Applications of Numerical Methods in Economics and Finance

APPLICATIONS OF NUMERICAL METHODS IN ECONOMICS AND FINANCE

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Abstract

This dissertation consists of three articles on the applications of numerical methods in economics and finance.

The first article investigates the performance of various estimators in estimating the continuous time short-term interest rate models under the assumption that the higher moment dynamics of the short rate series are misspecified. The Monte Carlo evidence suggests that volatility of a continuous time short-term interest rate model would be estimated with a serious bias if the higher moments of the series are misspecified. Furthermore, in the root mean square sense, computationally intensive simulation based estimators do not exhibit better finite sample performance than the conventional estimators.

The second article presents a numerical analysis of optimal initial and maintenance margin setting for futures commission merchants (futures brokers). The problem is analyzed in a profit maximization context using a Markov chain approach and it is shown that the uncertainty regarding the futures price process, the trader's attitude toward a negative margin account and the transaction costs associated with margin calls lead the futures brokers to set positive performance margins without the need of legal enforcements of the futures exchange.

The third article estimates the intertemporal allocation parameters using a new structural estimation technique, Simulated Residual Estimation. A series of Monte Carlo experiments which involve solving and simulating a life cycle model under both interest rate and income uncertainty are performed. The intertemporal allocation parameters, the elasticity of intertemporal substitution and the discount rate are repeatedly estimated using the exact Euler equation, the approximate (log-linearized) Euler equation and the Simulated Residual Estimation. The results of the experiments suggest that the Simulated Residual Estimation has superior finite sample properties in comparison to conventional GMM based estimators even under the circumstances in which the consumption data is measured with error.

To my dear father Resit Tayfun Alan without whom my accomplishments would not have been possible.

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Preface

The third article of this thesis was prepared with the intention of a joint publication with Professor Martin Browning. I had primary responsibility for the mathematical and empirical analysis.

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I

Introduction

The development of theoretical and applied economics and finance has been greatly influenced by the increasing availability of powerful computers. Advances in computing technology are offering economists the opportunity to examine issues in economics and finance with greater detail and precision. A great advantage of powerful computational facilities is that economists can explore the implications of extremely complicated non-linear models in far greater detail and perform structural estimation methods that may require sophisticated simulation algorithms.

With the introduction of the numerical dynamic optimization techniques, it has become possible to derive the functions of interest numerically with a reasonable accuracy. Until the last decade, the economic research dealing with the dynamic intertemporal optimization problems assumed simple and relatively unrealistic functional forms for consumer's preferences in order to obtain convenient linear closed form consumption functions. However, recently, the numerical dynamic programming is the main tool for the micro and macroeconomic research. (see Cocco et al. 1999, Gourinchas and Parker 2001). This is especially so for consumer problems where it is customary to assume that the rational agent maximizes his expected lifetime utility subject to his intertemporal budget constraint. The need for the numerical solution methods stems from the fact that these problems can get extremely complicated as we incorporate more realistic features into the standard model. For example, assuming labour income uncertainty or asset return uncertainty can rule out the possibility of obtaining an analytical expression for the consumption function.

Similarly, the newly introduced numerical optimization routines open the way for the empirical researchers to estimate the parameters of interest and obtain their estimated precision without knowing the analytical form of the estimator. The absence of the analytical expressions of criterion functions causes difficulties, usually due to the presence of high dimensional integrals. The recent development of the simulation based estimation techniques, indirect estimation (Gourieroux et al. 1993) and Efficient Method of Moments (Gallant and Tauchen 1996), circumvent these difficulties via simulation of the data for given values of the parameters.

This thesis consists of three articles on the application of numerical methods. Each article addresses important research questions in economics and finance by making use of the new numerical analysis and simulation based estimation techniques.

In the first article of the thesis, I examine the finite sample performance of simulation based estimators in comparison to the conventional methods in estimating continuous time short-term interest rate models when the higher moment dynamics of the short rate series are misspecified. Following the powerful Black and Scholes (1973) result, it has been customary in the derivative pricing literature to use the continuous time stochastic processes as modelling tools since it allows us to use the established theoretical results of stochastic calculus.

The parameter estimates of a continuous time model can be biased due to discretization of the continuous series since we cannot possibly obtain data for extremely small intervals. Moreover, even if we are able to collect data with very small intervals such as hourly stock prices or interest rates, unequally spaced data due to non-trading times will result in a different set of econometric problems. There has been extensive econometric research on estimating continuous time models using discrete data. Simulation based estimators such as the indirect estimator and the Efficient Method of Moments estimator were developed to correct for the bias arising from using discretely spaced data for a continuous time model.

In this paper the parameters of the Vasicek short term interest rate model are estimated using five different estimators: the maximum likelihood estimator based on the exact solution of the stochastic differential equation, a naive estimator using a first order approximation, the continuous-updating Generalized Method of Moments (GMM) estimator using the moment conditions of the approximate model, an indirect estimator using the discretized model as an auxiliary model, and an Efficient Method of Moments estimator (EMM) using the loglikelihood function of the approximate model as a score generator. The reason that the Vasicek model is chosen for this study is that it has a closed form solution that can be used as a benchmark for comparisons. The parameters of the model are the long run mean of the short-term interest rate, the parameter that controls the speed at which the series go back to its long run mean, and the volatility parameter.

I find that the parameters of a continuous time short term interest rate model that are of great importance in pricing contingent claims written on fixed income securities would be estimated with a serious bias if the higher moment behaviour of the series is misspecified. The bias is much more severe for the volatility parameter. Furthermore, in the root mean square sense, computationally burdensome simulation based estimators do not exhibit better finite sample performance than the conventional estimators.

In the second article of the thesis I analyze the problem of optimal performance margins (initial margin and maintenance margin) in the futures markets. Although the activity of hedging is the main rationale behind futures trading, traditionally, futures markets serve the needs of three groups of futures market users: those who want to hedge against the unwanted risk of their business venture, those who want to be informed about future prices of commodities and those who want to bear the unwanted risk of the hedgers for the purpose of a speculative profit. Undoubtedly, these markets serve a social purpose by helping private entrepreneurs to transfer the risk of their business and provide good estimates of the future prices of commodities.

The requirement for performance margins is one of the most distinguished features of futures markets. Initial margins represent good-faith deposits and they protect the brokers against the investor default. Serving the same purposes, the maintenance margin is the level of margin account at which the investor is required to replenish the deposit. The existence of an initial margin reduces the risk of investor's default. However, a high level of initial margin has an adverse effect on the volume of the trading activity since it imposes a liquidity cost on the traders. Similarly, while maintenance margins reduce the risk of broker's loss due to a default, it imposes certain type of transaction costs to the futures brokers. A high maintenance margin level results in more frequent margin calls, leading to high clerical cost as well as some unquantifiable costs due to customer annoyance.

In this paper, a complete numerical profit maximization model that rationalizes the existence of both maintenance and initial margins is set up in a partial equilibrium framework. The inclusion of the maintenance margin makes the analytical solution impossible. Therefore a new numerical solution technique is developed using the properties of the first order Markov process. The model generates positive initial and maintenance margins in the absence of a minimum margin requirement imposed by the futures exchange. The uncertainty regarding the futures price process, the trader's attitude toward a negative margin account and the transaction costs associated with margin calls leads the rational profit maximizing brokers to set positive performance margins without the need of legal enforcements of the futures exchange. This provides a strong support for the argument that the performance margins are the results of competitive market forces rather than legal enforcement.

In the third article, I propose a new simulation based estimation method to estimate the intertemporal allocation parameters of the standard Life Cycle model, the elasticity of intertemporal substitution and the subjective discount rate. Then I investigate the finite sample performance of the proposed estimator in comparison to the conventional GMM based estimators.

Estimation of these preference parameters has been one of the most studied issues in the empirical consumption and saving literature (Browning and Lusardi 1996). The key structural parameters of the consumer optimization problem are of interest, mainly for policy purposes. They are also crucial in testing the various implications of the underlying model. The intertemporal elasticity of substitution measures the agent's degree of willingness to substitute across periods, in other words, his willingness to smooth consumption over the life cycle. Since the intertemporal price is simply the prevailing interest rate, clearly, the magnitude of this parameter is important in understanding the variations in the life cycle savings in relation to interest rate movements.

Another important preference parameter is the rate of time preference or subjective discount rate. The difference between the agent's discount rate and the market rate (interest rate) reveals the degree of agent's impatience. Impatient agents are expected to decumulate their assets (or borrow if they do not have any assets) and exhibit a low rate of consumption growth (negative under certainty) over the life cycle, whereas a patient agent has a high rate of consumption growth due to a rapid asset accumulation. Therefore obtaining sensible discount rate estimates is also necessary in order to understand the variations in life cycle savings and consumption.

Traditionally, these parameters were estimated using the first order condition (Euler equation) of the standard life cycle model via GMM. The most appealing feature of Euler equation estimation is that the stochastic processes facing the agents need not be specified to identify the structural parameters. In other words, we do not need to model the labour income and the asset return processes to estimate the parameters of interest. Unfortunately, recent Monte Carlo evidence suggests that estimation based on the exact (non-linear) Euler equations yields seriously biased estimates when the consumption data is imperfectly measured. (see Carroll 2001). Moreover, the estimates obtained based on the linearized Euler equations also suffer from severe bias due to the fact that the instruments used in the estimation are not orthogonal to the model errors (Carroll 2001, Ludvigson and Paxson 2001).

The Monte Carlo experiment I perform involves solving and simulating a life cycle model under both interest rate and income uncertainty. Given the assumed functional form of the utility function and the uncertainty, a closed form solution for the consumption function cannot be derived. Instead, I use numerical dynamic programming techniques to solve for the consumption function for a generic consumer. After generating artificial consumption data using these consumption functions, I estimate the parameters of interest using the exact Euler equation, the approximate (log-linearized) Euler equation and the proposed estimation method, Simulated Residual Estimation. The results of the experiments suggest that this new method has superior finite sample properties in comparison to the conventional GMM based estimators even under circumstances in which the consumption data is measured with error.

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II

Estimating Continuous Time Interest Rate Models: Empirical Importance of Higher Moment Dynamics

Term structure models have become increasingly important since contingent claims written on fixed income securities became popular. In addition to their relevance to financial markets, term structure models (models that describe the behavior of the yield curve) are of interest to policy makers since understanding the dynamics of the yield curve is crucial for the government to determine its optimal debt strategy[1]. There has been a considerable amount of theoretical and empirical research on modeling the term structure of interest rates within the fields of economics and mathematical finance.

This paper focuses on term structure models driven solely by the short rate in the mathematical finance literature. The objective of the paper is to compare the finite sample performance of the various estimation techniques applied to continuous time short rate models under the circumstances in which the higher moment behavior of the short rate series is misspecified. The simple Vasicek short rate model is estimated using: a maximum likelihood estimator based on the exact discretization of the model, a maximum likelihood estimator based on a first order approximation, a GMM estimator based on a first order approximation by continuously updating the weighting matrix (Hansen, Heaton and Yaron 1996) and two simulation based estimators, the indirect (Gourieroux, Monfort and Renault, 1993) and the Efficient Method of Moments estimators (Gallant and Tauchen, 1998). We use the first order approximation as an auxiliary model for the

[1] For a detailed discussion on yield curve and government debt management see John Campbell (1995).

simulation based estimators. Assumed misspecification structures are based on the alternative short rate models proposed in the literature.

Although, by now, there appears to be a consensus among the researchers that neither the simple Vasicek nor any other single factor model can possibly replicate the dynamics of the yield curve, from the practitioner's point of view, these models are still very useful and analytically convenient tools for pricing certain type of derivative securities[2]. Following the powerful Black and Scholes (1973) result, it has been customary in the derivative pricing literature to use the diffusion processes as modelling tools since it allows us to exploit the already established results of the stochastic calculus. It seems only natural to benefit from the convenience of stochastic calculus for pricing claims written on fixed income securities where the interest rate can be modeled as a diffusion process. However, the main issue the literature has been trying to address recently is that the dynamics of the higher order moments of the short rate are much more complicated than a simple diffusion process can capture. Recent research efforts have resulted in alternative continuous time short rate diffusion models with richer higher moment dynamics. Even though there have been numerous attempts to model the short rate by modelling its higher order moment behavior, there appears to be no study examining the effects of misspecifying these higher order moment dynamics on the parameter estimates. This study addresses this particular issue.

Parameter estimates of a continuous time model can be biased due to discretization or

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2 See A. Pagan, A Hall and V. Martin (1996) for an excellent survey on term structure models.

misspecification or both. There has been extensive econometric research on estimating continuous time models (stochastic differential equations) using discrete data. Simulation based estimators such as the indirect estimator, the efficient method of moments and simulated method of moments were developed to correct for the bias arising from having to use discrete data for an estimation of a continuous time model[3]. These estimation methods are quite powerful in that they often do not need a sophisticated auxiliary model. In the Monte Carlo simulation environment we set up for this study, these estimators are the ones which measure the pure effects of misspecification since they correct for the discretization bias[4].

The results of this paper suggest that misspecifying the higher order moment dynamics of the short rate series in our modelling process and estimating the parameters of interest for the contingent claim pricing yields serious bias in the volatility estimates whereas it does not seem to have any serious effect on estimates of the parameter that controls the speed at which the short rate goes back to its long run mean. Moreover, the size and the magnitude of the bias depend on the omitted feature of the higher order moment dynamics of the short rate series as well as the estimation method utilized. Based on the

[3] Note that the term discretization bias is used to address two problems: (*i*) estimation of a stochastic differential equation when we do not have a closed form solution for the equation so that we are bound to use an approximate model based on some type of discretization scheme. This may be called an approximation bias. (*ii*) estimation of a stochastic differential equation with a discrete data series. In this study I refer to both of these isuues when I use the term discretization bias.

[4] The maximum likelihood estimator based on the exact discretization also measures the effects of misspecification in this study. But it should be noted that the majority of the interesting short rate models cannot be discretized exactly. Therefore, it may be more plausible to focus on the simulation based estimators. The reason that the simple Vasicek model is chosen for this study is that it does have a closed form solution so that it provides a useful benchmark case.

mean squared error criterion the simulation based estimators appear to perform no better than the estimators that use crude discretization schemes. However, the mean and the median estimates of these estimators do successfully recover the true parameter values in almost all cases considered.

The remainder of the paper is organized as follows. The next section is a limited overview of short rate models considered in this study. The third section describes the statistical characteristics of the short rate and section four discusses the estimation methods used. We discuss the main results of the experiments in section five and conclude in section six.

2. Modelling Short Term Interest Rates

Single factor term structure models have been the most studied since the publication of a very influential paper of Vasicek (1977). He derived a general form of the term structure of interest rates using a no-arbitrage argument assuming 1) the instantaneous rate follows a continuous Markov (diffusion) process, 2) the price of a discount bond depends only on the spot rate over its term, 3) the market is efficient. Cox, Ingersoll and Ross (CIR) (1985) proposed a single factor general equilibrium economy in which bond prices are determined by a no-arbitrage condition. Such an economy is driven by one factor, the short rate, which is assumed to follow a square-root diffusion process. Given the short rate process, the bond price process can be obtained via Itô's Lemma and the solution to the so-called fundamental partial differential equation. Despite some shortcomings, single factor models have been the most utilized models in the pricing literature due to their analytical tractability. In its relatively general form, a continuous time one factor short rate model is as follows:

$$dr_t = k(\theta - r_t)dt + \sigma r_t^{\gamma} dw_t \tag{1}$$

where k is a convexity parameter that determines the speed of mean reversion, θ is the long-run mean, σ is the constant volatility parameter, γ determines the degree of the level effect on volatility and dw_t is an increment of Brownian motion.[5] In the remainder of the paper this model will be referred to as the level effect model. Special cases where $\gamma = 0$ (Vasicek Model) and $\gamma = 1/2$ (CIR Model) lead to bond prices of analytically convenient exponential-affine form[6]:

$$P(\tau) = \exp(A(\tau) - B(\tau)r(t)) \tag{2}$$

where $\tau = T - t$, and $P(\tau)$ is the price of a discount bond that matures at time T at t, $r(t)$ is the instantaneous short rate (spot rate). The yield curve driven by the short rate is:

$$y(\tau) = \frac{-A(\tau) + B(\tau)r(t)}{\tau} \tag{3}$$

where $y(\tau)$ is the continuously compounded yield. In the case of the Vasicek model:

$$B(\tau) = \frac{e^{-k\tau} - 1}{k} \tag{4}$$

$$A(\tau) = R(\infty)(\tau + B(\tau)) - \frac{\sigma^2}{4k} B^2(\tau) \tag{5}$$

5 $dw_t \sim N(0, dt)$

6 This class of models are the most widely used due to their analytical convenience. When an affine term structure is assumed, the fundamental partial differential equation for the bond price can be written as several ordinary differential equations which can be solved in a relatively straightforward way.

$$R(\infty) = \theta - \frac{\lambda\sigma}{k} - \frac{1}{2}\left(\frac{\sigma}{k}\right)^2 \tag{6}$$

where λ is the so-called market price of risk, which is assumed to be independent of T .

Simple single factor Vasicek and CIR models poorly reproduce the observed yield curve. Hull and White (1987) extended these basic models by allowing for a time varying mean and speed of mean reversion in order to obtain a better fit, although their models have their own shortcomings.

In order to better fit the observed yield curve, some multifactor models have also been proposed. Two factor models, for example Longstaff and Schwartz (1992), usually specify that the short rate and its volatility follow independent diffusion processes. The authors developed a general equilibrium framework in which the economy is driven by two unobservable state variables. The short rate and its volatility, which determine the dynamics of the yield curve can be expressed by these factors via Itô's Lemma. Among other multifactor models, Brennan and Schwartz (1982) is worth mentioning. They identify two factors which determine the dynamics of the yield curve, the short rate and the long (consol) rate. Their approach is rooted in the risk management literature which decomposes the term structure dynamics into changes in "level", "slope" and "curvature" factors. Their model can be interpreted as being driven by the level (long yield) and slope (short rate-long rate).

One important stylized fact regarding the short rate process is that the distribution of the change in the short rate displays skewness and excess kurtosis. Clearly, standard

Gaussian single factor models fail to capture this fact since they all use Brownian motion. It is possible to generate thick tails with stochastic volatility models such as:

$$dr_t = k(\theta - r_t)dt + \sigma_t dw_{t,1} \tag{7}$$

$$d\sigma_t = \beta(\overline{\sigma} - \sigma_t)dt + \xi dw_{t,2}$$

where $\overline{\sigma}$ is the long run mean of short rate volatility, β is the speed of mean reversion for the volatility process, and $dw_{t,1}$ and $dw_{t,2}$ are independent Brownian motions.

Even though the evidence suggests that the models with time varying volatility perform better in terms of replicating the higher moment behavior of the data relative to single factor models, the data suggest that the excess kurtosis is more apparent in high frequency data, while stochastic volatility models imply the opposite (see S. Das 1999).

Jump-diffusion processes have been suggested to model stock return distributions which are also known to have much thicker tails than a normal distribution suggests. Merton (1976) priced a European call option in a closed form by modeling the underlying asset return as a jump-diffusion process, where the diffusion part was a geometric Brownian motion and the jump part was modeled with a Poisson process. The intuition behind incorporating the jumps into the standard model is as follows: Stock prices should reflect any anticipated relevant information about the future and they should not be predictable. Modeling the unanticipated information arrival with a normal distribution does not capture the fact that we do observe abnormal returns much more often than a normal distribution would suggest. This has very important implications for option pricing and risk management. It seems plausible that abnormal returns can be modeled by adding Poisson increments to the standard diffusion model which may or may not be independent of the diffusion (Gaussian) process. It should be noted that one can model the returns with a distribution which has thick tails such as the t-distribution, although this usually leads to analytically intractable models.

The literature on jump-diffusion models of interest rates is very new and fast growing. The real world facts such as discrete changes in central bank's target rates or interest rate regimes make these models more appealing than diffusion models. Moreover, it is possible to generate thicker tails by incorporating jumps into a standard gaussian process with or without a stochastic volatility component. Increasing excess kurtosis with high sampling frequency can be captured by these models since the instantaneous probability of jumps does not depend on sampling frequency. Another important feature of these models is that it is possible to incorporate unanticipated announcement effects into the continuous short rate process through Poisson jumps.

In a jump-diffusion process, the short rate can be modeled as:

$$dr_t = k(\theta - r_t)dt + \sigma dw_t + J d\pi(\lambda) \tag{8}$$

where π is a Poisson process, λ is the intensity of the Poisson process (mean number of abnormal information arrivals per unit of time), J is jump size which can be constant or drawn from a probability distribution. The rest of the model is the standard Vasicek model.

The estimation of the above model requires the probability density function of r_t.

There exists a known closed form likelihood function of the above model only for the case in which the jump size is distributed Bernoulli-exponential[7] (assuming dw_t, $d\pi$ and J are independent of each other).

3. Characteristics of Interest Rate Series

There is no doubt that the observed characteristics of the short rate should be the guide to any attempt at a model development. Even though this is a simulation study, it would be informative to briefly discuss what is known about the empirical behavior of the short rate. The plotting of most interest rate series suggests that interest rates move around what looks like a long-run mean. Another important characteristic that is apparent in most interest rate data is volatility clustering which means high volatility times are more likely to be followed by high volatility. This phenomenon led researchers to model interest rate behavior with time varying volatility. Another very important characteristic of the short rate is its higher moment behavior. The empirical distribution of change in the short rate typically has extremely heavy tails. This can definitely be captured by a stochastic volatility model. However, the fact that high frequency series display higher kurtosis (kurtosis of daily data is much higher than that of weekly and monthly data) suggests that something else is apparent in the data since time varying volatility implies the opposite. The facts that the mean reversion is faster and volatility is less persistent for interest rates compared to stock returns may imply that interest rates dynamics are strongly affected by economic shocks like central bank announcements. Another implication sug-

[7] See S. Das (1999) for the details.

gested by examining the statistical characteristics of the data is that the continuous time approach may not be as appropriate for modeling interest rates as it is for stock returns. Increasing excess kurtosis with increasing sampling frequency supports a discontinuous jumpy behavior rather than a continuous path as Brownian motion dictates.

The main features that one would like to observe in the short rate behavior predicted by any model and in the implied term structure can be roughly listed as follows:

(i) dispersion of rates should be reasonable and negative rates should not be allowed.

(ii) since historically, high rates tend to be followed by a decrease more frequently than an increase, the process should incorporate mean reversion.

(iii) rates of different maturity are imperfectly correlated. The correlation between more distant maturities is less than closer maturities and decrease in correlation is sharper at the short end of the maturity spectrum.

(iv) strong level effect (heteroscedasticity) and volatility clustering. Moreover, short rates display higher volatility and kurtosis than long rates.

No known term structure model captures all of these features at the same time. Perhaps instead of trying to capture all of the stylized facts apparent in the data, the approach should be to choose a model that captures only the essential features regarding the problem at hand. For example, in the case of pricing an option written on yield spreads, capturing the degree of correlation between the yields might be much more important than capturing volatility clustering or positive interest rates. It is known that sometimes seemingly 'poor' models can perform remarkably well in some respects compared to their more advanced and realistic alternatives. Simplicity, tractability and the ability to generate solutions quickly may be more desirable features of a model than its ability to replicate reality.

4. Estimation Methods for a Continuous Time Model of the Short Rate

The estimation of continuous time models has been one of the most studied areas in the financial econometrics literature. Issues surrounding the estimation of the parameters of the continuous time models have been extensively examined as the option pricing literature has become more sophisticated. Perhaps the most important and widely investigated statistical issue regarding the estimation of diffusion models is discretization bias, which arises from two sources. First, the available financial data is observed at discrete time points (time aggregation). This bias can be reduced by collecting data daily, instead of weekly or monthly for example. However in this case one has to deal with the problems of market micro structure such as non-trading days leading to unequally spaced data points. The second, and perhaps more important, source of discretization bias is the fact that only a small number of stochastic differential equations have a closed form solution, therefore the exact likelihood function cannot be derived for the majority of cases.

