Exact likelihood inference for multiple exponential populations under joint censoring

Exact likelihood inference for multiple exponential populations under joint censoring

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ABSTRACT:

The joint censoring scheme is of practical significance while conducting comparative life-tests of products from different units within the same facility. In this thesis, we derive the exact distributions of the maximum likelihood estimators (MLEs) of the unknown parameters when joint censoring of some form is present among the multiple samples, and then discuss the construction of exact confidence intervals for the parameters.

We develop inferential methods based on four different joint censoring schemes. The first one is when a jointly Type-II censored sample arising from k independent exponential populations is available. The second one is when a jointly progressively Type-II censored sample is available, while the last two cases correspond to jointly Type-I hybrid censored and jointly Type-II hybrid censored samples. For each one of these cases, we derive the conditional MLEs of the k exponential mean parameters, and derive their conditional moment generating functions and exact densities, using which we then develop exact confidence intervals for the k population parameters. Furthermore, approximate confidence intervals based on the asymptotic normality of the MLEs, parametric bootstrap intervals, and credible confidence regions from a Bayesian viewpoint are all discussed. An empirical evaluation of all these methods of confidence intervals is also made in terms of coverage probabilities and average widths. Finally, we present examples in order to illustrate all the methods of inference developed here for different joint censoring scenarios.

KEY WORDS: Exponential distribution; Joint Type-II censoring; Joint progressive Type-II censoring; Joint Type-II hybrid censoring; Joint Type-I hybrid censoring; Likelihood inference; Confidence bounds and intervals; Bayesian inference; Bias; Parametric bootstrap confidence intervals; Coverage probabilities; Conditional MLEs; Conditional confidence intervals; Mean square error.

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Chapter 1

Introduction

1.1 Order Statistics

In general, the failure times we observe from a life-testing experiment arise in a naturally increasing order, and so we can use the theory of order statistics to analyze these lifetime data. Extensive literature exists with regard to theory, methods and applications of order statistics. Interested readers may refer to the books by Arnold et al. (1992), Balakrishnan and Rao (1998a, b) and David and Nagaraja (2003) for exhaustive reviews on all these developments.

Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from a continuous population with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. The smallest of the X_i's is denoted by $X_{1:n}$, the second smallest is denoted by $X_{2:n}$, and so on, and the largest is denoted by $X_{n:n}$. The order statistics so obtained are $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$. To derive the density function of $X_{i:n}$, for $i = 1, 2, \cdots, n$, we consider the event $\{x < X_{i:n} \leq x + \delta_x\}$, which is the same as $X_r \leq x$ for $i-1$ of X_r's, $x < X_r \leq x + \delta_x$ for exactly one of X_r's, and $X_r > x + \delta_x$ for the remaining $n-i$ many of X_r's, except for terms of order $O(\delta_x^2)$. Evidently, we have $\frac{n!}{(i-1)!(n-i)!}$ many such possible events. Thus, we have

$$\begin{aligned}
& P(x < X_{i:n} \leq x + \delta_x) \\
= & \frac{n!}{(i-1)!(n-i)!}[F(x)]^{i-1}[1 - F(x + \delta_x)]^{n-i}(F(x + \delta_x) - F(x)) + O(\delta_x^2),
\end{aligned}$$

where $O(\delta_x^2)$ corresponding to all events with at least two X_r's in the interval $(x, x + \delta_x)$. The pdf of $X_{i:n}$ is then obtained as

$$\begin{aligned}
f_{i:n}(x) &= \lim_{\delta_x \to 0} \frac{P(x < X_{i:n} \leq x + \delta_x)}{\delta_x} \\
&= \frac{n!}{(i-1)!(n-i)!} f(x)[F(x)]^{i-1}[1 - F(x)]^{n-i}, \quad -\infty < x < \infty.
\end{aligned}$$

Similarly, the joint pdf of $X_{i:n}$ and $X_{j:n}$ can be obtained as

$$\begin{aligned}
& f_{i,j:n}(x_i, x_j) \\
= & \frac{n!}{(i-1)!(j-i-1)!(n-j)!} f(x_i)f(x_j)[F(x_i)]^{i-1}[F(x_j) - F(x_i)]^{j-i-1}[1 - F(x_j)]^{n-j}, \\
& \qquad -\infty < x_i < x_j < \infty.
\end{aligned}$$

The joint pdf of $X_{1:n}, \cdots, X_{n:n}$ can be obtained directly as

$$f_{1,\cdots,n:n}(x_1, \cdots, x_n) = n! \prod_{i=1}^{n} f(x_i), \quad -\infty < x_1 < \cdots < x_n < \infty.$$

The diverse applications of order statistics include robust estimation, detection of outliers, inference based on censored sample, survival analysis, and reliability theory. Of course, as mentioned earlier, order statistics play a very important role in the analysis of data obtained from life-testing experiments.

1.2 Common Censoring Schemes

Censoring is often encountered in reliability and life-testing experiments, since the experimenter may have to terminate the test before all items have failed due to time limit or economic reason. The two most common forms are Type-I and Type-II censoring schemes. In Type-I censoring scheme, the experimental time is fixed, but the number of observed failures is a random variable. In Type-II censoring scheme, the number of observed failures is fixed, but the experimental time is a random variable.

1.2.1 Type-I Censoring

Consider a life-testing experiment in which n units are placed on test. The ordered lifetimes of these units are denoted by $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$, respectively. The lifetimes of the sample units are assumed to be independent and identically distributed (i.i.d.)

random variables with cdf $F(x)$ and pdf $f(x)$.

Let T be a pre-fixed time. In Type-I censoring scheme, the experimenter will choose to terminate the experiment at the time point T. Let D be the random variable such that $X_{D:n} \leq T < X_{(D+1):n}$. Then, in this case, we have the likelihood function as

$$L(\theta|x_1, \cdots, x_D) = \frac{n!}{(n-D)!} \prod_{i=1}^{D} f(x_i)(1-F(T))^{n-D},$$
$$x_1 < x_2 < \cdots < x_D < T.$$

1.2.2 Type-II Censoring

Let r be a pre-fixed positive integer. In Type-II censoring scheme, the experimenter will choose to terminate the experiment when r failures have been observed. Thus, the experimental time here is $X_{r:n}$. In this case, we have

$$L(\theta|x_1, \cdots, x_r) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} f(x_i)(1-F(x_r))^{n-r},$$
$$x_1 < x_2 < \cdots < x_r.$$

1.2.3 Progressive Type-II Censoring

Progressive censoring has been discussed quite extensively recently in the literature. Suppose $(R_1, R_2, \cdots, R_r)$ is the pre-fixed progressive censoring scheme. Then, under the progressive Type-II censoring scheme, at the time of the first failure, R_1 of the

$n-1$ surviving units are withdrawn randomly from the life-test, at the time of the second failure, R_2 of the $n-R_1-2$ surviving units are withdrawn randomly from the life-test, and so on, until the r-th failure is observed, at which time, all remaining surviving units are withdrawn from the life-test. In this case, we have

$$L(\theta|x_1,\cdots,x_r)=\prod_{j=1}^{r}(n-j+1-\sum_{i=1}^{j-1}R_i)\prod_{i=1}^{r}f(x_i)(1-F(x_i))^{R_i},$$
$$x_1<x_2<\cdots<x_r.$$

As we can see, if $R_1=R_2=\cdots=R_{r-1}=0$, then $R_r=n-r$ which corresponds to the conventional Type-II censoring scheme. Of course, if $r=n$ with $R_1=R_2=\cdots=R_n=0$, then we will have the complete sample situation.

1.2.4 Type-I and Type-II Hybrid Censoring

In Type-I hybrid censoring scheme, the experimenter will choose to terminate the experiment at the time point $T^*=\min\{X_{r:n},T\}$. Thus, the test is terminated when a pre-fixed number, $r<n$, out of n items has been observed, or when a pre-fixed time T, has been reached. In this censoring, the experimental time is no more than T, but the number of observed failures is a random variable.

In Type-II hybrid censoring scheme, the experimenter will choose to terminate the experiment at the time point $T^{**}=\max\{X_{r:n},T\}$. Thus, if the r-th failure is observed before time T, the test will be terminated at time T; if the r-th failure is observed after time T, the test will be terminated at time $X_{r:n}$. In this censoring, the number of observed failures is at least r, but the experimental time is a random variable.

1.3 Joint Censoring and Background

The joint censoring scheme is of practical significance in conducting comparative life-tests of products from different lines within the same facility. Suppose products are being manufactured by k different lines within the same facility, and that k independent samples of sizes n_h, $1 \leq h \leq k$, are selected from these k lines and placed simultaneously on a life-testing experiment. In order to reduce the cost of the experiment as well as the experimental time, the experimenter may choose to terminate the experiment after a certain number (say, r) of failures has been observed altogether. In this situation, one may be interested in either point or interval estimation of the mean lifetimes of units produced by these k lines. In this thesis, we derive the maximum likelihood estimates of the scale parameters of k exponential populations under this set-up, and then develop exact conditional inferential methods based on these maximum likelihood estimates.

The joint censoring scheme has been considered before in the literature. For example, Epstein (1954) introduced the Type-I hybrid censoring scheme (Type-I HCS) in which the life-testing experiment is terminated as soon as a pre-specified number r out of n items has failed or a pre-fixed time T on test has been reached. The Type-I

HCS has been used as a reliability acceptance test in MIL-STD-781 C (1977). While Basu (1968) discussed a generalized Savage statistic, Johnson and Mehrotra (1972) studied a locally most powerful rank test under joint censoring. The problem of testing for the equality of two distributions, under the assumption of exponentiality, was discussed by Bhattacharyya and Mehrotra (1981). All these developments under the joint censoring scheme have focused on nonparametric and parametric tests of hypotheses; see Bhattacharyya (1995, Chapter 7 of Balakrishnan and Basu (1995)) for details. For the exact inference based on the MLEs, Chen and Bhattacharyya (1988) derived the exact distribution of the maximum likelihood estimator of the mean of an exponential distribution and an exact lower confidence bound for the mean based on a Type-I hybrid censored sample. Childs et al. (2003) obtained an alternative simpler form for the result of Chen and Bhattacharyya and also developed similar results for the case of Type-II hybrid censoring.

To study two or more competing products in regard to the duration of their service life, comparative lifetime experiments are quite useful. In this regard, Balakrishnan and Rasouli (2008) discussed exact inference for two exponential populations when Type-II censoring is implemented on the two samples in a joint manner. Balakrishnan and Rasouli (2010) subsequently extended their work to the case of two exponential populations when joint progressive Type-II censoring is implemented on the two samples in a combined manner. Recently, Shafay et al. (2012) developed exact inference for two exponential populations under joint Type-II hybrid censoring on the two samples. By assuming two Weibull populations, Parsi et al. (2012) discussed conditional maximum likelihood estimation and associated confidence intervals for the model parameters.

1.3.1 Joint Type-II Censoring

Let us suppose that $(X_1, \cdots, X_N)$ are N jointly distributed random variables, with $\{X_1, \cdots, X_N\} = \{X_{11}, \cdots, X_{1n_1}; X_{21}, \cdots, X_{2n_2}; \cdots; X_{k1}, \cdots, X_{kn_k}\}$, with $N = \sum_{h=1}^{k} n_h$. Suppose $X_{11}, X_{12}, \cdots, X_{1n_1}$ are the lifetimes of n_1 specimens from production line A_1, and are independent and identically distributed (iid) variables from a population with cdf $F_1(x)$ and pdf $f_1(x)$. Similarly, $X_{21}, X_{22}, \cdots, X_{2n_2}$ are the lifetimes of n_2 specimens from production line A_2, and are assumed to be a sample from pdf $f_2(x)$ and cdf $F_2(x)$, and so on, with $X_{k1}, X_{k2}, \cdots, X_{kn_k}$ denoting the lifetimes of n_k specimens from production line A_k being iid variables from pdf $f_k(x)$ and cdf $F_k(x)$. Denote the order statistics of these k random samples by $W_1 \leq W_2 \leq \cdots \leq W_N$, where N is the total sample size.

Let r denote a pre-fixed total number of failures to be observed. Then, under the joint Type-II censoring scheme for the k-samples, the observable data consist of $(\boldsymbol{\delta}, \mathbf{w})$, where $\mathbf{w} = (w_1, w_2, \cdots, w_r)$, $w_i \in \{X_{h_i1}, X_{h_i2}, \cdots, X_{h_in_i}\}$ for $1 \leq h_1, h_2, \cdots, h_r \leq k$, h_i indicating the production line where w_i is from. Moreover, associated to $(h_1, h_2, \cdots, h_r)$, let us define $\boldsymbol{\delta} = (\delta_1(h), \delta_2(h), \cdots, \delta_r(h))$ as

$$\delta_i(h) = \begin{cases} 1, & \text{if } h = h_i \\ 0, & \text{otherwise.} \end{cases} \tag{1.3.1}$$

Under such a joint Type-II censoring scheme, to derive the joint probability density function (pdf) of $W_1, W_2, \cdots, W_r$, we consider the event $\Omega = \{w_i < W_i \leq w_i + \varepsilon_{w_i}, 1 \leq i \leq r\}$ for $w_1 \leq w_2 \leq \cdots \leq w_r$, and $W_i \geq w_r + \varepsilon_{w_r}$ for the remaining $N - r$ of the W_i's. Let $M_r(h)$ be the number of units selected from the h-th sample $\{X_{h1}, \cdots, X_{hn_h}\}$ for $1 \leq h \leq k$. Evidently, we have $\prod_{h=1}^{k} \frac{n_h!}{(n_h - M_r(h))!}$ many such possible events. Consequently, we have

$$\begin{aligned} P(\Omega) &= \prod_{h=1}^{k} \frac{n_h!}{(n_h - M_r(h))!} \prod_{i=1}^{r} \prod_{h=1}^{k} [F_h(w_i + \varepsilon_{w_i}) - F_h(w_i)]^{\delta_i(h)} \\ &\quad \times \prod_{i=r+1}^{N} \prod_{h=1}^{k} [1 - F_h(w_r + \varepsilon_{w_r})]^{\delta_i(h)} + \sum_{i=1}^{r} O(\varepsilon_{w_i}^2). \end{aligned}$$

The joint pdf of $W_1, W_2, \cdots, W_r$ is then obtained as

$$\begin{aligned} f(w_1, w_2, \cdots, w_r) = c_r \prod_{i=1}^{r} \prod_{h=1}^{k} (f_h(w_i))^{\delta_i(h)} \prod_{h=1}^{k} (S_h(w_r))^{n_h - M_r(h)}, \\ -\infty < w_1 < w_2 < \cdots < w_r < \infty, \end{aligned}$$

where $S_h(w_r) = 1 - F_h(w_r)$ and $c_r = \frac{\prod_{h=1}^{k} n_h!}{\prod_{h=1}^{k} (n_h - M_r(h))!}$.

1.3.2 Joint Progressive Type-II Censoring

The joint progressive Type-II censoring is implemented as follows. At the time of the first failure $W_1 \in X_{h_1}$ for some $1 \leq h_1 \leq k$, R_1 units are randomly withdrawn from the remaining $N-1$ surviving units. Next, at the time of the second failure $W_2 \in X_{h_2}$ for some $1 \leq h_2 \leq k$, R_2 units are randomly withdrawn from the remaining $N-R_1-2$ surviving units, and so on. Finally, at the time of the r-th failure $W_r \in X_{h_r}$ for some $1 \leq h_r \leq k$, all remaining $R_r = N - r - \sum_{i=1}^{r-1} R_i$ surviving units are withdrawn from the life-testing experiment. One may refer to the book by Balakrishnan and Aggarwala (2000) and the discussion paper by Balakrishnan (2007) for elaborate details on progressive Type-II censoring and associated inferential results. Here, the joint progressive Type-II censoring scheme $\mathbf{R} = (R_1, R_2, \cdots, R_r)$ is pre-fixed and has the decomposition $R_i = \sum_{h=1}^{k} s_i(h)$, $1 \leq i \leq r$, where $s_i(h)$ is the number of units withdrawn at the time of the i-th failure belonging to the h-th sample, and these are unknown and are latent random variables. The data observed in this form will consist of $(\boldsymbol{\delta}, \mathbf{R}, \mathbf{w})$, where $\boldsymbol{\delta} = (\delta_1, \delta_2, \cdots, \delta_r)$ and $\mathbf{w} = (w_1, w_2, \cdots, w_r)$.

Under such a joint progressive Type-II censoring scheme, to derive the joint pdf of $W_1, W_2, \cdots, W_r$, we consider the event $\Omega = \{w_i < W_i \leq w_i + \varepsilon_{w_i}, 1 \leq i \leq r\}$ for $w_1 \leq w_2 \leq \cdots \leq w_r$, and for each i $(1 \leq i \leq r)$, R_i of the $N - i - \sum_{j=1}^{i-1} R_j$ surviving units are withdrawn randomly from the life-test, in which $s_i(h)$ of $n_h - M_i(h) - \sum_{j=1}^{i-1} s_j(h)$ many units are from h-th sample for $1 \leq h \leq k$. Where $M_{i-1}(h)$ is the number of units among $W_1, W_2, \cdots, W_{i-1}$, selected from the h-th sample $\{X_{h1}, \cdots, X_{hn_h}\}$ for

$1 \leq h \leq k$. Let $\mathcal{S}_{\mathbf{R}}$ be the set of all such possible $\mathbf{s}$. Evidently, for each specified $\mathbf{s}$, we have D_1 many possible choice for the observed data and the probability of $\mathbf{s} \in \mathcal{S}_{\mathbf{R}}$ is D_2, where

$$D_1 = \prod_{i=1}^{r} \left[\sum_{h=1}^{k} \left(n_h - M_{i-1}(h) - \sum_{j=1}^{i-1} s_j \right) \delta_i(h) \right],$$

$$D_2 = \prod_{i=1}^{r-1} \left\{ \frac{\prod_{h=1}^{k} \begin{pmatrix} n_h - M_i(h) - \sum_{j=1}^{i-1} s_j(h) \\ s_j(h) \end{pmatrix}}{\begin{pmatrix} N - i - \sum_{j=1}^{i-1} R_j \\ R_i \end{pmatrix}} \right\}.$$

Consequently, we have

$$f(w_1, w_2, \cdots, w_r) = c_r \prod_{i=1}^{r} \prod_{h=1}^{k} (f_h(w_i))^{\delta_i(h)} \prod_{i=1}^{r} \prod_{h=1}^{k} (S_h(w_i))^{s_i(h)},$$
$$-\infty < w_1 < w_2 < \cdots < w_r < \infty.$$

where $S_h(w_r) = 1 - F_h(w_r)$ and $c_r = D_1 D_2$.

1.3.3 Joint Type-I and Type-II Hybrid Censoring

Under the joint Type-I and Type-II hybrid censoring schemes for the k-samples, the observable data consist of $(\boldsymbol{\delta}, \mathbf{w})$, where $\boldsymbol{\delta}$ is the vector of indicators and $\mathbf{w}$ is the vector of ordered lifetimes, and these will be defined subsequently. Like conventional Type-I censoring scheme, the main disadvantage of Type-I hybrid censoring scheme is that most of the inferential results are obtained under the condition that the number of observed failures is at least one, and moreover, there may be very few failures at the termination point of the experiment. In that case, the efficiency of the estimator(s) may be low. For this reason, Childs et al. (2003) introduced the Type-II hybrid censoring scheme as an alternative to the Type-I hybrid censoring scheme. It has the advantage of guaranteeing that at least r failures will be observed at the end of the experiment. Of course, the disadvantage of this scheme is that the duration of the test is random.

Let D denote the number of failures up to time T. Then, D is a discrete random variable with support $\{0, 1, 2, \cdots, N\}$ with probability mass function

$$P(D = d) = \sum \begin{pmatrix} n_1 \\ l_1 \end{pmatrix} p_1^{l_1} q_1^{n_1 - l_1} \begin{pmatrix} n_2 \\ l_2 \end{pmatrix} p_2^{l_2} q_2^{n_2 - l_2} \cdots \begin{pmatrix} n_k \\ l_k \end{pmatrix} p_k^{l_k} q_k^{n_k - l_k},$$

where the summation is over $(l_1, l_2, \cdots, l_k)$ for which $\sum_{i=1}^{k} l_i = d$ for $1 \leq l_i \leq n_i$, and $p_j = F_j(T), q_j = 1 - F_j(T)$ $(1 \leq j \leq k)$. Therefore, under the joint Type-I hybrid censoring scheme described above, the observable data $(\boldsymbol{\delta}, \mathbf{w})$ is of the following form:

$$(\boldsymbol{\delta}, \mathbf{w}) = \begin{cases} (\delta_1, \delta_2, \cdots, \delta_D; w_1, w_2, \cdots, w_D), & \text{with } D = 0, 1, \cdots, r-1, \\ (\delta_1, \delta_2, \cdots, \delta_r; w_1, w_2, \cdots, w_r), & \text{with } D = r, r+1, \cdots, N. \end{cases}$$

Under the joint Type-II hybrid censoring scheme described above, the observable data

$(\boldsymbol{\delta}, \mathbf{w})$ is of the following form:

$$(\boldsymbol{\delta}, \mathbf{w}) = \begin{cases} (\delta_1, \delta_2, \cdots, \delta_r; w_1, w_2, \cdots, w_r), & \text{with } D = 0, 1, \cdots, r-1; \\ (\delta_1, \delta_2, \cdots, \delta_D; w_1, w_2, \cdots, w_D), & \text{with } D = r, r+1, \cdots, N, \end{cases}$$

where δ is as defined in (1.3.1).

Now, the joint pdf of $W_1, W_2, \cdots, W_r$ under joint Type-I hybrid censoring is obtained as

$$f(w_1, w_2, \cdots, w_r) = \begin{cases} c_D \prod_{i=1}^{D} \prod_{h=1}^{k} \left(f_h(w_i)\right)^{\delta_i(h)} \prod_{h=1}^{k} \left(S_h(T)\right)^{n_h - M_D(h)}, & T < w_r, \\ c_r \prod_{i=1}^{r} \prod_{h=1}^{k} \left(f_h(w_i)\right)^{\delta_i(h)} \prod_{h=1}^{k} \left(S_h(w_r)\right)^{n_h - M_r(h)}, & T > w_r, \\ & w_1 < w_2 < \cdots < w_r, \end{cases}$$

and the joint pdf of $W_1, W_2, \cdots, W_r$ under joint Type-II hybrid censoring is obtained as

$$f(w_1, w_2, \cdots, w_r) = \begin{cases} c_r \prod_{i=1}^{r} \prod_{h=1}^{k} \left(f_h(w_i)\right)^{\delta_i(h)} \prod_{h=1}^{k} \left(S_h(w_r)\right)^{n_h - M_r(h)}, & T < w_r, \\ c_D \prod_{i=1}^{D} \prod_{h=1}^{k} \left(f_h(w_i)\right)^{\delta_i(h)} \prod_{h=1}^{k} \left(S_h(T)\right)^{n_h - M_D(h)}, & T > w_r, \\ & w_1 < w_2 < \cdots < w_r, \end{cases}$$

where $S_h(w_r) = 1 - F_h(w_r)$, $S_h(T) = 1 - F_h(T)$, $c_r = \frac{\prod_{h=1}^{k} n_h!}{\prod_{h=1}^{k}(n_h - M_r(h))!}$ and $c_D = \frac{\prod_{h=1}^{k} n_h!}{\prod_{h=1}^{k}(n_h - M_D(h))!}$.

1.4 Scope of the Thesis

The main aim of this thesis is to consider k exponential distributions for the samples from k production lines, and to derive the exact distribution of the maximum likelihood estimators (MLEs) of the unknown parameters and to construct exact confidence intervals under various forms of joint censoring on the k samples.

In Chapter 2, when a jointly Type-II censored sample arising from k independent exponential populations is available, we derive the conditional MLEs of the k exponential mean parameters. The conditional moment generating functions and the exact densities of these MLEs are also obtained. By using these exact densities of the MLEs, we develop exact confidence intervals for the parameters. Moreover, approximate confidence intervals based on the asymptotic normality of the MLEs and credible confidence regions from a Bayesian viewpoint are also discussed. An empirical comparison of the exact, approximate, bootstrap and Bayesian intervals is also made in terms of coverage probabilities. Finally, an example is presented in order to illustrate all the methods of inference developed.

In Chapter 3, based on a joint progressively Type-II censored sample arising from k independent exponential populations, we discuss the conditional MLEs of the k exponential mean parameters and derive their conditional moment generating functions and exact densities. By using these exact densities of the MLEs, we develop exact confidence intervals for the exponential mean parameters. Furthermore, approximate confidence intervals based on the asymptotic normality of the MLEs and credible confidence regions from a Bayesian viewpoint are discussed. An empirical evaluation of the exact, approximate, bootstrap and Bayesian intervals is also made in terms of coverage probabilities and average widths. Finally, an example is presented in order to illustrate all the methods of inference developed.

In Chapter 4, inferential methods based on joint Type-II hybrid censoring scheme (HCS) are discussed. Based on a joint Type-II HCS arising from k independent exponential populations, we obtain the conditional MLEs of the k exponential mean parameters. We also derive their conditional moment generating functions and exact densities. Using these exact density functions, we then derive the means, variances and mean squared errors of these estimates, and also exact confidence intervals for the parameters. Moreover, approximate confidence intervals based on the asymptotic normality of the MLEs and credible confidence intervals from a Bayesian viewpoint are discussed. Finally, some simulation results are presented in order to illustrate all the methods of inference developed.

In Chapter 5, analogous results are developed for the joint Type-I HCS. We develop the exact results and also present some numerical results and an example to illustrate the established results.

Finally, in Chapter 6, we make some concluding remarks and also indicate some directions for possible future research.

Chapter 2

Inference Under Joint Type-II Censoring

2.1 Introduction

The joint censoring scheme is of practical significance in conducting comparative life-tests of products from different units within the same facility. Suppose products are being manufactured by k different lines within the same facility, and that k independent samples of sizes n_h, $1 \leq h \leq k$, are selected from these k lines and placed simultaneously on a life-testing experiment. Then, in order to reduce the cost of experiment and also to reduce the experimental time, the experimenter may choose to terminate the life-testing experiment as soon as a certain number (say, r) of failures occur. In this situation, one may be interested in either point or interval estimation of the mean lifetimes of units produced by these k lines. Here, exact results based on the maximum likelihood estimates are developed to facilitate this.

This joint censoring scheme has been considered before in the literature. Basu (1968) discussed a generalized Savage statistic. Johnson and Mehrotra (1972) studied locally most powerful rank test under joint censoring. The problem of testing for the equality of two distributions, under the assumption of exponentiality, was discussed by Bhattacharyya and Mehrotra (1981). All these developments under this joint censoring scheme have focused on nonparametric and parametric tests of hypotheses; see Bhattacharyya (1995, Chapter 7 of Balakrishnan and Basu (1995)). For the exact inference based on the MLEs, Chen and Bhattacharyya (1988) derived the exact distribution of the maximum likelihood estimator of the mean of an exponential distribution and an exact lower confidence bound for the mean based on a hybrid censored sample. Childs et al. (2003) obtained an alternative simple form which is equivalent to the results of Chen and Bhattacharyya. To study two or more competing products in regard to the duration of their service life, comparative lifetime experiments are of great importance. In this regard, Balakrishnan and Rasouli (2008) discussed exact inference for two exponential populations when Type-II censoring is implemented on the two samples in a joint manner. Here, we generalize their work by considering the k-sample problem. Suppose the test units from k lines under study are placed on a life-test simultaneously, that the successive failure times and the corresponding types (lines from which the failed units come from) are recorded, and that the experiment is terminated as soon as a specified total number of failures (say, r) occurred.

Suppose $X_{11}, X_{12}, \cdots, X_{1n_1}$ are the lifetimes of n_1 specimens from line A_1, and assumed to be independent and identically distributed (iid) variables from a population with cumulative distribution function (cdf) $F_1(x)$ and probability density function (pdf) $f_1(x)$. Similarly, $X_{21}, X_{22}, \cdots, X_{2n_2}$ are the lifetimes of n_2 specimens from line A_2 and assumed to be a sample from pdf $f_2(x)$ and cdf $F_2(x)$, and so on, with $X_{k1}, X_{k2}, \cdots, X_{kn_k}$ denoting the lifetimes of n_k specimens from line A_k being iid variables from pdf $f_k(x)$ and cdf $F_k(x)$.

Furthermore, let $N = \sum_{i=1}^{k} n_i$ denote the total sample size and r denote the total number of failures observed. Let $w_1 \leq w_2 \leq \cdots \leq w_N$ denote the order statistics of the N random variables $\{X_{ij}; 1 \leq i \leq k, 1 \leq j \leq n_i\}$.

Therefore, under the joint Type-II censoring scheme for the k-samples, the observable data consist of $(\boldsymbol{\delta}, \mathbf{w})$, where $\mathbf{w} = (w_1, w_2, \cdots, w_r)$, $w_i \in \{X_{h_i 1}, X_{h_i 2}, \cdots, X_{h_i n_i}\}$ for $1 \leq h_1, h_2, \cdots, h_r \leq k$, with r being a pre-fixed integer. Finally, associated to $(h_1, h_2, \cdots, h_r)$, let us define $\boldsymbol{\delta} = (\delta_1(h), \delta_2(h), \cdots, \delta_r(h))$ as

$$\delta_i(h) = \begin{cases} 1, & \text{if } h = h_i \\ 0, & \text{otherwise.} \end{cases} \tag{2.1.1}$$

2.2 MLEs, Exact Distributions and Inference

Let $M_r(h) = \sum_{i=1}^{r} \delta_i(h)$ denote the number of X_h-failures in $\mathbf{W}$ for $1 \le h \le k$ and $r = \sum_{h=1}^{k} M_r(h)$. Then, the likelihood of $(\boldsymbol{\delta}, \mathbf{W})$ is given by

$$\begin{aligned} & L(\theta_1, \theta_2, \cdots, \theta_k, \boldsymbol{\delta}, \mathbf{W}) \\ = \; & c_r \prod_{i=1}^{r} \prod_{h=1}^{k} (f_h(w_i))^{\delta_i(h)} \prod_{h=1}^{k} (S_h(w_r))^{n_h - M_r(h)}, \end{aligned} \tag{2.2.1}$$

where $S_h(w_r) = 1 - F_h(w_r)$ and $c_r = \frac{\prod_{h=1}^{k} n_h!}{\prod_{h=1}^{k}(n_h - M_r(h))!}$. When the k populations are exponential with cdf $F_h(x) = 1 - \exp(-\frac{x}{\theta_h})$, $x > 0$, and pdf $f_h(x) = \frac{1}{\theta_h} \exp(-\frac{x}{\theta_h})$, $x > 0$, for $1 \le h \le k$, the likelihood function in (2.2.1) becomes

$$\begin{aligned} & L(\theta_1, \theta_2, \cdots, \theta_k, \boldsymbol{\delta}, \mathbf{W}) \\ = \; & c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \prod_{i=1}^{r} \prod_{h=1}^{k} \left\{ \exp\left(-\frac{w_i}{\theta_h} \right) \right\}^{\delta_i(h)} \prod_{h=1}^{k} \left\{ \exp\left(-\frac{w_r}{\theta_h} \right) \right\}^{n_h - M_r(h)} \\ = \; & c_r \prod_{h=1}^{k} \exp\left\{ -M_r(h) \log \theta_h - \frac{\sum_{i=1}^{r} w_i \delta_i(h)}{\theta_h} - \frac{w_r(n_h - M_r(h))}{\theta_h} \right\} \qquad (2.2.2) \\ = \; & c_r \prod_{h=1}^{k} \exp\left\{ -M_r(h) \log \theta_h - \frac{\sum_{i=1}^{M_r(h)} x_{hi}}{\theta_h} - \frac{w_r(n_h - M_r(h))}{\theta_h} \right\}. \qquad (2.2.3) \end{aligned}$$

From (2.2.2), we can readily obtain the MLE of θ_h, for $1 \le h \le k$, as

$$\hat{\theta}_h = \frac{1}{M_r(h)} \left\{ \sum_{i=1}^{r} w_i \delta_i(h) + w_r(n_h - M_r(h)) \right\}, \tag{2.2.4}$$

or equivalently from (2.2.3) as

$$\hat{\theta}_h = \frac{1}{M_r(h)} \left\{ \sum_{i=1}^{M_r(h)} x_{hi} + w_r(n_h - M_r(h)) \right\}. \tag{2.2.5}$$

Remark 1 *From the likelihood function, we readily see that the MLE of θ_h does not exist when $M_r(h) = 0$. So, the MLEs in (2.2.5) are conditioned on $S = \{\sum_{h=1}^{k} M_r(h) = r$ and $M_r(h) \geq 1, \forall 1 \leq h \leq k\}$, or equivalently on the set $S = \{\sum_{h=1}^{k} M_r(h) = r$ and $\prod_{h=1}^{k} M_r(h) \neq 0\}$. We therefore need to discuss the exact distribution and other properties of the MLEs only conditional on the set S.*

Lemma 1 *Let $\mathbf{M}_r = (M_r(1), M_r(2), \cdots, M_r(k))$ and $\mathbf{t} = (t_1, t_2, \cdots, t_k)$ with $\sum_{j=1}^{k} t_j = r$. Further, let*

$$\begin{aligned} \mathbf{T} = \quad & \Big\{ \mathbf{t} : \max\{1, r - \sum_{h=1}^{k-1} n_j\} \leq t_k \leq \min\{r - k + 1, n_k\}\ , \\ & \max\{1, r - \sum_{h=1}^{k-2} n_j - t_k\} \leq t_{k-1} \leq \min\{r - k + 2 - t_k, n_{k-1}\}, \\ & \ldots \\ & \max\{1, r - n_1 - \sum_{h=3}^{k} t_h\} \leq t_2 \leq \min\{r - \sum_{h=3}^{k} t_h, n_2\} \Big\}. \end{aligned}$$

Then, we have

$$P(S) \quad = \quad \sum_{\mathbf{t} \in \mathbf{T}} P(\mathbf{M}_r = \mathbf{t}). \tag{2.2.6}$$

Proof Consider the set

$$S = \left\{ \sum_{h=1}^{k} M_r(h) = r, \prod_{h=1}^{k} M_r(h) \neq 0 \right\}.$$

Since $t_k \leq n_k$ and $r = \sum_{h=1}^{k} t_h \geq k - 1 + t_k$, we have $t_k \leq \min\{r - k + 1, n_k\}$. On the other hand, $t_k \geq 1$ and $r = \sum_{h=1}^{k} t_h \leq \sum_{h=1}^{k-1} n_h + t_k$, and so we have $t_k \geq \max\{1, r - \sum_{h=1}^{k-1} n_h\}$.

When we fix t_k in this way, we get $\max\{1, r - \sum_{h=1}^{k-2} n_j - t_k\} \leq t_{k-1} \leq \min\{r - k + 2 - t_k, n_{k-1}\}$, and therefore

$$S = \cup_{\mathbf{t} \in \mathbf{T}} \{\mathbf{M}_r = \mathbf{t}\}$$

which proves the lemma.

Remark 2 *In fact, the set* $\{M_r(1) = r - \sum_{h=2}^{k} t_h \stackrel{\triangle}{=} t_1, M_r(2) = t_2, \cdots,$ $M_r(k) = t_k\}$ *may be empty if any one of* $t_1, t_2, \cdots, t_k$ *is out of the range* $\{\mathbf{t} = (t_1, t_2 \cdots, t_k) : t_j \in \{1, 2, \cdots, n_j\}\}$. *In this case, the probability of such a set becomes*

$$P\left(\{M_r(1) = t_1, M_r(2) = t_2, \cdots, M_r(k) = t_k\}\right) = 0.$$

Theorem 1 **(a)** *The joint probability mass function of $\boldsymbol{\delta}$ is*

$$P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) = c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \prod_{i=1}^{r} \frac{1}{\sum_{h=1}^{k} \frac{n_h - M_{i-1}(h)}{\theta_h}}; \tag{2.2.7}$$

(b) *For $\mathbf{t} = (t_1, t_2, \cdots, t_k)$ such that $\sum_{j=1}^{k} t_j = r$ and $t_j \geq 1$ for all j, let $\boldsymbol{\delta}$ be as defined in (1.3.1), $M_r(h) = \sum_{i=1}^{r} \delta_i(h)$, and*

$$Q^*(\mathbf{t}) = \{\tilde{\boldsymbol{\delta}}(\mathbf{h}) = (\tilde{\delta}_1, \tilde{\delta}_2, \cdots, \tilde{\delta}_r) \text{ such that } M_r(h) = t_h \text{ for } 1 \leq h \leq k\}.$$

Then,

$$P(\mathbf{M}_r = \mathbf{t}) = \sum_{\tilde{\boldsymbol{\delta}} \in Q^*(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-t_h} \prod_{i=1}^{r} \frac{1}{\sum_{h=1}^{k} \frac{n_h - M_{i-1}(h)}{\theta_h}}; \tag{2.2.8}$$

(c) *Finally, we have*

$$P(S) = \sum_{\mathbf{t} \in \mathbf{T}} P(\mathbf{M}_r = \mathbf{t}). \tag{2.2.9}$$

Proof **(a)** From (2.2.2), we have the joint density of $(\boldsymbol{\delta}, \mathbf{W})$ as

$$\begin{aligned} & f(\boldsymbol{\delta}, \mathbf{W}) \\ = \; & c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \exp\left\{-\sum_{i=1}^{r-1}\sum_{h=1}^{k} \frac{w_i \delta_i(h)}{\theta_h}\right\} \exp\left\{-w_r \sum_{h=1}^{k} \frac{\delta_r(h) + n_h - M_r(h)}{\theta_h}\right\}. \end{aligned}$$

Let

$$
\begin{aligned}
a_j &= \sum_{h=1}^{k} \frac{\delta_j(h)}{\theta_h} = \frac{1}{\theta_{h_j}}, \\
b_r &= \sum_{h=1}^{k} \frac{\delta_r(h) + n_h - M_r(h)}{\theta_h} = \sum_{h=1}^{k} \frac{n_h - M_{r-1}(h)}{\theta_h}.
\end{aligned}
$$

Then, we have

$$
f(\boldsymbol{\delta}, \mathbf{W}) = c_r \left(\prod_{h=1}^{k} \theta_h^{-M_r(h)} \right) \exp\left\{ -\sum_{i=1}^{r-1} a_i w_i - b_r w_r \right\}. \tag{2.2.10}
$$

Upon integrating out $w_1, w_2, \cdots, w_r$ over $\{0 \le w_1 \le w_2 \le \cdots \le w_r < \infty\}$ in (2.2.10), by mapping it onto $\{0 \le u_i < \infty, 1 \le i \le k\}$ through the transformation

$$
\begin{cases}
u_1 &= w_1, \\
u_2 &= w_2 - w_1, \\
 & \cdots \\
u_r &= w_r - w_{r-1},
\end{cases}
$$

we obtain the joint probability mass function of $\boldsymbol{\delta}$ as

$$
P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) = c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \frac{1}{b_r} \prod_{i=1}^{r-1} \frac{1}{b_r + \sum_{j=i}^{r-1} a_j}. \tag{2.2.11}
$$

Observe that for $\tilde{\boldsymbol{\delta}}$, $M_r(h, \tilde{\boldsymbol{\delta}}) = \sum_{i=1}^{r} \tilde{\delta}_i(h)$ and

$$\begin{aligned} b_r + \sum_{j=i}^{r-1} a_j &= \sum_{h=1}^{k} \frac{\tilde{\delta}_r(h) + n_h - M_r(h)}{\theta_h} + \sum_{j=i}^{r-1} \sum_{h=1}^{k} \frac{\tilde{\delta}_j(h)}{\theta_h} \\ &= \sum_{h=1}^{k} \frac{n_h - M_{i-1}(h, \tilde{\boldsymbol{\delta}})}{\theta_h}, \end{aligned}$$

where $M_{i-1}(h, \tilde{\boldsymbol{\delta}}) = \sum_{l=1}^{i-1} \tilde{\delta}_l(h)$ depends on $\tilde{\boldsymbol{\delta}}$, and yet we denote it by $M_{i-1}(h)$ for simplicity, and take $M_0(h, \tilde{\boldsymbol{\delta}}) = 0$. Thus, (2.2.7) is obtained.

(b) It follows from (2.2.7).

(c) The required result follows immediately from Lemma 1 and Part (b).

