ decrease the value of $P(\underline{\phi},r_1)$ where

$$P(\underline{\phi},r_1) = U(\underline{\phi}) + r_1 \sum_{i=1}^{m} g_i^{-1}(\underline{\phi}) \qquad (36)$$

and $r_1 > 0$. Suppose that the feasible region R is never left. Assuming that our minimization method defines a continuous n-dimensional curve on which $P(\underline{\phi},r_1)$ is decreasing. Moving along this curve we should be able to find a point $\check{\underline{\phi}}(r_1) \in R^{\circ}$ which minimizes $P(\underline{\phi},r_1)$. Clearly the boundary can never be violated because $P(\underline{\phi},r_1) \to \infty$ as the boundary is approached, as shown from equations (30) and (32), so that the minimum $\check{\underline{\phi}}(r_1)$ (if it exists) must belong to the interior feasible region Figure (1). This procedure is repeated for $0 < r_{k+1} < r_k$, $k = 1,2,\ldots$ and the minimization of $P(\underline{\phi},r_k)$ can be achieved without violating the constraints. Ultimately it is hoped that a sequence of $U(\check{\underline{\phi}}(r_k))$ will converge to the minimum of $U(\underline{\phi})$ as the value of r_k tends to zero. We can expect that by taking r_k small enough we should be able to approach as close to the boundary as we wish Figure (2). A bad initial value of r_k will slow down convergence onto each response surface minimum. Too large a value of r_1 will cause the first few minima of P to be relatively independent of U, whereas too small a value will render the penalty term ineffective, except near the constraint boundaries where the surface rises very steeply. In this work the values chosen for r_k are,

$$1.0,\ 10^{-2},\ 10^{-4}, \ldots\ 10^{-8}.$$

3.2.2 New approach for constrained optimization problems

This is a new approach proposed by Bandler and Charalambous [13] to transform the constrained optimization problem into minimization of an unconstrained objective function. The original nonlinear programming problem is formulated as an unconstrained minimax problem.

The nonlinear programming problem can be stated as defined in equations (29) and (30)

Assuming a continuous function with continuous partial derivatives, then consider the problem of minimizing the unconstrained function

$$V(\underline{\phi},\underline{\alpha}) = \max_{1 \leq i \leq m} \left[U(\underline{\phi}),\ U(\underline{\phi}) - \alpha_i\, g_i\, (\underline{\phi})\right] \tag{37}$$

where

$$\underline{\alpha} \triangleq [\alpha_1\ \alpha_2\ \cdots\ \alpha_m]^T \tag{38}$$

and

$$\alpha_i > 0 \quad i = 1,2,\ldots, m \tag{39}$$

In this work the value of α is taken to be 1 and 50.

There are a number of advantages obtained by this new approach. The first is that the minimization of V can be regarded as an essentially unconstrained problem and a number of simple and suitable methods are available for its solution. The second is that the starting point can be anywhere. There is no need to distinguish between feasible and nonfeasible points. The third is that once suitable values for the α_i have been determined, one complete optimization yields the solution unlike penalty function methods.

3.3 Penalties for nonfeasible points

Inequality constraints only

Assuming that the initial solution is feasible, the simplest way of disallowing a constraint violation is by rejecting any set of parameter values which produces a non-feasible solution. This may be achieved by imposing a sufficiently large penalty on the objective function when any violation occurs. Thus, we may add the term

$$\sum_{i=1}^{m} w_i\ g_i^2(\underline{\phi}) \text{ where } w_i \begin{cases} = 0 & g_i(\underline{\phi}) \geqslant 0 \\ > 0 & g_i(\underline{\phi}) < 0 \end{cases} \tag{40}$$

to the objective function. As long as the constraints are satisfied the objective function is not penalized. However, nonfeasible points can be obtained with this formulation. An alternative which can prevent this is simply to set the objective function to its most unattractive value when $g_i(\underline{\phi}) < 0$. In practice such a value may be easy to determine on physical grounds.

In this work the value of w_i is taken to be very large when any $g_i(\underline{\phi}) < 0$ and this value is 10^{20}.

CHAPTER 4

RESULTS

4.1 Computational information

The computer used for all the problems is a CDC 6400. The terminating criterion for the Fletcher - Powell method, was set at 10^{-6} and the algorithm terminated if the change in the objective function or parameters was less than that number. The terminating criterion for the Fletcher method was also set at 10^{-6} and the algorithm terminated if the change in parameters was less than this number. In the Jacobson - Oksman algorithm there is the facility that the algorithm terminates when the change in the objective function is less than a number, set at 10^{-6}, and also when the norm of the gradient becomes smaller than a number set at 10^{-9}.

4.2 Cost - tolerances relation

A number of potentially useful and fairly well-behaved objective functions which might be used to represent the cost of a design can be formulated. In practice a suitable modelling problem would first have to be solved to determine the significant parameters involved partially or totally in the actual cost. Here we will assume either the absolute or the relative tolerances to be the main variables. Furthermore we assumed the total cost ($C(\underline{\phi}^\circ,\underline{\varepsilon})$ where $\underline{\varepsilon}$ is a set of tolerances) of the design is just the sum of the cost of the individual components.

It is intuitive to assume that

$$C(\underline{\phi}^\circ,\underline{\varepsilon})\to c\geq 0 \quad \text{as } \underline{\varepsilon}\to\underline{\infty} \tag{41}$$

and

$$C(\underline{\phi}^\circ,\underline{\varepsilon})\to\infty \qquad \text{for any } \varepsilon_i\to 0 \tag{42}$$

Two out of many possible functions which fulfil these requirements are

$$C = \sum_{i=1}^{n} \frac{k_i}{\varepsilon_i} \tag{43}$$

subject to

$$\underline{\varepsilon}\geq\underline{0} \tag{44}$$

where

C is the total cost of the design

ε_i is the tolerance of the i^{th} component $(i=1,2,\ldots,n)$

k_i is a constant value

n is the total number of components

This function is shown in Figure (3)