In order to overcome this difficulty, a number of approximation methods have been proposed in the literature. The Euler-Maruyama method is one of the most widely used first order explicit discretization methods and has proved to be highly convenient for the estimation of the diffusion models. In this paper, we examine the relative performance of five estimation methods used in the literature. It is important to note that there are indeed other methods we do not investigate in this study such as simulated maximum likelihood estimators and Bayesian techniques.[8] Instead, we concentrate on the methods that have been used extensively in the literature. Before discussing the Monte Carlo simulation results, brief explanations regarding the implementation of the methods are provided in the following.

Maximum Likelihood Estimation

When there is a closed form solution to the underlying stochastic differential equation, the likelihood function can be obtained in closed form. Consider the Vasicek model of the short rate:

$$dr_t = k(\theta - r_t)dt + \sigma dw_t \tag{9}$$

The solution to this stochastic differential equation (SDE) is:

$$r_t = \theta + (r_0 - \theta)e^{-kt} + \sigma \int_0^t e^{ks} dw_s \tag{10}$$

where $\int_0^t e^{ks} dw_s$ is the Itô integral with known properties[9]. The exact discretization for the Vasicek Model is:

$$r_{t+\Delta t} = e^{-k\Delta t} r_t + \theta(1 - e^{-k\Delta t}) + \epsilon_{t+\Delta t} \tag{11}$$

where ϵ_t is iid normal with zero mean and variance given by:

$$Var(\epsilon_{t+\Delta t}) = \frac{\sigma^2}{2k}(1 - e^{-2k\Delta t}) \tag{12}$$

[8] For Simulated Maximum Likelihood see Brant and Santa-Clara (1999). For Bayesian techniques see Elarian, Chib and Shephard (1999).

[9] See Elliot (1982) for an introductory treatment of stochastic calculus.

Given the above discretization, one can derive the closed form likelihood function and estimate the parameters θ, k and σ by maximizing the likelihood function.

Naive Estimation

Unfortunately, most of the stochastic differential equations do not have closed form solutions. In order to estimate the structural parameters one can discretize the continuous time model using first or higher order discretization schemes. The Euler-Maruyama discretization (first order explicit scheme) of the Vasicek model is:

$$r_{t+\Delta t} = r_t + k^*(\theta^* - r_t)\Delta t + \epsilon_{t+\Delta t} \tag{13}$$

where $\epsilon_{t+\Delta t} \sim N(0, \sigma^{*2}\Delta t)$. As $\Delta t \to 0$, the sample path of the discrete model approaches that of the continuous time model. The likelihood function for the discrete model is then straightforward to derive and the parameters can be estimated by maximizing the loglikelihood function:

$$\max_{\theta^*, k^*, \sigma^*} \sum_{t=1}^{T} \log f(r_{t+\Delta t}|r_t) \tag{14}$$

where,

$$\sum_{t=1}^{T} \log f(r_{t+\Delta t}|r_t) = Const - T \log \sigma^* - \frac{\sum_{t=1}^{T} (r_{t+\Delta t} - r_t - k^*(\theta^* - r_t)\Delta t)^2}{2\sigma^{*2}\Delta t} \tag{15}$$

Following Gourieroux et al (1993), maximum likelihood estimation based on the discretized model will be called naive estimation throughout this paper. Note that the naive estimator is based on an approximate model (a first order linear approximation of a convex function) and it will provide inconsistent estimates. There are many studies in the financial and statistics literature that address the problem of discretization bias. One way to improve the estimates is to use a higher order scheme. Unfortunately, as the discretization scheme gets more accurate, the derivation of a closed form likelihood function becomes more and more difficult.

Continuous-Updating GMM Estimation

The existence of a solution to an SDE will result in obtaining true population moments of the model to be estimated. If there is no closed form solution to the stochastic model, one can use population moments from the approximate model. For example the discretized model (Euler-Maruyama discretization) of the Vasicek model can be estimated using the following four orthogonality conditions:

$$\begin{aligned} E\left[\epsilon_{t+\Delta t}\right] &= 0 \\ E\left[\epsilon^2_{t+\Delta t} - \sigma^{*2}\right] &= 0 \\ E\left[r_t \epsilon_{t+\Delta t}\right] &= 0 \\ E\left[r_t(\epsilon^2_{t+\Delta t} - \sigma^{*2})\right] &= 0 \end{aligned}$$

where $[1, r_t]$ is the instrument set[10]. Following Hansen, Heaton and Yaron (1996), I programmed the GMM estimator in a way that the weighting matrix is changed for every hypothetical parameter value. The main attraction of this method is its insensitivity to how the moment conditions are scaled. Write the moment conditions as

$$E\left[\varphi(r_t, \rho)\right] = 0$$

[10] See W. H. Greene (1993) for a further discussion on GMM.

where ρ is q-dimensional parameter vector of interest. In this study $\rho \equiv \{\theta, k, \sigma\}$. Assume that $\left(1/\sqrt{T}\sum_{t=1}^{T}\varphi(r_t, \rho)\right)$ converges in distribution to a normally distributed random vector with mean zero and covariance matrix $V(\rho)$. In practice, the weighting matrix used in GMM estimation is $V(\widehat{\rho})^{-1}$ where $\widehat{\rho}$ is a consistent estimator of ρ.

Instead of taking the weighting matrix as given, the Continuous-Updating GMM estimator updates the weighting matrix as $\widehat{\rho}$ is changed in the minimization procedure. This estimator has been shown to perform better in the finite sample compared to other alternative GMM estimators[11].

Indirect Estimation

The absence of a closed form solution to the stochastic differential equation in hand has resulted in extensive research on simulation based estimation of diffusion models. Indirect estimation, developed by Gourieroux et al. (1993) is perhaps one of the most easily implemented simulation based estimation methods for continuous time Markovian models. It is simply a method of matching the stochastic process of the simulated data with that of the real data. Consider again the Vasicek model for spot rate:

$$dr_t = k(\theta - r_t) + \sigma dw_t \tag{16}$$

where we are interested in estimating the parameters $\rho = (\theta, k, \sigma)$.

The first step of indirect estimation is to estimate an auxiliary model using the data. The model estimated can be the one discretized using the Euler scheme. Fixing $\Delta t = 1$,

[11] See Hansen, Heaton and Yaron (1996) for the details of the method.

the model to be estimated is an AR(1) model :

$$r_t = k^*\theta^* + (1-k^*)r_{t-1} + \epsilon_t \tag{17}$$

where $\epsilon_t \sim N(0, \sigma^{*2})$.

This model can be estimated by maximum likelihood and the auxiliary model parameter estimates would be:

$$\widehat{\beta} = \arg\max \sum_{t=1}^{T} \ln f(r_t^d | r_{t-1}^d, \beta) \tag{18}$$

where $r_t^d \equiv \{r_1, r_2, ...r_t\}, \beta \equiv \{k^*, \theta^*, \sigma^*\}$, $f(.)$ is the likelihood function and d denotes data.

The second step is to simulate long data sets for a given parameter value of ρ via the true continuous time model and call it $r_t^s(\rho)$ and estimate $\widehat{\beta}$ (ρ) via maximum likelihood using the auxiliary model. The consistent estimates of ρ are obtained by minimizing the following quadratic function:

$$\min_{\rho}\{(\widehat{\beta} - \widehat{\beta}(\rho))'\Gamma(\widehat{\beta} - \widehat{\beta}(\rho))\} \tag{19}$$

where Γ is a positive definite matrix.

Note that in the above case, the auxiliary model contains the same number of parameters as the true model which makes the model just identified and the estimates are independent of the matrix Γ. Indirect estimation can be done by utilizing an auxiliary model that represents the structural model well in the sense that the parameters have a natural structural interpretation as in the above example. Another approach would be to use an auxiliary model with many more parameters than the structural model to provide a good approximation to a wider range of distributions. This is done to reach asymptotic efficiency by increasing the number of parameters. This approach of indirect estimation was developed by Gallant and Tauchen (1996) and is called Efficient Method of Moments.

Efficient Method of Moments Estimation

The Efficient Method of Moments (EMM) estimator is simply a generalized method of moments (GMM) estimator in which the criterion function to be minimized is the expectation of the score function of auxiliary model for the data. Efficiency is guaranteed with a good choice of an auxiliary model. In this paper, EMM is implemented using the simple first order approximation (AR(1) model) as the auxiliary model as in the indirect estimation[12]. Consider a stationary time series r_t where we are interested in estimating the continuous time parameter vector ρ as in the case of Indirect Estimation illustration. The first step is to estimate the parameters β of the auxiliary model with the data:

$$\widehat{\beta} = \arg\max \sum_{t=1}^{T} \ln f(r_t^d | r_{t-1}^d, \beta)$$

and the information matrix:

$$\widetilde{I} = \sum_{t=1}^{T} \left(\frac{\partial}{\partial \beta} \ln f(r_t^d | r_{t-1}^d, \widehat{\beta}) \right) \left(\frac{\partial}{\partial \beta} \ln f(r_t^d | r_{t-1}^d, \widehat{\beta}) \right)' \tag{20}$$

where $\ln f(r_t | r_{t-1}, \beta)$ is the score generator. Then compute the expectation of the score function with the long simulated series of $r_t^s(\rho)$:

$$m(\rho, \beta) = \frac{1}{T} \sum_{t=1}^{T} \frac{\partial}{\partial \beta} \ln f(r_t^s(\rho) | r_{t-1}^s(\rho), \beta) \tag{21}$$

[12] This is done in order to keep the two simulation based estimators comparable. We do not examine the effect of the auxiliary model choice in this study.

The last step is to minimize the quadratic function below. So, the EMM estimator is:

$$\widehat{\rho} = \arg\min m'(\rho, \widehat{\beta})(\widetilde{I})^{-1} m(\rho, \widehat{\beta}) \tag{22}$$

The EMM estimator is asymptotically as efficient as the MLE if the auxiliary model spans the structural model. Note that in both simulation based estimations (indirect and EMM) the structural parameter vector estimate $\widehat{\rho}$ is determined as:

$$\widehat{\rho} = \arg\min H'\Omega H$$

The indirect estimator differs from EMM in the choice of H and the weighting matrix Ω. Under regularity conditions given in both Gourieroux et al. and Gallant and Tauchen, $\widehat{\rho}$ is asymptotically normal.

5. Monte Carlo Results for Vasicek Short Rate Model

The objective of this study is to investigate the effects of misspecifying the higher moment dynamics of the short rate and examine the finite sample performance of the estimation techniques described in section 4 under such circumstances. All assumed misspecification structures imply some type of omitted higher moment behavior of the series used to estimate the Vasicek short rate model. The Monte Carlo experiments involve simulating the mean, the median and the root mean squared errors of the estimators considered. Although it is very convenient and a simple tool, the mean squared error is not a correct way to assess the finite sample performance of an estimator if the estimator does not have a finite variance. In the simulations none of the estimators yielded an implausible estimate for any of the parameters. Admittedly, this is not enough to justify the usage of the mean squared error criterion since one has to derive the analytical variances to reach such a conclusion. Consequently, we also report the median of the estimators, which is another measure for central tendency that has the attraction of insensitivity to the possible outliers. We also plot density functions.

For the experiments, we construct 500 Monte Carlo samples. The number of observations generated using the assumed data generating processes (true model that is not known to the econometrician) is 1000. For EMM and indirect estimation, the simulated data length is 10,000. In the first set of experiments we assumed that the data generating process (the true model) is in fact the Vasicek model. Then we perform the series of experiments assuming that the true model is: first, the level effect model, second, the jump diffusion model and third, the stochastic volatility model. These experiments imitate the world in which the econometrician estimates an empirical short rate model by misspecifying the higher moment dynamics of the data series. The econometrician estimates the simple Vasicek model when the actual data come from a more sophisticated process (namely the level effect or the jump diffusion or the stochastic volatility process). In the last set of experiments we investigate the functional relationship between the magnitude and the direction of the bias in the parameter estimates of the Vasicek model and the size of the parameters in the omitted feature of the series. For this, we repeat the entire experiment by changing the values assumed for the omitted parameters.

The true values of the unconditional standard deviations for the data generating models cannot be obtained in closed form since the models do not have known exact solutions[13]. In order to approximate the true unconditional standard deviations, we simulated long data sets (50,000) using a very small time interval ($dt = 0.001$).

Small Sample Performance Under the Null

It is important to pin down the small sample performance of the estimators considered under the null as a benchmark so that we can establish the pure effect of misspecification on the parameter estimates. For this purpose, first, we examine the estimators based on the mean squared error criterion under the hypothesis that the data generating process is in fact the Vasicek short rate model. The expected result is that the naive estimator and the GMM estimator based on the discretized model perform poorly even when there exist no misspecification. The bias in the estimates should be entirely due to discretization.

Table 1 and Figure 1 present the simulation results for all estimators under the null hypothesis. It can be immediately seen that the naive and GMM estimators perform poorly in estimating the speed parameter k and the volatility parameter σ. The magnitude of the discretization bias is quite large for these parameters compared to MLE and simulation based methods, EMM and indirect estimation. It is not surprising that the estimates obtained from the maximum likelihood estimation of the exact discretization yield the lowest root mean squared error (RMSE) (.055). The estimates of the unconditional mean θ do not seem to be affected by discretization. All estimators successfully recover this parameter and yield very small RMSE. Moreover, there appears to be no significant dif-

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[13] Except for the Vasicek model itself.

ference in performance between the indirect estimator and EMM. They yield almost the same RMSE for all parameters.

Small Sample Performance Under Misspecification

Table 2 presents the Monte Carlo results under the assumption that the actual data come from a more general diffusion model with conditional heteroscedasticity. Researchers who estimated this model, which nests the Vasicek, CIR and Brennan and Schwartz models, have found the parameter γ to be greater than 1.[14] Given the fact that interest rate data display a strong level effect (that is, the volatility of the series is higher at higher levels of the series) this may be a plausible data generating process. As Figure 2 (a) and Table 2 indicate, in terms of estimating the long-run mean, there seems to be no difference between the estimators. The simple estimators, naive and GMM, perform just as well as MLE and the more demanding methods, EMM and indirect estimation in estimating this parameter. It should be noted that the unconditional variance of r_t is very small and therefore the long run mean is estimated with great precision. However, for the mean reverting parameter k, the simple estimators perform very poorly. The mean and the median estimates of these estimators are not as close to the true value (0.5) as are the simulation estimators. The maximum likelihood estimator using the exact discretization performs the best, followed by the indirect estimator and EMM, based on the RMSE criterion. Notice that the assumed misspecification structure caused a slight increase in the RMSEs for the estimates of the parameter k, even though the unconditional variance

[14] For a detailed discussion of estimation and ergodic properties of the model see Broze, Scaillet and Zakoïan (1995)

of the data generating model is very small. Indirect estimation yields the largest RMSE in estimating the unconditional standard deviation of the data generating model whereas the smallest RMSEs come from the exact maximum likelihood and EMM (see Figure 4 and Table 2).

When the underlying data generating process is a jump-diffusion model as proposed by S. Das (1999), all estimators do equally well in estimating θ. The results are presented in Table 3, Figure 2 (b) and Figure 3 (b). The maximum likelihood estimator based on exact discretization yields the lowest RMSE (0.005) whereas the simulation based estimators result in slightly larger ones. The reason that the mean estimates of the long run mean θ is not effected by the misspecification is that the mean jump size is assumed to be 0. When we assume a non-zero mean for the jump size we expect a large increase in the mean and the median estimates as well as the RMSE (see Table 6). Similar to the level effect case discussed above, parameter k does not seem to be affected by this type of misspecification. Mean squared errors did not change much compared to the benchmark case. EMM, indirect estimator and MLE perform well in estimating k compared to GMM and naive estimator as it was in the benchmark case. Another noticeable result of this experiment is that the unconditional standard deviation of the model is estimated with large RMSEs by the naive and the GMM estimators (0.013 and 0.015 respectively) while the indirect estimator yields the lowest RMSE (0.005).

Under the assumption that the real data come from a stochastic volatility model, all estimators do equally well in estimating θ (Table 4 and Figure 2 (c)). Unlike the results obtained from the level effect and the jump diffusion models, in estimating the parameter k, the RMSEs obtained from the naive and GMM estimators are not larger than the ones obtained from EMM and the indirect estimator. In fact, GMM outperforms EMM based on the RMSE (.1 vs .11). The GMM estimator appears to have a smaller standard deviation leading to a smaller RMSE even though the estimate of its mean shows a sign of bias (.412 while the true value is 0.5). In estimating the unconditional standard deviation, the maximum likelihood estimator yields the lowest RMSE (0.010) whereas the naive and the GMM estimators yield the largest ones (0.017 and 0.019 respectively).

A word of caution regarding the sample size issues is in order. The RMSE is the square root of sum of the variance and the squared bias of an estimator. Naturally, the variance of an estimator decreases as the sample size increases, leading to a smaller RMSE regardless of whether the estimator is biased or not. However, since the bias will not disappear with an increase in the sample size, it is also natural to expect that the biased estimators (naive and GMM) will be outperformed by the unbiased ones (MLE, indirect estimator and EMM) as the sample size grows. The results for the parameter k presented in Table 4 are informative for this matter. Even though the size of the bias is larger for the naive and the GMM estimators, their RMSEs are not larger than the unbiased simulation based estimators. A larger sample size could easily change this result in favor of the unbiased estimators. Since the sample size we use in this study is already fairly large (1000) the gain from using the simulation based estimators may not be worthwhile.

Overall, the mean and the median estimates of the simulation based estimators are very close to the true values of the parameters although there appears to be no substantial difference in the estimated root mean squared errors in some cases (e.g. Table 2 for USTD, Table 4 for parameter k).

Severity of misspecification

It is not implausible to expect that there may be a one to one relationship between the magnitude of the assumed misspecification and the size of the bias in the parameter estimates. In order to have a sense of the magnitude of the specification error bias as a function of the magnitude of the omitted feature of the model we repeat the above experiments with the following changes, made one at a time:

(i) increasing the level effect parameter γ from 0.8 to 1.3 in the first data generation process (level effect model).

(ii) increasing the mean jump size μ in the jump-diffusion model from 0 to 0.05.

(iii) increasing ξ from 0.15 to 0.25 in the stochastic volatility model.

Table 5 presents the distribution of the parameter estimates of the Vasicek model when the underlying data generating process is the level effect model with larger γ. Results for the long run mean and the speed parameter are very similar to what we obtained with the lower γ which are presented in Table 2. A larger γ results in a lower unconditional variance and more precise parameter estimates for all estimation methods.

Table 6 presents the results for the underlying jump-diffusion model. It is not surprising that increasing the mean jump size increases the mean and the median estimates of the long-run mean. One noticeable result of this particular exercise is that the simulation based methods yield the largest RMSEs for all parameters, even though the mean and the median estimates seem fairly close to the true values except for the long-run mean parameter estimate.

Results for the stochastic volatility model are presented in Table 7. As in the jump-diffusion exercise, increasing the unconditional variance resulted in larger RMSEs for all estimators, particularly the simulation based estimators. One noticeable result is that the distributions of the estimates for the parameter k display more asymmetry than the previous cases. However, the median estimates of the simulation based estimators are very close to the true value of this parameter.

Overall, the results lead us to conclude first, the speed parameter k is not affected seriously by misspecification of the empirical model based on the central tendency measures. What is remarkable is that in estimating this parameter, maximum likelihood estimator based on the exact discretization of the Vasicek model yield the lowest RMSE under all misspecification assumptions even though the mean and the median estimates show some sign of bias. It appears that more demanding estimators (EMM and indirect estimators) do not perform as well as the others in estimating this parameter based on the RMSE. However the estimates of the mean and the median of these estimators seem to be very close to the true value of k.

Second, the volatility parameter will be estimated with a serious upward or downward bias depending on the omitted feature of the higher order moments of the short rate series. Ignoring the conditional heteroscedasticity if the actual data are generated by a process like the level effect model will result in severe downward bias in the volatility estimates whereas if the actual data come from a jump diffusion or a stochastic volatility model the parameter estimates will be biased upward.

6. Conclusion

In this study, we simulate the mean, median and the RMSEs of five estimators of continuous time short term interest rate models under three specification error structures related to higher order moment behavior of the interest rate series. The estimation methods which correct for the discretization bias, indirect estimation and EMM estimation, are compared with the ones that do not. The reason that the simple Vasicek short rate model is chosen for the experiments is that its stochastic differential equation has a closed form solution and therefore one can use an exact discretization scheme as a benchmark.

From a practitioner's point of view, it is of interest to understand how these estimators perform under different types of misspecification, since most of the researchers in the field agree that neither simple Vasicek nor any other single factor model can possibly replicate the dynamics of the higher moments of the real short rate series. Given the fact that these models are still very useful and analytically convenient tools for pricing certain types of options, estimating their parameters consistently may turn out to be important.

Overall, the results suggest that the important parameters for pricing fixed income securities would be estimated with a serious bias if one ignores certain features of the interest rate series. Particularly, misspecifying the dynamics of the higher moments may lead to a severe bias in the volatility estimate. The size and the direction of the bias in this parameter depend on the omitted features of the higher moment dynamics of the true data generating process and the estimation method used. Moreover, the simulation based estimation methods, which take the discretization bias into account, perform no better than the ones which do not under specification error. However, the simulation based estimators generally outperform the simple estimators based on some central tendency criteria (mean and median) leading to a conclusion that they are less biased estimators in the case of misspecified higher moment dynamics.

