LIFE CYCLE, UNCERTAIN LIFETIMES AND OVERLAPPING GENERATIONS

To my sister

Mussarat Almas

CONSUMPTION AND WORK HOURS IN LIFE CYCLE MODELS WITH UNCERTAIN LIFETIMES: AN INDIVIDUAL ANALYSIS AND AN EQUILIBRIUM ANALYSIS WITH OVERLAPPING GENERATIONS

by

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A Thesis

Submitted to the School of Graduate Studies

in Partial fulfillment of the Requirements

for the degree

Doctor of Philosophy

McMaster University

DOCTOR OF PHILOSOPHY (1988) McMASTER UNIVERSITY

(Economics) Hamilton, Ontario

TITLE: Consumption and Work Hours in Life Cycle Models with Uncertain Lifetimes: An Individual Analysis and an Equilibrium Analysis with Overlapping Generations

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NUMBER OF PAGES: xvii, 416

ABSTRACT

This thesis consists of two parts. In the first part, life cycle patterns of consumption, work hours, savings and assets of an individual under uncertain lifetimes with actuarially fair life insurance and annuities are studied and the effects on consumption and work hours of changes in survival probabilities, the age specific wage rates and the rate of interest are analyzed.

A substitutability relation between consumption and leisure and the availability of actuarially fair annuities can explain the lack of decumulation even after retirement.

The marginal saving rate out of anticipated labour income is positive. Under certain conditions, the marginal consumption rate could also be positive. Thus the Keynesian absolute income hypothesis may be fully supported in the life cycle context.

A general increase in survival rates creates only a wealth effect by changing the actuarially fair rates of interest. The resulting intertemporal substitution effect is offset by a change in discount factors applied to the utility function due to life uncertainty. On the other hand, an increase in survival probabilities for one individual only, creates an intertemporal substitution effect.

An evolutionary increase in wage rates creates an

intertemporal allocation effect by changing the relative price of expenditure (on consumption and leisure) at different ages and an intratemporal substitution effect by increasing the price of leisure relative to consumption. On the other hand, a parametric increase in wage rates creates a wealth effect, an intertemporal reallocation effect and an intratemporal substitution effect. The compensated effect of a parametric wage increase, consisting of the intratemporal substitution effect and the intertemporal reallocation effect, on work hours (consumption) is smaller (greater) than the effects of an equal evolutionary wage increase.

In the second part, a dynamic general equilibrium model based on the Samuelson-Diamond overlapping generations framework, is developed. The model is based on the standard continuous time life cycle model under life uncertainty and incorporates all the demographic aspects of overlapping generations.

The implications of mortality changes for aggregate economic behaviour are discussed. A mortality change carries a reallocation effect by affecting the lifetime decision of individuals, an age redistribution effect by changing the relative size of different cohorts, and a demographic effect by changing the growth rate of population. An increase in survival rates at the young or middle (old) age will reduce (increase) the capital-labour ratio and the wage rate and increase (decrease) the rate of interest.

ACKNOWLEDGEMENTS

First of all I must acknowledge the contribution of my sister, Mussarat Almas, to my education which is but only a part of her impression on my life. Without her persistent support I would never have acquired any education beyond high school.

In writing this thesis I had the privilege to work under the invaluable supervision of Professor Martin J. Browning, Professor John J. Burbidge and Professor A. Leslie Robb. I express my gratitude to Professor Browning (Chairman of the Committee) for his guidance and encouragement during research. I am also thankful to him for all the time he patiently spent on careful scrutiny of the thesis. Without his proper supervision this thesis would never have been completed.

Professor Robb's door was always open for me. His useful suggestions at various stages of research have tremendously enriched the quality of the thesis; for all that I am extremely grateful to him. I am also indebted to him for his help in the printing of this thesis.

Professor Burbidge's precise suggestions were very helpful in saving a lot of time in avoiding unproductive efforts. At certain stages in the research, his presence in the supervisory committee proved to be vital. I am very grateful to him.

I feel indebted to Professor Ernest H. Oksanen and Professor Frank T. Denton for their encouragement throughout my studies. I am also thankful to Aloma Munian, the department secretary, for her concern with my progress in studies and friendly co-operation.

Thanks are also due to Professor Aly Ercelawn of Quaid-i-Azam University, Islamabad who has been a great source of inspiration for me.

I am extremely thankful to all the members of my family, especially Ami, Munazza, Shehla and Babli, for their support and good wishes.

My education in Canada has been financed by McMaster University and the Government of Ontario, Canada. For this support I am thankful.

Finally, I cannot forget to thank many of my friends in Pakistan who offered a hand of help in supporting me in my move to Canada.

All the possible errors or omissions left in the thesis are my sole responsibility.

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LIST OF SYMBOLS USED

$a(x)$: assets held by an individual aged x,

$A(t)$: aggregate assets in the economy at time t,

$\hat{A}$: aggregate assets normalized by current births,

b: average rate of reproduction in the population,

$b(x)$: the rate of reproduction by the population aged x,

$B(t)$: number of births at time t,

$B(t,x)$: number of births by the population aged x at time t,

$c(x)$: the rate of consumption by an individual aged x,

$C(t)$: aggregate consumption at time t,

$\hat{C}$ aggregate consumption normalized by current births,

d: average death rate in the population,

$D(t)$: number of deaths at time t,

$D(t,x)$: number of deaths in the population aged x at time t,

$e(x)$: the rate of expenditure on consumption and leisure by an individual aged x measured in the units of consumption good,

f: the production function in intensive form,

g: the growth rate of population and of other aggregate variables in steady state,

$h(x)$: the rate of work hours by an individual aged x,

$H(t)$: aggregate labour supply (measured in hours) at time t,

$\hat{H}$: aggregate labour supply normalized by current births,

$j(x)$: an age productivity function showing the productivity of an hour of work at age x,

J: the entire age path of the productivity parameters, $j(x): 0 \leq x \leq T$,

$J(x)$: Jacobian matrix for the lower stage problem,

k: the capital labour ratio in steady state,

$K(t)$: capital stock at time t,

$l(x)$: the rate of leisure consumed by an individual aged x,

$m(x)$: labour income or earnings of an individual aged x,

$M(t)$: aggregate labour income at time t,

$n_1(x)$: the lower stage elasticity of demand for consumption with respect to wage rate,

$n_2(x)$: the lower stage elasticity of demand for leisure with respect to wage rate,

$p(x)$: the unconditional probability of survival (also equal to the actual survival rate in the population) to age x,

P: the entire age path of survival probabilities to various points in the life horizon,

$q(x)$: the unconditional probability of death (also equal to the actual death rate in the population) at age x,

r: the rate of interest on riskless bonds,

$r^*(x)$: the actuarially fair instantaneous rate of interest on annuities and life insured borrowings at age x,

R: the retirement age,

$s(x)$: a broad measure of savings rate by an individual aged x, defined as the excess of earnings plus net interest income over consumption,

$S(t)$: aggregate broad measure of savings at time t,

$\hat{S}$: aggregate broad measure of savings normalized by current births,

$S_1(x)$: the expenditure share of consumption,

$S_2(x)$: the expenditure share of leisure,

t: the time index,

T: upper limit on the uncertain life horizon,

$\bar{T}$: the expected lifetime,

$u(x)$: utility function defined over consumption at age x,

$U(x)$: utility function defined over consumption and leisure at age x,

$V(x)$: indirect utility function defined over the wage rate and expenditure rate at age x,

w: wage rate for a productive or an effective unit of labour,

$w(x)$: wage rate for an actual unit of labour at age x,

W: the entire age path of wage rates,

x: the age index,

y: the age index,

z: the age index,

$\alpha(n)$: the weighting factor attached with the wealth effect in Slutsky equation for the effect of an increase in the rate of interest on expenditure,

$\beta(n,x')$: the weighting factor attached with the wealth effect in Slutsky equation for the effect on expenditure of an increase in the wage rate for age x' anticipated at age n,

$\varepsilon_1(x)$: the lower stage elasticity of demand for consumption with respect to expenditure,

$\varepsilon_2(x)$: the lower stage elasticity of demand for leisure with respect to expenditure,

θ: the rate of endowment of leisure of an individual at each age,

$\lambda(x)$: the Lagrange multiplier in the lower stage optimizing problem,

$\xi(x)$ the rate of expenditure at constant prices by an individual aged x,

Π: the instantaneous conditional probability of death, $q(x)/p(x)$,

$\varsigma(x)$: a narrow measure of savings defined as the excess of earnings over consumption,

$\Sigma(t)$: aggregate narrow measure of savings at time t,

$\hat{\Sigma}$: aggregate narrow measure of savings normalized by current births,

Ω: the lifetime utility function,

$\zeta(x)$: the coefficient of relative risk aversion.

CHAPTER 1

INTRODUCTION

The main object of this thesis is to study the implications of life uncertainty and the positive economics consequences of life expectancy improvements in life cycle model of an individual and in a dynamic general equilibrium model with overlapping generations. A secondary object is the study of life cycle patterns of consumption, work hours, savings and asset holding of an individual in an endogenous work-leisure choice model with uncertain lifetimes.

The implications of life uncertainty for the life cycle consumption and savings behaviour of an individual have been studied in detail by Davies (1981), Hurd (1986), Levhari and Mirman (1977), Mirer (1987), Yaari (1965) and Zilcha and Friedman (1985). These studies do not include work-leisure choice problem into the analysis and assume an exogenously fixed income stream over the life cycle. In recent years, overlapping generations models have also been extended to incorporate uncertainty of life (Abel (1984, 1985, 1986a, 1986b), Eckstein, Eichenbaum, and Peled (1985), Hubbard (1984), Johnsten and Donaldson (1985), Karni and Zilcha (1984), Sheshinski and Weiss (1981), Sinha (1986)

Skinner (1985)). But the analyses have been limited to a two period discrete model with no uncertainty in the first period. For the study of life expectancy improvements, this model seems to be restrictive as it does not allow the study of various types of mortality changes.

Life expectancy in many countries has been improving for the past few decades. The recent trend in mortality improvements has led economists to study its economic aspects. The typical contribution of economists in this context has been to conduct the cost benefit analyses of different life saving schemes (For example, Arthur (1981) and Jones-Lee (1982, 1985)). But there are also a few studies on the positive economics aspects of the issue (Abel (1986), Hamermesh (1984), Nalebuff and Zeckhauser (1984), Sheshinski and Weiss (1981), Sinha (1986) and Skinner (1985)).

Hamermesh (1984) and Nalebuff and Zeckhauser (1984) discussed the effects of increasing the length of a certain life horizon on the life cycle consumption and work decision of an individual. Aside from being unrealistic, the assumption of a non-stochastic life horizon is not an appropriate choice for the study of mortality variations because under this assumption, the only way mortality can vary is by addition or subtraction of some years at the end of a certain life horizon. However, the observed mortality improvements appear to be the result of more people surviving to the later part of a given horizon rather than an increase in horizon (Fries (1980)). On the other hand, Abel (1986), Sheshinski and Weiss (1981)) assumed uncertain lifetimes in their analyses of the effect of an increase in life expectancy on the steady state levels of consumption and savings of an individual in an overlapping generations model. Skinner (1985) and Sinha (1986) also assumed uncertain lifetimes in analyzing the effect of increased longevity on capital-labour ratio. In all these studies the analysis has been simplified by choosing a two period model in which there is no uncertainty of survival through the first period. Thus life expectancy can be increased only by increasing the survival rate in the second period. This model is restrictive for the study of life expectancy improvements as it does not allow a distinctive analysis of changing survival prospects at different segments of life horizon and thus fails to relate increase in life expectancy to different saving and work habits of individuals across various segments of life.

Another commonly maintained assumption in a two period model is that all births take place in the first period when there is no uncertainty of survival. Due to this assumption growth rate of population becomes independent of survival rate. Thus the model ignores the possible demographic effects of an increase in life expectancy through a change in the growth rate of population.

The above discussion suggests that for a detailed study of mortality variations, a more general model is needed which permits sufficient variation in savings, work and reproductivity pattern across various segments of life. Therefore a continuous time horizon model is developed in which many generations overlap. Survival to each point in the horizon is assumed to be uncertain and reproductivity rates are assumed to vary smoothly with age. This framework has also been used by Arthur (1981) in his analysis of the value of life under the golden rule of accumulation. This framework is preferred to an alternative choice of a three period discrete model due to the following reason. In a discrete model the age pattern of consumption, savings, etc. is fixed in an arbitrarily pre-marked segment of life. The continuous time model, on the other hand, does not impose such behaviour. Rather it is the behavioural pattern which classifies the horizon into various segments.

Another consideration for choosing a continuous time model has to do with the study of life cycle model of an individual which also constitutes an important part of the thesis. A continuous time framework allows application of a convenient technique of optimization: the non-linear programming, to find a solution in an endogenous work-leisure choice model with the provision of retirement. Besides, this framework will enable an easy comparison of our findings with those of two basic studies on which our model is extended. These studies are Yaari's (1965) life cycle model of consumption under uncertain lifetimes and Heckman's (1974) life cycle model of consumption and work hours under certain lifetimes.

Since the time of death is not known, an individual can not plan a life cycle allocation scheme which guarantees zero asset holding at the time of death. This raises some accounting problems. In particular, if an individual dies with positive wealth, it is necessary to identify a distribution scheme by which this wealth is passed on to others. There are two approaches commonly adopted to deal with this issue. One of these is to assume that all the wealth of a deceased individual is passed on to his/her descendants. The other approach is to assume the existence of a perfect insurance market which intermediates in the risk sharing process among different members of each cohort. The wealth of a deceased individual is equally distributed among surviving members in the cohort. The first approach requires a detailed account of the distributional consequences of an uncertain amount of inheritance an individual may receive. Such an analysis does not seem to be possible in a continuous time model in which there is no identification of families. Therefore, throughout the thes's a perfect insurance market is assumed to exist.

Life cycle model of an individual is an integral part of an equilibrium model based on overlapping generations. While the life cycle model of an individual has been studied in detail in the literature, there seems to be a tendency to include a simplified version of it in the overlapping generations models due to the complex nature of the latter. In the present research the modelling of an individual's life cycle allocation decision is considered to be an important aspect of the equilibrium model. Due to this consideration, the thesis is divided into two parts: an individualistic life cycle model and an equilibrium model based on overlapping generations.

In first part, an individualistic life cycle model of consumption and work allocation decision under uncertain lifetimes is presented. In chapter 2, a survey of some related studies on the life cycle theory is presented. The general features of life cycle models with life uncertainty are described and previous contributions on the study of mortality variations are summarized. The treatment of work-leisure choice in the life cycle context is also discussed. The chapter concludes with a motivation for the proposed life cycle model.

In chapter 3, we adopt a two stage budgeting approach to explain the life cycle motive for consumption and work allocation over time under life uncertainty in the presence of actuarially fair life insurance and annuities. The analysis allows for retirement as a possible outcome of the optimal work allocation plan. The effects of evolutionary changes in wage rates over the life cycle on consumption and work hours are decomposed into intertemporal effects and an intratemporal effects. The relationships of consumption and savings with anticipated income are studied. The assets accumulation behaviour of the individual is also spelled out.

Chapter 4 deals with the comparative static analysis of the model. Of particular interest are the implications of mortality improvements for the life cycle allocation behaviour. Two types of mortality variations are discussed: an increase in survival probabilities equal to a general increase in survival rates in the society and an increase in subjective survival probabilities independent of the general mortality risks in the society. In the presence of actuarially fair life insurance and annuities, a general increase in survival probabilities carries a wealth effect by changing the actuarial rate of interest. The resulting intertemporal substitution effect is offset by an equivalent change in subjective discount rate applied to the lifetime utility function due to life uncertainty. On the other hand, if actuarial rates of interest are based on general survival risks in the society, an increase in subjective survival probabilities does not affect the actuarial rates of interest and therefore carries only an intertemporal substitution effect. The implications of an increase in risk to life are also studied by adopting the notion of a mean preserving spread in the distribution of life.

The effects of increasing the rate of interest are explained by Slutsky equations showing both the intertemporal substitution effects and the wealth effects. The effects of parametric changes in wage rates are accounted for by Slutsky equations showing the wealth effects, the intertemporal reallocation effects and the intratemporal substitution effects. The distinction between parametric and evolutionary changes in wage rates is also explained.

A summary of major findings of the individualistic life cycle model is presented in chapter 5.

Part II of the thesis starts with an introduction (chapter 6) which provides some motivation for the study of the effects of mortality variations on the aggregate economic variables.

Chapter 7 presents a selected review of the literature on overlapping generations model which is the basic tool of analysis for the general equilibrium effects of mortality changes. This review is not confined to studies on mortality changes. The general structure of overlapping generations models and some of the basic issues surrounding the literature are also discussed.

Due to its complex nature, the analysis of mortality changes is performed in a step by step process in the next three chapters. First, in chapter 3 an aggregate household model is presented. The main concern there is to form the aggregate household variables and relate them to their economic and demographic determinants. Then a production function which is linear in each of the two factor inputs, capital and labour, is added to the model. The purpose of this unrealistic but ad hoc linearity assumption is to make the two factor prices, the rate of interest and the wage rate, technologically fixed and thereby isolate the household model from endogenous changes in the factor prices which would otherwise take place due to mortality changes. The life cycle work allocation plan of an individual is also assumed to be fixed in this chapter.

The effects of a small increase in survival rates at various points of the life horizon on the aggregate capital stock, normalized by the number of current births, are calculated. Given the growth rate of population, capital stock is affected on two account, a reallocation of savings by individuals due to the wealth effect of a change in survival probabilities and an age redistribution effect brought about by the change in survival rates. The effects of mortality changes on the employment of labour are also obtained. An increase in survival rates during reproductive years of life will increase the growth rate of population. This, in turn, will increase the relative proportion of young individuals in the population. The Capital stock and the labour employment are affected according to the age pattern of asset holding and work hours.

The analysis of chapter 8 provides some groundwork for a more realistic study of mortality changes in chapter 9 where the assumption of fixed factor prices is relaxed by introducing a neo-classical <u>linear homogeneous</u> production activity into the model. The assumption of a fixed work plan of an individual is maintained in this chapter. The conditions for a Diamond (1965) type competitive equilibrium solution are established. Assuming that the equilibrium is stable, these conditions form the basis of analysis of mortality variations. The partial effects of mortality changes on the capital stock and labour employment are brought in from chapter 8 and the general equilibrium effects of mortality changes on the capital-labour ratio, the rate of interest and the wage rate are derived. The key result is that an increase in the survival rate for a young or middle age will reduce the capital-labour ratio, though an increase in survival rate for an old age will increase the capital-labour ratio.

In chapter 10, the assumption of a fixed work plan of an individual is relaxed. The life cycle model of an optimal consumption-work allocation decision, developed in chapters 3 and 4 is included in the general equilibrium model and the analysis of chapters 8 and 9 is repeated. An increase in survival rates for a young age or for the later part of middle age results in a lower capital-labour ratio, as found in chapter 9. Due to the complex nature of the model, the effects of an increase in survival rates for the other ages are not obvious. However, under certain conditions, the relaxation of the assumption of a fixed work plan of an individual does not change the qualitative nature of the results obtained in chapter 9.

Finally, a summary of all the findings of the general equilibrium analysis is presented in chapter 11.

PART I

AN INDIVIDUALISTIC LIFE CYCLE MODEL: THE AGE PATTERNS OF CONSUMPTION, WORK HOURS, SAVINGS AND ASSETS UNDER UNCERTAIN LIFETIMES

CHAPTER 2

SURVEY OF SELECTED LITERATURE ON THE LIFE CYCLE MODEL OF AN INDIVIDUAL

2.1 INTRODUCTION

In this chapter we will review briefly some of the studies on life cycle allocation theory. The main object of the first part of this thesis is to build a model of life cycle consumption and work decisions under life uncertainty and to study the implications of life expectancy and other parametric changes for life cycle allocation behaviour. Therefore this review is limited basically to the literature dealing with the treatment of life uncertainty in life cycle models and the study of life expectancy improvements.

The literature on life cycle models under life uncertainty is mainly in the context of a consumption profile decision with a given income stream over the life cycle. We plan to extend Yaari's (1965) uncertainty model to include work-leisure choice as a part of the life cycle problem. Therefore we will also review some of the literature dealing with the life cycle work allocation decision, though this literature is mainly in the context of certain lifetimes.

2.2 LIFE CYCLE MODELS OF CONSUMPTION ALLOCATION UNDER LIFE UNCERTAINTY

The fact that a consumer who is to make a choice between present and future consumption is uncertain about the length of his/her life, has long been recognized. At least both Marshall (1920) and Fisher (1930) acknowledged this fact in their discussion of the lifetime allocation problem of a consumer. But it was only after Yaari (1965) that an explicit treatment of this aspect was given by systematically introducing uncertainty of life in a standard life cycle model of a consumer.

Yaari (1965) studied the effect of life uncertainty on the life cycle consumption plan of a risk averse individual under four different circumstances, depending on the presence or absence of actuarially fair life insurance and annuities and of a bequest motive. His conclusions are as follows. In the absence of a bequest motive and life insurance, the effect of life uncertainty is to increase the rate of preference for present over future consumption. This type of behaviour for a consumer facing an uncertain life, had also been predicted earlier by Fisher (1930).

The fact that uncertainty of life increases the rate of preference for present over future consumption does not imply, however, that under uncertainty, the absolute level of consumption will be higher at early periods of life as compared to that under certainty. This is the basic theme of Levhari and Mirman (1977) which has also been picked up by Davies (1981) in his effort to explain the observed lack of decumulation by the elderly. We will discuss these two studies later in this chapter

Yaari showed that in the absence of a bequest motive, the effect of actuarially fair life insurance and annuities is to remove the impatience resulting from uncertainty of life.[1] However, the consumption plan does not, in general, coincide with the one chosen under certain lifetimes because of the difference in the budget constraints across the two situations.

For a consumer having a bequest motive, uncertainty of life in the absence of life insurance and annuities will increase (decrease) impatience if the marginal utility of consumption exceeds (falls short of) the marginal utility of bequests. This ambiguity had also been recognized by Fisher (1930), though he did not give the precise marginal conditions which determine the effect of life uncertainty on the degree of impatience.

In the final case considered by Yaari a bequest motive is present and actuarially fair life insurance and an-

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1 In next chapter we will discuss in detail, in what sense life insurance and annuities are actuarially fair.

nuities are available. Yaari showed that life insurance enables the consumer to separate the consumption decision from the bequest decision in the sense that saving and consumption plans affect each other globally, but not locally. Finally, Yaari showed that, if a choice is available, the consumer will opt for life insurance and annuities in the desired amount rather than going without them.

Following Yaari, there has been an increased interest in the literature in the study of various aspects of life uncertainty in life cycle models. Champernowne (1969), Davies (1981), Hurd (1986), Levhari and Mirman (1977), Miner (1987) and Zilcha and Friedman (1983, 1985) studied the effect of life uncertainty on the life cycle consumption pattern in the absence of life insurance and annuities.

Champernowne (1969) found that under these circumstances the effect of uncertainty is to reduce the level of consumption in the initial period of life. This result seems to contradict Yaari's (1965) assertion that under the circumstances considered, uncertainty of life increases consumer's preference for present over the future consumption. But, as Levhari and Mirman (1977) pointed out, there is, in fact, no contradiction because it does not follow from Yaari's result that uncertainty of life necessarily results in higher consumption during the early years of life. They argued that, if the risk averse consumer with uncertain life lives on the average for the same number of periods as the consumer with certain life, his/her desire to avoid risk by consuming more in the present and the desire to provide for a possible longer life, pull in opposite directions. The net effect of life uncertainty on consumption in the initial periods of life is uncertain.[2] According to Levhari and Mirman, Champernowne's result is not general as it depends on his specific parametric assumptions. Using the iso-elastic class of utility functions: $u(c) = c^{1-\gamma}/1-\gamma$, $\gamma > 0$, Levhari and Mirman concluded that the effect of uncertainty on consumption in the initial period of life depends, both in sign and magnitude, on the magnitudes of the coefficient of relative risk aversion, the rate of interest and the subjective rate of time preference (other than the one resulting from life uncertainty).

The assumption that life insurance is not available restricts the consumer to consume no more than his/her current wealth in any period of life. If this constraint is

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2 Instead of comparing the case of uncertainty with the case of perfect certainty, Levhari and Mirman studied the effect of increasing risk in the distribution of life on the consumption pattern of a consumer already facing an uncertain lifetime, with the assumption that the expected length of life does not change when the distribution of life becomes more risky. In chapter 4 we will adopt the concept of a mean preserving spread introduced by Rothschild and Stiglitz (1970) and also used by Levhari and Mirman (1977), to characterize increased riskiness.

binding in the initial period, an increase in uncertainty of life can in no circumstances result in higher consumption in that period. Levhari and Mirman's consumer, however, is not bound to this constraint because he/she is endowed with all of his/her lifetime resources at the time of birth.

Davies (1981) studied the effect of uncertainty on planned consumption at different ages. He examined whether the continued accumulation or mild decumulation observed among many elderly consumers can be explained by the uncertainty of life in the absence of life insurance and annuities.[3] Using the iso-elastic class of utility functions, $u(c) = c^{1-r}/1-r$, $r > 0$, Davies derived consumption paths both under certain and uncertain lifetimes and analyzed the age pattern of the ratio of consumption under uncertainty to that under certainty. His simulation results show that the age pattern of this ratio depends on the magnitudes of rate of interest, subjective time preference rate and the coefficient of relative risk aversion, r. However, Davies showed that, if r is large (greater than 3) then for a wide range

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3 The empirical evidence provided by many studies suggests that contrary to the traditional life cycle theory, retired households continue accumulating wealth. See, for example, Atkinson (1971), Hamermesh (1984b), Kurz (1984), Lydall (1955), Menchik and David (1983), Mirer (1979) and Shorrocks (1975). On the other hand, there are also some studies which suggest the opposite (Diamond and Hausman (1984), Hurd (1986, 1987) and King and Dicks-Mireaux (198[illegible]).

of the plausible values of interest rate and subjective time preference rate, uncertainty of life depresses consumption by a proportion increasing with age. He argued that the available empirical evidence suggests that the value of Γ is large enough to produce this result. Davies concluded from this result that uncertainty of life explains much of the lack of decumulation by the elderly.

Another explanation of this phenomenon is provided by Zilcha and Friedman (1983). They abandoned the assumption of a fixed utility function and a fixed probability distribution of life and analyzed the consumption and savings behaviour of an individual who is about to retire. They suggested that people replan their future consumption as they age in the light of changing tastes and subjective life distribution. Thus people may continue to accumulate as they age either because they become more optimistic about life expectancy or because they become more risk averse.

Like Davies (1981), Zilcha and Friedman also used the iso-elastic utility function, $u(c) = c^{1-\Gamma}/1-\Gamma$, $\Gamma > 0$. Their analysis shows that if the optimal consumption-age profile slopes downward, an increase in the coefficient of relative risk aversion, Γ, will flatten the consumption-age profile and therefore result in less consumption during some early years of planning horizon. This may explain the continued accumulation of wealth by the elderly. To the best of our knowledge there is no empirical study which suggests anything about the behaviour of a consumer's risk aversion over the life cycle. However, the other explanation for the continued accumulation by the elderly, provided by Zilcha and Friedman, seems to be compatible with the empirical facts. Life expectancy in many countries has in fact substantially increased during the last few decades (See, for example, Hamermesh (1985) and Fries (1980)). There is also some empirical evidence that people are aware of the current changes in the life tables and they incorporate this information in determining their subjective life horizon (Hamermesh (1985)). Therefore, as Zilcha and Friedman (1983) suggest, continued accumulation by the people as they age may be due to an increase in their expected horizons which may not have been anticipated at an earlier age.

In a subsequent study, Zilcha and Friedman (1985) used a concave utility function without specifying its parameters. Measuring an increase in risk aversion by a concave transformation of the original utility function, they reached the same conclusion as in their original study.

In a recent study, Mirer (1987) argued that the mere fact that lifetimes are uncertain and life insurance and annuities are not available, may explain the lack of dissavings in retirement. Since the conditional probability of death is expected to increase with age, the rate of discount applied to utility at a later age will be more than the one applied to utility of an earlier age. Therefore, if rate of interest is not too high to dominate the effect of discounting, at a certain point in life cycle when the conditional probability of survival has sufficiently increased, consumption will start decreasing with age. This may result in accumulation of assets by a retired individual.

Hurd (1986) discussed the life cycle allocation behaviour of a retired consumer under uncertain lifetimes. He showed that in the absence of annuities (or pensions, etc.) and a bequest motive, if the subjective rate of impatience exceeds the rate of interest, both consumption and wealth will decrease with age. If wealth were ever to increase, it would always increase due to declining consumption. But Hurd points out, this contradicts rational consumer behaviour requiring that all wealth be consumed by the maximum possible age if the individual survives. The subjective rate of impatience is the sum of the subjective discount rate (other than the one resulting from life uncertainty) and the instantaneous probability of death. Since the latter increases with age (approaching towards 1 at the maximum possible age), the subjective rate of impatience will eventually exceed the rate of interest at some age and remain so after that. Hurd concluded therefore that consumption and wealth will eventually decline with age no matter how high is the rate of interest.

Hurd then introduced a bequest motive in his model and showed that it causes the consumption trajectory to flatten. Since the consumer holds additional wealth for bequests, consumption in the initial periods will be smaller. Therefore, wealth will decline slower than without a bequest motive.

Barro and Friedman (1977) discussed the welfare aspects of life uncertainty under the assumption that actuarially fair life insurance and annuities are available. They considered, for comparison, the optimal lifetime utility in two alternative situations. In one situation life is uncertain with a known distribution and actuarially fair life insurance and annuities are available. In the other situation the length of life is randomly selected from the same distribution and announced to the consumer at the beginning of planning horizon. Barro and Friedman showed that, if the consumer is allowed to choose between the two alternatives, he/she will opt for not knowing the length of life. The reasoning goes as follows. The consumer with certain life does not escape the gamble because before the length of life is randomly selected, he/she is uncertain both about the time of death and the consumption plan he/she will choose after the knowledge of lifetime. The consumer with uncertain lifetime, on the other hand, does escape the part of gamble associated with life uncertainty. The consumer's ability to buy insurance allows him/her to pool the consequent monetary risk of uncertain death with others, although he/she is still uncertain about the length of life.

2.3 IMPLICATIONS OF IMPROVED LIFE EXPECTANCY IN A LIFE CYCLE MODEL

Another area of interest in the literature on life cycle models is the study of life expectancy improvements. Life expectancy has significantly increased during last fifty years in many countries.[4] There is also some evidence that people are aware of the current improvements in life expectancy. In a recent study, Hamermesh (1985) presents evidence from a survey of 650 Economists and 975 other people in the United States and suggests: "Most important, I find that people do extrapolate changing life tables when they determine their subjective horizons, and they are aware of levels of and improvements within current life tables."

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4 Life expectancy figures quoted in Hamermesh (1985) show that expected life in the United States and many European countries has substantially increased in the last few decades. Between 1939-41 and 1979-81 the expected life for 25 years old United States white males and females increased by 4.6 and 7.9 years respectively. The comparable figures for the United States nonwhites are 7.6 and 13,1 years respectively.

Hamermesh (1984) discussed the effect of an increase in longevity on consumption and labour supply in a life cycle model with certain lifetime. He concluded that, with an increase in life expectancy, consumption will decrease and work hours will increase in all the periods of life. Nalebuff and Zeckhauser (1984) further showed that under a certain life span, an increase in longevity will induce later retirement.

Both studies quoted above assumed that lifetime is certain. Therefore, the increase in longevity in their analyses is caused by adding more years to a given horizon. However, the observed improvements in life expectancy do not appear to be interpretable as being associated with an increased horizon. Rather, they are the result of more people surviving to the later part of a given horizon (Fries (1980)). Therefore a more realistic approach to study the increase in longevity would be to take life as uncertain and allow the survival probabilities to increase with a given upper bound on life (Skinner (1985)).

Abel (1986), Sheshinski and Weiss (1981) and Skinner (1985) adopt this approach in their analysis of the effect of an increase in longevity on the consumption allocation decision. All of them considered a two period model. Life is assumed to be certain in the first period and uncertain in the second. Wage income is assumed to be fixed and the consumer is assumed to have a bequest motive.

Sheshinski and Weiss (1981) also assume that in the second period of life the consumer receives no earnings. Actuarially fair annuities can be purchased in any amount through a social security scheme. In addition, the consumer is also able to carry his/her resources to the second period through a riskless bond. A simplified version of Yaari's uncertainty model is considered. The consumer is assumed to live a fraction θ of the second period which has a length equal to that of the first period. The fraction θ is assumed to be a random variable distributed over the range $0 \leq \theta \leq 1$ around mean $\bar{\theta}$, $0 < \bar{\theta} < 1$. Sheshinski and Weiss showed that an increase in the expected life $1+\bar{\theta}$ will result in a decrease in the flow of lifetime consumption.

In Abel's (1986) model the consumer also retires from work at the end of the first period. However, if the consumer survives in the second period, he/she receives a social security benefit in fixed amount for which he/she has paid a fixed social security tax in the first period. Actuarially fair life insurance and annuities as well as riskless bonds are also available. Abel showed that the effect of an increase in the probability of survival to the second period is to decrease (increase) consumption in both the periods and the level of bequests if consumption in the second period is more (less) than the social security income.

This result differs from the one found by Sheshinski and Weiss (1981) because of their different assumptions regarding the life cycle pattern of income. In that model the consumer earns no income in the second period of life. Therefore the consumer saves in the first period in order to provide for consumption in the second period. If life expectancy in the second period increases, the actuarially fair returns on social security savings will decrease and therefore the consumer will have to cut down lifetime consumption.[5] In Abel's model the consumer earns a fixed income in the second period of life through the mandatory social security scheme. If the planned consumption in the second period is more than the social security income, the consumer must have invested in annuities in the first period (see footnote above). In this case an increase in life expectancy will reduce actuarially fair return on annuities. This will force the consumer to reduce consumption in both the periods and the level of bequests. This is the same

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5 Sheshinski and Weiss also showed that in the presence of riskless bonds and actuarially fair annuities, bonds are reserved for bequests, while annuities are used exclusively for consumption during retirement. A similar result is obtained by Abel (1986).

result as found by Sheshinski and Weiss (1981). If, on the other hand, planned consumption in the second period is less than social security income, the consumer must have borrowed at the actuarially fair rate of interest in the first period. Now if the probability of survival into the second period increases, the actuarially fair interest cost on borrowings will decrease, thus permitting more consumption in both the periods of life and more bequests.

Finally, Skinner (1985) showed that in the absence of life insurance and annuities, the effect on consumption of an increase in the probability of survival into the second period is ambiguous. This ambiguity arises because in response to a better life expectancy the two motives for saving, namely the bequest motive and the life cycle motive work in opposite directions. With a better life expectancy, the probability of leaving bequests at an early age declines. Therefore there is less incentive to build up assets while young. As a result consumption in the first period will increase. The life cycle motive, on the other hand, calls for more savings and less consumption in the first period in order to provide for a longer expected life. The net effect of the two motives on consumption in the first period is uncertain.

Skinner then introduced in his model, life insurance as an additional method of providing for bequests. This life insurance policy is different from the one considered in the conventional models, like Yaari (1965), Sheshinski and Weiss (1981) and Abel (1986). Life insurance in Skinner's model guarantees bequests to the consumer's heirs in the agreed amount in the event of consumer's death at the end of first period. Since with an increase in the probability of survival into the second period the consumer can purchase the given amount of life insurance for less, he/she can spend more on consumption. Therefore an increase in the survival probability in this case will result in higher consumption in both periods.

Arthur (1981) and Katz (1979) studied the welfare implications of improvements in life expectancy. Using a two period model, Katz suggested that in the presence of actuarially fair life insurance and annuities, a consumer may be worse off by an increase in probability of survival into the future. But, as later pointed out by Pelzman and Rousslang (1982), Katz overlooked the implications of risk aversion in the utility function for his mathematical results. It turns out that a risk averse consumer will always be better off by an increase in probability of survival into the future. Arthur (1981) considered the same effect in his general equilibrium model. If we ignore the general equilibrium aspects of his model, his analysis also shows that an isolated risk averse consumer is better off with an increase in probability of survival to any age.

2.4 TREATMENT OF WORK-LEISURE CHOICE IN LIFE CYCLE MODELS

Life cycle models have also been used to study the lifetime work-leisure decision. An extreme approach frequently adopted is to assume that work-leisure choice is discrete in the sense that workers can choose only the age of retirement with fixed hours of work in each period before retirement (Burbidge and Robb (1980), Crawford and Lilien (1981), Hurd and Boskin (1984), Mitchell and Fields (1983), Nalebuff and Zeckhauser (1984), Sheshinski (1978)).

Crawford and Lilien (1981) also considered life uncertainty in analyzing the effect of social security on the retirement decision. They assumed that social security is actuarially fair and that an actuarially fair life insurance and annuities are not available in the private market. They also assumed that the social security contribution is less than the individual would choose if free to do so. The rate of return available on investment in social security, which is actuarially fair, is higher than the one on investment in bonds through a private market. Therefore, the provision of an increase in the social security contributions will increase the feasible consumption plan for the given retirement age, and will therefore create an income effect to induce earlier retirement. The increase in social security contributions will also create a substitution effect by reducing the marginal cost of consumption in terms of the leisure foregone and thus induce more consumption and less leisure. This substitution effect will encourage later retirement. Crawford and Lilien conclude that the net effect of an increase in social security contribution on the retirement decision is uncertain.

Life cycle models which discuss retirement behaviour assume that workers have only a limited choice. They can either work full time or retire from work. On the other hand, a large body of theoretical as well as empirical research on life cycle allocation problem is based on completely flexible work-leisure choice (Browning, Deaton and Irish (1985), Ghez and Becker (1975), Heckman (1974, 1976), Hurd (1971), King (1983), MaCurdy (1981, 1983, 1985), McCabe (1983), Moffitt (1985). No study on these lines could be found which would also discuss the implications of life uncertainty for the life cycle allocation of work hours. However, in the light of present research programme, it will be worthwhile to discuss the basic findings of these studies, in particular the ones by Browning, Deaton and Irish (1985), Heckman (1974), Ghez and Becker (1975) and MaCurdy (1981).

There are two main contributions of Heckman (1974) and Ghez and Becker (1975). First, the response of work hours to anticipated variations in the wage rate over the life cycle is unambiguous in sign, unlike the comparative static effect of a change in the wage rate on work hours in a one period work-leisure choice model. Other things being constant, work hours will vary in the same direction as the wage rate over the life cycle. This relationship has been empirically supported for the U.S.A. by Ghez and Becker (1975) and MaCurdy (1981, 1983, 1985) and for the U.K. by Browning, Deaton and Irish (1984). A survey of many other empirical studies in Killingsworth (1983) also seems to support this finding.

The other major contribution of Heckman (1974) and Ghez and Becker (1975) is that, unlike in the conventional life cycle models, consumption is not independent of current wage income unless the marginal utility of consumption is independent of leisure. The observed positive relationship between consumption and current income can be explained within the context of a life cycle model in which the wage rate is not constant over the life cycle and work hours can be varied. If consumption and leisure are substitutes for each other in the sense that the marginal utility of consumption increases with a reduction in leisure, then an anticipated increase in the wage rate will result in increased work hours and increased consumption.[6]

MaCurdy (1981) further exposed the behaviour of work hours in response to different types of changes in the wage rates. He distinguished among three different types of responses in work hours to changes in wage rates. First, the intertemporal substitution effect which explains the effect of anticipated or evolutionary changes in the wage rate over the life cycle on the life cycle pattern of work hours. Second, the uncompensated wage effect which captures the effect of a parametric change in wage rate on work hours. This effect shows the difference in work hours across individuals with similar preferences and wealth but different wage profiles. Finally, the compensated wage effect which shows the effect of a parametric change in the wage rate on work hours, holding lifetime utility constant. MaCurdy showed that the effect of an anticipated change in wage rate on work hours at any age is equivalent to the effect of a

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6 Prior to Heckman's result, Thurow (1969) attributed the observed positive relationship between consumption and current income to the credit market constraints which prevent consumers from borrowing against future income. Nagatani (1972) also found this relationship and attributed it to the uncertainty of future income. In a relatively recent study, however, Browning, Deaton and Irish (1985) found both substitutability and complemenarity relations between consumption and leisure in different estimations for the U.K. data.

parametric wage change at that age, holding constant the marginal utility of wealth.

Browning, Deaton and Irish (1985) showed that if prices of all the goods over the entire planning horizon are fully anticipated, the life cycle problem of an individual can be studied in the framework of a competitive profit maximizing firm. The individual can be viewed as maximizing profits for each period in the life cycle from the sales of utility (to himself/herself). The price of utility is the reciprocal of the marginal utility of wealth. The costs comprise of expenditure on purchase of goods. Since an anticipated change in wage rate is equivalent to a parametric change, holding constant the marginal utility of wealth, this change will not affect the price of utility (which is equal to the reciprocal of the marginal utility of wealth). Therefore the effect of an anticipated change in wage rate on leisure or consumption of a good is identical to the comparative static effect of a change in wage rate for a price taker individual who maximizes profits from the sales of utility in each period. Browning, Deaton and Irish showed that the life cycle demand system derived in this manner satisfies all the properties of the usual profit maximizing demand system. This approach is quite useful for empirical studies because it suppresses all the future information into a single parameter: the price of utility.

The empirical evidence by Dickens and Lundberg (1985), Gordon and Blinder (1980) and Gustman and Steinmeier (1982a, 1982b, 1984a, 1984b, 1985a, 1985b, 1986) suggests that neither of the two approaches, namely the discrete retirement choice and the fully flexible work-leisure choice models, is completely tenable. Gustman and Steinmeier concluded that retirement is not a discrete process. It involves a transition to partial retirement before full retirement. In many cases, partial retirement requires a job change and a wage reduction. This result was also found by Gordon and Blinder (1980). They concluded that a wage reduction may induce many workers to retire. Gustman and Steinmeier (1982a) provided evidence on hours restrictions on the jobs from the Michigan Panel Study of Income Dynamics, 1971-1975, suggesting that 56% of the employees face a minimum hours constraint on their jobs. They also quoted another survey of 267 firms, conducted by the Bureau of National Affairs in 1979, which shows that 15% of the firms allow their older employees to gradually reduce work hours and phase into partial retirement. Reid's (1986a) recent experiments with Canadian data show that 25% of workers want to reduce their work time with a proportional reduction in pay. He reached similar conclusions in another study for Canada (Reid (1986b)). On the other hand, using a sample of 555 employees from the Denver Income Maintenance

Experiment, 1972, Dickens and Lundberg (1985) concluded that a large proportion of the sample is working less than the desired number of hours.

2.5 CONCLUDING REMARKS

As our discussion in this chapter shows, the literature on life cycle allocation behaviour under uncertain lifetimes is mostly based on the assumption that the individual receives a fixed income stream over the life cycle, over which he/she has no control. On the other hand, there is a comprehensive literature on the treatment of work-leisure choice in the life cycle context, though it has been limited by the assumption that life is certain (except Crawford and Lilien's (1981) retirement model). One aspect of the present research is to develop a life cycle model under uncertain lifetimes in which both consumption and work hours are chosen by the individual.

We have also learned that there is a controversy in the literature regarding the extent of variability of work hours. Many studies allow only a discrete choice in which the individual can choose only the age of retirement with fixed hours of work in each period before retirement. On the other hand, there are also many studies which assume that the individual is free to choose hours of work in each period of life. But these studies do not consider retirement as a possible outcome of the life cycle work-leisure decision. Alternatively, these studies can be viewed as a model for the pre-retirement period of life. The empirical evidence seems to suggest that neither of these two extreme approaches is realistic. However, the institutional barriers which limit the choice of an individual, vary from one empirical situation to another. One cannot put forward a theoretical model which fits all the empirical situations.

In the present research we have chosen to consider a perfectly flexible work-leisure choice model. This type of model has already been developed by Heckman (1974), Ghez and Becker (1975), MaCurdy (1981) and Browning, Deaton and Irish (1985) under certain lifetimes. These models explain the life cycle work and consumption decisions with a direct utility maximizing approach and do not distinguish between the intertemporal and intratemporal effects of changes in wage rates. In the present research we plan to use a two stage budgeting approach which is expected to provide a clearer picture of life cycle behaviour. In addition, this approach offers a more explicit interpretation of the effects of anticipated and comparative static changes in various parameters, like survival probabilities, wage rates, and interest rate. The two stage budgeting approach provides a break down of these effects into intertemporal and intratemporal components.

Earlier studies on life cycle savings pattern are usually based on the assumption of a fixed income stream. The studies on life cycle consumption-work allocation decision, on the other hand, tend to ignore the analysis of savings and asset holding. We plan to study the savings-age relation in the consumption-work choice model under life uncertainty with actuarially fair life insurance and annuities. This study is expected to throw some more light on the relationship of consumption with earnings over the life cycle. We also plan to spell out the dynamic process of asset holding in the presence of actuarially fair life insurance and annuities.

We will adopt a continuous time framework which is analytically easier than a discrete model to study retirement as a possible outcome of the life cycle decision. This extended framework is expected to provide some information on the distinct consumption and savings patterns of an individual during working and retirement periods of life. It may explain the relationship of asset holding during retirement with the pre-retirement pattern of earnings and consumption.

With the proposed life cycle model we hope to obtain more general results than available in the existing literature, regarding the effects of changes in survival probabilities to various segments of life on consumption, work hours, retirement age and savings pattern.

This completes our discussion of the earlier studies on life cycle allocation behaviour. Now we formally proceed to construct a life cycle model of consumption and work allocation under uncertain lifetimes.

CHAPTER 3

CONSUMPTION, WORK HOURS, SAVINGS AND ASSET HOLDING IN LIFE CYCLE UNDER UNCERTAIN LIFETIMES WITH ACTUARIALLY FAIR LIFE INSURANCE AND ANNUITIES

3.1 INTRODUCTION

In this chapter we will develop a life cycle model of consumption and work hours under uncertain lifetimes in the presence of actuarially fair life insurance and annuities. A life cycle model of consumption and work hours under certainty has been presented by Heckman (1974) in a continuous time framework. We make use of this continuous time framework in our analysis.

As is well known from Yaari's (1965) model, the existence of life insurance and annuities eliminates the effect of life uncertainty on the consumer's preference for present over the future consumption. That is, the differential equation determining the age path of consumption looks similar to that under certainty, though the consumption plans are not identical in the two cases. This suggests that the differential equations determining the age paths of consumption and work hours in our model are likely to look similar to the ones obtained by Heckman (1974). In this sense our model is similar to Heckman's. But there are many aspects in which our analysis differs from Heckman's. First, unlike Heckman, we also allow retirement as a possible outcome of the life-cycle work allocation decision. This provision is expected to produce some additional results regarding the life cycle consumption, work and savings pattern of an individual. Second, we adopt a two stage budgeting approach in order to describe the characteristics of the life cycle solution. This allows an easier understanding of the model by giving new interpretations to the effects of various anticipated or parametric changes. Finally, the analysis is not limited to the study of life cycle consumption and work allocation decision. The life cycle pattern of savings and asset holding is also fully spelled out. This study of life cycle savings pattern provides some additional information on the relationships among consumption, earnings and assets accumulation.

3.2 ASSUMPTIONS AND FRAMEWORK OF THE ANALYSIS

In this section we will present the framework of the model which is standard in life cycle theory under uncertain lifetimes (See, for example, Arthur (1981), Barro and Friedman (1977) and Yaari (1965)). We start with the description of the distribution of life faced by the individual.

3.2.1 The Modelling of Life Uncertainty

Consider an individual who is assumed to live no more than T periods. The individual does not know the exact time of death, but has perfect knowledge of the probability distribution of life. This assertion can be supported by assuming that the individual lives in a sizable population of biologically identical members and therefore can calculate the distribution of life from the known vital statistics. As we have learned in chapter 2, there is some evidence that people are aware of the improvements in current life tables (Hamermesh (1985)).

Following the framework adopted by Yaari (1965), we now discuss the characteristics of the distribution of life. Let $q(x)$ be the unconditional probability of death at age x, perceived at age 0. Obviously, the density function $q(x)$ must satisfy the conditions:[1]

$$0 \leq q(x) < 1 \quad \text{for all } x, \text{ with } q(T) > 0 \tag{3.2.1}$$

$$\int_0^T q(x)\, dx = 1 \tag{3.2.2}$$

The probability of survival through age x, perceived at age 0, $p(x)$, can be calculated as follows.

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1 If T is the upper limit on the possible life horizon, the probability of death at age T, $q(T)$, must be non-zero.

$$p(x) = \int_x^T q(y)\,dy = 1 - \int_0^x q(y)\,dy \qquad (3.2.3)$$

It follows immediately that $0 < p(x) \leq 1$ for all $x < T$, $p(0) = 1$ and $p(T) = 0$. It also follows that

$$\dot{p}(x) = -q(x) \leq 0 \qquad (3.2.4)$$

That is, the probability of survival is non-increasing in age. The age paths of survival and death probabilities are illustrated in figure .3.2.1.[2]

Finally, we can also calculate the conditional probabilities as follows. The conditional probability of death at age x, given age y ($y \leq x$) and the conditional probability of survival through age x, given age y ($y \leq x$) are $q(x)/p(y)$ and $p(x)/p(y)$ respectively.

3.2.2 Feasibility Constraints With Life Insurance and Annuities

At each point of time during life, the individual receives a fixed endowment of leisure at the rate of θ per period which can be freely allocated between work and

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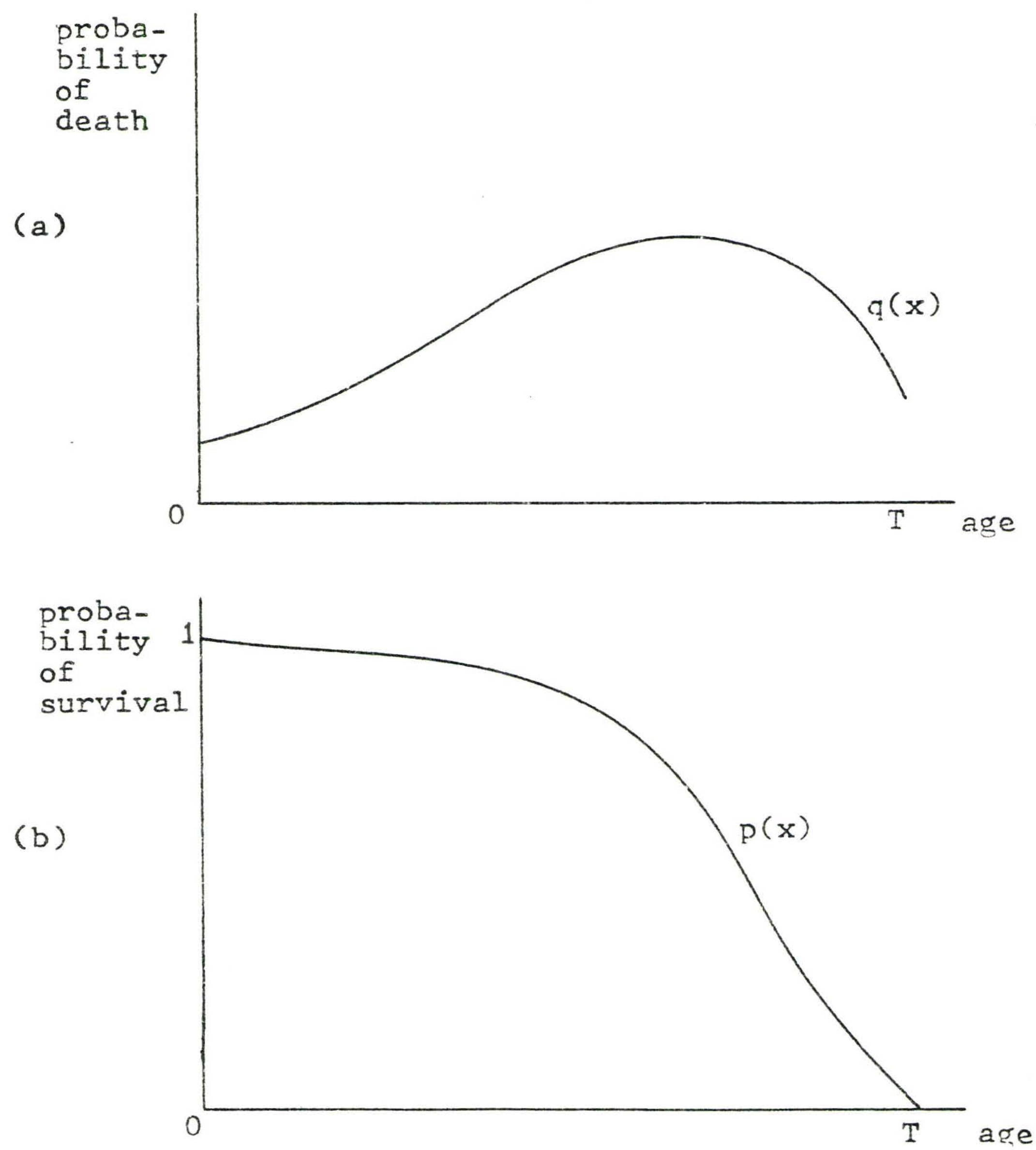
2 For the shapes of the schedules p(x) and q(x) see, for example, Davies (1981), Fries (1980) and Hamermesh (1985).

Figure 3.2.1 The Probability Distribution of Life and the Shape of Survival Schedule



retained leisure. In order to concentrate on the uncertainty of life, it is assumed that there is no other element of uncertainty present in the model. The rate of interest on riskless bonds is assumed to be fixed over time and known to the individual. The wage rate is also assumed to be known with certainty, although it may vary with age. The age pattern of wage rates will be discussed later in this chapter. The price of a single composite consumption good, assumed to be fixed and known to the individual, is normalized at 1.

We shall assume that the individual does not receive any transfer income in the form of inheritance, gifts, etc. and has no motive to leave bequests or to offer any other form of transfers to others. It is assumed that social security, pensions, health insurance and the like do not exist.

Finally, it is assumed that actuarially fair life insurance and annuities are available. Following Yaari (1965), we introduce life insurance and annuities by assuming that the individual can buy or sell actuarial notes with an insurance company. It is assumed that the insurance companies have perfect knowledge of the distribution of life and operate in a perfectly competitive market. The buyer of an actuarial note is a lender and receives an instantaneous return on it at each age so long as he/she keeps it. If he/she dies without redeeming the note, the insurance company is held free of any obligations. Likewise, the seller of a note is a borrower and pays an instantaneous interest on it to the insurance company at each age. If he/she dies as a borrower, the insurance company has no claim on his/her estate.

Perfect competition among insurance companies and the individuals' ability to buy or sell actuarial notes at any point of time, eliminate the possibility of expected profits or loss to the insurance companies on any actuarial note for any period of time. If an individual aged x buys a dollar worth of actuarial notes for a small period of time ϵ, then the expected profits of the insurance company from this transaction are zero if the expected value of principal and interest payments to the individual at age $x+\epsilon$, by the insurance company is equal to the certain revenue to the insurance company from the investment of that dollar in riskless bonds.[3] A similar condition should hold if the individual sells a dollar worth of actuarial notes.

For an individual aged x, the probability of survival to age $x+\epsilon$ is $p(x+\epsilon)/p(x)$. Therefore, if $r^*(x)$ and r are the instantaneous rates of return on actuarial notes and

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3 To avoid complications it is assumed that costs to run insurance industry are zero.

riskless bonds respectively, the condition of zero expected profits can be written as follows.

$$\lim_{\epsilon \to 0} [\{p(x+\epsilon)/p(x)\}\{1+\epsilon r^*(x)\} + \{1-p(x+\epsilon)/p(x)\}0 = 1 + \epsilon r]$$

This condition can also be interpreted as the requirement that rate of interest on actuarial notes is fair (Yaari (1965)). Writing $p(x+\epsilon)/p(x) = 1 + [p(x+\epsilon)-p(x)]/p(x)$, dividing through by ϵ, simplifying and taking the limit, the above equation implies

$$r^*(x) = r - \dot{p}(x)/p(x) \qquad (3.2.5)$$

Or, using (3.2.4)

$$r^*(x) = r + q(x)/p(x) \qquad (3.2.6)$$

That is, the actuarially fair rate of interest exceeds the rate on riskless bonds by the instantaneous conditional probability of death.

Yaari (1965) has shown that, if a risk averse individual is free to choose, he/she will opt for life insurance and annuities in the desired amount rather than bonds at rate r. Therefore our individual will hold life insurance and annuities to realize all intertemporal transactions.[4]

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4 Later in this chapter we will show how the individual is better off by the availability of actuarially fair life insurance and annuities.

Under these circumstances, the lifetime budget constraint facing the individual can be written as follows.

$$\int_0^T \exp\left[-\int_0^x r^*(y)\ dy\right] \{w(x)h(x) - c(x)\}\ dx \geq 0 \qquad (3.2.7)$$

where, h(x), c(x) and w(x) are the age specific rates of work hours, consumption and real wage respectively.

This formulation of the lifetime budget constraint is due to Yaari (1965). However, we can simplify it quite a bit. Using equation (3.2.5), the discount factor attached with the budget constraint can be written as

$$\exp\left[-\int_0^x r^*(y)\ dy\right] = \exp\left[-\int_0^x r\ dy + \int_0^x [\dot{p}(y)/p(y)]\ dy\right]$$

$$= \exp\left[-\int_0^x r\ dy\right] \exp\left[\int_0^x [d\log p(y)/dy]\ dy\right]$$

$$= \exp(-rx)\exp[\log p(x) - \log p(0)]$$

Since p(0) = 1 and therefore log p(0) = 0, the discount factor simplifies as

$$\exp\left[-\int_0^x r^*(y)\ dy\right] = e^{-rx}\ p(x) \qquad (3.2.8)$$

Substituting this result in (3.2.7), the lifetime budget constraint can be written as in Arthur (1981):

$$\int_0^T e^{-rx}\ p(x)\ [w(x)h(x)\ -\ c(x)]\ dx \geq 0 \tag{3.2.9}$$

The budget constraint written in this form states that, discounted by the rate of interest on riskless bonds, the present value of the stream of expected consumption should not exceed the present value of the stream of expected labour income.

There is another form in which we can express the budget constraint, as in Barro and Friedman (1977). Substituting from equation (3.2.3),

$$p(x) = \int_x^T q(y)\ dy,$$

into equation (3.2.9) and changing the order of integration we can write the budget constraint as follows.

$$\int_0^T q(x) \int_0^x e^{-ry}\ [w(y)h(y)\ -\ c(y)]\ dy\ dx \geq 0 \tag{3.2.10}$$

That is, discounted by the rate of interest on riskless bonds, the expected value of the discounted lifetime consumption should not exceed the expected value of the discounted lifetime income. Thus, due to risk sharing on an actuarially fair basis, the individual is required to satisfy the lifetime budget constraint only on average.

For our further analysis, we will use the simple form of the budget constraint given by equation (3.2.9). In addition to this constraint, the individual is also restricted to meet the following boundary conditions:

$$c(x) \geq 0, \quad 0 \leq h(x) \leq \theta \tag{3.2.11}$$

Any consumption-work plan is feasible if it is within the limits set by the budget constraint (3.2.9) and the boundary conditions (3.2.11).

3.2.3 The Lifetime Utility Function

The utility function defined over the rates of consumption and leisure over the entire life is assumed to be additively separable among sub-utility functions defined over the rates of consumption and leisure for each age. Without additive separability assumption marginal utilities at one age are related to the goods consumed at other ages. There are many possible patterns of inter-period related marginal utilities which one may like to study and, therefore, the analysis could end up with many possible patterns of the life cycle allocation behaviour. Thus, the additive separability assumption seems to be essentially indispensable for a positive outcome of the analysis. Besides, the general utility function also do not seem to be a suitable choice for modelling life uncertainty in the allocation problem.

Let, for an individual aged x, the utility rate associated with the rate of consumption $c(x)$ and the rate of retained leisure $\theta-h(x)$, be described by the function $U(c(x),\ \theta-h(x))$. Tastes are assumed to be independent of age. In addition the utility function is assumed to satisfy the following properties.[5]

Assumptions

1. $U(x)$ is continuous in $c(x)$ and $\theta-h(x)$ and has continuous first and second partial derivatives.
2. $U(x)$ is strictly increasing in $c(x)$ and $\theta-h(x)$, that is, $U_1(x)$ and $U_2(x)$ are positive.
3. $U_1(x)\big|_{c(x)=0} = U_2(x)\big|_{\theta-h(x)=0} = \infty$
4. $U(x)$ is strictly concave. Therefore $U_{11}(x) < 0,\ U_{22}(x) < 0,\ U_{11}(x)\ U_{22}(x) - U_{12}(x)^2 > 0.$

Assumption 1 enables us to make use of the tools of differential calculus in our analysis. Assumption 2 means

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5 Henceforth we will use the abbreviated notation $U(x)$ for the utility function $U(c(x),\ \theta-h(x))$. The first partial derivatives with respect to consumption and leisure will be denoted by $U_1(x)$ and $U_2(x)$ respectively. Similarly the conventional notation $U_{ij}(x)$ will be used for the second partial derivatives.

that the individual is never saturated with consumption or leisure. Assumption 3 implies that the individual cannot survive without consumption or leisure. In our optimization analysis this assumption will be useful to allow us to ignore the boundary conditions: $c(x) \geq 0$, $h(x) \leq \theta$. Finally, the assumption of strict concavity of the utility function allows application of nonlinear optimization techniques to obtain a well behaved solution to the life cycle allocation problem.[6]

The lifetime utility function can be obtained by integrating the age related utility functions over the entire life. Since the length of life is uncertain, we must consider the expected utility as the objective function of the individual. Thus, for an individual aged 0, the expected lifetime utility function can be written as:

$$\Omega = \int_0^T q(x) \int_0^x U(c(y), \theta-h(y))\, dy\, dx$$

Or, with a change in the order of integration

$$\Omega = \int_0^T U(c(x), \theta-h(x)) \int_x^T q(y)\, dy\, dx$$

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6 It may be noted that strict concavity of the instantaneous utility function is a necessary and sufficient condition for the lifetime utility function, considered below, to be quasi-concave.

Finally, using the relation (3.2.3), we can write

$$\Omega = \int_0^T p(x)\ U(c(x),\ \theta-h(x))\ dx \qquad (3.2.12)$$

Notice that the survival probability, p(x), is in general less than one and decreasing with age. Therefore, due to life uncertainty, the individual puts less weight on future utility as compared to the present. This behaviour of an individual facing an uncertain lifetime appears to be what Fisher (1930, pp.216-217) had in mind by his assertion: "Uncertainty of human life increases the rate of preference for present over future income for many people..."

As our lifetime utility function (3.2.12) shows, we have assumed that the individual does not discount future utility for any reason other than uncertainty of life. However, our results would not be much affected by introducing a discount rate in the lifetime utility function as long as it is less than the rate of interest on riskless bonds. As our analysis in sub-section 3.3.3 will show, the effect of discounting future utility due to life uncertainty is exactly offset by the excess of actuarial rate of interest over the rate on riskless bonds due to life insurance and annuities. Therefore, it is the difference between the rate of interest on riskless bonds and the subjective discount rate other than the one due to life uncertainty, which will determine the trade off between present and future spending. If the rate of interest is greater that the discount rate then, given that the wage rate and the price of the consumption good are constant over the life cycle, the individual will spend more in future periods than in the present. This result has also been established by Yaari (1965) in the context of life cycle consumption decision. It is in this sense that the assumption of zero discount rate and a positive rate of interest on riskless bonds is equivalent to assuming that the discount rate is less than the rate of interest.

The basic structure of our model is now complete. We have an objective function of the individual in the form of an expected lifetime utility function (3.2.12), a lifetime budget constraint (3.2.9), and the boundary conditions (3.2.11). We are now ready to carry out optimization analysis and discuss the properties of the optimal consumption-work plan. This is the subject matter of the next section.

3.3 THE OPTIMIZING SOLUTION AND THE LIFE CYCLE PATTERN OF CONSUMPTION AND WORK HOURS: A TWO-STAGE BUDGETING APPROACH

The lifetime allocation problem of the individual is to choose a feasible consumption-work plan in such a way as to maximize his/her expected lifetime utility. For a better understanding of the results, a two-stage budgeting approach is adopted. At the upper stage of the problem, lifetime resources are allocated among expenditure rates on consumption and retained leisure at various points of the life cycle. At the lower stage, the expenditure chosen from the upper stage for each age is optimally allocated between consumption and leisure.