The second possible objective function is

$$C = \sum_{i=1}^{n} k_i \log \frac{\phi_i^\circ}{\varepsilon_i} \tag{45}$$

subject to

$$\underline{\phi}^\circ\geq\underline{\varepsilon}\geq\underline{0} \tag{46}$$

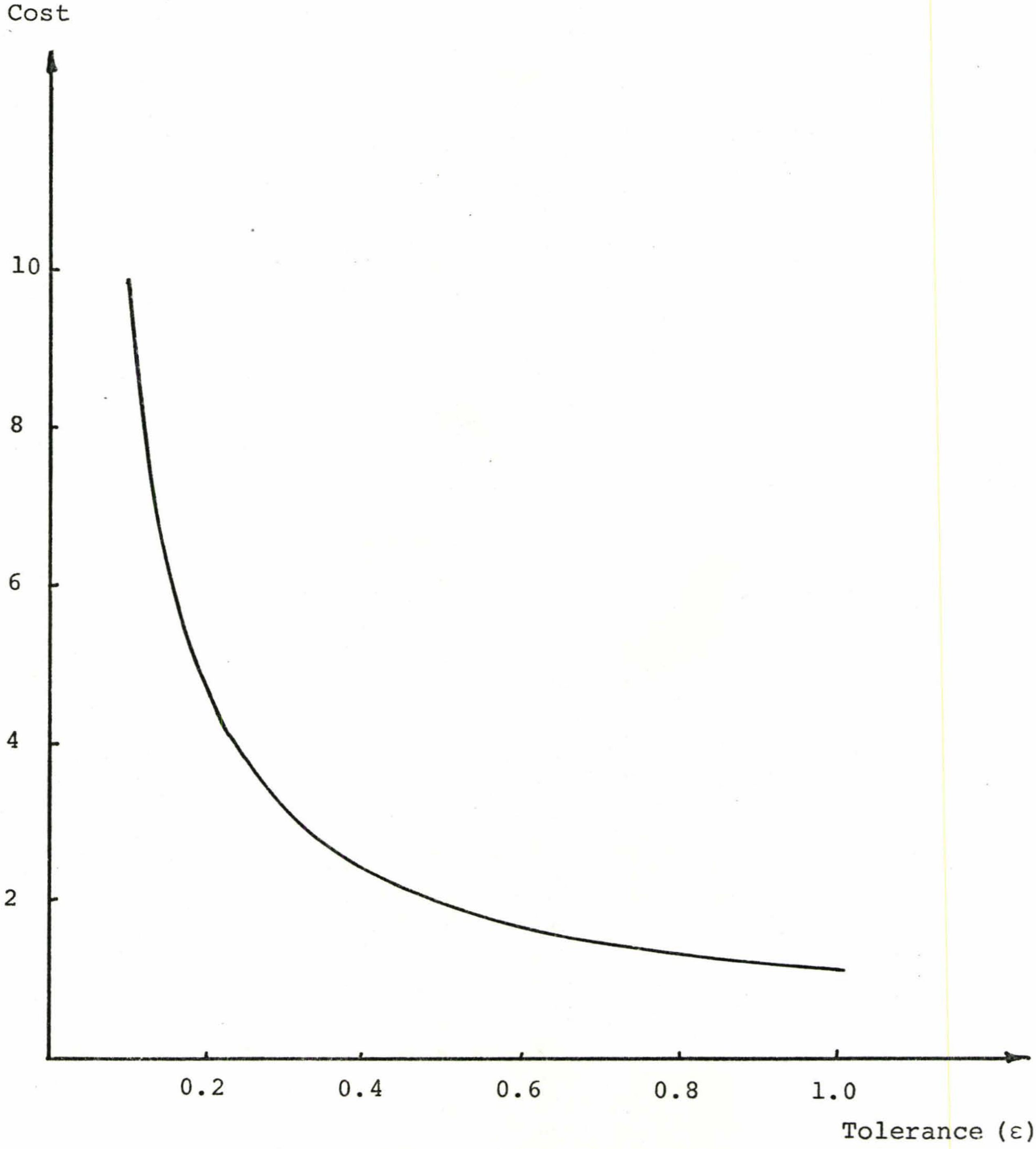

Figure (3)
The Cost - Tolerances Relationship (Equation (43)).

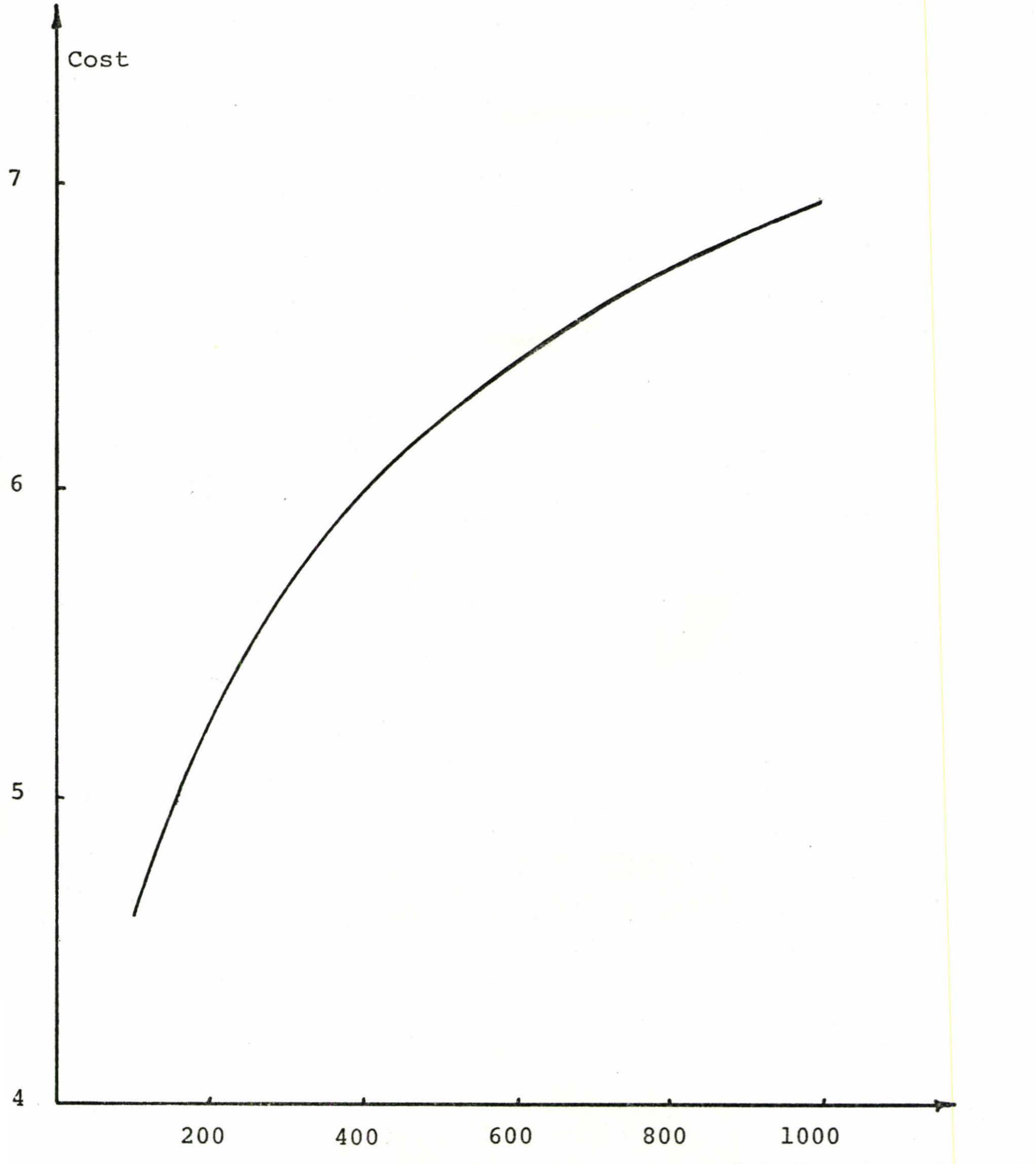

Figure (4)

The Cost - Tolerances Relationship (Equation (45)).