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TABLE 1

Small Sample Results Under Null

Estimation of Vasicek Model: $dr_t = k(\theta - r_t)dt + \sigma dw_t$

True Values: $\theta = 0.1 \quad k = 0.5 \quad \sigma = 0.06$

	θ			k			σ		
method	Mean	Median	RMSE	Mean	Median	RMSE	Mean	Median	RMSE
MLE	.101	.100	.005	.520	.516	.055	.061	.062	.002
NAIVE	.100	.100	.005	.405	.404	.099	.049	.049	.012
GMM	.100	.100	.005	.405	.403	.099	.049	.049	.012
EMM	.100	.100	.007	.487	.481	.067	.059	.059	.003
IND.E	.100	.100	.007	.505	.503	.069	.060	.060	.003

TABLE 2

Small Sample Results for Level Effect

Data Generating Model: $dr_t = k(\theta - r_t)dt + \sigma r_t^{\gamma} dw_t$

True Values: $\theta = 0.1 \quad k = 0.5 \quad \sigma = 0.06 \quad \gamma = 0.8$

Unconditional Standard Deviation of Data generating Model (USTD): 0.0095

Estimation of Vasicek Model: $dr_t = k(\theta - r_t)dt + \sigma dw_t$

	θ			k			USTD		
Method	Mean	Median	RMSE	Mean	Median	RMSE	Mean	Median	RMSE
MLE	.100	.100	.001	.520	.515	.060	.010	.010	.001
NAIVE	.100	.100	.001	.403	.404	.102	.008	.010	.002
GMM	.099	.100	.001	.405	.404	.100	.008	.010	.002
EMM	.099	.100	.001	.501	.496	.069	.010	.010	.001
IND.E	.100	.100	.001	.502	.496	.068	.008	.010	.005

TABLE 3

Small Sample Results for Jump-Diffusion

Data Generating Model: $dr_t = k(\theta - r_t)dt + \sigma dw_t + Jd\pi(\lambda)$

Jump size: $J \backsim N(\mu, \delta)$

True Values: $\theta = 0.1 \quad k = 0.5 \quad \sigma = 0.06 \quad \lambda = 0.2 \quad \mu = 0.0 \quad \delta = 0.12$

Unconditional Standard Deviation of Data generating Model (USTD): 0.08

Estimation of Vasicek Model: $dr_t = k(\theta - r_t)dt + \sigma dw_t$

	θ			k			USTD		
Method	Mean	Median	RMSE	Mean	Median	RMSE	Mean	Median	RMSE
MLE	.100	.100	.005	.521	.520	.049	.086	.086	.007
NAIVE	.100	.100	.007	.406	.404	.101	.068	.068	.013
GMM	.100	.100	.006	.405	.403	.100	.066	.065	.015
EMM	.100	.100	.009	.499	.495	.072	.084	.084	.007
IND.E	.101	.100	.009	.504	.499	.068	.081	.081	.005

TABLE 4

Small Sample Results for Stochastic Volatility

Data Generating Model: $dr_t = k(\theta - r_t)dt + \sigma_t^{0.8} dw_{t,1}$

$$d\sigma_t = \beta(\overline{\sigma} - \sigma_t)dt + \xi\sigma_t^{0.8} dw_{t,2}$$

True Values: $\theta = 0.1 \quad k = 0.5 \quad \overline{\sigma} = 0.06 \quad \beta = 0.1 \quad \xi = 0.15$

Unconditional Standard Deviation of Data generating Model (USTD): 0.075

Estimation of Vasicek Model: $dr_t = k(\theta - r_t)dt + \sigma dw_t$

	θ			k			USTD		
Method	Mean	Median	RMSE	Mean	Median	RMSE	Mean	Median	RMSE
MLE	.099	.100	.008	.527	.521	.063	.076	.075	.010
NAIVE	.101	.100	.006	.410	.411	.106	.060	.059	.017
GMM	.100	.100	.005	.412	.409	.100	.057	.056	.019
EMM	.099	.100	.009	.518	.504	.107	.075	.073	.012
IND.E	.101	.100	.008	.514	.507	.100	.071	.069	.012

TABLE 5

Small Sample Results for Severe Level Effect

Data Generating Model: $dr_t = k(\theta - r_t)dt + \sigma r_t^\gamma dw_t$

True Values: $\theta = 0.1 \quad k = 0.5 \quad \sigma = 0.06 \quad \gamma = 1.3$

Unconditional Standard Deviation of Data generating Model (USTD): 0.0027

Estimation of Vasicek Model: $dr_t = k(\theta - r_t)dt + \sigma dw_t$

	θ			k			USTD		
Method	Mean	Median	RMSE	Mean	Median	RMSE	Mean	Median	RMSE
MLE	.100	.100	.0002	.513	.509	.044	.003	.003	.0003
NAIVE	.100	.100	.0002	.403	.404	.100	.003	.003	.0003
GMM	.100	.100	.0003	.404	.404	.100	.002	.002	.002
EMM	.100	.100	.001	.463	.460	.078	.003	.002	.002
IND.E	.100	.100	.001	.466	.461	.076	.001	.0002	.003

TABLE 6

Small Sample Results for Severe Jump-Diffusion

Data Generating Model: $dr_t = k(\theta - r_t)dt + \sigma dw_t + Jd\pi(\lambda)$

Jump size: $J \backsim N(\mu, \delta)$

True Values: $\theta = 0.1 \quad k = 0.5 \quad \sigma = 0.06 \quad \lambda = 0.2 \quad \mu = 0.05 \quad \delta = 0.12$

Unconditional Standard Deviation of Data generating Model (USTD): 0.082

Estimation of Vasicek Model: $dr_t = k(\theta - r_t)dt + \sigma dw_t$

	θ			k			USTD		
Method	Mean	Median	RMSE	Mean	Median	RMSE	Mean	Median	RMSE
MLE	.120	.120	.021	.524	.521	.049	.087	.087	.007
NAIVE	.123	.123	.024	.406	.406	.101	.068	.068	.012
GMM	.120	.119	.021	.408	.405	.098	.068	.068	.012
EMM	.124	.122	.032	.472	.470	.145	.089	.085	.013
IND.E	.120	.121	.029	.483	.475	.139	.083	.084	.015

TABLE 7

Small Sample Results for Severe Stochastic Volatility

Data Generating Model: $dr_t = k(\theta - r_t)dt + \sigma_t^{0.8} dw_{t,1}$

$$d\sigma_t = \beta(\overline{\sigma} - \sigma_t)dt + \xi\sigma_t^{0.8} dw_{t,2}$$

True Values: $\theta = 0.1 \quad k = 0.5 \quad \overline{\sigma} = 0.06 \quad \beta = 0.1 \quad \xi = 0.25$

Unconditional Standard Deviation of Data generating Model (USTD): 0.094

Estimation of Vasicek Model: $dr_t = k(\theta - r_t)dt + \sigma dw_t$

	θ			k			USTD		
Method	Mean	Median	RMSE	Mean	Median	RMSE	Mean	Median	RMSE
MLE	.099	.099	.008	.525	.552	.063	.089	.091	.015
NAIVE	.100	.101	.008	.441	.437	.102	.076	.074	.028
GMM	.100	.100	.008	.416	.410	.117	.074	.071	.029
EMM	.098	.098	.023	.526	.502	.227	.095	.090	.025
IND.E	.100	.100	.027	.511	.484	.196	.094	.094	.040

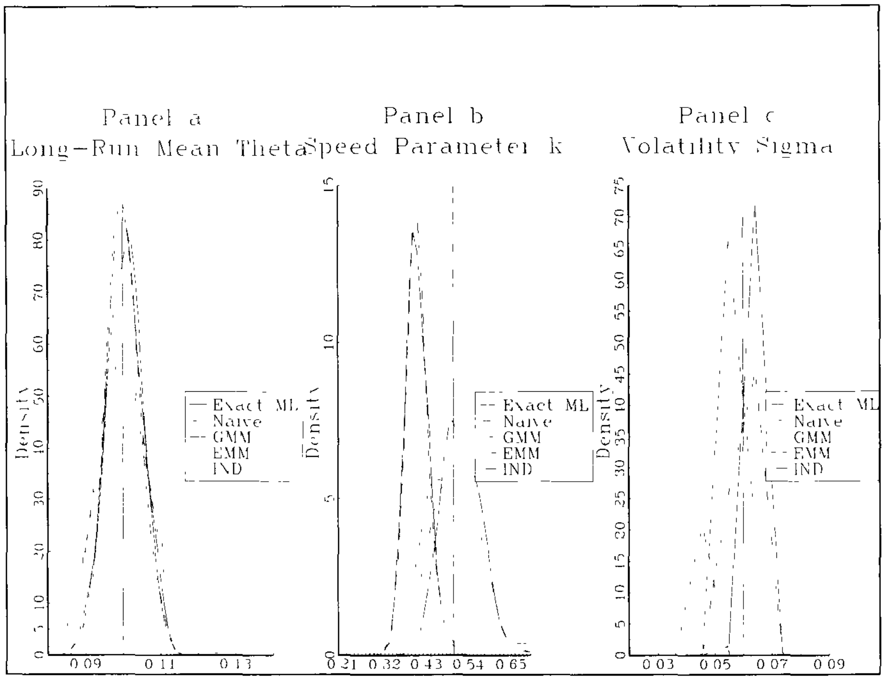

Figure 1: Densities of the Parameter Estimates of Vasicek Model

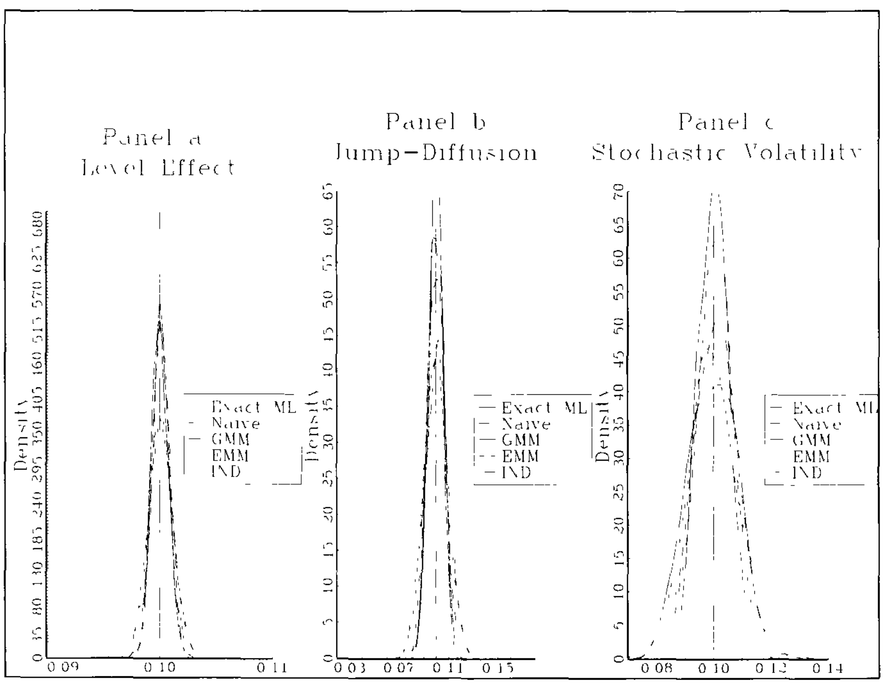

Figure 2: Densities of the Parameter Estimates of the Long-Run Mean θ

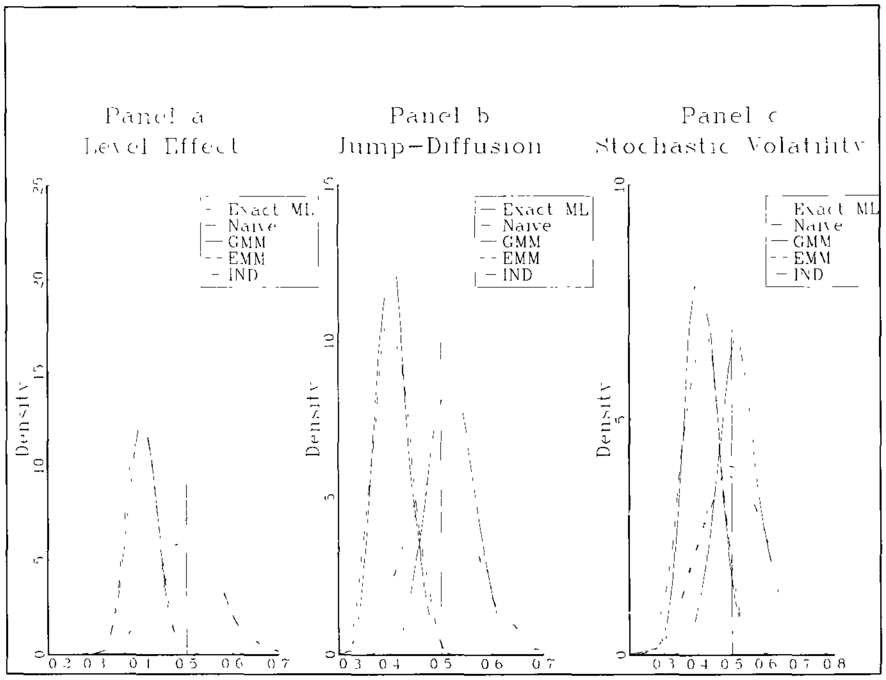

Figure 3: Densities of the Parameter Estimates of the Mean Reversion Speed k

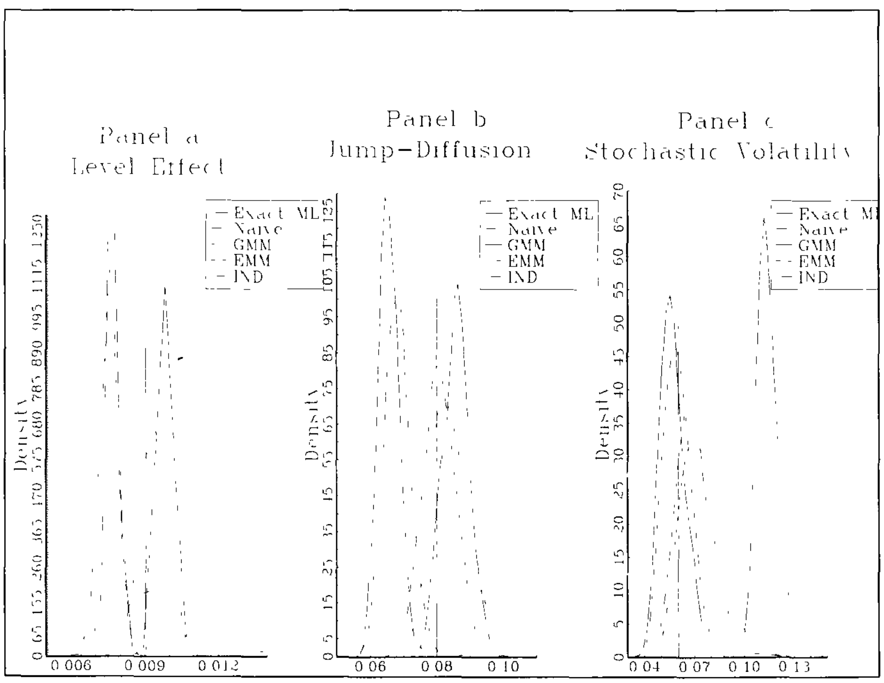

Figure 4: Densities of the Parameter Estimates of the Unconditional Standard Deviation

III

Determination of Optimal Margin Requirements for Futures Contracts: A Markov Chain Model

Margins on futures contracts are the most distinguishing safeguards of futures markets. *Initial margins* are security deposits that the traders must post with their brokers to be allowed to trade futures contracts. The *maintenance margin* is the level of the deposit at which the trader is required to replenish the margin and it arises from the fact that daily communication with traders for issuing and receiving transfers due to price fluctuations would be prohibitively costly. This paper establishes a theoretical rationale for positive initial and maintenance margins from the view point of futures commission merchants (brokers from here on). The margin setting problem is analyzed in a profit maximization context using a Markov chain approach. A version of the model is implemented numerically and for the values used, the optimal initial and maintenance margins set by the brokers are positive, as a result of transaction costs and trader default risk, and without any interference by an outside authority.

The existence of initial margins reduces the risk of a trader's default, and therefore the risk of a broker's loss. On the other hand, high initial margins reduce the attractiveness of the futures markets due to the costs they impose on the traders[1]. Researchers in finance and economics have investigated the determination of the optimum margin setting rule by exploiting this trade off. Figlewski (1984) uses a statistical model in order to set

[1] Telser (1981) shows that an exogenous increase in the initial margin on an asset reduces the weight of the asset in the optimal portfolio (substitution effect) and reduces the expected return from the optimal portfolio (total effect).

margin levels to yield a given level of protection against margin violation. In a more recent paper, Longin (1999) develops a statistical method to set the initial margin level based on extreme value theory. Gay et al. (1986) set up an analysis where a margin committee maximizes the aggregate utility of exchange members by considering price volatility, market liquidity, current and expected market conditions, etc. They do not assume explicit utility functions or an aggregation scheme. Instead, they derive testable hypotheses directly from distributional assumptions regarding the futures price process.

The system of maintenance margins is another tool to reduce the risk exposure of the broker since a high maintenance margin for a given level of initial margin is associated with a lower risk of default. However, frequent margin calls that are the consequences of a high maintenance margin (for a given level of initial margin) result in high transaction costs incurred by the broker. It should be noted that in this paper, both the quantifiable (clerical) and unquantifiable (customer annoyance) costs associated with the margin calls will be referred to as *transaction costs*. High maintenance margins may also have a negative impact on volume since it translates into more frequent communication with the broker regarding the necessary fund transfers.

Telser (1981) presents a detailed economic analysis where initial margins are the result of competitive market forces, not legal enforcements. In this article, following the same line of argument as Telser (1981), a complete optimization model that rationalizes the existence of both maintenance and initial margins is set up. A numerical profit maximization approach in which the levels of both initial and maintenance margins are decision variables of the futures broker is adopted and positive margins are derived. Instead of modeling a single trading day to set the margins, as has usually been done in the literature, the entire life span of the contract and the expected costs and profits associated with the trading of such a contract are considered.

This paper extends the existing literature on optimal margin setting in 3 major ways: First, although there has been extensive research on initial margin setting within the context of expected cost minimization, to our knowledge, there is no theoretical margin setting model that rationalizes a positive maintenance margin as another tool to reduce the broker's risk exposure. Although Figlewski (1984) incorporates the maintenance margin as well as the initial margin as tools to assure performance, he does not provide a structural model that gives a theoretical basis for positive margins. Similarly, Gay et al. (1986) do not have an explicit theoretical margin setting model even though they argue that the margins are results of an optimization decision of the margin committee. Second, this paper sets up an economic model in which the privately owned brokerage firms compete with each other for customers (traders) in a monopolistic market structure with restricted entry where they use initial and maintenance margins as tools for competition. It is important to note that unlike previous studies which tried to focus on hard-to-define objectives of the futures exchange, the model presented in this article is an economic model with well-defined objectives of the agents (brokerage firms). Finally, the inclusion of a maintenance margin and the consideration of an n-period contract make the analytical solution impossible. Therefore a new numerical solution technique is developed using the properties of first order Markov processes.

Two cost factors associated with futures contract trading from the broker's perspective are identified: (i) losses incurred by the broker in the case of a default, and (ii) transaction costs which result from margin calls. Intuitively, when the transaction cost approaches zero, the maintenance margin is expected to approach the initial margin level.

In this paper, futures brokers are viewed as private profit maximizing agents and it is assumed that the futures exchange does not provide any form of protection to its troubled member brokers, therefore it does not need to impose a minimum margin level.[2] It is important to note that this is a partial equilibrium model. Since the trader's preferences are not considered within the context of his optimal portfolio allocation, this study is not an attempt to derive equilibrium margins. Simply, a contract demand equation of a trader faced by the broker is defined as a function of initial and maintenance margins, instead of deriving such relationship through the investor's preferences, since otherwise the problem requires a complicated general equilibrium approach. In this basic setting, it is shown that both initial and maintenance margins are the consequences of optimizing behavior in a market where futures brokers compete for customers.

Most assumed functional relations within the model are supported by empirical evidence in the literature. It is assumed that there is a negative relationship between the number of contracts sold and the initial margin level. The vast majority of empirical

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2 In fact, in a competitive market the absence of protection of futures brokers by the exchange will reduce the broker's willingness to take unnecessary risks. Therefore, the minimum margin requirement is likely to become an unbinding constraint. See Telser (1981) for a detailed treatment of this issue.

research investigates the impact of margins on futures price volatility and trading volume. There seems to be a great deal of empirical evidence suggesting that high margins slow the market activity. Tomek (1985), Fishe and Goldberg (1986), Hartzmark (1986) and Adrangi and Chatrath (1999) find that high margins are associated with lower open interest.

Although there is a general agreement on the negative relationship between market activity and margin levels, the evidence regarding the relationship between margins and futures price volatility is somewhat mixed. The simple linear regression method was extensively utilized to establish the impact of margin changes on futures price volatility. While Hartzmark (1986) finds no evidence on positive relationship between margins and volatility, Hardouvelis and Kim (1996) finds a positive effect. Fishe et al (1990) find a negative but insignificant effect using ten CBT (Chicago Board of Trade) contracts. Even if evidence indicates a correlation between volatility and margin levels, it is hard to establish the direction of the influence empirically. The futures exchange board takes price volatility into account as an important factor in determining minimum margin requirements, therefore it is sensible to model the initial margin as a function of volatility. Fishe et al. (1990) find a positive and significant effect of volatility on margins. In the proposed model in this study, a positive effect of volatility on the optimum initial and maintenance margin levels is built in. It is natural to assume that high price volatility increases the likelihood of customer default, and hence increases the broker's risk exposure. Since the initial margins are security deposits, in our view, it is plausible to assume that a profit maximizing agent (broker) will react to increasing price volatility by increasing the margin requirements.

The remainder of the paper is organized as follows: The next section provides brief background information on some institutional details regarding futures markets and trading. In Section 3 the complete mathematical model is developed and analyzed. Finally, Section 4 summarizes our results.

2. Futures Contracts and Markets

Futures contracts can be thought of as highly institutionalized forward contracts. They differ from other forms of forward contracting on four principal grounds. First, they are traded in an organized *exchange* which has special institutional procedures designed to enable agents to trade efficiently. Second, each futures exchange operates a *clearinghouse* that stands behind the performance of the contracts, mainly protecting each trader against the risk of individual default by the other side of the trade. Third, these clearinghouses require that a certain amount of initial money is deposited into an account for each contract, known as a *margin account*. These accounts are resettled with respect to the current price of the futures commodity ("marking to market") daily. Finally, as opposed to most forward contracts, futures contracts are traded in *standardized* forms.

The functioning of the margins in futures markets is quite different from that of margins in stock markets. They are not credits as they are in stock markets. Instead, they are security deposits which are continuously updated throughout the life of the contract in order for the contract to be settled gradually. The amount of the margin varies from contract to contract and from broker to broker. It can be posted in cash or in short-term T-bills and it acts as a good faith deposit. The margin requirement is insurance for a future obligation, and also one of the most important regulatory tools that sustain the integrity of the markets.

The broker collects the initial margin amount from investors when the contract is sold and the margin accounts continuously reflect the daily futures price changes thereafter. Since there must be one long and one short position for each contract, the total net position will always be zero and at the delivery date the party with the short position must be prepared to make the necessary delivery to the party with the long position. While an increase in the futures price will increase the account level for the long position, it will decrease it for the short position by the same amount. When an account falls below the maintenance margin level, a margin call is issued by the broker. The investor is required to make the necessary payment to restore the account back to its initial deposit level. At that point, if the investor's account balance is negative, i.e., his loss due to the adverse price movement exceeded his margin account, the broker faces the risk of individual default. The contracts can be closed through either delivery of the underlying commodity at the maturity date or entering a reversing trade[3] any time before the maturity date.

Figure 1 represents a sample path of a margin account. On the first day, the account starts at the initial margin level (M) and its level changes according to the daily price movements. Since the account goes below the maintenance margin (m) at time t_5, there is

[3] This is the action that brings a trader's net position for a particular contract to zero. In fact, almost all futures contracts are closed just before the maturity date through a reversing trade instead of real delivery.

a margin call at that time and the account is restored back to the level M instantaneously because of the margin payment. At time t_{10} the account balance becomes negative in which case the investor can default and the amount shown in the graph would be the broker's loss if such default occurs. The model's assumptions regarding the timing of margin payments and the magnitude of broker's loss in case of a default are explained in Section 3.

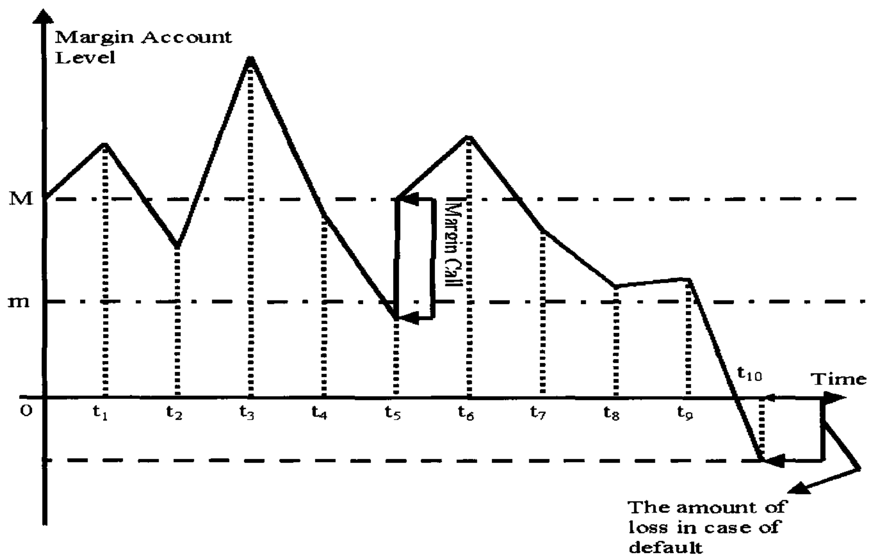

Figure 1: A sample path of margin account.

