NOETHERIAN PRIME RINGS OF

KRULL DIMENSION ONE

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By

RONALD WARREN RICHARDS, B.Sc., M.Sc.

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AUTHOR: Ronald Warren Richards, B.Sc. (University of Calgary)
M.Sc. (McMaster University)

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ABSTRACT

A right noetherian prime ring R is, by Goldie's Theorem, a right order in a simple artinian ring Q: Q is obtained from R by inverting all non-zero-divisors. Q can be described as the quotient ring of R at a torsion theory, the Goldie torsion theory. If R has right Krull dimension one, the Goldie torsion theory is generated by the class of all simple right R-modules.

In this thesis we develop a theory of localization for (right) noetherian prime rings of (right) Krull dimension one, based on the direct decompositions of the Goldie torsion theory. We characterize these decompositions, using a natural partition of the class of all simple modules, and show that the quotient rings at the components remain right noetherian, prime and of right Krull dimension one. Other desirable properties of these localizations are determined: they are perfect, they preserve the two-sidedness of ideals, and they are well behaved on the simple modules. We further show that they generalize the localizations at classical semiprime ideals.

A criterion is given for a right quotient ring at a component of a decomposition of the right Goldie torsion theory to be also a left quotient ring at a component of the left Goldie torsion theory. We show that this criterion is satisfied if the ring has global dimension no larger than two.

Finally, we study hereditary noetherian prime rings in the context of our localization theory.

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INTRODUCTION

The principal aim of this dissertation is the development of a suitable theory of localization for noetherian prime rings of Krull dimension one.

Localization has long been a powerful tool in the study of commutative rings. In recent years, a more general theory of localization has been developed for non-commutative rings, based on the work of Gabriel [13], Lambek [23], Dickson [9], Silver [33], and Goldman [16], among others. The result is a description of rings of quotients through the multifaceted concept of a torsion theory.

A prime purpose of a localization technique is to facilitate the analysis of a ring through the study of "simpler" rings, the localizations. With this aim, and within the general framework of torsion theories, Jategaonkar [20], Lambek and Michler [24], and Mueller [31] have studied the quotient rings at multiplicative sets determined by semiprime ideals in noetherian rings. When these semiprime ideals are <u>classical</u>, that is, if the corresponding multiplicative sets are sufficiently well behaved, the localizations closely parallel those in the commutative situation. A cornerstone in this programme has been the description by Goldie [15] of necessary and sufficient conditions for a ring to be an order in a semisimple or simple artinian ring.

In this thesis we restrict our attention to noetherian prime rings of Krull dimension one, and obtain a localization theory which is an extension of the concept of the localization at a classical semiprime ideal. We can give some justification for the restrictions on the ring. A noetherian prime ring, by Goldie's criteria, is an order in a simple artinian ring. The various localizations will be subrings of this simple artinian Goldie quotient ring, and we can therefore utilize its arithmetic to study them. The additional assumption of Krull dimension one guarantees that the torsion theory associated with this Goldie quotient ring, the Goldie torsion theory, is generated by the class of all simple modules. This property then allows us to directly decompose the Goldie torsion theory. It is this decomposition which is the basis of our localization technique. Finally, noetherian prime rings of Krull dimension one provide a natural generalization of the much studied hereditary noetherian prime (HNP) rings, the non-commutative analogues of Dedekind domains. A comprehensive theory of HNP rings has been developed by Eisenbud and Robson in [11] and [12], by Kuzmanovich in [21] and [22], by Goodearl [17], Lenagan [25] and others.

In the first chapter we give a characterization of the direct decompositions of an arbitrary torsion theory. Following a description of some properties of the components of such a decomposition, we use this characterization to construct a decomposition of the torsion theory generated by the class of all simple right modules over a right noetherian ring, a decomposition which refines all others.

The direct decomposition of the torsion theory generated by all simple modules has been studied by Dickson [8] and Albu ([1], [2], and [3]) in the case where all components are generated by single simple modules. The components of our decomposition are generated by classes of simple modules which are linked by a relation, denoted "$\sim$", which constitutes the obstruction to such an atomic decomposition.

The second chapter then applies the above decomposition to the Goldie torsion theory over a right noetherian prime ring of right Krull dimension one. The quotient rings at the components of the decomposition remain right noetherian, prime and of right Krull dimension one, and the embedding of the ring into each of these is a flat epimorphism. Several other desirable properties of the localizations are given.

Mueller [30] has introduced the concept of a <u>clan</u> of prime ideals, a minimal set of incomparable prime ideals whose intersection is a classical semiprime ideal. We show that over a noetherian prime ring of Krull dimension one, the class of simple modules annihilated by the elements of a clan generates a component of our decomposition. Thus, the localization at a component provides a generalization of the localization at a classical semiprime ideal.

In the third chapter we consider, for a noetherian prime ring of Krull dimension one, the problem: Under what conditions is the right quotient ring at a component of the decomposition of the right Goldie torsion theory also a left localization at a component of the left Goldie torsion theory? We give a technical criterion for this to occur and show that this criterion is met if we require the ring to have global dimension no larger than two. In this case, we can characterize those components corresponding to clans as being precisely those which are generated by finite sets of unfaithful simple modules. Even if the criterion fails, we can show the equality of the left and right quotient rings at the components corresponding to clans, and at the torsion theory cogenerated by all simple modules not associated with any clan.

In the fourth chapter we examine HNP rings in the context of our decomposition of the Goldie torsion theory. Here, the clans are just the <u>cycles</u> (in the sense of Eisenbud and Robson [12].) Using a result of Lenagan [25], we show that the cycles are precisely the sets of annihilators of those $\sim$-linked classes consisting entirely of unfaithful simple modules.

For HNP rings, the decomposition of the Goldie torsion theory into components associated with cycles, with an additional component generated by the simple modules not associated with cycles, has already been studied by Kuzmanovich in [21] and [22]. We do not know if our theory provides a generalization of this: that is, whether this additional component decomposes further.

We then analyze the $\sim$-links for HNP rings. One consequence we obtain is the preservation of cycles under arbitrary localizations, if we discard those prime ideals which are dense.

A final chapter lists, without proof, some examples.

CHAPTER I

DIRECT DECOMPOSITION OF TORSION THEORIES

A few preliminary remarks on notation and terminology are in order. All rings will have identities and all modules will be unitary, and will be assumed right modules unless stated otherwise. We denote by mod-R the category of right modules over the ring R. Unless modified by "right" or "left" an ideal will be assumed two-sided; noetherian (artinian) will mean both right and left noetherian (artinian). A regular element is a non-zero-divisor. E(X) will denote the injective hull of a module X. The reader is refered to [4] for a general reference on the theory of rings, and to [36] for a reference on module categories, torsion theories and quotient rings.

In this chapter we give a technical characterization of the direct decomposition of a torsion theory, and use it to describe the decomposition of the torsion theory generated by all simple modules over a right noetherian ring.

1. A CHARACTERIZATION OF THE DIRECT DECOMPOSITIONS

A torsion theory on the category of right R-modules can be described by any of the following:

(a) a torsion class $\mathcal{T}$: a class of modules closed under submodules, factor modules, direct sums and extensions,

(b) a torsion-free class $\mathcal{F}$: closed under submodules, products, essential extensions and extensions,

(c) a torsion radical $\tau(\_)$: a left exact, idempotent subfunctor of the identity functor on mod-R, with $\tau(X/\tau(X))=0$ for all $X \in$ mod-R, or

(d) a Gabriel filter $\mathcal{G}$ of right ideals: See Chapter VI of [36] for a definition; it is there called a Gabriel topology.

The reader is refered to [36], Chapter VI, for the details of the relationship between these concepts. Note that what we call a torsion theory is there refered to as a hereditary torsion theory. We list some crucial properties which we shall be using:

(a) $\mathrm{Hom}_R(A,B)=0$ for all $A\in\mathcal{T}$ and $B\in\mathcal{F}$. Given either one of $\mathcal{T}$ or $\mathcal{F}$, this relation determines the other.

(b) $\tau(A)\in\mathcal{T}$ and $A/\tau(A)\in\mathcal{F}$ for all $A \in$ mod-R.

(c) $I\in\mathcal{G}$ if and only if $R/I\in\mathcal{T}$.

(d) $a\in\tau(A)$ if and only if $\mathrm{ann}_R(a) \in \mathcal{G}$, for all $A \in$ mod-R.

Through an abuse of language, we will identify the torsion theory with the corresponding torsion class. If a torsion theory is distinguished by a sub- or superscript, the corresponding torsion-free class, torsion radical and Gabriel filter will be denoted by $\mathcal{F}$, τ and $\mathcal{G}$ respectively, amended by the same sub- or superscript.

We say that a torsion theory $\mathcal{T}$ is generated by a class $\mathcal{A}\subset$ mod-R if $\mathcal{T}$ is the smallest torsion class containing $\mathcal{A}$, and $\mathcal{T}$ is cogenerated by $\mathcal{A}$ if $\mathcal{F}$ is the smallest torsion-free class containing $\mathcal{A}$.

The set of torsion theories over a ring forms a complete lattice with the meet $T \wedge T'$ corresponding to the torsion class $T \cap T'$ and the join $T \vee T'$ corresponding to the torsion-free class $F \cap F'$. 1 is the torsion theory where all modules are torsion; 0 the torsion theory where all modules are torsion-free.

Definition. If T is a torsion class in mod-R and $\{T_\delta : \delta\varepsilon\Delta\}$ is a set of torsion sub-classes of T, then $T = \bigoplus_{\delta\varepsilon\Delta} T_\delta$ is a direct decomposition (or simply decomposition) of T if $A = \sum_{\delta\varepsilon\Delta}\oplus\tau_\delta(A)$ for all $A\varepsilon T$. The $T_\delta(\delta\varepsilon\Delta)$ are the components of the decomposition. (Of course we use the notation $T = T_0 \oplus T_1$ for a decomposition with the two components T_0 and T_1.) We call the decomposition trivial if $T=T_{\delta'}$ for some $\delta'\varepsilon\Delta$.

Lemma 1.1. Assume $T = \bigoplus_\Delta T_\delta$ and let $\Gamma\subset\Delta$. Then $\bigvee_\Gamma T_\gamma = \bigoplus_\Gamma T_\gamma$.

Proof. Let C be the class of all direct sums of elements of $\bigcup_\Gamma T_\gamma$. It is easy to check that C is a torsion class: if X is any submodule, factor module or extension of any modules in C, then $X = \sum_\Delta\oplus\ \tau_\delta(X)$, and one shows $\tau_\delta(X)=0$ if $\delta\not\varepsilon\Delta$, so $X\varepsilon C$. It follows that $C = \bigvee_\Gamma T$, and so, since $C = \sum_\Gamma\oplus\ \tau_\gamma(C)$ for all $C\varepsilon C$, $\bigvee_\Gamma T_\gamma = \bigoplus_\Gamma T_\gamma$. □

Lemma 1.2. Assume $T = \bigoplus_\Delta T_\delta$ and let P be a partition of Δ. Then $T = \bigoplus_{\Gamma\varepsilon P} (\bigoplus_\Gamma T_\gamma)$.

Proof. Let $\tau_\Gamma(\_)$ be the torsion radical associated with $\bigvee_\Gamma T_\gamma$ for each $\Gamma\varepsilon P$. Let $X\varepsilon T$. By Lemma 1.1, $\tau_\Gamma(X) = \sum_\Gamma\oplus\ \tau_\gamma(X)$, so $X = \sum_\Delta\oplus\ \tau_\delta(X) = \sum_{\Gamma\varepsilon P}\oplus\ \tau_\Gamma(X)$. □

Proposition 1.3. Let $\mathcal{T}$ be a torsion theory. $\mathcal{T} = \bigoplus_{\Lambda} \mathcal{T}_{\delta}$ if and only if:

i) $\mathcal{T} = \bigvee_{\Lambda} \mathcal{T}_{\delta}$ and

ii) $\bigvee_{\delta\neq\delta'} \mathcal{T}_{\delta} = F_{\delta'} \cap \mathcal{T}$ for all $\delta'\varepsilon\Delta$.

Proof. Assume i) and ii). Let $A\varepsilon\mathcal{T}$ and let $A_{\delta} = \tau_{\delta}(A)$ for each $\delta\varepsilon\Delta$. Consider $\sum_{\Delta} A_{\delta} \subset A$. This sum is direct, since for any $\delta'\varepsilon\Delta$,

$$A_{\delta'} \cap (\sum_{\delta\neq\delta'} A_{\delta}) \; \varepsilon \; \mathcal{T}_{\delta'} \cap (\bigvee_{\delta\neq\delta'} \mathcal{T}_{\delta})$$
$$= \mathcal{T}_{\delta'} \cap (F_{\delta'} \cap \mathcal{T}) \text{ by ii)}$$
$$= \{0\}.$$

Now $A/\sum_{\Delta}\oplus A_{\delta} \simeq (A/A_{\delta'})/(\sum_{\Delta}\oplus A_{\delta}/A_{\delta'})$ and $A/A_{\delta'} \; \varepsilon \bigvee_{\delta\neq\delta'} \mathcal{T}_{\delta}$ for all $\delta'\varepsilon\Delta$. Therefore

$$A/\sum_{\Delta}\oplus A_{\delta} \; \varepsilon \bigcap_{\delta'\varepsilon\Delta} (\bigvee_{\delta\neq\delta'} \mathcal{T}_{\delta})$$
$$= \bigcap_{\delta'\varepsilon\Delta} (F_{\delta'} \cap \mathcal{T}) \quad \text{by ii)}$$
$$= F \cap \mathcal{T} \quad \text{by i)}$$
$$= \{0\}.$$

We conclude that $A = \sum_{\Delta}\oplus A_{\delta}$ and thus $\mathcal{T} = \bigoplus_{\Lambda} \mathcal{T}_{\delta}$.