This function is shown in Figure (4). In both cases

$$k_i \geqslant 0 \qquad i = 1,2,\dots,n \tag{47}$$

4.3 The first objective function [equation (43)]

In this case it is assumed that the feasible region is defined by the following constraints and it is represented in Figure (5)

$$\begin{aligned} g_1(\underline{\phi}) &= -10 -2\,\phi_1 +5\,\phi_2 \geqslant 0 \\ g_2(\underline{\phi}) &= 6 - \phi_1 -2\,\phi_2 \geqslant 0 \\ g_3(\underline{\phi}) &= 182 +14\phi_1 -13\phi_2 \geqslant 0 \end{aligned} \tag{48}$$

In order to guarantee that all the points of the polytope, its vertices are defined as $(\phi_1-\varepsilon_1, \phi_2+\varepsilon_2)$, $(\phi_1+\varepsilon_1, \phi_2+\varepsilon_2)$, $(\phi_1+\varepsilon_1, \phi_2-\varepsilon_2)$ and $(\phi_1-\varepsilon_1, \phi_2-\varepsilon_2)$, will satisfy all the constraints, we have to modify the constraints as follows, for example the first constraint will be

$$\begin{aligned} g_{11} &= -10 -2(\phi_1-\varepsilon_1) +5(\phi_2+\varepsilon_2) \geqslant 0 \\ g_{12} &= -10 -2(\phi_1+\varepsilon_1) +5(\phi_2+\varepsilon_2) \geqslant 0 \\ g_{13} &= -10 -2(\phi_1+\varepsilon_1) +5(\phi_2-\varepsilon_2) \geqslant 0 \\ g_{14} &= -10 -2(\phi_1-\varepsilon_1) +5(\phi_2-\varepsilon_2) \geqslant 0 \end{aligned} \tag{49}$$

where ε_1, ε_2, ϕ_1 and ϕ_2 are the independent variables.

By using the Fiacco - McCormick transformation for the constrained objective function (discussed in 3.2.1) and the penalties for nonfeasible points (discussed in 3.3), now we have the unconstrained objective function as follows:

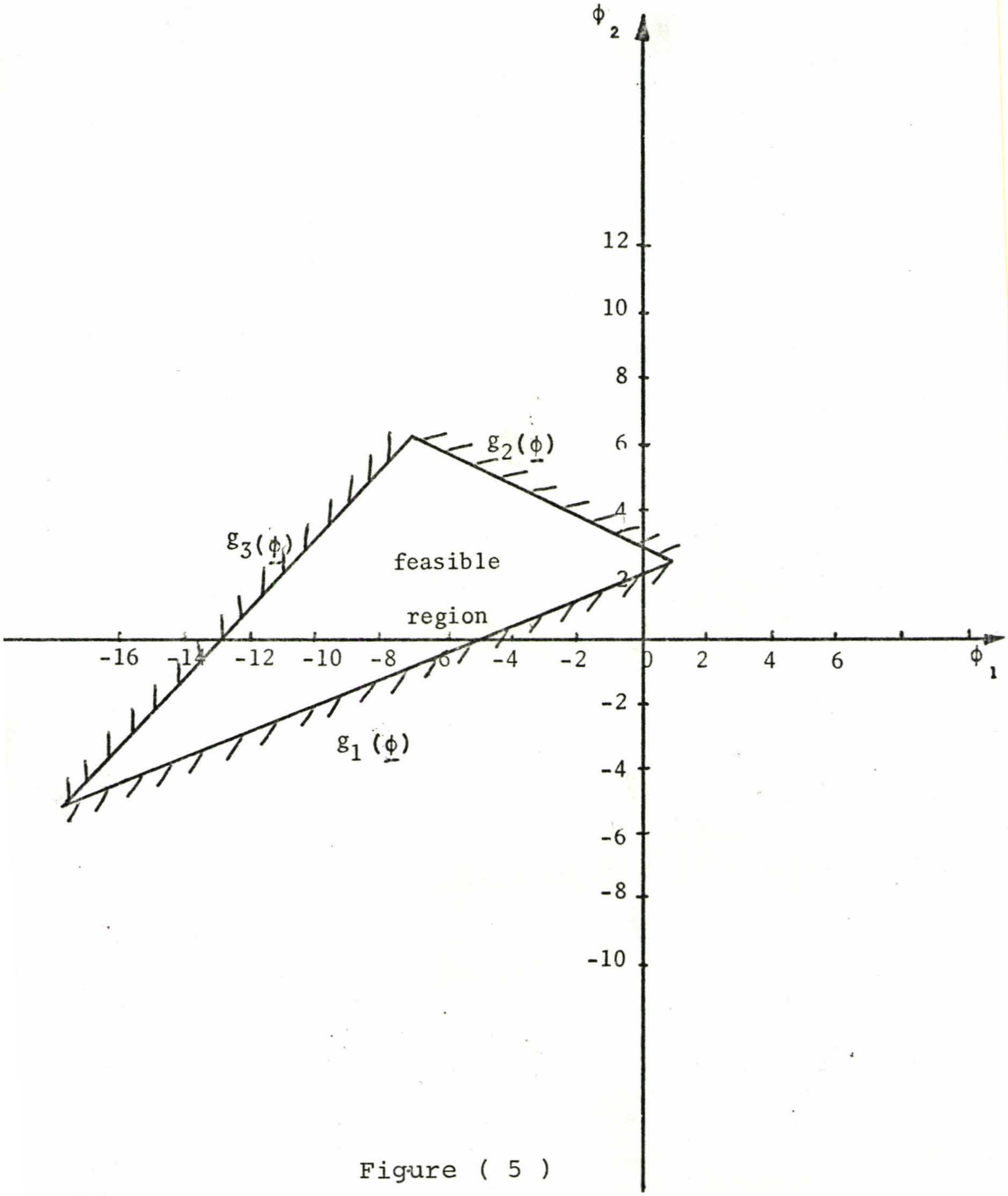

Figure (5)

The feasible region defined by equation (48).

$$P = \frac{k_1}{\varepsilon_1} + \frac{k_2}{\varepsilon_2} + r\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{g_{ij}(\underline{\phi})} + w\sum_{i=1}^{m}\sum_{j=1}^{n} g_{ij}^2(\underline{\phi}) \tag{50}$$

where

m is the number of constraints ($m=3$)

n is the number of independent variables ($n=4$)

$r>0$ (started at $r = 1.0$)

w is a large number to prevent nonfeasible points ($w=10^{10}$)

As explained before in 3.3 the term $w\sum_{i=1}^{m}\sum_{j=1}^{n} g_{ij}^2(\underline{\phi})$ will be active only when any violation of any constraint occured, i.e. when any $g_{ij}(\underline{\phi})<0$.

The gradients of the new objective function, [given in equation (50)], were calculated and the minimization process started using the three gradient methods discussed previously in 2.4.2, 2.4.3 and 2.4.4 repeating optimization process with decreasing sequence of the r values i.e.

$$r_1>r_2>\ldots>r_j>0$$

and each minimization being started at the previous point, e.g. minimization of $P(\underline{\phi},r_2)$ would be started at $\check{\underline{\phi}}(r_1)$. The values of r are chosen to range from 1.0 to 1.0×10^{-8}.

The solution of this problem is unique independent of the starting point.

