ORDERED TOPOLOGICAL SPACES AND TOPOLOGICAL SEMILATTICES

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By

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A Thesis

Submitted to the School of Graduate Studies

in Partial Fulfilment of the Requirements

for the Degree

Doctor of Philosophy

McMaster University

April 1977

DOCTOR OF PHILOSOPHY
(Mathematics)

MCMASTER UNIVERSITY
Hamilton, Ontario

TITLE: Ordered Topological Spaces and Topological Semilattices

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NUMBER OF PAGES: viii, 79

Abstract

In this thesis, we consider the relationship between ordered topological spaces and topological semilattices.

We study the structures of ordered topological spaces and give some characterizations of these spaces, one of which is an answer to a question raised in [5]. We also construct two equivalent Wallman type order compactifications.

In addition, we describe epireflective subcategories of categories of ordered topological spaces, and in particular compactifications.

ACKNOWLEDGEMENTS

I would like to express my sincerest appriciation to my supervisor, Professor T.H. Choe, for providing a very interesting topic for this thesis and his stimulating guidance. I appreciate also his patience in listening to and constructively criticizing every result as it was obtained.

I would like to thank McMaster University for its financial support which enabled me to devote all my time to this work.

My thanks are also due to Miss I. Bojkiwskyj for typing this thesis so carefully and cheerfully.

Finally, my deepst thanks go to my wife for her constant encouragement and moral support.

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INTRODUCTION

Ordered topological spaces, namely, topological spaces on which a partial order has been defined, were first systematically studied by Nachbin [27] and Ward Jr. [35].

In recent years, various structures of ordered topological spaces were introduced and investigated by many mathematicians ([4], [5], [12], [23], [29], [31], etc). In particular, there has been a growing interest in constructing order compactifications of completely regular ordered spaces ([5], [12], [31]), which are clean generalizations of Stone-Čech compactifications, that is, reduce to the Stone-Čech compactifications in the case of discrete order.

Our purpose in writing this work is to construct Wallman type order compactifications, and investigate structures of ordered topological spaces and relationships betwen ordered topological spaces and topological semilattices. A brief synopsis of the material presented here is given below:

Chapter 0 contains the basic definitions and theorems which we utilize in the ensuing chapters.

In Chapter I, we introduces topologies on the families of all closed decreasing (resp. increasing) subsets of an ordered topological space, and observe that if X is an ordered topological C-space with continuous order, then the set of all decreasing (resp. increasing) sets generated by singltons, is iseomorphic to X. This reduces to the Michael theorem [25] (see 0.15). Moreover, if X is also regularly ordered, then it can be embedded into a bounded topological semilattice.

In [30], Priestley has introduced the concepts of I_d - , I_i - and I-spaces (see also [12], [24]) and in [33], Ulmer has introduced the concept of relative adjointness. Using these concepts, we obtain relative adjointness between the category of compact zero-dimensional semilattices with least (resp. greatest) element and the category of compact zero-dimensional I_d (resp. I_i)-spaces with continuous order, which generalizes the Hofmann, Mislove and Stralka result [16] (see 0.24).

By observing that there exists enough characters on a compact zero-dimensional I_d-space (resp. I_i-space) with continuous order, we show that such a space is a projective limit of finite discrete topological ordered spaces.

In Chapter II, we give some characterizations of completely regular ordered spaces. In [5], Choe and Hong have raised the following open problem: What are the characterizations of $\mathbb{R}$-compact ordered space and $\mathbb{R}$-compact topological lattice? Here we answer this problem by giving a few characterizations for those spaces.

In [31] and [17], Rodriguez and Hommel have constructed the order compactification β_1X for a completely regular ordered space X, which generalizes Stone-Čech compactification. Moreover, in [5], Choe and Hong have also constructed another order compactification β_0X. It is shown in this chpater that the two order compactifications β_1X and β_0X are equivalent, i.e., they are iseomorphic. The main concerns of this chapter are to construct an order compactification for a convex ordered topological space with semicontinuous order which generalizes the well known Wallman compactification of a T_1 space. In order to do this, we introduce the concept of bi-filters on an ordered topological space, and obtain that for any convex ordered topological space X with semicontinuous order, there exists a T_1 compact ordered space $w_0(X)$, into which X is densely embedded, and $w_0(X)$ satisfies the following property: for any compact ordered space Y with continuous order and for any continuous increasing map $f: X \to Y$, there exists a unique continuous increasing map $\overline{f}: w_0(X) \to Y$ such that $\overline{f}|X = f$.

We consider the families of all nonegative increasing (resp. decreasing) lower semicontinuous functions on an ordered topological space. By using the concept of bi-ideals (see 2.4.6), we obtain another order compactification $\mathfrak{m}_0(X)$ for a convex ordered topological space X with semicontinuous order. This generalizes the Nielsen and Sloyer theorem [28] (see 0.19). In addition, we also observe that the above two order compactifications $w_0(X)$ and $\mathfrak{m}_0(X)$ are equivalent. This reduces to the Brümmer theorem [3] (see 0.20) in the case of discrete order.

Finally, in Chapter III, we study separation properties of ordered topological spaces, and generalizes the Hommel result [17].

We introduce the classes of E-completely regular ordered spaces and E-compact ordered spaces. It is shown that the category <u>ECRO</u>

(resp. ECOS) of all E-completely regular ordered (resp. E-compact ordered) spaces is an epi-reflective subcategory of category PTOP of all ordered topological spaces. Moreover, ECOS is also an epi-reflective subcategory of ECRO. Finally, we observe that if, in particular, $E = \mathbb{I}$ (or $\mathbb{R}$), the above facts reduce to the Rodriguez results [31].

CHAPTER 0

PRELIMINARIES

In this chapter we will give the basic definitions and some known results which will be needed in later chapters.

0.1 Definition: A partially ordered set is a set in which a binary relation $x \le y$ is defined, which satisfies for all x, y, z the following conditions:
(1) for all x, $x \le x$
(2) if $x \le y$ and $y \le x$, then $x = y$
(3) if $x \le y$ and $y \le z$, then $x \le z$.

A partially ordered set $(X, \le)$ is called discrete if $x \le y$ only when $x = y$. A map f from a partially ordered set X to a partially ordered set Y is said to be increasing (resp. decreasing) if $x \le y$ in X implies $f(x) \le f(y)$ (resp. $f(x) \ge f(y)$) in Y.

0.2 Notation and Definition: Let $(X, \le)$ be a partially ordered set and A a subset of X, then we write

$$d(A) = \{y \in X: y \le x \text{ for some } x \in A\},$$

$$i(A) = \{y \in X: x \le y \text{ for some } x \in A\}, \text{ and}$$

$$c(A) = d(A) \cap i(A).$$

In particular, if A is a singleton, say $\{x\}$, then we write $d(x)$ (resp. $i(x)$) instead of $d(\{x\})$ (resp. $i(\{x\})$).

A subset A of X is said to be decreasing (resp. increasing, resp. convex) if $A = d(A)$ (resp. $A = i(A)$, resp. $A = c(A)$).

By an ordered topological space (or topological ordered space) we mean a set endowed with both a topology and a partial order.

0.3 Definition: (Ward Jr. [35]). Let $(X, \mathcal{T}, \leq)$ be an ordered topological space, then $\leq$ is called

(1) lower semicontinuous if, whenever $a \nleq b$ in X, there exists an open neighbourhood U of a, such that, if $x \in U$, then $x \nleq b$.

(2) upper semicontinuous if for $a \nleq b$, there exists an open neighbourhood V of b, such that $x \in V$ implies $a \nleq x$.

(3) semicontinuous if it is both upper and lower semicontinuous.

(4) continuous if, whenever $a \nleq b$, there exists U open neighbourhood of a and V open neighbourhood of b, such that if x is in U and y is in V, $x \nleq y$.

0.4 Theorem: (McCartan [23]). 1. Let $(X, \mathcal{T}, \leq)$ be an ordered topological space, then the following conditions are equivalent:

(1) X is lower (upper) semicontinuous.

(2) If $a \nleq b$ in X, there exists an increasing (decreasing) neighbourhood N of a(b) such that $b \notin N$ $(a \notin N)$.

(3) For each $x \in X$, d(x) (i(x)) is closed.

2. Every ordered topological space with semicontinuous order is a T_1 space.

0.5 Lemma: (Ward Jr. [35]). 1. Let $(X, \mathcal{T}, \leq)$ be an ordered topological space, then the following conditions are equivalent:

(1) $\leq$ is continuous.

(2) The graph of the partial order $\leq$ is a closed set in XxX.

(3) If $a \nleq b$ in X, then there are neighbourhoods N and N' of a and b respectively, such that N is increasing, N' is decreasing and $N \cap N' = \emptyset$.

.2. Every ordered topological space with continuous order is a Hausdorff space.

0.6 Notation and Definition: Let $(X, \mathcal{T}, \leq)$ be an ordered topological space and let

$$\mathcal{U} = \{U \in \mathcal{T} : U = i(U)\},$$

$$\mathcal{L} = \{U \in \mathcal{T} : U = d(U)\},$$

then $\mathcal{U}$ and $\mathcal{L}$ are evidently topologies for X, which are called the upper, lower topologies respectively.

Furthermore, $(X, \mathcal{U})$ is said to be an upper topological space and $(X, \mathcal{L})$ a lower topological space.

We say that an ordered topological space X is convex if the set consisting of the sets in $\mathcal{L}$ and $\mathcal{U}$ is a subbase of the topology of X.

0.7 Proposition: (Nachbin [27]). Let X be a topological space equipped with a continuous order. If $K \subset X$ designates a compact subset of X, then the decreasing subset d(K) and the increasing subset i(K) are closed.

0.8 Definition: Let X be an ordered topological space and Y a subspace of X. Then Y is called an (order) subspace of X, if it has the induced order.

Let X_α be ordered topological spaces and $\prod_\alpha X_\alpha$ the product space of X_α. Then $\prod_\alpha X_\alpha$ is called an (order) product of X_α, if it has the cartesian order (i.e. for (x_α), (y_α) in $\prod_\alpha X_\alpha$ $(x_\alpha) \leq (y_\alpha)$ if and only if $x_\alpha \leq y_\alpha$ for each α).

0.9 Notation: $\mathbb{I} = [0, 1]$ (resp. $\mathbb{R}$) is the unit interval (resp. the set of real numbers) equipped with the usual topology and the usual order.

0.10 Definition: (Nachbin [27]). An ordered topological space $(X, \mathcal{J}, \leq)$ is completely regular ordered space if and only if $\leq$ is a continuous order, and

(1) whenever $x \nleq y$ in X, there exists a continuous increasing function $f: X \to [0, 1]$ such that $f(x) > f(y)$.

(2) for any $x \in X$ and any open neighbourhood V of x, there exists a continuous increasing function $f: X \to [0, 1]$ and a continuous decreasing function $g: X \to [0, 1]$ such that $f(x) = 1 = g(x)$ and $X - V \subseteq f^{-1}(0) \cup g^{-1}(0)$.

0.11 Proposition: The following statements hold:

(1) Any completely regular ordered space is completely regular as a topological space.

(2) Any order subspace of a completely regular ordered space is a completely regular ordered space.

(3) Any compact space equipped with a continuous order is a completely regular ordered space.

(4) An order product of completely regular ordered spaces is a completely regular ordered space.

0.12 Construction of $\beta_1 X$: (Rodriguez [31], Hommel [17]). Let $(X, \mathcal{J}, \leq)$ be a completely regular ordered space and let C_1X be the family of all continuous increasing maps of X into [0, 1]. Define $j: X \to [0, 1]^{|C_1X|}$ by $j(x)(f) = f(x)$ for all $f \in C_1X$ and all $x \in X$. Then j is an iseomorphism (i.e. order isomorphism and topological homeomorphism) from X into $[0, 1]^{|C_1X|}$. If we denote $\beta_1 X = \overline{j(X)}$, where the closure $\overline{}$ is taken in $[0,1]^{|C_1X|}$, then $\beta_1 X$ satisfies the following universal property: for every compact space Y equipped with a continuous order and for every continuous increasing map $f: X \to Y$, there exists a unique continuous increasing map $\bar{f}: \beta_1 X \to Y$ such that $\bar{f} \circ j = f$. We call $\beta_1 X$ the Stone-Čech order compactification.

0.13 The Embedding Theorem: (Mrówka [26]). Let $F = \{f_\alpha : \alpha \varepsilon \Gamma\}$ be a class of continuous functions with $f_\alpha : X \to X_\alpha$, where X and X_α, $\alpha \varepsilon \Gamma$, are topological spaces. Let h be the parametric map corresponding to the class F (i.e. h is a map of X into $\prod_{\alpha\varepsilon\Gamma} X_\alpha$ such that $h(p) = (f_\alpha(p))_{\alpha\varepsilon\Gamma}$ for each $p \varepsilon X$). We have

(1) h is continuous if and only if each f_α is continuous.

(2) h is one-to-one if and only if for every p, q in X with $p \neq q$, there is an $f_\alpha \varepsilon F$ with $f_\alpha(p) \neq f_\alpha(q)$.

(3) h is a homeomorphism if and only if h is continuous, one-to-one, and for every closed subset $A \subset X$ and for every $p \varepsilon X - A$, there exists a finite system $f_{\alpha_1}, \ldots, f_{\alpha_n}$ of F such that $(f_{\alpha_1}(p), \ldots, f_{\alpha_n}(p)) \notin$ cl $\{(f_{\alpha_1}(a), \ldots, f_{\alpha_n}(a)) : a \varepsilon A\}$, where the closure is taken in the product space $X_{\alpha_1} \times \ldots \times X_{\alpha_n}$.

(4) Assume that the spaces X_α are all Hausdorff and assume that h is a homeomorphism. Then h(X) is closed in $\prod_{\alpha\varepsilon\Gamma} X_\alpha$ if and only if the class F satisfies the following condition: there is no proper extension $\bar{X}$ of X such that every function $f_\alpha \varepsilon F$ admits a continuous extension $\bar{f} : \bar{X} \to X_\alpha$.

0.14 Definition: (Michael [25]). Let X be a topological space. Let 2^X denote the set of all non-empty closed subsets of X. For a subset A of X, we let $2^A = \{F \varepsilon 2^X : F \subseteq A\}$. We generate a topology on 2^X by taking all sets of the form 2^G and all sets of the form $2^X - 2^{X-G}$, for G open in X, as a subbasis. This topology on 2^X is called the finite topology (or exponential topology) and 2^X, endowed with this topology, is called the hyperspace of X.

0.15 Theorem: (Michael [25]). If X is Hausdorff, the set of singleton sets in 2^X is a closed subset of 2^X homeomorphic to X (see also Ginsberg [11]).

0.16 Definition: Let X be a topological space. A real valued function $f: X \to R$ is said to be lower semicontinuous, if for each $a \in R$, $f^{-1}((a, \infty))$ is an open set in X.

0.17 Theorem: Let X be a topological space, and let L(X) denote the set of all lower semicontinuous functions on X. Then the following hold:

(1) If $f \in L(X)$ and α is a nonnegative real number, then $\alpha f \in L(X)$.

(2) If f, $g \in L(X)$, then $\min\{f, g\} \in L(X)$.

(3) If D is a nonvoid subset of L(X), then $\sup\{f: f \in D\}$ is a function in L(X).

(4) If f, $g \in L(X)$, then $f + g \in L(X)$.

0.18 Definition: (Nielsen and Sloyer [28]). Let X be a T_1 space and consider the semi-ring $L^{\dagger}(X)$ of all nonnegative lower semicontinuous functions on X with the usual pointwise operations. By an ideal in $L^{\dagger}(X)$ we mean a proper subset I satisfying the following conditions:

(1) f, $g \in I$ imply $f + g \in I$.

(2) $f \in I$ and $g \in L^{\dagger}(X)$ imply $gf \in I$.

(3) $f \in I$ implies that there exists an idempotent g, $g \neq 1$, such that $g \cdot f = f$.

0.19 Theorem: (Nielsen and Sloyer [28]). Let X be a T_1 space and $\mathcal{M}(X)$ the family of all maximal ideals in $L^+(X)$. Then $\mathcal{M}(X)$ is a compactification of X under the Stone topology in which a subbase for the closed sets consists of the sets $\tilde{f} = \{I \varepsilon \mathcal{M}(X): f \varepsilon I\}$ with $f \varepsilon L^+(X)$.

0.20 Theorem: (Brümmer [3]). The above $\mathcal{M}(X)$ is equivalent to the Wallman compactification of X.

0.21 Definition: A topological semilattice is a commutative, idempotent topological semigroup. Equivalently, it is a Hausdorff topological space endowed with a partial order for which every two elements have a greatest lower bound and the function $(x, y) \to \text{glb}\{x, y\}$ is continuous.

0.22 Definition: Let A and B be categories. An adjunction from B to A is a triple (F, G, ϕ), where F and G are functors

$$\underline{B} \underset{G}{\overset{F}{\rightleftarrows}} \underline{A},$$

while ϕ is a function which assigns to each pair of objects $B \varepsilon \underline{B}$ and $A \varepsilon \underline{A}$ a bijection $\phi_{B,A}$: $\underline{A}(FB, A) \cong \underline{B}(B, GA)$ which is natural in B and A. In this case, the funotor F is said to be a left adjoint for G, while G is called a right adjoint for F.

0.23 Definition: A subcategory A of B is said to be reflective in B when the inclusion functor K: $\underline{A} \to \underline{B}$ has a left adjoint F: $\underline{B} \to \underline{A}$. This functor F may be called a reflector and the adjunction (F, K, ϕ): $\underline{B} \to \underline{A}$ a reflection of B in its subcategory A. Equivalently, for each $B \varepsilon \underline{B}$, there is an object RB of the subcategory A and a morphism η_B: $B \to RB$ such that every morphism g: $B \to A \varepsilon \underline{A}$ has the form $g = f o \eta_B$ for a unique morphism f: $RB \to A$ of A. If η_B is an epimorphism (resp. monomorphism) for each $B \in \underline{B}$, then $\underline{A}$ is called epi (resp. mono) reflective in $\underline{B}$.

0.24 Proposition: (Hofmann, Mislove and Stralka [16]). Let $\underline{ZC}$ be the category of zero-dimensional compact Hausdorff spaces, and $\underline{Z}$ the category of compact zero-dimensional semilattice with identity. For $X \in \underline{ZC}$, let $\Gamma(X) = 2^X$ be the space of closed subsets of X in the exponential topology and with union as operation. Then $\Gamma(X) \in \underline{Z}$ and $\Gamma\colon \underline{ZC} \to \underline{Z}$ is the left adjoint of the forgetful functor $|\ |\colon \underline{Z} \to \underline{ZC}$. The front adjunction $\eta\colon X \to |\Gamma(X)|$ is given by $\eta(x) = \{x\}$, while the back adjunction $\varepsilon\colon \Gamma(|S|) \to S$ is given by $\varepsilon(A) = \wedge A$ (g.l.b.A) for each $A \in \Gamma(|S|)$, for each $S \in \underline{Z}$.

0.25 Definition: (Priestley [30]). An ordered topological space $(X, \mathcal{T}, \leq)$ is said to be a C-space if, whenever a subset F of X is closed, d(F) and i(F) are closed.

CHAPTER I

RELATIONSHIPS BETWEEN ORDERED TOPOLOGICAL SPACES AND TOPOLOGICAL SEMILATTICES

Section 1. Embedding a regularly ordered space into a topological semilattice.

In this section, we shall discuss the relationship between an ordered topological space and a topological semilattice. Also, we generalize Michael's theorem (see 0.15).

Let $(X, \mathcal{T}, \leq)$ be an ordered topological space. Let $\mathcal{D}(X)$ denote the set of all closed decreasing subsets of X. For a subset A of X, we let

$$2^A = \{F \in \mathcal{D}(X) \colon F \subseteq A\}.$$

Consequently $\mathcal{D}(X) - 2^{X-A} = \{F \in \mathcal{D}(X) \colon F \cap A \neq \phi\}$.

Let $\mathcal{T}^*$ be the smallest topology on $\mathcal{D}(X)$ generated by the family $\{2^U \colon U$ an open decreasing subset of $X\} \cup \{\mathcal{D}(X) - 2^{X-U} \colon U$ an open increasing subset of $X\}$. We shall use the following notational convention. For subsets $A_0, A_1, \ldots, A_n$ of X, we let

$$B(A_0; A_1, \ldots, A_n) = 2^{A_0} \cap \bigcap_{i=1}^{n} (\mathcal{D}(X) - 2^{X-A_i}) = \{F \in \mathcal{D}(X) \colon F \subseteq A_0 \text{ and } F \cap A_i \neq \phi \text{ for all } i = 1, 2, \ldots, n\}.$$

Using this notation, we see that the sets $B(U_0; U_1, \ldots, U_n)$, where U_0 is open descreasing, and $U_1, \ldots, U_n$ are open increasing in X, form an open base of $\mathcal{D}(X)$. Thus $(\mathcal{D}(X), \mathcal{T}^*)$ is a topological space. The empty set is an isolated element of this space.

1.1.1 Remark: The following statements follow immediately:

(1) If A is a closed decreasing subset of X, then 2^A is closed

in $(\mathcal{E}(X), \mathcal{T}^*)$.

(2) If A is a closed increasing subset of X, then $\mathcal{E}(X) - 2^{X-A}$ is closed in $(\mathcal{E}(X), \mathcal{T}^*)$.

1.1.2 Remark: Dually, we define the following: Let $\mathcal{S}(X)$ denote the set of all closed increasing subsets of X. Let $\mathcal{T}_I^*$ be the smallest topology on $\mathcal{S}(X)$ generated by the family $\{2^U$: U an open increasing subset of X$\} \cup \{\mathcal{S}(X) - 2^{X-U}$: U an open decreasing subset of X$\}$. Then $(\mathcal{S}(X), \mathcal{T}_I^*)$ is a topological space.

1.1.3 Remark: We obtained two topological spaces $(\mathcal{E}(X), \mathcal{T}^*)$ and $(\mathcal{S}(X), \mathcal{T}_I^*)$ from the given ordered topological space X. But if the given order on X is discrete, then $(\mathcal{E}(X), \mathcal{T}^*)$ and $(\mathcal{S}(X), \mathcal{T}_I^*)$ coincide. These are equal to the hyperspace of X (see 0.13).

1.1.4 Theorem: Let $(X, \mathcal{T}, \leq)$ be an ordered topological space with continuous order. Then X can be embedded into $(\mathcal{E}(X), \mathcal{T}^*, \subseteq)$, where $\subseteq$ is the set inclusion relation.

Dually, the same statement holds for $(\mathcal{S}(X), \mathcal{T}_I^*, \supseteq)$.