Example 1 *When $k = 3$, we have*

$$P(S) = \sum_{t_3=1}^{r-2} \sum_{t_2=1}^{r-1-t_3} P(M_r(1) = t_1, M_r(2) = t_2, M_r(3) = t_3).$$

As an example, let us take $n_1 = n_2 = n_3 = 6$, and $r = 10$. Then, we can see that the sets $\{M_{10}(1) = 8, M_{10}(2) = 1, M_{10}(3) = 1\}$, $\{M_{10}(1) = 1, M_{10}(2) = 8, M_{10}(3) = 1\}$, $\{M_{10}(1) = 1, M_{10}(2) = 1, M_{10}(3) = 8\}$, $\{M_{10}(1) = 7, M_{10}(2) = 1, M_{10}(3) = 2\}$, etc. are all empty, and so the probability of these sets are 0. In this case, $P(S)$ has *the following form:*

$$\begin{aligned}
P(S) &= \sum_{t_2=3}^{6} P(M_{10}(1) = 9 - t_2, M_{10}(2) = t_2, M_{10}(3) = 1) \\
&\quad + \sum_{t_2=2}^{6} P(M_{10}(1) = 8 - t_2, M_{10}(2) = t_2, M_{10}(3) = 2) \\
&\quad + \sum_{t_2=1}^{6} P(M_{10}(1) = 7 - t_2, M_{10}(2) = t_2, M_{10}(3) = 3) \\
&\quad + \sum_{t_2=1}^{5} P(M_{10}(1) = 6 - t_2, M_{10}(2) = t_2, M_{10}(3) = 4) \\
&\quad + \sum_{t_2=1}^{4} P(M_{10}(1) = 5 - t_2, M_{10}(2) = t_2, M_{10}(3) = 5) \\
&\quad + \sum_{t_2=1}^{3} P(M_{10}(1) = 4 - t_2, M_{10}(2) = t_2, M_{10}(3) = 6).
\end{aligned}$$

Theorem 2 *Conditional on $\prod_{h=1}^{k} M_r(h) \neq 0$, the moment generating function (mgf) of $\hat{\theta}_l$ (for $l = 1, 2, \cdots, k$) is given by*

$$M_{\hat{\theta}_l}(t) = \frac{1}{P(\prod_{h=1}^{k} M_r(h) \neq 0)} \sum_{\mathbf{t}\in\mathbf{T}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{t})} P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) \prod_{i=1}^{r} (1 - \beta_{li}^* t)^{-1},$$

where $\mathbf{T}$ is the set defined in Lemma 1, and

$$\beta_{li}^* = \frac{n_l - M_{i-1}(l)}{M_r(l) \sum_{h=1}^{k} \frac{n_h - M_{i-1}(h)}{\theta_h}}, \qquad 1 \le l \le k, \quad 1 \le i \le r. \tag{2.2.12}$$

Proof Conditioning on the values of $\mathbf{M}_r$ for $\prod_{h=1}^{k} M_r(h) \neq 0$ and then on $\tilde{\boldsymbol{\delta}}$, we obtain

$$
\begin{aligned}
& E\left[\mathrm{e}^{t\hat{\theta}_l} | \prod_{h=1}^{k} M_r(h) \neq 0\right] P\left(\prod_{h=1}^{k} M_r(h) \neq 0\right) \\
= & \sum_{\mathbf{t}\in\mathbf{T}} E\left[\mathrm{e}^{t\hat{\theta}_l} | \mathbf{M}_r = \mathbf{t}\right] P(\mathbf{M}_r = \mathbf{t}) \\
= & \sum_{\mathbf{t}\in\mathbf{T}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{t})} \int_{0\le w_1\le w_2\le\cdots\le w_r<\infty} \mathrm{e}^{t\hat{\theta}_l} f(\tilde{\boldsymbol{\delta}}, \mathbf{W}) dw_1 \cdots dw_r \\
= & \sum_{\mathbf{t}\in\mathbf{T}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \\
& \times \int_{0\le w_1\le w_2\le\cdots\le w_r<\infty} \exp\left\{ \frac{t}{M_r(l)} \left(\sum_{i=1}^{r} w_i \tilde{\delta}_i(l) + w_r(n_l - M_r(l)) \right) \right. \\
& \left. - \sum_{i=1}^{r-1} a_i w_i - b_r w_r \right\} dw_1 \cdots dw_r \\
= & \sum_{\mathbf{t}\in\mathbf{T}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \\
& \times \int_{0\le w_1\le\cdots\le w_r<\infty} \exp\left\{ - \sum_{i=1}^{r-1} \left(a_i - \frac{t\tilde{\delta}_i(l)}{M_r(l)} \right) w_i \right. \\
& \left. - \left(b_r - \frac{t(n_l - M_{r-1}(l))}{M_r(l)} \right) w_r \right\} dw_1 \cdots dw_r \\
= & \sum_{\mathbf{t}\in\mathbf{T}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \\
& \times \left\{ \prod_{i=1}^{r-1} \frac{1}{b_r + \sum_{j'=i}^{r-1} a_{j'} - \left(\frac{n_l - M_{r-1}(l)}{M_r(l)} - \sum_{j'=i}^{r-1} \frac{\tilde{\delta}_{j'}(l)}{M_r(l)} \right) t} \right\} \frac{1}{b_r - \frac{t(n_l - M_{r-1}(l))}{M_r(l)}}.
\end{aligned}
$$

Now, observing that

$$\begin{aligned}
& b_r + \sum_{j'=i}^{r-1} a_{j'} - \left(\frac{n_l - M_{r-1}(l)}{M_r(l)} - \sum_{j'=i}^{r-1} \frac{\tilde{\delta}_{j'}(l)}{M_r(l)} \right) t \\
= \; & \sum_{h=1}^{k} \frac{n_h - M_{i-1}(h)}{\theta_h} - \left(\frac{n_l - M_{i-1}(l)}{M_r(l)} \right) t \\
= \; & (1 - \beta_{li}^* t) \sum_{h=1}^{k} \frac{n_h - M_{i-1}(h)}{\theta_h}
\end{aligned}$$

and $\beta_{li}^* = \frac{n_l - M_{i-1}(l)}{M_r(l) \sum_{h=1}^{k} \frac{n_h - M_{i-1}(h)}{\theta_h}}$, we obtain

$$\begin{aligned}
& E\left[\mathrm{e}^{t\hat{\theta}_l} | \prod_{h=1}^{k} M_r(h) \neq 0 \right] P\left(\prod_{h=1}^{k} M_r(h) \neq 0 \right) \\
= \; & \sum_{\mathbf{t} \in \mathbf{T}} \sum_{\boldsymbol{\delta} \in Q^*(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \prod_{i=1}^{r} \frac{1}{\sum_{h=1}^{k} \frac{n_h - M_{i-1}(h)}{\theta_h} - \frac{t}{M_r(l)} (n_l - M_{i-1}(l))} \\
= \; & \sum_{\mathbf{t} \in \mathbf{T}} \sum_{\tilde{\boldsymbol{\delta}} \in Q^*(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \prod_{i=1}^{r} \frac{1}{\sum_{h=1}^{k} \frac{n_h - M_{i-1}(h)}{\theta_h}} \prod_{i=1}^{r} (1 - \beta_{li}^* t)^{-1} \\
= \; & \sum_{\mathbf{t} \in \mathbf{T}} \sum_{\tilde{\boldsymbol{\delta}} \in Q^*(\mathbf{t})} P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) \prod_{i=1}^{r} (1 - \beta_{li}^* t)^{-1},
\end{aligned}$$

as required.

Remark 3 *For each fixed l, we may have some β_{li}^*'s to be the same. In this case, we resort the β_{li}^*'s and denote them by $\{\beta_{li}^*\}_{i=1}^{r'}$, and assume that α_i terms of β_{li}^*'s are equal, with $\sum_{i=1}^{r'} \alpha_i = r$. Thus, $\prod_{i=1}^{r} (1 - \beta_{li}^* t)^{-1}$ can be rewritten as $\prod_{i=1}^{r'} (1 - \beta_{li}^* t)^{-\alpha_i}$.*

In this case, we can rewrite the conditional mgf in Theorem 2.2 as

$$E\left[e^{t\hat{\theta}_l} | \prod_{h=1}^{k} M_r(h) \neq 0\right]$$
$$= \frac{1}{P(\prod_{h=1}^{k} M_r(h) \neq 0)} \sum_{\mathbf{t}\in\mathbf{T}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{t})} P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) \prod_{i=1}^{r'} (1-\beta_{li}^* t)^{-\alpha_i}. \quad (2.2.13)$$

Theorem 3 *Conditional on $\prod_{h=1}^{k} M_r(h) \neq 0$, the pdf of $\hat{\theta}_l$ (for $l = 1, 2, \cdots, k$) is given by*

$$f_{\hat{\theta}_l}(x) = \frac{1}{P(\prod_{h=1}^{k} M_r(h) \neq 0)} \sum_{\mathbf{t}\in\mathbf{T}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{t})} P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) g_{Y_{l,\boldsymbol{\delta}}}(x), \quad (2.2.14)$$

where $P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}})$ is as given in (2.2.7), $Y_{l,\boldsymbol{\delta}} \stackrel{d}{=} \sum_{i=1}^{r'} Y_{\alpha_i,i}^*$ with $Y_{1,i}^*$ being independent random variables having gamma $G(\alpha_i, \beta_{li}^*)$ distributions with shape parameter α_i and scale parameter β_{li}^*, and $g_{Y_{l,\boldsymbol{\delta}}}(x)$ is the pdf of $Y_{l,\boldsymbol{\delta}}$.

Remark 4 *From (2.2.14), it is clear that the distribution of the MLE $\hat{\theta}_l$ (for $l = 1, 2, \cdots, k$) is a weighted sum of distributions of random variables $Y_{l,\boldsymbol{\delta}}$, where $Y_{l,\boldsymbol{\delta}}$ itself is a sum of independent and non-identical gamma random variables.*

Corollary 1 *From (2.2.13), we immediately obtain the expressions for the first two moments of $\hat{\theta}_l$ as follows:*

$$E(\hat{\theta}_l) = \frac{1}{P(\prod_{h=1}^{k} M_r(h) \neq 0)} \sum_{\mathbf{t}\in\mathbf{T}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{t})} P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) \sum_{i=1}^{r'} \alpha_i \beta_{li}^*$$

and

$$E(\hat{\theta}_l^2) = \frac{1}{P(\prod_{h=1}^k M_r(h) \neq 0)} \sum_{\mathbf{t}\in\mathbf{T}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{t})} P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) \left\{ \sum_{i=1}^{r'} \alpha_i \beta_{li}^{*2} + \left(\sum_{i=1}^{r'} \alpha_i \beta_{li}^* \right)^2 \right\}.$$

Then, $\mathrm{Var}(\hat{\theta}_l)$ and $\mathrm{MSE}(\hat{\theta}_l)$ can be readily obtained from these two expressions.

It is convenient to rewrite the conditional mgf of $\hat{\theta}_l$ (for $l = 1, 2, \cdots, k$) in Theorem 2.2 as

$$\begin{aligned} M_{\hat{\theta}_l}(t) &= \frac{1}{P(\prod_{h=1}^k M_r(h) \neq 0)} \sum_{\mathbf{t}\in\mathbf{T}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{t})} P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) \prod_{i=1}^{r'} (1 - \beta_{li}^* t)^{-\alpha_i} \\ &= \frac{1}{P(\prod_{h=1}^k M_r(h) \neq 0)} \sum_{\mathbf{t}\in\mathbf{T}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{t})} P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) \sum_{i=1}^{r'} \sum_{j=1}^{\alpha_i} A_{li}^{(j)} (1 - \beta_{li}^* t)^{-j}, \end{aligned}$$

where A_{li}'s are coefficients obtained by writing the product $\prod_{i=1}^{r'}(1 - \beta_{li}^* t)^{-\alpha_i}$ in the partial fraction form $\sum_{i=1}^{r'} \sum_{j=1}^{\alpha_i} A_{li}^{(j)} (1 - \beta_{li}^* t)^{-j}$ which can be determined by the use of Lemma 5 in Appendix. Since $(1 - \beta_{li}^* t)^{-j}$ is the mgf of a gamma distribution with shape parameter j and scale parameter β_{li}^*, we can obtain the tail probability of $\hat{\theta}_l$ (for $l = 1, 2, \cdots, k$) from the above expression as

$$\begin{aligned} P(\hat{\theta}_l > b) &= \frac{1}{P(\prod_{h=1}^k M_r(h) \neq 0)} \sum_{\mathbf{t}\in\mathbf{T}} \\ &\quad \times \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{t})} P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) \sum_{i=1}^{r'} \sum_{j=1}^{\alpha_i} A_{li}^{(j)} \mathrm{e}^{-\frac{b}{\beta_{li}^*}} \sum_{j'=0}^{j-1} \frac{(\frac{b}{\beta_{li}^*})^{j'}}{j'!}. \end{aligned} \tag{2.2.15}$$

We shall assume that $P(\hat{\theta}_l > b)$ is an increasing function of θ_l when all other θ_j's are fixed for $j \neq l$. This assumption guarantees the invertibility of the pivotal quantities, and it has been verified to be true in this case through extensive computations; see Figure 2.1, for example. It should be mentioned that this approach has been used by a number of authors for constructing exact confidence intervals in a variety of contexts; see, for example, Childs et al. (2003), Chandrasekar et al. (2004), and Balakrishnan et al. (2007). We then have a $100(1-\alpha)\%$ lower confidence bound for θ_l as θ_{lL}, where θ_{lL} is such that $P_{\theta_{lL}}(\hat{\theta}_l > \hat{\theta}_{l,obs}) = \alpha$ with $\hat{\theta}_{l,obs}$ being the observed value of θ_l. Also, a $100(1-\alpha)\%$ confidence interval for θ_l is $(\theta_{lL}, \theta_{lU})$, where θ_{lL} and θ_{lU} are determined by $P_{\theta_{lL}}(\hat{\theta}_l > \hat{\theta}_{l,obs}) = \frac{\alpha}{2}$ and $P_{\theta_{lU}}(\hat{\theta}_l > \hat{\theta}_{l,obs}) = 1 - \frac{\alpha}{2}$.

By performing the same steps as done in the case of conditional marginal mgf, we can derive the conditional joint mgf of $(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k)$ as follows.

Theorem 4 *Conditional on $\prod_{h=1}^{k} M_r(h) \neq 0$, the joint mgf of $(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k)$ is given by*

$$\begin{aligned} & M_{\hat{\theta}_1,\hat{\theta}_2,\cdots,\hat{\theta}_k \mid \prod_{h=1}^{k} M_r(h)\neq 0}(t_1, \cdots, t_k) \\ = \; & \frac{1}{P(\prod_{h=1}^{k} M_r(h) \neq 0)} \sum_{\mathbf{t}\in\mathbf{T}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{t})} P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) \prod_{i=1}^{r} (1 - \sum_{l=1}^{k} \beta_{li}^* t_l)^{-1}, \end{aligned} \quad (2.2.16)$$

where

$$\beta_{li}^* = \frac{n_l - M_{i-1}(l)}{M_r(l) \sum_{h=1}^{k} \frac{n_h - M_{i-1}(h)}{\theta_h}}.$$

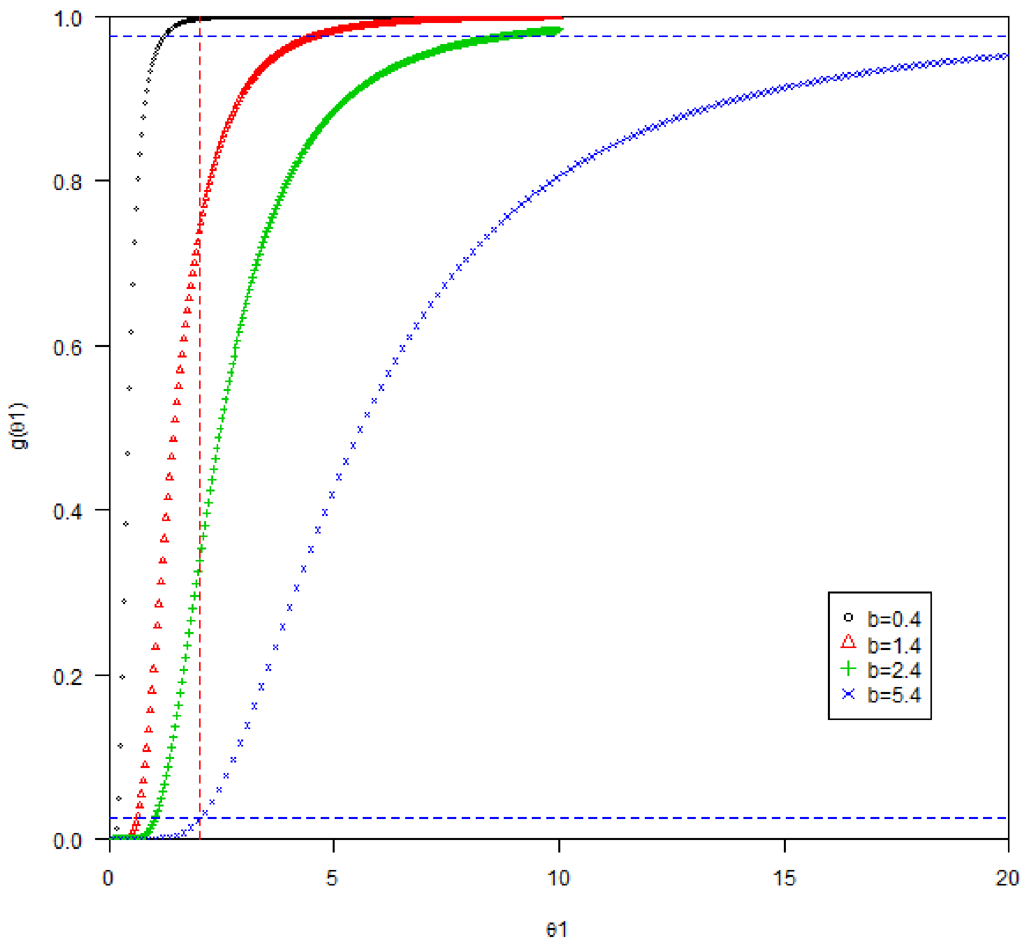

Figure 2.1: Plot of the function $g(\theta_1) = P_{\theta_1}(\hat{\theta}_l > b)$ for the choice of $b = 0.4, 1.4, 2.4, 5.4$, respectively.

Corollary 2 *From (2.2.16), we obtain the covariance of* $\hat{\theta}_{l_1}$ *and* $\hat{\theta}_{l_2}$ *to be*

$$\begin{aligned}
& Cov(\hat{\theta}_{l_1}, \hat{\theta}_{l_2}) \\
= \;& \frac{1}{P(\prod_{h=1}^{k} M_r(h) \neq 0)} \sum_{\mathbf{t}\in\mathbf{T}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{t})} P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) \left(\sum_{i=1}^{r} \beta^*_{l_1,i}\beta^*_{l_2,i} + \sum_{i=1}^{r} \beta^*_{l_1,i} \sum_{j=1}^{r} \beta^*_{l_2,j} \right) \\
& - \frac{1}{\left\{P(\prod_{h=1}^{k} M_r(h) \neq 0)\right\}^2} \sum_{\mathbf{t}\in\mathbf{T}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{t})} P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) \sum_{i=1}^{r} \beta^*_{l_1,i} \\
& \times \sum_{\mathbf{t}'\in\mathbf{T}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{t}')} P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) \sum_{i=1}^{r} \beta^*_{l_2,i}. \qquad (2.2.17)
\end{aligned}$$

2.3 Approximate Confidence Intervals

Let $I(\theta_1, \theta_2, \cdots, \theta_k) = (I_{i,j}(\theta_1, \theta_2, \cdots, \theta_k)), i, j = 1, 2, \cdots, k$, denote the Fisher information matrix of the parameters $\theta_1, \theta_2, \cdots, \theta_k$, where

$$I_{i,j}(\theta_1, \theta_2, \cdots, \theta_k) = -E\left(\frac{\partial^2 \ln L}{\partial\theta_i \partial\theta_j}\right). \qquad (2.3.1)$$

From the likelihood function in (2.2.3), we have $I_{i,j}(\theta_1, \theta_2, \cdots, \theta_k) = 0$ if $i \neq j$. Consequently, we have

$$\begin{aligned}
& I(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k) \\
= \;& -\mathrm{Diag}\left(\frac{\partial^2 \ln L}{\partial\theta_1^2}|_{\theta_1=\hat{\theta}_1}, \frac{\partial^2 \ln L}{\partial\theta_2^2}|_{\theta_2=\hat{\theta}_2}, \cdots, \frac{\partial^2 \ln L}{\partial\theta_k^2}|_{\theta_k=\hat{\theta}_k}\right), \qquad (2.3.2)
\end{aligned}$$

where

$$\begin{aligned}\frac{\partial^2 \ln L}{\partial \theta_h^2}\Big|_{\theta_h=\hat{\theta}_h} &= \frac{M_r(h)}{\hat{\theta}_h^2} - \frac{2\sum_{j=1}^{M_r(h)} x_{hj}}{\hat{\theta}_h^3} - \frac{2\left\{w_r(n_h - M_r(h))\right\}}{\hat{\theta}_h^3} \\ &= -\frac{M_r(h)}{\hat{\theta}_h^2}.\end{aligned}$$

Then, by using the asymptotic normality of the MLEs, we have $\hat{\theta}_h - \theta_h \sim N(0, I_{h,h}^{-1})$, using which we can express the approximate $100(1-\alpha)\%$ confidence interval for θ_h, $1 \le h \le k$, as

$$\hat{\theta}_h \pm Z_{\alpha/2}\frac{\sum_{i=1}^{M_r(h)} x_{hi} + w_r(n_h - M_r(h))}{(M_r(h))^{\frac{3}{2}}} = \hat{\theta}_h\left(1 \pm \frac{Z_{\alpha/2}}{\sqrt{M_r(h)}}\right),$$

where $Z_{\alpha/2}$ denotes the upper $\alpha/2$ percentage point of the standard normal distribution.

2.4 Bayesian Intervals

Let $u_h = \sum_{i=1}^{M_r(h)} x_{hi} + w_r(n_h - M_r(h))$. Then, we can rewrite the likelihood function as

$$L(\theta_1, \theta_2, \cdots, \theta_k, \boldsymbol{\delta}, \mathbf{W}) = c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \exp\left(-\frac{u_h}{\theta_h}\right). \tag{2.4.1}$$

Now, by assuming independent inverse gamma prior distributions, viz., $IG(a_h, b_h)$

for $1 \leq h \leq k$, we obtain from (2.4.1) the posterior joint density function as

$$L(\theta_1, \theta_2, \cdots, \theta_k | data) = c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)-a_h-1} \exp\left(-\frac{u_h + b_h}{\theta_h}\right).$$

Upon comparing this with (2.4.1), we see that the joint posterior density function of $(\theta_1, \theta_2, \cdots, \theta_k)$ is a product of k independent inverse gamma density functions. So, given the data, the posterior density function of $\hat{\theta}_h$ is simply $IG(M_r(h) + a_h, u_h + b_h)$. Thus, the Bayes estimator of θ_h under the squared-error loss function is

$$\hat{\theta}_{h,Bayes} = \frac{u_h + b_h}{M_r(h) + a_h - 1}, \qquad h = 1, 2, \cdots, k. \tag{2.4.2}$$

When we use Jeffreys' non-informative prior $I(\theta_h) \propto \frac{1}{\theta_h^2}$ corresponding to the special case when $a_h = 1$ and $b_h = 0$, for $1 \leq h \leq k$, the Bayes estimators in (2.4.2) coincide with the MLEs in (2.2.5).

Let $U_h = \frac{2(u_h+b_h)}{\theta_h}$ for $1 \leq h \leq k$. Then, the pivot U_h follows $\chi^2_{2(M_r(h)+a_h)}$ distribution, provided $2(M_r(h) + a_h)$ is a positive integer, for $1 \leq h \leq k$. In this case, the $100(1-\alpha)\%$ Bayes credible interval for θ_h becomes

$$\left(\frac{2(u_h + b_h)}{\chi^2_{2(M_r(h)+a_h),1-\alpha/2}}, \frac{2(u_h + b_h)}{\chi^2_{2(M_r(h)+a_h),\alpha/2}}\right), \qquad h = 1, 2, \cdots, k,$$

where $\chi^2_{v,\frac{\alpha}{2}}$ is the lower $\frac{\alpha}{2}$ percentage point of the chi-square distribution with v degrees of freedom.

2.5 Bootstrap Intervals

In this section, we consider confidence interval for θ_h $(h = 1, 2, \cdots, k)$ based on the Bootstrap-p and Bootstrap-t methods; see, for example, Efron and Tibshirani (1994).

To find the Bootstrap-p and Bootstrap-t intervals, in the first step, we generate original samples from k exponential populations with parameters θ_h of size n_h, $1 \leq h \leq k$. Next we sort the data, and determine to which population each failure belongs, and then estimate θ_h based on the conditional MLE in (2.2.5). In the second step, we generate a bootstrap sample $(\delta_1, \delta_2, \cdots, \delta_r; w_1, w_2, \cdots, w_r)$ by using the values $(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k)$, and then obtain the bootstrap estimates of θ_h, $1 \leq h \leq k$, say θ_h^*, $1 \leq h \leq k$, from the bootstrap sample. In the third step, we repeat the second step N-Boot times.

Boot-p: Suppose $K_{hB}(x) = P(\hat{\theta}_h^* \leq x)$, $1 \leq h \leq k$, is the cumulative distribution function of $\hat{\theta}_h^*$. Define $\hat{\theta}_{hB}(\alpha) = K_{hB}^{-1}(\alpha)$, $1 \leq h \leq k$, for a given α. Then, the $100(1-\alpha)\%$ Boot-p confidence interval for θ_h, $1 \leq h \leq k$, is given by

$$\left(\hat{\theta}_{hB}\left(\frac{\alpha}{2}\right), \hat{\theta}_{hB}\left(1 - \frac{\alpha}{2}\right)\right), \qquad 1 \leq h \leq k.$$

Boot-t: After generating the bootstrap samples in the second step and calculating $\hat{\theta}_h^*$, we need to use them to compute estimates of $Var(\hat{\theta}_h^*)$ from the observed Fisher information matrix in (2.3.2). Then, we determine the T_h^* statistic given by

$$T_h^* = \frac{\hat{\theta}_h^* - \hat{\theta}_h}{\sqrt{Var(\hat{\theta}_h^*)}}, \qquad 1 \leq h \leq k. \tag{2.5.1}$$

Now, suppose $L_{hB}(x) = P(T_h^* \leq x)$ is the cumulative distribution function of T_h^*, $1 \leq h \leq k$. Then, the $100(1-\alpha)\%$ Boot-t confidence interval for θ_h, $1 \leq h \leq k$, is given by

$$\left(\hat{\theta}_h - \sqrt{\mathrm{Var}(\hat{\theta}_h)}L_{hB}^{-1}\left(1-\frac{\alpha}{2}\right), \hat{\theta}_h - \sqrt{\mathrm{Var}(\hat{\theta}_h)}L_{hB}^{-1}\left(\frac{\alpha}{2}\right)\right), \qquad 1 \leq h \leq k.$$

2.6 Simulation Results and Discussion

A simulation study was carried out to evaluate the performance of the conditional MLEs and also all the confidence intervals discussed in the preceding sections. We considered different sample sizes for three populations (i.e., $k = 3$) as $n_1 = 18, 30, 35, 60, 75, 120$, $n_2 = 18, 35, 75$ and $n_3 = 18, 35, 75, 120$, and different choices of $r = 8, 12, 15, 30, 40, 60, 75, 80, 100, 120, 140, 180$. We also chose the parameters $(\theta_1, \theta_2, \theta_3)$ to be $(2, 4, 7)$ and $(3, 5, 9)$. For these cases, we computed the conditional MLEs for the parameters $(\theta_1, \theta_2, \theta_3)$. We also computed for $r = 8, 11$ the 95% confidence intervals for $(\theta_1, \theta_2, \theta_3)$ using exact, approximate, Boot-p and Boot-t methods (with N-Boot as 1000). For comparative purposes, we also computed the 95% credible intervals using Jeffreys' non-informative priors ($a_1 = a_2 = a_3 = 1$ and $b_1 = b_2 = b_3 = 0$).

We repeated this process 1000 times and computed the average values of the conditional MLEs as well as the coverage probabilities for all confidence intervals. The average values of the conditional MLEs are presented in Table 2.1. From these values, it is clear that the MLEs have a moderate bias when the essential sample size r is small even when the sample sizes (n_1, n_2, n_3) are not small. This bias also seems to affect the approximate confidence intervals based on normality as they are not centered properly in this case. However, the bias of the conditional MLEs become negligible when r increases, as is evident from Table 2.1.

In Table 2.2, the coverage probabilities of 95% confidence intervals of $(\theta_1, \theta_2, \theta_3)$ for all the methods are presented for some small and moderate values of n_1, n_2, n_3 and r. From these values, it is clear that the exact conditional method has its coverage probability to be very nearly 95% always, while the approximate method is not at all satisfactory (as low as 85% in some case). We also observe that between the two bootstrap methods, the Boot-p method performs better than the Boot-t method; the Bayesian method has very stable coverage probabilities (quite close to the nominal level of 95%). Moreover, we see that the approximate and bootstrap methods have lower coverage probabilities when n_1, n_2, n_3 are small. The importance of the exact method developed in the preceding sections becomes clear as it provides exact conditional confidence intervals with accurate coverage probabilities (compared to the nominal confidence levels) even for small sample sizes. However, the exact method becomes computationally quite intensive when r becomes large.

In Table 2.3 and 2.4, the coverage probabilities of 95% confidence intervals of $(\theta_1, \theta_2, \theta_3)$ for the approximate, Boot-p and Boot-t methods are presented for some large values of n_1, n_2, n_3 and r. Here again, we observe that the approximate method is not satisfactory unless r is rather large (at least 120). The two bootstrap methods and the Bayesian method all perform quite similar in terms of coverage probabilities whcih are nearly the same as the nominal confidence level.

From the results presented in these tables, we would recommend the use of the exact conditional confidence intervals for θ_h developed here whenever possible and especially when the sample sizes are small; but when the sample sizes get larger with a large r, the computational complexity increases in the exact conditional method, and in this case the Boot-p method and the Bayesian method are computationally simpler to use and they also possess good performance, for the interval estimation of parameters.

2.7 Illustrative Example

Nelson (1982, Ch. 10, Table 4.1) has given times to breakdown in minutes of an insulating fluid subjected to high voltage stress. The failure times were observed in the form of groups with each group reporting data on 10 insulating fluids. For the purpose of illustrating the methods of inference detailed in the preceding sections, let us consider the following three groups of samples of failure time data presented in Table 2.5.

Table 2.1: Average values of the conditional MLEs for different choices of (n_1, n_2, n_3) and r

(n_1, n_2, n_3)	r	$\theta_1 = 2$	$\theta_2 = 4$	$\theta_3 = 7$	$\theta_1 = 3$	$\theta_2 = 5$	$\theta_3 = 9$
$(18, 18, 18)$	8	2.49	5.03	6.69	3.75	6.49	9.05
$(18, 18, 18)$	11	2.08	4.50	7.64	3.06	5.83	10.42
$(18, 18, 18)$	15	2.15	4.66	8.67	3.26	5.71	11.28
$(30, 35, 35)$	15	2.20	4.88	8.87	3.36	5.85	11.28
$(30, 35, 35)$	30	2.09	4.32	8.20	3.09	5.32	10.21
$(30, 35, 35)$	40	2.06	4.22	7.60	3.04	5.29	9.64
$(30, 75, 75)$	30	2.15	4.24	7.91	3.23	5.18	10.03
$(30, 75, 75)$	60	2.03	4.09	7.36	3.13	5.08	9.40
$(30, 75, 75)$	80	2.06	4.04	7.13	3.08	4.98	9.19
$(60, 35, 35)$	30	2.05	4.64	8.89	3.10	5.72	11.14
$(60, 35, 35)$	60	2.00	4.16	7.70	3.02	5.23	9.92
$(60, 35, 35)$	80	2.01	4.11	7.41	3.00	5.05	9.28
$(60, 75, 75)$	40	2.02	4.22	7.73	3.08	5.22	10.19
$(60, 75, 75)$	75	2.02	4.10	7.28	3.07	5.11	9.40
$(60, 75, 75)$	120	2.01	4.05	7.07	3.03	5.03	9.19
$(120, 35, 35)$	40	2.04	5.05	9.16	3.02	6.00	11.78
$(120, 35, 35)$	100	2.00	4.23	7.65	3.02	5.21	9.93
$(120, 35, 35)$	140	2.00	4.13	7.28	3.02	5.09	9.43
$(35, 35, 120)$	40	2.07	4.41	7.11	3.16	5.45	9.30
$(35, 35, 120)$	100	2.02	4.09	7.03	3.04	5.15	9.09
$(35, 35, 120)$	140	2.00	4.04	7.03	3.02	5.02	9.07
$(120, 75, 75)$	60	2.02	4.28	7.81	3.03	5.21	9.97
$(120, 75, 75)$	120	2.01	4.10	7.34	3.01	5.08	9.36
$(120, 75, 75)$	180	2.01	4.01	7.08	3.01	5.01	9.16
$(75, 75, 120)$	60	2.02	4.19	7.33	3.07	5.18	9.42
$(75, 75, 120)$	120	2.01	4.10	7.10	3.02	5.08	9.15
$(75, 75, 120)$	180	2.02	4.03	7.03	3.01	5.01	9.07

Table 2.2: Coverage probabilities of different confidence intervals for some choices of (n_1, n_2, n_3) and small r

		$\theta = (2,4,7)$				
(n_1, n_2, n_3)	r	$Exact(\%)$	$Approximate(\%)$	$Bayes(\%)$	$Boot-\mathrm{t}(\%)$	$Boot-\mathrm{p}(\%)$
$(5,4,5)$	8	$(96,93,94)$	$(88,86,87)$	$(94,93,96)$	$(94,92,92)$	$(93,95,96)$
$(18,18,18)$	8	$(95,94,96)$	$(93,87,85)$	$(95,96,98)$	$(96,97,95)$	$(95,91,96)$
$(5,4,5)$	11	$(95,95,95)$	$(88,87,86)$	$(94,94,95)$	$(95,96,95)$	$(91,92,92)$
		$\theta = (3,5,9)$				
(n_1, n_2, n_3)	r	$Exact(\%)$	$Approximate(\%)$	$Bayes(\%)$	$Boot-\mathrm{t}(\%)$	$Boot-\mathrm{p}(\%)$
$(5,4,5)$	8	$(96,96,95)$	$(89,87,87)$	$(95,96,97)$	$(95,93,92)$	$(94,96,96)$
$(18,18,18)$	8	$(96,96,96)$	$(92,89,84)$	$(95,96,98)$	$(94,92,96)$	$(97,97,95)$
$(5,4,5)$	11	$(94,95,96)$	$(85,88,89)$	$(93,95,96)$	$(95,95,94)$	$(89,92,95)$

First of all, let us test the hypothesis $H_0 : \theta_1 = \theta_2 = \cdots = \theta_k$ versus $H_1 : \{$Not all θ's are equal$\}$. For using the likelihood ratio method (LRT), let us find

$$\lambda(\mathbf{W}) = \frac{\sup_{\Theta_0} L(\theta|\mathbf{W})}{\sup_{\Theta} L(\theta|\mathbf{W})}.$$

Asymptotically, $-2\log\lambda(\mathbf{W})$ is χ^2 distributed with $k-1$ degree of freedom, where

$$-2\log\lambda(\mathbf{W}) = 2\sum_{h=1}^{k} M_r(h)(\log\hat{\theta} - \log\hat{\theta}_h)$$

in this case, with

$$\hat{\theta} = \frac{1}{r}\left\{\sum_{i=1}^{r} w_i + w_r(n-r)\right\},$$

$$\hat{\theta}_h = \frac{1}{M_r(h)}\left\{\sum_{i=1}^{r} w_i\delta_i(h) + w_r(n_h - M_r(h))\right\}.$$

Table 2.3: Coverage probabilities of different confidence intervals for some choices of (n_1, n_2, n_3) and large r

$\theta = (2, 4, 7)$

(n_1, n_2, n_3)	r	*Approximate*(%)	*Bayes*(%)	*Boot*−t(%)	*Boot*−p(%)
(18, 18, 18)	15	(91, 92, 87)	(94, 94, 95)	(95, 100, 98)	(94, 96, 96)
(30, 35, 35)	15	(92, 91, 88)	(95, 95, 96)	(98, 100, 98)	(96, 96, 97)
(30, 35, 35)	30	(93, 93, 92)	(94, 95, 94)	(94, 97, 99)	(94, 95, 94)
(30, 35, 35)	40	(94, 91, 92)	(95, 95, 94)	(95, 96, 99)	(94, 94, 94)
(30, 75, 75)	30	(93, 93, 91)	(95, 96, 93)	(99, 95, 100)	(95, 96, 93)
(30, 75, 75)	60	(93, 94, 92)	(95, 96, 95)	(96, 100, 95)	(95, 95, 94)
(30, 75, 75)	80	(93, 94, 95)	(94, 94, 95)	(94, 100, 95)	(94, 95, 95)
(60, 35, 35)	30	(93, 90, 90)	(95, 93, 94)	(88, 100, 100)	(95, 93, 95)
(60, 35, 35)	60	(94, 92, 94)	(94, 95, 94)	(92, 99, 99)	(94, 95, 95)
(60, 35, 35)	80	(94, 96, 94)	(95, 94, 95)	(92, 100, 98)	(94, 95, 95)
(60, 75, 75)	40	(94, 93, 91)	(95, 96, 94)	(94, 96, 98)	(95, 95, 94)
(60, 75, 75)	75	(94, 94, 94)	(94, 95, 96)	(93, 100, 96)	(94, 95, 95)
(60, 75, 75)	120	(94, 94, 95)	(95, 94, 95)	(92, 100, 94)	(94, 94, 96)
(120, 35, 35)	40	(94, 91, 89)	(95, 93, 97)	(89, 100, 99)	(95, 93, 98)
(120, 35, 35)	100	(95, 94, 93)	(94, 94, 94)	(91, 100, 99)	(95, 93, 94)
(120, 35, 35)	140	(96, 95, 94)	(96, 96, 94)	(92, 100, 97)	(96, 96, 94)
(35, 35, 120)	40	(94, 93, 92)	(94, 94, 95)	(95, 100, 93)	(94, 94, 93)
(35, 35, 120)	100	(93, 92, 96)	(94, 95, 95)	(95, 100, 93)	(93, 95, 96)
(35, 35, 120)	140	(94, 95, 94)	(95, 94, 96)	(92, 100, 91)	(94, 94, 95)
(120, 75, 75)	60	(94, 95, 92)	(95, 95, 94)	(91, 98, 100)	(96, 95, 94)
(120, 75, 75)	120	(95, 94, 93)	(96, 95, 94)	(92, 100, 96)	(95, 95, 94)
(120, 75, 75)	180	(94, 93, 93)	(95, 94, 94)	(92, 100, 92)	(94, 94, 94)
(75, 75, 120)	60	(93, 94, 94)	(95, 93, 95)	(93, 100, 96)	(94, 93, 95)
(75, 75, 120)	120	(95, 93, 94)	(95, 95, 95)	(92, 100, 93)	(96, 95, 95)
(75, 75, 120)	180	(94, 94, 94)	(95, 94, 95)	(92, 100, 92)	(95, 93, 95)

Table 2.4: Coverage probabilities of different confidence intervals for some choices of (n_1, n_2, n_3) and large r

$\theta = (3, 5, 9)$

(n_1, n_2, n_3)	r	*Approximate*(%)	*Bayes*(%)	*Boot*−t(%)	*Boot*−p(%)
(18, 18, 18)	15	(90, 91, 87)	(95, 94, 96)	(96, 100, 99)	(94, 96, 97)
(30, 35, 35)	15	(92, 92, 90)	(95, 94, 96)	(98, 100, 99)	(95, 94, 97)
(30, 35, 35)	30	(93, 92, 91)	(95, 96, 95)	(97, 96, 99)	(94, 95, 95)
(30, 35, 35)	40	(94, 93, 92)	(96, 95, 95)	(96, 97, 99)	(96, 95, 95)
(30, 75, 75)	30	(91, 95, 92)	(94, 96, 94)	(100, 96, 99)	(94, 96, 94)
(30, 75, 75)	60	(94, 92, 94)	(96, 94, 94)	(98, 100, 96)	(95, 93, 94)
(30, 75, 75)	80	(93, 94, 94)	(93, 95, 94)	(98, 100, 95)	(93, 94, 94)
(60, 35, 35)	30	(94, 92, 92)	(96, 95, 92)	(92, 100, 100)	(95, 94, 93)
(60, 35, 35)	60	(94, 92, 92)	(95, 95, 96)	(93, 99, 98)	(94, 94, 96)
(60, 35, 35)	80	(94, 94, 93)	(94, 96, 95)	(93, 100, 96)	(94, 95, 95)
(60, 75, 75)	40	(93, 93, 93)	(94, 96, 94)	(97, 98, 100)	(94, 95, 94)
(60, 75, 75)	75	(92, 94, 95)	(94, 95, 96)	(95, 100, 97)	(93, 95, 96)
(60, 75, 75)	120	(95, 93, 95)	(95, 94, 94)	(92, 100, 94)	(95, 94, 95)
(120, 35, 35)	40	(94, 94, 90)	(95, 94, 96)	(88, 100, 100)	(95, 94, 97)
(120, 35, 35)	100	(94, 93, 94)	(96, 94, 94)	(91, 100, 100)	(96, 94, 94)
(120, 35, 35)	140	(92, 93, 94)	(92, 94, 95)	(92, 100, 97)	(92, 94, 95)
(35, 35, 120)	40	(92, 93, 92)	(95, 94, 96)	(98, 100, 91)	(95, 94, 95)
(35, 35, 120)	100	(94, 94, 94)	(95, 95, 96)	(94, 100, 92)	(94, 95, 95)
(35, 35, 120)	140	(94, 93, 94)	(95, 94, 95)	(95, 100, 91)	(94, 93, 94)
(120, 75, 75)	60	(93, 92, 91)	(95, 95, 94)	(93, 99, 98)	(94, 95, 94)
(120, 75, 75)	120	(94, 94, 94)	(95, 95, 94)	(93, 100, 96)	(95, 95, 95)
(120, 75, 75)	180	(95, 94, 95)	(95, 94, 96)	(92, 100, 94)	(95, 94, 96)
(75, 75, 120)	60	(94, 94, 93)	(96, 96, 95)	(95, 100, 94)	(96, 95, 94)
(75, 75, 120)	120	(95, 94, 95)	(95, 95, 96)	(93, 100, 93)	(95, 95, 96)
(75, 75, 120)	180	(96, 94, 96)	(96, 96, 95)	(91, 100, 94)	(97, 95, 96)

Table 2.5: Failure time data as three groups of insulating fluids

Group 1	0.31	0.66	1.54	1.70	1.82	1.89	2.17	2.24	4.03	9.99
Group 2	0.00	0.18	0.55	0.66	0.71	1.30	1.63	2.17	2.75	10.60
Group 3	0.49	0.64	0.82	0.93	1.08	1.99	2.06	2.15	2.57	4.75

Table 2.6: Bootstrap p-values for likelihood ratio statistic for testing $H_0 : \theta_1 = \theta_2 = \theta_3$

r	13	15
$-2\log\lambda(\mathbf{W})$	3.46	3.29
Bootstrap $p - value$	0.068	0.044

Table 2.7: Jointly Type-II censored data observed from Table 2.5 with $r = 15$

$\mathbf{W}$	0.00	0.18	0.31	0.49	0.55	0.64	0.66	0.66
	0.71	0.82	0.93	1.08	1.30	1.54	1.63	
h_i for which $\delta(h_i) = 1$	2	2	1	3	2	3	1	2
	2	3	3	3	2	1	2	

However, the asymptotic approximation may not be suitable here for small r. We therefore use the values of $(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$ to generate bootstrap samples $(\delta_1, \delta_2, \cdots, \delta_r;$ $w_1^*, w_2^*, \cdots, w_r^*)$, then obtain the bootstrap estimates of $(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$, say, $(\hat{\theta}_1^*, \hat{\theta}_2^*, \hat{\theta}_3^*)$, repeat this step 1000 Boot times, and get the Bootstrap p-value to be the proportion of times $-2\log\lambda(\mathbf{W}^*) > -2\log\lambda(\mathbf{W})$. Table 2.6 gives these Bootstrap p-values for different choices of r. We can see from Table 2.6 that we would reject H_0 by bootstrap method for a nominal level of 10% when $r = 13, 15$.