We now assume $\mathcal{T} = \bigoplus_{\Delta} \mathcal{T}_{\delta}$, and prove the reverse implication. i) follows trivially. To prove ii), assume $A \; \varepsilon \; F_{\delta'} \cap \mathcal{T}$. Then $\tau_{\delta'}(A)=0$, and so, applying Lemma 1.1, we see $A \; \varepsilon \bigoplus_{\delta\neq\delta'} \mathcal{T}_{\delta}$. To show the other inclusion, assume $B \; \varepsilon \bigvee_{\delta\neq\delta'} \mathcal{T}_{\delta}$. Then $\mathrm{Hom}_R(X,B)=0$ for all $X\varepsilon\mathcal{T}_{\delta'}$: if $\xi\varepsilon\mathrm{Hom}_R(X,B)$ then $\mathrm{Im}\xi \; \varepsilon \; (\bigvee_{\delta\neq\delta'} \mathcal{T}_{\delta}) \cap \mathcal{T}_{\delta'} = 0$, implying $\xi=0$. Therefore $B\varepsilon F_{\delta'}$. Obviously $B\varepsilon\mathcal{T}$ so the proof of ii) is complete. □

Remark. There is another useful characterization of the decomposition $\mathcal{T} = \mathcal{T}_0 \oplus \mathcal{T}_1$, in terms of the associated Gabriel filters: $\mathcal{T} = \mathcal{T}_0 \oplus \mathcal{T}_1$ if and only if

i) $I_0 + I_1 = R$ for all $I_0 \varepsilon \mathcal{G}_0$ and $I_1 \varepsilon \mathcal{G}_1$, and

ii) for all $I \varepsilon \mathcal{G}$ there exist $I_0 \varepsilon \mathcal{G}_0$ and $I_1 \varepsilon \mathcal{G}_1$ with $I_0 \cap I_1 = I$.

We omit the proof, which is straightforward, as we shall not need this criterion in the discussion that follows. Some other equivalent characterizations of this decomposition are given by Golan ([14]; 23.4).

Definition. Given a torsion theory $\mathcal{T}$, a module B is a $\mathcal{T}$-dense submodule of A if $A/B \varepsilon \mathcal{T}$, and a $\mathcal{T}$-closed submodule if $A/B \varepsilon \mathcal{F}$. A module D is $\mathcal{T}$-divisible if, for any $\mathcal{T}$-dense submodule B of a module A and any homomorphism $\xi \varepsilon \mathrm{Hom}_R(B,D)$, there is a mapping $\overline{\xi}$ in $\mathrm{Hom}_R(A,D)$ extending ξ. (We will use the terms dense, closed and divisible if $\mathcal{T}$ is evident from the context.)

Associated with each torsion theory $\mathcal{T}$, there is a full reflective subcategory $\mathcal{D}$ of mod-R, the category of torsion-free divisible modules. The reflector Q(_) is called the $\mathcal{T}$-quotient functor. For an elucidation and further characterizations of Q(_), the reader is refered to [36] (Chapter IX). We list some important properties:

a) Q(R), which we denote simply by Q, is a ring, the quotient ring or localization of R at $\mathcal{T}$.

b) $\mathcal{D}$ is isomorphic to a subcategory of mod-Q: every $D \varepsilon \mathcal{D}$ has a natural Q-module structure.

c) For any $A \varepsilon$ mod-R, we have the inclusions

$$A/\tau(A) \subset Q(A) \subset E(A/\tau(A))$$

with $A/\tau(A)$ dense in $Q(A)$ and $Q(A)$ closed in $E(A/\tau(A))$.

For a theory of localization to be effective, a procedure of globalization must be available: we must be able to conclude "global" properties of the original ring from the "local" properties of the various localizations. In our case, for example, if $\mathcal{T}$ is a torsion class in mod-R with a non-trivial decomposition $\mathcal{T} = \bigoplus_{\Delta} \mathcal{T}_\delta$ one can easily verify the following:

a) For any $X \varepsilon$ mod-R, $X=0$ if and only if $Q_\delta(X)=0$ for all $\delta \varepsilon \Delta$.

b) For any R-homomorphism ξ , ξ is a monomorphism if and only if $Q_\delta(\xi)$ is a monomorphism for all $\delta \varepsilon \Delta$.

(Compare these with the globalization results of Kuzmanovich ([21] and [22]) for the decomposition he gives of the Goldie torsion theory over an HNP ring.)

With a few restrictions imposed, we will show that the localizations at the components of a decomposition have other desirable properties. As a result of Lemma 1.2 we need only consider decompositions of the form $\mathcal{T} = \mathcal{T}_0 \oplus \mathcal{T}_1$.

Proposition 1.4. Let $\mathcal{T} = \mathcal{T}_0 \oplus \mathcal{T}_1$ and let $A \varepsilon \mathcal{T}_1$. Then $Q_0(A)=A$.

Proof. Since A is in $\mathcal{T}_1$, and hence $\mathcal{F}_0$, A is a submodule of $Q_0(A)$. We can thus consider the exact sequence

$$0 \to A \to Q_0(A) \to Q_0(A)/A \to 0$$

Now $Q_0(A) \varepsilon \mathcal{T}$ since $A \varepsilon \mathcal{T}_1$ and $Q_0(A)/A \ \varepsilon \ \mathcal{T}_0$. $Q_0(A)$ is $\mathcal{T}_0$-torsion-free divisible, so $\tau_0(Q_0(A))=0$. Therefore $Q_0(A) \varepsilon \mathcal{T}_1$, so

$Q_0(A)/A \in T_0 \cap T_1 = 0$, that is, $Q_0(A) = A$. □

Definition. A torsion theory T is faithful if $\tau(R)=0$. (In this case R is a subring of Q.)

Proposition 1.5. Assume $T = T_0 \oplus T_1$ where T_0 is faithful, and let $A \in T_1$. Then $Q_0(\mathrm{ann}_R(A)) = \mathrm{ann}_{Q_0}(A)$. (A is a Q_0-module by Proposition 1.4.)

Proof. Let $I = \mathrm{ann}_R(A)$ and $J = \mathrm{ann}_{Q_0}(A)$. If $I=0$ then $Q_0(I)=0$. But R is essential in Q_0 so $\mathrm{ann}_{Q_0}(A) = 0$.

Assume $I \neq 0$. $J \cap R \subset I$ since $A(J \cap R) = 0$. Also, $AQ_0(I) = 0$, for assume $aq=a'$ for some $a \in A$ and $q \in Q_0(I)$. As $Q_0(I)/I \in T_0$, there is a $L \in G_0$ with $qL \subset I$. Therefore $a'L = 0$ so $a' \in \tau_0(A) = 0$. Thus $Q_0(I) \subset J$. We then have the inclusions

$$J \cap R \subset I \subset Q_0(I) \cap R \subset J \cap R$$

and hence equality throughout.

Now $J/(J \cap R) \simeq (J + R)/R \in T_0$ so $J/Q_0(I) \in T_0$. Since $Q_0(I)$ is T_0-divisible, $Q_0(I)$ is a direct summand of J. But also $Q_0(I)$ is an essential R-submodule of J: if K is a submodule of J with $K \cap Q_0(I) = 0$, then $K \cap R = K \cap (J \cap R) = K \cap (Q_0(I) \cap R) = 0$, which implies $K = 0$ as R is essential in $R + J$. Therefore $Q_0(I) = J$. □

Remark. In the situation of Proposition 1.5, $A \in T_1$ is a faithful R-module if and only if it is a faithful Q_0-module.

The components of a decomposition retain some of the torsion theoretic properties of the original torsion theory, as we shall now illustrate.

Definition. A torsion theory is stable if the torsion class is closed under essential extensions.

Definition. A torsion theory $\mathcal{T}$ is perfect if the canonical embedding of $\mathcal{D}$ in mod-Q is an equivalence. In this case Q is a flat left R-module and Q(_), when considered as a functor on mod-R, is naturally equivalent to $\_ \otimes_R Q$, and hence is exact. (See [36], Chapter XI.)

Proposition 1.6. Let R be a right noetherian ring and let $\mathcal{T} = \bigoplus_{\Delta} \mathcal{T}_\delta$ be a decomposition of the torsion theory $\mathcal{T}$ over R. Then $\mathcal{T}$ is stable if and only if $\mathcal{T}_\delta$ is stable for all $\delta \varepsilon \Delta$.

Proof. a) Assume $\mathcal{T}$ is stable and let $A \varepsilon \mathcal{T}_{\delta'}$ for some $\delta' \varepsilon \Delta$. As A is essential in its injective hull E(A), $\tau_\delta(E(A)) \neq 0$ if and only if $\delta = \delta'$. Since $\mathcal{T}$ is stable, $E(A) \varepsilon \mathcal{T}$; it follows that $E(A) \varepsilon \mathcal{T}_{\delta'}$.

b) Assume $\mathcal{T}_\delta$ is stable for all $\delta \varepsilon \Delta$ and let $B \varepsilon \mathcal{T}$. We know $B = \sum_{\Delta} \oplus \tau_\delta(B)$. Now since R is right noetherian, a direct sum of injective modules is injective, so $E(B) = \sum_{\Delta} \oplus E(\tau_\delta(B)) \varepsilon \mathcal{T}$. □

We next show that a faithful torsion theory is perfect if and only if the components of a decomposition are all perfect. We first need two preliminary results.

Lemma 1.7. A torsion theory T is perfect if and only if every Q-module, when considered as an R-module, is in F.

Proof. One implication is trivial. To show the other, let us assume that every Q-module is torsion-free (as an R-module) and consider $X \in$ mod-Q. X is a Q-submodule of $Q(X)$ since $X \in F$, and thus by assumption, $Q(X)/X$, a Q-module, is in F. But we know $Q(X)/X \in T$. It follows that $Q(X) = X$, so X is torsion-free divisible for all X in mod-Q. Hence T is perfect. □

Lemma 1.8. Assume $T = T_0 \oplus T_1$, where T is a faithful torsion theory, and let $\hat{T}_1$ be the torsion class in mod-Q_0 consisting of all modules which are in T_1 as R-modules. Then $\hat{Q}_1(Q_0) = Q$.

Proof. As an R-module, Q/Q_0 is in $F_0 \cap T = T_1$, and $Q = Q_0(Q)$ is a Q_0-module. Therefore $Q/Q_0 \in \hat{T}_1$. But Q is $\hat{T}_1$-torsion-free divisible since it is T_1-torsion-free divisible. We can conclude that $Q = \hat{Q}_1(Q_0)$. □

Proposition 1.9. Let T be a faithful torsion theory and assume $T = \underset{\Delta}{\oplus} T_\delta$. Then T is perfect if and only if T_δ is perfect for all $\delta \in \Delta$.

Proof. a) Assume T_δ is perfect for all $\delta \in \Delta$ and let $X \in$ mod-Q. Then X is in mod-Q_δ for all $\delta \in \Delta$, so $X \in \bigcap_\Delta F_\delta = F$. By Lemma 1.7, T is perfect.

b) By Lemma 1.1, we need only consider the case $T = T_0 \oplus T_1$ and show that if T is perfect then T_0 is. Let X be a Q_0-module. By Lemma 1.8, and using the same notation, $\hat{Q}_1(Q_0) = Q$. Therefore $\hat{Q}_1(X)$ is a Q-module. Since T is perfect, $\hat{Q}_1(X) \in F$, and so

$X/\hat{\tau}_\gamma(X) \in F$. It follows that $\hat{\tau}_1(X) = \tau(X)$ and, by the decomposition of T, $\tau_0(X) = 0$: that is, $X \in F_0$. By Lemma 1.6, T_0 is perfect. □

Remark. a) actually proves that if $T = \bigvee_\Delta T_\delta$ and T_δ is perfect for each $\delta \in \Delta$, then T is perfect. (cf. [14], Proposition 17.4.)

2. DECOMPOSING THE TORSION THEORY GENERATED BY ALL SIMPLE MODULES

We proceed to decompose the torsion theory generated by the class of all simple modules over a right noetherian ring, using the criterion of Proposition 1.3.

Notation. We will denote by $S^\#$ the class of all simple right R-modules, and by $T^\#$ the torsion theory generated by $S^\#$.

Lemma 1.10. a) A module X is in the torsion class T generated by an isomorphism closed class S of simple modules if and only if every non-zero factor module of X has a submodule in S.

b) Every torsion sub-class of T is generated by a sub-class of S.

Proof. a) ([36], Chapter VIII, Proposition 3.1.)

b) Let T' be any torsion sub-class of T, and let S' be the class of all simple modules in T'. Assume $X \in T'$. As $X \in T$, every non-zero factor module of X has a submodule in S. But this module is also in T' and hence in S'. Thus, every non-zero factor of X has a submodule in S', so by a), S' generates T' . □

We now define an equivalence relation on $S^\#$ such that the

the equivalence classes generate the components of a decomposition of $T^{\#}$ when the ring R is right noetherian.

<u>Definition</u>. $\sim$ is the equivalence relation on $S^{\#}$ generated by the following relations:

i) $S \sim S'$ if $\mathrm{Ext}^1_R(S,S') \neq 0$, and

ii) $S \sim S'$ if $S \cong S'$.

<u>Remark</u>. Condition i) means that $S \sim S'$ if there is a non-split exact sequence

$$0 \to S' \to X \to S \to 0$$

This is precisely the obstruction to an atomic decomposition of $T^{\#}$ in the sense of Dickson [8].

<u>Notation</u>. We denote by S_λ $(\lambda \in \Lambda)$ the $\sim$-equivalence classes (<u>$\sim$-classes</u>) and by T_λ the torsion class generated by S_λ for each $\lambda \in \Lambda$.

<u>Theorem 1.11</u>. Let R be a right noetherian ring. Then $T^{\#} = \bigoplus_{\Lambda} T_\lambda$.

