Using Reputation in Repeated Selfish Routing with Incomplete Information

Using Reputation in Repeated Selfish Routing with Incomplete Information

By

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Abstract

We study the application of reputation as an instigator of beneficial user behavior in selfish routing and when the network users rely on the network coordinator for information about the network. Instead of using tolls or artificial delays, the network coordinator takes advantage of the users' insufficient data, in order to manipulate them through the information he provides. The issue that arises then is what can be the coordinator's gain without compromising by too much on the trust the users put on the information provided, i.e., by maintaining a reputation for (at least some) trustworthiness.

Our main contribution is the modeling of such a system as a repeated game of incomplete information in the case of single-commodity general networks. This allows us to apply known folk-like theorems to get bounds on the price of anarchy that are better than the well-known bounds without information manipulation.

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Chapter 1

Introduction

1.1 Selfish Routing

In the modern society where almost everyone travels on a day-to-day basis, the study of traffic networks plays an important role. For each traffic user, when facing more than one route that can lead her to the destination, she needs to choose one to travel on. In *selfish routing*, the traffic user always acts "selfishly" and picks the route which brings her the maximum profit (or minimum cost), without considering the payoff of any other users. If the traffic user takes the time cost on the route as the only measure, ignoring other factors (such as the landscape or personal feelings), then she will choose the one with the minimum time latency. The road is a public resource shared by many traffic users, and the more users travel on, the bigger travel latency tends to be. This phenomenon is called *congestion*.

This thesis is about the selfish routing with *homogeneous* and *infinitesimal* users, where "homogeneous" means that all traffic users are indistinct (for example, we do not distinguish motorcycles and cars), and "infinitesimal" implies that each traffic user's contribution to the congestion is negligible. So if some traffic user switches her choice of route from path p to path p', the increase of flow on p' and the decrease of the flow on p is too small to matter.

1.1.1 Routing Model

We will model the traffic network using a strongly connected, directed graph $G(V, E)$, with V being the set of nodes and E being the set of edges (links). In reality, the edges are exactly the roads, and the nodes are the intersection of the roads. All travelers in the network are starting from the nodes called origins(O) and going to their corresponding nodes called destinations(D). The origin and its corresponding destination is called an O-D pair. Let W be the set of such O-D pairs in the network, and *demand* d_w be the total amount of traffic users associated with the O-D pair $w \in W$. An O-D pair together with its associated demand is called a *commodity*. A path is a simple sequence of connected edges, and P_w denotes the set of all simple paths connecting an O-D pair $w \in W$. P is defined by $P := \bigcup_{w \in W} P_w$.

If an edge $e \in E$ is on a path $p \in P$, we will denote it by $e \in p$. *Incidence matrix* $\Delta_{|E| \times |P|}$ captures the relationship between edges and paths in the graph, which is defined by

$$\Delta_{ep} = \begin{cases} 1; \quad e \in p, \quad \forall e \in E, p \in P \\ 0; \quad e \notin p \quad \forall e \in E, p \in P \end{cases}$$

Each traffic user in the network will pick a path going from her origin to her destination. The aggregation of all traffic users' choice will induce a *flow* in the network. $\mathbf{h}$, a vector of dimension $|P|$, denotes a path flow where each item h_p denotes the amount of traffic users on path $p \in P$. $\mathbf{f}$, a vector of dimension $|E|$, denotes an edge flow where each item f_e denotes the amount of traffic user on edge $e \in E$.

Definition 1.1.1 ***(Feasible flow)*** *A path flow* $\mathbf{h}$ *is called feasible iff the* feasible conditions *hold:*

$$\sum_{p \in P_w} h_p = d_w, \quad \forall w \in W$$

$$h_p \geq 0, \quad \forall p \in P$$

An edge flow $\mathbf{f}$ is called feasible if there exists a feasible path flow $\mathbf{h}$, st. $\mathbf{f}$ and $\mathbf{h}$ satisfy the *compatible condition*:

$$\mathbf{f} = \Delta_{|E|\times|P|}\mathbf{h} \tag{1.1}$$

$\mathcal{F}$ denotes the set of all feasible edge flow, and $\mathcal{U}$ denotes the set of all feasible path flow.

If a feasible $\mathbf{f}$ and a feasible $\mathbf{h}$ satisfy the compatible condition, we call them *compatible* to each other, and we can get $f_e = \sum_{p \ni e} h_p, \forall e \in E$.

The traffic users are homogeneous, so they suffer the same latency on the same edge (or path). The common latency on an edge is related to the flow because the more traffic users travel on, the more congested it will be, and the users tend to suffer a bigger latency on it.

We will use *edge latency functions* to capture the relationship between a feasible edge flow $\mathbf{f} \in \mathcal{F}$ and the latency on each edge $e \in E$, defined by $l_e(\mathbf{f}) : \mathcal{F} \to \mathbb{R}, \forall e \in E$.

Similarly we will use *path latency functions* to capture the relationship between a feasible path flow $\mathbf{h} \in \mathcal{U}$ and the latency on each path $p \in P$, defined by $l_p(\mathbf{h}) : \mathcal{U} \to \mathbb{R}, \forall p \in P$.

We will use a tuple $(G, \mathbf{l}, \mathbf{d})$ to denote an instance of selfish routing, where G indicates the topology of the network, $\mathbf{l}$ indicates the latency functions and $\mathbf{d}$ indicates the demand.

We assume that the selfish routing problem satisfies the following properties.

Definition 1.1.2 ***(Separable property)*** *The latency on an edge $e \in E$ is only related to the edge flow on e, independently of the flow on any other edges.*

This makes sense because the congestion on any other edge has nothing to do with the latency on this edge. Thus later on, we will use $l_e(f_e)$ as the edge latency function.

Definition 1.1.3 ***(Additive property)*** *The latency on a path $p \in P$ is the summation of the latency on each edge on p: $l_p(\mathbf{h}) = \sum_{e \in p} l_e(f_e)$.*

This is also a normal attribute in real life. Written in matrix form, this is

$$\mathbf{l}(\mathbf{h}) = \Delta^T \mathbf{l}(\mathbf{f}) \tag{1.2}$$

Definition 1.1.4 ***(Social cost)*** *In a feasible flow $\mathbf{h}$ or $(\mathbf{f})$, each traffic user choosing path $p \in P$ will suffer the common latency $l_p(\mathbf{h})$. Define the* social cost *S to be the total latency suffered by all traffic users:*

$$S(\mathbf{h}) := \sum_{p \in P} h_p l_p(\mathbf{h})$$

We can also take the social cost as a function of the edge flow $\mathbf{f}$:

$$S(\mathbf{f}) := \sum_{e \in E} f_e l_e(f_e)$$

These two definitions are equivalent, i.e., if $\mathbf{f}$ and $\mathbf{h}$ are compatible to each other, then $S(\mathbf{h}) = S(\mathbf{f})$, since

$$\sum_{e \in E} f_e l_e(f_e) = \mathbf{f}^T \mathbf{l}(\mathbf{f}) = \mathbf{h}^T \Delta^T \mathbf{l}(\mathbf{f}) = \mathbf{h}^T \mathbf{l}(\mathbf{h}) = \sum_{p \in P} h_p l_p(\mathbf{h}) \tag{1.3}$$

Definition 1.1.5 ***(Optimal flow)*** *The optimal edge flow $\mathbf{f}^{opt}$ is the feasible edge flow that minimizes the social cost:*

$$\mathbf{f}^{opt} = \operatorname*{argmin}_{\mathbf{f} \in \mathcal{F}} S(\mathbf{f})$$

The optimal path flow $\mathbf{h}^{opt}$ is the one that is compatible to the optimal edge flow.

1.1.2 User Equilibrium

We can take the selfish routing problem as a game with all the infinitesimal users as the *players*. Each player's *strategy* is her choice of which path to travel on, and her *payoff* is the negative of the latency on the chosen path. In the selfish routing game, all the players are trying to maximize their payoff, which is to minimize the latency on her chosen path.

A *strategy profile* is a set of strategies which includes one and only one strategy for each player. In the selfish routing game, a strategy profile will induce a feasible path flow.

The *Nash Equilibrium*[20] of the game is the strategy profile in which no player has the incentive to switch her strategy, if all the other players do not switch their strategies. The *equilibrium flow* denotes the flow induced by the equilibrium strategy of the game. Denote $\mathbf{f}^{equ}$ and $\mathbf{h}^{equ}$ to be the equilibrium edge flow and path flow, respectively.

Studies about the equilibrium of selfish routing started early in the last century, when some basic principles were stated. The first step towards a mathematical investigation of the problem has been done by J. G. Wardrop [33] in 1952. He developed the so-called Wardrop's first principle of equilibrium. The principle states:

> *"The journey times in all routes actually used are equal and less than those which would be experienced by a single vehicle on any unused route".*

The traffic flow that satisfies this principle is referred to as *user equilibrium (UE)* flow (see [33][28]). It is an equilibrium state (or stable state) since each user chooses the route that is the best for her and no user may lower her transportation cost through unilateral switch.

Based on Wardrop's first principle, the equilibrium condition in mathematical language is:

Definition 1.1.6 ***(Equilibrium flow)*** *A feasible flow* $\mathbf{h}$ *is a traffic equilibrium flow for the instance* $(G, \mathbf{l}, \mathbf{d})$ *iff :* $\forall w \in W$, $\forall p_1, p_2 \in P_w$ *with* $p_1 > 0$, $l_{p_1}(\mathbf{h}) \leq l_{p_2}(\mathbf{h})$.

If we take the social cost as the benchmark of a feasible flow, then the optimal flow is the most efficient one since it minimizes the social cost. If all the traffic users act as a team instead of playing selfishly, then their choices will induce the optimal flow. But in selfish routing, every user plays selfishly and the equilibrium flow is the stable result of their choices.

Usually the equilibrium flow is different from the optimal one. Pigou's Example [24] is a simple example of this.

Example 1.1

Figure 1.1 shows a simple network with a single origin s, a single destination t, and total demand $d = 1$. There are two parallel candidate routes: path 1 has a fixed latency of 1, and the latency on path 2 is equal to the traffic flow on it. Obviously, at equilibrium all the traffic users will choose path 2, which is always a safe shot since path 2 is never worse than path 1. No traffic user would like to choose path 1, because in that case the latency on path 2 will be less than 1, and all those choosing path 1 have an incentive to switch to path 2.

The social cost of this equilibrium state is 1. But if they act under some imaginary coordinator, instead of "selfishly", then half of them will choose path 1 and the other half choose path 2, which induces the optimal flow with the social cost $3/4$.

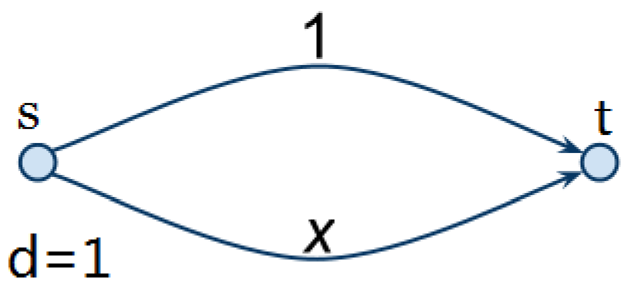

Figure 1.1: Pigou's Example

Beckmann et al. [2] observed that an equilibrium flow is an optimal solution to a related convex program, and analyzed the existence and uniqueness conditions for traffic equilibrium. It is proven that if the latency functions $l_e(f_e)$ are continuous and non-decreasing, then the equilibrium flow $\mathbf{f}^{equ}$ exists and, moreover, all such flow shares the same edge latency $\mathbf{l}(\mathbf{f})$.

Price of anarchy (PoA), proposed by Papadimitriou et al. [13][22], is a method to measure the inefficiency of selfish routing. It is defined by the ratio of the social cost at equilibrium flow to that at optimum flow.

Definition 1.1.7 ***(Price of anarchy)*** *If $\mathbf{f}^{equ}$ and $\mathbf{f}^{opt}$ are the equilibrium flow and the optimal flow in the network, respectively, then the price of anarchy is defined by*

$$PoA := \frac{S(\mathbf{f}^{equ})}{S(\mathbf{f}^{opt})} \tag{1.4}$$

Roughgarden et al. [28] [26] showed that the price of anarchy in user equilibrium is independent of the network topology, and they gave a tight upper bound of 4/3 if the latency functions are linear. Correa et al. [5] found an even simpler proof of this bound for linear latency functions, and generalized the result to general latency functions by categorizing the latency functions into classes.

The thesis is about methods to push the equilibrium flow towards the optimal flow, and reduce the "inefficiency" of the equilibrium flow. There are several previous works about methods to decrease the bound of PoA. Roughgarden [27] analyzed the ideas of *edge removal* and *imposing tolls*. Edge removal means that some of the edges are removed from the network and the users are not allowed to use them. Edge removal is a powerful method to decrease the social cost of equilibrium flow.

In the idea of imposing *tolls*, the the traffic users are forced to pay some extra tolls when using the network. The idea of *marginal tolls* appears early in 1920s [24]. If the latency functions $l_e(f_e), \forall e \in E$ are continuous and differentiable, then the marginal tolls are known to be [27]

$$\tau_e := f_e^{opt} \frac{dl_e}{df_e}(f_e^{opt})$$

The marginal tolls push the equilibrium flow to the optimal one.

Another idea, called *Stackelberg routing*, works under the assumption that some part of the traffic users would like to obey a central coordinator's command and the rest act selfishly in the routing problem. The coordinator controls a fraction $\alpha \in (0, 1]$ of the total amount of users and his strategy to control these part of users is referred to as *Stackelberg policy*. Roughgarden [29] defined two natural Stackelberg policies: SCALE and Largest Latency First (LLF). SCALE sets the flow on every path $p \in P$ equal to α times the optimal flow $\mathbf{h}_p^{opt}$. LLF (applied in parallel-link networks) orders the links by their latency in the optimal flow from largest to smallest, and saturates them one-by-one until there are no centrally controlled users remaining. Swammy [32] considered the Stackelberg policies with general latency functions. Karakostas et al. [12] generalized the result to multi-commodity networks with general latency functions, and got bounds of PoA for both the SCALE and LLF methods.

1.1.3 Stochastic User Equilibrium

Following McFadden [18] and Sheffi [30], we will consider the case where the traffic users are allowed to commit some (random) errors when estimating the path latency. In this case, users have some uncertainty (measured by the random error) about the latency, and their choices of path to travel on is also random. The model for this kind of selfish routing is referred to as *stochastic network loading model*. The strategy of each traffic user in commodity $w \in W$ is a probability distribution on every path $p \in P_w$, and the payoff is the negative of the expected latency on the paths they choose. The aggregation of all the users' strategy will induce a feasible flow, and the traffic flows where no user may lower her expected travel latency by unilaterally changing her current strategy are referred to as flows at *stochastic user equilibrium* (SUE).

In the stochastic network loading models, the distribution of the random error plays an important role. "Logit-based model"[3] and "probit-based model"[4] are the two most widely used models. In the logit-based model, the random errors on all paths are i.i.d (independent and identically distributed), following the Gumbel distribution, and in the probit-based model, the random errors are i.i.d. following the normal distribution. Guo and Yang [10] analyzed the stochastic user equilibrium in the "logit-based model" and they presented a bound of the price of anarchy which is related not only to the traffic network, but also to the Gumbel distribution.

1.1.4 Repeated Games

The kind of game above is referred to as *stage game*, or one-shot game, in which the players play the game once. A *repeated game* [16] is a sequence of stage games. In a repeated game, the stage game is played repeatedly in periods $0, 1, 2...$ and the players make their choices simultaneously in each period and observe that period's outcome before proceeding to the next one. Let a_i^t be the action chosen by player i in period t, and a^t be all the players' actions in period t. The payoff (stage payoff) for player i in period t is $u_i(a^t)$.

We consider two kinds of players, the *long-run players* and the *short-run players*. A long-run player i plays the game in every period and maximizes his *normalized discount sum of payoff*

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t u_i(a^t)$$

where $\delta \in [0, 1)$ is the *discount factor*. The discount factor δ indicates the importance of the stage payoffs in the future. The closer δ is to 1, the more equivalent the stage payoffs in the distant future are to the ones closer to the present.

A short-run player i is concerned only with his payoff in the current period t and hence is often referred to as myopic. He will maximize the stage payoff $u_i(a^t)$.

Fudenberg and Levine [8] [9] analyzed the case where one long-run player plays a simultaneous-move stage game with a sequence of short-run players who play only once in their current period but can observe all the previous play of the long-run player. They considered the case where the long-run player knows his payoff function but the short-run players do not. They represented short-run players' uncertainty about the long-run player' payoffs using Harsanyi's notion[11] of a game with *incomplete information*. Let the *Stackelberg strategy* be the pure strategy to which the long-run player would most like to commit himself. If there is a positive prior probability that the long-run player will always play the Stackelberg strategy in each period of the repeated game, then his payoff in any Nash equilibrium of the repeated game exceeds a bound that converges to the *Stackelberg payoff* in the stage game (the payoff he is expected to get when he commits himself to the Stackelberg strategy) as the discount factor approaches 1. Such results are known as *folk theorems*.

Liu et al. [15] discussed the repeated game where instead of observing the outcome of all the previous periods, the short-run players only have access to the outcome of the most recent K periods. They analyzed the equilibrium strategy and the payoff of the single long-run player in the repeated game.

1.2 Our Work and Applications

The thesis is about an idea to decrease the price of anarchy in the selfish routing. We presented several potential ideas to decrease PoA in Section 1.1.2, but there are some problems when we apply them in practice.

The idea of edge removal is unrealistic in practice, since given a traffic network, we can hardly remove an edge or find any reasonable excuse for the selfish users not to use an existing road.

The idea of imposing tolls is also hard to use in practice. First, the latency is in terms of time, and the only practical toll is in terms of money, while it is not easy to find the exchange rate of time and money because everybody has his own weight of these two. Besides, when the idea is applied in practice the only possible “person” to collect the tolls is the government. But the government needs to find a reason to collect tolls from the traffic users.