When k_1 and k_2 are taken to be equal 1, the result was as follows

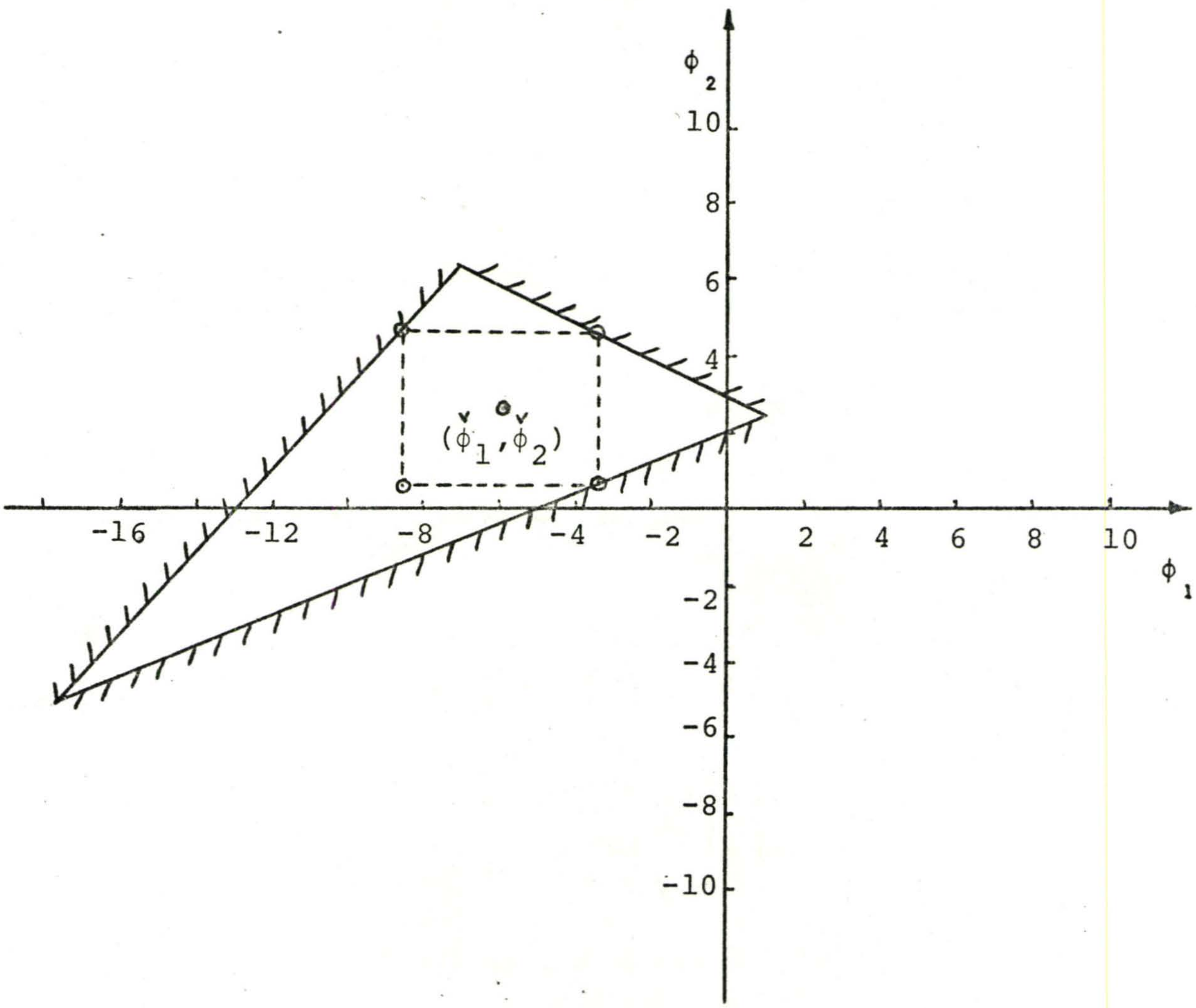

Figure (6)

The minimum for the feasible region shown in figure (5).

k_1	k_2	Min. Cost (P)	ϕ_1	ϕ_2	ε_1	ε_2
0.2	5.0	1.856987	-6.750497	2.781046	0.690973	3.190298
0.1	0.2	0.133234	-6.198300	2.702344	2.110575	2.332161
0.01	0.1	0.042967	-6.552011	2.764792	1.173904	2.910214
50.0	5.0	17.205863	-5.467933	2.634744	4.143075	0.975764
10.0	0.8	3.129029	-5.384869	2.551266	4.349907	0.964569
0.05	0.2	0.106900	-6.361567	2.732650	1.662812	2.606219
5.0	7.0	5.318757	-6.113290	2.687336	2.359197	2.188449
1.0	1.0	0.877085	-5.978965	2.662207	2.723588	1.962063

Table (1): The minimum cost for different values of k_i .
(for the feasible region shown in Figure (5)).

The minimum cost = 0.877 (Analytical solution = 0.8765)

The first tolerance = 2.72 (Analytical solution = 2.7210)

The second tolerance = 1.96 (Analytical solution = 1.9621)

and this result is shown in Figure (6).

The advantage of this method of optimization is, even starting with a nonfeasible point, the solution is feasible and unique.

The values of k_1 and k_2 are changed and the results are summarized in Table (1).

4.4 The second objective function [equation (45)]

In this case three different feasible regions are investigated.

4.4.1 The first feasible region

This feasible region was defined by three constraints which are

$$\begin{aligned} g_1(\underline{\phi}) &= 2 + 2\phi_1 - \phi_2 \geq 0 \\ g_2(\underline{\phi}) &= 143 - 11\phi_1 - 13\phi_2 \geq 0 \\ g_3(\underline{\phi}) &= -60 + 4\phi_1 + 15\phi_2 \geq 0 \end{aligned} \tag{51}$$

This feasible region is shown in Figure (7). Using the same procedure as before in 4.3 with

$$P = k_1 \log_e \frac{\phi_1}{\varepsilon_1} + k_2 \log_e \frac{\phi_2}{\varepsilon_2} + r \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{1}{g_{ij}(\underline{\phi})} + w \sum_{i=1}^{m} \sum_{j=1}^{n} g_{ij}^2(\underline{\phi}) \tag{52}$$

In this case we have a unique solution which is shown in Figure (8) for $k_1 = k_2 = 1$.