<u>Proof</u>. We show that the hypotheses of Proposition 1.3 are satisfied:

i) $T^{\#} = \bigvee_{\Lambda} T_\lambda$: Obviously $\bigvee_{\Lambda} T_\lambda \subset T^{\#}$. Since $T^{\#}$ is generated by $S^{\#}$, and $S^{\#} \subset \bigvee_{\Lambda} T_\lambda$, it follows that $T^{\#} = \bigvee_{\Lambda} T_\lambda$.

ii) $\bigvee_{\lambda \neq \lambda'} T_\lambda = F_{\lambda'} \cap T^{\#}$ for all $\lambda' \in \Lambda$: Let $\mathcal{U}_{\lambda'} = \bigvee_{\lambda \neq \lambda'} T_\lambda$. By Lemma 1.10, $A \in \mathcal{U}_{\lambda'}$ if and only if every non-zero factor module of A has a submodule in $S^{\#} \setminus S_{\lambda'}$ (the complement of $S_{\lambda'}$ in $S^{\#}$).

Let $A \in \mathcal{U}_{\lambda'}$. If $\xi \in \mathrm{Hom}_R(B,A)$ is a non-zero homomorphism for any $B \in \mathcal{T}_{\lambda'}$, then A has a submodule in $\mathcal{S}_{\lambda'}$, a contradiction. Hence $\mathrm{Hom}_R(B,A) = 0$ for all $B \in \mathcal{T}_{\lambda'}$ and thus A is in $\mathcal{F}_{\lambda'}$. Therefore $\mathcal{U}_{\lambda'} \subset \mathcal{F}_{\lambda'} \cap \mathcal{T}^{\#}$.

To show the reverse inclusion, let $A \in \mathcal{F}_{\lambda'} \cap \mathcal{T}^{\#}$. We will construct a sequence $(A_\beta)_{\beta \leq \nu}$ (for some ordinal number ν) of submodules of A satisfying, for all $\beta, \gamma \leq \nu$:

i) $A_\beta \subsetneq A_\gamma$ if and only if $\beta < \gamma$,

ii) $A_\beta \in \mathcal{U}_{\lambda'}$,

iii) $\mathrm{Ext}^1_R(S, A_\beta) = 0$ for all $S \in \mathcal{S}_{\lambda'}$, and

iv) $A_\nu = A$

It is then evident that $A \in \mathcal{U}_{\lambda'}$. The sequence is constructed recursively as follows:

Take $A_0 = 0$, and assume we have constructed A_β for all $\beta < \gamma$, such that i), ii) and iii) are satisfied.

If γ is a limit ordinal, let $A_\gamma = \bigcup_{\beta<\gamma} A$. Then $A_\gamma \in \mathcal{U}_{\lambda'}$ since a torsion class is closed under arbitrary colimits. $\mathrm{Ext}^1_R(S, A_\gamma) = 0$ for all $S \in \mathcal{S}_{\lambda'}$ since, as R is right noetherian, $\mathrm{Ext}^1_R(S,\_)$ preserves directed colimits ([35], Theorem 3.2).

If γ is not a limit ordinal, we can consider $X_{\gamma-1}$. If $X_{\gamma-1} = A$ we are done; otherwise $A/A_{\gamma-1} \in \mathcal{T}$ so, by Lemma 1.10 there is a module A_γ , where $A_{\gamma-1} \subset A_\gamma \subset A$, and such that $A_\gamma/A_{\gamma-1}$ is simple. In fact $A_\gamma/A_{\gamma-1}$ is in $\mathcal{S}^{\#} \setminus \mathcal{S}_{\lambda'}$, for assume the contrary. $A_\gamma/A_{\gamma-1} \in \mathcal{S}_{\lambda'}$ implies $\mathrm{Ext}^1_R(A_\gamma/A_{\gamma-1}, A_{\gamma-1}) = 0$, so A has a direct summand isomorphic to $A_\gamma/A_{\gamma-1}$. But this implies that A has a submodule in $\mathcal{T}_{\lambda'}$, a contradiction. Thus $A_\gamma/A_{\gamma-1} \in \mathcal{S}^{\#} \setminus \mathcal{S}_{\lambda'}$, and so $A_\gamma \in \mathcal{U}_{\lambda'}$.

To show $\mathrm{Ext}_R^1(S,A_\gamma) = 0$ for all $S \in \mathcal{S}_{\lambda}$, we apply the long exact sequence in $\mathrm{Ext}_R(S,\_)$ (cf. [26], Theorem 9.6) to the exact sequence

$$0 \to A_{\gamma-1} \to A_\gamma \to A_\gamma/A_{\gamma-1} \to 0$$

We obtain, for each $S \in \mathcal{S}_{\lambda}$, the exact sequence

$$\mathrm{Ext}_R^1(S,A_{\gamma-1}) \to \mathrm{Ext}_R^1(S,A_\gamma) \to \mathrm{Ext}_R^1(S,A_\gamma/A_{\gamma-1})$$

But $\mathrm{Ext}_R^1(S,A_{\gamma-1}) = 0$ by our construction, and $\mathrm{Ext}_R^1(S,A_\gamma/A_{\gamma-1}) = 0$ since S and $A_\gamma/A_{\gamma-1}$ are not $\sim$-equivalent. Therefore $\mathrm{Ext}_R^1(S,A_\gamma) = 0$.

It is clear that for some ordinal ν, $A = A_\nu$. □

We conclude the chapter with a proof that $\mathcal{T}^\# = \bigoplus_\Lambda \mathcal{T}_\lambda$ is the "best possible" direct decomposition of $\mathcal{T}^\#$.

<u>Definition</u>. Let $\mathcal{T}$ be a torsion theory with $\mathcal{T} = \bigoplus_\Delta \mathcal{T}_\delta = \bigoplus_\Gamma \mathcal{U}_\gamma$. The decomposition $\bigoplus_\Delta \mathcal{T}_\delta$ <u>refines</u> the decomposition $\bigoplus_\Gamma \mathcal{U}_\gamma$ if there is a partition $\mathcal{P}$ of Δ such that for each $\gamma \in \Gamma$ there is a $\Delta_\gamma \in \mathcal{P}$ with $\mathcal{U}_\gamma = \bigoplus_{\Delta_\gamma} \mathcal{T}_\delta$.

<u>Proposition 1.12</u>. $\bigoplus_\Lambda \mathcal{T}_\lambda$ refines any direct decomposition of $\mathcal{T}^\#$.

<u>Proof</u>. Assume $\mathcal{T}^\# = \bigoplus_\Gamma \mathcal{U}_\gamma$ and let S_γ be the class of simple modules in $\mathcal{U}_\gamma$. Let $S \in \mathcal{S}_\gamma$ and $S' \in \mathcal{S}^\#$ such that there is a non-split exact sequence

$$0 \to S \to X \to S' \to 0$$

Since X is indecomposable, and is in $\mathcal{T}^\#$ with a submodule in $\mathcal{U}_\gamma$, we must have $X\varepsilon\mathcal{U}_\gamma$. Hence $S'\varepsilon\mathcal{U}_\gamma$.

Assume there is a non-split exact sequence

$$0 \to S' \to X \to S \to 0 .$$

$S'\varepsilon\mathcal{S}_{\gamma'}$ for some γ'; therefore, by the above, $\gamma' = \gamma$. It follows that $\mathcal{S}_\gamma$ is $\sim$-closed for each $\gamma\varepsilon\Gamma$. Thus, there is a partition P of Λ with $\mathcal{S}_\gamma = \bigcup_{\Lambda_\gamma} \mathcal{S}_\lambda$ for some $\Lambda_\gamma\varepsilon P$. By Lemma 1.10, $\mathcal{U}_\gamma = \bigvee_{\Lambda_\gamma} \mathcal{T}_\lambda = \bigoplus_{\Lambda_\gamma} \mathcal{T}_\lambda$ for each $\gamma\varepsilon\Gamma$. $\square$

CHAPTER II

LOCALIZING NOETHERIAN PRIME RINGS OF KRULL DIMENSION ONE

1. DECOMPOSING THE GOLDIE TORSION THEORY

In this section we apply Theorem 1.11 to obtain a direct decomposition of the Goldie torsion theory for a right noetherian prime ring of right Krull dimension one. We then analyze the quotient rings at the components.

An ideal I of a ring R is called prime if, for any $a,b \in R$, $aRb \subset I$ implies $a \in I$ or $b \in I$; a ring is prime if 0 is a prime ideal.

The Krull dimension, $\kappa(A)$, of a module A is defined recursively as follows:

i) $\kappa(A) = 0$ if A is artinian.

ii) $\kappa(A) = n$ if $\kappa(A) \nless n$ and for any descending chain

$$A \supset A_1 \supset \ldots \supset A_i \supset \ldots$$

of submodules of A, $\kappa(A_i/A_{i+1}) < n$ for all but finitely many i. The reader is refered to [18] for the properties of κ.

We say a ring R has right Krull dimension n, and write right-$\kappa(R) = n$, if R has Krull dimension n as a right module. If the right and left Krull dimensions of R exist and are equal, we call this common value the Krull dimension of R, and denote it by $\kappa(R)$.

By Goldie's Theorem [15], a right noetherian prime ring R is a right order in a simple artinian ring Q. This ring Q is the quotient ring of R at the Goldie torsion theory, whose Gabriel filter consists of all essential right ideals. (cf. [36], Chapter VI, §6.) This coincides with the maximal torsion theory, the torsion theory cogenerated by E(R). As a further consequence of Goldie's Theorem, R has finite Goldie dimension: there is a finite bound on the number of non-zero components in a direct sum decomposition of a right ideal. Also, a right ideal is essential if and only if it contains a regular element of R. Finally, we note that the uniform right ideals of R are mutually sub-isomorphic: each is isomorphic to a submodule of any other. (A right ideal is <u>uniform</u> if every non-zero submodule is essential.)

Throughout this section R will be assumed to be right noetherian, prime and with right-$\kappa(R) = 1$.

<u>Proposition 2.1</u>. The Goldie torsion theory over a right noetherian prime ring R with right-$\kappa(R) = 1$ is generated by the class of all simple right modules.

<u>Proof</u>. By [18] (Proposition 6.1), for a right noetherian prime ring:

$$\text{right-}\kappa(R) = \sup\{\ \kappa(R/K) + 1 : K \text{ an essential right ideal}\}\ .$$

As R has right Krull dimension 1, R/K is artinian for every essential right ideal K. Therefore, every cyclic Goldie torsion module is artinian. Lemma 1.10 then implies that the Goldie torsion theory is generated by an isomorphism-closed class of simple modules. Since R is right noetherian and prime (and not simple artinian), every simple right module is Goldie torsion. Therefore, the class of all simple right modules generates the Goldie torsion theory. □

We collect some consequences for the Goldie torsion theory from the results of the first chapter:

Proposition 2.2. The Goldie torsion theory over R has a direct decomposition into components generated by the $\sim$-classes of simple modules. Every component of every decomposition of the Goldie torsion theory is generated by $\sim$-closed class of simple modules.

Proof. Proposition 2.1, Theorem 1.11, and Proposition 1.12. □

Notation. $T^{\#}$ will now denote the Goldie torsion theory, $Q^{\#}$ the Goldie quotient ring, and T_λ $(\lambda \in \Lambda)$ the components of the decomposition, where T_λ is generated by the $\sim$-class S_λ :

Proposition 2.3. Any component of any direct decomposition of $T^{\#}$ is perfect and stable.

Proof. $T^{\#}$ is perfect and stable ([36], Chapter VI, §7 and Chapter XI, §4). Propositions 1.6 and 1.9 then yield the result. □

We can give a complete description of the lattice of torsion theories over a right noetherian prime ring of right Krull dimension 1:

Proposition 2.4. The lattice of sub-torsion theories of $T^{\#}$ is a Boolean lattice, isomorphic to the power set lattice of $S^{\#}$. Furthermore, $T^{\#}$ is the unique co-atom in the lattice of all torsion theories over R.

Proof. By Proposition 2.1 and Lemma 1.10, every sub-torsion theory of $T^{\#}$ is completely determined by its simple torsion modules. Therefore, the lattice of sub-theories of $T^{\#}$ is isomorphic to the Boolean lattice of subsets of $S^{\#}$.

To show that $T^{\#}$ is the unique co-atom in the lattice of all torsion theories, we must show that if T is any torsion class, other than the torsion class of all modules, then $T \subset T^{\#}$. Since $T^{\#}$ is the maximal torsion theory, and as such contains all torsion theories for which R is torsion-free, we need only show that $\tau(R) = 0$.

Assume, conversly, that $\tau(R) \neq 0$. Then there is a uniform right ideal $U \varepsilon T$. Since the uniform right ideals are mutually sub-isomorphic, all uniform right ideals are in T. Because R has finite Goldie dimension, some finite direct sum of uniform right ideals is essential in R. Thus R has an essential right ideal K which is in T. But then K contains a regular element c, so $R \simeq cR$ is in T. We conclude that T = mod-R, a contradiction. □

We now come to the main result of this section, which describes the quotient ring of R at any component of a decomposition of $T^{\#}$.

Theorem 2.5. Let $T^{\#} = T_1 \oplus T_2$ be any non-trivial decomposition of $T^{\#}$ over a right noetherian prime ring R of right Krull dimension 1, with T_1 and T_2 generated by the $\sim$-closed classes S_1 and S_2 respectively. Let $\{S_\lambda : \lambda \varepsilon \Lambda_2\}$ be the partition of S_2 into $\sim$-classes. Then:

a) Q_1, the quotient ring at T_1, is right noetherian, prime and right-$\kappa(Q_1) = 1$,

b) S_2 is precisely the class of simple right Q_1-modules,

c) $\{S_\lambda : \lambda \in \Lambda_2\}$ is precisely the set of $\sim$-classes of simple Q_1-modules, and

d) $Q_1(\mathrm{ann}_R(A)) = \mathrm{ann}_{Q_1}(Q_1(A))$ for every unfaithful right module A.

<u>Proof</u>. a) Since T_1 is perfect, the right ideals of Q_1 are in a one-to-one, order preserving correspondence with the T_1-closed right ideals of R. Thus Q_1 is right noetherian since R is. Similarly, for any descending chain of right ideals of Q_1, all but finitely many of the factors are artinian, as this is the case in R. Noting that Q_1 is not artinian, as the decomposition was assumed non-trivial, we conclude that right-$\kappa(Q_1) = 1$.