Suppose the samples of sizes $n = (10, 10, 10)$ in Table 2.5 are from three exponential populations with means $(\theta_1, \theta_2, \theta_3)$, respectively. Suppose joint Type-II censoring with r as 12, 13 and 15 had been enforced on these data. For example, Table 2.7 presents the jointly Type-II censored data that would have been obtained from the data in Table 2.5 with $r = 15$.

We then computed the conditional MLEs of $(\theta_1, \theta_2, \theta_3)$ and the estimates of their standard deviations and mean square errors for the choices of $r = 12, 13, 15$ by using

Table 2.8: Conditional MLEs and the estimates of their standard deviations and mean square errors based on jointly Type-II censored data from Table 2.7

r	Mean	$\widehat{SD}$	$\widehat{MSE}$
	$(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$	$(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$	$(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$
12	$(4.805, 1.500, 1.872)$	$(3.7660, 0.8572, 1.2938)$	$(15.0964, 0.7629, 1.7577)$
13	$(5.685, 1.433, 2.092)$	$(4.4547, 0.7013, 1.3685)$	$(21.1555, 0.5062, 1.9611)$
15	$(4.640, 1.417, 2.422)$	$(4.2025, 0.6142, 1.4879)$	$(19.0674, 0.3837, 2.2964)$

Table 2.9: Estimates of the covariance matrix of the conditional MLEs based on jointly Type-II censored data from Table 2.7

r	Covariance matrix $(\rho(\theta_i, \theta_j))_{i,j}$
12	$\begin{pmatrix} 14.1827 & 0.0668 & 0.0431 \\ 0.0668 & 0.7348 & -0.0861 \\ 0.0431 & -0.0861 & 1.6738 \end{pmatrix}$
13	$\begin{pmatrix} 19.8440 & 0.0909 & 0.0579 \\ 0.0909 & 0.4918 & -0.0665 \\ 0.0579 & -0.0665 & 1.8728 \end{pmatrix}$
15	$\begin{pmatrix} 17.6612 & -0.0801 & -0.3429 \\ -0.0801 & 0.3773 & -0.0444 \\ -0.3429 & -0.0444 & 2.2139 \end{pmatrix}$

the expressions presented earlier in Section 2.2, and these are presented in Table 2.8. We have also computed the estimates of the covariance matrix of $(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$ from the expression in Corollary 2, and these are presented in Table 2.9. From the results in Tables 2.8 and 2.9, we find the estimates to be quite stable, and especially so for θ_2 since this population has smallest mean thus producing more failures in the joint censored data.

Table 2.10 presents the 95% confidence intervals for $(\theta_1, \theta_2, \theta_3)$ based on the exact, approximate, Bayes credible, Boot-p and Boot-t methods corresponding to the cases $r = 12$ and $r = 15$. From these results, we observe once again that the approximate

Table 2.10: 95% confidence intervals for $(\theta_1, \theta_2, \theta_3)$ corresponding to different methods based on jointly Type-II censored data from Table 2.7

	$r = 12$		
	CI for θ_1	CI for θ_2	CI for θ_3
Exact	$(1.55, 64.84)$	$(0.70, 4.20)$	$(0.83, 5.81)$
Approximate	$(0, 46.98)$	$(0, 7.31)$	$(0, 11.47)$
Boot-t	$(0, 40.31)$	$(0, 6.98)$	$(0, 11.08)$
Boot-p	$(0.57, 27.12)$	$(0.29, 13.19)$	$(0.33, 17.41)$
Bayes credible	$(0.67, 626.85)$	$(0.34, 16.44)$	$(0.39, 36.83)$
	$r = 15$		
	CI for θ_1	CI for θ_2	CI for θ_3
Exact	$(1.77, 22.57)$	$(0.86, 3.48)$	$(1.12, 7.04)$
Approximate	$(0, 48.83)$	$(0.07, 5.83)$	$(0, 13.08)$
Boot-t	$(0, 46.36)$	$(0, 5.25)$	$(0, 12.06)$
Boot-p	$(0.66, 33.20)$	$(0.29, 7.95)$	$(0.45, 22.50)$
Bayes credible	$(0.78, 771.24)$	$(0.34, 9.23)$	$(0.54, 30.60)$

and Boot-t confidence intervals are not satisfactory compared to the exact confidence intervals. Moreover, we note that the Bayesian credible intervals are quite wide compared to the exact confidence intervals. These demonstrate once again the importance of the exact results derived here for obtaining exact conditional confidence intervals in the case of small values of r.

Chapter 3

Inference Under Joint Progressive Type-II Censoring

3.1 Introduction

In life-testing experiments, the joint censoring scheme is of great importance while conducting comparative life-tests of products from different units within the same facility. Suppose that products are being produced by k different lines within the same facility, and that independent samples of sizes n_h, $1 \leq h \leq k$, are selected from these k lines and placed simultaneously on a life-testing experiment. For the sake of reducing cost and the experimental time, the experimenter may choose to terminate the experiment as soon as a certain number (say, r) of failures occur. In this situation, one may be interested in either point or interval estimation of the mean lifetimes of units produced by these k lines. Here, exact results based on maximum likelihood estimates are developed to facilitate this.

In the literature, this joint censoring scheme has been considered before. For example, a generalized Savage statistic has been discussed by Basu (1968). Johnson and Mehrotra (1972) studied locally most powerful rank test under joint censoring. The problem of testing the equality of two distributions, under the assumption of exponentiality, has been discussed by Bhattacharyya and Mehrotra (1981). All these developments under joint censoring scheme focused on nonparametric and parametric tests of hypotheses; see Bhattacharyya (1995, Chapter 7 of Balakrishnan and Basu (1995)). For inference based on the MLEs, Chen and Bhattacharyya (1988) derived the exact distribution of the maximum likelihood estimator of the mean of an exponential distribution and an exact lower confidence bound for the mean based on a hybrid censored sample. An alternative simple form, which is equivalent to the results of Chen and Bhattacharyya, has been given by Childs et al (2003). To study two or more competing products with regard to the duration of their service life, comparative lifetime experiments are quite useful. Balakrishnan and Rasouli (2008) discussed exact inference for two exponential populations when Type-II censoring is implemented on the two samples in a combined manner. Balakrishnan and Rasouli (2010) subsequently extended their work to the case of two exponential populations when joint progressive Type-II censoring is implemented on the two samples. In the present work, we generalize their work by considering the k-sample problem. Suppose the test units from k lines under study are placed on a life-test simultaneously, that the successive failure times and the corresponding types (lines from which the failed units come from) are recorded, and that the experiment is terminated as soon as a specified total number of failures (say, r) occurred.

Suppose $X_{11}, X_{12}, \cdots, X_{1n_1}$ are the lifetimes of n_1 specimens from line A_1, and they are assumed to be independent and identically distributed (iid) variables from a population with cumulative distribution function (cdf) $F_1(x)$ and probability density function (pdf) $f_1(x)$. Similarly, $X_{21}, X_{22}, \cdots, X_{2n_2}$ are the lifetimes of n_2 specimens from line A_2 and are assumed to be a sample from pdf $f_2(x)$ and cdf $F_2(x)$, and so on, with $X_{k1}, X_{k2}, \cdots, X_{kn_k}$ denoting the lifetimes of n_k specimens from line A_k and are assumed to be iid variables from pdf $f_k(x)$ and cdf $F_k(x)$.

Furthermore, let $N = \sum_{i=1}^{k} n_i$ denote the total sample size and r denote the total number of failures observed. Let $w_1 \leq w_2 \leq \cdots \leq w_N$ denote the order statistics of the N random variables $\{X_{ij}; 1 \leq i \leq k, 1 \leq j \leq n_i\}$.

Therefore, under the joint progressive Type-II censoring scheme for the k-samples, the observable data consist of $(\boldsymbol{\delta}, \mathbf{w})$, where $\mathbf{w} = (w_1, w_2, \cdots, w_r)$, $w_i \in \{X_{h_i 1}, X_{h_i 2}, \cdots, X_{h_i n_i}\}$ for $1 \leq h_1, h_2, \cdots, h_r \leq k$, with r being a pre-fixed integer. Finally, associated to $(h_1, h_2, \cdots, h_r)$, let us define $\boldsymbol{\delta} = (\delta_1(h), \delta_2(h), \cdots, \delta_r(h))$ as

$$\delta_i(h) = \begin{cases} 1, & \text{if } h = h_i \\ 0, & \text{otherwise.} \end{cases} \tag{3.1.1}$$

This progressive Type-II censoring is implemented as follows. At the time of the first failure $W_1 \in X_{h_1}$ for some $1 \leq h_1 \leq k$, R_1 units are randomly withdrawn from the remaining $N-1$ surviving units. Next, at the time of the second failure $W_2 \in X_{h_2}$ for some $1 \leq h_2 \leq k$, R_2 units are randomly withdrawn from the remaining $N - R_1 - 2$ surviving units and so on. Finally, at the time of the rth failure $W_r \in X_{h_r}$ for some $1 \leq h_r \leq k$, all remaining $R_r = N - r - \sum_{i=1}^{r-1} R_i$ surviving units are withdrawn from the life-testing experiment; see Balakrishnan and Aggarwala (2000) and the discussion paper by Balakrishnan (2007) for elaborate details on progressive Type-II censoring and associated inferential results. Here, the joint progressive Type-II censoring scheme $\mathbf{R} = (R_1, R_2, \cdots, R_r)$ is prefixed and has the decomposition $R_i = \sum_{h=1}^{k} s_i(h)$, $1 \leq i \leq r$, where $s_i(h)$ is the number of units withdrawn at the time of ith failure belonging to the $\mathbf{X}_h$-sample, and these are unknown and are random variables. The data observed in this form will consist of $(\boldsymbol{\delta}, \mathbf{R}, \mathbf{w})$.

Let $M_r(h) = \sum_{i=1}^{r} \delta_i(h)$ denote the number of $\mathbf{X}_h-$failures in $\mathbf{w}$, $1 \leq h \leq k$, and $r = \sum_{h=1}^{k} M_r(h)$. As we can see from the form of the progressive Type-II censoring, $\sum_{h=1}^{k} s_i(h) = R_i$, $\sum_{i=1}^{r} s_i(h) + M_r(h) = n_h$ and, of course, $\sum_{i=1}^{r} R_i = N - r$.

3.2 MLEs, Exact Distributions and Inference

Let $\mathbf{s} = \{\mathbf{s}(h)\}_{h=1}^{k} = \{(s_1(h), s_2(h), \cdots, s_r(h))\}_{h=1}^{k}$. The likelihood of $(\boldsymbol{\delta}, \mathbf{s}, \mathbf{w})$ is then given by

$$L(\theta_1, \theta_2, \cdots, \theta_k, \boldsymbol{\delta}, \mathbf{s}, \mathbf{w}) = c_r \prod_{i=1}^{r} \prod_{h=1}^{k} (f_h(w_i))^{\delta_i(h)} \prod_{i=1}^{r} \prod_{h=1}^{k} (S_h(w_i))^{s_i(h)}, \tag{3.2.1}$$

where $S_h(w_r) = 1 - F_h(w_r)$ and $c_r = D_1 D_2$, with

$$D_1 = \prod_{i=1}^{r} \left[\sum_{h=1}^{k} \left(n_h - M_{i-1}(h) - \sum_{j=1}^{i-1} s_j \right) \delta_i(h) \right],$$

$$D_2 = \prod_{i=1}^{r-1} \left\{ \frac{\prod_{h=1}^{k} \begin{pmatrix} n_h - M_i(h) - \sum_{j=1}^{i-1} s_j(h) \\ s_j(h) \end{pmatrix}}{\begin{pmatrix} N - i - \sum_{j=1}^{i-1} R_j \\ R_i \end{pmatrix}} \right\}.$$

The special case of joint Type-II censoring scheme is obtained when we set $R_1 = R_2 = \cdots = R_{r-1} = 0$ and $R_r = N - r$ in which case we will have $s_i(h) = 0$ when $1 \le i \le r-1$ and $s_r(h) = n_h - M_r(h)$ for all $1 \le h \le k$.

When the k populations are exponential with cdf $F_h(x) = 1 - \exp\left(-\frac{x}{\theta_h}\right)$, $x > 0$, and pdf $f_h(x) = \frac{1}{\theta_h} \exp\left(-\frac{x}{\theta_h}\right)$, $x > 0$, for $1 \le h \le k$, the likelihood function in (3.2.1) becomes

$$\begin{aligned} & L(\theta_1, \theta_2, \cdots, \theta_k, \boldsymbol{\delta}, \mathbf{s}, \mathbf{w}) \\ = \;& c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \prod_{i=1}^{r} \prod_{h=1}^{k} \left\{ \exp\left(-\frac{w_i}{\theta_h}\right) \right\}^{\delta_i(h)} \prod_{i=1}^{r} \prod_{h=1}^{k} \left\{ \exp\left(-\frac{w_i}{\theta_h}\right) \right\}^{s_i(h)} \\ = \;& c_r \prod_{h=1}^{k} \exp\left\{ -M_r(h) \log \theta_h - \frac{\sum_{i=1}^{r} (\delta_i(h) + s_i(h)) w_i}{\theta_h} \right\}. \end{aligned} \tag{3.2.2}$$

From (3.2.2), we readily obtain the MLE of θ_h, for $1 \leq h \leq k$, as

$$\hat{\theta}_h = \frac{1}{M_r(h)} \sum_{i=1}^{r} (\delta_i(h) + s_i(h))\, w_i. \tag{3.2.3}$$

Remark 5 *From the likelihood function in (3.2.2), we readily see that the MLE of θ_h does not exist when $M_r(h) = 0$. So, the MLEs in (3.2.3) are conditioned on $S = \{\sum_{h=1}^{k} M_r(h) = r$ and $M_r(h) \geq 1$ for $\forall 1 \leq h \leq k\}$, or equivalently on the set $S = \{\sum_{h=1}^{k} M_r(h) = r$ and $\prod_{h=1}^{k} M_r(h) \neq 0\}$. We, therefore, need to discuss the exact distribution and other properties of the MLEs only conditional on the set S.*

Lemma 2 *Let $\mathbf{M}_r = (M_r(1), M_r(2), \cdots, M_r(k))$ and $\mathbf{t} = (t_1, t_2, \cdots, t_k)$ with $t_h = n_h - \sum_{i=1}^{r} s_i(h)$, $1 \leq h \leq k$. Then,*

$$P\left(\prod_{h=1}^{k} M_r(h) \geq 1 | \mathbf{s} \right) = P\left(\mathbf{M}_r = \mathbf{t}\right). \tag{3.2.4}$$

Proof For fixed $\mathbf{s}$, observe that $\sum_{i=1}^{r} s_i(h) + Mr(h) = n_h$.

Remark 6 *For a scheme* $\mathbf{R}$*, let us denote*

$$\begin{aligned}\mathcal{S}_{\mathbf{R}} &= \left\{ \mathbf{s} = \{s_i(h)\}_{1\le h\le k, 1\le i\le r} : \quad \sum_{h=1}^{k} s_i(h) = R_i, 1 \le i \le r, \right. \\ &\left. \sum_{i=1}^{r} s_i(h) \le n_h - 1 \right\}.\end{aligned}$$

Then, we have

$$\begin{aligned}\mathcal{S}_{\mathbf{R}} = & \left\{ \mathbf{s} : 0 \le s_1(h) \le \min\left(R_1 - \sum_{j=1}^{h-1} s_1(j), n_h - 1 \right), 1 \le h \le k-1; \right. \\ & 0 \le s_2(h) \le \min\left(R_2 - \sum_{j=1}^{h-1} s_2(j), n_h - 1 - s_1(h) \right), 1 \le h \le k-1; \\ & \cdots; \\ & 0 \le s_{r-1}(h) \le \min\left(R_{r-1} - \sum_{j=1}^{h-1} s_{r-1}(j), n_h - 1 - \sum_{i=1}^{r-2} s_i(h) \right), 1 \le h \le k-1; \\ & 0 \le s_r(h) \le \min\left(R_r - \sum_{j=1}^{h-1} s_r(j), n_h - 1 - \sum_{i=1}^{r-1} s_i(h) \right), 1 \le h \le k-1, \\ & \left. \textit{such that } \sum_{h=1}^{k} s_i(h) = R_i \textit{ and } \sum_{i=1}^{r} s_i(h) \le n_h - 1 \right\}.\end{aligned}$$

Theorem 5 **(a)** *The joint probability mass function of* $\boldsymbol{\delta}$ *is given by*

$$P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) = c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \prod_{i=1}^{r} \frac{1}{\sum_{h=1}^{k} \frac{n_h - M_{i-1}(h,\tilde{\delta}) - \sum_{j=1}^{i-1} s_j(h)}{\theta_h}}; \tag{3.2.5}$$

(b) *For each* $\mathbf{s} \in \mathcal{S}_{\mathbf{R}}$*, denote* $\mathbf{t} = (t_1, t_2, \cdots, t_k) = (n_1 - \sum_{i=1}^{r} s_i(1), n_2 - \sum_{i=1}^{r} s_i(2), \cdots, n_k - \sum_{i=1}^{r} s_i(k))$*, with* $\sum_{j=1}^{k} t_j = r$ *and* $t_j \geq 1$ *for all* j*. Let* $\boldsymbol{\delta}$ *be as defined in (3.1.1),* $M_r(h) = \sum_{i=1}^{r} \delta_i(h)$*, and*

$$Q^*(\mathbf{s}) = \{\tilde{\boldsymbol{\delta}}(\mathbf{h}) = (\tilde{\delta}_1, \tilde{\delta}_2, \cdots, \tilde{\delta}_r) \text{ such that } M_r(h) = t_h \text{ for } 1 \leq h \leq k\}$$

Then,

$$P(\mathbf{M}_r = \mathbf{t}) = \sum_{\tilde{\boldsymbol{\delta}} \in Q^*(\mathbf{s})} c_r \prod_{h=1}^{k} \theta_h^{-t_h} \prod_{i=1}^{r} \frac{1}{\sum_{h=1}^{k} \frac{n_h - M_{i-1}(h, \tilde{\delta}) - \sum_{j=1}^{i-1} s_j(h)}{\theta_h}}; \qquad (3.2.6)$$

(c)

$$P\left(\prod_{h=1}^{k} M_r(h) \neq 0\right) = \sum_{\mathbf{s} \in \mathcal{S}_{\mathbf{R}}} P(\mathbf{M}_r = \mathbf{t}). \qquad (3.2.7)$$

Proof (a) From (3.2.2), we have the joint density of $(\boldsymbol{\delta}, \mathbf{s}, \mathbf{w})$ as

$$f(\boldsymbol{\delta}, \mathbf{s}, \mathbf{w}) = c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \exp\left(-\sum_{i=1}^{r} \sum_{h=1}^{k} \frac{w_i(\delta_i(h) + s_i(h))}{\theta_h}\right).$$

Let

$$a_j = \sum_{h=1}^{k} \frac{\delta_j(h) + s_j(h)}{\theta_h}.$$

Then, we have

$$f(\boldsymbol{\delta}, \mathbf{s}, \mathbf{w}) = c_r \left(\prod_{h=1}^{k} \theta_h^{-M_r(h)} \right) \exp\left(- \sum_{i=1}^{r} a_i w_i \right). \tag{3.2.8}$$

Upon integrating out $w_1, w_2, \cdots, w_r$ over $\{0 \le w_1 \le w_2 \le \cdots \le w_r < \infty\}$ in (3.2.8), after mapping it onto $\{0 \le u_i < \infty, 1 \le i \le k\}$ though the transformation

$$\begin{cases} u_1 & = w_1, \\ u_2 & = w_2 - w_1, \\ & \cdots \\ u_r & = w_r - w_{r-1}, \end{cases}$$

we obtain the joint probability mass function of $\boldsymbol{\delta}$ as follows:

$$P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) = c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \prod_{i=1}^{r} \frac{1}{\sum_{j=i}^{r} a_j}. \tag{3.2.9}$$

Observe that for $\tilde{\boldsymbol{\delta}}$, $\sum_{i=1}^{r} s_i(h) + M_r(h, \tilde{\boldsymbol{\delta}}) = n_h$, where $M_r(h, \tilde{\boldsymbol{\delta}}) = \sum_{i=1}^{r} \tilde{\delta}_i(h)$, and so

$$\begin{aligned}\sum_{j=i}^{r} a_j &= \sum_{j=i}^{r}\sum_{h=1}^{k}\frac{\delta_j(h)+s_j(h)}{\theta_h}\\ &= \sum_{h=1}^{k}\frac{M_r(h,\tilde{\boldsymbol{\delta}})-M_{i-1}(h,\tilde{\boldsymbol{\delta}})+\sum_{j=i}^{r}s_j(h)}{\theta_h}\\ &= \sum_{h=1}^{k}\frac{n_h-M_{i-1}(h,\tilde{\boldsymbol{\delta}})-\sum_{j=1}^{i-1}s_j(h)}{\theta_h},\end{aligned}$$

where $M_{i-1}(h,\tilde{\boldsymbol{\delta}})=\sum_{j=1}^{i-1}\tilde{\delta}_j(h)$ depends on $\tilde{\boldsymbol{\delta}}$, and yet we denote it by $M_{i-1}(h)$ for simplicity, and set $M_0(h)=0$. Thus, (3.2.5) is obtained.

(b) It readily follows from (3.2.5).

(c) The required result follows immediately from Remark 6 and Part (b).

Theorem 6 *Conditional on $\prod_{h=1}^{k} M_r(h)\neq 0$, the moment generating function (mgf) of $\hat{\theta}_l$ (for $l=1,2,\cdots,k$) is given by*

$$M_{\hat{\theta}_l}(t) = \frac{1}{P(\prod_{h=1}^{k}M_r(h)\neq 0)}\sum_{\mathbf{s}\in\mathcal{S}_{\mathbf{R}}}\sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{s})}P(\boldsymbol{\delta}=\tilde{\boldsymbol{\delta}})\prod_{i=1}^{r}(1-\beta_{li}^{*}t)^{-1},$$

where $\mathbf{R}$ is a scheme and $\mathcal{S}_{\mathbf{R}}$ is the sets defined in Remark 6, and

$$\beta_{li}^{*}=\frac{n_l-M_{i-1}(l)-\sum_{j=1}^{i-1}s_j(l)}{M_r(l)\sum_{h=1}^{k}\frac{n_h-M_{i-1}(h)-\sum_{j=1}^{i-1}s_j(h)}{\theta_h}},\quad 1\leq l\leq k, 1\leq i\leq r. \tag{3.2.10}$$

Proof Conditioning on the values of $\mathbf{M}_r$ for $\prod_{h=1}^{k} M_r(h) \neq 0$ and then on $\tilde{\boldsymbol{\delta}}$, we obtain

$$
\begin{aligned}
& E\left[\mathrm{e}^{t\hat{\theta}_l} | \prod_{h=1}^{k} M_r(h) \neq 0\right] P\left(\prod_{h=1}^{k} M_r(h) \neq 0\right) \\
= & \sum_{\mathbf{s}\in\mathcal{S}_{\mathbf{R}}} E(\mathrm{e}^{t\hat{\theta}_l} | \mathbf{M}_r = \mathbf{t}) P(\mathbf{M}_r = \mathbf{t}) \\
= & \sum_{\mathbf{s}\in\mathcal{S}_{\mathbf{R}}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{s})} \int_{0\le w_1\le w_2\le\cdots\le w_r<\infty} \mathrm{e}^{t\hat{\theta}_l} f(\tilde{\boldsymbol{\delta}}, \mathbf{w}) dw_1 \cdots dw_r \\
= & \sum_{\mathbf{s}\in\mathcal{S}_{\mathbf{R}}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{s})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \\
& \times \int_{0\le w_1\le w_2\le\cdots\le w_r<\infty} \mathrm{e}^{\frac{t}{M_r(l)}\left\{\sum_{i=1}^{r} w_i(\delta_i(l)+s_i(l))\right\}} \exp\left(-\sum_{i=1}^{r} a_i w_i\right) dw_1 \cdots dw_r
\end{aligned}
$$

$$
\begin{aligned}
= & \sum_{\mathbf{s}\in\mathcal{S}_{\mathbf{R}}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{s})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \\
& \times \int_{0\le w_1\le\cdots\le w_r<\infty} \exp\left\{-\sum_{i=1}^{r}\left(a_i - \frac{t(\tilde{\delta}_i(l) + s_i(l))}{M_r(l)}\right) w_i\right\} dw_1 \cdots dw_r \\
= & \sum_{\mathbf{s}\in\mathcal{S}_{\mathbf{R}}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{s})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \left\{\prod_{i=1}^{r} \frac{1}{\sum_{j=i}^{r}\left(a_j - \frac{t(\tilde{\delta}_j(l)+s_j(l))}{M_r(l)}\right)}\right\}.
\end{aligned}
$$

Now, we observe that

$$\begin{aligned}
& \sum_{j=i}^{r}\left(a_j - \frac{t(\tilde{\delta}_j(l)+s_j(l))}{M_r(l)}\right) \\
= \; & \sum_{j=i}^{r} a_j - \frac{t}{M_r(l)}\sum_{j=i}^{r}(\tilde{\delta}_j(l)+s_j(l)) \\
= \; & \sum_{h=1}^{k}\frac{n_h - M_{i-1}(h) - \sum_{j=1}^{i-1}s_j(h)}{\theta_h} - \frac{n_l - M_{i-1}(l) - \sum_{j=1}^{i-1}s_j(l)}{M_r(l)}t \\
= \; & (1-\beta_{li}^{*}t)\sum_{h=1}^{k}\frac{n_h - M_{i-1}(h) - \sum_{j=1}^{i-1}s_j(h)}{\theta_h},
\end{aligned}$$

where β_{li}^{*} is as given in Eq. (3.2.10). Thus, we obtain

$$\begin{aligned}
& E\left[\mathrm{e}^{t\hat{\theta}_l} | \prod_{h=1}^{k} M_r(h) \neq 0\right] P\left(\prod_{h=1}^{k} M_r(h) \neq 0\right) \\
= \; & \sum_{\mathbf{s}\in\mathcal{S}_{\mathbf{R}}}\sum_{\tilde{\boldsymbol{\delta}}\in Q^{*}(\mathbf{s})} c_r \prod_{h=1}^{k}\theta_h^{-M_r(h)}\prod_{i=1}^{r}\frac{1}{\sum_{h=1}^{k}\frac{n_h - M_{i-1}(h) - \sum_{j=1}^{i-1}s_j(h)}{\theta_h}}\prod_{i=1}^{r}(1-\beta_{li}^{*}t)^{-1} \\
= \; & \sum_{\mathbf{s}\in\mathcal{S}_{\mathbf{R}}}\sum_{\tilde{\boldsymbol{\delta}}\in Q^{*}(\mathbf{s})} P(\boldsymbol{\delta}=\tilde{\boldsymbol{\delta}})\prod_{i=1}^{r}(1-\beta_{li}^{*}t)^{-1},
\end{aligned}$$

as required.

Remark 7 *For fixed l and $\tilde{\boldsymbol{\delta}}$, some β_{li}^{*}'s may be the same. In this case, we resort the β_{li}^{*} values and still denote them by $\{\beta_{li}^{*}\}_{i=1}^{r'}$, and assume that there are r' distinct values with α_i of the β_{li}^{*}'s being equal, with $\sum_{i=1}^{r'}\alpha_i = r$. Consequently, the term $\prod_{i=1}^{r}(1-\beta_{li}^{*}t)^{-1}$ can be rewritten as $\prod_{i=1}^{r'}(1-\beta_{li}^{*}t)^{-\alpha_i}$, and so the conditional mgf in*

Theorem 6 can be expressed as follows:

$$\begin{aligned}
& E\left[e^{t\hat{\theta}_l} | \prod_{h=1}^{k} M_r(h) \neq 0\right] \\
= \; & \frac{1}{P\left(\prod_{h=1}^{k} M_r(h) \neq 0\right)} \sum_{\mathbf{s}\in\mathcal{S}_{\mathbf{R}}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{s})} P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) \\
& \times \prod_{i=1}^{r'} (1 - \beta_{li}^* t)^{-\alpha_i}. \qquad (3.2.11)
\end{aligned}$$

Theorem 7 *Conditional on* $\prod_{h=1}^{k} M_r(h) \neq 0$*, the pdf of* $\hat{\theta}_l$ *is given by*

$$\begin{aligned}
& f_{\hat{\theta}_l | \prod_{h=1}^{k} M_r(h)\neq 0}(x) \\
= \; & \frac{1}{P\left(\prod_{h=1}^{k} M_r(h) \neq 0\right)} \sum_{\mathbf{s}\in\mathcal{S}_{\mathbf{R}}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{s})} P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) g_{Y_{l,\delta}}(x), \qquad (3.2.12)
\end{aligned}$$

where $P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}})$ *is as given in (3.2.5),* $Y_{l,\boldsymbol{\delta}} \stackrel{d}{=} \sum_{i=1}^{r'} Y_{l,i}^*$ *with* $Y_{l,i}^*$ *being independent random variables having gamma* $G(\alpha_i, \beta_{li}^*)$ *distributions with shape parameters* α_i *and scale parameters* β_{li}^**, and* $g_{Y_{l,i}}(x)$ *is the pdf of* $Y_{l,i}$*.*

Remark 8 *From (3.2.12), it is clear that the distribution of the MLE* $\hat{\theta}_l$ *is a weighted sum of distributions of random variables of the type* $Y_{l,\delta}$*, where* $Y_{l,\delta}$ *itself is a sum of independent and non-identical gamma random variables.*

Corollary 3 *From (3.2.12), we immediately obtain the expressions for the first two* *conditional moments of $\hat{\theta}_l$ as follows:*

$$
\begin{aligned}
& E\left[\hat{\theta}_l | \prod_{h=1}^{k} M_r(h) \neq 0\right] \\
= & \frac{1}{P\left(\prod_{h=1}^{k} M_r(h) \neq 0\right)} \sum_{\mathbf{s}\in\mathcal{S}_{\mathbf{R}}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{s})} P(\boldsymbol{\delta}=\tilde{\boldsymbol{\delta}}) \sum_{i=1}^{r'} \alpha_i \beta_{li}^*
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[\hat{\theta}_l^2 | \prod_{h=1}^{k} M_r(h) \neq 0\right] = & \frac{1}{P\left(\prod_{h=1}^{k} M_r(h) \neq 0\right)} \sum_{\mathbf{s}\in\mathcal{S}_{\mathbf{R}}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{s})} P(\boldsymbol{\delta}=\tilde{\boldsymbol{\delta}}) \\
& \times \left\{ \sum_{i=1}^{r'} \alpha_i \beta_{li}^{*2} + \left(\sum_{i=1}^{r'} \alpha_i \beta_{li}^* \right)^2 \right\}.
\end{aligned}
$$

Then, $\mathrm{Var}(\hat{\theta}_l)$ and $\mathrm{MSE}(\hat{\theta}_l)$ can be readily obtained from these two expressions.

It is convenient to rewrite the conditional mgf of $\hat{\theta}_l$ (for $l = 1, 2, \cdots, k$) in Theorem 6 as

$$
\begin{aligned}
& M_{\hat{\theta}_l | \prod_{h=1}^{k} M_r(h) \neq 0}(t) \\
= & \frac{1}{P\left(\prod_{h=1}^{k} M_r(h) \neq 0\right)} \sum_{\mathbf{s}\in\mathcal{S}_{\mathbf{R}}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{s})} P(\boldsymbol{\delta}=\tilde{\boldsymbol{\delta}}) \prod_{i=1}^{r'} (1-\beta_{li}^* t)^{-\alpha_i} \\
= & \frac{1}{P\left(\prod_{h=1}^{k} M_r(h) \neq 0\right)} \sum_{\mathbf{s}\in\mathcal{S}_{\mathbf{R}}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{s})} P(\boldsymbol{\delta}=\tilde{\boldsymbol{\delta}}) \sum_{i=1}^{r'} \sum_{j=1}^{\alpha_i} A_{li}^{(j)} (1-\beta_{li}^* t)^{-j},
\end{aligned}
$$

where A_{li}'s are coefficients obtained by writing the product $\prod_{i=1}^{r}(1 - \beta_{li}^* t)^{-1} =$

$\prod_{i=1}^{r'}(1-\beta_{li}^{*}t)^{-\alpha_i}$ in the partial fraction form $\sum_{i=1}^{r'}\sum_{j=1}^{\alpha_i} A_{li}^{(j)}(1-\beta_{li}^{*}t)^{-j}$, which can be determined by the use of Lemma 5 in Appendix. Since $(1-\beta_{li}^{*}t)^{-j}$ is the mgf of a gamma distribution with scale parameter β_{li}^{*} and shape parameter j, we can obtain the conditional tail probability of $\hat{\theta}_l$ (for $l = 1, 2, \cdots, k$) from the above expression as

$$\begin{aligned} P\left(\hat{\theta}_l > b \,\middle|\, \prod_{h=1}^{k} M_r(h) \neq 0\right) &= \frac{1}{P\left(\prod_{h=1}^{k} M_r(h) \neq 0\right)} \sum_{\mathbf{s}\in\mathcal{S}_{\mathbf{R}}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{s})} P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) \\ &\quad \times \sum_{i=1}^{r'} \sum_{j=1}^{\alpha_i} A_{li}^{(j)} \mathrm{e}^{-\frac{b}{\beta_{li}^{*}}} \sum_{j'=0}^{j-1} \frac{\left(\frac{b}{\beta_{li}^{*}}\right)^{j'}}{j'!}. \end{aligned} \tag{3.2.13}$$

We shall now assume that the conditional tail probability of $\hat{\theta}_l$ in (3.2.13) is an increasing function of θ_l when all other θ_j's are fixed, for $j \neq l$. This assumption guarantees the invertibility of the pivotal quantities, and it has been verified to be true in this case through extensive computations under various of settings. It should be mentioned that this approach has been used by a number of authors for the construction of exact confidence intervals in a variety of contexts; see, for example, Childs et al. (2003) and Balakrishnan et al. (2007). We then have a $100(1-\alpha)\%$ lower confidence bound for θ_l as θ_{lL}, where θ_{lL} is such that $P_{\theta_{lL}}(\hat{\theta}_l > \hat{\theta}_{l,obs}) = \alpha$ with $\hat{\theta}_{l,obs}$ being the observed value of $\hat{\theta}_l$. Also, a $100(1-\alpha)\%$ confidence interval for θ_l is $(\theta_{lL}, \theta_{lU})$, where θ_{lL} and θ_{lU} are determined by $P_{\theta_{lL}}(\hat{\theta}_l > \hat{\theta}_{l,obs}) = \frac{\alpha}{2}$ and $P_{\theta_{lU}}(\hat{\theta}_l > \hat{\theta}_{l,obs}) = 1 - \frac{\alpha}{2}$.

By performing the same steps as done in the case of conditional marginal mgf, we can also derive the conditional joint mgf of $(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k)$ as follows.

Theorem 8 *Conditional on $\prod_{h=1}^{k} M_r(h) \neq 0$, the conditional joint mgf of $(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k)$ is given by*

$$
\begin{aligned}
& M_{\hat{\theta}_1,\hat{\theta}_2,\cdots,\hat{\theta}_k | \prod_{h=1}^{k} M_r(h)\neq 0}(t_1, \cdots, t_k) \\
= & \frac{1}{P\left(\prod_{h=1}^{k} M_r(h) \neq 0\right)} \sum_{\mathbf{s}\in\mathcal{S}_{\mathbf{R}}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{s})} P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) \\
& \times \prod_{i=1}^{r} \left(1 - \sum_{l=1}^{k} \beta_{li}^* t_l\right)^{-1}, \qquad (3.2.14)
\end{aligned}
$$

where β_{li}^* is as defined earlier in Eq. (3.2.10).

Corollary 4 *From (3.2.14), we find the covariance between $\hat{\theta}_{l_1}$ and $\hat{\theta}_{l_2}$ (for $l_1 \neq l_2$) as follows:*

$$
\begin{aligned}
Cov\left(\hat{\theta}_{l_1}, \hat{\theta}_{l_2}\right) = & \frac{1}{P\left(\prod_{h=1}^{k} M_r(h) \neq 0\right)} \sum_{\mathbf{s}\in\mathcal{S}_{\mathbf{R}}} \\
& \times \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{s})} P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) \left(\sum_{i=1}^{r} \beta_{l_1,i}^* \beta_{l_2,i}^* + \sum_{i=1}^{r} \beta_{l_1,i}^* \sum_{i=1}^{r} \beta_{l_2,i}^*\right) \\
& - \frac{1}{\left\{P\left(\prod_{h=1}^{k} M_r(h) \neq 0\right)\right\}^2} \sum_{\mathbf{s}\in\mathcal{S}_{\mathbf{R}}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{s})} P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) \sum_{i=1}^{r} \beta_{l_1,i}^* \\
& \times \sum_{\mathbf{s}\in\mathcal{S}_{\mathbf{R}}} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*(\mathbf{s})} P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}) \sum_{i=1}^{r} \beta_{l_2,i}^*. \qquad (3.2.15)
\end{aligned}
$$

3.3 Approximate Confidence Intervals

Let $I(\theta_1, \theta_2, \cdots, \theta_k) = (I_{i,j}(\theta_1, \theta_2, \cdots, \theta_k)), i, j = 1, 2, \cdots, k$, denote the Fisher information matrix of the parameters $\theta_1, \theta_2, \cdots, \theta_k$, where

$$I_{i,j}(\theta_1, \theta_2, \cdots, \theta_k) = -E\left(\frac{\partial^2 \ln L}{\partial \theta_i \partial \theta_j}\right). \tag{3.3.1}$$

From the likelihood function in (3.2.2), we have $I_{i,j}(\theta_1, \theta_2, \cdots, \theta_k) = 0$ if $i \neq j$. Consequently, we have

$$\begin{aligned} & I(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k) \\ = & -\mathrm{Diag}\left(\frac{\partial^2 \ln L}{\partial \theta_1^2}|_{\theta_1=\hat{\theta}_1}, \frac{\partial^2 \ln L}{\partial \theta_2^2}|_{\theta_2=\hat{\theta}_2}, \cdots, \frac{\partial^2 \ln L}{\partial \theta_k^2}|_{\theta_k=\hat{\theta}_k}\right), \end{aligned} \tag{3.3.2}$$

where

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \theta_h^2}|_{\theta_h=\hat{\theta}_h} &= \frac{M_r(h)}{\hat{\theta}_h^2} - \frac{2\sum_{i=1}^{r}(\delta_i(h) + s_i(h))w_i}{\hat{\theta}_h^3} \\ &= -\frac{M_r(h)}{\hat{\theta}_h^2}. \end{aligned}$$

Then, by using the asymptotic normality of the MLEs, we have $\hat{\theta}_h - \theta_h \sim N(0, I_{h,h}^{-1})$, so that we can express the approximate $100(1-\alpha)\%$ confidence interval for θ_h, $1 \le h \le k$, as

$$\hat{\theta}_h \pm Z_{\alpha/2} \frac{\sum_{i=1}^{r}(\delta_i(h) + s_i(h))w_i}{(M_r(h))^{\frac{3}{2}}} = \hat{\theta}_h \left(1 \pm \frac{Z_{\alpha/2}}{\sqrt{M_r(h)}}\right),$$

where $Z_{\alpha/2}$ denotes the upper $\alpha/2$ percentage point of the standard normal distribution.

3.4 Bayesian Intervals

Let $u_h = \sum_{i=1}^{r}(\delta_i(h) + s_i(h))w_i$. Then, we can rewrite the likelihood function as

$$L(\theta_1, \theta_2, \cdots, \theta_k, \boldsymbol{\delta}, \mathbf{s}, \mathbf{w}) = c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \exp\left(-\frac{u_h}{\theta_h}\right). \tag{3.4.1}$$

Now, by assuming independent inverse gamma prior distributions, viz., $IG(a_h, b_h)$ for $1 \le h \le k$, we obtain from (3.4.1) the posterior joint density function as

$$L(\theta_1, \theta_2, \cdots, \theta_k | \mathbf{w}) = c_r \prod_{h=1}^{k} \theta_h^{-M_r(h) - a_h - 1} \exp\left(-\frac{u_h + b_h}{\theta_h}\right).$$

Upon comparing this with (3.4.1), we see that the joint posterior density function of $(\theta_1, \theta_2, \cdots, \theta_k)$ is a product of k independent inverse gamma density functions. So, given the data, the posterior density function of $\hat{\theta}_h$ is simply $IG\left(M_r(h) + a_h, u_h + b_h\right)$.

Thus, the Bayes estimator of θ_h under the squared-error loss function is

$$\hat{\theta}_{h,Bayes} = \frac{u_h + b_h}{M_r(h) + a_h - 1}, \qquad h = 1, 2, \cdots, k. \tag{3.4.2}$$

When we use Jeffreys' non-informative prior $I(\theta_h) \propto \frac{1}{\theta_h^2}$ corresponding to the special case when $a_h = 1$ and $b_h = 0$, for $1 \leq h \leq k$, the Bayes estimators in (3.4.2) coincide with the MLEs in (3.2.3).

Let $U_h = \frac{2(u_h + b_h)}{\theta_h}$ for $1 \leq h \leq k$. Then, the pivot U_h follows $\chi^2_{2(M_r(h)+a_h)}$ distribution, provided $2(M_r(h) + a_h)$ is a positive integer, for $1 \leq h \leq k$. In this case, the $100(1-\alpha)\%$ Bayes credible interval for θ_h becomes

$$\left(\frac{2(u_h + b_h)}{\chi^2_{2(M_r(h)+a_h),1-\alpha/2}}, \frac{2(u_h + b_h)}{\chi^2_{2(M_r(h)+a_h),\alpha/2}} \right), \qquad h = 1, 2, \cdots, k,$$

where $\chi^2_{v,\frac{\alpha}{2}}$ is the lower $\frac{\alpha}{2}$ percentage point of the chi-square distribution with v degrees of freedom.

3.5 Bootstrap Intervals

In this section, we consider confidence interval for θ_h ($h = 1, 2, \cdots, k$) based on the Bootstrap-p and Bootstrap-t methods; see, for example, Efron and Tibshirani (1994).

