AN EQUIVARIANT MAIN CONJECTURE IN IWASAWA THEORY AND THE COATES-SINNOTT CONJECTURE

AN EQUIVARIANT MAIN CONJECTURE IN IWASAWA THEORY AND THE COATES-SINNOTT CONJECTURE

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Abstract

The classical Main Conjecture (MC) in Iwasawa Theory relates values of p-adic L-functions associated to 1-dimensional Artin characters over a totally real number field F to values of characteristic polynomials attached to certain Iwasawa modules. Wiles [47] proved the MC for odd primes p over arbitrary totally real base fields F and for the prime 2 over *abelian* totally real fields F. An equivariant version of the MC, which combines the information for all characters of the Galois group of a relative abelian extension E/F of number fields with F totally real, was formulated and proven for *odd* primes p by Ritter and Weiss in [33] under the assumption that the corresponding Iwasawa module is finitely generated over $\mathbb{Z}_p$ ("$\mu = 0$"). This assumption is satisfied for abelian fields and conjectured to be true in general.

In this thesis we formulate an Equivariant Main Conjecture (EMC) for *all* prime numbers p, which coincides with the version of Ritter and Weiss for odd p, and we provide a unified proof of the EMC for *all* primes p under the assumptions $\mu = 0$ and the validity of the 2-adic MC. The proof combines the approach of Ritter and Weiss with ideas and techniques used recently by Greither and Popescu [13] to give a proof of a slightly different formulation of an EMC under the same assumptions (p odd and $\mu = 0$) as in [33].

As an application of the EMC we prove the Coates-Sinnott Conjecture, again assuming $\mu = 0$. We also show that the p-adic version of the Coates-Sinnott Conjecture holds without the assumption $\mu = 0$ for abelian Galois extensions E/F of degree prime to p. These generalize previous results for odd primes due to Nguyen Quang Do in [27], Greither-Popescu [13], and Popescu in [30].

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بسم الله الرحمن الرحيم

Contents

Introduction

One of the most fascinating discoveries in Arithmetic Algebraic Geometry is the still mysterious relationship between certain algebraic and analytic data attached to a given arithmetic object. Classical examples include the Conjecture of Birch and Swinnerton-Dyer, which conjecturally relates the order of vanishing of the L-function attached to an elliptic curve at 1 to the rank of the algebraically defined Mordell-Weil group of the curve, and Dirichlets Analytic Class Number Formula, which gives a precise algebraic interpretation of the residue of the zeta-function of a number field at 1. This last example has been generalised to yield interpretations of special values of zeta-functions at arbitrary negative integers in terms of algebraic K-theory and motivic cohomology. One of the systematic approach to understand these deep relations between the algebraic and analytic objects is via Iwasawa Theory.

Iwasawa theory was initiated in the 1950s in order to study objects of arithmetic interest, e.g. class groups, elliptic curves, abelian varieties, motives, etc., over infinite towers of number fields. The prototype of Dirichlet's analytic class number formula, which relates certain analytic and arithmetic data in this theory, is called the main conjecture. We describe this conjecture in the classical form as follows:

Let p be a prime number, let F be a totally real number field, and let ψ be a 1-dimensional p-adic Artin character for F so that $F_\psi \cap F_\infty = F$, where F_ψ denotes the fixed field of the kernel of ψ and F_∞ is the cyclotomic $\mathbb{Z}_p$-extension of F. Let $\mathcal{O}_\psi$ denote the ring obtaining by adjoining all ψ-values to the ring $\mathbb{Z}_p$. We denote by S a finite set of primes of F containing the primes above p, and the infinite primes. In [7], Deligne and Ribet showed the existence of a p-adic L-function for the character ψ - following Kubota and Leopoldt for the case $F = \mathbb{Q}$ - which is continuous for $s \in \mathbb{Z}_p \setminus \{1\}$, and even at $s = 1$, if ψ is not trivial. This satisfies the following interpolation property for any integer $n \geq 1$:

$$L_p(1-n,\psi) = L_{E/F}(1-n,\psi\omega^{-n}) \prod_{\mathfrak{p}\in S_p} (1-\psi\omega^{-n}(\mathfrak{p})Nm(\mathfrak{p})^{1-n}).$$