The idea that part of the traffic users obey the coordinator’s command is also an unrealistic assumption, since in practice every user acts selfishly.

Our Model Description There will be a *coordinator* who can broadcast information about the traffic to all the users in the network via the public radio or through the Internet. He tells the users that there is some unexpected extra flow $\bar{\mathbf{h}}^{extra}$ in the network. Actually this is fabricated information, and the extra flow does not exist at all in the network. Broadcasting fabricated information is the coordinator's way to manipulate the traffic flow.

The following simple example shows how the extra flow helps to decrease the PoA if all the traffic users believe the information from the coordinator.

Example 1.2:

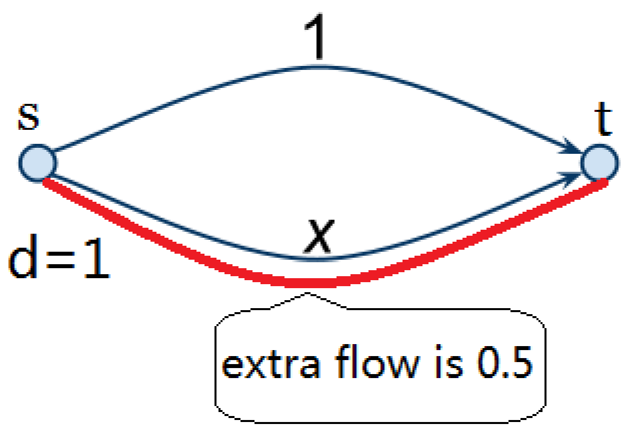

Figure 1.2: Virtual Flow Example

Figure 1.2 shows a simple network with a single origin s, a single destination t, and total demand $d = 1$. There are two parallel candidate routes: path 1 has a fixed latency of 1, and the latency on path 2 is equal to the traffic flow on it. If the coordinator broadcasts the information that on path 2 there is unexpected extra flow of amount 0.5 and all the users believe in the coordinator, then at equilibrium flow half of the traffic users will choose path 1, with the perceived travel latency 1, and the other half will choose path 2, also with the perceived travel latency 1.

The extra flow does not exist in reality, so the users choosing path 2 will suffer latency 0.5, although they are expected to suffer latency 1 when they make their choice. The social cost of the equilibrium flow is $0.5 \times 0.5 + 1 \times 0.5 = 3/4$, which is also the social cost of optimal flow. Thus the PoA goes from 4/3 (without the extra flow) to 1.

It is the coordinator's decision to choose how much extra flow to announce. We will consider the simple case where the coordinator picks an $x \in [0, 1]$ and sets $\bar{\mathbf{h}}^{extra} := (1 - x)\mathbf{h}^{opt}$ where $\mathbf{h}^{opt}$ is the optimal flow defined in Definition 1.1.5.

The traffic users do not have complete information about the network (they do not know how much the unexpected extra flow is). They will choose an amount of trust $y \in [0, 1]$ to put on the coordinator's information, and they will take the part $\mathbf{h}^{extra} = y\bar{\mathbf{h}}^{extra}$ into account when they make their choices of which path to travel on. We will call this part *virtual flow*, since it only exists in the users' mind.

We are going to introduce a bi-level game. The upper level is a trust game between the coordinator who decides how much advantage to take over users' incomplete information about the network, and the traffic users who decide how much to trust the coordinator. The lower level is the selfish routing game among all the traffic users, with the extra flow determined by the upper level. In the trust game, the coordinator's payoff is the negative of the price of anarchy of the selfish routing game in the lower level, and the traffic users's payoff is the negative of their perceived path latency in the equilibrium flow of the lower level.

Finally we consider the repeated games with the trust game being played repeatedly. The coordinator is a long-run player, and the traffic users are short-run since they do not care about the stage payoff in the future periods. The traffic users can look up the outcome of the previous periods, and then decide how much trust to put on the coordinator. In this way, the coordinator cannot be too greedy because he also needs to maintain his reputation so that the traffic users would like to put more trust in him when they observe the actions of the previous periods. We conclude that at the equilibrium of the repeated games, the price of anarchy of the selfish routing game is decreased from the well-known bounds (see [26][5]).

The thesis is about the traffic assignment. But the same analysis applies to an information network, which is about data traffic, instead of vehicle traffic.

1.3 Organization

Chapter 1 is the introduction part, which is about the basic background of the problem and presents some selected relevant works by the previous researchers. In Chapter 2 we consider the user equilibrium (UE) model and discuss the trust game between the traffic coordinator and traffic users. In Chapter 3, we focus on the stochastic user equilibrium (SUE) model, which takes traffic users' errors about the measured latency into account, and then discuss the trust game between the traffic coordinator and the users. Chapter 4 is about reputation and repeated games, where the traffic coordinator will play with a sequence of traffic users. The conclusion and some open problems will be included in Chapter 5.

Chapter 2

User Equilibrium

This chapter introduces a *trust game* between the traffic coordinator and the traffic users. The coordinator tries to manipulate the selfish users and push the user equilibrium (UE) flow towards the optimum, by broadcasting virtual flow to the users. The traffic users choose how much trust to put on the coordinator, and route selfishly in the network.

2.1 Trust Game between the Network Coordinator and the Users

The traffic coordinator knows everything about the network and he wants to minimize the PoA of the traffic network. The traffic users route selfishly in the network, i.e., they want to minimize the latency on their chosen path. Here we assume that the users know the topology of the network G, the demand $\mathbf{d}$ and latency function $\mathbf{l}$. This is reasonable because G and $\mathbf{l}$ are fixed for a certain traffic network, and we consider the case with fixed demand $\mathbf{d}$. We call $\mathbf{d}$ the *normal demand*, and the flow consisting of the users in $\mathbf{d}$ *normal flow*.

In the trust game, we introduce the coordinator who broadcasts the information of *unexpected extra flow* in the network to all the traffic users. The traffic users do not know anything about extra flow themselves, so the coordinator can take advantage of the users' *incomplete information* about the network (here the incomplete information is the extra flow) to manipulate the information he provides.

In our case, the extra flow that the coordinator broadcasts does not exist at all in reality and is referred to as "virtual flow". The coordinator provides this fabricated information to manipulate the normal flow and push it towards the optimum. If the traffic users believe the coordinator, they will take the extra flow into account when they choose the path to route on, because they believe the extra flow will also make a contribution to the congestion. Note that even though the information about the extra flow is fabricated, it is not easy for the traffic users to reveal this fact, because in the users' view, both the virtual flow and the normal flow are the aggregation of homogenous traffic users, and they are not distinguishable at all.

In the game, the coordinator decides how much advantage he will take of the users. In other words, he decides how much extra flow to broadcast to the users. We assume the maximum extra demand he will put is $\bar{\mathbf{d}}^{extra}$ and the corresponding extra path flow is $\bar{\mathbf{h}}^{extra}$.

Of course, the traffic users are not required to trust the coordinator's information. They are free to choose how much trust to put on the information the coordinator announces.

On the other hand, when the users distrust the coordinator's information, they will be confused about the amount of extra latency the extra flow will bring, and they will estimate it. Because of the homogeneity, we assume that each traffic user in commodity w will believe that the extra flow will increase her latency by m_w (the same amount for all the users in w), which is the *estimated extra latency*. Furthermore, we assume that the estimated extra latency is identical for all the paths the users may choose, i.e, the users do not have any personal feelings about the path $p \in P_w$, and they believe the extra flow will bring the same extra latency to all the paths.

Next, we will model this trust game formally.

Game Model

Players: In this game, there are $|W| + 1$ players. One is the network coordinator, and the other $|W|$ players are the W commodities. In the traffic model with homogenous traffic users, we do not distinguish the users in the same commodity, so we just pick a representative instead of considering each single user's choice.

In the following, we will focus on the simplified version where there is only a single O-D pair in the instance $(G, \mathbf{l}, \mathbf{d})$. In this case, all the traffic users share the same origin and destination, and $\mathbf{d}$ is one-dimensional, so we will use d instead. The trust game involves two players: the coordinator (denoted as *Player 1* in the following) and the representative of all the traffic users (denoted as *Player 2* in the following).

Strategies: Player 1's strategy s_1 is to pick an $x \in X = [0, 1]$, which decides how much advantage he will take of Player 2's incomplete information. In our model, Player 1 will tell Player 2 that the extra path flow is $\tilde{\mathbf{h}}^{extra} = (1 - x)\bar{\mathbf{h}}^{extra}$. $x = 0$ indicates that he is taking full advantage of Player 2 and the extra path flow is $\bar{\mathbf{h}}^{extra}$. $x = 1$ means that he takes no advantage of Player 2 and the extra edge flow is 0. In the model, Player 1 uses $\mathbf{h}^{opt}$ as $\bar{\mathbf{h}}^{extra}$, so the extra path flow he announces is $(1 - x)\mathbf{h}^{opt}$ (the corresponding extra edge flow is $(1 - x)\mathbf{f}^{opt}$). Clearly, $\bar{\mathbf{h}}^{extra}$ can become part of Player 1's strategy in the design of the game; we leave this ability to future work and just use $\mathbf{h}^{opt}$ in the following.

Player 2's strategy s_2 is to choose $y \in Y = [0, 1]$ which denotes her trust on Player 1. $y = 1$ means she completely trusts Player 1's information, and $y = 0$ implies that she totally distrusts Player 1. $1-y$ will be the corresponding weight she puts on her own estimation about the extra latency.

We consider the case where the two players move simultaneously. A *strategy profile* $s = (x, y) \in [0,1] \times [0,1]$ records Player 1 and Player 2's strategies.

Payoffs: Given a strategy profile $s = (x, y)$, Player 2's *perceived latency function* $\hat{\mathbf{l}}(\mathbf{h})$ will be

$$\hat{l}_p(\mathbf{h}) = l_p(\mathbf{h} + (1-x)y\mathbf{h}^{opt}) + (1-y)m, \forall p \in P \tag{2.1}$$

where $\mathbf{l}(\mathbf{h})$ is the latency function, $\mathbf{h}$ is the normal flow, $(1-x)\mathbf{h}^{opt}$ is the virtual flow Player 1 announces in which $\mathbf{h}^{opt}$ is the optimal flow, and m is Player 2's own estimation about the extra latency. Player 2 puts weight y on the information Player 1 announces, and $1-y$ on her own estimation.

The traffic users will route selfishly in the instance $(G, \hat{\mathbf{l}}, d)$. Denote the equilibrium flow of the instance by $\mathbf{f}^{equ}$ (and path flow $\mathbf{h}^{equ}$).

The result in Beckmann et al. [2] can also be applied here to prove that the instance with infinitesimal users admits at least one equilibrium flow and all the equilibrium flow shares the common latency.

We will use ρ, the price of anarchy (PoA) in Definition 1.1.7, to measure the inefficiency of the equilibrium flow. (1.4) implies $\rho := \frac{S(\mathbf{f}^{equ})}{S(\mathbf{f}^{opt})}$, where $S(\mathbf{f}) = \sum_{e\in E} f_e l_e(f_e)$ is the social cost function, $\mathbf{f}^{opt}$ is the optimal flow and $\mathbf{f}^{equ}$ is the equilibrium flow in the instance $(G, \hat{\mathbf{l}}, d)$.

Player 1 is the network coordinator and his payoff is defined by the negative of the price of anarchy.

Definition 2.1.1 ***(Player 1's payoff)*** *Player 1's payoff in the trust game is*

$$\Gamma_1 := -\rho = -\frac{S(\mathbf{f}^{equ})}{S(\mathbf{f}^{opt})}$$

The ratio ρ is lower bounded by 1 and $\rho = 1$ implies that the equilibrium flow coincides with the optimal one.

Player 2 represents the traffic users, and her payoff Γ_2 is defined by the negative of the perceived common latency at equilibrium $\mathbf{h}^{equ}$.

Definition 2.1.2 ***(Player 2's payoff)*** *Player 2's payoff in the trust game is*

$$\Gamma_2 := -\hat{l}_p(\mathbf{h}^{equ}), \quad \forall p \in P, \ \forall h_p^{equ} > 0$$

This definition is valid since, at equilibrium, Wardrop's Principle[33] guarantees that all the used paths p ($\forall p \in P, s.t. \ \ h_p^{equ} > 0$) bear the same latency. Hence, we can also use another equivalent definition:

$$\Gamma_2 = -\frac{1}{d} \sum_{p \in P} h_p^{equ} \hat{l}_p(\mathbf{h}^{equ}) \tag{2.2}$$

We have

$$\begin{aligned} \Gamma_2 &\overset{(1.2)(2.2)}{=} -\frac{1}{d} \sum_{p \in P} h_p^{equ} l_p(\mathbf{h}^{equ} + (1-x)y\mathbf{h}^{opt}) - (1-y)m \\ &\overset{(1.3)}{=} -\frac{1}{d} \sum_{e \in E} f_e^{equ} l_e(f_e^{equ} + (1-x)y f_e^{opt}) - (1-y)m \end{aligned} \tag{2.3}$$

2.2 User Equilibrium Flow

Lemma 2.2.1 $\mathbf{f}^{equ}$*(or* $\mathbf{h}^{equ}$*), the equilibrium flow for the instance* $(G, \hat{\mathbf{l}}, d)$*, is also the equilibrium flow for the instance* $(G, \tilde{\mathbf{l}}, d)$*, where*

$$\tilde{l}_e(f_e) = l_e(f_e + (1-x)y f_e^{opt}), \quad \forall e \in E \tag{2.4}$$

Proof:

$$\begin{aligned} \tilde{\mathbf{l}}(\mathbf{h}) &\overset{(1.2)}{=} \Delta_{|E| \times |P|}^T \tilde{\mathbf{l}}(\mathbf{f}) \\ &\overset{(2.4)}{=} \Delta_{|E| \times |P|}^T \mathbf{l}(\mathbf{f} + (1-x)y\mathbf{f}^{opt}) \end{aligned}$$

$$\overset{(1.1)}{=} \Delta^T_{|E|\times|P|}\mathbf{l}(\Delta_{|E|\times|P|}(\mathbf{h} + (1-x)y\mathbf{h}^{opt}))$$
$$\overset{(1.2)}{=} \mathbf{l}(\mathbf{h} + (1-x)y\mathbf{h}^{opt}) \tag{2.5}$$

Thus $\forall p \in P$, we have

$$\tilde{l}_p(\mathbf{h}) = l_p(\mathbf{h} + (1-x)y\mathbf{h}^{opt}) \overset{(2.1)}{=} \hat{l}_p(\mathbf{h}) - (1-y)m \tag{2.6}$$

From Definition 1.1.6, we can see that if $\mathbf{h}^{equ}$ is the equilibrium flow for the instance $(G, \hat{\mathbf{l}}, d)$, then it must be an equilibrium flow for the instance $(G, \tilde{\mathbf{l}}, d)$.

□

Lemma 2.2.1 states that the equilibrium flow $\mathbf{f}^{equ}$ for the instance $(G, \hat{\mathbf{l}}, d)$ is equal to the equilibrium flow for the instance $(G, \tilde{\mathbf{l}}, d)$. Next we will use the equilibrium flow for $(G, \tilde{\mathbf{l}}, d)$ as $\mathbf{f}^{equ}$, and furthermore analyze the payoff for Player 1 and Player 2 in the trust game.

Recall that $\tilde{l}_e(f_e) = l_e(f_e + (1-x)yf_e^{opt}), \forall e \in E$. If we denote $\alpha = (1-x)y$ for convenience, then

$$\tilde{l}_e(f_e) = l_e(f_e + \alpha f_e^{opt}), \forall e \in E \tag{2.7}$$

Furthermore, the part $\alpha\mathbf{f}^{opt}$ is virtual flow which does not contribute to the social cost.

2.2.1 Networks with Linear Latency Functions

Consider the case where the latency function of each edge is linear: $l_e(f_e) = a_e f_e + b_e, a_e, b_e \geq 0, \forall e \in E$.

Without the virtual flow $\alpha\mathbf{f}^{opt}$, Roughgarden and Tardos [26] proved that the price of anarchy of $(G, \mathbf{l}, d)$ is upper bounded by 4/3 for all kinds of network topology.

Next we will show that with the virtual flow $\alpha\mathbf{f}^{opt}$, the PoA is decreased.