Proof: Let us define a map $d: X \to \mathcal{E}(X)$ by $x \rightsquigarrow d(x)$, where $d(x) = \{y \in X: y \leq x\}$. Then we shall show that d is an iseomorphism from X into $\mathcal{E}(X)$. Indeed, it is easy to see that d is increasing and injective. Now, we show that d is continuous: let $x \in X$ and $B(U_0; U_1, \ldots, U_n)$ be a basic open neighbourhood of d(x). Then $d(x) \subseteq U_0$ and $d(x) \cap U_i \neq \phi$ for all $i = 1, 2, \ldots, n$. Let $p_i \in d(x) \cap U_i$, then $p_i \leq x$. Hence $x \in U_i$ for all $i = 1, 2, \ldots, n$. Let $U = \bigcap_{i=0}^{n} U_i$, then it is easy to see that U is an open neighbourhood of x and $d(U) \subseteq B(U_0; U_1, \ldots, U_n)$, and so the continuity follows immediately. Finally, we show that d is relatively closed: let V be a closed subset of X. Let $F \epsilon \overline{d(V)} \cap d(X)$, then $F \epsilon \overline{d(V)}$ and $F = d(x)$ for some $x \epsilon X$. Suppose that $d(x) \notin d(V)$, then $x \notin d(V)$, and hence $x \epsilon X - d(V)$. Since $X - d(V)$ is an open increasing set,

$$\{F' \epsilon \mathcal{D}(X): F' \cap (X-d(V)) \neq \phi\} \text{ is open in } \mathcal{D}(X)$$

and contains $d(x) = F$. Futhermore, we have $d(V) \cap \{F' \epsilon \mathcal{D}(X): F' \cap (X-d(V)) \neq \phi\} \neq \phi$. This contradicts the fact that $F \epsilon \overline{d(V)}$. Therefore, $d(x) \epsilon d(V)$, i.e., $F \epsilon d(V)$. It follows that $\overline{d(V)} \cap d(X) = d(V) \cap d(X)$. This completes proof.

1.1.5 <u>Remark</u>: In particular, if the given order on X is discrete, then Theorem 1.1.4 coincides with Michael's result (see 0.15).

1.1.6 <u>Theorem</u>: Let $(X, \mathcal{T}, \leq)$ be an ordered topological space and Y a topological space. Let $F: Y \to \mathcal{D}(X)$ be a mapping. Then F is continuous if and only if $F^{-1}(2^U)$ is open whenever U is open decreasing in X and $F^{-1}(2^K)$ is closed whenever K is closed decreasing in X.

<u>Proof</u>: Assume that F is continuous. Then $F^{-1}(G)$ is open whenever G is open in $\mathcal{D}(X)$ and $F^{-1}(H)$ is closed whenever H is closed in $\mathcal{D}(X)$. Now replacing G by 2^U with U open decreasing, and H by 2^K with K closed decreasing, the required result follows immediately from Remark 1.1.1. Conversely, let G be a subbasic open set in $\mathcal{D}(X)$. Then we may assume that either $G = 2^U$ or $G = \mathcal{D}(X) - 2^{X-V}$, where U is open decreasing and V is open increasing in X. By assumption, $F^{-1}(2^U)$ is obviously open in X. In order to show $F^{-1}(\mathcal{D}(X)-2^{X-V}) = \text{Int } F^{-1}(\mathcal{D}(X)-2^{X-V})$, let $y \epsilon F^{-1}(\mathcal{D}(X)-2^{X-V})$, then $F(y) \epsilon \mathcal{D}(X) - 2^{X-V}$ or $y \notin F^{-1}(2^{X-V})$. Now, since $X-V$ is closed decreasing in X, $F^{-1}(2^{X-V})$ is closed in Y. Hence $y \epsilon Y - F^{-1}(2^{X-V}) =$

$$\text{Int}\,(Y-F^{-1}(2^{X-V})) = \text{Int}\,(F^{-1}(\mathcal{D}(X)) - F^{-1}(2^{X-V}))$$
$$= \text{Int}\,[F^{-1}(\mathcal{D}(X) - 2^{X-V})],$$

therefore F is continuous. This completes the proof.

1.1.7 <u>Corollary</u>: Let $(X, \mathcal{T}, \leq)$ be an ordered topological space and Y a topological space. Let $F_j : Y \to \mathcal{D}(X)$ be a mapping for $j = 0, 1$. Then the union of the two continuous functions $F = F_0 \cup F_1$ is continuous, where $(F_0 \cup F_1)(y) = F_0(y) \cup F_1(y)$ for each $y \in Y$.

<u>Proof</u>: Since $F^{-1}(2^U) = (F_0 \cup F_1)^{-1}(2^U) = F_0^{-1}(2^U) \cap F_1^{-1}(2^U)$, the result follows from Theorem 1.1.6.

1.1.8 <u>Definition</u>: An ordered topological space $(X, \mathcal{T}, \leq)$ is said to be <u>lower</u> (<u>upper</u>) <u>regularly ordered</u> if and only if for each decreasing (increasing) closed set $F \subseteq X$ and each element $a \notin F$, there exist disjoint open neighbourhoods U of a and V of F such that U is increasing (decreasing) and V is decreasing (increasing) in X. $(X, \mathcal{T}, \leq)$ is said to be <u>regularly ordered</u> if and only if X is both lower and upper regularly ordered.

1.1.9 <u>Lemma</u>: (McCartan [23]). Let $(X, \mathcal{T}, \leq)$ be an ordered topological space, then the following conditions are equivalent:

(1) X is lower (upper) regularly ordered,

(2) For each $x \in X$ and each increasing (decreasing) open neighbourhood U of x, there exists an increasing (decreasing) open neighbourhood V of x such that $\overline{V} \subseteq U$.

1.1.10 <u>Proposition</u>: Let $(X, \mathcal{T}, \leq)$ be a regularly ordered space. Then $(\mathcal{D}(X), \mathcal{T}^*, \cup)$ is a topological semilattice with its operation as the set union $\cup$.

Proof: In order to show $\mathcal{L}(X)$ is Hausdorff, let K, F be in $\mathcal{L}(X)$ and $x \in K-F$. Since X is a regularly ordered space, there exist open increasing neighbourhood U of x and open decreasing neighbourhood V of F such that $U \cap V = \phi$. Hence it is easy to show that $F \in 2^V$, $K \in \mathcal{L}(X) - 2^{X-U}$ and $2^V \cap (\mathcal{L}(X) - 2^{X-U}) = \phi$. Hence $\mathcal{L}(X)$ is a Hausdorff space. It is immediate from Corollary 1.1.7 that $\cup$ is continuous. Thus $(\mathcal{L}(X), \mathcal{J}^*, \cup)$ is a topological semilattice.

1.1.11 Theorem: Any regularly ordered C-space with a semicontinuous order, can be embedded into a bounded topological semilattice.

Proof: This is immediate from Theorem 1.1.4 and Proposition 1.1.10.

1.1.12 Remark: Let $(X, \mathcal{J}, \vee)$ be a topological semilattice. It is well known that $(X, \mathcal{J}, \leq)$ has a continuous order, where $x \leq y$ if and only if $x \vee y = y$ for each x, y in X. Moreover, the same statement holds for a topological semilattice $(X, \mathcal{J}, \wedge)$.

Section 2. Relative adjointness of the category of compact zero-dimensional semilattices.

1.2.1 Definition: (Priestley [30]). An ordered topological space $(X, \mathcal{J}, \leq)$ is said to be an I_d- space (resp. I_i-space) if, whenever a subset U of X is open, d(U) (resp. i(U)) is open.

$(X, \mathcal{J}, \leq)$ is said to be an I-space if it is both I_d-and I_i-space. We note that the concept of I-space coincides with the concepts of $*$-space and continuous space, of Green [12] and McCartan [24] respectively.

1.2.2 Proposition: The following statements hold:

(1) Let $(X_\alpha, \mathcal{J}_\alpha, \leq_\alpha)$ be I_d-spaces. Then the order product space $\prod_\alpha X_\alpha$ is an I_d-space.

(2) Let $(X, \mathcal{T}, \leq)$ be an I_d-space and let A be an open subset of X. Then the order subspace A is an I_d- space.

<u>Proof</u>: (1) Let U be an open set in ΠX_α. Then $U = \underset{\alpha}{\cup} U_\alpha$, where U_α is basic open in ΠX_α. It is easy to show that $d(U) = d(\underset{\alpha}{\cup} U_\alpha) = \underset{\alpha}{\cup} d(U_\alpha)$. Since U_α is basic open, $U_\alpha = \prod_{i=1}^{n} V_{\alpha_i} \times \prod_{j \neq i} X_{\alpha_j}$. If follows that $d(U_\alpha) = \prod_{i=1}^{n} d(V_{\alpha_i}) \times \prod_{j \neq i} X_{\alpha_j}$. Hence $d(U_\alpha)$ is open for each α. Therefore $d(U)$ is open in ΠX_α.

(2) Let U be an open set in A. Then U is an open set in X. We can easily show that $d_A(U) = d(U) \cap A$, where $d_A(U)$ is a decreasing set in A. Hence A is an I_d-space.

1.2.3 <u>Remarks:</u> (1) Dually, the above proposition holds for I_i-spaces.

(2) The category of all I_d-spaces (resp. I_i-spaces) and continuous increasing maps is open hereditary and productive category.

1.2.4 <u>Proposition</u>: Let $(X, \mathcal{T}, \vee)$ (resp. $(X, \mathcal{T}, \wedge)$) be a topological semilattice. Then $(X, \mathcal{T}, \leq)$ is an I_d-space (resp. I_i-space).

<u>Proof</u>: Let A be an open subset of X. Let (x_α) be a net in $X - d(A)$ with $x_\alpha \to x$ in X. Suppose that $x \notin X - d(A)$, that is, $x \in d(A)$, then $x \leq a$ for some $a \in A$. Hence $x \vee a = a \in A$. Since $x_\alpha \to x$ and $\vee$ is continuous, $x_\alpha \vee a \to a$; hence there exists x_α such that $x_\alpha \vee a \in A$. It follows that $x_\alpha \in d(A)$, which contradicts that $x_\alpha \in X - d(A)$; hence $x \in X - d(A)$. Therefore $d(A)$ is open.

1.2.5 <u>Remark</u>: A compact zero-dimensional space equipped with a continuous order need not be an I_d-space (resp. I_i-space):

<u>Example</u>: Let 2^α be the cantor cube with the cartesian order, where $2 = \{0, 1\}$ is a discrete topological space with $0 < 1$. Let $Y = \{y_0\} \cup 2^\alpha$,

where $y_0 \notin 2^\alpha$ and $\{y_0\}$ is an isolated point. Define $y_0 > (0, \ldots, 0, \ldots) = 0$ (i.e. the least element of 2^α) and y_0 is incomparable to all members of 2^α except 0. Then Y is obviously a compact zero-dimensional space equipped with a continuous order, but it is not an I_d-space. For, $d(y_0) = \{y_0, 0\}$ in Y and $d(y_0)$ is not open in Y. Dually, one can easily construct a similar counter example for I_i-space.

1.2.6 Theorem: Let $(X, \mathcal{T}, \leq)$ be a compact zero-dimensional I-space with continuous order.

Then $(\mathcal{D}(X), \mathcal{T}^*, \cup)$ is a compact zero-dimensional semilattice.

Proof: Since X is a compact space with a continuous order, X is clearly regularly ordered space. Hence, by Proposition 1.1.10, $(\mathcal{D}(X), \mathcal{T}^*, \cup)$ is a topological semilattice. Firstly, we show that $(\mathcal{D}(X), \mathcal{T}^*)$ is compact. Let $\{W_\alpha: \alpha\varepsilon\Gamma\}$ be a subbasic open covering of $\mathcal{D}(X)$. Then we may assume that either $W_\alpha = 2^{U_\alpha}$ or $W_\alpha = \mathcal{D}(X) - 2^{X-U_\alpha}$ for some open decreasing U_α or for some open increasing U_α, respectively. Hence, it follows immediately that $\{U_\alpha: \alpha\varepsilon\Gamma\}$ is an open covering of X. Since X is compact, there exist a finite subcovering $\{U_{\alpha_i}: i = 1, 2, \ldots, n\}$ of $\mathcal{D}(X)$. Hence, it follows that $\{W_{\alpha_i}: i = 1, 2, \ldots, n\}$ is a finite subcovering of $\mathcal{D}(X)$. Therefore $\mathcal{D}(X)$ is a compact space. Finally, we show that $(\mathcal{D}(X), \mathcal{T}^*)$ is a zero-dimensional space: let $B(U_0; U_1, \ldots, U_n)$ be an open neighbourhood of F in $\mathcal{D}(X)$. Then $F \subseteq U_0$ and $F \cap U_j \neq \phi$ for each $j = 1, 2, \ldots, n$. Since X is a compact zero-dimensional Hausdorff space, there exist clopen sets V_0, and V_j such that $F \subseteq V_0 \subseteq U_0$, and $V_j \subsetneq U_j$, $F \cap V_j \neq \phi$ for each $j = 1, 2, \ldots, n$. Hence, $d(V_0)$ and $i(V_j)$ are clopen sets in X, since X is an I-space. Hence, we can easily show that $B(d(V_0); i(V_1), \ldots, i(V_n))$ is a clopen neighbourhood of F in $\mathcal{B}$ (X) and $B(d(V_0); i(V_1), \ldots, i(V_n)) \subseteq B(U_0; U_1, \ldots, U_n)$. Hence, $\mathcal{B}$(X) is a zero-dimensional space. This completes the Proof.

1.2.7. Definition: (Ulmer [33]). Let $\underline{M}'$, $\underline{M}$ and $\underline{N}$ be categories, possibly illegitimate, and let $F: \underline{M}' \to \underline{N}$, $G: \underline{N} \to \underline{M}$ and $J: \underline{M}' \to \underline{M}$ be functors. F is called left adjoint to G relative to J if for each pair $N \epsilon \underline{N}$, $M' \epsilon \underline{M}'$ there is given a (adjunction) bijection

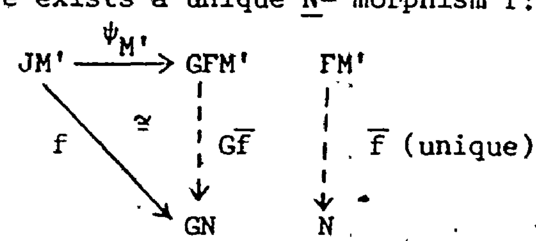
Ⓗ (M', N): $[FM', N] \cong [JM', GN]$, natural in N and M'. We may also say that F is J-left adjoint to G or that G has a J-left adjoint (namely F), or that F is the J-left adjoint of G.

If J is the identity functor of $\underline{M}$, then relative adjoints and adjoints coincide.

Ulmer [33] also showed that F is J-left adjoint to G if and only if there exists a natural transformation $\psi: J \to GF$ (the front adjunction) such that for each pair $M' \epsilon \underline{M}'$, $N \epsilon \underline{N}$ the composed map

$$[FM', N] \xrightarrow{G(FM', N)} [GFM', GN] \xrightarrow{[\psi(M'), GN]} [JM', GN]$$ is a bijective.

We note that this is equivalent to the following: For each $M' \epsilon \underline{M}'$ there exists $\underline{M}$-morphism $\psi_{M'}: JM' \to GFM'$ such that for any $N \epsilon N$ and any $\underline{M}$-morphism $f: JM' \to GN$, there exists a unique $\underline{N}$- morphism $\overline{f}: FM' \to N$ such that $G\overline{f} \circ \psi_{M'} = f$. i.e.



Let $\underline{CZI_d}$ = the category of all compact zero-dimensional I_d-spaces with continuous orders and continuous increasing maps.

CZI = the category of all compact zero-dimensional I-spaces with continuous orders and continuous increasing maps.

CZJS = the category of all compact zero-dimensional join semilattices with the least elements and continuous semilattice homomorphisms preserving the least elements to the corresponding ones.

Let J: CZI → CZI_d be the inclusion functor, and

let G: CZJS → CZI_d be the forgetful functor.

Define D: CZI → CZJS by the following:

X ⇝ D(X) = ($\mathcal{D}$(X), $\mathcal{J}^*$, ∪) for any X∈ob(CZI), and f: X → Y ⇝ D(f): D(X) → D(Y) defined by F ⇝ d(f(F)) for any F∈$\mathcal{D}$(X) = D(X) and for any CZI-morphism f: X → Y.

1.2.8 Lemma: D is a functor: CZI → CZJS.

Proof: By Theorem 1.2.6, D(X)∈ob(CZJS).

Firstly, we show that D(f) is a semilattice homomorphism preserving the least element. Let F, H be elements of $\mathcal{D}$(X), then D(f)(F∪H) =

$$d[f(F\cup H)] = d[f(F) \cup f(H)] = d(f(F)) \cup d(f(H))$$
$$= D(f)(F) \cup D(f)(H).$$

Secondly, we show that D(f) is continuous: let F∈$\mathcal{D}$(X) and $B(U_0;\ U_1, \ldots, U_n)$ basic open neighbourhood of D(f)(F), then $d(f(F)) \subseteq U_0$ and $d(f(F)) \cap U_i \neq \phi$ for each i = 1, 2, ..., n. Since f is continuous increasing map, it is immediate that $B(f^{-1}(U_0);\ f^{-1}(U_1), \ldots, f^{-1}(U_n))$ is an open set in $\mathcal{D}$(X). Since $d(f(F)) \subseteq U_0$, clearly $F \subseteq f^{-1}(U_0)$. Furthermore, $F \cap f^{-1}(U_i) \neq \phi$ for all i = 1, 2, ..., n. Hence, $F \in B(f^{-1}(U_0);\ f^{-1}(U_1), \ldots, f^{-1}(U_n))$. Also it is easy to see that

$D(f)\ [B(f^{-1}(U_0); f^{-1}(U_1), \ldots, f^{-1}(U_n))] \subseteq B(U_0; U_1, \ldots, U_n)$. Hence $D(f)$ is continuous. Therefore $D(f)$ is well defined. In order to show D is a functor, let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be continuous increasing maps, and let $F \in D(X)$. Then $(D(gf))(F) = d[g(f(F))]$, and $(D(g)D(f))(F) = D(g)[(D(f))(F)] = d[g(d(f(F)))]$. Let $z \in d[g(d(f(F)))]$. Then $z \leq g(k)$ for some $k \in d(f(F))$. Since $k \leq f(l)$ for some $l \in F$, $z \leq (gf)(l)$, that is, $z \in d[(gf)(F)]$. Therefore, $d[g(f(F))] = d[g(d(f(F)))]$. Hence, $D(gf) = D(g)\,D(f)$. Clearly, $D(1_X) = 1_{D(X)}$; thus D is a functor.

1.2.9. <u>Remark</u>: Let <u>COPS</u> be the category of all compact spaces equipped with continuous orders and continuous increasing maps, and let <u>CS</u> be the category of all compact join semilattices with least elements and continuous join semilattice homomorphisms preserving the least element. By the same argument as in the above lemma, we can show that D is also a functor: <u>COPS</u> $\rightarrow$ <u>CS</u>.

1.2.10. <u>Lemma</u>: (Lawson [21]): If S is a locally compact, totally diconnected topological semilattice, then S has small semilattices (i.e. every $s \in S$ has a neighbourhood basis consisting of subsemilattices).

1.2.11. <u>Lemma</u>: Let G: <u>CZJS</u> $\rightarrow$ <u>CZI_d</u> be the forgetful functor, and $S \in$ <u>CZJS</u> Then $\sup: \mathcal{D}(G(S)) \rightarrow S$ is continuous.

<u>Proof</u>: Let $F \in \mathcal{D}(G(S))$ and let $x = \sup(F)$ $(= \vee F)$. Let U be an open neighbourhood of x in S. Since S is a compact semilattice, there exist an open decreasing set V and an open increasing set W such that $x \in V \cap W \subset U$; we show that there exists an open subsemilattice L of S such that $x \in L \subset \overline{L} \subset V$. For, let O be a clopen set with $x \in O \subset V$. By Lemma 1.2.10, S has small semilattices. Hence, there exists an open

subsemilattice L of S such that $x \in L \subset O$. Hence it follows that $x \in L \subset \overline{L} \subset V$, where $\overline{L}$ is also a subsemilattice. Let K be a clopen set with $x \in K \subset L$ and let $V' = d(K)$. By Propositions 0.7 and 1.2.4, V' is clopen decreasing set and $x \in V' \subset d(\overline{L}) \subset V$. If follows that $F \in 2^{V'} (= \{B \in \mathcal{D}(G(S)) : B \subset V'\})$. Let $B \in \mathcal{D}(G(S))$ with $B \subset V'$, then $B \subset d(\overline{L})$. Since $\vee d(\overline{L}) = \vee\overline{L}$, $\vee B \leq \vee\overline{L}$. Now since $\vee\overline{L} \in \overline{L}$, it follows that $\vee B \in V$. Since W is an open decreasing neighbourhood of x, $F \in \{B \in \mathcal{D}(G(S)) : B \cap W \neq \phi\}$. Let $B \in \mathcal{D}(G(S))$ with $B \cap W \neq \phi$. Take $b \in B \cap W$. Since $b \leq \vee B$, $\vee B \in W$. Hence, we have shown that $B(V'; W)$ is an open neighbourhood of F and $\sup[B(V'; W)] \subset V \cap W \subset U$. Hence, sup is continuous.

1.2.12 <u>Theorem</u>: Let $J: \underline{CZI} \to \underline{CZI_d}$ be the inclusion functor, $G: \underline{CZJS} \to \underline{CZI_d}$ the forgetful functor, and let $D: \underline{CZI} \to \underline{CZJS}$ be the functor defined in Lemma 1.2.8.

Then D is a left adjoint to G relative to J.

<u>Proof</u>: Let $X \in \underline{CZI}$ and let $\eta_X: J(X) = X \to G(D(X))$ defined by $\eta_X(x) = d(x)$ for each $x \in X$. Then by Theorem 1.1.4, η_X is an embedding. For any $S \in \underline{CZJS}$ and any continuous increasing map $f: J(X) \to G(S)$, define $\overline{f}: D(X) \to S$ by $\overline{f}(F) = \vee(d(f(F)))$ for any $F \in D(X)$. Since S is compact semilattice, $\overline{f}$ is well defined and $\overline{f}$ is <u>CZJS</u>-morphism: $\overline{f}(F \cup H) =$

$$\vee[d(f(F \cup H))] = \vee[d(f(F)) \cup d(f(H))]$$
$$= (\vee d(f(F))) \vee (\vee d(f(H))) \text{ for } F, H \text{ in } D(X).$$

In order to show continuity of $\overline{f}$, we note that $\overline{f} = \sup \circ D(f)$, where $D(f): D(X) \to D(G(S))$ and $\sup: D(G(S)) \to S$.

By Lemmas 1.2.8 and 1.2.11, D(f) and sup are continuous. Hence, it follows immediately that $\overline{f}$ is continuous. We shall show that

$G(\overline{f})o\eta_X = f$, that is, $\overline{f}o\eta_X = f$. For, let $x\varepsilon X$, then it is easy to show that $d[f(d(x))] = d(f(x))$. Hence $(\overline{f}o\eta_X)(x) = \overline{f}(d(x)) = v(d[f(d(x))]) = v(d(f(x))) = f(x)$.