To find the Bootstrap-p and Bootstrap-t intervals, in the first step, we generate original samples from k exponential populations with parameters

θ_h of size n_h, $1 \leq h \leq k$. Next, we sort the data, and determine to which population each failure belongs, and then estimate θ_h by the conditional MLE in (3.2.3). In the second step, we generate a bootstrap sample $(\delta_1, \delta_2, \cdots, \delta_N; W_1, W_2, \cdots, W_N)$ by using the values $(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k)$, and then obtain the bootstrap estimates of $\hat{\theta}_h$, $1 \leq h \leq k$, say $\hat{\theta}_h^*$, $1 \leq h \leq k$, from the bootstrap sample. In the third step, we repeat the second step N-Boot times.

Boot-p: Suppose $K_{hB}(x) = P\left(\hat{\theta}_h^* \leq x\right)$, $1 \leq h \leq k$, is the cumulative distribution function of $\hat{\theta}_h^*$. Define $\hat{\theta}_{hB}(\alpha) = K_{hB}^{-1}(\alpha)$, $1 \leq h \leq k$, for a given α. Then, the $100(1-\alpha)\%$ Boot-p confidence interval for θ_h, $1 \leq h \leq k$, is given by

$$\left(\hat{\theta}_{hB}\left(\frac{\alpha}{2}\right), \hat{\theta}_{hB}\left(1 - \frac{\alpha}{2}\right)\right), \qquad 1 \leq h \leq k.$$

Boot-t: After generating the bootstrap samples in the second step and calculating $\hat{\theta}_h^*$, we need to use them to compute the estimate of $Var(\hat{\theta}_h^*)$ from the observed Fisher information matrix in (3.3.2). Then, we determine the T_h^* statistic given by

$$T_h^* = \frac{\hat{\theta}_h^* - \hat{\theta}_h}{\sqrt{Var(\hat{\theta}_h^*)}}, \qquad 1 \leq h \leq k. \tag{3.5.1}$$

Now, suppose $L_{hB}(x) = P(T_h^* \leq x)$ is the cumulative distribution function of T_h^*, $1 \leq h \leq k$. Then, the $100(1-\alpha)\%$ Boot-t confidence interval for θ_h, $1 \leq h \leq k$, is

given by

$$\left(\hat{\theta}_h - \sqrt{\mathrm{Var}(\hat{\theta}_h)}L_{hB}^{-1}\left(1-\frac{\alpha}{2}\right), \hat{\theta}_h - \sqrt{\mathrm{Var}(\hat{\theta}_h)}L_{hB}^{-1}\left(\frac{\alpha}{2}\right)\right), \quad 1 \le h \le k.$$

3.6 Simulation Results and Discussion

A simulation study was carried out to evaluate the performance of the conditional MLEs and also the average width of all confidence intervals discussed in the preceding sections for some choices of (n_1, n_2, n_3) and r. We considered different sample sizes for three populations (i.e, $k = 3$) as $n = (n_1, n_2, n_3)$, $n_1 = n_2 = n_3 = 5, 6, 7, 8, 9, 10$, and different choice for $r = 8, 10, 12, 18$. We list the different choices in Table 3.1 and 3.2 for which the simulation results are presented here. We also chose the parameters $(\theta_1, \theta_2, \theta_3)$ to be $(2, 4, 7)$ and $(3, 5, 9)$. For these cases, we computed the conditional MLEs for the parameters $(\theta_1, \theta_2, \theta_3)$ and the average width of 95% confidence intervals for $(\theta_1, \theta_2, \theta_3)$ using approximate, Boot-p and Boot-t methods (with N-Boot as 1000). For comparative purposes, we also computed the average widths of 95% credible intervals using Jeffreys' non-informative priors ($a_1 = a_2 = a_3 = 1$ and $b_1 = b_2 = b_3 = 0$). We repeated this process 1000 times and computed the average values of the conditional MLEs as well as the coverage probabilities for all confidence intervals, and these are presented in Table 3.3 and 3.4. From these values, it is clear that the MLEs have a moderate bias when the essential sample size r is small relative to the sample sizes (n_1, n_2, n_3) and become negligible when r increases relative to N, as is evident from Table 3.3 and 3.4.

In Table 3.3 and 3.4, the coverage probabilities of 95% confidence intervals of $(\theta_1, \theta_2, \theta_3)$ for all the methods are presented for the same choices of n_1, n_2, n_3 and r. We observe that between the two bootstrap methods, the Boot-p method performs better than the Boot-t method; the Bayesian method has very stable coverage probabilities (quite close to the nominal level of 95%); however, all these methods have lower coverage probabilities when the sample sizes are small. The exact method derived in Section 2 provides exact conditional confidence intervals with accurate coverage probabilities (compared to the nominal confidence levels) for small sample sizes.

3.7 Illustrative Example

Nelson (1982, Ch. 10, Table 4.1) has given times to breakdown in minutes of an insulating fluid subjected to high voltage stress. The failure times were observed in the form of groups with each group reporting data on 10 insulating fluids. For the purpose of illustrating the methods of inference detailed in the preceding sections, let us consider the following three groups of samples of failure time data presented in Table 3.7.

Suppose the samples of sizes $n = (10, 10, 10)$ in Table 3.7 are from three exponential populations with means $(\theta_1, \theta_2, \theta_3)$, respectively. Suppose a joint progressive Type-II censoring with $r = 12$ had been enforced on these data. For example, Tables 3.8 and 3.9 present the joint progressively Type-II censored data that obtained from

Table 3.1: Different choices of sample sizes and the joint progressive Type-II censoring scheme employed in the simulation study. (Here, $(1_7, 0)$, for example, means that the censoring scheme employed is (1,1,1,1,1,1,1,0))

(n_1, n_2, n_3)	r	$\mathcal{R}$	Scheme no. for $\theta = (2, 4, 7)$
$(5, 5, 5)$	8	$(1_7, 0)$	[111]
		$(2_3, 0_4, 1)$	[112]
$(6, 6, 6)$	8	$(1_7, 3)$	[121]
		$(2_5, 0_3)$	[122]
$(7, 7, 7)$	8	$(1_7, 6)$	[131]
		$(2_6, 0, 1)$	[132]
$(8, 8, 8)$	8	$(1_7, 9)$	[141]
		(2_8)	[142]
$(5, 5, 5)$	10	$(1_5, 0_5)$	[151]
		$(2_2, 0_7, 1)$	[152]
$(6, 6, 6)$	10	$(1_8, 0_2)$	[161]
		$(2_4, 0_6)$	[162]
$(7, 7, 7)$	10	$(1_9, 2)$	[171]
		$(2_5, 0_4, 1)$	[172]
$(8, 8, 8)$	10	$(1_9, 5)$	[181]
		$(2_7, 0_3)$	[182]
$(6, 6, 6)$	12	$(1_6, 0_6)$	[191]
		$(2_3, 0_9)$	[192]
$(7, 7, 7)$	12	$(1_9, 0_3)$	[1101]
		$(2_4, 1, 0_7)$	[1102]
$(8, 8, 8)$	12	(1_{12})	[1111]
		$(2_6, 0_6)$	[1112]
$(9, 9, 9)$	12	$(1_{11}, 4)$	[1121]
		$(2_7, 1, 0_4)$	[1122]
$(7, 7, 7)$	18	$(1_3, 0_{15})$	[1131]
		$(2, 1, 0_{16})$	[1132]
$(8, 8, 8)$	18	$(1_6, 0_{12})$	[1141]
		$(2_3, 0_{15})$	[1142]
$(9, 9, 9)$	18	$(1_9, 0_9)$	[1151]
		$(2_4, 1, 0_{13})$	[1152]
$(10, 10, 10)$	18	$(1_{12}, 0_6)$	[1161]
		$(2_6, 0_{12})$	[1162]

Table 3.2: Different choices of sample sizes and the joint progressive Type-II censoring scheme employed in the simulation study. (Here, $(1_7, 0)$, for example, means that the censoring scheme employed is (1,1,1,1,1,1,1,0))

(n_1, n_2, n_3)	r	$\mathcal{R}$	Scheme no. for $\theta = (3, 5, 9)$
$(5, 5, 5)$	8	$(1_7, 0)$	[211]
		$(2_3, 0_4, 1)$	[212]
$(6, 6, 6)$	8	$(1_7, 3)$	[221]
		$(2_5, 0_3)$	[222]
$(7, 7, 7)$	8	$(1_7, 6)$	[231]
		$(2_6, 0, 1)$	[232]
$(8, 8, 8)$	8	$(1_7, 9)$	[241]
		(2_8)	[242]
$(5, 5, 5)$	10	$(1_5, 0_5)$	[251]
		$(2_2, 0_7, 1)$	[252]
$(6, 6, 6)$	10	$(1_8, 0_2)$	[261]
		$(2_4, 0_6)$	[262]
$(7, 7, 7)$	10	$(1_9, 2)$	[271]
		$(2_5, 0_4, 1)$	[272]
$(8, 8, 8)$	10	$(1_9, 5)$	[281]
		$(2_7, 0_3)$	[282]
$(6, 6, 6)$	12	$(1_6, 0_6)$	[291]
		$(2_3, 0_9)$	[292]
$(7, 7, 7)$	12	$(1_9, 0_3)$	[2101]
		$(2_4, 1, 0_7)$	[2102]
$(8, 8, 8)$	12	(1_{12})	[2111]
		$(2_6, 0_6)$	[2112]
$(9, 9, 9)$	12	$(1_{11}, 4)$	[2121]
		$(2_7, 1, 0_4)$	[2122]
$(7, 7, 7)$	18	$(1_3, 0_{15})$	[2131]
		$(2, 1, 0_{16})$	[2132]
$(8, 8, 8)$	18	$(1_6, 0_{12})$	[2141]
		$(2_3, 0_{15})$	[2142]
$(9, 9, 9)$	18	$(1_9, 0_9)$	[2151]
		$(2_4, 1, 0_{13})$	[2152]
$(10, 10, 10)$	18	$(1_{12}, 0_6)$	[2161]
		$(2_6, 0_{12})$	[2162]

Table 3.3: Average values of the MLEs $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$ and coverage probabilities of different confidence intervals for some choices of (n_1, n_2, n_3), r and joint progressive Type-II censoring scheme.

Scheme no.	$\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$	$\theta = (2,4,7)$ *Approximate*	*Bayes*	*Boot*−t	*Boot*−p
111	$(2.13, 4.25, 7.74)$	$(88, 85, 84)$	$(96, 94, 91)$	$(84, 90, 91)$	$(91, 92, 91)$
112	$(2.14, 4.76, 9.02)$	$(85, 88, 90)$	$(94, 92, 90)$	$(82, 88, 90)$	$(89, 91, 95)$
121	$(2.33, 5.93, 10.90)$	$(91, 93, 93)$	$(94, 87, 85)$	$(86, 87, 87)$	$(90, 90, 95)$
122	$(2.21, 4.33, 7.71)$	$(88, 84, 85)$	$(94, 94, 93)$	$(85, 88, 91)$	$(91, 91, 92)$
131	$(2.52, 6.46, 11.82)$	$(94, 95, 94)$	$(92, 84, 80)$	$(86, 83, 83)$	$(91, 88, 92)$
132	$(2.21, 4.87, 9.60)$	$(90, 87, 91)$	$(95, 91, 88)$	$(86, 88, 90)$	$(92, 92, 96)$
141	$(2.88, 7.28, 11.46)$	$(97, 97, 94)$	$(91, 80, 83)$	$(87, 77, 83)$	$(87, 80, 93)$
142	$(2.24, 5.50, 11.05)$	$(92, 91, 94)$	$(96, 90, 86)$	$(89, 87, 85)$	$(93, 91, 96)$
151	$(2.01, 4.28, 7.21)$	$(86, 86, 82)$	$(96, 93, 94)$	$(83, 88, 90)$	$(88, 91, 88)$
152	$(2.08, 4.29, 7.95)$	$(87, 88, 89)$	$(96, 93, 94)$	$(84, 89, 93)$	$(89, 92, 94)$
161	$(2.14, 4.29, 7.69)$	$(90, 87, 86)$	$(96, 94, 92)$	$(86, 89, 93)$	$(92, 92, 93)$
162	$(2.03, 4.02, 7.44)$	$(88, 86, 85)$	$(95, 95, 93)$	$(84, 88, 92)$	$(91, 90, 90)$
171	$(2.25, 5.01, 10.73)$	$(92, 92, 94)$	$(93, 91, 85)$	$(85, 88, 83)$	$(90, 92, 94)$
172	$(2.10, 4.65, 9.02)$	$(88, 88, 93)$	$(95, 93, 90)$	$(83, 87, 90)$	$(90, 91, 96)$
181	$(2.34, 5.93, 12.30)$	$(94, 95, 94)$	$(94, 87, 77)$	$(86, 84, 72)$	$(91, 87, 86)$
182	$(2.15, 4.45, 7.77)$	$(91, 88, 85)$	$(96, 92, 92)$	$(87, 90, 93)$	$(93, 95, 93)$
191	$(1.96, 4.12, 7.15)$	$(87, 86, 86)$	$(96, 94, 94)$	$(82, 89, 94)$	$(90, 91, 91)$
192	$(2.01, 4.05, 7.10)$	$(87, 86, 85)$	$(95, 94, 93)$	$(83, 89, 91)$	$(90, 89, 89)$
1101	$(2.04, 4.24, 7.59)$	$(88, 89, 86)$	$(96, 94, 93)$	$(83, 90, 93)$	$(91, 92, 92)$
1102	$(2.07, 4.06, 6.96)$	$(89, 85, 85)$	$(94, 94, 93)$	$(85, 89, 92)$	$(91, 90, 90)$
1111	$(2.09, 4.95, 9.19)$	$(91, 92, 91)$	$(95, 89, 86)$	$(86, 89, 86)$	$(91, 93, 94)$
1112	$(2.06, 4.15, 7.27)$	$(89, 86, 86)$	$(95, 95, 92)$	$(84, 89, 93)$	$(91, 92, 91)$
1121	$(2.26, 5.38, 11.97)$	$(93, 95, 96)$	$(96, 88, 78)$	$(85, 84, 68)$	$(91, 89, 84)$
1122	$(2.15, 4.29, 7.73)$	$(90, 88, 88)$	$(95, 94, 92)$	$(86, 90, 94)$	$(92, 93, 94)$
1131	$(2.00, 3.92, 7.00)$	$(87, 88, 88)$	$(94, 94, 95)$	$(83, 91, 94)$	$(90, 90, 91)$
1132	$(2.02, 3.97, 6.93)$	$(89, 88, 87)$	$(95, 94, 95)$	$(85, 90, 94)$	$(92, 91, 90)$
1141	$(1.98, 4.14, 7.02)$	$(89, 90, 87)$	$(96, 94, 94)$	$(84, 91, 94)$	$(92, 91, 90)$
1142	$(2.00, 4.01, 7.01)$	$(87, 89, 88)$	$(95, 94, 95)$	$(83, 90, 95)$	$(90, 91, 92)$
1151	$(2.03, 4.10, 7.01)$	$(90, 90, 88)$	$(95, 94, 94)$	$(85, 91, 94)$	$(92, 93, 91)$
1152	$(2.00, 4.06, 7.07)$	$(87, 88, 88)$	$(95, 95, 94)$	$(83, 90, 95)$	$(91, 92, 91)$
1161	$(2.08, 4.00, 7.15)$	$(91, 88, 89)$	$(93, 94, 93)$	$(87, 91, 93)$	$(93, 92, 93)$
1162	$(2.02, 4.15, 7.22)$	$(90, 89, 89)$	$(96, 94, 93)$	$(86, 91, 93)$	$(93, 92, 93)$

Table 3.4: Average values of the MLEs $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$ and coverage probabilities of different confidence intervals for some choices of (n_1, n_2, n_3), r and joint progressive Type-II censoring scheme.

Scheme		$\theta = (3, 5, 9)$			
no.	$\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$	*Approximate*	*Bayes*	*Boot*−t	*Boot*−p
211	$(3.32, 5.50, 9.96)$	$(88, 87, 82)$	$(94, 93, 91)$	$(84, 88, 90)$	$(91, 93, 92)$
212	$(3.32, 5.80, 11.81)$	$(86, 86, 90)$	$(93, 92, 90)$	$(82, 87, 90)$	$(89, 90, 95)$
221	$(3.53, 7.24, 14.33)$	$(92, 94, 92)$	$(94, 89, 83)$	$(86, 89, 84)$	$(91, 93, 95)$
222	$(3.19, 5.61, 10.02)$	$(87, 86, 84)$	$(94, 93, 93)$	$(84, 89, 91)$	$(91, 93, 91)$
231	$(4.01, 7.92, 14.92)$	$(95, 95, 95)$	$(92, 84, 81)$	$(88, 83, 82)$	$(90, 88, 94)$
232	$(3.46, 6.19, 12.73)$	$(90, 89, 92)$	$(94, 91, 88)$	$(86, 88, 89)$	$(92, 93, 96)$
241	$(4.40, 8.61, 15.01)$	$(96, 97, 95)$	$(90, 83, 81)$	$(86, 80, 82)$	$(87, 83, 92)$
242	$(3.59, 6.96, 14.72)$	$(92, 92, 93)$	$(94, 89, 85)$	$(86, 88, 84)$	$(91, 91, 94)$
251	$(3.28, 5.14, 9.07)$	$(88, 85, 84)$	$(95, 96, 94)$	$(84, 88, 91)$	$(92, 91, 89)$
252	$(3.11, 5.46, 10.56)$	$(86, 88, 90)$	$(94, 92, 92)$	$(81, 88, 92)$	$(90, 90, 95)$
261	$(3.18, 5.45, 9.83)$	$(89, 88, 87)$	$(96, 93, 92)$	$(86, 89, 91)$	$(92, 94, 92)$
262	$(3.11, 5.20, 9.21)$	$(86, 86, 83)$	$(95, 95, 94)$	$(83, 88, 90)$	$(89, 91, 89)$
271	$(3.39, 6.21, 13.58)$	$(92, 92, 94)$	$(94, 93, 84)$	$(85, 89, 84)$	$(92, 92, 93)$
272	$(3.19, 5.76, 11.74)$	$(87, 88, 92)$	$(95, 92, 89)$	$(84, 87, 88)$	$(90, 90, 96)$
281	$(3.74, 7.34, 15.94)$	$(94, 96, 96)$	$(92, 89, 78)$	$(86, 83, 71)$	$(89, 88, 84)$
282	$(3.31, 5.49, 10.11)$	$(91, 88, 86)$	$(94, 94, 92)$	$(87, 91, 93)$	$(93, 93, 93)$
291	$(3.18, 5.28, 9.29)$	$(90, 87, 84)$	$(95, 94, 92)$	$(85, 88, 92)$	$(92, 92, 88)$
292	$(3.00, 5.11, 9.38)$	$(84, 85, 87)$	$(93, 94, 93)$	$(82, 87, 91)$	$(88, 89, 90)$
2101	$(3.15, 5.40, 9.92)$	$(92, 88, 86)$	$(96, 94, 92)$	$(87, 88, 92)$	$(94, 94, 93)$
2102	$(3.10, 5.10, 9.02)$	$(88, 86, 85)$	$(95, 93, 93)$	$(86, 87, 92)$	$(91, 89, 90)$
2111	$(3.26, 5.64, 12.00)$	$(91, 90, 90)$	$(96, 91, 88)$	$(86, 86, 87)$	$(92, 91, 94)$
2112	$(3.09, 5.40, 9.29)$	$(89, 89, 85)$	$(95, 94, 93)$	$(84, 90, 93)$	$(92, 93, 91)$
2121	$(3.45, 6.68, 15.41)$	$(94, 94, 95)$	$(94, 89, 75)$	$(87, 85, 68)$	$(92, 90, 81)$
2122	$(3.26, 5.50, 9.52)$	$(90, 88, 87)$	$(94, 94, 93)$	$(87, 89, 92)$	$(92, 93, 93)$
2131	$(3.00, 5.01, 9.16)$	$(89, 86, 87)$	$(95, 95, 94)$	$(84, 88, 93)$	$(91, 91, 89)$
2132	$(3.01, 4.96, 9.03)$	$(86, 87, 88)$	$(94, 95, 96)$	$(82, 88, 94)$	$(89, 89, 91)$
2141	$(3.01, 4.94, 9.06)$	$(88, 88, 87)$	$(95, 95, 93)$	$(84, 87, 92)$	$(91, 91, 90)$
2142	$(3.06, 5.08, 9.01)$	$(89, 89, 88)$	$(95, 96, 95)$	$(86, 90, 92)$	$(92, 92, 90)$
2151	$(3.11, 5.10, 9.20)$	$(91, 89, 88)$	$(96, 94, 93)$	$(86, 89, 94)$	$(93, 92, 92)$
2152	$(3.03, 4.86, 9.01)$	$(88, 87, 85)$	$(96, 94, 94)$	$(84, 87, 94)$	$(91, 91, 90)$
2161	$(3.11, 5.21, 9.19)$	$(91, 88, 88)$	$(95, 94, 92)$	$(88, 88, 93)$	$(93, 92, 92)$
2162	$(3.05, 5.22, 9.03)$	$(90, 88, 88)$	$(95, 93, 94)$	$(86, 88, 94)$	$(92, 92, 92)$

Table 3.5: The average widths for some choices of (n_1, n_2, n_3), r and joint progressive Type-II censoring scheme.

Scheme	$\theta = (2, 4, 7)$			
no.	*Approximate*	*Bayes*	*Boot*−t	*Boot*−p
111	(5.0, 9.7, 17.4)	(33.8, 68.6, 120.6)	(5.7, 12.6, 24.2)	(8.5, 60.5, 256.7)
112	(4.7, 10.6, 20.0)	(12.0, 26.8, 51.0)	(5.8, 14.6, 28.8)	(8.0, 86.1, 346.4)
121	(5.2, 13.2, 24.2)	(13.1, 33.3, 61.1)	(7.1, 23.7, 45.5)	(10.5, 194.9, 674.9)
122	(5.0, 9.5, 16.4)	(14.0, 27.7, 48.1)	(5.9, 12.2, 22.1)	(8.2, 53.9, 204.7)
131	(5.7, 13.9, 25.7)	(15.9, 41.1, 75.4)	(8.5, 27.0, 42.4)	(11.8, 215.7, 598.0)
132	(5.0, 10.6, 20.8)	(14.1, 31.2, 61.2)	(6.4, 17.1, 37.0)	(8.1, 99.6, 486.5)
141	(6.8, 16.6, 26.2)	(45.9, 113.3, 178.2)	(10.5, 29.5, 38.0)	(17.2, 240.3, 457.6)
142	(5.3, 12.3, 25.3)	(35.7, 84.5, 180.4)	(6.7, 22.1, 50.1)	(8.7, 176.3, 795.2)
151	(4.2, 8.3, 14.4)	(8.1, 17.2, 29.0)	(4.4, 9.8, 16.7)	(5.1, 40.7, 122.7)
152	(4.4, 8.6, 16.2)	(10.1, 20.8, 38.2)	(4.7, 10.6, 20.2)	(5.5, 38.3, 162.6)
161	(4.5, 8.6, 15.3)	(8.6, 17.3, 31.1)	(4.9, 10.9, 20.7)	(5.6, 44.4, 182.2)
162	(4.3, 8.1, 15.2)	(9.7, 19.0, 36.1)	(4.5, 9.3, 17.5)	(5.3, 36.0, 134.9)
171	(5.3, 6.2, 8.7)	(37.1, 86.1, 174.9)	(5.6, 17.4, 44.8)	(9.6, 103.8, 708.1)
172	(4.4, 9.2, 18.3)	(10.0, 22.1, 43.2)	(5.0, 13.2, 27.5)	(5.9, 86.2, 269.8)
181	(5.2, 8.2, 16.5)	(14.3, 35.5, 74.5)	(6.3, 24.5, 54.0)	(9.2, 183.9, 948.8)
182	(4.5, 8.8, 15.2)	(8.7, 18.0, 31.3)	(5.0, 11.7, 21.1)	(6.5, 54.8, 196.5)
191	(4.0, 5.9, 8.5)	(7.3, 15.3, 26.1)	(3.9, 8.5, 15.0)	(4.1, 29.2, 93.9)
192	(4.0, 6.7, 10.8)	(6.9, 13.9, 24.1)	(4.0, 8.2, 14.4)	(4.5, 26.0, 88.5)
1101	(4.1, 6.6, 11.6)	(7.0, 14.5, 25.8)	(4.1, 9.4, 17.8)	(4.3, 35.5, 140.8)
1102	(4.2, 6.1, 11.4)	(9.1, 17.7, 30.4)	(4.1, 8.5, 14.7)	(4.7, 29.6, 91.6)
1111	(4.6, 4.0, 11.2)	(31.1, 67.1, 130.8)	(4.3, 14.4, 31.7)	(5.0, 85.4, 364.3)
1112	(4.2, 5.8, 10.9)	(8.8, 18.1, 32.0)	(4.2, 8.9, 16.1)	(4.8, 33.4, 112.4)
1121	(4.5, 8.6, 17.1)	(7.7, 18.4, 40.8)	(5.0, 19.4, 55.8)	(5.7, 109.4, 956.6)
1122	(4.3, 6.0, 11.4)	(7.4, 14.7, 26.5)	(4.4, 9.8, 18.4)	(5.4, 39.4, 143.3)
1131	(3.2, 4.7, 8.0)	(4.4, 8.7, 15.5)	(3.2, 6.2, 11.2)	(3.4, 17.2, 60.1)
1132	(3.2, 5.2, 9.2)	(4.5, 8.8, 15.3)	(3.2, 6.3, 11.1)	(3.4, 17.6, 55.6)
1141	(3.3, 4.5, 5.6)	(4.8, 9.9, 16.8)	(3.1, 6.7, 11.5)	(3.2, 20.0, 59.9)
1142	(3.2, 4.3, 7.8)	(4.5, 9.0, 15.8)	(3.2, 6.4, 11.4)	(3.3, 18.8, 59.4)
1151	(3.2, 4.4, 9.0)	(4.5, 9.1, 15.5)	(3.2, 6.8, 11.8)	(3.4, 20.1, 62.3)
1152	(3.2, 4.7, 8.1)	(4.5, 9.2, 15.9)	(3.2, 6.6, 11.6)	(3.4, 19.4, 62.3)
1161	(3.4, 3.7, 7.1)	(4.9, 9.5, 16.9)	(3.3, 6.8, 12.7)	(3.4, 20.5, 72.3)
1162	(3.3, 5.0, 7.3)	(4.5, 9.3, 16.2)	(3.2, 6.9, 12.1)	(3.4, 20.9, 65.4)

Table 3.6: The average widths for some choices of (n_1, n_2, n_3), r and joint progressive Type-II censoring scheme.

Scheme no.	$\theta = (3, 5, 9)$ *Approximate*	*Bayes*	$Boot-$t	$Boot-$p
211	$(7.5, 11.8, 21.1)$	$(20.9, 34.6, 62.9)$	$(9.2, 16.2, 30.9)$	$(15.8, 77.4, 327.2)$
212	$(7.3, 12.9, 26.2)$	$(18.5, 32.5, 66.0)$	$(9.3, 17.6, 37.5)$	$(15.9, 77.4, 401.6)$
221	$(8.3, 15.7, 32.5)$	$(53.9, 109.0, 228.3)$	$(11.4, 28.4, 59.4)$	$(17.7, 184.1, 801.0)$
222	$(7.2, 12.1, 21.9)$	$(20.4, 35.7, 64.3)$	$(8.5, 15.8, 28.8)$	$(12.6, 67.8, 235.3)$
231	$(8.9, 17.6, 33.1)$	$(22.2, 44.2, 83.7)$	$(14.5, 32.8, 53.9)$	$(24.1, 211.6, 604.6)$
232	$(8.2, 14.0, 28.6)$	$(56.2, 99.9, 200.2)$	$(10.4, 21.5, 49.1)$	$(16.6, 106.5, 570.4)$
241	$(9.9, 18.5, 32.7)$	$(27.8, 54.2, 95.0)$	$(16.6, 35.4, 49.4)$	$(30.1, 225.5, 514.8)$
242	$(8.5, 15.7, 33.3)$	$(57.0, 110.0, 231.8)$	$(11.5, 27.3, 66.7)$	$(20.7, 177.8, 955.7)$
251	$(6.8, 10.3, 18.2)$	$(13.2, 20.7, 36.5)$	$(7.3, 11.7, 21.0)$	$(10.0, 32.5, 120.5)$
252	$(6.6, 11.1, 21.6)$	$(14.7, 25.9, 49.7)$	$(7.2, 13.4, 26.8)$	$(9.3, 46.3, 210.7)$
261	$(6.7, 11.1, 19.9)$	$(15.3, 26.0, 46.9)$	$(7.4, 13.8, 26.3)$	$(9.2, 52.0, 196.1)$
262	$(6.5, 10.5, 18.3)$	$(12.5, 21.1, 36.9)$	$(7.0, 12.0, 21.6)$	$(9.5, 37.4, 137.3)$
271	$(7.7, 9.3, 22.6)$	$(50.5, 95.8, 198.6)$	$(9.0, 21.2, 56.2)$	$(13.0, 95.9, 738.2)$
272	$(6.8, 11.6, 23.4)$	$(15.3, 27.3, 55.7)$	$(7.8, 16.2, 35.6)$	$(10.5, 66.0, 329.2)$
281	$(8.5, 13.0, 27.4)$	$(54.6, 115.8, 244.3)$	$(11.2, 29.3, 69.9)$	$(17.8, 204.7, 1085.1)$
282	$(7.0, 11.1, 19.9)$	$(15.7, 26.6, 48.0)$	$(7.9, 14.2, 27.3)$	$(11.7, 49.8, 219.6)$
291	$(6.4, 7.2, 11.8)$	$(11.8, 19.7, 34.1)$	$(6.3, 10.8, 19.4)$	$(8.0, 30.9, 111.4)$
292	$(6.0, 8.5, 15.5)$	$(10.3, 17.5, 32.0)$	$(6.0, 10.3, 19.1)$	$(7.3, 30.1, 110.4)$
2101	$(6.3, 8.3, 14.5)$	$(10.7, 18.4, 33.9)$	$(6.4, 11.9, 23.2)$	$(7.7, 35.8, 152.2)$
2102	$(6.3, 6.8, 12.4)$	$(11.5, 18.9, 33.5)$	$(6.3, 10.6, 19.0)$	$(7.9, 31.9, 104.0)$
2111	$(6.5, 8.7, 17.9)$	$(11.1, 19.1, 41.1)$	$(7.2, 15.5, 41.1)$	$(9.6, 57.2, 484.8)$
2112	$(6.2, 8.5, 15.3)$	$(10.5, 18.4, 31.7)$	$(6.3, 11.6, 20.6)$	$(8.5, 37.1, 123.6)$
2121	$(6.8, 12.3, 28.8)$	$(11.1, 21.5, 49.5)$	$(8.3, 23.6, 71.4)$	$(13.5, 113.1, 1210.0)$
2122	$(6.5, 8.5, 13.4)$	$(11.1, 18.7, 32.4)$	$(6.8, 12.4, 22.5)$	$(9.0, 44.6, 146.5)$
2131	$(4.9, 5.4, 8.4)$	$(7.1, 11.9, 21.8)$	$(4.8, 8.0, 14.6)$	$(5.3, 18.5, 67.4)$
2132	$(4.9, 5.5, 10.6)$	$(6.8, 11.1, 20.4)$	$(4.8, 7.9, 14.4)$	$(5.3, 18.1, 64.5)$
2141	$(4.8, 5.2, 11.0)$	$(6.8, 11.1, 20.4)$	$(4.8, 8.0, 14.8)$	$(5.2, 18.9, 71.2)$
2142	$(4.9, 5.1, 9.5)$	$(6.9, 11.4, 20.3)$	$(4.9, 8.2, 14.6)$	$(5.5, 18.8, 62.9)$
2151	$(5.1, 4.7, 8.2)$	$(7.2, 11.8, 21.3)$	$(5.0, 8.4, 15.5)$	$(5.5, 19.8, 73.3)$
2152	$(4.9, 5.1, 10.0)$	$(6.8, 11.0, 20.3)$	$(4.9, 7.9, 14.8)$	$(5.5, 19.0, 68.0)$
2161	$(5.0, 5.8, 5.0)$	$(7.0, 11.7, 20.7)$	$(5.0, 8.8, 16.3)$	$(5.5, 22.8, 86.6)$
2162	$(4.9, 5.9, 10.2)$	$(6.9, 11.8, 20.4)$	$(4.9, 8.6, 15.1)$	$(5.4, 21.7, 70.0)$

Table 3.7: Failure time data as three groups of insulating fluids

Group 1	0.31	0.66	1.54	1.70	1.82	1.89	2.17	2.24	4.03	9.99
Group 2	0.00	0.18	0.55	0.66	0.71	1.30	1.63	2.17	2.75	10.60
Group 3	0.49	0.64	0.82	0.93	1.08	1.99	2.06	2.15	2.57	4.75

Table 3.8: Joint progressively Type-II censored data observed from Table 3.7 with $r = 12$ and censoring scheme R_1

W	δ			$\mathbf{s}$		
0.00	0	1	0	1	0	0
0.18	0	1	0	1	0	0
0.31	1	0	0	1	0	0
0.49	0	0	1	0	0	1
0.55	0	1	0	0	0	1
0.64	0	0	1	0	1	0
0.66	1	0	0	0	1	0
0.66	0	1	0	0	0	1
1.08	0	0	1	0	1	0
1.54	1	0	0	1	0	0
1.63	0	1	0	0	0	1
2.17	1	0	0	0	0	1

the data in Table 3.7 with $r = 12$ and censoring schemes $R_1 = (1_{12})$ and $R_2 = (2_6, 0_6)$.

We then computed the conditional MLEs of $(\theta_1, \theta_2, \theta_3)$ and the estimates of their standard deviations and mean square errors for $r = 12$ from the expressions presented earlier in Section 2, and these are presented in Table 3.10. We have also computed the estimates of the covariance matrix of $(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$ from the expression in Corollary 3, and these are presented in Table 3.11. From the results in Tables 3.10 and 3.11, we find the estimates to be quite stable, and especially so for θ_2 since this population has smallest mean thus producing more failures in the joint progressively censored data.

Table 3.9: Joint progressively Type-II censored data observed from Table 3.7 with $r = 12$ and censoring scheme R_2

W	δ			$\mathbf{s}$		
0	1	0	0	0	0	2
0.18	1	0	0	1	1	0
0.31	1	0	0	1	1	0
0.55	1	0	0	1	0	1
0.64	1	0	0	1	0	1
0.66	1	0	0	0	1	1
0.82	1	0	0	0	0	0
1.89	1	0	0	0	0	0
2.15	0	1	0	0	0	0
2.17	0	0	1	0	0	0
2.17	0	0	1	0	0	0
4.03	0	1	0	0	0	0

Table 3.10: Conditional MLEs and the estimates of their standard deviations and mean square errors based on joint progressively Type-II censored data from Tables 3.8 and 3.9 according to the censoring schemes $R_1 = (1_{12})$ and $R_2 = (2_6, 0_6)$.

	Mean	$\widehat{SD}$	$\widehat{MSE}$
R	$(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$	$(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$	$(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$
R_1	$(1.68, 1.01, 1.41)$	$(1.1932, 0.5852, 0.9391)$	$(1.4692, 0.3476, 0.9049)$
R_2	$(2.35, 1.56, 2.22)$	$(1.6288, 0.9142, 1.5063)$	$(2.7313, 0.8490, 2.3316)$

Table 3.11: Estimates of the covariance matrix of the conditional MLEs based on joint progressively Type-II censored data from Tables 3.8 and 3.9

Covariance matrix $(\rho(\theta_i, \theta_j))_{i,j}$ for schemes R_1 and R_2
$\begin{pmatrix} 1.4237 & -0.0258 & -0.0494 \\ -0.02582 & 0.3424 & -0.0206 \\ -0.0494 & -0.0206 & 0.8820 \end{pmatrix}, \quad \begin{pmatrix} 2.6529 & -0.0566 & -0.1091 \\ -0.0566 & 0.8358 & -0.0525 \\ -0.1091 & -0.0525 & 2.2690 \end{pmatrix}$

Table 3.12: 95% confidence intervals for $(\theta_1, \theta_2, \theta_3)$ corresponding to different methods based on joint progressively Type-II censored data from Table 3.8 and 3.9 for censoring schemes R_1 and R_2.

	Scheme	$r = 12$		
	R	CI for θ_1	CI for θ_2	CI for θ_3
Exact	R_1	$(0.69, 6.59)$	$(0.46, 3.24)$	$(0.60, 5.13)$
	R_2	$(0.98, 8.90)$	$(0.71, 5.05)$	$(0.94, 8.20)$
Approximate	R_1	$(0.01, 4.35)$	$(0.92, 2.24)$	$(1.2, 3.55)$
	R_2	$(0.02, 4.94)$	$(0.61, 3.13)$	$(0.9, 4.59)$
Boot-t	R_1	$(0.82, 40.32)$	$(0.34, 6.12)$	$(0.63, 24.11)$
	R_2	$(0.66, 16.82)$	$(0.34, 5.27)$	$(0.60, 14.17)$
Boot-p	R_1	$(0.74, 7.65)$	$(0.40, 3.20)$	$(0.61, 5.91)$
	R_2	$(0.69, 6.08)$	$(0.47, 3.76)$	$(0.65, 5.60)$
Bayes credible	R_1	$(0.99, 8.42)$	$(0.51, 4.32)$	$(0.80, 6.89)$
	R_2	$(1.13, 9.51)$	$(0.71, 6.05)$	$(1.04, 8.89)$

Table 3.12 presents the 95% confidence intervals for $(\theta_1, \theta_2, \theta_3)$ based on the exact, approximate, Bayes credible, Boot-p and Boot-t methods corresponding to the case $r = 12$ with joint progressive Type-II censoring scheme $R_1 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ and $R_2 = (2, 2, 2, 2, 2, 2, 0, 0, 0, 0, 0, 0)$. From these results, we observe once again that the approximate and Boot-t confidence intervals are not satisfactory compared to the conditional exact confidence intervals. We also observe that the Boot-p and Bayesian methods yield results close to the exact confidence intervals, which once again reveals the importance of the results derived in Section 3.2 in obtaining exact conditional confidence intervals in the case of small values of r.

Chapter 4

Inference Under Joint Type-II Hybrid Censoring

4.1 Introduction

The joint censoring scheme is quite useful in conducting comparative life-tests of products from different units within the same facility. Suppose products are being manufactured by k different lines within the same facility, and that k independent samples of sizes n_h, $1 \leq h \leq k$, are selected from these k lines and placed simultaneously on a life-testing experiment. Then, in order to reduce the cost of experiment and also to reduce the experimental time, the life-testing experiment may be chosen to be terminated after a certain number (say, r) of failures has been observed. In this situation, one may be interested in either point or interval estimation of the mean lifetimes of units produced by these k lines. In addition, one may also place a condition based on time under test. In such a situation, here, if the termination of the life-testing experiment is done by a joint Type-I or Type-II hybrid censoring scheme, exact results based on the maximum likelihood estimates are developed for making point and interval inference on the parameters.

Epstein (1954) introduced the Type-I hybrid censoring scheme (Type-I HCS) in which the life-testing experiment is terminated as soon as a pre-specified number r out of n items has failed or a pre-fixed time T on test has been reached, which ever appears first. The Type-I HCS has been used as a reliability acceptance test in MIL-STD-781 C (1977). However, the Type-I HCS may result in very few failures at the termination point of the experiment. For this reason, Childs et al. (2003) introduced the Type-II hybrid censoring scheme (Type-II HCS) as an alternative to Type-I hybrid censoring scheme, in which the life-testing experiment terminates at the time when the latter of the above two stopping rules is reached. This censoring scheme has the advantage of guaranteeing that at least r failures are observed at the end of the experiment. The disadvantage in this scheme, however, is that it may take longer time to complete this life-test.

In the literature, joint Type-II censoring scheme and inferential methods based on such a scheme have been discussed earlier. For example, Basu (1968) discussed a generalized Savage statistic. Johnson and Mehrotra (1972) studied locally most powerful rank test under joint censoring. The problem of testing for the equality of two distributions, under the assumption of exponentiality, was discussed by Bhattacharyya and Mehrotra (1981). All these developments under this joint censoring scheme have focused on nonparametric and parametric tests of hypotheses; see Bhattacharyya (1995, Chapter 7 of Balakrishnan and Basu (1995)). For the exact inference based on the MLEs, Chen and Bhattacharyya (1988) derived the exact distribution of the maximum likelihood estimator of the mean of an exponential distribution and an exact lower confidence bound for the mean based on a Type-I hybrid censored sample. Childs et al. (2003) obtained an alternative simple form which is equivalent to the results of Chen and Bhattacharyya.

To study two or more competing products in regard to the duration of their service life, comparative lifetime experiments are of great importance. In this regard, Balakrishnan and Rasouli (2008) discussed exact inference for two exponential populations when Type-II censoring is implemented on the two samples in a joint manner. Here, we generalize their work by considering the k-sample problem. Suppose the test units from k lines under study are placed on a life-test simultaneously, that the successive failure times and the corresponding types (lines from which the failed units come from) are recorded, the experiment is terminated at time when the latter of a specified total number of failures has been observed and a pre-fixed time T on test has been reached. The main aim of this paper is to consider the parameters of k exponential distributions, and to derive the exact distributions of the maximum likelihood estimators of the unknown parameters based on a joint Type-II hybrid censored sample arising from these k independent exponential populations. The exact distributions of the maximum likelihood estimators and the exact confidence intervals for the exponential parameters are discussed.

Suppose $X_{11}, X_{12}, \cdots, X_{1n_1}$ are the lifetimes of n_1 specimens from line A_1, and they are assumed to be independent and identically distributed (iid) variables from a population with cumulative distribution function (cdf) $F_1(x)$ and probability density function (pdf) $f_1(x)$. Similarly, $X_{21}, X_{22}, \cdots, X_{2n_2}$ are the lifetimes of n_2 specimens from line A_2 and are assumed to be a sample from pdf $f_2(x)$ and cdf $F_2(x)$, and so on, with $X_{k1}, X_{k2}, \cdots, X_{kn_k}$ denoting the lifetimes of n_k specimens from line A_k and are assumed to be iid variables from pdf $f_k(x)$ and cdf $F_k(x)$.

Furthermore, let $N = \sum_{i=1}^{k} n_i$ denote the total sample size, T denote the pre-fixed test time, and r denote the pre-fixed number of failures to be observed. Let $w_1 \leq w_2 \leq \cdots \leq w_N$ denote the order statistics of the N random variables $\{X_{ij}; 1 \leq i \leq k, 1 \leq j \leq n_i\}$.