Lemma 2.2.2 *Let $\mathbf{f}^\alpha$ be the equilibrium flow for the instance $(G, \tilde{\mathbf{l}}, \mathbf{d})$ where $\tilde{\mathbf{l}}$ is defined in (2.7), and the latency functions are linear: $l_e(f_e) = a_e f_e + b_e, a_e, b_e \geq 0, \forall e \in E$. The price of anarchy $\rho(\alpha)$ is upper bounded by*

$$\rho(\alpha) := \frac{S(\mathbf{f}^\alpha)}{S(\mathbf{f}^{opt})} \leq \frac{4}{3+\alpha}, \forall \alpha \in [0,1] \tag{2.8}$$

Proof: Since $\mathbf{f}^\alpha$ is a traffic equilibrium, and $\tilde{l}_e(f_e) = l_e(f_e + \alpha f_e^{opt}), \forall e \in E$, the following variational inequality holds (see [26]):

$$\sum_{e\in E} \tilde{l}_e(f_e^\alpha)(f_e - f_e^\alpha) \geq 0, \quad \forall \mathbf{f} \in \mathcal{F} \tag{2.9}$$

By setting $\mathbf{f} := \mathbf{f}^{opt}$ in (2.9), we get that

$$\begin{aligned} S(\mathbf{f}^\alpha) &= \sum_{e\in E}(a_e f_e^\alpha + b_e) f_e^\alpha \\ &\leq \sum_{e\in E}(a_e f_e^\alpha + b_e) f_e^{opt} - \alpha \sum_{e\in E} a_e f_e^{opt}(f_e^\alpha - f_e^{opt}) \\ &\leq S(\mathbf{f}^{opt}) + (1-\alpha)\sum_{e\in E} a_e f_e^{opt}(f_e^\alpha - f_e^{opt}) \end{aligned} \tag{2.10}$$

If we set $c_e := a_e f_e^{opt}(f_e^\alpha - f_e^{opt}), \forall e \in E$, then

(a) if $f_e^\alpha \leq f_e^{opt}$, then obviously $c_e \leq 0$;

(b) if $f_e^{opt} < f_e^\alpha$, then we define $\lambda_e := f_e^{opt}/f_e^\alpha$, $0 \leq \lambda_e < 1$, and we have

$$\begin{aligned} c_e &= a_e (f_e^\alpha)^2(\lambda_e - \lambda_e^2) \\ &\leq [\frac{1}{4} - (\lambda_e - \frac{1}{2})^2] a_e (f_e^\alpha)^2 \\ &\leq \frac{1}{4}(a_e f_e^\alpha + b_e) f_e^\alpha \end{aligned}$$

To sum up, $c_e \leq \frac{1}{4}(a_e f_e^\alpha + b_e) f_e^\alpha$, so we have

$$\sum_{e\in E} a_e f_e^{opt}(f_e^\alpha - f_e^{opt}) \leq \frac{1}{4}\sum_{e\in E}(a_e f_e^\alpha + b_e) f_e^\alpha = \frac{1}{4} S(\mathbf{f}^\alpha) \tag{2.11}$$

From (2.10), we get

$$S(\mathbf{f}^\alpha) \leq S(\mathbf{f}^{opt}) + \frac{1}{4}(1-\alpha) S(\mathbf{f}^\alpha)$$

Then

$$\rho(\alpha) := \frac{S(\mathbf{f}^{\alpha})}{S(\mathbf{f}^{opt})} \leq \frac{1}{1 - 1/4(1-\alpha)} = \frac{4}{3+\alpha}, \quad 0 \leq \alpha \leq 1$$

□

$\rho(0) \leq \frac{4}{3}$ implies that if we do not introduce any virtual flow, the PoA is upper bounded by 4/3, which is the same result as the one Roughgarden and Tardos showed in [26].

Lemma 2.2.2 also shows that if the extra flow is equal to $\mathbf{f}^{opt}$, then the equilibrium flow will coincide with the optimum.

Lemma 2.2.3 *Let $\mathbf{f}^{\alpha}$ be the equilibrium flow for the instance $(G, \tilde{\mathbf{l}}, \mathbf{d})$ where $\tilde{\mathbf{l}}$ is defined in* (2.7)*, and the latency functions are linear: $l_e(f_e) = a_e f_e + b_e, a_e, b_e \geq 0, \forall e \in E$. We have*

$$\sum_{e \in E} f_e^{\alpha} \tilde{l}_e(f_e^{\alpha}) \leq \frac{4}{3-\alpha} S(\mathbf{f}^{opt}), \forall \alpha \in [0,1] \tag{2.12}$$

Proof: The variational inequality (2.9) for the instance $(G, \tilde{\mathbf{l}}, \mathbf{d})$ implies

$$\sum_{e \in E} f_e^{\alpha} \tilde{l}_e(f_e^{\alpha}) \leq \sum_{e \in E} f_e \tilde{l}_e(f_e^{\alpha}), \quad \forall \mathbf{f} \in \mathcal{F}$$

By setting $\mathbf{f} := \mathbf{f}^{opt}$, we get

$$\sum_{e \in E} f_e^{\alpha} l_e(f_e^{\alpha} + \alpha f_e^{opt}) \leq \sum_{e \in E} f_e^{opt} l_e(f_e^{\alpha} + \alpha f_e^{opt}) \tag{2.13}$$

Since the latency functions $l_e(x)$ are linear $l_e(x) = a_e x + b_e, a_e, b_e \geq 0, \forall e \in E$, and we have

$$\frac{1}{4}x^2 + y^2 - xy = (\frac{1}{2}x - y)^2 \geq 0, \forall x, y$$

we can get

$$y[l_e(x) - l_e(y)] = a_e(xy - y^2) \leq a_e \cdot \frac{1}{4}x^2 \leq \frac{1}{4} x l_e(x) \tag{2.14}$$

$$\Leftrightarrow \qquad yl_e(x) \leq yl_e(y) + \frac{1}{4}xl_e(x), \forall x, y \geq 0 \tag{2.15}$$

If we set $x := f_e^\alpha + \alpha f_e^{opt}, y := f_e^{opt}$, then (2.15) implies

$$f_e^{opt} l_e(f_e^\alpha + \alpha f_e^{opt}) \leq f_e^{opt} l_e(f_e^{opt}) + \frac{1}{4}(f_e^\alpha + \alpha f_e^{opt}) l_e(f_e^\alpha + \alpha f_e^{opt})$$

Thus

$$\begin{aligned}
&\sum_{e\in E} f_e^{opt} l_e(f_e^\alpha + \alpha f_e^{opt}) \\
\leq &\sum_{e\in E} f_e^{opt} l_e(f_e^{opt}) + \frac{1}{4}\sum_{e\in E}(f_e^\alpha + \alpha f_e^{opt}) l_e(f_e^\alpha + \alpha f_e^{opt}) \\
\leq &S(\mathbf{f}^{opt}) + \frac{1}{4}\sum_{e\in E} f_e^\alpha l_e(f_e^\alpha + \alpha f_e^{opt}) + \frac{1}{4}\alpha \sum_{e\in E} f_e^{opt} l_e(f_e^\alpha + \alpha f_e^{opt}) \\
\overset{(2.13)}{\leq} &S(\mathbf{f}^{opt}) + \frac{1}{4}(1+\alpha)\sum_{e\in E} f_e^{opt} l_e(f_e^\alpha + \alpha f_e^{opt})
\end{aligned} \tag{2.16}$$

Solving (2.16), we get

$$\sum_{e\in E} f_e^{opt} l_e(f_e^\alpha + \alpha f_e^{opt}) \leq \frac{4}{3-\alpha} S(\mathbf{f}^{opt})$$

and (2.13) implies

$$\sum_{e\in E} f_e^\alpha \tilde{l}_e(f_e^\alpha) \leq \frac{4}{3-\alpha} S(\mathbf{f}^{opt})$$

□

2.2.2 Networks with General Latency Functions

The analysis of the linear case in Section 2.2.1 can be extended to general latency functions that satisfy certain properties. These properties are described in the following assumptions.

Assumption 1 *$\forall e \in E, l_e(x)$ is a convex and non-decreasing continuous function of x, with the first and second derivative existing everywhere.*

Usually this assumption holds in practice. $l_e(x)$ is non-decreasing means that the more the traffic on a road, the greater the latency will be. $l_e(x)$ is convex means that if the traffic on an edge is already heavy, the same amount of additional flow coming in will make more serious congestion than in the case where the traffic is light. This is common in reality, because when the traffic is light, a small amount of extra flow will bring no congestion at all, but when a jam already exists, it will make a big difference.

Let $\mathcal{L}$ denote the family of the continuous and non-decreasing latency functions. Correa et al. [5] introduced the concept of β-function to denote the linearity of $\mathcal{L}$.

Definition 2.2.1 ***(Beta-function)*** *$\forall l \in \mathcal{L}$, define*

$$\beta(l) := \sup_{0<y<x} \frac{y[l(x)-l(y)]}{xl(x)} \tag{2.17}$$

Then $\beta(\mathcal{L})$ is

$$\beta(\mathcal{L}) := \sup_{l \in \mathcal{L}} \beta(l)$$

It is obvious that $0 \leq \beta(\mathcal{L}) \leq 1$. If $\mathcal{L}$ is the family of linear functions $l_e(f_e) = a_e f_e + b_e, a_e, b_e \geq 0, \forall e \in E$, then $\beta(\mathcal{L}) = \frac{1}{4}$ [5].

Assumption 2 *We assume that $\beta(\mathcal{L}) < \frac{1}{2}$, which indicates that these functions are not too "nonlinear".*

Recall that $\mathbf{l}(\mathbf{f})$ is $|E|-$dimensional. Define $\nabla L(\mathbf{f})$ to be the Jacobian matrix of $\mathbf{l}(\mathbf{f})$, with dimension $|E| \times |E|$

$$\nabla L(\mathbf{f})_{ij} = \frac{\partial l_i}{\partial f_j}, \quad \forall i, j \in E$$

If the separable property (see Definition 1.1.2) holds for the instance, then l_i is independent of $f_j, \forall j \neq i$, so $\nabla L(\mathbf{f})$ will be a diagonal matrix, with the diagonal entries $\nabla L(\mathbf{f})_{ii} = \frac{dl_i}{df_i}$. Note that $\nabla L(\mathbf{f})$ is positive semi-definite.

The *Jacobian similarity property* [23] implies that there exists a constant J satisfying that for all feasible flow $\mathbf{f}, \mathbf{f}'$, and for all $\mathbf{w} \in \mathbb{R}^{|E|}$

$$\frac{1}{J}\mathbf{w}^T \nabla L(\mathbf{f})\mathbf{w} \leq \mathbf{w}^T \nabla L(\mathbf{f}')\mathbf{w} \leq J\mathbf{w}^T \nabla L(\mathbf{f})\mathbf{w} \tag{2.18}$$

and the smallest J is referred to as the *Jacobian similarity factor.*

Since $\nabla L(\mathbf{f})$ is positive semi-definite, (2.18) indicates that $J \geq 1$. If the latency functions are linear, $J = 1$ since

$$\nabla L(\mathbf{f})_{ii} = \frac{dl_i}{df_i} = a_i = \nabla L(\mathbf{f}')_{ii}$$

Assumption 3 *The Jacobian similarity property holds for the instance* $(G, \mathbf{l}, \mathbf{d})$*, and the Jacobian similarity factor* J *satisfies*

$$J < \frac{1}{1 - \beta(\mathcal{L})}$$

Lemma 2.2.4 *Let* $\mathbf{f}^\alpha$ *be the equilibrium flow for the instance* $(G, \tilde{\mathbf{l}}, \mathbf{d})$ *where* $\tilde{\mathbf{l}}$ *is defined in* (2.7)*, and the latency functions* $\mathbf{l}$ *are convex and non-decreasing. If the Jacobian similarity property holds for the instance, then the price of anarchy is bounded by*

$$\rho(\alpha) := \frac{S(\mathbf{f}^\alpha)}{S(\mathbf{f}^{opt})} \leq \frac{1 + (J-1)\alpha}{1 - \beta + \beta\alpha} \tag{2.19}$$

where J *is the similarity factor and* $\beta := \beta(\mathcal{L})$*.*

Proof: Define

$$T_0 := \sum_{e \in E} f_e^{opt} l_e(f_e^\alpha) = [\mathbf{f}^{opt}]^T \mathbf{l}(\mathbf{f}^\alpha) \tag{2.20}$$

$$T_1 := \sum_{e \in E} f_e^\alpha l_e(f_e^\alpha + \alpha f_e^{opt}) \tag{2.21}$$

$$T_2 := \sum_{e \in E} f_e^{opt} l_e(f_e^\alpha + \alpha f_e^{opt}) \tag{2.22}$$

From Definition 2.2.1, we have

$$\beta \geq \frac{f_e^{opt}[l_e(f_e^\alpha) - l_e(f_e^{opt})]}{f_e^\alpha l_e(f_e^\alpha)}$$

Thus

$$\begin{aligned}
T_0 &= \sum_{e\in E} f_e^{opt} l_e(f_e^\alpha) \\
&\leq \sum_{e\in E} [\beta f_e^\alpha l_e(f_e^\alpha) + f_e^{opt} l_e(f_e^{opt})] \\
&= \beta \sum_{e\in E} f_e^\alpha l_e(f_e^\alpha) + \sum_{e\in E} f_e^{opt} l_e(f_e^{opt}) \\
&= \beta S(\mathbf{f}^\alpha) + S(\mathbf{f}^{opt}) \qquad (2.23)
\end{aligned}$$

Since $l_e(f_e)$ is continuous, if $l'_e(f_e)$ denotes the first derivative of $l_e(f_e)$ with respect to f_e, then the mean value theorem ([17]) indicates that $\exists t \in [0,1]$ such that if $f_e = {f_e}^\alpha + t\alpha {f_e}^{opt}$, then

$$l_e(f_e^\alpha + \alpha f_e^{opt}) = l_e(f_e^\alpha) + \alpha f_e^{opt} l'_e(f_e) \qquad (2.24)$$

Similarly, $\exists \hat{t} \in [0,1]$ such that if $\hat{f}_e = \hat{t} {f_e}^\alpha$, then

$$l_e(f_e^\alpha) = l_e(0) + f_e^\alpha l'_e(\hat{f}_e) \qquad (2.25)$$

The convexity of l_e guarantees $l'_e(f_e) \geq l'_e(\hat{f}_e)$, since $f_e \geq f_e^\alpha \geq \hat{f}_e$.
We can get

$$\begin{aligned}
l_e(f_e^\alpha + \alpha f_e^{opt}) f_e^\alpha &\overset{(2.24)}{=} l_e(f_e^\alpha) f_e^\alpha + \alpha f_e^{opt} l'_e(f_e) f_e^\alpha \\
&\overset{(2.25)}{\geq} l_e(f_e^\alpha) f_e^\alpha + \alpha f_e^{opt} [l_e(f_e^\alpha) - l_e(0)] \quad \forall e \in E
\end{aligned}$$

Thus we have

$$\begin{aligned}
T_1 &= \sum_{e\in E} l_e(f_e^\alpha + \alpha f_e^{opt}) f_e^\alpha \\
&\geq \sum_{e\in E} l_e(f_e^\alpha) f_e^\alpha + \alpha \sum_{e\in E} f_e^{opt} l_e(f_e^\alpha) - \alpha \sum_{e\in E} f_e^{opt} l_e(0) \\
&\geq S(\mathbf{f}^\alpha) + \alpha T_0 - \alpha \sum_{e\in E} f_e^{opt} l_e(0) \qquad (2.26)
\end{aligned}$$

Similarly, we have

$$T_2 = \sum_{e\in E} l_e(f_e^\alpha + \alpha f_e^{opt}) f_e^{opt}$$

$$=[\mathbf{f}^{opt}]^T \mathbf{l}(\mathbf{f}^{\alpha} + \alpha \mathbf{f}^{opt})$$

The mean value theorem implies that $\exists t \in [0,1]$ s.t. if $\mathbf{f} = \mathbf{f}^{\alpha} + t\alpha \mathbf{f}^{opt}$, then

$$\mathbf{l}(\mathbf{f}^{\alpha} + \alpha \mathbf{f}^{opt}) = \mathbf{l}(\mathbf{f}^{\alpha}) + \nabla L(\mathbf{f}) \cdot \alpha \mathbf{f}^{opt}$$

Thus

$$\begin{aligned} T_2 =& [\mathbf{f}^{opt}]^T [\mathbf{l}(\mathbf{f}^{\alpha}) + \alpha \nabla L(\mathbf{f}) \mathbf{f}^{opt}] \\ \stackrel{(2.20)}{=}& T_0 + \alpha [\mathbf{f}^{opt}]^T \nabla L(\mathbf{f}) \mathbf{f}^{opt} \end{aligned} \tag{2.27}$$

Besides, the mean value theorem implies that $\exists \hat{t} \in [0,1]$ such that if $\hat{\mathbf{f}} = \hat{t}\mathbf{f}^{opt}$, then

$$[\mathbf{f}^{opt}]^T [\mathbf{l}(\mathbf{f}^{opt}) - \mathbf{l}(\mathbf{0})] = [\mathbf{f}^{opt}]^T \nabla L(\hat{\mathbf{f}}) \mathbf{f}^{opt} \tag{2.28}$$

The similarity property indicates

$$[\mathbf{f}^{opt}]^T \nabla L(\mathbf{f}) \mathbf{f}^{opt} \leq J [\mathbf{f}^{opt}]^T \nabla L(\hat{\mathbf{f}}) \mathbf{f}^{opt}$$

Using this in (2.27) and (2.28), we get

$$\begin{aligned} T_2 &\leq T_0 + \alpha J [\mathbf{f}^{opt}]^T [\mathbf{l}(\mathbf{f}^{opt}) - \mathbf{l}(\mathbf{0})] \\ &= T_0 + \alpha J \sum_{e \in E} f_e^{opt} [l_e(f_e^{opt}) - l_e(0)] \\ &= T_0 + \alpha J S(\mathbf{f}^{opt}) - \alpha J \sum_{e \in E} f_e^{opt} l_e(0) \end{aligned} \tag{2.29}$$

The variational inequality (2.9) and (2.21), (2.22) imply $T_1 \leq T_2$. Thus

$$\begin{aligned} S(\mathbf{f}^{\alpha}) &\stackrel{(2.26)}{\leq} T_2 - \alpha T_0 + \alpha \sum_{e \in E} f_e^{opt} l_e(0) \\ &\stackrel{(2.29)}{\leq} (1-\alpha) T_0 + \alpha J S(\mathbf{f}^{opt}) - \alpha (J-1) \sum_{e \in E} f_e^{opt} l_e(0) \\ &\stackrel{(2.23)}{\leq} \beta(1-\alpha) S(\mathbf{f}^{\alpha}) + (\alpha J + 1 - \alpha) S(\mathbf{f}^{opt}) \end{aligned} \tag{2.30}$$

We can get from (2.30)

$$\rho(\alpha) = \frac{S(\mathbf{f}^{\alpha})}{S(\mathbf{f}^{opt})} \leq \frac{1 + (J-1)\alpha}{1 - \beta(1-\alpha)}$$

$\square$

In the case where the latency functions are linear, $J = 1$, $\beta = \frac{1}{4}$, the bound shown in Lemma 2.2.4 goes to $\frac{4}{3+\alpha}$, which coincides with the one in Lemma 2.2.2.

Lemma 2.2.5 *Let $\mathbf{f}^\alpha$ be the equilibrium flow for the instance $(G, \tilde{\mathbf{l}}, \mathbf{d})$ where $\tilde{\mathbf{l}}$ is defined in (2.7). If the latency function $\mathbf{l}$ satisfies Assumption 2, then we have*

$$\sum_{e\in E} f_e^\alpha \tilde{l}_e(f_e^\alpha) \leq \frac{1}{1-(\alpha+1)\beta} S(\mathbf{f}^{opt}) \tag{2.31}$$

where $\beta := \beta(\mathcal{L})$.