Now to see the uniqueness of $\overline{f}$, suppose that there exists $f'\colon D(X) \to S$ with $G(f')o\eta_X = f$, that is, $f'o\eta_X = f$. Then $\overline{f}o\eta_X = f'o\eta_X$. Let $F\varepsilon D(X)$. Then $F = \bigcup_{x\varepsilon F} d(x)$. Hence $\overline{f}(F) = \overline{f}(\bigcup_{x\varepsilon F} d(x)) = \bigcup_{x\varepsilon F} \overline{f}(d(x)) = \bigcup_{x\varepsilon F} \overline{f}(\eta_X(x))$

$$= \bigcup_{x\varepsilon F} f'(\eta_X(x)) = \bigcup_{x\varepsilon F} f'(d(x)) = f'(\bigcup_{x\varepsilon F} d(x)) = f'(F).$$

Therefore $\overline{f} = f'$. This completes the Proof.

1.2.13 Remark: Let CZMS denote the category of all compact zero-dimensional meet-semilattices with greatest elements and continuous semilattice homomorphisms preserving the greatest elements.

Let $E\colon$ CZI $\to$ CZMS be defined by the following:

$X \rightsquigarrow E(X) = (\mathcal{S}(X), \mathcal{T}_I^*, \cup)$ for each $X\varepsilon$CZI, and

$f\colon X \to Y \rightsquigarrow E(f)\colon \mathcal{S}(X) \to \mathcal{S}(Y)$ defined by $F \rightsquigarrow i(f(F))$ for $F\varepsilon\, \mathcal{S}(X)$ and any CZI-morphism f.

Then by analogy of Lemma 1.2.8, E is functor. Let $G'\colon$ CZMS $\to$ CZI_i be the forgetful functor, where CZI_i is the category of all compact zero-dimensional I_i spaces with continuous orders and continuous increasing maps. By a similar argument as that of the above theorem, we can dually show that E is a left adjoint to G' relative to J.

1.2.14 Remark: If the given order of all objects in the categories CZI_d, CZI_i and CZI, is in particular, discrete, then all these categories are the same as CZ, and J is the identity functor. Hence, D is left adjoint of G and E is left adjoint of G'. Hence, in this case these results reduce to the Hofmann, Mislove and Stralka's result (see 0.23).

Section 3. Existence of characters for compact zero-dimensional I_d-space.

1.3.1. Definition: For $X \in ZCI_d$, by a character of X we mean an element of $ZCI_d(X,2)$, where 2 is the two element discrete topological space $\{0,1\}$ with the order $0 < 1$.

1.3.2 Theorem: Let $(X, \mathcal{T}, \leq)$ be a compact zero-dimensional I_d-space with continuous order. Then there exists enough characters to separate the points of X.

Proof: Let $x \neq y$ in X. Then we may assume that $x \nleq y$. Since $\leq$ is a continuous order, there exist an open neighbourhood U of x and an open neighbourhood V of y such that $k \nleq \ell$ for any $k \in U$ and any $\ell \in V$. Since X is zero-dimensional, there is a clopen neighbourhood U' of x such that $U' \subseteq U$ and a clopen neighbourhood V' of y such that $V' \subseteq V$. It is easy to show that $k \nleq \ell$ for any $k \in U'$ and any $\ell \in V'$. Since X is compact I_d-space, $d(V')$ is a clopen set and does not contain x. Define $f: X \to 2$ by $f(x) = \begin{cases} 0 \text{ if } x \in d(V') \\ 1 \text{ otherwise} \end{cases}$

Then f is clearly continuous and f is increasing. For, let $a \leq b$ in X. Suppose that $f(a) = 1$, $f(b) = 0$. Then $b \in d(V')$, and hence $b \leq z$ for some $z \in V'$. Since $a \leq b$, $a \leq z$; hence $a \in d(V')$. Therefore $f(a) = 0$, which contradicts the assumption. If follows that f separates the points x and y.

1.3.3. Corollary: Let $(X, \mathcal{T}, \leq)$ be a compact zero-dimensional I-space with continuous order. Then there exists enough characters to separate the points of X.

1.3.4 <u>Corollary</u>: Let X be a compact zero-dimensional semilattice. Then there exist enough characters to separate the points of X.

1.3.5 <u>Theorem</u>: Let $(X, \mathcal{T}, \leq)$ be a compact zero-dimensional I_d-space with continuous order. Then X is a projective limit of finite discrete topological ordered spaces.

<u>Proof</u>: Let $\mathcal{F} = \{R_\alpha = Ker(f_\alpha) \mid f_\alpha: X \to Y_\alpha$ is a continuous increasing surjective mapping and Y_α is a finite discrete topological ordered space$\}$, where $Ker(f_\alpha) = \{(x,y) \in XxX: f_\alpha(x) = f_\alpha(y)\}$. Then $\mathcal{F}$ is a family of equivalence relations on X. Firstly, we show that $R_\alpha = Ker(f_\alpha)$ is closed and open for each f_α. For, R_α is obviously closed. We show that R_α is also open in XxX. Let (x_β, y_β) be a net in $XxX - R_\alpha$ with $(x_\beta, y_\beta) \to (x,y)$. Suppose that $(x,y) \notin XxX - R_\alpha$. Then $f_\alpha(x) = f_\alpha(y)$. It is easy to show that there exists an γ such that $f_\alpha(x_\gamma) = f_\alpha(x)$ and $f_\alpha(y_\gamma) = f_\alpha(y)$. Hence, $f_\alpha(x_\gamma) = f_\alpha(y_\gamma)$, that is, $(x_\gamma, y_\gamma) \in R_\alpha$; this is a contradiction. Therefore $XxX - R_\alpha$ is closed. Since X is compact Hausdorff, it follows immediately that X/R_α is a finite discrete space under the quotient topology. Define $[x] \leq_\alpha [y]$ if and only if $f_\alpha(x) \leq f_\alpha(v)$ for all [x] and [y] in X/R_α. Then it is easy to see that $\leq_\alpha$ is well defined and is a partial order. Hence $(X/R_\alpha, \leq_\alpha)$ is a finite discrete topological ordered space for all α. Secondly, let $\mathcal{F}'$ be the family of characters of X which separate the points of X. Then $\{R_\alpha = Ker(f_\alpha): f_\alpha \in \mathcal{F}'\}$ is a sub-family of $\mathcal{F}$. In order to show $\bigcap_{f_\alpha \in \mathcal{F}'} R_\alpha = \Delta$, let $(x,y) \notin \bigcap_{f_\alpha \in \mathcal{F}'} R_\alpha$. Suppose that $(x,y) \notin \Delta$, then $x \neq y$. Hence there exists an $f_\alpha \in \mathcal{F}'$ such that $f_\alpha(x) \neq f_\alpha(y)$. It follows that $(x,y) \notin Ker(f_\alpha) = R_\alpha$, that is, $(x,y) \notin \bigcap_{f_\alpha \in \mathcal{F}'} R_\alpha$, which is a contradiction.

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Hence it follows immediately that $\cap \{R_\alpha \mid R_\alpha \varepsilon \mathcal{F}\} = \Delta$. Finally, we show that $((R_\alpha)_{\alpha\varepsilon|\mathcal{F}|}, \subseteq)$ is a down directed set. For, let $R_\alpha, R_\beta \varepsilon \mathcal{F}$. Then $X/R_\alpha \cap R_\beta$ is a finite discrete space. Define $[x] \leq [y]$ if and only if $[x] \leq_\alpha [y]$ and $[x] \leq_\beta [y]$ for all $[x]$ and $[y]$ in $X/R_\alpha \cap R_\beta$. Then $\leq$ is well defined and $X/R_\alpha \cap R_\beta$ is clearly finite discrete topological space equipped with the partial order $\leq$. Hence the natural map f from X onto $X/R_\alpha \cap R_\beta$ is an increasing map and $R_\alpha \cap R_\beta = \mathrm{Ker}(f)$. Therefore $((R_\alpha)_{\alpha\varepsilon|\mathcal{F}|}, \subseteq)$ is a down directed set. Let X_α denote $(X/R_\alpha, \leq_\alpha)$. Then it is easy to see that X is a projective limit of $\{X_\alpha : \alpha\varepsilon|\mathcal{F}|\}$.

1.3.6 Remark: Dually, we can show that Theorems 1.3.2 and 1.3.5 hold for compact zero-dimensional I_i-space with continuous order.

CHAPTER II

WALLMAN TYPE ORDER COMPACTIFICATIONS

Section 1. Completely regular ordered spaces.

We recall that an ordered topological space $(X, \mathcal{J}, \leq)$ is completely regular ordered if and only if $\leq$ is a continuous order and the following two conditions are satisfied:

1. whenever $p \nleq q$ in X, there exists a continuous increasing function $f: X \to [0,1]$ such that $f(p) > f(q)$.

2. for every $p \in X$ and any open neighbourhood V of p, there exists a continuous increasing function $f: X \to [0,1]$ and a continuous decreasing function $g: X \to [0,1]$ such that $f(p) = 1 = g(p)$ and $X-V \subseteq f^{-1}(0) \cup g^{-1}(0)$.

2.1.1 Proposition: Let $(X, \mathcal{J}, \leq)$ be an ordered topological space with continuous order such that for every p, q in X with $p \nleq q$, there exists a continuous increasing map $f: X \to \mathbb{I}$ such that $f(p) > f(q)$. Then the following statements are equivalent:

1. $(X, \mathcal{J}, \leq)$ is a completely regular ordered space.

2. For every closed subset $A \subset X$ and $p \in X-A$, there exists a positive integer n and a continuous increasing function $f: X \to \mathbb{I}^n$ with $f(p) \notin \overline{f(A)}$ where the closure is taken in $\mathbb{I}^n$.

3. $(X, \mathcal{J}, \leq)$ is an $\mathbb{I}$-regular ordered space, i.e. it is iseomorphic to an (order) subspace of a power of $\mathbb{I}$.

Proof: (1) $\to$ (2): Let A be a closed subset of X and $p \in X-A$. Since X is completely regular ordered, there exist a continuous increasing function $f: X \to \mathbb{I}$ and a continuous decreasing function $g: X \to \mathbb{I}$

such that $f(p) = 1 = g(p)$ and $A \subseteq f^{-1}(0) \cup g^{-1}(0)$. Let $f' = 1 - g$, then f' is a continuous increasing function: $X \to \mathbb{I}$ such that $f'(p) = 0$. Define $h: X \to \mathbb{I}^2$ by $h(x) = (f(x), f'(x))$ for each $x \in X$. Then clearly h is a continuous increasing function. It is also easy to show that $h(p) \notin \overline{h(A)}$.

(2) → (3): Let $F = C_1(X,\mathbb{I})$ denote the set of all continuous increasing functions from X into $\mathbb{I}$. Let A be a closed subset of X with $P \in X-A$. By assumption, there exists a positive integer n and a continuous increasing function $f: X \to \mathbb{I}^n$ such that $f(p) \notin \overline{f(A)}$. We set $f_k = p_k \circ f$, where $1 \le k \le n$ and p_k is the projection of $\mathbb{I}^n$ onto its kth coordinate space $\mathbb{I}$. Hence, $f_k \in F$ for each $k = 1, 2, \ldots, n$. Furthermore, it is easy to see that $\{f_1, \ldots, f_n\}$ satisfies the following: $(f_1(p), \ldots, f_n(p)) \notin \overline{\{(f_1(a), \ldots, f_n(a)): a \in A\}}$, where the closure is taken in the product $\mathbb{I}^n$.

Define $\sigma: X \to \mathbb{I}^{|C_1(X, \mathbb{I})|}$ by $\sigma(x)(f) = f(x)$ for all $f \in F$ and each $x \in X$. Then it follows immediately that σ is an injective, increasing and continuous function. By theorem 0.12, $(X, \mathcal{J}, \le)$ is an $\mathbb{I}$-regular ordered space.

(3) → (1): Immediate from Proposition 0.10.

2.1.2 <u>Proposition</u>: Let $(X, \mathcal{J}, \le)$ be an ordered topological space with continuous order such that for every p, q in X with $p \not\le q$, there exists a continuous increasing function $f: X \to \mathbb{I}$ with $f(p) > f(q)$, then the following statements are equivalent:

1. $(X, \mathcal{J}, \le)$ is a completely regular ordered space.

2. For every net $\{x_\alpha: \alpha \in D\}$ in X, we have $x_\alpha \to x$ if and only if

$f(x_\alpha) \to f(x)$ for every $f \in C_1(X, \mathbb{I})$.

3. $(X, \mathcal{T}, \leq)$ is an $\mathbb{I}$-regular ordered space.

Proof: From the previous proposition, it is sufficient to show that (2) and (3) are equivalent:

(2) $\to$ (3): Let h be the parametric map corresponding to the family $C_1(X, \mathbb{I})$. Then it is easy to show that h is a continuous, one to one and increasing map. In order to show relative closedness of h, let A be a closed subset of X. Let $y \in \overline{h(A)} \cap h(X)$, then $y = h(x) \in \overline{h(A)}$ for some $x \in X$. Hence, there exists a net $\{y_\alpha\}$ in h(A) such that $y_\alpha \to y$. Since $y_\alpha = h(x_\alpha)$ for some $x_\alpha \in A$, $h(x_\alpha) \to h(x)$; hence it is easy to see that $f(x_\alpha) \to f(x)$ for every $f \in C_1(X, \mathbb{I})$. By hypothesis, we have $x_\alpha \to x$. Since A is closed in X, x belongs to A. Therefore, $h(x) \in h(A) \cap h(X)$. Hence, h is relatively closed. Consequently, h is an iseomorphism, and hence X is $\mathbb{I}$-regular ordered.

(3) $\to$ (2): Let $(X, \mathcal{T}, \leq)$ be an $\mathbb{I}$-regular ordered space. Then it is sufficient to show that for some net $\{x_\alpha : \alpha \in D\}$ in X, $x_\alpha \not\to x$ if and only if $f(x_\alpha) \not\to f(x)$ for some $f \in C_1(X, \mathbb{I})$. If $x_\alpha \not\to x$ then there exists a neighbourhood U of x such that for any $\alpha_0 \in D$, there exists a $\beta \geq \alpha_0$ such that $x_\beta \notin U$. Let E be the set of all $\beta \in D$ such that $x_\beta \notin U$. Then it is easy to see that E is a cofinal subset of D; hence $\{x_\beta : \beta \in E\}$ is a subnet of $\{x_\alpha\}$. Therefore $x \notin \overline{\{x_\beta\}}$ in X. Since $(X, \mathcal{T}, \leq)$ is $\mathbb{I}$-regular ordered, by the previous proposition, there exists a positive integer n and a continuous increasing function $f: X \to \mathbb{I}^n$ such that $f(x) \notin \overline{\{f(x_\beta)\}}$. Hence $f(x_\beta) \not\to f(x)$. Therefore, there exists

a k such that $1 \leq k \leq n$ and $(p_k o f)(x_\beta) \not\to (p_k o f)(x)$, where p_k is the projection on the kth coordinate space. But $p_k o f \in C_1(X, \mathbb{I})$. This completes the proof.

2.1.3 <u>Corollary</u>: Let $(X, \mathcal{T}, \leq)$ be an ordered topological space with continuous order. Then under the same assumptions of the previous Proposition on X, the following statements are equivalent:

1. $(X, \mathcal{T}, \leq)$ is a completely regular ordered space.

2. For every closed subset $A \subset X$ and $p \in X-A$, there exists a positive integer n and a continuous increasing function $f: X \to \mathbb{I}^n$ with $f(p) \notin \overline{f(A)}$.

3. For every net $\{x_\alpha : \alpha \in D\}$ of points of X, we have $x_\alpha \to x$ if and only if $f(x_\alpha) \to f(x)$ for every $f \in C_1(X, \mathbb{I})$.

4. $(X, \mathcal{T}, \leq)$ is an $\mathbb{I}$-regular ordered space.

5. $(X, \mathcal{T}, \leq)$ is an $\mathbb{R}$-regular ordered space, i.e. it is iseomorphic to a subspace of a power of $\mathbb{R}$.

6. For any point p of X and any open neighbourhood V of x, there exist finitely many continuous increasing maps $f_1, f_2, \ldots, f_n$: $X \to [-1,1]$ such that $f_i(p) = 0$ for each $i = 1, 2, \ldots, n$ and $X-V \subseteq \cup f_i^{-1}(\{-1,1\})$, where $[-1,1]$ is endowed with the usual order and the usual topology.

7. $\{f^{-1}(0)$: for all continuous increasing maps f from $(X, \mathcal{T}, \leq)$ into $\mathbb{R}\}$ forms a subbase of the closed sets of $\mathcal{T}$.

<u>Proof</u>: $(1) \leftrightarrow (2) \leftrightarrow (3) \leftrightarrow (4)$ is obvious from the previous two Propositions. $(4) \leftrightarrow (5) \leftrightarrow (6) \leftrightarrow (7) \leftrightarrow (1)$ are from Hong ([18]).

Section 2. $\mathbb{R}$ -compact ordered spaces.

In [5], Choe and Hong raised the following problem: What are the characterizations of $\mathbb{R}$ -compact ordered spaces and $\mathbb{R}$ -compact topological lattices? In this section, we answer this question by obtaining a few characterizations. Throughout this section, all ordered topological spaces which we consider are assumed to have a continuous order.

2.2.1 Definition: (Rodriquez [31], Choe and Hong [5]). An ordered topological space $(X, \mathcal{T}, \leq)$ is said to be an $\mathbb{R}$ -compact ordered space, if it is iseomorphic with a closed subspace of a power of $\mathbb{R}$.

2.2.2 Theorem: Let $(X, \mathcal{T}, \leq)$ be a completely regular ordered space. Then X is an $\mathbb{R}$ -compact ordered space if and only if there does not exist a completely regular ordered space $(X', \mathcal{T}', \leq')$ satisfying the following two conditions:

1. There exists an iseomorphism $e: X \to e(X) \subset X'$ such that $e(X) \neq \overline{e(X)} = X'$.

2. For every continuous increasing map $f: X \to \mathbb{R}$, there exists a continuous increasing map $\overline{f}: X' \to \mathbb{R}$ such that $\overline{f} \circ e = f$.

Proof: $(\Leftarrow)$: Since $(X, \mathcal{T}, \leq)$ is a completely regular ordered space, by Corollary 2.1.3, there exists an iseomorphism h of X into a subspace of a power of $\mathbb{R}$, that is, $h: X \to \mathbb{R}^m$, where $m = |C_1(X, \mathbb{R})|$ and $C_1(X, \mathbb{R})$ is the set of all continuous increasing maps from X into $\mathbb{R}$.

Hence by hypothesis, we can easily show that $h(X) = \overline{h(X)}$.

($\rightarrow$): Since X is an $\mathbb{R}$-compact ordered space, we assume that X is a closed subspace of a product $\prod_{k\in K} \mathbb{R}_k$ of $\mathbb{R}_k$, where $\mathbb{R}_k = \mathbb{R}$ for all $k\in K$. Let $e(: X \to X')$ denote embedding X into a completely regular ordered space X' which satisfies the condition (2). We may assume without loss of generality that $\overline{e(X)} = X'$. Hence, for every $k\in K$, there exists a continuous increasing map $P_k'\colon X' \to \mathbb{R}$ such that $P_k' \circ e = P_k$, where P_k is the kth projection of X. Let $F\colon X' \to \prod_{k\in K} \mathbb{R}_k$ be the parametric mapping determined by the family $\{P_k'\colon k\in K\}$. Then F is obviously a continuous and increasing function.

Since $(F\circ e)(x) = F(e(x)) = (P_k'(e(x)))_{k\in K}$

$$= (P_k' \circ e(x))_{k\in K} = (P_k(x))_{k\in K} = x,$$

$(F\circ e)(X) = X$. Hence $F(X') = F(\overline{e(X)}) \subseteq \overline{F(e(X))} = \overline{X} = X$, so that F is a function from X' into X. Hence, it is easy to show that $e\circ F\colon X' \to X'$ is equal to the identity on e(X). Since e(X) is dense in X', by the principle of the extension of identitites (Bourbaki [2]), $e\circ F = id_{X'}$. Since $F(X') \subseteq X$, $(e\circ F)(X') \subseteq e(X)$; therefore $X' \subseteq e(X)$, and hence $e(X) = X'$. Thus no completely regular ordered space X' satisfying the condition (2) satisfies the condition (1). This completes the proof.

2.2.3 <u>Definition</u>: Let $(X, \mathcal{J}, \leq)$ be an ordered topological space, and let A be a (order) subspace of X. Then a continuous increasing map $f\colon A \to \mathbb{R}$ is said to be <u>orderly extended over X</u> (or <u>order extendable on X</u>), if there exists a continuous increasing map $\overline{f}\colon X \to \mathbb{R}$ such that $\overline{f}|A = f$.

2.2.4 Lemma: Let $(X, \mathcal{T}, \leq)$ be an ordered topological space and let A be a subspace of $(X, \mathcal{T}, \leq)$. If every continuous increasing map $g: A \to \mathbb{R}$ such that either $g(x) \geq 1$ for all $x \in A$ or $g(x) \leq -1$ for all $x \in A$, is orderly extended over X, then every continuous increasing map $f: A \to \mathbb{R}$ can be orderly extended over X.

Proof: Let $f: A \to \mathbb{R}$ be an arbitrary continuous increasing function. Let $g_1(x) = 1 + \max(f(x), 0)$ and

$$g_2(x) = 1 - \min(f(x), 0).$$

Then g_1 is a continuous increasing and g_2 is a continuous decreasing map from A into $\mathbb{R}$. Moreover, $g_i(x) \geq 1$ for $i = 1, 2$ and for all $x \in A$. Hence $-g_2$ is a continuous increasing map and $-g_2(x) \leq -1$. By hypothesis, there exists continuous increasing maps $\overline{g}_1$ and $\overline{g}_2: X \to \mathbb{R}$ such that $\overline{g_1}|A = g_1$ and $\overline{g_2}|A = -g_2$. Since $f(x) = g_1(x) - g_2(x)$ for every $x \in A$, let us define $\overline{f}: X \to \mathbb{R}$ by $\overline{f}(x) = \overline{g_1}(x) + \overline{g_2}(x)$ for each $x \in X$, then $\overline{f}$ is a continuous increasing map and $\overline{f}|A = f$. Hence $\overline{f}$ is the required extension of f.

2.2.5 Remark: Let $(X, \mathcal{T}, \leq)$ be an ordered topological space, and let A be a subspace of X. If every continuous increasing map of A into $\mathbb{R}$ can be orderly extended over X, then we have that every continuous decreasing map of A into $\mathbb{R}$ can also be orderly extended over X.

2.2.6 Lemma: Let $(X, \mathcal{T}, \leq)$ be an ordered topological space, and let A be a subspace of $(X, \mathcal{T}, \leq)$. If every continuous increasing map $f: A \to \mathbb{R}$ is order extendable over X, then every continuous increasing map $f: A \to \prod_{k\in K} \mathbb{R}_k$, where $\mathbb{R}_k = \mathbb{R}$ for all $k\in K$, is also orderly extended over X.