Therefore, under the joint hybrid Type-II censoring scheme for the k-samples, the observable data consist of $(\boldsymbol{\delta}, \mathbf{w})$, where $\mathbf{w} = (w_1, w_2, \cdots, w_r)$, $w_i \in \{X_{h_i 1}, X_{h_i 2}, \cdots, X_{h_i n_i}\}$ for $1 \leq h_1, h_2, \cdots, h_r \leq k$. Finally, associated to $(h_1, h_2, \cdots, h_r)$, let us define $\boldsymbol{\delta} = (\delta_1(h), \delta_2(h), \cdots, \delta_r(h))$ as

$$\delta_i(h) = \begin{cases} 1, & \text{if } h = h_i \\ 0, & \text{otherwise.} \end{cases} \tag{4.1.1}$$

Let D denote the number of failures up to time T. Then, D is a discrete random variable with support $\{0, 1, 2, \cdots, N\}$ with probability mass function

$$P(D = d) = \sum \begin{pmatrix} n_1 \\ l_1 \end{pmatrix} p_1^{l_1} q_1^{n_1 - l_1} \begin{pmatrix} n_2 \\ l_2 \end{pmatrix} p_2^{l_2} q_2^{n_2 - l_2} \cdots \begin{pmatrix} n_k \\ l_k \end{pmatrix} p_k^{l_k} q_k^{n_k - l_k}$$

with $\sum_{i=1}^{k} l_i = d$ for $1 \leq l_i \leq n_i$, where $p_j = F_j(T)$, and $q_j = 1 - F_j(T)$ $(1 \leq j \leq k)$.

Therefore, under the joint Type-II hybrid censoring scheme described above, the observable data consist of $(\boldsymbol{\delta}, \mathbf{w})$ of the following form:

$$(\boldsymbol{\delta}, \mathbf{w}) = \begin{cases} (\delta_1, \delta_2, \cdots, \delta_r; w_1, w_2, \cdots, w_r), & \text{with } D = 0, 1, \cdots, r-1, \\ (\delta_1, \delta_2, \cdots, \delta_D; w_1, w_2, \cdots, w_D), & \text{with } D = r, r+1, \cdots, N, \end{cases}$$

where δ is as defined in (4.1.1).

4.2 MLEs, Exact Distributions and Inference

Let $M_r(h) = \sum_{i=1}^{r} \delta_i(h)$ denote the number of X_h-failures in $\mathbf{w}$ for $1 \leq h \leq k$, and $r = \sum_{h=1}^{k} M_r(h)$. Then, under the joint Type-II HCS, the likelihood of $(\boldsymbol{\delta}, \mathbf{W})$ is then given by

$$
\begin{aligned}
& L(\theta_1, \theta_2, \cdots, \theta_k, \boldsymbol{\delta}, \mathbf{w}) \\
= & \begin{cases} c_r \prod_{i=1}^{r} \prod_{h=1}^{k} \left(f_h(w_i)\right)^{\delta_i(h)} \prod_{h=1}^{k} \left(S_h(w_r)\right)^{n_h - M_r(h)}, & T < w_r, \\ c_D \prod_{i=1}^{D} \prod_{h=1}^{k} \left(f_h(w_i)\right)^{\delta_i(h)} \prod_{h=1}^{k} \left(S_h(T)\right)^{n_h - M_D(h)}, & T > w_r, \end{cases}
\end{aligned} \tag{4.2.1}
$$

where $S_h(w_r) = 1 - F_h(w_r)$, $S_h(T) = 1 - F_h(T)$ and $c_r = \frac{\prod_{h=1}^{k} n_h!}{\prod_{h=1}^{k}(n_h - M_r(h))!}$, and $c_D = \frac{\prod_{h=1}^{k} n_h!}{\prod_{h=1}^{k}(n_h - M_D(h))!}$.

When the k populations are exponential with cdf $F_h(x) = 1 - \exp(-\frac{x}{\theta_h})$, $x > 0$, and pdf $f_h(x) = \frac{1}{\theta_h}\exp(-\frac{x}{\theta_h})$, $x > 0$, for $1 \leq h \leq k$, the likelihood function in (4.2.1) simplifies as follows:

For $T < w_r$, $(D \leq r-1)$, the likelihood of $(\boldsymbol{\delta}, \mathbf{W})$ is given by

$$
\begin{aligned}
& L(\theta_1, \theta_2, \cdots, \theta_k, \boldsymbol{\delta}, \mathbf{w}) \\
= \; & c_r \exp\left\{ -\sum_{h=1}^{k} M_r(h) \log\theta_h - \sum_{h=1}^{k} \frac{1}{\theta_h}\left(\sum_{i=1}^{r} w_i \delta_i(h) + w_r(n_h - M_r(h)) \right) \right\};
\end{aligned} \tag{4.2.2}
$$

For $T > w_r$ $(D \geq r)$, the likelihood of $(\boldsymbol{\delta}, \mathbf{W})$ is given by

$$
\begin{aligned}
& L(\theta_1, \theta_2, \cdots, \theta_k, \boldsymbol{\delta}, \mathbf{w}) \\
= \; & c_D \exp\left\{ -\sum_{h=1}^{k} M_D(h) \log\theta_h - \sum_{h=1}^{k} \frac{1}{\theta_h}\left(\sum_{i=1}^{D} w_i \delta_i(h) + T(n_h - M_D(h)) \right) \right\}.
\end{aligned} \tag{4.2.3}
$$

Thus, under the Type-II HCS, for $1 \le h \le k$, the MLE of θ_h is given by

$$\hat{\theta}_h = \begin{cases} \frac{1}{M_r(h)} \left\{ \sum_{i=1}^{r} w_i \delta_i(h) + w_r(n_h - M_r(h)) \right\} & \text{when } D = 0, 1, \cdots, r-1, \\ \frac{1}{M_D(h)} \left\{ \sum_{i=1}^{D} w_i \delta_i(h) + T(n_h - M_D(h)) \right\} & \text{when } D = r, r+1, \cdots, N. \end{cases} \tag{4.2.4}$$

Remark 9 *From the likelihood function, we readily see that the MLE of θ_h does not exist if $T < W_r$ and $\prod_{h=1}^{k} M_r(h) = 0$; if $T > W_r$ and $\prod_{h=1}^{k} M_D(h) = 0$, then also the MLE of θ_h does not exist. So, the MLEs in (4.2.4) are conditioned on $S = \left\{ \prod_{h=1}^{k} M_D(h) \ge 1 \text{ or } \prod_{h=1}^{k} M_r(h) \ge 1 \right\}$. We, therefore, need to discuss the distribution and other properties of the MLEs only conditional on the set S.*

Lemma 3 *For $r \le d \le N$, let $\mathbf{M}_d = (M_d(1), M_d(2), \cdots, M_d(k))$ and $\mathbf{t} = (t_1, t_2, \cdots, t_k)$ with $\sum_{j=1}^{k} t_j = d$. Further, let us denote the set*

$$\begin{aligned} \mathbf{T}_d = \quad & \Bigg\{ \mathbf{t} : \max\{1, d - \sum_{h=1}^{k-1} n_j\} \le t_k \le \min\{d - k + 1, n_k\}, \\ & \max\{1, d - \sum_{h=1}^{k-2} n_j - t_k\} \le t_{k-1} \le \min\{d - k + 2 - t_k, n_{k-1}\}, \\ & \cdots, \\ & \max\{1, d - n_1 - \sum_{h=3}^{k} t_h\} \le t_2 \le \min\{d - \sum_{h=3}^{k} t_h, n_2\} \Bigg\}. \end{aligned}$$

Then, we have

$$P(S|D=d) = \sum_{\mathbf{t}\in\mathbf{T}_d} P(\mathbf{M}_d = \mathbf{t}) \tag{4.2.5}$$

Proof For $0 \le d < r$, consider the set

$$\{S|D=d\} = \left\{\sum_{h=1}^{k} M_d(h) = d, \prod_{h=1}^{k} M_d(h) \neq 0\right\}.$$

Since $t_k \le n_k$ and $d = \sum_{h=1}^{k} t_h \ge k-1+t_k$, we have $t_k \le \min\{d-k+1, n_k\}$. On the other hand, $t_k \ge 1$ and $d = \sum_{h=1}^{k} t_h \le \sum_{h=1}^{k-1} n_h + t_k$, and so we have $t_k \ge \max\{1, d - \sum_{h=1}^{k-1} n_h\}$.

When we fix t_k in this way, we get $\max\{1, d - \sum_{h=1}^{k-2} n_j - t_k\} \le t_{k-1} \le \min\{d - k + 2 - t_k, n_{k-1}\}$, and therefore

$$\{S|D=d\} = \cup_{\mathbf{t}\in\mathbf{T}_d}\{\mathbf{M}_r = \mathbf{t}\}$$

which proves the lemma.

Similarly, we have the following result for the case when $r \le d \le N$.

Lemma 4 *For* $0 \le d \le r-1$, *let* $\mathbf{M}_r = (M_r(1), M_r(2), \cdots, M_r(k))$ *and* $\mathbf{t} =$

$(t_1, t_2, \cdots, t_k)$ *with* $\sum_{j=1}^{k} t_j = r$. *Further, let us denote the set*

$$\begin{aligned}\mathbf{T}_r = \quad & \Big\{\mathbf{t} : \max\{1, r - \sum_{h=1}^{k-1} n_j\} \leq t_k \leq \min\{r - k + 1, n_k\}\ , \\ & \max\{1, r - \sum_{h=1}^{k-2} n_j - t_k\} \leq t_{k-1} \leq \min\{r - k + 2 - t_k, n_{k-1}\}, \\ & \cdots, \\ & \max\{1, r - n_1 - \sum_{h=3}^{k} t_h\} \leq t_2 \leq \min\{r - \sum_{h=3}^{k} t_h, n_2\}\Big\}.\end{aligned}$$

Then, we have

$$P(S|D = d) \quad = \quad \sum_{\mathbf{t} \in \mathbf{T}_r} P(\mathbf{M}_r = \mathbf{t}). \tag{4.2.6}$$

Theorem 9 *The joint probability mass function of* $\boldsymbol{\delta}$ *under the Type-II HCS is as follows:*

(a) *For* $0 \leq d \leq r - 1$,

$$P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}|D = d) = \frac{c_r}{P(D = d)} \prod_{h=1}^{k} \theta_h^{-M_r(h)} \sum_{l'=0}^{d} \varphi_{l'd}, \tag{4.2.7}$$

where

$$\begin{aligned}\varphi_{l'd} &= \left\{\prod_{\substack{j=0\\j\neq l'}}^{d}\sum_{h=1}^{k}\frac{(M_{d-l'}(h)-M_{d-j}(h))}{\theta_h}\right\}^{-1}\left\{\prod_{j=d+1}^{r}\sum_{h=1}^{k}\frac{n_h-M_{j-1}(h)}{\theta_h}\right\}^{-1}\\ &\quad\times\exp\left\{-T(\sum_{h=1}^{k}\frac{n_h-M_{d-l'}(h)}{\theta_h})\right\},\ 0\leq l'\leq d; \qquad (4.2.8)\end{aligned}$$

(b) *For* $d \geq r$*,*

$$P(\boldsymbol{\delta}=\tilde{\boldsymbol{\delta}}|D=d)=\frac{c_d}{P(D=d)}\prod_{h=1}^{k}\theta_h^{-M_d(h)}\sum_{l'=0}^{d}\psi_{l'd}, \qquad (4.2.9)$$

where

$$\begin{aligned}\psi_{l'd} &= \left\{\prod_{\substack{j=0\\j\neq l'}}^{d}\sum_{h=1}^{k}\frac{(M_{d-l'}(h)-M_{d-j}(h))}{\theta_h}\right\}^{-1}\\ &\quad\times\exp\left\{-T\sum_{h=1}^{k}\frac{n_h-M_{d-l'}(h)}{\theta_h}\right\},\ 0\leq l'\leq d; \qquad (4.2.10)\end{aligned}$$

(c) *For* $\mathbf{t}=(t_1,t_2,\cdots,t_k)$*, such that* $\sum_{j=1}^{k}t_j=d$ *and* $t_j\geq 1$ *for all* j*, let* $\boldsymbol{\delta}$ *be as defined in (4.1.1),* $M_d(h)=\sum_{i=1}^{d}\delta_i(h)$*, and*

$$Q_d^*(\mathbf{t})=\left\{\tilde{\boldsymbol{\delta}}(\mathbf{h})=(\tilde{\delta}_1,\tilde{\delta}_2,\cdots,\tilde{\delta}_r)\ \textit{such that}\ M_d(h)=t_h\ \textit{for}\ 1\leq h\leq k\right\}.$$

Then,

$$P(\mathbf{M}_d = \mathbf{t}) = \sum_{\tilde{\boldsymbol{\delta}} \in Q_d^*(\mathbf{t})} P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}); \tag{4.2.11}$$

(d) *Let* $\mathbf{T}_d$ *be the set as defined in Lemma 3, Then,*

$$P(S|D = d) = \begin{cases} \sum_{\mathbf{t} \in \mathbf{T}_r} P(\mathbf{M}_r = \mathbf{t}) & \text{if } d < r, \\ \sum_{\mathbf{t} \in \mathbf{T}_d} P(\mathbf{M}_d = \mathbf{t}) & \text{if } d \geq r. \end{cases} \tag{4.2.12}$$

As we can see, for some $\mathbf{t}$*, the probability* $P(\mathbf{M}_d = \mathbf{t}) = 0$*, when* $\mathbf{t} = (t_1, t_2, \cdots, t_k)$ *does not belong to* $\mathbf{T}_d$*.*

Proof (a) When $T < w_r$, let d be the integer such that $w_d < T < w_{d+1}$, from (4.2.2), we have the joint density of $(\boldsymbol{\delta}, \mathbf{W})$ as

$$f(\boldsymbol{\delta}, \mathbf{w}) = c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \exp\left(-\sum_{i=1}^{r-1} a_i w_i - b_r w_r \right), \tag{4.2.13}$$

$\{0 < w_1 < w_2 < \cdots < w_d < T < w_{d+1} < \cdots < w_r < \infty\}$, where $a_i = \sum_{h=1}^{k} \frac{\delta_i(h)}{\theta_h}$ and $b_r = \sum_{h=1}^{k} \frac{\delta_r(h) + n_h - M_r(h)}{\theta_h}$. Upon integrating out $w_1, w_2, \cdots, w_r$ over $\{0 < w_1 < \cdots < w_d < T < w_{d+1} < \cdots < w_r < \infty\}$ in (4.2.13), by using Lemma 7, we obtain the joint probability mass function of $\boldsymbol{\delta}$ as

$$
\begin{aligned}
& P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}} | D = d) P(D = d) \\
= \; & c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \int_{0 \le w_1 \le w_2 \le \cdots \le w_d < T} \exp\left(- \sum_{i=1}^{d} a_i w_i \right) dw_1 \cdots dw_d \\
& \times \int_{T \le w_{d+1} \le w_{d+2} \le \cdots \le w_r < \infty} \exp\left(- \sum_{i=d+1}^{r-1} a_i w_i - b_r w_r \right) dw_{d+1} \cdots dw_r \\
= \; & c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \sum_{l'=0}^{d} c_{l'}^{(d)} \exp\left(-T \sum_{i=d-l'+1}^{d} a_i \right) \tilde{c}_d^{(r)} \exp\left(-T(b_r + \sum_{i=d+1}^{r-1} a_i) \right) \\
= \; & c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \sum_{l'=0}^{d} c_{l'}^{(d)} \tilde{c}_d^{(r)} \exp\left(-T(b_r + \sum_{i=d-l'+1}^{r-1} a_i) \right),
\end{aligned}
$$

where

$$
\begin{aligned}
c_0^{(d)} &= \prod_{j=1}^{d} \frac{1}{\sum_{i=j}^{d} a_i}, \\
c_{l'}^{(d)} &= \frac{(-1)^{l'}}{\prod_{j=0}^{l'-1} \sum_{i=d-l'+1}^{d-j} a_i} \prod_{j=1}^{d-l'} \frac{1}{\sum_{i=j}^{d-l'} a_i}, \text{ for } 1 \le l' \le d, \\
\tilde{c}_d^{(r)} &= \frac{1}{b_r} \prod_{i=d+1}^{r-1} \frac{1}{b_r + \sum_{j=i}^{r-1} a_j},
\end{aligned}
$$

which are shown in Lemma 7 in the Appendix.

Observe that for $\tilde{\boldsymbol{\delta}}$, $M_r(h, \tilde{\boldsymbol{\delta}}) = \sum_{i=1}^{r} \tilde{\delta}_i(h)$ and

$$
\begin{aligned}
b_r + \sum_{i=j}^{r-1} a_i &= \sum_{h=1}^{k} \frac{\tilde{\delta}_r(h) + n_h - M_r(h)}{\theta_h} + \sum_{i=j}^{r-1} \sum_{h=1}^{k} \frac{\tilde{\delta}_i(h)}{\theta_h} \\
&= \sum_{h=1}^{k} \frac{n_h - M_{j-1}(h, \tilde{\boldsymbol{\delta}})}{\theta_h},
\end{aligned}
$$

$$\sum_{i=d-l'+1}^{d-j} a_i = \sum_{i=d-l'+1}^{d-j} \sum_{h=1}^{k} \frac{\tilde{\delta}_i(h)}{\theta_h} = \sum_{h=1}^{k} \frac{M_{d-j}(h,\tilde{\boldsymbol{\delta}}) - M_{d-l'}(h,\tilde{\boldsymbol{\delta}})}{\theta_h},$$

where $M_i(h,\tilde{\boldsymbol{\delta}}) = \sum_{l=1}^{i} \tilde{\delta}_l(h)$ depends on $\tilde{\boldsymbol{\delta}}$, and yet we denote it by $M_i(h)$, for simplicity, and set $M_0(h) = 0$. Thus, we have

$$c_0^{(d)} = \prod_{j=1}^{d} \frac{1}{\sum_{h=1}^{k} \frac{M_d(h)-M_{j-1}(h)}{\theta_h}}.$$

For $1 \leq l' \leq d$,

$$\begin{aligned}
c_{l'}^{(d)} &= \frac{(-1)^{l'}}{\prod_{j=0}^{l'-1} \sum_{i=d-l'+1}^{d-j} a_i} \prod_{j=1}^{d-l'} \frac{1}{\sum_{i=j}^{d-l'} a_i} \\
&= \frac{(-1)^{l'}}{\prod_{j=0}^{l'-1} \sum_{h=1}^{k} \frac{(M_{d-j}(h)-M_{d-l'}(h))}{\theta_h}} \prod_{j=1}^{d-l'} \frac{1}{\sum_{h=1}^{k} \frac{(M_{d-l'}(h)-M_{j-1}(h))}{\theta_h}}, \\
&= \prod_{\substack{j=0 \\ j\neq l'}}^{d} \frac{1}{\sum_{h=1}^{k} \frac{(M_{d-l'}(h)-M_{d-j}(h))}{\theta_h}}, \\
\tilde{c}_d^{(r)} &= \frac{1}{b_r} \prod_{j=d+1}^{r-1} \frac{1}{b_r + \sum_{i=j}^{r-1} a_i} \\
&= \prod_{j=d+1}^{r} \frac{1}{\sum_{h=1}^{k} \frac{n_h - M_{j-1}(h)}{\theta_h}}.
\end{aligned}$$

Then,

$$P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}} | D = d) = \frac{c_r}{P(D=d)} \prod_{h=1}^{k} \theta_h^{-M_r(h)} \sum_{l'=0}^{d} c_{l'}^{(d)} \tilde{c}_d^{(r)} \mathrm{e}^{-T \sum_{h=1}^{k} \frac{n_h - M_{d-l'}(h)}{\theta_h}},$$

from which (4.2.7) is obtained.

(b) When $T > w_r$, from (4.2.3), d is the integer such that $w_d < T < w_{d+1}$. We have the joint density of $(\boldsymbol{\delta}, \mathbf{W})$ as

$$\begin{aligned} f(\boldsymbol{\delta}, \mathbf{w}) &= c_d \prod_{h=1}^{k} \theta_h^{-M_d(h)} \exp\left\{-\sum_{i=1}^{d} a_i w_i\right\} \\ & \exp\left\{-\sum_{h=1}^{k} \frac{T(n_h - M_d(h))}{\theta_h}\right\} \end{aligned} \tag{4.2.14}$$

with $a_i = \sum_{h=1}^{k} \frac{\delta_i(h)}{\theta_h}$, for $1 \le i \le d$.

Upon integrating out $w_1, w_2, \cdots, w_d$ over $\{0 \le w_1 \le w_2 \le \cdots \le w_d < T\}$ in (4.2.14) by using Lemma 7, we obtain the joint probability mass function of $\boldsymbol{\delta}$ as follows:

$$\begin{aligned} & P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}} | D = d) P(D = d) \\ = \; & c_d \prod_{h=1}^{k} \theta_h^{-M_d(h)} \exp\left\{-\sum_{h=1}^{k} \frac{T(n_h - M_d(h))}{\theta_h}\right\} \\ & \times \int_0^T \cdots \int_0^{W_2} \exp(-\sum_{i=1}^{d} a_i w_i) dw_1 \cdots dw_d \\ = \; & c_d \prod_{h=1}^{k} \theta_h^{-M_d(h)} \exp\left\{-\sum_{h=1}^{k} \frac{T(n_h - M_d(h))}{\theta_h}\right\} \sum_{l'=0}^{d} c_{l'}^{(d)} \mathrm{e}^{-\sum_{i=d-l'+1}^{d} a_i T} \\ = \; & c_d \prod_{h=1}^{k} \theta_h^{-M_d(h)} \sum_{l'=0}^{d} c_{l'}^{(d)} \exp\left\{-T \sum_{h=1}^{k} \frac{n_h - M_{d-l'}(h)}{\theta_h}\right\} \end{aligned}$$

where $c_{l'}^{(d)} = \prod_{\substack{j=0 \\ j \neq l'}}^{d} \frac{1}{\sum_{h=1}^{k} \frac{(M_{d-l'}(h) - M_{d-j}(h))}{\theta_h}}$, $0 \le j \le d$.

Thus, (4.2.9) is obtained.

The results in (c) and (d) follow immediately from Lemma 3, Lemma 4 and Part (b).

Theorem 10 *Conditional on S, the moment generating function (mgf) of $\hat{\theta}_l$ (for $1 \leq l \leq k$) is given by*

$$
\begin{aligned}
M_{\hat{\theta}_l|S}(t) \;=\; & \sum_{d=0}^{r-1} \frac{1}{P(\prod_{h=1}^{k} M_r(h) \neq 0 | D = d)} \sum_{\mathbf{t} \in \mathbf{T}_r} \sum_{\tilde{\boldsymbol{\delta}} \in Q_r^*(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \sum_{l'=0}^{d} \varphi_{l'd} \\
& \times \prod_{j=0; j \neq l'}^{d} \frac{1}{1 - t\beta_{l,l'j}^{(d)}} \prod_{j=d+1}^{r} \frac{1}{1 - t\beta_{lj}^{*}} \exp\left\{ \frac{tT(n_l - M_{d-l'}(l))}{M_r(l)} \right\} \\
& + \sum_{d=r}^{N} \frac{1}{P(\prod_{h=1}^{k} M_d(h) \neq 0 | D = d)} \sum_{\mathbf{t} \in \mathbf{T}_d} \sum_{\tilde{\boldsymbol{\delta}} \in Q_d^*(\mathbf{t})} c_d \prod_{h=1}^{k} \theta_h^{-M_d(h)} \sum_{l'=0}^{d} \psi_{l'd} \\
& \times \prod_{j=0; j \neq l'}^{d} \frac{1}{1 - t\beta_{l,l'j}^{(d)}} \exp\left\{ \frac{tT(n_l - M_{d-l'}(l))}{M_d(l)} \right\}, \qquad (4.2.15)
\end{aligned}
$$

where $\varphi_{l'd}$ and $\psi_{l'd}$ are as given in (4.2.8) and (4.2.10), respectively. For $1 \leq l \leq k$, $0 \leq l' \leq d$ and $0 \leq j \leq d$,

$$
\beta_{l,l'j}^{(d)} = \frac{M_{d-l'}(l) - M_{j-1}(l)}{M_r(l) \sum_{h=1}^{k} \frac{(M_{d-l'}(h) - M_{j-1}(h))}{\theta_h}}, \qquad (4.2.16)
$$

$$
\beta_{lj}^{*} = \frac{n_l - M_{j-1}(l)}{M_r(l) \sum_{h=1}^{k} \frac{n_h - M_{j-1}(h)}{\theta_h}}. \qquad (4.2.17)
$$

Proof Conditioning on the set S and then on $\tilde{\boldsymbol{\delta}}$, we obtain

$$\begin{aligned} M_{\hat{\theta}_l|S}(t) &= E\left[\mathrm{e}^{t\hat{\theta}_l}|\prod_{h=1}^{k} M_r(h)\neq 0, \text{ or } \prod_{h=1}^{k} M_D(h)\neq 0\right] \\ &= \sum_{d=0}^{r-1} E\left(\mathrm{e}^{t\hat{\theta}_l}|D=d, \prod_{h=1}^{k} M_r(h)\neq 0\right) P(D=d) \\ &\quad + \sum_{d=r}^{N} E\left(\mathrm{e}^{t\hat{\theta}_l}|D=d, \prod_{h=1}^{k} M_D(h)\neq 0\right) P(D=d). \end{aligned} \tag{4.2.18}$$

When $d = 0, 1, 2, \cdots, r-1$, by using Lemma 7, we have

$$\begin{aligned} & E\left[\mathrm{e}^{t\hat{\theta}_l}|D=d, \prod_{h=1}^{k} M_r(h)\neq 0\right] P\left(\prod_{h=1}^{k} M_r(h)\neq 0|D=d\right) P(D=d) \\ = & \sum_{\mathbf{t}\in\mathbf{T}_r} \sum_{\tilde{\boldsymbol{\delta}}\in Q_r^*(\mathbf{t})} \int_{0\le w_1\le w_2\le\cdots\le w_d<T\le w_{d+1}\le w_{d+2}\le\cdots\le w_r<\infty} \mathrm{e}^{t\hat{\theta}_l} f(\tilde{\boldsymbol{\delta}}, \mathbf{w}) dw_1\cdots dw_r \\ = & \sum_{\mathbf{t}\in\mathbf{T}_r} \sum_{\tilde{\boldsymbol{\delta}}\in Q_r^*(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \int_{0\le w_1\le w_2\le\cdots\le w_d<T} \mathrm{e}^{-\sum_{i=1}^{d} a_i' w_i} dw_1\cdots dw_d \\ & \times \int_{T\le w_{d+1}\le w_{d+2}\le\cdots\le w_r<\infty} \mathrm{e}^{-\sum_{i=d+1}^{r-1} a_i' w_i - b_r' w_r} dw_{d+1}\cdots dw_r \\ = & \sum_{\mathbf{t}\in\mathbf{T}_r} \sum_{\tilde{\boldsymbol{\delta}}\in Q_r^*(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \left(\sum_{l'=0}^{d} b_{l'}^{(d)} \mathrm{e}^{-T\sum_{i=d-l'+1}^{d} a_i'}\right) \tilde{b}_d^{(r)} \mathrm{e}^{-T(b_r' + \sum_{i=d+1}^{r-1} a_i')} \\ = & \sum_{\mathbf{t}\in\mathbf{T}_r} \sum_{\tilde{\boldsymbol{\delta}}\in Q_r^*(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \sum_{l'=0}^{d} b_{l'}^{(d)} \tilde{b}_d^{(r)} \mathrm{e}^{-T(b_r' + \sum_{i=d-l'+1}^{r-1} a_i')}, \end{aligned}$$

where $b'_r = b_r - \frac{t}{M_r(l)}(\tilde{\delta}_r(l) + n_l - M_r(l))$, $a'_i = a_i - \frac{t\tilde{\delta}_i(l)}{M_r(l)}$,

$$
\begin{aligned}
b_0^{(d)} &= \prod_{j=1}^{d} \frac{1}{\sum_{i=j}^{d} a'_i}, \\
b_{l'}^{(d)} &= \frac{(-1)^{l'}}{\prod_{j=0}^{l'-1} \sum_{i=d-l'+1}^{d-j} a'_i} \prod_{j=1}^{d-l'} \frac{1}{\sum_{i=j}^{d-l'} a'_i}, \text{ for } 1 \leq l' \leq d, \\
\tilde{b}_d^{(r)} &= \frac{1}{b'_r} \prod_{j=d+1}^{r-1} \frac{1}{b'_r + \sum_{i=j}^{r-1} a'_i}.
\end{aligned}
$$

Now, observing that

$$
\begin{aligned}
& b'_r + \sum_{i=j}^{r-1} a'_i \\
= \ & b_r + \sum_{i=j}^{r-1} a_i - \frac{t(\tilde{\delta}_r(l) + n_l - M_r(l))}{M_r(l)} - \sum_{i=j}^{r-1} \frac{t\tilde{\delta}_i(l)}{M_r(l)} \\
= \ & \sum_{h=1}^{k} \frac{n_h - M_{j-1}(h)}{\theta_h} - \frac{t(n_l - M_{j-1}(l))}{M_r(l)},
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i=j}^{d} a'_i &= \sum_{i=j}^{d} a_i - \sum_{i=j}^{d} \frac{t\tilde{\delta}_j(l)}{M_r(l)} \\
&= \sum_{h=1}^{k} \frac{M_d(h) - M_{j-1}(h)}{\theta_h} - \frac{t(M_d(l) - M_{j-1}(l))}{M_r(l)},
\end{aligned}
$$

we obtain

$$
b_0^{(d)} = \prod_{j=1}^{d} \frac{1}{\sum_{h=1}^{k} \frac{M_d(h) - M_{j-1}(h)}{\theta_h}} \prod_{j=1}^{d} \frac{1}{1 - t\beta_{l,0j}^{(d)}},
$$

where $\beta_{l,0j}^{(d)} = \frac{M_d(l)-M_{j-1}(l)}{M_r(l)\sum_{h=1}^{k}\frac{M_d(h)-M_{j-1}(h)}{\theta_h}}$.

For $1 \leq l' \leq r$, we also have

$$
\begin{aligned}
b_{l'}^{(d)} &= \frac{(-1)^{l'}}{\prod_{j=0}^{l'-1}\sum_{i=d-l'+1}^{d-j} a_i'} \prod_{j=1}^{d-l'} \frac{1}{\sum_{i=j}^{d-l'} a_i'} \\
&= \frac{(-1)^{l'}}{\prod_{j=0}^{l'-1}\left(\sum_{h=1}^{k}\frac{\left(M_{d-j}(h)-M_{d-l'}(h)\right)}{\theta_h} - \frac{t\left(M_{d-j}(l)-M_{d-l'}(l)\right)}{M_r(l)}\right)} \\
&\quad \times \prod_{j=1}^{d-l'} \frac{1}{\sum_{h=1}^{k}\frac{\left(M_{d-l'}(h)-M_{j-1}(h)\right)}{\theta_h} - \frac{t\left(M_{d-l'}(l)-M_{j-1}(l)\right)}{M_r(l)}}, \\
&= \frac{1}{\prod\limits_{\substack{j=0\\ j\neq l'}}^{d}\sum_{h=1}^{k}\frac{\left(M_{d-l'}(h)-M_{d-j}(h)\right)}{\theta_h}} \prod_{\substack{j=0\\ j\neq l'}}^{d} \frac{1}{1-t\beta_{l,l'j}^{(d)}}, \\
\tilde{b}_d^{(r)} &= \frac{1}{b_r'} \prod_{j=d+1}^{r-1} \frac{1}{b_r' + \sum_{i=j}^{r-1} a_i'} \\
&= \frac{1}{\prod_{j=d+1}^{r}\left(\sum_{h=1}^{k}\frac{n_h-M_{j-1}(h)}{\theta_h} - \frac{t(n_l-M_{j-1}(l))}{M_r(l)}\right)} \\
&= \frac{1}{\prod_{j=d+1}^{r}\sum_{h=1}^{k}\frac{n_h-M_{j-1}(h)}{\theta_h}} \prod_{j=d+1}^{r} \frac{1}{1-t\beta_{lj}^{*}},
\end{aligned}
$$

where $\beta_{l,l'j}^{(d)} = \frac{M_{d-l'}(l)-M_{d-j}(l)}{M_r(l)\sum_{h=1}^{k}\frac{(M_{d-l'}(h)-M_{d-j}(h))}{\theta_h}}$ and $\beta_{lj}^{*} = \frac{n_l-M_{j-1}(l)}{M_r(l)\sum_{h=1}^{k}\frac{n_h-M_{j-1}(h)}{\theta_h}}$. Thus, when

$d = 0, 1, \cdots, r-1$,

$$
\begin{aligned}
& E\left[\mathrm{e}^{t\hat{\theta}_l} | D = d, \prod_{h=1}^{k} M_r(h) \neq 0\right] P\left(\prod_{h=1}^{k} M_r(h) \neq 0 | D = d\right) P(D = d) \\
= & \sum_{\mathbf{t}\in\mathbf{T}_r} \sum_{\tilde{\boldsymbol{\delta}}\in Q_r^*(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \sum_{l'=0}^{d} \frac{1}{\prod\limits_{\substack{j=0\\ j\neq l'}}^{d} \sum_{h=1}^{k} \frac{\left(M_{d-l'}(h) - M_{d-j}(h)\right)}{\theta_h}} \prod_{\substack{j=0\\ j\neq l'}}^{d} \frac{1}{1 - t\beta_{l,l'j}^{(d)}} \\
& \times \frac{1}{\prod_{j=d+1}^{r} \sum_{h=1}^{k} \frac{n_h - M_{j-1}(h)}{\theta_h}} \prod_{j=d+1}^{r} \frac{1}{1 - t\beta_{lj}^{*}} \\
& \times \exp\left\{-T\left(\sum_{h=1}^{k} \frac{n_h - M_{d-l'}(h)}{\theta_h} - \frac{t(n_l - M_{d-l'}(l))}{M_r(l)}\right)\right\} \\
= & \sum_{\mathbf{t}\in\mathbf{T}_r} \sum_{\tilde{\boldsymbol{\delta}}\in Q_r^*(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \sum_{l'=0}^{d} \varphi_{l'd} \prod_{\substack{j=0\\ j\neq l'}}^{d} \frac{1}{1 - t\beta_{l,l'j}^{(d)}} \prod_{j=d+1}^{r} \frac{1}{1 - t\beta_{lj}^{*}} \\
& \times \exp\left\{\frac{tT\left(n_l - M_{d-l'}(l)\right)}{M_r(l)}\right\},
\end{aligned}
\tag{4.2.19}
$$

where $\varphi_{l'd}$ is as given in (4.2.8).

Similarly, when $d = r, r+1, \cdots, N$,

$$
\begin{aligned}
& E\left[\mathrm{e}^{t\hat{\theta}_l} | D = d, \prod_{h=1}^{k} M_r(h) \neq 0\right] P\left(\prod_{h=1}^{k} M_r(h) \neq 0 | D = d\right) P(D = d) \\
= & \sum_{\mathbf{t}\in\mathbf{T}_d} \sum_{\tilde{\boldsymbol{\delta}}\in Q_d^*(\mathbf{t})} \int_{0\leq w_1\leq w_2\leq\cdots\leq w_d<T} \mathrm{e}^{t\hat{\theta}_l} f(\tilde{\boldsymbol{\delta}}, \mathbf{W}) dw_1 \cdots dw_r \\
= & \sum_{\mathbf{t}\in\mathbf{T}_d} \sum_{\tilde{\boldsymbol{\delta}}\in Q_d^*(\mathbf{t})} c_d \prod_{h=1}^{k} \theta_h^{-M_d(h)} \exp\left\{-\sum_{h=1}^{k} \frac{T\left(n_h - M_d(h)\right)}{\theta_h} + \frac{tT\left(n_l - M_d(l)\right)}{M_d(l)}\right\} \\
& \times \int_{0\leq w_1\leq w_2\leq\cdots\leq w_d<T} \exp\left\{-\sum_{i=1}^{d} a_i' w_i\right\} dw_1 \cdots dw_d,
\end{aligned}
$$

where $a_i' = a_i - \frac{t\tilde{\delta}_i(l)}{M_r(l)}$. Then, by using Lemma 7, we get

$$
\begin{aligned}
& E\left[\mathrm{e}^{t\hat{\theta}_l}|D=d, \prod_{h=1}^{k} M_r(h) \neq 0\right] P\left(\prod_{h=1}^{k} M_r(h) \neq 0 | D=d\right) P(D=d) \\
= & \sum_{\mathbf{t}\in\mathbf{T}_d} \sum_{\tilde{\boldsymbol{\delta}}\in Q_d^*(\mathbf{t})} c_d \prod_{h=1}^{k} \theta_h^{-M_d(h)} \left(\sum_{l'=0}^{d} b_{l'}^{(d)} \mathrm{e}^{-T\sum_{i=d-l'+1}^{d} a_i'}\right) \\
& \times \exp\left\{-\sum_{h=1}^{k} \frac{T\left(n_h - M_d(h)\right)}{\theta_h} + \frac{tT\left(n_l - M_d(l)\right)}{M_d(l)}\right\} \\
= & \sum_{\mathbf{t}\in\mathbf{T}_d} \sum_{\tilde{\boldsymbol{\delta}}\in Q_d^*(\mathbf{t})} c_d \prod_{h=1}^{k} \theta_h^{-M_d(h)} \sum_{l'=0}^{d} b_{l'}^{(d)} \\
& \times \exp\left\{-T\sum_{h=1}^{k} \frac{n_h - M_{d-l'}(h)}{\theta_h} + \frac{tT\left(n_l - M_{d-l'}(l)\right)}{M_d(l)}\right\} \\
= & \sum_{\mathbf{t}\in\mathbf{T}_d} \sum_{\tilde{\boldsymbol{\delta}}\in Q_d^*(\mathbf{t})} c_d \prod_{h=1}^{k} \theta_h^{-M_d(h)} \sum_{l'=0}^{d} \psi_{l'd} \prod_{\substack{j=0\\ j\neq l'}}^{d} \frac{1}{1-t\beta_{l,l'j}^{(d)}} \\
& \times \exp\left\{\frac{tT\left(n_l - M_{d-l'}(l)\right)}{M_d(l)}\right\}. \qquad (4.2.20)
\end{aligned}
$$

Upon substituting from (4.2.19) and (4.2.20) into (4.2.18), we obtain the expression in (4.2.15) as required.

Remark 10 *For fixed l, l' and $\tilde{\boldsymbol{\delta}}$, some $\beta_{l,l'i}^{(d)}$'s and β_{li}^*'s may be the same. In this case, for $d < r$, we resort the $\beta_{l,l'i}^{(d)}$ and β_{li}^* values and denote them by $\{\beta_{l,l'i}^{(1)}\}_{i=1}^{r'}$, and assume that there are r' distinct values with $\alpha_{l,l'i}^{(1)}$ of the $\beta_{l,l'i}^{(1)}$'s being equal, with $\sum_{i=1}^{r'} \alpha_{l,l'i}^{(1)} = r$. Consequently, the term $\prod_{j=0;j\neq l'}^{d}\left(1-t\beta_{l,l'j}^{(d)}\right)^{-1} \times \prod_{j=d+1}^{r}\left(1-t\beta_{lj}^{*}\right)^{-1}$ can be rewritten as $\prod_{j=1}^{r'}\left(1-t\beta_{l,l'j}^{(1)}\right)^{-\alpha_{l,l'j}^{(1)}}$; For $d \geq r$, similarly, $\prod_{j=0;j\neq l'}^{d}\left(1-t\beta_{l,l'j}^{(d)}\right)^{-1}$ can be rewritten as $\prod_{j=1}^{r''}\left(1-t\beta_{l,l'j}^{(2)}\right)^{-\alpha_{l,l'j}^{(2)}}$ with $\sum_{i=1}^{r''} \alpha_{l,l'i}^{(2)} = d$, and thus, the conditional mgf* *in Theorem 10 can be expressed as follows:*

$$\begin{aligned} M_{\hat{\theta}_l|S}(t) &= \sum_{d=0}^{r-1} \frac{1}{P(\prod_{h=1}^{k} M_r(h) \neq 0 | D = d)} \sum_{\mathbf{t}\in\mathbf{T}_r} \sum_{\tilde{\boldsymbol{\delta}}\in Q_r^*(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \sum_{l'=0}^{d} \varphi_{l'd} \\ &\quad \times \prod_{j=1}^{r'} \left(1 - t\beta_{l,l'j}^{(1)}\right)^{-\alpha_{l,l'j}^{(1)}} \exp\left\{\frac{tT(n_l - M_{d-l'}(l))}{M_r(l)}\right\} \\ &\quad + \sum_{d=r}^{N} \frac{1}{P(\prod_{h=1}^{k} M_d(h) \neq 0 | D = d)} \sum_{\mathbf{t}\in\mathbf{T}_d} \sum_{\tilde{\boldsymbol{\delta}}\in Q_d^*(\mathbf{t})} c_d \prod_{h=1}^{k} \theta_h^{-M_d(h)} \sum_{l'=0}^{d} \psi_{l'd} \\ &\quad \times \prod_{j=1}^{r''} \left(1 - t\beta_{l,l'j}^{(2)}\right)^{-\alpha_{l,l'j}^{(2)}} \exp\left\{\frac{tT(n_l - M_{d-l'}(l))}{M_d(l)}\right\} \qquad (4.2.21) \end{aligned}$$

Remark 11

1. $(1 - ct)^{-\alpha}$ *is the mgf of the gamma* $G(\alpha, c)$ *distribution with scale parameter* c *and shape paprameter* α*;*

2. e^{ct} *is the mgf of the degenerate distribution localized at the point* c*.*

Theorem 11 *Conditional on the set* S*, the pdf of* $\hat{\theta}_l$ *(for* $1 \leq l \leq k$*) is given by*

$$\begin{aligned} f_{\hat{\theta}_l|S}(x) &= \sum_{d=0}^{r-1} \frac{1}{P(\prod_{h=1}^{k} M_r(h) \neq 0 | D = d)} \sum_{\mathbf{t}\in\mathbf{T}_r} \sum_{\tilde{\boldsymbol{\delta}}\in Q_r^*(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \sum_{l'=0}^{d} \varphi_{l'd} g_{Y_{l'd}^{(1)}} \\ &\quad + \sum_{d=r}^{N} \frac{1}{P(\prod_{h=1}^{k} M_d(h) \neq 0 | D = d)} \sum_{\mathbf{t}\in\mathbf{T}_d} \\ &\quad \times \sum_{\tilde{\boldsymbol{\delta}}\in Q_d^*(\mathbf{t})} c_d \prod_{h=1}^{k} \theta_h^{-M_d(h)} \sum_{l'=0}^{d} \psi_{l'd} g_{Y_{l'd}^{(2)}}, \qquad (4.2.22) \end{aligned}$$

where $Y_{l'd}^{(1)} \stackrel{d}{=} \sum_{i=1}^{r'} Y_{l'd,i}^{(1)} + Y_{l'r}$ *with* $Y_{l'd,i}^{(1)}$ *being independent random variables having gamma* $G(\alpha_{l,l'i}^{(1)}, \beta_{l,l'i}^{(1)})$ *distributions with scale parameters* $\beta_{l,l'i}^{(1)}$ *and shape parameters*

$\alpha^{(1)}_{l,l'i}$, $Y_{l'r}$ *being independent random variables having degenerate distribution localized at the point* $\frac{T(n_l - M_{d-l'}(l))}{M_r(l)}$, *and* $g_{Y^{(1)}_{l'd}}(x)$ *is the pdf of* $Y^{(1)}_{l'd}$; $Y^{(2)}_{l'd} \overset{d}{=} \sum_{i=1}^{r''} Y^{(2)}_{l'd,i} + Y_{l'd}$ *with* $Y^{(2)}_{l'd,i}$ *being independent random variables having gamma* $G(\alpha^{(2)}_{l,l'i}, \beta^{(2)}_{l,l'i})$ *distributions with scale parameters* $\beta^{(2)}_{l,l'i}$ *and shape parameters* $\alpha^{(2)}_{l,l'i}$, $Y_{l'd}$ *being independent random variables having degenerate distribution localized at the point* $\frac{T(n_l - M_{d-l'}(l))}{M_d(l)}$, *and* $g_{Y^{(2)}_{l'd}}(x)$ *is the pdf of* $Y^{(2)}_{l'd}$.