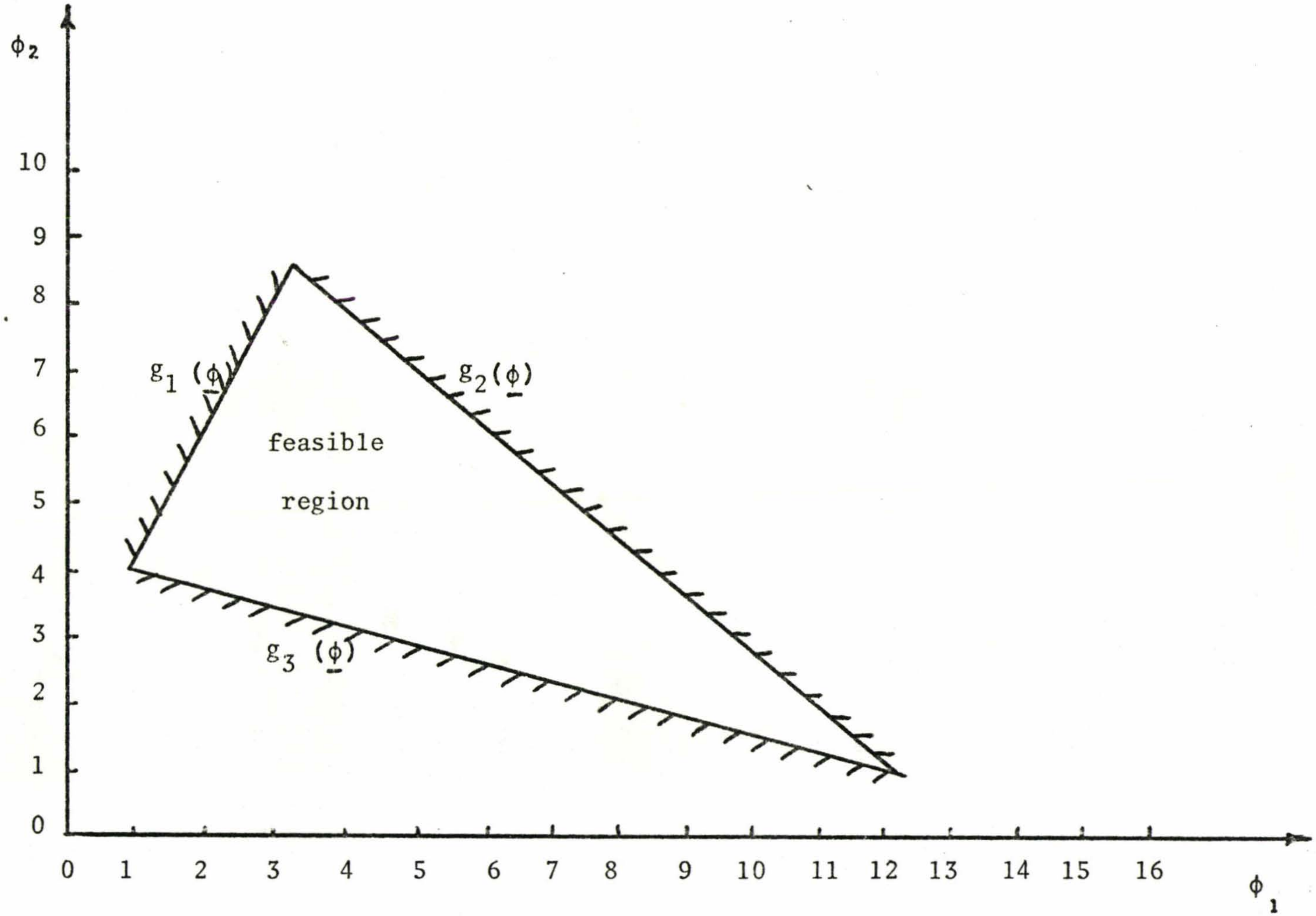

Figure (7)

The feasible region defined by equation (51).

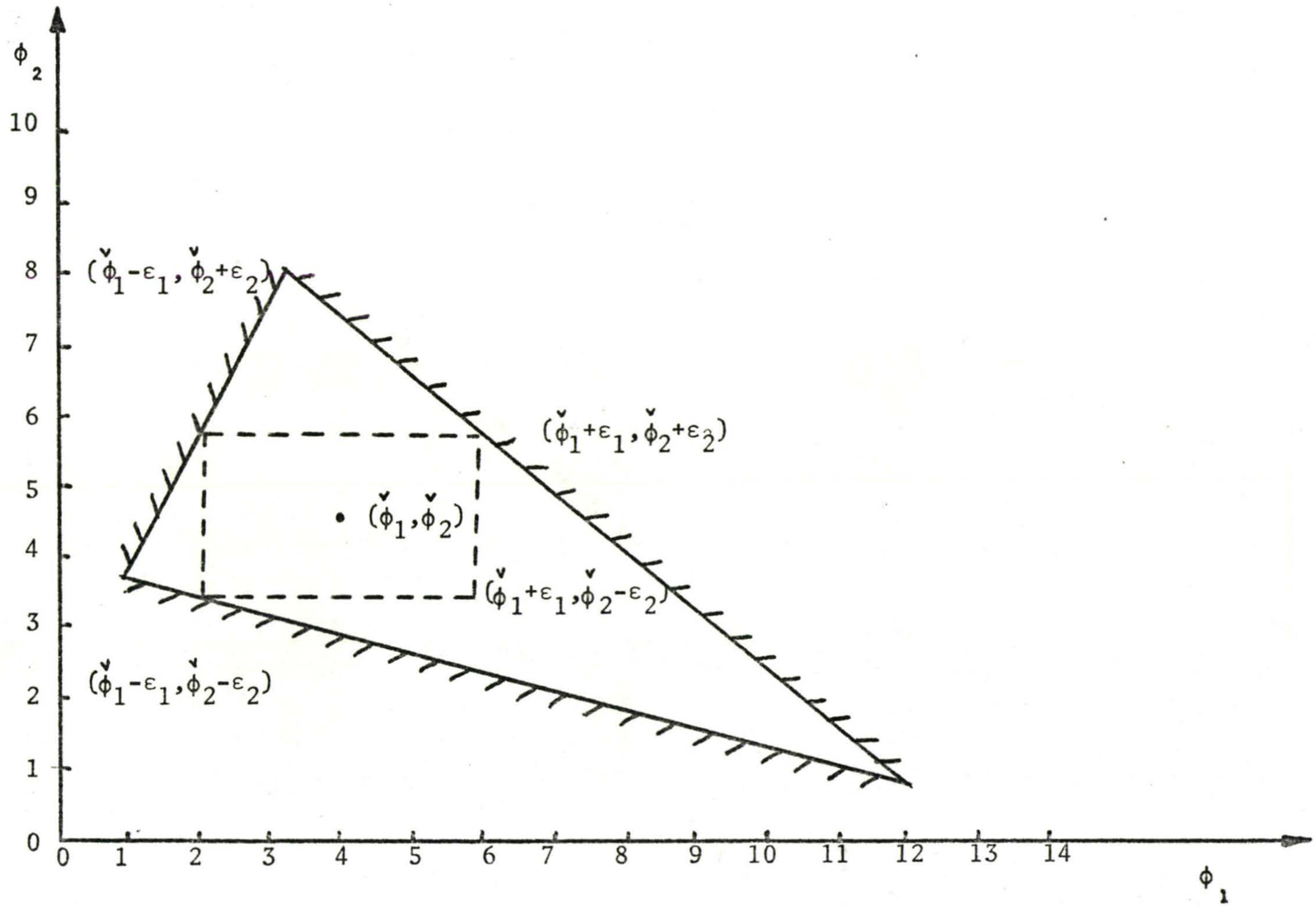

Figure (8)

The minimum for the feasible region shown in Figure (7).

k_1	k_2	Min. Cost (P)	ϕ_1	ϕ_2	ε_1	ε_2
20.0	0.3	5.70341	4.68316	3.80718	3.75177	0.05554
1.5	0.02	0.41713	4.68673	3.80264	3.76049	0.04958
3.0	1.5	3.93812	4.16222	4.46937	2.46671	0.92148
0.02	0.5	0.47261	3.24815	5.63151	0.211067	2.44133
0.5	2.0	2.64611	3.53298	5.26925	0.91443	1.96750
1.0	1.0	2.02289	3.94371	4.74714	1.92763	1.28473

Table (2) : The minimum cost for different values of k_i .

(for feasible region shown in Figure (7)).

Values of r	New Fletcher Method	Jacobson-Oksman Method	Fletcher-Powell Method
1.0	5.6702	5.9784*	5.6702
10^{-1}	2.6529	2.8630*	2.6529
10^{-2}	2.1741	2.5628*	2.2259*
10^{-3}	2.0659	2.5328*	2.1282*
10^{-4}	2.0359	2.5298*	2.0974*
10^{-5}	2.0269	2.5295*	2.0885*
10^{-6}	2.0241	2.5295*	2.0883*
10^{-7}	2.0232	2.5295*	2.0883*
10^{-8}	2.0229	2.5295*	2.0882*

Table (3) : The minimum value of the function for different values of r for the three gradient methods.