3. Mathematical Model

The problem considered is to determine the optimal initial and maintenance margins as well as the optimum average commission set by the futures broker for a particular $n-$period contract under futures price uncertainty. The steady state probabilities of a margin violation and a margin call are determined given the trader's probability of default and distributional assumptions regarding the futures price process. The probability of default for a given trader for a given trading day is a function of his account balance, his expectations about the futures price movements, the time to expiry date and his personal characteristics. For example a trader with a long position whose balance becomes negative today may choose to default if he expects a further price decrease and if the contract is near its end.

Ideally, one would like to observe all the relevant variables so that these probabilities can be estimated for each trader. Unfortunately, researchers cannot observe traders' characteristics nor can they observe their account balance for each trading period. Since the number of margin violations is not observed either, it is not possible to estimate these probabilities. But brokers have access to most of these variables, therefore it is feasible for them to attach an estimated probability of default to a particular trader (or, more realistically, to a particular type of trader). Conditional on all other relevant variables, the probability of default is expected to be higher for a negative account balance that is larger in absolute value.

Definitions and Assumptions

The assumptions and definitions of the model are as follows:

1- The broker is a risk neutral, profit maximizing agent.

2 -There is no minimum margin requirement imposed by the exchange.

3- The stochastic process F_t represents the futures price at the end of period t. Then $\Delta F_t, t = 1, 2, \ldots$ represents the daily price change and it is assumed to be independently and identically distributed (i.i.d.). Although the proposed Markov chain approach allows for the use of any type of price change distribution provided that ΔF_t are i.i.d. random variables, in this study ΔF_t is assumed to be distributed normally with zero mean and standard deviation σ.

4- Investors can always borrow or lend money at the prevailing market interest rate.

5- An investor can default if his account level hits a negative value as a result of daily price changes. If a default occurs, the broker incurs a loss and the magnitude of this loss is equal to the trader's current account balance[4]. There is no economic incentive for the investor to default if he has a positive account balance.

6- Margin payments are made instantaneously when the call is issued, otherwise the default occurs. In other words, there is no grace period.

7- All investors have the same risk of default conditional on their account balance, so there is no reason for the broker to impose different margins for different customers[5]. The trader's choice of a particular broker is given and the brokers are not regarded by customers as perfect substitutes.

8- The broker is assumed to have a fixed number of traders. In the case of a trader default or position closing another trader is assumed to buy the contract.

9- The number of brokers is fixed by barrier to entry. Brokers have two tools for competition: maintenance and initial margins. Every broker faces his own demand equa-

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4 In fact, the cost incurred by the broker in case of a default may be different than the amount shown in Figure 1 since there will be legal follow ups in order to recover the loss. The assumption is solely for computational simplicity.

5 In fact, the traders are typically speculators who hope to make a profit by taking risk and hedgers who use futures markets to reduce some risk. Even though our model can easily be extended to allow for this fact, for computational simplicity, we assume only one type of trader.

tion for a particular trader. He sets initial and maintenance margins to maximize profits. The number of contracts sold is assumed to be a decreasing function of the initial and maintenance margins.

The Markov Chain Model

The level of an investor's margin account balance per contract, A_t, is the amount of deposit per contract in his account at the end of period t. The margin account fluctuates daily according to the daily price fluctuations. The model is set up in a way that the Markovian nature of the futures price process translates into a Markovian account balance process with upper and lower bounds. In order to have a tractable margin setting model, the realizations of the random variable A_t are represented on a defined grid. In other words, even though, A_t can take any value on the real line in a given trading period, by discretizing the state space, it is allowed to take only a finite number of specified values.

Moreover, to create the Markov matrix for the necessary calculations, the possible margin account levels are rounded down to the lower end of its interval. If, for instance, A_t is between 1.5 and 2 the account level is assumed to be 1.5. Let A_t^* be the rounded value of A_t. The following equation describes the level of the margin account for an investor with the long position:

$$A_t = \begin{cases} A_{t-1}^* + \Delta F_t & \text{if } m < A_{t-1} + \Delta F_t \\ M & \text{if } 0 \leq A_{t-1} + \Delta F_t \leq m \\ M & \text{if } A_{t-1} + \Delta F_t < 0 \text{ with probability } 1 - q(A_{t-1} + \Delta F_t) \\ \text{default} & \text{if } A_{t-1} + \Delta F_t < 0 \text{ with probability } q(A_{t-1} + \Delta F_t) \end{cases}$$

Similarly, the account level for a short position is given by the following equation :

$$A_t = \begin{cases} A^*_{t-1} - \Delta F_t, & \text{if } m < A_{t-1} - \Delta F \\ M, & \text{if } 0 \leq A_{t-1} - \Delta F \leq m \\ M, & \text{if } A_{t-1} - \Delta F_t < 0 \text{ with probability } 1 - q(A_{t-1} - \Delta F_t) \\ \text{default}, & \text{if } A_{t-1} - \Delta F_t < 0 \text{ with probability } q(A_{t-1} - \Delta F_t) \end{cases}$$

where q (default probability) is a step function of the account level. Remember that a negative account balance that is larger in absolute value implies a higher default probability for a given trader.

Allowing only for discrete values for the level of margin account, it is possible to generate a Markov matrix that contains one-period transition probabilities for each state. Each possible attainable account level represents a state of the Markov chain. The transition probabilities are determined by the probability distribution of futures prices and the exogenous default probabilities. Suppose that we define $2k + 1$ grid points to represent the attainable account levels where the first k are positive, the $(k + 1)$st is zero and the remaining k are negative with the same magnitude as the positive ones in absolute value. Let $g_{\max}$ and $g_{\min}$ denote the maximum and minimum grid points. Then, $(g_{\max} + |g_{\min}|)/(2k)$ is the step size for the discretization. By dividing the interval between $g_{\max}$ and $g_{\min}$ into $2k$ sub-intervals, we obtain $2k + 1$ grid points and hence $2k$ states.

Example 1: *Suppose* $k = 2$, $g_{\max} = 2$ *and* $g_{\min} = -2$. *Then, the discretization step size is* $(2 + 2)/4 = 1$. *Hence we obtain* $2k + 1 = 5$ *grid points as* $[2, 1, 0, -1, -2]$. *The* *account level states are then defined as follows:*

State	**Interval**
1	$1 < A_t \leq 2$
2	$0 < A_t \leq 1$
3	$-1 < A_t \leq 0$
4	$-2 \leq A_t \leq -1$

$\triangleleft$

Let the sth and ℓth grid points, g_s and g_ℓ represent the initial and maintenance margins levels respectively so that $M = g_s$ and $m = g_\ell$ satisfying the condition: $1 \leq s \leq \ell \leq k$. Note that the grid points are subscripted in descending order. The process will never stay in the states between ℓ and 0 due simply to the assumption of an instantaneous payment after a margin call. There is no economic incentive for the investor to default when the account balance is positive since he can instead settle the contract by taking an opposite position and receive his remaining deposit back. Hence, he will make the margin payment and restore his margin account back to its original level M with probability 1 if the account drops to a value between m and 0.

Other than the states from 1 to k, additional states are necessary to represent the default event by which we can determine the amount of loss incurred by the broker when a trader defaults. Each of the k negative account levels represents a different default state. Moreover, each default state has another exogenous default probability attached to it. Even though the default probabilities are treated as exogenous and assumed to be the same for all the traders, they take different values for each default state since the probability of a default must be higher at the lower account levels for a given trader.

The broker also incurs a transaction cost every time a margin call is issued. This can be thought of as clerical costs or any type of costs associated with communicating with the customer (trader) to obtain the necessary funds for the margin account. In order to calculate this cost, it is necessary to know the frequency of the margin calls during the lifetime of a contract and the magnitude of the cost associated with each call. The account level can attain the initial margin level either because of the daily price changes or because of the margin payments. It is necessary to distinguish those two cases, since the transaction cost is incurred by the broker only when a margin call is issued. This can be achieved by defining an extra state (Z) that keeps track of the margin calls. The process will go to state Z with probability 1 if the margin account hits any level between m and 0 since there will be a margin call when the trader's account balance is still positive. The process will go to state Z with probability $(1 - q)$ if the margin account hits a negative value since there will be a margin call but the trader may default with the probability q where q is $k \times 1$ vector whose elements are indexed from $k + 1$ to $2k$. It is important to note that after attaining the state Z, the behavior of the process will be identical to that after the initial margin state. With this additional state, the state transition matrix will be of size $2k + 1$.

If the trader defaults we assume that the contract is taken up by another trader so that the process goes to the initial margin state.

For an account that is in a long position, the state transition probabilities can be written as

$$P_{ij} = \begin{cases} p_{ij} & \text{if } 1 \leq i < \ell \text{ and } 1 \leq j < k+1 \\ q_j p_{ij} & \text{if } 1 \leq i < \ell \text{ and } k+1 \leq j \leq 2k \\ \sum\limits_{j=\ell}^{k} p_{ij} + \sum\limits_{j=k+1}^{2k} p_{ij}(1-q_j) & \text{if } 1 \leq i < \ell \text{ and } j = 2k+1 \\ 0 & \text{if } \ell \leq i < k+1 \text{ and } 1 \leq j \leq 2k \\ 1 & \text{if } \ell \leq i < k+1 \text{ and } j = 2k+1 \\ 0 & \text{if } k+1 \leq i \leq 2k \text{ and } j \neq \ell \\ 1 & \text{if } k+1 \leq i \leq 2k \text{ and } j = \ell \\ p_{sj} & \text{if } i = 2k+1 \end{cases}$$

where P_{ij} is the probability that the process goes to the state j given it is in the state i.

(i) If the process is between $g_{\max}$ and the maintenance margin level $(1 \leq i < \ell)$, it will go from state i to a default state $(k+1 \leq j \leq 2k)$ with the probability $q_j p_{ij}$, and it will go to the state Z with the probability $\sum\limits_{j=\ell}^{k} p_{ij} + \sum\limits_{j=k+1}^{2k} p_{ij}(1-q_j)$. Remember that the process can go to the state Z due only to a margin call. A margin call will be executed if the account level goes to a state lower than the maintenance margin. In the case of a positive account level at the time of the call the account will be restored back to its initial margin level with probability one. The first term $\sum\limits_{j=\ell}^{k} p_{ij}$ corresponds to such a case. However, if the account level is negative at the time of the margin call then the account will be replenished with the probability $(1-q_j)$. This case corresponds to the second term $\sum\limits_{j=k+1}^{2k} p_{ij}(1-q_j)$.

(ii) If the process is at a margin call state but still positive $(\ell \leq i < k+1)$, it will go to the state Z with probability 1. It cannot go anywhere else.

(iii) If the process is at a default state ($k+1 \leq i \leq 2k$), (the case in which the trader chooses to default) the contract is assumed to be taken up by someone else, so the process goes back to initial margin state s.

(iv) If the process is at the state Z ($i = 2k+1$), it evolves as if it is at the initial margin state s.

In order to calculate the expected default and transaction costs, define the states of the transition matrix as $\{g_1, \ldots, g_s, \ldots, g_\ell, ...g_{k,}d_{k+1}, \ldots, d_{2k}, Z\}$. Here, g_i represents positive account levels with $g_1 = g_{\max}$, and d_i represents negative account levels (default states) with $d_{2k} = g_{\min}$. Finally, Z represents the state to which the process goes after a margin call.

Before proceeding to the calculation of expected values (i.e., costs and profits) and the analysis of the model it would be informative to illustrate the use of the model in practice with an example:

Example 2: *Assume that we set the maximum and minimum grid points as* 2 *and* -2, *respectively. Moreover, assume that* $k = 4$. *Then the step size for the discretization is*

$$\frac{g_{\max} + |g_{\min}|}{2k} = \frac{4}{8} = 0.5.$$

Thus, the grid points for the account level are given as

$$[2, 1.5, 1, 0.5, 0, -0.5, -1, -1.5, -2]$$

which are indexed as in the following table describing the 8 states:

State	Interval
1	$1.5 < A_t \leq 2$
2	$1 < A_t \leq 1.5$
$\vdots$	$\vdots$
8	$-2 \leq A_t \leq -1.5.$

For example, if the account level reaches, say, 1.355 *the process will be in the state* 2, *whereas if it hits the value* -1.8, *the process will be in state* 8 *and so on.*

Now, assume that at period $t = 0$ *we set the initial margin level to* 1.5 *and maintenance margin level to* 0.5, *i.e.,* $M = g_s = 1.5$ *(implying* $s = 2$*) and* $m = g_\ell = 0.5$ *(implying* $\ell = 4$*). The default states are the ones that correspond to negative account values as follows:*

Default State	Interval	State
d_5	$-0.5 \leq A_t < 0$	5
d_6	$-1 \leq A_t < -0.5$	6
d_7	$-1.5 \leq A_t < -1$	7
d_8	$-2 \leq A_t < -1.5$	8

To construct the transition probability matrix, we recall that P_{ij} *is the probability of the process going into state* j *given it was in state* i *in the previous period. Hence,*

$$p_{11} = \Pr(1.5 < A_t \leq 2 \mid 1.5 < A_{t-1} \leq 2) = \Pr(0 \leq \Delta F_t < \infty).$$

show the relationship between the average margin call cost (per contract sold) and the initial and maintenance margins respectively, or equivalently

$$p_{11} = \Pr(A_t^* = 1.5 \mid A_{t-1}^* = 1.5\,) = \Pr(0 \leq \Delta F_t < \infty).$$

Since we assume $\Delta F_t \sim N(\mu, \sigma)$ *this value can be obtained easily when* μ *and* σ *are specified.*

Similarly, we find

$$p_{12} = \Pr(1 < A_t \leq 1.5 \mid 1.5 < A_{t-1} \leq 2) = \Pr(-0.5 \leq \Delta F_t < 0).$$

or

$$p_{12} = \Pr(A_t^* = 1 \mid A_{t-1}^* = 1.5) = \Pr(-0.5 \leq \Delta F_t < 0).$$

if the account level attains a value between 0.5 *and* 0 *(state* 5*) there will be a margin* *call and an instantaneous payment will be made with probability 1. Together with* k *default states and the additional state* Z*, the size of the transition matrix is* $2k+1=9$.

If the account level hits a value say between -0.5 *and* -1 *(i.e., the default state* d_5*), there will be a margin call followed by either an instantaneous payment or a default. Hence, the account level will be restored to the level* $M=1.5$ *and the process will go to state* Z *with probability* $(1-q_5)$ *or the investor will default and the process will go to the state* s *with probability* q_5.

A few of the other probabilities in the transition matrix are obtained as

$$\begin{aligned} p_{16} &= q_6 \Pr(-1 \le A_t < -0.5 \mid 1.5 < A_{t-1} \le 2) \\ &= q_6 \Pr(A_t^* = -1 | \, A_{t-1}^* = 1.5) \end{aligned}$$

and

$$\begin{aligned} p_{26} &= q_6 \Pr(-1 \le A_t < -0.5 \mid 1 \le A_{t-1} < 1.5) \\ &= q_6 \Pr(A_t^* = -1 | \, A_{t-1}^* = 1). \end{aligned}$$

The remaining probabilities can be obtained in a similar manner. $\triangleleft$

After defining the transition matrix, one can calculate the steady state probabilities and derive the expected cost and profit functions. The speed at which the matrix converges to the steady state depends on the relative magnitude of the standard deviation of the price change to the discretization step size. In order to justify the use of the steady state probabilities for cost and profit calculations, parameters of the model should be chosen carefully. Too small variation in the change in price process relative to the step size of the discretization will rule out the desired convergence.

Expected Cost and Profit Functions

The steady state probabilities are the probabilities of being in a given state when the steady state is reached, i.e., unconditional probabilities. Steady state probabilities associated with each state for the long and the short positions are denoted as π_i^L and π_i^S respectively where $i = 1, ..., 2k+1$. Since it is assumed that all traders have the same risk of default and the price change distribution is symmetric with zero mean, steady state probabilities for both positions become equal and the superscripts can be ignored. The steady state probability π_{2k+1} corresponding to the state Z is the probability of a margin call for a position in the steady state.

If c is the unit cost of each margin call then the expected transaction cost per period, per contract sold will be:

$$E(C_{Tr}) = c\pi_{2k+1}.$$

Whenever a default occurs, the amount lost by the broker is the account's current negative balance. Then, the expected cost of default per period, per contract that broker incurs is given by:

$$E(C_D) = -\sum_{i=k+1}^{2k} d_i \pi_i.$$

Thus, the total per period expected cost incurred by the broker is:

$$E(C_T) = -\sum_{i=k+1}^{2k} d_i \pi_i + c\pi_{2k+1}$$

Note that since the transition probabilities are determined by the parameters of the underlying distribution, default probabilities of traders and the levels of M and m, steady state probability π is some function of $M, m, \ \sigma, \ \mu$ and q.

If γ denotes the fixed commission charged by the broker per period per contract, then the revenue function for the broker is:

$$R = n\gamma Q(M, m)$$

where, n is the maximum number of periods of the contract and $Q(M, m)$ can be thought of as a particular trader's demand function for a futures contract[6]. This function is assumed to be decreasing in all its arguments.

Combining the above expressions, for an n period contract, the broker's expected profit is given by:

$$E(\Pi) = Q\left(M, m\right) n \left[\gamma - E(C_{Tr}) + E(C_D)\right]$$

or,

$$E(\Pi) = Q\left(M, m\right) \left[n \left(\gamma - \sum_{i=k+1}^{2k} d_i \pi_i + c\pi_{2k+1}\right)\right]$$

[6] Commission is assumed to be charged per period per contract by the broker. Every conract has n periods. In the case of a default another identical trader is assumed to take up this position (see assumption 8). Therefore, even if there is a default every contract lives for exactly n periods.

Analysis of the Model

Functional Forms

The revenue function is assumed to be decreasing in M and m. The expected cost function, which is sum of the expected default and expected transaction costs, is constructed such that:

$$
\begin{aligned}
\frac{\partial E(C_D)}{\partial M} &< 0 \\
\frac{\partial E(C_{Tr})}{\partial M} &< 0 \\
\frac{\partial E(C_{Tr})}{\partial m} &> 0 \\
\frac{\partial E(C_D)}{\partial m} &< 0
\end{aligned}
$$

Unlike the previous optimum margin setting literature, which is mainly based on a cost minimization assumption of the clearinghouse, the expected cost function constructed in this study is decreasing everywhere with respect to the initial margin. In fact, it is clear that high initial margins impose an opportunity cost on traders. In the proposed model, this cost is represented in the revenue function of the broker through a contract demand equation. In practice, it is easier to quantify this cost in terms of the number of contracts sold by estimating a simple demand equation.

A numerical illustration

The expected profit function can be obtained numerically by calculating the profit for each possible initial and maintenance margin levels. This can be done by constructing a simple loop that calculates the expected profit for every possible combination of the decision variables within the grid after constructing the transition matrix. The levels that give the maximum expected profit are the optimal initial and maintenance margins. Clearly, a finer discretization leads to longer computation time. As the discretization step size approaches zero, the numerical solution is expected to approach the analytical one. The simulation results suggest that the proposed model is stable in the sense that as we define a finer grid the solution does not change very much. It is very important to keep in mind that for desired convergence, the parameters of the model should be chosen wisely in relation to the discretization step size.

In this section, a simple hypothetical example is illustrated where the daily price change of a unit of a futures commodity is normally distributed with zero mean and standard deviation 0.7. The contract demand function is assumed to be linear:

$$Q = \alpha + \beta_1 M + \beta_2 m$$

The grid range $(2.5, -2.5)$ is divided into 100 points. The demand parameters and the default probabilities are taken as $\alpha = 300$, $\beta_1 = -10$, $\beta_2 = -1$, $q_i = \{0.005, \ldots, 0.01\}$ where $i = k + 1, \ldots, 2k$, $c = 0.005$ and $\gamma = 0.045$. Finally the life of the contract $n = 100$. The solution to the numerical profit maximization is $M^* = 1.55$, $m^* = 0.8$, for the initial margin and the maintenance margin respectively (per contract). Figure 2 presents the expected total profit surface for our hypothetical contract. The flat region in the graph corresponds to implausible combinations of initial and maintenance margins (where the maintenance margin is greater than the initial margin). The behavior of the expected transaction cost functions with respect to both decision variables is presented in Figures 3 and 4. Figures 5 and 6 display the behavior of the average default cost with respect to the initial and maintenance margins respectively. Finally, the behavior of the average expected cost (average transaction cost plus average default cost) is presented in Figures 7 and 8. Notice that expected average cost with respect to the maintenance margin is U-shaped since the transaction cost is increasing in the level of the maintenance margin whereas the average default cost is decreasing. On the other hand, the average expected cost is decreasing everywhere with respect to the initial margin. In all figures, the left out variable is fixed to its optimum level (0.8 for the maintenance margin, 1.55 for the initial margin).

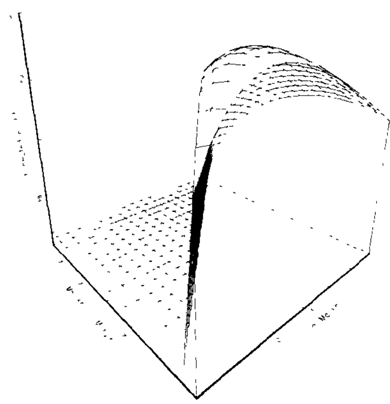

Figure 2: Expected Profit surface

4. Summary and Conclusion

Using the institutional and statistical features of the futures markets a concave profit function is constructed for the futures broker in order to set optimal initial and maintenance margin levels for a particular n-period contract. Naturally, the model can be applied to any type of futures contract. The power of the model stems from the fact that it is extremely easy to incorporate almost any institutional or/and statistical features of the futures markets. As mentioned before, distributional assumptions on futures price changes are by no means restricted to the conventional normal distribution. As long as the change in futures price is assumed to be identically and independently distributed, one can utilize a distribution which has thicker tails than the normal distribution in order to capture the extreme price movements. One important institutional feature of the futures markets is the fact that the margin requirements for hedgers and speculators are different. It may be plausible to think that default risk of different type of traders are different since the hedgers are in the market to reduce some unwanted risk whereas the speculators trade in order to make profit by taking this risk. The model can easily be extended by attaching smaller default probabilities to hedgers than speculators, implying lower optimal margin levels for the hedgers.

This article presents a model that provides support for the argument that the performance margins in the futures markets are results of competitive market forces rather than legal enforcements such as minimum margin requirements of futures exchanges. A simple numerical implementation of the model shows that in the absence of such requirements, brokers still impose positive margins to their customers due simply to the fact that in executing trade for customers, they assume a position of risk. The fact that exchanges may provide protection to its troubled member brokers leads to a divergence from the competitive outcome. This alone can justify the need for minimum margin requirements since protection of any form will alter the broker's behavior, causing him to impose lower margins than what would be suggested in markets with no interference.