Proof:

From the variational inequality (2.9), we have

$$\sum_{e\in E} f_e^\alpha l_e(f_e^\alpha + \alpha f_e^{opt}) \leq \sum_{e\in E} f_e^{opt} l_e(f_e^\alpha + \alpha f_e^{opt}) \tag{2.32}$$

In (2.17), if we set $x := f_e^\alpha + \alpha f_e^{opt}, y := f_e^{opt}$, then we will get

$$\frac{f_e^{opt} l_e(f_e^\alpha + \alpha f_e^{opt}) - f_e^{opt} l_e(f_e^{opt})}{(f_e^\alpha + \alpha f_e^{opt}) l_e(f_e^\alpha + \alpha f_e^{opt})} \leq \beta,$$

which implies

$$f_e^{opt} l_e(f_e^\alpha + \alpha f_e^{opt}) \leq f_e^{opt} l_e(f_e^{opt}) + \beta(f_e^\alpha + \alpha f_e^{opt}) l_e(f_e^\alpha + \alpha f_e^{opt})$$

Thus

$$\begin{aligned}
&\sum_{e\in E} f_e^{opt} l_e(f_e^\alpha + \alpha f_e^{opt}) \\
\leq &\sum_{e\in E} f_e^{opt} l_e(f_e^{opt}) + \beta \sum_{e\in E} f_e^\alpha l_e(f_e^\alpha + \alpha f_e^{opt}) + \alpha\beta \sum_{e\in E} f_e^{opt} l_e(f_e^\alpha + \alpha f_e^{opt}) \\
\overset{(2.32)}{\leq} &S(\mathbf{f}^{opt}) + (\beta + \alpha\beta) \sum_{e\in E} f_e^{opt} l_e(f_e^\alpha + \alpha f_e^{opt})
\end{aligned} \tag{2.33}$$

Assumption 2 indicates that $\beta < \frac{1}{2}$, so $(\alpha+1)\beta < 1$ when $0 \leq \alpha \leq 1$. Thus (2.33) implies

$$\sum_{e\in E} f_e^{opt} l_e(f_e^\alpha + \alpha f_e^{opt}) \leq \frac{1}{1-(\alpha+1)\beta} S(\mathbf{f}^{opt})$$

From (2.32), we have

$$\sum_{e\in E} \tilde{f}_e^\alpha l_e(f_e^\alpha) = \sum_{e\in E} f_e^\alpha l_e(f_e^\alpha + \alpha f_e^{opt}) \leq \frac{1}{1-(\alpha+1)\beta} S(\mathbf{f}^{opt})$$

$\square$

In the case where the latency functions are linear, $\beta = \frac{1}{4}$, and the bound here turns to $\sum_{e\in E} \tilde{f}_e^\alpha l_e(f_e^\alpha) \leq \frac{4}{3-\alpha} S(\mathbf{f}^{opt})$, which coincides with the one in Lemma 2.2.3.

2.3 Stackelberg Payoff in the Trust Game

In the trust game, Player 1's payoff is defined using the price of anarchy.

Lemma 2.3.1 *Player 1's payoff in the trust game is lower bounded by*

$$\Gamma_1(x,y) \geq -\frac{1+(J-1)(1-x)y}{1-\beta+\beta(1-x)y} \tag{2.34}$$

where J denotes the Jacobian similarity factor and $\beta := \beta(\mathcal{L})$.

Proof: From Definition 2.1.1, we have $\Gamma_1(x,y) = -\frac{S(\mathbf{f}^{equ})}{S(\mathbf{f}^{opt})}$ where $\mathbf{f}^{equ}$ is the equilibrium flow for the instance $(G, \hat{\mathbf{l}}, d)$, which is equal to the equilibrium for $(G, \tilde{\mathbf{l}}, d)$ as Lemma 2.2.1 implies.

By setting $\alpha := (1-x)y, x \in [0,1], y \in [0,1]$, we can use Lemma 2.2.4 to get

$$\frac{S(\mathbf{f}^{equ})}{S(\mathbf{f}^{opt})} \leq \frac{1+(J-1)(1-x)y}{1-\beta+\beta(1-x)y}$$

Thus

$$\Gamma_1(x,y) \geq -\frac{1+(J-1)(1-x)y}{1-\beta+\beta(1-x)y}$$

□

In the following, we will refer to the upper bound in Lemma 2.3.1 as $\bar{\Gamma}_1$.

Definition 2.3.1 $\bar{\Gamma}_1$ *is a lower bound of* Γ_1*, defined by*

$$\bar{\Gamma}_1(x,y) = -\frac{1+(J-1)(1-x)y}{1-\beta+\beta(1-x)y} \tag{2.35}$$

Note that in the case where the latency functions are linear, we have $J=1, \beta=\frac{1}{4}$ and

$$\bar{\Gamma}_1(x,y) = -\frac{4}{3+(1-x)y} \tag{2.36}$$

Lemma 2.3.2 $\mathbf{f}^{equ}$*, the equilibrium flow for* $(G,\hat{\mathbf{l}},d)$*, is continuous on* (x,y)*.*

Proof: Lemma 2.2.1 implies $\mathbf{f}^{equ}$ is also the equilibrium flow for $(G,\tilde{\mathbf{l}},d)$. We denote $a=(x,y)$, $a\in\Phi=[0,1]\times[0,1]$. Given an $a\in\Phi$, from [28] we know that $\mathbf{h}^{equ}$, the equilibrium path flow for $(G,\tilde{\mathbf{l}},d)$, is the solution to the parametric mathematical problem

$$\min_{\mathbf{h}} \quad z(\mathbf{h},a) = \sum_{e\in E}\int_0^{f_e}\tilde{l}_e(\omega)d\omega \qquad s.t.$$

$$\begin{aligned} \sum_{p\in P} h_p &= d, \\ f_e &= \sum_{p\ni e} h_p, \quad \forall e\in E \\ h_p &\geq 0, \qquad \forall p\in P \end{aligned}$$

The feasible set of the minimization problem is $\mathcal{U}$, which is non-empty and bounded. Besides, the feasible set (which is $\mathcal{U}$) is independent of $a\in\Phi$, so Theorem 2.2 in [34] indicates that when a belongs to a neighborhood of a^* and $a\to a^*$, we have $\mathbf{h}^{equ}(a)\to\mathbf{h}^{equ}(a^*)$.

So, the equilibrium flow $\mathbf{h}^{equ}$ is continuous at (x,y), and therefore $\mathbf{f}^{equ}$ is also continuous. □

Lemma 2.3.3 *1) Function $\bar{\Gamma}_1(x,y)$ and $\Gamma_1(x,y)$ are continuous on $x \in [0,1]$ and $y \in [0,1]$;*

2) If Assumption 3 holds for the instance $(G, \hat{\mathbf{l}}, d)$, then $\bar{\Gamma}_1(x,y)$ is continuously decreasing in $x \in [0,1]$ and increasing in $y \in [0,1]$.

Proof:

1) When $x, y \in [0,1]$, $1-\beta+\beta(1-x)y$ is positive, and Definition 2.3.1 implies the continuity of $\bar{\Gamma}_1(x,y)$.

Lemma 2.3.2 implies that $\mathbf{f}^{equ}$ is continuous on x and y. Also

$$\Gamma_1(x,y) = -\frac{S(\mathbf{f}^{equ})}{S(\mathbf{f}^{opt})} = -\frac{\sum_{e\in E} f_e^{equ} l_e(f_e^{equ})}{S(\mathbf{f}^{opt})},$$

and $l_e(f_e^{equ}), e \in E$ is continuous, so $\Gamma_1(x,y)$ is continuous on $\mathbf{f}^{equ}$, and furthermore, is continuous on x and y.

2) By taking the first derivative of $\bar{\Gamma}_1$ on x, we have

$$\frac{\partial \bar{\Gamma}_1}{\partial x} = -\frac{[\beta-(J-1)(1-\beta)]y}{[1-\beta+\beta(1-x)y]^2}$$

Assumption 3 indicates $(1-\beta)J < 1$, thus $[\beta-(J-1)(1-\beta)]y > 0$, which implies $\frac{\partial \bar{\Gamma}_1}{\partial x} < 0$. So $\bar{\Gamma}_1(x,y)$ is decreasing in $x \in [0,1]$.

By taking the first derivative of $\bar{\Gamma}_1$ on y, we have

$$\frac{\partial \bar{\Gamma}_1}{\partial y} = -\frac{[(J-1)(1-\beta)-\beta](1-x)}{[1-\beta+\beta(1-x)y]^2}$$

Assumption 3 implies $[(J-1)(1-\beta)-\beta]y < 0$, so $\bar{\Gamma}_1(x,y)$ is increasing in $y \in [0,1]$.

$\square$

Lemma 2.3.4 *Function $\Gamma_2(x,y)$ is continuous on x and y.*

Proof:

In Definition 2.1.2, we have

$$\Gamma_2 = -\frac{1}{d}\sum_{e\in E} f_e^{equ} l_e(f_e^{equ} + (1-x)y f_e^{opt}) - (1-y)m$$

$l_e(f_e^{equ}), e \in E$ is continuous, and Lemma 2.3.2 implies $\mathbf{f}^{equ}$ is continuous on x and y, thus $\Gamma_2(x, y)$ is continuous on x and y. □

Definition 2.3.2 *(**Best response**) In the trust game, given Player 1's strategy $x \in [0,1]$, $B(x) \subseteq [0,1]$ is the set which satisfies*

$$\Gamma_2(x, y^*) \geq \Gamma_2(x, y), \quad \forall y \in [0,1], \forall y^* \in B(x)$$

Furthermore, we defined Player 2's best response to Player 1's strategy x to be

$$y^*(x) = \min_{y\in B(x)} y$$

Given an $x \in [0,1]$, Lemma 2.3.4 indicates that $\Gamma_2(x, y)$ is continuous on the closed set $Y = [0,1]$, so the boundedness theorem[25] implies Γ_2 is bounded and the set $B(x)$ is non-empty.

From the definition we can see that, in the trust game, if Player 1 plays $x \in [0,1]$, then Player 2 always has an incentive to play an element in $B(x)$. When $B(x)$ is not a singleton set, we assume that Player 2 always picks the least trust in $B(x)$ (the worst case).

Definition 2.3.3 *(**Stackelberg strategy**) [31] In the trust game, Player 1's Stackelberg strategy is defined by*

$$x_s := \arg\max_{x\in[0,1]} \Gamma_1(x, y^*(x))$$

Player 1's Stackelberg payoff is

$$\Gamma_1^s = \Gamma_1(x_s, y^*(x_s)) \geq \Gamma_1(x, y^*(x)), \forall x \in [0,1] \tag{2.37}$$

Next we are going to analyze Player 1's payoff at equilibrium and his Stackelberg payoff.

2.3.1 Linear Latency Functions

We start from the simple case with linear latency functions $l_e(f_e) = a_e f_e + b_e (a_e, b_e \geq 0, \forall e \in E)$. In this case, $J = 1$ and $\beta = \frac{1}{4}$, and Player 1's payoff function is lower bounded by $\bar{\Gamma}_1$

$$\bar{\Gamma}_1(x, y) = -\frac{4}{3 + y(1-x)} \tag{2.38}$$

Lemma 2.3.5 *When m is positive, Player 1's Stackelberg payoff $\Gamma_1(x_s, y^*(x_s))$ is lower bounded away from $-\frac{4}{3}$.*

Proof:

By Definition 2.3.3, $\Gamma_2(x, y^*(x)) \geq \Gamma_2(x, y), \forall x, y \in [0, 1]$, so if we pick $y = 1$, then (2.3) implies

$$-\frac{1}{d}\sum_{e\in E} f_e^* l_e(f_e^* + (1-x)y^*(x)f_e^{opt}) - [1 - y^*(x)]m \geq -\frac{1}{d}\sum_{e\in E} f_e^1 l_e(f_e^1 + (1-x)f_e^{opt}) \tag{2.39}$$

where $\mathbf{f}^*$ is the equilibrium flow when Player 1 plays x and Player 2 plays $y^*(x)$ in the trust game, and $\mathbf{f}^1$ is the equilibrium flow when Player 1 plays x and Player 2 plays 1 in the trust game.

If we set $\alpha = 1 - x$, then Lemma 2.2.3 implies

$$\sum_{e\in E} f_e^1 l_e(f_e^1 + (1-x)f_e^{opt}) \leq \frac{4}{2+x} S(\mathbf{f}^{opt}) \tag{2.40}$$

Besides, we have

$$\sum_{e\in E} f_e^* l_e(f_e^* + (1-x)y^*(x)f_e^{opt}) \geq \sum_{e\in E} f_e^* l_e(f_e^*) = S(\mathbf{f}^*) \tag{2.41}$$

Combining (2.39), (2.40) and (2.41), we get

$$y^*(x) \geq 1 - \frac{4}{2+x}\frac{S(\mathbf{f}^{opt})}{md} + \frac{S(\mathbf{f}^*)}{md} \tag{2.42}$$

If we pick a positive $\epsilon, 0 < \epsilon < 1$, then $\forall x \in [0, 1)$, we have

1. **Case a**: $1 \geq y^*(x) \geq \epsilon$. In this case,

$$\begin{aligned}\Gamma_1(x, y^*(x)) &\overset{(2.34)}{\geq} \bar{\Gamma}_1(x, y^*(x)) \\ &\overset{Lemma\ 2.3.3}{\geq} \bar{\Gamma}_1(x, \epsilon) \\ &= -\frac{4}{3 + (1-x)\epsilon}\end{aligned}$$

   Thus

$$\Gamma_1^s \overset{(2.37)}{\geq} \Gamma_1(x, y^*(x)) \geq -\frac{4}{3 + (1-x)\epsilon} \tag{2.43}$$

2. **Case b**: $0 \leq y^*(x) < \epsilon$. In this case, if we denote $k := \frac{md}{S(\mathbf{f}^{opt})}$, then (2.42) implies

$$\begin{aligned}\Gamma_1(x, y^*(x)) = -\frac{S(\mathbf{f}^*)}{S(\mathbf{f}^{opt})} &\geq -\frac{4}{2+x} + [1 - y^*(x)]k \\ &> -\frac{4}{2+x} + (1-\epsilon)k\end{aligned} \tag{2.44}$$

   Thus

$$\begin{aligned}\Gamma_1^s &\overset{(2.37)}{\geq} \Gamma_1(x, y^*(x)) \\ &\overset{(2.44)}{>} -\frac{4}{2+x} + (1-\epsilon)k\end{aligned} \tag{2.45}$$

Combining (2.43) and (2.45), we can see that Player 1's Stackelberg payoff satisfies

$$\Gamma_1^s \geq \min\{-\frac{4}{3+(1-x)\epsilon}, -\frac{4}{2+x} + (1-\epsilon)k\}, \forall x \in [0,1), \forall \epsilon \in (0,1)$$

Furthermore, we have

$$\begin{aligned}\Gamma_1^s &\geq \max_{x\in[0,1),\epsilon\in(0,1)} \min\{-\frac{4}{3+(1-x)\epsilon}, -\frac{4}{2+x} + (1-\epsilon)k\} \\ &\geq \max_{(x,\epsilon)\in\mathcal{C}} -\frac{4}{3+(1-x)\epsilon}\end{aligned} \tag{2.46}$$

where $\mathcal{C}$ is defined by the system

$$-\frac{4}{3+(1-x)\epsilon} \leq -\frac{4}{2+x} + (1-\epsilon)k \tag{2.47}$$

$$0 \leq x < 1, 0 < \epsilon < 1$$

(2.47) is equivalent to

$$f(x) := \frac{4}{2+x} - \frac{4}{3+(1-x)\epsilon} \leq (1-\epsilon)k$$

$f(x)$ is strictly decreasing in x, and $f(1) = 0 < (1-\epsilon)k, \forall \epsilon \in (0,1)$, so $\exists \epsilon_0 > 0$, and $x_0 < 1$ s.t. (2.47) holds $\forall x \in [x_0, 1)$, $\epsilon = \epsilon_0$. Hence $\mathcal{C}$ is non-empty.

To sum up, (2.46) indicates that Player 1's Stackelberg payoff is lower bounded by the mathematical program

$$\max_{(x,\epsilon)} -\frac{4}{3+(1-x)\epsilon} \qquad s.t.$$

$$0 < k = \frac{md}{S(\mathbf{f}^{opt})}$$
$$\frac{4}{2+x} - \frac{4}{3+(1-x)\epsilon} \leq (1-\epsilon)k$$
$$0 \leq x < 1, 0 < \epsilon < 1$$

Since (x_0, ϵ_0), $(\epsilon_0 > 0, x_0 < 1)$ is a feasible point, Γ_1 is lower bounded by

$$-\frac{4}{3+(1-x_0)\epsilon_0},$$

which is strictly greater than $-\frac{4}{3}$.

$\square$

Therefore, in the trust game, Player 1's Stackelberg payoff is always bounded away from $-\frac{4}{3}$.

Note that all the hope that the price of anarchy in our model is less than the one in [26] lies on the chance that Player 1 will play an $x < 1$, Player 2 will play a $y > 0$, otherwise the "virtual flow" we introduced is 0. But in the Nash Equilibrium of the trust game, $(0, 0)$ may happen.

Eample 2.3

The graph in Figure 2.1 is a simple network with a single origin s, a single destination t, and total demand $d = 1$. There are 2 feasible paths: path 1, choosing the upper edge from s to o and then to t, and path 2, choosing the lower edge from s to o and then to t. The optimal flow is $h_1^{opt} = \frac{1}{2}$ and $h_2^{opt} = \frac{1}{2}$, and the social cost at optimum is $a + \frac{3}{4}$.