Moreover, if $\overline{A} = X$, then every continuous increasing map $f: A \to B = \overline{B} \subset \prod_{k\in K} \mathbb{R}_k$ into a closed subspace of this product is also orderly extended over X.

Proof: Let $f: A \to \prod_{k\in K} \mathbb{R}_k$ be an arbitrary continuous increasing function. Then obviously $P_k of: A \to \mathbb{R}_k$ is a continuous increasing map, where P_k is the projection of $\prod_{k\in K} \mathbb{R}_k$. By hypothesis, there exists a continuous increasing map $\overline{f}_k: X \to \mathbb{R}_k$ such that $\overline{f}_k|A = P_k of$. Let $\overline{f}: X \to \prod_{k\in K} \mathbb{R}_k$ be parametric map determined by the family $\{\overline{f}_k: k\in K\}$. Then $\overline{f}$ is obviously a continuous increasing function and $\overline{f}|A = f$.

If $\overline{A} = X$ and $f(A) \subset B = \overline{B} \subset \prod_{k\in K} \mathbb{R}_k$, then there exists a continuous increasing map $\overline{f}: X \to \prod_{k\in K} \mathbb{R}_k$ such that $\overline{f}|A = f$. Hence, we have $\overline{f}(X) = \overline{f}(\overline{A}) \subset \overline{\overline{f}(A)} = \overline{f(A)} \subset \overline{B} = B$, so that $\overline{f}: X \to B$ is the required function.

Let $\beta_1 X$ be the Stone-Čech order compactification (see 0.11) and let $\beta_1: X \to \beta_1 X$ be a dense embedding. Then we have the following theorem:

2.2.7 Theorem: Let $(X, \mathcal{T}, \leq)$ be a completely regular ordered space. Then X is an $\mathbb{R}$-compact ordered space if and only if for every $x_o \in \beta_1 X - \beta_1(X)$, there exists a continuous decreasing map $h: \beta_1 X \to [-1,1]$ such that either $h(x_o) = 0$ and $h(x) > 0$ for all $x\in\beta_1(X)$ or $h(x_o) = 0$ and $h(x) < 0$ for all $x\in\beta_1(X)$, where $[-1,1]$ is an ordered topological space equipped with the usual topology and the usual order.

<u>Proof</u>: $(\rightarrow)$: Suppose that X is an $\mathbb{R}$-compact ordered space. Take an arbitrary point $x_o \varepsilon \beta_1 X - \beta_1(X)$. Since $X' = \beta_1(X) \cup \{x_o\} \subset \beta_1 X$ is a completely regular ordered space satisfying condition (1) in Theorem 2.2.2, there exists a continuous increasing map $f: \beta_1(X) \rightarrow \mathbb{R}$ which is not orderly extended over X'. By Lemma 2.2.4, we may assume that either $f(x) \geq 1$ for all $x \varepsilon \beta_1(X)$ or $f(x) \leq -1$ for all $x \varepsilon \beta_1(X)$. Hence, we consider the following two cases:

Case 1: $f(x) \geq 1$ for all $x \varepsilon \beta_1(X)$. Let $h_o: \beta_1(X) \rightarrow [-1,1]$ be the function determined by $h_o(x) = [f(x)]^{-1}$. Then h_o is a continuous decreasing map. Hence, by Remark 2.2.5, h_o is orderly extended over $\beta_1 X$. Let $h: \beta_1 X \rightarrow [-1,1]$ be its extension. Assume that $h(x_o) \neq o$. Then the continuous increasing map $f: \beta_1(X) \rightarrow \mathbb{R}$ would be orderly extended to the function $f': X' \rightarrow \mathbb{R}$ defined by means of the formula $f'(x) = [h(x)]^{-1}$, which is a contradiction. Hence, $h(x_o) = o$. Moreover, $h(x) > o$ for all $x \varepsilon \beta_1(X)$.

Case 2. $f(x) \leq -1$ for all $x \varepsilon \beta_1(X)$. By a similar method to that of case 1, we can obtain a continuous decreasing map $h: \beta_1 X \rightarrow [-1,1]$ such that $h(x_o) = o$ and $h(x) < o$ for all $x \varepsilon \beta_1(X)$.

$(\leftarrow)$: Assume that X is not an $\mathbb{R}$-compact ordered space. Then by Theorem 2.2.2, there exists a completely regular ordered space X' containing X as a dense proper ordered subset and such that every continuous increasing map $g: X \rightarrow \mathbb{R}$ can be orderly extended over X'. We may assume without loss of generality that $X' - X = \{x'\}$. By the construction of $\beta_1 X$ (see 0.11), $\beta_1 X \subset \mathbb{I}^{|C_1(X,\mathbb{I})|} \subset \prod_{k \varepsilon K} \mathbb{R}_k$, where $\mathbb{R}_k = \mathbb{R}$ for every $k \varepsilon K$ and $|K| = |C_1(X,\mathbb{I})|$. Thus by Lemma 2.2.6, there exists an extension $\beta_1': X' \to \beta_1 X$ (i.e. β_1' is continuous increasing and $\beta_1' | X = \beta_1$) of the continuous increasing map $\beta_1: X \to \beta_1 X$, and it is easy to show that $\beta_1'(x') \in \beta_1 X - \beta_1(X)$. Let x_o denote $\beta_1'(x')$, then we shall show that for any continuous decreasing map $h: \beta_1 X \to [-1,1]$ such that either $h(x) > o$ for all $x \in \beta_1(X)$ or $h(x) < o$ for all $x \in \beta_1(X)$, then $h(x_o) \neq o$. We consider the following two cases:

Case 1. $h(x) > o$ for all $x \in \beta_1(X)$: For every continuous decreasing map $h: \beta_1(X) \to [-1,1]$ which satisfies $h(x) > o$ for all $x \in \beta_1(X)$, the formula $g(x) = [h \circ \beta_1(x)]^{-1}$ defines a continuous increasing map $g: X \to \mathbb{R}$. Since $g(x) \geq 1$ for $x \in X$, we have $g'(x) \geq 1$ for the extension $g': X' \to \mathbb{R}$. But the continuous decreasing maps $(g')^{-1}$ and $h \circ \beta_1'$ are identical on the space X. Hence $h(x_o) = h(\beta_1'(x')) = [g'(x')]^{-1} > o$.

Case 2. $h(x) < o$ for all $x \in \beta_1(X)$: By the same arguments as in case 1, we can obtain $h(x_o) < o$. In either case, $h(x_o) \neq o$. Thus X is an $\mathbb{R}$-compact ordered space.

2.2.8 Remark: In [5], Choe and Hong constructed a compact ordered space $\beta_o X$ for any completely regular ordered space X, as follows: $\beta_o X$ is the set of all maximal o-completely regular filters[1] on X, endowed with the topology generated by $\{U^*: U^* = \{\mathfrak{m} \in \beta_o X: U \in \mathfrak{m}\}$, U is an open set of X$\}$ and a relation $\leq$ defined by: $\mathfrak{m} \leq \mathfrak{n}$ in $\beta_o X$ if and only if $\lim f(\mathfrak{m}) \leq \lim f(\mathfrak{n})$ for all $f \in C_o(X)$, where $C_o(X)$ is the family of all

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[1] Let X be a completely regular ordered space. A filter $\mathcal{F}$ on X is said to be o-completely regular if $\mathcal{F}$ has an open base $\mathcal{B}$ with the property that for each $U \in \mathcal{B}$, there exists a $V \in \mathcal{B}$, with $V \subseteq U$ and there exist finitely many continuous increasing maps $f_1, \ldots, f_n: X \to [-1,1]$ such that $f_i(V) = o$ for each $i = 1, 2, \ldots, n$ and X-U is contained in $\cup f_i^{-1}(\{-1,1\})$.
By a maximal o-completely regular filter on X is meant an o-completely regular filter not contained in any other o-completely regular filter.

continuous increasing maps from X into [-1,1]. They showed that $\beta_0 X$ is a [-1,1]-extendable[2] compactification of X.

2.2.9 <u>Proposition</u>: The two constructions $\beta_1 X$ and $\beta_0 X$ are order equivalent in the following sense: given

$$\begin{array}{ccc} X & \xrightarrow{\beta_1} & \beta_1 X \\ {\scriptstyle \beta_0}\downarrow & \swarrow {\scriptstyle \overline{\beta}_0} & \\ \beta_0 X & & \end{array}$$

where β_0, β_1 are dense embeddings, there exists an iseomorphism $\overline{\beta}_0 \colon \beta_1 X \to \beta_0 X$ such that $\overline{\beta}_0 o \beta_1 = \beta_0$.

<u>Proof</u>: Given

$$\begin{array}{ccc} X & \xrightarrow{\beta_1} & \beta_1 X \\ {\scriptstyle \beta_0}\downarrow & & \\ \beta_0 X & & \end{array}$$

, by (0.11) there exists a unique continuous increasing map $\overline{\beta}_0 \colon \beta_1 X \to \beta_0 X$ such that $\overline{\beta}_0 o \beta_1 = \beta_0$. Since $\beta_1 X$ is a compact ordered space, $\beta_1 X$ is iseomorphic to a closed subspace of a power of $\mathbb{I}$. Since $\mathbb{I}$ and [-1,1] are obviously isemorphic, $\beta_1 X$ is iseomorphic to a closed subspace of a power of $[-1,1]^{|C_1(\beta_1 X)|}$, where $C_1(\beta_1 X)$ is the family of all continuous increasing maps from $\beta_1 X$ into [-1,1]. Since $P_f o \beta_1 \colon X \to [-1,1]$ is a continuous increasing map (where P_f is the fth projection) and $\beta_0 X$ is [-1,1]-extendable, there exists a unique continuous increasing map $\overline{P}_f \colon \beta_0 X \to [-1,1]$ for each $f \in C_1(\beta_1 X)$ such that $\overline{P}_f | X = P_f o \beta_1$ and $\beta_0 o \overline{P}_f = P_f o \beta_1$. Let $\overline{\beta}_1 \colon \beta_0 X \to [-1,1]^{|C_1(\beta_1 X)|}$ be the parametric map determined by the family $\{\overline{P}_f \colon f \in C_1(\beta_1(X))\}$.

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[2] Let <u>A</u> be a category and C an object of <u>A</u>. An <u>A</u>-morphism f: A → B is said to be <u>C-extendable</u> if for any <u>A</u>-morphism g: A → C, there is an <u>A</u>-morphism $\overline{g}$: B → C with g = $\overline{g}$of.

Let $\overline{\beta}_1$ denote $\prod_f \overline{P}_f$, that is, $\overline{\beta}_1(x) = (\prod_f \overline{P}_f)(x) = (\overline{P}_f(x))_{f\in C_1(\beta_1 X)}$ for each $x\in X$. Hence we have $\prod_f (P_f o \beta_1) = \prod_f (\overline{P}_f|X)$. Since

$$\overline{\beta}_1(\beta_0(X)) \subset \overline{\overline{\beta}_1(\beta_0(X))} = \overline{(\prod_f \overline{P}_f)(\beta_0(X))} = \overline{\prod_f (\overline{P}_f o \beta_0(X))}$$
$$= \overline{\prod_f P_f o \beta_1(X)} = \overline{\beta_1(X)} = \beta_1 X,$$

it follows that $\overline{\beta}_1 \colon \beta_0 X \to \beta_1 X$ is a continuous increasing function.

Since $(\overline{\beta}_1 o \overline{\beta}_0)(X) = (\prod_f \overline{P}_f)(\beta_0(X)) = \prod_f ((\overline{P}_f o \beta_0)(X))$

$$= \prod_f ((P_f o \beta_1)(X) = \beta_1(X) = 1_{\beta_1 X}(X),$$

we have $(\overline{\beta}_1 o \overline{\beta}_0)|X = 1_{\beta_1 X}|X$. Similarly, $(\overline{\beta}_0 o \overline{\beta}_1)|X = 1_{\beta_0 X}|X$. It follows that $\beta_0 X$ and $\beta_1 X$ are iseomorphic.

2.2.10 <u>Definition</u>: A <u>topological lattice</u> is a lattice which is a Hausdorff topological space in such a way that the binary operations sup, $\vee$, and inf, $\wedge$, are continuous.

The following definitions are due to Choe and Hong [5].

2.2.11 <u>Definition</u>: A topological lattice L is said to be <u>completely regular</u>, if for any point x of L and an open neighbourhood U of x, there exist finitely many continuous homomorphisms $f_1, f_2, \ldots, f_n$: $L \to [-1,1]$ such that $f_i(x) = o$ for each $i = 1, 2, \ldots, n$ and $X-U \subseteq \cup \{f_i^{-1}(\{-1,1\}) \colon 1 \le i \le n\}$.

2.2.12 <u>Definition</u>: Let L be a topological lattice. Then a topological lattice is said to be <u>L-regular</u> (resp. <u>L-compact</u>), if it is iseomorphic with a (resp. closed) sublattice of a power of L.

2.2.13 <u>Lemma</u>: (Choe and Hong [5]). For a topological lattice L, these are equivalent:

1. L is a completely regular topological lattice.

2. $\{f^{-1}(o): f\varepsilon \mathrm{Hom}(L, \mathbb{R})\}$ forms a subbase for closed sets, where Hom $(L, \mathbb{R})$ is the set of all continuous lattice homomorphisms of L into $\mathbb{R}$.

3. L is an $\mathbb{R}$-regular topological lattice

4. L is an $\mathbb{I}$-regular topological lattice.

2.2.14 Theorem: Let $(X, \vee_X, \wedge_X, \mathcal{T}_X)$ be a completely regular topological lattice. Then X is an $\mathbb{R}$-compact topological lattice if and only if there does not exist a completely regular topological lattice $(X', \vee_X', \wedge_X', \mathcal{T}_X')$ which satisfies the following two conditions:

1. There exists an iseomorphism $e: X \to e(X) \subset X'$ such that $e(X) \neq \overline{e(X)} = X'$.

2. For every continuous homomorphism $f: X \to \mathbb{R}$, there exists a continuous homomorphism $\overline{f}: X' \to \mathbb{R}$ such that $\overline{f}oe = f$.

Proof: By using the same arguments as those in Theorem 2.2.2, we can prove the theorem.

Section 3. Wallman type order compactifications.

Let $(X, \mathcal{T}, \leq)$ be an ordered topological space. Let $\mathcal{U}$ and $\mathcal{L}$ be the upper and lower topologies on X, namely, $\mathcal{U} = \{U\varepsilon \mathcal{T}: U = i(U)\}$ and $\mathcal{L} = \{U\varepsilon \mathcal{T}: U = d(U)\}$ (see 0.6).

We recall that a filter $\mathcal{F}$ in a topological space $(X, \mathcal{T})$ is an open (resp. closed) filter if $\mathcal{F}$ has a filter base consisting only of open (resp. closed) sets.

2.3.1 Definition: Let $(X, \mathcal{T}, \leq)$ be an ordered topological space. Let $\mathcal{F}$ be a closed filter in the upper topological space $(X, \mathcal{U})$ and $\mathcal{G}$ a closed filter in the lower topological space $(X, \mathcal{L})$. A pair $(\mathcal{F}, \mathcal{G})$ of filters $\mathcal{F}$ and $\mathcal{G}$ is said to be a bi-filter on X, if $F \cap G \neq \phi$ for any $F \in \mathcal{F}$ and any $G \in \mathcal{G}$.

For given two bi-filters $(\mathcal{F}_1, \mathcal{G}_1)$ and $(\mathcal{F}_2, \mathcal{G}_2)$, we define a relation $(\mathcal{F}_1, \mathcal{G}_1) \subseteq (\mathcal{F}_2, \mathcal{G}_2)$ if and only if $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and $\mathcal{G}_1 \subseteq \mathcal{G}_2$.

By a maximal bi-filter on X we mean a bi-filter not contained in any other bi-filter under the above relation.

2.3.2 Remark: It is easy to see, by Zorn's lemma, that every bi-filter is contained in a maximal bi-filter.

For an ordered topological space $(X, \mathcal{T}, \leq)$, we write

$$\Gamma_{\mathcal{U}} X = \{A \subseteq X: A \text{ is a closed decreasing set}\}, \text{ and}$$

$$\Gamma_{\mathcal{L}} X = \{A \subseteq X: A \text{ is a closed increasing set}\}.$$

2.3.3 Lemma: Let $(\mathcal{F}, \mathcal{G})$ be a maximal bi-filter, and $A \in \Gamma_{\mathcal{U}} X$. Then $A \in \mathcal{F}$ if and only if given $F \in \mathcal{F}$, $G \in \mathcal{G}$, we have $A \cap F \cap G \neq \phi$. A dual statement holds for $\mathcal{G}$.

Proof: $(\rightarrow)$: Trivial

$(\leftarrow)$: Let $\mathcal{B}_{\mathcal{F}}$ be a filter base of $\mathcal{F}$, which consists only of decreasing closed sets. By the hypothesis, $A \cap B \neq \phi$ for all $B \in \mathcal{B}_{\mathcal{F}}$. Let $\mathcal{F}'$ be the filter generated by a family $\{A \cap B: B \in \mathcal{B}_{\mathcal{F}}\}$. Then $\mathcal{F}'$ is evidently a closed filter on $(X, \mathcal{U})$ and $\mathcal{F} \subseteq \mathcal{F}'$. Since $(\mathcal{F}', \mathcal{G})$ is clearly a bi-filter and $(\mathcal{F}', \mathcal{G}) \supseteq (\mathcal{F}, \mathcal{G})$, it follows that $(\mathcal{F}, \mathcal{G}) = (\mathcal{F}', \mathcal{G})$

by the maximality of $(\mathcal{F}, \mathcal{G})$. Hence $\mathcal{F} = \mathcal{F}'$, this is $A \varepsilon \mathcal{F}$.

2.3.4 Lemma: Let $(\mathcal{F}, \mathcal{G})$ be a maximal bi-filter. Then the following statements hold:

1. Let A_1 and A_2 be in $\Gamma_{\mathcal{U}} X$ and $A_1 \cup A_2 \varepsilon \mathcal{F}$.

   Then either $A_1 \varepsilon \mathcal{F}$ or $A_2 \varepsilon \mathcal{F}$.

   A dual statement holds for $\mathcal{G}$.

2. Let $A \varepsilon \Gamma_{\mathcal{U}} X$, $B \varepsilon \Gamma_{\mathcal{L}} X$ and $A \cup B = X$.

   Then either $A \varepsilon \mathcal{F}$ or $B \varepsilon \mathcal{G}$.

Proof: (1): Assume that $A_1 \notin \mathcal{F}$ and $A_2 \notin \mathcal{F}$. By Lemma 2.3.3, there exist $F_1, F_2 \varepsilon \mathcal{F}$ and $G_1, G_2 \varepsilon \mathcal{G}$ such that $A_1 \cap F_1 \cap G_1 = \phi$ and $A_2 \cap F_2 \cap G_2 = \phi$. Hence $(A_1 \cup A_2) \cap (F_1 \cap F_2) \cap (G_1 \cap G_2) = \phi$. It follows that $(A_1 \cup A_2) \notin \mathcal{F}$.

(2): Suppose that $A \notin \mathcal{F}$ and $B \notin \mathcal{G}$. Then $A \cap F_1 \cap G_1 = \phi$ and $B \cap F_2 \cap G_2 = \phi$ for some F_1, F_2 in $\mathcal{F}$ and for some G_1, G_2 in $\mathcal{G}$. Since $(\mathcal{F}, \mathcal{G})$ is a bi-filter, $(F_1 \cap F_2) \cap (G_1 \cap G_2) \neq \phi$. It is then easy to see that $A \cup B \neq X$.

2.3.5 Remark: Let $(X, \mathcal{T}, \leq)$ be an ordered topological space with semicontinuous order. For each $x \varepsilon X$, we write

$$\varphi(d(x)) = \{A \text{ is a subset of } X: d(x) \subseteq A\}, \text{ and}$$

$$\varphi(i(x)) = \{A \text{ is a subset of } X: i(x) \subseteq A\}.$$

Then every $\varphi(d(x))$ is clearly a closed filter, but it need not be a maximal closed filter in $(X, \mathcal{U})$ under the inclusion relation. See the example below. Moreover, a dual statement holds for $\varphi(i(x))$.

Example: Let $N = \{0, 1, 2\}$ be an ordered topological space equipped with the usual order and discrete topology, i.e. $\mathcal{T} = \{\phi, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$. Then $\mathcal{U} = \{\phi, \{2\}, \{1,2\}, \{0,1,2\}\}$, and $\mathcal{L} = \{\phi, \{0\}, \{0,1\}, \{0,1,2\}\}$.

Since $\varphi(d(0)) = \varphi(\{0\}) = \{\{0\}, \{0,1\}, \{0,2\}, \{0,1,2\}\}$,

$\varphi(d(1)) = \varphi(\{0,1\}) = \{\{0,1\}, \{0,1,2\}\}$, and

$\varphi(d(2)) = \varphi(\{0,1,2\}) = \{\{0,1,2\}\}$, it follows that $\varphi(d(2)) \subsetneq \varphi(d(1)) \subsetneq \varphi(d(0))$. Hence $\varphi(d(2))$ and $\varphi(d(1))$ are not maximal closed filters in $(N, \mathcal{U})$. We note that, if the given order on N is discrete, then $\varphi(d(x))$ is a maximal closed filter.

2.3.6 Lemma: Let $(X, \mathcal{T}, \leq)$ be an ordered topological space with semicontinuous order. Then for each $x \in X$, $(\varphi(d(x)), \varphi(i(x)))$ is a maximal bi-filter.

Proof: Let $A \in \varphi(d(x))$ and $B \in \varphi(i(x))$. Then $d(x) \subseteq A$ and $i(x) \subseteq B$. Hence $A \cap B \neq \phi$. Therefore $(\varphi(d(x)), \varphi(i(x)))$ is a bi-filter. Suppose that there exists a bi-filter $(\mathcal{F}, \mathcal{G})$ such that $(\varphi(d(x)), \varphi(i(x))) \subsetneq (\mathcal{F}, \mathcal{G})$. It follows that either $\varphi(d(x)) \subsetneq \mathcal{F}$ or $\varphi(i(x)) \subsetneq \mathcal{G}$.
Suppose that $\varphi(d(x)) \subsetneq \mathcal{F}$, then there exists an $F \in \mathcal{F}$ such that $F \notin \varphi(d(x))$; hence $d(x) \nsubseteq F$. Since $\mathcal{F}$ is a closed filter in $(X, \mathcal{U})$, there exists a decreasing closed set A such $A \in \mathcal{F}$ and $A \subseteq F$. Hence $d(x) \nsubseteq A$ or $x \notin A$. Therefore $i(x) \subseteq X-A$ or $(X-A) \in \varphi(i(x))$. It follows that $(X-A) \in \mathcal{G}$, and thus $A \cap (X-A) = \phi$. This is a contradiction, since $(\mathcal{F}, \mathcal{G})$ is a bi-filter. Also in the case that $\varphi(i(x)) \subsetneq \mathcal{G}$, we obtain a contradiction by a similar argument as the above. Therefore $(\varphi(d(x)), \varphi(i(x)))$ is a maximal bi-filter.