Corollary 5 *From (4.2.15), we immediately obtain the expressions for the first two moments of* $\hat{\theta}_l$ *as follows:*

$$
\begin{aligned}
E(\hat{\theta}_l|S) \;=\; & \sum_{d=0}^{r-1} \frac{1}{P(\prod_{h=1}^{k} M_r(h) \neq 0 | D = d)} \sum_{\mathbf{t} \in \mathbf{T}_r} \sum_{\tilde{\boldsymbol{\delta}} \in Q_r^*(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \sum_{l'=0}^{d} \varphi_{l'd} \\
& \times \left(\sum_{\substack{j=0 \\ j \neq l'}}^{d} \beta^{(d)}_{l,l'j} + \sum_{j=d+1}^{r} \beta^*_{lj} + \frac{T(n_l - M_{d-l'}(l))}{M_r(l)} \right) \\
& + \sum_{d=r}^{N} \frac{1}{P(\prod_{h=1}^{k} M_d(h) \neq 0 | D = d)} \sum_{\mathbf{t} \in \mathbf{T}_d} \sum_{\tilde{\boldsymbol{\delta}} \in Q_d^*(\mathbf{t})} c_d \prod_{h=1}^{k} \theta_h^{-M_d(h)} \sum_{l'=0}^{d} \psi_{l'd} \\
& \times \left(\sum_{\substack{j=0 \\ j \neq l'}}^{d} \beta^{(d)}_{l,l'j} + \frac{T(n_l - M_{d-l'}(l))}{M_d(l)} \right)
\end{aligned}
$$

and

$$\begin{aligned}
E(\hat{\theta}_l^2|S) &= \sum_{d=0}^{r-1} \frac{1}{P(\prod_{h=1}^{k} M_r(h) \neq 0|D=d)} \sum_{\mathbf{t}\in\mathbf{T}_r} \sum_{\tilde{\boldsymbol{\delta}}\in Q_r^*(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \sum_{l'=0}^{d} \varphi_{l'd} \\
&\quad \times \left\{ \left(\sum_{\substack{j=0\\ j\neq l'}}^{d} \beta_{l,l'j}^{(d)} + \sum_{j=d+1}^{r} \beta_{lj}^{*} + \frac{T(n_l - M_{d-l'}(l))}{M_r(l)} \right)^2 \right. \\
&\quad \left. + \sum_{\substack{j=0\\ j\neq l'}}^{d} \left(\beta_{l,l'j}^{(d)}\right)^2 + \sum_{j=d+1}^{r} \left(\beta_{lj}^{*}\right)^2 \right\} \\
&\quad + \sum_{d=r}^{N} \frac{1}{P(\prod_{h=1}^{k} M_d(h) \neq 0|D=d)} \sum_{\mathbf{t}\in\mathbf{T}_d} \sum_{\tilde{\boldsymbol{\delta}}\in Q_d^*(\mathbf{t})} c_d \prod_{h=1}^{k} \theta_h^{-M_d(h)} \sum_{l'=0}^{d} \psi_{l'd} \\
&\quad \times \left\{ \left(\sum_{\substack{j=0\\ j\neq l'}}^{d} \beta_{l,l'j}^{(d)} + \frac{T(n_l - M_{d-l'}(l))}{M_d(l)} \right)^2 + \sum_{\substack{j=0\\ j\neq l'}}^{d} \left(\beta_{l,l'j}^{(d)}\right)^2 \right\}.
\end{aligned}$$

Then, $\mathrm{Var}(\hat{\theta}_l)$ and $\mathrm{MSE}(\hat{\theta}_l)$ can be readily obtained from these two expressions.

It is convenient to rewrite the conditional mgf of $\hat{\theta}_l$ (for $l = 1, 2, \cdots, k$) in Theorem 10 as

$$\begin{aligned}
M_{\hat{\theta}_l|S}(t) &= \sum_{d=0}^{r-1} \frac{1}{P(\prod_{h=1}^{k} M_r(h) \neq 0|D=d)} \sum_{\mathbf{t}\in\mathbf{T}_r} \sum_{\tilde{\boldsymbol{\delta}}\in Q_r^*(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \sum_{l'=0}^{d} \varphi_{l'd} \\
&\quad \times \sum_{j=1}^{r'} \sum_{j'=0}^{\alpha_{l,l'j}^{(1)}} A_{l,l'jj'}^{(1)} \left(1 - t\beta_{l,l'j}^{(1)}\right)^{-j'} \exp\left\{ \frac{tT(n_l - M_{d-l'}(l))}{M_r(l)} \right\} \\
&\quad + \sum_{d=r}^{N} \frac{1}{P(\prod_{h=1}^{k} M_d(h) \neq 0|D=d)} \sum_{\mathbf{t}\in\mathbf{T}_d} \sum_{\tilde{\boldsymbol{\delta}}\in Q_d^*(\mathbf{t})} c_d \prod_{h=1}^{k} \theta_h^{-M_d(h)} \sum_{l'=0}^{d} \psi_{l'd} \\
&\quad \times \sum_{j=1}^{r''} \sum_{j'=0}^{\alpha_{l,l'j}^{(2)}} A_{l,l'jj'}^{(2)} \left(1 - t\beta_{l,l'j}^{(2)}\right)^{-j'} \exp\left\{ \frac{tT(n_l - M_{d-l'}(l))}{M_d(l)} \right\}, \qquad (4.2.23)
\end{aligned}$$

where $A^{(1)}_{l,l'jj'}$'s are coefficients obtained by writing the product $\prod_{j=1}^{r'}\left(1-t\beta^{(1)}_{l,l'j}\right)^{-\alpha^{(1)}_{l,l'j}}$ in the partial fraction form $\sum_{j=1}^{r'}\sum_{j'=1}^{\alpha^{(1)}_{l,l'j}} A^{(1)}_{l,l'jj'}\left(1-t\beta^{(1)}_{l,l'j}\right)^{-j'}$ and $A^{(2)}_{l,l'jj'}$'s are coefficients obtained by writing the product $\prod_{j=1}^{r''}\left(1-t\beta^{(2)}_{l,l'j}\right)^{-\alpha^{(2)}_{l,l'j}}$ in the partial fraction form $\sum_{j=1}^{r''}\sum_{j'=1}^{\alpha^{(2)}_{l,l'j}} A^{(2)}_{l,l'jj'}\left(1-t\beta^{(2)}_{l,l'j}\right)^{-j'}$, and the coefficients $A^{(1)}_{l,l'jj'}, A^{(2)}_{l,l'jj'}$ can be readily determined by the use of Lemma 5 in Appendix. Since $(1-ct)^{-j}\mathrm{e}^{At}$ is the mgf of the random variable $X+A$, where X has the gamma distribution with scale parameter c and shape parameter j, we readily obtain the tail probability of $\hat{\theta}_l$ (for $l=1,2,\cdots,k$) from the above expression as

$$
\begin{aligned}
& P(\hat{\theta}_l > b|S) \\
= \; & \sum_{d=0}^{r-1}\frac{1}{P(\prod_{h=1}^{k} M_r(h)\neq 0|D=d)}\sum_{\mathbf{t}\in\mathbf{T}_r}\sum_{\tilde{\boldsymbol{\delta}}\in Q_r^*(\mathbf{t})} c_r\prod_{h=1}^{k}\theta_h^{-M_r(h)}\sum_{l'=0}^{d}\varphi_{l'd} \\
& \times\sum_{j=1}^{r'}\sum_{j'=0}^{\alpha^{(1)}_{l,l'j}}\frac{A^{(1)}_{l,l'jj'}}{j'!}\Gamma\left(j', \frac{1}{\beta^{(1)}_{l,l'j}}\langle b-\frac{T(n_l-M_{d-l'}(l))}{M_r(l)}\rangle\right) \\
& +\sum_{d=r}^{N}\frac{1}{P(\prod_{h=1}^{k} M_d(h)\neq 0|D=d)}\sum_{\mathbf{t}\in\mathbf{T}_d}\sum_{\tilde{\boldsymbol{\delta}}\in Q_d^*(\mathbf{t})} c_d\prod_{h=1}^{k}\theta_h^{-M_d(h)}\sum_{l'=0}^{d}\psi_{l'd} \\
& \times\sum_{j=1}^{r''}\sum_{j'=0}^{\alpha^{(2)}_{l,l'j}}\frac{A^{(2)}_{l,l'jj'}}{j'!}\Gamma\left(j', \frac{1}{\beta^{(2)}_{l,l'j}}\langle b-\frac{T(n_l-M_{d-l'}(l))}{M_d(l)}\rangle\right),
\end{aligned}
\tag{4.2.24}
$$

where $\langle x\rangle=\max\{x,0\}$ and $\Gamma(a,z)=\int_z^{\infty}t^{a-1}\mathrm{e}^{-t}dt$ is the upper incomplete gamma function.

We shall assume that $P(\hat{\theta}_l > b|S)$ is an increasing function of θ_l when all other θ_j's are fixed for $j\neq l$. This assumption guarantees the invertibility of the pivotal quantities, and it has been verified to be true in this case through extensive computations under various setting. It should be mentioned that this approach has been used by a number of authors for constructing exact confidence intervals in a variety of contexts; see, for example, Childs et al. (2003) and Balakrishnan et al. (2007). We then have a $100(1-\alpha)\%$ lower confidence bound for θ_l as θ_{lL}, where θ_{lL} is such that $P_{\theta_{lL}}(\hat{\theta}_l > \hat{\theta}_{l,obs}|S) = \alpha$ with $\hat{\theta}_{l,obs}$ being the observed value of θ_l. Also, a $100(1-\alpha)\%$ confidence interval for θ_l is $(\theta_{lL}, \theta_{lU})$, where θ_{lL} and θ_{lU} are determined by $P_{\theta_{lL}}(\hat{\theta}_l > \hat{\theta}_{l,obs}|S) = \frac{\alpha}{2}$ and $P_{\theta_{lU}}(\hat{\theta}_l > \hat{\theta}_{l,obs}|S) = 1 - \frac{\alpha}{2}$.

By performing the same steps as done in the case of conditional marginal mgf, we can derive the conditional joint mgf of $(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k)$ as follows.

Theorem 12 *Conditional on the set S, the joint mgf of $(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k)$ is given by*

$$
\begin{aligned}
& M_{\hat{\theta}_1,\hat{\theta}_2,\cdots,\hat{\theta}_k|S}(t_1,\cdots,t_k) \\
= & \sum_{d=0}^{r-1} \frac{1}{P(\prod_{h=1}^{k} M_r(h) \neq 0 | D=d)} \sum_{\mathbf{t}\in\mathbf{T}_r} \sum_{\tilde{\boldsymbol{\delta}}\in Q_r^*(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \sum_{l'=0}^{d} \varphi_{l'd} \\
& \times \prod_{\substack{j=0\\ j\neq l'}}^{d} \frac{1}{1-\sum_{l=1}^{k} t_l \beta_{l,l'j}^{(d)}} \prod_{j=d+1}^{r} \frac{1}{1-\sum_{l=1}^{k} t_l \beta_{lj}^{*}} \exp\left\{ \frac{\sum_{l=1}^{k} t_l T(n_l - M_{d-l'}(l))}{M_r(l)} \right\} \\
& + \sum_{d=r}^{N} \frac{1}{P(\prod_{h=1}^{k} M_d(h) \neq 0 | D=d)} \sum_{\mathbf{t}\in\mathbf{T}_d} \sum_{\tilde{\boldsymbol{\delta}}\in Q_d^*(\mathbf{t})} c_d \prod_{h=1}^{k} \theta_h^{-M_d(h)} \sum_{l'=0}^{d} \psi_{l'd} \\
& \times \prod_{\substack{j=0\\ j\neq l'}}^{d} \frac{1}{1-\sum_{l=1}^{k} t_l \beta_{l,l'j}^{(d)}} \exp\left\{ \frac{\sum_{l=1}^{k} t_l T(n_l - M_{d-l'}(l))}{M_d(l)} \right\}, \qquad (4.2.25)
\end{aligned}
$$

where $\varphi_{l'd}$ and $\psi_{l'd}$ are defined by (4.2.8) and (4.2.10), respectively; also for $1 \leq l \leq k$,

$0 \le l' \le d$, *and* $0 \le j \le d$, $\beta^{(d)}_{l,l'j}$ *and* β^*_{lj} *are as defined in (4.2.16) and (4.2.17).*

Corollary 6 *From (5.1.15), we obtain* $E(\hat{\theta}_{l_1}\hat{\theta}_{l_2})$ *to be*

$$
\begin{aligned}
& E(\hat{\theta}_{l_1}\hat{\theta}_{l_2}) \\
= \;& \sum_{d=0}^{r-1} \frac{1}{P(\prod_{h=1}^{k} M_r(h) \neq 0 | D = d)} \sum_{\mathbf{t}\in\mathbf{T}_r} \sum_{\tilde{\boldsymbol{\delta}}\in Q_r^*(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \sum_{l'=0}^{d} \varphi_{l'd} \\
& \times \left\{ \left(\sum_{\substack{j=0\\ j\neq l'}}^{d} \beta^{(d)}_{l_1,l'j} + \sum_{j=d+1}^{r} \beta^*_{l_1 j} + \frac{T(n_l - M_{d-l'}(l_1))}{M_r(l_1)} \right) \right. \\
& \times \left(\sum_{\substack{j=0\\ j\neq l'}}^{d} \beta^{(d)}_{l_2,l'j} + \sum_{j=d+1}^{r} \beta^*_{l_2 j} + \frac{T(n_2 - M_{d-l'}(l_2))}{M_r(l_2)} \right) \\
& \left. + \sum_{\substack{j=0\\ j\neq l'}}^{d} \beta^{(d)}_{l_1,l'j}\beta^{(d)}_{l_2,l'j} + \sum_{j=d+1}^{r} \beta^*_{l_1 j}\beta^*_{l_2 j} \right\} \\
& + \sum_{d=r}^{N} \frac{1}{\left\{P(\prod_{h=1}^{k} M_d(h) \neq 0 | D = d)\right\}^2} \sum_{\mathbf{t}\in\mathbf{T}_d} \sum_{\tilde{\boldsymbol{\delta}}\in Q_d^*(\mathbf{t})} c_d \prod_{h=1}^{k} \theta_h^{-M_d(h)} \sum_{l'=0}^{d} \psi_{l'd} \\
& \times \left\{ \left(\sum_{\substack{j=0\\ j\neq l'}}^{d} \beta^{(d)}_{l_1,l'j} + \frac{T(n_{l_1} - M_{d-l'}(l_1))}{M_d(l_1)} \right) \left(\sum_{\substack{j=0\\ j\neq l'}}^{d} \beta^{(d)}_{l_2,l'j} + \frac{T(n_{l_2} - M_{d-l'}(l_2))}{M_d(l_2)} \right) \right. \\
& \left. + \sum_{\substack{j=0\\ j\neq l'}}^{d} \beta^{(d)}_{l_1,l'j}\beta^{(d)}_{l_2,l'j} \right\}.
\end{aligned}
\tag{4.2.26}
$$

From the above corollary, the covariance and correlation coefficient between MLEs $\hat{\theta}_{l_1}$ and $\hat{\theta}_{l_2}$ can also be readily obtained.

4.3 Approximate Confidence Intervals

Let $I(\theta_1, \theta_2, \cdots, \theta_k) = (I_{i,j}(\theta_1, \theta_2, \cdots, \theta_k)), i, j = 1, 2, \cdots, k$, denote the Fisher information matrix of the parameters $\theta_1, \theta_2, \cdots, \theta_k$, where

$$I_{i,j}(\theta_1, \theta_2, \cdots, \theta_k) = -E\left(\frac{\partial^2 \ln L}{\partial\theta_i \partial\theta_j}\right) \tag{4.3.1}$$

From the likelihood function in (4.2.4), we have $I_{i,j}(\theta_1, \theta_2, \cdots, \theta_k) = 0$ if $i \neq j$. Consequently, we have

$$\begin{aligned} & I(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k) \\ = \; & -\mathrm{Diag}\left(\frac{\partial^2 \ln L}{\partial\theta_1^2}|_{\theta_1=\hat{\theta}_1}, \frac{\partial^2 \ln L}{\partial\theta_2^2}|_{\theta_2=\hat{\theta}_2}, \cdots, \frac{\partial^2 \ln L}{\partial\theta_k^2}|_{\theta_k=\hat{\theta}_k}\right), \end{aligned} \tag{4.3.2}$$

where, for $d = 0, 1, 2, \cdots, r-1$,

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial\theta_h^2}|_{\theta_h=\hat{\theta}_h} &= \frac{M_r(h)}{\hat{\theta}_h^2} - \frac{2\left\{\sum_{i=1}^{r} w_i\delta_i(h) + w_r(n_h - M_r(h))\right\}}{\hat{\theta}_h^3} \\ &= -\frac{M_r(h)}{\hat{\theta}_h^2}, \end{aligned}$$

and for $d = r, r+1, \cdots, N$,

$$\begin{aligned}\frac{\partial^2 \ln L}{\partial \theta_h^2}\Big|_{\theta_h=\hat{\theta}_h} &= \frac{M_d(h)}{\hat{\theta}_h^2} - \frac{2\left\{\sum_{i=1}^{d} w_i \delta_i(h) + T(n_h - M_d(h))\right\}}{\hat{\theta}_h^3} \\ &= -\frac{M_d(h)}{\hat{\theta}_h^2}.\end{aligned}$$

Then, by using the asymptotic normality of the MLEs, we have $\hat{\theta}_h - \theta_h \sim N(0, I_{h,h}^{-1})$. With $\tilde{d} = \max\{d, r\}$, we can express the approximate $100(1-\alpha)\%$ confidence interval for θ_h, $1 \le h \le k$, as

$$\begin{cases} \hat{\theta}_h \pm Z_{\alpha/2} \frac{\sum_{i=1}^{\tilde{d}} w_i \delta_i(h) + w_r(n_h - M_{\tilde{d}}(h))}{M_{\tilde{d}}(h)^{\frac{3}{2}}} = \hat{\theta}_h(1 \pm \frac{Z_{\alpha/2}}{M_{\tilde{d}}(h)^{\frac{1}{2}}}), \\ \qquad\qquad \text{when } d = 0, 1, 2, \cdots, r-1 \\ \hat{\theta}_h \pm Z_{\alpha/2} \frac{\sum_{i=1}^{\tilde{d}} w_i \delta_i(h) + T(n_h - M_{\tilde{d}}(h))}{M_{\tilde{d}}(h)^{\frac{3}{2}}} = \hat{\theta}_h(1 \pm \frac{Z_{\alpha/2}}{M_{\tilde{d}}(h)^{\frac{1}{2}}}), \\ \qquad\qquad \text{when } d = r, r+1, \cdots, N, \end{cases}$$

where $Z_{\alpha/2}$ denotes the upper $\alpha/2$ percentage point of the standard normal distribution.

4.4 Bayesian Intervals

Let

$$u_h = \begin{cases} \sum_{i=1}^{r} w_i \delta_i(h) + w_r(n_h - M_r(h)), & \text{when } d = 0, 1, 2, \cdots, r-1, \\ \sum_{i=1}^{d} w_i \delta_i(h) + T(n_h - M_d(h)), & \text{when } d = r, r+1, \cdots, N. \end{cases}$$

Then, we can rewrite the likelihood function as

$$L(\theta_1, \theta_2, \cdots, \theta_k, \boldsymbol{\delta}, \mathbf{w}) = c_r \prod_{h=1}^{k} \theta_h^{-M_{\tilde{d}}(h)} \exp(-\frac{u_h}{\theta_h}) \tag{4.4.1}$$

Now, by assuming independent inverse gamma prior distributions, viz., $IG(a_h, b_h)$ for $1 \leq h \leq k$, we obtain from (5.3.1) the posterior joint density function as

$$L(\theta_1, \theta_2, \cdots, \theta_k | data) = c_{\tilde{d}} \prod_{h=1}^{k} \theta_h^{-M_r(h) - a_h - 1} \exp\left(-\frac{u_h + b_h}{\theta_h}\right).$$

Upon comparing this with (5.3.1), we see that the joint posterior density function of $(\theta_1, \theta_2, \cdots, \theta_k)$ is a product of k independent inverse gamma density functions. So, given the data, the posterior density function of $\hat{\theta}_h$ is simply $IG(M_{\tilde{d}}(h) + a_h, u_h + b_h)$. Thus, the Bayes estimator of θ_h under the squared-error loss function is

$$\hat{\theta}_{h,Bayes} = \frac{u_h + b_h}{M_{\tilde{d}}(h) + a_h - 1}, \qquad h = 1, 2, \cdots, k. \tag{4.4.2}$$

When we use Jeffreys' non-informative prior $I(\theta_h) \propto \frac{1}{\theta_h^2}$ corresponding to the special case when $a_h = 1$ and $b_h = 0$, for $1 \leq h \leq k$, the Bayes estimators in (5.3.2) coincide with the MLEs in (4.2.3).

Let $U_h = \frac{2(u_h+b_h)}{\theta_h}$ for $1 \leq h \leq k$. Then, the pivot U_h follows $\chi^2_{2(M_r(h)+a_h)}$ distribution, provided $2(M_{\tilde{d}}(h) + a_h)$ is a positive integer, for $1 \leq h \leq k$. In this case, the $100(1-\alpha)\%$ Bayes credible interval for θ_h becomes

$$\left(\frac{2(u_h + b_h)}{\chi^2_{2(M_{\tilde{d}}(h)+a_h),1-\alpha/2}}, \frac{2(u_h + b_h)}{\chi^2_{2(M_{\tilde{d}}(h)+a_h),\alpha/2}} \right), \qquad h = 1, 2, \cdots, k,$$

where $\chi^2_{v,\frac{\alpha}{2}}$ is the lower $\frac{\alpha}{2}$ percentage point of the chi-square distribution with v degrees of freedom.

4.5 Bootstrap Intervals

In this section, we consider confidence interval for θ_h $(h = 1, 2, \cdots, k)$ based on the Bootstrap-p and Bootstrap-t methods; see, for example, Efron and Tibshirani (1994).

To find the Bootstrap-p and Bootstrap-t intervals, in the first step, we generate original samples from k exponential populations with parameters θ_h of size n_h, $1 \leq$

$h \leq k$. Next we sort the data, and determine to which population each failure belongs, and then estimate θ_h using the conditional MLE in (4.2.3). In the second step, we generate a bootstrap sample $(\delta_1, \delta_2, \cdots, \delta_r; w_1, w_2, \cdots, w_r)$ by using the values $(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k)$, and then obtain the bootstrap estimates of θ_h, $1 \leq h \leq k$, say θ_h^*, $1 \leq h \leq k$, from the bootstrap sample. In the third step, we repeat the second step N-Boot times.

Boot-p: Suppose $K_{hB}(x) = P(\hat{\theta}_h^* \leq x)$, $1 \leq h \leq k$, is the cumulative distribution function of $\hat{\theta}_h^*$. Define $\hat{\theta}_{hB}(\alpha) = K_{hB}^{-1}(\alpha)$, $1 \leq h \leq k$, for a given α. Then, the $100(1-\alpha)\%$ Boot-p confidence interval for θ_h, $1 \leq h \leq k$, is given by

$$\left(\hat{\theta}_{hB}\left(\frac{\alpha}{2}\right), \hat{\theta}_{hB}\left(1-\frac{\alpha}{2}\right)\right), \qquad 1 \leq h \leq k.$$

Boot-t: After generating the bootstrap samples in the second step and calculating $\hat{\theta}_h^*$, we need to use them to compute the estimate of $Var(\hat{\theta}_h^*)$ from the observed Fisher information matrix in (5.2.2). Then, we determine the T_h^* statistic given by

$$T_h^* = \frac{\hat{\theta}_h^* - \hat{\theta}_h}{\sqrt{\widehat{\mathrm{Var}}(\hat{\theta}_h^*)}}, \qquad 1 \leq h \leq k. \tag{4.5.1}$$

Now, suppose $L_{hB}(x) = P(T_h^* \leq x)$ is the cumulative distribution function of T_h^*, $1 \leq h \leq k$. Then, the $100(1-\alpha)\%$ Boot-t confidence interval for θ_h, $1 \leq h \leq k$, is

given by

$$\left(\hat{\theta}_h - \sqrt{\widehat{\mathrm{Var}}(\hat{\theta}_h)}L_{hB}^{-1}\left(1-\frac{\alpha}{2}\right), \hat{\theta}_h - \sqrt{\widehat{\mathrm{Var}}(\hat{\theta}_h)}L_{hB}^{-1}\left(\frac{\alpha}{2}\right)\right), \qquad 1 \le h \le k.$$

4.6 Simulation Results and Discussion

A simulation study was carried out to evaluate the performance of the conditional MLEs discussed in the preceding sections. We considered sample sizes for the three populations as $n = (6, 6, 6)$, and different choice for r and T. We also chose the parameters $(\theta_1, \theta_2, \theta_3)$ to be $(2, 4, 7)$ and $(3, 5, 9)$. For these cases, we computed the conditional MLEs for the parameters $(\theta_1, \theta_2, \theta_3)$ and the empirical values of their means, standard deviations, mean square errors and covariance matrices for different choices of r and T. The results of these obtained from 10,000 Monto Carlo simulations are presented in Tables 4.1-4.4. From the results presented in these tables, we observe that while the estimate of θ_1 is very stable even for small r and T, the estimate of θ_2 and θ_3 become stable only for larger values of r and T. This is to be expected since when θ_1 is smaller than θ_2 and θ_3, when r and T are small, most of the failures observed would have resulted from the exponential population with parameter θ_1 and very few failures would have come from the exponential populations with parameters θ_2 and θ_3. This does get rectified when r and T are increased, as one would expect.

Table 4.1: The average values of the conditional MLEs and the estimates of their standard deviations and mean square errors when $\theta = (2, 4, 7)$ and $n = (6, 6, 6)$ for different choices of r and T.

		$Mean$	$\widehat{SD}$	$\widehat{RMSE}$
r	T	$(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$	$(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$	$(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$
6	2.5	$(2.53, 5.09, 7.53)$	$(1.5698, 3.4305, 4.2068)$	$(1.6554, 3.5980, 4.2407)$
6	3.0	$(2.43, 5.01, 7.64)$	$(1.4171, 3.4997, 4.6224)$	$(1.4811, 3.6439, 4.6660)$
6	3.5	$(2.34, 4.90, 7.71)$	$(1.2956, 3.3444, 4.9020)$	$(1.3406, 3.4624, 4.9529)$
6	4.0	$(2.32, 4.81, 7.78)$	$(1.1448, 3.2937, 5.1209)$	$(1.1880, 3.3926, 5.1806)$
6	4.5	$(2.27, 4.73, 7.82)$	$(1.0764, 3.0720, 5.1843)$	$(1.1088, 3.1571, 5.2487)$
9	3.5	$(2.20, 4.98, 8.47)$	$(1.1600, 3.5002, 5.5129)$	$(1.1766, 3.6342, 5.7046)$
9	4.5	$(2.16, 4.82, 8.65)$	$(1.0674, 3.2816, 5.9640)$	$(1.0795, 3.3819, 6.1880)$
9	5.5	$(2.12, 4.67, 8.42)$	$(0.9777, 2.9850, 5.9351)$	$(0.9852, 3.0595, 6.1029)$
9	6.5	$(2.10, 4.51, 8.18)$	$(0.9500, 2.5552, 5.5583)$	$(0.9553, 2.6055, 5.6823)$
9	7.5	$(2.08, 4.41, 7.97)$	$(0.9092, 2.4285, 5.2019)$	$(0.9128, 2.4628, 5.2910)$
12	4.5	$(2.08, 4.43, 8.41)$	$(0.9536, 2.5099, 5.7856)$	$(0.9569, 2.5468, 5.9545)$
12	5.5	$(2.06, 4.48, 8.49)$	$(0.9318, 2.4932, 5.9014)$	$(0.9340, 2.5399, 6.0866)$
12	6.5	$(2.05, 4.44, 8.53)$	$(0.9187, 2.4467, 5.9894)$	$(0.9203, 2.4854, 6.1815)$
12	7.5	$(2.03, 4.39, 8.37)$	$(0.8863, 2.3359, 5.9232)$	$(0.8869, 2.3681, 6.0799)$
12	8.5	$(2.02, 4.27, 8.20)$	$(0.8759, 2.1185, 5.4961)$	$(0.8762, 2.1350, 5.6260)$
15	7.0	$(2.01, 4.22, 7.60)$	$(0.8443, 1.9259, 3.7038)$	$(0.8443, 1.9380, 3.7525)$
15	9.0	$(2.02, 4.14, 7.64)$	$(0.8533, 1.8750, 3.7717)$	$(0.8535, 1.8805, 3.8263)$
15	11.0	$(2.01, 4.14, 7.62)$	$(0.8352, 1.8857, 3.8115)$	$(0.8352, 1.8909, 3.8619)$
15	13.0	$(2.00, 4.11, 7.53)$	$(0.8270, 1.8359, 3.7921)$	$(0.8270, 1.8390, 3.8293)$
15	15.0	$(2.02, 4.06, 7.52)$	$(0.8270, 1.7836, 3.7710)$	$(0.8272, 1.7848, 3.8067)$

Table 4.2: The average values of the conditional MLEs and the estimates of their standard deviations and mean square errors when $\theta = (3, 5, 9)$ and $n = (6, 6, 6)$ for different choices of r and T.

		$Mean$	$\widehat{SD}$	$\widehat{RMSE}$
r	T	$(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$	$(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$	$(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$
6	3.5	$(3.82, 6.48, 9.86)$	$(2.5574, 4.5146, 5.7666)$	$(2.6850, 4.7496, 5.8307)$
6	4.5	$(3.59, 6.26, 9.99)$	$(2.1984, 4.3946, 6.3885)$	$(2.2770, 4.5713, 6.4654)$
6	5.5	$(3.49, 5.97, 10.20)$	$(1.9237, 3.9316, 6.7351)$	$(1.9861, 4.0500, 6.8407)$
6	6.5	$(3.40, 5.79, 10.07)$	$(1.6890, 3.5709, 6.8513)$	$(1.7362, 3.6581, 6.9341)$
6	7.5	$(3.33, 5.65, 9.93)$	$(1.5446, 3.3313, 6.7011)$	$(1.5785, 3.3947, 6.7657)$
9	7.0	$(3.23, 5.74, 10.64)$	$(1.5819, 3.4843, 7.3324)$	$(1.5989, 3.5623, 7.5145)$
9	8.5	$(3.18, 5.60, 10.56)$	$(1.4721, 3.1107, 7.2800)$	$(1.4826, 3.1672, 7.4445)$
9	10.0	$(3.13, 5.46, 10.18)$	$(1.3925, 2.9112, 6.5140)$	$(1.3990, 2.9472, 6.6197)$
9	11.5	$(3.11, 5.36, 9.89)$	$(1.3616, 2.6037, 5.9283)$	$(1.3657, 2.6286, 5.9953)$
9	13.0	$(3.07, 5.30, 9.78)$	$(1.3082, 2.5403, 5.7143)$	$(1.3100, 2.5580, 5.7671)$
12	10.5	$(3.07, 5.39, 10.63)$	$(1.3505, 2.7520, 7.2666)$	$(1.3525, 2.7801, 7.4472)$
12	12.0	$(3.05, 5.30, 10.37)$	$(1.3111, 2.6630, 6.7338)$	$(1.3122, 2.6796, 6.8710)$
12	13.5	$(3.02, 5.19, 10.07)$	$(1.2897, 2.4252, 6.1649)$	$(1.2898, 2.4325, 6.2575)$
12	15.0	$(3.01, 5.21, 9.94)$	$(1.2587, 2.4328, 5.8890)$	$(1.2588, 2.4418, 5.9638)$
12	16.5	$(3.00, 5.15, 9.80)$	$(1.2508, 2.3231, 5.2514)$	$(1.2508, 2.3281, 5.3114)$
15	15.5	$(3.02, 5.15, 9.92)$	$(1.2544, 2.3242, 4.9150)$	$(1.2546, 2.3290, 5.0004)$
15	18.0	$(3.00, 5.12, 9.65)$	$(1.2514, 2.2585, 4.6868)$	$(1.2514, 2.2619, 4.7318)$
15	20.5	$(3.02, 5.06, 9.56)$	$(1.2336, 2.1882, 4.6098)$	$(1.2337, 2.1891, 4.6442)$
15	23.0	$(3.00, 5.00, 9.44)$	$(1.2327, 2.1518, 4.4514)$	$(1.2327, 2.1518, 4.4736)$
15	25.5	$(3.01, 5.01, 9.37)$	$(1.2450, 2.1120, 4.3390)$	$(1.2450, 2.1120, 4.3544)$

Table 4.3: The average values of the estimates of the covariance matrix of the conditional MLEs when $\theta = (2,4,7)$ and $n = (6,6,6)$ for different choices of r and T.

r	T	$\theta = (2,4,7)$ Covariance matrix $(\rho(\theta_i,\theta_j))_{i,j}$	T	$\theta = (3,5,9)$ Covariance matrix $(\rho(\theta_i,\theta_j))_{i,j}$
6	2.5	$\begin{pmatrix} 2.464 & -0.018 & 0.283 \\ -0.018 & 11.768 & 1.152 \\ 0.283 & 1.152 & 17.698 \end{pmatrix}$	3.0	$\begin{pmatrix} 6.540 & -0.202 & 0.590 \\ -0.202 & 20.382 & 2.230 \\ 0.590 & 2.230 & 33.254 \end{pmatrix}$
6	3.0	$\begin{pmatrix} 2.008 & 0.056 & 0.288 \\ 0.056 & 12.248 & 1.522 \\ 0.288 & 1.522 & 21.367 \end{pmatrix}$	4.0	$\begin{pmatrix} 4.833 & 0.267 & 0.998 \\ 0.267 & 19.312 & 2.196 \\ 0.998 & 2.196 & 40.813 \end{pmatrix}$
6	3.5	$\begin{pmatrix} 1.678 & 0.121 & 0.500 \\ 0.121 & 11.185 & 1.088 \\ 0.500 & 1.088 & 24.030 \end{pmatrix}$	5.0	$\begin{pmatrix} 3.701 & 0.179 & 0.593 \\ 0.179 & 15.458 & 1.961 \\ 0.593 & 1.961 & 45.362 \end{pmatrix}$
6	4.0	$\begin{pmatrix} 1.311 & 0.229 & 0.352 \\ 0.229 & 10.848 & 1.199 \\ 0.352 & 1.199 & 26.224 \end{pmatrix}$	6.0	$\begin{pmatrix} 2.853 & 0.260 & 0.562 \\ 0.260 & 12.751 & 2.014 \\ 0.562 & 2.014 & 46.940 \end{pmatrix}$
6	4.5	$\begin{pmatrix} 1.159 & 0.135 & 0.229 \\ 0.135 & 9.437 & 0.823 \\ 0.229 & 0.823 & 26.877 \end{pmatrix}$	7.0	$\begin{pmatrix} 2.386 & 0.217 & 0.274 \\ 0.217 & 11.098 & 1.143 \\ 0.274 & 1.143 & 44.905 \end{pmatrix}$
9	3.5	$\begin{pmatrix} 1.345 & -0.227 & -0.094 \\ -0.227 & 12.251 & -0.396 \\ -0.094 & -0.396 & 30.392 \end{pmatrix}$	6.5	$\begin{pmatrix} 2.502 & -0.087 & 0.220 \\ -0.087 & 12.140 & 0.915 \\ 0.220 & 0.915 & 53.765 \end{pmatrix}$
9	4.5	$\begin{pmatrix} 1.139 & -0.105 & 0.147 \\ -0.105 & 10.769 & 0.436 \\ 0.147 & 0.436 & 35.570 \end{pmatrix}$	8.0	$\begin{pmatrix} 2.167 & 0.013 & 0.236 \\ 0.013 & 9.677 & 1.046 \\ 0.236 & 1.046 & 52.999 \end{pmatrix}$
9	5.5	$\begin{pmatrix} 0.956 & -0.012 & 0.224 \\ -0.012 & 8.911 & 0.715 \\ 0.224 & 0.715 & 35.225 \end{pmatrix}$	9.5	$\begin{pmatrix} 1.939 & -0.002 & 0.189 \\ -0.002 & 8.475 & 0.415 \\ 0.189 & 0.415 & 42.432 \end{pmatrix}$
9	6.5	$\begin{pmatrix} 0.903 & -0.029 & 0.117 \\ -0.029 & 6.529 & 0.578 \\ 0.117 & 0.578 & 30.895 \end{pmatrix}$	11.0	$\begin{pmatrix} 1.854 & -0.073 & 0.098 \\ -0.073 & 6.779 & 0.531 \\ 0.098 & 0.531 & 35.144 \end{pmatrix}$
9	7.5	$\begin{pmatrix} 0.827 & -0.067 & 0.125 \\ -0.067 & 5.897 & 0.498 \\ 0.125 & 0.498 & 27.060 \end{pmatrix}$	12.5	$\begin{pmatrix} 1.711 & -0.009 & 0.338 \\ -0.009 & 6.453 & 0.596 \\ 0.338 & 0.596 & 32.653 \end{pmatrix}$

Table 4.4: The average values of the estimates of the covariance matrix of the conditional MLEs when $\theta = (3, 5, 9)$ and $n = (6, 6, 6)$ for different choices of r and T.

r	T	$\theta = (2, 4, 7)$ Covariance matrix $(\rho(\theta_i, \theta_j))_{i,j}$	T	$\theta = (3, 5, 9)$ Covariance matrix $(\rho(\theta_i, \theta_j))_{i,j}$
12	4.5	$\begin{pmatrix} 0.909 & -0.088 & -0.303 \\ -0.088 & 6.300 & -1.443 \\ -0.303 & -1.443 & 33.474 \end{pmatrix}$	10.0	$\begin{pmatrix} 1.824 & 0.006 & 0.103 \\ 0.006 & 7.574 & 0.142 \\ 0.103 & 0.142 & 52.803 \end{pmatrix}$
12	5.5	$\begin{pmatrix} 0.868 & -0.054 & -0.063 \\ -0.054 & 6.216 & -1.302 \\ -0.063 & -1.302 & 34.827 \end{pmatrix}$	11.5	$\begin{pmatrix} 1.719 & 0.011 & 0.058 \\ 0.011 & 7.092 & 0.084 \\ 0.058 & 0.084 & 45.344 \end{pmatrix}$
12	6.5	$\begin{pmatrix} 0.844 & -0.046 & -0.097 \\ -0.046 & 5.986 & -0.773 \\ -0.097 & -0.773 & 35.873 \end{pmatrix}$	13.0	$\begin{pmatrix} 1.663 & 0.055 & 0.048 \\ 0.055 & 5.881 & 0.023 \\ 0.048 & 0.023 & 38.006 \end{pmatrix}$
12	7.5	$\begin{pmatrix} 0.786 & -0.004 & 0.065 \\ -0.004 & 5.456 & -0.425 \\ 0.065 & -0.425 & 35.085 \end{pmatrix}$	14.5	$\begin{pmatrix} 1.584 & -0.043 & 0.064 \\ -0.043 & 5.919 & 0.188 \\ 0.064 & 0.188 & 34.681 \end{pmatrix}$
12	8.5	$\begin{pmatrix} 0.767 & -0.029 & 0.008 \\ -0.029 & 4.488 & -0.067 \\ 0.008 & -0.067 & 30.208 \end{pmatrix}$	16.0	$\begin{pmatrix} 1.564 & 0.019 & 0.075 \\ 0.019 & 5.397 & 0.290 \\ 0.075 & 0.290 & 27.577 \end{pmatrix}$
15	7.0	$\begin{pmatrix} 0.713 & -0.017 & -0.066 \\ -0.017 & 3.709 & -0.368 \\ -0.066 & -0.368 & 13.718 \end{pmatrix}$	15.0	$\begin{pmatrix} 1.573 & -0.009 & -0.109 \\ -0.009 & 5.402 & -0.389 \\ -0.109 & -0.389 & 24.157 \end{pmatrix}$
15	9.0	$\begin{pmatrix} 0.728 & -0.029 & -0.025 \\ -0.029 & 3.516 & -0.376 \\ -0.025 & -0.376 & 14.226 \end{pmatrix}$	17.5	$\begin{pmatrix} 1.566 & -0.056 & -0.020 \\ -0.056 & 5.101 & -0.033 \\ -0.020 & -0.033 & 21.966 \end{pmatrix}$
15	11.0	$\begin{pmatrix} 0.698 & -0.011 & -0.023 \\ -0.011 & 3.556 & -0.148 \\ -0.023 & -0.148 & 14.528 \end{pmatrix}$	20.0	$\begin{pmatrix} 1.522 & -0.068 & -0.014 \\ -0.068 & 4.788 & -0.045 \\ -0.014 & -0.045 & 21.250 \end{pmatrix}$
15	13.0	$\begin{pmatrix} 0.684 & 0.020 & -0.042 \\ 0.020 & 3.370 & 0.002 \\ -0.042 & 0.002 & 14.380 \end{pmatrix}$	22.5	$\begin{pmatrix} 1.519 & 0.007 & 0.095 \\ 0.007 & 4.630 & -0.082 \\ 0.095 & -0.082 & 19.815 \end{pmatrix}$
15	15.0	$\begin{pmatrix} 0.684 & 0.015 & 0.026 \\ 0.015 & 3.181 & -0.048 \\ 0.026 & -0.048 & 14.221 \end{pmatrix}$	25.0	$\begin{pmatrix} 1.550 & -0.030 & -0.006 \\ -0.030 & 4.460 & -0.048 \\ -0.006 & -0.048 & 18.827 \end{pmatrix}$

Chapter 5

Inference Under Joint Type-I Hybrid Censoring

5.1 MLEs, Exact Distributions and Inference

In this section, we drive analogous results for Type-I HCS, wherein the experiment is terminated at the random time $T^* = \min\{W_r, T\}$. As in section 4.2, let D be the number of observed failures up to time T.