Here $\omega : F(\mu_{2p}) \to \mathbb{Z}_p^\times$ is the Teichmüller character, and S_p is the set of primes in F sitting above p. Let $H_\psi \in \mathcal{O}_\psi[T]$ be defined as $\psi(\gamma)(T+1)-1$ if $F_\psi \subseteq F_\infty$, and 1 otherwise. They also showed that there exists a power series $G_{\psi,S}(T) \in \mathcal{O}_\psi[[T]]$ so that

$$L_p^S(1-s,\psi) = \frac{G_{\psi,S}(\kappa(\gamma)^s - 1)}{H_\psi(\kappa(\gamma)^s - 1)},$$

where $L_p^S(1-s,\psi)$ denotes the p-adic L-function with Euler factors removed at the primes in S. This power series represents the analytic object in the main conjecture. Let $F_{\psi,\infty}$ denote the cyclotomic $\mathbb{Z}_p$-extension of F_ψ, with Galois group $\Gamma := Gal(F_{\psi,\infty}/F_\psi) =< \gamma >$. Let $M_{\psi,\infty}^S$ be the maximal abelian pro-p-extension

of $F_{\psi,\infty}$, which is unramified outside the primes in S, with Galois group $\mathfrak{X}_\infty^S := Gal(M_{\psi,\infty}^S/F_{\psi,\infty})$. The pro-$p$-group $\mathfrak{X}_\infty^S$ is equipped with a (torsion) $\Lambda := \mathbb{Z}_p[[\Gamma]]$-module structure, as well as a $Gal(F_\psi/F)$-action given by inner automorphisms. Serre showed that the completed group ring Λ can be identified with the one variable power series $\mathbb{Z}_p[[T]]$, by mapping $\gamma - 1$ to T. By the Structure Theorem 1.1.4 for $\mathbb{O}_\psi[[T]]$-modules, the ψ-eigenspace

$$\mathfrak{X}_\infty^{S,\psi} = \left\{x \in \mathfrak{X}_\infty^S \otimes_{\mathbb{Z}_p} \mathcal{O}_\psi | \sigma(x) = \psi(\sigma)x \quad \text{for all } \sigma \in Gal(F_\psi/F)\right\}$$

of the Λ-module $\mathfrak{X}_\infty^S$ is pseudo-isomorphic, as an $\mathcal{O}_\psi[[T]]$-module, to a unique $\mathcal{O}_\psi[[T]]$-module of the form

$$\bigoplus_{i=1}^{m} \mathcal{O}_\psi[[T]]/{\mathfrak{p}_\mathfrak{i}}^{n_i}$$

for $m \geq 1$ and $n_i \geq 1$. Here $\mathfrak{p}_\mathfrak{i}$ is the ideal generated by either a fixed uniformizer $\pi \in \mathcal{O}_\psi$ or a monic irreducible polynomial in $\mathcal{O}_\psi[T]$. Let $F_{\psi,S}(T) \in \mathcal{O}_\psi[T]$ be a generator of the ideal $\prod_{i=1}^m {\mathfrak{p}_\mathfrak{i}}^{n_i}$. By the Weierstrass Preparation Theorem (cf. Theorem 1.1.3) we have the decompositions

$$F_{\psi,S}(T) = \pi^{\mu(F_{\psi,S})} f_{\psi,S}^*(T) \quad \text{and} \quad G_{\psi,S}(T) = \pi^{\mu(G_{\psi,S})} g_{\psi,S}^*(T) u_{\psi,S}(T),$$

where $f_{\psi,S}^*(T)$ and $g_{\psi,S}^*(T)$ are monic polynomials in $\mathcal{O}_\psi[T]$, and $u_{\psi,S}(T)$ is a unit power series in $\mathcal{O}_\psi[[T]]$. The polynomial $f_{\psi,S}^*(T)$ is called the characteristic polynomial. The classical Main Conjecture in Iwasawa theory is formulated as follows:

$$f_{\psi,S}^*(T) = g_{\psi,S}^*(T).$$

This was proven by Mazur and Wiles in [24] in the case $F = \mathbb{Q}$ and p odd, and more generally by Wiles in [47] for any totally real number field F and an odd prime p. He also proved the conjecture for the prime 2 provided F is an abelian extension of $\mathbb{Q}$. He also showed the equality of the μ-invariants $\mu(F_{\psi,S}) = \mu(G_{\psi,S})$ for any odd prime p, where both sides have been conjectured to be zero at odd primes p and further has been verified for the case $F/\mathbb{Q}$ is abelian (cf. [9]). For odd primes p we will see that the Λ-torsion module $\mathfrak{X}_\infty^{S,\psi}$ is of projective dimension at most 1 and has a principal Fitting ideal generated by $F_{\psi,S}(T)$ (cf. Proposition 1.3.4). Therefore, another formulation of the Main Conjecture for odd primes reads as follows:

$$Fitt_{\mathcal{O}_\psi[[T]]}(\mathfrak{X}_\infty^{S,\psi}) = (G_{\psi,S}(T)).$$

Let E/F be an abelian extension of totally real number fields with Galois group G. One can write the classical Main Conjecture with respect to any (even) character ψ of G, and obtain the corresponding relationship between the algebraic and the analytic data for any ψ as above. Combining all these relationships, based on the classical Main Conjecture, for all characters ψ of G leads to the formulation of the so-called

Equivariant Main Conjecture in Iwasawa theory. Ritter and Weiss formulated such a conjecture in [33] for any odd prime p, and proved it assuming that a certain Iwasawa module is a finitely generated $\mathbb{Z}_p$-module. To explain this we need the following set-up:

Let E/F be an abelian extension of totally real number fields, and let E_∞ be the cyclotomic extension of E. Let M_∞^S denote the maximal abelian pro-p-extension of E_∞, unramified outside the primes in S, and let $\mathfrak{X}_\infty := Gal(M_\infty^S/E_\infty)$. We denote by G_∞ the Galois group of E_∞/F, by H the Galois group of E_∞/F_∞, and by $\mathbb{A}$ the completed group ring $\mathbb{Z}_p[[G_\infty]]$. Assuming that G_∞ is abelian, we conclude the existence of a group $\Gamma \leq G_\infty$ so that $G_\infty = H \times \Gamma$. Again by inner automorphisms $\mathfrak{X}_\infty$ has a (torsion) $\mathbb{A}$-module structure, whose projective dimension is not necessarily at most one. However, to formulate an Equivariant Main Conjecture, similar to the classical Main Conjecture, one needs a finitely generated $\mathbb{A}$-torsion module of projective dimension at most one. Let d_∞ be a non-zero divisor of the augmentation ideal ΔG_∞ of $\mathbb{A}$, let c_∞ be an invertible element of the total ring of fraction of $\mathbb{A}$ so that $d_\infty = c_\infty((\gamma-1)e + (1-e))$, where e is the idempotent attached to the trivial character of H. We denote by L the fixed field of E/F under the action of the p-Sylow subgroup of G, by $\mathcal{G}$ the Galois group of the maximal algebraic extension Ω_L^S of L, unramified outside the primes in S, over L, and by $\mathcal{H}$ the Galois group of Ω_L^S/E_∞. There is a commutative diagram of $\mathbb{A}$-modules

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbb{A} & = & \mathbb{A} & & \\
 & & & & \downarrow \Psi & & \downarrow \psi & & \\
0 & \to & \mathfrak{X} & \to & \mathcal{Y}_\infty & \to & \Delta G_\infty & \to & 0 \\
 & & || & & \downarrow & & \downarrow & & \\
0 & \to & \mathfrak{X}_\infty & \to & \mathcal{Z}_\infty & \to & z_\infty & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0, & &
\end{array}$$

where ψ maps 1 to d_∞, Ψ maps 1 to a pre-image y_∞ of d_∞, and $\mathcal{Y}_\infty = H_0(\mathcal{H}, \Delta\mathcal{G})$ is the coinvariant of the augmentation ideal $\Delta\mathcal{G}$ of $\mathbb{Z}_p[[\mathcal{G}]]$ respect to the group $\mathcal{H}$. The $\mathbb{A}$-torsion module $\mathcal{Z}_\infty$ in the diagram above, whose projective dimension is at most one, shows up as the algebraic object in the Equivariant Main Conjecture of Ritter-Weiss. We note that the construction of $\mathcal{Z}_\infty$ depends on the choice of d_∞. The analytic object is defined as follows:

$$G_S = \sum_{\psi \in \hat{H}} G_{\psi,S}(\gamma - 1) \cdot e_\psi \in \frac{1}{|H|}\mathcal{O}[H][[\Gamma]],$$

where e_ψ is the idempotent attached to any character ψ of H, i.e.