To explain precisely how the two stages work, we first define the rate of normalized expenditure, $e(x)$, on consumption and leisure, measured in the units of the consumption good, as

$$e(x) = c(x) + w(x)\{\theta - h(x)\} \tag{3.3.1}$$

Given the rate of expenditure at age x, $e(x)$, the lower stage problem is to choose the rates of consumption, $c(x)$, and leisure, $\theta-h(x)$, in such a way as to maximize the rate of utility at age x, $U(c(x), \theta-h(x))$. The solution thus obtained gives the conditional demand functions for consumption and leisure, given the rate of expenditure. From these demand functions can be derived the indirect utility function which describes the utility rate at age x as a function of expenditure and the wage rate at age x. The indirect utility function shows the utility derived from a given rate of expenditure under the assumption of optimal behaviour at the lower stage. Using this indirect utility function, the expected lifetime utility can then be expressed as a function of expenditure and wage rates at various points of the life cycle. Next, moving to the upper stage, the problem is to allocate lifetime resources among expenditure rates at various points of the life cycle. Now we formally proceed to the problem.

3.3.1 The Problem at the Lower Stage

The lower stage problem at age x can be outlined as follows.

$$\underset{(c(x),\,h(x))}{\text{Maximize}}\; U(c(x),\ \theta-h(x))$$

$$\text{Subject to:}\quad e(x) - c(x) - w(x)\{\theta-h(x)\} \geq 0$$

$$c(x) \geq 0, \quad 0 \leq h(x) \leq \theta$$

The assumption of positive marginal utilities (assumption 2) implies that the budget constraint in the above problem will always hold with strict equality under optimal behaviour.

In addition, assumption 3 on the utility function guarantees that the boundary conditions $c(x) \geq 0$ and $h(x) \leq \theta$ will not be binding. As a result, the problem can be converted into the following Lagrange format.

$$\underset{(c(x),h(x))}{\text{Maximize}} \; U(c(x),\ \theta-h(x)) + \lambda(x)[e(x)-c(x)-w(x)\{\theta-h(x)\}]$$

such that $h(x) \geq 0$ and $\lambda(x) > 0$.

The Kuhn-Tucker conditions for maximization are:

(i) $U_1(x) - \lambda(x) = 0$

(ii) $-U_2(x) + \lambda(x)w(x) \leq 0$

(iii) $h(x) \geq 0$

(iv) $h(x)\ [U_2(x) + \lambda(x)w(x)] = 0$

(v) $e(x) - c(x) - w(x)\{\theta-h(x)\} = 0$

If the solution exists, it will be either an interior solution: $c(x) > 0$, $h(x) > 0$, or a corner solution: $c(x) > 0$, $h(x) = 0$. When the maximizing condition (ii) holds with strict equality, there can be an interior solution. In this case the maximizing conditions after eliminating $\lambda(x)$ can be expressed as follows.

$$U_2(x) - w(x)\ U_1(x) = 0 \tag{3.3.2}$$

$$e(x) - c(x) - w(x)\{\theta-h(x)\} = 0 \tag{3.3.3}$$

Since the utility function is assumed to have continuous first and second partial derivatives, it is obvious that the above two equations have continuous first partial derivatives. Later in our comparative static analysis, we will prove that the relevant Jacobian matrix for this set of equations is non-singular. Therefore the Implicit Function Theorem guarantees that there exists a continuous solution:

$$c(x) = c(w(x), e(x)) \tag{3.3.4}$$

$$h(x) = h(w(x), e(x)) \tag{3.3.5}$$

which has continuous first partial derivatives with respect to $w(x)$ and $e(x)$.

Next, if the maximizing condition (iii) holds with strict equality, we have an explicit solution:

$$c(x) = e(x) - w(x)\} \tag{3.3.6}$$

$$h(x) = 0 \tag{3.3.7}$$

Obviously, this solution is continuous and has continuous first partial derivatives with respect to $w(x)$ and $e(x)$.

Finally, we must also consider the border line case in which both the maximizing conditions (ii) and (iii) hold with strict equality. If a solution occurs in the neighbourhood of such a point, it may switch from interior to corner or vice versa in response to small changes in any parameter of the problem. Therefore, we call such a point a switching point. At such point, the boundary condition $h(x) \geq 0$ is effective just at the margin. In other words, the maximizing conditions (3.3.2) and (3.3.3) describing the interior solution yield the result: $c(x) = e(x) - w(x)\vartheta$ and $h(x) = 0$, which is also the corner solution at that particular switching point. Therefore the solution is continuous in the neighbourhood of a switching point.

The above discussion can be summarized in the following result.

Theorem 3.3.1

A solution to the lower stage problem exists in the entire domain: $w(x) > 0$, $e(x) > 0$. The resulting consumption demand and labour supply functions are everywhere continuous. In addition, the two functions have continuous first partial derivatives with respect to the two arguments $w(x)$ and $e(x)$ at all the points except at the possible switching points.

We now discuss the comparative static effects in the lower stage solution which will be useful later, not only to relate the solution at the two stages, but also to perform the comparative static analysis of the entire solution.

The comparative static effects for the interior solution can be obtained by differentiating equations (3.3.2) and (3.3.3) with respect to all the variables, $c(x)$, $h(x)$, $w(x)$ and $e(x)$. The result in compact form is:

$$J(x)\begin{bmatrix} dc(x) \\ dh(x) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} de(x) + \begin{bmatrix} U_1(x) \\ \theta - h(x) \end{bmatrix} dw(x) \qquad (3.3.8)$$

where, $J(x)$ is the Jacobian matrix defined below.

$$J(x) = \begin{bmatrix} U_{21}(x)-w(x)U_{11}(x) & -U_{22}(x)+w(x)U_{12}(x) \\ -1 & w(x) \end{bmatrix} \qquad (3.3.9)$$

Substituting $w(x) = U_2(x)/U_1(x)$ from equation (3.3.2), the determinant of matrix $J(x)$ can be solved as

$$|J(x)| = -Q(x)/U_1(x)^2 \qquad (3.3.10)$$

where, $Q(x)$ is the quadratic form:

$$Q(x) = U_2(x)^2 U_{11}(x) - 2U_1(x)U_2(x)U_{12}(x) + U_1(x)^2 U_{22}(x) \qquad (3.3.11)$$

Under the assumption of the strict concavity of the utility function, $Q(x)$ is negative.

Solving (3.3.8) with $dw(x) = 0$ and using (3.3.10), we obtain the following partial derivatives:

$$\partial c(x)/\partial e(x) = \mu_2(x)/Q(x) \tag{3.3.12}$$

$$\partial h(x)/\partial e(x) = -\mu_1(x)/w(x)Q(x) \tag{3.3.13}$$

where,

$$\mu_2(x) = U_1(x)^2 U_{22}(x) - U_1(x)U_2(x)U_{12}(x) \tag{3.3.14}$$

$$\mu_1(x) = U_2(x)^2 U_{11}(x) - U_1(x)U_2(x)U_{12}(x) \tag{3.3.15}$$

The assumptions on the utility function are not sufficient to determine the signs of $\mu_1(x)$ and $\mu_2(x)$. However $\mu_1(x)$ and $\mu_2(x)$ satisfy the following obvious restriction.

$$\mu_1(x) + \mu_2(x) = Q(x) < 0 \tag{3.3.16}$$

From this restriction follows the adding-up property of the 'income effects', known as the Engel aggregation condition.[7]

$$[\partial c(x)/\partial e(x)] - w(x)\,[\partial h(x)/\partial e(x)] = 1 \tag{3.3.17}$$

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[7] This property can also be obtained directly from the budget equation: $e(x) = c(x) + w(x)\,\{\bar{t}-h(x)\}$.

If both consumption and leisure are normal, $\partial c(x)/\partial e(x) > 0$ and $\partial h(x)/\partial e(x) < 0$. If consumption is inferior and, therefore, $\partial c(x)/\partial e(x) < 0$, we must have $\partial h(x)/\partial e(x) < 0$, that is, leisure must be normal. Likewise, if leisure is inferior and, therefore, $\partial h(x)/\partial e(x) > 0$, we must have $\partial c(x)/\partial e(x) > 0$, that is, consumption must be normal.

Next holding $de(x) = 0$, we obtain the following price effects by solving equations (3.3.8):

$$\partial c(x)/\partial w(x) = -[w(x)U_1(x)^3/Q(x)] - \{\partial - h(x)\}[\mu_2(x)/Q(x)]$$
$$\partial h(x)/\partial w(x) = -[U_1(x)^3/Q(x)] + \{\partial - h(x)\}[\mu_1(x)/w(x)Q(x)]$$

It is easy to confirm that the first part in the two price effects represents the compensated price effect where as the second part corresponds to the income effect. Therefore, we can write the effects of an increase in the wage rate in terms of the following Slutsky equations:

$$\partial c(x)/\partial w(x) = [\partial c(x)/\partial w(x)|_{U(x)}] - [\partial - h(x)][\partial c(x)/\partial e(x)] \tag{3.3.18}$$

$$\partial h(x)/\partial w(x) = [\partial h(x)/\partial w(x)|_{U(x)}] - [\partial - h(x)][\partial h(x)/\partial e(x)], \tag{3.3.19}$$

where,

$$\partial c(x)/\partial w(x)\Big|_{U(x)} = -\, w(x)U_1(x)^3/Q(x) \tag{3.3.20}$$

$$\partial h(x)/\partial w(x)\Big|_{U(x)} = -\, U_1(x)^3/Q(x) \tag{3.3.21}$$

The compensated effect of an increase in the wage rate is to increase both consumption and work hours as expected. As before, the signs of the income effects are not certain. In order to be more definite about the possible signs of the two price effects, we will use the following identity which can be obtained either by using equations (3.3.16), (3.3.18) and (3.3.19), or directly from the budget equation.[8]

$$[\partial c(x)/\partial w(x)] - w(x)\,[\partial h(x)/\partial w(x)] = -\,\{\theta - h(x)\} \tag{3.3.22}$$

This identity can be translated in terms of elasticities as follows:

$$n_1(x) = -\,[S_2(x)/S_1(x)]\,[1+n_2(x)] \tag{3.3.23}$$

$S_1(x)$ and $S_2(x)$ are respectively the expenditure shares of consumption and leisure and $n_1(x)$ and $n_2(x)$ are respectively

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8 This identity is known as the Cournot aggregation condition.

the elasticities of consumption and leisure with respect to wage rate. That is,

$$S_1(x) = c(x)/e(x)$$

$$S_2(x) = w(x)\{\theta-h(x)\}/e(x)$$

$$n_1(x) = [\partial c(x)/\partial w(x)]\ [w(x)/c(x)]$$

$$n_2(x) = [\partial\{\theta-h(x)\}/\partial w(x)]\ [w(x)/\{\theta-h(x)\}]$$

If, by any chance, leisure is a 'Giffen' good, that is, $n_2(x)$ is positive then, as equation (3.3.23) implies, $n_1(x)$ must be negative. In this case an increase in the wage rate will result in a decrease both in consumption and work hours. The more realistic case, however, is $n_2(x) < 0$, that is, a higher wage rate will result in increased work hours. Now if the demand for leisure is inelastic, that is, $|n_2(x)| < 1$, then $n_1(x)$ will be negative and therefore an increase in the wage rate will result in less consumption. On the other hand, if the demand for leisure is elastic, that is, $|n_2(x)| > 1$, then $n_1(x)$ will be positive and therefore a wage increase will result in more consumption.

Turning now to the corner solution (not at a switching point), the comparative static effects are quite obvious from the explicit solution given by equations (3.3.6) and (3.3.7).

$$\partial c(x)/\partial e(x) = 1 \tag{3.3.24}$$

$$\partial h(x)/\partial e(x) = 0 \tag{3.3.25}$$

$$\partial c(x)/\partial w(x) = -\theta \tag{3.3.26}$$

$$\partial h(x)/\partial w(x) = 0 \tag{3.3.27}$$

It may be noted here that the effect of a small change in the wage rate in the case of a corner solution contains only an income effect. The substitution effect is absent because after the change in the wage rate, work hours still remain at zero. The income effect is zero on work hours and negative on consumption, because with a given expenditure and leisure being fixed at its maximum, an increase in the wage rate must result in lower consumption. Finally, it may be noted that the effect of an increase in the wage rate in a corner solution can also be written in terms of the Slutsky equations (3.3.18) and (3.3.19). In the Slutsky equation for work hours both the substitution effect and the income effect are zero. In the Slutsky equation for consumption the substitution effect is zero where as the income effect of a wage increase is negative.

Next, we explore the conditions under which the solution may switch from an interior to a corner segment or vice versa. The following results are in order:

Theorem 3.3.2

A corner solution: $h(x) = 0$ is not possible so long as $e(x) \leq w(x)\theta$.

Proof

Since $e(x) = c(x) + w(x)(\theta - h(x))$, $e(x) \leq w(x)\theta$ implies that $c(x) \leq w(x)h(x)$. However $c(x) > 0$ by assumption 3, therefore $h(x)$ must be positive.

Theorem 3.3.3

If, with an increase in expenditure, the solution switches from an interior (corner) to a corner (interior) segment, leisure must be normal (inferior) in the neighbourhood of that switching point.

Proof

Let $e'(x)$ be the level of expenditure at which the solution switches from an interior to a corner segment. Then for a small number ε, at $e'(x)-\varepsilon$ the work hours must be positive. That is, in response to a small increase in expenditure from $e'(x)-\varepsilon$ to $e'(x)$, work hours decrease from a positive number to zero. Therefore leisure must be normal in the neighbourhood of $e'(x)$. Similarly, if at $e'(x)$, the solution switches from a corner to an interior segment, then for a small number ε, work hours must become positive at $e'(x)+\varepsilon$. This is possible only if leisure is inferior in the neighbourhood of $e'(x)$.

Theorem 3.3.4

If, with an increase in the wage rate, the solution switches from a corner (interior) to an interior (corner) segment, leisure must be non-Giffen (Giffen) in the neighbourhood of that switching point.

Proof

If at $w'(x)$, solution switches from a corner (interior) to an interior (corner) segment then for a small number ε, work hours must be positive at the wage rate $w'(x)+\varepsilon$ ($w'(x)-\varepsilon$). That is, as the wage rate increases from $w'(x)$ to $w'(x)+\varepsilon$ (from $w'(x)-\varepsilon$ to $w'(x)$), work hours increase from zero to a positive number (decrease from a positive number to zero). This is possible only if leisure is non-Giffen (Giffen) in the neighbourhood of $w'(x)$.

A final point to be noted here is that the first partial derivatives of the two functions $c(w(x), e(x))$ and $h(w(x), e(x))$ are not continuous at a switching point. First, notice that, if we approach a switching point from the interior segment, both the derivatives $\partial h(x)/\partial e(x)$ and $\partial h(x)/\partial w(x)$ are non-zero in general (Theorems 3.3.3 and

3.3.4). Therefore equations (3.3.17) and (3.3.22) imply that $\partial c(x)/\partial e(x) \neq 1$ and $\partial c(x)/\partial w(x) \neq -\theta$. But, if we approach the same switching point from the corner segment, we get $\partial h(x)/\partial e(x) = 0$, $\partial h(x)/\partial w(x) = 0$, $\partial c(x)/\partial e(x) = 1$ and $\partial c(x)/\partial w(x) = -\theta$ (See equations (3.3.24) through (3.3.27)) which are different from their respective values obtained above. Thus, all the partial derivatives of $c(w(x), e(x))$ and $h(w(x), e(x))$ are discontinuous at a switching point.

This completes our discussion of the lower stage solution. In next sub-section we will derive and discuss the properties of the indirect utility function which will be used to find a solution at the upper stage.

3.3.2 The Indirect Utility Function and its Properties

The indirect utility function can be obtained by substituting the consumption demand and the labour supply functions into the utility function. From the last sub-section we have learned that the consumption demand and the labour supply functions can be written in their general form as:

$$c(x) = c(w(x), e(x))$$
$$h(x) = h(w(x), e(x))$$

For a corner solution, these two functions degenerate to an explicit form as noted in equations (3.3.6) and (3.3.7). Substituting the above two functions into the utility function, we can write the indirect utility function in its general form as follows.

$$U(x) = U[c(w(x),e(x)) \ , \ \theta - h(w(x),e(x))] \qquad (3.3.28)$$

For a corner solution, this function takes a simple form:

$$U(x) = U[e(x)-w(x)\theta, \ \theta] \qquad (3.3.29)$$

But in any case, we can write the indirect utility function in its compact form as

$$V(x) = V(w(x), \ e(x)) \qquad (3.3.30)$$

In order to obtain a well behaved solution at the upper stage of our problem and to perform the comparative static analysis, we have to make sure that the indirect utility function satisfies the necessary properties analogous to our assumptions on the direct utility function. Fortunately, all the essential properties are met, as listed in the theorem. The proof is relegated to the appendix (section 3.A.1.)

Theorem 3.3.5

(i) $V(x)$ is continuous in $w(x)$ and $e(x)$ and has continuous first partial derivatives in the entire domain: $w(x) > 0$, $e(x) > 0$. Except at a switching point, the function $V(x)$ also has continuous second partial derivatives.

(ii) $V(x)$ is increasing in $e(x)$, that is, $V_2(x) > 0$.

(iii) $V_2(x)\,|_{e(x)=0} = \infty$

(iv) $V(x)$ is strictly concave in $e(x)$.

Properties (i) and (iv) will allow us to apply the tools of differential calculus to find a solution to the upper stage problem. Property (ii) implies that the budget constraint at the upper stage will hold with strict equality. Finally, property (iii) implies that the individual cannot survive without a positive expenditure on consumption and leisure.

Using the indirect utility function, we can now formulate our problem at the upper stage.

3.3.3 The Problem at the Upper Stage

The problem at the upper stage is to allocate lifetime resources among expenditure rates at various points of the horizon in such a way as to maximize lifetime utility. By using the indirect utility function, the expected lifetime utility can be expressed as a function of expenditure rates at various points of the horizon as follows.

$$\Omega = \int_0^T p(x)\ V(w(x),\ e(x))\ dx \tag{3.3.31}$$

Likewise, since $e(x) = c(x) + w(x)\{\theta - h(x)\}$, the lifetime budget constraint can also be expressed in terms of expenditure rates as follows.

$$\int_0^T e^{-rx}\ p(x)\ [w(x)\theta - e(x)]\ dx \geq 0 \tag{3.3.32}$$

The lower stage bounds on consumption and leisure: $c(x) \geq 0$ and $\theta - h(x) \geq 0$, imply an upper stage bound on expenditure: $e(x) \geq 0$. Theorem 3.3.5(iii) guarantees, however, that this boundary condition is not binding. Thus, the objective is to maximize the expected lifetime utility function (3.3.31) subject to the lifetime budget constraint (3.3.32), by choosing $e(x)$ for all x. The solution is characterized by the following equations.

$$V_2(x) - e^{-rx}\ V_2(0) = 0, \quad 0 \leq x \leq T \tag{3.3.33}$$

$$\int_0^T e^{-rx}\ p(x)\ [w(x)\theta - e(x)]\ dx = 0 \tag{3.3.34}$$

From these equations follows the first result of this sub-section.

Theorem 3.3.6

There exists a solution for the planned expenditure rate:

$$e(x) = e(x, P, r, W) \qquad (3.3.35)$$

where, P stands for the entire age path of the survival function: $p(y)$, $0 \leq y \leq T$ and W stands for the entire age path of the function: $w(x)$, $0 \leq y \leq T$. The function $e(x, P, r, W)$ is everywhere continuous in all its arguments. In addition, all the first partial derivatives of this function are continuous except at the possible switching points.

The proof of this theorem is long and therefore is given in the appendix (section 3.A.2).

Although equations (3.3.33) and (3.3.34) determine the entire solution at the upper stage, we can make use of equation (3.3.33) alone to trace out the age profile of expenditure. Differentiating this equation with respect to x, we obtain the following differential equation which describes the age path of expenditure.

$$\dot{e}(x) = -r\ [V_2(x)/V_{22}(x)] - [V_{12}(x)/V_{22}(x)]\ \dot{w}(x) \qquad (3.3.36)$$

Notice that the age pattern of expenditure, according to this differentia equation, does not show any impatience due to life uncertainty. The reason is that this source of impatience has been fully removed by the existence of actuarially fair life insurance and annuities. The discount factor which appears in the lifetime utility function due to life uncertainty, also appears symmetrically in the lifetime budget constraint because of the actuarially fair rate of interest at which the individual carries his/her resources from present to future or vice versa. Therefore, if present spending is more desirable because of an uncertain future, it is also more expensive, to the same extent, because of the higher rate of interest. The net effect of uncertainty on relative preference for present over the future spending is zero. This result can also be seen in the fixed income life cycle consumption models, considered by Yaari (1965) and Barro and Friedman (1977). In fact, if the subjective rate of time preference in Yaari's model is set equal to zero and the wage rate in our model is held constant, the differential equation governing the solution in our model coincides with the one obtained by Yaari

(See equation (33) in Yaari (1965)).[9]

The fact that the availability of life insurance and annuities removes the impatience resulting from uncertainty of life does not, however, imply that our solution coincides with the one that would be obtained in a model with certain lifetime if the expected length of life is the same in the two models. As Yaari (1965) puts it, while the differential equation governing the solution will be the same in the two models, the constants of integration which determine the solution to these differential equations, will be different because of the different budget constraints. In other words, although the intertemporal substitution effect of the higher rate of interest due to life insurance and annuities is completely offset by the effect of impatience due to life uncertainty, the income effect of a higher rate of interest is still present.

Earlier in sub-section 3.2.3 we have asserted that the individual is better off by the availability of actuarially fair life insurance and annuities. Now we can qualify that statement in the context of our model. Accord-

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9 It is appropriate to compare our expenditure schedule with Yaari's consumption schedule. The reason is that consumption is the only good in Yaari's model and therefore is synonymous with expenditure, where as, in our model consumption is only a part of expenditure.

ing to the lifetime budget constraint (3.2.10), we can write the upper stage budget constraint (3.3.32) as follows.

$$\int_0^T q(x) \int_0^x e^{-ry} [w(y)\theta - e(y)] \, dy \, dx \geq 0$$

Or, using E_x, the conventional notation for expectation over the probability distribution of the length of life x, we can write the budget constraint as

$$E_x \left[\int_0^x e^{-ry} [w(y)\theta - e(y)] \, dy \right] \geq 0 \qquad (3.3.37)$$

Next consider the lifetime budget constraint without life insurance and annuities. If life insurance and annuities are not available, the individual will have to bear the burden of local borrowing constraints which restrict him/her from borrowing against future expected income. In terms of our problem at the upper stage, this means that the lifetime budget constraint must hold for each possible length of life, not just for the maximum possible length of life, T. Thus the lifetime budget constraint without life insurance and annuities will be as follows.

$$\int_0^x e^{-ry} [w(y)\theta - e(y)] \, dy \geq 0 \qquad \text{for all } x \leq T \qquad (3.3.38)$$

The budget constraint with actuarially fair life insurance and annuities, given by (3.3.37) is less restrictive than the one without life insurance and annuities, given by (3.3.38). The obvious reason is that the budget constraint (3.3.38) requires a certain inequality to hold for each possible length of life where as the budget constraint (3.3.37) requires the same inequality to hold only on the average. Thus, as also shown by Yaari (1965), the individual is better off by the availability of actuarially fair life insurance and annuities as compared to the situation in which these are not available. The underlying reason can be explained as follows. If, without life insurance and annuities, assets are positive at the time of death, these are thrown away from the point of view of the individual concerned and thus generate no utility. With actuarially fair life insurance and annuities there is no such wastage. If, on the other hand, assets are equal to zero at the time of death, the individual may have suffered loss of utility due to his/her inability to borrow and thereby increase consumption before death. There is no such borrowing constraint in the presence of life insurance and annuities.

Now we come back to equation (3.3.36) which will be crucial in deriving the age profiles of consumption and work hours. The first part of this equation shows that expenditure is increasing with age if the rate of interest is positive. With a positive rate of interest and a constant wage rate, expenditure is more expensive in the early years of life as compared to later years. On the other hand, present utility is as desirable as the future utility because the effect of life uncertainty on the preference for present over future spending is fully offset by the higher rate of interest due to actuarially fair life insurance and annuities and there is no other discounting applied to the utility function. Therefore, under the optimal allocation scheme expenditure is increasing with age. Notice that in case we allow discounting of future utility for any reason other than life uncertainty, expenditure would be increasing with age if the rate of interest is greater than the discount rate. This is the conventional result found by Yaari (1965) and others.

The second part of equation (3.3.36) measures the effect of anticipated or evolutionary changes in the wage rate on expenditure. If $V_{12}(x)$ is positive (negative), that is, if, with an increase in wage rate, the marginal utility of expenditure increases (decreases), the rate of expenditure will increase (decrease) with age in response to an anticipated increase in the wage rate.

Notice that in the approach used by MaCurdy (1981), the second term in equation (3.3.36) also represents the marginal utility of wealth constant effect of changes in the wage rate.[10] This can be easily seen in our model. In deriving the differential equation (3.3.36) we have used the tangency condition (3.3.33) holding constant the marginal utility of initial wealth which is equal to the marginal utility of expenditure at age 0, that is, $V_2(0)$. Therefore, the coefficient attached to $\dot{w}(x)$ in equation (3.3.36) can also be obtained as:

$$de(x)/dw(x) \Big|_{V_2(0)} = - V_{12}(x)/V_{22}(x) \qquad (3.3.39)$$

Therefore, the differential equation (3.3.36) can be written as

$$\dot{e}(x) = -r\ [V_2(x)/V_{22}(x)] + [de(x)/dw(x) \Big|_{V_2(0)}]\ \dot{w}(x) \qquad (3.3.40)$$

For completeness, we express the first and second partial derivatives of the indirect utility function appearing in equation (3.3.36), in terms of the derivatives of the direct utility function. The resulting expression for equation (3.3.36) will also be used to relate the solution at the upper stage with that at the lower stage.

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10 The marginal utility of wealth constant effect of changes in wage rate on work hours are discussed by MaCurdy (1981). Since MaCurdy did not use the two stage budgeting approach, there is no concept of expenditure in his model. Therefore, we can not compare his results with ours at this stage. However, when we will come to the discussion of work hours, we will find some interesting comparisons.

First consider the interior solution. The expression for $V_{12}(x)$ for an interior solution, given by (3.A.4) of appendix 3.A.1, is reproduced below.

$$V_{12}(x) = U_{11}(x) \ [\partial c(x)/\partial w(x)] - U_{12}(x) \ [\partial h(x)/\partial w(x)]$$

But this expression can be made more concise. We can substitute for $\partial c(x)/\partial w(x)$ and $\partial h(x)/\partial w(x)$ from equations (3.3.18) and (3.3.19) respectively in this expression. With a few manipulations, the final result is

$$V_{12}(x) = - [U_1(x)\psi_1(x)/w(x)Q(x)] - [\{\theta-h(x)\}U_1(x)^2\Delta(x)/Q(x)] \tag{3.3.41}$$

where, $\psi_1(x)$ is given by (3.3.15) and $\Delta(x)$ is given below.

$$\Delta(x) = U_{11}(x) \ U_{22}(x) - U_{12}(x)^2 > 0 \tag{3.3.42}$$

Now substitute into the differential equation (3.3.36), the expressions for $V_2(x)$, $V_{22}(x)$ and $V_{12}(x)$ from equations (3.A.2), (3.A.6) and (3.3.41) respectively. The result after simplification is as follows.

$$\dot{e}(x) = -r \ [Q(x)/U_1(x)\Delta(x)] + [\{\psi_1(x)/U_2(x)\Delta(x)\} + \{\theta-h(x)\}] \ \dot{w}(x) \tag{3.3.43}$$

Next, consider the corner solution. Substituting for $V_2(x)$, and $V_{12}(x)$ and $V_{22}(x)$ from equations (3.A.9), (3.A.11) and (3.A.12) respectively into equation (3.3.36), the result is

$$\dot{e}(x) = -r\ [U_1(x)/U_{11}(x)] + \theta\ \dot{w}(x) \qquad (3.3.44)$$

As before, the first term in the above two equations is positive, indicating that expenditure is increasing with age if the real rate of interest is positive (or greater than the subjective discount rate). The second term, which represents the effect of an anticipated wage increase on expenditure, can be decomposed into two parts which we shall call a cost of living effect and an intertemporal allocation effect. The cost of living effect shows the effect of an anticipated wage increase on expenditure on a given basket of consumption and leisure. The intertemporal allocation effect shows the effect of an anticipated wage increase on expenditure due to the optimal allocation of consumption and leisure over the life cycle, holding the wage rate constant. This decomposition will be quite useful to integrate the upper stage solution with the lower stage solution.

To find the intertemporal allocation effect, consider the definition of expenditure for period $x-\epsilon$:

$$e(x-\epsilon) = c(x-\epsilon) + w(x-\epsilon)\{\theta-h(x-\epsilon)\}$$

Now define real expenditure for period x, $\xi(x)$, as the value of consumption and leisure measured in period $x-\epsilon$ prices (in this case wage rate).

$$\xi(x) = c(x) + w(x-\epsilon)\{\theta-h(x)\}$$

Notice that, if period $x-\epsilon$ is considered as the base period, then the expenditure in period $x-\epsilon$, defined above also represents real expenditure. Subtracting the constant price expenditure for period $x-\epsilon$ from that for period x, we get

$$\begin{aligned}\xi(x) - \xi(x-\epsilon) &= [c(x) + w(x-\epsilon)\{\theta-h(x)\}] - e(x-\epsilon)\\ &= e(x) - e(x-\epsilon) - \{\theta-h(x)\}[w(x) - w(x-\epsilon)]\end{aligned}$$

Dividing by ϵ and taking the limit as ϵ approaches zero, we find

$$\dot{\xi}(x) = \dot{e}(x) - \{\theta-h(x)\}\dot{w}(x) \tag{3.3.45}$$

This equation describes the relationship between the instantaneous rate of change of constant price expenditure and that of current price expenditure. Substituting equations (3.3.43) and (3.3.44) into equation (3.3.45) and setting $h(x) = 0$ for a corner solution, we can find the relationship of expenditure with anticipated change in wage rate, holding the wage rate constant. The results of these substitutions are:

$$\dot{\xi}(x) = -r\ [Q(x)/U_1(x)\Delta(x)] + [\mu_1(x)/U_2(x)\Delta(x)]\ \dot{w}(x) \qquad \text{interior solution} \qquad (3.3.46)$$

$$\dot{\xi}(x) = -r\ [U_1(x)/U_{11}(x)] \qquad \text{corner solution} \qquad (3.3.47)$$

The second term in equation (3.3.46) which is equal to the first component of second term in equation (3.3.43), represents the effect of an increase in the wage rate on the constant price expenditure, or what we call an intertemporal allocation effect of an anticipated increase in the wage rate. We know from (3.3.13) that the expression $\mu_1(x)$ is negative (positive) if leisure is normal (inferior). This gives us the following result.

Theorem 3.3.7

With an anticipated increase in the wage rate, real expenditure will decrease (increase) in an interior solution if leisure is normal (inferior).

This behaviour of a rational individual is analogous to that of a profit maximizing firm which will decrease (increase) output when the price of a normal (inferior) factor increases. It can be explained as follows. If leisure is normal, then it is economical to spend less when its price (wage rate) is higher. On the other hand, if leisure is inferior, then the individual can economize on it, when the wage rate is higher, by increasing expenditure and thus reducing the consumption of leisure time. This result could have been obtained more easily in the framework of profit maximizing approach used by Browning, Deaton and Irish (1985).

Equation (3.3.47) implies that along the corner solution real expenditure is independent of wage rate. This trivial result can also be compared with the behaviour of a profit maximizing firm which will not change its output in response to a change in the price of a fixed factor.

The cost of living effect of an anticipated change in the wage rate can be obtained by taking the age derivative of expenditure: $e(x) = c(x) + w(x)\ \{\theta - h(x)\}$, holding constant the rate of consumption, $c(x)$, and the rate of leisure, $\theta - h(x)$. The result is obvious:

$$\dot{e}(x)\ \Big|_{c(x),h(x)} = \{\theta - h(x)\}\ \dot{w}(x) \quad \text{interior solution} \qquad (3.3.48)$$

$$\dot{e}(x)\ \Big|_{c(x),h(x)} = \theta\ \dot{w}(x) \quad \text{corner solution} \qquad (3.3.49)$$

With the rates of consumption and leisure being fixed, for a one dollar increase in the wage rate for age x, expenditure at that age will increase by the number of hours retained for leisure.

To recapitulate the above ideas we would like to restate the differential equations (3.3.43) and (3.3.44) in terms of decomposition effects of an anticipated change in wage rates. Thus with a more precise notation, we write:

$$\dot{e}(x) = -r\ [Q(x)/U_1(x)\Delta(x)] + [de(x)/dw(x)\big|_{c,l}]\ \dot{w}(x)$$
$$+ [de(x)/dw(x)\big|_{i,a}]\ w(x) \qquad \text{interior solution} \qquad (3.3.50)$$
$$\dot{e}(x) = -r\ [U_1(x)/U_{11}(x)] + [de(x)/dw(x)\big|_{c,l}]\ \dot{w}(x)$$
$$\text{corner solution} \qquad (3.3.51)$$

where, the subscripts c,l and i,a stand for the cost of living effect and intertemporal allocation effect respectively, and

$$de(x)/dw(x)\big|_{c,l} = \theta - h(x) \qquad \text{interior solution} \qquad (3.3.52)$$
$$de(x)/dw(x)\big|_{c,l} = \theta \qquad \text{corner solution} \qquad (3.3.53)$$
$$de(x)/dw(x)\big|_{i,a} = \mu_1(x)/U_2(x)\Delta(x)$$
$$\text{interior solution} \qquad (3.3.54)$$
$$de(x)/dw(x)\big|_{i,a} = 0 \qquad \text{corner solution} \qquad (3.3.55)$$

These results will be useful for the study of the age patterns of consumption and work hours in next subsection and to compare the effects of different types of changes in the wage rate considered in chapter 4.

It will be useful to relate the growth rate of expenditure with the rate of interest (which is equal to the growth rate of prices if wage rate is constant over the life cycle) and the growth rate of wage rate. For this purpose, we express equations (3.3.43) and (3.3.44) in a slightly different form. First consider equation (3.3.43). Using equation (3.A.2) and (3.A.6), we can write equation (3.3.43) as follows.

$$\dot{e}(x) = -r\,[V_2(x)/V_{22}(x)] + [\{V_2(x)/V_{22}(x)\}\{\mu_1(x)/w(x)Q(x)\} + \{\theta - h(x)\}]\,\dot{w}(x)$$

Substituting $\mu_1(x)/w(x)Q(x) = -\,\partial h(x)/\partial e(x)$ from equation (3.3.13) and noting that $-\,\partial h(x)/\partial e(x) = \partial l(x)/\partial e(x)$, where $l(x)$ is the rate of leisure retained at age x, the above equation can be written:

$$\dot{e}(x) = -r\,[V_2(x)/V_{22}(x)] + [\{V_2(x)/V_{22}(x)\}\{\partial l(x)/\partial e(x)\} + l(x)]\,\dot{w}(x) \qquad (3.3.56)$$

Now define

$$S_2(x) = w(x)l(x)/e(x) \qquad (3.3.57)$$

$$\varepsilon_2(x) = [\partial l(x)/\partial e(x)]\,[e(x)/l(x)] \qquad (3.3.58)$$

$$\xi(x) = -\,e(x)V_{22}(x)/V_2(x) \qquad (3.3.59)$$

where $S_2(x)$ is the expenditure share of leisure, $\varepsilon(x)$ is the elasticity of demand for leisure with respect to expenditure and $\xi(x)$ is the coefficient of relative risk aversion.

Notice also that $S_2(x)\epsilon_2(x) = w(x)[\partial l(x)/\partial e(x)]$ is also equal to the marginal expenditure share of leisure. With a few manipulations, equation (3.3.56) can be simplified as:

$$\dot{e}(x)/e(x) = [r/\xi(x)] + S_2(x)\ [1 - \epsilon_2(x)/\xi(x)]\ [\dot{w}(x)/w(x)] \tag{3.3.60}$$

Next, consider equation (3.3.44) for the corner solution. Since $U_1(x) = V_2(x)$ and $U_{11}(x) = V_{22}(x)$, we can write:

$$\dot{e}(x)/e(x) = [r/\xi(x)] + S_2(x)\ [\dot{w}(x)/w(x)] \tag{3.3.61}$$

The two equations differ because in case of a corner solution, the elasticity of demand for leisure with respect to expenditure is equal to zero. Based on equations (3.3.60) and (3.3.61), we can write the following theorems:

Theorem 3.3.8

If the wage rate is held constant, the growth rate of expenditure is proportional to reciprocal of the coefficient of relative risk aversion.

Theorem 3.3.9

If the rate of interest is equal to zero (or equal to the subjective discount rate), the growth rate of expenditure can be expressed as a product of three factors: the expenditure share of leisure, one minus the ratio of elasticity of demand for leisure with respect to expenditure to the coefficient of relative risk aversion, and the growth rate of wage rate.

This completes our discussion of the solution at the upper stage. Now we can combine the solution at lower stage with that at the upper stage to find an unconditional solution for consumption and work plans.

3.3.4 Synthesis of the Lower and the Upper Stage Solutions: Life Cycle Pattern of Consumption and Work Hours

In this sub-section we are interested in finding the age pattern of consumption and work hours. Our solution at the lower stage can in general be summarized in the following equations.

$c(x) = c(w(x), e(x))$

$h(x) = h(w(x), e(x))$

The solution at the upper stage, given by equation (3.2.35), is also reproduced below.

$e(x) = e(x, P, r, W)$

These three equations taken together determine the entire solution to our life cycle allocation problem. The solution at the lower stage is conditional upon expenditure which is determined at the upper stage. Thus, substituting the upper stage solution for expenditure in the solution for consumption and work hours obtained at the lower stage, we obtain the following unconditional solution for consumption and work hours.

$$c(x) = c[w(x), e(x, P, r, W)] \tag{3.3.62}$$

$$h(x) = h[w(x), e(x, P, r, W)] \tag{3.3.63}$$

Or, in compact form,

$$c(x) = c(x, P, r, W) \tag{3.3.64}$$

$$h(x) = h(x, P, r, W) \tag{3.3.65}$$

Theorems 3.3.1 and 3.3.6 imply the following result on the consumption and work schedules.

Theorem 3.3.10

The functions $c(x, P, r, W)$ and $h(x, P, r, W)$ are everywhere continuous in all their arguments, x, P, r and W. In addition, the two functions have continuous first partial derivatives with respect to x, P, r and W except at a switching point.

Next, the differential equations governing the unconditional solution for consumption and work schedules can be obtained by differentiating equations (3.3.62) and (3.3.63) with respect to x. The results are:

$$\dot{c}(x) = [\partial c(x)/\partial w(x)]\ \dot{w}(x) + [\partial c(x)/\partial e(x)]\ \dot{e}(x) \tag{3.3.66}$$

$$\dot{h}(x) = [\partial h(x)/\partial w(x)]\ \dot{w}(x) + [\partial h(x)/\partial e(x)]\ \dot{e}(x) \tag{3.3.67}$$

To appreciate these equations more fully, we substitute the Slutsky equations for the first terms in these equations from the lower stage (equations (3.3.18) and (3.3.19) and equations (3.3.50) and (3.3.51) for $\dot{e}(x)$. The results of these substitutions are:

$$\begin{aligned}\dot{c}(x) = {} & [\partial c(x)/\partial w(x)\,|_{U(x)}]\ \dot{w}(x) - [\theta-h(x)][\partial c(x)/\partial e(x)]\ \dot{w}(x) \\ & + [\partial c(x)/de(x)]\ [de(x)/dw(x)\,|_{c,l}]\ \dot{w}(x) \\ & + [\partial c(x)/\partial e(x)]\ [de(x)/dw(x)\,|_{i,a}]\ \dot{w}(x) \\ & - r\ [\partial c(x)/\partial e(x)]\ [Q(x)/U_1(x)\Delta(x)]\end{aligned} \tag{3.3.68}$$

$$\begin{aligned}\dot{h}(x) = {} & [\partial h(x)/\partial w(x)\,|_{U(x)}]\ \dot{w}(x) - [\theta-h(x)][\partial h(x)/\partial e(x)]\ \dot{w}(x) \\ & + [\partial h(x)/de(x)]\ [de(x)/dw(x)\,|_{c,l}]\ \dot{w}(x) \\ & + [\partial h(x)/\partial e(x)]\ [de(x)/dw(x)\,|_{i,a}]\ \dot{w}(x) \\ & - r\ [\partial h(x)/\partial e(x)]\ [Q(x)/U_1(x)\Delta(x)]\end{aligned} \tag{3.3.69}$$

where, in the last term in these equations the expression $[Q(x)/U_1(x)\Delta(x)]$ should be replaced by $[U_1(x)/U_{11}(x)]$ in case of a corner solution.

Theorem 3.3.11

The cost of living effects on consumption and work hours: $[\partial c(x)/\partial e(x)][de(x)/dw(x)|_{c,l}]\ \dot{w}(x)$ and $[\partial h(x)/\partial e(x)][de(x)/dw(x)|_{c,l}]\ \dot{w}(x)$ are exactly offset by the two income effects: $-[\theta-h(x)][\partial c(x)/\partial e(x)]\dot{w}(x)$ and $-[\theta-h(x)][\partial h(x)/\partial e(x)]\dot{w}(x)$ respectively.

Proof

The cost of living effects can be written, according to equations (3.3.52) and (3.3.53), as

$$[\partial c(x)/\partial e(x)]\ [de(x)/dw(x)|_{c,l}]\ \dot{w}(x) = [\theta-h(x)]\ [\partial c(x)/de(x)]\ \dot{w}(x)$$

$$[\partial h(x)/\partial e(x)]\ [de(x)/dw(x)|_{c,l}]\ \dot{w}(x) = [\theta-h(x)]\ [\partial h(x)/de(x)]\ \dot{w}(x)$$

which are obviously equal to the negative of the respective income effects. This completes the proof.

This result has a straightforward interpretation. Given work hours and consumption at age x, a one dollar increase in the wage rate requires an increase in expenditure by the amount of leisure retained, that is, $\theta-h(x)$. If expenditure is increased by this amount, the individual can buy the same bundle of leisure and consumption as purchased before the wage increase. This increase in expenditure is exactly equal to the compensating variation in expenditure necessary to eliminate the income effect of a wage increase at the lower stage solution. Thus the differential equations (3.3.68) and (3.3.69) describing the age paths of consumption and work hours can be written as

$$
\begin{aligned}
\dot{c}(x) = {} & [\partial c(x)/\partial w(x) \big|_{U(x)}]\ \dot{w}(x) \\
& + [\partial c(x)/\partial e(x)]\ [de(x)/dw(x) \big|_{i,a}]\ \dot{w}(x) \\
& - r\ [\partial c(x)/\partial e(x)]\ [\Omega(x)/U_1(x)\lambda(x)]
\end{aligned} \tag{3.3.70}
$$

$$
\begin{aligned}
\dot{h}(x) = {} & [\partial h(x)/\partial w(x) \big|_{U(x)}]\ \dot{w}(x) \\
& + [\partial h(x)/\partial e(x)]\ [de(x)/dw(x) \big|_{i,a}]\ \dot{w}(x) \\
& - r\ [\partial h(x)/\partial e(x)]\ [\Omega(x)/U_1(x)\lambda(x)]
\end{aligned} \tag{3.3.71}
$$

The first term in these two equations represents the effect of substitution between consumption and leisure at a given age. The second term shows the effect of optimal allocation of expenditure over the life cycle on consumption and work hours. Therefore the effect of an anticipated increase in the wage rate can be split into two parts, an intratemporal substitution effect and an intertemporal allocation effect. The intratemporal substitution effect implies that an anticipated increase in the wage rate at age x before retirement will result in more work hours and more consumption at that age. A wage increase in case of a corner solution has no intratemporal substitution effect.

The intertemporal allocation effect says that in case of an interior solution, a wage increase at age x makes the commodity bundle at that age relatively more expensive. If leisure is a normal good, the individual will allocate less expenditure for age x. This implies that work hours will increase and consumption will decrease (increase) at age x if it is normal (inferior). If leisure is inferior, then expenditure will increase at age x. With leisure being inferior, work hours and consumption both will increase. Thus, we can write the following result.

Theorem 3.3.12

The intertemporal allocation effect of an anticipated increase in the wage rate results in more work hours, irrespective of the expenditure elasticity of labour supply. Consumption will be effected in the opposite direction if and only if both consumption and leisure are normal. If any of these two goods is inferior, consumption will be affected in the same direction as work hours.

Finally, a wage increase in case of a corner solution does not have any intertemporal allocation effect. The reason is that a wage increase in this case does not affect the allocation of expenditure over the life cycle.

The fact that there is no income or wealth effect of an anticipated change in wage rate is well known in the literature (See, for example, Browning, Deaton and Irish (1985) and MaCurdy (1981)). The reason is that at the beginning of the planning horizon the individual knows the age pattern of future wage rates and plans his/her lifetime allocation of work hours and consumption in light of anticipated variations in the wage rate. Therefore these changes in wage rate over the life cycle are not parametric changes that would create any income or wealth effects.[11]

The relationship of work hours with changes in the anticipated wage rate has been regarded as the intertemporal substitution effect by MaCurdy (1981). The two stage budgeting analysis, conducted above, however, shows that this relationship does not fully represent substitution over time. Part of the substitution takes place at each age between consumption and leisure as a result of anticipated change in wage rate. Therefore it would be misleading to identify the full effect of an anticipated change in the wage rate with the intertemporal substitution effect. Equa-

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11 The distinction between the evolutionary and the parametric changes in the wage rates will be explained in more detail in chapter 4.

tions (3.3.70) and (3.3.71) clearly explain how this effect is a result of both the intratemporal substitution and the intertemporal substitution. The intratemporal substitution effect measures the substitution between leisure and consumption at a given age, holding constant the utility rate at that age. The intertemporal substitution effect measures the change in consumption and leisure (therefore work hours) across the life cycle, holding constant the wage rate, or the marginal rate of substitution between consumption and leisure at each age. This allocation results from the consideration that expenditure at some ages is cheaper than at other ages as a result of a changing wage rate.

For a more definite inference regarding the age paths of consumption and work hours we substitute for the partial derivatives appearing in equations (3.3.70) and (3.3.71). For an interior solution, substitute for the derivatives: $\partial c(x)/\partial e(x)$, $\partial h(x)/\partial e(x)$, $\partial c(x)/\partial w(x)\big|_{U(x)}$, $\partial h(x)/\partial w(x)\big|_{U(x)}$ and $de(x)/dw(x)\big|_{i,a}$ from equations (3.3.12), (3.3.13), (3.3.20), (3.3.21) and (3.3.54) respectively. The results after simplification are:

$$\dot{c}(x) = - r\,[U_2(x)/U_1(x)\Delta(x)] - [U_1(x)U_{12}(x)/\Delta(x)]\,\dot{w}(x) \tag{3.3.72}$$

$$\dot{h}(x) = r\,[U_1(x)/U_2(x)\Delta(x)] - [U_1(x)U_{11}(x)/\Delta(x)]\,\dot{w}(x) \tag{3.3.73}$$

Next, for a corner solution, we substitute for $\partial c(x)/\partial e(x)$, $\partial h(x)/\partial e(x)$ and $de(x)/dw(x)\,|_{i,a}$ from equations (3.3.24), (3.3.25) and (3.3.55) respectively. Also recall that in case of a corner solution we have to replace the expression $[\theta(x)/U_1(x)\Delta(x)]$ by $[U_1(x)/U_{11}(x)]$. With $\partial c(x)/\partial w(x)\,|_{U(x)} = \partial h(x)/\partial w(x)\,|_{U(x)} = 0$, the following results are found.

$$\dot{c}(x) = - r\ [U_1(x)/U_{11}(x)] \tag{3.3.74}$$

$$\dot{h}(x) = 0 \tag{3.3.75}$$

Unless otherwise stated, for the rest of the analysis we will restrict ourselves to the assumption that both consumption and leisure are normal. Therefore, it follows from equations (3.3.12) and (3.3.13) that, both $\mu_1(x)$ and $\mu_2(x)$ are negative. Over the life cycle the segments of interior and corner solutions will be referred to as the working and retirement periods of life. Some of the results regarding the age pattern of consumption and work hours directly follow from the four differential equations above.

Theorem 3.3.13

If the wage rate is constant and the rate of interest is positive (or greater than the subjective discount rate), consumption will increase and work hours will decrease with age during the working years of life.

This result is well known in the literature (Browning, Deaton and Irish (1985), Heckman (1974), Ghez and Becker (1975)). With a positive rate of interest, a constant price of the consumption good (normalized at 1) and a constant wage rate, the discounted price of the consumption good and the discounted wage rate are decreasing with age. Therefore consumption is more expensive and work is better rewarded in the earlier periods of life. If future utility is not discounted, the optimal life cycle allocation under these circumstances will call for less consumption and more work in the early periods of life than in the later periods.

Theorem 3.3.14

If the rate of interest is zero (or equal to the subjective discount rate), then, during the working period(s) of life, work hours will increase (decrease) with age whenever the wage rate is increasing (decreasing) with age. Again, with the rate of interest equal to zero (or equal to the subjective discount rate), if $U_{12}(x)$ is negative, then during the working period(s) of life consumption will also increase (decrease) with age whenever the wage rate is increasing (decreasing) with age. If $U_{12}(x)$ is positive, the relationship of consumption with the wage rate is reversed. Finally, if $U_{12}(x)$ is zero, consumption is independent of age-wise variations in the wage rate.

This result was first established by Heckman (1974) and Ghez and Becker (1975) for the life cycle model under certain lifetime. The result that work hours will respond positively to the variations in the wage rate over the life cycle, is unambiguous and does not depend on any assumptions regarding the usual income or price elasticities. The reason is that the response of work hours to the anticipated changes in the wage rate is a result of the optimal allocation of labour time over the life cycle and cannot be confused with the response of work hours to alternative wage offers within a period.[12]

The reason that consumption may also vary with the wage rate is explained by the complementarity or substitutability relationship between consumption and leisure. If $U_{12}(x)$ is negative, the marginal utility of leisure decreases with an increase in consumption or the marginal utility of consumption decreases with an increase in leisure. In other words consumption and leisure are sub-

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12 This behaviour of work pattern has been empirically tested by Browning, Deaton and Irish (1985), Ghez and Becker (1975), MaCurdy (1981, 1983, 1985) and many other studies reported in Killingsworth (1983).

stitutes for each other. Therefore, if work hours are increasing with age due to an increasing wage rate, leisure will be decreasing and consumption will be increasing. By a similar reasoning, if $U_{12}(x)$ is positive, consumption will be decreasing when leisure is decreasing due to an increasing wage rate. Finally, if $U_{12}(x) = 0$, consumption is independent of the wage rate.[13] The observed positive relationship between consumption and current income is compatible with the life cycle theory if there is a substitutability relationship between consumption and leisure, that is, $U_{12}(x) < 0$. In this case an anticipated increase in the wage rate will call for an increase in work hours and labour income as well as an increase in consumption.

Theorem 3.3.15

If the rate of interest is positive (or greater than the subjective discount rate), consumption is increasing with age and is independent of wage rate during retirement. Work hours are constant at

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13 The utility function considered by MaCurdy (1981) is additively separable in consumption and leisure and therefore $U_{12}(x) = 0$. In this case the intratemporal substitution effect of an anticipated change in wage rate on consumption is exactly offset by the intertemporal allocation effect.

zero irrespective of the rate of interest and wage rate.

This is an obvious result. Since the individual is not working, his/her life time decision is independent of the wage rate. If the rate of interest is positive, consumption must be increasing with age because expenditure is increasing and leisure is constant at its maximum.

For further results we here need to know the age profile of the wage rate. It is generally assumed that the wage schedule is concave, increasing with age up to some point and decreasing thereafter. For empirical evidence on this wage-age relationship, see Browning, Deaton and Irish (1984), Ghez and Becker (1975), Gordon and Blinder (1980) and Killingsworth (1983). This wage-age relationship implies that, once the wage rate starts declining, work hours will decrease with age if the rate of interest is non-negative. But if the wage rate is increasing and the rate of interest is positive, work hours may increase or decrease with age, depending on whether the effect of an increasing wage rate dominates the effect of a positive rate of interest or vice versa. This wage-age relationship yields the following additional results.

Theorem 3.3.16

If the rate of interest is non-negative (or not less than the subjective discount rate) and the wage rate is decreasing, or, the rate of interest is positive (or greater than the subjective discount rate) and wage rate is non-increasing, the individual will never re-enter the labour force once retired.

Proof

Suppose the individual retires at age R. Then, if the individual enters the labour force at some age, say x' after age R, work hours must be increasing in the neighbourhood of age x' due to the continuity of the labour supply function. But this is not possible because, if the rate of interest is non-negative and the wage rate is decreasing, or, if the rate of interest is positive and the wage rate is non-increasing, then work hours must also be decreasing during the working period(s) of life according to the differential equation (3.3.73).

Theorem 3.3.17

At the retirement age there is an upward (downward) jump in the <u>slope of</u> consumption schedule, c(x), if $U_{12}(x)$ is negative (positive). If $U_{12}(x)$ is equal to zero, then there is no sudden change in the slope of c(x).

Proof

The difference between the right side and the left side limits of the slope of c(x) as x approaches the retirement age, R, can be found by using the differential equations (3.3.72) and (3.3.74) as follows.

$$\lim_{x \to R+} \dot{c}(x) - \lim_{x \to R-} \dot{c}(x) = - r\ [U_1(R)/U_{11}(R)]$$

$$+ r\ [\mu_2(R)/U_1(R)\Delta(R)] + [U_1(R)U_{12}(R)/\Delta(R)]\ \dot{w}(R)$$

This difference can be solved as:

$$\lim_{x \to R+} \dot{c}(x) - \lim_{x \to R-} \dot{c}(x) = - [U_{12}(R)/U_{11}(R)]$$

$$[r\{\mu_1(R)/U_2(R)\Delta(R)\} - \{U_1(R)U_{11}(R)/\Delta(R)\}\dot{w}(R)]$$

Or, using equation (3.3.73), we end up with:

$$\lim_{x \to R+} \dot{c}(x) - \lim_{x \to R-} \dot{c}(x) = - [U_{12}(R)/U_{11}(R)]\ \lim_{x \to R-} \dot{h}(R)$$

The sign of this expression is opposite of the sign of cross derivative $U_{12}(x)$. If $U_{12}(x) = 0$, then this expression is also equal to zero. Hence the proof.

Theorem 3.3.18

If the rate of interest is positive (or greater than the subjective discount rate) and there is an interior peak in work hours, it will occur before the peak in wage rate.

Proof

First of all, work hours are constant at zero after retirement and, therefore, cannot peak. Let x^w be the age at which wage rate peaks. Then, at each working age $x \geq x^w$, $h(x)$ must be decreasing according to the differential equation (3.3.73). Therefore, if there is a peak in work hours, it must be at some age, called x^h, below age x^w.

Theorem 3.3.19

If the rate of interest is positive (or greater than the subjective discount rate) and $U_{12}(x)$ is negative, then an interior peak in consumption can occur only after the peak in the wage rate and before retirement.

Proof

At age $x \leq x^w$, the wage rate is non-decreasing. With $U_{12}(x) < 0$, consumption must also be non-decreasing if the rate of interest is zero. Since the rate of interest is positive, consumption must be increasing at age $x \leq x^w$. Therefore, if there is a peak in consumption, it must be at some age, called x^c, after age x^w. During retirement consumption is strictly increasing because of a positive rate of interest. Therefore an interior peak in consumption is not possible during retirement.

We can now discuss the general shapes of the consumption and work schedules. First consider the work schedule. Ignoring for the moment, the possibility of retirement, we can see from the differential equation (3.3.73) that at very early years of life when the wage rate is increasing, its effect in increasing work hours may dominate the opposite effect of a positive rate of interest. But, since the wage rate is assumed to be increasing with age at a diminishing rate, the effect of a positive rate of interest in depressing work hours will eventually dominate the opposite effect of a slow growth in the wage rate and therefore work hours will start decreasing with age before the wage rate is at peak (theorem 3.3.18). After the peak, the wage rate starts decreasing with age, reinforcing the effect of a positive rate of interest in decreasing work hours. It is possible that work hours reach zero before the end of horizon, that is, age T. If this happens, the individual will retire from work for the rest of his/her life (theorem 3.3.16). Since during the early years of life, work hours may be increasing with age, it seems possible that, at the beginning of the horizon work hours are equal to zero. But this possibility can be ignored by assuming that the planning horizon starts at a later age when the wage rate has increased to a sufficient level at which work hours are positive.

For the age pattern of consumption let us assume that $U_{12}(x)$ is negative, that is, consumption and leisure are substitutes for each other. Therefore, consumption is increasing with age due to the anticipated increase in the wage rate. Thus, if the rate of interest is positive, at the early years of life when the wage rate is increasing, consumption is also increasing with age. Consumption will continue to increase even after the wage rate starts decreasing with age if the rate of decrease in the wage rate is small. However, since the wage rate is assumed to be decreasing with age at an increasing rate, consumption may eventually start decreasing with age at the later segment of life. During retirement, consumption does not respond to changes in the wage rate. Since the rate of interest is positive, consumption will be increasing with age during retirement.

Now we are in a position to discuss the age pattern of earnings. Earnings at age x, m(x), are defined as

$$m(x) = w(x)h(x) \tag{3.3.76}$$

Differentiating this equation with respect to x, we obtain the differential equation describing the age profile of earnings:

$$\dot{m}(x) = w(x)\dot{h}(x) + h(x)\dot{w}(x) \qquad (3.3.77)$$

First of all, it is obvious that earnings are constant at zero during retirement when $h(x) = \dot{h}(x) = 0$. Before retirement, earnings will display the following age pattern:

Theorem 3.3.20

If there is an interior peak in work hours, earnings will also peak. If the rate of interest is positive (or greater than the subjective discount rate), earnings are increasing with age before the peak in work hours and decreasing after the peak in the wage rate. Finally, earnings peak later in life than work hours and earlier than the wage rate.

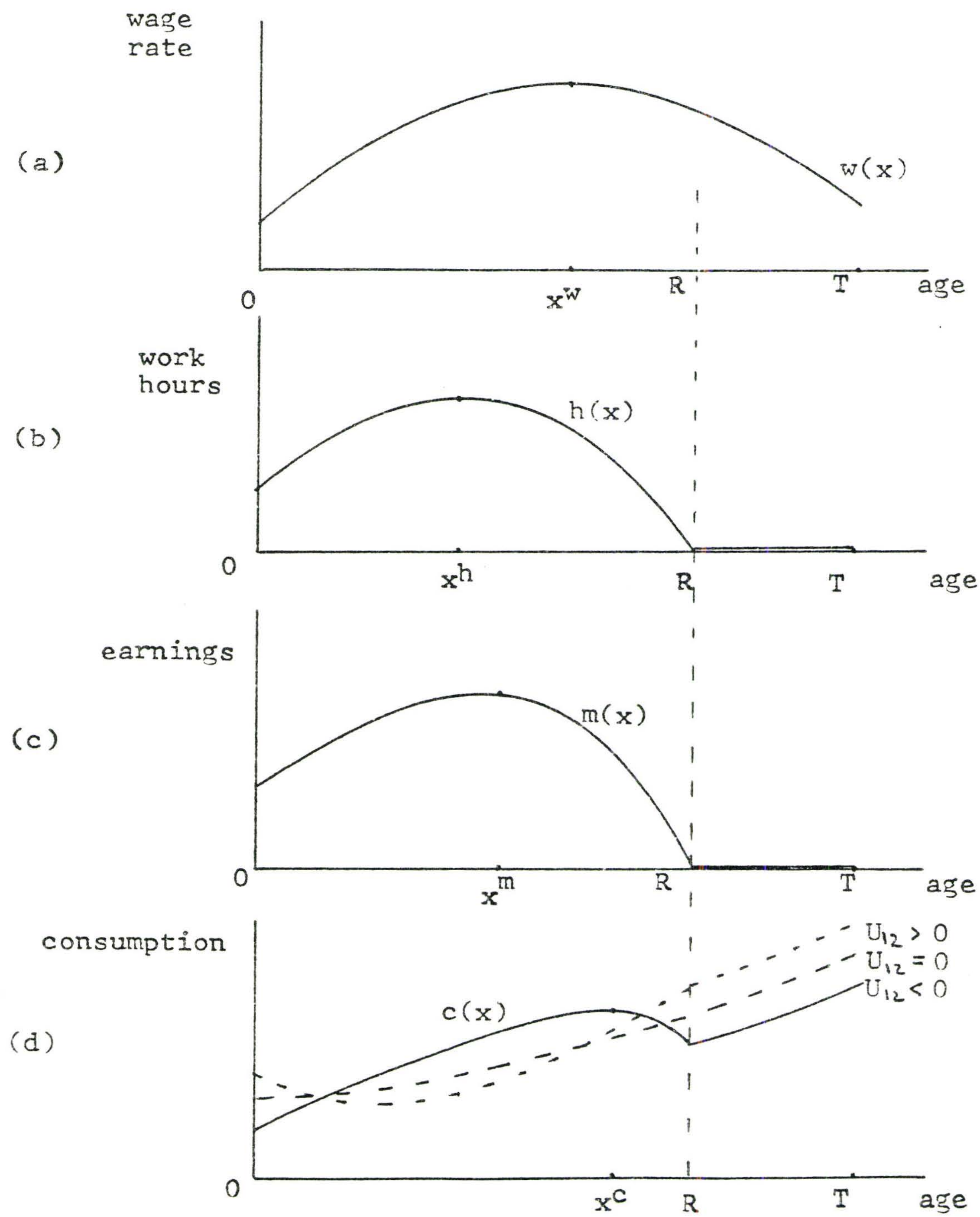
Proof

Before age x^h when work hours are at peak, the wage rate must be increasing according to theorem 3.3.18. Equation (3.3.77) implies that earnings are increasing with age during this segment of life. Similarly, after age x^w, when the wage rate is at a peak, work hours must be decreasing with age. Equation (3.3.77) implies that earnings are decreasing with age after age x^w. Finally, at age x^h, $h(x)$ is constant and $w(x)$ is increasing with age. Therefore $m(x)$ is also increasing with age. At age x^w, $w(x)$ is constant but $h(x)$ is decreasing with age. Therefore $m(x)$ must be decreasing with age. Then, earnings must peak at some age, called x^m, between age x^h and age x^w.

The "typical" age patterns of the wage rate, work hours, earnings and consumption for a positive interest rate are shown in figure 3.3.1. In part (a) we draw a wage profile which is concave, increasing with age up to age x^w and decreasing thereafter. In part (b) we show the age path of work hours. It is assumed that the individual does retire from work at age R. The age profiles of earnings and consumption are displayed in part (c) and (d) respectively. Three consumption profiles are shown in the figure, one for each case: $U_{12}(x) < 0$, $U_{12}(x) > 0$ and $U_{12}(x) = 0$. The consumption profile for the case $U_{12}(x) < 0$ is drawn with a solid line. For the other two cases the consumption profile is drawn with a broken line. The shapes of these profiles follows from our earlier results.

We have drawn the earnings and consumption profiles in two different diagrams. The reason is, that at this stage we do not know their relative positions. We will

Figure 3.3.1 The Age Profiles of Work Hours, Earnings and Consumption with a Hump Shaped Wage-Age Relation and a Positive Rate of Interest



consider this in the next section where we discuss the age pattern of savings and asset holding.