(Feasible region shown in figure (7)).

* Cubic interpolation seems to have failed to work, function value does not change significantly.

Feasible region shown in Figure (7)	New Approach	Fiacco - McCormick
Independent Variables		
ϕ_1	3.94368	3.94371
ϕ_2	4.74715	4.74715
ε_1	1.92771	1.92763
ε_2	1.28474	1.28473
Objective Function	2.02277	2.02289
No. of Function Evaluations	196	37
Initial Values of Parameters	α=1.0	r_k=1.0
Final Values of Parameters	α=1.0	r_k=10^{-8}

Table (4) : Comparison between the results using the Fiacco-McCormick technique [12] and the New approach [13], using the Fletcher method.

The minimum cost = 2.02 (Analytical solution = 2.0237)

The first parameter = 3.94 (Analytical solution = 4.0026)

The first tolerance = 1.93 (Analytical solution = 1.9368)

The second parameter = 4.75 (Analytical solution = 4.7425)

The second tolerance = 1.28 (Analytical solution = 1.2786)

Different values of k_1 and k_2 are chosen and the results are summarized in Table (2).

4.4.2 The second feasible region

This feasible region is defined by the three constraints which are

$$
\begin{aligned}
g_1(\underline{\phi}) &= -3 - \phi_1 + \phi_2 \geq 0 \\
g_2(\underline{\phi}) &= 12 - \phi_1 - 2\phi_2 \geq 0 \\
g_3(\underline{\phi}) &= \phi_1 \geq 0
\end{aligned}
\tag{53}
$$

This feasible region is shown in Figure (9). Using the same objective function and the same procedure as in 4.4.1, we get the unique solution for $k_1 = k_2 = 1$

The minimum cost = 1.106 (Analytical solution = 1.1)

The first parameter = 2.01×10^{-3} (Analytical solution = 0)

The first tolerance = 2.01×10^{-3} (Analytical solution = 0)

The second parameter = 4.501 (Analytical solution = 4.5)

The second tolerance = 1.497 (Analytical solution = 1.5)

Different values of k_i are chosen and the results are summarized in Table (5).

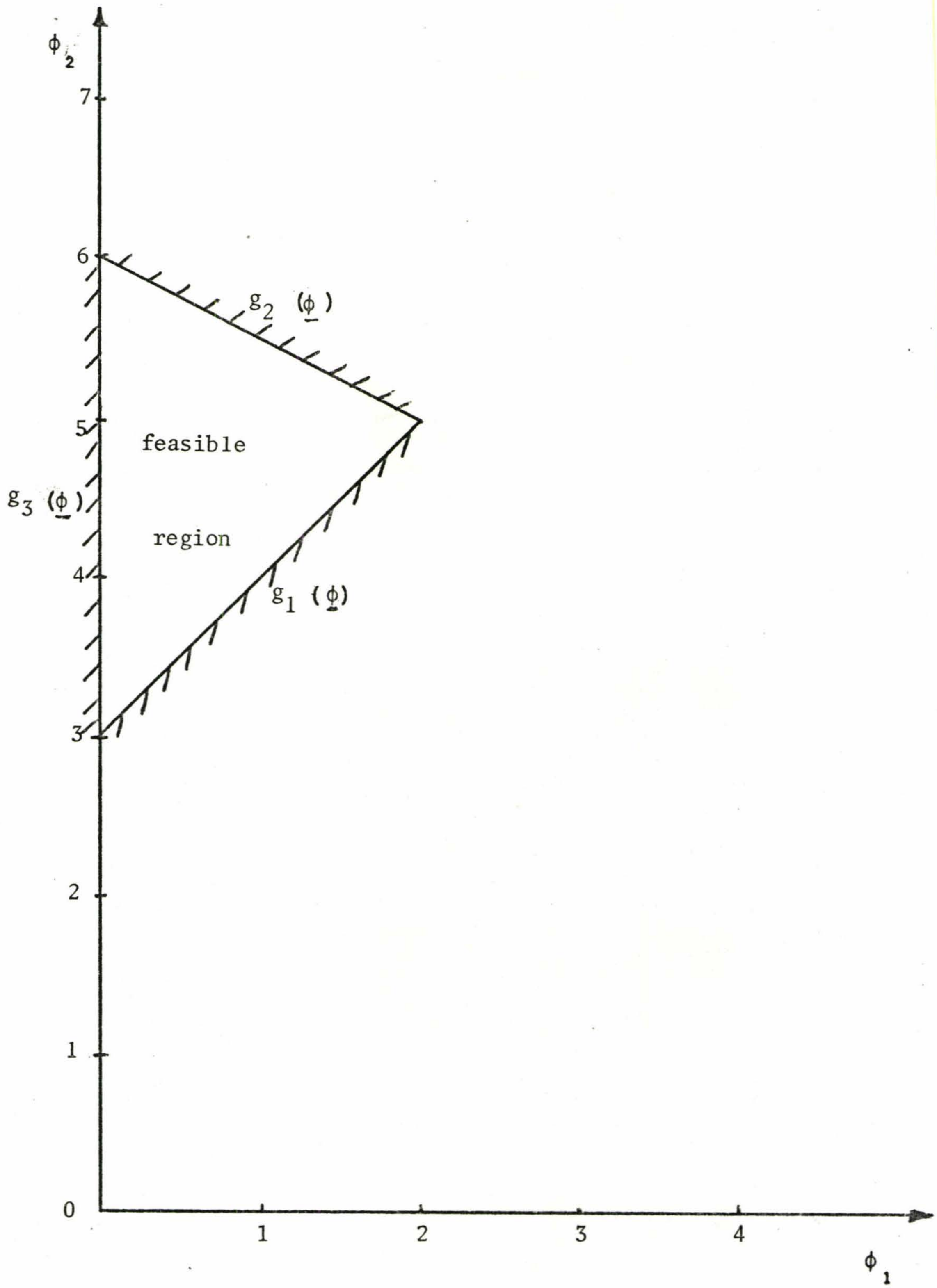

Figure (9)

The feasible region defined by equation (53).

k_1	k_2	Min. Cost (P)	ϕ_1	ϕ_2	ε_1	ε_2
20.0	0.3	0.34188	0.01208	4.50604	0.01208	1.48161
1.5	0.02	0.2409	0.03064	4.51532	0.03063	1.45300
3.0	1.5	1.65923	0.00221	4.50111	0.00221	1.49656
0.02	0.5	0.55091	0.00090	4.50044	0.00088	1.49845
0.5	2.0	2.20422	0.00101	4.50050	0.00101	1.49838
1.0	1.0	1.10552	0.00201	4.50101	0.00201	1.49683

Table (5) : The minimum cost for different values of k_i .

(for the feasible region shown in Figure (9)).