In what follows, we assume that $(X, \mathcal{J}, \leq)$ is a convex ordered topological space with semicontinuous order.

Let $w_0(X)$ denote the collection of all maximal bi-filters $(\mathcal{F}, \mathcal{G})$ on X. For a given closed decreasing set A and a given increasing closed set B in X, define $A^d = \{(\mathcal{F}, \mathcal{G}) \in w_0(X): A \in \mathcal{F}\}$, and

$$B^i = \{(\mathcal{F}, \mathcal{G}) \in w_0(X): B \in \mathcal{G}\},$$

then the family $\{A^d: A \in \Gamma_{\mathcal{U}} X\}$ forms a closed base for a topology on $w_0(X)$, since $(A_1 \cup A_2)^d = A_1^d \cup A_2^d$ holds for any A_1 and A_2 in $\Gamma_{\mathcal{U}} X$. Similarly, the family $\{B^i: B \in \Gamma_{\mathcal{L}} X\}$ forms a closed base for a topology on $w_0(X)$.

Let $\mathcal{W}_{\mathcal{U}}$ and $\mathcal{W}_{\mathcal{L}}$ be the topologies on $w_0(X)$ which have the above families as their closed bases, respectively.

Let $\mathcal{W}$ be the smallest topology on $w_0(X)$ containing $\mathcal{W}_{\mathcal{U}}$ and $\mathcal{W}_{\mathcal{L}}$, then every basic open set in $(w_0(X), \mathcal{W})$ can be written in the form $w_0(X) - (A^d \cup B^i)$ for some $A \in \Gamma_{\mathcal{U}} X$ and for some $B \in \Gamma_{\mathcal{L}} X$. We also note that $(A_1 \cap A_2)^d = A_1^d \cap A_2^d$ and $(B_1 \cap B_2)^i = B_1^i \cap B_2^i$, where A_1, A_2 in $\Gamma_{\mathcal{U}} X$ B_1, B_2 in $\Gamma_{\mathcal{L}} X$.

Let us define an order relation $\leq$ on $w_0(X)$ as follows:

$(\mathcal{F}_1, \mathcal{G}_1) \leq (\mathcal{F}_2, \mathcal{G}_2)$ if and only if $\mathcal{F}_1 \supseteq \mathcal{F}_2$ and $\mathcal{G}_1 \subseteq \mathcal{G}_2$ for any $(\mathcal{F}_1, \mathcal{G}_1)$ and $(\mathcal{F}_2, \mathcal{G}_2)$ in $w_0(X)$. Then $\leq$ is obviously a partial order on $w_0(X)$. Hence $(w_0(X), \mathcal{W}, \leq)$ is an ordered topological space.

2.3.7 <u>Remark</u>: Let $(w_0(X), \mathcal{W}, \leq)$ be the ordered topological space given in the above, and let $A \in \Gamma_{\mathcal{U}} X$ and $B \in \Gamma_{\mathcal{L}} X$. Then A^d is a closed decreasing set and B^i is a closed increasing set in $w_0(X)$.

Moreover, $w_0(X)$ is a convex ordered topological space. For, in order to show that A^d is decreasing, let $(\mathcal{F}_1, \mathcal{J}_1) \in A^d$ and $(\mathcal{F}_2, \mathcal{J}_2) \leq (\mathcal{F}_1, \mathcal{J}_1)$ for $(\mathcal{F}_2, \mathcal{J}_2)$ in $w_0(X)$, then $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Since $A \in \mathcal{F}_1$, $A \in \mathcal{F}_2$. Hence $(\mathcal{F}_2, \mathcal{J}_2) \in A^d$. Therefore A^d is a decreasing set. Dually, B^i is an increasing set.

2.3.8 <u>Lemma</u>: Let $(X, \mathcal{J}, \leq)$ be a convex ordered topological space with semicontinuous order. Let us define a map $\Phi: X \to w_0(X)$ by $\Phi(x) = (\varphi(d(x)), \varphi(i(x)))$ for each $x \in X$. Then Φ is a dense embedding from X into $w_0(X)$.

<u>Proof</u>: Firstly, we show that Φ is an isomorphism. To show that Φ is one to one, let $x \nleq y$ in X. Then $x \nleq y$ or $y \nleq x$. If $x \nleq y$, then $y \notin i(x)$ or $i(y) \nsubseteq i(x)$. It follows that $i(x) \notin \varphi(i(y))$ or $\varphi(i(x)) \nsubseteq \varphi(i(y))$. Hence $(\varphi(d(x)), \varphi(i(x))) \nleq (\varphi(d(y)), \varphi(i(y)))$, i.e., $\Phi(x) \nleq \Phi(y)$. Similarly, if $y \nleq x$, then we have $\Phi(x) \nleq \Phi(y)$; therefore Φ is one to one. Let $x \leq y$ in X. Then $y \in i(x)$ and $x \in d(y)$. It is easy to see that $\varphi(d(y)) \subseteq \varphi(d(x))$ and $\varphi(i(x)) \subseteq \varphi(i(y))$. Hence $(\varphi(d(x)), \varphi(i(x))) \leq (\varphi(d(y)), \varphi(i(y)))$, that is, $\Phi(x) \leq \Phi(y)$. Therefore Φ is an increasing map. It is also immediate that if $\Phi(x) \leq \Phi(y)$, then $x \leq y$. This proves that Φ is an isomorphism from $(X, \leq)$ into $(w_0(X), \leq)$. Secondly, we show that Φ is a dense homeomorphism from $(X, \mathcal{J})$ into $(w_0(X), \mathcal{W})$. We observe the following: for given closed decreasing set A,

$$A^d \cap \Phi(X) = \{(\varphi(d(x)), \varphi(i(x))): A \in \varphi(d(x))\}$$
$$= \{(\varphi(d(x)), \varphi(i(x))): d(x) \subseteq A\}$$
$$= \{\Phi(x): x \in A\} = \Phi(A).$$

Similarly, for a given closed increasing set B, $B^i \cap \Phi(X) = \Phi(B)$. Since X is a convex ordered topological space, it is easy to see that Φ is a homeomorphism from X onto $\Phi(X)$.

To show that $\Phi(X)$ is a dense subset of $w_0(X)$, let $w_0(X) - (A^d \cup B^i)$ be a non-empty basic open set, where A is a closed decreasing set and B is a closed increasing set. Take a maximal bi-filter $(\mathcal{F}, \mathcal{G}) \in w_0(X) - (A^d \cup B^i)$. Then $(\mathcal{F}, \mathcal{G}) \notin A^d$ and $(\mathcal{F}, \mathcal{G}) \notin B^i$. Hence $A \notin \mathcal{F}$ and $B \notin \mathcal{G}$. By Lemma 2.3.4, $A \cup B \neq X$. Hence $(X-A) \cap (X-B) \neq \phi$. Pick a $y \in (X-A) \cap (X-B)$, then $\Phi(y) = (\varphi(d(y)), \varphi(i(y))) \in \Phi(X)$. Assume that $\Phi(y) \notin w_0(X) - (A^d \cup B^i)$. Then $\Phi(y) \in A^d$ or $\Phi(y) \in B^i$. If $\Phi(y) \in A^d$, i.e., $(\varphi(d(y)), \varphi(i(y))) \in A^d$, then $A \in \varphi(d(y))$. Hence $d(y) \subseteq A$. Therefore $y \in A$, which contradicts the fact that $y \in X-A$. If $\Phi(y) \in B^i$, then we again have a contradiction. Hence $\Phi(y) \in w_0(X) - (A^d \cup B^i)$. Therefore $\Phi(X) \cap (w_0(X) - (A^d \cup B^i)) \neq \phi$, that is, $\Phi(X)$ is a dense subset of $w_0(X)$. Hence, we have proved that Φ is a dense embedding from $(X, \mathcal{T}, \leq)$ into $(w_0(X), \mathcal{W}, \leq)$.

2.3.9 Lemma: $(w_0(X), \mathcal{W}, \leq)$ is a T_1 compact ordered space.

Proof: To show that $w_0(X)$ is a T_1 space, let $(\mathcal{F}_1, \mathcal{G}_1) \neq (\mathcal{F}_2, \mathcal{G}_2)$ in $w_0(X)$. First we assume that $\mathcal{F}_1 \not\subseteq \mathcal{F}_2$. Take $F_1 \in \mathcal{F}_1$ such that $F_1 \notin \mathcal{F}_2$. Since $\mathcal{F}_1$ is a closed filter in $(X, \mathcal{U})$, there exists a closed decreasing set A_1 such that $A_1 \in \mathcal{F}_1$ and $A_1 \subseteq F_1$, and hence $A_1 \notin \mathcal{F}_2$. It follows that $(\mathcal{F}_1, \mathcal{G}_1) \in A_1^d$ and $(\mathcal{F}_2, \mathcal{G}_2) \notin A_1^d$. Therefore, $w_0(X)-A_1^d$ is an open neighbourhood of $(\mathcal{F}_2, \mathcal{G}_2)$ in $w_0(X)$ such that $(\mathcal{F}_1, \mathcal{G}_1) \notin w_0(X)-A_1^d$. Next, if $\mathcal{F}_2 \not\subseteq \mathcal{F}_1$, then by a similar argument as that in the case of $\mathcal{F}_1 \nsubseteq \mathcal{F}_2$, there exists an open neighbourhood of $(\mathcal{F}_1, \mathcal{G}_1)$, which does not contain $(\mathcal{F}_2, \mathcal{G}_2)$. If $\mathcal{F}_2 \subseteq \mathcal{F}_1$, then $\mathcal{G}_2 \nsubseteq \mathcal{G}_1$, because $(\mathcal{F}_1, \mathcal{G}_1)$ and $(\mathcal{F}_2, \mathcal{G}_2)$ are distinct maximal bi-filters. Hence, there exists a closed increasing set B_2 such that $B_2 \in \mathcal{G}_2$ and $B_2 \notin \mathcal{G}_1$. Therefore $(\mathcal{F}_2, \mathcal{G}_2) \in B_2^i$ and $(\mathcal{F}_1, \mathcal{G}_1) \notin B_2^i$. It follows that $w_0(X)-B_2^i$ is an open neighbourhood of $(\mathcal{F}_1, \mathcal{G}_1)$ which does not contain $(\mathcal{F}_2, \mathcal{G}_2)$. This proves that $w_0(X)$ is a T_1 space. Now, we show that $w_0(X)$ is a compact space. Let $\{A_\alpha^d, B_\beta^i: \alpha\in\Gamma, \beta\in\Delta\}$ be a family of subbasic closed sets having a finite intersection property. Hence, let $\mathcal{A}$ be the filter generated by $\{A_\alpha: \alpha\in\Gamma\}$, and $\mathcal{B}$ the filter generated by $\{B_\beta: \beta\in\Delta\}$, then $(\mathcal{A}, \mathcal{B})$ is evidently a bi-filter. By Remark 2.3.2, there exists a maximal bi-filter $(\mathcal{F}, \mathcal{G})$ containing $(\mathcal{A}, \mathcal{B})$. It follows that $A_\alpha \in \mathcal{F}$ and $B_\beta \in \mathcal{G}$ for all $\alpha\in\Gamma$ and all $\beta\in\Delta$. Therefore, $(\mathcal{F}, \mathcal{G}) \in \cap \{A_\alpha^d \cap B_\beta^i: \alpha\in\Gamma, \beta\in\Delta\}$. By Alexandroff subbasis theorem, $(w_0(X), \mathcal{W})$ is a compact space.▪

2.3.10 Remark: In the proof of the above lemma, we note that $(w_0(X), \mathcal{W}, \leq)$ is an ordered topological space which has either lower semicontinuous order or upper semicontinuous order.
We also note that a compact ordered space with lower semicontinuous order need not have a semicontinuous order. To see this, let $\mathbb{Z}^+$ be the set of all natural numbers equipped with the usual ordering and the cofinite topology. Then $\mathbb{Z}^+$ is compact and its order is lower semicontinuous, since $d(n)$ is closed for each $n\in\mathbb{Z}^+$. But its order is not semicontinuous because it is not upper semicontinuous. In particular,

this shows that a T_1 compact ordered space need not have a semicontinuous order.

Combining Lemma 2.3.6, Lemma 2.3.8 and Lemma 2.3.9, we obtain the following theorem:

2.3.11 Theorem: Let $(X, \mathcal{T}, \leq)$ be a convex ordered topological space with semicontinuous order, then $(w_0(X), \mathcal{W}, \leq)$ is a T_1 compact ordered space in which X is densely embedded.

2.3.12 Remark: We note that if the given order on $(X, \mathcal{T}, \leq)$ in the above theorem is discrete, then the Theorem reduces to the Wallman compactification of general topology.

Let $(X, \mathcal{T}, \leq)$ be an ordered topological space and $(Y, \mathcal{T}', \leq')$ a compact ordered space with continuous order, and $(\mathcal{F}, \mathcal{G})$ a maximal bi-filter on X. Let $f: X \to Y$ be a continuous increasing map. Let us define by $\mathcal{F}^*$ the filter generated by the family $\{A | A$ is a closed decreasing set in Y and $f^{-1}(A) \in \mathcal{F}\}$, and by $\mathcal{G}^*$ the filter generated by the family $\{B | B$ is a closed increasing set in Y and $f^{-1}(B) \in \mathcal{G}\}$.

2.3.13 Lemma: Under the above assumptions, $(\mathcal{F}^*, \mathcal{G}^*)$ is a bi-filter on Y and there exists a unique y in Y such that $y \in \cap \{F \cap G: F \in \mathcal{F}^*$ and $G \in \mathcal{G}^*\}$.

Proof: Since f is a continuous increasing map, it is easy to show that $\mathcal{F}^*$ is a closed filter in the upper space $(Y, \mathcal{U})$ and $\mathcal{G}^*$ is a closed filter in the lower space $(Y, \mathcal{L})$. It is also straightforward that $(\mathcal{F}^*, \mathcal{G}^*)$ is a bi-filter in Y. Since Y is compact, the filter

$\{F \cap G: F \in \mathcal{F}^*, G \in \mathcal{G}^*\}$ has an accumulation point y.

Hence $y \in \cap \{\overline{F \cap G}: F \in \mathcal{F}^*, G \in \mathcal{G}^*\} \subseteq \cap \{\overline{F} \cap \overline{G}: F \in \mathcal{F}^*, G \in \mathcal{G}^*\}$

$$\subseteq \cap \{\overline{A} \cap \overline{B}: A \in \mathcal{B}_{\mathcal{F}^*}, B \in \mathcal{B}_{\mathcal{G}^*}\}$$
$$= \cap \{A \cap B: A \in \mathcal{B}_{\mathcal{F}^*}, B \in \mathcal{B}_{\mathcal{G}^*}\}$$
$$\subseteq \cap \{F \cap G: F \in \mathcal{F}^*, G \in \mathcal{G}^*\},$$

where $\mathcal{B}_{\mathcal{F}^*}$ and $\mathcal{B}_{\mathcal{G}^*}$ are the filter bases of $\mathcal{F}^*$ and $\mathcal{G}^*$ respectively. Hence there exists a y in Y such that $y \in \{F \cap G: F \in \mathcal{F}^*$ and $G \in \mathcal{G}^*\}$. In order to see the uniqueness of y, suppose that there exists $x \neq y$ in Y such that x and y are elements of $\cap \{F \cap G: F \in \mathcal{F}^*, G \in \mathcal{G}^*\}$. Then we may assume that $x \nleq y$. Hence $i(x) \cap d(y) = \phi$. Since Y is a compact ordered space with continuous order, we can show that there exist an open increasing neighbourhood U of x and an open decreasing neighbourhood V of y such that $U \cap V = \phi$; thus $(Y-U) \cup (Y-V) = Y$. Therefore $f^{-1}(Y-U) \cup f^{-1}(Y-V) = X$. Since f is a continuous increasing map, by Lemma 2.3.4, $f^{-1}(Y-U) \in \mathcal{F}$ or $f^{-1}(Y-V) \in \mathcal{G}$. By the definition of $\mathcal{F}^*$ and $\mathcal{G}^*$, $(Y-U) \in \mathcal{F}^*$ or $(Y-V) \in \mathcal{G}^*$. If $(Y-U) \in \mathcal{F}^*$, then $x \in Y-U$, and hence $x \notin U$, which contradicts the fact that $x \in U$. Similarly, if $(Y-V) \in \mathcal{G}^*$, then we obtain a contradiction. Hence $x = y$.

2.3.14 <u>Theorem</u>: Let $(X, \mathcal{T}, \leq)$ be a convex ordered topological space with semicontinuous order and $(Y, \mathcal{T}', \leq')$ a compact ordered space with continuous order. Let $f: X \to Y$ be any continuous increasing function. Then there exists a unique continuous increasing function $\overline{f}$ from $w_0(X)$ into Y such that $\overline{f} \circ \Phi = f$.

Proof: Let $f: X \to Y$ be a continuous increasing function. For given $(\mathcal{F}, \mathcal{G}) \in w_0(X)$, let $\mathcal{F}^*$ and $\mathcal{G}^*$ be the filters given as before. By Lemma 2.3.13, there exists a unique point of $\cap \{F \cap G: F \in \mathcal{F}^*, G \in \mathcal{G}^*\}$. Now, let us define $\overline{f}: w_0(X) \to Y$ as follows: for given $(\mathcal{F}, \mathcal{G}) \in w_0(X)$, $\overline{f}((\mathcal{F}, \mathcal{G}))$ is the unique point of $\cap \{F \cap G: F \in \mathcal{F}^*, G \in \mathcal{G}^*\}$, that is, $\overline{f}((\mathcal{F}, \mathcal{G})) \in \cap \{F \cap G: F \in \mathcal{F}^*, G \in \mathcal{G}^*\}$, then we show that $\overline{f}$ is the required map.

Firstly, we show that $\overline{f} \circ \Phi = f$: Let x be any point of X. Then we can easily show that $[\varphi(d(x))]^* = \varphi(d(f(x)))$ and $[\varphi(i(y))]^* = \varphi(i(f(x)))$. Hence $([\varphi(d(x))]^*, [\varphi(i(x))]^*) = (\varphi(d(f(x))), \varphi(i(f(x)))$. It follows that $(\overline{f} \circ \Phi)(x) = \overline{f}((\varphi(d(x)), \varphi(i(x)))) = f(x)$. Thus $\overline{f} \circ \Phi = f$. Now, we show that $\overline{f}$ is continuous map: Since $w_0(X)$ and Y are convex ordered topological spaces, it is sufficient to show that $\overline{f}$ is continuous from $(w_0(X), \mathcal{W}_{\mathcal{U}})$ into $(Y, \mathcal{L})$, and $\overline{f}$ is continuous from $(w_0(X), \mathcal{W}_{\mathcal{L}})$ into $(Y, \mathcal{U})$. For a fixed point $(\mathcal{F}, \mathcal{G}) \in w_0(X)$, let U be an open decreasing neighbourhood of $\overline{f}((\mathcal{F}, \mathcal{G}))$ in Y. Hence $Y-U$ is a closed increasing set and does not contain $\overline{f}((\mathcal{F}, \mathcal{G}))$. It follows that $d(\overline{f}((\mathcal{F}, \mathcal{G}))) \cap i(Y-U) = \phi$. Since Y is a compact ordered space with continuous order, there exist an open decreasing set W and an open increasing set V such that $d(\overline{f}((\mathcal{F}, \mathcal{G})) \subseteq W$, $Y-U \subseteq V$ and $W \cap V = \phi$. Hence $(Y-W) \cup (Y-V) = Y$; therefore $f^{-1}(Y-W) \cup f^{-1}(Y-V) = X$. By Lemma 2.3.4, $f^{-1}(Y-W) \in \mathcal{G}$ or $f^{-1}(Y-V) \in \mathcal{F}$ for any maximal bi-filter $(\mathcal{F}, \mathcal{G})$. Hence $(\mathcal{F}, \mathcal{G}) \in [f^{-1}(Y-W)]^i$ or $(\mathcal{F}, \mathcal{G}) \in [f^{-1}(Y-V)]^d$. Therefore $(\mathcal{F}, \mathcal{G}) \in [f^{-1}(Y-W)]^i \cup [f^{-1}(Y-V)]^d$. It follows that $[f^{-1}(Y-W)]^i \cup [f^{-1}(Y-V)]^d = w_0(X)$. But, since $d(\overline{f}((\mathcal{F}, \mathcal{G})) \subseteq W$,

$\overline{f}((\mathcal{F}, \mathcal{G})) \notin Y-W$. Hence it is easy to see that $(\mathcal{F}, \mathcal{G}) \notin [f^{-1}(Y-W)]^i$. Thus $w_0(X) - [f^{-1}(Y-W)]^i$ is an open decreasing neighbourhood of $(\mathcal{F}, \mathcal{G})$ in $w_0(X)$. Let $(\mathcal{F}', \mathcal{G}')$ be a member of $w_0(X) - [f^{-1}(Y-W)]^i$. Then $(\mathcal{F}', \mathcal{G}') \notin [f^{-1}(Y-W)]^i$. Since we have already shown that $w_0(X) = [f^{-1}(Y-W)]^i \cup [f^{-1}(Y-V)]^d$, $(\mathcal{F}', \mathcal{G}') \in [f^{-1}(Y-V)]^d$. Hence $f^{-1}(Y-V) \in \mathcal{F}'$, and therefore $(Y-V) \in (\mathcal{F}')^*$. It follows that $\overline{f}((\mathcal{F}', \mathcal{G}')) \in Y-V \subseteq U$. Thus we have $\overline{f}((w_0(X) - [f^{-1}(Y-W)]^i)) \subseteq U$. Hence, $\overline{f}$ is a continuous function from $(w_0(X), \mathcal{W}_{\mathcal{U}})$ into $(Y, \mathcal{L})$. Similarly, we can show that $\overline{f}$ is continuous from $(w_0(X), \mathcal{W}_{\mathcal{L}})$ into $(Y, \mathcal{U})$. Therefore $\overline{f}$ is a continuous function.