Now we relate the growth rates of demands for consumption and leisure with the rate of interest and the growth rate of wage rate. First we rewrite equations (3.3.66) and (3.3.67) as follows.

$$\dot{c}(x)/c(x) = [\partial c(x)/\partial w(x)]\ [w(x)/c(x)]\ [\dot{w}(x)/w(x)] + [\partial c(x)/\partial e(x)]\ [e(x)/c(x)]\ [\dot{e}(x)/e(x)] \quad (3.3.78)$$

$$\dot{l}(x)/l(x) = [\partial l(x)/\partial w(x)]\ [w(x)/l(x)]\ [\dot{w}(x)/w(x)] + [\partial l(x)/\partial e(x)]\ [e(x)/l(x)]\ [\dot{e}(x)/e(x)] \quad (3.3.79)$$

where, we have converted equation (3.3.67) in terms of leisure, $l(x) = \theta - h(x)$. Now we define the following elasticities:

$$\eta_1(x) = [\partial c(x)/\partial w(x)]\ [w(x)/c(x)] \quad (3.3.80)$$

$$\eta_2(x) = [\partial l(x)/\partial w(x)]\ [w(x)/l(x)] \quad (3.3.81)$$

$$\epsilon_1(x) = [\partial c(x)/\partial e(x)]\ [e(x)/c(x)] \quad (3.3.82)$$

$$\epsilon_2(x) = [\partial l(x)/\partial e(x)]\ [e(x)/l(x)] \quad (3.3.83)$$

Using these notations for the elasticities, we can express the growth rates of consumption and leisure in terms of the growth rates of wage rate and expenditure as follows.

$$\dot{c}(x)/c(x) = \eta_1(x)\ [\dot{w}(x)/w(x)] + \epsilon_1(x)\ [\dot{e}(x)/e(x)] \quad (3.3.84)$$

$$\dot{l}(x)/l(x) = \eta_2(x)\ [\dot{w}(x)/w(x)] + \epsilon_2(x)\ [\dot{e}(x)/e(x)] \qquad (3.3.85)$$

This gives us the following results:

Theorem 3.3.21

Given the rate of expenditure, the growth rate of consumption (leisure) is equal to the growth rate of wage rate multiplied by the elasticity of demand for consumption (leisure) with respect to the wage rate.

Theorem 3.3.22

If the wage rate is held constant over the life cycle, the growth rate of consumption (leisure) is equal to the growth rate of expenditure multiplied by the elasticity of demand for consumption (leisure) with respect to expenditure.

The growth rates of consumption and leisure can alternatively be expressed in terms of the rate of interest and the growth rate of wage rate. Substituting equation (3.3.60), and using the above notations for elasticities, equations (3.3.78) and (3.3.79) can be written in terms of elasticities, etc. as follows.

$$\dot{c}(x)/c(x) = [n_1(x) + \epsilon_1(x)S_2(x) \{1-\epsilon_2(x)/\delta(x)\}] \{\dot{w}(x)/w(x)\}] + r\ [\epsilon_1(x)/\delta(x)] \tag{3.3.86}$$

$$\dot{l}(x)/l(x) = [n_2(x) + \epsilon_2(x)S_2(x) \{1-\epsilon_2(x)/\delta(x)\}] \{\dot{w}(x)/w(x)\}] + r\ [\epsilon_2(x)/\delta(x)] \tag{3.3.87}$$

Following results are implied by these equations.

Theorem 3.3.23

If the wage rate is held constant, the growth rate of consumption (leisure) is proportional to the ratio of the elasticity of demand for consumption (leisure) with respect to expenditure to the coefficient of relative risk aversion.

Theorem 3.3.24

If the rate of interest is equal to zero (or equal to the subjective discount rate), the growth rates of consumption and leisure can be expressed as the products of the growth rate of wage rate by a factor involving the elasticities of demands with respect to expenditure and wage rate, the expenditure shares and the coefficient of relative risk aversion.

This concludes our analysis of the life cycle consumption and work allocation behaviour of the individual. We can now move to the study of life cycle savings and assets accumulation behaviour of the individual.

3.4 SAVING BEHAVIOUR AND THE DYNAMICS OF ASSET HOLDING

In this section we discuss the life cycle savings and asset holding behaviour of the individual. There are two measures of savings we are interested in. First, there is a <u>narrow measure</u> of savings defined as the excess of earnings over consumption. Although this concept does not really measure actual savings because it ignores interest income on the accumulated wealth or the interest costs on the accumulated net borrowings, it bears great operational significance for our analysis. This narrow measure of savings will be useful to understand the dynamics of asset holding. In addition, in our analysis of mortality variations, this measure of savings will play the key role. The <u>broad measure</u> of savings is defined as the excess of income, including both earnings and possible interest income, over current spending on consumption and possible interest payments on the net borrowings.

Let us first discuss the narrow measure of savings. The rate of these savings is defined as

$$\sigma(x) = m(x) - c(x) \tag{3.4.1}$$

Recalling the definition of expenditure and earnings from equations (3.3.1) and (3.3.76), notice that the schedule $\sigma(x)$ is closely related to the schedule $e(x)$ as follows.

$$\sigma(x) = w(x)\theta - e(x) \tag{3.4.2}$$

Thus, the savings rate $\sigma(x)$ can also be interpreted as the excess of the value of the current endowment of time over the expenditure on consumption and leisure. This relationship also allows the following inference from theorem 3.3.6.

Theorem 3.4.1

There exists a solution for the schedule $\sigma(x)$

$$\begin{aligned} \sigma(x) &= w(x)\theta - e(x, P, r, W) \\ &= \sigma(x, P, r, W) \end{aligned} \tag{3.4.3}$$

The function $\sigma(x, P, r, W)$ is everywhere continuous in all its arguments and has continuous first partial derivatives except at a switching point.

The age pattern of $\sigma(x)$ can be studied by differentiating equation (3.4.2) with respect to x:

$$\dot{\gamma}(x) = \theta\ \dot{w}(x) - \dot{e}(x) \tag{3.4.4}$$

Substituting for $\dot{e}(x)$ from equation (3.3.43), we obtain the differential equation for $\gamma(x)$ for an interior solution:

$$\dot{\gamma}(x) = r\ [\Theta(x)/U_1(x)\Delta(x)] + [h(x) - \{\mu_1(x)/U_2(x)\Delta(x)\}]\ \dot{w}(x) \tag{3.4.5}$$

Likewise, substituting for $\dot{e}(x)$ from equation (3.3.44), we obtain the differential equation for $\gamma(x)$ for a corner solution:

$$\dot{\gamma}(x) = r\ [U_1(x)/U_{11}(x)] \tag{3.4.6}$$

Two obvious results follow from these differential equations.

Theorem 3.4.2

> Holding the wage rate constant, if the rate of interest is positive (or greater than the subjective discount rate), the schedule $\gamma(x)$ will be decreasing with age throughout the life cycle.

This is a conventional result. With preference for present over future spending being exactly offset by life insurance and annuities, a positive rate of interest will encourage a delay in spending in the life cycle. This will produce the savings behaviour stated above.

Theorem 3.4.3

> In case of an interior solution, if leisure is a normal good then $g(x)$ is positively correlated with the anticipated wage rate. In case of a corner solution, the schedule $g(x)$ is independent of the wage rate.

This result complements the earlier finding by Heckman (1974) and Ghez and Becker (1975) that under certain condition ($U_{12}(x) < 0$), the life cycle model could explain a positive relationship between consumption and anticipated earned income over the life cycle. The above result implies that over the life cycle, wage induced variations in consumption will always be smaller than wage induced variations in earnings both measured in the absolute terms. Thus, our model predicts that while the marginal propensity to consume out of anticipated earned income may be positive under certain condition, it will definitely be less than one. This result suggests that the Keynesian absolute income hypothesis is supported even in the life cycle context.

For the illustration of this point, consider the wage profile in figure 3.4.1(a). If the rate of interest is equal to zero (or equal to the subjective discount rate), the earnings profile will follow the path as shown in figure 3.4.1(b). If $U_{12}(x) < 0$, consumption will follow a similar path as shown in the figure. Theorem 3.4.3 and the lifetime budget constraint

$$\int_0^T p(x)\ \sigma(x)\ dx = 0$$

imply that the savings schedule $\sigma(x)$ will also follow a similar hump-shaped age path as shown in figure 3.4.1(b).[14] This age profile of savings clarifies the implied restrictions on the placement of consumption and earnings profiles. In particular, at the age of the peak wage rate where both consumption and earnings are also at peak, consumption not only cannot exceed earnings but also the former will fall short of the latter by the maximum margin.

Now, we can discuss the general shape of the schedule $\sigma(x)$ over the life cycle when the rate of interest is positive. During the retirement period of life, the schedule $\sigma(x)$ is independent of the wage rate. Therefore,

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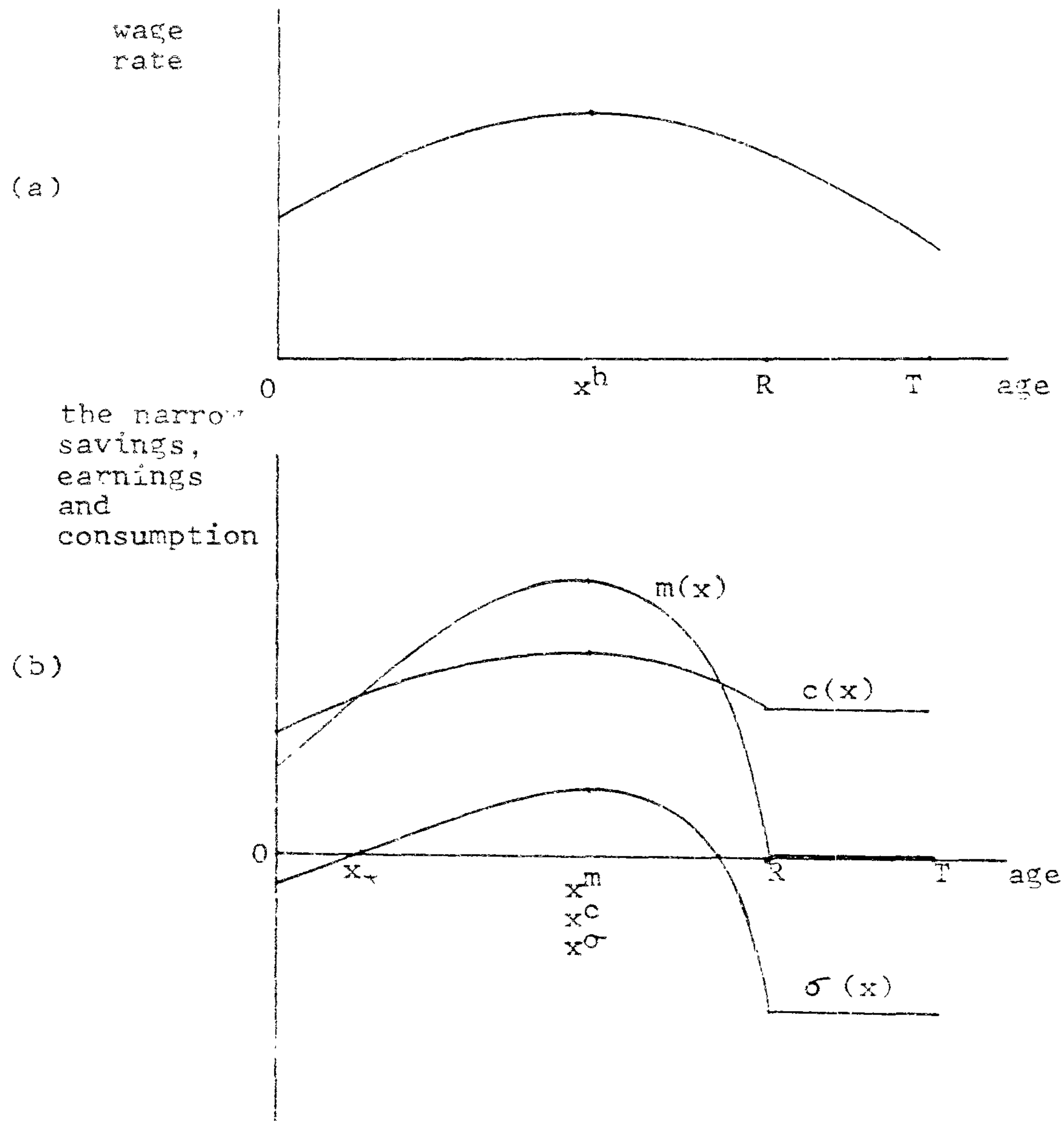
14 During the early years of life the savings schedule need not be negative. The consumption and earnings schedules shown in figure 3.4.1 differ from the ones in Heckman (1974) and Ghez and Becker (1975) because they did not consider retirement in their analysis.

Figure 3.4.1 The Relationship Among the Anticipated Wage Rate, the Planned Consumption, Earnings and the Narrow Measure of Savings with a Zero Rate of Interest



savings will unambiguously decrease with age after retirement. In view of our assumption that the wage rate is increasing with age at a diminishing rate up to age x^w, during some early years of life the increase in the wage rate may increase savings fast enough to outweigh the opposite effect of a positive rate of interest. Therefore, during some early years of life the savings rate $\sigma(x)$ may be increasing with age. However, early on the schedule $\sigma(x)$ will start decreasing with age and will continue to do so since the wage rate is increasing with age at a diminishing rate up to some age and decreasing thereafter. If we assume that during some early years of life the savings rate $\sigma(x)$ does increase with age, then the age profile of $\sigma(x)$ will follow an inverted J-shaped path. There are two more results regarding the shape of $\sigma(x)$ which we would like to report.

Theorem 3.4.4

> If the rate of interest is positive (or greater than the subjective discount rate), the savings schedule $\sigma(x)$ can peak only before age x^m at which earnings peak.

The proof is somewhat involved and is given in appendix 3.A.3.

Theorem 3.4.5

If the rate of interest is positive (or greater than the subjective discount rate) and the wage rate is non-increasing at the retirement age or, the rate of interest is non-negative (or not less than the subjective discount rate) and wage rate is decreasing at the retirement age, then there will be an upward jump in the <u>slope of</u> savings schedule $\sigma(x)$ at the retirement age.

Proof

The left side and the right side limits of the slope of $\sigma(x)$ as x approaches toward R, the retirement age, can be obtained from the differential equations (3.4.5) and (3.4.6) respectively. The difference between the two limits is

$$\lim_{x \to R-} \dot{\sigma}(x) - \lim_{x \to R+} \dot{\sigma}(x) = - [\mu_1(R)/U_2(R)\delta(R)]\, \dot{w}(R)$$
$$+ r\, [\mu_1(R)^2 / U_1(R)U_2(R)^2 U_{11}(R)\delta(R)]$$

This difference is negative under any of the two sufficient conditions stated. Hence the proof.

There is more we can say about the age path of $\sigma(x)$. The lifetime budget constraint can be written as:

$$\int_0^T e^{-rx}\, p(x)\, \sigma(x)\, dx = 0 \qquad (3.4.7)$$

Thus, the weighted sum of lifetime savings is equal to zero. We know that during retirement earnings are zero but consumption is positive. Therefore during retirement $\sigma(x)$ is negative. Since the schedule $\sigma(x)$ follows an inverted J-shaped path, $\sigma(x)$ must be positive during some middle years of life. Continuity of the earnings and consumption schedules implies that age x^* at which $\sigma(x)$ changes from a positive value to a negative, must be less than the retirement age R. Since $\sigma(x)$ is increasing with age during early years of life, it may also be negative during some early years.

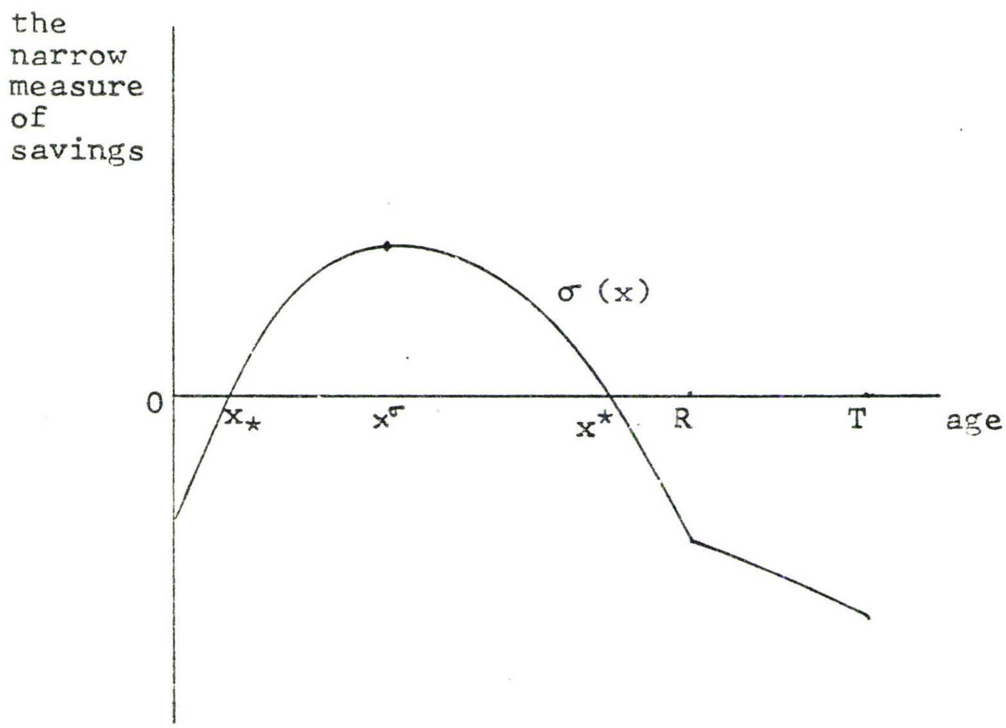
The age pattern of $\sigma(x)$ is illustrated in figure 3.4.2. It is assumed that $\sigma(x)$ does start negative. The age at which $\sigma(x)$ changes from negative to positive and the age at which it changes from positive to negative are marked x_* and x^* respectively.

Now we turn to the broad measure of savings which can formally be defined as

$$\begin{aligned} s(x) &= w(x)h(x) + r^*(x)a(x) - c(x) \\ &= w(x)h(x) + [r + q(x)/p(x)]\, a(x) - c(x) \qquad (3.4.8) \end{aligned}$$

where, $r^* = r + q(x)/p(x)$ is the actuarially fair rate of interest and $a(x)$ is the stock of assets held at age x. If

Figure 3.4.2 The Age Profile of the Narrow Measure of Savings $\sigma(x)$



assets at age x are positive, this definition is interpreted as earned income plus interest income minus consumption. On the other hand, if $a(x)$ is negative, the definition should be interpreted as earned income minus interest payments on borrowings minus consumption. Substituting $w(x)h(x) - c(x) = \sigma(x)$ in equation (3.4.8), we can also write

$$s(x) = \sigma(x) + [r + q(x)/p(x)]\ a(x) \qquad (3.4.9)$$

We know the general shape of the schedule $\sigma(x)$ but so far we have not said anything about the shape of the assets schedule $a(x)$. Therefore, at this stage we cannot discuss the savings schedule $s(x)$. In fact, the two schedule $a(x)$ and $s(x)$ are closely related to each other. Defined by equation (3.4.9), the rate of savings $s(x)$ is also equal to the net addition to assets at age x. Therefore we can write the differential equation describing the age path of asset holding.

$$s(x) = \dot{a}(x) = \sigma(x) + [r + q(x)/p(x)]\ a(x) \qquad (3.4.10)$$

Since the individual does not receive any transfer income, his/her assets at age zero are equal to zero. Given this initial condition, the differential equation (3.4.10) can be solved for $a(x)$. First recall from equation (3.2.4) that $q(x) = -\dot{p}(x)$. Therefore with a different arrangement, equation (3,4,10) can be written

$$\dot{a}(y) - r\,a(y) + [\dot{p}(y)/p(y)]\,a(y) = \sigma(y)$$

where, we have used y rather than x for the age index for a reason that will become obvious as we proceed. Multiplying both the sides by the discount factor $e^{-ry}\,p(y)$, we can write this equation as follows:

$$e^{-ry}p(y)\dot{a}(y) - r\,e^{-ry}p(y)a(y) + e^{-ry}\dot{p}(y)a(y) = e^{-ry}p(y)\sigma(y)$$

Or, $d[e^{-ry}p(y)a(y)]/dy = e^{-ry}p(y)\sigma(y)$

Integrating over the interval [0,x], we find

$$e^{-rx}\,p(x)\,a(x) - p(0)\,a(0) = \int_0^x e^{-ry}\,p(y)\,\sigma(y)\,dy$$

Since $a(0) = 0$, we can solve for $a(x)$ as:

$$a(x) = [e^{rx}/p(x)]\int_0^x e^{-ry}\,p(y)\,\sigma(y)\,dy \qquad (3.4.11)$$

This relationship between assets and savings implies the following result.

Theorem 3.4.6

The solution for a(x) can be written

$$a(x) = a(x, P, r, W) \qquad (3.4.12)$$

The function $\dot{a}(x, P, r, W)$ is continuous in all its arguments and has a continuous age derivative at all the points including the possible switching points. In addition the first partial derivatives of this function with respect to other arguments are also continuous except at a switching point.

Proof

Substituting the solution for $\sigma(x)$ given by equation (3.4.3) into equation (3.4.11), we can write

$$a(x) = [e^{rx}/p(x)] \int_0^x e^{-ry}\, p(y)\, \sigma(y, P, r, W)\, dy$$

Since $\sigma(x, P, r, W)$ is continuous in all its argument, so is the function $a(x, P, r, W)$. Next, continuity of the first partial derivatives of $a(x, P, r, W)$ follows from the continuity of the first partial derivatives of the function $\sigma(x, P, r, W)$. Finally, according to equation (3.4.10), the age derivative of $a(x)$ can be expressed in terms of the following function.

$$\dot{a}(x) = \sigma(x, P, r, W) + [r + q(x)/p(x)]\, a(x, P, r, W)$$

Since the functions $\sigma(x, P, r, W)$ and $a(x, P, r, W)$ are continuous at a switching point, the age derivative of $a(x, P, r, W)$ given above is also continuous at such point.

The relationship between savings and assets given by equation (3.4.11) has a straight forward interpretation. First, rewrite this equation as

$$a(x) = \int_0^x [e^{-ry}p(y)/e^{-rx}p(x)] \, \sigma(y) \, dy \qquad (3.4.13)$$

From equation (3.2.8) we know that

$$e^{-ry} \, p(y) = \exp\left[- \int_0^y r^*(z) \, dz\right]$$

$$e^{-rx} \, p(x) = \exp\left[- \int_0^x r^*(z) \, dz\right]$$

where, $r^*(z) = r + q(z)/p(z)$ is the actuarial rate of interest. These two equations imply

$$e^{-ry}p(y)/e^{-rx}p(x) = \exp\left[- \int_0^y r^*(z) \, dz + \int_0^x r^*(z) \, dz\right]$$

$$= \exp\left[\int_y^x r^*(z) \, dz\right]$$

Using this result in equation (3.4.13), we obtain the desired expression

$$a(x) = \int_0^x \exp\left[\int_y^x r^*(z)\ dz\right] \sigma(y)\ dy \tag{3.4.14}$$

Equation (3.4.14) says that the level of assets at age x are obtained by accumulating the excess of earnings over consumption (positive or negative) incurred at each previous age $y < x$, inflated by the appropriate interest factor.

Now we can discuss the age pattern of the assets schedule. First, as we have already noted, assets at age zero are equal to zero, that is $a(0) = 0$. Second, if the individual survives, his/her assets at age T are also equal to zero. If we look at the definition of a(x) given by equation (3.4.11), it seems that $a(T) = 0/0$ which is an undefined number. But applying the appropriate (L'Hospital) rule for the limit of a(x) as x approaches T, we obtain a definite answer.

$$\underset{x \to T}{\text{Limit}}\ a(x) = \frac{\underset{x \to T}{\text{Limit}} \dfrac{d}{dx}\left[e^{rx} \displaystyle\int_0^x e^{-ry}\ p(y)\ \sigma(y)\ dy\right]}{\underset{x \to T}{\text{Limit}}\ dp(x)/dx}$$

Upon solving, we obtain

$$\operatorname*{Limit}_{x \to T} a(x) = [1/\dot{p}(T)] \left[p(T)\sigma(T) + r\ e^{rT} \int_0^T e^{-ry} p(y)\sigma(y)dy \right]$$

The second term inside the square brackets is equal to zero because of the lifetime budget constraint. With $p(T) = 0$, the first term is also equal to zero. Since $\dot{p}(T) = - q(T) \neq 0$ (see conditions (3.2.1)), the whole expression is equal to zero. Therefore

$$\operatorname*{Limit}_{x \to T} a(x) = a(T) = 0$$

The age pattern of asset holding at other segments of life can be summarized in the following theorem.

Theorem 3.4.7

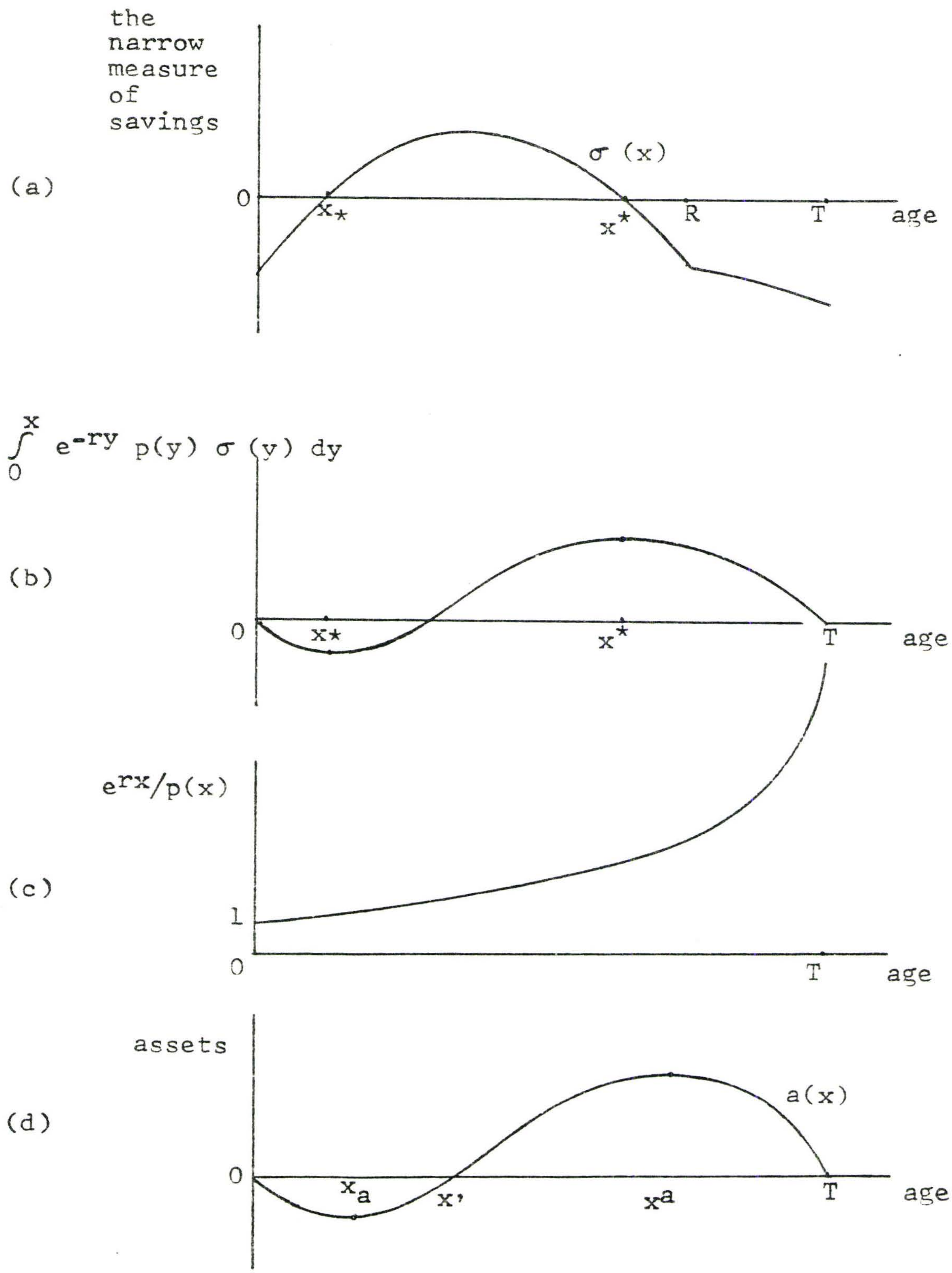
If during some early years of life the savings rate $\sigma(x)$ is negative, the assets schedule $a(x)$ will also be negative. The age at which $a(x)$ reaches its minimum, called x_a, comes after age x_*. At some point between age x_a and age x^*, $a(x)$ becomes zero and after this age $a(x)$ is positive except at age T where it is again equal to zero. The age at which $a(x)$ peaks, called x^a, comes after age x^*.

Proof

Since $a(0) = 0$ and $\sigma(x)$ is negative until age x_*, equation (3.4.11) implies that $a(x)$ must also be negative. The differential equation (3.4.10) shows that at age x_*, at which $\sigma(x)$ is equal to zero and $a(x)$ is negative, $a(x)$ is decreasing with age. This implies that the age at which $a(x)$ will reach its minimum, called x_a, will come after age x_*. Since $a(T) = 0$ and $\sigma(x) < 0$ after age x^*, equation (3.4.11) shows that $a(x^*)$ must be positive. Continuity of $a(x)$ implies that $a(x)$ will become zero at some age, called x', after age x_a and before age x^*. Since at age x', $a(x) = 0$ and $\sigma(x) > 0$, equation (3.4.10) implies that $a(x)$ must be increasing with age. Equation (3.4.11) also implies that $a(x)$ will remain positive in the interval: $x' < x < T$. Finally, at age x^*, $\sigma(x) = 0$ and $a(x) > 0$. Therefore, according to the differential equation (3.4.10), $a(x)$ must be increasing at age x^*. Thus $a(x)$ will peak after age x^*.

The age profile of assets schedule, $a(x)$ is sketched in figure 3.4.3. In part (a) we plot the age profile of the narrow savings schedule, $\sigma(x)$. In part (b) we plot the function:

Figure 3.4.3 The Age Profile of Asset Holding



$$\int_0^x e^{-ry}\, p(y)\, \sigma(y)\, dy$$

According to the age pattern of savings schedule $\sigma(x)$ shown in part (a) of the figure, the value of this function is decreasing up to age x_*, when $\sigma(x)$ is negative, increasing in the interval $x_* < x < x^*$, when $\sigma(x)$ is positive and decreasing thereafter, when $\sigma(x)$ is again negative. This function reaches at its minimum and maximum at age x_* and age x^* respectively. In part (c) of the figure we plot the function $e^{rx}/p(x)$ which is everywhere increasing in age. The plot of a(x) is obtained in part (d) by multiplying the two functions drawn in part (b) and part (c).

The age pattern of asset holding shows that during some early years of life the individual is a net borrower and in the later part of life he/she is a net lender. In defining an actuarially fair rate of interest we have assumed that at the time of death, all the business of the individual with the insurance company is finished. The insurance company assumes the individual's assets or liabilities when he/she dies. Someone who dies in debt is bailed out and while alive has paid an interest rate higher than that on riskless debt and someone who dies with positive assets forfeits them but has enjoyed a rate of return above the rate on riskless bonds while alive. Let us explore this point a little further. The actuarial rate of interest for an individual aged x exceeds the rate on bonds by the instantaneous conditional probability of death $q(x)/p(x)$. Therefore, if the individual dies at age x, the present value of his/her extra interest income (positive or negative depending on the sign patter of $a(y)$ during his/her life) due to the actuarial rate of interest is

$$\int_0^x e^{-ry}\, p(y)\, [\{q(y)/p(y)\}\, a(y)]\, dy$$

Since the probability of death at age x is $q(x)$, the expected present value of the extra interest income is

$$\int_0^T q(x) \int_0^x e^{-ry}\, p(y)\, [\{q(y)/p(y)\} a(y)]\, dy\, dx$$

Changing the order of integration and simplifying, we can write this expression as follows.

$$\int_0^T q(x)\, [e^{-rx} p(x) a(\ )]\, dx$$

Since $e^{-rx}p(x)a(x)$ is the present value of assets held at age x and $q(x)$ is the probability of death at age x, this expression measures the expected present value of assets the individual will lose in case of death. Thus the expected present value of extra interest income is exactly equal to the expected present value of assets left by the individual at the time of death. This means that the provision of actuarially fair life insurance and annuities allows the individual to pool the monetary risk of an uncertain death with others. It is in this sense that Barro and Friedman (1977) suggest that in the presence of actuarially fair life insurance and annuities an individual will prefer an uncertain life horizon as compared to the situation in which the length of life is randomly selected and announced to the individual at the beginning of planning horizon.

Now we come back to the broad measure of savings $s(x)$ defined as the rate of change in asset holding (equation (3.4.10)). First we write the following obvious result.

Theorem 3.4.8

The savings schedule $s(x)$ can be expressed in term of the following function

$$s(x) = s(x, q(x), P, r, W) \tag{3.4.15}$$

The function $s(x, q(x), P, r, W)$ is everywhere continuous in all its arguments and has continuous first partial derivatives except at a switching point.

This result follows directly from the definition of $s(x)$ given by equation (3.4.10) and continuity of the functions $\gamma(x, P, r, W)$ and $a(x, P, r, W)$ and their first partial derivatives.

Next, we reproduce below the relationship among the two savings rates and assets given by equation (3.4.10).

$$s(x) = \dot{a}(x) = \gamma(x) + [r + q(x)/p(x)]\ a(x) \qquad (3.4.16)$$

The two savings schedules and assets schedule satisfy the following relationships. The savings schedule $s(x)$ is negative in the intervals $0 \le x < x_a$ and $x^a < x \le T$ when $a(x)$ is decreasing and positive in the interval $x_a < x < x^a$ when $a(x)$ is increasing. The broad measure of the savings rate, $s(x)$, is smaller (greater) than the narrow measure $\gamma(x)$ in the interval $0 \le x \le x'$ $(x' \le x \le T)$ when assets are negative (positive).

For further results, we differentiate the above savings schedule with respect to x. The result is as follows:

$$\dot{s}(x) = \dot{\gamma}(x) + [r + q(x)/p(x)]\ \dot{a}(x) + a(x)\ \dot{\pi}(x)$$

where, $\pi(x)$ is the instantaneous conditional probability of death $q(x)/p(x)$. Substituting the short notation $r^*(x)$ for the instantaneous actuarial rate of interest, $r + q(x)/p(x)$ and $s(x)$ for $\dot{a}(x)$, we can also write this equation:

$$\dot{s}(x) = \dot{\sigma}(x) + r^*(x)\ s(x) + a(x)\ \dot{\pi}(x) \qquad (3.4.17)$$

Again substituting for s(x) from equation (3.4.16), we find:

$$\dot{s}(x) = \dot{\sigma}(x) + r^*(x)\ \sigma(x) + r^*(x)^2\ a(x) + a(x)\dot{\pi}(x) \qquad (3.4.18)$$

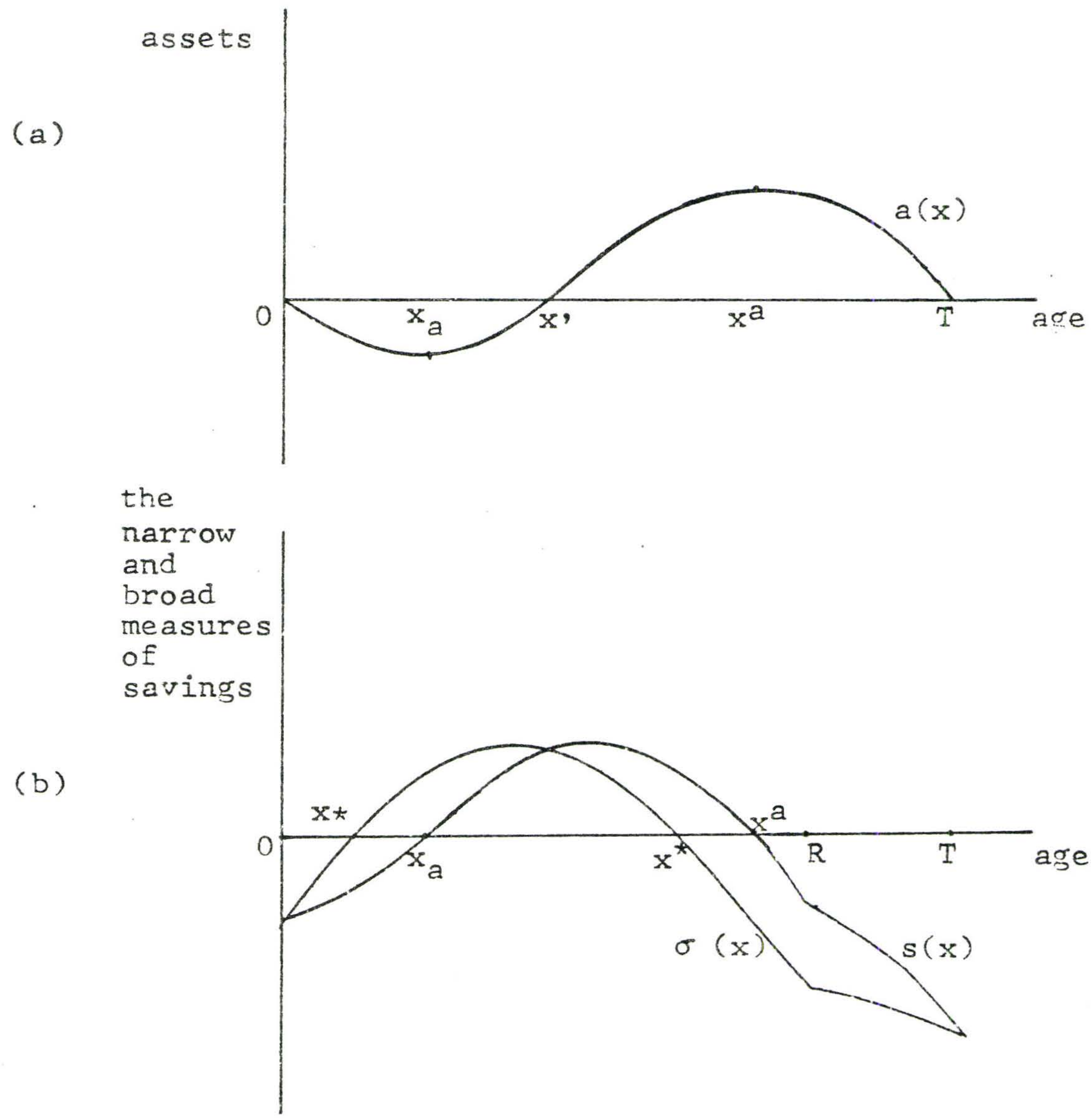
These relations imply the following result.

Theorem 3.4.9

Given that the conditional probability of death is increasing in the relevant range, if $a(x) \geq 0$ at the age when $s(x)$ is at a peak or when $\sigma(x)$ is at a peak, then the age at which $s(x)$ peaks, called x^s, will come later in life than age x^σ at which $\sigma(x)$ peaks.

Figure 3.4.4 illustrates all the relationships among the two measures of savings and assets. In part (a) is shown the age profile of assets. The two measures of savings are sketched in part (b). As shown in the figure, at the retirement age there is an upward jump in the slope of the broad measure of savings, that is, $s(x)$. This follows from equation (3.4.17), continuity of the schedules $s(x)$ and $\dot{\pi}(x)$ and theorem 3.4.5.

Figure 3.4.4 The Relationship Among the Two Measures of Savings and Assets



It is interesting to note the placement of age x^a, at which the individual starts decumulating, in comparison to the retirement age R. As shown in the figure, age x^a comes earlier than the retirement age R. However, the structure of our model does not necessarily imply this relation. In other words, within the framework of our analysis it is quite possible that the individual may be accumulating assets even during some early years of retirement.

Since at the retirement age earnings are zero, $s(R) = - c(R)$. Therefore the condition that $s(x)$ is positive at this age implies the following relation:

$$s(R) > 0 \quad \text{if and only if} \quad [r + \pi(R)]a(R) > c(R) \qquad (3.4.19)$$

That is, savings are positive if and only if the interest income is more than consumption. If $s(R)$ is positive, then assets must be increasing with age. If the conditional probability of death, $\pi(x)$, is non-decreasing, the interest income must also be increasing with age. Then consumption should also be increasing with age fast enough to make sure that savings will eventually become negative because if the individual survives to the last instant of the horizon, he/she would have consumed all the assets.

In any case, we can see from condition (3.4.19) that, if the actuarial rate of interest is high, the individual may be accumulating even after retirement. It has been suggested in the literature that, under life uncertainty, if life insurance and annuities are not available, the individual may continue to accumulate after retirement in order to avoid the risk of being left with little to consume if he/she survives longer (Davies (1981), and Zilcha and Friedman (1983)).

This result can be obtained in the present model due to a different reason. If $U_{12}(x) < 0$, consumption during early years of retirement will be increasing faster than the later years of working period of life (see Theorem 3.3.17 and diagram 3.3.1). For a few years before retirement consumption may have even started decreasing with age provided the wage rate is decreasing. This implies that at the beginning of the retirement period, consumption is expected to be lower due to the endogenous work allocation decision. On the other hand, income could be higher due to the actuarial rate of interest on assets which is higher than the rate on bonds. This implies that during some early years of retirement the broad measure of savings could be positive, and therefore, the individual may still be accumulating. Thus, besides other reasons, an endogenous work-leisure choice and uncertainty of life in the presence of life insurance and annuities could also explain the continued accumulation or mild decumulation by the elderly.

This completes the analysis of an individual's life cycle model. After summarizing the main findings of this chapter we will move to the comparative static analysis of this model.

3.5 CONCLUDING REMARKS

A two stage budgeting approach was adopted to explain consumption, work and retirement decisions in a life cycle model under uncertain lifetimes with actuarially fair life insurance and annuities. Due to life insurance and annuities, the effect of uncertainty on relative time preference is removed. Therefore, the differential equations describing the age paths of consumption and work hours were found to be the same as under certainty. In that respect many of the results regarding the age pattern of consumption and work hours found in this research are comparable with those already established in the literature for certain lifetimes. However, the application of the two stage budgeting approach, endogenization of retirement decision and studying in detail savings and assets holding provided many additional results for the life cycle hypothesis. In addition, the two stage budgeting approach better explained the life cycle allocation process. Some of the findings regarding life cycle allocation behaviour are summarized below.

The effect of an anticipated increase in wage rate on expenditure (on consumption and leisure) was decomposed into two parts. First, there is a cost of living effect which results in more expenditure on a given basket of consumption and leisure as the wage rate increases. Second, there is an intertemporal allocation effect which arises due to the reason that a wage increase makes expenditure more expensive. Given the marginal rate of substitution between consumption and leisure over the life cycle, an anticipated increase in wage rate will result in an increase (decrease) in expenditure if leisure is a normal (inferior) good.

The effects of an anticipated increase in wage rate on consumption and work hours were decomposed into two components: an intertemporal allocation effect and an intratemporal substitution effect. First, an anticipated increase in wage rate over the life cycle will make expenditure more expensive in future than in present. This will result in a decreasing (increasing) expenditure over the life cycle if leisure is a normal (inferior) good. This age pattern of expenditure will, in turn, result in an increasing work hours over the life cycle whether leisure is inferior or normal. If consumption and leisure are normal goods, consumption will be decreasing with age. But, if any of consumption and leisure are inferior, consumption will be increasing with age. This substitution over the life cycle may be interpreted as the intertemporal substitution effect. Second, as a result of an anticipated increase in the wage rate, consumption and work will increase due to substitution between leisure and consumption at each age. This effect has an interpretation of an intratemporal substitution effect. Thus the full effect of an anticipated increase in wage rate should not be regarded, like MaCurdy (1981), as an intertemporal substitution effect.

The life cycle motive explains the retirement decision. It was shown that, if the wage rate is non-increasing at retirement age and the rate of interest is positive, the individual will never re-enter the labour market once retired.

Before retirement, consumption was shown to be increasing and work hours decreasing with age due to a positive rate of interest. Work hours are positively correlated with anticipated variations in the wage rate over the life cycle. If consumption and leisure are substitutes for each other, consumption also positively responds to these variations in the wage rate. After retirement, consumption is increasing with age due to a positive rate of interest and is independent of the wage rate.

The relative position of the peak ages of different variables was studied. If the wage rate is increasing with age earlier in life and decreasing thereafter, consumption and work hours can have interior peaks. If work hours peak, then earnings will also peak. Work hours peak earlier in life than earnings which in turn peak earlier than the wage rate. Consumption can peak only after the peak in the wage rate.

The study of savings and assets accumulation behaviour enabled us to put additional restrictions on the relationship between consumption and earnings over the life cycle. In particular, over the life cycle, wage induced variations in consumption will always be smaller than the wage induced variations in earnings both measured in absolute terms. Thus, while the marginal propensity to consume out of anticipated earned income may be positive under certain condition, it will definitely be less than one. This result suggests that the Keynesian absolute income hypothesis may be supported even in the life cycle context.

It was also shown that the individual saves the maximum part of earnings before earnings are at peak (assuming, as usual, that wage rate is increasing earlier in life and decreasing thereafter). During early years of life consumption may exceed earnings. During middle years, consumption is less than earnings and finally, during the later years, consumption again exceeds earnings.

The life cycle behaviour of two measures of savings was studied, a narrow measure defined as the excess of earnings over consumption and a broad measure defined as the excess of full income (earnings plus net interest income) over consumption. At early years of life, when asset holdings are negative, interest income is also negative. Therefore the broad measure of savings is smaller than the narrow measure. In later years, when asset holding, and therefore interest income, is positive, the broad measure of savings exceeds the narrow measure.

While savings out of earnings are definitely negative at retirement, the savings out of full income could be positive or negative. Consumption during some early years of retirement is expected to be lower if consumption and leisure are substitutes. On the other hand, interest income is higher due to actuarially fair annuities which yield a higher rate of return than bonds. Therefore, with an endogenous work-leisure decision and actuarially fair annuities, it is possible that during some early years of retirement, the individual may still accumulate wealth.

The growth rates of expenditure, consumption and leisure were related with the rate of interest and the growth rate of wage rate. These relations depend on the coefficient of relative risk aversion, the lower stage elasticities of demand for consumption and leisure with respect to expenditure and wage rate and the lower stage expenditure shares of consumption and leisure.

3.A APPENDIX TO CHAPTER 3

3.A.1 Proof of Theorem 3.3.5

We will prove properties (i) through (iv) first for an interior solution, then for a corner solution and finally for a switching point solution.

Interior Solution

(i) By assumption 1, $U(x)$ is continuous in $c(x)$ and $l(x)$ which in turn are continuous in $w(x)$ and $e(x)$ by theorem 3.3.1. Therefore, by definition, $V(x)$ is continuous in $w(x)$ and $e(x)$. The first partial derivatives of $V(x)$ are

$$V_1(x) = -\{\theta - h(x)\} U_1(x) \qquad (3.A.1)$$

$$V_2(x) = U_1(x) \qquad (3.A.2)$$

Since by assumption 1, $U_1(x)$ is continuous in $c(x)$ and $h(x)$ which in turn are continuous in $w(x)$ and $e(x)$ by theorem 3.3.1, both the partial derivatives $V_1(x)$ and $V_2(x)$ are continuous in $w(x)$ and $e(x)$. Next, consider the second partial derivatives which can be obtained from equations (3.A.1) and (3.A.2) as follows.

$$V_{11}(x) = -\{\theta - h(x)\}[U_{11}(x)\{\partial c(x)/\partial w(x)\} - U_{12}(x)\{\partial h(x)/\partial w(x)\}] + U_1(x)\ [\partial h(x)/\partial w(x)] \qquad (3.A.3)$$

$$V_{12}(x) = U_{11}(x)[\partial c(x)/\partial w(x)] - U_{12}(x)[\partial h(x)/\partial w(x)] \quad (3.A.4)$$

$$V_{22}(x) = U_{11}(x)[\partial c(x)/\partial e(x)] - U_{12}(x)[\partial h(x)/\partial e(x)] \quad (3.A.5)$$

All the first and second partial derivatives of the direct utility function appearing in the above equations are continuous in $c(x)$ and $h(x)$ by assumption 1. Similarly $c(x)$ and $h(x)$ and their first partial derivatives with respect to $w(x)$ and $e(x)$ are continuous by theorem 3.3.1. Therefore all the second partial derivatives $V_{ij}(x)$ given above are continuous.

(ii) Since $U_1(x)$ is positive by assumption 2, equation (3.A.2) implies that $V_2(x)$ is also positive.

(iii) Since $c(x)$ or θ-$h(x)$ cannot be negative, both must be equal to zero when $e(x) = c(x) + w(x)\{\theta-h(x)\}$ is equal to zero. Therefore, using (3.A.2) and assumption 3, we conclude

$$V_2(x)\big|_{e(x)=0} = U_1(x)\big|_{c(x)=0} = \infty$$

(iv) Substituting for $\partial c(x)/\partial e(x)$ and $\partial h(x)/\partial e(x)$ from equations (3.3.12) and (3.2.13) respectively in equation (3.A.5) and simplifying, we find

$$V_{22}(x) = U_1(x)^2\Delta(x)/Q(x) \quad (3.A.6)$$

$Q(x)$, defined by equation (3.2.11) is negative and $\Delta(x)$, defined below is positive under the assumption of strict concavity of the utility function.

$$\Delta(x) = U_{11}(x)U_{22}(x) - U_{12}(x)^2 \tag{3.A.7}$$

Since $Q(x) < 0$ and $\Delta(x) > 0$, $V_{22}(x)$ must be negative. Therefore the indirect utility function $V(x)$ is strictly concave in $e(x)$.

Corner Solution

(i) Continuity of the indirect utility function follows from assumption 1 and theorem 3.3.1 in the same way as under an interior solution. The first and second partial derivatives of $V(x)$, which can be easily obtained, are

$$V_1(x) = -\theta U_1(x) \tag{3.A.8}$$

$$V_2(x) = U_1(x) \tag{3.A.9}$$

$$V_{11}(x) = \theta^2 U_{11}(x) \tag{3.A.10}$$

$$V_{12}(x) = -\theta U_{11}(x) \tag{3.A.11}$$

$$V_{22}(x) = U_{11}(x) \tag{3.A.12}$$

All these partial derivatives are continuous in $c(x)$ and $h(x)$ by assumption 1. In turn, $c(x)$ and $h(x)$ are continuous in $w(x)$ and $e(x)$. Therefore all the first and second partial derivatives of $V(x)$ given above are continuous in $w(x)$ and $e(x)$.

(ii) Equation (3.A.9) implies that $V_2(x) > 0$ by assumption 2.

(iii) This property is not relevant for a corner solution because, as theorem 3.3.2 says, a corner solution is not possible at $e(x) = 0$.

(iv) Equation (3.A.12) implies that $V_{22}(x) < 0$ by assumption 4. Therefore $V(x)$ is strictly concave.

Switching Point Solution

(i) Since $c(x)$ and $h(x)$ are continuous in $w(x)$ and $e(x)$ at a switching point by theorem 3.3.1, the indirect utility function $V(x) = U[c(w(x),e(x))\ ,\ \theta - h(w(x),e(x))]$ is also continuous at such point. As we approach a switching point from the interior solution, the first partial derivatives $V_1(x)$ and $V_2(x)$, given by equations (3.A.1) and (3.A.2), approach to $-\theta\ U_1(e(x)-w(x)\theta,\ \theta)$ and $U_1(e(x)-w(x)\theta,\ \theta)$ respectively. If we approach that point from the corner solution, the two derivatives, now given by equations (3.A.8) and (3.A.9), take the same values as before. Therefore $V_1(x)$ and $V_2(x)$ are also continuous at a switching point.

(ii) At a switching point $V_2(x) = U_1(e(x)-w(x)\theta,\ \theta)$ which is positive by assumption 2.

(iii) As before there cannot be a corner solution and therefore a switching point at $e(x) = 0$.

(iv) If we approach a switching point from the interior solution, $V_2(x)$ is decreasing in $e(x)$ at the rate $U_1(x)^2\delta(x)/Q(x)$, as shown by equation (3.A.6). If we approach that point from the corner solution, $V_2(x)$ is again decreasing in $e(x)$, though at a different rate, $U_{11}(x)$ (equation (3.A.12)). Therefore $V(x)$ is concave in the neighbourhood of a switching point.

3.A.2 Proof of Theorem 3.3.6

Equation (3.3.33), written in more elaborate form,

$$V_2(w(x),e(x)) - e^{-rx} V_2(w(0),e(0)) = 0 \qquad (3.A.13)$$

can be imagined as an implicit function involving $e(x)$, $e(0)$, x, r, $w(0)$, and $w(x)$ as its arguments. Since this function is continuous in all of its arguments and $V_{22}(w(x),e(x)) \neq 0$, according to the Implicit Function Theorem $e(x)$ can be solved as a continuous function of $e(0)$, x, r, $w(0)$ and $w(x)$. Let us write this function as

$$e(x) = E^x(e(0),\ x,\ r,\ w(0),\ w(x)) \qquad (3.A.14)$$

where, the function takes the value e(0) at x = 0. Using this function in equation (3.3.34), we obtain an implicit function in e(0), P, r, and W , where P and W contain the entire domains of $p(\dot{x})$ and w(x), $0 \le x \le T$.

$$\theta \int_0^T e^{-rx}\, p(x)\, w(x)\, dx - E(e(0), P, r, W) \tag{3.A.15}$$

where,

$$E(e(0), P, r, W) = \int_0^T e^{-rx}\, p(x)\, E^x(e(0),x,r,w(0),w(x))\, dx \tag{3.A.16}$$

The implicit function given by (3.A.15) is obviously continuous in all of its arguments since the function: $E^x(e(0), x, r, w(0), w(x))$ is continuous. Consider the following derivative.

$$dE(e(0), P, r, W)/de(0) =$$

$$\int_0^T e^{-rx}\, p(x)\, [dE^x(e(0),x,r,w(0),w(x))/de(0)]\, dx$$

$$= \int_0^T e^{-2rx}\, p(x)\, [V_{22}(0)/V_{22}(x)]\, dx$$

Since $V_{22}(0)/V_{22}(x) \neq 0$, therefore $dE(e(0), P, r, W)/de(0) \neq 0$. It follows that e(0) can be solved as a continuous function of P, r and W from equation (3.A.15). We can write this function as

$$e(0) = e(P, r, W) \tag{3.A.17}$$

Finally, substituting this function in equation (3.A.14), we find the solution for e(x).

$$\begin{aligned} e(x) &= E^{x}(e(P, r, W), x, r, w(0), w(x)) \\ &= e(x, P, r, W) \end{aligned} \tag{3.A.18}$$

Since the function $E^{x}(e(0), x, r, w(0), w(x))$ is continuous in e(0), x, r, w(0) and w(x) and e(0) = e(P, r, W) is continuous in P, r and W, it follows that the function e(x, P, r, W) is continuous in all its arguments.

Since within a segment of interior or corner solution, $V_{12}(x)$ and $V_{22}(x)$ are continuous by theorem 3.3.5(i), the implicit function (3.A.13) has continuous first partial derivatives. This implies, according to the Implicit Function Theorem, that the function given by (3.A.14) which is obtained from (3.A.13) also has continuous first partial derivatives. This in turn implies that the implicit function (3.A.15) and its explicit counterpart (3.A.17) also have continuous first partial derivatives. It follows, therefore, that the solution for planned expenditure given by the function (3.A.18) has continuous first partial derivatives with respect to x, P, r and W within a segment of interior or corner solutions.

3.A.3 Proof of Theorem 3.4.4

The differential equation (3.4.5) implies that $\sigma(x)$ is decreasing after age x^W. We can show that $\sigma(x)$ is also decreasing in the interval $x^m < x < x^W$. The definition of savings given by equation (3.4.1) implies

$$\dot{\sigma}(x) = \dot{m}(x) - \dot{c}(x) \tag{3.A.19}$$

Consider the differential equation (3.3.73) which implies that

$\dot{h}(x) < 0$ if and only if

$$r\,[\mu_1(x)/U_2(x)\Delta(x)] < [U_1(x)U_{11}(x)/\Delta(x)]\,\dot{w}(x)$$

Multiplying both sides by $-\,U_2(x)\mu_2(x)/U_1(x)\mu_1(x) < 0$ and subtracting $[U_1(x)U_{12}(x)/\Delta(x)]\,\dot{w}(x)$ we can write

$\dot{h}(x) < 0$ if and only if

$$-\,r\,[\mu_2(x)/U_1(x)\Delta(x)] - [U_1(x)U_{12}(x)/\Delta(x)]\,\dot{w}(x) >$$
$$-\,[\{U_2(x)U_{11}(x)\mu_2(x)/\mu_1(x)\Delta(x)\} + \{U_1(x)U_{12}(x)/\Delta(x)\}]\,\dot{w}(x)$$

The left hand side of this inequality is equal to $\dot{c}(x)$ (see equation (3.3.72)). The right hand side can be simplified to $-\,[U_1(x)^2U_2(x)/\mu_1(x)]\,\dot{w}(x)$. Thus the above inequality implies

$\dot{h}(x) < 0$ if and only if $\dot{c}(x) > -[U_1(x)^2 U_2(x)/\mu_1(x)]\,\dot{w}(x)$.

Since $\mu_1(x) < 0$, this relationship implies that in the interval $x^h < x < x^w$ when $h(x)$ is decreasing and $w(x)$ is increasing, $c(x)$ must be increasing. This means that in the sub-interval $x^m < x < x^w$ when $m(x)$ is decreasing, $c(x)$ is increasing. Thus equation (3.A.19) implies that $\sigma(x)$ must be decreasing in this interval. We already know that $\sigma(x)$ is decreasing in the interval $x^w \leq x \leq T$. Therefore $\sigma(x)$ can peak only before age x^m.

CHAPTER 4

COMPARATIVE STATIC EFFECTS ON THE LIFE CYCLE ALLOCATION DECISION

4.1 INTRODUCTION

In the previous chapter we have concluded, apart from other things, that the lifetime allocation decision of the individual depends on certain parameters like the survival probabilities, interest rate and the wage rates. In this chapter we will discuss in detail how the individual reacts to small changes in these parameters. The effect of an increase in the probability of survival to various points of life cycle is of particular interest in this research. This exercise will show how the individual's lifetime allocation decision is affected by different types of mortality improvements all of which result in an equal increase in expected lifetime. This analysis will be useful to study the implications of mortality improvements for the aggregate economic behaviour in part II of this thesis.

In addition, we will also study the effects of an increase in the rate of interest and wage rates on the life cycle allocation decision. The type of changes in wage rates considered in this chapter is different from the type discussed in chapter 3. In particular, the changes in wage rates across life cycle, discussed in chapter 3 are known to the individual at the beginning of planning horizon and, therefore, are already incorporated into the lifetime budget constraint. The changes in wage rates to be discussed in this chapter, on the other hand, are not incorporated into the lifetime budget constraint. These changes in wage rates are the parametric changes unlike the evolutionary or anticipated changes discussed earlier. This distinction will become clearer in section 4.5.

In the next section we discuss some preliminary matters relating to comparative static analysis in the life cycle model. In particular, we restate the life cycle model for an individual who had implemented a part of his/her initial life cycle plan and faces the problem of replanning due to a change in some parameter(s) of the problem. The effect of changes in the distribution of life are studied in section 4.3. First we shall discuss the effects of an increase in the general survival rates for various segments of life. These changes in the survival rates affect both the expected lifetime utility function of the individual as well as the lifetime budget constraint due to the implied changes in the actuarial rates of interest. Then, we shall discuss the implications of an increase in a mean preserving spread in the distribution of life for the lifetime allocation decision of the individual. Finally, we shall also discuss briefly the implications of an increase in the subjective survival probabilities, independent of general survival risks in the society. These subjective mortality changes will alter the expected lifetime utility function of the individual but will leave the lifetime budget constraint unchanged, the reason being that the actuarial rates of interest are based on general survival risks in the society rather than the subjective life distribution of some particular individual. The effects of an increase in the rate of interest and the age specific wage rates are studied in section 4.4 and 4.5 respectively. The main mathematical results used for the analysis are appended to the chapter.

4.2 PRELIMINARIES

The solution to the life cycle allocation problem is determined in two stages. The conditional solution for consumption and work hours at age x, determined at the lower stage, depends on the wage rate and the expenditure rate at age x. In turn, the expenditure rate at age x is determined at the upper stage and depends on all the parameters of the problem. The logical way to proceed with the comparative static analysis is to start at the upper stage. The solution of the upper stage problem is characterized by equations (3.3.33) and (3.3.34) which are reproduced below.

$$V_2(x) = e^{-rx} V_2(0), \quad 0 \le x \le T \tag{4.2.1}$$

$$\int_0^T e^{-rx} p(x) [w(x)\theta - e(x)] dx = 0 \tag{4.2.2}$$

It will be assumed that at a certain age the individual realizes that some parameters of the problem have changed. This change in the parameters will force the individual to replan his/her allocation of expenditure for the rest of life. Let us suppose that at age n the individual is informed of a change in certain parameters of the problem. Obviously, the initial plan of expenditure allocation for any age below age n has been implemented. All the individual is left to do is to replan the allocation of expenditure for the rest his/her of life. In order to capture this aspect, we restate the optimizing conditions (4.2.1) and (4.2.2) in terms of the life cycle problem of an individual aged n. First consider the tangency condition (4.2.1). For $x = n$, this condition can be written as

$$V_2(n) = e^{-rn} V_2(0)$$

Dividing equation (4.2.1) by this equation, we can write, for $x \ge n$,

$$V_2(x) = e^{-r(x-n)} V_2(n) \tag{4.2.3}$$

Next, we can partition equation (4.2.2) as follows.

$$\int_0^n e^{-rx}\ p(x)[w(x)\theta-e(x)]\ dx + \int_n^T e^{-rx}\ p(x)[w(x)\theta-e(x)]\ dx = 0$$

Dividing this equation by the discount factor $e^{-rn}\ p(n)$, we can write

$$a(n) + \int_n^T e^{-r(x-n)}\ [p(x)/p(n)]\ [w(x)\theta-e(x)]\ dx = 0 \qquad (4.2.4)$$

where, a(n) is asset holding at age n (equation (3.4.11)).

Equations (4.2.3) and (4.2.4) are the optimizing conditions for the allocation of expenditure by an individual aged n who holds initial assets in the amount a(n) as a result of his/her past optimizing behaviour. If there is any change in the parameters of the problem which the individual is informed of at age n, then the past behaviour is unaffected by this change. Therefore we can treat a(n) as a given parameter. However, the individual may change his/her future plan from age n onwards.

We are now ready to study the effects of changes in different parameters anticipated at age n.

4.3 THE EFFECTS OF CHANGES IN THE DISTRIBUTION OF LIFE

Unlike in the conventional two period model, life expectancy in our model can be increased by increasing the probability of survival to any age. This is an important departure from the earlier work by Sheshinski and Weiss (1981), Abel (1986) and Skinner (1985). In their models life is uncertain only in the second period. Therefore life expectancy can increase only if there is an increase in the probability of survival in the second period. The present model is more general as it recognizes the fact that human life at any age is never certain. Thus in studying the effect of longevity on life cycle behaviour one may consider the effect of an increase in the probability of survival to various points of horizon.

First we discuss how different types of mortality improvements can be brought about by changing the distribution of life which is described by the density function $q(x)$. Consider the simplest form of a change in the distribution of life. Let us suppose that the probability of death at some age, say x_1, decreases by $dq(x_1)$. Since the density function $q(x)$ must sum to unity, there must be a compensating increase in the probability of death at some other age(s). Suppose that this compensating increase in the probability of death takes place at age x_2. Obviously $dq(x_2) = - dq(x_1) > 0$ since $dq(x_1) < 0$. This change in the distribution of life implies a mortality improvement if and only if $x_2 > x_1$. To be more explicit, consider the expected life of the individual, $\bar{T}$.

$$\bar{T} = \int_0^T x \, q(x) \, dx \tag{4.3.1}$$

With the above described change in the distribution of life, the expected life of the individual will increase by

$$d\bar{T} = (x_2 - x_1) \, dq(x_2) > 0 \tag{4.3.2}$$

The individual's consumption and work schedules etc. have been expressed as functions of survival probabilities. Therefore to study the effect of above change in the distribution of life, we first study how this change relates to the survival probabilities. Thus consider the definition of probability of survival to age x, p(x).

$$p(x) = 1 - \int_0^x q(y) \, dy \tag{4.3.3}$$

It follows immediately that

$$\begin{aligned} dp(x) &= dq(x_2) > 0 \quad \text{for all } x\text{: } x_1 \leq x < x_2 \\ &= 0 \quad \text{for all } x\text{: } x < x_1 \text{ or } x \geq x_2 \end{aligned} \tag{4.3.4}$$

To express the increase in expected life associated with the change in the distribution of life under consideration, in terms of the resulting changes in the survival probabilities, we now express the expected life in terms of survival probabilities. The expected life is given by equation (4.3.1). Substituting $q(x) = -\dot{p}(x)$ in (4.3.1), solving the definite integral and noting that $p(T) = 0$, we obtain

$$\bar{T} = \int_0^T p(x)\ dx \tag{4.3.5}$$

The increase in expected life resulting from the mortality improvement is given as,

$$d\bar{T} = \int_{x_1}^{x_2} dp(x)\ dx \tag{4.3.6}$$

where, $dp(x) = dq(x_2) > 0$ for all x: $x_1 \leq x < x_2$.