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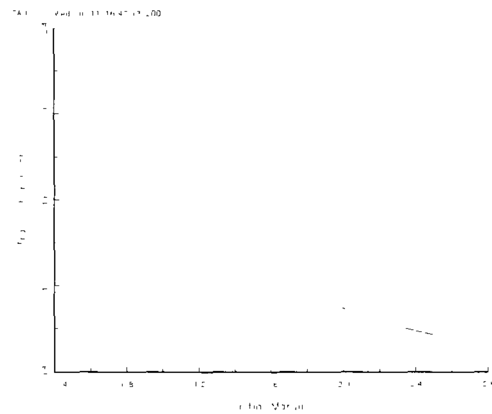

Figure 3: Transaction cost and initial margin

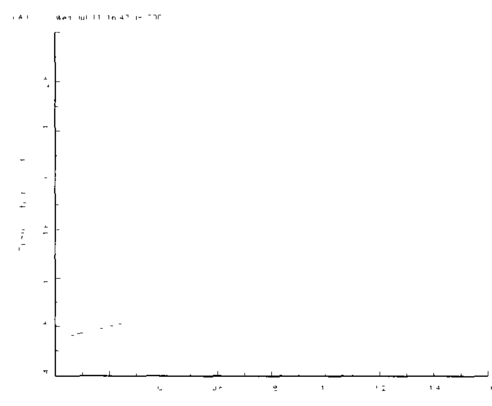

Figure 4: Trancastion cost and maintenance margin

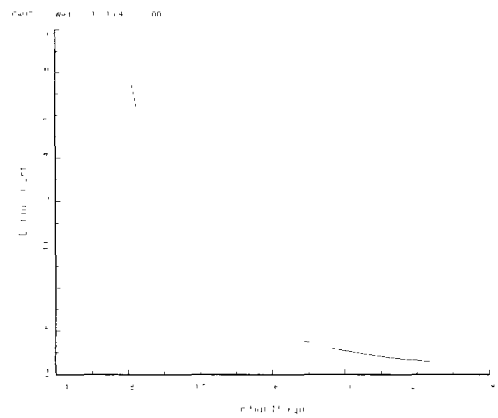

Figure 5: Default cost and initial margin

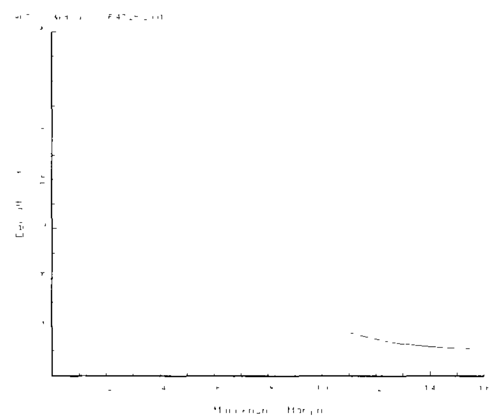

Figure 6: Default cost and maintenance margin

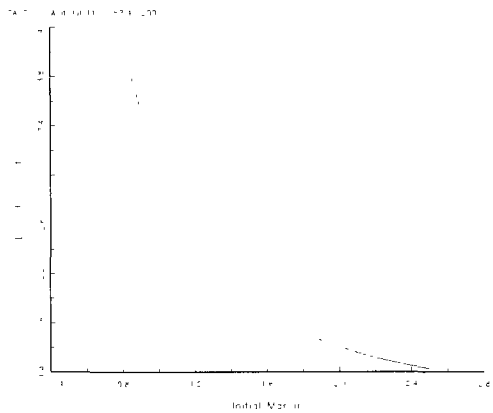

Figure 7: Average expected cost and initial margin

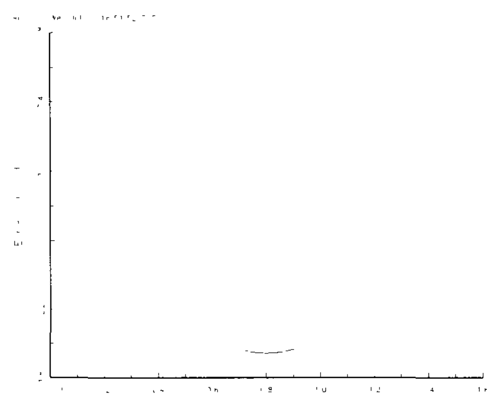

Figure 8: Average expected cost and maintenance margin

IV
Estimating Intertemporal Allocation Parameters Using Simulated Residual Estimation

Over the past quarter century many attempts have been made to estimate the parameters governing intertemporal allocation using Euler equation techniques applied to micro data; Browning and Lusardi (1996) discuss the results of 25 studies using micro data and conclude that the results are disappointing. A number of recent Monte Carlo based papers have investigated why we experience this failure (Carroll 2001, Ludvigson and Paxson, 2001, Attanasio and Low, 2000). The problems identified are manifold but the most important seem to be the paucity of appropriate data (long panels on consumption) and the problem of dealing with the substantial measurement error in consumption (see Shapiro (1984), Altonji and Siow (1987) and Runkle (1991)). The latter means that we cannot use the exact Euler equation for estimation if the equation is non-linear in parameters (a point first made by Hansen and Singleton (1982)). Approximate Euler equations (whether first order or second order) have severe problems in that they introduce latent variables that lead to violations of the orthogonality conditions exploited by GMM methods. Thus Carroll (2001) concludes that "empirical estimation of consumption Euler equations should be abandoned". On the other hand, Attanasio and Low (2000) present results that suggest that the Carroll conclusion is overly pessimistic if we have long panels (40 periods, say) and substantial time series variation in real rates. We do not find this conclusion too comforting for empirical work since we do not have long consumption panels. Thus the emerging consensus seems to be that we must give up on empirical Euler equations and return to estimating consumption functions ('structural models') based on specifying the environment agents face (see Carroll and Samwick (1997), Gourinchas and Parker (2001) and Attanasio, Banks, Meghir and Weber (1999)). In practice these methods are very similar to calibration (as used in, for example, Hubbard, Skinner and Zeldes (1995)). The problems with this approach are that it is very cumbersome and can only accommodate very limited sources of uncertainty and heterogeneity. Moreover, results may not be robust to small changes in the specification of the structural model (for example, Browning and Ejrnæs (2001) show that the Gourinchas and Parker and Attanasio *et al.* (1999) results are very sensitive to how we account for family composition).

In this paper we focus on estimating the parameters of intertemporal allocation (rather than testing for the implications of particular models). We propose an alternative approach to GMM estimation of Euler equations that is based on simulating the distribution of expectations errors. We show that this provides a half-way house between Euler equation methods using orthogonality conditions and full structural estimation. We first note that associated with every structural model there is an unconditional expectations error distribution. We show that if we know this distribution and observe consumption paths and interest rates, then we can identify utility parameters (the discount factor and the elasticity of intertemporal substitution) without having to specify the underlying stochastic environment. Without extra information the underlying model is not identified, but this is a strength rather than a weakness if we are only interested in preference parameters since it gives the method robustness as compared with full-fledged structural estimation.

We present an analysis of the unconditional distributions associated with models that are widely used in the literature (for example, Deaton's (1991) buffer stock models with explicit liquidity constraints, models with impatient agents with self-imposed liquidity constraints and nearly patient agents with random walk income processes). This serves to develop intuition and to illustrate many of the points we wish to make.

To estimate, we use a simulation based method that is in the class of Simulated Minimum Distance (SMD) estimators. This involves the specification of 'auxiliary parameters'; we find the conventional linearized Euler equation provides a very simple and convenient vehicle to do this. The method suggested is many orders of magnitude faster than full structural estimation. Above we stated that we can recover the utility parameters if we know the expectations error distribution. Since we never do know the distribution, we address the problem of testing whether the distribution chosen for the estimation procedure is a good approximation using goodness-of-fit tests applied to the predicted distribution.

We present Monte Carlo evidence on our estimator and exact and approximate (GMM based) estimators. We take as designs for these simulations the designs used in the recent papers alluded to above. In line with previous investigators we find that if consumption is measured with even moderate error, exact Euler equation estimation performs poorly. We also replicate the previous finding that approximate methods do poorly if we have short panels. By contrast, our SMD estimator works well when other estimators do not. In particular, when there is considerable measurement error (for example, half the observed consumption growth variance is due to noise) our estimator works reasonably well even for moderate sample sizes.

We briefly discuss and analyze the issues of the use of income and asset information in identification and improving precision; accounting for heterogeneity (in preferences and income processes).

The remainder of the paper is organized as follows: The next section provides a detailed analysis of Euler equation for consumption and econometric issues regarding the estimation of such an equation. In Section 3 we present a discussion of the expectational error distributions associated with various models in the literature. Section 4 presents the estimation method we use in our Monte Carlo experiments. Finally, in Section 5, we discuss the small sample properties of the estimator we propose as well as the properties of the traditional GMM based estimators.

2. Euler Equation Estimation

Exact Euler equation estimation

We consider a standard intertemporal optimization problem for which agent h has expected utility at time t of:

$$E_{h,t}\left[\sum_{j=0}^{T-t}\frac{v(C_{h,t+j})}{(1+\delta)^j}\right] \tag{1}$$

where C is non-durable consumption, $v(.)$ is an increasing, concave sub-utility function, δ is a discount rate and $E_{ht}\,(.)$ denotes the expectations operator conditional on the information that agent h has at time t. The evolution of assets over time is given by:

$$A_{h,t+j+1} = (1 + r_{h,t+j})A_{h,t+j} + Y_{h,t+j} - C_{h,t+j} \tag{2}$$

where A is assets, Y is stochastic labor income and r is the stochastic real rate of interest. The first order condition for the optimization problem gives the Euler equation for consumption:

$$v'(C_{h,t}) = \frac{1}{(1+\delta)} E_{ht}\left[(1 + r_{h,t+1})v'(C_{h,t+1})\right] \tag{3}$$

A widely used functional form for the sub-utility function is the iso-elastic form:

$$v\left(C_{h,t}\right) = \frac{\left(C_{h,t}\right)^{(1-\gamma)}}{(1-\gamma)} \tag{4}$$

where the parameter γ is the coefficient of relative risk aversion (CRRA), which we assume is the same for everyone. Interest usually centres on the inverse of this parameter, the elasticity of intertemporal substitution (EIS):

$$\phi = \frac{1}{\gamma} \tag{5}$$

Low values of the EIS indicate an aversion to fluctuating consumption streams.[1] For the iso-elastic case with exponential discounting the only other preference parameter in this program is the discount rate δ. We have the *exact Euler equation*:

$$\left(\frac{C_{h,t+1}}{C_{h,t}}\right)^{-\gamma} \frac{(1 + r_{h,t+1})}{(1+\delta)} = 1 + \varepsilon_{h,t+1} \text{ with } E_{h,t}\left(\varepsilon_{h,t+1}\right) = 0 \tag{6}$$

This relationship has been the basis of very many estimates of the preference parameters and tests for the validity of the standard orthogonality assumptions in general and for the

[1] We prefer to emphasise the role of this parameter as representing aversion to fluctuations rather than to risk since it is operative even when there is no uncertainty.

"excess sensitivity" of consumption to predictable income growth in particular. GMM estimation is based on the assumed orthogonality of the error term $\varepsilon_{h,t+1}$ to all variables dated t or before, such as lagged consumption, interest rate and income variables. As originally emphasized by Hall (1978), this is a very attractive procedure since one can estimate the preference parameters without explicitly parameterizing the stochastic environment that agents face.

The above equation represents the equilibrium condition over time, not across individuals. In other words, there is no reason to believe that the expectational errors will sum to zero for a cross section at any particular time. This result is generally due to the presence of aggregate shocks. However, the assumption of complete markets will change these results. In an environment in which the agents can hedge against all the idiosyncratic risk, consumption of the individuals will not differ due to individual-specific shocks, so individual consumption will respond only to aggregate risk, which can be captured by aggregate consumption (see Mace, 1991 and Cochrane 1991). Hence, only under the assumption of complete markets is the estimation of the allocation parameters using cross section data possible.

Problems for GMM estimation on micro data arise if the consumption data are measured with error. For example, if we allow for a multiplicative measurement error so that observed consumption $C^o_{h,t}$ is given by:

$$C^o_{h,t} = C_{h,t}\eta_{h,t} \text{ with } E\left(\eta_{h,t}\right) = 1 \tag{7}$$

then the exact Euler equation for observable consumption becomes

$$\left(\frac{C^o_{h,t+1}}{C^o_{h,t}}\right)^{-\gamma}\frac{(1+r_{h,t+1})}{(1+\delta)}=\left(\frac{\eta_{h,t+1}}{\eta_{h,t}}\right)^{-\gamma}(1+\varepsilon_{h,t+1}) \tag{8}$$

The problem this gives is that the composite error term does not have a conditional expectation of unity, even if we assume that $\eta_{h,t+1}$ and $\varepsilon_{h,t+1}$ are independent:

$$E_t\left[\left(\frac{\eta_{h,t+1}}{\eta_{h,t}}\right)^{-\gamma}(1+\varepsilon_{h,t+1})\right]=E_t\left(\frac{\eta_{h,t+1}}{\eta_{h,t}}\right)^{-\gamma}E_t\left(1+\varepsilon_{h,t+1}\right)$$

$$E_t\left[\left(\frac{\eta_{h,t+1}}{\eta_{h,t}}\right)^{-\gamma}(1+\varepsilon_{h,t+1})\right]=\left(\eta_{h,t}\right)^{\gamma}E_t\left(\eta_{h,t+1}\right)^{-\gamma}\neq 1 \tag{9}$$

It is now widely accepted that household level consumption data information is likely to be very noisy. For example, Runkle (1991) estimates that 76% of the variation in the growth rate of food consumption in the PSID is noise. Dynan (1993) reports that the standard deviation of changes in log consumption in the CEX (American Consumer Expenditure Survey) is 0.2, which seems too large for 'true' variations. This is particularly worrying since the CEX is a diary-based expenditure survey and it might be hoped that such information is less noisy than data from panel surveys. The other widely used data resource are quasi-panels, constructed from cross-section expenditure survey information by taking within-period means following the same population (e.g. means over all the 25 year olds in one year and all the 26 year olds in the next year). Although this averaging reduces the effect of measurement error, the construction of quasi-panels from samples which change over time induces sampling error which is very much like measurement error.

The presence of measurement error when estimating non-linear equations is problematic. In our context, the basic problem is that measurement error makes it appear as though consumption is less smooth over time that it actually is, which results in too low an estimate for the CRRA (with a consequent bias of the EIS away from zero). That is, measurement error makes it appear as though agents are less averse to fluctuations with a consequent bias in the estimate of the EIS away from zero. Carroll (2001) shows this in simulations with only cross-section variation in interest rates ($r_{h,t} = r_h$ for all t). To show the extent of the problem when we have time varying interest rates, we take a similar environment to Carroll (2001)[2] with the polar case in which everyone faces the same stochastic interest rate ($r_{h,t} = r_t$ for all h). We construct optimal consumption paths with a CRRA of 4 and then add a multiplicative error on the consumption values. Taking a measurement error variance such that 50% (respectively, 75%) of the time series variation is noise, the average CRRA estimate is 2.13 (0.9 respectively) which indicates substantial bias. Increasing the number of cross section units does not affect the bias.

Approximate Euler equations

A natural alternative to GMM estimation of the exact Euler equation is GMM estimation of the first or second order approximation to the nonlinear Euler equation (the first derivation is due to Hansen and Singleton(1983); see, for example, Carroll (2001) for the derivations we now present). From equation (6) we have the following (log) quadratic

[2] Fuller details of our simulation procedures will be given below.

consumption growth equation:

$$\Delta \log C_{h,t+1} - \alpha - \frac{1}{\gamma} r_{h,t+1} - \frac{\gamma}{2} \left(\Delta \log C_{h,t+1}\right)^2 = e_{h,t+1} \tag{10}$$

where the constant term α contains the discount rate and means of the third and higher order unconditional moments of the error term $\varepsilon_{h,t+1}$. The error term $e_{h,t+1}$ contains the expectational error and also time varying components of the higher conditional moments (conditional on past information).[3] The first order log-linear approximation (equation (10) without the squared term) has been used very extensively in the applied micro literature due to the fact that a multiplicative measurement error becomes additive as a result of log linearization. The usual (and uncontroversial) assumption is that the instruments other than consumption that are used in the estimation are uncorrelated with the measurement error. The $MA(1)$ error structure induced in the errors due to the measurement error is easily accounted for in GMM. Most researchers use twice (or more) lagged variables for instruments (but note that we could use first lags of any variable other than consumption since these are assumed uncorrelated with the measurement error). The problem with this approach is that the movements in the higher order moments (for example, the skewness) that are subsumed into the error term will generally cause it to be correlated with lagged variables, which leaves us without any instruments for GMM.

Here we present a brief discussion of the findings of Carroll (2001), Ludvigson and

[3] This brings out clearly that the one parameter γ controls attitudes to fluctuating consumption paths (through the coefficient on the real rate) and prudence (through the coefficient on the squared term). This close identification of fluctuation aversion and prudence is solely a result of using the iso-elastic form; other forms break the link between aversion to fluctuations and prudence (for example, the quadratic utility function has fluctuation aversion but no prudence).

Paxson (2001) and Attanasio and Low (2000); in our simulations we shall replicate many of their results and discuss them in greater detail. Ludvigson and Paxson (2001) solve and simulate a life cycle model with stochastic income and an additively separable iso-elastic utility function assuming a fixed interest rate of 3% and a discount rate of 5% (so that agents are assumed to be impatient)[4]. They then follow Dynan (1993) and use the simulated data to estimate relative prudence using the second order approximation to the Euler equation (equation (10) with no interest rate)[5]. They find that the estimate of the CRRA is downward biased; that is, it is estimated that agents are less averse to fluctuations than they actually are. Carroll (2001) performs a similar analysis allowing for cross-section variation in the interest rate, but no time series variation. He finds that the estimate of the CRRA is upward biased. Below we shall argue that the differences between the two sets of analyses can be traced to the different income processes used.

Neither Ludvigson and Paxson nor Carroll allow for time variation in interest rates to identify the EIS. Our own feeling is that trying to estimate a price elasticity (the EIS) without some price variation is almost certainly doomed to failure. Attanasio and Low (2000) present results allowing for time series variation in interest rates. They solve and simulate a simple life cycle model with stochastic income and interest rates and then estimate first and second order approximations to the Euler equation. They argue that

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[4] The term 'impatient' here and henceforth refers to the condition $\delta - r > 0$. Note that, if income grows over time, consumers can be impatient even if $\delta = r$. But for all the models considered in this paper zero income growth is assumed.

[5] In the case of iso-elastic utility, the relative prudence parameter is $\frac{\gamma+1}{2}$. Ludvison and Paxson (2001) assume a fixed interest rate and estimate the equation

$\Delta \log C_{h,t+1} = \alpha + \frac{\gamma+1}{2} \left(\Delta \log C_{h,t+1}\right)^2 + e_{h,t+1}.$

one can estimate the EIS consistently if the time period of the sample is long enough[6]. However, for panel lengths of, say, 20 periods there is still considerable bias, so that the Attanasio and Low results are not very encouraging empirically. Attanasio and Low also show in their Monte Carlo study that the precision of the estimates increases considerably with the variance of the interest rate. A potential problem they identify is that even moderately impatient agents will typically hold net wealth stocks that are close to any borrowing limit they face. In this case consumption becomes very sensitive to the income shocks and it is difficult to extract the relatively small variations in consumption growth due to interest rate changes. Note however that this problem is not special to the approximate Euler equations; our simulations presented below suggest that the same problem arises for exact Euler equation with no measurement error.

The implications of these analyses

We draw the following implications from the analyses of Ludvigson and Paxson (2001), Carroll (2001), Attanasio and Low (2000) and our own supplementary investigations.

1- There is not much point in trying to estimate price elasticities (such as the EIS) without some variation in the price (in this context, variations in the real interest rate).

2- Attempts to gauge the extent of prudence are more successful if agents are impatient since then we will be observing the buffer stock savings due to income uncertainty.

[6] They use the term 'consistent' as $T \to \infty$. They experiment with different T and show that the mean estimate of the EIS approaches its true value while the estimated standard errors become smaller as T increases.

3- Attempts to measure the EIS (reactions to interest rate changes) are more successful if agents are patient and build up wealth. In this case, temporary changes in income do not lead agents to vary consumption (since they have assets to smooth their consumption) so that the extraction of the consumption growth/ interest rate signal is easier.

4- If there is a liquidity constraint and agents are sometimes constrained, then the Euler equation does not hold in periods of constraint and consumption is not affected at all by the interest rate (an effective EIS of zero). This leads to a downward bias in the estimate of the EIS if we do not observe whether or not the agent is constrained and proceed as if they never are.

5- Measurement error introduces considerable bias into GMM exact Euler equation estimates of the EIS, even if there is substantial variation in the interest rate. This problem does not reduce if we have a large number of cross-section units.

6- Generally, approximate Euler equation methods do badly, but the results of Attanasio and Low show clearly that this is a small sample (small T) problem. This requires a small-T solution, for the typical case in which we have short and noisy panels.

The net result of the above is that we agree with Carroll (2001) and Ludvigson and Paxson (2001) that the econometric methods we currently have on hand are not up to estimating the EIS (or the discount rate) on short and noisy panels. What alternatives remain? One is to revert to old style consumption studies that are only loosely linked to conventional life-cycle theory. This is not very attractive to a generation raised on dynamic general equilibrium models and empirical modelling that stays close to the theory.

A second alternative is to move to estimation based on structural models. Thus Carroll and Samwick (1997) perform a structural estimation in which they identify the discount rate; all of the other parameters are fixed at 'reasonable' values. Gourinchas and Parker (2001) use structural estimation to estimate the EIS and the discount factor. They use CEX information and Method of Simulated Moments estimation which matches the moments generated by the data with that of simulated data. This procedure involves the numerical solution of the dynamic programming problem for every parameter value that the estimation procedure considers. The procedure is extremely slow. An obvious problem regarding this approach is the fact that one needs to specify the underlying stochastic process (income process in their case since they use a fixed interest rate) which is not necessary for Euler equation estimation (whether exact or approximate). It is not clear whether a slight misspecification of the income process will not completely change the results. To examine this would require that the estimation procedure be analyzed under misspecification, which would be extremely time consuming. Although full structural modelling is extremely promising, an alternative is needed that reduces substantially the computational burden without sacrificing the close link to the theory. We present here an alternative that relies on simulating the distribution of expectations errors directly; this can be seen as a half-way house between Euler equation estimation and structural modelling.

3. Unconditional and Conditional Distributions of Expectation Errors

Below we shall present an alternative approach to GMM which is based on the distribution of the expectations error. In this section we present an extended discussion of the distributions associated with various models in the literature. In order to illustrate our point, we present a wide range of models with different sets of parameters and different income processes within the time separable iso-elastic utility framework. We consider both fixed and stochastic interest rate models. We assume a finite lifetime of 60 periods with no bequest motive and we start all agents off with the same wealth. After generating a 60-period consumption path for an individual, we remove the first and the last 10 periods. Further details of the simulation methods are given in the Appendix. Table 1 presents the features of all the 10 models we consider.

Model 1 is our benchmark model with a standard CRRA (of 4), the discount rate equal to the real rate of interest, a simple income process and no liquidity constraints. Model 2 allows for less aversion to fluctuations (risk). Model 3 allows for impatience. Model 4 is the same as the benchmark model except that we have a higher income variance. Model 5 also modifies the benchmark income process, but in this case the income process is given a small probability of zero income in any period. Model 6 imposes a no-borrowing liquidity constraint and model 7 imposes the liquidity constraint and impatience (a Deaton buffer stock environment). Model 8 allows for a stochastic real interest rate over time. The final two models allow for some heterogeneity. Model 9 allows that there may be heterogeneity in the CRRA; specifically we assume a mixing model in which agents have $\gamma = 4$ or $\gamma = 2$ with probability one half. Model 10 allows that agents have different income processes; specifically they have a low variance (model 1) process or a high variance (model 4) process with probability half.

For models 1 to 8, consumption paths were generated for 500 ex-ante identical individuals. Given time paths $C_{h,t}$, we generate errors for agent h in period $t+1$ by:

$$\varepsilon_{h,t+1} = \left(\frac{C_{h,t+1}}{C_{h,t}}\right)^{-\gamma} \frac{(1+r_{t+1})}{(1+\delta)} \tag{11}$$

For model 9 we mix the errors from models 1 and 2 with a 50% probability and for model 10 we mix the errors from models 1 and 4 with a 50% probability. Distributional features of the expectations errors for our ten models are presented in Table 2. In each case we use only the observations from periods 10 to 50 (to give expectations errors for periods 11 to 50) to minimize the impact of starting and end effects.