Finally, we show that $\overline{f}$ is an increasing map. Let $(\mathcal{F}_1, \mathcal{G}_1) \leq (\mathcal{F}_2, \mathcal{G}_2)$ in $w_0(X)$. Suppose that $\overline{f}((\mathcal{F}_1, \mathcal{G}_1)) \not\leq \overline{f}((\mathcal{F}_2, \mathcal{G}_2))$. Since Y is a comapct ordered space with continuous order, there exist an open increasing neighbourhood U of $\overline{f}((\mathcal{F}_1, \mathcal{G}_1))$ and an open decreasing neighbourhood V of $\overline{f}((\mathcal{F}_2, \mathcal{G}_2))$ such that $U \cap V = \phi$. Hence $\overline{f}((\mathcal{F}_1, \mathcal{G}_1)) \notin V$. Since $\overline{f}$ is a continuous function from $(w_0(X), \mathcal{W}_{\mathcal{U}})$ into $(Y, \mathcal{L})$, there exists a closed increasing set A in X such that $w_0(X) - A^i$ is an open decreasing set containing $(\mathcal{F}_2, \mathcal{G}_2)$ and $\overline{f}((w_0(X)-A^i)) \subseteq V$. Hence, we have $(\mathcal{F}_1, \mathcal{G}_1) \in w_0(X) - A^i$, since $(\mathcal{F}_1, \mathcal{G}_1) \leq (\mathcal{F}_2, \mathcal{G}_2)$. It follows that $\overline{f}((\mathcal{F}_1, \mathcal{G}_1)) \in V$, which contradicts $\overline{f}((\mathcal{F}_1, \mathcal{G}_1)) \notin V$. Hence, we have $\overline{f}((\mathcal{F}_1, \mathcal{G}_1)) \leq \overline{f}((\mathcal{F}_2, \mathcal{G}_2))$. This proves that $\overline{f}$ is an increasing function. In particular, the uniqueness of $\overline{f}$ is clear.

2.3.15 <u>Theorem</u>: Let $(X, \mathcal{J}, \leq)$ be a convex compact ordered space with semicontinuous order, then $(X, \mathcal{T}, \leq)$ is iseomorphic with $(w_0(X), \mathcal{W}, \leq)$.

Proof: Let $(\mathcal{F}, \mathcal{G})$ be a maximal bi-filter on X. Then the filter $\{F \cap G: F \in \mathcal{F}, G \in \mathcal{G}\}$ has an accumulation point, say y, in X. Hence, we have

$$\{x\} \subseteq \cap \{\overline{F \cap G}: F \in \mathcal{F}, G \in \mathcal{G}\} \subseteq \cap \{\overline{F} \cap \overline{G}: F \in \mathcal{F}, G \in \mathcal{G}\}$$
$$\subseteq \cap \{\overline{A} \cap \overline{B}: A \in \mathcal{B}_{\mathcal{F}}, B \in \mathcal{B}_{\mathcal{G}}\}$$
$$= \cap \{A \cap B: A \in \mathcal{B}_{\mathcal{F}}, B \in \mathcal{B}_{\mathcal{G}}\}$$

where $\mathcal{B}_{\mathcal{F}}$ and $\mathcal{B}_{\mathcal{G}}$ are the filter bases of $\mathcal{F}$ and $\mathcal{G}$ respectively. Since d(x) is a closed decreasing set and $\{x\} \subseteq \cap \{A: A \in \mathcal{B}_{\mathcal{F}}\}$, we have $d(x) \subseteq \cap \{A: A \in \mathcal{B}_{\mathcal{F}}\}$. Similarly, we have $i(x) \subseteq \cap \{B: B \in \mathcal{B}_{\mathcal{G}}\}$. Hence, it immediately follows that $(\mathcal{F}, \mathcal{G}) \subseteq (\varphi(d(x)), \varphi(i(x)))$. By the maximality of $(\mathcal{F}, \mathcal{G})$, we have $(\mathcal{F}, \mathcal{G}) = (\varphi(d(x)), \varphi(i(x)))$. Hence, we have $\Phi(X) = w_0(X)$. Therefore $(X, \mathcal{T}, \leq)$ is iseomorphic with $(w_0(X), \mathcal{W}, \leq)$.

2.3.16 Definition: (Nachbin [27]): An ordered topological space $(X, \mathcal{T}, \leq)$ is said to be normally ordered if, for every two disjoint closed subsets A and B of X such that A is decreasing and B is increasing, there exist two disjoint open sets U and V such that U contains A and is decreasing, and V contains B and is increasing.

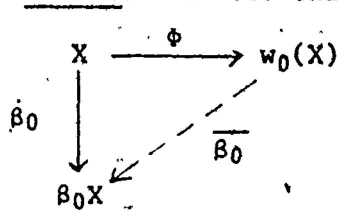
2.3.17 Theorem: Let $(X, \mathcal{T}, \leq)$ be a convex ordered topological space with semicontinuous order. If $w_0(X)$ has a continuous order, then X is a normally ordered space.

Proof: If $w_0(X)$ has a continuous order, then $w_0(X)$ is clearly a normally ordered space. Let A and B be two disjoint closed decreasing and increasing sets in X, respectively. Suppose that $A^d \cap B^i \neq \phi$, and pick $(\mathcal{F}, \mathcal{G}) \in A^d \cap B^i$. Then $A \in \mathcal{F}$ and $B \in \mathcal{G}$. Since $(\mathcal{F}, \mathcal{G})$ is a bi-filter, $A \cap B \neq \phi$, which is a contradiction. Hence $A^d \cap B^i = \phi$. Since $w_0(X)$ is normally ordered, there exists an open decreasing set W and an open increasing set W' in $w_0(X)$ such that $A^d \subseteq W$, $B^i \subseteq W'$ and $W \cap W' \neq \phi$. While, W and W' can be written in the following forms: $W = \bigcup_j (w_0(X)-B_j^i)$ and $W' = \bigcup_j (w_0(X)-A_j^d)$, where B_j in $\Gamma_{\mathcal{L}} X$ and A_j in $\Gamma_{\mathcal{U}} X$. Since A^d and B^i are compact, we have

$$A^d \subseteq \bigcup_{j=1}^{n} (w_0(X)-B_j^i) = w_0(X) - \bigcap_{j=1}^{n} B_j^i = w_0(X) - (\bigcap_{j=1}^{n} B_j)^i.$$

Similarly, $B^i \subseteq \bigcup_{j=1}^{m} (w_0(X)-A_j^d) = w_0(X) - (\bigcap_{j=1}^{m} A_j)^d$. Let $U = X - (\bigcap_{j=1}^{n} B_j)$ and $V = X - (\bigcap_{j=1}^{m} A_j)$. Then U is an open decreasing set and V is an open increasing set. Let $x \in A$, then $d(x) \subseteq A$, and hence $(\varphi(d(x)), \varphi(i(x))) \in A^d$. Since $A^d \subseteq w_0(X) - (\bigcap_{j=1}^{n} B_j)^i$, we have $(\varphi(d(x)), \varphi(i(x))) \notin (\bigcap_{j=1}^{n} B_j)^i$. Hence $\bigcap_{j=1}^{n} B_j \notin \varphi(i(x))$; therefore $i(x) \nsubseteq \bigcap_{j=1}^{n} B_j$. Hence, we have $x \in X - \bigcap_{j=1}^{n} B_j$. It follows that $A \subseteq U$. Similarly, we have $B \subseteq V$. We also note that $[w_0(X) - (\bigcap_{j=1}^{n} B_j)^i] \cap [w_0(X) - (\bigcap_{j=1}^{m} A_j)^d] = \phi$. Hence, it is easy to see that $U \cap V = \phi$. This proves that X is a normally ordered space.

2.3.18 <u>Remark</u>: We note that given the following diagram:



, where X is a completely regular

ordered space, and $\beta_0 X$ is a compact ordered space with continuous order described in Remark 2.2.8, there exists a unique continuous increasing function $\overline{\beta_0}$ such that $\overline{\beta_0} \circ \Phi = \beta_0$. However, we do not know whether or not the converse of Theorem 2.3.17 holds. If the converse of the theorem 2.3.17 holds, then $\beta_0 X$ and $w_0(X)$ are order equivalent, in other words, there exists an iseomorphism from $w_0(X)$ onto $\beta_0 X$ such that the above diagram commutes.

Section 4. Order compactifications of lower semicontinuous functions.

We recall that a function from a topological space X into $\mathbb{R}$ is lower semicontinuous if and only if for each $a \in \mathbb{R}$, $f^{-1}((a, \infty))$ is open in X (see 0.16).

Let $(X, \mathcal{J}, \leq)$ be an ordered topological space and let:

$\mathcal{L}(X) = \{f: f \text{ is a lower semicontinuous function on } X\}$.

$\mathcal{L}^+(X) = \{f \in \mathcal{L}(X): f \text{ is nonnegative}\}$, and

$\mathcal{L}^i(X) = \{f \in \mathcal{L}^+(X): f \text{ is increasing}\}$, and

$\mathcal{L}^d(X) = \{f \in \mathcal{L}^+(X): f \text{ is decreasing}\}$.

We also denote by $\mathcal{U}$ and $\mathcal{L}$ the upper and lower topologies, respectively,.

2.4.1 Remark: It is easy to see that the following statements hold:

1. U is an open increasing set in $(X, \mathcal{J}, \leq)$ if and only if its characteristic function χ_U belongs to $\mathcal{L}^i(X)$, and U is an open decreasing set in X if and only if its characteristic function χ_U belongs to $\mathcal{L}^d(X)$.

2. $\mathcal{L}^+(X)$, $\mathcal{L}^i(X)$ and $\mathcal{L}^d(X)$ form semi-rings under the usual pointwise operations (cf. 0.17).

2.4.2 <u>Remark</u>: The idempotent set of $\mathcal{L}^i(X)$ is equal to the family $\{\chi_U: U\varepsilon\mathcal{U}\}$ of all characteristics functions of open increasing sets in X. For, since every characteristic function is obviously idempotent, the former is clearly contained in the latter, from the above Remark 2.4.1. In order to see the other inclusion, let f be a non-negative lower semicontinuous increasing function and $f^2 = f$. Let $F = \{x\varepsilon X: f(x) = 0\}$, then F is a closed decreasing set. Hence $X-F\varepsilon\mathcal{U}$ and $f = \chi_{X-F}$.

Given f, g in $\mathcal{L}(X)$, define $(f\vee g)(x) = \max\{f(x), g(x)\}$, and $(f\wedge g)(x) = \min\{f(x), g(x)\}$ for each $x\varepsilon X$, then $(f\vee g)$ and $(f\wedge g)$ are in $\mathcal{L}(X)$. In fact, $\mathcal{L}(X)$ is a complete join semilattice. In particular, $\mathcal{L}^i(X)$ and $\mathcal{L}^d(X)$ are lattices under the above operations $\vee$ and $\wedge$.

The following definition is due to Nielsen and Sloyer [28]:

2.4.3 <u>Definition</u>: A proper subset I of $\mathcal{L}^i(X)$ is called an <u>ideal</u> if and only if it satisfies the following conditions:

1. If f and g are in I, then $f + g\varepsilon I$.
2. If $f\varepsilon I$ and $g\varepsilon\mathcal{L}^i(X)$, then $g\cdot f\varepsilon I$
3. If $f\varepsilon I$, then there exists an idempotent g, $g \neq 1$ in $\mathcal{L}^i(X)$ such that $g\cdot f = f$, where 1 is defined by $1(X) = 1$.

Ideals in $\mathcal{L}^d(X)$ are defined analogously.

Let I be an ideal in $\mathcal{L}^i(X)$, and let $f\varepsilon\mathcal{L}^i(X) - I$. Then the ideal generated by $I \cup \{f\}$, denoted by (I,f) is clearly the ideal $\{m+\ell\cdot f: m\varepsilon I, \ell\varepsilon\mathcal{L}^i(X)$ and $g\cdot(m+\ell\cdot f) = m+\ell f$ for some idempotent $g(\neq 1)$ in $\mathcal{L}^i(X)\}$.

2.4.4 <u>Remark</u>: Let $(X, \mathcal{T})$ be a T_1 space. Then each point of X can be associated with a maximal ideal in $L^+(X)$ (see Nielsen and Sloyer [28] and 0.18). But, this statement need not be true in an ordered topological space $(X, \mathcal{T}, \leq)$ with semicontinuous order: For example, let $X = [0,1]$ be a topological space equipped with the usual topology and the usual order. For each $x \in X$, let $I_x = \{f \in \mathcal{L}^i(X): f(x) = 0\}$. Then it can be shown that I_x is an ideal of $\mathcal{L}^i(X)$. In fact, let $f \in I_x$. Then $X-d(x) \in \mathcal{U}$, since $d(x)$ is a closed decreasing set in X. By Remark 2.4.1, it follows that $\chi_{X-d(x)} \in \mathcal{L}^i(X)$, $\chi_{X-d(x)} \neq 1$ and $\chi_{X-d(x)} \cdot f = f$. Hence I_x is an ideal for each $x \in X$. Given $x \in X$ $(x \neq 0)$, $I_x \subsetneq I_0$, so that I_x is not maximal for each $x(\neq 0)$. We also note that if the order on $(X, \mathcal{T}, \leq)$ is discrete, then $L^+(X) = \mathcal{L}^i(X)$.

2.4.5 <u>Remark:</u> The following statements are easy to see:

1. Let $\Gamma_{\mathcal{U}}(X)$ denote the family of all closed decreasing sets in X and $\Gamma_{\mathcal{L}}(X)$ the family of all closed increasing sets in X, then $\Gamma_{\mathcal{U}}(X) = \{Z(f): f \in \mathcal{L}^i(X)\}$ and $\Gamma_{\mathcal{L}}(X) = \{Z(f): f \in \mathcal{L}^d(X)\}$, where $Z(f) = \{x \in X: f(x) = 0\}$.

2. If f and g are elements of $\mathcal{L}^+(X)$, then $Z(f+g) = Z(f) \cap Z(g)$ $Z(f \cdot g) = Z(f) \cup Z(g)$.

The following definition is due to Canfell [4].

2.4.6 <u>Definition</u>: Let I and J be ideals in $\mathcal{L}^i(X)$ and $\mathcal{L}^d(X)$ respectively. The pair (I,J) is said to be a <u>bi-ideal</u> in $(\mathcal{L}^i(X), \mathcal{L}^d(X))$, if given $i \in I$ and $j \in J$, $Z(i) \cap Z(j) \neq \phi$.

For given two bi-ideals (I_1,J_1) and (I_2,J_2), we define a relation $(I_1,J_1) \subseteq (I_2,J_2)$ if and only if $I_1 \subseteq I_2$ and $J_1 \subseteq J_2$.

By a <u>maximal bi-ideal</u> on X we mean a bi-ideal not contained in any other bi-filter under the above relation.

2.4.7 <u>Remark</u>: We note, by Zorn's lemma, that every bi-ideal is contained in a maximal bi-ideal.

Let $\mathfrak{M}_0(X)$ denote the set of all maximal bi-ideals in ($\mathcal{L}^i(X)$, $\mathcal{L}^d(X)$).

2.4.8 <u>Lemma</u>: Let $(M,N) \in \mathfrak{M}_0(X)$ and let $f \in \mathcal{L}^i(X)$.

Then $f \in M$ if and only if for given $m \in M$ and $n \in N$, $Z(f) \cap Z(m) \cap Z(n) \neq \phi$.

A dual statement holds for N.

<u>Proof</u>: ($\rightarrow$): Let $f \in M$. For given $m \in M$ and $n \in N$, since M is ideal, $m+f \in M$. Hence $\phi \neq Z(f+m) \cap Z(n) = Z(f) \cap Z(m) \cap Z(n)$.

($\leftarrow$): Assume that $f \notin M$. Let (M,f) be the ideal in $\mathcal{L}^i(X)$ generated by M and f, then $(M,f) = \{m+\ell f: m \in M, \ell \in \mathcal{L}^i(X)$ and $g\cdot(m+\ell f) = m+\ell f$ for some idempotent $g(\neq 1)$ in $\mathcal{L}^i(X)\}$. For each $n \in N$, $Z(m+\ell f) \cap Z(n) = Z(m) \cap Z(\ell f) \cap Z(n) \supseteq Z(m) \cap Z(f) \cap Z(n) \neq \phi$, that is, $Z(m+\ell f) \cap Z(n) \neq \phi$. Hence ((M,f), N)) is a bi-ideal containing (M,N). But, this is a contradiction to the maximality of (M,N); therefore $f \in M$.

2.4.9 <u>Lemma</u>: Let $(M,N) \in \mathfrak{M}_0(X)$. Then the following statements hold:

1. Let f and f' be elements of $\mathcal{L}^i(X)$ and $f\cdot f' \in M$, then either $f \in M$ or $f' \in M$.

A dual statement holds for N

2. Let $f \in \mathcal{L}^i(X)$ and $g \in \mathcal{L}^d(X)$, then $f\cdot g = 0$ implies either $f \in M$ or $g \in N$.

Proof: (1) and (2) are immediate from Lemma 2.4.8.

Let us define $f^d = \{(M,N) \in \mathfrak{M}_0(X)\colon f \in M\}$ for given $F \in \mathcal{L}^i(X)$, and

$$g^i = \{(M,N) \in \mathfrak{M}_0(X)\colon g \in N\} \text{ for given } g \in \mathcal{L}^d(X),$$

then $\{f^d\colon f \in \mathcal{L}^i(X)\}$ form a base for the closed sets in $\mathfrak{M}_0(X)$, since $f^d \cup f'^d = (f \cdot f')^d$. Similarly, $\{g^i\colon g \in \mathcal{L}^d(X)\}$ also forms a base for the closed sets in $\mathfrak{M}_0(X)$. We denote the topologies in $\mathfrak{M}_0(X)$ which have $\{f^d\colon f \in \mathcal{L}^i(X)\}$ and $\{g^i\colon g \in \mathcal{L}^d(X)\}$ as basis respectively, by $\mathfrak{M}_{\mathcal{L}}$ and $\mathfrak{M}_{\mathfrak{u}}$, and let $\mathfrak{M}$ be the smallest topology containing $\mathfrak{M}_{\mathcal{L}}$ and $\mathfrak{M}_{\mathfrak{u}}$. Define an order relation $\le$ on $\mathfrak{M}_0(X)$ by the following: $(M,N) \le (M',N')$ if and only if $M \supseteq M'$ and $N \subseteq N'$ for each (M,N) and (M',N') in $\mathfrak{M}_0(X)$. Then $\le$ is obviously a partial order on $\mathfrak{M}_0(X)$ and $(\mathfrak{M}_0(X), \mathfrak{M}, \le)$ is an ordered topological space.

2.4.10 Remark: It is immediate that f^d and g^i are closed decreasing and increasing sets in $(\mathfrak{M}_0(X), \mathfrak{M}, \le)$, respectively, for given $f \in \mathcal{L}^i(X)$ and $g \in \mathcal{L}^d(X)$. Hence, we note that $\mathfrak{M}_0(X)$ is a convex ordered topological space.

2.4.11 Proposition: The convex ordered topological space $(\mathfrak{M}_0(X), \mathfrak{M}, \le)$ is T_1-compact.

Proof: By analogous arguments to those in Lemma 2.3.9, we can show that $\mathfrak{M}_0(X)$ is a T_1 space. In order to show compactness of $\mathfrak{M}_0(X)$, let $\{f_\alpha^d, g_\beta^i\colon \alpha \in \Gamma \text{ and } \beta \in \Lambda\}$ be a family of subbasic members with the finite intersection property. We note that if $f_\alpha^d \cap g_\beta^i \neq \phi$, then $Z(f_\alpha) \cap Z(g_\beta) \neq \phi$. Hence $\{Z(f_\alpha), Z(g_\beta)\colon \alpha \in \Gamma \text{ and } \beta \in \Lambda\}$ has the finite intersection property.

Let us define $I = \{f\varepsilon\ \mathcal{L}^i(X): Z(f) \supseteq \bigcap_{j=1}^{n} Z(f_{\alpha j})$ for any finite number of $f_\alpha\}$ and $J = \{g\varepsilon\ \mathcal{L}^d(X): Z(g) \supseteq \bigcap_{j=1}^{m} Z(g_{\beta j})$ for any finite number of $g_\beta\}$. Then it is easy to see that I and J are ideals in $\mathcal{L}^i(X)$ and $\mathcal{L}^d(X)$, respectively. Therefore, the pair (I,J) is obviously a bi-ideal. By Remark 2.4.7, there exists a maximal bi-ideal (M,N) containg (I,J). Hence $f_\alpha \varepsilon M$ and $g_\beta \varepsilon N$ for all $\alpha\varepsilon\Gamma$ and all $\beta\varepsilon\Lambda$. It follows that $(M,N)\ \varepsilon\ f_\alpha^d$ and $(M,N)\ \varepsilon\ g_\beta^i$. Hence $(M,N)\ \varepsilon \cap \{f_\alpha^d, g_\beta^i: \alpha\varepsilon\Gamma$ and $\beta\varepsilon\Lambda\}$, and therefore $\mathcal{M}_0(X)$ is a compact space by the Alexandroff subbasis theorem.

2.4.12 Definition: Let (I,J) be a bi-ideal in ($\mathcal{L}^i(X)$, $\mathcal{L}^d(X)$). Then the bi-ideal (I,J) is said to be fixed if there exists a point $p\varepsilon X$ such that $p\ \varepsilon \cap\{Z(i), Z(j): i\varepsilon I$ and $j\varepsilon J\}$. Otherwise, it is said to be free.

2.4.13 Lemma: Let$(X, \mathcal{T}, \leq)$ be an ordered topological space with semicontinuous order. For each $p\varepsilon X$, define $M_p^i = \{f\varepsilon\ \mathcal{L}^i(X): f(p) = 0\}$ and $M_p^d = \{g\varepsilon\ \mathcal{L}^d(X): g(p) = 0\}$.
Then (M_p^i, M_p^d) is a fixed bi-ideal with a point p.

Proof: This is immediate from the definition.

2.4.14 Proposition: Let $(X, \mathcal{T}, \leq)$ be an ordered topological space with semicontinuous order. Then the fixed maximal bi-ideals in ($\mathcal{L}^i(X)$, $\mathcal{L}^d(X)$) are precisely the pairs (M_p^i, M_p^d) for $p\varepsilon X$.
Moreover, these bi-ideals are distinct for distinct points in X.

Proof: Let (M,N) be a fixed maximal bi-ideal with a point p in X. Then it is easy to see that $(M,N) \subseteq (M_p^i, M_p^d)$. By Lemma 2.4.13, (M_p^i, M_p^d) is a fixed bi-ideal with a point p. To show maximality of (M_p^i, M_p^d), let $f \in \mathcal{L}^i(X)$ and $f \notin M_p^i$; then $f(p) > 0$. Since $i(p)$ is a closed increasing set, $X-i(p)$ is an open decreasing set. By Remark 2.4.1, $\chi_{X-i(p)} \in \mathcal{L}^d(X)$. Suppose that $Z(f) \cap Z(\chi_{X-i(p)}) \neq \phi$. Take $x \in Z(f) \cap Z(\chi_{X-i(p)})$, then $f(x) = 0$ and $\chi_{X-i(p)}(x) = 0$. So, $x \notin X-i(p)$. Hence $x \in i(p)$, i.e. $p \leq x$. It follows that $f(p) \leq f(x)$. Hence $f(p) = 0$, which is a contradiction. Therefore, we have $Z(f) \cap Z(\chi_{X-i(p)}) = \phi$. We also note that $\chi_{X-i(p)}$ belongs to M_p^d. Hence $((M_p^i, f), M_p^d)$ is not a bi-ideal. We can easily observe that a dual result holds for $g \in \mathcal{L}^d(X)$ and $g \notin M_p^d$. Therefore (M_p^i, M_p^d) is a maximal bi-ideal. Hence, the fixed maximal bi-ideals in $(\mathcal{L}^i(X), \mathcal{L}^d(X))$ are precisely the pairs (M_p^i, M_p^d) for $p \in X$.