Q_1 is prime, for if I and I' are ideals of Q_1 with $II' = 0$, then $I \cap R = 0$ or $I' \cap R = 0$ since R is prime. Therefore $I = 0$ or $I' = 0$, as R is essential in Q_1.

b) By Proposition 1.4, if $S \in S_2$ then $Q_1(S) = S$, so S_2 is a subclass of the class of all simple Q_1-modules

Let X be any simple right Q_1-module. Then $X \simeq Q_1/M$ for a maximal right ideal M of Q_1. Since Q_1 is right noetherian, prime and with right Krull dimension 1, M is an essential right ideal. Therefore $M \cap R$ is an essential right ideal of R. Now

$$R/(M \cap R) \simeq (M+R)/M \subset Q_1/M$$

so X has a simple R-submodule S. As T_1 is perfect, $X \in F_1$. Thus

$X \in \mathcal{F}_1 \cap \mathcal{T}^\# = \mathcal{T}_2$ so $S \in \mathcal{T}_2$. It follows that $S = Q_1(S) = X$. We conclude that every simple right Q_1-module is in $\mathcal{S}_2$. Therefore $\mathcal{S}_2$ is precisely the class of simple Q_1-modules.

c) To show that $Q_1(\_)$ preserves the $\sim$-classes in $\mathcal{S}_2$ we prove the stronger result: $\mathrm{Ext}^1_R(S,S') \neq 0$ if and only if $\mathrm{Ext}^1_{Q_1}(S,S') \neq 0$ for S and S' in $\mathcal{S}_2$.

Let S and S' be in $\mathcal{S}_2$ and assume $\mathrm{Ext}^1_R(S,S') \neq 0$. Then there is a non-split exact sequence of R-modules:

$$0 \to S' \to X \to S \to 0$$

By Proposition 1.4, $S = Q_1(S)$, $X = Q_1(X)$ and $S' = Q_1(S')$, so this is also an exact sequence of Q_1-modules, which is obviously non-split as such.

Conversely, if $\mathrm{Ext}^1_{Q_1}(S,S') \neq 0$, then there is a non-split exact sequence of Q_1-modules

$$0 \to S' \to X \to S \to 0$$

If this were split as an R-exact sequence it would also be as a Q_1-exact sequence, since $Q_1(\_)$ would preserve the splitting. Thus $\mathrm{Ext}^1_R(S,S') \neq 0$.

d) Let $I = \mathrm{ann}_R(A)$. Since R is prime, I is essential in R, and hence $A \in \mathcal{T}^\#$. Now $R/I \in \mathcal{T}^\#$, so by the decomposition, $R/I = I_1/I \oplus I_2/I$, where $I_2/I \in \mathcal{T}_1$ and $I_1/I \in \mathcal{T}_2$. Then $R/I_2 \simeq I_1/I \in \mathcal{T}_2$ so $I_2 \in \mathcal{G}_2$. One readily verifies that, in fact, $I_2 = \mathrm{ann}_R(\tau_2(A))$ as I_2 is the minimal element of $\mathcal{G}_2$ containing I.

By Proposition 1.5, $Q_1(I_2) = \mathrm{ann}_{Q_1}(\tau_2(A))$. But, as $I_2/I \in \mathcal{T}_1$, $Q_1(I) = Q_1(I_2) = \mathrm{ann}_{Q_1}(\tau_2(A)) = \mathrm{ann}_{Q_1}(Q_1(A))$. $\square$

Corollary 2.6. In the situation of the above Theorem, $Q_1(I)$ is an ideal of Q_1 whenever I is an ideal of R.

Proof: d) of the Theorem. □

Theorem 2.5 shows that the localizations at the components of a decomposition of $T\#$ preserve certain crucial properties of the ring, properties which are preserved by the localizations at classical semiprime ideals. (cf. [19], Theorem 9.) In fact, the localizations at components generalize the localizations at classical semiprime ideals, as we shall show in the next section. We note also that Michler [29] has shown that if R is a fully bounded noetherian prime ring of Krull dimension one, then every over-ring of R which is contained in Q# is noetherian and of Krull dimension one.

2. ∼-CLASSES AND CLANS

We open with some remarks on the concepts discussed in this section. An ideal I of a ring R is semiprime if $aRa \subset I$ implies $a \in I$. Associated with a semiprime ideal I over a right noetherian ring R there is a multiplicative set $C(I) = \{c \in R: \bar{c} \text{ is regular in } R/I\}$. This set determines a torsion theory T_I with Gabriel filter $G_I = \{K \text{ a right ideal: } r^{-1}K \cap C(I) = \phi \text{ for all } r \in R\}$. By [24] (Corollary 3.10), this is the same as the torsion theory cogenerated by E(R/I).

A right Ore set is a multiplicative set C such that for each $x \in R$ and $c \in C$ there exist $x' \in R$ and $c' \in C$ with $xc' = cx'$. A semiprime ideal I of a (right) noetherian ring is <u>right localizable</u> if $C(I)$ is a right Ore set. If, in addition, the Jacobson radical J of Q_I has the right Artin-Rees property (i.e. for each right ideal K, there is an n such that $K \cap J^n \subset KJ$), then I is called <u>right classical</u>. I is called <u>classical</u> (<u>localizable</u>) if it is both right and left classical (respectively localizable). We will see that over a right noetherian prime ring R, where right-$\kappa(R) = 1$, the concepts of right classical and right localizable coincide.

If R is a right noetherian prime ring and right-$\kappa(R) = 1$, the quotient ring Q_I at the right classical semiprime ideal I is a semilocal noetherian prime ring with Jacobson radical $Q_I(I) = IQ_I$. KQ_I is an ideal of Q_I whenever K is an ideal of R, and the non-zero prime ideals of Q_I are in one-to-one correspondence with the prime (i.e. maximal) ideals of R which contain I. (cf. [20], Theorem 2.1.) Compare this with our Theorem 2.5.

A semiprime ideal over a right noetherian ring can be expressed uniquely as a finite irredundant intersection of prime ideals. If $\mathcal{C}$ is such a set of prime ideals, with $I = \bigcap \mathcal{C}$, we say that $\mathcal{C}$ is (right) classical if I is. We will use the alternate label $T_{\mathcal{C}}$ to denote the torsion theory T_I. By Lemma 2.4, $T_{\mathcal{C}}$ is generated by an isomorphism-closed class of simple modules, which we will call $S_{\mathcal{C}}$.

A <u>clan</u> (cf. [30]) is a non-empty finite classical set of mutually incomparable prime ideals, no subset of which is classical.

The reader is referred to [20], [24], [30], and [31] for references on these classical localizations.

In this section, noting a one-to-one correspondence between non-zero prime ideals and isomorphism types of unfaithful simple modules, we show that each clan corresponds to a $\sim$-class of simple modules, over a noetherian prime ring of Krull dimension 1.

<u>Lemma 2.7</u>. Let R be a right noetherian prime ring with right-$\kappa(R) = 1$. There is a one-to-one correspondence between non-zero prime ideals of R and isomorphism types of unfaithful simple right modules, associating to each unfaithful simple module its annihilator and to each prime ideal P the unique (up to isomorphism) simple submodule of R/P.

<u>Proof</u>. The verification is straightforward, given the observation that, since right-$\kappa(R) = 1$, every prime ideal P is maximal (or 0), so R/P is a simple artinian ring and thus all simple modules annihilated by P are isomorphic. □

<u>Lemma 2.8</u>. If R is a right noetherian prime ring and right-$\kappa(R) = 1$, then every right localizable semiprime ideal of R is right classical.

<u>Proof</u>. We use the criterion that a right localizable semiprime ideal I is right classical if, for every $x \in E(R/I)$, there is a natural number n such that $xI^n = 0$. ([20], Theorem 4.5.) Since xR is a cyclic Goldie torsion module, it has finite composition length, so $Q_I(xR)$ is a right Q_I-module of finite length. But Q_I is

semilocal having Jacobson radical $IQ_I = Q_I I Q_I$. Thus $(IQ_I)^n$ annihilates xQ_I . Therefore $xI^n = 0$. □

For the following lemmas, until we specify otherwise, R is a right noetherian prime ring, right-$\kappa(R) = 1$, and $\mathcal{C}$ is a right classical set of prime ideals with $I = \bigcap \mathcal{C}$. (For Lemma 2.13 and the results that follow it we must have these conditions on R on both the right and left.)

Lemma 2.9. Let S be a simple R-module. Then $S \in \mathcal{T}_{\mathcal{C}}$ if and only if $\mathrm{ann}_R(S) \notin \mathcal{C}$.

Proof. Assume $\mathrm{ann}_R(S) = P$, where $P \in \mathcal{C}$. S is isomorphic to a submodule of R/P and hence to a submodule of $E(R/I) = \bigoplus_{\mathcal{C}} E(R/P)$. Thus $S \in \mathcal{F}_{\mathcal{C}}$, so $S \notin \mathcal{T}_{\mathcal{C}}$.

Conversely, if $\mathrm{ann}_R(S) \notin \mathcal{C}$, then S is not isomorphic to any submodule of E(R/I). Thus $\mathrm{Hom}_R(S,E(R/I)) = 0$ so $S \in \mathcal{T}_{\mathcal{C}}$. □

Definition. Let $\mathcal{T}$ be a torsion theory in mod-R. A right ideal K of R is $\mathcal{T}$-critical if K is maximal among the $\mathcal{T}$-closed right ideals.

Lemma 2.10. Every $\mathcal{T}_{\mathcal{C}}$-critical right ideal of R is a maximal right ideal.

Proof. Let K be any $\mathcal{T}_{\mathcal{C}}$-critical right ideal. Then $Q_{\mathcal{C}}(K)$ is a maximal right ideal of $Q_{\mathcal{C}}$, since $\mathcal{T}_{\mathcal{C}}$ is perfect.

Now $IQ_{\mathcal{C}}$ is the Jacobson radical of $Q_{\mathcal{C}}$. Therefore $Q_{\mathcal{C}}(I)$ annihilates $Q_{\mathcal{C}}(R/K) \simeq Q_{\mathcal{C}}/Q_{\mathcal{C}}(K)$ and so $Q_{\mathcal{C}}(I) \cap R$ annihilates R/K. Thus $I \subset K$.

Assume K' is a maximal right ideal of R such that $K \subset K'$. Then $I \subset K'$, so $I \subset \mathrm{ann}_R(R/K')$. By Lemma 2.9, $R/K' \in \mathcal{F}_C$. Since K was assumed $\mathcal{T}_C$-critical, we conclude that $K = K'$ is a maximal right ideal of R. □

Remark. The above lemma shows that the simple modules annihilated by elements of C are precisely the simple right Q_C-modules: If S is a simple module with $\mathrm{ann}_R(S) \in C$, then there is a cyclic module X with $S \subset X \subset Q_C(S)$ and X/S a simple module in $\mathcal{T}_C$. Lemma 2.10 then implies that $X = S$. But any simple Q_C-module is the $\mathcal{T}_C$-quotient of some simple R-module, and hence the assertion.

Lemma 2.11. Let S and S' be simple R-modules such that $\mathrm{ann}_R(S) \in C$. If $\mathrm{Ext}^1_R(S',S) \neq 0$, then $\mathrm{ann}_R(S') \in C$.

Proof. Since $\mathrm{Ext}^1_R(S',S) \neq 0$, there is a non-split exact sequence

$$0 \to S \to X \to S' \to 0 \quad .$$

X is cyclic; say $X \simeq R/K$. We then see that $\mathrm{ann}_R(S') \in C$, for otherwise K would be $\mathcal{T}_C$-critical and hence maximal by Lemma 2.10. □

Lemma 2.12. Let X be an R-module of finite composition length. Then every composition factor of $X/\tau_C(X)$ is annihilated by an element of C.

Proof. Assume the contrary. Let

$$0 = X_0 \subset X_1 \subset \ldots \subset X_n = X/\tau_C(X)$$

be a composition series such that X_{m+1}/X_m is not annihilated by any element of C, and m is minimal with this property for all composition series of $X/\tau_C(X)$.

Now $X_m \neq 0$ by Lemma 2.9, since $X/\tau_C(X) \notin \Gamma_C$. We can therefore consider the exact sequence

$$0 \to X_m/X_{m-1} \to X_{m+1}/X_{m-1} \to X_{m+1}/X_m \to 0 .$$

By the minimality of m, $\mathrm{ann}_R(X_m/X_{m-1}) \in C$. But then, by Lemma 2.11, the above sequence must split, for otherwise $\mathrm{ann}_R(X_{m+1}/X_m) \in C$, contrary to assumption. The splitting of this sequence, however, contradicts the minimality of m. We therefore conclude that all composition factors of $X/\tau_C(X)$ are annihilated by elements of C. □

We must here strengthen the hypotheses on the ring R. R will henceforward be assumed to be a (two-sided) noetherian prime ring with $\kappa(R) = 1$. Note also that we use the notation p.dim(A) for the projective dimension of a module A.

<u>Lemma 2.13</u>. $\mathrm{Ext}^1_R(\_,R)$ is a duality between the categories of right and left R-modules of finite composition length and projective dimension one.

<u>Proof</u>. Let X be a right R-module of finite length and projective dimension 1. We will show that $X_* = \mathrm{Ext}^1_R(X,R)$ is a left R-module of finite length and projective dimension 1, and X_{**} is naturally isomorphic to X.

Let

$$0 \to P \to R^n \to X \to 0$$

be a projective resolution of X. Applying the long exact sequence in $\mathrm{Ext}_R(\_,R)$ we obtain the exact sequence

$$0 \to X^* \to (R^n)^* \to P^* \to X_* \to 0$$

(where $A^* = \mathrm{Hom}_R(A,R)$ for any module A). But $X \in T\#$, so $X^* = 0$. Thus $\mathrm{p.dim}(X_*) = 1$.

X_* is finitely generated since P^* is. We will show that X_* is left Goldie torsion, and it will follow that X_* has finite length. (Every cyclic Goldie torsion module has finite length, so every finitely generated one does.)

Given any $\theta \in \mathrm{Hom}_R(P,R)$, the result will be proven if we can find a regular element $c \in R$ such that $c\theta$ extends to a homomorphism $\hat{\theta}$ in $\mathrm{Hom}_R(R^n,R)$.