$$e_\psi = \frac{1}{\mid G \mid} \sum_{\sigma \in G} \psi(\sigma)\sigma^{-1}.$$

One version of the Equivariant Main Conjecture of Ritter-Weiss for the odd primes is as follows:

$$Fitt_{\mathbb{A}}(\mathcal{Z}_\infty) = (c_\infty G_S),$$

which was verified under the assumption of the vanishing of the μ-invariant of $\mathfrak{X}_\infty$, i.e. assuming that $\mathfrak{X}_\infty$ is a finitely generated $\mathbb{Z}_p$-module. It is worth mentioning that they have generalized and proven their Equivariant Main Conjecture in the non-commutative case, still assuming the vanishing of the μ-invariant of a certain Iwasawa module (cf. [37]).

We now describe our Equivariant Main Conjecture for an arbitrary prime p. For an abelian extension E/F, by applying the algebraic construction of the Equivariant Main Conjecture of Ritter-Weiss to the set S_f of finite primes in S, we construct the $\mathbb{A}$-torsion module $\mathcal{Z}_\infty^f$, which is of projective dimension at most 1. We show that it satisfies the following exact sequence:

$$0 \to \mathbb{Z}_p \to \mathbb{A}/d_\infty\mathbb{A} \to \alpha(\mathcal{Z}_\infty^f)^\# \to \alpha(\mathfrak{X}_\infty^f)^\# \to 0,$$

in which

$$pd_{\mathbb{A}}(\mathbb{A}/d_\infty\mathbb{A}) \leq 1 \quad \text{and} \quad pd_{\mathbb{A}}(\alpha(\mathcal{Z}_\infty^f)) \leq 1,$$

where $\mathfrak{X}_\infty^f$ is the Galois group of the maximal abelian pro-p-extension of E_∞, unramified outside the primes in S_f, over E_∞, α is the adjoint functor in Iwasawa theory and $\#$ denotes the inverse action given by $\gamma \cdot m = m^\gamma$. The Equivariant Main Conjecture is then formulated in Chapter 2 (cf. Conjecture 2.1.6) as follows:

$$Fitt_{\mathbb{A}}(\mathcal{Z}_\infty^f) = (c_\infty G_S^*)$$

for the power series

$$G_S^* = \sum_{\psi \in \hat{H}} G_{\psi,S}^*(\gamma - 1) \cdot e_\psi \in \frac{1}{|H|}\mathcal{O}[H][[\Gamma]],$$

where $G_{\psi,S}(T) = \pi^{\mu(G_{\psi,S})}G_{\psi,S}^*(T) = \pi^{\mu(G_{\psi,S})}g_{\psi,S}^*(T)u_{\psi,S}(T)$. In Section 2.1.2 we prove this conjecture follows from the classical Main Conjecture under the assumption $\mu = 0$, i.e. assuming that $\mathfrak{X}_\infty^f$ is a finitely generated $\mathbb{Z}_p$-module, by taking advantage of the idea of determinantal ideals used by Greither and Popescu in [13] in a recent proof of a slightly different formulation of an Equivariant Main Conjecture under the same assumptions (p odd and $\mu = 0$) as Ritter-Weiss in [33].

As an application of this Equivariant Main Conjecture we verify the Coates-Sinnott Conjecture. This conjecture is a generalization of the classical Stickelberger Theorem, which provides elements annihilating the class group of a cyclotomic field, using special values of certain analytic functions. To make it more precise let E/F be an abelian extension with Galois group G, and let S be a finite set of primes in F containing the primes ramified in E and the infinite primes. Let

$$\Theta^S_{E/F}(s) = \sum_{\chi \in \hat{G}} L^S_{E/F}(s, \chi^{-1}) \cdot e_\chi \in \mathbb{C}[G]$$

be the S-incomplete equivariant L-function, where e_χ is the idempotent attached to any character χ of G. In [7], Deligne and Ribet proved that