Values of r	New Fletcher Method	Jacobson-Oksman Method	Fletcher-Powell Method
1.0	20.0612	24.1595*	20.0612
10^{-1}	4.5404	5.1519*	4.7778*
10^{-2}	2.1609	2.7878*	2.8352*
10^{-3}	1.5075	2.5514*	2.1051*
10^{-4}	1.2698	2.5277*	1.9476*
10^{-5}	1.1733	2.5253*	1.9220*
10^{-6}	1.1320	2.5251*	1.9215*
10^{-7}	1.1137	2.5251*	1.9214*
10^{-8}	1.1055	2.5251*	1.9214*

Table (6) : The minimum value of the function for different values of r for the three gradient methods. (Feasible region shown in figure (9)).

* Cubic interpolation seems to have failed to work, function value does not change significantly.

Feasible Region Shown in Figure (9)	New Approach	Fiacco-McCormick
Independent Variables		
ϕ_1	0.00270	0.00210
ϕ_2	4.4813	4.50100
ε_1	0.00270	0.00210
ε_2	1.48069	1.49683
Objective Function	1.10823	1.10551
No. of Function Evaluations	293	58
Initial Values of Parameters	$\alpha=50$	$r_k=1.0$
Final Values of Parameters	$\alpha=5000$	$r_k=10^{-8}$

Table (7) : Comparison between the results using the Fiacco-McCormick technique [12] and the new approach [13], using the Fletcher method.

It was found, as shown in Table (5) that one of the components of the device can be removed, which will give minimum cost, and the device still functions properly as defined by the constraints because none of constraints has been violated.

4.4.3 The third feasible region

This region is defined by four constraints which are

$$
\begin{aligned}
g_1(\underline{\phi}) &= 5 - \phi_1 + \phi_2 \geq 0 \\
g_2(\underline{\phi}) &= 2 - \phi_2 \geq 0 \\
g_3(\underline{\phi}) &= -8 + 2\phi_1 + \phi_2 \geq 0 \\
g_4(\underline{\phi}) &= \phi_2 \geq 0
\end{aligned}
\tag{54}
$$

This feasible region is shown in Figure (10). Using the same objective function and the same procedure as in 4.4.1, we get the unique solution for $k_1 = k_2 = 1$.

The minimum cost = 2.144 (Analytical solution = 2.1437)

The first parameter = 4.57 (Analytical solution = 4.5664)

The first tolerance = 0.69 (Analytical solution = 0.6992)

The second parameter = 1.13 (Analytical solution = 1.1328)

The second tolerance = 0.87 (Analytical solution = 0.8672)

This result is shown in Figure (11). Different values of k_1 and k_2 are chosen and the results are summarized in Table (8).

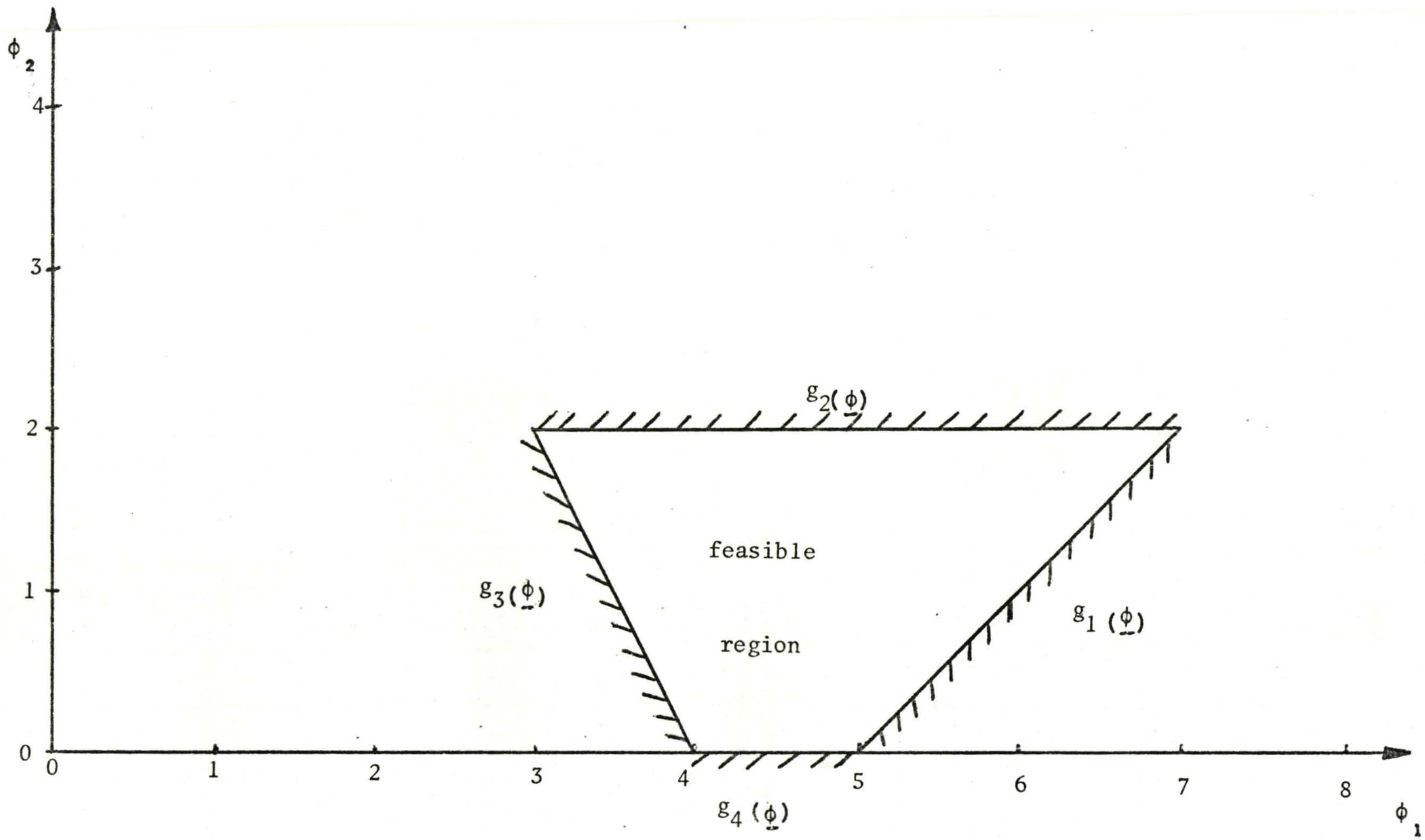

Figure (10)

The feasible region defined by equation (54).