Now $R^n/P \in T\#$, so applying the Goldie quotient functor $Q\#(\_)$, we obtain $\theta\# = Q\#(\theta) \in \mathrm{Hom}_{Q\#}(Q\#(P),Q\#)$, and $Q\#(P) = Q\#(R^n) = (Q\#)^n$. Thus $\theta\#$ is determined by left multiplication by an element $(q_i)_{i=1}^{n}$ in $(Q\#)^n$. Take c to be a common left denominator of the q_i $(i=1,\ldots,n)$, which exists since R is an order in $Q\#$. If we restrict $c\theta\#$ to R, we have the required extension $\hat{\theta}$ of $c\theta$.

Finally, X_{**} is naturally isomorphic to X since $P^{**} \simeq P$ and $(R^n)^* \simeq R$ naturally. □

<u>Notation</u>. Henceforward we shall denote $\mathrm{Ext}^1_R(A,R)$ by A_* for any R-module A.

<u>Remark</u>. A similar duality was utilized in other contexts by Auslander and Bridger [5], Cohn [7], and Zaks [38].

Notation. If $\mathcal{C}$ is a clan, we will denote by $\mathcal{S}'_{\mathcal{C}}$ the class of all simple modules annihilated by elements of $\mathcal{C}$, and, as usual, by $\mathcal{T}'_{\mathcal{C}}$ the torsion theory generated by $\mathcal{S}'_{\mathcal{C}}$, with corresponding quotient functor $Q'_{\mathcal{C}}$. By Lemma 2.9, $\mathcal{S}'_{\mathcal{C}}$ is the complement of $\mathcal{S}_{\mathcal{C}}$ in $\mathcal{S}^{\#}$ and $\mathcal{T}'_{\mathcal{C}}$ is the complement of $\mathcal{T}_{\mathcal{C}}$ in the lattice of sub-torsion theories of $\mathcal{T}^{\#}$.

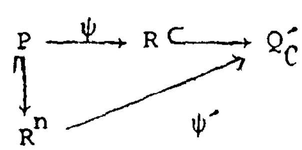
Lemma 2.14. Let $\mathcal{C}$ be a clan. Let X be a right R-module of finite length and projective dimension 1, and with every composition factor annihilated by a prime ideal in $\mathcal{C}$. Then every composition factor of X_* is annihilated by a prime ideal in $\mathcal{C}$.

Proof. Let $I = \bigcap \mathcal{C}$ and let

$$0 \to P \to R^n \to X \to 0$$

be a projective resolution of X. We may, in fact, assume that P is a submodule of R^n. We must show that there is a natural number m, such that for any $\theta \in \mathrm{Hom}_R(P,R)$ and any $p \in I^m$, there is a homomorphism $\hat{\theta} \in \mathrm{Hom}_R(R^n,R)$ extending $p\theta$. This is precisely the condition that I^m annihilates $X_* \simeq P^*/(R^n)^*$, and thus it implies that every composition factor of X_* is annihilated by an element of $\mathcal{C}$.

For any $\psi \in \mathrm{Hom}_R(P,R)$ there exists $\psi' \in \mathrm{Hom}_R(R^n,Q'_{\mathcal{C}})$ such that the diagram



commutes, since $R^n/P \simeq X \in \mathcal{T}'_{\mathcal{C}}$ and $Q'_{\mathcal{C}}$ is $\mathcal{T}'_{\mathcal{C}}$-divisible. In particular, any $p\theta$ can be extended by $p\theta'$.

Now consider the commuting diagram

$$\begin{array}{ccc} P & \xrightarrow{\theta} & R \\ \big\downarrow & & \big\downarrow \\ Q_{C}(P) & \xrightarrow[\theta_{C}]{} & Q_{C} \end{array}$$

where $\theta_{C} = Q_{C}(\theta)$. Q_{C} is a semilocal noetherian prime ring, $\kappa(Q_{C}) = 1$, and the Jacobson radical of Q_{C} is $Q_{C}(I) = IQ_{C}$. $Q_{C}(P)$ is a projective right Q_{C}-module and $Q_{C}(X) \simeq Q_{C}^{n}/Q_{C}(P)$ is a right Q_{C}-module of finite length and projective dimension 1. As Q_{C} is semilocal, all simple Q_{C}-modules are unfaithful, so $Q_{C}(I)^{k}$ annihilates $Q_{C}(X)$ for some k. We can therefore apply Lemma 2.13 to obtain that $\mathrm{Ext}^{1}_{Q_{C}}(Q_{C}(X),Q_{C})$ is a left Q_{C}-module of finite length and projective dimension 1, and hence there is some m such that $Q_{C}(I)^{m}$ annihilates $\mathrm{Ext}^{1}_{Q_{C}}(Q_{C}(X),Q_{C})$. But this implies that for every $p \varepsilon I^{m}$ there is a $\tilde{\theta}_{C} \; \varepsilon \; \mathrm{Hom}_{Q_{C}}(Q_{C}^{n},Q_{C})$ such that the diagram

$$\begin{array}{ccc} Q_{C}(P) & \xrightarrow{p\theta_{C}} & Q_{C} \\ \big\downarrow & \nearrow_{\tilde{\theta}_{C}} & \\ Q_{C}^{n} & & \end{array}$$

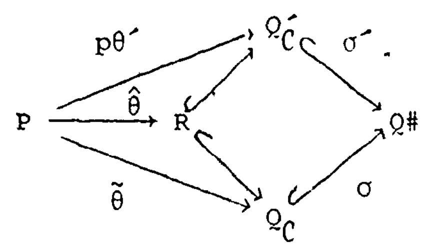
commutes.

Let $\tilde{\theta}$ be the restriction of $\tilde{\theta}_{C}$ to R^{n} and note that $p\theta$ is simply the restriction of $p\theta_{C}$ to P. We obtain the commuting diagram:

$$\begin{array}{ccc} P & \xrightarrow{p\theta} & Q_{C} \\ \big\downarrow & \nearrow_{\tilde{\theta}} & \\ R^{n} & & \end{array}$$

We have thus extended $p\theta$ to a homomorphism $p\theta' \in \mathrm{Hom}_R(P^n, Q'_C)$ and to a homomorphism $\tilde{\theta} \in \mathrm{Hom}_R(R^n, Q_C)$. Since $Q_C \cap Q'_C = R$, as T_C and T'_C are complementary in the lattice of sub-torsion theories of $T\#$, we have an extension of $p\theta$ to some $\hat{\theta} \in \mathrm{Hom}_R(R^n, R)$.

This can be illustrated by the following commuting diagram:



where $\hat{\theta}$ exists since R is the intersection of Q_C and Q'_C in $Q\#$ (and $\sigma' p\theta' = \sigma\tilde{\theta}$ since both extend to $Q\#(p\theta)$). □

<u>Lemma 2.15</u>. Let C be a clan and let $X \notin T_C$ be a right R-module of finite length. If $\mathrm{p.dim}(X) = 1$, then $\mathrm{p.dim}(X/\tau_C(X)) = 1$.

<u>Proof</u>. Let $X' = X/\tau_C(X)$. We first show that $Q_C(X') = X'$.

As $X' \in F_C$, we can consider $Q_C(X')/X' \in T_C$. If $Q_C(X')/X'$ is non-zero then it has a simple submodule $Y/X' \in T_C$. By Lemma 2.9, $\mathrm{ann}_R(Y/X') \notin C$. But by Lemma 2.12, every composition factor of $Y/\tau_C(Y)$ is annihilated by an element of C. Now $\tau_C(Y) \cap X' = 0$, and X' is essential in Y, so $\tau_C(Y) = 0$. Thus $\mathrm{ann}_R(Y/X') \in C$, a contradiction. It follows that $X' = Q_C(X')$.

Now consider an exact sequence:

$$0 \to K \to R^n \to X' \to 0$$

Since $X' = Q_C(X') = Q_C(X)$, X' has projective dimension 1 as a Q_C-module, and hence flat dimension at most 1 as an R-module (as Q_C is a flat

R-module). Therefore, K is a finitely generated flat module. As R is noetherian, K is projective. Consequently, $p.\dim(X') \leq 1$. But X' is Goldie torsion, so we must have $p.\dim(X') = 1$. □

Lemma 2.16. Let X be a (direct sum) indecomposable right module of finite length and projective dimension 1. If some composition factor of X is in $\mathcal{S}_{\mathcal{C}}'$ then X is in $\mathcal{T}_{\mathcal{C}}'$.

Proof. Let $X' = X/\tau_{\mathcal{C}}(X)$. $X' \in \mathcal{T}_{\mathcal{C}}'$ by Lemma 2.12, and $X' \neq 0$ since some composition factor of X is in $\mathcal{S}_{\mathcal{C}}'$. By Lemma 2.15, $p.\dim(X') = 1$. We will show that $X_* \simeq X'_*$. The result will follow, since then $X \simeq X'$.

Consider the exact sequence:

$$0 \to \tau_{\mathcal{C}}(X) \to X \to X' \to 0$$

Unless $\tau_{\mathcal{C}}(X) = 0$, in which case $X = X'$, $p.\dim(\tau_{\mathcal{C}}(X)) = 1$. We can apply $\mathrm{Ext}^1_R(\_,R)$ to obtain the exact sequence of left modules:

$$0 \to X'_* \to X_* \to \tau_{\mathcal{C}}(X)_* \to 0$$

If $\tau_{\mathcal{C}}(X)_* = 0$, we are done. Otherwise, let $C = \tau_{\mathcal{C}}(X)_*$. We show that C is in $\mathcal{T}^{o}_{\mathcal{C}}$, the left torsion class determined by $\mathcal{C}$.

Let $C/\tau^{o}_{\mathcal{C}}(C) = C'$. By Lemma 2.12, every composition factor of C' is annihilated by an element of $\mathcal{C}$. By Lemma 2.15, $p.\dim(C') = 1$ so C'_* is isomorphic to a submodule of $C_* = \tau_{\mathcal{C}}(X)$ and hence $C'_* \in \mathcal{T}_{\mathcal{C}}$. This contradicts Lemma 2.14 unless $C'_* = 0$. Hence $C' = 0$ and $C \in \mathcal{T}^{o}_{\mathcal{C}}$.

By Lemma 2.12, every composition factor of $X_*/\tau^{o}_{\mathcal{C}}(X_*)$ is annihilated by an element of $\mathcal{C}$. Therefore every composition factor of

$X_*/(X'_* \oplus \tau^o_{\mathcal{C}}(X_*))$ is annihilated by an element of $\mathcal{C}$. But $X_*/(X'_* \oplus \tau^o_{\mathcal{C}}(X_*))$ is an epimorphic image of C, and as such is in $\mathcal{T}^o_{\mathcal{C}}$. By Lemma 2.9, it follows that $X_* = X'_* \oplus \tau^o_{\mathcal{C}}(X_*)$. As X was assumed indecomposable we must have $X_* = X'_*$. □

Theorem 2.17. Let R be a noetherian prime ring with $\kappa(R) = 1$, and let $\mathcal{C}$ be a clan in R. The class $\mathcal{S}'_{\mathcal{C}}$ of all simple modules annihilated by elements of $\mathcal{C}$ is a $\sim$-equivalence class.

Proof. Let $S \in \mathcal{S}'_{\mathcal{C}}$ and suppose

$$0 \to S \to X \to S' \to 0$$

is a non-split exact sequence, where S' is a simple right R-module. Lemma 2.12 implies that $S' \in \mathcal{S}'_{\mathcal{C}}$.

Now, suppose

$$0 \to S'' \to Y \to S \to 0$$

is a non-split exact sequence, with S" simple. Since Y is cyclic, $Y \simeq R/K$ for some essential right ideal K. But then K contains a regular element c, and thus Y is an epimorphic image of the cyclic Goldie torsion module R/cR. Since Y has a unique non-zero proper submodule S", there must be some indecomposable summand Z of R/cR which has Y as an epimorphic image. p.dim(Z) = 1 since p.dim(R/cR)=1. As $S \in \mathcal{S}'_{\mathcal{C}}$, we can apply Lemma 2.16 to conclude that $Z \in \mathcal{T}'_{\mathcal{C}}$, and hence $S'' \in \mathcal{S}'_{\mathcal{C}}$. We have thus shown that $\mathcal{S}'_{\mathcal{C}}$ is $\sim$-closed.

Now, $Q_{\mathcal{C}}$ is a semilocal noetherian prime ring, $\kappa(Q_{\mathcal{C}}) = 1$ and the prime ideals of $Q_{\mathcal{C}}$ are precisely the $PQ_{\mathcal{C}}$ for $P \in \mathcal{C}$. ([20], Theorem 2.1.) In fact, $\mathcal{C} = \{PQ_{\mathcal{C}} : P \in \mathcal{C}\}$ is a clan in $Q_{\mathcal{C}}$, for if $\tilde{I}$

is any classical semiprime ideal of Q_C containing IQ_C, then $\tilde{I} \cap R$ is a classical semiprime ideal of R which contains I, contradicting that C is a clan unless $I = IQ_C$ (where, as usual, $I = \bigcap C$).

We note that since S_C and S'_C are $\sim$-closed, $T^{\#} = T_C \oplus T'_C$. Thus S'_C is precisely the class of simple right Q_C-modules, by Theorem 2.5.

Define the relation $\rightsquigarrow$ on the prime ideals of Q_C by: $PQ_C \rightsquigarrow P'Q_C$ if there exist simple right modules S and S', $\text{ann}_R(S) = P$, $\text{ann}_R(S') = P'$, and a non-split exact sequence of Q_C-modules

$$0 \to S \to X \to S' \to 0$$

(cf. [31], Proposition 2). By [31], Theorem 5, as Q_C is a fully bounded noetherian ring, C is generated under $\rightsquigarrow$ by any of its elements. Thus S'_C is a $\sim$-class of Q_C-modules.