It is obvious from (4.3.6) that a given increase in survival probability by $dq(x_2)$ in a given interval x_1 to x_2 will increase the expected life by the same margin irrespective of the placement of that interval in the life horizon. But, as we shall shortly notice in this section, the implications of this mortality improvement for the life cycle allocation behaviour crucially depend on the placement of this interval. In other words, the effect of mortality improvement on the life cycle allocation of consumption and work hours depends on the segment of the life horizon over which the survival probabilities are assumed to have increased.

We are now ready to study the effects of improvements in life expectancy on the individual's life cycle allocation decision. There are two types of life expectancy improvements which can be studied in our model. First, there can be an improvement in the subjective probability distribution of life for the individual concerned, with the general mortality prospects in the society being unchanged. This subjective improvement in life expectancy may be due to personal reasons, like improvement in health prospects or a change in the degree of optimism or pessimism. The other type of improvement in life expectancy is due to a general mortality improvement in the society which will also affect the subjective life distribution of the individual concerned.

In our discussion so far we have not distinguished between the subjective and the actual distributions of life. It has been assumed that the two distributions are identical and therefore insurance companies carry their transactions with the individual at an actuarially fair rates of interest based on the probability distribution of life perceived by the individual. In analyzing the effects of subjective improvements in life expectancy, we need to specify the behaviour of insurance companies towards such improvements. It will be assumed that insurance companies calculate actuarial rates of interest on the basis of general mortality risks and therefore do not recognized changes in the subjective distribution of life of any particular individual.

This assumption implies that, while a general mortality improvement will change both the expected lifetime utility function and the budget constraint, a subjective improvement in life expectancy will affect only the expected lifetime utility function. Now we discuss these two types of improvements in the distribution of life.

4.3.1 The Effects of General Mortality Improvements

As we have discussed above, a general mortality improvement will affect the allocation decision of the individual due to two factor, namely the change in expected lifetime utility function and the change in lifetime budget constraint. The budget constraint will change due to changes in the actuarial rates of interest implied by the mortality improvement. The individual will realize this change in the actuarial rates of interest only after the mortality improvement has been known and fully acknowledged by the insurance market. In the following analysis we assume that a small increase in probability of survival to a specific age, say x', takes place and that this change is known by the individual as well as acknowledged by the insurance market when the individual is aged n. If $x^* < n$, then, as already explained, this mortality improvement is of no consequence for the life cycle allocation decision of the individual. Therefore we assume that $x^* \geq n$.[1]

In the appendix (section 4.A.1, equation (4.A.5)) it is shown that the effect of an increase in the probability of survival to age x^* anticipated at age n on expenditure at age x is as follows.

$$de(x)/dp(x^*) = [e(x)/\xi(x)\phi(n)]\ [e^{-rx^*}\ \sigma(x^*)] \qquad (4.3.7)$$

where,

$$\phi(n) = \int_n^T e^{-rx}\ p(x)\ [e(x)/\xi(x)]\ dx > 0 \qquad (4.3.8)$$

and,

$$\xi(x) = -\ e(x)V_{22}(x)/V_2(x) > 0 \qquad (4.3.9)$$

---

1 In the long run, or what may be called a steady state, all the changes in survival rates are expected to become a public knowledge. But, then we should be comparing the life cycle allocation decision of two individuals aged zero, rather than aged n, one facing the initial distribution of life and the other facing the changed distribution. It would be an interesting exercise to look at the transitional issue involved in passing from one steady state to another. We plan to take on this issue in our future research. The present analysis, however, is more suitable for the study of such mortality changes which are recognized even in a short period.

It is known that the coefficient of relative risk aversion $\xi(x)$ and hence $\phi(n)$ are positive. Therefore the sign and magnitude of the effect of an increase in the probability of survival to age x' anticipated at age $n < x'$ on expenditure at age $x \geq n$ is directly related to the sign and magnitude of the present value of savings planned for age x'. Thus the main result of this section can be summarized as

Theorem 4.3.1

> Anticipated at age n, an increase in the probability of survival to an age in the interval x_* to x^* (below x_* or above x^*) when the savings rate $\sigma(x)$ is positive (negative), will result in more (lesser) expenditure at each future age $x \geq n$,

where x_* and x^* are the ages at which the savings schedule $\sigma(x)$ changes its sign from negative to positive and from positive to negative respectively.

A similar result has been obtained by Abel (1986) and Sheshinski and Weiss (1981) in their analysis of the effect of a better life expectancy for the second period of a two period consumption model. Our result, however, is more general as it distinguishes among different types of improvements in life expectancy.

The result can be interpreted as follows. With an increase in the probability of survival to an age at which the savings rate $\gamma(x)$ is positive (negative), the present value of lifetime savings will increase (decrease) for the given savings plan. To bring the budget in balance, the individual will increase (decrease) expenditure in all the future periods of life.

To be more specific, we now study the type of mortality improvement which increases the probability of survival in the interval x_1 to x_2 by $dq(x_2)$ (See equations (4.3.4)). First note that, as given in equation (4.3.7), the effect of an increase in the probability of survival to age x^* on expenditure depends on age x^* only through a factor of proportionality: $e^{-rx^*}\gamma(x^*)$. Therefore, treating x^* as a continuous variable and integrating equation (4.3.7) over the interval $[x_1, x_2]$, the effect of mortality improvement represented by equations (4.3.4) on the allocation of expenditure can be obtained as follows.

$$de(x) = [a(x)/\xi(x)\phi(n)] \int_{x_1}^{x_2} e^{-ry}\,\gamma(y)\,dy \qquad (4.3.10)$$

This mortality improvement will lower the instantaneous conditional probability of death $q(y)/p(y)$ and hence the actuarially fair rate of interest for the interval x_1 to x_2. With this lower rate of interest, the present value of lifetime savings will increase (decrease) if the savings rate during that segment of life is positive (negative). The rest of the argument goes as discussed above.

An important thing to notice is that the resulting decrease in the actuarially fair rate of interest carries no intertemporal substitution effects. The reason is that this effect is fully offset by the impatience due to life uncertainty. Thus, if an increase in life expectancy lowers the actuarial rate of interest by lowering the instantaneous conditional probability of death, it will also lower the instantaneous rate of time preference by the same margin. Thus the only effect of a mortality improvement on the lifetime allocation decision is due to the 'wealth effect'.

The sensitivity of expenditure with respect to mortality variation also depends on the time lag between the age of anticipation and the actual age of the mortality variation. To see this point, we differentiate the derivative $de(x)/dp(x^*)$ given in equation (4.3.7) with respect to n.

$$d[de(x)/dp(x^*)]/dn = - [e(x)/\bar{\xi}(x)\phi(n)^2]\,[e^{-rx^*}\,\sigma(x^*)]\,[d\phi(n)/dn]$$

Or, substituting

$$d\phi(n)/dn = - e^{-rn}\,p(n)\,[e(n)/\bar{\xi}(n)], \tag{4.3.11}$$

we can write:

$$d[de(x)/dp(x^*)]/dn = [e^{-rn}\ p(n)e(n)/\xi(n)\phi(n)]\ de(x)/dp(x^*) \qquad (4.3.12)$$

Since $\xi(n)$ and $\phi(n)$ are positive, the sign of the derivative $d[de(x)/dp(x^*)]/dn$ will be the same as that of the derivative $de(x)/dp(x^*)$. Therefore, if a mortality improvement results in more (less) expenditure, then a delay in the anticipation of a change in survival probability will increase (decrease) the extent of this effect. This gives us the next result:

Theorem 4.3.2

The later the mortality improvement is anticipated, the shorter will be the remaining horizon for replanning, and therefore, the more will be the adjustment in expenditure.

Next, we discuss the effect of mortality improvements on consumption and work hours. Since the mortality improvement does not disturb the relative price of leisure in terms of consumption for any age, these effects can be easily inferred from the lower stage solution as

$$dc(x)/dp(x^*) = [\partial c(x)/\partial e(x)]\ [de(x)/dp(x^*)] \qquad (4.3.13)$$

$$dh(x)/dp(x') = [\partial h(x)/\partial e(x)]\,[de(x)/dp(x')] \qquad (4.3.14)$$

If both consumption and leisure are normal, a mortality improvement which results in more (less) expenditure will also result in more (less) consumption at each age and less (more) work hours at each age before retirement. The reason is that with the real wage rate given, when the expenditure increases (decreases) we move along the expansion path and, therefore, before retirement both consumption and leisure also increase (decrease) as both of these are assumed to be normal. During retirement the change in expenditure induced by the mortality improvement is fully absorbed by an equal change in consumption because leisure is fixed at its maximum possible level. Thus theorem 4.3.1 implies the following result regarding the effect of a mortality improvement on the allocation of consumption and work hours.

Theorem 4.3.3

If consumption and leisure are normal, then with an anticipated (at age n) increase in the probability of survival to an age in the interval x_* to x^* (below x_* or above x^*) when the savings rate $\sigma(x)$ is positive (negative), consumption will increase (decrease) at each future age $x \geq n$ and work hours will decrease (increase) at each future working age x: $n \le x \le R$.

Let us now examine how the retirement decision is affected by mortality improvements. The retirement age is defined by the condition that the solution for work hours for the pre-retirement segment of the horizon approaches zero as the age index approaches the retirement age. In other words, the individual will retire at age R if

$$\lim_{x \to R^-} h(x,\ P,\ r,\ W) = 0 \qquad (4.3.15)$$

Differentiating this condition with respect to $p(x^*)$, while treating x as the endogenous variable, we can write

$$\lim_{x \to R^-} [\dot{h}(x)\ \{dx/dp(x^*)\} + \{dh(x)/dp(x^*)\}] = 0$$

Taking the limit and solving for $dR/dp(x^*)$, we obtain

$$dR/dp(x^*) = \frac{\lim_{x \to R^-} [dh(x)/dp(x^*)]}{-\lim_{x \to R^-} \dot{h}(x)} \qquad (4.3.16)$$

The denominator in this expression is positive because work hours must be decreasing in the left side neighbourhood of the retirement age. The numerator could be positive or negative, depending on normality of leisure and the nature of mortality improvement. Therefore, we can write the following result regarding the effect of mortality improvement on the retirement decision.

Theorem 4.3.4

> If leisure is normal, then an anticipated increase in the probability of survival to a future age in the interval x_* to x^* (below x_* or above x^*) when the savings rate $\sigma(x)$ is positive (negative), will induce an earlier (later) retirement.

The signs of all the effects of an increase in the probability of survival to different ages are summarized in table 4.3.1.

At this stage we can also do some ground work for the general equilibrium analysis of mortality improvements to be conducted in second part of the thesis. For this purpose we discuss the effects of mortality variations on the narrow measure of savings $\sigma(x)$. Since, by definition

$$\sigma(x) = w(x)\theta - e(x)$$

and $w(x)$ and θ are fixed, it follows that

$$d\sigma(x)/dp(x') = - de(x)/dp(x') \tag{4.3.17}$$

Table 4.3.1

The Effects of an Increase in

General Survival Rate to Age x'

	$x_* < x' < x^*$	$x' < x_*$ or $x' > x^*$	$x' = x_*$ or $x' = x_*$
de(x)	+	-	0
	Consumption and Leisure are Normal		
dc(x)	+	-	0
dh(x)	-	+	0
dR	-	+	0
	Consumption is Inferior		
dc(x)	-	+	0
dh(x)	-	+	0
dR	-	+	0
	Leisure is Inferior		
dc(x)	+	-	0
dh(x)	+	-	0
dR	+	-	0

Therefore we can infer the following result from theorem 4.3.1.

Theorem 4.3.5

An anticipated increase in the probability of survival to an age in the interval x_* to x^* (below x_* or above x^*) when the savings rate $\sigma(x)$ is positive (negative), will result in less (more) savings in the narrow sense at each future age.

This result will be very important in studying the effect of mortality improvements in the general equilibrium model. Therefore, we explicitly write the expression for the derivative in equation (4.3.17) by substituting for $de(x)/dp(x')$ from equation (4.3.7).

$$d\sigma(x)/dp(x') = - [e(x)/\xi(x)\phi(n)] [e^{-rx'} \sigma(x')]$$

This equation shows that the effect of an increase in the probability of survival to age x' depends on age x' only through a factor of proportionality $[e^{-rx'} \sigma(x')]$. In our equilibrium model we will be interested in the study of mortality improvements in the steady state. Therefore, we will be considering changes in the mortality schedule which are known at the beginning of the horizon. The corresponding expression for $d\sigma(x)/dp(x^*)$ is

$$d\sigma(x)/dp(x^*) = -\,[e(x)/\xi(x)\phi(0)]\,[e^{-rx^*}\,\sigma(x^*)] \qquad (4.3.18)$$

Next we move to the study of another type of change in the distribution of life which increases risk of life.

4.3.2 The Implications of Increasing General Mortality Risk

In this section we discuss the effect of increasing risk in the distribution of life on life cycle allocation behaviour. We will use the definition of a mean preserving spread given by Rothschild and Stiglitz (1970) to characterize the riskiness of the distribution of life. According to the notion of a mean preserving spread a probability density function $G(x)$ represents more spread than $F(x)$ if $G(x)$ has more weights in the tails of the distribution of x than the function $F(x)$. This implies that the cumulative distribution functions should satisfy the relation:

$$\int_0^y G(x)\,dx - \int_0^y F(x)\,dx \quad \begin{cases} \geq 0 & \text{for small values of } y \\ \leq 0 & \text{for large values of } y \end{cases}$$

where, it is assumed that the lower bound on x is 0. This relation, together with the condition that mean of x obtained with the two distributions is identical, gives the following conditions for $G(x)$ to represent more mean preserving spread than $F(x)$.

$$\int_0^z \int_0^y [G(x) - F(x)]\, dx\, dy \geq 0 \quad \text{for } 0 \leq z < T$$
$$= 0 \quad \text{for } z = T$$

where, T is the upper bound on x.

In the context of life uncertainty, a distribution of life represented by the density function $q'(x)$ is riskier than another distribution represented by the density function $q(x)$ if the two density functions satisfy the conditions:

$$\int_0^z \int_0^y [q'(x)-q(x)]\, dx\, dy \geq 0 \quad \text{for } 0 \leq z < T$$
$$= 0 \quad \text{for } z = T$$

(See equations (6) and (7) in Rothschild and Stiglitz (1970)). If the distribution $q'(x)$ represents more risk, there must be a strict inequality at least for one $z < T$.

We can express these integral conditions in terms of survival probabilities. Consider the relationship of a survival probability with probabilities of death given by equation (3.2.3):

$$p(y) = 1 - \int_0^y q(x)\, dx$$

Using this relation, the integral conditions can be written as

$$\int_0^z [p'(y)-p(y)]\, dy \leq 0 \qquad \text{for } 0 \leq z < T$$
$$= 0 \qquad \text{for } z = T$$

Or, if the difference $p'(x) - p(x)$ is small, we can write the integral conditions in terms of the cumulative density function as follows.

$$cdp(z) \leq 0 \qquad \text{for } 0 \leq z < T \qquad (4.3.19)$$

$$cdp(z) = 0 \qquad \text{for } z = T \qquad (4.3.20)$$

with at least one strict inequality, and where $cdp(z)$ is the cumulative density function defined below.

$$cdp(z) = \int_0^z dp(y)\, dy \qquad (4.3.21)$$

Now we can discuss the implications of increasing risk for the life cycle allocation decision. Consider equation (4.3.7) which implies that the effect of a shift in the mortality schedule on planned expenditure at age x of an individual aged 0 can be written as follows:

$$de(x) = [e(x)/\xi(x)\phi(\hat{\ })] \int_0^T \delta(z) dp(z)\ dz \qquad (4.3.22)$$

and, where $\delta(z) = e^{-rz}\ \gamma(z)$ can be written as

$$\delta(z) = \int_0^z \dot{\delta}(y)\ dy + \delta(0)$$

This relationship implies that

$$\int_0^T \delta(z)\ dp(z)\ dz = \int_0^T dp(z) \int_0^z \dot{\delta}(y)\ dy\ dz + \delta(0) \int_0^T dp(z)\ dz$$

The second term on the right hand side of this equation is equal to zero according to the integral condition (4.3.20). Therefore, with a change in the order of integration, we can write:

$$\int_0^T \delta(z)\ dp(z)\ dz = \int_0^T \dot{\delta}(z) \int_z^T dp(y)\ dy\ dz$$

Or, using the definition of the cumulative density function given by (4.3.21) and the integral condition (4.3.20),

$$\int_0^T \delta(z)\ dp(z)\ dz = - \int_0^T \dot{\delta}(z)\ cdp(z)\ dz$$

Substituting this result in equation (4.3.22) we obtain the desired relationship:

$$de(x) = - [e(x)/\xi(x)\phi(0)] \int_0^T \dot{\delta}(z)\ cdp(z)\ dz \qquad (4.3.23)$$

Clearly, the factor $- [e(x)/\xi(x)\phi(n)]$ is negative. We also know by the integral condition (4.3.19) that the cumulative density function cdp(z) is non-positive for all z < T, with at least one negative term. Therefore the sign of the above differential depends on the pattern of the age derivative of the function $\delta(x)$. According to our analysis in chapter 3, $\sigma(x)$ is increasing with age up to some point x^{σ} and decreasing thereafter. This implies that the variable $\delta(x) = e^{-rx}\ \sigma(x)$ must be increasing up to some age $x^{\delta} < x^{\sigma}$ and decreasing thereafter. Under these conditions, the above differential has an indeterminate sign. However, we can discuss the implications of increasing risk after age x^{δ}. Thus we assume that

$$
\begin{aligned}
cdp(z) &= 0 && \text{for } 0 \le z < x^{\delta} \\
&\le 0 && \text{for } x^{\delta} \le z < T \\
&= 0 && \text{for } z = T
\end{aligned}
$$

with at least one strict inequality. Equation (4.3.23) accordingly simplifies to

$$de(x) = - [e(x)/\xi(x)\phi(n)] \int_{x^{\delta}}^T \dot{\delta}(z)\ cdp(z)\ dz \qquad (4.3.24)$$

Since $-[e(x)/\xi(x)\phi(n)]$ and $\dot{\delta}(z)$ are negative and $cdp(z)$ is also negative at least for one z, and non-positive for all z, the change in expenditure, $de(x)$ must be negative.

Theorem 4.3.6

> With actuarially fair life insurance and annuities, a change in survival probabilities for ages above age x^{δ} which makes the distribution of life more risky, will result in less expenditure at each age.

The reasoning is as follows. An increase in risk results in higher probabilities of surviving to the later part of the horizon in exchange for the lower survival probabilities at younger ages. Since the saving rate $\sigma(x)$ is decreasing with age, the present value of life time savings will decrease for the given savings plan and thus result in less expenditure at each age.

The implications of increasing risk of life on the life cycle behaviour in the absence of life insurance and annuities has been discussed by Levhari and Mirman (1977). In their analysis a change in the distribution of life means a change in the stream of discount factors attached with the utility function for different ages. In our model this effect is exactly offset by the intertemporal substitution effects due to the corresponding changes in the age specific actuarial rates of interest. However, the life cycle allocation will be affected due to the wealth effect associated with changes in the actuarial rates of interest. Thus the effect of increasing risk in our model works through a different channel than in Levhari and Mirman (1977).

The effect of increasing risk on the allocation of consumption and work hours is straight forward. If both consumption and leisure are normal, then increasing risk will result in lesser consumption at each age and more work at each working age.

4.3.3 The Effects of Subjective Mortality Improvements

In order to study such life expectancy improvements, we must distinguish between the distribution of life perceived by the individual and the general life distribution in the society on the basis of which insurance companies calculate the actuarial rates of interest. Our analysis so far has been conducted under the assumption that the two distributions coincide. Now we assume that the two distributions of life are not necessarily identical and study the effect of small changes in the subjective survival probabilities, with the general mortality rates being fixed.

For the proposed analysis, we will use different notations for the subjective survival probabilities and the actual survival rates. The reason is that in the comparative static analysis we will allow changes only in the subjective survival probabilities with the actual mortality rates being fixed. Under these circumstances, the lifetime allocation problem of the individual at age n can be written as follows.

Maximize

$$\int_n^T p(x)\ V(w(x),e(x))\ dx \qquad (4.3.25)$$

Subject to,

$$a(n) + \int_n^T e^{-r(x-n)}[p^*(x)/p^*(n)][w(x)\theta-e(x)]\ dx = 0 \qquad (4.3.26)$$

where p(x) is the subjective survival probability and $p^*(x)$ is the general survival rate in the society. The maximizing conditions for this problem are:

$$V_2(x) = e^{-r(x-n)}\ [p^*(x)p(n)/p(x)p^*(n)]\ V_2(n) \qquad (4.3.27)$$

$$\int_n^T e^{-rx}\ p^*(x)\ [w(x)\theta - e(x)]\ dx = \text{constant} \qquad (4.3.28)$$

Differentiating these two equations with respect to $p(x')$, we can find the effect of an increase in subjective survival probability on the allocation of expenditure. In appendix 3.A.2, this effect is shown by equations (4.A.13) and (4.A.14) to be as follows.

$$de(x)/dp(x') = - [e(x)/\xi(x)]\ [e^{-rx'}\ p^{*}(x')e(x')/\xi(x')\phi^{*}(n)p(x')] + [e(x')/\xi(x')p(x')] \qquad \text{for } x = x' \qquad (4.3.29)$$

$$de(x)/dp(x') = - [e(x)/\xi(x)]\ [e^{-rx'}\ p^{*}(x')e(x')/\xi(x')\phi^{*}(n)p(x')] \qquad \text{for } x \neq x' \qquad (4.3.30)$$

where, $\xi(x)$, the coefficient of relative risk aversion, is defined by equation (4.3.9) to be as follows:

$$\xi(x) = - e(x)V_{22}(x)/V_{2}(x) > 0 \qquad (4.3.31)$$

and $\phi^{*}(n)$, which is analogous to $\phi(n)$ is as follows.

$$\phi^{*}(n) = \int_{n}^{T} e^{-rx}\ p^{*}(x)\ [e(x)/\xi(x)]\ dx > 0 \qquad (4.3.32)$$

For the interpretation of equations (4.3.29) and (4.3.30), first notice that $\xi(x)$ and hence $\phi^{*}(n)$ are positive. In addition, by the definition of $\phi^*(n)$, the expression: $e^{-rx'} p^*(x')e(x')/\xi(x')/\phi^*(n)$ is a positive fraction. Thus equations (4.3.29) and (4.3.30) imply the following result.

Theorem 4.3.7

> If the actuarial rates of interest are held constant, an increase in the probability of survival to age x' will result in more expenditure at age x' and less expenditure at each other age $x \neq x'$.

The reasoning goes as follows. With an increase in the probability of survival to age x', the subjective weighting factor attached with the utility rate for age x' increases. Therefore, the utility rate for age x' is discounted less heavily. The result is that the individual will allocate more expenditure for age x'. With the actuarial rates of interest being fixed, the lifetime budget constraint is unchanged. Therefore expenditure for age x' can increase if and only if there is a compensating decrease in expenditure at some other age. With additively separable lifetime utility function, expenditure for all the ages $x \neq x'$ will decrease because the relative price of expenditure rate at all these ages is unchanged.

An increase in the probability of survival in the present context carries only the substitution effect. This result is in contrast with the result obtained in subsection 4.3.1 where an increase in the probability of survival had only the wealth effect. The substitution effect of the type obtained in this section was fully offset by an opposite substitution effect due to the resulting changes in the actuarial rates of interest. The net effect therefore was due to the wealth effect associated with the changes in the actuarial rates of interest.

We can also discuss briefly the effects of a subjective increase in the probability of survival on the allocation of consumption and work hours. The following result is implied by theorem 4.3.7 and equations (4.3.13) and (4.3.14).

Theorem 4.3.8

If the actuarial rates of interest are held constant, an increase in the probability of survival to age $x' \leq R$ will result in more consumption and less work hours at age x', less consumption at each other age $x \neq x'$ and more work hours at each other working age $x \neq x'$, $x \leq R$, provided both consumption and leisure are normal goods. An increase in the probability of survival at age $x' \geq R$ will result in more consumption at age x^*, less consumption at each other age $x \neq x^*$ and more work hours at each working age $x \leq R$.

Next, we analyze the effect on the retirement decision of the individual. Equation (4.3.16) shows that the individual will retire earlier (later) if the increase in survival probability results in less (more) work hours in the left side neighbourhood of the retirement age. We know by theorem 4.3.8 that an increase in probabilities of survival to any ages not in left side neighbourhood of the retirement age will result in more work hours in that neighbourhood of the retirement age (provided leisure is normal). On the other hand, an increase in survival probabilities to ages in the left side neighbourhood of retirement age will result in less work hours in that neighbourhood (provide leisure is normal). This implies the following result regarding the retirement decision.

Theorem 4.3.9

If the actuarial rates of interest are held constant, an increase in probabilities of survival to ages in the neighbourhood (not in the neighbourhood) of the retirement age will result in an earlier (later) retirement, provided leisure is a normal good.

The signs of the effects of subjective mortality improvements on consumption, work hours and retirement age are summarized in table 4.3.2. This completes our discussion of the effects of different types of mortality changes on the lifetime allocation decision of the individual.[2] Now we move to the analysis of a change in the rate of interest.

4.4 THE EFFECTS OF AN INCREASE IN THE RATE OF INTEREST

We first discuss the effect of a small increase in the rate of interest on the allocation of expenditure. This effect can be obtained by differentiating the optimizing conditions (4.2.3) and (4.2.4) with respect to r and solving for de(x)/dr. As shown in the appendix (equation (4.A.25)), the full effect of an increase in the rate of interest can be split into two parts according to the Slutsky equation:

$$de(x)/dr = de(x)/dr\Big|_{\Omega} + \alpha(n)\ [de(x)/da(n)] \qquad (4.4.1)$$

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2 Like the effects of increasing risk in the general mortality schedule, we can also discuss the effects of increasing subjective risk to life. these effects have been found to be too general to postulate anything about the individual's behaviour towards risk to life. The results depend on the behaviour of the coefficient of relative risk aversion over time as well as on the precise pattern of changes in survival probabilities. It is in our future research plan to conduct a detailed study of such changes in subjective survival risk.

TABLE 4.3.2

The Effects of an Increase in the Subjective Probability of Survival to Age x^*

		Consumption and Leisure are Normal	Consumption is Inferior	Leisure is Inferior
de(x)	$x = x^*$	+	+	+
	$x \neq x^*$	-	-	-
dc(x)	$x = x^*$	+	-	+
	$x \neq x^*$	-	+	-
dh(x)	$x = x^*$	-	-	+
	$x \neq x^*$	+	+	-
dR	Limit $_{x^* \to R^-}$	-	-	+
	all other x^*	+	+	-

The first term in this equation represents the lifetime utility held constant effect or the compensated effect of an increase in the rate of interest where as the second term has the interpretation of the wealth effect associated with an increase in the rate of interest.[3] The term $\alpha(n)$ will be defined below. We will discuss these effects one by one.

Consider first the compensated effect which is shown to be as follows in the appendix (equation (4.A.22)).

$$de(x)/dr \Big|_{\Omega} = - [e(x)/\xi(x)] \ (\bar{x}_n - x) \qquad (4.4.2)$$

where, $\bar{x}_n$ is the weighted average of x defined as follows.

$$\bar{x}_n = \frac{\int_n^T x \, e^{-rx} \, p(x) \ [e(x)/\xi(x)] \ dx}{\int_n^T e^{-rx} \, p(x) \ [e(x)/\xi(x)] \ dx} \qquad (4.4.3)$$

With $n < \bar{x}_n < T$ and $\xi(x) > 0$, this compensated effect implies the following result on the allocation of expenditure.

---

3 The effects of an increase in rate of interest on demands for goods in a life cycle model have been derived long ago by Tintner (1938b). The above decomposition of the full effects of an increase in the rate of interest into the compensated and wealth effects is a simplified version of the decomposition effects obtained by Tintner for a non-additive lifitime utility function.

Theorem 4.4.1

With lifetime utility held constant, the effect of an increase in the rate of interest is that at earlier ages (below $\bar{x}_n$) expenditure will decrease and at the later ages (above $\bar{x}_n$) it will increase.

The reason is as follows. With an increase in the rate of interest, the price of future expenditure falls relative to the present. Therefore it is cheaper to spend more in the future than in the present.

Next, consider the second part in the Slutsky equation (4.4.1). This part represents the wealth effect associated with an increase in the rate of interest. The derivative $de(x)/da(n)$ represents the effect of an increase in the initial wealth on expenditure. In the appendix (equation (4.A.24)) this effect is shown to be as follows.

$$de(x)/da(n) = e^{-rn}\, p(n)\, [e(x)/\xi(x)\phi(n)] \qquad (4.4.4)$$

where, $\phi(n)$ and $\xi(x)$ are defined by equations (4.3.8) and (4.3.9) respectively. With $\xi(x)$ and $\phi(x)$ being positive, this wealth effect shows that due to an increase in the initial wealth, expenditure will increase at each age.

Finally, the weighting factor $\alpha(n)$ is shown in the appendix (equation (4.A.27)) to be as follows.

$$\alpha(n) = \int_{n}^{T} e^{-r(x-n)} [p(x)/p(n)] a(x) dx \tag{4.4.5}$$

The sign of this weighting factor depends on the age pattern of asset holding. If the savings rate $\sigma(x)$ is positive during the early years and negative later on, then assets are positive at each age. In this case the weighting factor $\alpha(n)$ will be positive for all the values of n. Therefore the wealth effect of an increase in the rate of interest will be positive. The reason is that, if savings are positive during the early years and negative during the later years, then an increase in the rate of interest will increase the interest income on lendings. On the other hand, if savings are negative during early years and positive later on, then asset holding is always negative. In that case an increase in the rate of interest will increase interest cost on borrowing, and thus create a negative wealth effect.

According to the age pattern of savings posited in chapter 3, the savings rate $\sigma(x)$ is assumed to be negative during some early years. During the middle years, savings are positive and during the later years savings are again negative. Accordingly, asset holding is negative during some early years, up to age x' and positive later on. Thus, when the rate of interest increases, the individual bears some additional interest costs, provided the rate of interest increases before age x'. But later on, the individual becomes a net lender and therefore gains some additional interest income. The net effect, in principle, could be in any direction. This discussion can be summarized in the following result.

Theorem 4.4.2

> An increase in the rate of interest after age x', when assets become positive, carries a positive wealth effect, implying more expenditure at each future age.

For further discussion, we assume that the negative asset holding at the early years of life is small enough such that the wealth effect of an increase in the rate of interest at any age is positive. The full effect of an increase in the rate of interest is predictable only for the later years of life (above age $\bar{x}_n$) when both the compensated effect and the wealth effect are positive. Therefore, an increase in the rate of interest implies more expenditure after age $\bar{x}_n$. On the other hand, at any age below $\bar{x}_n$, the compensated and wealth effects work in the opposite directions, and therefore the effect of an increase in the rate of interest is uncertain. This ambuigity has also been pointed out by Feldstein and Tsiang (1968) in the context of a two period medel.

The effects of a higher rate of interest on the allocation of consumption and work hours can be easily inferred from the above result. Thus using the lower stage solution, we find

$$dc(x)/dr = [\partial c(x)/\partial e(x)]\,[de(x)/dr] \tag{4.4.6}$$

$$dh(x)/dr = [\partial h(x)/\partial e(x)]\,[de(x)/dr] \tag{4.4.7}$$

Both consumption and leisure are assumed to be normal, that is, $\partial c(x)/\partial e(x) > 0$ and $\partial h(x)/\partial e(x) < 0$. Therefore before retirement, consumption will move in the same direction as expenditure and work hours will change in the opposite direction. After retirement work hours are fixed at zero and therefore consumption will move in the same direction as expenditure. Thus, the compensated effect of a higher rate of interest is to decrease consumption and increase work hours during the early years (below age $\bar{x}_n$) and to increase consumption and decrease work hours during the later years. If the wealth effect of a higher rate of interest on expenditure is assumed to be positive, then consumption will increase and work hours will decrease at each age due to this wealth effect.

Finally, we can also study the effect of a higher rate of interest on the retirement decision of the individual. The condition that the individual will retire at age R is given by equation (4.3.15). Differentiating this condition with respect to r, we can write

$$\lim_{x \to R^-} [\dot{h}(x) \{dx/dr\} + \{dh(x)/dr\}] = 0$$

Taking the limit and solving for dR/dr, we obtain

$$dR/dr = \frac{\lim_{x \to R^-} [dh(x)/dr]}{- \lim_{x \to R^-} \dot{h}(x)} \qquad (4.4.8)$$

Since the denominator of this expression is positive, we have the following result.

Theorem 4.4.3

An increase in the rate of interest will result in an earlier (a later) retirement if it results in less (more) work hours in the neighbourhood the of retirement age.

Table 4.4.1 provides the details of the sings of various effects of an increase in the rate of interest.

Table 4.4.1

The Effects of an Increase in the Rate of Interest

		Compensated Effects	Wealth Effects	Full Effects
de(x)	$x < \bar{x}_n$	-	+	?
	$x = \bar{x}_n$	0	+	+
	$x > \bar{x}_n$	+	+	+
		Consumption and Leisure are Normal		
dc(x)	$x < \bar{x}_n$	-	+	?
	$x = \bar{x}_n$	0	+	+
	$x > \bar{x}_n$	+	+	+
dh(x)	$x < \bar{x}_n$	+	-	?
	$x = \bar{x}_n$	0	-	-
	$x > \bar{x}_n$	-	-	-
		Consumption is Inferior		
dc(x)	$x < \bar{x}_n$	+	-	?
	$x = \bar{x}_n$	0	-	-
	$x > \bar{x}_n$	-	-	-
dh(x)	$x < \bar{x}_n$	+	-	?
	$x = \bar{x}_n$	0	-	-
	$x > \bar{x}_n$	-	-	-
		Leisure is Inferior		
dc(x)	$x < \bar{x}_n$	-	+	?
	$x = \bar{x}_n$	0	+	+
	$x > \bar{x}_n$	+	+	+
dh(x)	$x < \bar{x}_n$	-	+	?
	$x = \bar{x}_n$	0	+	+
	$x > \bar{x}_n$	+	+	+

4.5 INCREASES IN WAGE RATES

4.5.1 The Effects of age specific changes in wage rates

For a better understanding of the comparative static analysis we discuss the effects of a small increase in the wage rate for a specific age, rather than a shift in wage profile. Let us assume, therefore, that a small increase in the wage rate for a specific age x' is anticipated by the individual at age n, with the wage rate for each other age being constant. We start our analysis with the effect of this wage increase on the allocation of expenditure. As derived in the appendix (equation (4.A.40)), this effect can be written in terms of a Slutsky equation as follows.

$$de(x)/dw(x') = [de(x)/dw(x')\big|_{\Omega}] + B(n, x')\ [de(x)/da(n)] \qquad (4.5.1)$$

where, the first term, $de(x)/dw(x')\big|_{\Omega}$, is the lifetime utility held constant effect or the compensated effect where as the second term, $B(n, x')\ [de(x)/da(n)]$, has the interpretation of the wealth effect associated with the wage increase. The derivative $de(x)/da(n)$ measures the effect of an increase in initial wealth on expenditure, given by equation (4.4.4) and the weighting factor $B(n, x')$ is the effect on wealth at age n of an increase in the wage rate for age x^* anticipated at age n.

Now we discuss these effects one by one. The compensated effect has been shown in the appendix to be as follows.

$$de(x)/dw(x^*)\Big|_{\Omega} =$$
$$- [e(x)/\xi(x)\phi(n)]\ e^{-rx^*}\ p(x^*)\ [\mu_1(x^*)/U_2(x^*)\Delta(x^*)]$$
$$+ [\mu_1(x^*)/U_2(x^*)\Delta(x^*) + \theta - h(x^*)]$$
$$\text{for } x = x^*,\ x^* \le R \qquad (4.5.2)$$

$$de(x)/dw(x^*)\Big|_{\Omega} =$$
$$- [e(x)/\xi(x)\phi(n)]\ e^{-rx^*}\ p(x^*)\ [\mu_1(x^*)/U_2(x^*)\Delta(x^*)]$$
$$\text{for } x \ne x^*,\ x^* \le R \qquad (4.5.3)$$

$$de(x)/dw(x^*)\Big|_{\Omega} = \theta \qquad \text{for } x = x^*,\ x^* \ge R \qquad (4.5.4)$$

$$de(x)/dw(x^*)\Big|_{\Omega} = 0 \qquad \text{for } x \ne x^*,\ x^* \ge R \qquad (4.5.5)$$

where, $\phi(n)$ and $\xi(x)$ are defined by equations (4.3.8) and (4.3.9) respectively. In order to interpret these compensated effects we further decompose them into two parts as follows. Recall from equation (3.3.52) of chapter 3 that $\theta - h(x^*)$ in equation (4.5.2) represents the cost of living effect of a wage increase at age x^* on expenditure at age x^* for a working age. Likewise, according to equation

(3.3.53), θ in equation (4.5.4) represents the cost of living effect of a wage increase during retirement. The remaining effects in the above four equations must be due to reallocation of consumption and work hours over the life cycle as a result of the wage increase.

Accordingly, we can rewrite the above equations as:

$$de(x)/dw(x^*)\big|_{\Omega} = de(x)/dw(x^*)\big|_{c,l} + de(x)/dw(x^*)\big|_{\Omega,ir} \tag{4.5.6}$$

where we use the notations c,l and Ω,ir for the cost of living effect and the intertemporal reallocation effect of an increase in the wage rate. These effects on expenditure are as follows.

$$de(x)/dw(x^*)\big|_{c,l} = \theta - h(x) \qquad \text{for } x = x^*,\ x^* \le R \tag{4.5.7}$$

$$de(x)/dw(x^*)\big|_{c,l} = 0 \qquad \text{for } x \ne x^*,\ x^* \le R \tag{4.5.8}$$

$$de(x)/dw(x^*)\big|_{c,l} = \theta \qquad \text{for } x = x^*,\ x^* \ge R \tag{4.5.9}$$

$$de(x)/dw(x^*)\big|_{c,l} = 0 \qquad \text{for } x \ne x^*,\ x^* \ge R \tag{4.5.10}$$

$$\begin{aligned} de(x)/dw(x^*)\big|_{\Omega,ir} = \\ &- [e(x)/\xi(x)\phi(n)]\ e^{-rx^*}\ p(x^*)\ [\mu_1(x^*)/U_2(x^*)\Delta(x^*)] \\ &+ [\mu_1(x^*)/U_2(x^*)\Delta(x^*)] \qquad \text{for } x = x^*,\ x^* \le R \end{aligned} \tag{4.5.11}$$

$$\begin{aligned} de(x)/dw(x^*)\big|_{\Omega,ir} = \\ &- [e(x)/\xi(x)\phi(n)]\ e^{-rx^*}\ p(x^*)\ [\mu_1(x^*)/U_2(x^*)\Delta(x^*)] \\ &\qquad \text{for } x \ne x^*,\ x^* \le R \end{aligned} \tag{4.5.12}$$

$$de(x)/dw(x^*)\big|_{\Omega,ir} = 0 \qquad \text{for } x = x^*,\ x^* \geq R \qquad (4.5.13)$$

$$de(x)/dw(x^*)\big|_{\Omega,ir} = 0 \qquad \text{for } x \neq x^*,\ x^* \geq R \qquad (4.5.14)$$

From the definition of $\phi(n)$ and $\xi(x)$, given by (4.3.8) and (4.3.9) we know that both of these are positive and the expression: $e^{-rx^*}\ p(x^*)e(x^*)/\xi(x^*)\phi(n)$ is a positive fraction. Therefore following results are implied by the above equations:

Theorem 4.5.1

The compensated effect of a wage increase on expenditure can be broken into two parts. A cost of living effect and an intertemporal reallocation effect. The cost of living effect implies that, given the rate of consumption and leisure, a one dollar increase in the wage rate at age x will result in an increase in expenditure at that age by the amount of leisure retained, $\{\theta-h(x)\}$. The cost of living effect of a wage increase at some age on expenditure at other ages is zero. The intertemporal reallocation effect implies that, if leisure is normal, then an increase in the wage rate during a working age will result in less expenditure at that age and more at others. A change in the wage rate during retirement is irrelevant for the life cycle allocation decision. Therefore the intertemporal reallocation effect of a wage increase during retirement is zero.

These results are similar to those found in chapter 3. But, as we shall explain later in this chapter, the compensated effects of parametric changes in wage rates are not identical to the effects of evolutionary changes in wage rates. The decomposition of compensated effects into the cost of living and intertemporal reallocation effects will be useful to find the effects of a parametric wage increase on consumption and work hours.

Now we move to the wealth effect associated with a wage increase, given by the second term in equation (4.5.1). The expression for the wealth effect de(x)/da(n) is given by equation (4.4.4). It shows that an increase in initial wealth will result in more expenditure at each age. The weighting factor B(n, x') which measures the effect on wealth at age n of an increase in the wage rate for age x' anticipated at age n, is given below (See equation (4.A.41) in the appendix).

$$B(n, x') = e^{-r(x'-n)} [p(x')/p(n)]\ h(x') \qquad \text{for } x \leq R \tag{4.5.15}$$

$$B(n, x') = 0 \quad \text{for } x' \leq R \tag{4.5.16}$$

Theorem 4.5.2

The wealth effect of a wage increase implies that an increase in the wage rate before retirement results in more expenditure at all ages since for each one dollar increase in the wage rate at a working age, wealth at age n will increase by the number of hours worked at that age discounted by the interest factor. There is no wealth effect associated with a wage increase after retirement.

Notice that the part of the endowment of time $\theta-h(x)$ which is retained for leisure does not enter into the wealth effect because the effect of increase in its value is offset by the corresponding increase in spending. Thus, only that part of the endowment θ will effect wealth which is sold in the market, that is, $h(x)$. That is why there is no wealth effect of an increase in the wage rate for a retirement age.

For our later analysis we substitute equation (4.5.6) which shows the decomposition of compensated effect into the cost of living effect and the intertemporal reallocation effect, into the Slutsky equation (4.5.1). Thus, in light of equations (4.5.7) through (4.5.10) we obtain:

$$de(x)/dw(x^*) = de(x)/dw(x^*)\big|_{c,1} + de(x)/dw(x^*)\big|_{\Omega,ir} + B(n, x^*)\,[de(x)/da(n)] \qquad \text{for } x = x^* \qquad (4.5.17)$$

$$de(x)/dw(x^*) = de(x)/dw(x^*)\big|_{\Omega,ir} + B(n, x^*)\,[de(x)/da(n)] \qquad \text{for } x \neq x^* \qquad (4.5.18)$$

These equation are useful to study the effects of a change in wage rates on the allocation of work hours and consumption over the life cycle to which we now turn our attention. Recall from chapter 3 that the lower stage solution for consumption and work hours can in general be written as

$$c(x) = c(w(x), e(x)) \qquad (4.5.19)$$

$$h(x) = h(w(x), e(x)), \qquad (4.5.20)$$

though for age $x \geq R$ this solution degenerates to

$$c(x) = e(x) - w(x)\theta \qquad \text{for } x \geq R \qquad (4.6.21)$$

$$h(x) = 0 \qquad \text{for } x \geq R \qquad (4.6.22)$$

The full effects of an increase in the wage rate for age x^* anticipated at age $n \leq x^*$ on the life cycle allocation of consumption and work hours at age $x \geq n$ can be represented by the following equations.

$$dc(x)/dw(x^*) = \partial c(x)/\partial w(x^*) + [\partial c(x)/\partial e(x)][de(x)/dw(x^*)] \quad \text{for } x = x^* \qquad (4.5.23)$$

$$dc(x)/dw(x^*) = [\partial c(x)/\partial e(x)]\,[de(x)/dw(x^*)] \quad \text{for } x \neq x^* \qquad (4.5.24)$$

$$dh(x)/dw(x^*) = \partial h(x)/\partial w(x^*) + [\partial h(x)/\partial e(x)][de(x)/dw(x^*)] \quad \text{for } x = x^* \qquad (4.5.25)$$

$$dh(x)/dw(x^*) = [\partial h(x)/\partial e(x)]\,[de(x)/dw(x^*)] \quad \text{for } x \neq x^* \qquad (4.5.26)$$

Theorem 4.5.3

The full effect of an increase in the wage rate on the allocation of consumption and work hours can be decomposed into two parts, a one period effect and a life cycle effect. The one period effect represented by the first term in equations (4.5.23) and (4.5.25), measures the effect of an increase in the current wage rate, given the life cycle allocation of expenditure. The one period effect of an increase in the wage rate at age x^* is zero on expenditure at age $x \neq x^*$. The life cycle effect, represented by the second term in equations (4.5.23) and (4.5.25) and the only term in equations (4.5.24) and (4.5.26) measures the change in consumption and work hours solely due to the reallocation of expenditure over the life cycle as a result of the wage increase.

For further details, we make use of the Slutsky equations both for the lower stage effects of an increase in the wage rate on consumption and work hours and for the upper stage effect on expenditure. The Slutsky equations for the lower stage effects, given by equations (3.3.18) and (3.3.19) are reproduced below:

$$\partial c(x)/\partial w(x^*) = \partial c(x)/\partial w(x^*)\Big|_{U(x^*)} - [\vartheta-h(x^*)]\ [\partial c(x)/\partial e(x^*)] \tag{4.5.27}$$

$$\partial h(x)/\partial w(x^*) = \partial h(x)/\partial w(x^*)\Big|_{U(x^*)} - [\vartheta-h(x^*)]\ [\partial h(x)/\partial e(x^*)] \tag{4.5.28}$$

The Slutsky equations for the upper stage effects on expenditure are given by equations (4.5.17) and (4.5.18). Using these Slutsky equations, we can write equations (4.5.23) through (4.5.26) as follows.

$$\begin{aligned} dc(x)/dw(x^*) = {} & \partial c(x)/\partial w(x^*) \big|_{U(x^*)} \\ & - [\vartheta - h(x^*)]\ [\partial c(x)/\partial e(x^*)] \\ & + [\partial c(x)/\partial e(x)]\ [de(x)/dw(x^*) \big|_{\Omega,cl}] \\ & + [\partial c(x)/\partial e(x)]\ [de(x)/dw(x^*) \big|_{\Omega,ir}] \\ & + [\partial c(x)/\partial e(x)]\ B(n,x^*)\ [de(x)/da(n)] \\ & \qquad\qquad \text{for } x = x^* \end{aligned} \tag{4.5.29}$$

$$\begin{aligned} dc(x)/dw(x^*) = {} & [\partial c(x)/\partial e(x)]\ [de(x)/dw(x^*) \big|_{\Omega,ir}] \\ & + [\partial c(x)/\partial e(x)]\ B(n,x^*)\ [de(x)/da(n)] \\ & \qquad\qquad \text{for } x \neq x^* \end{aligned} \tag{4.5.30}$$

$$\begin{aligned} dh(x)/dw(x^*) = {} & \partial h(x)/\partial w(x^*) \big|_{U(x^*)} \\ & - [\vartheta - h(x^*)]\ [\partial h(x)/\partial e(x^*)] \\ & + [\partial h(x)/\partial e(x)]\ [de(x)/dw(x^*) \big|_{\Omega,cl}] \\ & + [\partial h(x)/\partial e(x)]\ [de(x)/dw(x^*) \big|_{\Omega,ir}] \\ & + [\partial h(x)/\partial e(x)]\ B(n,x^*)\ [de(x)/da(n)] \\ & \qquad\qquad \text{for } x = x^* \end{aligned} \tag{4.5.31}$$

$$\begin{aligned} dh(x)/dw(x^*) = {} & [\partial h(x)/\partial e(x)]\ [de(x)/dw(x^*) \big|_{\Omega,ir}] \\ & + [\partial h(x)/\partial e(x)]\ B(n,x^*)\ [de(x)/da(n)] \\ & \qquad\qquad \text{for } x \neq x^* \end{aligned} \tag{4.5.32}$$

As in chapter 3, the first term in equations (4.5.29) and (4.5.31) can be referred to as the intratemporal substitution effect of a wage increase because it relates to substitution between consumption and leisure at a given age due to an increase in the current wage rate. The second term in these two equations can be called the income effect of a wage increase as it shows the effect of an equivalent decrease in expenditure at age x necessary to realize the same level of utility at age x as attained due to the wage increase. The third term shows the cost of living effect working through the increase in expenditure necessary to retain a given bundle of consumption and leisure as the wage rate increases. The next two terms in these equations as well as the only two terms in equations (4.5.30) and (4.5.32) have been referred to earlier as the intertemporal reallocation effect and the wealth effect respectively.

Theorem 4.5.4

The income effect components of the one period effects of a parametric wage increase on consumption and work hours, that is, $-[\theta-h(x^*)]\,[\partial c(x)/\partial e(x^*)]$ and $-[\theta-h(x^*)]\,[\partial h(x)/\partial e(x^*)]$ are exactly offset by the respective cost of living effect components of the life cycle effect: $[\partial c(x)/\partial e(x)]\,[de(x)/dw(x^*)\,|_{\Omega,c}]$ and $[\partial h(x^*/\partial e(x)]\,[de(x)/dw(x^*)\,|_{\Omega,c}]$.

Proof

The cost of living effects can be written according to equations (4.5.7) and (4.5.9) as

$$[\partial c(x)/\partial e(x)]\ [de(x)/dw(x')\,|_{\Omega,c1}] = [\theta - h(x')]\ [\partial c(x)/de(x')]$$

$$[\partial h(x)/\partial e(x)]\ [de(x)/dw(x')\,|_{\Omega,c1}] = [\theta - h(x')]\ [\partial h(x)/de(x')]$$

which are obviously equal to the negative of the respective income effects.

The fact that there is no net income effect associated with a wage increase is exactly what one should expect in a life cycle model. A change in income at any age due to a parametric change is converted into the resulting change in wealth at the beginning of the planning horizon (age n in this case) and this additional wealth is then allocated among different commodities at different ages.

Equations (4.5.29) through (4.5.32) and theorem 4.5.4 imply the following result.

Theorem 4.5.5

The full effect of an increase in wage rate on the life cycle allocation of consumption and work hours can be decomposed as follows:

$$dc(x)/dw(x^*) = [\partial c(x)/\partial w(x^*) \big|_{U(x^*)}]$$
$$+ [\partial c(x)/\partial e(x)] [de(x)/dw(x^*) \big|_{\Omega, ir}]$$
$$+ [\partial c(x)/\partial e(x)] B(n,x^*) [de(x)/da(n)]$$
$$\text{for } x = x^* \qquad (4.5.33)$$

$$dc(x)/dw(x^*) = [\partial c(x)/\partial e(x)] [de(x)/dw(x^*) \big|_{\Omega, ir}]$$
$$+ [\partial c(x)/\partial e(x)] B(n,x^*) [de(x)/da(n)]$$
$$\text{for } x \neq x^* \qquad (4.5.34)$$

$$dh(x)/dw(x^*) = [\partial h(x)/\partial w(x^*) \big|_{U(x^*)}]$$
$$+ [\partial h(x)/\partial e(x)] [de(x)/dw(x^*) \big|_{\Omega, ir}]$$
$$+ [\partial h(x)/\partial e(x)] B(n,x^*) [de(x)/da(n)]$$
$$\text{for } x = x^* \qquad (4.5.35)$$

$$dh(x)/dw(x^*) = [\partial h(x)/\partial e(x)] [de(x)/dw(x^*) \big|_{\Omega, ir}]$$
$$+ [\partial h(x)/\partial e(x)] B(n,x^*) [de(x)/da(n)]$$
$$\text{for } x \neq x^* \qquad (4.5.36)$$

As in chapter 3, the first term in equations (4.5.33) and (4.5.35) will be referred to as the intratemporal substitution effect of an increase in the wage rate at age x^*. The second term in equations (3.5.33) and (3.5.35) and the first term in equations (3.5.34) and (3.5.36), will be called the intertemporal reallocation effect of an increase in the wage rate at age x^*. Finally, the last term will be called the wealth effect of the wage increase. The behaviour of consumption and work hours due to these three effects can be summarized as follows.

Theorem 4.5.6

The intratemporal substitution effect, implies that an increase in the wage rate at age x^* before retirement will result in more consumption and more work hours at age x^*. A wage increase after retirement has no intratemporal substitution effect.

Theorem 4.5.7

The intertemporal reallocation effect of an increase in the wage rate at age x^* before retirement, will result in more work hours at that age and less at others, irrespective of its expenditure elasticity. Consumption will be affected in the opposite direction if and only if both consumption and leisure are normal. If any one of these two goods is inferior, consumption will be affected in the same direction as work hours. Finally, a wage increase after retirement has no intertemporal reallocation effect.

Proof

If leisure is normal, then in response to an increase in the wage rate for age x^* before retirement the individual will allocate less expenditure for age x^* and more expenditure for each age $x \neq x^*$ (See equations (4.5.11) and (4.5.12) and theorem 4.5.1). This implies that, with leisure being normal, work hours will increase and consumption will decrease (increase) at age x^* if it is normal (inferior) and the opposite will result for each age $x \neq x^*$. On the other hand, if leisure is inferior, then expenditure will increase at age x^* and decrease at each age $x \neq x^*$. With leisure being inferior, work hours will again increase, but consumption will definitely increase at age x^* and the opposite will happen at each age $x \neq x^*$.

This means that the intertemporal effect of an increase in the wage rate at age x^* will definitely result in more work hours at that age and less at others, irrespective of its expenditure elasticity. In addition, consumption will be affected in the opposite direction if and only if both consumption and leisure are normal. If any one of these two goods is inferior, consumption will be affected in the same direction as work hours. The result that a wage increase after retirement has no intertemporal reallocation effect is obvious from equations (4.5.13) and (4.5.14).

Theorem 4.5.8

The wealth effect, says that an increase in the wage rate at age $x' \leq R$ will increase wealth at the beginning of the planning horizon, n, by the number of hours worked at age x' discounted by the interest factor (equation 4.5.15) and theorem 4.5.2). Therefore, if consumption and leisure are normal, consumption will increase at each age and work hours will decrease at each at each age before retirement. The wealth effect of an increase in the wage rate after retirement is zero (equation (4.5.16) and theorem 4.5.2).

Theorem 4.5.9

The intratemporal substitution effects of a wage increase on consumption and work hours are independent of the age at which the change in the wage rate is anticipated. The earlier the wage increase at age x', $x' \leq R$, anticipated, the larger (smaller) will be the intertemporal reallocation effects on work hours and consumption at age x' (at any age $x \neq x'$). Finally, the earlier the wage increase at age x', $x' \leq R$, anticipated, the smaller will be the associated wealth effects on work hours and consumption.

Thus, whether the change in wage rate at age x^* is anticipated at an earlier age, or it comes as a shock at age x^* is of no consequence for the intratemporal substitution effect. The reason is that this is a one period effect and has nothing to do with the life cycle considerations. Therefore it is independent of the anticipation lag.

The intertemporal reallocation effect does, on the other hand, depend on the anticipation lag. The smaller the age n at which the wage increase is anticipated, the larger will be the value of $\phi(n)$ (as can be seen from equation (4.3.8)). This implies that the intertemporal reallocation effect of an increase in the wage rate for age x^* on expenditure at age x^*, given by equations (4.5.11), will be larger in absolute terms. On the other hand, the intertemporal reallocation effect on expenditure at any age $x \neq x^*$, given by equation (4.6.12), will be smaller in absolute terms.

In other words, the earlier the age at which the wage increase is anticipated, the longer will be the remaining horizon for replanning. With a longer period for adjustment, expenditure at age x^* can be more easily changed. But with a longer period for adjustment, the compensating changes in each other period will be smaller.

Finally, the relationship of the wealth effect with the anticipation lag can be explained due to two factors. First, the earlier the wage increase anticipated, the longer the remaining planning horizon for readjustment and therefore the smaller the effect of a given increase in initial wealth on expenditure at any age. Second, an earlier anticipation of a wage increase also results in a smaller weighting factor attached to the wealth effect, $B(n,x')$, as can be seen from equation (4.5.15). The reason is that the longer the anticipation lag, that is, $x'-n$, the more heavily will be the increased wage income at age x' discounted to obtain the effect on initial wealth.

Finally, we study briefly the effect of a wage increase on the retirement decision of the individual. Consider the retirement condition (4.3.15). Differentiating this condition with respect to $w(x')$, we can write

$$\lim_{x \to R^-} [\dot{h}(x) \{dx/dw(x')\} + \{dh(x)/dw(x')\}] = 0$$

Taking the limit and solving for $dR/dw(x')$, we obtain

$$dR/dw(x') = \frac{\lim_{x \to R^-} [dh(x)/dw(x')]}{- \lim_{x \to R^-} \dot{h}(x)} \qquad (4.5.37)$$

Thus the obvious result is that, if the wage increase results in less (more) work at an age slightly below the retirement age, the individual will also retire earlier (later).

The signs of different components of the comparative static effects of an increase in the wage rate for some age before retirement on consumption and work hours at different ages are reported in tables 4.5.1. and 4.5.2.

4.5.2 The Comparison of Parametric and Evolutionary Changes in Wage Rates

The parametric changes in wage rates discussed above should be distinguished from the anticipated changes across the life cycle, also called the evolutionary changes, discussed in chapter 3. To understand this distinction it should be realized that in the context of a life cycle problem the wage profile is a set of distinct parameters, namely, the age specific wage rates. These should not be interpreted as a time series of a single variable. Thus, the fact that wage rate may vary across the life cycle means that these distinct parameters take their own values which may differ from one another. Then, evolutionary changes in the wage rate involve having some distinct parameters differing from one another. On the other hand, a change in the wage rate from its initially anticipated value is clearly a parametric change. As MaCurdy (1981) interprets, the effect of such a parametric change in the wage rate on the life

Table 4.5.1

Decomposition of the Compensated Effects of a Parametric Increase in the Wage Rate for Age x' < R into the Intratemporal and intertemporal Effects

		Intratemporal Substitution Effects	Intertemporal Reallocation Effects	Compensated Effects
		Consumption and Leisure are Normal		
dc(x)	$x = x'$	+	-	?
	$x \neq x'$	0	+	+
dh(x)	$x = x'$	+	+	+
	$x \neq x'$	0	-	-
		Consumption is Inferior		
dc(x)	$x = x'$	+	+	+
	$x \neq x'$	0	-	-
dh(x)	$x = x'$	+	+	+
	$x \neq x'$	0	-	-
		Leisure is Inferior		
dc(x)	$x = x'$	+	+	+
	$x \neq x'$	0	-	-
dh(x)	$x = x'$	+	+	+
	$x \neq x'$	0	-	-

Table 4.5.2

Decomposition of the Effects of a Parametric Increase in the Wage Rate for Age x^* < R into the Compensated and Wealth Effects

		Compensated Effects	Wealth Effects	Full Effects
		Consumption and Leisure are Normal		
dc(x)	$x = x^*$	?	+	?
	$x \neq x^*$	+	+	+
dh(x)	$x = x^*$	+	-	?
	$x \neq x^*$	-	-	-
		Consumption is Inferior		
dc(x)	$x = x^*$	+	-	?
	$x \neq x^*$	-	-	-
dh(x)	$x = x^*$	+	-	?
	$x \neq x^*$	-	-	-
		Leisure is Inferior		
dc(x)	$x = x^*$	+	+	+
	$x \neq x^*$	-	+	?
dh(x)	$x = x^*$	+	+	+
	$x \neq x^*$	-	+	?

cycle allocation decision shows the difference in allocation across two individuals who are identical in every respect but have different wage profiles. As a consequence, the effects of a parametric change in the wage rate are different from those of an evolutionary change.

First, as explained by MaCurdy (1981), a parametric change in the wage rate carries a wealth effect which is absent in the case of an evolutionary change. The reason is that an evolutionary change in the wage rate means that the wage rate across different ages is not identical. This age pattern of wage rates does not create any wealth effect. On the other hand, a parametric change in the wage rate will also change wealth if the individual is working at the age for which the wage rate has changed.

To be more precise, consider the way the two effects have been derived. The derivation of the effects of evolutionary changes in the wage rates is based on the tangency condition (3.3.33) alone, ignoring the lifetime budget constrain. The reason is that these type of changes in wage rates do not alter the lifetime budget constraint which already incorporates the fact that wage rate may vary across the life cycle. On the other hand, the derivation of the effects of parametric changes in wage rates is based on both the tangency condition (3.3.33) as well as the budget constraint (3.3.34). The parametric changes in wage rates are not anticipated at the beginning of the planning horizon and hence these changes are not incorporated into the budget constraint initially perceived by the individual. The obvious outcome of this difference is that the evolutionary changes in wage rates do not create any wealth effect where as the parametric changes in wage rates will also change wealth if the individual is working at the age for which the wage rate has changed.

The fact that there is no wealth effect associated with an evolutionary change in the wage rate, does not make the effects of this type of change in the wage rate comparable with the compensated effects of a parametric change in the wage rate. The effects of an evolutionary change in the wage rate consist of the intratemporal substitution effects and the intertemporal allocation effects. The compensated effects of a parametric change in the wage rate are split into the intratemporal substitution effects and the intertemporal reallocation effects. While the intratemporal effects are identical across the two types of changes in the wage rate, the intertemporal effects are different. The intertemporal allocation effects of an evolutionary change in the wage rate are the results of optimal <u>allocation</u> of expenditure over the life cycle in the light of a known wage profile. On the other hand, the intertemporal reallocation effects, represent the effect of optimal <u>reallocation</u> of expenditure in the light of changes in wage rates away from the initially anticipated wage profile.

To be more specific, recall from chapter 3 that the effects of an evolutionary change in the wage rate are identical to the effects of a parametric change holding constant the marginal utility of wealth. The compensated effects, on the other hand, are based on a parametric change holding constant lifetime utility, rather than marginal utility. The difference between the two types of affects of a change in the wage rate on the allocation of expenditure is explained by the following result proved in appendix 4.A.5.

Theorem 4.5.10

In the context of life cycle allocation problem of an individual aged n, the compensated effect of a parametric change in the wage rate on expenditure: $de(x)/dw(x^*)\big|_{\Omega}$ is related to the effect of an evolutionary change in the wage rate: $de(x)/dw(x^*)\big|_{V_2(n)}$ as follows.

$$de(x)/dw(x^*)\big|_{\Omega} = de(x)/dw(x^*)\big|_{V_2(n)}$$
$$+ [de(x)/V_2(n)]\,[dV_2(n)/dw(x^*)\big|_{\Omega}] \qquad (4.5.38)$$

where, $de(x)/V_2(n)$, the effect of an increase in the marginal utility of wealth on expenditure at age x is:

$$de(x)/V_2(n) = e^{-r(x-n)}/V_{22}(x) < 0 \qquad (4.5.39)$$

and $dV_2(n)/dw(x^*)|_\Omega$, the compensated effect of an increase in the wage rate for age x on the marginal utility of wealth is:

$$dV_2(n)/dw(x^*)|_\Omega = - e^{r(x-n)} V_{22}(x)$$
$$[e(x)/\xi(x)\phi(n)] e^{-rx^*} p(x^*) [\psi_1(x^*)/U_2(x^*)\delta(x^*)]$$
$$\text{for } x^* \leq R \qquad (4.5.40)$$

$$dV_2(n)/dw(x^*)|_\Omega = 0 \qquad \text{for } x^* \geq R \qquad (4.5.41)$$

Equation (4.5.39) says that with an increase in the marginal utility of wealth, expenditure at each age must decrease. This is an obvious outcome of a utility function concave in expenditure. This result is also analogous to the behaviour of a profit maximizing firm which will decrease its expenditure on inputs as the price of output decreases.[4] Equation (4.5.40) says that if leisure is a

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4 It may be noted that reciprocal of the marginal utility of wealth is equal to the marginal cost of utility. In their profitable approach to the life cycle problem, Browning, Deaton and Irish (1985) interpret the reciprocal of the marginal utility of wealth as the price of utility for a price taker 'profit maximizing' individual since under profit maximization marginal cost is equal to the price of output.

normal (inferior) good, an increase in the wage rate at a working age will decrease (increase) the marginal utility of nominal wealth. Again, this result is analogous to the effects of an increase in the price of a normal (inferior) factor input on the marginal cost of production.[5] Equation (4.5.41) says that an increase in the wage rate after retirement has no effect on the marginal utility of wealth. Once again, we find an analogy. An increase in the price of a fixed factor input has no effect on marginal cost. The above discussion implies:

Theorem 4.5.11

The compensated effect on expenditure of an increase in the wage rate for some age before retirement is greater (smaller) than the effect of an equal evolutionary change if leisure is a normal (an inferior) good. The compensated effect of an increase in the wage rate for some age after retirement is equal to the effect of an evolutionary change.