Suppose $X_{11}, X_{12}, \cdots, X_{1n_1}$ are the lifetimes of n_1 specimens from line A_1, and assumed to be independent and identically distributed (iid) variables from a population with cumulative distribution function (cdf) $F_1(x)$ and probability density function (pdf) $f_1(x)$. Similarly, $X_{21}, X_{22}, \cdots, X_{2n_2}$ are the lifetimes of n_2 specimens from line A_2 and assumed to be a sample from pdf $f_2(x)$ and cdf $F_2(x)$, and so on, with $X_{k1}, X_{k2}, \cdots, X_{kn_k}$ denoting the lifetimes of n_k specimens from line A_k being iid variables from pdf $f_k(x)$ and cdf $F_k(x)$.

Furthermore, let $N = \sum_{i=1}^{k} n_i$ denote the total sample size and r denote the total number of failures observed. Let $w_1 \le w_2 \le \cdots \le w_N$ denote the order statistics of the N random variables $\{X_{ij}; 1 \le i \le k, 1 \le j \le n_i\}$.

Therefore, under the joint hybrid Type-II censoring scheme for the k-samples, the observable data consist of $(\boldsymbol{\delta}, \mathbf{W})$, where $\mathbf{W} = (w_1, w_2, \cdots, w_r)$, $w_i \in \{X_{h_i 1}, X_{h_i 2}, \cdots, X_{h_i n_i}\}$ for $1 \le h_1, h_2, \cdots, h_r \le k$, with r being a pre-fixed integer. Finally, associated to $(h_1, h_2, \cdots, h_r)$, let us define $\boldsymbol{\delta} = (\delta_1(h), \delta_2(h), \cdots, \delta_r(h))$ as

Therefore, under the joint Type-I hybrid censoring scheme described above, the observable data consist of $(\boldsymbol{\delta}, \mathbf{W})$ of the following form:

$$(\boldsymbol{\delta}, \mathbf{W}) = \begin{cases} (\delta_1, \delta_2, \cdots, \delta_D; W_1, W_2, \cdots, W_D), & T < W_r \text{ with } D = 0, 1, \cdots, r-1, \\ (\delta_1, \delta_2, \cdots, \delta_r; W_1, W_2, \cdots, W_r), & T > W_r \text{ with } D = r, r+1, \cdots, N, \end{cases}$$

Then, under the Type-I HCS, the likelihood of $(\boldsymbol{\delta}, \mathbf{W})$ is given by

$$\begin{aligned} & L(\theta_1, \theta_2, \cdots, \theta_k, \boldsymbol{\delta}, \mathbf{W}) \\ = & \begin{cases} c_D \prod_{i=1}^{D} \prod_{h=1}^{k} (f_h(w_i))^{\delta_i(h)} \prod_{h=1}^{k} (S_h(T))^{n_h - M_D(h)}, & T < W_r, \\ c_r \prod_{i=1}^{r} \prod_{h=1}^{k} (f_h(w_i))^{\delta_i(h)} \prod_{h=1}^{k} (S_h(w_r))^{n_h - M_r(h)}, & T > W_r, \end{cases} \end{aligned} \tag{5.1.1}$$

where $S_h(w_r) = 1 - F_h(w_r)$, $S_h(T) = 1 - F_h(T)$ and $c_r = \frac{\prod_{h=1}^{k} n_h!}{\prod_{h=1}^{k} (n_h - M_r(h))!}$, and $c_D =$

$\frac{\prod_{h=1}^{k} n_h!}{\prod_{h=1}^{k}(n_h - M_D(h))!}$.

When the k populations are exponential with cdf $F_h(x) = 1 - \exp(-\frac{x}{\theta_h})$, $x > 0$, and pdf $f_h(x) = \frac{1}{\theta_h}\exp(-\frac{x}{\theta_h})$, $x > 0$, for $1 \leq h \leq k$, the likelihood function in (5.1.1) simplifies as follows:

For $T < W_r$, $(D \leq r-1)$, the likelihood of $(\boldsymbol{\delta}, \mathbf{W})$ is given by

$$\begin{aligned} & L(\theta_1, \theta_2, \cdots, \theta_k, \boldsymbol{\delta}, \mathbf{W}) \\ = \; & c_D \exp\left\{-\sum_{h=1}^{k} M_D(h)\log\theta_h - \sum_{h=1}^{k}\frac{1}{\theta_h}\left(\sum_{i=1}^{D} w_i\delta_i(h) - T(n_h - M_D(h))\right)\right\}; \end{aligned} \tag{5.1.2}$$

For $T > W_r$, $(D \geq r)$, the likelihood of $(\boldsymbol{\delta}, \mathbf{W})$ is given by

$$\begin{aligned} & L(\theta_1, \theta_2, \cdots, \theta_k, \boldsymbol{\delta}, \mathbf{W}) \\ = \; & c_r \exp\left\{-\sum_{h=1}^{k} M_r(h)\log\theta_h - \sum_{h=1}^{k}\frac{1}{\theta_h}\left(\sum_{i=1}^{r} w_i\delta_i(h) - w_r(n_h - M_r(h))\right)\right\}. \end{aligned} \tag{5.1.3}$$

Thus, under the Type-I HCS, for $1 \leq h \leq k$, the MLE of θ_h is given by

$$\hat{\theta}_h = \begin{cases} \frac{1}{M_D(h)}\left\{\sum\limits_{i=1}^{D} w_i\delta_i(h) + T(n_h - M_D(h))\right\} & \text{when } D = 0, 1, \cdots, r-1, \\ \frac{1}{M_r(h)}\left\{\sum\limits_{i=1}^{r} w_i\delta_i(h) + w_r(n_h - M_r(h))\right\} & \text{when } D = r, r+1, \cdots, N. \end{cases} \tag{5.1.4}$$

Theorem 13 *The joint probability mass function of $\boldsymbol{\delta}$ under the Type-I HCS is as follows:*

(a) *For $T < W_r$ $(d \leq r-1)$,*

$$P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}} | D = d) = \frac{c_d}{P(D=d)} \prod_{h=1}^{k} \theta_h^{-M_d(h)} \sum_{l'=0}^{d} \psi_{l'd}, \tag{5.1.5}$$

where, for $0 \leq l' \leq d$, $\psi_{l'd}$ determined by (4.2.10).

(b) *For $T \geq W_r$ $(d \geq r)$,*

$$P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}} | D = d) = \frac{c_r}{P(D=d)} \prod_{h=1}^{k} \theta_h^{-M_r(h)} \sum_{l'=0}^{r} \tilde{\varphi}_{l'r}, \tag{5.1.6}$$

where

$$\begin{aligned}
\tilde{\varphi}_{l'r} &= c_{l'}^{(r)} \exp\left\{ -T \sum_{h=1}^{k} \frac{n_h - M_{r-l'}(h)}{\theta_h} \right\}, \ 0 \leq l' \leq r; \qquad (5.1.7) \\
c_0^{(r)} &= \prod_{j=1}^{r} \frac{1}{\sum_{h=1}^{k} \frac{n_h - M_{j-1}(h)}{\theta_h}}, \\
c_{l'}^{(r)} &= \frac{-1}{\sum_{h=1}^{k} \frac{n_h - M_{r-l'}(h)}{\theta_h}} \prod_{\substack{j=0 \\ j \neq l'}}^{r} \frac{1}{\sum_{h=1}^{k} \frac{(M_{r-l'}(h) - M_{r-j}(h))}{\theta_h}}, 1 \leq l' \leq r.
\end{aligned}$$

(c) *For $\mathbf{t} = (t_1, t_2, \cdots, t_k)$, such that $\sum_{j=1}^{k} t_j = d$ and $t_j \geq 1$ for all j, let $\boldsymbol{\delta}$ be as defined in (4.1.1), $M_d(h) = \sum_{i=1}^{r} \delta_i(h)$, and*

$$Q_d^*(\mathbf{t}) = \left\{ \tilde{\boldsymbol{\delta}}(\mathbf{h}) = (\tilde{\delta}_1, \tilde{\delta}_2, \cdots, \tilde{\delta}_r);\ \textit{such that } M_d(h) = t_h \textit{ for } 1 \leq h \leq k \right\}.$$

Then,

$$P(\mathbf{M}_r = \mathbf{t}) = \sum_{\tilde{\boldsymbol{\delta}} \in Q_d^*(\mathbf{t})} P(\boldsymbol{\delta} = \tilde{\boldsymbol{\delta}}); \tag{5.1.8}$$

(d) *Let $\mathbf{T}_d$ be the set as defined in Lemma 3, Then,*

$$P(S|D = d) = \begin{cases} \sum_{\mathbf{t} \in \mathbf{T}_d} P(\mathbf{M}_d = \mathbf{t}) & \text{if } d \leq r-1, \\ \sum_{\mathbf{t} \in \mathbf{T}_r} P(\mathbf{M}_r = \mathbf{t}) & \text{if } d \geq r. \end{cases} \tag{5.1.9}$$

As we can see, for some $\mathbf{t}$, the probability $P(\mathbf{M}_d = \mathbf{t}) = 0$, when $\mathbf{t} = (t_1, t_2, \cdots, t_k)$ does not belong to T_d.

Theorem 14 *Conditional on the S, the mgf of $\hat{\theta}_l$ $(1 \leq l \leq k)$ is given by*

$$\begin{aligned} M_{\hat{\theta}_l|S}(t) &= \sum_{d=0}^{r-1} \frac{1}{P(\prod_{h=1}^k M_r(h) \neq 0 | D = d)} \sum_{\mathbf{t} \in \mathbf{T}_d} \sum_{\tilde{\boldsymbol{\delta}} \in Q_d^*(\mathbf{t})} c_d \prod_{h=1}^k \theta_h^{-M_d(h)} \sum_{l'=0}^{d} \psi_{l'd} \\ &\times \prod_{\substack{j=0 \\ j \neq l'}}^{d} \frac{1}{1 - t\beta_{l,l'j}^{(d)}} \exp\left\{ \frac{tT\left(n_l - M_{d-l'}(l)\right)}{M_d(l)} \right\}. \\ &+ \sum_{d=r}^{N} \frac{1}{P(\prod_{h=1}^k M_d(h) \neq 0 | D = d)} \sum_{\mathbf{t} \in \mathbf{T}_r} \sum_{\tilde{\boldsymbol{\delta}} \in Q_r^*(\mathbf{t})} c_r \prod_{h=1}^k \theta_h^{-M_r(h)} \sum_{l'=0}^{r} \tilde{\varphi}_{l'r} \\ &\times \frac{1}{1 - t\beta_{l,r-l'+1}^*} \prod_{\substack{j=0 \\ j \neq l'}}^{r} \frac{1}{1 - t\beta_{l,l'j}^{(r)}} \exp\left\{ \frac{tT\left(n_l - M_{r-l'}(l)\right)}{M_r(l)} \right\}, \end{aligned} \tag{5.1.10}$$

where $\psi_{l'd}$ and $\tilde{\varphi}_{l'r}$ are as defined in (4.2.10) and (5.1.8), for $1 \leq l \leq k$ and $0 \leq l' \leq d$, $0 \leq j \leq d$, $\beta_{l,l'j}^{(d)}$ determined by (4.2.16), β_{lj}^* determined by (4.2.17).

Remark 12 *For fixed l, l' and $\tilde{\boldsymbol{\delta}}$, some $\beta^{(d)}_{l,l'i}$'s and β^*_{li}'s may be the same. In this case, for $d < r$, we resort the $\beta^{(d)}_{l,l'i}$ values and denote them by $\{\beta^{(1)}_{l,l'i}\}^{r'}_{i=1}$, and assume that there are r' distinct values with $\alpha^{(1)}_{l,l'i}$ of the $\beta^{(1)}_{l,l'i}$'s being equal, with $\sum\limits_{i=1}^{r'} \alpha^{(1)}_{l,l'i} = d$. Consequently, the term $\prod\limits_{\substack{j=0\\ j\neq l'}}^{d} \left(1 - t\beta^{(d)}_{l,l'j}\right)^{-1}$ can be rewritten as $\prod\limits_{j=1}^{r'} \left(1 - t\beta^{(1)}_{l,l'j}\right)^{-\alpha^{(1)}_{l,l'j}}$; For $d \geq r$, similarly, $\prod\limits_{\substack{j=0\\ j\neq l'}}^{r} \left(1 - t\beta^{(r)}_{l,l'j}\right)^{-1} \left(1 - t\beta^*_{l,r-l'+1}\right)^{-1}$ can be rewritten as $\prod_{j=1}^{r''} \left(1 - t\beta^{(2)}_{l,l'j}\right)^{-\alpha^{(2)}_{l,l'j}}$ with $\sum_{i=1}^{r''} \alpha^{(2)}_{l,l'i} = r+1$, and thus, the conditional mgf in Theorem 10 can be expressed as follows:*

$$
\begin{aligned}
M_{\hat{\theta}_l|S}(t) \;=\; & \sum_{d=0}^{r-1} \frac{1}{P(\prod_{h=1}^{k} M_r(h) \neq 0 | D = d)} \sum_{\mathbf{t}\in\mathbf{T}_d} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*_d(\mathbf{t})} c_d \prod_{h=1}^{k} \theta_h^{-M_d(h)} \sum_{l'=0}^{d} \psi_{l'd} \\
& \times \prod_{j=1}^{r'} \left(1 - t\beta^{(1)}_{l,l'j}\right)^{-\alpha^{(1)}_{l,l'j}} \exp\left\{\frac{tT(n_l - M_{d-l'}(l))}{M_d(l)}\right\} \\
& + \sum_{d=r}^{N} \frac{1}{P(\prod_{h=1}^{k} M_d(h) \neq 0 | D = d)} \sum_{\mathbf{t}\in\mathbf{T}_r} \sum_{\tilde{\boldsymbol{\delta}}\in Q^*_r(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \sum_{l'=0}^{d} \tilde{\varphi}_{l'r} \\
& \times \prod_{j=1}^{r''} \left(1 - t\beta^{(2)}_{l,l'j}\right)^{-\alpha^{(2)}_{l,l'j}} \exp\left\{\frac{tT(n_l - M_{r-l'}(l))}{M_r(l)}\right\} \qquad (5.1.11)
\end{aligned}
$$

Theorem 15 *Conditional on the set S, the pdf of $\hat{\theta}_l$ (for $1 \leq l \leq k$) is given by*

$$
\begin{aligned}
f_{\hat{\theta}_l|S}(x) \;=\; & \sum_{d=0}^{r-1} \frac{1}{P(\prod_{h=1}^{k} M_r(h) \neq 0 | D = d)} \sum_{\mathbf{t}\in\mathbf{T}_d} \sum_{\tilde{\delta}\in Q^*_d(\mathbf{t})} c_d \prod_{h=1}^{k} \theta_h^{-M_d(h)} \sum_{l'=0}^{d} \psi_{l'd} h_{Y^{(1)}_{l'd}} \\
& + \sum_{d=r}^{N} \frac{1}{P(\prod_{h=1}^{k} M_d(h) \neq 0 | D = d)} \sum_{\mathbf{t}\in\mathbf{T}_r} \sum_{\tilde{\delta}\in Q^*_r(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \\
& \times \sum_{l'=0}^{r} \tilde{\varphi}_{l'r} g^{(2)}_{Y_{l'r}}, \qquad (5.1.12)
\end{aligned}
$$

where $Y^{(1)}_{l'd} \stackrel{d}{=} \sum_{i=1}^{r'} Y^{(1)}_{l'd,i} + Y_{l'r}$ *with* $Y^{(1)}_{l'd,i}$ *being independent random variables having gamma* $G(\alpha^{(1)}_{l,l'i}, \beta^{(1)}_{l,l'i})$ *distributions with scale parameters* $\beta^{(1)}_{l,l'i}$ *and shape parameters* $\alpha^{(1)}_{l,l'i}$*,* $Y_{l'r}$ *being independent random variables having degenerate distribution localized at the point* $\frac{T(n_l-M_{d-l'}(l))}{M_r(l)}$*, and* $g_{Y^{(1)}_{l'd}}(x)$ *is the pdf of* $Y^{(1)}_{l'd}$*;* $Y^{(2)}_{l'r} \stackrel{d}{=} \sum_{i=1}^{r''} Y^{(2)}_{l'd,i} + Y_{l'd}$ *with* $Y^{(2)}_{l'd,i}$ *being independent random variables having gamma* $G(\alpha^{(2)}_{l,l'i}, \beta^{(2)}_{l,l'i})$ *distributions with scale parameters* $\beta^{(2)}_{l,l'i}$ *and shape parameters* $\alpha^{(2)}_{l,l'i}$*,* $Y_{l'd}$ *being independent random variables having degenerate distribution localized at the point* $\frac{T(n_l-M_{d-l'}(l))}{M_d(l)}$*, and* $g_{Y^{(2)}_{l'd}}(x)$ *is the pdf of* $Y^{(2)}_{l'd}$*.*

Corollary 7 *From (4.2.15), we immediately obtain the expressions for the first two moments of* $\hat{\theta}_l$ *as follows:*

$$\begin{aligned} E(\hat{\theta}_l) &= \sum_{d=0}^{r-1} \frac{1}{P(\prod_{h=1}^{k} M_r(h) \neq 0|D=d)} \sum_{\mathbf{t}\in\mathbf{T}_d} \sum_{\tilde{\delta}\in Q_d^*(\mathbf{t})} c_d \prod_{h=1}^{k} \theta_h^{-M_d(h)} \sum_{l'=0}^{d} \psi_{l'd} \\ &\quad \times \left(\sum_{j=0;j\neq l'}^{d} \beta^{(d)}_{l,l'j} + \frac{T(n_l - M_{d-l'}(l))}{M_d(l)} \right) \\ &\quad + \sum_{d=r}^{N} \frac{1}{P(\prod_{h=1}^{k} M_d(h) \neq 0|D=d)} \sum_{\mathbf{t}\in\mathbf{T}_r} \sum_{\tilde{\delta}\in Q_r^*(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \sum_{l'=0}^{r} \tilde{\varphi}_{l'r} \\ &\quad \times \left(\sum_{j=0;j\neq l'}^{r} \beta^{(r)}_{l,l'j} + \beta^*_{l,r-l'+1} + \frac{T(n_l - M_{d-l'}(l))}{M_r(l)} \right) \end{aligned}$$

and

$$\begin{aligned}
E(\hat{\theta}_l^2) &= \sum_{d=0}^{r-1} \frac{1}{P(\prod_{h=1}^{k} M_r(h) \neq 0 | D = d)} \sum_{\mathbf{t} \in \mathbf{T}_d} \sum_{\tilde{\delta} \in Q_d^*(\mathbf{t})} c_d \prod_{h=1}^{k} \theta_h^{-M_d(h)} \sum_{l'=0}^{d} \psi_{l'd} \\
&\times \left\{ \left(\sum_{j=0; j \neq l'}^{d} \beta_{l,l'j}^{(d)} + \frac{T(n_l - M_{d-l'}(l))}{M_d(l)} \right)^2 + \sum_{j=0; j \neq l'}^{d} \left(\beta_{l,l'j}^{(d)} \right)^2 \right\} \\
&+ \sum_{d=r}^{N} \frac{1}{P(\prod_{h=1}^{k} M_d(h) \neq 0 | D = d)} \sum_{\mathbf{t} \in \mathbf{T}_r} \sum_{\tilde{\delta} \in Q_r^*(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \sum_{l'=0}^{r} \tilde{\varphi}_{l'r} \\
&\times \left\{ \left(\sum_{j=0; j \neq l'}^{r} \beta_{l,l'j}^{(r)} + \beta_{l,r-l'+1}^{*} + \frac{T(n_l - M_{d-l'}(l))}{M_r(l)} \right)^2 \right. \\
&\left. + \sum_{j=0; j \neq l'}^{r} \left(\beta_{l,l'j}^{(r)} \right)^2 + \left(\beta_{l,r-l'+1}^{*} \right)^2 \right\}
\end{aligned}$$

Then, $\mathrm{Var}(\hat{\theta}_l)$ and $\mathrm{MSE}(\hat{\theta}_l)$ can be readily obtained from these two expressions.

It is convenient to rewrite the conditional mgf of $\hat{\theta}_l$ (for $l = 1, 2, \cdots, k$) in Theorem 10 as

$$\begin{aligned}
M_{\hat{\theta}_l|S}(t) &= \sum_{d=0}^{r-1} \frac{1}{P(\prod_{h=1}^{k} M_r(h) \neq 0 | D = d)} \sum_{\mathbf{t} \in \mathbf{T}_d} \sum_{\tilde{\delta} \in Q_d^*(\mathbf{t})} c_d \prod_{h=1}^{k} \theta_h^{-M_d(h)} \sum_{l'=0}^{d} \psi_{l'd} \\
&\times \sum_{j=1}^{r'} \sum_{j'=0}^{\alpha_{l,l'j}^{(1)}} A_{l,l'jj'}^{(1)} \left(1 - t\beta_{l,l'j}^{(1)} \right)^{-j'} \exp\left\{ \frac{tT(n_l - M_{d-l'}(l))}{M_r(l)} \right\} \\
&+ \sum_{d=r}^{N} \frac{1}{P(\prod_{h=1}^{k} M_d(h) \neq 0 | D = d)} \sum_{\mathbf{t} \in \mathbf{T}_r} \sum_{\tilde{\delta} \in Q_r^*(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \sum_{l'=0}^{r} \tilde{\varphi}_{l'r} \\
&\times \sum_{j=1}^{r''} \sum_{j'=0}^{\alpha_{l,l'j}^{(2)}} A_{l,l'jj'}^{(2)} \left(1 - t\beta_{l,l'j}^{(2)} \right)^{-j'} \exp\left\{ \frac{tT(n_l - M_{d-l'}(l))}{M_d(l)} \right\}, \qquad (5.1.13)
\end{aligned}$$

where $A^{(1)}_{l,l'jj'}$'s are coefficients obtained by writing the product $\prod_{j=1}^{r'}\left(1-t\beta^{(1)}_{l,l'j}\right)^{-\alpha^{(1)}_{l,l'j}}$ in the partial fraction form $\sum_{j=1}^{r'}\sum_{j'=1}^{\alpha^{(1)}_{l,l'j}} A^{(1)}_{l,l'jj'}\left(1-t\beta^{(1)}_{l,l'j}\right)^{-j'}$ and $A^{(2)}_{l,l'jj'}$'s are coefficients obtained by writing the product $\prod_{j=1}^{r''}\left(1-t\beta^{(2)}_{l,l'j}\right)^{-\alpha^{(2)}_{l,l'j}}$ in the partial fraction form $\sum_{j=1}^{r''}\sum_{j'=1}^{\alpha^{(2)}_{l,l'j}} A^{(2)}_{l,l'jj'}\left(1-t\beta^{(2)}_{l,l'j}\right)^{-j'}$, and the coefficients $A^{(1)}_{l,l'jj'}, A^{(2)}_{l,l'jj'}$ can be readily determined by the use of Lemma 5 in Appendix. Since $(1-ct)^{-j}\mathrm{e}^{At}$ is the mgf of the random variable $X+A$, where X has the gamma distribution with scale parameter c and shape parameter j, we readily obtain the tail probability of $\hat{\theta}_l$ (for $l=1,2,\cdots,k$) from the above expression as

$$
\begin{aligned}
& P(\hat{\theta}_l > b|S) \\
= \; & \sum_{d=0}^{r-1}\frac{1}{P(\prod_{h=1}^{k}M_r(h)\neq 0|D=d)}\sum_{\mathbf{t}\in\mathbf{T}_d}\sum_{\tilde{\delta}\in Q^*_d(\mathbf{t})} c_d\prod_{h=1}^{k}\theta_h^{-M_d(h)}\sum_{l'=0}^{d}\psi_{l'd} \\
& \times\sum_{j=1}^{r'}\sum_{j'=0}^{\alpha^{(1)}_{l,l'j}}\frac{A^{(1)}_{l,l'jj'}}{j'!}\Gamma\left(j',\frac{1}{\beta^{(1)}_{l,l'j}}\langle b-\frac{T(n_l-M_{d-l'}(l))}{M_r(l)}\rangle\right) \\
& +\sum_{d=r}^{N}\frac{1}{P(\prod_{h=1}^{k}M_d(h)\neq 0|D=d)}\sum_{\mathbf{t}\in\mathbf{T}_r}\sum_{\tilde{\delta}\in Q^*_r(\mathbf{t})} c_r\prod_{h=1}^{k}\theta_h^{-M_r(h)}\sum_{l'=0}^{r}\tilde{\varphi}_{l'r} \\
& \times\sum_{j=1}^{r''}\sum_{j'=0}^{\alpha^{(2)}_{l,l'j}}\frac{A^{(2)}_{l,l'jj'}}{j'!}\Gamma\left(j',\frac{1}{\beta^{(2)}_{l,l'j}}\langle b-\frac{T(n_l-M_{d-l'}(l))}{M_d(l)}\rangle\right),
\end{aligned}
\tag{5.1.14}
$$

where $\langle x\rangle=\max\{x,0\}$ and $\Gamma(a,z)=\int_z^\infty t^{a-1}\mathrm{e}^{-t}dt$ is the upper incomplete gamma function.

We shall assume that $P(\hat{\theta}_l > b|S)$ is an increasing function of θ_l when all other θ_j's are fixed for $j\neq l$. This assumption guarantees the invertibility of the pivotal quantities, and it has been verified to be true in this case through extensive computations under various setting. It should be mentioned that this approach has been used by a number of authors for constructing exact confidence intervals in a variety of contexts; see, for example, Childs et al. (2003) and Balakrishnan et al. (2007). We then have a $100(1-\alpha)\%$ lower confidence bound for θ_l as θ_{lL}, where θ_{lL} is such that $P_{\theta_{lL}}(\hat{\theta}_l > \hat{\theta}_{l,obs}|S) = \alpha$ with $\hat{\theta}_{l,obs}$ being the observed value of θ_l. Also, a $100(1-\alpha)\%$ confidence interval for θ_l is $(\theta_{lL}, \theta_{lU})$, where θ_{lL} and θ_{lU} are determined by $P_{\theta_{lL}}(\hat{\theta}_l > \hat{\theta}_{l,obs}|S) = \frac{\alpha}{2}$ and $P_{\theta_{lU}}(\hat{\theta}_l > \hat{\theta}_{l,obs}|S) = 1 - \frac{\alpha}{2}$.

By performing the same steps as done in the case of conditional marginal mgf, we can derive the conditional joint mgf of $(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k)$ as follows.

Theorem 16 *Conditional on the set S, the joint mgf of $(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k)$ is given by*

$$
\begin{aligned}
& M_{\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k|S}(t_1, \cdots, t_k) \\
= & \sum_{d=0}^{r-1} \frac{1}{P(\prod_{h=1}^k M_r(h) \neq 0 | D = d)} \sum_{\mathbf{t} \in \mathbf{T}_d} \sum_{\tilde{\delta} \in Q_d^*(\mathbf{t})} c_d \prod_{h=1}^k \theta_h^{-M_d(h)} \sum_{l'=0}^{d} \psi_{l'd} \\
& \times \prod_{\substack{j=0 \\ j \neq l'}}^{d} \frac{1}{1 - \sum_{l=1}^k t_l \beta_{l,l'j}^{(d)}} \exp\left\{ \frac{\sum_{l=1}^k t_l T(n_l - M_{d-l'}(l))}{M_d(l)} \right\} \\
& + \sum_{d=r}^{N} \frac{1}{P(\prod_{h=1}^k M_d(h) \neq 0 | D = d)} \sum_{\mathbf{t} \in \mathbf{T}_r} \sum_{\tilde{\delta} \in Q_r^*(\mathbf{t})} c_r \prod_{h=1}^k \theta_h^{-M_r(h)} \sum_{l'=0}^{r} \tilde{\varphi}_{l'r} \\
& \times \frac{1}{1 - \sum_{l=1}^k t_l \beta_{lj}^*} \prod_{\substack{j=0 \\ j \neq l'}}^{r} \frac{1}{1 - \sum_{l=1}^k t_l \beta_{l,l'j}^{(r)}} \exp\left\{ \frac{\sum_{l=1}^k t_l T(n_l - M_{d-l'}(l))}{M_r(l)} \right\},
\end{aligned}
\tag{5.1.15}
$$

where $\psi_{l'd}$ *and* $\tilde{\varphi}_{l'r}$ *are as defined in (4.2.10) and (5.1.8), for* $1 \le l \le k$ *and* $0 \le l' \le d$, $0 \le j \le d$, $\beta^{(d)}_{l,l'j}$ *determined by (4.2.16),* β^*_{lj} *determined by (4.2.17).*

Corollary 8 *From (5.1.15), we obtain* $E(\hat{\theta}_{l_1}\hat{\theta}_{l_2})$ *to be*

$$\begin{aligned}
E(\hat{\theta}_{l_1}\hat{\theta}_{l_2}) \;=\; & \sum_{d=0}^{r-1} \frac{1}{P(\prod_{h=1}^{k} M_r(h) \neq 0 | D = d)} \sum_{\mathbf{t}\in\mathbf{T}_d} \sum_{\tilde{\delta}\in Q^*_d(\mathbf{t})} c_d \prod_{h=1}^{k} \theta_h^{-M_d(h)} \sum_{l'=0}^{d} \psi_{l'd} \\
& \times \left\{ \left(\sum_{j=0; j\neq l'}^{d} \beta^{(d)}_{l_1,l'j} + \frac{T(n_{l_1} - M_{d-l'}(l_1))}{M_d(l_1)} \right) \right. \\
& \left. \times \left(\sum_{j=0; j\neq l'}^{d} \beta^{(d)}_{l_2,l'j} + \frac{T(n_{l_2} - M_{d-l'}(l_2))}{M_d(l_2)} \right) + \sum_{j=0; j\neq l'}^{d} \beta^{(d)}_{l_1,l'j}\beta^{(d)}_{l_2,l'j} \right\} \\
& + \sum_{d=r}^{N} \frac{1}{P(\prod_{h=1}^{k} M_d(h) \neq 0 | D = d)} \sum_{\mathbf{t}\in\mathbf{T}_r} \sum_{\tilde{\delta}\in Q^*_r(\mathbf{t})} c_r \prod_{h=1}^{k} \theta_h^{-M_r(h)} \sum_{l'=0}^{r} \tilde{\varphi}_{l'r} \\
& \times \left\{ \left(\sum_{\substack{j=0 \\ j\neq l'}}^{r} \beta^{(r)}_{l_1,l'j} + \beta^*_{l_1,r-l'+1} + \frac{T(n_{l_1} - M_{d-l'}(l_1))}{M_r(l_1)} \right) \right. \\
& \times \left(\sum_{\substack{j=0 \\ j\neq l'}}^{r} \beta^{(r)}_{l_2,l'j} + \beta^*_{l_2,r-l'+1} + \frac{T(n_{l_2} - M_{d-l'}(l_2))}{M_r(l_2)} \right) \\
& \left. + \sum_{\substack{j=0 \\ j\neq l'}}^{r} \beta^{(r)}_{l_1,l'j}\beta^{(r)}_{l_2,l'j} + \beta^*_{l_1,r-l'+1}\beta^*_{l_2,r-l'+1} \right\}. \qquad (5.1.16)
\end{aligned}$$

From the above corollary, the covariance and correlation coefficient between MLEs $\hat{\theta}_{l_1}$ and $\hat{\theta}_{l_2}$ can also be readily obtained.

5.2 Approximate Confidence Intervals

Let $I(\theta_1, \theta_2, \cdots, \theta_k) = (I_{i,j}(\theta_1, \theta_2, \cdots, \theta_k)), i, j = 1, 2, \cdots, k$, denote the Fisher information matrix of the parameters $\theta_1, \theta_2, \cdots, \theta_k$, where

$$I_{i,j}(\theta_1, \theta_2, \cdots, \theta_k) = -E\left(\frac{\partial^2 \ln L}{\partial\theta_i \partial\theta_j}\right) \tag{5.2.1}$$

From the likelihood function in (4.2.4), we have $I_{i,j}(\theta_1, \theta_2, \cdots, \theta_k) = 0$ if $i \neq j$. Consequently, we have

$$\begin{aligned} & I(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k) \\ = \; & -\text{Diag}\left(\frac{\partial^2 \ln L}{\partial\theta_1^2}|_{\theta_1=\hat{\theta}_1}, \frac{\partial^2 \ln L}{\partial\theta_2^2}|_{\theta_2=\hat{\theta}_2}, \cdots, \frac{\partial^2 \ln L}{\partial\theta_k^2}|_{\theta_k=\hat{\theta}_k}\right), \end{aligned} \tag{5.2.2}$$

where, for $d = 0, 1, 2, \cdots, r-1$,

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial\theta_h^2}|_{\theta_h=\hat{\theta}_h} &= \frac{M_d(h)}{\hat{\theta}_h^2} - \frac{2\left\{\sum\limits_{i=1}^{d} w_i\delta_i(h) + T(n_h - M_d(h))\right\}}{\hat{\theta}_h^3} \\ &= -\frac{M_d(h)}{\hat{\theta}_h^2}, \end{aligned}$$

and for $d = r, r+1, \cdots, N$,

$$\begin{aligned}
\frac{\partial^2 \ln L}{\partial \theta_h^2}\Big|_{\theta_h=\hat{\theta}_h} &= \frac{M_r(h)}{\hat{\theta}_h^2} - \frac{2\left\{\sum_{i=1}^{r} w_i \delta_i(h) + w_r(n_h - M_r(h))\right\}}{\hat{\theta}_h^3} \\
&= -\frac{M_r(h)}{\hat{\theta}_h^2}.
\end{aligned}$$

Then, by using the asymptotic normality of the MLEs, we have $\hat{\theta}_h - \theta_h \sim N(0, I_{h,h}^{-1})$. With $\tilde{d} = \min\{d, r\}$, we can express the approximate $100(1-\alpha)\%$ confidence interval for θ_h, $1 \le h \le k$, as

$$\begin{cases}
\hat{\theta}_h \pm Z_{\alpha/2} \frac{\sum_{i=1}^{\tilde{d}} w_i \delta_i(h) + T(n_h - M_{\tilde{d}}(h))}{M_{\tilde{d}}(h)^{\frac{3}{2}}} = \hat{\theta}_h\left(1 \pm \frac{Z_{\alpha/2}}{M_{\tilde{d}}(h)^{\frac{1}{2}}}\right), \\
\qquad\qquad \text{when } d = 0, 1, 2, \cdots, r-1 \\
\hat{\theta}_h \pm Z_{\alpha/2} \frac{\sum_{i=1}^{\tilde{d}} w_i \delta_i(h) + w_r(n_h - M_{\tilde{d}}(h))}{M_{\tilde{d}}(h)^{\frac{3}{2}}} = \hat{\theta}_h\left(1 \pm \frac{Z_{\alpha/2}}{M_{\tilde{d}}(h)^{\frac{1}{2}}}\right), \\
\qquad\qquad \text{when } d = r, r+1, \cdots, N,
\end{cases}$$

where $Z_{\alpha/2}$ denotes the upper $\alpha/2$ percentage point of the standard normal distribution.

5.3 Bayesian Intervals

Let

$$u_h = \begin{cases} \sum_{i=1}^{d} w_i\delta_i(h) + T(n_h - M_d(h)), & \text{when } d = 0, 1, 2, \cdots, r-1, \\ \sum_{i=1}^{r} w_i\delta_i(h) + w_r(n_h - M_r(h)), & \text{when } d = r, r+1, \cdots, N. \end{cases}$$

Then, we can rewrite the likelihood function as

$$L(\theta_1, \theta_2, \cdots, \theta_k, \boldsymbol{\delta}, \mathbf{w}) = c_r \prod_{h=1}^{k} \theta_h^{-M_{\tilde{d}}(h)} \exp(-\frac{u_h}{\theta_h}) \tag{5.3.1}$$

Now, by assuming independent inverse gamma prior distributions, viz., $IG(a_h, b_h)$ for $1 \le h \le k$, we obtain from (5.3.1) the posterior joint density function as

$$L(\theta_1, \theta_2, \cdots, \theta_k | data) = c_{\tilde{d}} \prod_{h=1}^{k} \theta_h^{-M_r(h) - a_h - 1} \exp\left(-\frac{u_h + b_h}{\theta_h}\right).$$

Upon comparing this with (5.3.1), we see that the joint posterior density function of $(\theta_1, \theta_2, \cdots, \theta_k)$ is a product of k independent inverse gamma density functions. So, given the data, the posterior density function of $\hat{\theta}_h$ is simply $IG(M_{\tilde{d}}(h) + a_h, u_h + b_h)$. Thus, the Bayes estimator of θ_h under the squared-error loss function is

$$\hat{\theta}_{h,Bayes} = \frac{u_h + b_h}{M_{\tilde{d}}(h) + a_h - 1}, \qquad h = 1, 2, \cdots, k. \tag{5.3.2}$$

When we use Jeffreys' non-informative prior $I(\theta_h) \propto \frac{1}{\theta_h^2}$ corresponding to the special case when $a_h = 1$ and $b_h = 0$, for $1 \leq h \leq k$, the Bayes estimators in (5.3.2) coincide with the MLEs in (4.2.3).

Let $U_h = \frac{2(u_h+b_h)}{\theta_h}$ for $1 \leq h \leq k$. Then, the pivot U_h follows $\chi^2_{2(M_r(h)+a_h)}$ distribution, provided $2(M_{\tilde{d}}(h) + a_h)$ is a positive integer, for $1 \leq h \leq k$. In this case, the $100(1-\alpha)\%$ Bayes credible interval for θ_h becomes

$$\left(\frac{2(u_h + b_h)}{\chi^2_{2(M_{\tilde{d}}(h)+a_h),1-\alpha/2}}, \frac{2(u_h + b_h)}{\chi^2_{2(M_{\tilde{d}}(h)+a_h),\alpha/2}} \right), \qquad h = 1, 2, \cdots, k,$$

where $\chi^2_{v,\frac{\alpha}{2}}$ is the lower $\frac{\alpha}{2}$ percentage point of the chi-square distribution with v degrees of freedom.

5.4 Bootstrap Intervals

In this section, we consider confidence interval for θ_h $(h = 1, 2, \cdots, k)$ based on the Bootstrap-p and Bootstrap-t methods; see, for example, Efron and Tibshirani (1994).

To find the Bootstrap-p and Bootstrap-t intervals, in the first step, we generate original samples from k exponential populations with parameters θ_h of size n_h, $1 \leq$

$h \le k$. Next we sort the data, and determine to which population each failure belongs, and then estimate θ_h using the conditional MLE in (4.2.3). In the second step, we generate a bootstrap sample $(\delta_1, \delta_2, \cdots, \delta_r; w_1, w_2, \cdots, w_r)$ by using the values $(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k)$, and then obtain the bootstrap estimates of θ_h, $1 \le h \le k$, say θ_h^*, $1 \le h \le k$, from the bootstrap sample. In the third step, we repeat the second step N-Boot times.

Boot-p: Suppose $K_{hB}(x) = P(\hat{\theta}_h^* \le x)$, $1 \le h \le k$, is the cumulative distribution function of $\hat{\theta}_h^*$. Define $\hat{\theta}_{hB}(\alpha) = K_{hB}^{-1}(\alpha)$, $1 \le h \le k$, for a given α. Then, the $100(1-\alpha)\%$ Boot-p confidence interval for θ_h, $1 \le h \le k$, is given by

$$\left(\hat{\theta}_{hB}\left(\frac{\alpha}{2}\right), \hat{\theta}_{hB}\left(1 - \frac{\alpha}{2}\right)\right), \qquad 1 \le h \le k.$$

Boot-t: After generating the bootstrap samples in the second step and calculating $\hat{\theta}_h^*$, we need to use them to compute the estimate of $Var(\hat{\theta}_h^*)$ from the observed Fisher information matrix in (5.2.2). Then, we determine the T_h^* statistic given by

$$T_h^* = \frac{\hat{\theta}_h^* - \hat{\theta}_h}{\sqrt{\widehat{\mathrm{Var}}(\hat{\theta}_h^*)}}, \qquad 1 \le h \le k. \tag{5.4.1}$$

Now, suppose $L_{hB}(x) = P(T_h^* \le x)$ is the cumulative distribution function of T_h^*, $1 \le h \le k$. Then, the $100(1-\alpha)\%$ Boot-t confidence interval for θ_h, $1 \le h \le k$, is

given by

$$\left(\hat{\theta}_h - \sqrt{\widehat{\mathrm{Var}}(\hat{\theta}_h)}L_{hB}^{-1}\left(1-\frac{\alpha}{2}\right), \hat{\theta}_h - \sqrt{\widehat{\mathrm{Var}}(\hat{\theta}_h)}L_{hB}^{-1}\left(\frac{\alpha}{2}\right)\right), \qquad 1 \le h \le k.$$

5.5 Simulation Results and Discussion

A simulation study was carried out to evaluate the performance of the conditional MLEs discussed in the preceding sections. We considered sample sizes for the three populations as $n = (6, 6, 6)$, and different choice for r and T. We also chose the parameters $(\theta_1, \theta_2, \theta_3)$ to be $(2, 4, 7)$ and $(3, 5, 9)$. For these cases, we computed the conditional MLEs for the parameters $(\theta_1, \theta_2, \theta_3)$ and the empirical values of their means, standard deviations, mean square errors and covariance matrices for different choices of r and T. The results of these obtained from 10,000 Monto Carlo simulations are presented in Tables 5.1-5.4. From the results presented in these tables, we observe that while the estimate of θ_1 is very stable even for small r and T, the estimate of θ_2 and θ_3 become stable only for larger values of r and T. This is to be expected since when θ_1 is smaller than θ_2 and θ_3, when r and T are small, most of the failures observed would have resulted from the exponential population with parameter θ_1 and very few failures would have come from the exponential populations with parameters θ_2 and θ_3. This does get rectified when r and T are increased, as one would expect.