$$Ann_{\mathbb{Z}[G]}(H^0(E, \mathbb{Q}/\mathbb{Z}(n))) \cdot \Theta^S_{E/F}(1-n) \subset \mathbb{Z}[G]$$

for any integer $n \geq 1$. Stickelberger's Theorem shows that the following analytic object is in the annihilator ideal of the class group $Cl(\mathcal{O}_E)$ of the field E in the case $F = \mathbb{Q}$:

$$Ann_{\mathbb{Z}[G]}(H^0(E, \mathbb{Q}/\mathbb{Z}(1))) \cdot \Theta^S_{E/F}(0) \subseteq Ann_{\mathbb{Z}[G]}(Cl(\mathcal{O}_E)).$$

This setup has been generalized in two directions: First of all one looks at an arbitrary relative abelian extension E/F of number fields. Here the analogue of Stickelberger's Theorem (Brumer's Conjecture) is still not completely known. In a different direction one replaces again the class group by algebraic K-groups or motivic cohomology groups and studies annihilators of these groups as Galois modules for relative abelian extensions. In [5], Coates and Sinnott formulated the relevant conjecture in terms of higher Quillen K-groups as

$$Ann_{\mathbb{Z}[G]}(H^0(E, \mathbb{Q}/\mathbb{Z}(n))) \cdot \Theta^S_{E/F}(1-n) \subseteq Ann_{\mathbb{Z}[G]}(K_{2n-2}(\mathcal{O}_E))$$

for any integer $n \geq 2$. As a result of the recent works of Voevodsky [45], now the relation between algebraic K-theory, étale cohomology for all prime numbers and motivic cohomology is known. This yields the motivic formulation of the Coates-Sinnott Conjecture, which implies the K-theoretic version. Moreover, it enables us to study each p-primary part of the conjecture separately for any prime number p as follows:

$$Ann_{\mathbb{Z}_p[G]}(H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))) \cdot \Theta^S_{E/F}(1-n) \subseteq Ann_{\mathbb{Z}_p[G]}(H^2_{\acute{e}t}(\mathcal{O}'_E, \mathbb{Z}_p(n))$$

for any integer $n \geq 2$. After some fundamental work of Coates-Sinnott in [5] and more recent results by Ritter-Weiss, Nguyen Quang Do, Burns-Greither, Greither-Popescu et al., the Coates-Sinnott Conjecture is completely proven for any odd prime p, assuming $\mu = 0$. However, the 2-primary information has been neglected more or less completely due to various technical problems. For example, there was no formulation of an Equivariant Main Conjecture in Iwasawa theory for the prime 2.

At the end of Chapter 2, as an application of our Equivariant Main Conjecture, we prove the the Coates-Sinnott Conjecture for any totally real extension E/F, again under the assumption $\mu = 0$ (cf. Theorem 2.2.8).

In the last chapter of the thesis we deduce the p-adic version of the Coates-Sinnott Conjecture *without* the assumption $\mu = 0$ from the classical Main Conjecture in Iwasawa theory in the following situations (see Theorems 3.1.3 and 3.2.4):

- $n \geq 2$ is an even number and E/F is a finite abelian extension of number fields of order prime to p, where E is a totally real number field.
- $n \geq 2$ is an odd number and E/F is a finite abelian extension of number fields of degree $2m$, where m is not divisible by p, E is a CM-field and F is a totally real number field.

For odd primes p and any totally real field E this has been done by Popescu in [30]. In the proof for the prime 2 we have to assume the classical Main Conjecture in the case $E/\mathbb{Q}$ is not abelian, as well as the equality of the algebraic and the analytic μ-invariants. It is worth mentioning that the restriction on the parity of n simply avoids the trivial cases, where the S-incomplete equivariant L-functions vanish.

Chapter 1

Background Material

1.1 Iwasawa theory

1.1.1 Introduction

Let F be a number field with $r_1(F)$ real embeddings and $r_2(F)$ pairs of complex embeddings, and let p be a fixed prime. Let F_∞ be a $\mathbb{Z}_p$-extension of F, i.e. an extension with Galois group $Gal(F_\infty/F) \simeq \Gamma$ where Γ is a multiplicative group isomorphic to the additive group $\mathbb{Z}_p$. We fix a topological generator γ of Γ. For $n \geq 0$ let $\Gamma_n := \Gamma/\Gamma^{p^n} \simeq \mathbb{Z}/p^n\mathbb{Z}$ and let F_n denote the fixed field of F_∞ under the action of Γ^{p^n}. Then we have the tower of fields

$$F := F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots \subseteq F_\infty$$

with $Gal(F_n/F) \simeq \Gamma_n \simeq \mathbb{Z}/p^n\mathbb{Z}$ for any $n \geq 0$.