To analyze the relationships between the effects of evolutionary and parametric changes in the wage rate on con-

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5 It should be noted again, that the 'marginal cost of utility' is reciprocal of the marginal utility of wealth (See Browning, Deaton and Irish (1985)).

sumption and work hours, we substitute in (4.5.38) for the compensated effect: $de(x)/dw(x') |_{\Omega}$ in terms of the cost of living effect and the intertemporal reallocation effect from equation (4.5.6). According to our analysis in chapter 3 (subsection 3.3.3), we can also substitute for the effect of an evolutionary change in the wage rate in terms of a cost of living effect and an intertemporal allocation effect. Since the cost of living effects are identical on both the sides, the result of this substitution is:

$$de(x)/dw(x') |_{\Omega, ir} = + de(x)/dw(x') |_{i,a} + [de(x)/V_2(n)] \, [dV_2(n)/dw(x') |_{\Omega}] \tag{4.5.42}$$

Multiplying both the sides by the derivative $\partial c(x)/\partial e(x)$ and $\partial h(x)/\partial e(x)$ alternatively, we can find the relationship of the effects of evolutionary and compensated changes in the wage rate on consumption as well as on work hours. This relationship is summarized in the following result.

Theorem 4.5.12

The intertemporal reallocation effects of a parametric change in the wage rate are related with the intertemporal allocation effects of an evolutionary change as follows.

$$[\partial c(x)/\partial e(x)]\ [de(x)/dw(x^*)\ |_{\Omega,ir}] =$$
$$[\partial c(x)/\partial e(x)]\ [de(x)/dw(x^*)\ |_{i,a}]$$
$$+\ [\partial c(x)/\partial e(x)]\ [de(x)/V_2(n)]\ [dV_2(n)/dw(x^*)\ |_{\Omega}] \tag{4.5.43}$$

$$[\partial h(x)/\partial e(x)]\ [de(x)/dw(x^*)\ |_{\Omega,ir}] =$$
$$[\partial h(x)/\partial e(x)]\ [de(x)/dw(x^*)\ |_{i,a}]$$
$$+\ [\partial h(x)/\partial e(x)]\ [de(x)/V_2(n)]\ [dV_2(n)/dw(x^*)\ |_{\Omega}] \tag{4.5.44}$$

Adding the intratemporal effects on both the sides of equations (4.5.43) and (4.5.44), we obtain the following result:

$$\partial c(x)/\partial w(x^*)\ |_{U(x)}\ +\ [\partial c(x)/\partial e(x)]\ [de(x)/dw(x^*)\ |_{\Omega,ir}] =$$
$$[\partial c(x)/\partial w(x^*)\ |_{U(x)}]$$
$$+\ [\partial c(x)/\partial e(x)]\ [de(x)/dw(x^*)\ |_{i,a}]$$
$$+\ [\partial c(x)/\partial e(x)]\ [de(x)/V_2(n)]\ [dV_2(n)/dw(x^*)\ |_{\Omega}]$$

$$\partial h(x)/\partial w(x^*)\ |_{U(x)}\ +\ [\partial h(x)/\partial e(x)]\ [de(x)/dw(x^*)\ |_{\Omega,ir}] =$$
$$[\partial h(x)/\partial w(x^*)\ |_{U(x)}]$$
$$+\ [\partial h(x)/\partial e(x)]\ [de(x)/dw(x^*)\ |_{i,a}]$$
$$+\ [\partial h(x)/\partial e(x)]\ [de(x)/V_2(n)]\ [dV_2(n)/dw(x^*)\ |_{\Omega}]$$

The left hand sides of the above equations are equal to the compensated effects of a parametric change in the wage rate. Similarly the first two components on the right hand sides are equal to the effects of an evolutionary change in the wage rate . Thus we obtain the following result.

Theorem 4.5.13

The compensated effects of a parametric change in the wage rate are related with the effects of an evolutionary change as follows.

$$\partial c(x)/\partial w(x^*)\,|_{\Omega} = [\partial c(x)/\partial w(x^*)\,|_{V_2(n)}] + [\partial c(x)/\partial e(x)]\,[de(x)/V_2(n)]\,[dV_2(n)/dw(x^*)\,|_{\Omega}] \tag{4.5.45}$$

$$\partial h(x)/\partial w(x^*)\,|_{\Omega} = [\partial h(x)/\partial w(x^*)\,|_{V_2(n)}] + [\partial h(x)/\partial e(x)]\,[de(x)/V_2(n)]\,[dV_2(n)/dw(x^*)\,|_{\Omega}] \tag{4.5.46}$$

From these relations we can also infer the following result.

Theorem 4.5.14

The compensated effect on work hours of a parametric increase in the wage rate for some age before retirement is less than the effect of an evolutionary increase. The compensated effect on consumption of a parametric increase in the wage rate is more (less) than the effect of an evolutionary increase if both consumption and leisure are normal (any one of consumption and leisure is inferior). The two types of effects of an increase in the wage rate for some age after retirement are equal to each other, both being equal to zero.

Proof

The derivative: $[de(x)/V_2(n)]$ is known to be negative by equation (4.5.39). If leisure is normal, the derivatives: $[dV_2(n)/dw(x^*)|_{\Omega}]$ and $[\partial h(x)/\partial e(x)]$ will be negative, (see equation (4.5.40)). If, on the other hand leisure is inferior, both of these derivatives are positive. Therefore, whether leisure is normal or inferior, the difference between the two types of effects of a changes in the wage rate:

$$\partial h(x)/\partial w(x^*)|_{\Omega} - [\partial h(x)/\partial w(x^*)|_{V_2(n)}] = [\partial h(x)/\partial e(x)]\,[de(x)/V_2(n)]\,[dV_2(n)/dw(x^*)|_{\Omega}]$$

is negative. Next consider the effects on consumption. If both consumption and leisure are normal, the difference between the two types of effects of a change in the wage rate:

$$\partial c(x)/\partial w(x^*)\big|_{\Omega} - [\partial c(x)/\partial w(x^*)\big|_{V_2(n)}] =$$
$$[\partial c(x)/\partial e(x)]\ [de(x)/V_2(n)]\ [dV_2(n)/dw(x^*)\big|_{\Omega}]$$

is positive. If any of the two goods is inferior, this difference is negative. Finally, the derivative: $dV_2(n)/dw(x^*)\big|_{\Omega}$ equals zero for $x^* \geq R$ (see equation (4.5.41)). This completes the proof.

The break down of the compensated effects given by equations (4.5.45) and (4.5.46) can be referred to as the break down into the 'specific substitution' effects and the 'general substitution' effects due to Houthakker (1960). In analyzing the comparative static effects of price changes with additive utility function, Houthakker has referred the cross compensated effects to be the general substitution effects. Due to additive separability of the utility function, these effects come only through change in the marginal utility of income brought about by the change in price. If the utility function is not additive, the cross substitution effects will be different. The additional components in the cross substitution effects due to non-additivity of the utility function are referred to be the specific substitution effects by Houthakker. This means that the compensated effects with non additive utility function can be decomposed into two components, the specific substitution effects and the general substitution effects. The specific substitution effects can be obtained by evaluating the effects of a price change, holding constant the marginal utility. The general substitution effects can be obtained by deriving the effects of a price change on the demands for goods due to the resulting compensated change in the marginal utility.

This is precisely what we have done in obtaining the break down of the compensated effects given by equations (4.5.45) and (4.5.46). Thus the effects of an evolutionary change in the wage rate or the marginal utility held constant effects of a parametric change in the wage rate can also be called the 'specific substitution' effects of a parametric change in the wage rate. Similarly the effects of a parametric change in the wage rate through the compensated change in the marginal utility of wealth can be called the general substitution effects. To recapitulate these ideas, we now report the signs pattern of the break down of the compensated effects of a parametric increase in the wage rates in table 4.5.3.

Table 4.5.3

Decomposition of the Compensated Effects of a Parametric Increase in the Wage Rate for Age x^* < R into the Specific and the General Substitution Effects

		Compensated Effects of a Wage Increase	The Specific Sub. Effects or the Marginal Utility Constant Effects Or Effects of an Evolutionary Wage Increase	Col. (1) minus Col. (2) The General Sub. Effects
			Consumption and Leisure are Normal	
$dc(x)$	$x = x^*$	?	?	+
	$x \neq x^*$	+	0	+
$dh(x)$	$x = x^*$	+	+	-
	$x \neq x^*$	-	0	-
			Consumption is Inferior	
$dc(x)$	$x = x^*$	+	+	-
	$x \neq x^*$	-	0	-
$dh(x)$	$x = x^*$	+	+	-
	$x \neq x^*$	-	0	-
			Leisure is Inferior	
$dc(x)$	$x = x^*$	+	+	-
	$x \neq x^*$	-	0	-
$dh(x)$	$x = x^*$	+	+	-
	$x \neq x^*$	-	0	-

4.6 CONCLUDING REMARKS

In this chapter we studied the implications of various parametric changes on the life cycle consumption and work allocation decision. The parametric changes included the survival probabilities, the rate of interest and the wage rates. The analysis shows that the effects of these parametric changes are sensitive to the anticipation lag, that is, the lag between the actual time of a parametric change and the time when this change was anticipated. The earlier the parametric change is anticipated, the longer will be the horizon for replanning and, therefore, the smaller will be the adjustment for each future age.

Two types of mortality changes were discussed, an increase in general survival rates in the society and a subjective increase in the survival probabilities for the individual concerned. It was shown that an increase in general survival rates affects the individual's lifetime allocation decision due to two reasons. First, this mortality improvement will affect the expected lifetime utility function of the individual due to the changes in survival probabilities. Second, an increase in survival rates in the society in general will also affect the lifetime budget constraint of the individual due to the changes in the actuarially fair rates of interest which are calculated on the basis of general survival risks in the society. This type of mortality improvement carries wealth effects by changing the actuarial rates of interest. But the resulting intertemporal substitution effects are fully offset by an equivalent change in discount factors attached with the lifetime utility function. The wealth effects depend on the savings pattern during the segment of life for which the survival rates have increased. An increase in the survival rates to a young or an old age, when savings are negative, will result in less consumption, more work, more savings and a later retirement (assuming that consumption and leisure are normal). On the other hand, the effects of an increase in the survival rates to a middle age, when savings are positive, are the opposite. This result cannot be obtained in a two period model without assuming, like Abel (1986), that savings in old age could be positive.

The implications of an increase in the mean preserving spread in the distribution of life were also noted. If the survival probabilities through the earlier segment of the horizon, when savings out of earnings are increasing, are held constant, then the effect of increasing risk is to reduce consumption and increase work hours for all the future ages (assuming that consumption and leisure are normal).

Since the effects of general mortality changes come from the resulting changes in the actuarial rates of interest, it is important to note that for the realization of these effects, the insurance companies must also fully recognize the current mortality variations. An alternative type of mortality variation could be due to a changes in survival prospects for the individual concerned, independent of general survival rates in the society. We showed that this type of mortality change affects only the expected lifetime utility function and, therefore, carries only an intertemporal substitution effect. The actuarial rates of interest which are calculated on the basis of general survival risks in the society rather than the subjective distribution of life of a particular individual, are unchanged. Therefore this subjective mortality change will leave the lifetime budget constraint unchanged. It was shown that an increase in the subjective probability of survival to some age will result in more consumption and less work hours at that age and less consumption and more work hours at each other age.

The effect on consumption and work hours of an increase in the rate of interest was shown to consist of a compensated effect and a wealth effect. The compensated effect results in higher consumption and lower work hours at later ages in compensation for lower consumption and higher work hours at earlier ages. The wealth effect depends on the sign pattern of asset holding over the life cycle. If the effect of negative asset holding at an earlier age is small, then a higher rate of interest will create a positive wealth effect by increasing the interest income on assets. This will result in more consumption and less work (assuming both consumption and leisure are normal). The full effect of an increase in the rate of interest is predictable only for the later segment of life at which both the compensated and wealth effects are in the same direction.

Parametric changes in the wage rate were studied in detail. The effects on consumption and work hours are split into three parts, an intratemporal substitution effect, an intertemporal reallocation effect and a wealth effect.

The intratemporal substitution effect of an increase in the wage rate at some specific age relates to substitution between consumption and leisure, given the rate of utility at that age. This effect results in more consumption and more work at that age. The intertemporal reallocation effect of a higher wage rate relates to the resulting reallocation of expenditure over the life cycle due to the changes in relative prices of expenditure for different ages brought about by the increase in the wage rate. This will result in more work at the age for which the wage rate is increased and less at the other ages. If both consumption and leisure are normal, consumption will be affected in the opposite direction. If, on the other hand, either one of these two goods is inferior, consumption will be affected in the same direction as work hours. An increase in the wage rate during working age will increase wage income for a given work plan. The consequent wealth effect is to decrease work hours and increase consumption, provided both consumption and leisure are normal.

A parametric change in the wage rate has a distinctly different effect than an evolutionary change. First, a parametric change will carry a wealth effect which is absent in the evolutionary change. Second, the effect of an evolutionary change in the wage rate is not even comparable with the compensated effect of a parametric change. While the former is identical to the effect of a parametric change holding constant the marginal utility of wealth, the latter is based on a parametric change holding constant lifetime utility, rather than the marginal utility. While the intratemporal effects are identical across the two types of changes in the wage rate, the intertemporal effects are different. The intertemporal allocation effect of an evolutionary change is a result of optimal allocation of expenditure over the life cycle in the light of a known wage profile. On the other hand, the intertemporal reallocation effect results from the reallocation of expenditure in the light of a change in the wage rate away from the initially anticipated wage profile.

To further explain this distinction, the compensated effects of an increase in the wage rate were decomposed into two components, the marginal utility held constant effects or the 'specific substitution' effects and the effects through the compensated change in the marginal utility of wealth or the 'general substitution' effects. It was shown that the compensated effects of a parametric increase in the wage rate on work hours are smaller than the effects of an equal evolutionary wage increase or the specific substitution effects. Therefore, the general substitution effects of a parametric wage increase on work hours are negative. In addition, the compensated effects of a parametric wage increase on consumption are greater (smaller) than the effects of an equal evolutionary wage increase or the specific substitution effects if both consumption and leisure are normal (any one of consumption and leisure is inferior). This implies that the general substitution effects of a parametric wage increase on consumption are positive (negative) if consumption and leisure are normal (either consumption or leisure is inferior).

4.A APPENDIX TO CHAPTER 4

4.A.1 The effects of General Mortality Improvements on the Allocation of Expenditure (Equation 4.3.7)

In this appendix we will derive an expression for the effect of a small increase in the probability of survival to age x' anticipated at age $n \le x'$ on expenditure at age x, $n \le x \le T$. Differentiation the optimizing conditions (4.2.3) and (4.2.4) with respect to $p(x')$, we find

$$de(x)/dp(x') = e^{-r(x-n)} [V_{22}(n)/V_{22}(x)]\ de(n)/dp(x') \quad (4.A.1)$$

$$\int_n^T e^{-r(x-n)} [p(x)/p(n)][de(x)/dp(x')]\ dx = e^{-r(x'-n)} \sigma(x')/p(n)$$

Multiplying the second equation by $e^{-rn}\ p(n)$, we write:

$$\int_n^T e^{-rx}\ p(x)\ [de(x)/dp(x')]\ dx = e^{-rx'}\ \sigma(x')$$

Substituting $de(x)/dp(x')$ from equation (4.A.1) into this equation, we obtain the following equation in one unknown, namely $de(n)/dp(x')$.

$$V_{22}(n)\ [de(n)/dp(x')] \int_n^T e^{-rx} p(x)\ e^{-r(x-n)}\ [1/V_{22}(x)]\ dx = e^{-rx'}\ \sigma(x')$$

Or substituting $e^{-r(x-n)} = V_2(x)/V_2(n)$, we obtain:

$$[V_{22}(n)/V_2(n)]\ [de(n)/dp(x')]\ \int_n^T e^{-rx} p(x)\ [V_2(x)/V_{22}(x)]\ dx = e^{-rx'}\ \sigma(x')$$

Now we introduce coefficient of relative risk aversion, $\xi(x)$, defined as follows.

$$\xi(x) = -\ e(x)V_{22}(x)/V_2(x) \tag{4.A.2}$$

Using this coefficient, we can write the above equation as:

$$-\ [V_{22}(n)/V_2(n)]\ \phi(n)\ [de(n)/dp(x')] = e^{-rx'}\ \sigma(x') \tag{4.A.3}$$

where,

$$\phi(n) = \int_n^T e^{-rx}\ p(x)\ [e(x)/\xi(x)]\ dx > 0 \tag{4.A.4}$$

Equation (4.A.3) can be solved for $de(n)/dp(x')$ as follows:

$$de(n)/dp(x') = -\ [V_2(n)/V_{22}(n)\phi(n)]\ [e^{-rx'}\ \sigma(x')]$$

Finally, substituting this result in equation (4.A.1) and simplifying, we obtain the solution for $de(x)/dp(x')$:

$$de(x)/dp(x') = -\ [e^{-r(x-n)}V_2(n)/V_{22}(x)\phi(n)]\ [e^{-rx'}\ \sigma(x')]$$

Substituting $e^{-r(x-n)}V_2(n) = V_2(x)$, and using (3.A.2), we end up with:

$$de(x)/dp(x') = [e(x)/\xi(x)\phi(n)]\ [e^{-rx'}\ \eta(x')] \qquad (4.A.5)$$

4.A.2 The Effects of Subjective Mortality Improvements on Expenditure (Equations (4.3.29), (4.3.30)

For the problem under consideration, the optimizing conditions are give by equations (4.3.27) and (4.3.28) of the text. These equations are reproduced below.

$$V_2(x) = e^{-r(x-n)}\ [p^*(x)p(n)/p(x)p^*(n)]\ V_2(n) \qquad (4.A.6)$$

$$\int_n^T e^{-rx}\ p^*(x)\ [w(x)\theta - e(x)]\ dx = \text{constant} \qquad (4.A.7)$$

Differentiating equation (4.A.6) with respect to p(x'), we can write:

$$\begin{aligned} de(x)/dp(x') = {} & \\ & e^{-r(x-n)}\ [p^*(x)p(n)/p(x)p^*(n)]\ [V_{22}(n)/V_{22}(x)]\ de(n)/dp(x') \\ & \quad - e^{-r(x'-n)}\ [p^*(x')p(n)/p(x')^2p^*(n)]\ V_2(n)/V_{22}(x') \\ & \qquad\qquad \text{for } x = x' \end{aligned}$$

$$\begin{aligned} de(x)/dp(x') = {} & \\ & e^{-r(x-n)}\ [p^*(x)p(n)/p(x)p^*(n)]\ [V_{22}(n)/V_{22}(x)]\ de(n)/dp(x') \\ & \qquad\qquad \text{for } x \neq x' \end{aligned}$$

Or substituting $e^{-r(x-n)}\,[p^*(x)p(n)/p(x)p^*(n)] = V_2(x)/V_2(n)$ from equation (4.A.6),

$$de(x)/dp(x') = [V_2(x)/V_2(n)]\,[V_{22}(n)/V_{22}(x)]\,de(n)/dp(x') - [V_2(x')/V_{22}(x')p(x')] \quad \text{for } x = x' \qquad (4.A.8)$$

$$de(x)/dp(x') = [V_2(x)/V_2(n)]\,[V_{22}(n)/V_{22}(x)]\,de(n)/dp(x') \quad \text{for } x \neq x' \qquad (4.A.9)$$

Next we differentiate equation (4.A.7) with respect to $p(x')$. The result is:

$$\int_n^T e^{-rx}\,p^*(x)\,[de(x)/dp(x')]\,dx = 0$$

Substituting equations (4.A.8)) and (4.A.9) into the above equation, we obtain:

$$[V_{22}(n)/V_2(n)]\,[de(n)/dp(x')] \int_n^T e^{-rx}\,p^*(x)\,[V_2(x)/V_{22}(x)]\,dx - e^{-rx'}\,p^*(x')\,[V_2(x')/V_{22}(x')p(x')] = 0$$

Using equation (4.A.2), we can write this equation as:

$$[V_{22}(n)/V_2(n)]\,[de(n)/dp(x')]\,\phi^*(n) = e^{-rx'}\,p^*(x')\,[e(x')/\xi(x')p(x')]$$

Or,

$$de(n)/dp(x') = [V_2(n)/V_{22}(n)]\ [e^{-rx'}\ p^*(x')e(x')/\xi(x')\phi^*(n)]\ [1/p(x')] \quad (4.A.10)$$

where,

$$\phi^*(n) = \int_n^T e^{-rx}\ p^*(x)\ [e(x)/\xi(x)]\ dx > 0$$

Substituting equation (4.A.10) back into equations (4.A.8) and (4.A.9), we obtain:

$$de(x)/dp(x') = [V_2(x)/V_{22}(x)]\ [e^{-rx'}\ p^*(x')e(x')/\xi(x')\phi^*(n)p(x')] - V_2(x')/V_{22}(x')p(x') \quad \text{for } x = x' \quad (4.A.11)$$

$$de(x)/dp(x') = [V_2(x)/V_{22}(x)]\ [e^{-rx'}\ p^*(x')e(x')/\xi(x')\phi^*(n)p(x')] \quad \text{for } x \neq x' \quad (4.A.12)$$

Or,

$$de(x)/dp(x') = -\ [e(x)/\xi(x)]\ [e^{-rx'}\ p^*(x')e(x')/\xi(x')\phi^*(n)p(x')] + [e(x')/\xi(x')p(x')] \quad \text{for } x = x' \quad (4.A.13)$$

$$de(x)/dp(x') =$$

$$- [e(x)/\xi(x)]\ [e^{-rx'}\ p^{*}(x')e(x')/\xi(x')\phi^{*}(n)p(x')] \quad \text{for } x \neq x' \qquad (4.A.14)$$

4.A.3 The Effects of an Increase in the Rate of Interest on the Allocation of Expenditure

In order to find the effect of an increase in the rate of interest on the allocation of expenditure we differentiate the optimizing conditions (4.2.3) and (4.2.4) with respect to r. The result of this differentiation for equation (4.2.3) is as follows.

$$de(x)/dr = e^{-r(x-n)}\ [V_{22}(n)/V_{22}(x)]\ de(n)/dr$$
$$- (x - n)\ e^{-r(x-n)}\ [V_2(n)/V_{22}(x)]$$

Or substituting $e^{-r(x-n)} = V_2(x)/V_2(n)$, we obtain:

$$de(x)/dr = [V_2(x)V_{22}(n)/V_2(n)V_{22}(x)]\ de(n)/dr$$
$$- (x - n)\ [V_2(x)/V_{22}(x)] \qquad (4.A.15)$$

Next, differentiating equation (4.2.4) with respect to r and multiplying the resulting expression by e^{-rn} p(n), we get

$$- \int_n^T e^{-rx}\ p(x)\ [de(x)/dr]\ dx - \int_n^T (x-n)\ e^{-rx}\ p(x)\ e(x)\ dx = 0$$

Substituting equation (4.A.15) into this equation, we find

$$- [V_{22}(n)/V_2(n)]\ [de(n)/dr] \int_n^T e^{-rx}\ p(x)\ [V_2(x)/V_{22}(x)]\ dx\ +$$

$$\int_n^T (x-n)e^{-rx}p(x)[V_2(x)/V_{22}(x)]dx\ -\ \int_n^T (x-n)e^{-rx}p(x)\sigma(x)dx\ =\ 0$$

Or, using (4.A.2) and (4.A.4),

$$[V_{22}(n)/V_2(n)]\ \phi(n)\ [de(n)/dr]$$

$$- \int_n^T (x-n)e^{-rx}p(x)[e(x)/\xi(x))]dx\ -\ \int_n^T (x-n)e^{-rx}p(x)\sigma(x)dx\ =\ 0$$

where, $\phi(n)$ is defined by equation (4.A.4). The above equation can be solved for $de(n)/dr$ as follows.

$$de(n)/dr = [V_2(n)/V_{22}(n)]\ (\bar{x}_n - n)$$

$$+ [V_2(n)/V_{22}(n)\phi(n)] \int_n^T (x-n)\ e^{-rx}\ p(x)\ \sigma(x) \qquad (4.A.16)$$

where, $\bar{x}_n$ is the weighted average of x defined as follows.

$$\bar{x}_n = \frac{\int_n^T x\ e^{-rx}\ p(x)\ [e(x)/\xi(x)]\ dx}{\int_n^T e^{-rx}\ p(x)\ [e(x)/\xi(x)]\ dx} \qquad (4.A.17)$$

Finally, substituting for $de(n)/dr$ from equation (4.A.16) back into equation (4.A.15), we obtain the desired solution.

$$de(x)/dr = [V_2(x)/V_{22}(x)]\ (\bar{x}_n - x)$$

$$+ [V_2(x)/V_{22}(x)\phi(n)] \int_n^T (x-n)\ e^{-rx}\ p(x)\ \sigma(x)\ dx$$

Or, using (4.A.2),

$$de(x)/dr = -\ [e(x)/\xi(x)]\ (\bar{x}_n - x)$$

$$-\ [e(x)/\xi(x)\phi(n)] \int_n^T (x-n)\ e^{-rx}\ p(x)\ \sigma(x)\ dx \qquad (4.A.18)$$

The effect of an increase in the rate of interest can be expressed in terms of a Slutsky equation representing the compensated and wealth effects. The compensated effect can be obtained by differentiating the following equations with respect to r.

$$V_2(x) = e^{-r(x-n)}\ V_2(n) \qquad (4.A.19)$$

$$\int_0^n p(x)\ V(x)\ dx + \int_n^T p(x)\ V(x)\ dx = \Omega \qquad (4.A.20)$$

where, equation (4.A.19) is the tangency condition for the optimizing problem at the upper stage. With a fixed value of Ω, equation (4.A.20) holds constant the lifetime utility. For an individual at age n at which the rate of interest is assumed to have changed, the rate of utility derived at each previous age x < n is fixed. Therefore, differentiating the two equations with respect to r, we can write

$$de(x)/dr\,\big|_{\Omega} = e^{-r(x-n)}\,[V_{22}(n)/V_{22}(x)]\; de(n)/dr\,\big|_{\Omega}$$
$$- (x-n)\; e^{-r(x-n)}\,[V_2(n)/V_{22}(x)] \qquad (4.A.21)$$

$$\int_n^T p(x)\; V_2(x)\; [\dot{d}e(x)/dr\,\big|_{\Omega}]\; dx = 0$$

Substituting for $de(x)/dr\,\big|_{\Omega}$ from equation (4.A.21) into the above equation and simplifying, we can write:

$$- e^{rn}\; V_{22}(n)\; \phi(n)\; de(n)/dr\,\big|_{\Omega}$$
$$+ e^{rn}\; V_2(n) \int_n^T (x-n)\; e^{-rx}\; p(x)\; [e(x)/\xi(x)]\; dx = 0$$

Or, simplifying further

$$de(n)/dr\,\big|_{\Omega} = [V_2(n)/V_{22}(n)]\; (\bar{x}_n - n)$$

where, $\bar{x}_n$ is the weighted average of x, defined by equation (4.A.17). Substituting for $de(n)/dr\,\big|_{\Omega}$ from the above equation back into equation (4.A.21), we find

$$de(x)/dr\,\big|_{\Omega} = e^{-r(x-n)}\; [V_2(n)/V_{22}(x)]\; (\bar{x}_n - x)$$

Or substituting $e^{-r(x-n)}\; V_2(n) = V_2(x)$ and then $V_2(x)/V_{22}(x) = -e(x)/\xi(x)$, we obtain the compensated effect of an increase in the rate of interest.

$$de(x)/dr\,\big|_{\Omega} = -\ [e(x)/\xi(x)]\ (\bar{x}_n - x) \tag{4.A.22}$$

Next, we find an expression for the wealth effect on expenditure. Differentiating equations (4.2.3) and (3.2.4) which represent the optimizing conditions, with respect to initial wealth a(n) and simplifying, we obtain

$$de(x)/da(n) = e^{-r(x-n)}\ [V_{22}(n)/V_{22}(x)]\ de(n)/da(n), \tag{4.A.23}$$

$$\int_n^T e^{-r(x-n)}\ [p(x)/p(n)]\ [de(x)/da(n)]\ dx = 1$$

Substituting for de(x)/da(n) from equation (4.A.23) into the above equation, using (4.A.4) and simplifying, we find

$$de(n)/da(n) = -\ e^{-rn}\ p(n)\ [V_2(n)/V_{22}(n)\phi(n)]$$

Substituting this result back into equation (4.A.23), simplifying and using (4.A.2), we obtain:

$$de(x)/da(n) = e^{-rn}\ p(n)\ [e(x)/\xi(x)\phi(n)] \tag{4.A.24}$$

Using the expressions (4.A.22) and (4.A.24), the full effect of an increase in the rate of interest as given by equation (4.A.18), can be represented by the Slutsky equation:

$$de(x)/dr = de(x)/dr\big|_{\Omega} + \chi(n)\ [de(x)/da(n)] \qquad (4.A.25)$$

where, $de(x)/dr\big|_{\Omega}$ is the compensated effect given by (4.A.22), $de(x)/da(n)$ is the wealth effect given by (4.A.24) and $\chi(n)$ is the weighting factor defined below.

$$\chi(n) = -\int_n^T (x-n)\ e^{-r(x-n)}\ [p(x)/p(n)]\ \tau(x)\ dx$$

The weighting factor $\chi(n)$ can be written in a more interpretable form as:

$$\chi(n) = -\int_n^T e^{-r(x-n)}\ [p(x)/p(n)]\ \tau(x)\ \int_n^x dy\ dx$$

Or, changing the order of integration,

$$\chi(n) = -\ [e^{rn}/p(n)] \int_n^T \int_x^T e^{-ry}\ p(y)\ \tau(y)\ dy\ dx \qquad (4.A.26)$$

Since $\int_0^T e^{-ry}\ p(y)\ \tau(y)\ dy = 0$, we can write:

$$\int_x^T e^{-ry}\ p(y)\ \tau(y)\ dy = -\int_0^x e^{-ry}\ p(y)\ \tau(y)\ dy$$

Substituting this relation in equation (4.A.26) and using the definition of assets given by (3.4.11), we obtain

$$x(n) = \int_n^T e^{-r(x-n)} [p(x)/p(n)]\ a(x)\ dx \qquad (4.A.27)$$

4.A.4 The Effects of an Increase in the Wage Rate on the Allocation of Expenditure

We will derive an expression for the effect of a small increase in the wage rate for a specific age x', anticipated at age $n \le x'$, on expenditure at age x. Again we start with the optimizing conditions (4.2.3) and (4.2.4). Differentiating equation (4.2.3) with respect to $w(x')$, we can write:

$$\begin{aligned} de(x)/dw(x') = {} & e^{-r(x-n)} [V_{22}(n)/V_{22}(x)]\ [de(n)/dw(x')] \\ & - [V_{12}(x')/V_{22}(x')] \qquad \text{for } x = x' \qquad (4.A.28) \end{aligned}$$

$$\begin{aligned} de(x)/dw(x') = {} & e^{-r(x-n)} [V_{22}(n)/V_{22}(x)]\ [de(n)/dw(x')] \\ & \text{for } x \ne x' \qquad (4.A.29) \end{aligned}$$

Next, we differentiate equation (4.2.4) with respect to $w(x')$. The result is:

$$\begin{aligned} & \int_n^T e^{-r(x-n)} [p(x)/p(n)]\ [de(x)/dw(x')]\ dx \\ & \qquad = \div\ e^{-r(x'-n)} [p(x')/p(n)] \end{aligned}$$

Substituting for $de(x)/dw(x')$ from equations (4.A.28) and (4.A.29) into the above equation and simplifying, we obtain

$$- [e^{rn}/p(n)]\ [V_{22}(n)/V_2(n)]\ \phi(n)\ [de(n)/dw(x^*)]$$

$$= e^{-r(x^*-n)}\ [p(x^*)/p(n)]\ [\{V_{12}(x^*)/V_{22}(x^*)\} + \theta]$$

From this equation we can solve for $de(n)/dw(x^*)$ as

$$de(n)/dw(x^*) =$$

$$- [V_2(n)/V_{22}(n)\phi(n)]\ e^{-rx^*}p(x^*)\ [\{V_{12}(x^*)/V_{22}(x^*)\} + \theta]$$

Now we substitute this solution for $de(n)/dw(x^*)$ back into equations (4.A.28) and (4.A.29). The result is as follows:

$$de(x)/dw(x^*) =$$

$$- [V_2(x)/V_{22}(x)\phi(n)]\ e^{-rx^*}p(x^*)\ [\{V_{12}(x^*)/V_{22}(x^*)\} + \theta]$$

$$- [V_{12}(x^*)/V_{22}(x^*)] \qquad \text{for } x = x^*$$

$$de(x)/dw(x^*) =$$

$$- [V_2(x)/V_{22}(x)\phi(n)]\ e^{-rx^*}p(x^*)\ [\{V_{12}(x^*)/V_{22}(x^*)\} + \theta]$$

$$\text{for } x \neq x^*$$

Or using (4.A.2),

$$de(x)/dw(x^*) =$$

$$[e(x)/\xi(x)\phi(n)]\ e^{-rx^*}p(x^*)\ [\{V_{12}(x^*)/V_{22}(x^*)\} + \theta]$$

$$- [V_{12}(x^*)/V_{22}(x^*)] \quad \text{for } x = x^* \quad (4.A.30)$$

$$de(x)/dw(x^*) =$$

$$[e(x)/\xi(x)\phi(n)]\ e^{-rx^*}p(x^*)\ [\{V_{12}(x^*)/V_{22}(x^*)\} + \partial] \quad \text{for } x \neq x^* \qquad (4.A.31)$$

For the compensated effect we differentiate equations (4.A.19) and (4.A.20) with respect to $w(x^*)$. The results are:

$$de(x)/dw(x^*)\big|_{\Omega} = e^{-r(x-n)}\ [V_{22}(n)/V_{22}(x)]\ [de(n)/dw(x^*)\big|_{\Omega}] - [V_{12}(x^*)/V_{22}(x^*)] \quad \text{for } x = x^* \qquad (4.A.32)$$

$$de(x)/dw(x^*)\big|_{\Omega} = e^{-r(x-n)}\ [V_{22}(n)/V_{22}(x)]\ [de(n)/dw(x^*)\big|_{\Omega}] \quad \text{for } x \neq x^* \qquad (4.A.33)$$

$$\int_n^T p(x)\ V_2(x)\ [de(x)/dw(x^*)\big|_{\Omega}]\ dx = -\ p(x^*)\ V_1(x^*) \qquad (4.A.34)$$

Substituting for $de(x)/dw(x^*)\big|_{\Omega}$ from equations (4.A.32) and (4.A.33) into equation (4.A.34), we get

$$-\ [e^{rn}\ V_{22}(n)]\ [de(n)/dw(x^*)\big|_{\Omega}]\ \phi(n)$$
$$= p(x^*)\ [\{V_2(x^*)V_{12}(x^*)/V_{22}(x^*)\} - V_1(x^*)] \qquad (4.A.35)$$

Equations (3.A.1) and (3.A.2) of last chapter imply that for working age

$$V_1(x) = -\{\theta - h(x)\}\,V_2(x) \tag{4.A.36}$$

Likewise for the retirement period, we can write according to equations (3.A.8) and (3.A.9), $V_1(x) = -\theta\, V_2(x)$. Equation (4.A.36) however covers both cases since in retirement $h(x) = 0$. Substituting this relation into equation (4.A.35) and solving for $de(n)/dw(x^*)\,|_\Omega$, the result is as follows:

$$de(n)/dw(x^*)\,|_\Omega = -\,[e^{-rn}\,p(x^*)V_2(x^*)/V_{22}(n)\phi(n)]$$
$$[\{V_{12}(x^*)/V_{22}(x^*)\} + \{\theta - h(x^*)\}] \tag{4.A.37}$$

Substituting for $de(n)/dw(x^*)\,|_\Omega$ back into equation (4.A.32) and (4.A.33) and simplifying, we get

$$de(x)/dw(x^*)\,|_\Omega = -\,[V_2(x)/V_{22}(x)\phi(n)]$$
$$e^{-rx^*}\,p(x^*)\,[\{V_{12}(x^*)/V_{22}(x^*)\} + \{\theta - h(x^*)\}]$$
$$-\,[V_{12}(x^*)/V_{22}(x^*)] \qquad \text{for } x = x^*$$

$$de(x)/dw(x^*)\,|_\Omega = -\,[V_2(x)/V_{22}(x)\phi(n)]$$
$$e^{-rx^*}\,p(x^*)\,[\{V_{12}(x^*)/V_{22}(x^*)\} + \{\theta - h(x^*)\}]$$
$$\text{for } x \neq x^*$$

Using (4.A.2), we can write these equations as follows.

$$de(x)/dw(x^*)\big|_{\Omega} =$$
$$[e(x)/\xi(x)\phi(n)]\ e^{-rx^*}\ p(x^*)\ [\{V_{12}(x^*)/V_{22}(x^*)\} + \{\theta-h(x^*)\}]$$
$$- [V_{12}(x^*)/V_{22}(x^*)] \quad \text{for } x = x^* \qquad (4.A.38)$$

$$de(x)/dw(x^*)\big|_{\Omega} =$$
$$[e(x)/\xi(x)\phi(n)]\ e^{-rx^*}\ p(x^*)\ [\{V_{12}(x^*)/V_{22}(x^*)\} + \{\theta-h(x^*)\}]$$
$$\text{for } x \neq x^* \qquad (4.A.39)$$

Comparing these compensated effects with the full effects of a wage increase, given by equations (4.A.30) and (4.A.31), we can write the latter as:

$$de(x)/dw(x^*) = [de(x)/dw(x^*)\big|_{\Omega}]$$
$$+ [e^{-rx^*}\ p(x^*)e(x)/\xi(x)\phi(n)]\ h(x^*)$$

Or, using the expression for the wealth effect, given by equation (4.A.24), we can write the Slutsky equation for a wage increase as follows:

$$de(x)/dw(x^*) = [de(x)/dw(x^*)\big|_{\Omega}] + \beta(n,\ x^*)\ [de(x)/da(n)]$$
$$(4.A.40)$$

where, $de(x)/dw(x^*)\big|_{\Omega}$ is the compensated effect of an increase in the wage rate for age x^* given by equations (4.A.38) and (4.A.39), $de(x)/da(n)$ is the wealth effect, given by equation (4.A.24) and $B(n, x^*)$, the weighting factor which measures the effect of an increase in the wage rate for age x^* on wealth at age n, is given below.

$$B(n, x^*) = e^{-r(x^*-n)} \, [p(x^*)/p(n)] \; h(x^*) \tag{4.A.41}$$

The compensated effect, $de(x)/dw(x^*)\big|_{\Omega}$ may be written in a more interpretable form by substituting for $V_{12}(x^*)$ and $V_{22}(x^*)$ in terms of the corresponding expressions in the direct utility function. Thus, for the pre-retirement period we substitute for $V_{12}(x^*)$ and $V_{22}(x^*)$ from equations (3.3.41) and (3.A.6) respectively into equations (4.A.38) and (4.A.39). For the retirement period we substitute for these partial derivatives from equations (3.A.11) and (3.A.12) respectively. The results after simplification are:

$$de(x)/dw(x^*)\big|_{\Omega} =$$

$$- [e(x)/\xi(x)\phi(n)] \; e^{-rx^*} \; p(x^*) \; [\mu_1(x^*)/U_2(x^*)\delta(x^*)]$$

$$+ [\mu_1(x^*)/U_2(x^*)\delta(x^*) + \{\theta - h(x^*)\}]$$

$$\text{for } x = x^*,\; x^* \le R \tag{4.A.42}$$

$$de(x)/dw(x^*)\big|_{\Omega} =$$

$$- [e(x)/\xi(x)\phi(n)] \; e^{-rx^*} \; p(x^*) \; [\mu_1(x^*)/U_2(x^*)\delta(x^*)]$$

$$\text{for } x \ne x^*,\; x^* \le R \tag{4.A.43}$$

$$de(x)/dw(x')\big|_{\Omega} = \theta \qquad \text{for } x = x',\ x' \geq R \qquad (4.A.44)$$

$$de(x)/dw(x')\big|_{\Omega} = 0 \qquad \text{for } x \neq x',\ x' \geq R \qquad (4.A.45)$$

4.A.5 Proof of Theorem 4.5.10

In chapter 3, we have shown in equation (3.3.39) that the effect of an evolutionary change in the wage rate on expenditure can also be obtained as the effect of a parametric change holding constant the marginal utility of initial wealth. In the present context of the life cycle allocation problem of an individual aged n, the effect of an evolutionary change in the wage rate given by equation (3.3.39) can be written as follows.

$$de(x)/dw(x')\big|_{V_2(n)} = -\,V_{12}(x)/V_{22}(x) \qquad \text{for } x = x' \qquad (4.A.46)$$

$$de(x)/dw(x')\big|_{V_2(n)} = 0 \qquad \text{for } x \neq x' \qquad (4.A.47)$$

Substituting for $V_{12}(x')$ and $V_{22}(x')$ from equations (3.3.41) and (3.A.6) respectively for the pre-retirement period and from equations (3.A.11) and (3.A.12) respectively for the retirement period we can write this effect as follows.

$$de(x)/dw(x')\big|_{V_2(n)} = [(U_1(x')/U_2(x')s(x')) + (\theta-h(x'))]$$

$$\text{for } x = x',\ x' \leq R \qquad (4.A.48)$$

$$de(x)/dw(x^*)\big|_{V_2(n)} = 0 \qquad \text{for } x \neq x^*,\ x^* \leq R \qquad (4.A.49)$$

$$de(x)/dw(x^*)\big|_{V_2(n)} = \theta \qquad \text{for } x = x^*,\ x^* \geq R \qquad (4.A.50)$$

$$de(x)/dw(x^*)\big|_{V_2(n)} = 0 \qquad \text{for } x \neq x^*,\ x^* \geq R \qquad (4.A.51)$$

We have to prove that the compensated effects given by equations (4.A.42) through (4.A.45) are related with these marginal utility held constant effects as follows.

$$de(x)/dw(x^*)\big|_{\Omega} = de(x)/dw(x^*)\big|_{V_2(n)} + [de(x)/V_2(n)]\,[dV_2(n)/dw(x^*)\big|_{\Omega}] \qquad (4.A.52)$$

First we find an expression for $de(x)/V_2(n)$, the effect of an increase the marginal utility of wealth on expenditure at age x. Given the tangency condition:

$$V_2(x) = e^{-r(x-n)}\ V_2(n) \qquad (4.A.53)$$

we can find:

$$de(x)/V_2(n) = e^{-r(x-n)}/V_{22}(x) < 0 \qquad (4.A.54)$$

From the same tangency condition we also have a relation between the effects of a compensated change in the wage rate on expenditure and on the marginal utility of wealth:

$$V_{22}(x)\ [de(x)/dw(x^*)\,|_{\Omega}] + V_{12}(x^*) = e^{-r(x-n)}\ [dV_2(n)/dw(x^*)\,|_{\Omega}] \qquad \text{for } x = x^*$$

$$V_{22}(x)\ [de(x)/dw(x^*)\,|_{\Omega}] = e^{-r(x-n)}\ [dV_2(n)/dw(x^*)\,|_{\Omega}] \qquad \text{for } x \neq x^*$$

Therefore,

$$dV_2(n)/dw(x^*)\,|_{\Omega} = e^{r(x-n)}\ V_{22}(x)\ [de(x)/dw(x^*)\,|_{\Omega}] + e^{r(x^*-n)}\ V_{12}(x^*) \qquad \text{for } x = x^*$$

$$dV_2(n)/dw(x^*)\,|_{\Omega} = e^{r(x-n)}\ V_{22}(x)\ [de(x)/dw(x^*)\,|_{\Omega}] \qquad \text{for } x \neq x^*$$

Substituting for $de(x)/dw(x^*)\,|_{\Omega}$ from equations (4.A.38) and (4.A.39), we find:

$$dV_2(n)/dw(x^*)\,|_{\Omega} = e^{r(x-n)}\ V_{22}(x)\ [e(x)/\xi(x)\phi(n)]\ e^{-rx^*}\ p(x^*)\ [\{V_{12}(x^*)/V_{22}(x^*)\} + \{1-h(x^*)\}] \qquad \text{for all } x$$

Substituting for $V_{12}(x^*)$ and $V_{22}(x^*)$ from equations (3.3.41) and (3.A.6) respectively for the pre-retirement period and from equations (3.A.11) and (3.A.12) for the post retirement period, we obtain:

$$dV_2(n)/dw(x^*)\,\Big|_\Omega = -\,e^{r(x-n)}\,V_{22}(x)$$
$$[e(x)/\xi(x)\phi(n)]\;e^{-rx^*}\;p(x^*)\;[\mu_1(x^*)/U_2(x^*)\Delta(x^*)] \qquad \text{for } x \le R \qquad (4.A.55)$$

$$dV_2(n)/dw(x^*)\,\Big|_\Omega = 0 \qquad \text{for } x \ge R \qquad (4.A.56)$$

Multiplying these expressions by $de(x)/V_2(n)$, given by equation (4.A.54), we find:

$$[de(x)/V_2(n)]\;[dV_2(n)/dw(x^*)\,\Big|_\Omega] =$$
$$-\,[e(x)/\xi(x)\phi(n)]\;e^{-rx^*}\;p(x^*)\;[\mu_1(x^*)/U_2(x^*)\Delta(x^*)] \qquad \text{for } x^* \le R \qquad (4.A.57)$$

$$[de(x)/V_2(n)]\;[dV_2(n)/dw(x^*)\,\Big|_\Omega] = 0 \qquad \text{for } x^* \ge R \qquad (4.A.58)$$

Combining these effects with the marginal utility held constant effects given by equations (4.A.48) through (4.A.51), we find:

$$de(x)/dw(x^*)\big|_{V_2(n)} + [de(x)/V_2(n)]\,[dV_2(n)/dw(x^*)\big|_{\Omega}] =$$
$$- [e(x)/\xi(x)\phi(n)]\; e^{-rx^*}\; p(x^*)\; [\mu_1(x^*)/U_2(x^*)\Delta(x^*)]$$
$$+ [\mu_1(x^*)/U_2(x^*)\Delta(x^*) + \{\theta - h(x^*)\}]$$
$$\text{for } x = x^*,\; x^* \leq R \qquad (4.A.59)$$

$$de(x)/dw(x^*)\big|_{V_2(n)} + [de(x)/V_2(n)]\,[dV_2(n)/dw(x^*)\big|_{\Omega}] =$$
$$- [e(x)/\xi(x)\phi(n)]\; e^{-rx^*}\; p(x^*)\; [\mu_1(x^*)/U_2(x^*)\Delta(x^*)]$$
$$\text{for } x \neq x^*,\; x^* \leq R \qquad (4.A.60)$$

$$de(x)/dw(x^*)\big|_{V_2(n)} + [de(x)/V_2(n)]\,[dV_2(n)/dw(x^*)\big|_{\Omega}] = \theta$$
$$\text{for } x = x^*,\; x^* \geq R \qquad (4.A.61)$$

$$de(x)/dw(x^*)\big|_{V_2(n)} + [de(x)/V_2(n)]\,[dV_2(n)/dw(x^*)\big|_{\Omega}] = 0$$
$$\text{for } x \neq x^*,\; x^* \geq R \qquad (4.A.62)$$

These effects are exactly identical to the compensated effects given by equations (4.A.42) through (4.A.48). Hence the proof.

CHAPTER 5

SUMMARY OF THE INDIVIDUALISTIC LIFE CYCLE MODEL

A life cycle model of consumption and work hours under uncertain lifetime with actuarially fair life insurance and annuities has been developed. A two stage budgeting approach was used to model the individual's life cycle allocation behaviour. This new approach offers a clearer picture of the life cycle allocation process and therefore provides some additional results regarding the life cycle hypothesis which have not been recognized in the existing literature.

The effects of anticipated variations in the wage rate on work hours are interpreted as an intertemporal substitution effect by MaCurdy (1981). The two stage budgeting analysis, on the other hand, decomposes this effect into two parts interpretable as an intratemporal substitution effect and an intertemporal allocation effect. The intratemporal effect is the result of substitution between consumption and leisure, given the age specific utility rate. The intertemporal effect, on the other hand, relates to substitution over the life cycle, given the marginal rate of substitution between consumption and leisure for each age. The net effect of an anticipated increase in the wage rate over a segment of the life cycle is to increase work hours over that segment. The net affect results in increased consumption if consumption and leisure are substitutes. This explains a positive relation between current consumption and current earnings. This relation was further refined by the study of savings behaviour. It was shown that wage induced changes in consumption are always smaller than the changes in earnings. Therefore the marginal propensity to consume out of the planned earnings is less than one.

As expected, consumption is increasing with age and work hours decreasing due to a positive rate of interest (or a rate of interest greater than the subjective discount rate due to any reason other than life uncertainty). This relation, coupled with the assumption of a concave wage profile, which is first increasing and then decreasing with age, also explains the retirement process. It was shown that, if the wage rate is non-increasing at the retirement age and the rate of interest is positive, the individual will never re-enter the labour market once retired.

The relative position of the peak ages of different variables has been studied. If the wage rate is increasing with age in the earlier segment of life and decreasing thereafter, consumption and work hours can have interior peaks. If work hours peak, then earnings will also peak. Work hours peak earlier in life than earnings which in turn peak earlier than wage rate. Consumption can peak only after the peak in wage rate. Finally, savings out of earnings can peak only before earnings are at a peak.

Savings and asset holding behaviour of the individual was also studied in detail. A narrow measure of savings was defined as the excess of earnings over consumption while a broad measure was defined as the excess of full income (earnings plus net interest income) over consumption. In early years of life when asset holding is negative, interest income is also negative. Therefore the broad measure of savings is smaller than the narrow measure. In the later years when asset holding, and therefore interest income, is positive, the broad measure of savings exceeds the narrow measure.

The life cycle model also explains the possibility of continuing asset accumulation by the retired. If consumption and leisure are substitutes, consumption may be reduced at retirement due to a declining wage rate. Interest income may be higher due to an actuarially fair rates of interest on assets which exceed the rate on bonds (by the instantaneous conditional probabilities of death). These two factors combined may result in positive savings (in broad sense) and therefore an accumulation of wealth during some early years of retirement.

The life cycle effects of various parametric changes were then studied. Different types of mortality variations were analyzed. It was shown that an increase in the general survival rates in the society carries a wealth effect, by changing the actuarial rates of interest. But the resulting intertemporal substitution effect is fully offset by an equivalent change in impatience due to life uncertainty. The wealth effect depends on the savings pattern. An increase in the probability of survival to a young or an old age, when savings are negative, will result in lesser consumption, more work, more savings and a later retirement. On the other hand, the effect of an increase in the probability of survival to a middle age, when savings are positive, is the opposite. If the survival rates through the earlier segment of the horizon are held constant, then the effect of an increase in a mean preserving spread in the distribution of life is to reduce consumption and increase work hours for all the future ages.

The effects of subjective mortality improvements, independent of the general survival rates in the society, were also studied. Since the actuarial rates of interest are based on the general survival risks in the society, these mortality improvements do not affect the actuarial rates of interest. Therefore the lifetime budget constraint of the individual is not affected due to these mortality improvements. The lifetime allocation decision of the individual, however, is affected due to the change in the expected lifetime utility function implied by the changes in subjective survival probabilities. Thus, unlike in the case of a general mortality improvement, a subjective mortality improvement carries only an intertemporal substitution effect on the lifetime allocation decision. An increase in the subjective probability of survival to some age results in more consumption and less work hours at that age and less consumption and more work hours at the other ages.

The effect of an increase in the rate of interest was decomposed into a compensated effect and a wealth effect. A higher rate of interest on assets creates a positive wealth effect and thus results in an increase in the stream of consumption and a decrease in the stream of work hours. The compensated effect results in a higher consumption and lower work hours at later ages, in compensation for lower consumption and higher work hours at earlier ages. The full effect is predictable only for the later segment of life at which both the compensated and wealth effects are in the same direction.

A parametric change in the wage rate was studied in detail. The effects on consumption and work hours were split into three parts, an intratemporal substitution effect, an intertemporal reallocation effect and a wealth effect. The intratemporal substitution effect relates to substitution between consumption and leisure at each age for which the wage rate is increased, given the rate of utility at that age. This results in more consumption and more work at that age. The intertemporal reallocation effect relates to the resulting reallocation over the life cycle, given the marginal rate of substitution between consumption and leisure at each age and the lifetime utility. This will result in more work at the age for which the wage rate is increased and less at other working ages. Consumption is affected in the opposite (same) direction to the change in work hours if both consumption and leisure are normal (any of these two goods is inferior). Finally, the wealth effect of an increase in the wage rate results in a decrease in work hours and an increase in consumption at each age.

The implications of a parametric change in the wage rate for the life cycle allocation process are different from those of an evolutionary change due to two reasons. First, the wealth effects are present in a parametric change but absent in an evolutionary change. Second, even the compensated effects of a parametric change are not comparable with the effects of an evolutionary change. The compensated effects are based on a parametric change in the wage rate holding constant the lifetime utility. The effects of an evolutionary change, on the other hand, are identical to the effects of a parametric change in the wage rate holding constant the marginal utility of wealth, not lifetime utility.

To further explain this distinction, the compensated effects of a parametric increase in the wage rate were decomposed into two parts, the 'specific substitution' effects and the 'general substitution' effects. The specific substitution effects are the marginal utility held constant effects and, therefore, are identical to the effects of an equal evolutionary change in the wage rate. The general substitution effects arise due to a compensating change in the marginal utility of wealth as a result of the change in the wage rate.

It was shown that the compensated effects of an increase in the wage rate for some age on work hours at different ages are smaller than the specific substitution effects or the effects of an equal evolutionary change in the wage rate. The compensated effects on consumption are greater (smaller) than the effects of an equal evolutionary change in the wage rate if consumption and leisure are normal (any one of consumption and leisure is inferior). This implies that the general substitution effects of a parametric wage increase on work hours are negative. The general substitution effects on consumption are positive (negative) if both consumption and leisure are normal (either consumption or leisure is inferior).

PART II

OVERLAPPING GENERATIONS, DYNAMIC GENERAL EQUILIBRIUM AND THE STUDY OF LIFE EXPECTANCY

CHAPTER 6

INTRODUCTION TO PART II

This part of thesis considers the implications of mortality improvements for aggregate economic behaviour. In particular, the effects of an increase in survival rates along various segments of life, on capital intensity, the rate of interest and the wage rate are studied. The analysis is conducted in the framework of a continuous time version of a standard overlapping generations model.

The only available studies which have discussed the positive effects of longevity in an aggregate model are by Skinner (1985) and Sinha (1986). Skinner calculated the effect of longevity on capital intensity in the absence of annuities in a simulation model and found that this effect is positive, though negligible. Sinha showed that an increase in longevity will increase the equilibrium capital-labour ratio if actuarially fair annuities are available. The analyses in these two studies are based on a two period model in which longevity can be increased only by increasing the survival rate into the second period when an individual's savings are negative.

An examination of survival rates at various ages in different countries reported in various issues of World Health Statistics Annual (World Health Organization), shows that the actual mortality improvements in the past have been a result of an increase in survival rates at all the segments of life horizon. Therefore to find the effects of an increased longevity on the aggregate economic variables, one would have to go into the details of changes in the life tables and calculate the net effect of all the changes in survival rates at various points in the horizon.

These statistics also show that the survival rates at various ages differ from country to country. In particular, while there seems to be little scope of further improvements in survival rates at young ages in the developed countries, there is enough room for such mortality improvements in the underdeveloped countries.[1]

In the light of these observations it seems important to distinguish among improvements in survival rates at different segment of life horizon while analyzing the effects of longevity on aggregate economic behaviour. In particular, this analysis is expected to provide distinct im-

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1 According to World Health Statistics Annual, survival rates in Japan in 1981 at the age of 1 year, 15 years, 45 yeras and 65 years were 0.992. 0.988. 0.953 and 0.808 respectively. The comarable rates for Mouratius were 0.962, 0.950, 0.875 and 0.580 respectively.

plications of different types of mortality improvements for the classification of a dynamic equilibrium as efficient or inefficient.

In the present research survival risks are assumed to be present at all instants of the life horizon. Due to a continuous time framework, many generations of different ages co-exist at any instant of time. A hump shaped savings-age relationship is derived from the underlying life cycle model of an individual. The effects of mortality improvements are calculated in the steady state competitive solution. Thus, if an increase in survival rates for a specific segment of the horizon is assumed to have taken place, its effect on aggregate savings or assets will depend on the savings rates of those in that segment of life and the impact of life cycle reallocation decisions on the savings rates of different generations in response to the mortality improvement.

The two studies on longevity, quoted above, also assume that all the births take place in an individual's first period of life and therefore increases in longevity cannot affect the growth rate of population. In the present analysis, on the other hand, the possible effect of mortality improvements on the growth rate of population is also recognized. If the growth rate is increased, the relative proportion of the young population will increase in the economy. This will also affect the aggregate economic variables in accordance with the savings and work habits of the young population relative to the old.

After presenting a selected review of literature on overlapping generations models and the studies on life expectancy in chapter 7, we will build-up our analysis in the next three chapters. In chapter 8 we assume that the factor prices, the rate of interest and the wage rate, and the life cycle work plan of an individual are fixed and analyze the effects of life expectancy improvements on the aggregate capital stock and the employment of labour. In chapter 9, we relax the assumption of fixed factor prices and analyze the effects of life expectancy improvements on the equilibrium capital-labour ratio, the rate of interest and the wage rate. Finally, in chapter 10 we also relax the assumption of a fixed life cycle work plan of an individual and repeat the analysis of chapter 9. A summary of all the findings of the general equilibrium model is presented in chapter 11.

CHAPTER 7

DYNAMIC ECONOMIC MODELS WITH OVERLAPPING GENERATIONS: A SURVEY OF SELECTED LITERATURE

7.1 INTRODUCTION

The rate of interest plays a crucial role in dynamic economic theory. The basic principles of the theory of interest were long age laid down by Fisher (1930). But Fisher was mainly concerned with household behaviour rather than the general equilibrium analysis of the entire economy as were later investigators of life cycle consumption theory such as Modigliani and Brumberg (1954) and Friedman (1957). The traditional neo-classical theory of economic growth, on the other hand, relies heavily on the production sector of the economy and tends to ignore the role of household sector in the determination of interest rates in spite of its intention to determine a dynamic general equilibrium (See, for example, Solow (1956) and Swan (1956)).

The latest theory of the determination of the interest rate based on overlapping generations models has attracted a lot of attention in the literature. The original version of overlapping generations model put forward by Samuelson (1958) does not include a production activity and hence any role of capital as a productive factor in the dynamic equilibrium. Yet, it suggests a realistic approach to deal with the capital market. It fully spells out the dynamic process of assets supply by the household sector in an infinite horizon model while recognizing the fact that human life is finite. Samuelson's consumption loan model opened new avenues for research in different areas of economics like capital theory, public finance, monetary economics and demographic economics.

7.2 THE ISSUE OF DYNAMIC INEFFICIENCY

Samuelson demonstrated that in a pure exchange dynamic model, the free market system does not guarantee a socially optimum solution which requires that rate of interest should be equal to the growth rate of population, or what is known as the golden rule of accumulation. In addition, a free market solution which results in an interest rate less than the growth rate of population is dynamically inefficient. This aspect of inefficiency of the competitive solution had also been recognized in the traditional growth framework by Malinvaud (1953), Koopman (1957) and Phelps (1965).

Samuelson made extreme assumptions in his analysis in order to make sure that the resulting competitive equilibrium solution is dynamically inefficient. In his model people receive income in the early period of their lives and none at the end. This assumption is exactly the opposite of what Fisher (1930) had assumed in his impatience argument for a positive rate of interest. As a result, in a pure exchange model based on the Samuelson (Fisher) assumption, net assets of the society on the golden rule path are positive (negative). This distinction between the two types of exchange models has been pointed out by Gale (1973). He called an exchange model based on Fisher's assumption a classical model.

Gale demonstrated that the rate of interest in a competitive market solution, or what he called, a balanced equilibrium, is less (greater) than the growth rate of population if the model is Samuelson (classical). He also showed that, in a two period model a competitive equilibrium is Pareto optimal in the classical case and not in Samuelson case. This aspect of efficiency in an exchange model has also been pointed out by Starrett (1972). He stressed that the golden rule path is always efficient and more desirable that any other steady state path if and only if it is followed by all the generations in the indefinite past and the indefinite future. If, however, the existing path is not the golden rule path, a transition to the golden rule path may not lead to a Pareto improvement.

Diamond (1965) introduced production into Samuelson's model and re-examined the issue of inefficiency. The single good in his model can be consumed or used as a productive factor, capital. Therefore, individuals can transfer their resources from the first period to the second by lending their savings to firms. If the marginal product of capital is positive, a competitive solution guarantees a positive rate of interest which in Samuelson's exchange model was always negative. But the interest rate may still fall short of the growth rate of population. Therefore, the introduction of a production activity into the model does not eliminate the possibility of an inefficient competitive solution.

Again, like in Gale's (1973) exchange model, we can have a classical or a Samuelson model in an economy with a production activity. According to Burbidge (1983a), the model is Samuelson (classical) if in the golden rule household assets exceed (fall short of) the stock of capital held by the firms, that is the aggregate credit balance of households and firms combined is positive (negative). Or alternatively, the model is Samuelson (classical) if in the balanced equilibrium, the interest rate is smaller (greater) than the growth rate of population. Willis (1985) further generalized the analysis by considering a model with many overlapping generations and a production activity. He derived similar conclusions to those obtained by Burbidge (1983a).

Several explanations have been given for the possible inefficiency of a competitive equilibrium solution in a dynamic model. Samuelson (1958) suggested that the source of inefficiency is the limited opportunity of exchange among different generations. Cass and Yaari (1966) showed that this source of inefficiency is nothing different from the usual absence of double coincidence of wants in a barter exchange and, therefore, can be observed in a static model if there is no clearing house or what they called an intermediary. The real source of inefficiency, according to Cass and Yaari, is the existence of dead weight in the form of inventories. Cass and Yaari assigned an additional role to the intermediary in the dynamic model which is not needed in a static model. The intermediary must carry negative net worth to balance the net assets (or dead weight) of the private sector. However, Starrett (1972) provided a counter example in which there is no dead weight in the sense of Cass and Yaari but the competitive equilibrium is still inefficient.

Samuelson (1958, 1959) has also suggested that the source of inefficiency is the infinite dimension of the dynamic model. Shell (1971) further elaborated this point and demonstrated that a double infinity of traders and dated commodities can result in an inefficient competitive solution. He also showed that, if the number of traders and the number of goods traded are both infinite, a competitive equilibrium may not be efficient even in a static model in which all the traders are present in the Walrasian market.

7.3 THE ROLE OF SOCIAL SECURITY IN OVERLAPPING GENERATIONS MODEL

A substantial amount of effort in the literature has also been directed to sort out some device to restore efficiency in an otherwise inefficient competitive solution. Cass and Yaari (1966) suggested that there should be an intermediary to support the golden rule path. The intermediary would hold a credit balance (positive or negative) equal to the negative of the credit balance of private sector. In a Samuelson model, society is a net creditor and therefore the intermediary must hold net debt to support golden rule. Cass and Yaari argued that such an intermediary, which always remains under debt, cannot be private. As regards the classical model, Cass and Yaari suggested that golden rule path can be attained in this model with the help of a privately operated intermediary because the intermediary in a classical model will always hold a positive credit balance.

But Willis (1985) has shown that the golden rule path in the classical model cannot be supported by private market institutions because of the following reason. If the intermediary is privately owned, shares in its assets constitute additional wealth for the members of society. With this additional wealth, the optimal consumption plan will exceed the one permissible under the social budget constraint. Obviously, this higher consumption plan is not feasible.