Table 5.1: The average values of the conditional MLEs and the estimates of their standard deviations and mean square errors when $\theta = (2, 4, 7)$ and $n = (6, 6, 6)$ for different choices of r and T.

		$Mean$	$\widehat{SD}$	$\widehat{RMSE}$
r	T	$(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$	$(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$	$(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$
6	2.5	$(2.64, 4.85, 6.02)$	$(1.8448, 3.1517, 3.2607)$	$(1.9524, 3.2639, 3.4057)$
6	3.0	$(2.64, 4.86, 6.03)$	$(1.8575, 3.1810, 3.3663)$	$(1.9656, 3.2945, 3.5039)$
6	3.5	$(2.60, 4.90, 6.08)$	$(1.8095, 3.2342, 3.3920)$	$(1.9057, 3.3581, 3.5153)$
6	4.0	$(2.62, 4.92, 6.08)$	$(1.7921, 3.2811, 3.4768)$	$(1.8968, 3.4078, 3.5968)$
6	4.5	$(2.65, 4.90, 6.13)$	$(1.8234, 3.2431, 3.4588)$	$(1.9343, 3.3650, 3.5673)$
9	3.5	$(2.23, 5.10, 8.00)$	$(1.3573, 3.8210, 5.0298)$	$(1.3775, 3.9772, 5.1282)$
9	4.5	$(2.25, 4.97, 8.18)$	$(1.2793, 3.6423, 5.3800)$	$(1.3025, 3.7687, 5.5075)$
9	5.5	$(2.24, 5.07, 8.15)$	$(1.2429, 3.7526, 5.5242)$	$(1.2650, 3.9022, 5.6432)$
9	6.5	$(2.25, 5.04, 8.12)$	$(1.3006, 3.7318, 5.4163)$	$(1.3240, 3.8743, 5.5306)$
9	7.5	$(2.22, 5.00, 8.13)$	$(1.2211, 3.6687, 5.4763)$	$(1.2414, 3.8035, 5.5910)$
12	4.5	$(2.11, 4.66, 8.74)$	$(1.0505, 3.3205, 6.3170)$	$(1.0567, 3.3860, 6.5516)$
12	5.5	$(2.10, 4.57, 8.65)$	$(0.9916, 2.9646, 6.4824)$	$(0.9965, 3.0197, 6.6886)$
12	6.5	$(2.10, 4.50, 8.67)$	$(1.0110, 2.7102, 6.5214)$	$(1.0159, 2.7552, 6.7313)$
12	7.5	$(2.09, 4.52, 8.63)$	$(0.9705, 2.8129, 6.5465)$	$(0.9744, 2.8606, 6.7456)$
12	8.5	$(2.09, 4.42, 8.64)$	$(0.9915, 2.6112, 6.5933)$	$(0.9957, 2.6442, 6.7942)$
15	7.0	$(2.03, 4.40, 8.52)$	$(0.8845, 2.4737, 6.3454)$	$(0.8852, 2.5052, 6.5238)$
15	9.0	$(2.04, 4.21, 8.00)$	$(0.8814, 2.0746, 5.4620)$	$(0.8822, 2.0856, 5.5524)$
15	11.0	$(2.03, 4.19, 7.68)$	$(0.8727, 2.0325, 4.6932)$	$(0.8733, 2.0417, 4.7426)$
15	13.0	$(2.03, 4.20, 7.58)$	$(0.8661, 1.9593, 4.4530)$	$(0.8665, 1.9692, 4.4904)$
15	15.0	$(2.05, 4.17, 7.59)$	$(0.8864, 1.9444, 4.2571)$	$(0.8878, 1.9522, 4.2982)$

Table 5.2: The average values of the conditional MLEs and the estimates of their standard deviations and mean square errors when $\theta = (3, 5, 9)$ and $n = (6, 6, 6)$ for different choices of r and T.

		$Mean$	$\widehat{SD}$	$\widehat{RMSE}$
r	T	$(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$	$(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$	$(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$
6	3.5	(3.96, 6.27, 8.10)	(2.8759, 4.1993, 4.5740)	(3.0328, 4.3866, 4.6615)
6	4.5	(3.94, 6.30, 8.09)	(2.8265, 4.2987, 4.7074)	(2.9794, 4.4921, 4.7944)
6	5.5	(3.98, 6.34, 8.18)	(2.8381, 4.3232, 4.7552)	(3.0018, 4.5270, 4.8253)
6	6.5	(3.94, 6.38, 8.15)	(2.7565, 4.3702, 4.7941)	(2.9134, 4.5838, 4.8683)
6	7.5	(3.92, 6.35, 8.12)	(2.7691, 4.3649, 4.7673)	(2.9175, 4.5697, 4.8475)
9	7.0	(3.42, 6.10, 10.47)	(2.0110, 4.3496, 7.0004)	(2.0540, 4.4856, 7.1521)
9	8.5	(3.39, 6.15, 10.55)	(1.9791, 4.3885, 7.1121)	(2.0168, 4.5358, 7.2800)
9	10.0	(3.41, 6.14, 10.53)	(2.0024, 4.4430, 7.0558)	(2.0436, 4.5872, 7.2192)
9	11.5	(3.39, 6.18, 10.54)	(1.9653, 4.5188, 7.1686)	(2.0027, 4.6706, 7.3318)
9	13.0	(3.39, 6.14, 10.67)	(2.0154, 4.4742, 7.2024)	(2.0522, 4.6173, 7.3944)
12	10.5	(3.17, 5.56, 11.07)	(1.5196, 3.2933, 8.4661)	(1.5289, 3.3411, 8.7165)
12	12.0	(3.18, 5.53, 11.05)	(1.5083, 3.1974, 8.1906)	(1.5193, 3.2412, 8.4436)
12	13.5	(3.16, 5.48, 11.14)	(1.5565, 3.0776, 8.4199)	(1.5646, 3.1142, 8.6866)
12	15.0	(3.15, 5.55, 11.13)	(1.5225, 3.2389, 8.5639)	(1.5302, 3.2847, 8.8255)
12	16.5	(3.13, 5.53, 11.03)	(1.4681, 3.1161, 8.3438)	(1.4741, 3.1602, 8.5862)
15	15.5	(3.07, 5.23, 9.98)	(1.3277, 2.4516, 5.9225)	(1.3294, 2.4625, 6.0033)
15	18.0	(3.05, 5.23, 9.73)	(1.3260, 2.4083, 5.2926)	(1.3270, 2.4189, 5.3433)
15	20.5	(3.08, 5.20, 9.70)	(1.3282, 2.3761, 5.1418)	(1.3304, 2.3843, 5.1889)
15	23.0	(3.06, 5.16, 9.67)	(1.3306, 2.4062, 4.9658)	(1.3320, 2.4115, 5.0109)
15	25.5	(3.06, 5.19, 9.63)	(1.3362, 2.3993, 4.9349)	(1.3376, 2.4068, 4.9747)

Table 5.3: The average values of the estimates of the covariance matrix of the conditional MLEs when $\theta = (2, 4, 7)$ and $n = (6, 6, 6)$ for different choices of r and T.

r	T	$\theta = (2,4,7)$ Covariance matrix $(\rho(\theta_i, \theta_j))_{i,j}$	T	$\theta = (3,5,9)$ Covariance matrix $(\rho(\theta_i, \theta_j))_{i,j}$
6	2.5	$\begin{pmatrix} 3.403 & 0.173 & 1.155 \\ 0.173 & 9.933 & 3.651 \\ 1.155 & 3.651 & 10.632 \end{pmatrix}$	3.0	$\begin{pmatrix} 8.271 & 0.020 & 2.310 \\ 0.020 & 17.634 & 6.583 \\ 2.310 & 6.583 & 20.921 \end{pmatrix}$
6	3.0	$\begin{pmatrix} 3.450 & -0.044 & 1.040 \\ -0.044 & 10.118 & 4.017 \\ 1.040 & 4.017 & 11.332 \end{pmatrix}$	4.0	$\begin{pmatrix} 7.989 & -0.106 & 2.504 \\ -0.106 & 18.479 & 6.577 \\ 2.504 & 6.577 & 22.159 \end{pmatrix}$
6	3.5	$\begin{pmatrix} 3.274 & -0.024 & 1.067 \\ -0.024 & 10.460 & 4.092 \\ 1.067 & 4.092 & 11.506 \end{pmatrix}$	5.0	$\begin{pmatrix} 8.055 & -0.569 & 2.464 \\ -0.569 & 18.690 & 6.927 \\ 2.464 & 6.927 & 22.612 \end{pmatrix}$
6	4.0	$\begin{pmatrix} 3.212 & 0.095 & 1.103 \\ 0.095 & 10.765 & 4.320 \\ 1.103 & 4.320 & 12.088 \end{pmatrix}$	6.0	$\begin{pmatrix} 7.598 & -0.205 & 2.282 \\ -0.205 & 19.098 & 6.948 \\ 2.282 & 6.948 & 22.984 \end{pmatrix}$
6	4.5	$\begin{pmatrix} 3.325 & -0.046 & 1.059 \\ -0.046 & 10.518 & 4.249 \\ 1.059 & 4.249 & 11.963 \end{pmatrix}$	7.0	$\begin{pmatrix} 7.668 & -0.145 & 1.997 \\ -0.145 & 19.053 & 7.154 \\ 1.997 & 7.154 & 22.727 \end{pmatrix}$
9	3.5	$\begin{pmatrix} 1.842 & -0.169 & 0.286 \\ -0.169 & 14.600 & 1.371 \\ 0.286 & 1.371 & 25.299 \end{pmatrix}$	6.5	$\begin{pmatrix} 4.044 & -0.700 & 0.056 \\ -0.700 & 18.919 & 0.622 \\ 0.056 & 0.622 & 49.006 \end{pmatrix}$
9	4.5	$\begin{pmatrix} 1.637 & -0.275 & 0.218 \\ -0.275 & 13.266 & 1.085 \\ 0.218 & 1.085 & 28.944 \end{pmatrix}$	8.0	$\begin{pmatrix} 3.917 & -0.549 & 0.398 \\ -0.549 & 19.259 & 0.537 \\ 0.398 & 0.537 & 50.582 \end{pmatrix}$
9	5.5	$\begin{pmatrix} 1.545 & -0.292 & 0.245 \\ -0.292 & 14.082 & 0.612 \\ 0.245 & 0.612 & 30.517 \end{pmatrix}$	9.5	$\begin{pmatrix} 4.009 & -0.572 & 0.030 \\ -0.572 & 19.740 & 0.108 \\ 0.030 & 0.108 & 49.785 \end{pmatrix}$
9	6.5	$\begin{pmatrix} 1.692 & -0.397 & 0.187 \\ -0.397 & 13.926 & 0.224 \\ 0.187 & 0.224 & 29.337 \end{pmatrix}$	11.0	$\begin{pmatrix} 3.862 & -0.690 & 0.384 \\ -0.690 & 20.420 & 0.494 \\ 0.384 & 0.494 & 51.389 \end{pmatrix}$
9	7.5	$\begin{pmatrix} 1.491 & -0.362 & 0.271 \\ -0.362 & 13.459 & 0.502 \\ 0.271 & 0.502 & 29.990 \end{pmatrix}$	12.5	$\begin{pmatrix} 4.062 & -0.584 & 0.309 \\ -0.584 & 20.018 & 0.540 \\ 0.309 & 0.540 & 51.875 \end{pmatrix}$

Table 5.4: The average values of the estimates of the covariance matrix of the conditional MLEs when $\theta = (3, 5, 9)$ and $n = (6, 6, 6)$ for different choices of r and T.

r	T	$\theta = (2, 4, 7)$ Covariance matrix $(\rho(\theta_i, \theta_j))_{i,j}$	T	$\theta = (3, 5, 9)$ Covariance matrix $(\rho(\theta_i, \theta_j))_{i,j}$
12	4.5	$\begin{pmatrix} 1.104 & -0.035 & -0.014 \\ -0.035 & 11.026 & 0.513 \\ -0.014 & 0.513 & 39.905 \end{pmatrix}$	10.0	$\begin{pmatrix} 2.309 & -0.176 & -0.382 \\ -0.176 & 10.846 & -1.070 \\ -0.382 & -1.070 & 71.675 \end{pmatrix}$
12	5.5	$\begin{pmatrix} 0.983 & -0.028 & 0.009 \\ -0.028 & 8.789 & -0.504 \\ 0.009 & -0.504 & 42.022 \end{pmatrix}$	11.5	$\begin{pmatrix} 2.275 & -0.167 & -0.433 \\ -0.167 & 10.223 & -1.759 \\ -0.433 & -1.759 & 67.087 \end{pmatrix}$
12	6.5	$\begin{pmatrix} 1.022 & -0.087 & -0.245 \\ -0.087 & 7.345 & -0.576 \\ -0.245 & -0.576 & 42.529 \end{pmatrix}$	13.0	$\begin{pmatrix} 2.423 & -0.162 & -0.547 \\ -0.162 & 9.472 & -1.635 \\ -0.547 & -1.635 & 70.896 \end{pmatrix}$
12	7.5	$\begin{pmatrix} 0.942 & -0.067 & -0.098 \\ -0.067 & 7.912 & -1.508 \\ -0.098 & -1.508 & 42.857 \end{pmatrix}$	14.5	$\begin{pmatrix} 2.318 & -0.207 & -0.502 \\ -0.207 & 10.490 & -2.052 \\ -0.502 & -2.052 & 73.340 \end{pmatrix}$
12	8.5	$\begin{pmatrix} 0.983 & -0.074 & -0.242 \\ -0.074 & 6.818 & -1.227 \\ -0.242 & -1.227 & 43.471 \end{pmatrix}$	16.0	$\begin{pmatrix} 2.155 & -0.122 & -0.329 \\ -0.122 & 9.710 & -1.781 \\ -0.329 & -1.781 & 69.619 \end{pmatrix}$
15	7.0	$\begin{pmatrix} 0.782 & 0.004 & -0.008 \\ 0.004 & 6.119 & 0.179 \\ -0.008 & 0.179 & 40.265 \end{pmatrix}$	15.0	$\begin{pmatrix} 1.763 & -0.061 & -0.215 \\ -0.061 & 6.010 & -0.369 \\ -0.215 & -0.369 & 35.076 \end{pmatrix}$
15	9.0	$\begin{pmatrix} 0.777 & -0.040 & 0.006 \\ -0.040 & 4.304 & 0.070 \\ 0.006 & 0.070 & 29.834 \end{pmatrix}$	17.5	$\begin{pmatrix} 1.758 & -0.107 & -0.122 \\ -0.107 & 5.800 & -0.199 \\ -0.122 & -0.199 & 28.011 \end{pmatrix}$
15	11.0	$\begin{pmatrix} 0.762 & -0.031 & -0.047 \\ -0.031 & 4.131 & 0.064 \\ -0.047 & 0.064 & 22.027 \end{pmatrix}$	20.0	$\begin{pmatrix} 1.764 & -0.120 & -0.138 \\ -0.120 & 5.646 & -0.289 \\ -0.138 & -0.289 & 26.438 \end{pmatrix}$
15	13.0	$\begin{pmatrix} 0.750 & -0.017 & -0.042 \\ -0.017 & 3.839 & 0.040 \\ -0.042 & 0.040 & 19.829 \end{pmatrix}$	22.5	$\begin{pmatrix} 1.771 & -0.039 & -0.022 \\ -0.039 & 5.790 & -0.421 \\ -0.022 & -0.421 & 24.659 \end{pmatrix}$
15	15.0	$\begin{pmatrix} 0.786 & -0.020 & -0.007 \\ -0.020 & 3.781 & -0.212 \\ -0.007 & -0.212 & 18.123 \end{pmatrix}$	25.0	$\begin{pmatrix} 1.786 & -0.069 & -0.109 \\ -0.069 & 5.757 & -0.370 \\ -0.109 & -0.370 & 24.354 \end{pmatrix}$

Chapter 6

Concluding Remarks

6.1 Summary of Work

In this thesis, by considering k exponential populations, we have developed exact inferential methods based on four different joint censoring schemes— (i) jointly Type-II censored sample, (ii) jointly progressively Type-II censored sample, (iii) jointly Type-I hybrid censored sample, and (iv) jointly Type-II hybrid censored sample. For each of these cases, we have derived the conditional MLEs of the k exponential mean parameters, and have derived their conditional moment generating functions and exact densities, using which we have then developed exact confidence intervals for the k exponential parameters. Furthermore, approximate confidence intervals based on the asymptotic normality of the MLEs, parametric bootstrap intervals, and credible confidence regions from a Bayesian viewpoint have all been discussed.

For different censoring schemes, a simulation study has been carried out to evaluate the performance of the conditional MLEs of the parameters $(\theta_1, \theta_2, \theta_3)$ for different choices of r and (n_1, n_2, n_3). From these results, it is clear that the MLEs have a moderate bias when the essential sample size r is small even when the sample sizes (n_1, n_2, n_3) are not small. However, the bias of the conditional MLEs become negligible when r increases.

An empirical evaluation is also made of all the confidence intervals. We also computed exact 95% confidence intervals for some small values of r. From these results, it is clear that the exact conditional method has its coverage probability to be quite close to the nominal level of 95% always, while the approximate method is not at all satisfactory. We also observe that between the two bootstrap methods, the Boot-p method performs better than the Boot-t method; the Bayesian credible interval has very stable coverage probabilities (quite close to the nominal level of 95%). Moreover, we observe that the approximate and bootstrap methods have lower coverage probabilities when n_1, n_2, n_3 are small. Also, we have presented examples to illustrate all the methods of inference developed here for different joint censoring scenarios. The importance of the exact method developed in this thesis becomes clear as it provides exact conditional confidence intervals with accurate coverage probabilities (compared to the nominal confidence levels) even for small sample sizes and small values of r.

However, the exact method becomes computationally quite intensive when r is large. Hence, we would recommend the use of the exact conditional confidence intervals for θ_h developed here whenever possible and especially when the sample sizes are small; but when the sample sizes get larger with a large r, the computational complexity increases in the exact conditional method, and in this case the Boot-p method and the Bayesian method are computationally simpler to use and they also possess good performance, in the interval estimation of parameters. We, therefore would recommend one of these two methods for the interval estimation.

6.2 Possible Further Research

From the research work carried out in this thesis, we identify the following problems that will be of great interest for further research:

- In the development of exact confidence intervals, we assumed that $P(\hat{\theta}_h > b)$ is an increasing function of θ_h when all other θ_j's are fixed for $j \neq h$ $(1 \leq h \leq k)$. It would be useful to establish this result formally;

- Type-I and Type-II censoring schemes are the most common and popular censoring schemes. As we have seen, Type-I censoring scheme has the advantage of fixed experimental time, but may end up with very few observed failures at the end of the experiment. Type-II censoring scheme has the advantage of having at least a certain number of observed failures, but may take a long time to terminate the experiment. For this reason, some Generalized HCS and Unified HCS have been proposed in the literature; see, for example, Balakrishnan and

Kundu (2013). It will be of interest to develop exact inferential results analogous to those developed in the thesis for these Generalized HCS and Unified HCS when they are jointly implemented in the k samples;

- Recently, Childs et al. (2012) considered two-parameter exponential distribution and developed exact inferential results under HCS. Following this line, it will be of interest to consider the case of two-parameter exponential distribution and then develop exact inferential results analogous to those developed in the thesis for various forms of jointly censored data.

Appendix A

Proof of Some Lemmas

Lemma 5 *For fixed* l*,* $1 \leq l \leq k$*, suppose* $\{\beta^*_{li}\}_{i=1}^{r'}$ *are distinct. Let us define the functions* $h_{i'}(t) = \prod_{i \neq i'} (1 - \beta^*_{li} t)^{-\alpha_i}$*, for* $i, i' = 1, 2, \cdots, r'$*. Then, the coefficients* $A^{(j)}_{li}$ *in expression (2.2.15) are determined as follows:*

$$\prod_{i=1}^{r'} (1 - \beta^*_{li} t)^{-\alpha_i} = \sum_{i=1}^{r'} \sum_{j=1}^{\alpha_i} A^{(j)}_{li} (1 - \beta^*_{li} t)^{-j},$$

where

$$
\begin{aligned}
A_{li'}^{(\alpha_{i'})} &= h_{i'}\left(\frac{1}{\beta_{li'}^*}\right) = \prod_{i\neq i'}\left(\frac{\beta_{li'}^*}{\beta_{li'}^* - \beta_{li}^*}\right)^{\alpha_i}, & \text{(A.0.1)}\\
A_{li'}^{(\alpha_{i'}-1)} &= -\sum_{i\neq i'}\frac{\alpha_i\beta_{li}^*}{\beta_{li'}^* - \beta_{li}^*}h_{i'}\left(\frac{1}{\beta_{li'}^*}\right), & \text{(A.0.2)}\\
A_{li'}^{(\alpha_{i'}-2)} &= \frac{1}{2}\left\{\left(\sum_{i\neq i'}\frac{\alpha_i\beta_{li}^*}{\beta_{li'}^* - \beta_{li}^*}\right)^2 + \sum_{i\neq i'}\frac{\alpha_i(\beta_{li}^*)^2}{(\beta_{li'}^* - \beta_{li}^*)^2}\right\}h_{i'}\left(\frac{1}{\beta_{li'}^*}\right), & \text{(A.0.3)}\\
A_{li'}^{(\alpha_{i'}-3)} &= -\frac{1}{3!}\left\{\left(\sum_{i\neq i'}\frac{\alpha_i\beta_{li}^*}{\beta_{li'}^* - \beta_{li}^*}\right)^3 + 3\sum_{i\neq i'}\frac{\alpha_i\beta_{li}^*}{\beta_{li'}^* - \beta_{li}^*}\sum_{i\neq i'}\frac{\alpha_i(\beta_{li}^*)^2}{(\beta_{li'}^* - \beta_{li}^*)^2}\right.\\
&\quad \left. + 2\sum_{i\neq i'}\frac{\alpha_i(\beta_{li}^*)^3}{(\beta_{li'}^* - \beta_{li}^*)^3}\right\}h_{i'}\left(\frac{1}{\beta_{li'}^*}\right), & \text{(A.0.4)}\\
A_{li'}^{(\alpha_{i'}-4)} &= \frac{1}{4!}\left\{\left(\sum_{i\neq i'}\frac{\alpha_i\beta_{li}^*}{\beta_{li'}^* - \beta_{li}^*}\right)^4 + 3\left(\sum_{i\neq i'}\frac{\alpha_i\beta_{li}^*}{\beta_{li'}^* - \beta_{li}^*}\right)^2\sum_{i\neq i'}\frac{\alpha_i(\beta_{li}^*)^2}{(\beta_{li'}^* - \beta_{li}^*)^2}\right.\\
&\quad +8\sum_{i\neq i'}\frac{\alpha_i\beta_{li}^*}{\beta_{li'}^* - \beta_{li}^*}\sum_{i\neq i'}\frac{\alpha_i(\beta_{li}^*)^3}{(\beta_{li'}^* - \beta_{li}^*)^3} + 6\left(\sum_{i\neq i'}\frac{\alpha_i(\beta_{li}^*)^2}{(\beta_{li'}^* - \beta_{li}^*)^2}\right)^2\\
&\quad \left. +3!\sum_{i\neq i'}\frac{\alpha_i(\beta_{li}^*)^4}{(\beta_{li'}^* - \beta_{li}^*)^4}\right\}h_{i'}\left(\frac{1}{\beta_{li'}^*}\right), & \text{(A.0.5)}
\end{aligned}
$$

and so on.

Proof The coefficients $A_{li}^{(j)}$ are determined from the following partial function identity:

$$\prod_{i=1}^{r'}(1-\beta_{li}^* t)^{-\alpha_i} = \sum_{i=1}^{r'}\sum_{j=1}^{\alpha_i} A_{li}^{(j)}(1-\beta_{li}^* t)^{-j}.$$

For fixed l and i', we have

$$\begin{aligned} A_{li'}^{(\alpha_{i'})} &= \prod_{i\neq i'}(1-\beta_{li}^* t)^{-\alpha_i} - (1-\beta_{li'}^* t)^{\alpha_{i'}} \sum_{i\neq i'}\sum_{j=1}^{\alpha_i} A_{li}^{(j)}(1-\beta_{li}^* t)^{-j} \\ &\quad - \sum_{j=1}^{\alpha_{i'}-1} A_{li'}^{(j)}(1-\beta_{li'}^* t)^{\alpha_{i'}-j}. \end{aligned}$$

Taking $t = \frac{1}{\beta_{li'}^*}$, we have

$$A_{li'}^{(\alpha_{i'})} = \prod_{i\neq i'}\left(1-\frac{\beta_{li}^*}{\beta_{li'}^*}\right)^{-\alpha_i} = \prod_{i\neq i'}\left(\frac{\beta_{li'}^*}{\beta_{li'}^*-\beta_{li}^*}\right)^{\alpha_i} = h_{i'}\left(\frac{1}{\beta_{li'}^*}\right).$$

Moreover,

$$\begin{aligned} A_{li'}^{(\alpha_{i'}-1)} &= \frac{1}{1-\beta_{li'}^* t}\prod_{i\neq i'}(1-\beta_{li}^* t)^{-\alpha_i} - \frac{A_{li'}^{(\alpha_{i'})}}{1-\beta_{li'}^* t} \\ &\quad -(1-\beta_{li'}^* t)^{\alpha_{i'}-1}\sum_{i\neq i'}\sum_{j=1}^{\alpha_i} A_{li}^{(j)}(1-\beta_{li}^* t)^{-j} - \sum_{j=1}^{\alpha_{i'}-2} A_{li'}^{(j)}(1-\beta_{li'}^* t)^{\alpha_{i'}-1-j}. \end{aligned}$$

When $\alpha_{i'} \geq 2$, the last two terms have factor $1-\beta_{li'}^* t$ and so vanish when taking limit $t \to \frac{1}{\beta_{li'}^*}$. Thus, we get

$$\begin{aligned} A_{li'}^{(\alpha_{i'}-1)} &= \lim_{t\to\frac{1}{\beta_{li'}^*}} \frac{1}{1-\beta_{li'}^* t}\left(\prod_{i\neq i'}(1-\beta_{li}^* t)^{-\alpha_i} - A_{li'}^{(\alpha_{i'})}\right) \\ &= \lim_{t\to\frac{1}{\beta_{li'}^*}} \frac{1}{1-\beta_{li'}^* t}\left(h_{i'}(t) - A_{li'}^{(\alpha_{i'})}\right) \\ &= -\frac{1}{\beta_{li'}^*} h_{i'}'\left(\frac{1}{\beta_{li'}^*}\right). \end{aligned}$$

Note that $h_{i'}(t) = \prod_{i \neq i'} (1 - \beta^*_{li} t)^{-\alpha_i}$, and so

$$\begin{aligned} \log h_{i'}(t) &= -\sum_{i \neq i'} \alpha_i \log(1 - \beta^*_{li} t), \\ \frac{h'_{i'}(t)}{h_{i'}(t)} &= \sum_{i \neq i'} \frac{\alpha_i \beta^*_{li}}{1 - \beta^*_{li} t}, \\ h'_{i'}(t) &= \sum_{i \neq i'} \frac{\alpha_i \beta^*_{li}}{1 - \beta^*_{li} t} h_{i'}(t), \end{aligned} \tag{A.0.6}$$

and consequently,

$$A^{(\alpha_{i'}-1)}_{li'} = -\sum_{i \neq i'} \frac{\alpha_i \beta^*_{li}}{\beta^*_{li'} - \beta^*_{li}} h_{i'} \left(\frac{1}{\beta^*_{li'}} \right).$$

In a similar way, when $\alpha_{i'} \geq 3$, we have

$$\begin{aligned} A^{(\alpha_{i'}-2)}_{li'} &= \frac{1}{(1 - \beta^*_{li'} t)^2} \prod_{i \neq i'} (1 - \beta^*_{li} t)^{-\alpha_i} - \frac{A^{(\alpha_{i'})}_{li'}}{(1 - \beta^*_{li'} t)^2} - \frac{A^{(\alpha_{i'}-1)}_{li'}}{1 - \beta^*_{li'} t} \\ &\quad -(1 - \beta^*_{li'} t)^{\alpha_{i'}-2} \sum_{i \neq i'} \sum_{j=1}^{\alpha_i} A^{(j)}_{li} (1 - \beta^*_{li} t)^{-j} - \sum_{j=1}^{\alpha_{i'}-3} A^{(j)}_{li'} (1 - \beta^*_{li'} t)^{\alpha_{i'}-2-j}, \end{aligned}$$

which yields

$$\begin{aligned} A^{(\alpha_{i'}-2)}_{li'} &= \lim_{t \to \frac{1}{\beta^*_{li'}}} \frac{1}{(1 - \beta^*_{li'} t)^2} \left\{ \prod_{i \neq i'} (1 - \beta^*_{li} t)^{-\alpha_i} - A^{(\alpha_{i'})}_{li'} - A^{(\alpha_{i'}-1)}_{li'} (1 - \beta^*_{li'} t) \right\} \\ &= \lim_{t \to \frac{1}{\beta^*_{li'}}} \frac{1}{(1 - \beta^*_{li'} t)^2} \left\{ h_{i'}(t) - A^{(\alpha_{i'})}_{li'} - A^{(\alpha_{i'}-1)}_{li'} (1 - \beta^*_{li'} t) \right\} \\ &= \frac{1}{2!(\beta^*_{li'})^2} h''_{i'} \left(\frac{1}{\beta^*_{li'}} \right). \end{aligned}$$

Now to find $h''_{i'}\left(\frac{1}{\beta^*_{li'}}\right)$, we take derivatives on both sides of (A.0.6) to get

$$\begin{aligned} h''_{i'}(t) &= \sum_{i\neq i'} \frac{\alpha_i(\beta^*_{li})^2}{(1-\beta^*_{li}t)^2} h_{i'}(t) + \sum_{i\neq i'} \frac{\alpha_i\beta^*_{li}}{1-\beta^*_{li}t} h'_{i'}(t) \\ &= \left\{ \sum_{i\neq i'} \frac{\alpha_i(\beta^*_{li})^2}{(1-\beta^*_{li}t)^2} + (\sum_{i\neq i'} \frac{\alpha_i\beta^*_{li}}{1-\beta^*_{li}t})^2 \right\} h_{i'}(t). \end{aligned} \tag{A.0.7}$$

Thus, we obtain

$$A^{(\alpha_{i'}-2)}_{li'} = \frac{1}{2!}\left\{ \left(\sum_{i\neq i'} \frac{\alpha_i\beta^*_{li}}{\beta^*_{li'}-\beta^*_{li}} \right)^2 + \sum_{i\neq i'} \frac{\alpha_i(\beta^*_{li})^2}{(\beta^*_{li'}-\beta^*_{li})^2} \right\} h_{i'}\left(\frac{1}{\beta^*_{li'}}\right).$$

Finally, when $\alpha_{i'} \geq 4$, we have

$$\begin{aligned} A^{(\alpha_{i'}-3)}_{li'} \;&=\; \frac{1}{(1-\beta^*_{li'}t)^3} \prod_{i\neq i'} (1-\beta^*_{li}t)^{-\alpha_i} - \frac{A^{(\alpha_{i'})}_{li'}}{(1-\beta^*_{li'}t)^3} - \frac{A^{(\alpha_{i'}-1)}_{li'}}{(1-\beta^*_{li'}t)^2} \\ &-\frac{A^{(\alpha_{i'}-2)}_{li'}}{1-\beta^*_{li'}t} - (1-\beta^*_{li'}t)^{\alpha_{i'}-3} \sum_{i\neq i'} \sum_{j=1}^{\alpha_i} A^{(j)}_{li}(1-\beta^*_{li}t)^{-j} \\ &- \sum_{j=1}^{\alpha_{i'}-4} A^{(j)}_{li'}(1-\beta^*_{li'}t)^{\alpha_{i'}-3-j}, \end{aligned}$$

which yields

$$
\begin{aligned}
& A_{li'}^{(\alpha_{i'}-3)} \\
= \quad & \lim_{t \to \frac{1}{\beta^*_{li'}}} \frac{1}{(1-\beta^*_{li'}t)^3} \left\{ \prod_{i \neq i'} (1-\beta^*_{li}t)^{-\alpha_i} - A_{li'}^{(\alpha_{i'})} - A_{li'}^{(\alpha_{i'}-1)}(1-\beta^*_{li'}t) \right. \\
& \left. -A_{li'}^{(\alpha_{i'}-2)}(1-\beta^*_{li'}t)^2 \right\} \\
= \quad & \lim_{t \to \frac{1}{\beta^*_{li'}}} \frac{1}{(1-\beta^*_{li'}t)^3} \left\{ h_{i'}(t) - A_{li'}^{(\alpha_{i'})} - A_{li'}^{(\alpha_{i'}-1)}(1-\beta^*_{li'}t) - A_{li'}^{(\alpha_{i'}-2)}(1-\beta^*_{li'}t)^2 \right\} \\
= \quad & -\frac{1}{3!(\beta^*_{li'})^3} h'''_{i'} \left(\frac{1}{\beta^*_{li'}} \right).
\end{aligned}
$$

Upon taking derivatives on both sides of (A.0.7), we get

$$
\begin{aligned}
h'''_{i'}(t) &= \left\{ 2\sum_{i \neq i'} \frac{\alpha_i (\beta^*_{li})^3}{(1-\beta^*_{li}t)^3} + 2\sum_{i \neq i'} \frac{\alpha_i \beta^*_{li}}{1-\beta^*_{li}t} \sum_{i \neq i'} \frac{\alpha_i (\beta^*_{li})^2}{(1-\beta^*_{li}t)^2} \right\} h_{i'}(t) \\
& \quad + \left\{ \sum_{i \neq i'} \frac{\alpha_i (\beta^*_{li})^2}{(1-\beta^*_{li}t)^2} + \left(\sum_{i \neq i'} \frac{\alpha_i \beta^*_{li}}{1-\beta^*_{li}t} \right)^2 \right\} h'_{i'}(t) \\
&= \left\{ \left(\sum_{i \neq i'} \frac{\alpha_i \beta^*_{li}}{1-\beta^*_{li}t} \right)^3 + 3\sum_{i \neq i'} \frac{\alpha_i \beta^*_{li}}{1-\beta^*_{li}t} \sum_{i \neq i'} \frac{\alpha_i (\beta^*_{li})^2}{(1-\beta^*_{li}t)^2} \right. \\
& \quad \left. +2\sum_{i \neq i'} \frac{\alpha_i (\beta^*_{li})^3}{(1-\beta^*_{li}t)^3} \right\} h_{i'}(t).
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
A_{li'}^{(\alpha_{i'}-3)} &= -\frac{1}{3!} \left\{ \left(\sum_{i \neq i'} \frac{\alpha_i \beta^*_{li}}{\beta^*_{li'} - \beta^*_{li}} \right)^3 + 3\sum_{i \neq i'} \frac{\alpha_i \beta^*_{li}}{\beta^*_{li'} - \beta^*_{li}} \sum_{i \neq i'} \frac{\alpha_i (\beta^*_{li})^2}{(\beta^*_{li'} - \beta^*_{li})^2} \right. \\
& \quad \left. +2\sum_{i \neq i'} \frac{\alpha_i (\beta^*_{li})^3}{(\beta^*_{li'} - \beta^*_{li})^3} \right\} h_{i'} \left(\frac{1}{\beta^*_{li'}} \right).
\end{aligned}
$$

In general, for $1 \leq j \leq \alpha_{i'}$, we have

$$A_{li'}^{(\alpha_{i'}-j)} = \frac{(-1)^j}{j!(\beta_{li'}^*)^j} h_{i'}^{(j)} \left(\frac{1}{\beta_{li'}^*} \right).$$

In the above, $\beta_{li}^* = \frac{n_l - M_{i-1}(l)}{M_r(l) \sum_{h=1}^k \frac{n_h - M_{i-1}(h)}{\theta_h}}$ for $1 \leq l \leq k$ and $1 \leq i \leq r'$, as given earlier in (2.2.12).

Lemma 6 *Let*

$$\begin{aligned} f_m(w) &= c_0^{(m)} + c_1^{(m)} e^{-a_m w} + \cdots + c_m^{(m)} e^{-\sum_{i=1}^m a_i w} \\ &= \sum_{j=0}^m c_j^{(m)} e^{-\sum_{i=m-j+1}^m a_i w}. \end{aligned}$$

such that $f_m(w_m) = \int_0^{w_m} f_{m-1}(w_{m-1}) e^{-a_m w_{m-1}} dw_{m-1}$. *Then, the coefficients* $c_j^{(m)}$ *can be found as follows:*

$$c_0^{(m)} = \sum_{j=0}^{m-1} \frac{c_j^{(m-1)}}{\sum_{i=m-j}^m a_i}, c_1^{(m)} = -\frac{c_0^{(m-1)}}{a_m}, \cdots, c_m^{(m)} = -\frac{c_{m-1}^{(m-1)}}{\sum_{i=1}^m a_i},$$

in which we adopt the usual conventions that $\prod_{j=1}^0 d_j = 1$ *and* $\sum_{j=i}^{i-1} d_j = 0$.

Proof We have

$$\begin{aligned} f_m(w_m) &= \int_0^{w_m} f_{m-1}(w_{m-1})\mathrm{e}^{-a_m w_{m-1}} dw_{m-1} \\ &= \int_0^{w_m} \sum_{j=0}^{m-1} c_j^{(m-1)} \mathrm{e}^{-\sum_{i=m-j}^{m-1} a_i w_{m-1}} \mathrm{e}^{-a_m w_{m-1}} dw_{m-1} \\ &= \sum_{j=0}^{m-1} \frac{c_j^{(m-1)}}{\sum_{i=m-j}^{m} a_i} \left(1 - \mathrm{e}^{-\sum_{i=m-j}^{m} a_i w_m}\right) \\ &= \sum_{j=0}^{m-1} \frac{c_j^{(m-1)}}{\sum_{i=m-j}^{m} a_i} - \sum_{j=0}^{m-1} \frac{c_j^{(m-1)}}{\sum_{i=m-j}^{m} a_i} \mathrm{e}^{-\sum_{i=m-j}^{m} a_i w_m}. \end{aligned}$$

Comparing the coefficients on both sides, we have

$$c_0^{(m)} = \sum_{j=0}^{m-1} \frac{c_j^{(m-1)}}{\sum_{i=m-j}^{m} a_i},$$

$$c_1^{(m)} = -\frac{c_0^{(m-1)}}{a_m}, \cdots, c_l^{(m)} = -\frac{c_{l-1}^{(m-1)}}{\sum_{i=m-l+1}^{m} a_i}, \cdots, c_m^{(m)} = -\frac{c_{m-1}^{(m-1)}}{\sum_{i=1}^{m} a_i}.$$

Lemma 7 *Under the assumptions of Lemma 6 and $f_0(w) = 1$, we have*

$$c_0^{(m)} = \prod_{j=1}^{m} \frac{1}{\sum_{i=j}^{m} a_i}$$

and

$$c_l^{(m)} = \frac{(-1)^l}{\prod_{j=0}^{l-1} \sum_{i=m-l+1}^{m-j} a_i} \prod_{j=1}^{m-l} \frac{1}{\sum_{i=j}^{m-l} a_i}, \ \text{for } 1 \le l \le m.$$

Proof First, it is easy to see that $c_0^{(m)} = f_m(+\infty) = \lim_{w \to +\infty} f_m(w)$. Nest, we have

$$
\begin{aligned}
f_1(w_1) &= \int_0^{w_1} \mathrm{e}^{-a_1 w_0} dw_0, \\
f_2(w_2) &= \int_0^{w_2} f_1(w_1)\mathrm{e}^{-a_2 w_1} dw_1 = \int_0^{w_2}\int_0^{w_1} \mathrm{e}^{-(a_1 w_0 + a_2 w_1)} dw_0 dw_1, \\
&\cdots \\
f_m(w_m) &= \int_0^{w_m} f_{m-1}(w_{m-1})\mathrm{e}^{-a_m w_{m-1}} dw_{m-1} \\
&= \int_{0<w_0<\cdots<w_{m-1}<w_m} \mathrm{e}^{-(a_1 w_0 + a_2 w_1 + \cdots + a_m w_{m-1})} dw_0 \cdots dw_{m-1}.
\end{aligned}
$$

Thus, we find

$$
c_0^{(m)} = \int_{0<w_0<\cdots<w_{m-1}<\infty} \mathrm{e}^{-(a_1 w_0 + a_2 w_1 + \cdots + a_m w_{m-1})} dw_0 \cdots dw_{m-1}.
$$

Upon integrating out $w_0, w_1, \cdots, w_{m-1}$ over $\{0 \le w_0 \le w_1 \le \cdots \le w_{m-1} < \infty\}$ in the above integral, after mapping it onto $\{0 \le u_i < \infty, 1 \le i \le m-1\}$ through the transformation

$$
\begin{cases}
u_1 & = w_0, \\
u_2 & = w_1 - w_0, \\
 & \cdots \\
u_m & = w_{m-1} - w_{m-2},
\end{cases}
$$

we obtain $c_0^{(m)}$. Also, by using Lemma 6,

$$
\begin{aligned}
c_1^{(m)} &= -\frac{c_0^{(m-1)}}{a_m} = -\frac{1}{a_m}\prod_{j=1}^{m-1}\frac{1}{\sum_{i=j}^{m-1} a_i}, \\
c_2^{(m)} &= -\frac{c_1^{(m-1)}}{a_m + a_{m-1}} = \frac{1}{(a_m + a_{m-1})a_{m-1}}\prod_{j=1}^{m-2}\frac{1}{\sum_{i=j}^{m-2} a_i},
\end{aligned}
$$

and in general, for $1 \le l \le m$,

$$
c_l^{(m)} = -\frac{c_{l-1}^{(m-1)}}{\sum\limits_{i=m-l+1}^{m} a_i} = \frac{(-1)^l}{\prod_{j=0}^{l-1}\sum_{i=m-l+1}^{m-j} a_i}\prod_{j=1}^{m-l}\frac{1}{\sum_{i=j}^{m-l} a_i}.
$$

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