The main features of the expectational error distribution for the benchmark model (model 1) are that it has unit mean (as we would expect) and some skewness. The right (positive) skewness is observed for all of our models; it reflects the concavity of the consumption function in cash-on-hand (total labour plus non-labour wealth) and the symmetry of the income processes we use in all models except for model 5. Although the skewness is small and the density of the distribution looks like a Normal, a formal test of Normality rejects decisively[7]. Instead we find that the expectations errors for this simple model appear to be lognormally distributed. Model 2 differs from the benchmark in having a lower risk aversion. As can be seen, a lower CRRA is associated with a lower standard deviation for the unconditional expectations errors.[8] The mapping from

[7] We use the software Stata for the normality and lognormality tests. The test is simply to check whether the third and the fourth moments are close to normal. The lognormality test is the normality test of transformed variable.

[8] The variances of consumption growth for the two models (not shown) are very similar so that the

the CRRA to the expectations error variance seen in this simple comparison forms a partial basis for identification in the estimation scheme we present below. Of course, the variance of the distribution will also depend on environmental factors such as the amount of interest rate variation and the underlying earnings process, so that more is required for identification. Note as well that a lower aversion to fluctuations leads to lower skewness, so that Normality is closer to being accepted. This reflects the fact that a lower CRRA is closer to linear (risk neutrality which in turn implies no prudence) so that the consumption function is less concave. Still, however, we formally reject Normality but not lognormality.

A comparison of models 1 and 3 reveals that higher impatience is associated with a higher variance and slightly more skewness. Once again we do not reject lognormality. Model 4 indicates that a higher income variance leads to a higher error variance distribution with larger skewness and kurtosis. In model 5 agents are sometimes (but rarely) hit by a really bad income draw. In this case, current consumption relative to the previous and following period is small (except in the very rare case in which the agent receives two consecutive disastrous income draws). Since we then take (the inverse of) fourth powers, the associated expectations errors are very large, hence the very pronounced skewness and kurtosis. Although such a process is not very realistic, it serves to illustrate that expectations errors can be very far from lognormal, as can be seen from the test statistic. Models 6 and 7 introduce a no-borrowing constraint. These constraints rarely bind for

lower value in model 2 is because we are taking the inverse of a square rather than the inverse of the fourth power (see equation (11)).

model 6 agents and the consequent error distribution is very similar to the benchmark distribution. However, the same is not true for the model 7 in which the agents are impatient. They quickly run down their assets and often bump up against the borrowing limit which gives them a lower current consumption than they would wish, relative to the future. The effect on the expectations error distribution for the model 7 is interesting. First we see that the mean is less than unity.[9] Second, the other moments are all very similar to the benchmark model (so that lognormality is not rejected).

Finally we turn to the effect of heterogeneity. From model 9 we see that introducing heterogeneity in the CRRA leads to lower skewness and fatter tails with a consequent decisive rejection of lognormality. For income process heterogeneity (model 10) we obtain the same result.

In this section we have presented the expectations error distributions associated with a wide range of models. A number of points emerge. First, different underlying environments may give rise to very similar distributions (compare models 1 and 7). As we shall discuss below, this will impact on the data needs for identification. Second, all of the expectations error distributions we have derived are right skewed; it is not clear how general this property is. Third, we do not reject lognormality for the unconditional error distribution for a wide range of models. This will be used extensively in our estimation procedure. Finally, some deviations from the benchmark model, such as Carroll income processes or heterogeneity in preference parameters, lead to strong rejections

[9] This is for the error distribution. The ratio of current to lagged consumption also should be below unity since the agent is impatient.

of lognormality. To test for the underlying assumptions or to make allowance for, say, heterogeneity may require more information.

4. Estimation methods

Simulated Minimum Distance

Our estimation procedure is simulation based. Following Hall and Rust (1999) we refer to the general technique as Simulated Minimum Distance (SMD) since it is based on matching (minimizing the distance between) statistics from the data and from a simulated model. The class of SMD estimators includes the EMM procedure of Gallant and Tauchen (1996) and the Indirect Inference methods of Gouriéroux, Monfort and Renault (1993). Here we present a short account of the method as applied generally to panel data; see Hall and Rust (1999) and Alvarez, Browning and Ejrnæs (2001) for details.

Suppose that we observe $h = 1, 2..H$ units over $t = 1, 2...T$ periods recording the values on a set of Y variables that we wish to model and a set of X variables that are to be taken as conditioning variables. Thus we record $\{(Y_1, X_1), ...(Y_H, X_H)\}$ where Y_h is a $T \times l$ matrix and X_h is a $T \times k$ matrix. For modelling we assume that Y given X is identically and independently distributed over units with the parametric conditional distribution $F(Y_h|X_h;\theta)$, where θ is an m-vector of parameters.[10] If this distribution is tractable enough we could derive a likelihood function and use either maximum likelihood estimation or simulated maximum likelihood estimation. Alternatively, we might derive some moment implications of this distribution for observables and use GMM to

[10] This could be generalised to allow for dependence on the initial values of the Y variables, as in Alvarez *et al.* (2001).

recover estimates of a subset of the parameter vector. Sometimes, however, deriving the likelihood function is extremely onerous; in that case, we can use SMD if we can simulate Y_h given the observed X_h and the parameters of the model. Thus we choose an integer S for the number of replications and then generate $S * H$ simulated outcomes $\{(Y_1^1, X_1), ...(Y_H^1, X_H), (Y_1^2, X_1), ...(Y_H^S, X_H)\}$; these outcomes, of course, depend on the model chosen $(F(.))$ and the value of θ taken in the model.

Thus we have some 'real' data on H units and some simulated data on $S * H$ units that have the same form. The obvious procedure is to choose a value for the parameters which minimizes the distance between some features of the real data and the same features of the simulated data. To do this, define a set of auxiliary parameters that are used for matching. Gallant and Tauchen (1996) suggest first finding a 'score generator' (flexible quasi-likelihood function) which nests the true model, and then using the score vector from this as auxiliary parameters. In the Gouriéroux *et al.* (1993) Indirect Inference procedure, the auxiliary parameters are maximizers of a given data dependent criterion which constitutes an approximation to the true DGP. Both of these approaches are motivated by attempts to derive estimators that have efficiency properties that are close to MLE. In Hall and Rust (1999), the auxiliary parameters are simply statistics that describe important aspects of the data; this is very close to calibration. We follow this approach. Thus we first define a set of auxiliary parameters (below we shall discuss in detail how to do this for the intertemporal problem):

$$\gamma_j^D = \frac{1}{H}\sum_{h=1}^{H} g^j\left(Y_h, X_h\right),\ j = 1, 2...J \tag{12}$$

where $J \geq m$ so that we have at least as many auxiliary parameters as model parameters. Denote the J-vector of auxiliary parameters derived from the data by γ^D. Using the same functions $g^j\left(.\right)$ we can also calculate the corresponding values for the simulated data:

$$\gamma_j^S = \frac{1}{S * H}\sum_{s=1}^{S}\sum_{h=1}^{H} g^j\left(Y_h^s, X_h\right),\ j = 1, 2...J \tag{13}$$

and denote the corresponding vector by $\gamma^S\left(\theta\right)$ where the notation explicitly shows the dependence on the model parameter values (but not the dependence on the X variables observed). Identification requires that the Jacobian of the mapping from model parameters to auxiliary parameters has full rank:

$$rank\left(\nabla_\theta \gamma^S\left(\theta\right)\right) = m \text{ with probability } 1 \tag{14}$$

This effectively requires that the model parameters be 'relevant' for the auxiliary parameters.

Given sample and simulated auxiliary parameters we take a $J \times J$ positive definite matrix W and define the SMD estimator:

$$\hat{\theta}_{SMD} = \arg\min_\theta \left(\gamma^S\left(\theta\right) - \gamma^D\right)' W \left(\gamma^S\left(\theta\right) - \gamma^D\right) \tag{15}$$

Alvarez *et al.* (2001) perform a small Monte Carlo study and argue that it is best to work with just identified models ($J = m$). This is largely because the objective function typically has many local minima and we can only be sure we have converged to a global minimum (not necessarily unique) if the model is just identified. In the just identified case the choice of W is irrelevant (except for computational reasons) and the minimized criterion should be zero. For just identified models, we would conclude that the model is 'well-specified' (relative to a particular choice of m auxiliary parameters) if and only if there is some value of the model parameters such that $\gamma^S\left(\hat{\theta}_{SMD}\right) = \gamma^D$. Typically we have $J > m$; in this case we use m of the auxiliary parameters to fit the model and the remaining $J - m$ auxiliary parameters to test for the goodness of fit.

Simulated Residual Estimation

Suppose we have observations on H households followed for T years. We begin by assuming that we only observe household consumption in each period and real rates between periods (which we assume to be time varying but common across agents); thus we observe $\{r, C_h\}_{h=1,\dots H}$. Below we shall consider the case where we also observe earnings and asset levels. For the moment we assume that we observe consumption with no measurement error; we shall deal with this in the next sub-section. Since we introduce two innovations in modelling (SMD and the use of simulated residuals) we begin by considering how we would use SMD to estimate preference parameters if we used full structural modelling with each agent having the same finite horizon Υ (with Υ chosen to be somewhat larger than T to be able to remove the beginning and end effects). We proceed in a number of steps.

1- First we define (perhaps joint) processes for income and the real rate; usually these would be estimated using data taken from the population from which we draw our sample.

2- Next we take parameter values for preferences (typically, the EIS and the discount rate).

3- Then we derive Υ period specific consumption (policy) functions conditional on current state variables (typically, the current realization of income and the real rate for dependent processes and current cash on hand). It is rarely possible to do this analytically, so that we need to use numerical policy (or value) iteration methods. The last period consumption function is trivial (consume everything) and the consumption functions for the earlier periods are obtained by backward induction.

4- At this point we are ready to start simulation. To simplify the exposition we assume that we set S (the number of replications of the panel in the SMD estimation procedure) equal to unity and draw H first period income and real rate values (conditional on 'period 0' values for income and the real rate) and give each of the synthetic H agents a starting value for assets. From the consumption function for period 1 we calculate first period consumption for each agent. We then draw new values of income and the real rate and calculate period 2 consumption and so on. The end result of this is a set of Υ real rate, consumption, income and asset realizations for each synthetic unit. We then trim these to remove starting and end effects and to give a time series of length T for each household $\{r^s, C_h^s\}_{h=1,\ldots H}$ (where now the s superscript reminds us that this is simulated data).

5- We now need to choose auxiliary parameters. Since we have two parameters (the EIS and the discount rate) we need two auxiliary parameters. Our choice are the OLS coefficients in the simple regression of consumption growth on the real rate:

$$\Delta \log C_{h,t+1} = \alpha + \phi r_{h,t+1} + e_{h,t+1} \tag{16}$$

Note that, in general, these are *not* unbiased estimates of the interesting parameters. But (under weak assumptions) they are unbiased estimates of something and that something is the same for the true data and the simulated data if we have the 'true' model. It is this property that makes SMD so useful. Note that we could equally well take the GMM estimates of the two parameters (with the constant and lagged interest rates as instruments) as auxiliary parameters; we prefer the OLS since it is simpler and quicker. We present results below that indicate that the choice of auxiliary parameters is not too important, provided the identification condition is satisfied.

6- The last step gives two sets of estimates: $\left(\hat{\alpha}^D_{OLS}, \widehat{\phi}^D_{OLS}\right)$ for the data and $\left(\hat{\alpha}^S_{OLS}, \widehat{\phi}^S_{OLS}\right)$ from the simulated data. We now compare the two sets of estimates. If they are the same, we stop. If they differ, we go back to step 2 and choose new parameter values. In practice, of course, we would embed this in an optimization routine or perform a grid search over the EIS and discount factor parameters.

It will be seen that steps 3 and 4 are very time consuming and estimation will be very slow. We now present a technique which cuts out these steps. After step 4 we could define simulated expectation errors:

$$\varepsilon^S_{h,t+1} = \left(\frac{C^S_{h,t+1}}{C^S_{h,t}}\right)^{-\gamma} \frac{(1 + r_{t+1})}{(1 + \delta)} \tag{17}$$

Conversely, if we knew the distribution of the expectations errors we could simulate expectations errors, $\varepsilon^S_{h,t+1}$ and then we could construct paths of consumption ratios using:

$$\frac{C^S_{h,t+1}}{C^S_{h,t}} = \left\{ \frac{(1+\delta)}{(1+r_{t+1})} \varepsilon^S_{h,t+1} \right\}^{-\frac{1}{\gamma}} \tag{18}$$

This is, of course, very fast (as compared to steps 3 and 4 above). We could then use the simulated paths in steps 5 and 6. The error simulation step requires a specification of the distribution of the expectations errors. One part of this easy: it should be serially uncorrelated with an unconditional mean of unity. If we now choose a simple two parameter form such as the lognormal then we have one extra model parameter to estimate. This in turn requires an extra auxiliary parameter; the obvious choice in step 5 above is to use the variance of OLS errors. Using a more flexible distribution with a large number of parameters will result in better identification of the structural parameters but will require more auxiliary parameters for identification. We refer to our estimation procedure as Simulated Residual Estimation (SRE).

Note that we simulate expectational errors $\varepsilon^S_{h,t+1}$ as if they are conditional on $C^S_{h,t}$ even though they come from an independent distribution. This may raise some concerns since, in principle, the conditional distribution may differ from the unconditional distribution. To determine whether this approximation is good within the framework of our data generating model, we check the possible dependence between $\varepsilon_{h,t+1}$ and $C_{h,t}$ generated by the underlying model.

Table 3 presents some correlations between the true expectational errors $\varepsilon_{h,t+1}$ and $C_{h,t}$ for 50 agents with 40 period consumption, all pooled so the number of observations is $(T-1)N = 1950$. As we observe from the table, approximating the conditional expectational error distribution with an unconditional distribution is in fact acceptable given the low correlations between the higher moments. This is not to say that the dependence is not a concern in general, but within the framework of our data generating model it is plausible to proceed with an unconditional distribution.

Before presenting the full optimization algorithm, we have to digress a little and discuss how to simulate draws from a lognormal distribution with a mean of unity. In the optimization routine for any simulation estimator it is important to keep the draws constant from iteration to iteration, otherwise the optimization routine becomes unstable. We can simulate a lognormal by taking:

$$X \backsim \exp\left(a + bN\left(0,1\right)\right)$$

where $N\left(0,1\right)$ denotes the standard Normal. The mean and variance of X are given by:

$$\begin{aligned} \mu_X &= \exp\left(a\right)\sqrt{\exp\left(b^2\right)} \\ \sigma_X^2 &= \exp\left(2a\right)\exp\left(b^2\right)\left(\exp\left(b^2\right)-1\right) \end{aligned}$$

To ensure that the mean is unity we need to impose:

$$a = -\frac{b^2}{2}$$

Thus if we simulate draws from a lognormal with mean 1 and a standard deviation of σ_X we use:

$$X \backsim \exp\left(-\frac{\ln\left(1+\sigma_X^2\right)}{2} + \sqrt{\ln\left(1+\sigma_X^2\right)}N\left(0,1\right)\right)$$

In the algorithm below, the procedure is to draw a matrix vector of standardized Normal variables and then to use this formula to give a lognormal with unit mean and varying standard deviation.

The algorithm for this simple case is:

1- Run OLS of consumption growth on the real rate and record the estimates of the constant, the slope parameter and the variance of the error term.

2- From the standard Normal, draw standardized simulated residuals $\nu^S_{h,t}$ for $t = 2, ...T$ and $h = 1, ...H$. These standardized errors are kept constant from iteration to iteration.

3- Choose a standard deviation for the expectations error distribution σ_ε and construct simulated expectations errors:

$$\varepsilon^S_{h,t} = \exp\left(-\frac{\ln(1+\sigma^2_\varepsilon)}{2} + \sqrt{\ln(1+\sigma^2_\varepsilon)}\nu^S_{h,t}\right) \tag{19}$$

where $\nu^S_{h,t}$ is a standard Normal variable. Choose values for the intertemporal allocation parameters (γ, δ). Construct consumption ratios using equation (18).

4- Repeat step 1 for the simulated data.

5- If the values from steps 1 and 4 are the same, stop. Otherwise, go to step 3 (so that we keep the same expectations error from iteration to iteration) and revise the choice of $(\gamma, \delta, \sigma_\varepsilon)$.

6- In practice we would once again use either an optimization algorithm to revise parameter values or perform a grid search. In either case the computational time is much lower than for full structural estimation.

Accounting for measurement error

In the account of SRE given in the last sub-section we ignored the possibility that consumption is measured with error. The log-linearized equation was introduced largely to take account of measurement error since any multiplicative measurement error is incorporated into the error term and as long as it is uncorrelated with the instruments used in the estimation it does not distort the parameter estimates. The only complication arising for GMM estimation of the approximate Euler equation is that the error terms in the consumption growth equation will have an MA(1) structure since we are first differencing the noise. This suggests an auxiliary parameter that will allow us to take account of measurement error in SRE. If we assume that the measurement error is multiplicative lognormal with unit mean then we need to estimate one extra parameter, the standard deviation of the measurement error, σ_η. Since the only source of autocorrelation in the error term is the measurement error, we can simply use the extent of the first order autocorrelation as an auxiliary parameter. Specifically, we run the OLS equation (16) and record the OLS parameters and the error first order autocorrelation. We emphasize once again that the latter is not a consistent estimator for anything without very strong assumptions; nevertheless, it is useful. To estimate, we modify the algorithm above to:

1- Run OLS of consumption growth on the real rate and record the estimates of the constant, the slope parameter, the variance of the error term and the autocorrelation parameter of the regression errors.

2- From the standard Normal, draw standardized simulated residuals $\nu_{h,t}^S$ and measurement errors $\tilde{\eta}_{h,t}^S$ for $t = 2, ...T$ and $h = 1, ...H$.

3- Choose standard deviations for the expectations error distribution, σ_ε, and the measurement error, σ_η, and construct simulated expectations errors, $\varepsilon^S_{h,t}$, and measurement errors, $\eta^S_{h,t}$, as above. Choose values for the intertemporal allocation parameters (γ, δ). Construct consumption ratios using equation (18). Introduce measurement error by multiplying the consumption ratio by measurement error ratios to define 'observed' simulated consumption ratios:

$$\frac{C^S_{h,t+1}}{C^S_{h,t}} \frac{\eta^S_{h,t+1}}{\eta^S_{h,t}} \tag{20}$$

4- Repeat step 1 for the simulated data with the 'observed' simulated consumption ratios.

5- If the values from steps 1 and 4 are the same, stop. Otherwise, go to step 3 and revise the choice of $(\gamma, \delta, \sigma_\eta, \sigma_\varepsilon)$.

Thus a simple model with two preference parameters can be estimated using data on (noisy) consumption levels and interest rates. What if we now observe more?

Using income and asset information

SRE relies on specifying the expectations error distribution. As we saw in section 2, the lognormality assumption is a poor one if there are sometimes very low income realizations. Our own feeling is that identification in non-standard situations requires more information. For example, suppose we observe income realizations and sometimes there is very low income. We can then model that current low income realizations are associated with low consumption ratios relative to the last period and high consumption ratios relative to the next period. In particular, if income follows a random walk then the current shock is permanent and should have a powerful effect on consumption. Similarly, if we think that agents are sometimes liquidity constrained so that the Euler equation does not hold, then we need to observe asset information. In general, the Euler equation will only hold if positive assets are carried forward. We can model this in an SRE framework. Heterogeneity poses more difficult problems, since there is no obvious auxiliary parameter to help identify heterogeneity.

All of the above only uses consumption and interest rate information. There is no doubt that one must achieve a better identification by using the observed income series for each household. In a typical household survey, income information is available although its quality has always been questionable. In our experiments, households end up realizing different incomes each period. It is only natural to use this information in the SMD estimation by conditioning the expectation errors distribution on income. Note that the strength of the method is that we do not have to specify the income or interest rate processes. All we need is to observe the realized values of such stochastic variables. It is also important to note that this is an approximation. We do not know the real distribution. However, the use of any extra information on households will make the approximation more accurate.

5. Small sample properties

In this section we present small sample results on GMM estimation of exact and approximate Euler equations and our Simulated Residual Estimation (SRE) method. We remind the reader that one of the most important conclusions that we take from the recent literature is that the estimation problem here is inherently a small sample one (see the discussion at the end of section 2); hence we do not present any asymptotic results and rely on Monte Carlo simulations alone. We use the same simulation environment as described in section 3.

In our Monte Carlo work, we generate data using a standard life cycle model where a generic consumer maximizes his expected utility subject to his intertemporal budget constraint. We assume the utility function is intertemporally separable and the sub-utilities are iso-elastic (no durables and demographic effects). The problem of the generic consumer h is

$$\begin{aligned} & \max E_t \left[\sum_{j=0}^{T-t} \frac{(C_{h,t+j})^{1-\gamma}}{1-\gamma} \frac{1}{(1+\delta)^j} \right] \\ s.t.\ A_{h,t+j+1} &= (1+r_{h,t+j})A_{h,t+j} + Y_{h,t+j} - C_{h,t+j} \end{aligned}$$

where C is non-durable consumption, A is assets, Y is stochastic labor income and r is stochastic real interest rate. We assume finite life and end of life T is certain. The discount rate δ and the coefficient of risk aversion γ are positive. Our generic consumer has no bequest motive i.e. $A_{T+1} = 0$. The stochastic process driving labor income is assumed to be the following

$$Y_{h,t+1} = Y_{h,t} z_{h,t+1} \tag{21}$$

where z_{t+1} is iid with mean 1 and a constant variance σ_z^2. We assume that the innovations to income are independent over time and across individuals i.e. we assume away aggregate shocks to income. Individuals can use only one asset to smooth their consumption against these idiosyncratic income shocks. The return on this asset (interest rate) is generated by an AR(1) process:

$$r_{h,t+1} = (1-\rho)\mu + \rho r_{h,t} + \epsilon_{h,t+1} \tag{22}$$

where μ is the unconditional mean, ρ is AR(1) coefficient with $0 < \rho < 1$, and ϵ_{t+1} is assumed to be *iid* normal with mean 0 and standard deviation σ_ϵ.

Following Deaton(1991), the budget constraint is re-defined as

$$X_{h,t+j+1} = (1 + r_{h,t+j+1})(X_{h,t+j} - C_{h,t+j}) + Y_{h,t+j+1} \tag{23}$$

where $X_{h,t+j} = A_{h,t+j} + Y_{h,t+j}$ (cash on hand).

Having a nonstationary income process makes the problem harder to solve since the range of possible income values is too large. Instead, we redefine all the relevant variables in terms of their ratios to current income and solve for the consumption to income ratio. By doing this we reduced the number of state variables to two, namely cash on hand to income ratio and interest rate. Moreover, we obtain an iid income process which can be approximated by standard Quadrature methods. Given this redefinition of the relevant variables, the Euler equation can be written as

$$\theta_t(w_t, r_t)^{-\gamma} - \frac{1}{(1+\delta)} \int\int \theta_{t+1}(w_{t+1}, r_{t+1})^{-\gamma} z_{t+1}^{-\gamma} dF(z_{t+1}) dG(r_{t+1}) = 0 \tag{24}$$

where $\theta_t = \frac{C_t}{Y_t}$, $w_t = \frac{X_t}{Y_t}$.

The problem is solved via policy function iteration using the terminal value condition. At the terminal date T, consumption is function of only cash on hand and since the bequest motive is assumed away, so that $\theta_T = w_T$.

For the income process, we use 10 point Gaussian Quadrature and following Tauchen (1986) we approximate the interest rate process by forming a 10 point first order discrete Markov process. We use a cubic spline to approximate the consumption function at each iteration. Since we solve a finite life problem, we obtain T consumption-to-income ratio functions $\{\theta_1(w_1, r_1), ...\theta_t(w_t, r_t), ...\theta_T(w_T)\}$.

Table 4 reports the parameter values used in the solution and the simulation of the model described above. The agent is allowed to borrow the amount he can pay back with certainty. In the infinite life case this would correspond to the borrowing limit of $\frac{\min Y}{\max r}$. The discount rate and the mean interest rate are chosen to be equal in order to prevent consumers to quickly go towards the borrowing constraint. When the discount rate is large relative to the interest rate, consumers borrow close to the maximum possible amount. Then the movement of consumption is largely driven by income and the identification of interest rate impact on consumption growth becomes very difficult.