Aside from the credit position of the intermediary, there is a basic difficulty in realizing the transition from balanced equilibrium to the golden rule path in the classical model. In the classical model, both the balanced and golden rule paths are efficient. Therefore, the transition to the golden rule path must make at least one generation worse off (Starrett (1972)). This transition is not possible in a free market system. Thus, the golden rule path cannot be realized by privately organized market institutions, both in the Samuelson and the classical model.

Samuelson (1958) suggested two procedures to attain efficiency in his model. One of these procedures is to introduce paper money into the economy. Samuelson insisted that money is valued for what it will fetch in exchange, otherwise it has no utility by itself. Obviously, Samuelson meant by money a legal tender which has no real backing of gold or any other valuable durable good. This is essentially the assumption of the existence of an intermediary, namely the monetary authority, which is supposed to hold a net debt equal to the value of the stock of money in circulation. It has already been concluded that such an intermediary cannot be private.

The other procedure which Samuelson (1958) recommended is a social security scheme through which young generations provide for the old. In return, the young generations expect that when they get old, they will be supported by the young generations at that time. For the enforcement of this social security scheme Samuelson allowed government intervention in the economy.

The above discussion suggests that there is room for government intervention in an otherwise free market system. The government intervention can take the usual form of fiscal tools, like taxes and subsidies, or a simple one through a social security system.

Diamond (1965) analyzed the effect of an increase in national debt on the welfare of a representative individual in the steady state competitive solution. With the debt growing at the rate of population growth in steady state, the excess of interest payments on debt over the increase in debt is assumed to be balanced by a labour income tax, which will be positive (negative) if the rate of interest is greater (smaller) than the growth rate of population. Diamond showed that an increase in internal debt which also creates social capital, lowers (raises) utility if the solution is efficient (inefficient). An increase in external debt which does not create social capital, lowers utility in the efficient case and may lower or raise it in the inefficient case.

Stein (1969) showed that the mere existence of social capital and a social rule that a generation may not consume the social capital or its imputed rent, may produce efficiency. If the competitive solution is initially inefficient and therefore the interest rate is less than the growth rate, the stock of social capital will grow less rapidly than the labour force. As long as rate of interest falls short of the growth rate, the social capital per worker will continue to decline and the rate of interest will continue to rise towards the growth rate. It is possible that social capital will vanish before the realization of the golden rule. Stein suggested that a sufficient amount of internal debt may prevent this situation. The existence of social capital also guarantees that the interest rate will never exceed the growth rate because in that situation social capital per worker will rise and therefore the interest rate will decline towards the growth rate.

Samuelson (1975) showed that there exists an infinity of social security programmes that can support the golden rule steady state path. The reason is that any fully funded increase in social capital will drive out an equal amount of private capital. Therefore the allocation between social and private capital is a matter of indifference.

Bowley (1981) has taken the position that the golden rule is not the only optimal allocation. All interest rates greater than or equal to the growth rate of population are Pareto efficient by this view as they all maximize some social welfare function. Therefore the rate of interest is essentially determined by the government policy. If government does not intervene, there may not be a Pareto optimal equilibrium. But if government intervenes by creating debt and choosing the resulting interest rate, it must also have chosen the underlying welfare function.

7.4 THE STUDY OF LIFE UNCERTAINTY

In recent years, overlapping generations models have been extended to recognize the fact that human life is uncertain. Sheshinski and Weiss (1981) examined the effect of social security under life uncertainty on the savings behaviour of an individual who has a bequest motive. They showed that, if the level of social security is optimally chosen by the individual, a fully funded and a pay as you go social security are equivalent in terms of all the real aggregates. They also showed that, contrary to Barro's (1974) neutrality theorem, imposed changes in social security may have real effects because bequests are not perfect substitutes for actuarially fair social security benefits.[1] Sheshinski and Weiss considered life uncertainty in a limited sense. In their model all the people born at the same time have the same life span.

Karni and Zilcha (1984) considered a 3 period overlapping generations model with uncertain lifetimes. In their model people receive fixed income in the first two periods of life and none in the third period. Fiat money is also included in the model in order to avoid the well known possibility of dynamic inefficiency of the type discovered by Samuelson (1958). The growth rate of population is assumed to be zero. If life insurance and annuities are not available, the optimal steady state consumption plan in a free market results in a zero rate of interest (equal to the growth rate of population) which, however, is not Pareto optimal because of uncertainty of life. With actuarially fair

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1 Barro (1974) has shown that in the presence of a bequest motive an increase in government debt has no net-wealth effect on private capital, the rate of interest or current and future consumption. The increased tax burden on the future generations will be offset by an appropriate change in the level of bequests by the present generations.

life insurance, people can share the monetary risk of uncertain life. The optimal steady state consumption plan in this case also results in a zero rate of interest but the solution now is Pareto optimal.

Abel (1984b, 1985) considered a two period model in which survival through the second period is assumed to be uncertain. Each consumer earns a fixed labour income in the first period and none in the second. Private annuities and life insurance are not available. In case the individual dies after the first period, all his/her wealth is distributed to his/her children but the individual has no bequest motive. Abel introduced an actuarially fair fully funded social security, the amount of which is chosen by the government rather than by the individual. The introduction of a small amount of such social security causes aggregate private capital and aggregate national capital to fall. If the rate of return on capital is equal to the growth rate of population, aggregate consumption is unaffected by social security. If the rate of return on capital is greater (less) than the growth rate of population, aggregate consumption will decrease (increase) because of social security. The introduction of social security also narrows the inter-cohort distribution of consumption.

Abel then modified the analysis by allowing actuarially fair private annuities in the model, which are perfect substitutes for actuarially fair social security. In this case the introduction of social security has no effect on the economy. However, the introduction of private annuities which an individual can choose in the desired amount has different effects than the social security. If the rate of return on capital is equal to the growth rate of population, the introduction of private annuities will reduce aggregate consumption of the young generation and increase the aggregate consumption of the old generation.

In a subsequent paper Abel (1986) considered a situation in which consumers have different survival probabilities and they can buy actuarially fair private annuities offering different rates of return. A bequest motive is also assumed to be present. Abel showed that, if social security does not discriminate on the basis of different mortality risks, then the introduction of social security will narrow the cross sectional distribution of consumption and bequests.

Hubbard (1984) also discussed the effect of social security on savings, consumption and welfare in a model in which life span is certain in the pre-retirement period and uncertain in the post-retirement period. Wage income, the individual's retirement age and work hours are assumed to be fixed. A production activity is also included in the model. Hubbard simulated the effect of social security for a range of values for the elasticity of intertemporal substitution and concluded that, for interest rates greater than the growth rate of the economy, consumption per head, capital stock per worker, the savings ratio, output per worker and the level of welfare will fall and the interest rate will increase.

7.5 THE IMPLICATIONS OF A BETTER LIFE EXPECTANCY

There are some studies which also discuss the effects of an increase in life expectancy in an overlapping generations model. The implications of increased longevity for an individual's lifetime allocation behaviour have been studied by Abel (1986), Sheshinski and Weiss (1981) and Skinner (1985) and we have already discussed these studies in chapter 2.

Skinner (1985) also discussed the effect of an increase in the probability of survival into the second period on the aggregate assets-labour supply ratio in the presence of a bequest motive and the absence of life insurance or annuities. Using a consumer expenditure survey for the U.S.A., he showed that his model predicts a very small increase in the assets-labour supply ratio in response to an increase in life expectancy. The primary reason for the increase in the assets-labour ratio is an increase in the relative number of older people who hold more assets but work lesser than the young. The induced change in assets because of the reallocation of consumption by individuals is negligible. The reason is that an increase in survival probability will induce more savings to finance consumption for a possible longer life, but it will also reduce the need to save for contingent bequests, the net effect being uncertain.

In a recent study Sinha (1986) concluded that if actuarially fair annuities are available, then in the absence of a bequest motive an increase in longevity will result in a higher capital-labour ratio. In Sinha's model the primary reason for this result is the decrease in actuarially fair rate of interest on annuities brought about by the increase in survival rate for the second period of life. This lower rate of interest will reduce the income from annuities in the second period of life. Therefore the young generation will increase their savings in the first period.

Arthur (1981) discussed the general equilibrium effects of an increase in the survival rate at various points of the life horizon on the welfare of a representative individual under the golden rule of accumulation. He concluded that, given the rate of interest and the wage rate, an increase in the probability of survival will increase the well being of an individual, as discussed in chapter 2. However, an increase in the survival rate during the reproductive years of life will also increase the growth rate of population. Under the golden rule of accumulation, this increase in the growth rate will imply an equal increase in the rate of interest and a decrease in the wage rate. Arthur argued that in assessing the value of life measured by the increase in welfare of a representative individual because of an increase in the survival rate, the effect of resulting changes in the rate of interest and the wage rate should also be incorporated. But, Arthur did not introduce social capital, or the like, to enforce the golden rule. Therefore, in measuring the value of life he did not acknowledge the consequent implications of changes in social capital when the growth rate of population increases due to increased life expectancy.

7.6 CONCLUDING REMARKS

Overlapping generations models have been widely used in different areas of economics, specially in the public finance literature. In the recent past, the framework has been extended to random life horizon models. The implications of private life insurance and annuities and social security have been studied under alternative assumptions regarding the fairness (in actuarial sense) of such intertemporal transactions.

There are a few studies which discuss the implications of changes in the distribution of the random life horizon. Skinner's (1985) finding that increases in life expectancy may increase or decrease capital accumulation, is of particular interest for the present research. The possibility of a decline in capital accumulation in response to a better life expectancy in Skinner's model, rests on the assumption that individuals have a strong bequest motive. They must build up assets for bequests at an early age because survival in the future is uncertain. In the absence of contingent bequests, increasing life expectancy cannot reduce capital accumulation. In Sinha's model there is no bequest motive and actuarially fair annuities are available. An increase in longevity in his model results in a higher capital-labour ratio.

Like many others who have studied overlapping generations models, Skinner (1985) and Sinha (1986) also used a two period model for their analyses. As usual, consumption in the second period is more than earnings. Therefore increasing the survival probability in the second period will induce more savings in the first period.

The two studies also assume that all births take place in the first period when there is no uncertainty. This model, therefore, can not incorporate the effect of a higher growth rate of population resulting from a mortality improvement. On the other hand, this effect is over emphasized in Arthur's (1981) analysis of the value of life. Arthur considered only the golden rule steady state. Therefore unless the mortality improvement effects the growth rate of population, it has no general equilibrium consequences in his analysis.

Arthur's (1981) study does not relate directly to our research programme as it is confined to the welfare implications of mortality improvements. But it provides a suitable framework to study the positive effects of mortality variations. In particular, Arthur used a continuous time overlapping generations model in which many generations co-exist. In the present research we will follow his framework without, however, resorting to the golden rule of accumulation.

We shall assume that survival is uncertain not only at an old age but also at a middle age when savings are positive. There are many different types of possible changes in the life tables which may all result in an equal increase in expected horizon. But some of these changes are associated with a higher probability of survival to middle age when an individual's savings rate is positive, rather than to old age when the savings rate is negative. Such a mortality improvement will carry a positive wealth effect on an individual's allocation decision and therefore reduce the need for savings. Thus an improvement in life expectancy may increase or decrease capital accumulation even in the absence of contingent bequests.

The possible effect of mortality improvements on the growth rate of population will also be recognized. If the growth rate is effected, it will disturb the age distribution of population. This in turn may also effect economic variables if the behaviour of young and old individuals is different from one another.

Now we move to the next chapter where we formulate the aggregate household model and study the effects of mortality improvements on the aggregate capital stock and the employment of labour.

CHAPTER 8

AN AGGREGATE HOUSEHOLD MODEL AND THE STUDY OF MORTALITY IMPROVEMENTS WITH FIXED FACTOR PRICES AND FIXED WORK SCHEDULE OF AN INDIVIDUAL

8.1 INTRODUCTION

The main purpose of this chapter is to prepare some groundwork for the study of the effects of mortality variations on the equilibrium capital-labour ratio, the rate of interest and the wage rate. First an aggregate household model is developed. The main concern in this context is to form aggregate economic variables like consumption, labour supply, savings and assets and to relate these economic aggregates to their economic and demographic determinants.

We will adopt a continuous time framework of an overlapping generations model with uncertain lifetimes in the presence of actuarially fair life insurance and annuities. This framework has also been used by Arthur (1981) in his analysis of the value of life under the golden rule of accumulation. We follow his framework without, however, resort to the golden rule assumption which he employed. In other words, we will consider only the competitive equilibrium for our analysis as in Diamond's (1965) model. The benefit of adopting a continuous time framework is that it allows to study the implications of mortality improvements which are caused by an increase in survival rates at alternative points of the life horizon.

A simplified version of a general equilibrium model is considered which is based on aggregate household behaviour and a production activity. It is assumed that the production function relating output to the two factor inputs, capital and labour is <u>linear in each of the two factor inputs</u>. This assumption implies that both the rate of interest and the wage rate are constant (in terms of output) and given by technology. With technologically given factor prices, the levels of the two factor inputs are fully supply determined. Thus the analysis in this chapter focuses essentially on the household model.

Another outcome of the linear production function is that the capital and labour markets are isolated from each other. Therefore one can study each of these two markets separately without reference to the other.

The analysis is further simplified by assuming that an individual's work schedule is fixed. This is the conventional assumption in the literature on overlapping generations models (See, for example, Diamond (1965), Skinner (1985) and Willis (1985)). This assumption will be carried over to the next chapter where we will consider a neo-classical linear homogeneous production function and discuss the effects of mortality improvements on the equilibrium capital-labour ratio, the rate of interest and the wage rate. This assumption will be relaxed, however, in chapter 10 where we explore the effects of mortality improvements in the framework of a perfectly flexible work-leisure choice decision. This step by step build-up of the analysis will help us to understand the complex nature of our results.

The effects of small increases in survival rates at various points of the life cycle on the employment of the two factor inputs are then studied. Because of the assumed technology, which implies that the two factor prices are fixed, the effects of these mortality improvements on household assets and labour supply are fully realized in the respective markets and therefore represent the actual changes in the employment of the two factor inputs. In other words, the effects of mortality improvements on the household decision can be studied in isolation from the rest of the economy.

8.2 AN INDIVIDUAL'S LIFE CYCLE MODEL WITH ACTUARIALLY FAIR LIFE INSURANCE AND ANNUITIES

The individual's life cycle model with an endogenous work-leisure choice has been discussed in chapters 3 and 4. The results of these chapters will be used in our more general model to be considered in chapter 10. But in the present chapter we need a simplified version of the life cycle model in which the individual's work schedule is fixed. In all other respects, the life cycle model considered in this chapter will be similar to the one discussed in chapters 3 and 4.

As before, we assume that actuarially fair life insurance and annuities are available. We must emphasize at this point that this assumption is crucial for the general equilibrium analysis under uncertain lifetimes. If risk pooling on an actuarially fair basis is not assumed, we must allow involuntary intergenerational transfers because at the time of death an individual may be holding positive net wealth. The presence of such random involuntary transfers requires a detailed consideration of the distributional implications of life uncertainty. Although we are not typically interested in this aspect of life uncertainty, a good discussion on this subject can be found in Desai and Shah (1981). A convenient way to bypass this issue is to assume that there exists a perfect insurance market. With a perfect insurance market, at the time of death, the net assets of an individual (positive or negative) are equally distributed among all the members of his/her cohort as a part of agreement.

An alternative approach would be to assume that life insurance and annuities do not exist and at the time of death, all the assets of an individual (which cannot be negative) are passed on to his/her offspring. But under this assumption, as Abel (1984, 1985) has shown, individuals receive different levels of inheritance, depending on the life spans of their parents and forefathers. While it is easy to trace such family history in Abel's two period model, this exercise is not possible in our continuous time model in which many generations co-exist. Besides, in our framework it is not possible to identify any individual with his/her parents or children.

Another approach could be to assume that assets of a deceased individual are passed on to worthy institutions by the will of the deceased. This approach also seems to result in complications in our model. Therefore, to avoid all the issues relating to the distributional implications of life uncertainty, we assume that a perfect insurance market exists through which individuals pool the monetary risk of uncertain lifetimes.

Under the circumstances considered, the life cycle allocation problem of an individual is to maximize the lifetime utility function:

$$\int_0^T p(x)\ u(c(x))\ dx \qquad (8.2.1)$$

subject to the lifetime budget constraint:

$$\int_0^T e^{-rx}\ p(x)\ [w(x)h(x)\ -\ c(x)]\ dx \geq 0 \qquad (8.2.2)$$

where, we have assumed, as before, that the lifetime utility function is additively separable among age related sub-utility functions, $u(c(x))$. As usual, the utility function $u(c(x))$ is assumed to be strictly increasing and concave in $c(x)$. Therefore $u'(c(x)) > 0$ and $u''(c(x)) < 0$. It is also assumed that $u'(c(x))|_{c(x)=0} = \infty$.

The maximizing conditions for this problem are:

$$u'(c(x))\ -\ e^{-rx}\ u'(c(0)) = 0,\ \text{and} \qquad (8.2.3)$$

$$\int_0^T e^{-rx}\ p(x)\ [w(x)h(x)\ -\ c(x)]\ dx = 0 \qquad (8.2.4)$$

The solution for $c(x)$ can be expressed in terms of the following continuously differentiable function.

$$c(x) = c(x,\ P,\ r,\ W) \qquad (8.2.5)$$

where, P and W represents respectively the entire age paths of survival probabilities, $p(x)$: $0 \leq x \leq T$ and wage rates, $w(x)$, $0 \leq x \leq T$. Throughout this chapter, we will assume that the individual's work schedule , $h(x)$, is fixed and we have, therefore, suppressed it from the function (8.2.5). This has been done to keep the notation consistent with that in chapter 10, where the labour supply schedule is endogenous and therefore not an independent argument in the consumption demand function.

The age profile of consumption can be obtained by differentiating equation (8.2.3) with respect to x. The result is:

$$\dot{c}(x) = - r\ [u'(c(x))/u''(c(x))] \tag{8.2.6}$$

Since $u'(c(x))$ is positive and $u''(c(x))$ is negative, consumption is increasing with age, provided the rate of interest is positive.

We can now discuss the savings behaviour of the individual in this simple model. First, consider the narrow measure of savings, $\sigma(x)$. The solution for $\sigma(x)$ can be inferred from the solution for $c(x)$, given by (8.2.5), as follows.

$$\begin{aligned} \sigma(x) &= m(x) - c(x, P, r, W) \\ &= \sigma(x, P, r, W) \end{aligned} \tag{8.2.7}$$

where, $m(x) = w(x)h(x)$ is the labour income schedule.

Similarly, the age profile of the savings schedule, $\dot{\sigma}(x)$, can be obtained by using (8.2.6), as follows.

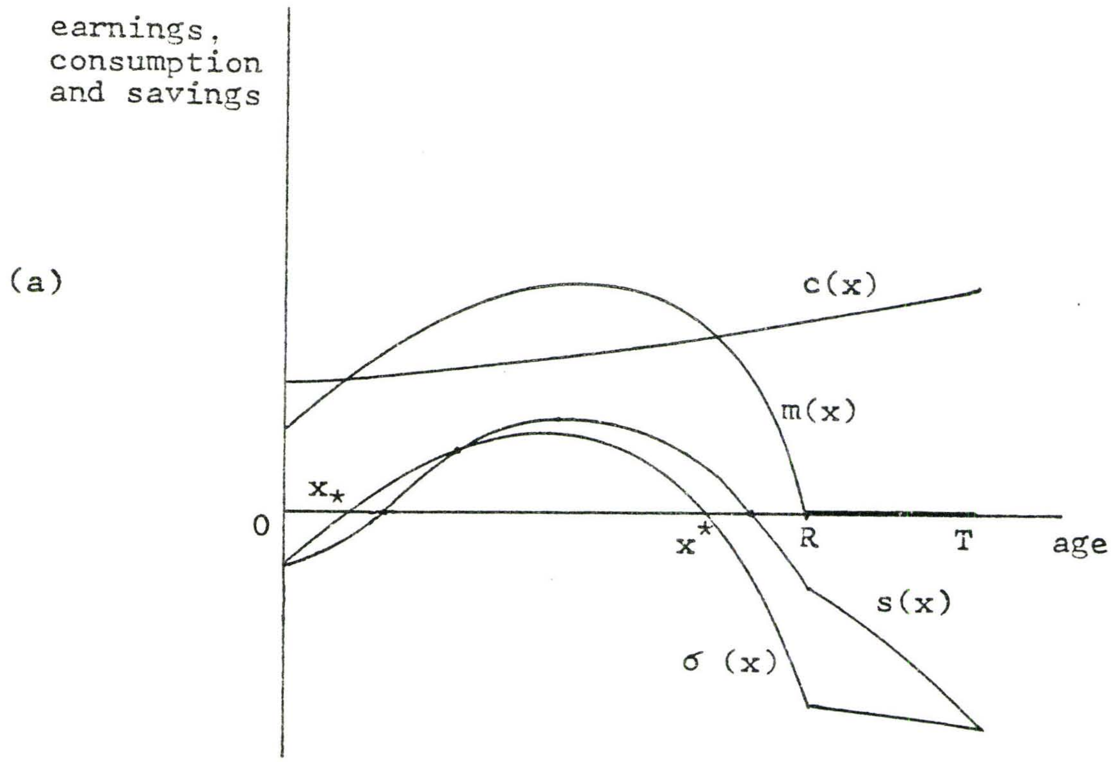
$$\dot{\sigma}(x) = \dot{m}(x) - \dot{c}(x) \tag{8.2.8}$$

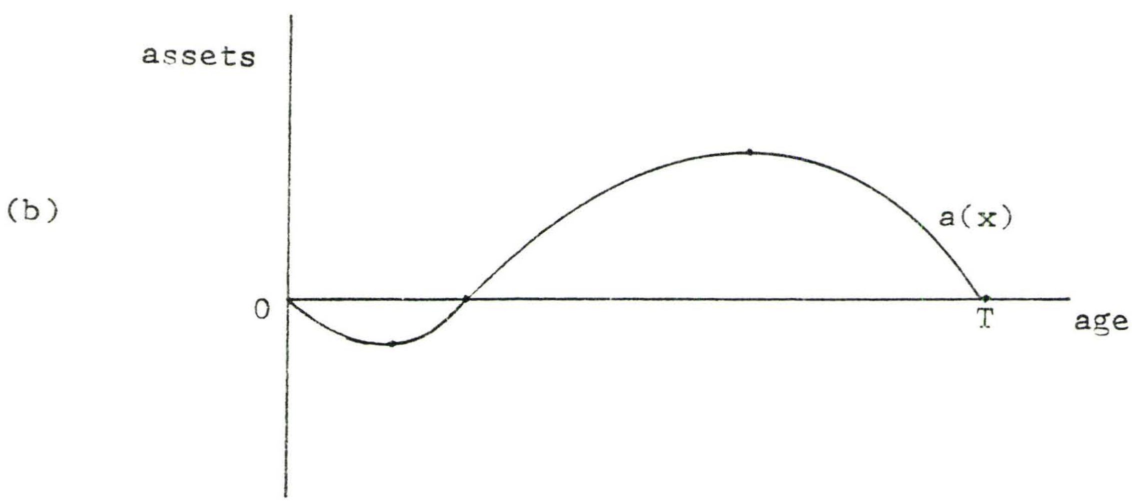
The lifetime budget constraint (8.2.4) implies that savings must be positive at some ages and negative at others (except in the trivial case: $\sigma(x) = 0$ for all x). To determine the age pattern of savings, we must also know the age path of labour income, $m(x)$. It is assumed that work hours and wage schedules follows the same age paths as the ones obtained in chapter 3. Therefore labour income is increasing in age up to some point and decreasing thereafter, eventually becoming zero towards the end of life horizon.

The age profiles of the schedules $m(x)$, $c(x)$ and $\sigma(x)$ are displayed in figure 8.2.1(a). During the early years of life up to age x_*, the savings rate is assumed to be negative (although it could be positive as well, depending on the rate of interest and the labour income pattern). During the middle years of life, between age x_* and age x^*, the savings rate is positive. Finally, during the later years of life, after age x^*, the savings rate is again negative. The retirement age at which labour income becomes zero, is marked R.

Next, consider the broad measure of savings, $s(x)$, and asset holding, $a(x)$. According to our analysis in chapter 3, we can write these two schedules as follows:

Figure 8.2.1 The Age Profiles of Labour Income, Consumption, the Two Measures of Savings and Assets with a Fixed Work Schedule





$$a(x) = [e^{rx}/p(x)] \int_0^x e^{-ry}\, p(y)\, \gamma(y)\, dy \tag{8.2.9}$$

$$s(x) = \gamma(x) + [r + \{q(x)/p(x)\}]\, a(x) \tag{8.2.10}$$

The age profiles of the schedules s(x) and a(x) are displayed in figures 8.2.1(a) and 8.2.1(b) respectively.

Let us now study the effect of a small increase in the probability of survival to a specific age, say x', on the life cycle patterns of consumption and savings. The results of this exercise will be useful to study the implications of mortality improvements for the aggregate economic behaviour. Differentiating the maximizing conditions (8.2.3) and (8.2.4) with respect to p(x'), we can write:

$$\partial c(x)/\partial p(x') = e^{-rx}\, [u'(c(0))/u''(c(x))]\, \partial c(0)/\partial p(x') \tag{8.2.11}$$

$$\int_0^T e^{-rx}\, p(x)\, [\partial c(x)/\partial p(x')]\, dx = e^{-rx'}\, \gamma(x') \tag{8.2.12}$$

Substituting for $\partial c(x)/\partial p(x')$ from equation (8.2.11) into (8.2.12) and rearranging the terms, we obtain:

$$\partial c(0)/\partial p(x') = \frac{e^{-rx'}\, \gamma(x')\, [1/u''(c(0))]}{\int_0^T e^{-2rx}\, p(x) [1/u''(c(x))]\, dx}$$

Finally, substituting for $\partial c(0)/\partial p(x^*)$ from this equation back into equation (8.2.11), we obtain the desired result:

$$\partial c(x)/\partial p(x^*) = \frac{e^{-rx}[1/u''(c(x))]\ e^{-rx^*}\ \gamma(x^*)}{\int_0^T e^{-2rx}\ p(x)[1/u''(c(x))]\ dx} \qquad (8.2.13)$$

Since $u''(c(x))$ is negative for all x, this derivative will be positive (negative) for all x if $\gamma(x^*)$ is positive (negative). Therefore with an increase in the probability of survival to an age between x_* and x^* (below x_* or above x^*) when the savings rate, $\gamma(x)$, is positive (negative), consumption at all the ages will increase (decrease). This result can be explained in the same manner as in chapter 4 and therefore there is no need to repeat the argument.

Next, consider the effect on the savings schedule, $\gamma(x)$. Since $\gamma(x) = m(x) - c(x)$ and $m(x)$ is given exogenously, the effect of an increase in the probability of survival to age x^* on the savings schedule can be inferred from (8.2.13) as follows:

$$\partial \gamma(x)/\partial p(x^*) = \frac{-\ e^{-rx}[1/u''(c(x))]\ e^{-rx^*}\ \gamma(x^*)}{\int_0^T e^{-2rx}\ p(x)[1/u''(c(x))]\ dx} \qquad (8.2.14)$$

Therefore in response to an increase in the probability of survival to an age between x_* and x^* (below x_* or above x^*)

when the savings rate, $g(x)$, is positive (negative), the individual will decrease (increase) savings in all the periods of life.

This completes our discussion of the life cycle allocation behaviour of an individual.

8.3 THE CHARACTERISTICS OF A STABLE POPULATION

In this section we will discuss the dynamics of population and other demographic variables under the assumption that population is stable. The population is called stable if the age specific rates of reproduction and mortality and the normalized age distribution of population are all constant over time (Lotka (1956)). This assumption is implicit (if not explicit) in all studies which assume a constant growth rate of population.

Let $B(t)$, $D(t)$ and $N(t)$ be the size of births, deaths and population respectively in period t. Likewise, let $B(t,x)$, $D(t,x)$ and $N(t,x)$ denote respectively the number of births by the population aged x, deaths in the population aged x and the size of the population aged x in period t. If the population is stable, we can write

$$B(t,x)/N(t,x) = b(x) \tag{8.3.1}$$

$$D(t,x)/B(t-x) = q(x) \tag{8.3.2}$$

$$N(t,x)/N(t) = n(x) \tag{8.3.3}$$

That is, the age specific rates of reproduction, the age specific death rates and the age distribution of population are constant over time. With T being the upper bound on age, the following relation holds:

$$N(t,x) = \int_x^T D[t+(y-x), y] \, dy = B(t-x) - \int_0^x D[t+(y-x), y] \, dy$$

In other word, the size of aged x population in period t is equal to the sum of future deaths of that population. Or, the size of aged x population in period t is equal to the number of births in period t-x minus the past deaths of those born in period t-x. Dividing through by the number of births in period t-x, the above relation can be written as follows

$$N(t,x)/B(t-x) = \int_x^T q(y) \, dy = 1 - \int_0^x q(y) \, dy$$

Because of the assumption that the age specific death rates are constant over time, we have dropped the time variable which would otherwise appear in the death rates, q(y).

Notice too that the ratio N(t,x)/B(t-x) is the survival rate for age x out of the total born. The above relation says that the age specific survival rates are also constant in a stable population. Therefore we can write

$$p(x) = \int_x^T q(y)\ dy = 1 - \int_0^x q(y)\ dy \qquad (8.3.4)$$

where,

$$p(x) = N(t,x)/B(t-x) \qquad (8.3.5)$$

It may be recalled that at the beginning of chapter 3 we assumed that an individual living in a sizable population of identical members can calculate the probability distribution of his/her life. Because of this assumption we do not distinguish between the probability of death at age x perceived at age 0 and the actual death rate at age x out of the total born in that age group. That is why we have used the symbol q(x) to represent both the probability of death and the actual death rate. For the same reason, the symbol p(x) denotes both the probability of survival through age x and the actual survival rate through age x.

We can show that under the assumption of a stable population, the number of births and the number of deaths are proportional to the size of population. The number of births in period t can be written as

$$B(t) = \int_0^T B(t,x)\ dx = N(t) \int_0^T [B(t,x)/N(t,x)]\ [N(t,x)/N(t)]\ dx$$

Or,

$$B(t) = b\ N(t) \tag{8.3.6}$$

where,

$$b = \int_0^T b(x)\ n(x)\ dx \tag{8.3.7}$$

Likewise, the number of deaths in period t can be written as

$$D(t) = \int_0^T D(t,x)\ dx =$$

$$= N(t) \int_0^T [D(t,x)/B(t-x)][N(t,x)/N(t)]/[N(t,x)/B(t-x)dx]$$

Or,

$$D(t) = d\ N(t) \tag{8.3.8}$$

where,

$$d = \int_0^T [q(x)n(x)/p(x)]\ dx \tag{8.3.9}$$

Obviously, the growth rate of population which is,

$[B(t) - D(t)]/N(t) = b - d,$

is constant over time. Since the number of births and deaths are proportional to the size of population, these also grow at the constant rate at which the population is growing. Let us denote the growth rate by g.

Now we can relate the growth rate of population with the rates of survival and reproduction. Births in period t can also be written as follows:

$$B(t) = \int_0^T B(t,x)\ dx = \int_0^T B(t-x)\ p(x)\ b(x)\ dx$$

With g being the constant growth rate of population, we can write $B(t) = e^{gt}\ B(0)$ and $B(t-x) = e^{g(t-x)}\ B(0)$. Dividing $B(t-x)$ by $B(t)$ we can write: $B(t-x)/B(t) = e^{-gx}$ or $B(t-x) = e^{-gx}\ B(t)$. Thus, dividing both sides of the above equation by B(t), we obtain

$$\int_0^T e^{-gx}\ p(x)\ b(x)\ dx = 1 \qquad (8.3.10)$$

This is an implicit function which shows how the growth rate of population depends on the age path of survival rates P and the age path of reproduction rates b. The partial derivative of this function with respect to g is

$$-\int_0^T x\ e^{-gx}\ p(x)\ b(x)\ dx$$

which is non-zero. Therefore we can solve for g as a function of P and b. Let us write this function as

$$g = g(P,\ b) \qquad (8.3.11)$$

The effects of variations in survival and reproduction rates on the growth rate of population are then as follows:

$$g_p \, dP = \frac{\int_0^T e^{-gx} \, b(x) \, dp(x) \, dx}{\int_0^T x \, e^{-gx} \, p(x) \, b(x) \, dx} \qquad (8.3.12)$$

$$g_p \, db = \frac{\int_0^T e^{-gx} \, p(x) \, db(x) \, dx}{\int_0^T x \, e^{-gx} \, p(x) \, b(x) \, dx} \qquad (8.3.13)$$

Equation (8.3.12) says that the growth rate of population will increase if the survival rates increase for those periods of life when the rate of reproduction is positive. Similarly equation (8.3.13) shows that the effect of an increase in the rates of reproduction on the growth rate of population is positive.

8.4 AN AGGREGATE HOUSEHOLD MODEL

In section 8.2 we have discussed the life cycle allocation behaviour of an individual. Now we can aggregate the household behaviour over the entire society in which many generations co-exist. It is assumed that all the members of the society have identical preferences. We also assume that the economy is on a steady state path and therefore the rate of interest and the age specific wage rates are constant over time. Since all the individuals face the identical distribution of life, their expected lifetime labour income is also identical.

Under these circumstances the age specific rates of consumption will also be constant over time. Therefore aggregate consumption in period t can be written as

$$C(t) = \int_0^T N(t,x)\ c(x)\ dx = \int_0^T B(t-x)\ p(x)\ c(x)\ dx$$

or,

$$C(t) = B(t) \int_0^T e^{-gx}\ p(x)\ c(x)\ dx \qquad (8.4.1)$$

Similarly, aggregate labour supply in period t can be written as follows:

$$H(t) = B(t) \int_0^T e^{-gx}\ p(x)\ j(x)\ h(x)\ dx \qquad (8.4.2)$$

The function j(x) which measures the age productivity of labour, translates an actual unit of labour into an 'effective unit'. With the normalization j(0) = 1, the function j(x) transforms the work hours at age x in the equivalent hours of work at age 0. We assume that work is rewarded according to effective units, not actual units of labour. Therefore, if w is the wage rate for a unit of labour at age 0, the wage rate for an actual unit of labour (or, j(x) 'effective units' of labour) at age x will be wj(x). That is, the wage rate for an actual unit of labour at age x, w(x), can be written as

$$w(x) = w\, j(x) \tag{8.4.3}$$

It may be noted that w(x) is to be interpreted as the same wage rate schedule which we used in our individualistic model in section 8.2. Notice that the age profile of wage rate is proportional to the age profile of productivity function, j(x). Thus, if work is rewarded according to effective units of labour, the age path of j(x) can be inferred from the age path of wage rate. Thus our assumption in chapter 3 (which is also carried over to section 8.2) that wage rate is increasing with age at a diminishing rate up to some age and decreasing thereafter, implies a similar age pattern of productivity function, j(x).

Next, we find the relationship between aggregate labour income and aggregate labour supply. Aggregate labour income at time t, M(t) can be obtained by adding labour incomes of all the individuals of different ages at time t. That is,

$$M(t) = B(t) \int_0^T e^{-gx} \, p(x) \, m(x) \, dx \tag{8.4.4}$$

Or, substituting $m(x) = w(x)\ h(x) = w\ j(x)\ h(x)$, we can write aggregate labour income in terms of aggregate labour supply as follows:

$$M(t) = w \, B(t) \int_0^T e^{-gx} \, p(x) \, j(x) \, h(x) \, dx = w \, H(t)$$

Now we can write the two measures of aggregate savings. The narrow measure, $\Sigma(t)$, is defined as the excess of aggregate labour income, $M(t)$, over aggregate consumption, $C(t)$. That is,

$$\Sigma(t) = M(t) - C(t) \tag{8.4.5}$$

Substituting for $M(t)$ and $C(t)$ from equations (8.4.1) and (8.4.4) respectively and recalling that $m(x) - c(x) = \sigma(x)$, we can express aggregate savings, $\Sigma(t)$, as the sum of individual savings over all the members of the society. That is,

$$\Sigma(t) = B(t) \int_0^T e^{-gx} \, p(x) \, \sigma(x) \, dx \tag{8.4.6}$$

Next consider the broad measure of savings, $S(t)$. These savings can be defined as the net rate of increase in aggregate assets. Therefore we can write

$$S(t) = dA(t)/dt \tag{8.4.7}$$

where, $A(t)$ denotes the stock of aggregate assets in period t, defined as follows:

$$A(t) = B(t) \int_0^T e^{-gx}\, p(x)\, a(x)\, dx \tag{8.4.8}$$

Or, substituting for $a(x)$ from equation (8.2.9)

$$A(t) = B(t) \int_0^T e^{(r-g)x} \int_0^x e^{-ry}\, p(y)\, \sigma(y)\, dy\, dx \tag{8.4.9}$$

Since births, $B(t)$ grow at a constant rate g, it follows from (8.4.7) and (8.4.8) that the broad measure of aggregate savings, $S(t)$ can be written as

$$S(t) = g\, A(t) \tag{8.4.10}$$

To further elucidate the broad concept of savings we now develop an accounting relationship between the narrow measure of savings, $\Sigma(t)$ and assets, $A(t)$. This relationship will also be useful to study the effects of mortality improvements on aggregate capital stock in section 8.6. The aggregate savings, $\Sigma(t)$, defined by (8.4.6) can be written as follows.

$$\Sigma(t) = B(t) \int_0^T e^{-rx} p(x) \sigma(x) e^{(r-g)x} dx \qquad (8.4.11)$$

It is convenient to write $e^{(r-g)x}$ in the following form:

$$e^{(r-g)x} = -(r-g) \int_x^T e^{(r-g)y} dy + e^{(r-g)T}$$

Substituting this relation in equation (8.4.11), we obtain

$$\Sigma(t) = -(r-g) B(t) \int_0^T e^{-rx} p(x) \sigma(x) \int_x^T e^{(r-g)y} dy\, dx$$
$$+ e^{(r-g)T} B(t) \int_0^T e^{-rx} p(x) \sigma(x) dx \qquad (8.4.12)$$

The second term in the above expression is equal to zero because of the lifetime budget constraint of the individual (See equation (8.2.4)). Therefore, with a change in the order of integration, equation (8.4.12) becomes.

$$\Sigma(t) = (g-r) B(t) \int_0^T e^{(r-g)x} \int_0^x e^{-ry} p(y) \sigma(y) dy\, dx \qquad (8.4.13)$$

Or, using (8.4.9), we obtain our desired result:

$$\Sigma(t) = (g-r) A(t) \qquad (8.4.14)$$

This leads us to the following well known relationship between savings and assets:

Theorem 8.4.1:

> If aggregate assets, $A(t)$, are positive (negative) then the narrow measure of aggregate savings, $\Sigma(t)$, will be positive (negative) in the case $g > r$ and non-positive (non-negative) in the $g \leq r$.

The case $g > r$ has come to be known as the 'inefficient' case while $g \leq r$ is known as the 'efficient' case (See, for example, Gale (1973), Shell (1971) and Starrett (1972)). The above relationship between savings and assets has been established by Gale (1973) for an exchange economy and by Willis (1985) for a production economy under certain lifetimes. Notice that this classification of competitive equilibrium is based on the assumption of given tastes and technology. In the context of life uncertainty tastes also include survival probabilities since the lifetime utility function of an individual has been obtained by integrating age related utility rates weighted by the survival probabilities over the entire life horizon. Later in this chapter we will be interested in the implications of mortality improvements for the classification of competitive equilibrium as efficient or inefficient. Since mortality improvements will also change the utility function, these implications should not be taken as normative consequences of mortality improvements.

Combining (8.4.10) and (8.4.14) we can also write the broad measure of aggregate savings, S(t), as follows.

$$S(t) = \Sigma(t) + r\,A(t) \qquad (8.4.15)$$

This relationship among the two measures of savings and assets implies the following result.

Theorem 8.4.2:

The effect of extra interest income (or costs) resulting from actuarially fair life insurance and annuities which was present in the individual's savings schedule, s(x), disappears from the corresponding aggregate measure of savings, S(t).

It may be recalled from chapter 3 (equation (3.4.9)) that the broad measure of savings for an individual includes extra interest income from assets due to the actuarially fair rate of interest. The fact that this effect disappears from the aggregate broad measure of savings is exactly what one would expect if the life insurance and annuities are actuarially fair. To further explain this point, we express aggregate savings, S(t), in terms of the individual savings schedule, $s(x)$. Therefore, we substitute for $\Sigma(t)$ and $A(t)$ from equations (8.4.6) and (8.4.8) respectively into equation (8.4.15). The result is

$$S(t) = B(t) \int_0^T e^{-gx}\, p(x)\, [\sigma(x) + r\, a(x)]\, dx$$

Since $s(x) = \sigma(x) + [r + q(x)/p(x)]\, a(x)$, the above expression can be written in terms of $s(x)$ as follows:

$$S(t) = B(t) \int_0^T e^{-gx}\, p(x)\, [s(x) - \{q(x)/p(x)\}a(x)]\, dx$$

If $a(x)$ is positive, then the term $\{q(x)/p(x)\}a(x)$ can be interpreted as the per capita expenditure on insurance premium by people aged x. The full amount of this insurance premium is paid, however, only by those who die at age x. Since this is an expenditure item, it will cause per capita savings of people aged x to be reduced by the amount $\{q(x)/p(x)\}a(x)$. This is also exactly the amount the insurance companies distribute as an extra per capita interest income to people aged x. The full amount of this interest income is distributed among those who survive to age x. Therefore the extra interest income of the survivors is exactly offset by the payment of insurance premiums by those who died at age x. If, on the other hand, $a(x)$ is negative then the term $\{q(x)/p(x)\}a(x)$ constitutes an extra per capita income for people aged x because it is their liability to the insurance companies which will not be paid. Accordingly, per capita savings of the people aged x will increase by the amount $-\{q(x)/p(x)\}a(x)$. With $a(x)$ being negative, the extra interest costs by the survivors in this case are exactly offset by the unpaid liabilities of the dead.

At this point we can also confirm the fact that insurance companies do not earn any profits. Consider the balance sheet of the insurance companies. The cost of interest payments by insurance companies in period t is

$$B(t) \int_0^T e^{-gx}\, p(x)\, [r + \{q(x)/p(x)\}]a(x)\, dx$$

The interest income on capital held by insurance companies is

$$r\, B(t) \int_0^T e^{-gx}\, p(x)\, a(x)\, dx = B(t) \int_0^T e^{-rx}\, p(x)\, r\, a(x)\, dx$$

Finally, the income from insurance premiums is

$$B(t) \int_0^T e^{-gx} q(x) a(x)\, dx = B(t) \int_0^T e^{-gx} p(x) [q(x)/p(x)] a(x)\, dx$$

It is obvious that sum of these two sources of income is equal to the cost of interest payments. Therefore insurance companies earn no profits (positive or negative) and serve only as an intermediary between households and the ultimate users of capital, that is, firms.

Let us now see what factors determine the levels of the aggregate variables consumption, labour supply, savings and assets. The individual consumption and savings schedules, that is, $c(x)$ and $\sigma(x)$ have been shown to be the functions of age, the age path of survival probabilities, the interest rate and the age path of wage rates. These functions, given by (8.2.5) and (8.2.7) respectively, are reproduced below:

$$c(x) = c(x, P, r, W)$$

$$\sigma(x) = \sigma(x, P, r, W)$$

The age path of wage rates, W, appearing in these functions can be written as follows:

$$W = w\,J$$

where, w is the wage rate for a unit of labour at age zero and J represents the entire age path of the age productivity parameters $j(y) : 0 \leq y \leq T$. Throughout the analysis we assume that the age productivity function, $j(x)$, is fixed. Therefore, we suppress it from the above two functions, which can now be written as:

$c(x) = c(x, P, r, w)$

$\sigma(x) = \sigma(x, P, r, w)$

Substituting these two functions into the appropriate places, we can write the aggregate variables with all their arguments as follows.

$$C(t) = B(t) \int_0^T e^{-gx}\, p(x)\, c(x, P, r, w)\, dx \qquad (8.4.16)$$

$$H(t) = B(t) \int_0^T e^{-gx}\, p(x)\, j(x)\, h(x)\, dx \qquad (8.4.17)$$

$$\Sigma(t) = B(t) \int_0^T e^{-gx}\, p(x)\, \sigma(x, P, r, w)\, dx \qquad (8.4.18)$$

$$A(t) = B(t) \int_0^T e^{(r-g)x} \int_0^x e^{-ry}\, p(y)\, \sigma(y,P,r,w)\, dy\, dx \qquad (8.4.19)$$

$$S(t) = g\, B(t) \int_0^T e^{(r-g)x} \int_0^x e^{-ry}\, p(y)\, \sigma(y,P,r,w)\, dy\, dx$$
(8.4.20)

Notice that all these aggregate variables depend on time only through a factor of proportionality B(t), current births, which grows at a constant rate, g. Therefore all the aggregates grow at a common constant rate, g, an outcome of the assumption of steady state. With normalization by the current births, the above functions can be written in the following compact form.

$$\hat{C} = \hat{C}(g, P, r, w) \tag{8.4.21}$$

$$\hat{H} = \hat{H}(g, P) \tag{8.4.22}$$

$$\hat{\Sigma} = \hat{\Sigma}(g, P, r, w) \tag{8.4.23}$$

$$\hat{S} = \hat{S}(g, P, r, w) \tag{8.4.24}$$

$$\hat{A} = \hat{A}(g, P, r, w) \tag{8.4.25}$$

where, ^ indicates that the variable has been divided by the number of current births, B(t).

8.5 A FEASIBLE COMPETITIVE SOLUTION

First we assume that the production function for a single good, which can be consumed or used as capital, is linear in each of the two inputs, capital and labour. Under the assumed competitive conditions, the rate of interest and the wage rate are equated to the marginal products of capital and labour respectively. Since, with a linear production function, both marginal products are constant, the rate of interest and the wage rate are also constant, given by the technology. Let us call the fixed rate of interest and the fixed wage rate as $\bar{r}$ and $\bar{w}$ respectively. Under these circumstances we can write the production function as

$$Y = \bar{r}\ K + \bar{w}\ H \tag{8.5.1}$$

Consequently, profits of the firms $(Y - \bar{r}\ K - \bar{w}\ H)$ are identically equal to zero irrespective of the levels of capital and labour employed. In other words, the firms are willing to employ as much capital and labour as available in the market. Thus, the level of employment of the two factors are completely determined by their respective supplies.

The supply of labour is independent of the rate of interest and the wage rate while the supply of assets depends on the two factor prices. The equilibrium levels of labour and capital, normalized by the number of current births, are given as follows.

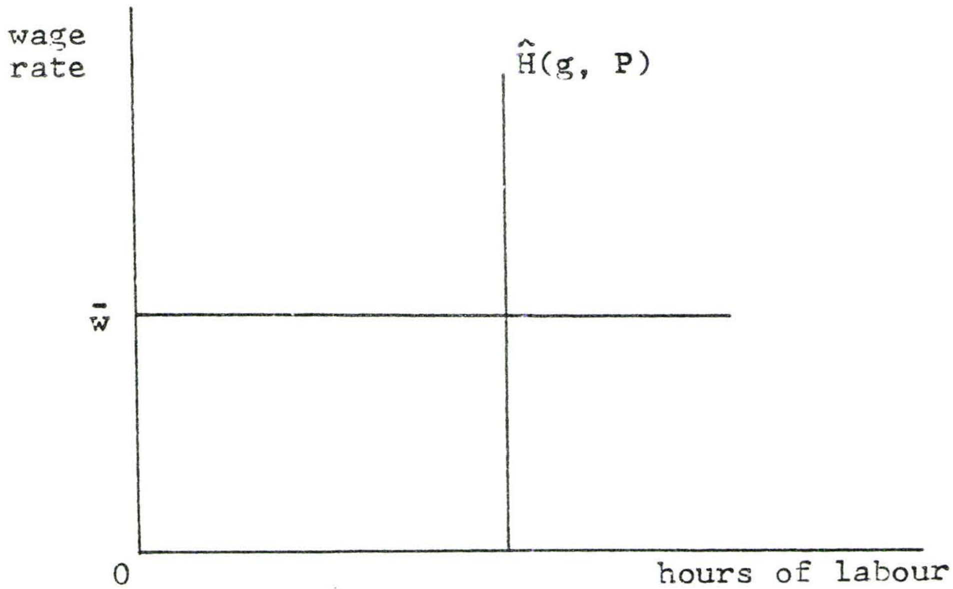
$$\hat{H} = \hat{H}(g,\ P) \tag{8.5.2}$$

$$\hat{A} = \hat{A}(g,\ P,\ \bar{r},\ \bar{w}) \tag{8.5.3}$$

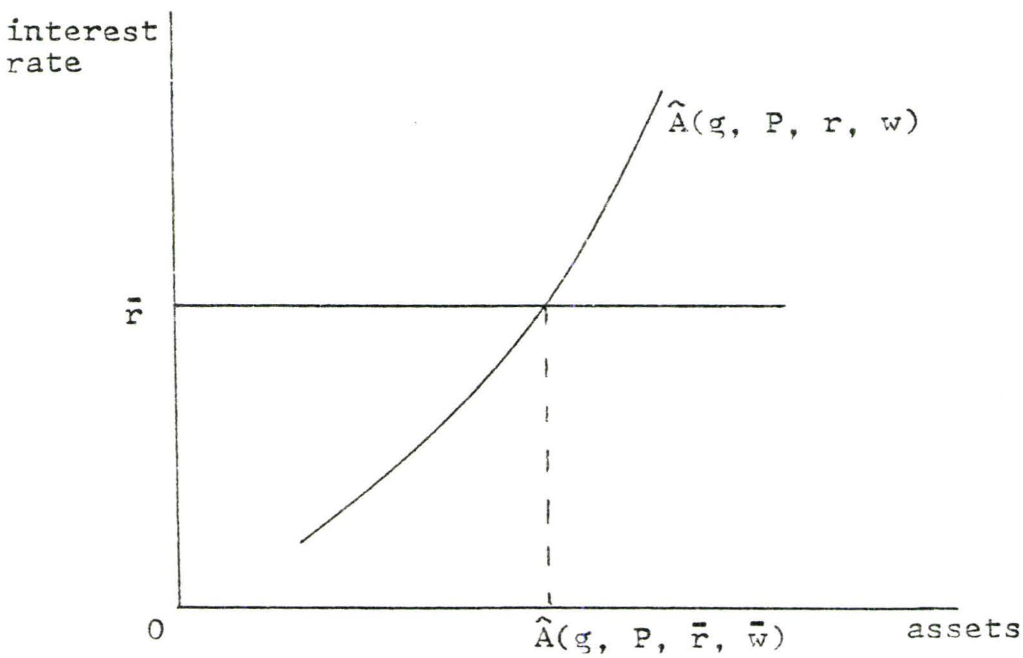
The solution is illustrated in figures 8.5.1(a) and 8.5.1(b). Notice that in figure 8.5.1(b) we have assumed that the assets supply function responds positively to the rate of interest. But, with a horizontal demand function for capital, this assumption is not vital for the analysis. In any event, in the next chapter we discuss the possible slope of this function.

Figure 8.5.1 Equilibrium in the Labour and Assets Markets with Fixed Wage Rate and Rate of Interest

(a)



(b)



This completely describes the nature of general equilibrium for the simple economy under consideration. We are now ready to study the implications of mortality improvements in this economy. This analysis is conducted in the next section.

8.6 GENERAL EQUILIBRIUM EFFECTS OF MORTALITY IMPROVEMENTS

In this section we discuss the effects of increased life expectancy on the aggregate capital stock and aggregate labour. This will be done by introducing a small change in the survival rate (or, survival probability) at a specific age, say x^* and calculating its effect on aggregate assets and labour supply. Then, we will study how this effect depends on the age at which the survival rate is assumed to increase. Since in the present model the equilibrium levels of capital and labour are fully determined by their respective supplies, the resulting change in the supply of assets and labour will represent the effect of the mortality improvement on the equilibrium levels of capital and labour respectively.

8.6.1 The Effects of Mortality Improvements on Capital Stock

To find the effect of an increase in the survival rate at age x^* on the equilibrium level of capital stock, we need to differentiate the function $\hat{A} = \hat{A}(g, P, \bar{r}, \bar{w})$ with respect to $p(x^*)$. But, first recall from section 8.3 that the growth rate of population, g, can be expressed in term of more basic variables, namely the age path of survival rates and the age path of reproduction rates, by the following function:

$$g = g(P, b) \tag{8.6.1}$$

Substituting this function into the equilibrium level of capital stock, we can write

$$\hat{A} = \hat{A}(g(P,b), P, \bar{r}, \bar{w}) \tag{8.6.2}$$

The derivative of this function with respect to $p(x^*)$ can be written as follows.

$$d\hat{A}/dp(x^*) = \partial\hat{A}/\partial p(x^*) + [\partial\hat{A}/\partial g]\ [\partial g/\partial p(x^*)] \tag{8.6.3}$$

Thus, the full effect of an increase in the survival rate at age x^* on the capital stock can be split into two parts. The first part in equation (8.6.3), $\partial\hat{A}/\partial p(x^*)$, can be referred to as the pure economic effect of an increase in the survival rate, as will later become obvious. The second part, $[\partial\hat{A}/\partial g][\partial g/\partial p(x^*)]$, represents a pure demographic effect. It measures the change in capital stock resulting purely from the shift in the growth rate of population induced by the increase in survival rate. We will discuss these two effects one by one.

8.6.2 The Economic Effect on the Capital Stock

To evaluate this effect, given by the derivative $\partial\hat{A}/\partial p(x')$, consider the relationship between savings and assets given by (8.4.14) which implies that for $g \neq r$, $\hat{A} = \hat{\Sigma}/(g-r)$. Therefore, for $g \neq r$, we can write

$$\partial\hat{A}/\partial p(x') = [\partial\hat{\Sigma}/\partial p(x')]/(g-r) \qquad (8.6.4)$$

Differentiating $\hat{\Sigma} = \int_0^T e^{-gx}\, p(x)\, \sigma(x)\, dx$

with respect to $p(x')$, we obtain

$$\partial\hat{\Sigma}/\partial p(x') = e^{-gx'}\, \sigma(x') + \int_0^T e^{-gx}\, p(x)\, [\partial\sigma(x)/\partial p(x')]\, dx \qquad (8.6.5)$$

We know that at age x_* and age x^*, $\sigma(x) = 0$. We also know by equation (8.2.14) that $[\partial\sigma(x)/\partial p(x')] = 0$ for all x, if $\sigma(x') = 0$. Therefore, equation (8.6.5) implies that at $x' = x_*$ and $x' = x^*$, the derivative $\partial\hat{\Sigma}/\partial p(x')$ is equal to zero. This in turn implies, according to equation (8.6.4), that $\partial\hat{A}/\partial p(x')$ is equal to zero. For other values of x', consider the second part in equation (8.6.5). Dividing and multiplying by

$\int_0^T e^{-rx}\, p(x)\, [\partial\sigma(x)/\partial p(x^*)]\, dx$, we can write this term as:

$$\int_0^T e^{-gx}\, p(x)\, [\partial\sigma(x)/\partial p(x^*)]\, dx =$$

$$\frac{\int_0^T e^{(r-g)x} e^{-rx} p(x)[\partial\sigma(x)/\partial p(x^*)]dx \;\; \int_0^T e^{-rx} p(x)[\partial\sigma(x)/\partial p(x^*)]dx}{\int_0^T e^{-rx}\, p(x)\, [\partial\sigma(x)/\partial p(x^*)]\, dx}$$

$$= \overline{e^{(r-g)x}} \int_0^T e^{-rx}\, p(x)\, [\partial\sigma(x)/\partial p(x^*)]\, dx \qquad (8.6.6)$$

where, we use the notation $\overline{e^{(r-g)x}}$ for the weighted average of the variable $e^{(r-g)x}$, defined below.

$$\overline{e^{(r-g)x}} = \frac{\int_0^T e^{(r-g)x}\, e^{-rx}\, p(x)\, [\partial\sigma(x)/\partial p(x^*)]\, dx}{\int_0^T e^{-rx}\, p(x)\, [\partial\sigma(x)/\partial p(x^*)]\, dx} \qquad (8.6.7)$$

Notice from equation (8.2.14) that the derivative $\partial\sigma(x)/\partial p(x^*)$ depends on the age x^* only through a factor of proportionality, $e^{-rx^*}\, \sigma(x^*)$. This factor cancels out from the numerator and the denominator of the weighted average defined above. Therefore this weighted average is independent of x^*, the age at which the survival rate is assumed to increase.

Using equation (8.2.12) and noting that $\partial\sigma(x)/\partial p(x')$ $= -\partial c(x)/\partial p(x')$, we can also write equation (8.6.6) as

$$\int_0^T e^{-gx}\ p(x)\ [\partial\sigma(x)/\partial p(x')]\ dx = -\ \overline{e^{(r-g)x}}\ e^{-rx'}\ \sigma(x')$$

Substituting this result in equation (8.6.5), we obtain

$$\partial\hat{\Sigma}/\partial p(x') = e^{-gx'}\ \sigma(x') - \overline{e^{(r-g)x}}\ e^{-rx'}\ \sigma(x')$$

Or, $\partial\hat{\Sigma}/\partial p(x') = e^{-rx'}\ \sigma(x')\ [e^{(r-g)x'} - \overline{e^{(r-g)x}}]$

Substituting this result in equation (8.6.4) yields

$$\partial\hat{A}/\partial p(x') = e^{-rx'}\ \sigma(x')[\{\overline{e^{(r-g)x}}/(r-g)\} - \{e^{(r-g)x'}/(r-g)\}] \qquad (8.6.8)$$

The above expression has been derived for $g \neq r$. For $g = r$, we can take its limit as g approaches towards r. Applying L'Hospital Rule, the result is found to be as follows:

$$\partial\hat{A}/\partial p(x') = e^{-rx'}\ \sigma(x')[\bar{x} - x'] \qquad (8.6.9)$$

where, $\bar{x}$ is the weighted average of x obtained by applying the same weighting scheme as used in (8.6.7).

The derivatives given by (8.6.8) and (8.6.9) have been evaluated for a specific age, x'. Replacing this specific age by the age index, x, we can express these derivatives as functions of the age at which the survival rate is assumed to increase. Therefore, we can write

$$\partial\hat{A}/\partial p(x^*) = e^{-rx} \phi(x) [\bar{z} - z(x)] \qquad (8.6.10)$$

where,

$$\bar{z} = \overline{e^{(r-g)x}}/(r-g) \quad \text{for} \quad g \neq r$$
$$= \bar{x} \quad \text{for} \quad g = r \qquad (8.6.11)$$

$$z(x) = e^{(r-g)x}/(r-g) \quad \text{for} \quad g \neq r$$
$$= x \quad \text{for} \quad g = r \qquad (8.6.12)$$

To determine the age pattern of the derivative $\partial\hat{A}/\partial p(x^*)$, we first determine the age path of the function $z(x)$. The slope and curvature of this function are as follows.

$$\dot{z}(x) = e^{(r-g)x} > 0 \quad \text{for } g \neq r$$
$$= 1 > 0 \quad \text{for } g = r \qquad (8.6.13)$$

$$\ddot{z}(x) = (r - g)\, e^{(r-g)x} < 0 \quad \text{for } g > r$$
$$> 0 \quad \text{for } g < r \qquad (8.6.14)$$
$$= 0 \quad \text{for } g = r$$

Next, since $\bar{z}$ is the weighted average of the variable $z(x)$, the latter being monotonic in x, the value of $\bar{z}$ must lie somewhere between $z(0)$ and $z(T)$. Although it is possible to have $\bar{z} < z(x_*)$ or $\bar{z} > z(x^*)$ under certain age patters of savings, for the detailed study we will concentrate only on the more likely case $z(x_*) \leq \bar{z} \leq z(x^*)$.[1] However the general characteristics of our results are not dependent on this assumption (see footnote 2 of this chapter).

The age pattern of the derivative $\partial\hat{A}/\partial p(x')$ can be studied with the help of a diagram. In part (a) of figure 8.6.1 are drawn the age paths of the two functions $e^{-rx}\ \sigma(x)$ and $\bar{z} - z(x)$. The function $\bar{z} - z(x)$ has been drawn under the assumption: $g < r$. If $g > r$, the function $\bar{z} - z(x)$ will be convex and with $g = r$, it will be linear. Multiplying the height of the two functions $\bar{z} - z(x)$ and $e^{-rx}\ \sigma(x)$, we obtain the age path of the derivative $\partial\hat{A}/\partial p(x')$ in part (b) of the figure. This age path summarizes the main result of this chapter.

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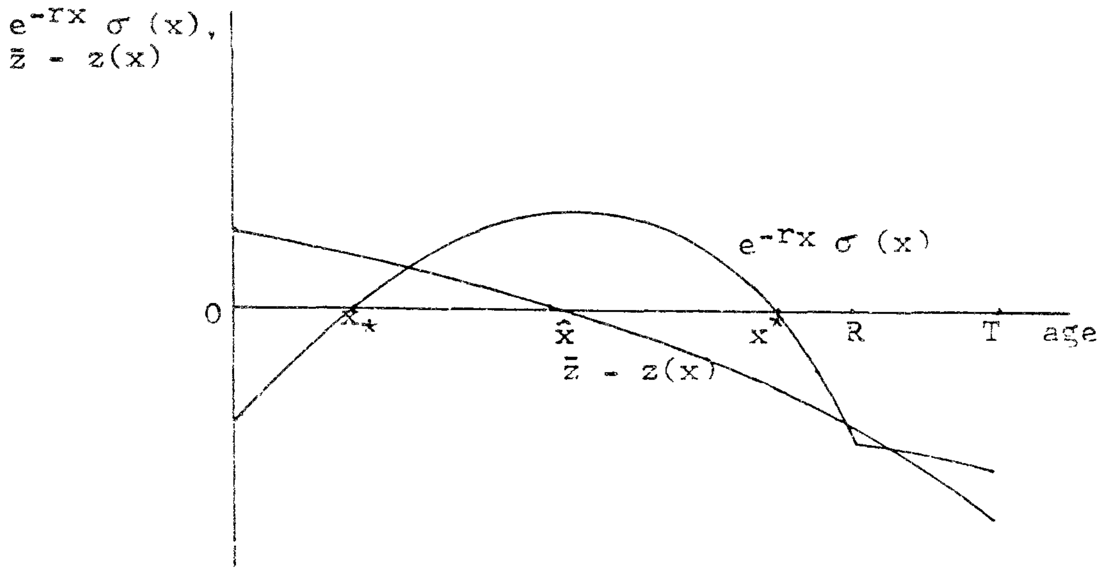
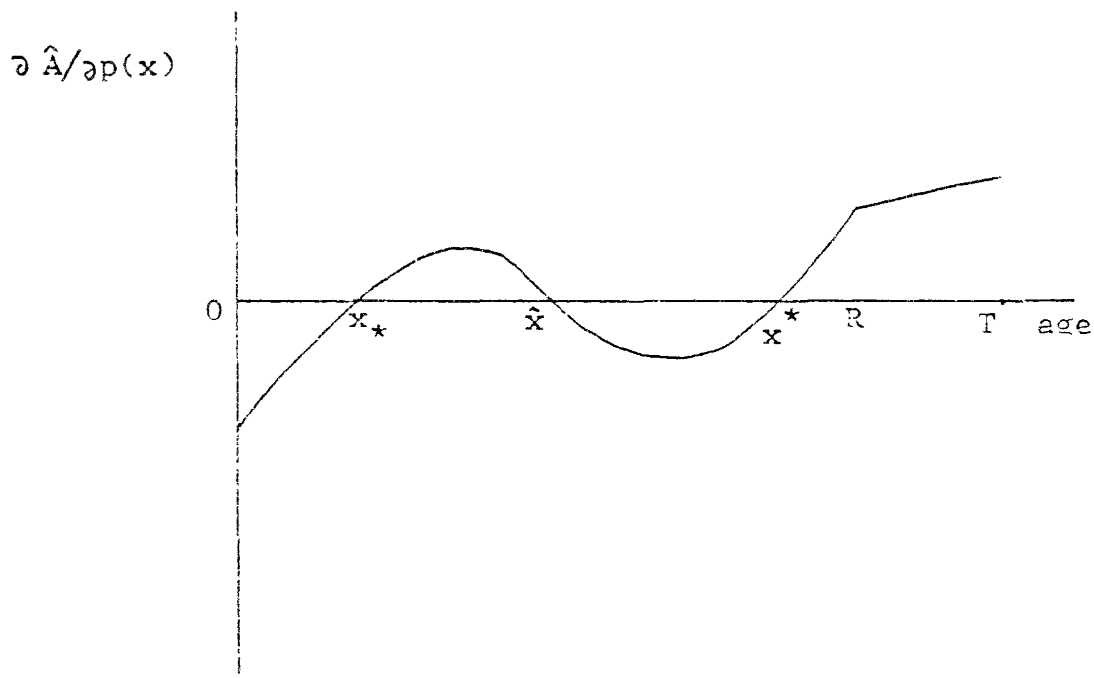
1 The age x_* at which individuals start consuming less than current labour income, is expected to come quite early in life. On the other hand, the age x^* at which individuals start consuming more than current labour income, is expected to come later in life, a few years before retirement. Therefore it seems more likely that the weighted average $\bar{z}$ will lie somewhere between $z(x_*)$ and $z(x^*)$.