If the coordinator claims that the extra flow on path 1 and path 2 is $h_1^{extra} = \frac{1}{2}(1 - x)$ and $h_2^{extra} = \frac{1}{2}(1 - x)$, $x \in [0, 1]$ and Player 2's trust is $y \in [0, 1]$, then the equilibrium flow will be $h_1^{equ} = 1 - \frac{(1-x)y}{2}$ and $h_2^{equ} = \frac{(1-x)y}{2}$. Player 1's payoff in the trust game is

$$\Gamma_1(x, y) = -\frac{S(\mathbf{h}^{equ})}{S(\mathbf{h}^{opt})} = -\frac{\frac{1}{4}(1 - y + xy)^2 + \frac{3}{4} + a}{\frac{3}{4} + a}$$

Player 2's payoff is

$$\Gamma_2(x, y) = -[1 + a(1 - x)y + (1 - y)m]$$

where m is her own estimation about the extra latency.

Player 1 always has an incentive to play $x = 0$, no matter what Player 2 plays. When $m < a$, Player 2's best response to $x = 0$ is $y^*(0) = 0$, thus the equilibrium of the trust game is $(0, 0)$.

At equilibrium, the "virtual flow" does not help, and the price of anarchy is $\frac{1+a}{\frac{3}{4}+a}$, which converges to $\frac{4}{3}$ when a goes to 0.

2.3.2 General Latency Functions

We will consider the general latency functions which satisfy Assumptions 1-3 in Section 2.2.2. For general latency functions, the well-known bound of PoA is $\frac{1}{1-\beta}$ (see [5]), where β is defined in (2.2.1).

$(0, 0)$ might be the Nash Equilibrium of the trust game, which means Player 1 will win nothing from the "virtual flow", and in that case Player 1's payoff can be as bad as $\bar{\Gamma}_1(0, 0)$. Next we will analyze Player 1's Stackelberg

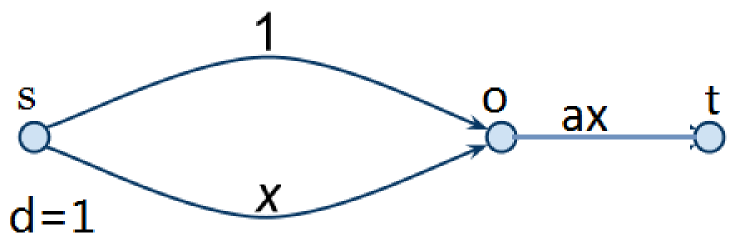

Figure 2.1: A Traffic Example

payoff Γ_1^s in the trust game and show that it is bounded away from $\bar{\Gamma}_1(0,0)$. Note that

$$\bar{\Gamma}_1(0,0) = -\frac{1}{1-\beta}$$

Lemma 2.3.6 *When m is positive, under Assumptions 1, 2 and 3 Player 1's Stackelberg payoff Γ_1^s is lower bounded away from $\bar{\Gamma}_1(0,0)$.*

Proof: The same method in Section 2.3.1 applies here. (2.39)(2.41) also hold for general latency functions. Lemma 2.2.5 implies

$$\sum_{e\in E} f_e^1 l_e(f_e^1 + (1-x)f_e^{opt}) \leq \frac{1}{1-(2-x)\beta} S(\mathbf{f}^{opt})$$

Thus we can get

$$y^*(x) \geq 1 - \frac{1}{1-(2-x)\beta}\frac{S(\mathbf{f}^{opt})}{md} + \frac{S(\mathbf{f}^*)}{md} \tag{2.48}$$

If we pick a positive $\epsilon, 0 < \epsilon < 1$, then $\forall x \in [0,1)$, we have

1. **Case a**: $1 \geq y^*(x) \geq \epsilon$. In this case,

$$\begin{aligned}\Gamma_1(x, y^*(x)) &\overset{(2.34)}{\geq} \bar{\Gamma}_1(x, y^*(x)) \\ &\overset{Lemma\ 2.3.3}{\geq} \bar{\Gamma}_1(x,\epsilon) \\ &\overset{(2.35)}{=} -\frac{1+(J-1)(1-x)\epsilon}{1-\beta+\beta(1-x)\epsilon}\end{aligned}$$

Thus

$$\Gamma_1^s \overset{(2.37)}{\geq} \Gamma_1(x, y^*(x)) \geq -\frac{1+(J-1)(1-x)\epsilon}{1-\beta+\beta(1-x)\epsilon} \tag{2.49}$$

2. **Case b**: $0 \leq y^*(x) < \epsilon$. In this case, if we denote $k := \frac{md}{S(\mathbf{f}^{opt})}$, then inequality (2.48) implies

$$\Gamma_1(x, y^*(x)) = -\frac{S(\mathbf{f}^*)}{S(\mathbf{f}^{opt})} > -\frac{1}{1-(2-x)\beta} + (1-\epsilon)k \tag{2.50}$$

Furthermore, we have

$$\begin{aligned} \Gamma_1^s &\overset{(2.37)}{\geq} \Gamma_1(x, y^*(x)) \\ &> -\frac{1}{1-(2-x)\beta} + (1-\epsilon)k \end{aligned} \tag{2.51}$$

Combining (2.49)(2.51), we can see that Player 1's Stackelberg payoff satisfies

$$\Gamma_1^s \geq \min\{-\frac{1+(J-1)(1-x)\epsilon}{1-\beta+\beta(1-x)\epsilon}, -\frac{1}{1-(2-x)\beta}+(1-\epsilon)k\}, \forall x \in [0,1), \forall \epsilon \in (0,1)$$

Furthermore, we have

$$\begin{aligned} \Gamma_1^s &\geq \max_{x\in[0,1),\epsilon\in(0,1)} \min\{-\frac{1+(J-1)(1-x)\epsilon}{1-\beta+\beta(1-x)\epsilon}, -\frac{1}{1-(2-x)\beta} + (1-\epsilon)k\} \\ &\geq \max_{(x,\epsilon)\in\mathcal{C}} -\frac{1+(J-1)(1-x)\epsilon}{1-\beta+\beta(1-x)\epsilon} \end{aligned} \tag{2.52}$$

where $\mathcal{C}$ is defined by the system

$$\begin{aligned} -\frac{1+(J-1)(1-x)\epsilon}{1-\beta+\beta(1-x)\epsilon} &\leq -\frac{1}{1-(2-x)\beta} + (1-\epsilon)k \\ 0 \leq x &< 1, 0 < \epsilon < 1 \end{aligned} \tag{2.53}$$

(2.53) is equivalent to

$$f(x) := \frac{1}{1-(2-x)\beta} - \frac{1+(J-1)(1-x)\epsilon}{1-\beta+\beta(1-x)\epsilon} \leq (1-\epsilon)k$$

When Assumptions 2 and 3 hold, $\beta < \frac{1}{2}$ and $J < \frac{1}{1-\beta}$, $f(x)$ is strictly decreasing in x, and $f(1) = 0 < (1-\epsilon)k$, so $\exists \epsilon_0 > 0$, and $x_0 < 1$ s.t. (2.47) holds $\forall x \in [x_0, 1)$ and $\epsilon = \epsilon_0$. Hence $\mathcal{C}$ is non-empty.

To sum up, (2.52) indicates that Player 1's Stackelberg payoff is lower bounded by the mathematical problem

$$\max_{(x,\epsilon)} -\frac{1+(J-1)(1-x)\epsilon}{1-\beta+\beta(1-x)\epsilon} \qquad s.t.$$

$$0 < k = \frac{md}{S(\mathbf{f}^{opt})}$$
$$-\frac{1+(J-1)(1-x)\epsilon}{1-\beta+\beta(1-x)\epsilon} \leq -\frac{1}{1-(2-x)\beta} + (1-\epsilon)k$$
$$0 \leq x < 1, 0 < \epsilon < 1$$

Since (x_0, ϵ_0), $(\epsilon_0 > 0, x_0 < 1)$ is a feasible point, Γ_1 is lower bounded by

$$-\frac{1+(J-1)(1-x_0)\epsilon_0}{1-\beta+\beta(1-x_0)\epsilon_0}$$

When Assumption 3 holds, $J < \frac{1}{1-\beta}$, and the bound is strictly greater than $-\frac{1}{1-\beta}$.

□

2.4 Networks with Multi-Commodities

If we consider the general network with multiple commodities, i.e., the instance $(G, \mathbf{l}, \mathbf{d})$ with $|W|$ commodities, then there will be $|W|+1$ players in the trust game. One player is the network coordinator and the other $|W|$ players are the representatives for the traffic users, each one for a commodity. In the strategy profile $s = (s_1, s_w) \in [0,1]^{|W|+1}$, s_1 is the strategy for the coordinator (denoted as Player 1 in the following) and $s_w, w \in W$ is the strategy for the representative of traffic users in commodity $w \in W$ (denoted as Player w in the following).

Player 1's strategy s_1 is to pick an $x \in [0,1]$ which denotes the amount of advantage he will take of all the traffic users' incomplete information. He broadcasts the information of extra flow to all the traffic users in the network, $\bar{\mathbf{f}}^{extra} = (1-x)\mathbf{f}^{opt}$.

Player w's strategy is to pick a $y_w \in [0,1]$ to denote the trust she would put on Player 1, and $1 - y_w$ indicates the weight she would put on her own estimation about the extra latency. The *estimated extra latency* is denoted by m_w for commodity $w \in W$.

When the strategy profile in the trust game is $s = (s_1, s_w) \in [0,1]^{|W|+1}$, we denote the induced equilibrium flow by $\mathbf{f}$, and Player w's payoff is the negative of the perceived cost

$$\Gamma_w = -\hat{l}_p(\mathbf{f}) = -\sum_{e \in p} l_e(f_e + (1-x)y_w f_e^{opt}) - (1-y_w)m_w, \forall w \in W, \forall p \in P_w,\ s.t.\ h_p > 0$$

Player 1's payoff is the negative of the price of anarchy

$$\Gamma_1 = -\frac{S(\mathbf{f})}{S(\mathbf{f}^{opt})}$$

Assumption 4 *$\exists x_0 \in [0,1)$ s.t. when Player 1 plays some x_0, Player w's ($w \in W$) best response is to put a positive trust, i.e., $y_w^*(x_0) > 0, \forall w \in W$.*

If Assumption 4 holds, then $\exists x \in [0,1)$ such that when Player 1 plays x, Player w's best response is $y_w^* > 0, \forall w \in W$. We denote

$$y_{min} := \min_{w \in W} y_w^* \qquad y_{max} := \max_{w \in W} y_w^*$$

We analyze the PoA of the induced selfish routing instance $(G, \hat{\mathbf{l}}, \mathbf{d})$. The analysis in Section 2.2.2 also works here, and we can get Player 1's payoff

$$\Gamma_1 = -\frac{S(\mathbf{f}^{equ})}{S(\mathbf{f}^{opt})} \geq -\frac{1 + (1-x)(y_{max}J - y_{min})}{1 - \beta + \beta(1-x)y_{min}}$$

where J is the Jacobian similarity factor and $\beta = \beta(\mathcal{L})$.

When all the players $w \in W$ put the same trust, i.e., $\exists y$ s.t. $y = y_w$, $\forall w \in W$, we have $y_{max} = y_{min} = y$, and the bound of Player 1's payoff converges to

$$\Gamma_1 = -\frac{S(\mathbf{f}^{equ})}{S(\mathbf{f}^{opt})} \geq -\frac{1+(1-x)y(J-1)}{1-\beta+\beta(1-x)y}$$

which coincides with the result in Lemma 2.3.1.

Here we did some basic work in the case of networks with multiple commodities.

We hope to prove that Assumption 4 holds (or find the condition which makes the assumption hold). To do this, we need to consider all the $|W|+1$ players' payoff functions and analyze the trust game with multiple players (instead of two). Until now, we have not been able to find a way to get rid of Assumption 4, or get the payoff for Player 1 if the assumption fails.

Chapter 3

Stochastic User Equilibrium

Chapter 2 is about the user equilibrium (UE), in which traffic users choose the path with minimum time cost from their origin to the destination. In that model, we assume that all traffic users have perfect information regarding travel time over the entire network when they choose the paths to travel on. In other words, the travel latency function is deterministic.

This chapter is about a new kind of traffic equilibrium, called *stochastic user equilibrium (SUE)*. In SUE, there is a random error in traffic user's perception of the travel latency. In this chapter, we are still going to deal with the bi-level problem, with the trust game between the traffic coordinator and the users being the upper level, and the selfish routing with random perception error being the lower level.

3.1 Stochastic Network Loading Models

We consider the case of selfish routing where the traffic users do not have perfect information of the travel cost. In other words, travelers have some uncertainty about the latency functions. More realistically, the perceived travel latency may be considered as a random variable for all the traffic users. Each user may perceive a different travel time over the same path. The kind of traffic model is referred to as stochastic network loading model (SNLM).

We consider the traffic model with single commodity and infinitesimal traffic users, and we will use $(G, \mathbf{l}, d, \epsilon)$ to represent the SNLM. G indicates the topology of the network, d indicates the demand and $\mathbf{l}$ denotes the (measured) latency functions, as defined in Chapter 1.ϵ denotes the random error of users' perception of the latency.

Recall that $\mathcal{U}$ denotes the set of all feasible path flow, i.e., $\mathbf{h} \in \mathcal{U}$ iff $\sum_{p \in P} h_p = d, h_p \geq 0, \forall p \in P$. $\mathcal{F}$ denotes the set of all feasible edge flow, i.e., $\mathbf{f} \in \mathcal{F}$ iff $\exists \mathbf{h} \in \mathcal{U}$ s.t. $\mathbf{f} = \Delta \mathbf{h}$, where Δ is the incidence matrix defined in Chapter 1.

In the stochastic network loading model, when a traffic user chooses a path $p \in P$, the perceived latency she will bear is a stochastic variable

$$c_p = l_p + \epsilon_p, \quad \forall p \in P$$

where l_p is the measured travel latency on path $p \in P$, and ϵ_p is the random error.

Furthermore, assume that the expectation $E[\epsilon_p] = 0$, or $E[c_p] = l_p$, which indicates that the average random error is 0.

Since $c_p(\forall p \in P)$ is a stochastic variable, each traffic user may have a different perception of the path latency. In selfish routing, each user will pick the path that bears the shortest perceived latency. In the traffic model with infinitesimal users, given the measured latency l_p of all the paths $p \in P$, we can get q_p, the ratio of users choosing path p (see [30])

$$\begin{aligned} q_p =& Pr(c_p \leq c_t, \forall t \in P), && \forall p \in P \\ =& Pr(l_p + \epsilon_p \leq l_t + \epsilon_t, \forall t \in P) && \forall p \in P \end{aligned} \tag{3.1}$$

In other words, the probability that a certain path $p \in P$ is chosen is the probability that its travel latency is perceived to be the shortest of all the alternatives.

Definition 3.1.1 ***(Stochastic user equilibrium)*** *In the instance* $(G, \mathbf{l}, d, \epsilon)$*, a path flow* $\mathbf{h}$ *reaches the* stochastic user equilibrium *iff*

$$h_p = q_p d, \forall p \in P \tag{3.2}$$

where q_p *is the probability that path* $p \in P$ *is chosen, given the measured travel latency* $\mathbf{l}$ *defined in* (3.1)*.*

When every single selfish traffic user chooses the path with shortest perceived latency for her, the aggregation of all users' choice induces the equilibrium flow. At equilibrium flow, no single user has an incentive to unilaterally switch her current choice, which she has already selected based on her perceived latency.

When the model satisfies the additive and separable attributes, we have $l_p = \sum_{e \in p} l_e(f_e)$ where $l_e(f_e)$ is the edge latency function. From the definition, we know the SUE flow $\mathbf{h}^{sue}, \mathbf{f}^{sue}$ satisfies:

$$\frac{h_p^{sue}}{d} = Pr[\sum_{e \in p} l_e(f_e^{sue}) + \epsilon_p \leq \sum_{e \in t} l_e(f_e^{sue}) + \epsilon_t, \forall t \in P], \quad \forall p \in P$$

3.1.1 Logit-based Model

A widely used route-choice model is called the "logit-based model". In the logit-based model, the $\epsilon_p, \forall p \in P$ are identically and independently distributed (i.i.d.) variables of the Gumbel distribution $Gumbel(\theta)$ [3]:

Definition 3.1.2 ***(Gumbel distribution)*** *Gumbel distribution is the distribution with the probability density function* $f_\theta(x) = \theta e^{-(\theta x + \gamma)} e^{-e^{-(\theta x+\gamma)}}, \forall x \in (-\infty, \infty)$*, where* γ *is the Euler-Mascheroni constant,*
$\gamma := \lim_{n \to \infty}[(\sum_{k=1}^{n} \frac{1}{k}) - ln(n)] \approx 0.57721566$.

If $X \sim Gumbel(\theta)$, then we have $E[X] = 0$, and $\sigma^2 = \frac{\pi^2}{6\theta^2}$.

Following [30], we know that in the logit-based model, when the random error terms ϵ_p are i.i.d of $Gumbel(\theta)$, $\mathbf{h}^{sue}$ (or $\mathbf{f}^{sue}$) is the the stochastic user equilibrium flow iff

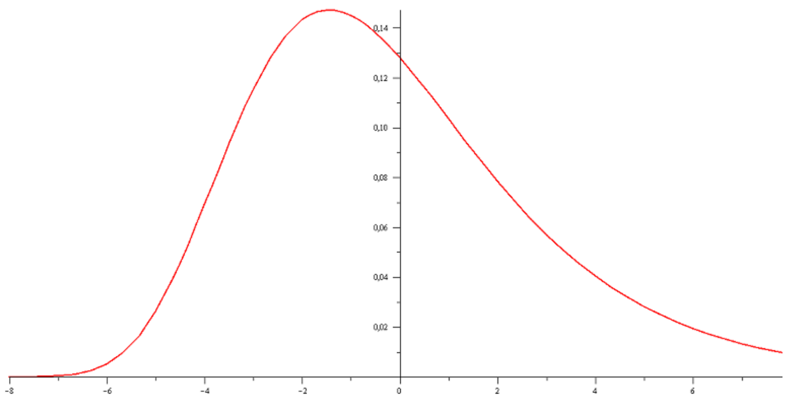

Figure 3.1: Probability Density Function of the Gumbel Distribution ($\theta = 0.4$)

$$q_p = \frac{h_p^{sue}}{d} = \frac{e^{-\theta l_p(\mathbf{h}^{sue})}}{\sum_{t\in P} e^{-\theta l_t(\mathbf{h}^{sue})}}, \quad \forall p \in P \tag{3.3}$$

(3.3) implies $h_p^{sue} > 0, \forall p \in P$, which means that in SUE, all the paths will be used with a positive probability.