But if we have a non-split exact sequence

$$0 \to S \to X \to S' \to 0$$

of Q_C-modules, this must also be a non-split exact sequence of R-modules: otherwise, the quotient functor $Q_C(\_)$ applied to this sequence would induce a splitting as a Q_C-exact sequence. Thus S'_C is a $\sim$-class. □

We isolate the following:

<u>Corollary 2.18</u>. For any clan C, $T^{\#} = T_C \oplus T'_C$, and every Goldie torsion module X has a direct sum decomposition $X = X_C \oplus X'_C$ where every simple subfactor of X'_C is annihilated by an element of C, and no simple subfactor of X_C is. □

We can obtain a partial converse of Theorem 2.17.

Proposition 2.19. Let $\mathcal{S}$ be a $\sim$-class consisting solely of unfaithful simple modules, and containing only finitely many isomorphism types. Then $\mathcal{C} = \{\mathrm{ann}_R(S) : S \in \mathcal{S}\}$ is a right classical set of prime ideals.

Proof. Let $I = \bigcap \mathcal{C}$ and recall that $\mathcal{C}(I)$ is the multiplicative set associated with I. By [20], Theorem 3.9, I is a right classical semiprime ideal if $\mathcal{C}(I)$ operates regularly on $E(R/I)$.

Let $x \in E(R/I)$. xR is a cyclic Goldie torsion module and so has finite length. Thus, there exist submodules $X' \subset X \subset E(R/I)$ such that $0 \neq x \in X \setminus X'$ and X/X' is a simple right module. But $X/X' \in \mathcal{S}$ since $E(R/I)$ is in the (stable) torsion class generated by $\mathcal{S}$. Assume $\mathrm{ann}_R(X/X') = P$. Then X/X' is isomorphic to a submodule of R/P and hence to a submodule of R/I, since $E(R/I) \simeq \bigoplus_{P \in \mathcal{C}} E(R/P)$. Since $\mathcal{C}(I)$ operates regularly on R/I, it operates regularly on X/X'. Therefore $xc \neq 0$ for all $0 \neq x \in E(R/I)$ and $c \in \mathcal{C}(I)$. Thus $\mathcal{C}(I)$ operates regularly on $E(R/I)$. $\square$

In Proposition 3.8, we will show that with some additional hypotheses (satisfied, for example, if the ring has global dimension no larger than 2), $\mathcal{C}$ is, in fact, a clan.

CHAPTER III

TWO-SIDED QUOTIENT RINGS

1. A CRITERION FOR TWO-SIDEDNESS

Throughout this chapter R will be assumed to be noetherian, prime and of Krull dimension 1. We will show that in certain cases, $\mathrm{Ext}^1_R(\_,R)$ provides a correspondence between right and left $\sim$-closed classes of simple modules in such a way that the corresponding quotient rings are identical. In particular, we show that this is true for those $\sim$-classes corresponding to clans, the $\sim$-closed class of all simples not associated with clans (and the complements of these classes), and for all $\sim$-classes if the global dimension of R is no greater than 2. The use of $\mathrm{Ext}^1_R(\_,R)$ to provide this correspondence is based on a suggestion by B. Mueller. The correspondence is defined as follows:

__Definition__. Let S be a $\sim$-closed class of simple right (or left) R-modules. Define A_S to be the class of all right (left) modules of finite composition length, projective dimension 1, and with every composition factor in S. We then define S^{o} to be the class of all simple composition factors of $X_* = \mathrm{Ext}^1_R(X,R)$ for all X in A_S. (Recall that by Lemma 2.13, $\mathrm{Ext}^1_R(\_,R)$ provides a duality between the categories of right and left modules of finite length and projective dimension 1.)

Lemma 3.1. Let $\mathcal{S}$ be a $\sim$-closed class of simple right modules. For any $S \in \mathcal{S}$ there exists $X \in A_{\mathcal{S}}$ such that S is an epimorphic image of X. X can be chosen indecomposable.

Proof. $S \simeq R/M$ for a maximal right ideal M of R. M contains a regular element c, so S is an epimorphic image of $Y = R/cR$, and p.dim (Y) = 1. But S must be an epimorphic image of some indecomposable direct summand X of Y. Since $\mathcal{S}$ is $\sim$-closed and X is indecomposable, all composition factors of X are in $\mathcal{S}$. Thus $X \in A_{\mathcal{S}}$. □

Lemma 3.2. Let $\mathcal{S}$ be a $\sim$-closed class of simple right modules. If $\mathcal{S}^{\circ}$ is $\sim$-closed then $\mathcal{S}^{\circ\circ} \supset \mathcal{S}$.

Proof. By Lemma 3.1, for any $S \in \mathcal{S}$ there is a X in $A_{\mathcal{S}}$ with S as an epimorphic image. S is a composition factor of $X_{**} \simeq X$ so $S \in \mathcal{S}^{\circ\circ}$. □

Lemma 3.3. Let $\mathcal{S}_1$ be a $\sim$-closed class of simple right modules such that $\mathcal{S}_1^{\circ}$ is $\sim$-closed and $\mathcal{S}_1 = \mathcal{S}_1^{\circ\circ}$, and let $\mathcal{S}_2$ be the complement of $\mathcal{S}_1$ in $\mathcal{S}\#$. Then $\mathcal{S}_2^{\circ}$ is $\sim$-closed, $\mathcal{S}_2 = \mathcal{S}_2^{\circ\circ}$ and $\mathcal{S}_2^{\circ}$ is the complement of $\mathcal{S}_1^{\circ}$ in $(\mathcal{S}\#)^{\circ}$, the class of all simple left modules.

Proof. We first show that any simple left module is in either $\mathcal{S}_1^{\circ}$ or $\mathcal{S}_2^{\circ}$. Let S be a simple left module. Then it is an epimorphic image of some indecomposable left module X of finite length and projective dimension 1. As X is indecomposable, X_* is, so either $X_* \in A_{\mathcal{S}_1}$ or $X_* \in A_{\mathcal{S}_2}$. Therefore S, as a composition factor of $X \simeq X_{**}$, is in either $\mathcal{S}_1^{\circ}$ or $\mathcal{S}_2^{\circ}$.

Now $\mathcal{S}_1^{\circ} \cap \mathcal{S}_2^{\circ} = \phi$, for assume S is in this intersection. Then

S is a composition factor of X_* for some indecomposable $X \in A_{S_2}$. But, as S_1^o is $\sim$-closed and X_* is indecomposable, every composition factor of X_* is in S_1^o. This implies $X \simeq X_{**} \in A_{S_1}$, a contradiction. Thus S_1^o and S_2^o are complementary subsets of $(S^\#)^o$. It follows that, since S_1^o is $\sim$-closed, S_2^o is.

It remains to show that $S_2' = S_2^{oo}$. By Lemma 3.2, it suffices to show that $S_2^{oo} \subset S_2$. Let $S \in S_2^{oo}$. Then S is a composition factor of Y, for some indecomposable $Y \in A_{S_2^o}$. Y_* is indecomposable, so is in either A_{S_1} or A_{S_2}, as S_1 and S_2 are complementary $\sim$-closed classes. But if $Y_* \in A_{S_1}$, then, since $S_1 = S_1^{oo}$, every composition factor of $Y \cong Y_{**}$ is in S_1, a contradiction. Therefore $Y_* \in A_{S_2}$ and hence $S \in S_2$. □

Lemma 3.4. a) If C is a clan and S_C' is the class of all simple right modules S where $\mathrm{ann}_R(S)$ is in C, then $(S_C')^o$ is the $\sim$-class of all simple left modules S' where $\mathrm{ann}_R(S')$ is in C. Furthermore, $(S_C')^{oo} = S_C'$.

b) If S' is the class of all simple right modules S where $\mathrm{ann}_R(S)$ is in no clan, then $(S')^o$ is the $\sim$-closed class of all simple left modules S' where $\mathrm{ann}_R(S')$ is in no clan. Furthermore, $(S')^{oo} = S'$.

Proof. Theorem 2.17, Lemma 2.14 and Lemma 3.3. □

Theorem 3.5. Let S be a $\sim$-closed class of simple right modules such that S^o is $\sim$-closed and $S^{oo} = S$, and let Q and Q^o be the quotient rings at the torsion theories generated by S and S^o

respectively. Then $Q = Q^o$ (as subrings of $Q\#$).

Proof. It is sufficient to show that $Q \otimes_R Q^o/R = 0$, for assume this is the case. Then, for any $q \in Q$ and $q_0 \in Q^o$ there exist $r \in R$ and $q' \in Q$ such that $q'r = q$ and $rq_0 \in R$. Therefore, $qq_0 = q'rq_0 \in Q$. If we take, in particular, $q = 1$, we obtain that q_0 is in Q. Thus $Q^o \subset Q$. By symmetry, $Q \subset Q^o$ so $Q = Q^o$.

To show that $Q \otimes_R Q^o/R = 0$, we first observe that the left R-module Q^o/R is the direct limit of its submodules of finite length and projective dimension 1: if $\bar{x} \in Q^o/R$, then $x = rc^{-1}$ where c is a regular element of R. Thus $\bar{x} \in Rc^{-1}/R$, which has finite length and projective dimension 1. But $R\bar{x} \in \mathcal{T}^o$, the torsion class generated by $\mathcal{S}^o$, so $\bar{x} \in \tau^o(Rc^{-1}/R)$. Now $\tau^o(Rc^{-1}/R)$ has finite length, projective dimension 1 (as it is a direct summand of Rc^{-1}/R), and is a submodule of Q^o/R (as $Q^o/R = \tau^o(Q\#/R) \supset \tau^o(Rc^{-1}/R)$).

As a result, we need only show that $Q \otimes_R X = 0$ for all $X \in \mathcal{A}_{\mathcal{S}^o}$. But, for any such X, $X_* \in \mathcal{A}_\mathcal{S}$ and $X \simeq X_{**}$. We must thus show that $Q \otimes_R \mathrm{Ext}^1_R(Y,R) = 0$ for all $Y \in \mathcal{A}_\mathcal{S}$.

The proof will be complete if we know that

$$Q \otimes_R \mathrm{Ext}^1_R(Y,R) \simeq \mathrm{Ext}^1_R(Y,Q)$$

for assume this is the case and let

$$0 \to Q \to A \to Y \to 0$$

be an exact sequence with $Y \in \mathcal{A}_\mathcal{S}$. Since $Y \in \mathcal{T}$ and Q is $\mathcal{T}$-divisible, this sequence splits. Thus $\mathrm{Ext}^1_R(Y,Q) = 0$ and hence the result.

All that remains to show is that, for $Y \in \mathcal{A}_S$:

$$Q \otimes_R \mathrm{Ext}_R^1(Y,R) \simeq \mathrm{Ext}_R^1(Y,Q) .$$

Take a projective resolution

$$0 \to P_1 \to P_0 \to Y \to 0$$

of Y. Applying the long exact sequence in $\mathrm{Ext}_R(\_,R)$, we obtain the exact sequence:

$$0 \to \mathrm{Hom}_R(P_0,Q) \xrightarrow{\alpha} \mathrm{Hom}_R(P_1,Q) \to \mathrm{Ext}_R^1(Y,Q) \to 0$$

By [26], Proposition 4.2,

$$Q \otimes_R \mathrm{Hom}_R(P_i,R) \simeq \mathrm{Hom}_R(P_i,Q) \qquad (i=0,1)$$

via the isomorphism ι_i with $\iota_i(q \otimes \theta) = q\theta$. Noting that the monomorphism $\beta: Q \otimes_R \mathrm{Hom}_R(P_0,R) \to Q \otimes_R \mathrm{Hom}_R(P_1,R)$ induced by ι_i is just the map $Q \otimes \alpha$, we obtain that $\mathrm{Ext}_R^1(Y,R)$ is the cokernel of $Q \otimes \alpha$. Therefore it is isomorphic to $Q \otimes_R \mathrm{Ext}_R^1(Y,R)$. □

<u>Corollary 3.6</u>. Let S be as in Theorem 3.5 and let S' be the complement of S in $S^{\#}$. Let Q' and $(Q')^{\circ}$ be the quotient rings at the torsion theories generated by S' and $(S')^{\circ}$ respectively. Then $Q' = (Q')^{\circ}$.

<u>Proof</u>. Lemma 3.3 and the above Theorem. □

<u>Corollary 3.7</u>. a) Let C be a clan. Then (in the terminology of Chapter II, 2) $Q_C = (Q_C)^{\circ}$ and $Q'_C = (Q'_C)^{\circ}$.

b) Let S' be the class of all simple modules not annihilated by any element of any clan, and let S'' be the complement of S' in $S^{\#}$. Then $Q' = (Q')^{\circ}$ and $Q'' = (Q'')^{\circ}$.

<u>Proof</u>. Lemma 3.3, Lemma 3.4 and the above Theorem. □

In the situation of Theorem 3.5, we can strengthen Proposition 2.19.

Proposition 3.8. Let $\mathcal{S}$ be a $\sim$-class of simple modules containing only finitely many isomorphism types, and such that every $S \in \mathcal{S}$ is unfaithful. If $\mathcal{S}^{o}$ is $\sim$-closed and $\mathcal{S} = \mathcal{S}^{\infty}$, then $\mathcal{C} = \{ann_R(S) : S \in \mathcal{S}\}$ is a clan.

Proof. With Q and Q^{o} defined as in Theorem 3.5, $Q = Q^{o}$, a noetherian prime ring of Krull dimension 1. By Proposition 2.19, $\mathcal{C}$ is a right classical set, and by Lemma 2.9, Q is the quotient ring of R at $\mathcal{C}$. Thus Q is a semilocal ring.

Now, by Theorem 2.5, $\mathcal{S}^{o}$ is precisely the class of all simple left Q-modules, and hence contains only finitely many isomorphism types, since Q is a semilocal ring. Applying Proposition 2.19 again, we obtain that $\mathcal{C}$ is a left classical set of prime ideals. Thus $\mathcal{C}$ is a classical set. But $\mathcal{S}$ is a $\sim$-class, so by Theorem 2.17, $\mathcal{C}$ must be a minimal classical set: that is, $\mathcal{C}$ is a clan. □

2. GLOBAL DIMENSION TWO

We will denote by gl.dim(R) the global dimension of R. We retain the assumptions that R is noetherian, prime and of Krull dimension 1. It will be shown in this section that the hypotheses of Theorem 3.5 are satisfied by every $\sim$-closed class if we impose the condition that $gl.dim(R) \leq 2$.