For GMM we use continuous updating GMM (see Hansen, Heaton and Yaron (1996)) to remove any dependence on the normalization. For the exact form, we estimate the preference parameters β and γ using the following orthogonality condition on the error term:

$$E_{h,t}\left[\left(\frac{C_{h,t+1}}{C_{h,t}}\right)^{-\gamma}(1+r_{h,t+1})\beta - 1\right] = E_{h,t}\left[\varepsilon_{h,t+1}\right] = 0$$

The instruments taken are the constant, lagged income and the real rate, so that we have one degree of overidentification. Our second empirical model is the approximate Euler equation:

$$\ln\left(\frac{C_{h,t+1}}{C_{h,t}}\right) = \alpha + \frac{1}{\gamma}r_{t+1} + \varepsilon_{h,t+1}$$

The final estimator we use is the Simulated Minimum Distance (SMD) estimator. We use the approximate Euler equation as the auxiliary model and the errors used to generate simulated consumption paths are obtained from imposing a mean 1 lognormal distribution. In the first set of experiments we do not condition on realized income in order to see how far we can go with such a crude approximation. Then we model the conditional mean of the expectation errors via a linear specification.

SMD and GMM

In our Monte Carlo experiments, we investigate the small sample properties of SMD, GMM on the exact Euler Equation and GMM on the first order approximation, both with and without measurement error. We perform two sets of experiments. First, we assume that the econometrician has panel data on consumption and estimates the preference parameters by pooling all individuals together. These estimations were performed for 50 ex-ante identical households followed for 40 periods. The number of replications is 500. We repeat the same exercise by decreasing the number of households to 20 holding the number of periods at 40, and then we perform the same exercise by holding the number of households at 20 and decreasing the time period to 20, in order to have an idea about T and N consistency.

Table 5 presents the distributional features of the CRRA and discount factor estimates for various T and N with and without measurement error. It is not surprising that in case of no measurement error and a long panel (environment 1), GMM using the exact Euler Equation performs the best. The estimator captures the true parameter values fairly precisely. SMD performs much better than the GMM estimator using the first order approximation. GMM on the approximate equation (AGMM) results in an upward/downward bias in the estimates of CRRA/EIS even with a very large T and no measurement error in the data. As a by-product of the SMD estimation we obtain the estimates of the standard deviation of the measurement error. In the case of no measurement error (true standard deviation is zero), our SMD procedure recovers the zero standard deviation.

Results for $T = 40$, $N = 20$ and $T = 20$, $N = 20$ are as expected. The standard deviations obtained from of all estimators have gone up. The most dramatic increase of the standard deviation comes from the GMM estimation on the approximate Euler equation. A decrease in the number of individuals from 50 to 20 results in an increase in the standard deviation of the CRRA estimates from 1.39 to 17.87. This figure goes up to 21.7 in the case of a reduction in the number of time periods from 40 to 20.

A lognormally distributed measurement error with the standard deviation of 0.02 corresponds to a case in which approximately 50% of the variation in consumption growth is noise. It goes up to 85% when the standard deviation is 0.05. (mean standard deviation of consumption growth under no measurement error is 0.016 after 500 simulations). In order to establish the small sample properties of each estimator under the measurement error we experiment with lognormally distributed multiplicative measurement error. When 50 percent of variation of consumption growth is noise ($\sigma_\eta = 0.02$), an immediate observation from the last row of the Table 5 is that the exact GMM yields serious bias in the estimates whereas GMM on the approximate equation does not seem to be affected by the extra noise added to the model. Most importantly, the mean of SMD is by far the closest to the true value of 4. It is not surprising that the measurement error increases the standard deviation of the parameter estimates for all estimators. Finally, SMD appears to recover the measurement error variance on average very precisely.

6. Conclusions

There seems to be widespread agreement that given currently available data, we cannot accurately estimate the parameters of intertemporal allocation using GMM on Euler equations, whether they be exact or approximate. Our reading of this literature and our own results is that this is a small sample (strictly, short panel) problem. The alternative seems to be to move to full structural modelling. In the current state of the art this is very cumbersome, fragile and unable to deal with significant heterogeneity. We present a novel estimation procedure that represents a half-way house between Euler equation methods and full structural modelling. This is based on simulating expectation errors and we refer to it as Simulated Residual Estimation (SRE). We also show how to allow for measurement error in consumption. An investigation of the small sample properties of the SRE estimator indicate that it dominates GMM estimation of both exact and approximate Euler equations in the case when we have short panels and noisy consumption data.

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Appendix: Consumption Functions

We initialize the algorithm with the consumption rule at the end of life $c_T(x_T) = x_T$. So we have assumed away the bequest motive. The constraint on borrowing is that at the end of life the individual should pay back all his outstanding debt. In practice this constraint will never bind because the functional form of the utility function implies that zero consumption is assumed away, since it results in marginal utility going to infinity. Instead we will observe very impatient individuals getting very close to the borrowing limit, whereas it will be irrelevant for the patient ones. Since we do not assume an explicit borrowing limit as in Deaton (1991), the consumption functions are continuously differentiable. In fact, in our case where agents have iso-elastic preferences and income uncertainty, consumption functions are strictly concave (see Carroll and Kimball 1996). In order to solve the problem, we define an exogenous grid for the cash on hand to income ratio: $\{x_j\}_{j=1}^{J}$. It is important to adjust the grid as the solution goes back in time. The algorithm finds the consumption that makes the standard Euler equation hold for each value of x and r. In practice, we took 100 points for x and 10 points for r. After obtaining c_{T-1}, we use a cubic spline to approximate $c_{T-1}(x_{T-1})$ for each r. After obtaining the consumption functions for each age, we simulate life time consumption paths using the intertemporal budget constraint and generating random draws for income and interest rate. Generated paths differ due to different realizations of income and interest rates for each individual.

For our Monte Carlo experiment we generate 60 period consumption paths for ex ante identical consumers. Individuals are assumed to face the same interest rate series. Therefore individuals' consumption paths differ due only to different income realizations. Although it is possible to allow for cross section variation in the interest rate we believe it is more plausible to assume only time series variation. We do several experiments by using different time period (T) and number of individuals (N) in the estimations. For example in the first case we estimate the model using $T = 40$ and $N = 50$. This corresponds to a case where the researcher has a 40 period panel on 50 individuals. The measurement error is assumed to be multiplicative and single parameter log normal.

TABLE 1

Models

Model	**CRRA** γ	**Real rate** r	**Discount** rate, δ	**Income** process	**Liquidity** constraint
1	4	$0.05(0)$	0.05	RW, $\sigma = 0.02$	No
2	2	$0.05(0)$	0.05	RW, $\sigma = 0.02$	No
3	4	$0.03(0)$	0.05	RW, $\sigma = 0.02$	No
4	4	$0.05(0)$	0.05	RW, $\sigma = 0.05$	No
5	4	$0.05(0)$	0.05	Carroll process*	Implicit
6	4	$0.05(0)$	0.05	RW, $\sigma = 0.02$	Yes
7	4	$0.03(0)$	0.05	RW, $\sigma = 0.02$	Yes
8	4	$0.05(.011)$	0.05	RW, $\sigma = 0.02$	No
9	2 or 4	$0.05(0)$	0.05	RW, $\sigma = 0.02$	No
10	4	$0.05(0)$	0.05	RW, $\sigma = 0.02$ or 0.05	No

Note: variance of r in parentheses. 'RW' refers to 'random walk in logs without drift'. Errors of income process are distributed log-normally with mean 1, std σ.

* Constant permanent income with 1% probability of zero current income.

TABLE 2

Distributions of Expectational Errors for Different Models

Model	mean	median	std	skew	kurt	N test	L test
1	1.001	.998	.077	.201	3.00	26.5	1.5
2	1.001	1.000	.040	.097	2.94	6.8	1.3
3	1.001	.998	.086	.237	3.01	36.6	1.3
4	1.006	.984	.184	.740	4.02	160.0	3.9
5	1.006	.993	.285	24.95	950.5	9723	2377
6	1.001	.998	.077	.201	3.00	26.5	1.5
7	0.986	.983	.077	.208	3.01	28.4	1.4
8	0.999	.998	.065	.202	3.21	33.1	4.0
9	1.001	.999	.061	.133	3.87	76.2	72.6
10	1.005	0.998	.137	.500	4.83	319.8	168.4

Note. N (L) test is a test for Normality (lognormality), both χ^2 (2)

Number of observations $N * (T-1) = 19,500$

TABLE 3

Dependence of Expectational Errors on

Lagged Consumption (correlation coefficients)

	$\varepsilon_{h,t+1}$	$\varepsilon^2_{h,t+1}$	$\varepsilon^3_{h,t+1}$	$\varepsilon^4_{h,t+1}$
$C_{h,t}$	$-.012$	$-.013$	$-.011$	$-.016$
$C^2_{h,t}$	$-.030$	$-.025$	$-.019$	$-.014$

TABLE 4

Parameter Values

Parameter	Value
Coefficient of Relative Risk Aversion (γ)	4
Discount rate (δ)	.05
mean r (μ)	0.05
AR(1) coefficient of r	0.6
Standard deviation of interest rate shocks (σ_ϵ)	0.011
mean income innovation z	1
Standard deviation of income innovation(σ_z)	0.02
Standard deviation of Measurement Error (σ_η)	0.02, 0.05

TABLE 5

Small Sample Results: Means and Standard Deviations

of Sampling Distributions

Environment				Exact GMM		AGMM	SRE (SMD)		
	T	N	σ_η	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\sigma}_\eta$
1	40	50	0	0.951 (.002)	3.84 (0.72)	5.12 (1.39)	0.952 (.002)	3.51 (0.76)	.002 (.0015)
2	40	20	0	0.950 (.013)	4.83 (2.73)	6.80 (17.87)	0.951 (.004)	3.70 (1.29)	.002 (.002)
3	20	20	0	0.941 (.024)	5.56 (6.02)	6.48 (21.8)	0.950 (.023)	4.76 (5.36)	.0025 (.0023)
4	40	50	0.02	0.954 (.001)	2.13 (0.44)	6.00 (7.50)	0.952 (.005)	3.67 (1.79)	0.015 (0.01)

Note: True values are $\gamma = 4$ and $\beta = 0.952$

Values in brackets are standard deviations.

V

Conclusion

In this dissertation, three research questions in economics and finance are addressed using newly developed numerical analysis and the simulation based estimation techniques.

In the first article, I assess the finite sample performance of two simulation based estimators, the indirect estimator and the Efficient Method of Moments estimator, relative to the conventional estimators (Generalized Method of Moments and Maximum Likelihood) within a specific application. I perform a series of Monte Carlo experiments that involve estimating a continuous time short-term interest rate model under the assumption that the higher moments of the short rate series are misspecified. The results of my Monte Carlo simulations suggest that the misspecification bias can be serious for some important parameters of the model and the simulation based methods are about as vulnerable as the conventional methods to such circumstances.

The second article of my thesis presents an application of a new numerical optimization technique that uses the properties of the first order discrete Markov series. The optimum performance margin setting problem of futures markets is analyzed in a profit maximization context where a futures commission merchant is the decision maker. He operates under circumstances in which margin violations by his customers are costly, as are communications regarding futures trading. The implications of my model provide a support for the argument that positive performance margins can be the consequences of competitive market forces rather than legal enforcements.

The third article of my thesis combines dynamic programming techniques with a specific simulation based estimation method to address a long standing research question in microeconomics: identification of the intertemporal allocation parameters of the Life Cycle Model of consumption. For this, I propose a new simulation based estimation technique, Simulated Residual Estimation. In order to assess the finite sample properties of Simulated Residual Estimation, I perform a series of Monte Carlo experiments which involve numerically solving and simulating a standard life cycle model under labour income and interest rate uncertainty. I find that the finite sample performance of the proposed estimator is superior to the conventional Generalized Method of Moments based estimators when consumption data is measured imperfectly.