Let $p \neq q$ in X. Then we may assume without loss of generality that $q \nleq p$. Hence $p \notin i(q)$ or $p \in X-i(q)$. By Remark 2.4.1, $\chi_{X-i(q)} \in \mathcal{L}^d(X)$, and hence $\chi_{X-i(q)}(q) = 0$ and $\chi_{X-i(q)}(p) = 1$. Hence $\chi_{X-i(q)} \in M_q^d$, but $\chi_{X-i(q)} \notin M_p^d$. It follows that $(M_p^i, M_p^d) \neq (M_q^i, M_q^d)$. This completes the proof.

2.4.15 Proposition: Let $(X, \mathcal{T}, \leq)$ be a compact ordered space with semicontinuous order, then every bi-ideal in $(\mathcal{L}^i(X), \mathcal{L}^d(X))$ is fixed.

Proof: Let (I,J) be a bi-ideal in $(\mathcal{L}^i(X), \mathcal{L}^d(X))$. Then the family $\{Z(i), Z(j): i \in I, j \in J\}$ has the finite intersection property. By compactness of X, the result immediately follows.

2.4.16 Remark: From Propositions 2.4.14 and 2.4.15, we note that if $(X, \mathcal{T}, \leq)$ is a compact ordered space with semicontinuous order, then every maximal bi-ideal is of the form (M_p^i, M_p^d) for $p \in X$.

2.4.17 Theorem: Let $(X, \mathcal{T}, \leq)$ be a convex ordered topological space with semicontinuous order, then $(X, \mathcal{T}, \leq)$ is iseomorphic to a dense subspace of $(\mathcal{M}_0(X), \mathcal{M}, \leq)$.

Proof: Let us define a map $e: (X, \mathcal{T}, \leq) \to (\mathcal{M}_0(X), \mathcal{M}, \leq)$ by $e(p) = (M_p^i, M_p^d)$ for each $p \in X$. Then, by Proposition 2.4.11, e is injective. Firstly, we show that e is an order isomorphism: If $p \leq q$ in X, then $M_p^i \supseteq M_q^i$ and $M_p^d \subseteq M_q^d$. Hence $(M_p^i, M_p^d) \leq (M_q^i, M_q^d)$, that is, $e(p) \leq e(q)$. So e is an increasing function. To show that if $e(p) \leq e(q)$ in $\mathcal{M}_0(X)$, then $p \leq q$ in X; assume that $p \nleq q$; then $p \notin d(q)$ or $p \in X-d(q)$. Hence $\chi_{X-d(q)} \in \mathcal{L}^i(X)$, $\chi_{X-d(q)}(p) = 1$ and $\chi_{X-d(q)}(q) = 0$. It follows that $\chi_{X-d(q)} \in M_q^i$ and $\chi_{X-d(q)} \notin M_p^i$. Hence $M_q^i \nsubseteq M_p^i$, which is a contradiction; therefore $p \leq q$. Hence e is an order isomorphism. We can therefore identify X with e(X).

Secondly, we show that X is dense in $\mathcal{M}_0(X)$: it suffices to show that if $f \in \mathcal{L}^i(X)$ and $g \in \mathcal{L}^d(X)$, then $\overline{[Z(f)]} = f^d$ in $(\mathcal{M}_0(X), \mathcal{M}_{\mathcal{L}})$ and $\overline{[Z(g)]} = g^i$ in $(\mathcal{M}_0(X), \mathcal{M}_{\mathcal{U}})$, [here-denotes closure in the given spaces respectively] because if $f = g = 0$, then $\overline{X} = \mathcal{M}_0(X)$. Since $Z(f) \subseteq f^d$ and f^d is $\mathcal{M}_{\mathcal{L}}$-closed, $\overline{[Z(f)]} \subseteq f^d$. On the other hand, suppose that $f'^d \supseteq Z(f)$ for some $f' \in \mathcal{L}^i(X)$, then $Z(f') = X \cap f'^d \supseteq Z(f)$. Let $(M,N) \in f^d$, then $f \in M$. Since $Z(f') \supseteq Z(f)$, by Lemma 2.4.8, we have $f' \in M$; hence $(M,N) \in f'^d$. It follows that $f'^d \supseteq f^d$. Hence $\overline{[Z(f)]} = f^d$

in ($\mathcal{M}_0(X)$, $\mathcal{M}_{\mathcal{L}}$).

Finally, we show that $\mathcal{M}|X = \mathcal{T}$. By convexity of the topologies $\mathcal{T}$ and $\mathcal{M}$, it is sufficient to show that $\mathcal{M}_{\mathcal{L}}|X = \Gamma_{\mathcal{U}}(X)$ and $\mathcal{M}_{\mathcal{U}}|X = \Gamma_{\mathcal{L}}(X)$ (see Remark 2.4.5). Let $f \in \mathcal{L}^i(X)$. Then $p \in f^d \cap X$ iff $(M_p^i, M_p^d) \in f^d$ iff $f \in M_p^i$ iff $p \in Z(f)$. Hence it is easy to see that $\mathcal{M}_{\mathcal{L}}|X = \Gamma_{\mathcal{U}}(X)$. Similarly, we have $\mathcal{M}_{\mathcal{U}}|X = \Gamma_{\mathcal{L}}(X)$. The proof is thus complete.

2.4.18 <u>Corollary</u>: Let $(X, \mathcal{T}, \le)$ be a convex compact ordered space with semicontinuous order, then $(X, \mathcal{T}, \le)$ is isomorophic to $(\mathcal{M}_0(X), \mathcal{M}, \le)$.

<u>Proof</u>: By Remark 2.4.16, the mapping $e: X \to \mathcal{M}_0(X)$ given by $e(p) = (M_p^i, M_p^d)$ is obviously onto. Hence the result immediately follows from the above theorem.

2.4.19 <u>Remark</u>: If the given order on X in Theorem 2.4.17 is discrete, then the theorem reduces to the main result of Nielsen and Sloyer [28]. (see Theorem 0.19).

<u>Section 5. Equivalence of the two order compactifications $w_0(X)$ and $\mathcal{M}_0(X)$.</u>

In the sections 3 and 4, we constructed two compactifications $w_0(X)$ and $\mathcal{M}_0(X)$ for a convex ordered topological space $(X, \mathcal{T}, \le)$ with semicontinuous order.

In this section, we investigate relations between $w_0(X)$ and $\mathcal{M}_0(X)$. In fact, it turns out that they are order equivalent. Throughout this section, we use the same notations as those given in the sections 3 and 4. Let $(X, \mathcal{T}, \le)$ be an ordered topological space with semicontinuous order. Let I be an ideal in $\mathcal{L}^i(X)$ and $\mathcal{F}$ a closed filter in $(X, \mathcal{U})$. We denote $Z(I) = \{Z(f): f\varepsilon I\}$, and

$$Z^{-1}(\mathcal{F}) = \{f \varepsilon \mathcal{L}^i(X): Z(f) \varepsilon \mathcal{F}\}.$$

2.5.1 Lemma: Let $(X, \mathcal{T}, \leq)$ be an ordered topological space with semicontinuous order. For any ideal I in $\mathcal{L}^i(X)$, let $\mathcal{F}$ be the filter generated by Z(I); (that is, $\mathcal{F}$ = {F is a subset of X: $F \supseteq Z(f)$ for some $f\varepsilon I$} $\equiv \mathcal{G}([Z(I)])$). Then $\mathcal{F}$ is a closed filter in $(X, \mathcal{U})$.

Proof: We show that Z(I) is a filter base for $\mathcal{F}$, consisting only of decreasing closed sets. Obviously, Z(f) is a decreasing closed set for each $f\varepsilon I$. Let $Z(f) \varepsilon Z(I)$. Then there exists an idempotent $g \neq 1$ in $\mathcal{L}^i(X)$ such that $g \cdot f = f$. Hence $Z(g) \subseteq Z(f)$. Suppose that $Z(g) = \phi$. Then $g(x) \neq 0$ for all $x\varepsilon X$. Since g is idempotent, $g(x) = 1$ for all $x\varepsilon X$. Hence $g = 1$, but this is a contradiction. Therefore $Z(g) \neq \phi$. It follows that $Z(f) \neq \phi$. If Z(f) and Z(g) belong to Z(I), then by Remark 2.4.5, $Z(f) \cap Z(g) \varepsilon Z(I)$. Hence Z(I) is a filter base for $\mathcal{F}$, that is, $\mathcal{F}$ is a closed filter in $(X, \mathcal{U})$.

2.5.2 Lemma: Let $(X, \mathcal{T}, \leq)$ be an ordered topological space with semicontinuous order, and let $\mathcal{F}$ be a closed filter in $(X, \mathcal{U})$. Then $Z^{-1}(\mathcal{F})$ is an ideal in $\mathcal{L}^i(X)$. Moreover, $\mathcal{F} = \mathcal{G}([Z(Z^{-1}(\mathcal{F}))])$.

Proof: It is easy to see that $Z^{-1}(\mathcal{F})$ is an ideal in $\mathcal{L}^i(X)$. To show the equality $\mathcal{F} = \mathcal{G}([Z(Z^{-1}(\mathcal{F}))])$, let $F \varepsilon \mathcal{G}([Z(Z^{-1}(\mathcal{F}))])$. Then there exists $B \varepsilon Z(Z^{-1}(\mathcal{F}))$ such that $B \subseteq F$. Since $B \varepsilon Z(Z^{-1}(\mathcal{F}))$, $B = Z(f)$ for some $f \varepsilon Z^{-1}(\mathcal{F})$; hence $B\varepsilon\mathcal{F}$, and thus $F\varepsilon\mathcal{F}$. Conversely, let $F\varepsilon\mathcal{F}$.

Since $\mathcal{F}$ is a closed filter in $(X, \mathcal{U})$, there exists a decreasing closed set A such that $A \varepsilon \mathcal{F}$ and $A \subsetneq F$. We note that $A = Z(\chi_{X-A})$. Hence $\chi_{X-A} \varepsilon Z^{-1}(\mathcal{F})$, and hence $A \varepsilon Z(Z^{-1}(\mathcal{F}))$. If follows that $F \varepsilon \mathcal{G}([Z(Z^{-1}(\mathcal{F}))])$ and hence the equality follows immediately.

2.5.3 Remark: We note that the above two Lemmas 2.5.1 and 2.5.2 hold dually for $\mathcal{L}^d(X)$ and $(X, \mathcal{L})$.

2.5.4 Lemma: Let $(X, \mathcal{T}, \leq)$ be an ordered topological space with semicontinuous order. Let (M,N) be a maximal bi-ideal in $(\mathcal{L}^i(X), \mathcal{L}^d(X))$. Let $\mathcal{F}$ and $\mathcal{G}$ be the filters generated by the families $\{Z(f): f\varepsilon M\}$ and $\{Z(g): g\varepsilon N\}$ respectively; that is, $\mathcal{F} = \mathcal{G}(Z(M))$ and $\mathcal{G} = \mathcal{G}(Z(N))$. Then $(\mathcal{F}, \mathcal{G})$ is a maximal bi-filter in X.

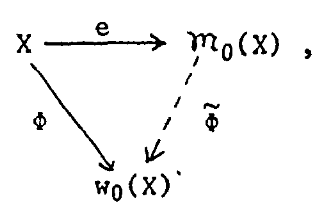
Proof: By Lemma 2.5.1 and Remark 2.5.3, $\mathcal{F}$ and $\mathcal{G}$ are closed filters in $(X, \mathcal{U})$ and $(X, \mathcal{L})$ respectively. If $F\varepsilon \mathcal{F}$ and $G\varepsilon \mathcal{G}$, then there exist $f\varepsilon M$ and $g\varepsilon N$ such that $Z(f) \subseteq F$ and $Z(g) \subseteq G$. Since (M,N) is a bi-ideal, $Z(f) \cap Z(g) \neq \phi$. Hence $F \cap G \neq \phi$. Therefore $(\mathcal{F}, \mathcal{G})$ is a bi-filter in X. To show the maximality of $(\mathcal{F}, \mathcal{G})$, suppose that there exists a bi-filter $(\mathcal{F}', \mathcal{G}')$ such that $(\mathcal{F}, \mathcal{G}) \subsetneq (\mathcal{F}', \mathcal{G}')$. Then $\mathcal{F} \subsetneq \mathcal{F}'$ or $\mathcal{G} \subsetneq \mathcal{G}'$. If $\mathcal{F} \subsetneq \mathcal{F}'$, then we have $Z^{-1}(\mathcal{F}) \subsetneq Z^{-1}(\mathcal{F}')$. It is now easy to see that $(M,N) = (Z^{-1}(\mathcal{F}), Z^{-1}(\mathcal{G})) \subsetneq (Z^{-1}(\mathcal{F}'), Z^{-1}(\mathcal{G}'))$, which contradicts the maximality of (M,N). Similarly, if $\mathcal{G} \subsetneq \mathcal{G}'$, then we have a contradiction. Thus $(\mathcal{F}, \mathcal{G})$ is a maximal bi-filter.

2.5.5 Lemma: Let $(X, \mathcal{T}, \leq)$ be an ordered topological space with semicontinuous order. Let $(\mathcal{F}, \mathcal{G})$ be a maximal bi-filter in X. Then

$(Z^{-1}(\mathcal{F}), Z^{-1}(\mathcal{G}))$ is a maximal bi-ideal in $(\mathcal{L}^{i}(X), \mathcal{L}^{d}(X))$.

Proof: By Lemma 2.5.2 and Remark 2.5.3, $Z^{-1}(\mathcal{F})$ and $Z^{-1}(\mathcal{G})$ are ideals in $\mathcal{L}^{i}(X)$ and $\mathcal{L}^{d}(X)$ respectively. Obviously, $(Z^{-1}(\mathcal{F}), Z^{-1}(\mathcal{G}))$ is a bi-ideal. Assume that there exists a bi-ideal (I,J) such that $(Z^{-1}(\mathcal{F}), Z^{-1}(\mathcal{G})) \subsetneq (I,J)$. Then $Z^{-1}(\mathcal{F}) \subsetneq I$ or $Z^{-1}(\mathcal{G}) \subsetneq J$. If $Z^{-1}(\mathcal{F}) \subsetneq I$, then we have $\varphi[Z(Z^{-1}(\mathcal{F}))] \subsetneq \varphi[Z(I)]$. It is now easy to see that $(\mathcal{F}, \mathcal{G}) \subsetneq (\varphi[Z(I)], \varphi[Z(J)])$. This contradicts the maximality of $(\mathcal{F}, \mathcal{G})$. If $Z^{-1}(\mathcal{G}) \subsetneq J$, then we have also a contradiction. Thus, $(Z^{-1}(\mathcal{F}), Z^{-1}(\mathcal{G}))$ is a maximal bi-ideal.

2.5.6 Theorem: Let $(X, \mathcal{T}, \leq)$ be a convex ordered topological space with semicontinuous order. Then given the following diagram:



there exists an iseomorphism $\tilde{\Phi}: \mathcal{M}_0(X) \to w_0(X)$ such that $\tilde{\Phi} o e = \Phi$; that is, $\mathcal{M}_0(X)$ and $w_0(X)$ are order equivalent.

Proof: For any $(M,N) \in \mathcal{M}_0(X)$, define $\tilde{\Phi}$: $\mathcal{M}_0(X) \to w_0(X)$ by $\tilde{\Phi}((M,N)) = (\varphi[Z(M)], \varphi[Z(N)])$. This is well defined by Lemma 2.5.4. Now, we shall show that $\tilde{\Phi}$ is the required map:

Firstly, we show that $\tilde{\Phi} o e = \Phi$: Let $x \in X$, and let $A \in \varphi(d(x))$, then $d(x) \subseteq A$. Since $d(x) = Z(\chi_{X-d(x)})$, we have $\chi_{X-d(x)} \in \mathcal{L}^{i}(X)$. Hence $\chi_{X-d(x)} \in M_x^i$. It immediately follows that $A \in \varphi[Z(M_x^i)]$. Hence

$\varphi(d(x)) \subseteq \varphi[Z(M_x^i)]$. Conversely, let $A \in \varphi[Z(M_x^i)]$. Then $A \supseteq Z(f)$ for some $f \in M_x^i$; hence $x \in Z(f)$. Since $Z(f)$ is a closed decreasing set, $d(x) \subseteq Z(f)$. Therefore $d(x) \subseteq A$ or $A \in \varphi(d(x))$. Thus $\varphi[Z(M_x^i)] \subseteq \varphi(d(x))$. Hence we have show that $\varphi(d(x)) = \varphi[Z(M_x^i)]$ for each $x \in X$. Similarly, we can show that $\varphi(i(x)) = \varphi[Z(M_x^d)]$. Thus we have $(\varphi(d(x)), \varphi(i(x))) = (\varphi[Z(M_x^i)], \varphi[Z(M_x^d)]$. Therefore for each $x \in X$,

$$\begin{aligned}(\tilde{\Phi} o e)(x) &= \tilde{\Phi}(e(x)) \\ &= \tilde{\Phi}((M_x^i, M_x^d)) \\ &= (\varphi[Z(M_x^i)], \varphi[Z(M_x^d)]) \\ &= (\varphi(d(x)), \varphi(i(x))) = \Phi(x).\end{aligned}$$

Hence $\tilde{\Phi} o e = \Phi$.

Secondly, we show that Φ is an order isomorphism: Let (M_1,N_1) and (M_2,N_2) be in $\mathfrak{M}_0(X)$, and let $\tilde{\Phi}((M_1,N_1)) = \tilde{\Phi}((M_2,N_2))$, that is, $(\varphi[Z(M_1)], \varphi[Z(N_1)]) = (\varphi[Z(M_2)], \varphi[Z(N_2)])$. We can easily see that $M_1 \subseteq Z^{-1}(\varphi[Z(M_1)])$ and $N_1 \subseteq Z^{-1}(\varphi[Z(N_1)])$. Since (M_1,N_1) is a maximal bi-ideal, $(M_1,N_1) = (Z^{-1}(\varphi[Z(M_1)]), Z^{-1}(\varphi[Z(N_1)]))$. Similarly, we have $(M_2,N_2) = (Z^{-1}(\varphi[Z(M_2)]), Z^{-1}(\varphi[Z(N_2)]))$. Since $\tilde{\Phi}((M_1,N_1)) = \tilde{\Phi}((M_2,N_2))$, $(Z^{-1}(\varphi[Z(M_1)]), Z^{-1}(\varphi[Z(N_1)])) = (Z^{-1}(\varphi[Z(M_2)]), Z^{-1}(\varphi[Z(N_2)]))$, that is, $(M_1,N_1) = (M_2,N_2)$. Hence $\tilde{\Phi}$ is one to one.

Let $(\mathcal{F}, \mathcal{G}) \in W_0(X)$. Then by Lemma 2.5.5, $(Z^{-1}(\mathcal{F}), Z^{-1}(\mathcal{G})) \in \mathfrak{M}_0(X)$. Hence $\tilde{\Phi}((Z^{-1}(\mathcal{F}), Z^{-1}(\mathcal{G})) = (\varphi[Z(Z^{-1}(\mathcal{F}))], \varphi[Z(Z^{-1}(\mathcal{G}))]) = (\mathcal{F}, \mathcal{G})$ by Lemma 2.5.2. Thus $\tilde{\Phi}$ is onto

Let $(M_1,N_1) \leq (M_2,N_2)$ in $\mathfrak{M}_0(X)$. Then $M_1 \supseteq M_2$ and $N_1 \subseteq N_2$. Thus, we have $\varphi[Z(M_1)] \supseteq \varphi[Z(M_2)]$ and $\varphi[Z(N_1)] \subseteq \varphi[Z(N_2)]$. Hence $\tilde{\Phi}((M_1,N_1)) \leq \tilde{\Phi}((M_2,N_2))$ and thus $\tilde{\Phi}$ is increasing.

Let (M_1,N_1) and (M_2,N_2) be in $\mathfrak{M}_0(X)$ and let $\tilde{\Phi}((M_1,N_1)) \leq \tilde{\Phi}((M_2,N_2))$ in $w_0(X)$. Then it is easy to show that $(M_1,N_1) \leq (M_2,N_2)$. Therefore $\tilde{\Phi}$ is an order isomorphism.

Finally, we show that $\tilde{\Phi}$ is a homeomorphism: For given $f \in \mathcal{L}^i(X)$, let $(M,N) \in f^d$; then $f \in M$ or $Z(f) \in Z(M)$. Since $Z(M) \subseteq \varphi[Z(M)]$, $Z(f) \in \varphi[Z(M)]$. Hence $\tilde{\Phi}((M,N)) = (\varphi[Z(M)], \varphi[Z(N)]) \in Z(f)^d$. Thus we have $\tilde{\Phi}(f^d) \subseteq Z(f)^d$. Conversely, let $(\mathcal{F}, \mathcal{G}) \in Z(f)^d$. Then $Z(f) \in \mathcal{F}$ or $f \in Z^{-1}(\mathcal{F})$; hence $(Z^{-1}(\mathcal{F}), Z^{-1}(\mathcal{G})) \in f^d$. Therefore $\tilde{\Phi}((Z^{-1}(\mathcal{F}), Z^{-1}(\mathcal{G})) = (\mathcal{F}, \mathcal{G}) \in \tilde{\Phi}(f^d)$. Thus $Z(f)^d \subseteq \tilde{\Phi}(f^d)$. Hence we have $\tilde{\Phi}(f^d) = Z(f)^d$ for given $f \in \mathcal{L}^i(X)$. Dually, $\tilde{\Phi}(g^i) = Z(g)^i$ for given $g \in \mathcal{L}^d(X)$. Since $\mathfrak{M}_0(X)$ and $w_0(X)$ are convex ordered topological spaces, $\tilde{\Phi}$ is clearly a homeomorphism. Hence $\tilde{\Phi}$ is an iseomorphism from $\mathfrak{M}_0(X)$ onto $w_0(X)$. This completes the proof.

2.5.7 <u>Remark</u>: If the given order on X in the above theorem 2.5.6, is discrete, then this reduces to the main results of Brümmer's paper [3] (see Theorem 0.20), that is, $\mathfrak{M}_0(X)$ is the Wallman compactification of a T_1 space X.

CHAPTER III

E-COMPACT ORDERED SPACES

Section 1. Separation Properties of ordered topological spaces.

In this section, all ordered topological spaces which we consider are assumed to have a continuous order.