Lemma 3.9. If gl.dim(R) = 1 then every Goldie torsion module has projective dimension 1.

Proof. A Goldie torsion module cannot be projective and thus must have projective dimension 1. $\square$

Lemma 3.10. If gl.dim(R) = 2 and X is a Goldie torsion module of projective dimension 1, then every non-zero submodule of X has projective dimension 1.

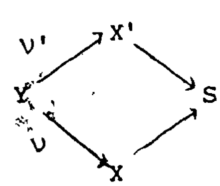
Proof. Let Y be a non-zero submodule of X. Y cannot have projective dimension 0, since it is Goldie torsion. Applying the long exact sequence in $\mathrm{Ext}_R(\_,A)$, for any module A, we obtain the exact sequence:

$$\mathrm{Ext}_R^2(X,A) \to \mathrm{Ext}_R^2(Y,A) \to \mathrm{Ext}_R^3(X/Y,A)$$

$\mathrm{Ext}_R^3(X/Y,A) = 0$ since gl.dim(R) = 2, and $\mathrm{Ext}_R^2(X,A) = 0$ since p.dim(X) = 1. Thus $\mathrm{Ext}_R^2(Y,A) = 0$ for all modules A, and hence p.dim(Y) = 1. $\square$

Lemma 3.11. Assume gl.dim(R) $\leq$ 2. Let $\mathcal{S}$ be a $\sim$-class of simple right (or left) modules and let $S \in \mathcal{S}$. If S is an epimorphic image of both the indecomposable modules X and X', where X and X' are in $A_{\mathcal{S}}$, then all composition factors of X_* and X'_* are in the same $\sim$-class.

Proof. Let the following diagram be a pullback of the epimorphisms:

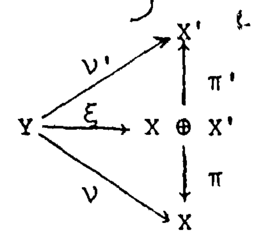


Then Y is isomorphic to a submodule of $X \oplus X'$, so by Lemma 3.10, p.dim(Y) = 1. Applying the duality $\mathrm{Ext}^1_R(\_,R)$ we obtain the monomorphisms $\nu_*: X_* \to Y_*$ and $\nu'_*: X'_* \to Y_*$.

We will assume that X_* and X'_* have composition factors in different $\sim$-classes and obtain a contradiction. As X_* and X'_* are indecomposable, if they contain composition factors which are not $\sim$-equivalent we have the induced monomorphism:

$$\nu_* \oplus \nu'_*: X_* \oplus X'_* \to Y_*$$

Applying the duality again we obtain the commuting diagram:



where ξ is the epimorphism $(\nu_* \oplus \nu'_*)_*$, and π and π' are the projections. But then ξ must simply be the embedding σ of Y into $X \oplus X'$, as $\pi'\xi = \pi'\sigma$ and $\pi\xi = \pi\sigma$. We conclude that $Y = X \oplus X'$, which contradicts the construction of Y. $\square$

<u>Lemma 3.12</u>. Assume gl.dim(R) ≤ 2 and let S and S' be simple right (left) R-modules such that $\mathrm{Ext}_R(S',S) \neq 0$. If X and X' are indecomposable modules of finite length and projective dimension 1, such that S is an epimorphic image of X and S' is an epimorphic image of X', then all composition factors of X_* and X'_* are $\sim$-equivalent.

Proof. Let

$$0 \to S \to A \to S' \to 0$$

be a non-split exact sequence. Since A is cyclic with unique simple submodule S, we can find a module X", indecomposable, of finite length and projective dimension 1, with an epimorphism $\nu'' \in \mathrm{Hom}_R(X'',A)$. Then S' is an epimorphic image of X" so, by Lemma 3.11, all composition factors of X'_* and X''_* are $\sim$-equivalent.

Consider the following pullback diagram:

$$\begin{array}{ccc} X_0 & \longrightarrow & X'' \\ {\scriptstyle \nu_0}\downarrow & & \downarrow{\scriptstyle \nu''} \\ S & \longrightarrow & A \end{array}$$

By Lemma 3.10 (or Lemma 3.9 if gl.dim(R) = 1), p.dim(X) = 1. Now some indecomposable summand Y of X_0 maps epimorphically onto S. By Lemma 3.11, all composition factors of Y_* and X_* are $\sim$-equivalent. But $Y \subset X''$, so Y_* is an epimorphic image of X''_*. Therefore, all composition factors of Y_* and X''_* are $\sim$-equivalent, and hence this is so for all composition factors of X_* and X'_*. □

<u>Proposition 3.13</u>. Assume $\mathrm{gl.dim}(R) \leq 2$ and let $\mathcal{S}$ be a right $\sim$-class. Then $\mathcal{S}^{o}$ is a left $\sim$-class and $\mathcal{S} = \mathcal{S}^{oo}$.

<u>Proof</u>. Let S and S' be in $\mathcal{S}^{o}$. Then S and S' are composition factors of X_* and X'_* respectively, for some X and X' in $\mathcal{A}_{\mathcal{S}}$. As a result of Lemma 3.12, all composition factors of X_* and X'_* are in the same $\sim$-equivalence class. In particular, $S \sim S'$.

It remains to show that $\mathcal{S}^{o}$ is $\sim$-closed. Let $T \in \mathcal{S}^{o}$, where, say, T is a composition factor of Z_* ($Z \in \mathcal{A}_{\mathcal{S}}$). Then T is an epimorphic image of some submodule Y of Z_*. By Lemma 3.10, p.dim(Y) = 1. Assume that for some simple left module T', $\mathrm{Ext}^1_R(T,T') \neq 0$ or $\mathrm{Ext}^1_R(T',T) \neq 0$. Let T' be an epimorphic image of the indecomposable module Y' of finite length and projective dimension 1. Then, by Lemma 3.12, all composition factors of Y_* and Y'_* are $\sim$-equivalent. Thus $Y' \in \mathcal{A}_{\mathcal{S}}$ and so $T' \in \mathcal{S}^{o}$. We conclude that $\mathcal{S}^{o}$ is a $\sim$-class.

By Lemma 3.2, $\mathcal{S}^{oo} \supset \mathcal{S}$, so since $\mathcal{S}^{oo}$ is a $\sim$-class, $\mathcal{S}^{oo} = \mathcal{S}$. □

Corollary 3.14. If R is a noetherian prime ring with Krull dimension 1 and global dimension no greater than 2, then the right quotient ring at the torsion theory generated by a $\sim$-closed class $\mathcal{S}$ is also a left quotient ring at the torsion theory generated by the left $\sim$-closed class $\mathcal{S}^{o}$.

Proof. Theorem 3.5 and the above Proposition. □

We pose the question: To what extent can the hypothesis $\mathrm{gl.dim}(R) \leqq 2$ be weakened?

CHAPTER IV

HEREDITARY NOETHERIAN PRIME RINGS

1. $\sim$-CLASSES AND CYCLES

Hereditary noetherian prime (HNP) rings are the non-commutative analogues of Dedekind domains. In this chapter we present some results on HNP rings in the context of the theory developed in the preceding chapters.

We must first mention some of the key concepts involved in the study of HNP rings. For an ideal I of the HNP ring R we define:

$$O_r(I) = \{q \in Q\#: Iq \subset I\} \text{ and}$$
$$O_\ell(I) = \{q \in Q\#: qI \subset I\}.$$

A cycle is a finite set $\{M_1, \ldots, M_n\}$ of maximal ideals, where $O_r(M_1) = O_\ell(M_2), \ldots, O_r(M_i) = O_\ell(M_{i+1}), \ldots, O_r(M_n) = O_\ell(M_1)$. A cycle has an alternate description as a minimal set of maximal ideals whose intersection is an invertible ideal. (An ideal is invertible if $II^* = (I^*)I = R$ where $I^* = \{q \in Q\#: qI \subset R\}$.) The concept of a cycle was defined and studied in [12]. By Theorem 11 of [30], the cycles are precisely the clans.

Lenagan [25] has shown that for a (right) bounded HNP ring, every non-zero ideal contains an invertible ideal, and hence every maximal ideal is in a cycle. (A ring is right bounded if every essential right ideal contains a non-zero two-sided ideal.)

We also note that an artinian HNP ring is simple artinian. Throughout this chapter, R will be a non-artinian HNP ring. As a consequence, R has global dimension 1.

We open by presenting some well known results which allow us to apply the results of the preceding chapters to HNP rings.

Lemma 4.1. A non-artinian HNP ring has Krull dimension 1.

Proof. Let R be an HNP ring. By [37], Theorem 4, R/K is artinian for every essential right ideal K. Therefore, by [18], Proposition 6.1, the Krull dimension of R is 1. □

Proposition 4.2. In an HNP ring the clans are precisely the cycles.

Proof. [30], Theorem 11. □

Corollary 4.3. Let R be an HNP ring and let $\{C_\gamma : \gamma \in \Gamma\}$ be the set of cycles of R. Then every Goldie torsion module X has a decomposition: $X = \bigoplus_{\Gamma} X_\gamma \oplus X'$ where every simple subfactor of X_γ is annihilated by an element of C_γ, and no simple subfactor of X' is annihilated by any element of any cycle.

Proof. This follows from Theorem 2.17, which can be applied on the basis of the above Proposition. □

This decomposition has been demonstrated by Kuzmanovich ([22], Theorem 3.7), together with a corresponding globalization.

The localizations at the components of a decomposition of the Goldie torsion theory are again hereditary. In fact:

Proposition 4.4. Let R be an HNP ring with Goldie quotient ring Q#, and let R' be any over-ring of R contained in Q#. Then R' is HNP, and is the quotient ring of R for some torsion theory. Further, every torsion theory over R is perfect.

Proof. [17], Proposition 2 and Theorem 5, and [36], Corollary 3.6 of Chapter XI. □

In the HNP case we can further strengthen Propositions 2.19 and 3.8.

Proposition 4.5. Let R be a non-artinian HNP ring and let $\mathcal{S}$ be a $\sim$-class of simple right modules containing only unfaithful modules. Then $\mathcal{S}$ has only finitely many isomorphism types and $\{ann_R(S) : S \in \mathcal{S}\}$ is a cycle.

Proof. Let Q be the quotient ring of R at the torsion theory cogenerated by $\mathcal{S}$. By Theorem 2.5, $\mathcal{S}$ is precisely the class of simple right Q-modules, and all $S \in \mathcal{S}$ are unfaithful as Q-modules. Thus Q is right bounded, so by [25], every maximal ideal of Q is in a cycle. Therefore, since $\mathcal{S}$ is a $\sim$-class, $\{ann_Q(S) : S \in \mathcal{S}\}$ is a cycle of Q, as a result of Theorem 2.17. But then $\mathcal{S}$ contains only finitely many isomorphism types, so by Propositions 3.8 and 3.13, $\{ann_R(S) : S \in \mathcal{S}\}$ is a cycle in R. □

For HNP rings, the nature of the $\sim$-links between simple modules annihilated by ideals in a cycle can be described explicitely.

We must first strengthen our notation.

Definition. Let S and S' be simple right modules. Then $S \rightsquigarrow S'$ if $Ext^1_R(S',S) \neq 0$.

Proposition 4.6. Let $\mathcal{C}$ be a cycle and let $\mathcal{S} = \{S_1,\ldots,S_n\}$ be a set of isotypic representatives of the class of modules annihilated by elements of $\mathcal{C}$. Then, under a suitable reindexing, $S_1 \rightsquigarrow S_2 \rightsquigarrow \ldots \rightsquigarrow S_n \rightsquigarrow S_1$, and these are the only $\rightsquigarrow$ relations between elements of $\mathcal{S}$. Further, if $P_i = ann_R(S_i)$ for $i = 1,\ldots,n$, then $O_r(P_i) = O_\ell(P_{i+1})$ (i.e. the $\rightsquigarrow$ correspond to the links between ideals in a cycle as defined in [12].)

Proof. Noting that, by [10], every proper factor ring is serial, we determine that there is a unique generalized composition series of $E(S_1)$:

$$0 \subset S_1 \subset X_1 \subset X_2 \subset \ldots \subset E(S_1)$$

where $X_{k+1}/X_k \simeq X_{j+1}/X_j$ if and only if $k \equiv j \pmod{n}$. This has been constructed in [34], and follows from the facts that X_i has a unique composition series and $E(S)/X_i \simeq E(X_{i+1}/X_i)$ for all i. Also (from [34]), S_i is a composition factor of $E(S_1)$ for all $i = 1,\ldots n$, again by the uniseriality of X_i, since all S_i are $\sim$-linked and there are only finitely many of them.

It follows that $S_1 \rightsquigarrow X_1/S_1 \simeq S_2$ and inductively that $S_{i+1} \simeq X_i/X_{i-1} \rightsquigarrow X_{i+1}/X_i \simeq S_{i+2}$.

It remains to show that the $\rightsquigarrow$-links correspond to the links between the associated maximal ideals. Let $P_1 = ann_R(S_1)$. Then, by [32], Corollary 5.4, $R_1 = O_r(P_1)$ is a minimal over-ring of R in Q#. Further, $R = I_{R_1}(P_1)$, the idealizer of P_1 in R_1, and R_1/P_1 is semisimple with all simple summands isomorphic. Let Y_1 be a representative of this isomorphism class. Then S_1 is an R-submodule of Y_1.