Figure 8.6.1 The Age Pattern of the Economic Effect of an Increase in Survival Rates on Aggregate Assets

(a)



(b)



Theorem 8.6.1:

If the individual's savings schedule follows a hump shaped age path, an increase in survival rates at a young age (below age x_*) or at the later part of middle age (between the ages $\hat{x}$ and x^*) will result in a decrease in the capital stock, where $\hat{x}$ is the age such that $\bar{z} = z(\hat{x})$. On the other hand, an increase in survival rates at the earlier part of the middle age (between the ages x_* and $\hat{x}$) or at the old age (above age x^*) will result in an increase in the capital stock.[2]

Let us now see how we can explain this rather complex pattern of the effect of mortality improvement on the capital stock. Consider the definition of assets held by an individual aged x

$$a(x) = [e^{rx}/p(x)] \int_0^x e^{-ry}\, p(y)\, g(y)\, dy \qquad (8.6.15)$$

Given the savings rate, $g(y)$, an increase in survival rate at age x will decrease the actuarially fair return on all

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2 If we relax the assumption: $z(x_*) \leq \bar{z} \leq z(x^*)$, the age path of the derivative $\partial A/\partial p(x^*)$ will be similar to the one shown in figure 8.6.1, except that in case $\bar{z} > z(x^*)$, points x^* and $\hat{x}$ will be reversed and in case $\bar{z} < z(x_*)$, points x_* and $\hat{x}$ will be reversed.

previous savings by increasing the conditional survival rate at age x given age y, $p(x)/p(y)$. Therefore, the absolute level of assets accumulated by an individual aged x will decrease. However, the aggregate level of assets held by all the individuals aged x will not be affected because the size of the population aged x, $N(t, x) = p(x)\ B(t-x)$, will increase by the same factor by which the absolute level of assets of an individual decreases. Thus aggregate assets of all individuals aged x, normalized by the current births,

$$N(t,x)a(x)/B(t) = e^{(r-g)x} \int_0^x e^{-ry}\ p(y)\ \sigma(y)\ dy \qquad (8.6.16)$$

are independent of the survival rate at age x, $p(x)$. This is an obvious outcome of risk sharing on an actuarially fair basis.

Next, a higher survival rate at age $y < x$ implies a lower conditional survival rate at age x given age y, $p(x)/p(y)$ and therefore, a higher actuarially fair return on savings at age y. With this higher return, the assets held by an individual aged x, which are obtained by accumulating all the previous savings (positive or negative) inflated by the actuarially fair interest factor will increase (decrease) if the savings rate at age y, $\sigma(y)$, is positive (negative). Also notice that any change in a survival rate at age, say x', will have no effect on assets held by an individual aged less than x'. Therefore, the earlier the age at which the survival rate is assumed to increase, the higher will be the number of individuals affected by the resulting increase in the actuarially fair return on savings.

Finally, as we have concluded in section 8.2, in response to an increase in the survival rate at the age at which the savings rate, $\sigma(y)$, is positive (negative), an individual will decrease (increase) savings in all periods of life. Thus, this reallocation effect on aggregate assets is opposite in sign to the effect of a higher return on the given savings plan discussed above.

Combining the two effects, we expect that if the survival rates increase at some earlier segment of life, the effect of a higher return on the given savings plan will dominate the reallocation effect and the opposite will happen if the survival rates increase at some later segment of life. In other words, if the age at which this pattern is reversed is called $\hat{x}$, then with an increase in the survival rates in the segment below age $\hat{x}$, aggregate assets will change in the same direction as the effect of a higher return on the given savings plan. If the age $\hat{x}$ lies somewhere in the interval x_* to x^* then with an increase in survival rates in the segment: 0 to x_* (x_* to $\hat{x}$), aggregate assets will decrease (increase) because in this segment, the savings rate, $\sigma(y)$, and therefore, the effect of a higher return on assets with a given savings plan, on aggregate savings is negative (positive).

On the other hand, with an increase in survival rates in the segment above age $\hat{x}$, the resulting change in aggregate assets will be of the same sign as the realloca- tion effect. Therefore, with an increase in survival rates in the segment $\hat{x}$ to x^* (x^* to T) aggregate assets will decrease (increase) because in this segment, the savings rate, $s(y)$, is positive (negative) and therefore the real- location effect is negative (positive). This age pattern of the effect of mortality improvement on aggregate assets and, therefore, on the capital stock is exactly what is shown in figure 8.6.1.

Let us now see how this result compares with some earlier findings. Sheshinski and Weiss (1981) showed that in the presence of actuarially fair annuities, an increase in the fraction of the retirement period an individual ex- pects to survive, will result in more savings in the working period of life. Since in their two period model, savings of the young (during the working period) are equal to aggregate assets, their conclusion is analogous to our result $\partial\hat{A}/\partial p(x) > 0$ for $x^* < x \leq T$.

Skinner's (1985) simulation analysis, also in the context of a two period model, shows that an increase in the survival rate in the second period of life will raise the assets-labour supply ratio, though he also concluded that this effect is negligible in magnitude. Since the work-leisure choice in Skinner's analysis is exogenous and work hours of an old worker are positive (almost 50% of the work hours of a young worker), the increase in the survival rate in the second period will definitely increase aggregate labour supply. This implies that in Skinner's calculations the absolute level of assets must have increased by a greater percentage than the assets-labour supply ratio. In that sense, Skinner's conclusion is also in agreement with our result: $\partial\hat{A}/\partial p(x) > 0$ for $x^* < x \leq T$, though his model is not exactly comparable with ours since he did not allow life insurance and annuities in his analysis.[3] Finally, the result $\partial\hat{A}/\partial p(x) > 0$ for $x^* < x \leq T$ has also been obtained by Sinha (1986).

The two period model considered in these studies, however, is restrictive to study mortality improvements because it assumes that survival through the first period of life is certain and therefore increases in life expectancy cannot be associated with an increase in survival rates at an earlier age when the savings rate is positive. In the

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3 Recall from chapter 7 that the life insurance considered by Skinner is different from that considered in our model and by Arthur (1981), Barro and Friedman (1977) and Yaari (1965).

present model, on the other hand, mortality improvements can be brought about by increasing survival rates at any segment of life during which the savings rate may be positive as well as negative.

Another common assumption in these studies as well as in other two period models is that all the births take place during the first period of life when there in no uncertainty. Therefore an increase in life expectancy has no demographic effect associated with an increased growth rate of the population. In the present model this effect has been fully recognized and we now turn our attention to that.

8.6.3 The Demographic Effect on the Capital Stock

This effect, given by the second term in equation (8.6.3), is reproduced below.

$$[\partial\hat{A}/\partial g]\ [\partial g/\partial p(x^*)] \qquad (8.6.17)$$

First consider the factor $\partial\hat{A}/\partial g$. With $\hat{A}$ defined as

$$\hat{A} = \int_0^T e^{-gx}\ p(x)\ a(x)\ dx \qquad (8.6.18)$$

the derivative $\partial\hat{A}/\partial g$ is clearly given by.

$$\partial\hat{A}/\partial g = -\int_0^T x\ e^{-gx}\ p(x)\ a(x)\ dx \qquad (8.6.19)$$

Substituting $x = \int_0^x dy$, we can write this derivative as

$$\partial\hat{A}/\partial g = - \int_0^T e^{-gx}\, p(x)\, a(x) \int_0^x dy\, dx$$

With a change in the order of integration we can also write this expression as follows:

$$\partial\hat{A}/\partial g = - \int_0^T \int_x^T e^{-gy}\, p(y)\, a(y)\, dy\, dx \tag{8.6.20}$$

Aggregate assets, defined by equation (8.6.18), are positive because in equilibrium assets are equal to the capital stock which is positive. Since, in addition, assets of an individual are negative at an early age and positive at later ages, the following relation must hold:

$$\int_x^T e^{-gy}\, p(y)\, a(y)\, dy > 0 \quad \text{for all } x \tag{8.6.21}$$

This implies that the derivative $\partial\hat{A}/\partial g$ given by equation (8.6.20) must be negative. The reason is quite straightforward. With an increase in the growth rate of population, the age distribution of population is disturbed from its initial position. In particular the proportion of younger people increases and the proportion of older people decreases. From the age path of assets holding of an individual predicted in section 8.2, we know that in a broad sense young people hold less assets than the old. Therefore, with the change in the age distribution of population in the direction of relatively more young people, brought about by the increase in the growth rate of population, assets per new born will decrease.

Next, consider the factor $\partial g/\partial p(x^*)$ in equation (8.6.17). This derivative follows directly from equation (8.3.12).

$$\partial g/\partial p(x^*) = \frac{e^{-gx^*}\ b(x^*)}{\int_0^T x\ e^{-gx}\ p(x)\ b(x)\ dx} \qquad (8.6.22)$$

This expression shows that the growth rate of population will increase if the rate of reproduction, $b(x^*)$, is positive at the age at which the survival rate is assumed to increase. If, on the other hand, $b(x^*)$ is zero, the growth rate of population will not be affected.

Replacing x^* by the age index x, we can write the demographic effect as a function of age at which the survival rate is assumed to increase, as $[\partial\hat{A}/\partial g][\partial g/\partial p(x)]$. Since $\partial g/\partial p(x) > 0$ $(= 0)$ if $b(x) > 0$ $(= 0)$ and $\partial\hat{A}/\partial g < 0$, we conclude the following result.

$$[\partial\hat{A}/\partial g]\,[\partial g/\partial p(x)] < 0 \quad \text{if } b(x) > 0$$
$$= 0 \quad \text{if } b(x) = 0 \qquad (8.6.23)$$

Figure 8.6.2 illustrates the age path of this demographic effect. In part (a) we plot the functions e^{-gx} and $b(x)$. It is assumed that the rate of reproduction, $b(x)$, is positive and increasing in age up to some point in middle age and decreasing thereafter, eventually becoming zero in late age. In part (b) we plot the demographic effect, $[\partial\hat{A}/\partial g][\partial g/\partial p(x)]$ by multiplying the functions e^{-gx} and $b(x)$, dividing by

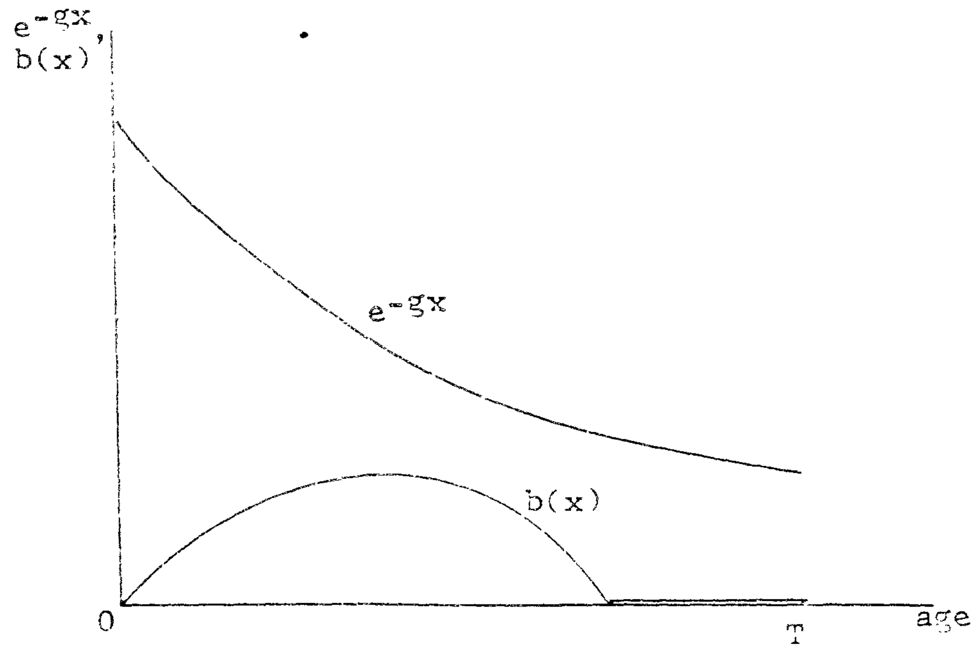
$$\int_0^T x\, e^{-gx}\, p(x)\, b(x)\, dx$$

and then multiplying by the derivative $\partial\hat{A}/\partial g < 0$ given by equation (8.6.20).

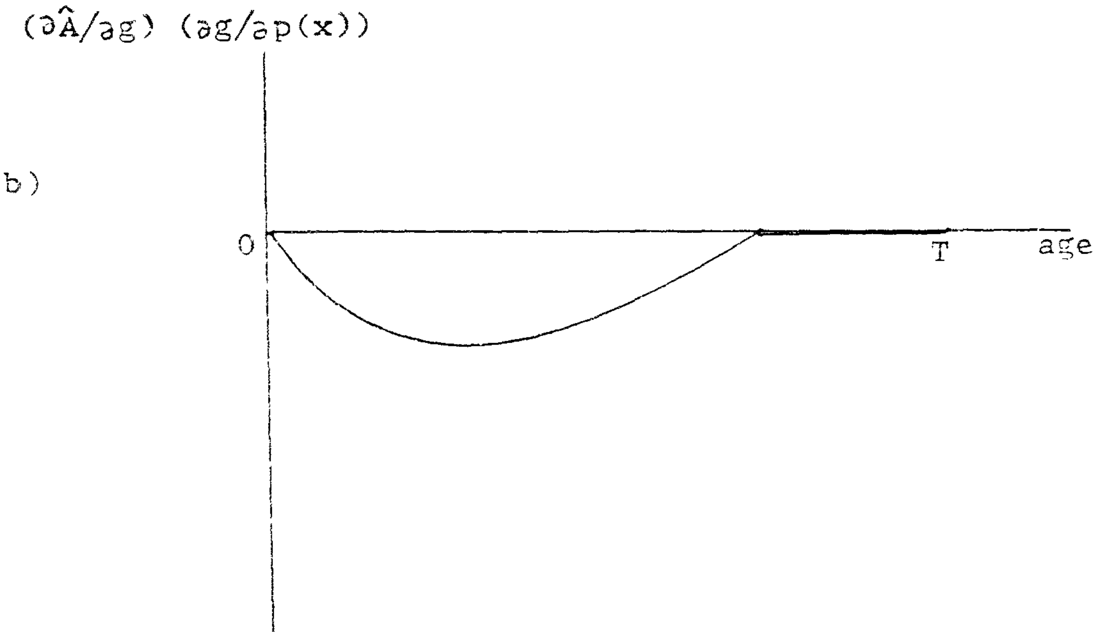
An increase in survival rate at an age at which the reproductive rate, $b(x)$, is zero, will leave the growth rate of population unchanged and, therefore, will have no effect on aggregate assets. Thus, as shown in the diagram, an increase in survival rate at the beginning or towards the end of the horizon where the reproductive rate is assumed to be equal to zero, has no effect on aggregate assets. The diagram also shows that the effect of an increase in survival rate during the reproductive years of life is to decrease aggregate assets and therefore the capital stock.

Figure 8.6.2 The Age Pattern of the Demographic Effect of an Increase in Survival Rates on Aggregate Assets

(a)



(b)



Now we can combine the two effects to find the full effect of mortality improvement on capital stock.

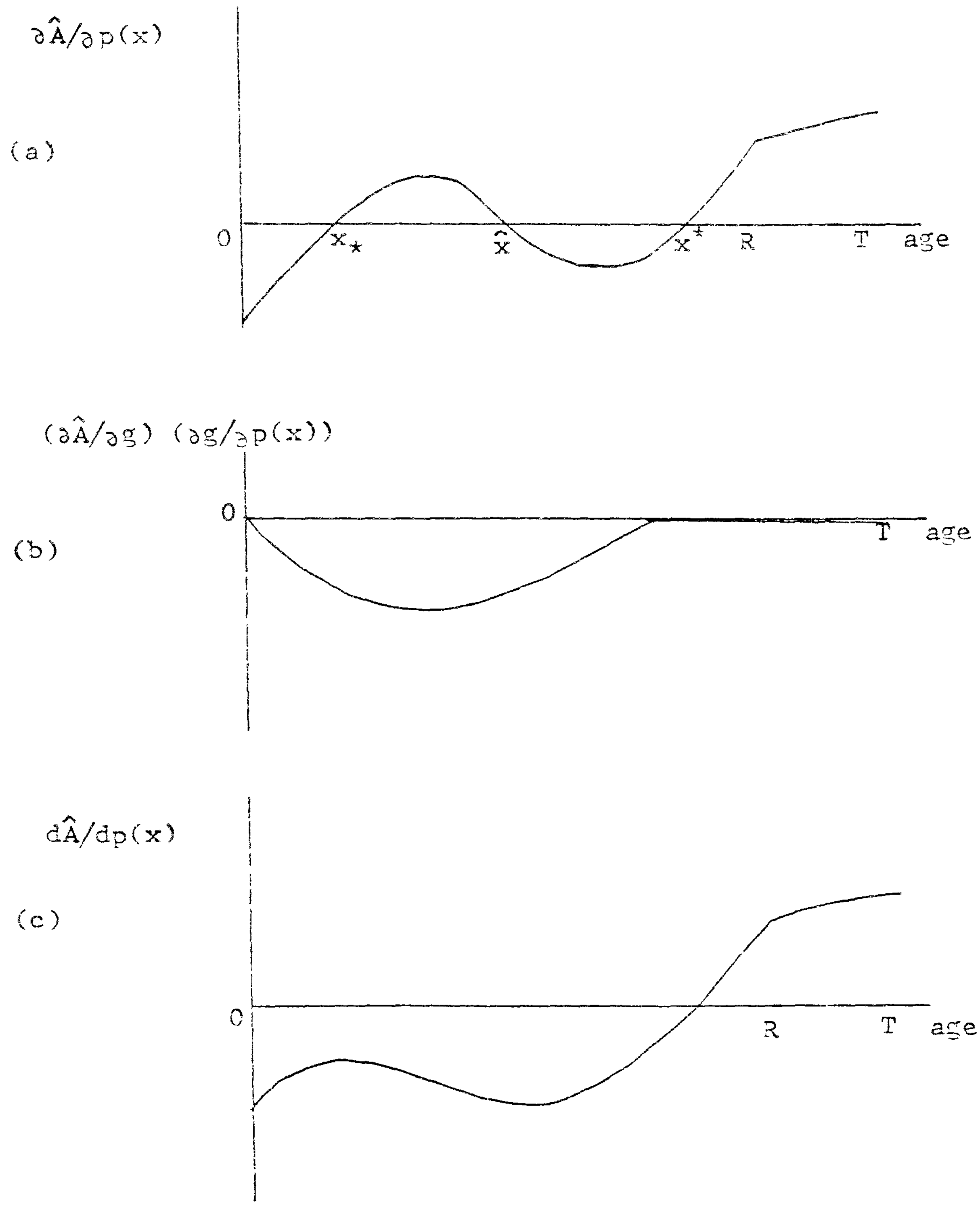
8.6.4 Full Effect of Mortality Improvements on the Capital Stock

To find the full effect of mortality improvement on capital stock, we simply have to add the two components $\partial\hat{A}/\partial p(x)$ and $[\partial\hat{A}/\partial g][\partial g/\partial p(x)]$ as in equation (8.6.3). Thus, by adding the two functions drawn in figures 8.6.1(b) and 8.6.2(b) we find the relationship between the age at which the survival rate is assumed to increase and the resulting change in capital stock. This relationship, as drawn in figure 8.6.3, shows that the effect of increase in survival rates during young or middle (old) age is to decrease (increase) the capital stock. However, the effect of an increase in survival rates at some points in the middle of age x_* and age $\hat{v}$ is in fact ambiguous. But for simplification we assume that this effect is negative as shown in the figure.

8.6.5 The Effects of Mortality Improvements on the Employment of Labour

We first substitute $g = g(P, b)$ in the aggregate labour supply function (8.4.22) to get

Figure 8.6.3 The Age Pattern of the Full Effect of an Increase in Survival rates on Aggregate Assets



$$\hat{H} = \hat{H}(g(P,b), P) \tag{8.6.24}$$

As before, the derivative of this function with respect to p(x') can be expressed in terms of two additive components as follows.

$$\partial\hat{H}/\partial p(x) = \partial\hat{H}/\partial p(x) + [\partial\hat{H}/\partial g]\ [\partial g/\partial p(x)] \tag{8.6.25}$$

With the individual's work schedule given, there is no economic effect of mortality improvement on aggregate labour supply. However, an increase in survival rates during the working years of life will increase aggregate labour supply by increasing the number of workers who survive during those years. This effect is represented by the first component in the above equation. It can be calculated from the aggregate labour supply function,

$$\hat{H} = \int_0^T e^{-gx}\ p(x)\ j(x)\ h(x)\ dx \tag{8.6.26}$$

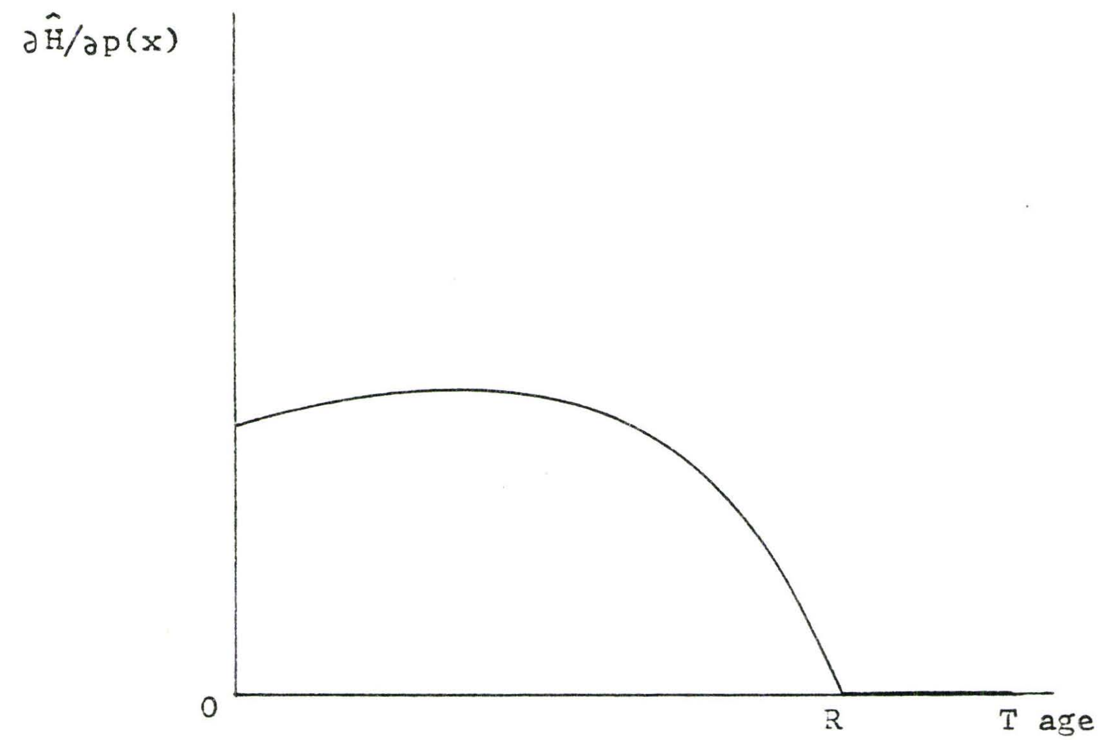
as follows:

$$\partial\hat{H}/\partial p(x) = e^{-gx}\ m(x)/w \tag{8.6.27}$$

In accordance with the age pattern of labour income of an individual m(x), the age path of this derivative is shown in figure 8.6.4.

Next, consider the effect of mortality improvement associated with a possible increase in the growth rate of

Figure 8.6.4 The Age Pattern of the Age Redistributional Effect of an Increase in Survival Rates on the Employment of Labour



population given by the second component in equation (8.6.25). The factor $\partial g/\partial p(x)$ has already been evaluated in terms of equation (8.6.22). The other factor $\partial\hat{H}/\partial g$ can be evaluated from the aggregate labour supply function (8.6.26) as follows:

$$\partial\hat{H}/\partial g = - \int_0^T x\, e^{-gx}\, p(x)\, j(x)\, h(x)\, dx$$

Substituting $x = \int_0^x dy$

and changing the order of integration, we can write this derivative as follows:

$$\partial\hat{H}/\partial g = - \int_0^T \int_x^T e^{-gy}\, p(y)\, j(y)\, h(x)\, dy\, dx \qquad (8.6.28)$$

Since work hours are positive or zero at various points of the life cycle, the above derivative must be negative. The reason again lies in the change in the age distribution of population brought about by the increase in the growth rate of population. With the proportion of the younger population being higher, there are fewer people per new born. Therefore, the higher the growth rate of population, the lower will be the aggregate labour supply normalized by the current births.

Multiplying this derivative by $\partial g/\partial p(x)$ which is positive (zero) if at age x at which the survival rate, p(x), is supposed to have increased, the rate of reproduction, b(x), is positive (zero), we find the demographic effect of mortality improvement:

$$[\partial\hat{H}/\partial g]\,[\partial g/\partial p(x)] \begin{cases} < 0 & \text{if } b(x) > 0 \\ = 0 & \text{if } b(x) = 0 \end{cases} \tag{8.6.29}$$

This demographic effect is illustrated in figure 8.6.5.

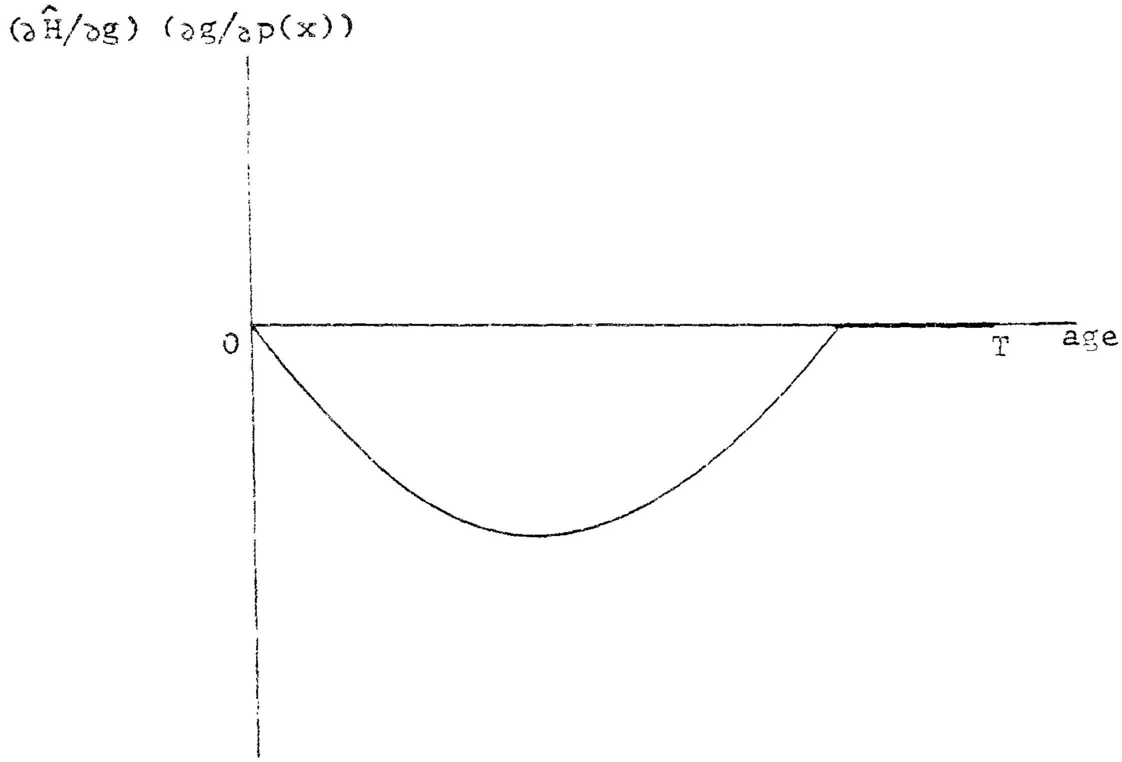
The full effect of the mortality improvement on the employment of labour is ambiguous in sign because there is no a priori reason to decide which of the two effects, namely $\partial\hat{H}/\partial p(x)$ or $[\partial\hat{H}/\partial g][\partial g/\partial p(x)]$ is the stronger.

8.7 CONCLUDING REMARKS

The effects of mortality improvements on the capital stock and the employment of labour have been studied under the assumption that the two factor prices, that is, the wage rate and rate of interest are exogenously given. The main conclusions are outlined below.

An increase in life expectancy may increase or decrease the capital stock depending on the nature of the mortality improvement. This result is obtained in a pure life cycle model and does not depend on any bequest motive. Thus Skinner's (1985) result that an increase in longevity

Figure 8.6.5 The Age Pattern of the Demographic Effect of an Increase in Survival Rates on the Employment of Labour



may reduce capital stock, is possible in a pure life cycle model without a contingent bequest motive.

Given the growth rate of population, with a hump shaped savings schedule, increase in survival rates at an old age will increase the normalized level of capital stock, as expected. However, the effect of an increase in survival rates at young age is to decrease the capital stock. Finally, the effect of an increase in survival rates at the earlier (later) part of middle age is to increase (decrease) the capital stock. Actual mortality improvements will, in general, increase survival rates at all the segments of life. Therefore, to find the effect of increased longevity on the capital stock, one would have to go into the details of changes in the life tables and calculate the net effect of all the changes in survival rates at various points of the horizon.

With the individual's labour supply schedule exogenously given, increases in survival rates during the working (retirement) period of life will increase (have no effect on) the normalized level of the employment of labour.

Traditional two period models, in which all the births take place in the first period and survival is uncertain only in the second period, ignore the demographic effect of a possible increase in the growth rate of population resulting from the increase in longevity. If the rates of reproduction are independent of the survival rates, this demographic effect in our model implies that increases in survival rates during the reproductive years of life will decrease the normalized levels of capital stock and labour. This adds further doubt to the presumed positive relation between longevity and capital accumulation.

CHAPTER 9

A NEO-CLASSICAL PRODUCTION FUNCTION, ENDOGENOUS FACTOR PRICES AND THE EFFECTS OF MORTALITY IMPROVEMENTS

9.1 INTRODUCTION

In this chapter we generalize our model by introducing a well behaved neo-classical production function. In all other respects the model considered is the same as that developed in the previous chapter. With this more realistic production technology, the two factor prices, the wage rate and rate of interest, become endogenous and are determined by the interaction of supply and demand in the factor markets. The consequence of this modification for the analysis of mortality improvements is that the changes in the supply of assets and labour brought about by the mortality improvements are somewhat offset by actions in the markets. With endogenous factor prices, the changes push the equilibrium wage rate and the rate of interest from their initial equilibrium positions. As a result, the equilibrium levels of employment of capital and labour will not change by as much as dictated by the changes in their respective supplies.

This more general model also enable us to study the effects of mortality improvements on the equilibrium wage rate and the rate of interest.

9.2 DETERMINATION OF FACTOR DEMANDS UNDER A NEO-CLASSICAL PRODUCTION ACTIVITY

The production function relating the two factor inputs, capital and labour to a singe output, $Y(t)$, which can be consumed or used for capital accumulation, is assumed to be linear homogeneous. Therefore we can write

$$Y(t) = H(t)\ f(k(t)) \tag{9.2.1}$$

where, $k(t)$ is the capital-labour ratio: $K(t)/H(t)$. The production function is assumed to satisfy the usual neo-classical properties of continuity. That is, the function $f(k)$ is continuous and has continuous first and second derivatives. The marginal products of capital and labour are assumed to be positive and diminishing. Therefore:

$$\partial Y(t)/\partial K(t) = f'(k(t)) > 0 \tag{9.2.2}$$

$$\partial Y(t)/\partial H(t) = f(k(t)) - k(t)\ f'(k(t)) > 0 \tag{9.2.3}$$

$$\partial^2 Y(t)/\partial K(t)^2 = f''(k(t))/H(t) < 0 \tag{9.2.4}$$

$$\partial^2 Y(t)/\partial H(t)^2 = k(t)^2\ f''(k(t))/H(t) < 0 \tag{9.2.5}$$

In addition, the production function is assumed to satisfy the following properties.

$$Y(t) = H(t)\ f(k(t)) = 0 \text{ if } H(t) = 0 \text{ or } K(t) = 0 \tag{9.2.6}$$

$$\lim_{k(t)\to 0} f'(k(t)) = \infty \tag{9.2.7}$$

$$\lim_{k(t)\to \infty} f'(k(t)) = 0 \tag{9.2.8}$$

The markets for output, capital and labour are assumed to be perfectly competitive. Under these circumstances, the demands for capital and labour are determined by equating the values of their marginal products to their respective prices. Therefore, normalizing the price of output at 1, we can write

$$f'(k) = r \tag{9.2.9}$$

$$f(k) - k\ f'(k) = w \tag{9.2.10}$$

Because of the assumption of a steady state, the two factor prices, and therefore, the capital-labour ratio are constant over time. That is why the time variable has been dropped from the above equations.

It is obvious that the two marginal products are functions of the capital-labour ratio. Therefore, for any pair of factor prices, it is not possible to determine the absolute levels of capital and labour demanded. However, we can find the capital-labour ratio demanded. The pair of factor prices which is compatible with a unique capital-labour ratio is given by the factor price frontier, obtained as follows. From equation (9.2.9) we can solve for the demand for the capital-labour ratio k as a function of r, that is,

$$k = k(r) \qquad (9.2.11)$$

with

$$dk(r)/dr = 1/f''(k) < 0 \qquad (9.2.12)$$

Substituting the demand function for k, (9.2.11), in equation (9.2.10), we obtain the factor price frontier:

$$w = f(k(r)) - k(r)\ f'(k(r)) \qquad (9.2.13)$$

This equation determines all possible pairs of w and r which are compatible with the given production technology. A specific pair of w and r which satisfies this equation, corresponds to a unique capital-labour ratio, whether determined by equation (9.2.9) or (9.2.10). The slope and curvature of factor price frontier are

$$dw/dr = -\ k(r) < 0 \qquad (9.2.14)$$

$$d^2w/dr^2 = -\ 1/f''(r) > 0 \qquad (9.2.15)$$

This completely describes the production side of the economy. The household side has already been discussed in detail in chapter 8. We can now discuss the possible existence of a general equilibrium and its properties.

9.3 A FEASIBLE COMPETITIVE SOLUTION

As we have already discussed, the absolute levels of demands for capital and labour by firms cannot be determined because of the linear homogeneity of the production function. However, a unique capital-labour ratio can be determined at any feasible pair of rate of interest and wage rate given by the factor price frontier. Under these circumstances, the equilibrium levels of the rate of interest and the wage rate can be determined simultaneously by the following conditions:

$$A^{*}(g,\ P,\ r,\ w) = k(r) \tag{9.3.1}$$

$$w = w(r) \tag{9.3.2}$$

where, $A^{*}(g, P, r, w)$ is the assets-labour supply ratio: $\hat{A}(g, P, r, w)/\hat{H}(g, P)$, $k(r)$ is the demand function for the capital-labour ratio (9.2.11) and $w = w(r)$ is the factor price frontier (9.2.13), written in compact form.[1]

The two equations (9.3.1) and (9.3.2) can be combined as follows.

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1 Conditions (9.3.1) and (9.3.2) are analogues of the functions: $w = \phi(r)$ and $r_{t+1} = \psi(w_t)$ in Diamond's (1965) two period model.

$$A^{*}(g, P, r, w(r)) = k(r) \tag{9.3.3}$$

Notice that this equation contains only one unknown, the rate of interest. If a solution exists, we can find the equilibrium rate of interest. Using this equilibrium rate of interest, we can also find the equilibrium wage rate from the factor price frontier (9.3.2) and the equilibrium capital-labour ratio from equation (9.2.11).

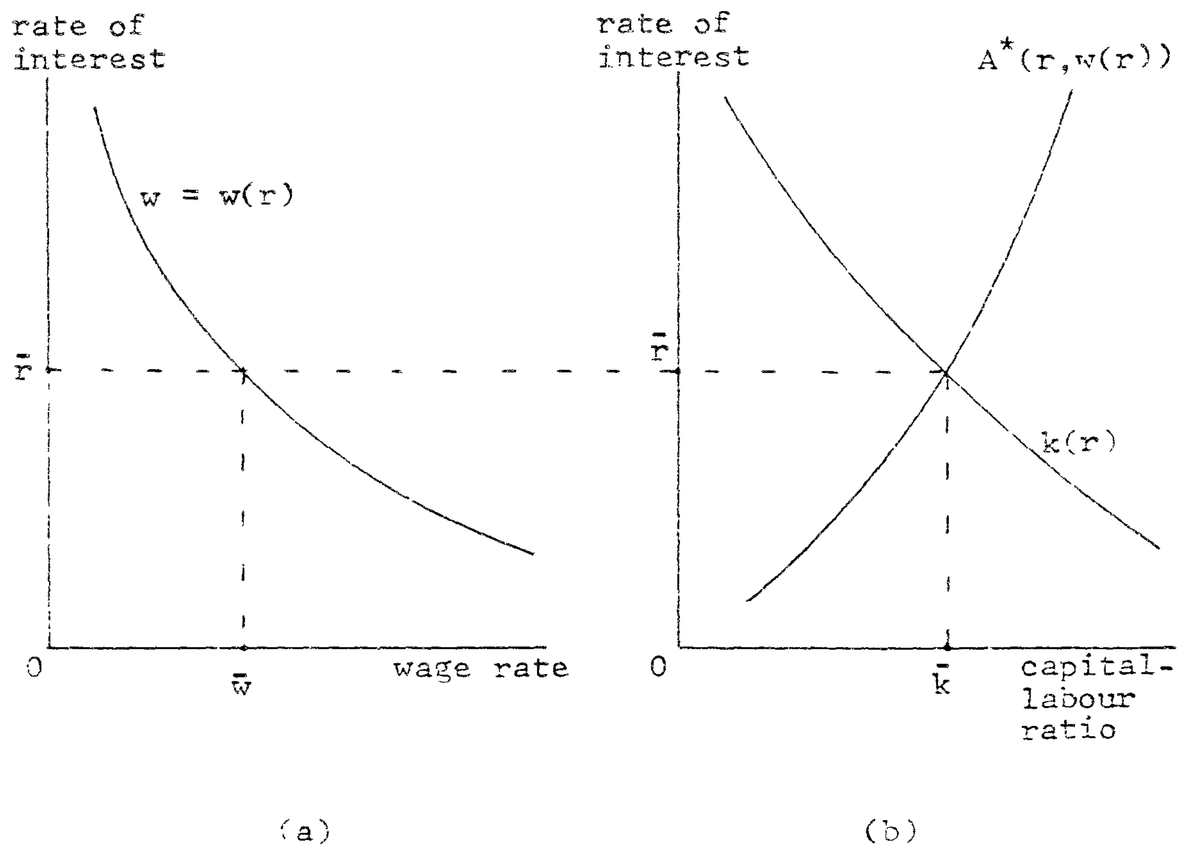
It is assumed that a unique and stable solution to equation (9.3.3) exists. The solution is illustrated in figure 9.3.1. The equilibrium levels of rate of interest, wage rate and capital-labour ratio are marked as $\bar{r}$, $\bar{w}$ and $\bar{k}$ respectively. In the following analysis we will provide a sufficient condition for the stability of equilibrium and discuss what this condition implies for our model.

We will follow Samuelson's (1943) correspondence principle to explore stability of equilibrium, assuming that a Walrasian adjustment mechanism is operative in the market. The excess demand function for the capital-labour ratio can be written as a function of rate of interest as follows:

$$E(r) = A^{*}(r, w(r)) - k(r) \tag{9.3.4}$$

Since g and P are only shift parameters, we have suppressed these from the excess demand function for the present

Figure 9.3.1 Determination of the Equilibrium Capital-Labour Ratio, the Rate of Interest and the Wage Rate with a Neo-Classical Production Function



analysis. Next, the Walrasian price adjustment equation can be written as

$$\dot{r} = R(E) \tag{9.3.5}$$

with $R'(E) > 0$.

Following the standard procedure, we now formulate an equation of motion for the endogenous variable, r, which describes the necessary and sufficient condition for stability in the Walrasian market.

$$d\dot{r}/dr < 0 \tag{9.3.6}$$

Using equations (9.3.4) and (9.3.5) this equation of motion can be solved as follows.

$$d\dot{r}/dr = R'(E)\ (dE/dr) \tag{9.3.7}$$

Since $R'(E) > 0$, $d\dot{r}/dr$ will be negative if and only if $dE/dr < 0$. That is, the excess demand function must be negatively sloped for Walrasian stability.

Let us now explore the conditions that are imposed on our model by this stability condition. In terms of the supply and demand functions for the capital-labour ratio, the stability condition can be written as follows.

$$dA^*/dr = (\partial A^*/\partial r) + (\partial A^*/\partial w)(dw/dr) > (dk/dr) \tag{9.3.8}$$

Since aggregate labour supply is independent of the rate of interest and the wage rate, and recalling that $A^{*} = \hat{A}/\hat{H}$, this condition can be further simplified as follows.

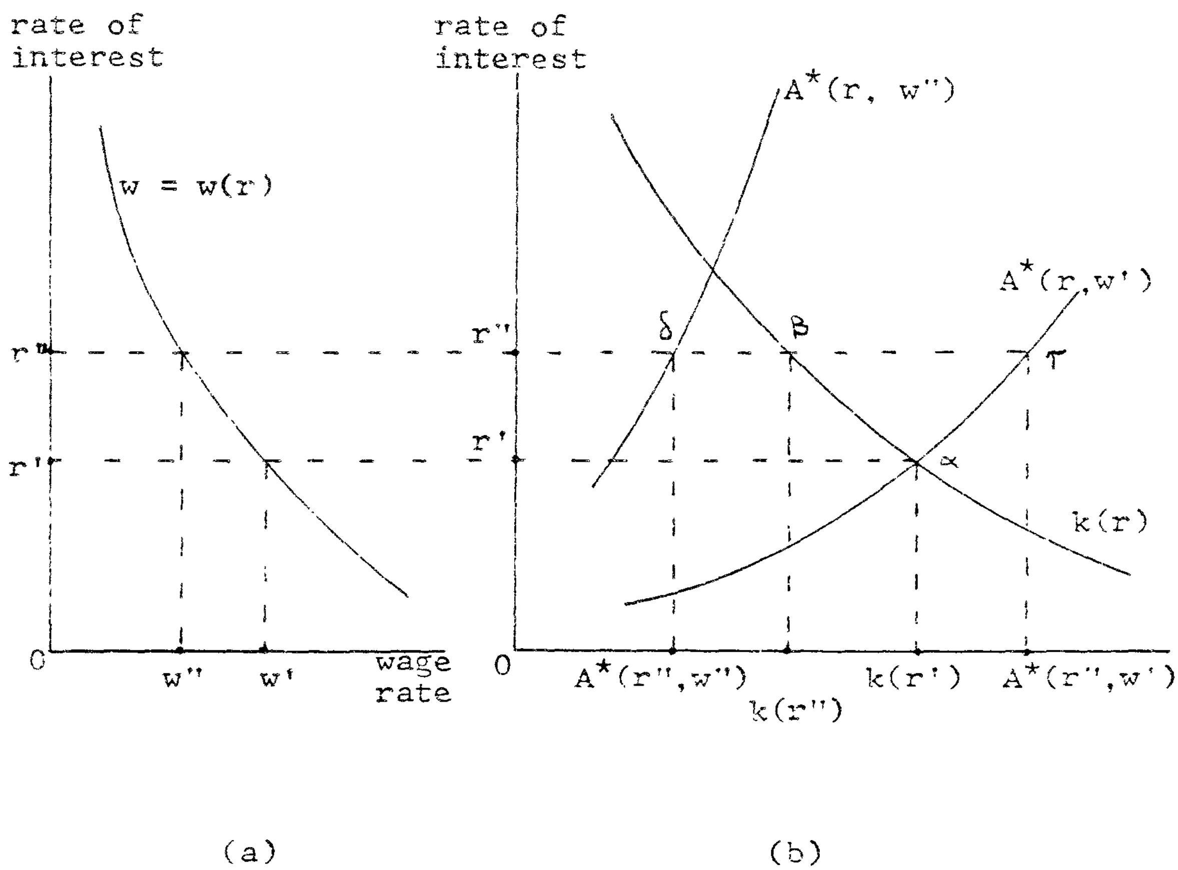
$$[1/\hat{H}]\ [(\partial\hat{A}/\partial r) + (\partial\hat{A}/\partial w)(dw/dr)] > (dk/dr) \qquad (9.3.9)$$

This means that the stability condition involves the total effect of an increase in rate of interest on the assets supply function including the partial effect $\partial\hat{A}/\partial r$ as well as the indirect effect through the change in wage rate which is implied by a higher rate of interest with the neoclassical production technology, that is, $(\partial\hat{A}/\partial w)(dw/dr)$. The derivatives dw/dr and dk/dr in the above stability condition are known to be negative. However, it does not seem possible to determine the signs of the other two derivatives $\partial\hat{A}/\partial r$ and $\partial\hat{A}/\partial w$ without parametrizing the household model. We can assume that $\partial A^{*}/\partial r$ is positive, but this condition alone is not sufficient for stability.[2] Figure 9.3.2 illustrates the situation. Suppose that the system is initially in an equilibrium position at point α, with the equilibrium rate of interest r' the wage rate w' and the

---

2 Using a continuous time framework such as ours, but with a different income pattern, Summers (1981) has performed a simulation analysis with a constant elasticity utility function and concluded that aggregate assets respond positively to the rate of interest, that is, $\partial A^{*}/\partial r$ is positive, for a wide range of the parameter values.

Figure 9.3.2 An Example of Unstable Equilibrium with a Positively Sloped Assets Supply Function



capital-labour ratio $k(r') = A^*(r', w')$. The function $A^*(r, w')$ has been drawn for the given wage rate w'. Now suppose that rate of interest is raised arbitrarily to r''. At this rate of interest, the demand for the capital-labour ratio, as determined at point G, is lowered to the level $k(r'')$. If the wage rate is held constant at w', the assets-labour supply ratio, as determined at point F, would be raised to the level $A^*(r'',w')$. One may argue that, since supply is more than demand, the rate of interest should be pushed down towards the equilibrium position. But, the argument is not complete.

The higher rate of interest r'' and the lower capital-labour ratio $k(r'')$ are compatible with a lower wage rate w'', not with w'. In other words, given the competitive conditions and linear homogeneous technology, firms will offer the wage rate w'' if they plan to hire capital and labour in the proportion $k(r'')$. This higher wage offer may shift the assets-labour supply function either way. For the sake of argument, we assume that the new assets-labour supply function $A^*(r, w'')$ crosses the horizontal line at r'' to the left of point G, at point H. This means that adjusting for the change in the wage rate, the full effect of an arbitrary increase in rate of interest to r'' is to decrease the assets-labour supply ratio to the level $A^*(r'', w'')$ which is less than the demand for capital-labour ratio $k(r'')$. Thus the rate of interest will be pushed even further away from the equilibrium position. The reason is that in this illustrative example, the assets-labour supply function with an endogenous wage rate, passing through points α and δ, is negatively sloped and less steep than the demand function for the capital-labour ratio. This illustrates the point that even if the derivative $\partial A^*/\partial r$ is positive, which would be a sufficient condition for stability if assets were not responsive to wage rate, there is no guarantee of stability.

9.4 GENERAL EQUILIBRIUM EFFECTS OF MORTALITY IMPROVEMENTS

We can now study the effects of mortality improvements on the equilibrium capital-labour ratio, the rate of interest and the wage rate, assuming that the equilibrium is stable. The equilibrium condition for the aggregate model is reproduced below:

$$A^*(g,\ P,\ r,\ w(r)) = k(r) \qquad (9.4.1)$$

It is obvious that the demand side of the model is independent of survival rates. This means that all the effect of a change in survival rates will come through a shift in the assets-labour supply ratio. Therefore, we first discuss the effect of an increase in survival rates on assets-labour supply ratio.

As in the previous chapter, the full effect of an increase in survival rate at any age x, on the assets-labour supply ratio can be split into two parts, the economic effect and the demographic effect, as follows:

$$dA^*/dp(x) = \partial A^*/\partial p(x) + [\partial A^*/\partial g][\partial g/\partial p(x)] \qquad (9.4.2)$$

We will discuss the two effects sequentially.

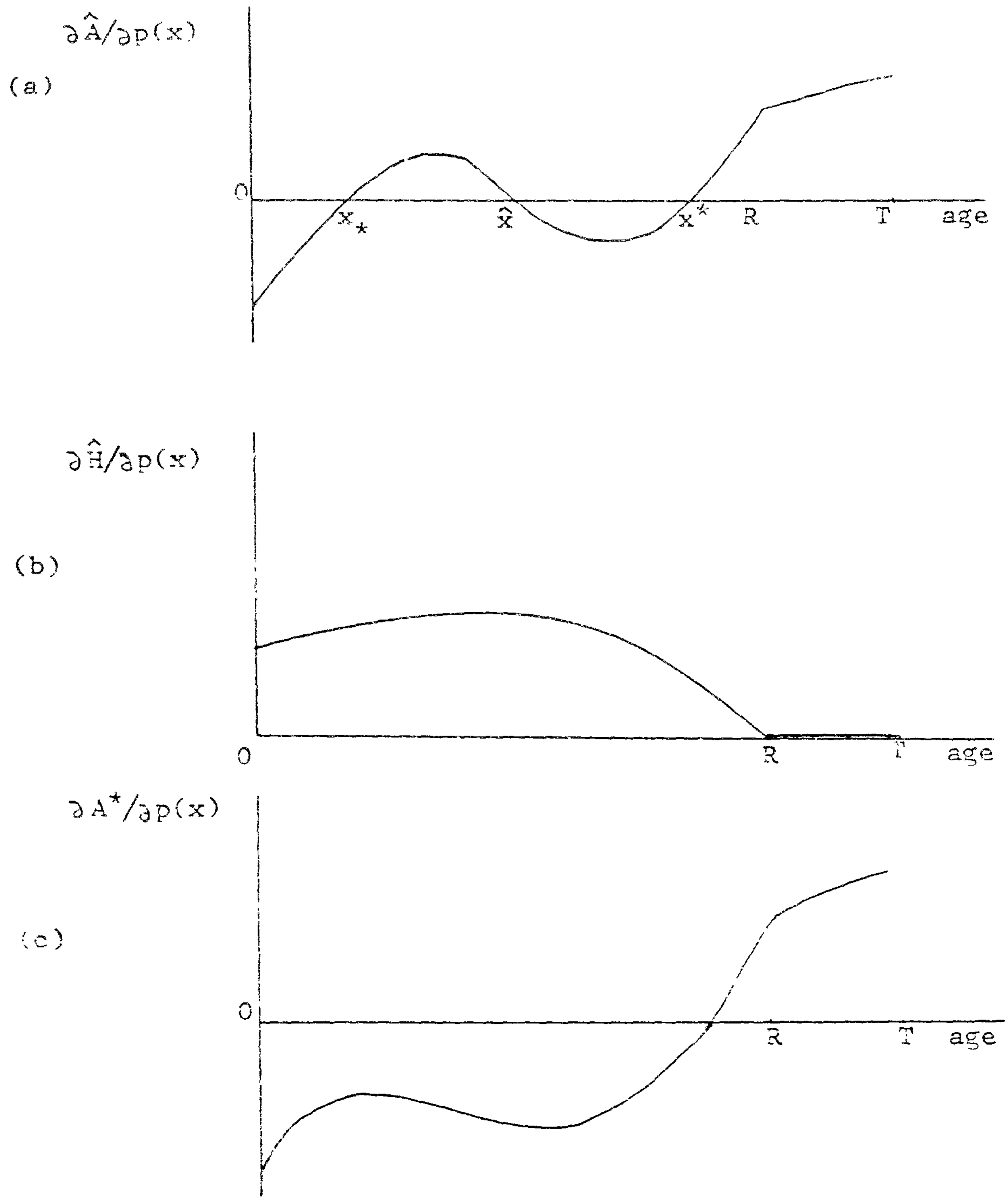
9.4.1 The Economic Effect of Mortality Improvements on the Assets-Labour Supply Ratio

Since $A^* = \hat{A}/\hat{H}$, the economic effect of an increase in survival rates at age x on A^* can be written as follows:

$$\partial A^*/\partial p(x) = A^* [\{\partial\hat{A}/\partial p(x)/\hat{A}\} - \{\partial\hat{H}/\partial p(x)/\hat{H}\}] \qquad (9.4.3)$$

The two derivatives $\partial\hat{A}/\partial p(x)$ and $\partial\hat{H}/\partial p(x)$ were evaluated in the previous chapter and their age patterns were displayed in figures 8.6.1(b) and 8.6.4 respectively. From these two derivatives we can obtain the age pattern of the derivative $\partial A^*/\partial p(x)$ by using the relation (9.4.3). This is done in figure 9.4.1. As shown in the figure, the age pattern of this derivative suggests that the effect of an increase in a survival rate at a young or middle age (an old age) is to decrease (increase) the assets-labour supply ratio. However, the effect of an increase in a survival

Figure 9.4.1 The Age Pattern of the Economic Effect of an Increase in Survival Rates on the Assets-Labour Supply Ratio



rate at some point in the middle of age x_* and age $\hat{x}$ is, in fact ambiguous.[3] But to simplify the diagram we assume that this effect is negative.[4]

9.4.2 The Demographic Effect of Mortality Improvements on the Assets-Labour Supply Ratio

This effect, given by the second term in equation (9.4.2), is reproduced below:

$$[\partial A^*/\partial g]\ [\partial g/\partial p(x)] \tag{9.4.4}$$

First consider the factor $\partial A^*/\partial g$. With $A^* = \hat{A}/\hat{H}$, we can write,

$$dA^*/dg = A^*\ [\{\partial\hat{A}/\partial g)/\hat{A}\} - \{\partial\hat{H}/\partial g)/\hat{H}\}] \tag{9.4.5}$$

The normalized levels of aggregate assets and labour supply, given by equations (8.6.18) and (8.6.26) respectively, are:

$$\hat{A} = \int_0^T e^{-gx}\ p(x)\ a(x)\ dx$$

$$\hat{H} = \int_0^T e^{-gx}\ p(x)\ j(x)\ h(x)\ dx$$

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3 Recall from chapter 8 (theorem 8.6.1 and figure 8.6.1) that at age $\hat{x}$, $\bar{z} = z(\hat{x})$.

4 For consistency, this simplification will be carried over to all the subsequent figures.

Taking the derivatives and substituting in equation (9.4.5), we find the following result.

$$\partial A^{*}/\partial g = - A^{*} (\bar{x}_a - \bar{x}_m) \tag{9.4.6}$$

where $\bar{x}_a$ and $\bar{x}_m$ are respectively the average ages of asset holding and labour income, defined as follows:

$$\bar{x}_a = \frac{\int_0^T x\, e^{-gx}\, p(x)\, a(x)\, dx}{\int_0^T e^{-gx}\, p(x)\, a(x)\, dx} \tag{9.4.7}$$

$$\bar{x}_m = \frac{\int_0^T x\, e^{-gx}\, p(x)\, m(x)\, dx}{\int_0^T e^{-gx}\, p(x)\, m(x)\, dx} \tag{9.4.8}$$

Notice that $\bar{x}_m$ is also the weighted average of the 'effective units' of labour supply. In defining this average we have multiplied the numerator and the denominator of equation (9.4.8) by the age zero wage rate, w, and used the relation $wj(x)h(x) = w(x)h(x) = m(x)$.

The two averages $\bar{x}_a$ and $\bar{x}_m$ differ because of their different weighting schemes. The weighting scheme based on labour income assigns more weight to the younger age as compared to the weighting scheme based on assets holding. Therefore, we expect that $\bar{x}_a$ will be greater than $\bar{x}_m$. Consequently, the derivative given by (9.4.6) is expected to be negative. A common sense explanation goes as follows. With an increase in the growth rate of population, the proportion of younger population increases and the proportion of older population decreases. From the age paths of assets holding and labour income shown in figure 8.2.1, we know that, in a broad sense, young people earn more labour income and hold less assets than the old. Therefore, with the change in the age distribution of population in the direction of relatively more young people, the assets-labour supply ratio will decrease.

Next, consider the factor $\partial g/\partial p(x)$. In the previous chapter, we concluded that this factor is positive (equal to zero) if the rate of reproduction, $b(x)$, is positive (equal to zero) at the age at which the survival rate is assumed to have increased. In particular, this derivative was shown by equation (8.6.22) to be as follows:

$$\partial g/\partial p(x') = \frac{e^{-gx'}\ b(x')}{\int_0^T x\ e^{-gx}\ p(x)\ b(x)\ dx}$$

Combining the two factors: $\partial A^*/\partial g$ and $\partial g/\partial p(x)$, we can conclude that

$$[\partial A^*/\partial g]\,[\partial g/\partial p(x)] \begin{cases} < 0 & \text{if } b(x) > 0 \\ = 0 & \text{if } b(x) = 0 \end{cases} \qquad (9.4.9)$$

Figure 9.4.2 illustrates the age pattern of this demographic effect. This figure has been drawn in the same way as figure 8.6.2 in the last chapter. In part (a) we plot the functions e^{-gx} and $b(x)$. The demographic effect, $[\partial A^*/\partial g][\partial g/\partial p(x)]$ is obtained in figure (b) by multiplying the derivative $\partial A^*/\partial g < 0$ (equation (9.4.6)) by the functions e^{-gx} and $b(x)$, and dividing by

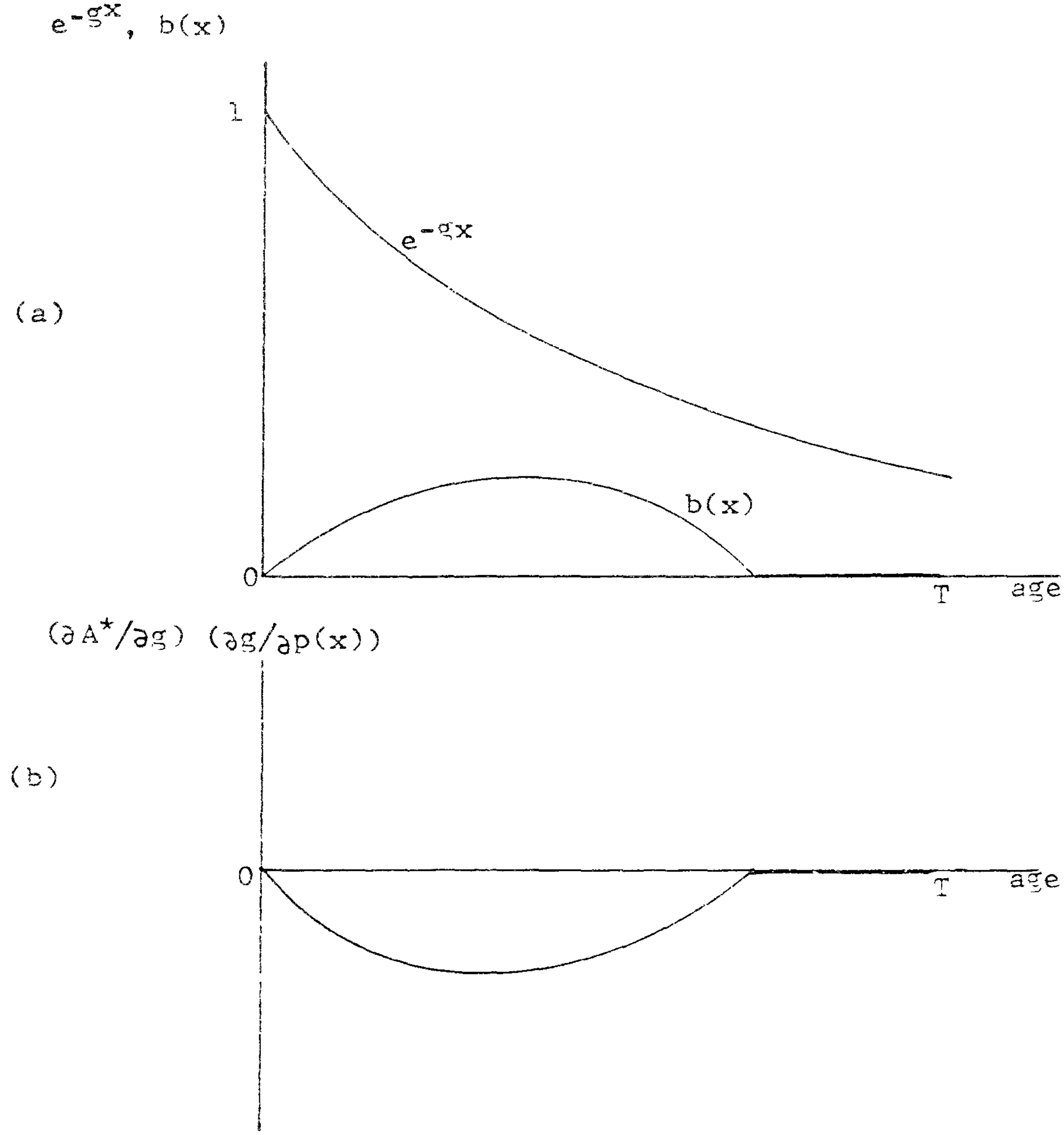
$$\int_0^T x\, e^{-gx}\, p(x)\, b(x)\, dx$$

Now we are in a position to combine the two effects to find the full effect of mortality improvement on assets-labour supply ratio.

9.4.3 The Full Effect of Mortality Improvements on the Assets-Labour Supply Ratio

To find the full effect of mortality improvements on the assets-labour supply ratio, we simply have to add the two components $\partial A^*/dp(x)$ and $[\partial A^*/\partial g][\partial g/\partial p(x)]$ as in equation (9.4.2). Thus by adding the two functions drawn in figures 9.4.1(c) and 9.4.2(b), we find the relationship between the age at which the survival rate is assumed to

Figure 9.4.2 The Age Pattern of the Demographic Effect of an Increase in Survival Rates on the Assets-Labour Supply Ratio



increase and the resulting change in the assets-labour supply ratio. This relationship, drawn in figure 9.4.3, shows that the effect of an increase in the survival rate at a young or middle (an old) age is to lower (raise) the assets-labour supply ratio.[5] In this figure we have assumed that the rates of reproduction become zero at age x^* at which the individual starts consuming more than labour income (that is, savings schedule $\sigma(x)$ becomes non-positive). But the general shape of the function $dA^*/dp(x)$ does not depend on this assumption as long as it is assumed that the rates of reproduction become zero somewhere during late years of life.