Guo and Yang gave the *variational inequality* formulation of SUE in the logit-based model [10], which indicates that $\mathbf{h}^{sue}$ is SUE flow for the instance $(G, \mathbf{l}, d, \epsilon)$ iff

$$\sum_{p\in P} (l_p^{sue} + \frac{1}{\theta} \ln h_p^{sue})(h_p - h_p^{sue}) \geq 0, \forall \mathbf{h} \in \mathcal{U} \tag{3.4}$$

We will use the price of anarchy ρ^{sue} to capture the inefficiency of SUE

$$\rho^{sue} = \frac{S(\mathbf{f}^{sue})}{S(\mathbf{f}^{opt})} \tag{3.5}$$

where $S(\mathbf{f})$ is the social cost function $S(\mathbf{f}) = \sum_{e\in E} f_e l_e(f_e), \forall \mathbf{f} \in \mathcal{F}$, and $\mathbf{f}^{opt} = \arg\min_{\mathbf{f}\in\mathcal{F}} S(\mathbf{f})$.

Since at SUE each traffic user is choosing the path with minimum perceived latency, we can get the expected perceived latency for the traffic users

$$W := E[\min_{p\in P} c_p^{sue}] = E[\min_{p\in P}(l_p^{sue} + \epsilon_p)]$$

where the expectation is taken with respect to the random variables ϵ_p.

Theorem 2.2 in [21] indicates that in the logit-based model, we have

$$W = -\frac{1}{\theta} \ln \sum_{p \in P} e^{-\theta l_p^{sue}} \tag{3.6}$$

Combining (3.3) and (3.6), we can get

$$W = l_p^{sue} + \frac{1}{\theta} \ln h_p^{sue} - \frac{1}{\theta} \ln d, \quad \forall p \in P \tag{3.7}$$

3.2 Trust Game between the Traffic Coordinator and the Users

The trust game is between the traffic coordinator and the traffic users. In this model, the coordinator (Player 1) broadcasts the fabricated information about the extra flow to all the traffic users, and Player 2 (the representative of all the traffic users) chooses how much trust to put on the information. The only difference with the model in Chapter 2 is that Player 2's own estimation about the extra latency on each path $p \in P$ due to the extra traffic is not a deterministic factor m, but a stochastic term $m + \epsilon_0$, where m is the expected extra latency due to the extra traffic and ϵ_0 is the error of the estimation. Furthermore, we assume the errors follow i.i.d $Gumbel(\theta_0)$.

Game Model

Players: In this game there are two players. One is the network coordinator (Player 1) and the other is the representative of all the traffic users (Player 2). In the traffic model with single commodity and homogenous traffic users, we just pick a representative instead of considering each single user's choice.

Strategies: Player 1's strategy s_1 is to pick an $x \in X = [0, 1]$, which indicates the amount of advantage he will take on Player 2's incomplete information (here the incomplete information means Player 2 does not know about the extra flow). In other words, Player 1 will tell Player 2 that the extra flow is $(1-x)\mathbf{h}^{opt}$. $x = 0$ indicates that he is taking full advantage of Player 2 and the extra flow is $\mathbf{h}^{opt}$. $x = 1$ indicates that he takes no advantage of Player 2 and the extra flow is 0.

Player 2's strategy s_2 is to pick $y \in Y = [0, 1]$ which indicates her trust in Player 1. $y = 1$ means she completely trusts Player 1's information, and $y = 0$ implies that she totally distrusts Player 1. $1-y$ will be the corresponding weight she puts on her own estimation about the extra latency, which is $m+\epsilon_0$.

A *strategy profile* $s = (x, y) \in [0, 1] \times [0, 1]$ records both of the players' strategies in the trust game. A strategy profile will induce a selfish routing instance between all the traffic users.

Payoffs: In the selfish routing instance induced by $s = (x, y)$, the perceived latency on path p will be

$$\begin{aligned} c_p(\mathbf{h}) =& l_p(\mathbf{h} + (1-x)y\mathbf{h}^{opt}) + (1-y)(m + \epsilon_0) \\ =& l_p(\mathbf{h} + (1-x)y\mathbf{h}^{opt}) + (1-y)m + (1-y)\epsilon_0 \end{aligned} \tag{3.8}$$

where $\mathbf{l}$ is the latency function, $\mathbf{h}$ is the normal flow, $(1-x)\mathbf{h}^{opt}$ is the extra flow Player 1 announces, $m + \epsilon_0$ is Player 2's own estimation about the extra latency, and y is Player 2's trust in Player 1.

If we set $\epsilon = (1-y)\epsilon_0$, then $y = 1$ implies $\epsilon = 0$, and the users have no estimation error about the perceived latency. In this case the equilibrium flow induced by the trust game is the UE flow discussed in Chapter 2.

When $y \in [0, 1)$, from Definition 3.1.2, we can see that if $\epsilon_0 \sim Gumbel(\theta_0)$, then $\epsilon \sim Gumbel(\theta)$ where $\theta = \frac{1}{1-y}\theta_0$.

In our model, the users' measured latency $\hat{\mathbf{l}}$ is

$$\hat{\mathbf{l}}(\mathbf{h}) = \mathbf{l}(\mathbf{h} + (1-x)y\mathbf{h}^{opt}) + (1-y)m$$

ϵ is the estimation error about the path latency, which follows $Gumbel(\theta)$ in our case.

When there is random error in traffic users' perceived latency, we will use the stochastic network loading model $(G, \hat{\mathbf{l}}, d, \epsilon)$ to model the selfish routing problem.

Let $\mathbf{h}^{sue}$ (or $\mathbf{f}^{sue}$) denote the SUE of the SNLM $(G, \hat{\mathbf{l}}, d, \epsilon)$ induced by the trust game profile $s = (x, y)$.

Definition 3.2.1 *Player 1's payoff $\Gamma_1(x, y)$ in the trust game is the negative of price of anarchy of $(G, \hat{\mathbf{l}}, d, \epsilon)$*

$$\Gamma_1(x, y) := -\rho^{sue} = -\frac{S(\mathbf{f}^{sue})}{S(\mathbf{f}^{opt})} \tag{3.9}$$

Definition 3.2.2 *Player 2's payoff $\Gamma_2(x, y)$ in the trust game is the negative of users' expected perceived latency at SUE of $(G, \hat{\mathbf{l}}, d, \epsilon)$*

$$\begin{aligned} \Gamma_2(x, y) :=& -E[\min_{p \in P} c_p(\mathbf{h}^{sue})] \\ \overset{(3.6)}{=}& \frac{1}{\theta} \ln \sum_{p \in P} e^{-\theta \hat{l}_p^{sue}} \end{aligned} \tag{3.10}$$

3.3 Stochastic User Equilibrium Flow

Lemma 3.3.1 $\mathbf{f}^{sue}$*(or* $\mathbf{h}^{sue}$*), the equilibrium flow for the logit-based SNLM $(G, \hat{\mathbf{l}}, d, \epsilon)$, is also the equilibrium flow for the logit-based SNLM $(G, \tilde{\mathbf{l}}, d, \epsilon)$, where*

$$\tilde{l}_e(f_e) = l_e(f_e + (1 - x) y f_e^{opt}), \quad \forall e \in E \tag{3.11}$$

Proof:

For all $p \in P$, (2.6) states

$$\tilde{l}_p(\mathbf{h}) = \hat{l}_p(\mathbf{h}) - (1 - y) m \tag{3.12}$$

If $\mathbf{h}^{sue}$ is the stochastic user equilibrium flow for the logit-based SNLM $(G, \hat{\mathbf{l}}, d, \epsilon)$, then (3.3) indicates

$$\begin{aligned}
\frac{h_p^{sue}}{d} &= \frac{e^{-\theta \hat{l}_p(\mathbf{h}^{sue})}}{\sum_{t \in P} e^{-\theta \hat{l}_t(\mathbf{h}^{sue})}}, \quad \forall p \in P \\
&\overset{(3.12)}{=} \frac{e^{-\theta[\tilde{l}_p(\mathbf{h}^{sue})+(1-y)m]}}{\sum_{t \in P} e^{-\theta[\tilde{l}_t(\mathbf{h}^{sue})+(1-y)m]}} \\
&= \frac{e^{-\theta \tilde{l}_p(\mathbf{h}^{sue})} e^{-(1-y)m}}{\sum_{t \in P} e^{-\theta \tilde{l}_t(\mathbf{h}^{sue})} e^{-(1-y)m}} \\
&= \frac{e^{-\theta \tilde{l}_p(\mathbf{h}^{sue})}}{\sum_{t \in P} e^{-\theta \tilde{l}_t(\mathbf{h}^{sue})}}
\end{aligned}$$

Thus $\mathbf{h}^{sue}$ is also the stochastic user equilibrium flow for the logit-based SNLM $(G, \tilde{\mathbf{l}}, d, \epsilon)$.

□

Lemma 3.3.1 implies that the SUE flow $\mathbf{f}^{sue}$ for the instance $(G, \hat{\mathbf{l}}, d, \epsilon)$ is equal to the SUE flow for the instance $(G, \tilde{\mathbf{l}}, d, \epsilon)$. Next we will use the equilibrium flow for $(G, \tilde{\mathbf{l}}, d, \epsilon)$ as $\mathbf{f}^{sue}$, and furthermore analyze the payoff for Player 1 and Player 2 in the trust game.

3.3.1 Networks with Linear Latency Functions

We start from the simple case, where the measured latency functions $l_e(f_e)$ are linear $l_e(f_e) = a_e f_e + b_e (a_e, b_e \geq 0, \forall e \in E)$.

Lemma 3.3.2 *Let $\mathbf{h}^{sue}$ be the SUE flow for instance $(G, \tilde{\mathbf{l}}, d, \epsilon)$ where $\tilde{\mathbf{l}}$ is defined in* (3.11) *and latency functions are linear. The price of anarchy in the SNLM is bounded by*

$$\rho^{sue} = \frac{S(\mathbf{f}^{sue})}{S(\mathbf{f}^{opt})} \leq \frac{4}{3 + y(1-x)} (1 + \frac{k}{\theta c})$$

where k solves $ke^k = \frac{1}{e}(|P| - 1)$, $c = \frac{S(\mathbf{f}^{opt})}{d}$ is the average travel latency of all network users at optimal flow, and θ is the parameter of ϵ.

Proof: The SUE flow $\mathbf{h}^{sue}$ satisfies the variational inequaility (3.4)

$$\sum_{p\in P}[\tilde{l}_p(\mathbf{h}^{sue}) + \frac{1}{\theta}\ln h_p^{sue}](h_p - h_p^{sue}) \geq 0, \forall \mathbf{h} \in \mathcal{U}$$

If we set $\mathbf{h} := \mathbf{h}^{opt}$, then we get

$$\sum_{p\in P}[l_p(\mathbf{h}^{sue} + (1-x)y\mathbf{h}^{opt}) + \frac{1}{\theta}\ln h_p^{sue}](h_p^{opt} - h_p^{sue}) \geq 0 \tag{3.13}$$

which implies

$$\sum_{p\in P} l_p(\mathbf{h}^{sue} + (1-x)y\mathbf{h}^{opt})(h_p^{opt} - h_p^{sue}) + \frac{1}{\theta}\sum_{p\in P}(h_p^{opt} - h_p^{sue})\ln h_p^{sue} \geq 0 \tag{3.14}$$

Besides, we have

$$\begin{aligned}
&\sum_{p\in P} l_p(\mathbf{h}^{sue} + (1-x)y\mathbf{h}^{opt})(h_p^{opt} - h_p^{sue}) \\
\overset{(1.3)}{=}&\sum_{e\in E} l_e(f_e^{sue} + (1-x)yf_e^{opt})(f_e^{opt} - f_e^{sue}) \\
=&\sum_{e\in E}[l_e(f_e^{sue}) + (1-x)ya_e f_e^{opt}](f_e^{opt} - f_e^{sue}) \\
=&\sum_{e\in E} l_e(f_e^{sue})f_e^{opt} - \sum_{e\in E} l_e(f_e^{sue})f_e^{sue} + (1-x)y\sum_{e\in E} a_e f_e^{opt}(f_e^{opt} - f_e^{sue})
\end{aligned}$$

Then (3.14) implies

$$\begin{aligned}
\sum_{e\in E} l_e(f_e^{sue})f_e^{sue} \leq& \sum_{e\in E} l_e(f_e^{sue})f_e^{opt} + (1-x)y\sum_{e\in E} a_e f_e^{opt}(f_e^{opt} - f_e^{sue}) \\
&+ \frac{1}{\theta}\sum_{p\in P}(h_p^{opt} - h_p^{sue})\ln h_p^{sue} \\
\leq& \sum_{e\in E} l_e(f_e^{opt})f_e^{opt} + [1-(1-x)y]\sum_{e\in E} a_e f_e^{opt}(f_e^{sue} - f_e^{opt}) \\
&+ \frac{1}{\theta}\sum_{p\in P}(h_p^{opt} - h_p^{sue})\ln h_p^{sue}
\end{aligned} \tag{3.15}$$

For the term $\sum_{e\in E} a_e f_e^{opt}(f_e^{sue} - f_e^{opt})$, we have

$$\sum_{e\in E} a_e f_e^{opt}(f_e^{sue} - f_e^{opt}) \overset{(2.11)}{\leq} \frac{1}{4}\sum_{e\in E} l_e(f_e^{sue})f_e^{sue} \tag{3.16}$$

Using Lemma 3 in Guo and Yang [10], we have

$$\sum_{p\in P}(h_p^{opt} - h_p^{sue})\ln h_p^{sue} \leq kd \tag{3.17}$$

Combining (3.15), (3.16) and (3.17), we have

$$S(\mathbf{f}^{sue}) \leq \frac{1}{1-\frac{1}{4}[1-y(1-x)]}[S(\mathbf{f}^{opt}) + \frac{1}{\theta}kd]$$

Then, the price of anarchy of this stochastic user system is:

$$\rho^{sue} = \frac{S(\mathbf{f}^{sue})}{S(\mathbf{f}^{opt})} \leq \frac{4}{3+y(1-x)}(1+\frac{k}{\theta c})$$

$\square$

3.3.2 Networks with General Latency Functions

We will consider the logit-based SNLMs $(G, \mathbf{l}, d, \epsilon)$ where the measured latency functions $l_e(f_e), \forall e \in E$ are non-linear, and Assumptions 1-3 are satisfied.

Lemma 3.3.3 *Let $\mathbf{h}^{sue}$ be the SUE flow for $(G, \tilde{\mathbf{l}}, d, \epsilon)$ with $\tilde{\mathbf{l}}$ defined in* (2.4)*. If Assumption 1 holds for $\mathbf{l}$, then the price of anarchy of $(G, \tilde{\mathbf{l}}, d, \epsilon)$ is bounded by*

$$\rho^{sue} = \frac{S(\mathbf{f}^{sue})}{S(\mathbf{f}^{opt})} \leq \frac{1+(J-1)(1-x)y+\frac{k}{\theta c}}{1-\beta[1-(1-x)y]}$$

where k solves $ke^k = \frac{1}{e}(|P|-1)$, $c = \frac{S(\mathbf{f}^{opt})}{d}$, J is the Jacobian similarity factor, $\beta = \beta(\mathcal{L})$, and θ is the parameter of ϵ.

Proof:

We start again from the variational inequality (3.4) to get (3.14), and define

$$T_0 := \sum_{e\in E} f_e^{opt} l_e(f_e^{sue}) = \sum_{p\in P} h_p^{opt} l_p(\mathbf{h}^{sue})$$

$$T_1 := \sum_{e\in E} f_e^{sue} l_e(f_e^{sue} + (1-x)yf_e^{opt}) = \sum_{p\in P} h_p^{sue} l_p(\mathbf{h}^{sue} + (1-x)y\mathbf{h}^{opt})$$

$$T_2 := \sum_{e\in E} f_e^{opt} l_e(f_e^{sue} + (1-x)yf_e^{opt}) = \sum_{p\in P} h_p^{opt} l_p(\mathbf{h}^{sue} + (1-x)y\mathbf{h}^{opt})$$

Combining (3.14) and (3.17), we have

$$T_1 \leq T_2 + \frac{1}{\theta}kd \tag{3.18}$$

Since (2.23),(2.26) and (2.29) also hold here, we have

$$T_0 \overset{(2.23)}{\leq} \beta S(\mathbf{f}^{sue}) + S(\mathbf{f}^{opt}) \tag{3.19}$$

$$T_1 \overset{(2.26)}{\geq} S(\mathbf{f}^{sue}) + (1-x)yT_0 - (1-x)y \sum_{e\in E} f_e^{opt} l_e(0) \tag{3.20}$$

$$T_2 \overset{(2.29)}{\leq} T_0 + (1-x)yJS(\mathbf{f}^{opt}) - (1-x)yJ \sum_{e\in E} f_e^{opt} l_e(0) \tag{3.21}$$

From (3.18),(3.19),(3.20) and (3.21), we get

$$S(\mathbf{f}^{sue}) - \beta[1-(1-x)y]S(\mathbf{f}^{sue}) \leq 1 + (J-1)(1-x)yS(\mathbf{f}^{opt}) + \frac{1}{\theta}kd$$

Then

$$\rho^{sue} = \frac{S(\mathbf{f}^{sue})}{S(\mathbf{f}^{opt})} \leq \frac{1 + (J-1)(1-x)y + \frac{k}{\theta c}}{1 - \beta[1-(1-x)y]}$$

□

In the simple case with linear latency functions, $J = 1$ and $\beta = \frac{1}{4}$, and the bound converges to the one in Lemma 3.3.2.