Now R_1 is a localization of R at some torsion theory, which, since R_1 is minimal, must be generated by a single simple module. But this single simple module is isomorphic to a submodule of R_1/R and hence it must be Y_1/S_1 (by [32], Corollary 2.4), which, by uniseriality, is just S_2. We thus obtain the non-split exact sequence:

$$0 \to S_1 \to Y_1 \to S_2 \to 0$$

We know that $R_1 = O_\ell(P_2)$ for some P_2. Now P_2P_1 annihilates Y_1 since $Y_1 \subset R_1/P_1$ and $P_2P_1 \subset P_1$. Thus P_2P_1 annihilates S_2. We conclude that $P_2 = ann_R(S_2)$. By the cyclic nature of the $\rightsquigarrow$-links, we have actually demonstrated that $O_r(P_i) = O_\ell(P_{i+1})$ where $P_i = ann_R(S_i)$ and $P_{i+1} = ann_R(S_{i+1})$. □

2. ARBITRARY LOCALIZATIONS

We now analyze the behavior of classes of simple modules under arbitrary localizations of an HNP ring. Our first result shows that if $\mathcal{S}$ is an arbitrary class of simple modules and Q is the quotient ring of R at the torsion theory generated by $\mathcal{S}$, then Q is also the left quotient ring at the torsion theory generated by $\mathcal{S}^{\circ} = \{\mathrm{Ext}^1_R(S,R) : S \in \mathcal{S}\}$. This generalizes Corollary 3.14 for HNP rings. The basis of the proof is the following:

<u>Lemma 4.7.</u> Let S be a simple right R-module and let $Q(\_)$ be any quotient functor on <u>mod-R</u> (with quotient ring Q) such that $Q(S) \neq 0$. Then $\mathrm{Ext}^1_R(S,R)$ is a left R-submodule of $\mathrm{Ext}^1_Q(Q(S),Q)$.

<u>Proof</u>. Suppose $S \simeq R/M$. Then $\mathrm{Ext}^1_R(S,R) \simeq \mathrm{Hom}_R(M,R)/\mathrm{End}_R(R)$ as can be seen by applying the long exact sequence in $\mathrm{Ext}_R(\_,R)$ to the exact sequence:

$$0 \to M \to R \to S \to 0$$

Similarly, noting that $Q(S) \simeq Q/Q(M)$ (as the localization is perfect),

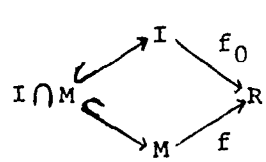
$$\mathrm{Ext}^1_Q(Q(S),Q) \simeq \mathrm{Hom}_Q(Q(M),R)/\mathrm{End}_Q(Q)$$

Consider the R-homomorphism $\gamma\colon \mathrm{Hom}_R(M,R) \to \mathrm{Hom}_Q(Q(M),Q)$ defined by: $\gamma(f) = Q(f)$. γ induces the R-homomorphism

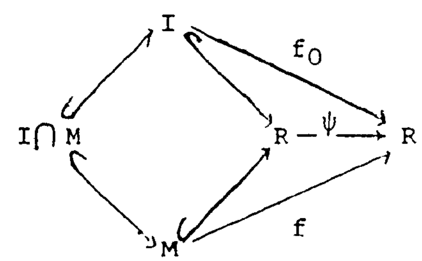
$$\bar{\gamma}\colon \mathrm{Ext}^1_R(S,R) \to \mathrm{Ext}^1_Q(Q(S),Q)$$

(for if $f \in \mathrm{Hom}_R(M,R)$ extends to an endomorphism of R then $Q(f)$ extends to an endomorphism of Q).

Now, if $\bar{\gamma}(\bar{f}) = 0$ for some $f \in \mathrm{Hom}_R(M,R)$ then there is a $\xi \in \mathrm{End}_Q(Q)$ extending $\gamma(f)$. But then $\xi = Q(f_0)$ for some $f_0 \in \mathrm{Hom}_R(I,R)$ for some right ideal I which is dense in R (with respect to the given torsion theory). We thus have $Q(f_0)$ extending $\gamma(f)$ and hence the commuting diagram:



But $I + M = R$ since I is dense and M is critical. Therefore, there is a ψ such that the diagram



commutes. Therefore $\bar{f} = 0$. It follows that γ is a monomorphism. □

<u>Proposition 4.8</u>. Let Q be the right quotient ring of R at the torsion theory generated by an arbitrary class $\mathcal{S}$ of simple right modules. Then Q is also the left quotient ring of R at the torsion theory generated by $\mathcal{S}^o = \{S_*: S\varepsilon\mathcal{S}\}$.

<u>Proof</u>. Let $\mathcal{S}'$ be the complement of $\mathcal{S}$ in $\mathcal{S}^\#$. Then $\{Q(S): S\varepsilon\mathcal{S}'\}$ is the class of all simple right Q-modules. Since Q is HNP, $\{Ext^1_Q(Q(S),Q): S\varepsilon\mathcal{S}'\}$ is the the class of all simple left Q-modules. By Lemma 4.7, $Ext^1_R(S,R) \subset Ext^1_Q(Q(S),Q)$ for all $S \varepsilon \mathcal{S}'$. Now, by Proposition 4.4, Q is the left quotient ring of R for some torsion theory, so this must be the torsion theory generated by $\mathcal{S}^o = \{Ext_R(S,R): S\varepsilon\mathcal{S}\}$ (noting that $\mathcal{S}^o$ is the complement of $(\mathcal{S}')^o$ in the class of all simple left modules). □

We now analyze the behavior of the relation $\rightsquigarrow$ with respect to arbitrary localizations of R.

<u>Lemma 4.9</u>. Let S and S' be simple right R-modules with $S \rightsquigarrow S'$, and let Q(_) be any quotient functor on <u>mod-R</u> such that $Q(S), Q(S') \neq 0$.

Then $Q(S) \rightsquigarrow Q(S')$ as Q-modules.

Proof. Consider the non-split exact sequence:

$$0 \to S \to X \to S' \to 0$$

Apply the functor Q(_) to obtain the exact sequence

$$0 \to Q(S) \to Q(X) \to Q(S') \to 0$$

which is non-split since S is essential in X and hence in Q(X). □

Lemma 4.10. Let S and S' be simple right R-modules and let Q(_) be any quotient functor such that $Q(S), Q(S') \neq 0$. If $Q(S) \sim Q(S')$ as Q-modules then $S \sim S'$.

Proof. We need only show that if $Q(S) \rightsquigarrow Q(S')$ as Q-modules, then $S \sim S'$. Assume $S \not\sim S'$ and let

$$0 \to Q(S) \to X \to Q(S') \to 0$$

be a non-split Q-exact sequence. X is Goldie torsion, so $X = \sum_{\Lambda} \oplus \tau_{\lambda}(X)$ (as defined in Chapter I). Now, since $S \not\sim S'$, the simple subfactors of Q(S) are not $\sim$-equivalent to the simple subfactors of Q(S'). Thus, the sequence splits, so $Q(S) \not\rightsquigarrow Q(S')$ as Q-modules. □

Lemma 4.11. Let S_0, S_1, and S_2 be simple right R-modules with $S_0 \rightsquigarrow S_1 \rightsquigarrow S_2$, and let Q(_) be any quotient functor with $Q(S_1) = 0$ and $Q(S_0), Q(S_2) \neq 0$. Then $Q(S_0) \rightsquigarrow Q(S_2)$ as Q-modules.

Proof. Suppose we have non-split exact sequences:

$$0 \to S_0 \to X \to S_1 \to 0$$

and $$0 \to S_1 \to Y \to S_2 \to 0$$

Then $E(S_0)/S_0$ has a submodule isomorphic to S_1. Since R is hereditary, $E(S_0)/S_0$ is injective. Thus $E(S_1) \subset E(S_0)/S_0$, and hence $Y \subset E(S_0)/S_0$.

We can then apply $Q(\_)$ to obtain $Q(S_2) = Q(Y) \subset E(S_0)/Q(S_0)$.

Taking the pullback:

$$\begin{array}{ccc} E(S_0) & \twoheadrightarrow & E(S_0)/Q(S_0) \\ \uparrow & & \uparrow \\ B & \longrightarrow & Q(S_2) \end{array}$$

and the pullback:

$$\begin{array}{ccc} Q(S_0) & \longrightarrow & E(S_0) \\ \uparrow & & \uparrow \\ A & \longrightarrow & B \end{array}$$

we obtain the following commuting diagram:

$$\begin{array}{ccccccccc} 0 & \to & Q(S_0) & \to & E(S_0) & \to & E(S_0)/Q(S_0) & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & A & \longrightarrow & B & \longrightarrow & Q(S_2) & \longrightarrow & 0 \end{array}$$

Applying $Q(\_)$ to the bottom row, we obtain the exact sequence:

$$0 \to Q(A) \to Q(B) \to Q(S_2) \to 0$$

which is non-split since S_0 is essential in B. Now $Q(A) \simeq Q(S_0)$, so $Q(S_0) \rightsquigarrow Q(S_2)$ as Q-modules. $\square$

<u>Proposition 4.12</u>. Let $Q(\_)$ be an arbitrary quotient functor on <u>mod-R</u> (not the Goldie quotient functor). Then $\tilde{C}$ is a cycle in Q

if and only if there is a cycle $\mathcal{C}$ in R such that

$$\hat{\mathcal{C}} = \{ann_Q(Q(S)) : S \text{ simple}, ann_R(S) \in \mathcal{C}, Q(S) \neq 0\} .$$

Proof. a) Let $\mathcal{C}$ be a cycle in R and let $\hat{\mathcal{C}} = \{ann_Q(Q(S)) : S \text{ simple}, ann_R(S) \in \mathcal{C}, Q(S) \neq 0\}$. By Proposition 4.6, there is a set of isotypic representatives $\{S_1, \ldots, S_n\}$ of simple modules annihilated by ideals in $\mathcal{C}$, such that $S_1 \rightsquigarrow S_2 \rightsquigarrow \ldots \rightsquigarrow S_n \rightsquigarrow S_1$. By the preceding three Lemmas, applying Q(_) to this set yields a set of simple right Q-modules $\{Q(S_{i_1}), \ldots, Q(S_{i_k})\}$ with $Q(S_{i_1}) \rightsquigarrow Q(S_{i_2}) \rightsquigarrow \ldots \rightsquigarrow Q(S_{i_k}) \rightsquigarrow Q(S_{i_1})$.

These $Q(S_{i_j})$ are unfaithful as Q-modules since, by [6], Theorem 3.5, the Jacobson radical J_j of the localization of R at the prime ideal $ann_R(S_{i_j})$ is non-zero. Hence $Q(S_{i_j})$ is annihilated by $J_j \cap Q$. We can thus apply Proposition 4.5 to obtain that $\{ann_Q(Q(S_{i_j}) : j=1, \ldots, k\}$ is a cycle

b) Let $\hat{\mathcal{C}}$ be a cycle in Q. Then $\hat{\mathcal{C}} = \{ann_Q(Q(S)) : S \in \mathcal{S}\}$ for some class $\mathcal{S}$ of simple right R-modules. By [6], Theorem 3.5, the Jacobson radical J_P of Q_P (the localization of Q at the prime ideal P) is non-zero, for P in $\hat{\mathcal{C}}$. Now Q_P is the localization of R at the prime ideal $ann_R(S)$ where $P = ann_Q(Q(S))$. Thus, by [6], Theorem 3.5, $ann_R(S)$ is in some cycle $\mathcal{C}$. Part a) now implies that $\{ann_Q(Q(S)) : S \text{ simple}, ann_R(S) \in \mathcal{C}, Q(S) \neq 0\} = \hat{\mathcal{C}}$. □

CHAPTER V

EXAMPLES

We list here, without proof, some examples.

1) Commutative noetherian prime rings of Krull dimension 1:

A commutative ring which is noetherian, prime and of Krull dimension 1 is either a Dedekind domain or a noetherian domain of infinite global dimension.

2) A non-commutative Dedekind prime ring with one maximal invertible ideal and infinitely many faithful simple modules:

Let F be a field and σ an automorphism of infinite order. Then $D = F[x;\sigma]$, the ring of polynomials subject to $ax = x\sigma(a)$ for all $a \in F$, has the required properties (cf. [11]). The maximal ideal is xD.

3) An HNP ring with one maximal invertible ideal, a maximal ideal not in a cycle, and infinitely many faithful simple modules:

Let D be as in 2) and let K be any maximal right ideal which is not a two-sided ideal, Then $H = I_D(K)$ (the idealizer of K in D) is an HNP ring with the required properties. K is a maximal ideal of H which is in no cycle. (cf. [32].)

4) A simple HNP ring with all simple modules $\sim$-equivalent:

Let F be a field of characteristic 0, and let $B = F(y)[x]$, subject to $xy-yx = 1$. Then B is a simple principal ideal domain and all simple modules are equivalent (cf. [27]).

5) An HNP ring which is not simple and with all simple modules $\sim$-equivalent:

With F and B as in 4), let $R_0 = F + xB$. Then R_0 has the required properties (cf. [32]). All simple modules are $\sim$-equivalent since B is a localization of R_0 (and applying the results of Chapter IV (2)).

6) A noetherian prime ring of Krull dimension 1 and global dimension 2, and with all simple modules $\sim$-equivalent:

With F and B as in 4), $R_1 = F + x^2B$ has the required properties (cf. [32]).

7) A construction of new HNP rings from old, preserving the $\sim$-structure:

Let R be an HNP ring with maximal ideal M.

$$T = \begin{pmatrix} R & M \\ R & R \end{pmatrix}$$

is an HNP ring and there is a one-to-one correspondence between the simple modules of R not annihilated by M and the simple modules of T not annihilated by

$$\begin{pmatrix} M & M \\ R & M \end{pmatrix}$$

The simple module of R which is annihilated by M corresponds to two simples of T, which are $\sim$-linked (cf. [32]). The above correspondence provides a one-to-one correspondence between the $\sim$-classes of R and T.

8) An interesting noetherian prime ring of Krull dimension 1, which is not HNP:

Let R be an HNP ring and I an arbitrary non-zero ideal.

$$U = \begin{pmatrix} R & I \\ R & R \end{pmatrix}$$

is a noetherian prime ring of Krull dimension 1, which is HNP if and only if I is semiprime (as follows from [32]). It appears that there is a one-to-one correspondence between the $\sim$-classes of U and those of R.

9) Michler ([28], Remark 6.6, and [29]) has offered a construction of noetherian prime rings of Krull dimension 1 and arbitrary global dimension.

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