Overall, this thesis shows that we now have very powerful numerical methods and quite sophisticated estimation techniques to address a wide variety of basic problems. Researchers in economics and finance have started to benefit from these new developments by attacking the problems previously considered intractable.

```
/***********************************************************************
****/
/****************This program Estimates Vasicek Model****************/
/***using ML when data comes from True model******/
new;
_pset="/home/labour/sk/paper1/CEAsver";
loadm path=^_pset;
save path=^_pset;
output file=mle1.out reset;
format /m1 /ro 10,3;
m=1;
routine4:
mu=0.1; k=0.5; sig=0.06; /*true parameter values*/
truep=mu|k|sig;
dt=0.1;

r0=mu;
i=1;
routine1:
dw=sqrt(dt)*rndn(1,1);
r1=r0+k*(mu-r0)*dt+sig*dw;
  if i eq 1;
   r=r1;
  else;
  r=r | (r1);
   endif;
  if i eq 7000; goto final1;
  endif;
i=i+1;
r0=r1;
goto routine1;
final1:
rnew=r;
/***********************************************************/
gap=10;
routine3:
intr=rnew[seqa(1,gap,rows(rnew)/gap),.];
rt=intr[1:rows(intr)-1,.];
rt1=intr[2:rows(intr),.];
T=rows(rt)*dt;
step=0.5;
eps=0.0001;

proc lhf(parvec);
     local a,k,sig,l1, l2,l3, l31, l32,p1,p2,p3;
     a=parvec[1,.];
     k=parvec[2,.];
     sig=parvec[3,.];
     p1=1-exp(-k);
     p2=exp(-k);
     p3=(1-exp(-2*k))/(2*k);
     l1=-ln(sig);
     l2=-ln(sqrt(p3));
     l31=-(rt1-a*p1-p2*rt)^2;
     l32=2*(sig^2)*p3;
     l3=l31./l32;
     retp(l1+l2+l3);
```

```
endp;

/*****************************************/
a0=0.07; k0=0.6; sig0=0.1;
parvec0=a0|k0|sig0;
i=0;
routine2:
gradien=gradp(&lhf,parvec0);
hessian=-gradien'*gradien;
parvec1=parvec0+invpd(-hessian) * sumc(gradien) * step;
     if (meanc(abs(parvec1-parvec0)) < eps) or (i eq 20); goto final2;
     endif;
     i=i+1;
     a0=parvec1[1,1];
     k0=parvec1[2,1];
     sig0=parvec1[3,1];
     parvec0=a0|k0|sig0;
     goto routine2;
final2:
if m eq 1;
parvec=parvec1';
else;
parvec=parvec|(parvec1');
endif;
if m eq 500; goto final4;
endif;
print parvec1;
print m;
print i;
m=m+1;
goto routine4;
final4:
mle0=parvec;
save mle0;
meanc(mle0);
truep;
```

```
/**********************************************************************
****/
/***************This program Estimates Vasicek Model***************/
/***using GMM Estimator when data comes from True model******/
new;
_pset="/home/labour/sk/paper1/CEAsver";
loadm path=^_pset;
save path=^_pset;
output file=mle1.out reset;
format /m1 /ro 10,3;
library optmum;
optset;
_opalgr=5;
_opmiter=100;
pe=10;
m=1;
routine4:
mu=0.1; k=0.5; sig=0.06;/*true parameter values*/
truep=mu|k|sig;
dt=0.1;

r0=mu;
i=1;
routine1:
dw=sqrt(dt)*rndn(1,1);
r1=r0+k*(mu-r0)*dt+sig*dw;
  if i eq 1;
   r=r1;
  else;
  r=r | (r1);
   endif;
  if i eq 7000; goto final1;
  endif;
i=i+1;
r0=r1;
goto routine1;
final1:
rnew=r;
/************************************************/
gap=10;
routine3:
intr=rnew[seqa(1,gap,rows(rnew)/gap),.];
int1=intr[1:rows(intr)-1,.];
int2=intr[2:rows(intr),.];
T=rows(int1)*dt;
c1=ones(rows(int1),1);
step=0.5;
eps=0.001;


proc min(x,y);
local result;
if x gt y;
result=y;
else;
result=x;
endif;
```

```
retp(result);
endp;


proc gmm11(parvec); /* 1,rt1 are the instruments */
  local alfa, beta,sigma,orth1,orth2,orth3,orth4, orth5,orth6,mat,omega;
  alfa=parvec[1,.];
  beta=parvec[2,.];
  sigma=parvec[3,.];
orth1=c1'*(int2-alfa*beta-int1*(1-beta));
orth2=int1'*(int2-alfa*beta-int1*(1-beta));
orth3=c1'*((int2-alfa*beta-int1*(1-beta))^2-sigma^2);
orth4=int1'*((int2-alfa*beta-int1*(1-beta))^2-sigma^2);
orth5=c1'*((int2-alfa*beta-int1*(1-beta))^3);
orth6=int1'*((int2-alfa*beta-int1*(1-beta))^3);
mat=orth1|orth2|orth3|orth4;
retp(mat'*eye(4)*mat+pe*(min(sigma,0))^2);
endp;
parvec0=0.1|0.4|0.05;
{theta,f,g,retcode}=optmum(&gmm11,parvec0);
alfa1=theta[1,.];
beta1=theta[2,.];
sigma1=theta[3,.];

orth1r=c1.*(int2-alfa1*beta1-int1*(1-beta1));
orth2r=int1.*(int2-alfa1*beta1-int1*(1-beta1));
orth3r=c1.*((int2-alfa1*beta1-int1*(1-beta1))^2)-sigma1^2;
orth4r=int1.*((int2-alfa1*beta1-int1*(1-beta1))^2)-sigma1^2;
orth5r=c1.*((int2-alfa1*beta1-int1*(1-beta1))^3);
orth6r=int1.*((int2-alfa1*beta1-int1*(1-beta1))^3);
v=orth1r~orth2r~orth3r~orth4r;
VA=1/(rows(v))*v'v;
/******************************/;
/*******************************/;

proc gmm22(parvec); /* 1,rt1 are the instruments */
  local alfa, beta,sigma,orth1,orth2,orth3,orth4, orth5,orth6,mat,omega;
  alfa=parvec[1,.];
  beta=parvec[2,.];
  sigma=parvec[3,.];
orth1=c1'*(int2-alfa*beta-int1*(1-beta));
orth2=int1'*(int2-alfa*beta-int1*(1-beta));
orth3=c1'*((int2-alfa*beta-int1*(1-beta))^2-sigma^2);
orth4=int1'*((int2-alfa*beta-int1*(1-beta))^2-sigma^2);
orth5=c1'*((int2-alfa*beta-int1*(1-beta))^3);
orth6=int1'*((int2-alfa*beta-int1*(1-beta))^3);
mat=orth1|orth2|orth3|orth4;
retp(mat'*invpd(VA)*mat+pe*(min(sigma,0))^2);
endp;

{parvec1,f1,g1,retcode}=optmum(&gmm22,theta);
if m eq 1;
parvec=parvec1';
else;
parvec=parvec|(parvec1');
endif;
if m eq 500; goto final4;
```

```
endif;
print parvec1;
print m;
m=m+1;
goto routine4;
final4:
gmm0=parvec;
save gmm0;
meanc(gmm0);
truep;
```

```
/***********************************************************************
****/
/****************This program does simulations for EMM Estimation*/
/*for Vasicek when true model is Vasicek*******/
/*********************for O-U process**********************/
@EULER DATA GENERATOR@
new;
_pset="/home/labour/sk/paper1/CEAsver";
loadm path=^_pset;
save path=^_pset;
output file=emOU1.out reset;
format /m1 /ro 10,3;
library optmum;
optset;
declare _opmiter=20;
l=1;
routine6:
mu=0.1; k=0.5; sigma=0.06; /*true values*/
gap=10;
step=0.5;
eps=0.0001;
n=1;
dt=0.1;
i0=0.1;
var=sigma^2/(2*k)*(1-exp(-2*k*dt));
routine7: /*creating true path with true parameters*/
e=sqrt(var)*rndn(1,1);
i1=i0*exp(-k*dt)+mu*(1-exp(-k*dt))+e; /*real data generator*/
  if n eq 1;
  in=i1;
  else;
  in=in|(i1);
  endif;
 if n eq 7000; goto final7;
 endif;
 n=n+1;
 i0=i1;
 goto routine7;
 final7:
inti=in[seqa(1,gap,rows(in)/gap),.];
int1=inti[1:rows(inti)-1,.];
int2=inti[2:rows(inti),.];
T=rows(int1);

a0=0.07; k0=0.3; sig0=0.1; /*initial values*/
parvec0=a0|k0|sig0;
i=0;
routine2:
gradien=gradp(&lhf,parvec0);
hessian=-gradien'*gradien;
parvec1=parvec0+invpd(-hessian) * sumc(gradien) * step;
     if (meanc(abs(parvec1-parvec0)) < eps) or (i eq 20); goto final2;
     endif;
     i=i+1;
     a0=parvec1[1,1];
     k0=parvec1[2,1];
```

```
        sig0=parvec1[3,1];
        parvec0=a0|k0|sig0;
        goto routine2;
    final2:
    parvec=parvec1;
    m=parvec[1,.];
    p=parvec[2,.];
    si=parvec[3,.];
    momen=meanc(gradien);
    omega=-hessian;
    dw=sqrt(dt)*rndn(7000,1);
    /****************************************************/

    proc simax(theta);
    local a,q,s,j, r1, r, r0,rnew, intr,rt, rt1,
    deriva,derivq,derivs,deriv,score;
    a=theta[1,.];
    q=theta[2,.];
    s=theta[3,.];
    j=1;
    r0=0.1; /*play with this later!!!!*/
    routine1:
    r1=r0+q*(a-r0)*dt+s*dw[j,.];
       if j eq 1;
       r=r1;
       else;
       r=r|(r1);
       endif;
       if j eq 7000; goto final1;
       endif;
       j=j+1;
       r0=r1;
       goto routine1;
    final1:
    rnew=r;
    intr=rnew[seqa(1,gap,rows(rnew)/gap),.];
    rt=intr[1:rows(intr)-1,.];
    rt1=intr[2:rows(intr),.];
    deriva=p/si^2*(rt1-rt-p*(m-rt));
    derivq=1/si^2*(rt1-rt-p*(m-rt)).*(m-rt);
    derivs=(-1/si)+(1/si^3)*(rt1-rt-p*(m-rt))^2;
    deriv=deriva~derivq~derivs;
    score=meanc(deriv);
    retp(score'*inv(omega)*score);
    endp;

    /*************************************/;
    m0=0.09; q0=0.4; s0=0.05; /*initial values*/;
    x0=parvec;
    /**************************************/
    { thetas,f,g,retcode } = optmum(&simax,x0);
    if l eq 1;
    params=thetas';
    else;
    params=params|(thetas');
  - endif;
    if l eq 500; goto final6;
```

```
endif;
l=l+1;
print l;
goto routine6;
final6:
emm00=params;
save emm00;

proc lhf(parvec);
     local d,k,sig,l1, l2, l21;
     d=parvec[1,.];
     k=parvec[2,.];
     sig=parvec[3,.];
     l1=-ln(sig);
     l21=-(int2-k*d-int1*(1-k))^2;
     l2=1/(2*sig^2)*l21;
     retp(l1+l2);
endp;

/*******************************************/
```

```
/*********************************************************************
****/
/****************This program Estimates Vasicek Model****************/
/***using IND E. when data comes from true model and *****/
/*******auxiliary model is AR(1)******/
new;
library optmum;
optset;
_pset="/home/labour/sk/paper1/CEAsver";
loadm path=^_pset;
save path=^_pset;
output file=ind11.out reset;
format /m1 /ro 10,3;
n=1;
routine4:
mu=0.1; k=0.5; sig=0.06; /*true parameter values*/
truep=mu|k|sig;
dt=0.1;

r0=mu;
i=1;
routine1:
dw=sqrt(dt)*rndn(1,1);
r1=r0+k*(mu-r0)*dt+sig*dw;
  if i eq 1;
   r=r1;
  else;
  r=r | (r1);
   endif;
  if i eq 7000; goto final1;
  endif;
i=i+1;
r0=r1;
goto routine1;
final1:
in=r;
/************************************************/
gap=10;
inti=in[seqa(1,gap,rows(in)/gap),.];
int1=inti[1:rows(inti)-1,.];
int2=inti[2:rows(inti),.];
T=rows(int1);
c1=ones(rows(int1),1);
int1=c1~int1;
lset1=inv(int1'*int1)*int1'*int2; /* this is k*a and (1-k) */
lsed2=1-lset1[2,.];
lsed1=lset1[1,.]/lsed2;
pred1=int1*lset1;
sid=sqrt((int2-pred1)'*(int2-pred1)/rows(int1));
lsed=lsed1|lsed2|sid; /*what data gives*/

dw=sqrt(dt)*rndn(7000,1);
/****************************************************/

proc simax(theta);
local a,q,s,j, r1, r,r0,rnew, intr,rt, rt1,
lset,lse1,lse2,si,c,pred,betas;
```

```
a=theta[1,.];
q=theta[2,.];
s=theta[3,.];
j=1;
r0=lsed1; /*play with this later!!!!*/
routine1:
r1=r0+q*(a-r0)*dt+s*dw[j,.];
   if j eq 1;
   r=r1;
   else;
   r=r|(r1);
   endif;
   if j eq 7000; goto final1;
   endif;
   j=j+1;
   r0=r1;
   goto routine1;
final1:
rnew=r;
intr=rnew[seqa(1,gap,rows(rnew)/gap),.];
rt=intr[1:rows(intr)-1,.];
rt1=intr[2:rows(intr),.];
c=ones(rows(rt),1);
rt=c~rt;
lset=inv(rt'*rt)*rt'*rt1; /* this is k*a and (1-k) */
lse2=1-lset[2,.];
lse1=lset[1,.]/lse2;
pred=rt*lset;
si=sqrt((rt1-pred)'*(rt1-pred)/rows(rt));
betas=lse1|lse2|si;
retp((lsed-betas)'*eye(3)*(lsed-betas));
endp;

/************************************/;
a0=0.05; k0=0.1; s0=0.04; /*initial values*/;
x0=a0|k0|s0;
x0=lsed;
/*************************************/
{ thetas,f,g,retcode } = optmum(&simax,x0);
if n eq 1;
parvec=thetas';
else;
parvec=parvec|(thetas');
endif;
if n eq 500; goto final4;
endif;
print n;
n=n+1;
goto routine4;
final4:
ind00=parvec;
save ind00;
meanc(ind00);
truep;
```

```
/*This program creates Markov Matrix to calculate Cost and Profit*/
/*IN THIS PROGRAM, I DIVIDED HYPOTHETICAL ACCOUT LEVEL RANGE NOT THE
NORMAL DIST*/
/* this program gets the values for 2 dimensional graphs for Maintenance
Margin*/
/*MOST IMPORTANTLY THIS PROGRAM HAS THE MOST RECENT VERSION OF THE
MATRIX*/
/*NO ABSORBING STATE, SMALLER MATRIX*/
/*MOST MOST RECENT VERSION, SHOR-LONG ICIN AYRI MATRISLER*/
new;
_pset="d:\\paper2\\summer00";
loadm path=^_pset;
save path=^_pset;
output file=prog1.out reset;
format /m1 /ro 10,5 ;
maxr=2500; /*maximium account level for a unit ($)*/
minr=-2500; /*minimum account level for a unit ($)*/
mdt=0; /* mean of futures price change ($)*/
sdt=700; /*sdt deviation of futures price change ($) */
dmax=50;  /* number of discrete points (2k)*/
inc=maxr/(dmax/2);
tran=1;  /* hypothesized unit transaction cost */

k=1;
do until k eq dmax+1;
  if k eq 1;
  vect=maxr|(maxr-k*inc);
  else;
  vect=vect|(maxr-k*inc);
  endif;
  k=k+1;
endo;
/*vector "vect" contains account levels*/
n=11;
routine6:
l=19;
routine:
M=vect[n,.];
mar=vect[l,.];
S=indnv(mar,vect)+dmax/2+1; /*size of the matrix (changes acc to
location of mar)*/;
routine1:
mat=zeros(S,S); /* matrix for long position */
mats=zeros(S,S); /* matrix for short position*/
for i (1, indnv(mar,vect),1);
    mat[i,1]=cdfnc((((i-1)*inc)-mdt)/sdt);  *ilk ust column for long*/
    mats[i,1]=cdfn((((1-i)*inc)-mdt)/sdt);  *ilk ust column for short*/
 for j (2, indnv(mar,vect), 1);
    mat[i,j]=cdfn2(((i-j)*inc-mdt)/sdt,inc/sdt); /* rows mar'a, columns
mar a kadar, long*/
    mats[i,j]=cdfn2(((j-i)*inc-inc-mdt)/sdt,inc/sdt); /* rows mar'a,
columns mar a kadar, short */
endfor;
endfor;
mat1=zeros(S,S); /* default olmayan margin calls. prob(marin altina
gitme ama - degil)*/
```

```
mat2t=zeros(S+S,S+S);
mat2ts=zeros(S+S,S+S);
mat2=zeros(dmax/2,dmax/2); /* Z nin default probabiltileri icin*/
mat1s=zeros(S,S);
mat2s=zeros(dmax/2,dmax/2); /* Z nin default probabiltileri icin*/;
for i(1,indnv(mar,vect),1);
  for j(indnv(mar,vect)+1,indnv(0,vect),1);
mat1[i,j]=cdfn2(((i-j)*inc-mdt)/sdt,inc/sdt);
mat1s[i,j]=cdfn2(((j-i)*inc-inc-mdt)/sdt,inc/sdt);
mat11=sumc(mat1');
mat11s=sumc(mat1s');
endfor;
endfor;
for i(1,indnv(mar,vect),1);
   for j(indnv(0,vect)+1,indnv(0,vect)+dmax/2-1,1);
mat2t[i,j]= cdfn2(((i-j)*inc-mdt)/sdt,inc/sdt);
mat2ts[i,j]= cdfn2(((j-i)*inc-inc-mdt)/sdt,inc/sdt);
endfor;
endfor;

for i(1,indnv(mar,vect),1);
  mat2t[i,indnv(0,vect)+dmax/2]=cdfn((((i-(indnv(0,vect)+dmax/2-
1))*inc)-mdt)/sdt);
  mat2ts[i,indnv(0,vect)+dmax/2]=cdfnc((((indnv(0,vect)+dmax/2-1-
i)*inc)-mdt)/sdt);
endfor;
@x=sumc(mats');
y=sumc(mat2ts');
x+y[1:rows(x),1]+mat11s[1:rows(x),1];@

mat2=mat2t[1:dmax/2,indnv(0,vect)+1:indnv(0,vect)+1+dmax/2-1];
mat2s=mat2ts[1:dmax/2,indnv(0,vect)+1:indnv(0,vect)+1+dmax/2-1];
/*default probabilities*/
h=1;
prob=0.005;
routine2:   /* default probabilities*/
if h eq 1;
probd=prob;
else;
probd=probd|(prob);
endif;
if h eq dmax/2; goto final2;
endif;
h=h+1;
prob=prob+0.005;
goto routine2;
final2:
probd=1-probd;
mat22=mat2*probd;
mat22s=mat2s*probd;
if rows(mat)-rows(mat11) gt 0;
temp1=zeros(rows(mat)-rows(mat11),1); /* matrisleri ayni size 'a
getiriyorum*/
       mat11=mat11|temp1;
       mat11s=mat11s|temp1;
    elseif rows(mat)-rows(mat11) lt 0;
```

```
         temp1=zeros(rows(mat11)-rows(mat),1); /* Bu if olayi, sadece
matrisleri ayni size a getirmek icin*/
         mat11=mat11[1:rows(mat),1];
         mat11s=mat11s[1:rows(mat),1];
    endif;
if rows(mat)-rows(mat22) gt 0;
   temp2=zeros(rows(mat)-rows(mat22),1);
   mat22=mat22|temp2;
   mat22s=mat22s|temp2;
     elseif rows(mat)-rows(mat11) lt 0;
       temp2=zeros(rows(mat22)-rows(mat),1);
       mat22=mat22[1:rows(mat),1];
       mat22s=mat22s[1:rows(mat),1];
endif;
mat3=mat11+mat22; /* Z probabilities for long*/
mat3s=mat11s+mat22s; /* Z probabilities for short*/
mat[.,S]=mat3;
mats[.,S]=mat3s; /* Extra stat'in probabilitileri tamamlandi*/

mat4=zeros(rows(mat2),cols(mat2));    /* default probabilities for long*/
mat4s=zeros(rows(mat2s),cols(mat2s)); /*for short*/
for i(1,rows(mat2),1);
  mat4[i,.]=(1-probd)';
  mat4s[i,.]=(1-probd)';
 endfor;
mat44=mat2.*mat4;
mat44s=mat2s.*mat4s;
add1=zeros(S,indnv(mar,vect));     /* add' ler mat4 'u mat size'ina
getirmek icin, boylece sadece toplama ile hallolur*/
add2=zeros(rows(mat)-rows(mat44),cols(mat44));
mat44=mat44|add2;
mat44s=mat44s|add2;
mat44=add1~mat44;
mat44s=add1~mat44s;
mu=zeros(S,1);
mat44=mat44~mu;
mat44s=mat44s~mu;
mat=mat+mat44;
mats=mats+mat44s;
for j (1, S,1);
   mat[S,j]=mat[indnv(M,vect),j];
   mats[S,j]=mats[indnv(M,vect),j];
endfor;

for i (indnv(mar,vect)+1,S-1,1);
  mat[i,S]=1;
  mats[i,S]=1;
endfor;

/* no values between mar and M1 state, also no negative values (instant
adjustment assump.)*/

longmat=mat*mat*mat*mat*mat*mat*mat*mat*mat*mat*mat*mat*mat*mat*mat*mat*
mat*
mat*mat*mat*mat*mat*mat*mat*mat*mat*mat*mat*mat*mat*mat*mat*mat*mat*mat*
mat*mat;
```

```
longmats=mats*mats*mats*mats*mats*mats*mats*mats*mats*mats*mats*mats*mat
s*mats*mats*mats*mats*
mats*mats*mats*mats*mats*mats*mats*mats*mats*mats*mats*mats*mats*mats*ma
ts*mats*mats*mats*mats*mats;

a=indnv(0,vect)+1;
kayip=vect[a:rows(vect),.];
defcost=(longmat[indnv(M,vect),indnv(mar,vect)+1:S-1])*(-kayip);
trancost=longmat[indnv(M,vect),S]*tran;
expcost=defcost+trancost;
defcosts=(longmats[indnv(M,vect),indnv(mar,vect)+1:S-1])*(-kayip);
trancosts=longmats[indnv(M,vect),indnv(mar,vect)+1]*tran;
expcosts=defcosts+trancosts;
totcost=expcost+expcosts;
alfa=300;
beta=-0.01;
C=25; /*commision*/
revenue=(alfa+beta*M)*C;
Profit=revenue-totcost*(alfa+beta*M);
totalcost=totcost*(alfa+beta*M);
if l eq 19;
MM=vect[l,.];
Excost=totalcost;
Expro=Profit;
else;
MM=MM|(vect[l,.]);
Expcost=Expcost|(totalcost);
Expro=Expro|(Profit);
endif;
if l eq n+1; goto final;
endif;
l=l-1;
goto routine;
final:
data2=MM~Expro;
data2;
```

```
@Simple CRRA model, Continuous Dynamic Programming @
new;
/*THIS PROGRAM REPLICATES C. Carroll's lecture Notes on solving dynamic
micro models */
/*USING EULER EQUATION ITERATION AND FITTING MARTIN'S CUBIC SPLINE*/
/*INCOME Random Walk R SERIALLY CORRELATED ,SOLUTION METHOD IS
CARROLL'S*/
/*FINITE LIFE SOLUTION AND SIMULATION, NO LIQUIDTY CONSTRAINT*/
/*only gamma is different 2,4,1.5 (1.5 is for replicating A&L)*/
/*polnonc2, 4,1.fmt is generated as a consumption function 1 corresponds
gamma 1.5*/
_pset="c:\\sule\\copenhag";
loadm path=^_pset;
save path=^_pset;
output file=junk5.out reset;
format /m1 /ro 10,5;
library optmum;
optset;
/*PARAMETERS FOR THE MODEL*/
sigma=1.5;
delt=0.02;
beta=1/(1+delt);
std=0.02; /*STD OF INCOME*/
mu=0; /*MEAN INCOME*/
N=100;
rho=0.6;
sigr=0.011; /*std of white noise in AR(1)*/
thet2=sigr^2/(1-rho^2);
thet=sqrt(thet2);
meanr=0.015;

/*CREATING MARKOV MATRIX FOR AR(1) INTEREST RATE PROCESS*/
m=10; /*number of states for interest rate process*/
rhigh=2.5*thet+meanr;
rlow=-2.5*thet+meanr;
irate=seqa(rlow,(rhigh-rlow)/(m-1),m);
j=1;
roj:
k=2;
routinek:
w=irate[k,1]-irate[k-1,1];
condm=(1-rho)*meanr+rho*irate[j,1];
if k eq 2;
pr=cdfn2((irate[k,1]-condm-w/2)/sigr, w/sigr);
else;
pr=pr~(cdfn2((irate[k,1]-condm-w/2)/sigr, w/sigr));
endif;
if k eq m-1; goto finalk;
endif;
k=k+1;
goto routinek;
finalk:
if j eq 1;
prob=pr;
else;
prob=prob|(pr);
endif;
```

```
if j eq m; goto fij;
endif;
j=j+1;
goto roj;
fij:
i=1;
roi:
w=irate[m,1]-irate[m-1,1];
condi=(1-rho)*meanr+rho*irate[i,1];
if i eq 1;
pr1=cdfn((irate[1,1]-condi+w/2)/sigr);
prn=1-cdfn((irate[m,1]-condi-w/2)/sigr);
else;
pr1=pr1|(cdfn((irate[1,1]-condi+w/2)/sigr));
prn=prn|(1-cdfn((irate[m,1]-condi-w/2)/sigr));
endif;
if i eq m; goto fii;
endif;
i=i+1;
goto roi;
fii:
prob=pr1~prob~prn;

/*NODES AND WEIGHTS FOR THE QUADRATURE TAKEN FROM TAUCEN'S PROGRAM*/
y={-4.85946282833230
   -3.58182348355192
    -2.48432584163895
    -1.46598909439116
    -0.48493570751550
     0.48493570751550
     1.46598909439115
     2.48432584163895
     3.58182348355192
     4.85946282833231};
y=y';

 weight={0.00000431065263
    0.00075807093431
    0.01911158050077
    0.13548370298027
    0.34464233493202
    0.34464233493202
    0.13548370298027
    0.01911158050077
    0.00075807093431
    0.00000431065263 };
weight=weight';
nod=10; /*number of nodes*/
income=-std^2/2+y*std;
income=exp(income);
/*calculating the lower bound for cash on hand*/
a=1/(1+maxc(irate));
qw= -(1-a^80)/(1-a)*minc(income);  /*maximum you can borrow*/
xmin= 0;
xmax=10;
x=seqa(xmin,(xmax-xmin)/(N-1),N);
/*STATE SPACE FOR saving */
```

```
smin=xmin;
smax=xmax;
s=seqa(smin,(smax-smin)/(N-1),N);
x11=x;
po=1;
routinepo:
/*CASH ON HAND*/
xmin=0; @-minc(income)/(1+maxc(irate));@
xmax=10-0.0;
x=seqa(xmin,(xmax-xmin)/(N-1),N);
/*STATE SPACE FOR saving */
smin=xmin;
smax=xmax;
s=seqa(smin,(smax-smin)/(N-1),N);
/*LAST PERIOD POLICY FUNCTION*/
i=1;
routinei:
proc fun(c);
local eul, pr;
eul=(c^(-sigma)-beta*(sumc((1+irate[1,1])*(((1+irate[1,1])*(x[i,1]-
c)+income)^(-sigma)).*weight)*prob[po,1]+
sumc((1+irate[2,1])*(((1+irate[2,1])*(x[i,1]-c)+income)^(-
sigma)).*weight)*prob[po,2]+
sumc((1+irate[3,1])*(((1+irate[3,1])*(x[i,1]-c)+income)^(-
sigma)).*weight)*prob[po,3]+
sumc((1+irate[4,1])*(((1+irate[4,1])*(x[i,1]-c)+income)^(-
sigma)).*weight)*prob[po,4]+
sumc((1+irate[5,1])*(((1+irate[5,1])*(x[i,1]-c)+income)^(-
sigma)).*weight)*prob[po,5]+
sumc((1+irate[6,1])*(((1+irate[6,1])*(x[i,1]-c)+income)^(-
sigma)).*weight)*prob[po,6]+
sumc((1+irate[7,1])*(((1+irate[7,1])*(x[i,1]-c)+income)^(-
sigma)).*weight)*prob[po,7]+
sumc((1+irate[8,1])*(((1+irate[8,1])*(x[i,1]-c)+income)^(-
sigma)).*weight)*prob[po,8]+
sumc((1+irate[9,1])*(((1+irate[9,1])*(x[i,1]-c)+income)^(-
sigma)).*weight)*prob[po,9]+
sumc((1+irate[10,1])*(((1+irate[10,1])*(x[i,1]-c)+income)^(-
sigma)).*weight)*prob[po,10]))^2;
retp(eul);
endp;
{c1,f,gretcode}=optmum(&fun,0.01);


@if i eq 1;
optc=c1;
else;
optc=optc|c1;
endif;
if i eq n; goto finali;
endif;
i=i+1;
goto routinei;
finali:
x1=x;
/*CASH ON HAND*/
```

```
/*numerically calculating invomega in carroll's notes eq=69,70*/
q=1;
routq:
invom=zeros(n,1);
for i(1,n,1);
invom[i,1]=(sumc(((1+irate[1,1])*((fspline(x,optc,cspline(x,optc),((1+ir
ate[1,1])*s[i,1]+income)))^(-sigma))).*weight)*prob[po,1]+
sumc(((1+irate[2,1])*((fspline(x,optc,cspline(x,optc),((1+irate[2,1])*s[
i,1]+income)))^(-sigma))).*weight)*prob[po,2]+
sumc(((1+irate[3,1])*((fspline(x,optc,cspline(x,optc),((1+irate[3,1])*s[
i,1]+income)))^(-sigma))).*weight)*prob[po,3]+
sumc(((1+irate[4,1])*((fspline(x,optc,cspline(x,optc),((1+irate[4,1])*s[
i,1]+income)))^(-sigma))).*weight)*prob[po,4]+
sumc(((1+irate[5,1])*((fspline(x,optc,cspline(x,optc),((1+irate[5,1])*s[
i,1]+income)))^(-sigma))).*weight)*prob[po,5]+
sumc(((1+irate[6,1])*((fspline(x,optc,cspline(x,optc),((1+irate[6,1])*s[
i,1]+income)))^(-sigma))).*weight)*prob[po,6]+
sumc(((1+irate[7,1])*((fspline(x,optc,cspline(x,optc),((1+irate[7,1])*s[
i,1]+income)))^(-sigma))).*weight)*prob[po,7]+
sumc(((1+irate[8,1])*((fspline(x,optc,cspline(x,optc),((1+irate[8,1])*s[
i,1]+income)))^(-sigma))).*weight)*prob[po,8]+
sumc(((1+irate[9,1])*((fspline(x,optc,cspline(x,optc),((1+irate[9,1])*s[
i,1]+income)))^(-sigma))).*weight)*prob[po,9]+
sumc(((1+irate[10,1])*((fspline(x,optc,cspline(x,optc),((1+irate[10,1])*
s[i,1]+income)))^(-sigma))).*weight)*prob[po,10])^(-1/sigma);
endfor;
/*where does liquidty constraint binds*/
xbind=beta^(-1/sigma)*fspline(s,invom,cspline(s,invom),0);
xmin=0; xmin-minc(income)*(1/(1+maxc(irate))^(q+1));
xmax=50-(q+1)*0.0;
x=seqa(xmin,(xmax-xmin)/(N-1),N);
/*STATE SPACE FOR saving */
smin=xmin;
smax=xmax;
s=seqa(smin,(smax-smin)/(N-1),N);
/*SOLVING EARLIER PERIODS OF CONSUMPTION*/
i=1;
routier:
proc fun1(c);
local equ;
equ=(c-beta^(-1/sigma)*fspline(s,invom,cspline(s,invom),(x[i,1]-c)));
retp(equ);
endp;
c00=0.001;
{c11,retcode}=EqSolve(&fun1,c00);

if i eq 1;
optc1=c11;
else;
optc1=optc1|c11;
endif;
if i eq rows(x); goto fini;
endif;
i=i+1;
goto routier;
fini:
/*GOING BACK MULTIPERIODS*/
```

```
if q eq 1;
policy=x11~optc~optc1;
diff=maxc(abs((optc-optc1)./optc1));
else;
policy=policy~optc1;
diff=diff|maxc(abs((optc-optc1)./optc1));
endif;
if q eq 78; goto finq;
endif;
optc=optc1;
q=q+1;
q;
goto routq;
finq:
if po eq 1;
policy1=policy;
else;
policy1=policy1|policy;
endif;
if po eq 10; goto finalpo;
endif;
po=po+1;
po;
goto routinepo;
finalpo:
polall=policy1;
save polall;@

     /* run fspline after to fit the interpolation */

        /* Reference: Numerical recipes, section 3.3 */

proc cspline(x,y);

   local nin, cout, u, utemp, sig, p, iter;
      nin = rows(x);
     if  minc(x[2:nin]-x[1:nin-1]) <= 0;   "x variables need to be sorted
and unique";    end;  endif;

     cout = 0;
        u = 0;

iter = 2;
   do until iter > nin-1+0.5;
           sig = (x[iter]-x[iter-1])/(x[iter+1]-x[iter-1]);
              p = 2+sig*cout[iter-1];
         cout = cout|((sig-1)/p);
      utemp = ((y[iter+1]-y[iter])/(x[iter+1]-x[iter])) - ((y[iter]-
y[iter-1])/(x[iter]-x[iter-1]));
              u = u|(6*utemp/(x[iter+1]-x[iter-1])-sig*u[iter-1])/p;
    iter=iter+1;
endo;

cout = cout|0;

iter = nin-1;
```

```
do until iter < 0.5;
       cout[iter] = cout[iter]*cout[iter+1]+u[iter];
             iter = iter-1;
endo;

retp(cout);
endp;



          /* interpolates xint, given x,y */
        /* run cpline first to give cp = cparam */

    /* Reference: Numerical recipes, section 3.3 */

proc fspline(x,y,cp,xint);
       local klo, khi, k, nin, h, a, b, nint, yint, niter;

           nin = rows(x);
          nint = rows(xint);
          yint = zeros(nint,1);

if minc(x[2:nin]-x[1:nin-1]) <= 0;
"Error: must have distict values of input x";   end;  endif;
@if minc(xint) < minc(x);    "Extrapolating below";     endif;
if maxc(xint) > maxc(x);   "Extrapolating above";     endif;@

niter = 1;
    do until niter > nint;

    klo = 1;  khi = nin;

do while khi-klo > 1;
   k = floor((klo+khi)/2);
   if x[k] > xint[niter];  khi = k;  else;  klo = k;   endif;
endo;
           h = x[khi]-x[klo];
           a = (x[khi]-xint[niter])/h;
           b = (xint[niter]-x[klo])/h;
 yint[niter] =
          a*y[klo]+b*y[khi]+((a*a*a-a)*cp[klo]+(b*b*b-
b)*cp[khi])*(h*h)/6;
niter = niter+1;
endo;

 retp(yint);

endp;
```