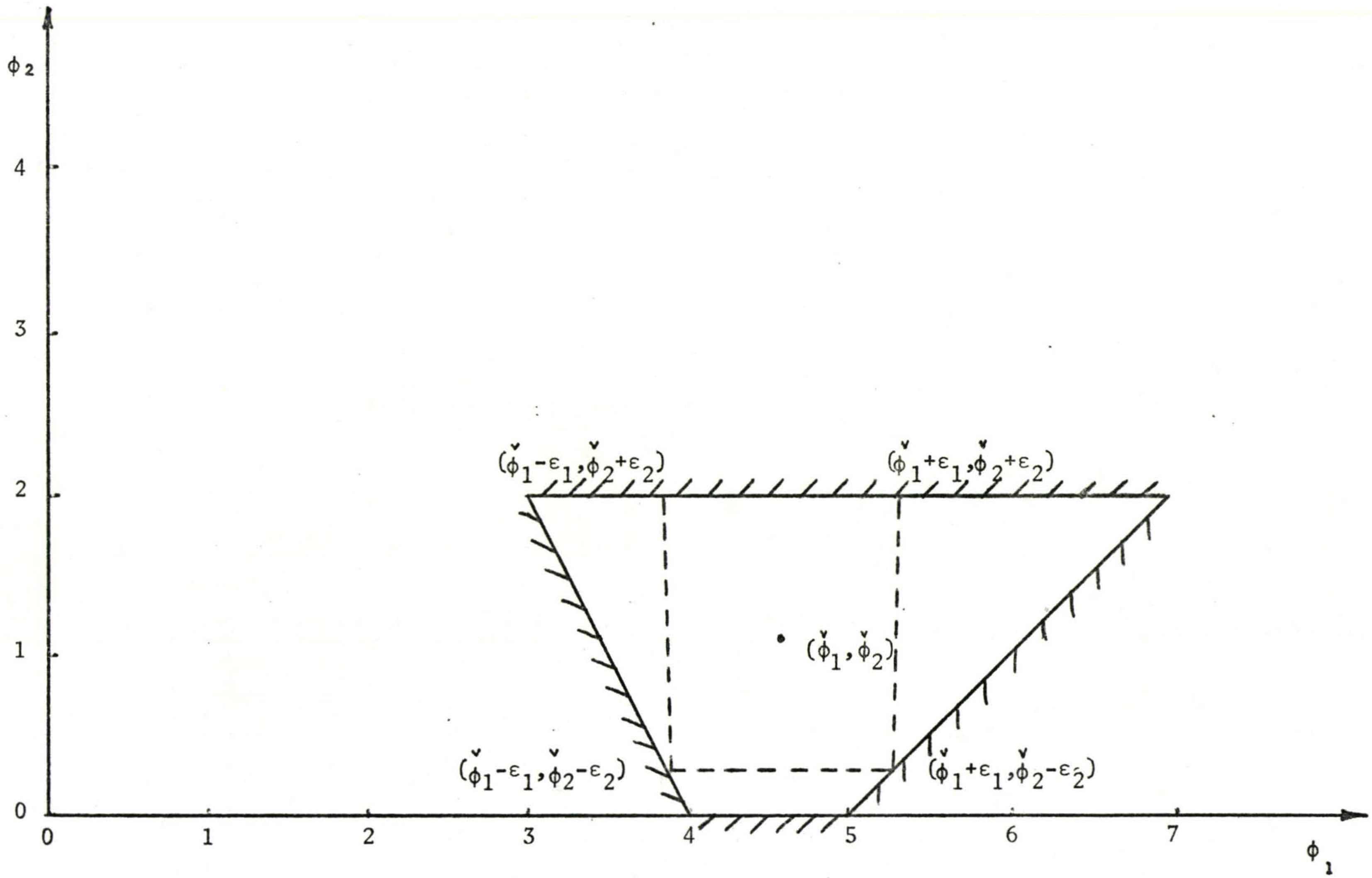

Figure (11)

The minimum for the feasible region shown in Figure (10).

k_1	k_2	Min. Cost (P)	ϕ_1	ϕ_2	ε_1	ε_2
20.0	0.3	19.96516	4.98853	1.97707	1.96559	0.02289
1.5	0.02	1.48634	4.98973	1.97949	1.96919	0.02036
3.0	1.5	5.62602	4.71116	1.42231	1.13348	0.57757
0.02	0.5	0.04413	4.49990	0.99184	0.49950	0.99170
0.5	2.0	1.09908	4.49999	0.99475	0.49994	0.99466
1.0	1.0	2.14408	4.56637	1.13265	0.69908	0.86708

Table (8) : The minimum cost for different values of k_i .
(for the feasible region shown in Figure (10)).

Values of r	New Fletcher Method	Jacobson-Oksman Method	Fletcher-Powell Method
1.0	16.7352	16.7352	16.7352
10^{-1}	4.2891	4.2891	4.5374*
10^{-2}	2.6064	2.6064	2.6064
10^{-3}	2.2675	2.2675	2.2675
10^{-4}	2.1811	2.1811	2.2047*
10^{-5}	2.1551	2.1551	2.1975*
10^{-6}	2.1473	2.1488*	2.1968
10^{-7}	2.1449	2.1488*	2.1968*
10^{-8}	2.1441	2.1488*	2.1968*

Table (9) : The minimum value of the function for different values of r for the three gradient methods. (Feasible region shown in figure (10)).

* Cubic interpolation seems to have failed to work, function value does not change significantly.

Feasible Region shown in Figure (10)	New Approach	Fiacco - McCormick
Independent Variables		
ϕ_1	4.56211	4.56637
ϕ_2	1.13143	1.13265
ε_1	0.69324	0.69908
ε_2	0.86532	0.86708
Objective Function	2.13415	2.14408
No. of Function Evaluations	150	30
Initial Values of Parameters	$\alpha=1.0$	$r_k=1.0$
Final Values of Parameters	$\alpha=1.0$	$r_k=10^{-8}$

Table (10) : Comparison between the results using the Fiacco-McCormick technique [12] and the New approach [13], using the Fletcher method.

4.5 Concluding remarks

It was found from the computer results that the higher the value of w the steeper the objective function in the non-feasible region resulting in the failure of the cubic interpolation as shown in tables (3), (6), and (9) for the Jacobson - Oksman and Fletcher - Powell methods. It was found also that theuse of the Fiacco - McCormick technique generally requires fewer function evaluations to reach the minimum accurately compared with the new approach using the Fletcher method.

CHAPTER 5

CONCLUSIONS

Minimum cost of any design based on reasonable tolerance, under certain constraints (in some special cases), have been obtained using the transformation of the constrained problem, by two methods [12] and [13], to an unconstrained minimization problem in conjunction with three efficient gradient minimization problems.

In general the use of gradient techniques have been found efficient for solving the cost-tolerance problem.

In this work it was considered, if the algorithm converged to a unique point, starting from n arbitrary starting points, where n is the number of independent variables (n = 4 for this work), then this point was considered an optimum.

From the minimization techniques used, Fletcher - Powell [7] was found to be reliable in the sense that the algorithm never diverged. But it is relatively slow compared to the method proposed by Fletcher [8], and the one by Jacobson and Oksman [9].

The method proposed by Fletcher was found the most efficient of the methods used. In most cases it requires the least number of function evaluations to reach the optimum.

In this work it was found that, if we have a feasible region similar to that in Figure (9), we can omit one of the components of this device, and hence lower the cost but the device will still function properly, because none of the constraints has been violated. For this feasible region the result was

$$\phi_1 = 2.01 \times 10^{-3} \quad , \quad \varepsilon_1 = 2.01 \times 10^{-3} \quad .$$

So

$$\phi_1 - \varepsilon_1 \simeq 0 \text{ and } \phi_1 + \varepsilon_1 = 4.02 \times 10^{-3}$$

which will be still relatively small. Also it was found that starting with any point, feasible or nonfeasible point, the algorithms still converge to the same minimum. This is due to the penalty function [3.3] which was added to the objective function every time any of the constraint is violated,i.e. when any $g_i(\underline{\phi}) < 0$.

If, as is usual in the design of systems, the optimal design is obtained, then a fairly large number of inequality constraints usually define the acceptable region. For any particular set of reasonable tolerances one could exploit the likelihood of the worst case (point most likely to violate a given constraint) being predictible by a local linearization or higher-order approximation of the constraints to greatly reduce the actual cost of the necessary computations that is implied by the 2^n vertices of the tolerance region.

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