We will refer to the upper bound in Lemma 3.3.3 as $\bar{\Gamma}_1$,

$$\bar{\Gamma}_1(x, y) = -\frac{1 + (J-1)(1-x)y + \frac{k}{\theta c}}{1 - \beta[1-(1-x)y]} \tag{3.22}$$

Note that in the case where the latency functions are linear, we have

$$\bar{\Gamma}_1(x, y) = -\frac{4}{3 + y(1-x)}(1 + \frac{k}{\theta c}) \tag{3.23}$$

3.4 Stackelberg Payoff in the Trust Game

Lemma 3.4.1 $\mathbf{f}^{sue}$*, the SUE flow for* $(G, \hat{\mathbf{l}}, d, \epsilon)$*, is continuous on* (x, y)*.*

Proof: Lemma 3.3.1 implies that $\mathbf{f}^{sue}$ is also the SUE flow for $(G, \tilde{\mathbf{l}}, d, \epsilon)$.

The analysis in the proof in Lemma 2.3.2 also works here. We denote $a = (x, y)$, $a \in \Phi = [0, 1] \times [0, 1]$. Given an $a \in \Phi$, from [10] we know that $\mathbf{h}^{sue}$, the equilibrium path flow for $(G, \tilde{\mathbf{l}}, d, \epsilon)$, is the solution to the parametric mathematical problem

$$\min_{\mathbf{h}} \quad z(\mathbf{h}, a) = \frac{1}{\theta} \sum_{p \in P} h_p \ln h_p + \sum_{e \in E} \int_0^{f_e} \tilde{l}_e(\omega) d\omega \quad s.t.$$

$$\sum_{p \in P} h_p = d,$$

$$f_e = \sum_{p \ni e} h_p, \qquad \forall e \in E$$

$$h_p \geq 0. \quad \forall p \in P$$

The feasible set of the minimization problem is $\mathcal{U}$, which is non-empty and bounded. Besides, the feasible set (which is $\mathcal{U}$) is independent of $a \in \Phi$, so Theorem 2.2 in [34] indicates that when a belongs to a neighborhood of a^* and $a \to a^*$, we have $\mathbf{h}^{equ}(a) \to \mathbf{h}^{equ}(a^*)$.

So, the equilibrium flow $\mathbf{h}^{equ}$ is continuous at (x, y), and therefore $\mathbf{f}^{equ}$ is also continuous.

$\square$

Lemma 3.4.2 *1) Function* $\bar{\Gamma}_1(x, y)$ *and* $\Gamma_1(x, y)$ *are continuous on* $x \in [0, 1]$ *and* $y \in [0, 1]$*;*

2) If Assumptions 1-3 hold for the instance $(G, \hat{\mathbf{l}}, d, \epsilon)$*, then* $\bar{\Gamma}_1(x, y)$ *is continuously decreasing in* $x \in [0, 1]$ *and increasing in* $y \in [0, 1]$*.*

Proof: The analysis in the proof of Lemma 2.3.3 also works here. □

Lemma 3.4.3 $\Gamma_2(x, y)$ *is continuous.*

Proof: The analysis in the proof of Lemma 2.3.4 also works here. □

Assumption 5 *When Player 2's estimation about the extra latency is* $m + \epsilon_0$*, we assume that* $\theta_0 m > \ln|P|$ *holds, where* θ_0 *is the parameter of* ϵ_0*, and* P *is the set of paths connecting the O-D pair.*

The expectation of Player 2's own estimation about the extra latency is m, and the variance σ_0^2 satisfies $\sigma_0^2 = \frac{6\pi^2}{\theta_0^2}$, thus $\theta_0 m = \frac{\pi}{\sqrt{6}}\frac{m}{\sigma_0}$. So Assumption 5 indicates $\frac{m}{\sigma_0} > \frac{\sqrt{6}}{\pi}\ln|P|$.

Lemma 3.4.4 *If Assumptions 1, 2, 3 and 5 hold, then Player 1's Stackelberg payoff* Γ_1^s *is lower bounded away from* $\bar{\Gamma}_1(0, 0)$*.*

Proof:

We start from the simple case where the latency functions are linear. By definition, $\Gamma_2(x, y^*(x)) \geq \Gamma_2(x, y), \forall y \in [0, 1]$.

If we set $y := 1$, then the random error is 0, and the SUE converges to the UE. If we set $\alpha = 1 - x$, then Lemma 2.2.3 implies

$$\Gamma_2(x, 1) \geq -\frac{4}{2+x}\frac{S(\mathbf{f}^{opt})}{d} \tag{3.24}$$

(3.7) implies that if $y^*(x) \neq 1$, and $\mathbf{h}^{sue}$ denotes the induced SUE flow, then

$$\Gamma_2(x, y^*(x)) = -W = -\hat{l}_p(\mathbf{h}^{sue}) - \frac{1}{\theta}\ln h_p^{sue} + \frac{1}{\theta}\ln d, \forall p \in P$$

Furthermore we have

$$\Gamma_2(x, y^*(x)) = -\frac{1}{d}\sum_{p\in P} h_p^{sue}\hat{l}_p(\mathbf{h}^{sue}) - \frac{1}{\theta d}\sum_{p\in P} h_p^{sue}\ln h_p^{sue} + \frac{1}{\theta}\ln d \tag{3.25}$$

Note that in the minimization problem

$$\begin{aligned} \min Z(\mathbf{h}) =& \sum_{p\in P} h_p \ln h_p \qquad s.t. \\ \sum_{p\in P} h_p =& d, \\ h_p \geq & 0 \end{aligned}$$

the KKT conditions indicate that the solution is $h_p = d/|P|, \forall p \in P$ and the minimum $Z_{min} = d\ln d - d\ln|P|$ [14].

$\mathbf{h}^{sue}$ is a feasible solution for the problem, so we have

$$\frac{1}{\theta d}\sum_{p\in P} h_p^{sue} \ln h_p^{sue} \geq \frac{1}{\theta}(\ln d - \ln|P|) \tag{3.26}$$

Then we have

$$\begin{aligned} \Gamma_2(x, y^*(x)) \overset{(3.25)(3.26)}{\leq} & -\frac{1}{d}\sum_{p\in P} h_p^{sue}\hat{l}_p(\mathbf{h}^{sue}) + \frac{1}{\theta}\ln|P| \\ \overset{(3.12)}{\leq} & -\frac{1}{d}\sum_{p\in P} h_p^{sue}\tilde{l}_p(\mathbf{h}^{sue}) + \frac{1}{\theta}\ln|P| - [1-y^*(x)]m \\ \overset{(3.11)}{=} & -\frac{1}{d}\sum_{p\in P} h_p^{sue} l_p(\mathbf{h}^{sue} + (1-x)y^*(x)\mathbf{h}^{opt}) + [1-y^*(x)](\frac{1}{\theta_0}\ln|P| - m) \\ \overset{(2.5)(1.3)}{=} & -\frac{1}{d}\sum_{e\in E} f_e^{sue} l_e(f_e^{sue} + (1-x)y^*(x)f_e^{opt}) + [1-y^*(x)](\frac{1}{\theta_0}\ln|P| - m) \\ \leq & -\frac{1}{d}\sum_{e\in E} f_e^{sue} l_e(f_e^{sue}) + [1-y^*(x)](\frac{1}{\theta_0}\ln|P| - m) \\ = & -\frac{1}{d}S(\mathbf{f}^{sue}) + [1-y^*(x)](\frac{1}{\theta_0}\ln|P| - m) \end{aligned} \tag{3.27}$$

$\Gamma_2(x,1) \leq \Gamma_2(x, y^*(x))$, thus if Assumption 5 holds, then combining (3.24) and (3.27), we have

$$y^*(x) \geq 1 - \frac{4}{2+x}\frac{\theta_0 S(\mathbf{f}^{opt})}{(\theta_0 m - \ln|P|)d} + \frac{\theta_0 S(\mathbf{f}^{sue})}{(\theta_0 m - \ln|P|)d} \tag{3.28}$$

The same analysis in the proof of Lemma 2.3.5 also works here, and we can prove that Γ_1^s is lower bounded by

$$\begin{aligned}\Gamma_1^s &\geq \max_{x\in[0,1),\lambda\in(0,1)} \min\{-\frac{4}{3+(1-x)\lambda}[1+\frac{k}{\theta_0 c}], -\frac{4}{2+x}+(1-\lambda)r\} \\ &\geq \max_{(x,\lambda)\in\mathcal{C}} -\frac{4}{3+(1-x)\lambda}[1+\frac{k}{\theta_0 c}]\end{aligned}$$

where $r := (m - \frac{\ln|P|}{\theta_0})d/S(\mathbf{f}^{opt})$, and $\mathcal{C}$ is the set of (x,λ) that are the solutions of the system

$$\begin{gathered}\frac{4}{2+x} - \frac{4}{3+(1-x)\lambda}[1+\frac{k}{\theta_0 c}] \leq (1-\lambda)r \\ x\in[0,1), \lambda\in(0,1]\end{gathered} \tag{3.29}$$

Note that when Assumption 5 holds, $r > 0$. Since

$$f(x) = \frac{4}{2+x} - \frac{4}{3+(1-x)\lambda}[1+\frac{k}{\theta_0 c}]$$

is continuous and strictly decreasing in $x \in [0,1)$, $\mathcal{C}$ is non-empty if $\exists\lambda\in(0,1]$ s.t.

$$f(1) < (1-\lambda)r \tag{3.30}$$

Since $f(1) = -\frac{4}{3}\frac{k}{\theta_0 c} < 0$, (3.30) holds, and $\mathcal{C}$ is non-empty. Furthermore $\exists\lambda_0 > 0$, and $x_0 < 1$ such that (3.29) holds $\forall x \in [x_0, 1)$, and $\lambda = \lambda_0$.

To sum up, Player 1's Stackelberg payoff Γ_1^s is lower bounded by

$$\begin{gathered}\max_{x,\lambda} -\frac{4}{3+(1-x)\lambda}[1+\frac{k}{\theta_0 c}] \qquad s.t. \\ r = \frac{md - \frac{\ln|P|}{\theta_0}d}{S(\mathbf{f}^{opt})} \\ \frac{4}{2+x} - \frac{4}{3+(1-x)\lambda}[1+\frac{k}{\theta_0 c}] \leq (1-\lambda)r \\ 0 \leq x < 1, 0 < \lambda \leq 1\end{gathered}$$

(x_0, λ_0) is a feasible point for the maximum program, so Γ_1^s is lower bounded by

$$-\frac{4}{3+(1-x_0)\lambda_0}[1+\frac{k}{\theta_0 c}]$$

which is strictly greater than $\bar{\Gamma}_1(0,0) = -\frac{4}{3}(1+\frac{k}{\theta_0 c})$.

We can generalize the analysis from the linear latency functions to general latency functions satisfying Assumptions 1-3. Following the same method, we have

$$y^*(x) \geq 1 - \frac{1}{1-(2-x)\beta}\frac{\theta_0 S(\mathbf{f}^{opt})}{(\theta_0 m - \ln|P|)d} + \frac{\theta_0 S(\mathbf{f}^{sue})}{(\theta_0 m - \ln|P|)d} \tag{3.31}$$

Also we can get the bound of Γ_1^s which is

$$\begin{aligned} &\max_{x,\lambda} -\frac{1+(J-1)(1-x)\lambda}{1-\beta+\beta(1-x)\lambda}[1+\frac{k}{\theta_0 c}] \qquad s.t. \\ &r = \frac{m - \frac{\ln|P|}{\theta_0}d}{S(\mathbf{f}^{opt})} \\ &\frac{1}{1-(2-x)\beta} - \frac{1+(J-1)(1-x)\lambda}{1-\beta+\beta(1-x)\lambda}[1+\frac{k}{\theta_0 c}] \leq (1-\lambda)r \\ &0 \leq x < 1, 0 < \lambda \leq 1 \end{aligned} \tag{3.32}$$

Since

$$f(x) := \frac{1}{1-(2-x)\beta} - \frac{1+(J-1)(1-x)\lambda}{1-\beta+\beta(1-x)\lambda}[1+\frac{k}{\theta_0 c}]$$

is continuous and strictly decreasing in $x \in [0,1)$ when Assumption 2 and 3 hold, $\mathcal{C}$ is non-empty if $\exists \lambda \in (0,1]$ such that

$$f(1) < (1-\lambda)r \tag{3.33}$$

Since $f(1) = -\frac{4}{3}\frac{k}{\theta_0 c} < 0$, (3.33) holds and $\mathcal{C}$ is non-empty. Furthermore $\exists \lambda_0 > 0$, and $x_0 < 1$ s.t. (3.32) holds $\forall x \in [x_0, 1)$ and $\lambda = \lambda_0$.

(x_0, λ_0) is a feasible point for the maximization problem, so Γ_1^s is lower bounded by

$$-\frac{1+(J-1)(1-x_0)\lambda_0}{1-\beta+\beta(1-x_0)\lambda_0}[1+\frac{k}{\theta_0 c}]$$

When Assumption 2 and 3 hold, it is strictly greater than $\bar{\Gamma}_1(0,0) = -\frac{1}{1-\beta}[1+\frac{k}{\theta_0 c}]$.

$\square$

Note that $(0,0)$ might be the Nash Equilibrium of the trust game, so Player 1's Stackelberg payoff is bounded away from his equilibrium payoff.

Chapter 4

Reputation and Repeated Games

In the trust game mentioned in the previous chapters, Player 1 (coordinator) and Player 2 (representative of traffic users) play this game only once. We call such a game a "stage game", or "one-shot" game.

If the stage game is played repeatedly, but the players do not record any information about previous periods, then all the players will insist on playing the Nash Equilibrium strategy of the stage game repeatedly. It is exactly the fact that the players have a record of the past history of the game that allows Player 1 to achieve a higher payoff than the equilibrium payoff in the stage game by exploiting a *reputation* that he can build in his interaction with Player 2. We formulate this new setting using the standard notions of repeated games, as they are used in game theory and economics.

4.1 Repeated Games

A *repeated game* is an infinite repetition of the playing of a stage game in periods or times $t = 0, 1, 2, \ldots, \infty$. In our case the stage game is the trust game defined in Section 2 (the induce traffic equilibrium is UE) and Section 3 (the induced traffic equilibrium is SUE).

Of central importance in order to escape the stage game Nash equilibrium is the notion of *history* $h^t = \{(x_0, y_0), (x_1, y_1), \ldots, (x_{t-1}, y_{t-1})\}$, defined for every time length t as the sequence of pure strategies played by the two players in the first t periods ($h^0 = \emptyset$ at the beginning of the game). Each player always records all his or her past actions (has *perfect recall*), and we consider the case where the history is also available to all the players (they have access to all players' previous action). But we will later distinguish between a Player 2 with unlimited memory who has a perfect record of Player 1's actions from the very beginning (period 0), and a Player 2 that has a limited memory and can only record the last K actions of Player 1. In reality, the former means that the traffic coordinator publishes the records of all the previous periods; the latter means that the traffic users only have access to the records of the most recent K periods.

Player 1 is a *long-run* player, i.e., his total payoff is a summation of his stage payoff over all periods discounted by a *discount factor* $\delta \in [0, 1)$:

$$g_1(x, y) = (1 - \delta) \sum_{t=0}^{\infty} \delta^t \Gamma_1^t(x^t, y^t) \tag{4.1}$$

The factor δ indicates the importance of the payoff in the future. The closer δ is to 1, the more equivalent (in terms of importance) stage payoffs in the distant future are to the ones closer to the present. The factor $(1 - \delta)$ is a normalization factor that brings the repeated game payoff to the same units as the stage payoff. In our case, Player 1 (the coordinator) is almost equally interested in the payoffs of all periods, i.e., $\delta \to 1$.

Let $\mathcal{H}^t = (X \times Y)^t$ be the set of all possible histories of length $t \geq 0$ ($\mathcal{H}^0 = \emptyset$), and $\mathcal{H} = \cup_{t=0}^{\infty} \mathcal{H}^t$ be the set of all possible histories. Then the *behavioral strategy* of (long-run) Player 1 is defined as $\sigma_1 : \mathcal{H} \to X$.

Player 2 can be simulated as a sequence of *short-run* players. She can be replaced by an infinite sequence of players $i_0, i_1, i_2, \ldots$, each with a behavioral strategy of $\sigma_2^{i_t} : \mathcal{H}^t \to Y$ and payoff Γ_2^t; each such player enters the game in only one specific period and acts myopically, but has the whole history available to Player 2 in that period.

A *Nash equilibrium* of the repeated game then is defined in the usual way, as a behavioral strategy profile $\sigma = (\sigma_1, \sigma_2^{i_0}, \sigma_2^{i_1}, \ldots)$ with the property that no deviation by any player will improve his (or her) payoff if the other players' strategies remain the same.

In order to exploit *reputation phenomena* in repeated games, we define two types for Player 1's strategy profile:

- **committed type ω_c:** If Player 1 is of this type, he always plays $c \in [0,1]$ in every period, independently of the history of the repeated game. The strategy c is the Stackelberg strategy defined as follows:

  $$c := \arg\max_{x \in [0,1]} \Gamma_1(x, y^*(x))$$

  The Stackelberg strategy is the one Player 1 would like the most to commit himself to.

- **rational type ω_r:** Player 1 is not restricted in playing any strategy in every period (he is opportunistic), and the payoff for the moves of this type of Player 1 is given by $g_1(x, y)$ defined in (4.1).

In the repeated game, Player 1's type is a private information that Player 2 does not know. Player 2's *perception* of the type of Player 1 is captured by $\mu(\omega_c|h)$ which indicates the probability Player 2 assigns to Player 1 being of committed type when she sees history h. $\mu^* = \mu(\omega_c|h^0)$ is the *initial belief* Player 2 assigns to Player 1 being of committed type when she has no history available (and, hence, probability $1 - \mu^*$ of being of rational type ω_r).