3.1.1 Definition: (Priestley [29]). An ordered topological space $(X, \mathcal{T}, \leq)$ is said to be an N-space if given closed sets F_1, F_2 such that $a_1 \nleq a_2$ for all $a_1 \varepsilon F_1$, $a_2 \varepsilon F_2$, there exist disjoint open sets G_1, G_2 respectively decreasing, increasing such that $F_i \subseteq G_i$, $i = 1, 2$. An ordered topological space $(X, \mathcal{T}, \leq)$ is said to have property N if and only if X is an N-space.

3.1.2 Definition: (Hommel [17]). An ordered topological space $(X, \mathcal{T}, \leq)$ is said to have property:

T_I iff, with respect to the ordering and the dual ordering of X, one has: Given two disjoint subsets K, F of X, such that K is compact and F is closed and decreasing, there is a decreasing neighbourhood U of F, and an increasing neighbourhood V of K, such that $U \cap V = \phi$.

T_{II} iff given $x \varepsilon X$ and a neighbourhood U of x, there exists continuous functions f, g: $X \to [0,1]$ such that f is increasing, g is decreasing and furthermore, $f(x) = g(x) = 1$, and $\inf (f,g) = 0$ on X-U.

T_{III} iff for any x, $y \varepsilon X$ with $x \nleq y$, there is a continuous increasing function $f: X \to \mathbb{R}$ for which $f(x) > f(y)$.

T_{IV} iff for any closed decreasing (resp. increasing) subset F of X and any point $x \notin F$, there is an increasing continuous function $f: X \to \mathbb{R}$

such that $f = 0$ on F and $f(x) = 1$ (resp. $f = 1$ on F and $f(x) = 0$).

3.1.3 <u>Proposition</u>: Let $(X, \mathcal{T}, \leq)$ be an ordered topological space. If X has property N, then X has property T_I.

<u>Proof</u>: Let K, F be disjoint subsets such that K is compact and F is closed and decreasing. Then K is closed and $b \nleq a$ for any $a \in K$ and any $b \in F$. Since X has property N, there exist disjoint open sets G, H respectively decreasing, increasing such that $G \supseteq F$ and $H \supseteq K$. Hence X has property T_I.

3.1.4 <u>Remark</u>: The converse of the above proposition 3.1.3 is not true in general. For example, let $[0,\Omega]$ and $[0,\omega]$ be ordinal spaces, where Ω is the first uncountable ordinal and ω is the first infinite ordinal. Then $E = [0,\omega] \times [0,\Omega]$ is a compact ordered space with the cartesian order: $(m,\xi) \leq (n,\gamma)$ if and only if $m \leq n$ and $\xi \leq \gamma$ for (m,ξ) and (n,γ) in E. Let $X = [0,\omega] \times [0,\Omega] - \{(\omega,\Omega)\}$ with the relative topology and the relative order; then X has property T_I but does not have property N. For, X is a locally compact ordered space (cf. Dugundji [7]) and by Corollary 3.1.8, X has property T_I. But X is not an N-space, since there exist closed sets $F_1 = \{(n,\Omega): 0 \leq n < \omega\}$ and $F_2 = \{(\omega,\xi): 0 \leq \xi < \Omega\}$ such that $a_1 \nleq a_2$ for all $a_1 \in F_1$, all $a_2 \in F_2$, and that F_1, F_2 cannot be separated by disjoint open decreasing, increasing neighbourhoods, respectively.

We note that a locally compact ordered space need not be an N-space.

3.1.5 Remark: Priestley [29] showed that if X is an N-space, then X is a normally ordered space; but the following example will show that the converse is not true in general:

Let $X = [0,1] \times [0,1] - \{(0,1)\}$ with the usual topology and with the following ordering: $(x_1,y_1) \leq (x_2,y_2)$ if and only if $x_1 \leq x_2$ and $y_1 = y_2$ for (x_1,y_1) and (x_2,y_2) in X. Then X is a normally ordered space, but X is not an N-space; for, obviously X is a locally compact ordered space with a countable base, and thus X is a normally ordered space. But X is not an N-space, because there exist closed sets $F_1 = (\{1\} \times [0,1]) - \{(1,0)\}$, $F_2 = \{(0,0)\}$ such that $a_1 \nleq a_2$ for all $a_1 \varepsilon F_1$, all $a_2 \varepsilon F_2$. Moreover, F_1 and F_2 cannot be separated by disjoint open decreasing, increasing neighbourhoods respectively. In what follows, we assume that the concept of a regular ordered space is the same one in the McCartan [23].

3.1.6 Lemma: (McCartan [23]). Every locally compact ordered space $(X, \mathcal{T}, \leq)$ is regularly ordered.

3.1.7 Theorem: Let $(X, \mathcal{T}, \leq)$ be a regularly ordered space. Then X has property T_I.

Proof: Let K, F be disjoint subsets of X such that K is compact and F is closed and decreasing. Then X-F is open decreasing. For each xεK, we have xεX-F. Since X is a regularly ordered space, there exists an increasing neighbourhood U_x of x such that $\overline{U}_x \subseteq X-F$, or $F \subseteq X-\overline{U}_x$, for each xεK. Since K is compact there exist $x_1, x_2, \ldots, x_n$ in K such that $K \subsetneq \bigcup_{i=1}^{n} Ux_i$ and $F \subsetneq \bigcap_{i=1}^{n} (X-\overline{U}x_i)$. Let $U = \bigcup_{i=1}^{n} Ux_i$ and

$V = \bigcap_{i=1}^{n} (X-\overline{U}x_i)$. Then we have $U \cap V = \phi$.

We set $W = d(V)$. Assume that $U \cap W \neq \phi$, and so let $y \in U \cap W$. Then $y \in Ux_i$ for some i and there exists $z \in V$ such that $y \leq z$. Hence $z \in Ux_i$ for some i and therefore $U \cap V \neq \phi$, which is a contradiction. Thus we have $U \cap W \neq \phi$. Hence U is an increasing neighbourhood of K and W is a decreasing neighbourhood of F, with U and W disjoint. It follows that X has property T_I.

3.1.8 <u>Corollary</u>: (Hommel [17]). Any locally compact ordered space $(X, \mathcal{T}, \leq)$ has property T_I.

<u>Proof</u>: This follows immediately from Lemma 3.1.6 and Theorem 3.1.7.

3.1.9 <u>Remark</u>: 1. A regularly ordered space need not be an N-space. This follows immediately from the example of Remark 3.1.5 and the fact that every normally ordered space is obviously a regularly ordered space.
2. A regularly ordered space need not have any of the properties T_{II}, T_{III} and T_{IV}: The following example is due to Hommel [17]: Let Ω be the first uncountable ordinal, take $\Omega_0 = [1,\Omega)$ and $\Omega' = [1,\Omega]$ with their order topology. Then $X = \Omega_0 \times \Omega'$ is locally compact and $G = \Delta \cup (A \times B) \cup \{((x,x), (y,y)): x, y \in \Omega_0, x \leq y\} \cup \{((x,\Omega),(y,\Omega)): x, y \in \Omega_0, x \leq y\}$, where $A = \{(x,x): x \in \Omega_0\}$ and $B = \{(x,\Omega): x \in \Omega_0\}$, is the closed graph of an ordering on X. Then Hommel [17] showed that X didn't have properties T_{II}, T_{III} and T_{IV}. But by Lemma 3.1.6, X is a regularly ordered space; thus it shows that a regularly ordered space need not have any of the properties T_{II}, T_{III} and T_{IV}.

3.1.10 Remark: Combining the results in this section and the results in Hommel [17], we obtain the following table, where "+" indicates that the property in the left hand column implies the corresponding property in the top line.

	N	T_I	T_{II}	T_{III}	T_{IV}
locally compact ordered space	-	+	-	-	-
regularly ordered space	-	+	-	-	-
completely regularly ordered space	-	-	+	+	-
normally ordered space	-	+	-	+	+
compact ordered space	+	+	+	+	+

Section 2. E-completely regular ordered spaces.

Throughout section 2 and section 3, we assume that $E = (E, \mathcal{T}_E, \leq_E)$ is a fixed ordered topological space, unless otherwise indicated.

3.2.1 Definition: Let $(X, \mathcal{T}_X, \leq_X)$ be an ordered topological space. Then X is said to be E-completely regular ordered if X is iseomorphic to a subspace of a power of E.

3.2.2. Remark: 1. If the given orders on X and E are discrete, then $(X, \mathcal{T}_X, \leq_X)$ is E-completely regular ordered if and only if $(X, \mathcal{T}_X)$ is E-completely regular. In other words, the concept of E-completely regular ordered spaces has the concept of E-completely regular spaces as a special case (cf. Mrówka [26]).

2. Let $(X, \mathcal{T}_X, \leq_X)$ be an ordered topological space with continuous order, and let $E = \mathbb{I}$ (or $\mathbb{R}$). Then X is E-completely regular ordered if and only if X is completely regular ordered.

3.2.3 <u>Lemma</u>: Let $(X, \mathcal{T}_X, \leq_X)$ be an ordered topological space. Then the following statements hold:

1. $(E, \mathcal{T}_E, \leq_E)$ is E-completely regular ordered.

2. If $(X, \mathcal{T}_X, \leq_X)$ is E-completely regular ordered and $(X', \mathcal{T}_{X'}, \leq_{X'})$ is iseomorphic to a subspace of $(X, \mathcal{T}_X, \leq_X)$, then X' is a E-completely regular ordered.

3. The product of an arbitrary collection of E-completely regular ordered spaces is E-completely regular ordered.

<u>Proof</u>: These are immediate consequences of the definition.

Let <u>ECRO</u> denote the category of all E-completely regular ordered spaces and continuous increasing maps, and

<u>PTOP</u>: the category of all ordered topological spaces, and continuous increasing maps, and

<u>HOTS</u>: the category of all ordered topological spaces, whose order is continuous, and continuous increasing maps.

Then, by Lemma 3.2.3, <u>ECRO</u> is clearly a hereditary, productive category.

3.2.4 <u>Proposition</u>: <u>ECRO</u> is an epireflective subcategory of <u>PTOP</u>.

<u>Proof</u>: Given X ε <u>PTOP</u>, let $C_1(X,E)$ denote the family of all continuous increasing maps from X into E. Define $\phi: X \to E^{|C_1(X,E)|}$ by $\phi(x)(f) = f(x)$ for each $f \in C_1(X,E)$ and each $x \in X$. Then ϕ is obviously a continuous increasing map. Let $R(X) = \phi(X)$. Then by Lemma 3.2.3, R(X) is an E-completely regular ordered space with the relative topology and the relative order.

Firstly, we observe that for every $f \in C_1(X,E)$, there exists a unique

$\overline{f}$: $R(X) \to E$ such that $\overline{f}o\phi = f$. To show this, let $\overline{f} = p_f oe$, where e: $R(X) \to E^{|C_1(X,E)|}$ is the inclusion map and p_f: $E^{|C_1(X,E)|} \to E$ is the fth projection. Then $\overline{f}$ is continuous and increasing, and $\overline{f}o\phi = f$. Furthermore, since ϕ is onto, $\overline{f}$ is unique.

Secondly, we show that for any $Y \in$ ECRO and for any PTOP-morphism f: $X \to Y$, there exists a unique ECRO-morphism $\overline{f}$: $R(X) \to Y$ such that $\overline{f}o\phi = f$. To show this, we identify Y with a subspace of $E^{|S|}$, where S is a set, since Y is an E-completely regular ordered space. For every $s \in S$, let $f_s = p_s of$; then by the first observation, there exists a unique continuous increasing map $\overline{f}_s$: $R(X) \to E$ such that $\overline{f}_s o\phi = f_s$ for each $s \in S$. Define $\overline{f}$: $R(X) \to E^{|S|}$ by $\overline{f}(y) = (\overline{f}_s(y))_{s\in S}$ for each $y \in R(X)$. We denote this map by $\overline{f} = \prod_{s\in S} \overline{f}_s$. Then $\overline{f}$ is a continuous increasing map.

Since
$$\begin{aligned}\overline{f}(R(X)) = \overline{f}(\phi(X)) &= (\prod_{s\in S} \overline{f}_s)(\phi(X)) \\ &= (\prod_{s\in S} \overline{f}_s o\phi)(X) = (\prod_{s\in S} f_s)(X) \\ &= (\prod_{s\in S} p_s of)(X) \\ &= f(X),\end{aligned}$$

$\overline{f}$ is a continuous increasing map from $R(X)$ into Y. Moreover, the uniqueness of $\overline{f}$ is immediate from the surjectivity of ϕ.

3.2.5 Corollary: Let E be an ordered topological space with continuous order. Then ECRO is an epireflective subcategory of HOTS.

Proof: This follows immediately from the same argument as that given in the proof of the above Proposition 3.2.4.

3.2.6 Corollary: (Rodriquez [31]) CγORR is an epireflective subcategory of HOTS, where CγORR is the category of all completely regular ordered spaces and continuous increasing maps.

Proof: Let $E = \mathbb{I}$ (or $\mathbb{R}$). Then by Proposition 2.1.1, ECRO = CγORR. Hence the proof is immediate from Corollary 3.2.5.

3.2.7 Proposition: An ordered topological space $(X, \mathcal{T}_X, \leq_X)$ is an E-completely regular ordered space if and only if the following conditions hold:

1. For every p, q in X with $p \nleq_X q$, there exists a continuous increasing map $f: X \to E$ such that $f(p) \nleq_E f(q)$.

2. For every closed subset $A \subset X$ and every point $p \in X-A$, there exists a positive integer n and a continuous increasing function $f: X \to E^n$ with $f(p) \notin \overline{f(A)}$.

Proof: $(\rightarrow)$: Let X be an E-completely regular ordered space. Then there exists an iseomorphism h such that $h(X) \subset E^{|S|}$ for some set S. For every p, q in X with $p \nleq_X q$, we have $h(p) \nleq h(q)$. Hence $(p_s o h)(p) \nleq_E (p_s o h)(q)$ for some p_s which is the projection of $E^{|S|}$ into the sth coordinate space. Thus (1) holds. Let A be a closed subset of X and $p \in X-A$. Then $h(p) \in h(X) - \overline{h(A)}$ in $E^{|S|}$. Hence there exists a positive integer n such that $U = p_1^{-1}(U_1) \cap p_2^{-1}(U_2) \cap \ldots \cap p_n^{-1}(U_n)$ is a basic open set containing h(p) and $U \cap h(A) = \phi$. Define $f: X \to E^n$ by $f(x) = ((p_i o h)(x))_{i=1}^n$ for each $x \in X$. Then f is continuous and increasing, and $f(p) = ((p_1 o h)(p), \ldots, (p_n o h)(p)) \in U_1 \times U_2 \times \ldots \times U_n$. It is easy to show that $f(A) \cap (U_1 \times U_2 \times \ldots \times U_n) = \phi$. Hence $f(p) \notin \overline{f(A)}$. Thus (2) holds.

($\Leftarrow$): Define $h: X \to E^{|C_1(X,E)|}$ by $h(x)(f) = f(x)$ for each $x \varepsilon X$ and each $f \varepsilon C_1(X,E)$. Then h is continuous and increasing. Also h is one to one. For, let $p \neq q$ in X. Then $p \nleq_X q$ or $q \nleq_X p$. If $P \nleq_X q$, then, by (1), there exists a continuous increasing map $f: X \to E$ such that $f(p) \nleq_E f(q)$. Hence $f(p) \neq f(q)$. Similarly, we have $f(p) \neq f(q)$ whenever $q \nleq_X p$, and hence h is one to one. Let $h(x) \leq h(y)$ in $E^{|C_1(X,E)|}$, and suppose that $x \nleq_X y$. Then, by (1), $g(x) \nleq_E g(y)$ for some continuous increasing map $g: X \to E$. Hence $h(x) \nleq h(y)$. But this is a contradiction; hence $x \leq_X y$. Thus h is continuous and an order isomorphism. To show that h is a homeomorphism, let A be a closed subset of X and $p \varepsilon X-A$. Then, by (2), there exists a positive integer n and a continuous increasing map $f: X \to E^n$ such that $f(p) \notin \overline{f(A)}$. Let $f_k = p_k \circ f$, where $1 \leq k \leq n$. Then $f_k \varepsilon C_1(X,E)$ for each k, and clearly $f_1, f_2, \ldots, f_n$ satisfies the following condition: $(f_1(p), \ldots, f_n(p)) \notin \overline{\{(f_1(a), \ldots, f_n(a)) : a \varepsilon A\}}$, where the closure is taken in E^n. Hence by Theorem 0.12, h is an iseomorphism; that is, X is an E-completely regular ordered space.

Using the same argument as in Proposition 2.1.12, we have the following:

3.2.8 <u>Proposition</u>: Let $(X, \mathcal{T}_X, \leq_X)$ be T_0-ordered space such that for every x, y in X with $x \nleq_X y$, there exists a continuous increasing function $f: X \to E$ with $f(x) \nleq_E f(y)$. Then X is an E-completely regular ordered space if and only if for every net $\{x_\alpha : \alpha \varepsilon D\}$ in X, we have $x_\alpha \to x$ if and only if $f(x_\alpha) \to f(x)$ for every $f \in C_1(X,E)$.

3.2.9 Remark: 1. If an ordered topological spaces has a semicontinuous order (or continuous order), then its underlying topological space is clearly a T_0-space. Hence Proposition 3.2.8 holds for such ordered topological spaces as well.

2. Let 2 denote the two point discrete space {0,1} with the usual order. If X is a 2-completely regular ordered space, then X is a compact zero-dimensional ordered space with continuous order.

Section 3. E-compact ordered spaces.

3.3.1 Definition: Let $(X, \mathcal{T}_X, \leq_X)$ be an ordered topological space. Then X is said to be E-compact ordered if X is iseomorphic to a closed subspace of a power of E.

3.3.2 Remark: 1. If the given orders on X and E are both discrete, then $(X, \mathcal{T}_X, \leq_X)$ is E-compact ordered if and only if $(X, \mathcal{T}_X)$ is an E-compact space. In other words, the concept of an E-compact ordered space contains that of an E-compact space as a special case (cf. Engelking and Mrówka [9]).

2. Let X be an ordered topological space with continuous order. If $E = \mathbb{I}$, then X is E-compact ordered if and only if X is compact ordered with a continuous order.

If $E = \mathbb{R}$, then X is E-compact ordered if and only if X is $\mathbb{R}$-compact ordered.

3. Every E-compact ordered space is obviously E-completely regular ordered.

3.3.3 Lemma: Let $(X, \mathcal{T}_X, \leq_X)$ be an ordered topological space. Then the following statements hold:

1. $(E, \mathcal{T}_E, \leq_E)$ is E-compact ordered.

2. If $(X, \mathcal{T}_X, \leq_X)$ is E-compact ordered and $(X', \mathcal{T}_{X'}, \leq_X')$ is iseomorphic to a closed subspace of $(X, \mathcal{T}_X, \leq_X)$, then X' is E-compact ordered.

3. The product of an arbitrary collection of E-compact ordered spaces is E-compact ordered.

Proof: This follows immediately from the definition.

Let ECOS denote the category of all E-compact ordered space and continuous increasing maps.

3.3.4 Theorem: Let E be an ordered topological space with continuous order. Then ECOS is an epireflective subcategory of ECRO.

Proof: Given $X \in$ ECRO, let $C_1(X,E)$ denote the family of all continuous increasing maps from X into E. Define $\sigma: X \to E^{|C_1(X,E)|}$ by $\sigma(x)(f) = f(x)$ for all $x \in X$ and all $f \in C_1(X,E)$. Then by the same arguments as those in Proposition 3.2.7, σ is an iseomorphism of X into $E^{|C_1(X,E)|}$. Let $\beta_{oE}X = \overline{\sigma(X)}$, where the closure is taken in $E^{|C_1(X,E)|}$ and $\overline{\sigma(X)}$ has the relative topology and the relative order. By the same methods as in Proposition 3.2.4, we obtain that for every continuous increasing map $f: X \to E$, there exists a unique continuous increasing map $\overline{F}: \beta_{oE} X \to E$ such that $\overline{f}|X = f$. Let Y be an E-compact ordered space, i.e. $Y \in$ ECOS, and $g: X \to Y$ be any continuous increasing function. As in Proposition 3.2.4, there exists a unique continuous increasing function h: $\beta_{oE} X \to Y$ such that $h|X = g$. To show this, let g be a given continuous increasing map from X into Y. Since Y is an E-compact ordered space, Y is iseomorphic to a closed subspace of $\Pi\{E_\alpha: \alpha\varepsilon\Gamma\}$, where $E_\alpha = E$ for every $\alpha\varepsilon\Gamma$. For each projection map p_α, put $g_\alpha = p_\alpha o g$. Then g_α is a continuous increasing map: $X \to E$. Hence there exists a unique continuous increasing extension $\overline{g}_\alpha: \beta_{oE} X \to E$. Now, define h: $\beta_{oE} X \to \Pi E_\alpha$ by $h(p)(\overline{g}_\alpha) = \overline{g}_\alpha(p)$ for each $p \in \beta_{oE} X$, and for all $\overline{g}_\alpha$, $\alpha\varepsilon\Gamma$. Then h is a continuous, increasing map. If $p\varepsilon X$, then $\overline{g}_\alpha(p) = g_\alpha(p) = p_\alpha o g(p)$. Hence $h(p) = g(p)$ for all $p\varepsilon X$. Finally, since Y is dense in $\beta_{oE} X$, h(X) is dense in $h(\beta_{oE} X)$. But, Y is closed in ΠE_α, and hence $h(X) \subset Y$. It follows that $h(\beta_{oE} X) \subset Y$. Therefore, h is obviously the required extension of g. Hence ECOS is an epireflective subcategory of ECRO.

3.3.5 Remark: One easily observes that if Y is an E-compact ordered space containing X densely and inducing the order of X and such that every continuous increasing map f: $X \to E$ admits a continuous increasing extension to Y, then Y is iseomorphic to $\beta_{oE} X$ under an iseomorphism that is the identity on X.

We call this $\beta_{oE} X$ in the above theorem the E-order compactification of X.

3.3.6 Coroliary: (Rodriquez [31]). Let $\mathbb{I}$cOT and $\mathbb{R}$cOT be the categories of $\mathbb{I}$-compact ordered and $\mathbb{R}$-compact ordered spaces respectively. Then $\mathbb{I}$cOT and $\mathbb{R}$cOT are both epireflective subcategories of CγORR.

Proof: Take $E = \mathbb{I}$ (or $\mathbb{R}$), then ECRO = $\mathbb{I}$cOT (or $\mathbb{R}$cOT) and ECRO = CγORR. Hence the proof is straightforward.

3.3.7 Corollary: Let E be an ordered topological space with continuous order. Let X be an E-completely regular ordered space. Then X is E-compact ordered if and only if $X = \beta_{oE} X$.

Proof: This follows immediately from Theorem 3.3.4 and Remark 3.3.5.

3.3.8 Remark: Let E be an ordered topological space with continuous order. Then, combining Corollary 3.2.5 and Theorem 3.3.4, we obtain that ECOS is also an epireflective subcategory of HOTS.

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