9.4.4 The Economic Effect of Mortality Improvements on the Equilibrium Rate on Interest, the Wage Rate and the Capital-Labour Ratio

The economic effects of mortality improvements on the equilibrium rate of interest can be found by differentiating the equilibrium condition (9.4.1) with respect

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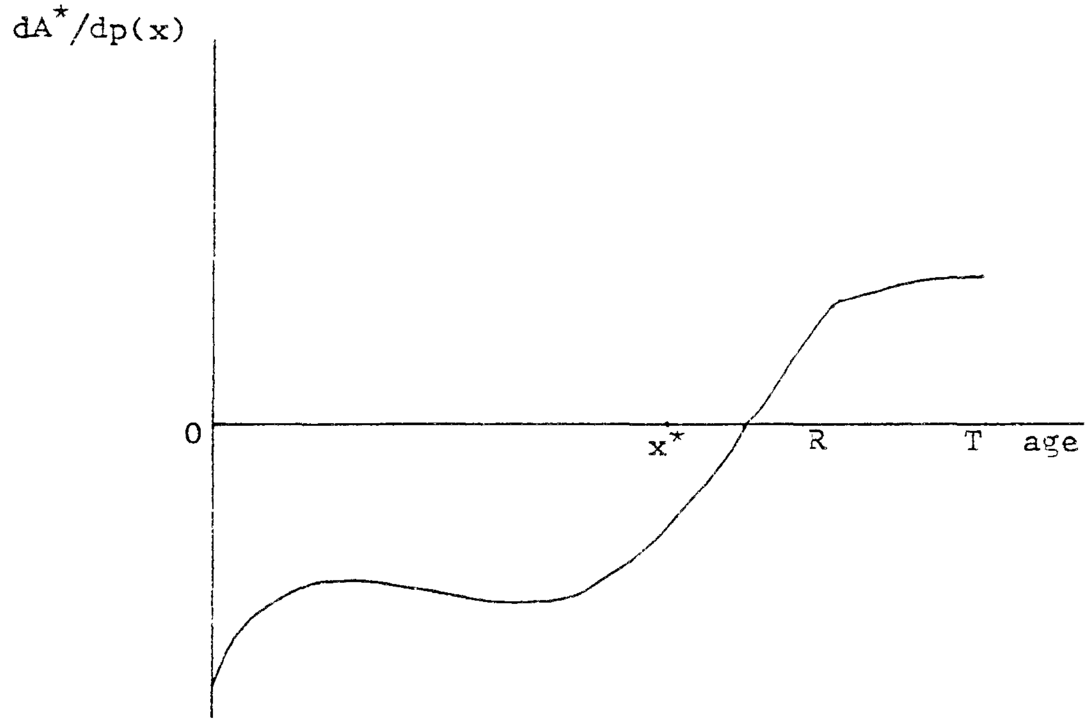
5 It may be recalled from sub-section 9.4.1 that the economic effect of an increase in a survival rate at some point in the middle of age x_* and age $\hat{x}$ is ambiguous. However, with the demographic effect being negative, the full effect of an increase in survival rate at such a point is more likely to be negative than just the economic effect.

Figure 9.4.3 The Age Pattern of the Full Effect of an Increase in Survival Rates on the Assets-Labour Supply Ratio



to the survival rate, $p(x)$, holding constant the growth rate of population, g. The result is as follows:

$$\partial r/\partial p(x) = \frac{-\ \partial A^*/\partial p(x)}{(dA^*/dr) - (dk/dr)} \tag{9.4.10}$$

Next, the effect on the equilibrium wage rate can be obtained by multiplying the slope of the factor price frontier, $dw/dr = -k$ (equation (9.2.14)), by the derivative $\partial r/\partial p(x)$, given above. The result is

$$\partial w/\partial p(x) = \frac{k(r)\ [\partial A^*/\partial p(x)]}{(dA^*/dr) - (dk/dr)} \tag{9.4.11}$$

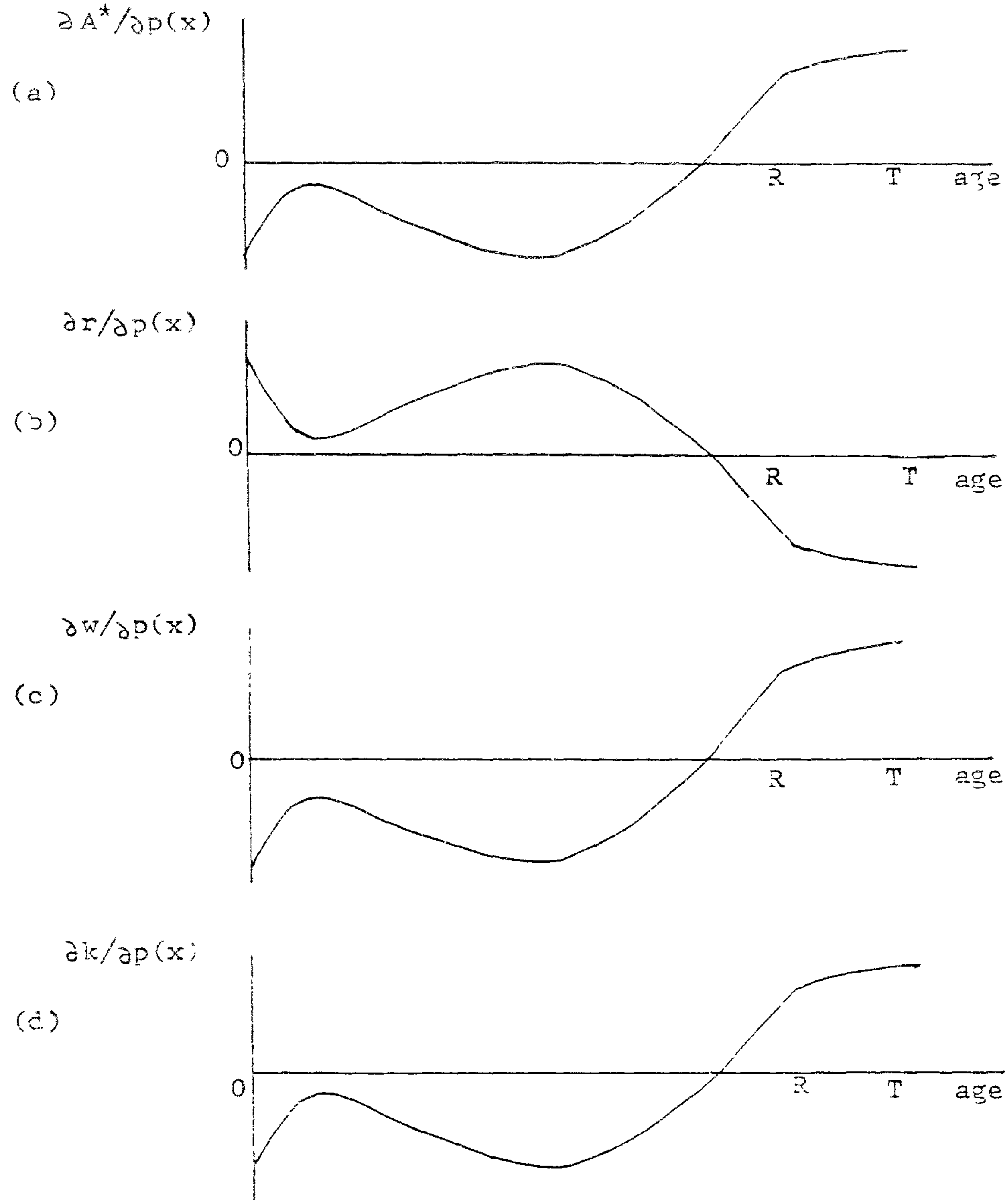
Finally, to find the effect on the equilibrium capital-labour ratio, we differentiate the demand function for the capital-labour ratio, $k(r)$, with respect to r and multiply the resulting derivative, by $\partial r/\partial p(x)$. Thus we obtain

$$\partial k/\partial p(x) = \frac{-(dk/dr)\ [\partial A^*/\partial p(x)]}{(dA^*/dr) - (dk/dr)} \tag{9.4.12}$$

The denominator on the right hand side of each of the above three equations, $(dA^*/dr) - (dk/dr)$, is positive by the stability condition (9.3.8). It is also know that $k(r)$ is positive and dk/dr is negative. It follows, therefore, from equations (9.4.10), (9.4.11) and (9.4.12) that the equilibrium rate of interest will change in the opposite direction to the change in assets-labour supply ratio and the equilibrium wage rate and capital-labour ratio will change in the same direction as the change in the assets-labour supply ratio. Thus, as shown in figure 9.4.4, the effect of an increase in the survival rate at a young or middle age is to increase the equilibrium rate of interest and decrease the equilibrium wage rate and capital-labour ratio. If, on the other hand, survival rate increases at an old age, the rate of interest will decrease and the wage rate and capital-labour ratio will increase.

It may be noted that the effect of mortality improvement on the equilibrium capital-labour ratio will be less (greater), in absolute terms, than the effect on the assets-labour supply ratio if the assets-labour supply function is positively (negatively) sloped. The reason is that the effect on the assets-labour supply ratio was evaluated under the assumption that the rate of interest and the wage rate are constant. Once these variables are allowed to change, a rightward (leftward) shift in assets-labour supply function will push down (up) the rate of interest. If the assets-labour supply function is positively sloped, the assets-labour supply ratio will decrease (increase), thus partly compensating the initial change in the assets-labour supply ratio. If, on the other hand, the assets-labour supply function is negatively sloped, the effect of a change

Figure 9.4.4 The Age Pattern of the Economic Effect of an Increase in Survival Rates on the Equilibrium Rate of Interest, the Wage Rate and the Capital-Labour Supply Ratio



in rate of interest on the assets-labour supply ratio will further re-enforce the initial change in the assets-labour supply ratio. But in any case, if the solution is stable in a Walrasian sense, endogenizing factor prices will not reverse the direction of the effect of a mortality improvement on the capital-labour ratio.

9.4.5 The Demographic Effect of Mortality Improvements on the Equilibrium Rate of Interest, the Wage Rate and the Capital-Labour Ratio

The demographic effect of a mortality improvement on the equilibrium rate of interest, the wage rate and the capital-labour ratio can be obtained in the same way as the economic effect. The only difference is that we start with differentiating the equilibrium condition (9.4.1) with respect to g and then multiply the resulting differential by the derivative $\partial g/\partial p(x)$. The results are as follows:

$$[\partial r/\partial g][\partial g/\partial p(x)] = \frac{-\,[\partial A^*/\partial g][\partial g/\partial p(x)]}{(dA^*/dr) - (dk/dr)} \qquad (9.4.13)$$

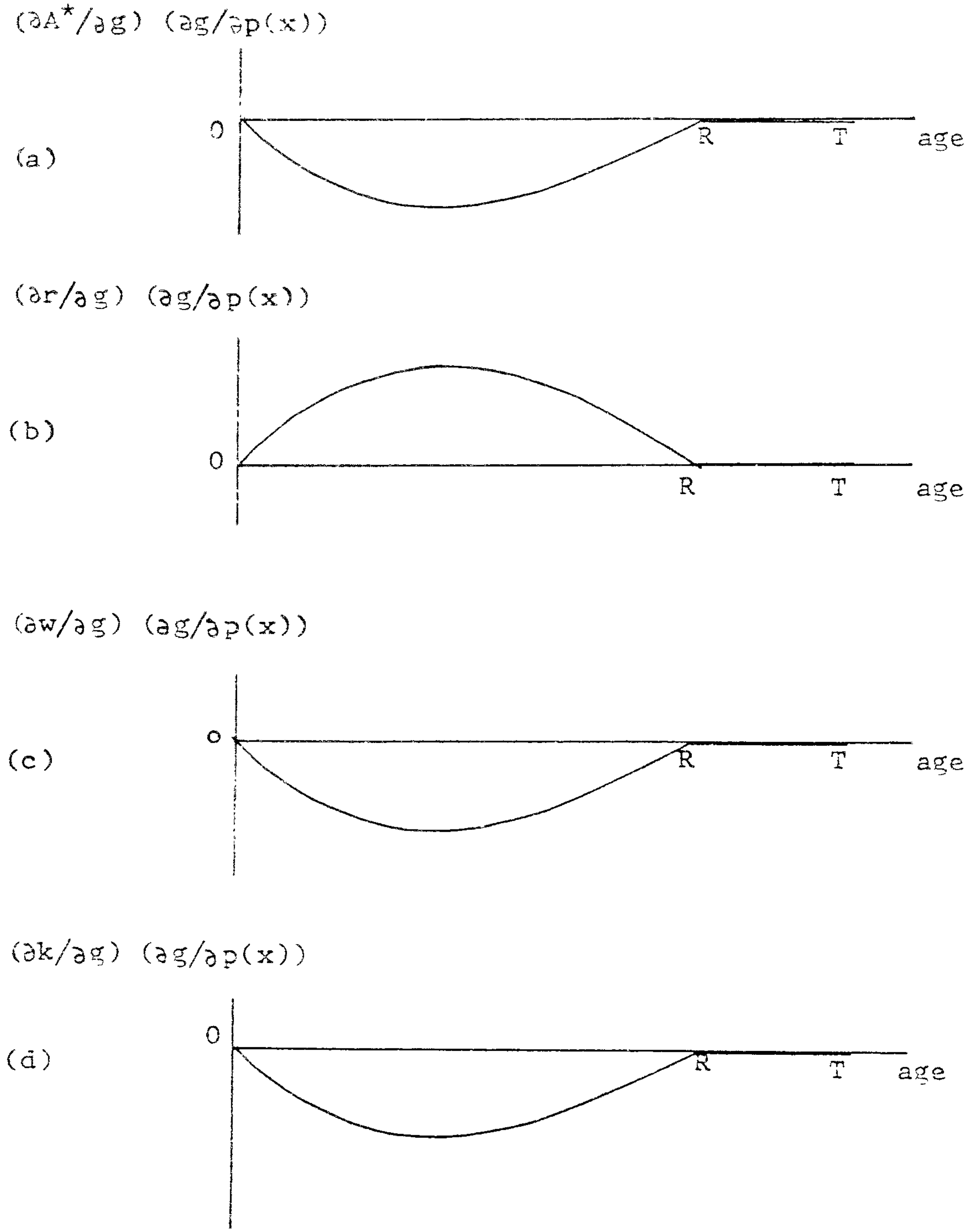
$$[\partial w/\partial g][\partial g/\partial p(x)] = \frac{k(r)\;[\partial A^*/\partial g][\partial g/\partial p(x)]}{(dA^*/dr) - (dk/dr)} \qquad (9.4.14)$$

$$[\partial k/\partial g][\partial g/\partial p(x)] = \frac{-\,[dk/dr][\partial A^*/\partial g][\partial g/\partial p(x)]}{(dA^*/dr) - (dk/dr)} \qquad (9.4.15)$$

The age profiles of these effects of mortality improvement are shown in figure 9.4.5. It may be noted that the demographic effect of a mortality improvement on the equilibrium rate of interest, the wage rate and the capital-labour ratio depends on the age at which the survival rate is assumed to have increased (which determines the resulting change in the growth rate of population), the difference between the mean ages of asset holding and labour income (which determines the effect of a given increase in the growth rate of population on the assets-labour supply function), and the slopes of the functions $A^*(g, P, r, w(r))$ and $k(r)$. Figure 9.4.5 illustrates the relationship between the age at which survival rate is assumed to increase and the resulting change in the rate of interest, the wage rate and the capital-labour ratio, given the magnitudes of the other parameters involved. This relationship essentially depends on the effect of an increase in the survival rate on the growth rate of population.

In the following analysis we discuss how a given increase in the growth rate of population relates to the resulting change in the equilibrium rate of interest. We argue that for a given increase in the growth rate of population, the change in the equilibrium rate of interest could be small or big in size depending upon the magnitudes of the parameters involved. This aspect has an important bearing

Figure 9.4.5 The Age Pattern of the Demographic Effect of an Increase in Survival Rates on the Equilibrium Rate of Interest, the Wage Rate and the Capital-Labour Supply Ratio



on the classification of a competitive equilibrium as efficient or inefficient. Figure 9.4.6 shows how a given increase in the growth rate of population may convert an inefficient equilibrium into an efficient one and vice versa.[6] In figures (a) and (b) the competitive equilibrium is initially inefficient with the growth rate of population g' and the rate of interest r'. In both cases the growth rate increases by the same margin (to g*) but the assets-labour supply function $A^*(g, P, r, w(r))$ shifts by a bigger margin in figure (b) than in figure (a) because of a wider difference between the mean ages of asset holding and labour income. The result is that in figure (b), the increase in the rate of interest is big enough to overshoot the increase in the growth rate and therefore the final competitive equilibrium solution becomes efficient. In figure (a), on the other hand, with a modest increase in the rate of interest, the solution remains inefficient.

In figure (c) the solution is initially efficient and remains so after the increase in the growth rate of population. Finally, in figure (d) the solution is initially efficient and becomes inefficient after the increase in the

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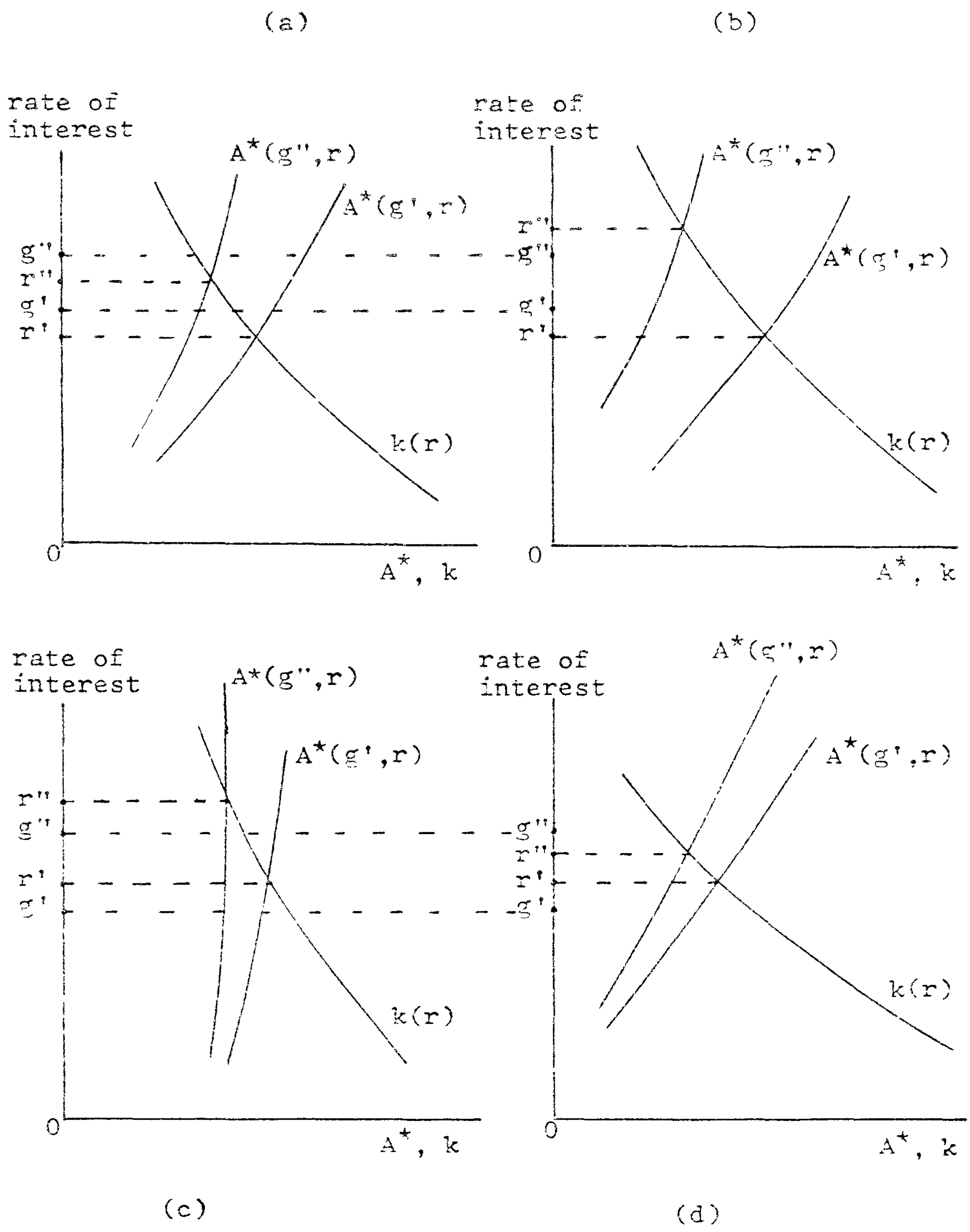
6 Recall from chapter 8 that a competitive equilibrium is classified as inefficient if $g > r$ and as efficient if $g \leq r$ (Gale (1973), Shell (1971) and Starrett (1972)).

Figure 9.4.6 The Demographic Effect of Mortality Improvements and the Classification of a Competitive Equilibrium as efficient or Inefficient



growth rate of population. For illustrative purpose the different outcomes in figures (c) and (d) are accounted for by different slopes of the two functions $A^*(g, P, r, w(r))$ and $k(r)$ across the two cases. The shift in the function $A^*(g, P, r, w(r))$, measured by the horizontal distance is considered equal across the two cases.

In the above analysis we have discussed the relationship between the growth rate of population and the equilibrium rate of interest. We know that the growth rate of population can be changed only by changing the survival rates or the rates of reproduction. In the above analysis we have considered only the demographic effect of an increase in survival rates on the equilibrium rate of interest. Alternatively, the increase in the growth rate of population can be viewed as being the result of an increase in reproduction rates rather than an increase in survival rates.

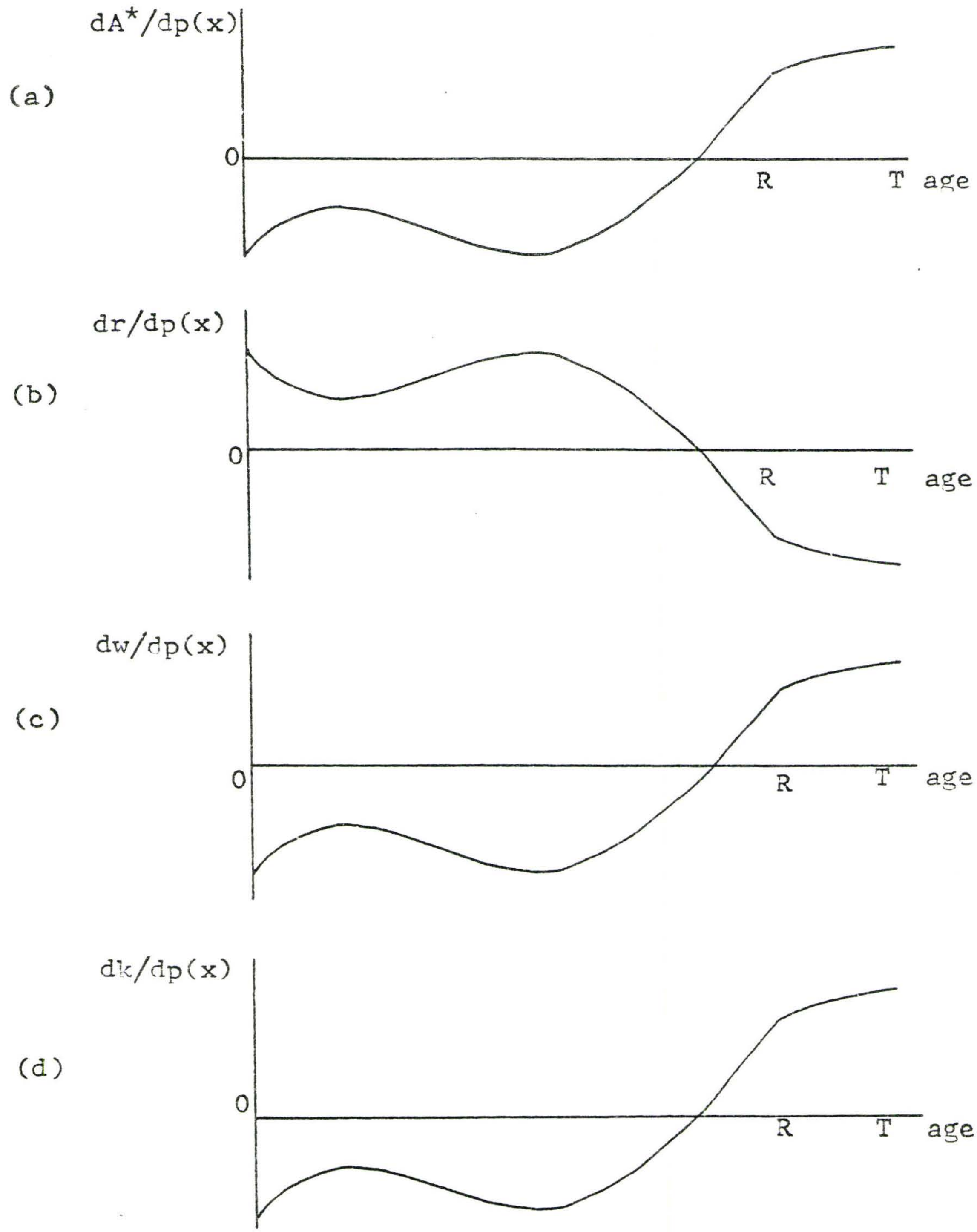
Another reason to consider only the demographic effect of an increase in survival rates in the above analysis is to isolate the analysis from normative implications of mortality changes. The economic effect of mortality improvements also involves changes in the lifetime utility function due to the changes in survival probabilities. Thus, if we consider the full effects of mortality improvements, then, for example, an efficient solution may have been converted into an inefficien one while at the same time the steady state utility may have increased for all generations due to the changes in lifetime utility function as a result of better survival probabilities. In this case a transition into an inefficient equilibrium from an efficient one may have occurred along with a Pareto improvement rather than a Pareto deterioration. We will discuss this issue in more detail in the next sub-section.

In next section we combine the economic and demographic effects of mortality improvements and study the implications of mortality improvements for the classification of a competitive equilibrium as efficient or inefficient.

9.4.6 The Full Effect of Mortality Improvements on the Equilibrium Rate of Interest, the Wage Rate and the Capital-Labour Ratio

The full effect of mortality improvements can be obtained by adding equations (9.4.10) and (9.4.13) for the effects on the equilibrium rate of interest, equations (9.4.11) and (9.4.14) for the effects on the wage rate and equations (9.4.12) and (9.4.15) for the effects on the capital-labour ratio. Figure 9.4.7 shows the age pattern of these effects.

Figure 9.4.7 The Age Pattern of the Full Effect of an Increase in Survival Rates on the Equilibrium Rate of Interest, the Wage Rate and the Capital-Labour Supply Ratio



Once again we can discuss the implications of mortality improvements for the classification of competitive equilibrium as efficient or inefficient. An increase in survival rates at an old age, when the rates of reproduction are equal to zero, has no effect on the growth rate of population, but it will result in a lower equilibrium rate of interest. Therefore, if the competitive equilibrium is initially inefficient, it must remain so after the mortality improvement but if the equilibrium is initially efficient, the mortality improvement may carry it into the inefficient region. On the other hand, an increase in survival rates at an earlier age, when the rates of reproduction are positive, will raise both the equilibrium rate of interest and the growth rate of population. Since the change in the rate of interest can be smaller or larger than the change in the growth rate depending on the parameters involved, the mortality improvement has no predictable consequence on the classification of the competitive equilibrium as efficient or inefficient.

Again we remind ourselves that the classification of a competitive equilibrium as efficient or inefficient holds with given tastes (which also include probabilities of survival) and technology, etc. If a mortality improvement converts an efficient equilibrium into an inefficient one, it opens the possibility of a Pareto improvement (for example, by an intergenerational transfer scheme), although the mortality improvement by itself may have resulted in all generations being better off. Likewise, if a mortality improvement converts an inefficient equilibrium into an efficient one, it eliminates the possibility of a Pareto improvement, irrespective of the effect of mortality improvement on the welfare of different generations.

This discussion implies that a conversion of an efficient equilibrium into an inefficient one due to a mortality improvement does not necessarily imply a welfare loss. The new equilibrium is inefficient in the new state (after the change in survival probabilities and, hence, in the utility function). Similarly a conversion of an inefficient equilibrium into an efficient one due to a mortality improvement does not necessarily imply a welfare gain. The new equilibrium is efficient in the new state (after the change in the utility function).

It seems that no definite welfare implications can be attached with the analysis of mortality improvements. The change in the equilibrium rate of interest (or the capital-labour ratio) cannot be associated with a welfare change unless an explicit analysis of the effects of mortality improvements on the welfare of different generations is conducted. At this point, we leave this subject for future research.

9.5 CONCLUDING REMARKS

A linear homogeneous production function has been added to the household model to examine the competitive equilibrium. Due to the linear homogeneity of the production function, the market equilibrium condition is defined only in terms of the capital-labour ratio. Equilibrium is described by a technologically feasible pair of rate of interest and the wage rate at which the demand for the capital-labour ratio is equal to the assets-labour supply ratio.

The effects of mortality improvements on the equilibrium rate of interest, the wage rate and the capital-labour ratio have been studied under competitive conditions, assuming that the equilibrium is stable in Walrasian sense. As in the previous chapter, we have found that these effects depend, among other things, on the age at which the survival rate is assumed to increase. Given the growth rate of population, with a hump-shaped savings schedule, an increase in survival rates at an old age will increase the equilibrium capital-labour ratio and the wage rate and decrease the equilibrium rate of interest. On the other hand, the effect of an increase in survival rates at earlier segments of life is the opposite.[7] Thus Skinner's (1985) result that an increase in life expectancy may reduce the capital-labour ratio, can be found in a pure life cycle model and there is no need to introduce contingent bequests to obtain this result.

Like in the previous chapter, we have also studied the effect of an increase in the growth rate of population induced by mortality improvements. This effect shows that an increase in survival rates during the reproductive years of life will decrease the equilibrium capital-labour ratio and the wage rate and increase the equilibrium rate of interest.

Finally, the implications of mortality improvements for the classification of competitive equilibrium as efficient or inefficient are noted. An intermediate result is that, ignoring the economic effect of variations in survival rates, a higher growth rate of population has no implications for the classification of equilibrium as efficient or inefficient. In particular, an economy with a slower growth rate of population may be even further away from an effi-

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7 If work hours at certain points in the middle of life horizon are low, an increase in the survival rates at that segment may also result in an increase in the equilibrium capital-labour ratio. But this segment is likely to be a small fraction of the entire horizon.

cient equilibrium than an economy with a faster growth rate, though the reverse may also be true.

An increase in survival rates at an old age leaves the growth rate of population unchanged, but it lowers the equilibrium rate of interest. Thus, if the competitive equilibrium is initially inefficient, it will remain so after the mortality improvement and if the equilibrium is initially efficient, the mortality improvement may carry it into the inefficient region. An increase in survival rates at an earlier age raises both the growth rate of population and the rate of interest. In this case the mortality improvement carries no implications for the classification of equilibrium as efficient or inefficient.

The classification of a competitive equilibrium as efficient or inefficient holds with given tastes (which also include survival probabilities) and technology, etc. A conversion of an efficient equilibrium into an inefficient one due to a mortality improvement does not necessarily imply a welfare loss. The new equilibrium is inefficient in the new state (after the change in survival probabilities and, hence, in the utility function). Similarly a conversion of an inefficient equilibrium into an efficient one due to a mortality improvement does not necessarily imply a welfare gain. The new equilibrium is efficient in the new state (after the change in the utility function).

CHAPTER 10

THE MODEL WITH ENDOGENOUS FACTOR PRICES, ENDOGENOUS WORK HOURS AND THE EFFECTS OF MORTALITY IMPROVEMENTS

10.1 INTRODUCTION

In this chapter we consider a general equilibrium model in which the individual's labour supply schedule is endogenous. In all other respects the model is similar to the one presented in the last chapter. This is a significant departure from the conventional literature on overlapping generations model in which an individual's work habits are assumed to be fixed. The study of mortality variations is a long run issue and it seems to be a strong assumption that these demographic changes will not induce individuals to change their work habits.

The construction of our model in this chapter differs only slightly from previous chapters. However, the individual's life cycle model which underlies the analysis is different because, unlike in the previous analysis, it contains the work-leisure choice as an endogenous decision. Due to this difference we will have to restate, briefly, the behaviour of the household model and the nature of general equilibrium.

10.2 THE HOUSEHOLD MODEL WITH ENDOGENOUS WORK HOURS AND A FEASIBLE COMPETITIVE SOLUTION

The formation of aggregate household variables and their accounting identities are independent of any assumption regarding household behaviour. However their economic determination does depend on the underlying household behaviour. In the present context we have to modify the household model developed in chapter 8 to make it consistent with the individual's life cycle model of chapter 3.

According to our analysis in chapter 3, an individual's consumption, work and narrow savings schedules can be written in terms of the following functions.

$$c(x) = c(x, P, r, W)$$
$$h(x) = h(x, P, r, W)$$
$$\varsigma(x) = \varsigma(x, P, r, W)$$

Since $W = w\,J$ and J can be suppressed from these functions as being an age path of fixed parameters, we can write these functions as follows.

$$c(x) = c(x, P, r, w)$$
$$h(x) = h(x, P, r, w)$$
$$\varsigma(x) = \varsigma(x, P, r, w)$$

Substituting these function into the aggregate variables we get the same results as in chapter 8 except that the aggregate labour supply function becomes:

$$H(t) = B(t) \int_0^T e^{-gx}\, p(x)\, j(x)\, h(x, P, r, w)\, dx \qquad (10.2.1)$$

Or, dividing by the number of current births, B(t), we can write this function in compact form as follows:

$$\hat{H} = \hat{H}(g, P, r, w) \qquad (10.2.2)$$

As in chapter 9, we assume that production possibilities are described by a neo-classical linear homogeneous production function. Under these conditions an equilibrium in the two factor markets is characterized by the conditions:

$$A^*(g, P, r, w(r)) = k(r) \qquad (10.2.3)$$

$$w = w(r) \qquad (10.2.4)$$

As before we assume that there exists a unique and stable equilibrium. Notice that the stability condition in the present model is a little more complicated than before. In chapter 9 we concluded that in a Walrasian market the equilibrium is stable if and only if

$$dA^*/dr = (\partial A^*/\partial r) + (\partial A^*/\partial w)(dw/dr) > (dk/dr) \qquad (10.2.5)$$

With a fixed work schedule of an individual, this condition contained only the derivatives of the assets supply function with respect to rate of interest and wage rate (see equation

(9.3.9)). Now with an endogenous work decision, this condition must be spelled out in terms of the derivatives of both assets and labour supply functions as follows:

$$A^{*} \, [\{(\partial\hat{A}/\partial r) + (\partial\hat{A}/\partial w)(dw/dr)\}/\hat{A}$$
$$- \{(\partial\hat{H}/\partial r) + (\partial\hat{H}/\partial w)(dw/dr)\}/\hat{H}] > (dk/dr) \qquad (10.2.6)$$

Due to complex nature of the comparative static results of the individualistic model in chapter 4 it does not seem possible to determine the signs of the partial derivatives $\partial\hat{A}/\partial r$, $\partial\hat{A}/\partial w$, $\partial\hat{H}/\partial r$ and $\partial\hat{H}/\partial w$. Therefore, we cannot examine the feasibility of the above stability condition. We simply assume that this condition is met by our model. We are now ready to conduct the analysis of mortality improvements in this model.

10.3 THE EFFECTS OF MORTALITY IMPROVEMENTS

The full effect of an increase in survival rate at age x on the assets-labour supply ratio has been expressed in terms of two additive components by equation (9.4.2). This equation is reproduced below:

$$dA^{*}/dp(x) = \partial A^{*}/\partial p(x) + [\partial A^{*}/\partial g][\partial g/\partial p(x)] \qquad (10.3.1)$$

First we find the component: $\partial A^{*}/\partial p(x)$ which we refer to as the economic effect.

With an endogenous work schedule of an individual, the analysis becomes more complicated than before. We are now unable to determine the direction of change in the assets-labour supply ratio in response to an increase in survival rates at most ages by directly evaluating the relevant derivatives. In the following analysis we discuss the basic problems involved in this regard. However later in this section we will adopt an indirect approach in our effort to solve this problem by using the accounting relationships among the household variables.

As in equation (9.4.3), the economic effect of an increase in survival rate at age x on the assets-labour supply ratio, that is, $\partial A^*/\partial p(x)$ can be written as

$$\partial A^*/\partial p(x) = A^* \, [\{\partial \hat{A}/\partial p(x)/\hat{A}\} - \{\partial \hat{H}/\partial p(x)/\hat{H}\}] \qquad (10.3.2)$$

As in chapter 8, the derivative $\partial \hat{A}/\partial p(x)$ can be expressed in terms of the following function.

$$\partial \hat{A}/\partial p(x) = e^{-rx} \, \varepsilon(x) \, [\bar{z} - z(x)] \qquad (10.3.3)$$

with the same properties as before (see expressions (8.6.10) through (8.6.14) and the related discussion in chapter 8)). In particular, the weighted average $\bar{z}$ is independent of the age at which the survival rate is assumed to have increased. This average was defined in chapter 8 as below (see equation (8.6.11)).

$$\bar{z} = \overline{e^{(r-g)x}}/(r-g) \qquad \text{for} \quad g \neq r$$
$$\quad = \bar{x} \qquad\qquad\qquad \text{for} \quad g = r \tag{10.3.4}$$

where, $\overline{e^{(r-g)x}}$ is the weighted average defined below (see equation (8.6.7)):

$$\overline{e^{(r-g)x}} = \frac{\int_0^T e^{(r-g)x}\, e^{-rx}\, p(x)\, [\partial\sigma(x)/\partial p(x^*)]\, dx}{\int_0^T e^{-rx}\, p(x)\, [\partial\sigma(x)/\partial p(x^*)]\, dx} \tag{10.3.5}$$

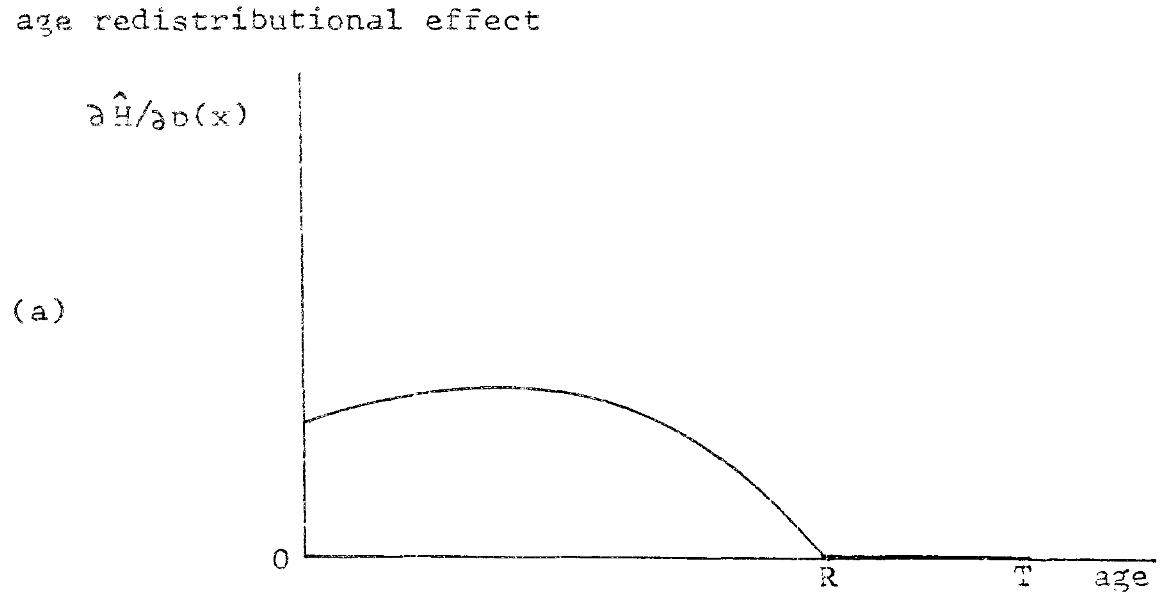
For $g = r$, $\bar{z}$ converges to $\bar{x}$ which is calculated by using the same weighting scheme as used in (10.3.5). From equation (4.3.18) of chapter 4 we can see that the derivative $\partial\sigma(x)/\partial p(x^*)$ depends on the age x^* only through a factor of proportionality : $e^{-rx^*}\, \sigma(x^*)$ which cancels out from the numerator and the denominator of (10.3.5). Therefore, as in chapter 8, this weighted average is independent of x^*, the age at which the survival rate is assumed to have increased.

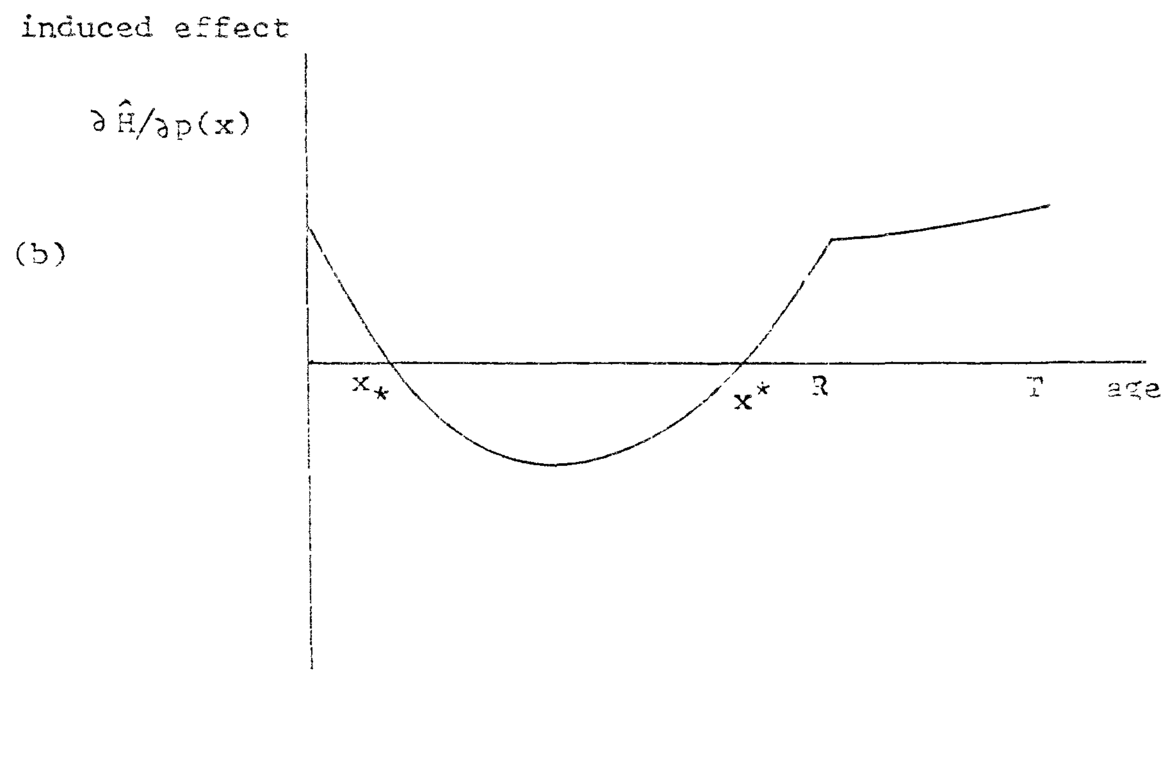
The derivative $\partial\hat{A}/\partial p(x) = e^{-rx}\, \sigma(x)\, [\bar{z} - z(x)]$ follows the same age pattern as in chapter 8. Therefore there does not seem to be any problem in evaluating the effect of mortality improvements on assets supply.

The effect of an increase in survival rates at some ages on labour supply is uncertain in sign, however. The reason is as follows. Given the work schedule of an individual, an increase in the survival rate during the working period of life will increase aggregate labour supply by increasing the size of working population, as shown in figure 8.6.4. This effect may be called the age redistributional effect. With an endogenous work schedule, the increase in the survival rate (or survival probability) will also induce individuals to change their work schedule. According to our analysis in chapter 4 an increase in survival rates below age x_* or above age x^* (between age x^* and age x_*) when the savings rate $\tau(x)$ is negative (positive), will induce more (lesser) work hours throughout the life cycle. We call this effect the induced effect. The age pattern of the two effects is shown in figure 10.3.1. It is obvious that, for an increase in survival rates at the early or later part of life, the sign of these two effects combined is to increase aggregate labour supply. However, the effect of an increase is survival rates in the middle of the segment x_* to x^* is ambiguous in sign.

By directly evaluating the derivatives, it does not seem to be possible to determine the effect of mortality improvement on assets-labour supply ratio except for an increase in survival rates below age x_* which will result in less assets and more labour supply. Therefore we need some additional information to determine the age pattern of the

Figure 10.3.1 The Age Pattern of the Age Redistributional Effect and the Induced Effect of an Increase in Survival Rates on Aggregate Labour Supply





derivative $\partial A^*/\partial p(x)$. The following two results will be helpful in this regard. The proofs are given in the appendix (section 10.A).

Theorem 10.3.1

The sum of the derivative $\partial A^*/\partial p(x)$, weighted by the probability of survival over the entire life cycle is zero. That is

$$\int_0^T p(x)\,[\partial A^*/\partial p(x)]\,dx = 0 \qquad (10.3.6)$$

Theorem 10.3.2

The sign of the derivative $\partial\{\hat{A}/\hat{H}\}/\partial p(x)$ is the same as that of $\partial\{\hat{A}/\hat{C}\}/\partial p(x)$. In particular the two derivatives are interrelated as follows.

$$\partial\{\hat{A}/\hat{H}\}/\partial p(x) = w\{\hat{C}/w\hat{H}\}^2\ \partial\{\hat{A}/\hat{C}\}/\partial p(x) \qquad (10.3.7)$$

Now we proceed to determine the sign of the derivative $\partial\{\hat{A}/\hat{H}\}/\partial p(x)$. We do this in a step by step process for various segments of life.

Theorem 10.3.3

$\partial\{\hat{A}/\hat{H}\}/\partial p(x) < 0$ for $x = x_*$, $x = x^*$ and $x = \dot{x}$, where $\dot{x}$ is the age such that $\bar{z} = z(\dot{x})$.

Proof

At age x_* and age x^* the savings rate $j(x) = 0$. Also, by definition at age $\hat{x}$, $\bar{z} = z(\hat{x})$. Equation (10.3.3) implies that $\partial\hat{A}/\partial p(x) = 0$ at all these ages. Next, consider the effect on labour supply. According to the comparative static results found in chapter 4, any change in the survival rate at age x_* and age x^* will not induce an individual to change his/her work plan. The only effect remaining, therefore, is due to the resulting change in the age distribution of population. This effect is given by $e^{-gx}\ j(x)\ h(x)$. At each of these two ages the savings rate $j(x)$ is zero. Since consumption is always positive, current labour income and therefore work hours must also be positive. Therefore, the effect of an increase in the survival rate on the labour supply $e^{-gx}\ j(x)\ h(x)$ is positive. Finally, at age $\hat{x}$, $\partial\hat{A}/\partial p(x) = 0$. Since $\hat{\Sigma} = (g - r)$ $\hat{A}$, we also get $\partial\hat{\Sigma}/\partial p(x) = 0$ at $x = \hat{x}$. Therefore $w\partial\hat{H}/\partial p(x)$ is equal to $\partial\hat{C}/\partial p(x)$ which can be calculated as follows:

$$\partial\hat{C}/\partial p(x) = e^{-gx}\ c(x) + \int_0^T e^{-gy}\ p(y)\{\partial c(y)/\partial p(x)\}\ dy$$

The first term on the right hand side is obviously positive. Since $\hat{x}$ lies between age x_* and age x^*, the savings rate $g(x)$ is positive at age $\hat{x}$ and therefore according to our analysis in chapter 4, $\partial c(y)/\partial p(x)$ is also positive. Thus both the terms in above expression are positive. Therefore $\partial\hat{C}/\partial p(x) = w\partial\hat{H}/\partial p(x)$ is positive.

Since at age x_*, x^* and $\hat{x}$, $\partial\hat{A}/\partial p(x) = 0$ and $\partial\hat{H}/\partial p(x) > 0$, $\partial A^*/\partial p(x)$ must be negative.

Since in the neighbourhood of points x_*, x^* and $\hat{x}$ the derivative $\partial\hat{A}/\partial p(x)$ is small, the above theorem also implies that $\partial A^*/\partial p(x)$ will be negative in a small neighbourhood around these three points.

Theorem 10.3.4

$\partial A^*/\partial p(x) < 0$ for $x < x_*$.

Proof

We already know that in this range $\partial\hat{A}/\partial p(x)$ is negative. Next, the derivative $\partial\hat{H}/\partial p(x)$ is as follows:

$$\partial\hat{H}/\partial p(x) = e^{-gx}\ j(x)\ h(x)$$

$$+ \int_0^T e^{-gy}\ p(y)\ j(y)\ \{\partial h(y)/\partial p(x)\}\ dy$$

The first term in this equation is obviously positive. The second term is also positive because for $x < x_*$,

$\sigma(x)$ is negative and, therefore, $\partial h(y)/\partial p(x)$ is positive. With $\partial\hat{A}/\partial p(x) < 0$ and $\partial\hat{H}/\partial p(x) > 0$ we must have

$$\partial A^*/\partial p(x) = A^* [\{\partial\hat{A}/\partial p(x)\}/\hat{A} - \{\partial\hat{H}/\partial p(x)\}/\hat{H}] < 0$$

This completes the proof.

Theorem 10.3.5

$\partial A^*/\partial p(x) < 0$ for $\hat{x} < x < x^*$.

Proof

By theorem 10.3.2 the sign of the derivative $\partial A^*/\partial p(x)$ is the same as that of $\partial\{\hat{A}/\hat{C}\}/\partial p(x)$. We also know by theorem 8.6.1 that $\partial\hat{A}/\partial p(x) < 0$ for $\hat{x} < x < x^*$. Next consider the derivative $\partial\hat{C}/\partial p(x)$

$$\partial\hat{C}/\partial p(x) = e^{-gx} c(x) + \int_0^T e^{-gy} p(y)\{\partial c(y)/\partial p(x)\} dy$$

The first term in this equation is positive. Next, for $\hat{x} < x < x^*$, the savings rate $\sigma(x)$ is positive. Therefore $\partial c(y)/\partial p(x)$ and hence the second term is also positive. Thus $\partial\hat{C}/\partial p(x)$ is positive. Since $\partial\hat{A}/\partial p(x) < 0$ and $\partial\hat{C}/\partial p(x) > 0$, we must have

$$\partial\{\hat{A}/\hat{C}\}/\partial p(x) = \hat{C}^{-2} [\hat{C}\{\partial\hat{A}/\partial p(x)\} - \hat{A} \{\partial\hat{C}/\partial p(x)\}] < 0$$

It has been established that $\partial A^*/\partial p(x)$ is negative in the neighbourhood of the points x_*, x^* and $\hat{x}$ and in the intervals $x < x_*$ and $\hat{x} < x < x^*$. We still have to examine the sign of this derivative in the intervals $x_* < x < \hat{x}$ and $x^* < x \leq T$ which seems to be uncertain. However, we can show the following result.

Theorem 10.3.6

If $x_* = \hat{x}$, then $\partial A^*/\partial p(x)$ must be positive at least at some points in the interval $x^* < x \leq T$.

Proof

$$\text{Since } \int_0^T p(x)\ \{\partial A^*/\partial p(x)\}\ dx = 0$$

by theorem 10.3.1 and $\partial A^*/\partial p(x) < 0$ for $x \leq x^*$, $\partial A^*/\partial p(x)$ must be positive at least somewhere in the interval $x^* < x \leq T$.

Even if age $\hat{x}$ is slightly above age x_*, this result can be expected to hold. The reason is that the derivative $\partial A^*/\partial p(x)$ is also negative in the neighbourhood of points x_* and $\hat{x}$.

The above discussion suggests that, if the interval: $[x_*, \hat{x}]$ is small, we obtain more or less the same age pattern of the economic effect of an increase in survival rates on the assets-labour supply ratio as obtained in chapter 9 where the work schedule of an individual was assumed to be fixed.

The effects on the equilibrium rate of interest, the wage rate and the capital-labour ratio can be obtained in the same manner as in chapter 9 and therefore there is no need to repeat that exercise.

The demographic effect of mortality improvements does not depend on whether the work schedule of an individual is fixed or endogenous. It depends, among other things, on the age pattern of asset holdings and the labour income of an individual. This age pattern is same in the present chapter as in chapters 8 and 9. Therefore the demographic effects of mortality improvements will also be the same as before.

10.4 CONCLUDING REMARKS

With an endogenous work schedule of an individual the analysis of mortality improvements becomes quite complicated. In particular the effect of an increase in survival rates at the earlier part of middle age and at old age cannot be easily signed. However, one can find a sufficient and conceivable condition under which the signs of these effects can be determined. It turns out that the effects of mortality improvements in this extended model are more or less the same as found in chapter 9.

10.A APPENDIX TO CHAPTER 10

10.A.1 Proof of Theorem 10.3.1

First we reproduce equation (10.3.2) in a slightly different form

$$\{\partial A^*/\partial p(x)\}/A^* = \{\partial \hat{A}/\partial p(x)\}/\hat{A} - \{\partial \hat{H}/\partial p(x)\}/\hat{H} \qquad (10.A.1)$$

According to this equation we can write

$$(1/A^*) \int_0^T p(x) \, \{\partial A^*/\partial p(x)\} \, dx =$$

$$(1/\hat{A}) \int_0^T p(x)\{\partial \hat{A}/\partial p(x)\}dx - (1/\hat{H}) \int_0^T p(x)\{\partial \hat{H}/\partial p(x)\}dx \qquad (10.A.2)$$

This equation contains two components involving the derivatives $\partial \hat{A}/\partial p(x)$ and $\partial \hat{H}/\partial p(x)$. Consider the first component. According to equation (10.3.3), the derivative $\partial \hat{A}/\partial p(x)$ can be written as

$$\partial \hat{A}/\partial p(x) = e^{-rx} \, \sigma(x) \, [\bar{z} - z(x)]$$

Since $\bar{z}$ is a constant, we can write

$$\int_0^T p(x) \, \{\partial \hat{A}/\partial p(x)\} \, dx = \bar{z} \int_x^T e^{-rx} \, p(x) \, \sigma(x) \, dx$$

$$- \int_0^T e^{-rx} \, p(x) \, \sigma(x) \, z(x) \, dx \qquad (10.A.3)$$

The first term in this equation vanishes because of the lifetime budget constraint of an individual (see equation (3.4.7)). Therefore, substituting for z(x) from equation (8.6.12) for $g \neq r$, this equation becomes

$$\int_0^T p(x)\ \{\partial\hat{A}/\partial p(x)\}\ dx = [1/(g-r)] \int_0^T e^{-gx}\ p(x)\ \sigma(x)\ dx$$

$$= [1/(g-r)]\ \hat{\Sigma}$$

Or, using the identity (8.4.14) we find, for $g \neq r$

$$\int_0^T p(x)\ \{\partial\hat{A}/\partial p(x)\}\ dx = \hat{A} \qquad (10.A.4)$$

Next, substituting for z(x) from equation (8.6.12) for $g = r$, equation (10.A.3) becomes

$$\int_0^T p(x)\ \{\partial\hat{A}/\partial p(x)\}\ dx = - \int_0^T x\ e^{-rx}\ p(x)\ \sigma(x)\ dx \qquad (10.A.5)$$

Since $x = T - \int_x^T dy$

equation (10.A.5) can be written as

$$\int_0^T p(x)\ \{\partial\hat{A}/\partial p(x)\}\ dx =$$

$$- T \int_0^T e^{-rx}\ p(x)\ \sigma(x)\ dx + \int_0^T e^{-rx}\ p(x)\ \sigma(x) \int_x^T dy\ dx$$

The first term vanishes because of the lifetime budget constraint of an individual. With a change in the order of integration, we can write this equation as

$$\int_0^T p(x)\ \{\partial\hat{A}/\partial p(x)\}\ dx = \int_0^T \int_0^x e^{-ry}\ p(y)\ \sigma(y)\ dy\ dx$$

With g = r, the right hand side of the above equation is equal to the level of aggregate normalized assets, $\hat{A}$. Therefore for g = r also we get the same relation as given by equation (10.A.4).

Next, consider the second component in equation (10.A.2). Let us first write the derivative of aggregate normalized labour supply $\hat{H}$ with respect to survival rate at age x. With $\hat{H}$ defined by equation (8.6.26), this derivative can be written as follows.

$$\partial\hat{H}/\partial p(x) = e^{-gx}j(x)h(x) + \int_0^T e^{-gy}\ p(y)\ j(y)\ \{\partial h(y)/\partial p(x)\}\ dy$$

We know from our comparative static analysis in chapter 4 that for all y, the derivative $\partial h(y)/\partial p(x)$ depends on age x only through a factor of proportionality: $e^{-rx}\ \sigma(x)$. Therefore the second term in the above equation as a whole can be written as: $\Gamma\ e^{-rx}\ \sigma(x)$ where Γ is a constant definite integral of some function of y over the interval [0, T]. Thus we can write

$$\partial\hat{H}/\partial p(x) = e^{-gx}\ j(x)\ h(x) + r\ e^{-rx}\ j(x)$$

Multiplying by p(x) and integrating over x for the interval [0, T], we can write

$$\int_0^T p(x)\ \{\partial\hat{H}/\partial p(x)\}\ dx = \int_0^T e^{-gx}\ p(x)\ j(x)\ h(x)\ dx$$

$$+\ r \int_0^T e^{-rx}\ p(x)\ j(x)\ dx$$

The first term in this equation is equal to aggregate normalized labour supply $\hat{H}$ and the second term is equal to zero because of the lifetime budget constraint of an individual. Therefore

$$\int_0^T p(x)\ \{\partial\hat{H}/\partial p(x)\}\ dx = \hat{H} \qquad (10.A.6)$$

Substituting equations (10.A.4) and (10.A.6) into equation (10.A.2), we get

$$(1/A^*) \int_0^T p(x)\ \{\partial A^*/\partial p(x)\}\ dx = \{\dot{A}/A\} - \{H/H\}$$

and therefore,

$$\int_0^T p(x) \ \{\partial A^*/\partial p(x)\} \ dx = 0$$

This completes the proof.

10.A.2 Proof of Theorem 10.3.2

Consider the following identity

$$w\hat{H} - \hat{C} = \hat{\Sigma} = (g - r) \ \hat{A}$$

which implies

$$w\hat{H} = (g - r) \ \hat{A} + \hat{C} \qquad (10.A.7)$$

Therefore we can write the ratio $\hat{A}/\hat{H}$ as follows.

$$\hat{A}/\hat{H} = w\hat{A}/w\hat{H} = w\hat{A}/[(g - r)\hat{A} + \hat{C}]$$

$$= w \ (\hat{A}/\hat{C})/[(g - r)(\hat{A}/\hat{C}) + 1]$$

Now differentiating both sides with respect to p(x), we obtain

$$\partial(\hat{A}/\hat{H})/\partial p(x) = w \ [(g - r)(\hat{A}/\hat{C}) + 1]^{-2}$$

$$[\{(g - r)(\hat{A}/\hat{C}) + 1\} \ \{\partial(\hat{A}/\hat{C})/\partial p(x)\}$$

$$- \ (\hat{A}/\hat{C})(g - r)\{\partial(\hat{A}/\hat{C})/\partial p(x)\}]$$

$$= w \ \hat{C}^2 \ [(g-r)\hat{A} + \hat{C}]^{-2} \ \partial(\hat{A}/\hat{C})/\partial p(x)$$

Or using (10.A.7),

$$\partial\{\hat{A}/\hat{H}\}/\partial p(x) = w\{\hat{C}/w\hat{L}\}^2 \; \partial\{\hat{A}/\hat{C}\}/\partial p(x) \qquad (10.A.8)$$

This completes the proof.

CHAPTER 11

SUMMARY AND CONCLUSION OF PART II[1]

The implications of alternative types of mortality improvements have been studied in an aggregate model under competitive conditions. A continuous time version of an overlapping generations model was used for the analysis. Actuarially fair life insurance and annuities were assumed to be available. Individuals were assumed to be selfish. Therefore, bequests or gifts were not present in the model. It was also assumed that pensions or social security do not exist. Except for the uncertainty of life, there was no other element of uncertainty present in the model. The main findings of the research are as follows.

Given the rate of interest and the wage rate, an increase in life expectancy may increase or decrease the capital stock depending on the nature of the mortality improvement. This result is obtained in a pure life cycle model and does not depend on any bequest motive. Thus, Skinner's result that an increase in longevity may increase or

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1 For the summary and conclusion of part I of the thesis see chapter 5.

decrease capital stock, is possible in a pure life cycle model.

Given the factor prices and the growth rate of population, with a hump shaped savings schedule, an increase in survival rates at an old age will increase the normalized level of the capital stock, as expected. However, the effect of an increase in survival rates at a young age is to decrease the capital stock. Finally, the effect of an increase in survival rates at an earlier (later) part of middle age is to increase (decrease) the capital stock.

The effects of mortality improvements with an endogenous rate of interest and wage rate, but fixed work schedule of an individual, were also studied under competitive conditions, assuming that the equilibrium is stable. Once again we find that these effects depend on the age at which the survival rate is assumed to increase. Given the growth rate of population, with a hump-shaped savings schedule, an increase in survival rates at an old age will increase the equilibrium capital-labour ratio and the wage rate and decrease the equilibrium rate of interest. The effect of an increase in survival rates at the earlier segments of life is the opposite. Thus Skinner's result that increase in life expectancy may reduce the capital-labour ratio, can be found in a pure life cycle model without a contingent bequest motive.

The traditional two period models, in which all births take place in the first period and survival is uncertain only in the second period, ignore the demographic effect of a possible increase in the growth rate of population resulting from the increase in longevity. This effect has been recognized in the present research. If the rates of reproduction are independent of survival rates, this effect shows that an increase in survival rates during the reproductive years of life will decrease the equilibrium capital-labour ratio and the wage rate and increase the equilibrium rate of interest. This further supports our suspicion about the presumed positive relation between longevity and capital accumulation.

The implications of mortality improvements for the classification of competitive equilibrium as efficient or inefficient were noted. An intermediate result is that, ignoring the economic effect of variations in survival rates, a higher growth rate of population has no implications for the classification of equilibrium as efficient or inefficient. In particular, an economy with a slower growth rate of population may even be further away from an efficient equilibrium than an economy with a faster growth rate.

An increase in survival rates at an old age will leave the growth rate of population unchanged, but it will lower the equilibrium rate of interest. Thus, if the competitive equilibrium is initially inefficient, it will remain so after the mortality improvement and if the equilibrium is initially efficient, the mortality improvement may carry it into the inefficient region. An increase in survival rates at earlier age will raise both the growth rate of population and the rate of interest. In this case the mortality improvement carries no implications for the classification of equilibrium as efficient or inefficient.

The classification of a competitive equilibrium as efficient or inefficient holds with given tastes (which also include survival probabilities) and technology, etc. A conversion of an efficient equilibrium into an inefficient one due to a mortality improvement does not necessarily imply a welfare loss. The new equilibrium is inefficient in the new state (after the change in survival probabilities and, hence, in the utility function). Similarly a conversion of an inefficient equilibrium into an efficient one due to a mortality improvement does not necessarily imply a welfare gain. The new equilibrium is efficient in the new state (after the change in the utility function).

The analysis was then repeated for a more complicated model in which the work schedule of an individual was assumed to be endogenous. The results were shown to be more or less the same as obtained with a fixed work schedule.

Aside from these results relating to the basic issue involved in this research, we also noted some other results.

The effect of a higher rate of interest due to actuarially fair life insurance and annuities present in an individual's savings or assets is absent from aggregate savings or assets.

The fundamental accounting identities between savings and assets which had been obtained for an overlapping generations model under certainty (Gale (1973), Willis (1985)), also hold in a model with life uncertainty and actuarially fair life insurance and annuities.

In a new-classical model with a linear homogeneous production function, the stability of a Walrasian equilibrium is not guaranteed by a positive relationship between assets supply and the rate of interest. An arbitrary increase in the rate of interest from the equilibrium level will also imply a lower wage rate under the neo-classical production technology. This decrease in the wage rate may reduce the level of assets enough to compensate the increase due to a higher rate of interest. Under these conditions the equilibrium solution may be unstable despite a positive relationship between assets supply and rate of interest.

The present analysis has been conducted for a free competitive model. In particular no mandatory social security programme is considered and the life insurance and annuities market is assumed to be competitive and free of any distortions. In addition, no bequest motive is allowed in the model. The research can be easily extended to consider the effects of these special environments surrounding the individual's life cycle allocation decision. The analysis of mortality changes can also be repeated for a model in which life insurance and annuities do not exist. In addition, one can also study the welfare implications of mortality changes under alternative assumptions regarding life insurance and annuities. To study the short term effects of mortality changes it would be useful to examine the transition paths in response to changes in survival rates. We plan to search in these areas in future.

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