4.2 Payoff in the Repeated Game

We first consider the case where there is no limit to the length of history, which means that at period t, Player 2 has access to the full history from period 0 to period $t-1$.

Let Γ_1^s be the Stackelberg payoff of Player 1 in the stage game, and Γ_1^{min} be the minimum possible payoff he would get in the stage game.

Let $\underline{V}_1(\delta, \mu^*)$ be the least payoff achievable by Player 1 in the repeated game with discount factor δ and initial belief μ^* for the type of Player 1 held by Player 2.

Since the payoff functions $\Gamma_1(x, y)$ and $\Gamma_2(x, y)$ in the stage game are proved to be continuous in Lemma 2.3.3 and Lemma 2.3.4 when the perceived latency functions are deterministic, and in Lemma 3.4.2 and Lemma 3.4.3 when the perceived latency functions are stochastic, we can use Theorem 4 in Fundenberg and Levine [8]:

Theorem 4.2.1 ([8]) *If $0 < \mu^* < 1$, then for all $\epsilon > 0$ there exists a $\underline{\delta} < 1$ such that for all $\delta \in (\underline{\delta}, 1)$*

$$\underline{V}_1(\delta, \mu^*) \geq (1-\epsilon)\Gamma_1^s + \epsilon\Gamma_1^{min}.$$

This Folk theorem implies that Player 1 can almost achieve the payoff Γ_1^s in the repeated game when $\delta \to 1$. We have shown that if the equilibrium of the stage game is $(0, 0)$, then Γ_1^s is bounded away from $\bar{\Gamma}_1(0, 0)$, which is the bound of $\Gamma_1(0, 0)$.

4.3 A New Payoff Function

Instead of considering the exact price of anarchy for each instance, plenty of work (see [28][12][10]) is about the bound of price of anarchy which applies for a certain class of latency functions.

So, in our model we assume that Player 1 uses the bound $\bar{\Gamma}_1$ in (2.35) (in the case of UE) and (3.22) (in the case of SUE) as the payoff function of the stage game, instead of Γ_1.

In the following we focus on the case where the perceived latency is stochastic and latency functions are the general functions satisfying Assumptions 1, 2 and 3. Note that linear functions are special cases of such general functions. Besides, when the stochastic part ϵ in the perceived latency goes to 0, the instance goes to the deterministic case.

Fact 1 *The following are true for function* $\bar{\Gamma}_1(x, y)$*:*

1. *(myopic incentive of Player 1)* $\bar{\Gamma}_1(x, y)$ *is strictly decreasing in* x *if* $y > 0$ *and constant if* $y = 0$ *(see Lemmas 2.3.3, 3.4.2).*

2. *(Player 1 wants to be trusted)* $\bar{\Gamma}_1(x, y)$ *is strictly increasing in* y*, unless* $x = 1$ *in which case it is constant (see Lemmas 2.3.3, 3.4.2).*

Assumption 6 *In the case where Player 2's perceived latency is stochastic and the latency functions satisfy Assumptions 1, 2 and 3, we assume that*

$$\frac{[\beta + (J-1)(1-\beta)](1-2\beta)}{\beta} > \frac{k}{\theta_0 c} \tag{4.2}$$

holds for the instance $(G, \mathbf{l}, d, \epsilon)$ *where* k *solves* $ke^k = \frac{1}{e}(|P| - 1)$, $c = \frac{S(\mathbf{f}^{opt})}{d}$ *and* θ_0 *is the parameter of* ϵ_0.

Note that Assumptions 2 indicates $[\beta + (J-1)(1-\beta)](1-2\beta) > 0$, and furthermore, in the deterministic case, $\frac{1}{\theta_0}$ goes to 0, and the assumption holds.

Besides, in the case where latency functions are linear, $J = 1$, $\beta = \frac{1}{4}$, and (4.2) goes to

$$\frac{k}{\theta_0 c} < \frac{1}{2}$$

Lemma 4.3.1 *(sub-modularity of Player 1)* $\bar{\Gamma}_1(x, y) - \bar{\Gamma}_1(x', y)$ *is strictly increasing in* y *for any* $x < x'$.

Proof: (3.22) indicates

$$\bar{\Gamma}_1(x, y) = -\frac{1 + (J-1)(1-x)y + \frac{k}{\theta c}}{1 - \beta[1 - (1-x)y]}$$

Thus $\forall x, x' \in [0,1]$ and $x < x'$,

$$f(x,y) = \bar{\Gamma}_1(x,y) - \bar{\Gamma}_1(x',y) = \frac{(x'-x)y[\beta(1+\frac{k}{\theta c}+(J-1)(1-\beta)]}{[1-\beta+\beta(1-x')y][1-\beta+\beta(1-x)y]}$$
$$= \frac{(x'-x)y[\beta+\beta(1-y)\frac{k}{\theta_0 c}+(J-1)(1-\beta)]}{[1-\beta+\beta(1-x')y][1-\beta+\beta(1-x)y]}$$

$f(x,y)$ is increasing in y for all $0 \leq x < x' \leq 1$, if

$$\bar{f}(y) = \frac{y[\beta+\beta(1-y)\frac{k}{\theta_0 c}+(J-1)(1-\beta)]}{[1-\beta+\beta y]^2}$$

is increasing in y.

By taking the derivative of $\bar{f}$ on y, we have

$$\frac{d\bar{f}}{dy} = \frac{[\beta+\beta(1-y)\frac{k}{\theta_0 c}+(J-1)(1-\beta)][1-\beta-\beta y]-\beta y\frac{k}{\theta_0 c}}{(1-\beta-\beta y)^3}$$

Assumption 6 implies

$$\frac{[\beta+(J-1)(1-\beta)](1-2\beta)}{\beta} > \frac{k}{\theta_0 c}$$

Then $\frac{d\bar{f}}{dy} > 0$ holds for all $0 \leq y \leq 1$. Thus $\bar{\Gamma}_1(x,y) - \bar{\Gamma}_1(x',y)$ is strictly increasing in y for any $x < x'$.

□

Assumption 7 *In the case where the perceived latency is stochastic and the latency functions satisfy Assumptions 1, 2 and 3, we assume that Player 2's own estimation about the extra latency, $m+\epsilon_0$, satisfies*

$$m > \frac{\beta}{1-\beta}\frac{S(\mathbf{f}^{opt})}{d} + \frac{\ln|P|}{\theta_0} \tag{4.3}$$

where $\beta = \beta(\mathcal{L})$, $S(\mathbf{f}^{opt})$ is the social cost of the optimal flow, and θ_0 is the parameter of ϵ_0.

Assumption 7 is about Player 2's own estimation about the extra latency. Note that $\frac{S(\mathbf{f}^{opt})}{d}$ is the average latency in the optimal flow.

In the case where latency functions are linear, $\beta = \frac{1}{4}$, and $\frac{\beta}{1-\beta} = \frac{1}{3}$. Then (4.3) is

$$m > \frac{1}{3}\frac{S(\mathbf{f}^{opt})}{d} + \frac{\ln|P|}{\theta_0}$$

Besides, Definition 3.1.2 implies that the variance of ϵ_0 satisfies $\sigma_0^2 = \frac{\pi^2}{6\theta_0^2}$, so the part $\frac{\ln|P|}{\theta_0}$ is equal to $\frac{\pi \ln|P|}{\sqrt{6}}\sigma_0$. When σ_0 goes to 0, the perceived latency tends to be deterministic, the part $\frac{\ln|P|}{\theta_0}$ goes to 0, and (4.3) is

$$m > \frac{\beta}{1-\beta}\frac{S(\mathbf{f}^{opt})}{d}$$

Lemma 4.3.2 *(valuable reputation for Player 1) $\exists c \in [0,1]$, s.t. $\bar{\Gamma}_1(c, y^*(c)) > \bar{\Gamma}_1(0,0)$. In other words, Player 1 prefers Player 2 to believe that he is a commitment type and to act accordingly rather than play $(0,0)$ in the stage game.*

Proof: We consider the case where latency functions are the general functions satisfying Assumptions 1, 2 and 3, and the perceived latency is stochastic.

Note that Lemma 3.4.4 shows that $\Gamma_1^s > \Gamma_1(0,0)$ holds under Assumption 5. The same result also applies for $\bar{\Gamma}_1^s$, if we make a stronger assumption (Assumption 7) about Player 2's estimation of extra latency, instead of Assumption 5.

From (3.31) we have

$$\begin{aligned} y^*(x) &\geq 1 - \frac{1}{1-(2-x)\beta}\frac{\theta_0 S(\mathbf{f}^{opt})}{(\theta_0 m - \ln|P|)d} + \frac{\theta_0 S(\mathbf{f}^{sue})}{(\theta_0 m - \ln|P|)d} \\ &\geq 1 - \frac{1}{1-(2-x)\beta}\frac{\theta_0 S(\mathbf{f}^{opt})}{(\theta_0 m - \ln|P|)d} + \frac{\theta_0 S(\mathbf{f}^{opt})}{(\theta_0 m - \ln|P|)d} \\ &= 1 - \frac{(2-x)\beta}{1-(2-x)\beta}\frac{S(\mathbf{f}^{opt})}{d}\frac{\theta_0}{\theta_0 m - \ln|P|} \end{aligned}$$

Note that

$$\frac{(2-x)\beta}{1-(2-x)\beta} \in (\frac{\beta}{1-\beta}, \frac{2\beta}{1-2\beta}] \quad \Leftrightarrow \quad x \in [0,1)$$

Thus if

$$\frac{\beta}{1-\beta}\frac{S(\mathbf{f}^{opt})}{d}\frac{\theta_0}{\theta_0 m - \ln|P|} < 1$$

$$\Leftrightarrow m > \frac{\beta}{1-\beta}\frac{S(\mathbf{f}^{opt})}{d} + \frac{\ln|P|}{\theta_0},$$

which is implied in Assumption 7, then $\exists x_0 < 1$ such that $y^*(x_0) > 0$, and

$$\bar{\Gamma}_1^s \geq \bar{\Gamma}_1(x_0, y^*(x_0)) \overset{(2.35)}{=} -\frac{1+(J-1)(1-x_0)y^*(x_0)}{1-\beta+\beta(1-x_0)y^*(x_0)}$$

Note that $\bar{\Gamma}_1(0,0) = -\frac{1}{1-\beta}$, and $\bar{\Gamma}_1^s > \bar{\Gamma}_1(0,0)$ holds when $J < \frac{1}{1-\beta}$ as Assumption 3 indicates.

□

Note that the case where latency functions are linear is a special case of the general latency functions satisfying Assumptions 1, 2 and 3, where $\beta = \frac{1}{4}$ and $J = 1$.

Besides, when $\frac{1}{\theta_0}$ goes to 0, the stochastic case goes to the deterministic one.

Assumption 8 *(myopic incentive of Player 2) For any strategy by Player 1* $x \in [0,1]$, *Player 2's best response* $y^*(x)$ *is increasing in* x.

The assumption implies that the less Player 1 takes the advantage over Player 2, the more trust Player 2 would like to put on Player 1.

Fact 1, Lemmas 4.3.1, 4.3.2 and Assumption 8 allow us to use a more powerful result by Liu and Skrzypacz [15] in the case where Player 2 is of *bounded rationality*, in the sense that Player 2's record keeping is limited (e.g., by memory limitations) to recording only the K most recent actions of Player 1, for some parameter K (Player 2 still has perfect recall of her own actions in all past history).

Unlike the Folk theorem of [8], this limitation allows [15] to describe exactly the equilibrium strategies for the two players, and prove a payoff bound for Player 1's payoff similar to the bound in Theorem 4.2.1 *at any point of the game* (not just at the beginning of the game as the bound in Theorem 4.2.1

does). This is important for the study of games that have already been played for a number of periods which we do not know (or do not care about), and we want to evaluate the quality of Player 1's payoff at the moment we start our observation.

Let $P(t), \mu(\omega|h)$ be Player 2's prior belief of whether the current period is t (i.e., she does not keep track of time, so she must have a prior belief on which is the current period), and her posterior belief over Player 1's type being $\omega(\omega_c$ or $\omega_0)$ given a history h (truncated to the most recent K periods for Player 1's actions). Assume $P(t)$ is *improper uniform prior* which indicates that Player 2 assigns identical probability on $t = 0, 1, 2....$ Note that μ^* is Player 2's initial belief over Player 1's type being ω_c with the history $h^0 = \emptyset$.

If h contains a non-commitment action $x \neq c$, then

$$\mu(\omega_c|h) = 1 - \mu(\omega_0|h) = 0$$

In this case, the notion of equilibrium used is that of *stationary Perfect Bayesian Equilibrium (PBE)* which is more sophisticated than the simple Nash equilibrium considered above since it takes into account Player 2's beliefs μ . μ is updated using Bayes' rule[1].

Then Theorem 2 in [15] holds in our case:

Theorem 4.3.1 ([15]) *For any $\epsilon > 0$, $\mu^* \in (0, 1)$, there exists an integer $K(\epsilon, \mu^*)$ such that Player 1's payoff in the repeated game with limited records of length $K > K(\epsilon, \mu^*)$ is lower bounded at any time by*

$$(1 - \delta^K)\bar{\Gamma}_1(0, 0) + \delta^K \bar{\Gamma}_1^s - \epsilon$$

which converges to $\bar{\Gamma}_1^s$ as δ goes to 1.

Theorem 4.3.1 indicates that in the repeated game, the bound of Player 1's payoff converges to his Stackelberg payoff in the stage game. In the

[1]See [15] for a formal definition of the Bayes' rule applied here.

previous sections, we proved that the Stackelberg payoff is bounded away from $\bar{\Gamma}_1(0,0)$, which might be the equilibrium payoff.

Given a history h of length K, we define the *index* of h to be the number of commitment actions since the most recent non-commitment action in h. Theorem 1 in [15] describes the the equilibrium strategies for the players in the repeated game with limited-length history.

Theorem 4.3.2 ([15]) *For any $\delta > \bar{\delta}, K > 0$ and any prior $\mu^* > 0$, any stationary PBE takes the following form:*

1. There exists a strictly increasing sequence $\{\beta_k\}_{k=0}^{K-1} \subset (0,1)$ such that, if the index of the history is $k < K$, Player 1 plays c with probability β_k and 0 with probability $1 - \beta_k$; Player 2 plays $y = y^*(\beta_k)$.

2. If the index of history is K, Player 1 plays 0 (with probability 1); Player 2 plays $y^*(\mu_K)$ where $\mu_K > \beta_{K-1}$.

The theorem indicates that when the history is totally "clear" (Player 1 plays c in all the previous K periods), Player 1 will exploit his *reputation*, and play his dominant strategy 0 in the current period. When the history h is not clear (Player 1 plays something other than c in the history), the index k indicates the distance from h to the clear history (the greater k is, the closer h is to the clear history). The greater k is, with the higher probability Player 1 will play c (he wants to accumulate his reputation).

Chapter 5

Conclusion and Open Problems

The thesis is about a method to decrease the price of anarchy of a selfish routing problem. We designed a trust game between the traffic coordinator and the users, in which the coordinator takes advantage of the users' incomplete information and provides them with fabricated information of unexpected extra traffic. Each strategy profile in the trust game will induce a selfish routing instance, and Player 1 wants to decrease the price of anarchy of the induced selfish routing instance.

We concluded that Player 1's Stackelberg payoff in the trust game is bounded away from his equilibrium payoff (in cases where the equilibrium of the trust game is $(0, 0)$). To entice the coordinator to play the Stackelberg strategy instead of the Nash Equilibrium strategy in the trust game, we considered the repeated game with the trust game to be the stage game in each period. In the repeated game with infinite memory of the previous actions, using the result in [8], we concluded that the bound of the coordinator's payoff in the repeated game converges to the Stackelberg payoff in the stage game. In the repeated game with limited-length memory, the result in [15] implies that the coordinator's payoff in the repeated game converges to the Stackelberg payoff in the stage game <u>at any time</u>.

In conclusion, in the repeated game with each stage game being the trust game between the coordinator and the users, the coordinator can push the equilibrium traffic flow towards the optimal flow and decrease the PoA.

There are still several open problems in the study.

- We made several assumptions in the thesis to make the conclusion work. Assumptions 1-3 are about the latency functions. Linear functions satisfy all the three. Assumption 1 works for polynomial latency functions $l(x) = \sum_{i=0}^{\infty} a_i x^i, a_i \geq 0$. Assumption 2 works for linear, quadratic and cubic functions, and fails for higher level polynomial functions. Assumption 3 is widely used (see [23]), but we need to look more into it and figure out the circumstances where it can be applied.

  Assumption 4 is an assumption about the instance with multiple commodities. Assumptions 5 and 6 are about the Player 2's estimation of the extra latency. Assumption 7 is a bound for m that we need when we use $\bar{\Gamma}_1$ as Player 1's payoff. Assumption 8 is about the best response in the trust game which is a basic property for any reasonable trust game (the less advantage Player 1 takes of Player 2, the more trust Player 2 should put), but we have not yet found an explicit formula for Player 2's best response, so we have to put that into an assumption. Future work may focus on the relaxation of the assumptions.

- In this thesis, Player 1 picks $x \in [0,1]$ and then tells all the users that the unexpected extra flow is $(1-x)\mathbf{h}^{opt}$. Actually he can be more flexible when choosing the extra flow to announce. Future work can be on some other clever choices of the extra flow.

- The main part of the thesis is about the selfish routing problem with a single commodity. In the case of multiple commodities, we made Assumption 4 in the thesis, and in the future, we may try to get rid of it or make some other natural assumptions instead.

- When we use the negative of the price of anarchy as Player 1's payoff function, we can use the conclusion in [9] to get the payoff in the repeated game with infinite-length history. Also in the thesis we got bounds of PoA, and when using the bounds to define Player 1's payoff function, we can apply the conclusion in [15] under several additional assumptions (Assumptions 6, 7 and 8). To find a better (tighter) bound to avoid the assumptions remains an open problem.

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