Example 1.1.1. *Let $L := F(\zeta_{2p})$ be the field obtained by adjoining a primitive $2p$-th root of unity ζ_{2p} to F, and let $L_\infty := \bigcup_{n\geq 0} F(\zeta_{2p^n})$ be the field obtained by adjoining all p-power roots of unity to F. Then $G(L_\infty/F) \simeq \mathbb{Z}_p \times \Delta$, where Δ is finite. The Δ-fixed points of L_∞ give us a $\mathbb{Z}_p$-extension of F, which is called the* **cyclotomic** *$\mathbb{Z}_p$-extension of F.*

Remark 1.1.2. *One can show that any $\mathbb{Z}_p$-extension F_∞/F is p-ramified, i.e. unramified outside the primes above p, and also that there is an integer n, so that F_∞/F_n is totally ramified (cp. [46], §13.1). Using class field theory, one can further show that the number of independent $\mathbb{Z}_p$-extensions of F is $1 + r_2(F) + \delta_F$, where $\delta_F \geq 0$ is the so-called Leopoldt defect (cf. [14], 2.3). We say that the* **Leopoldt conjecture**

holds if δ_F vanishes. Leopoldt conjecture holds when F is an abelian extension of $\mathbb{Q}$ or of an imaginary quadratic number field (see [29], Theorem 10.3.16).

Let $\mathcal{O}$ denote the integral closure of $\mathbb{Z}_p$ in some finite extension of $\mathbb{Q}_p$ and let π be a uniformizer in $\mathcal{O}$. Let

$$\mathcal{O}[[\Gamma]] := \varprojlim_n \mathcal{O}[\Gamma_n]$$

be the completed group ring. Serre showed that one obtains an isomorphism between $\mathcal{O}[[\Gamma]]$ and the ring of formal power series $\Lambda := \mathcal{O}[[T]]$ by mapping γ to $1+T$:

$$\Lambda \simeq \mathcal{O}[[\Gamma]]. \tag{1.1}$$

We recall that Λ is a Notherian local domain of Krull dimension 2 with maximal ideal $< \pi, T >$, whose height one prime ideals are either the ideal generated by $\pi \in \mathcal{O}$ or the ideals generated by irreducible polynomials in $\mathcal{O}[T]$ (cf. [4], Chap. VII).

In Iwasawa theory, the study of finitely generated Λ-modules is of special interest. We first review some definitions. A monic polynomial

$$T^n + b_{n-1}T^{n-1} + b_{n-2}T^{n-2} + \cdots + b_0 \in \Lambda$$

is called **distinguished** if $b_i \in (\pi)$ for all i. As an example, $\omega_n := (1+T)^{p^n} - 1$ is distinguished for any $n \geq 0$.

Proposition 1.1.3. *(Weierstrass Preparation Theorem, cf. [14], §7.1). Let $F \in \Lambda$. Then there exists an integer $\mu \geq 0$ such that F can be expressed uniquely as*

$$F(T) = \pi^{\mu(F)} F^*(T)$$

for $F^*(T) := f^*(T)u(T)$, where $f^*(T) \in \mathcal{O}[T]$ is a distinguished polynomial of degree λ, and $u(T) \in \Lambda^\times$ is a unit.

In Proposition 1.1.3, the exponent $\mu(F)$ is defined to be the **μ-invariant** of the power series F. Before we state the Structure Theorem for Λ-modules, we recall that a morphism $\phi : M \to N$ is called a pseudo-isomorphism and denoted by $M \sim N$ if the kernel and the cokernel of ϕ are both finite.

Theorem 1.1.4. *(Structure Theorem for Λ-modules, cf. [46], §13.2). Let M be a finitely generated Λ-module. There exist $r \geq 0$, $m \geq 1$ and $n_i \geq 1$ so that M is pseudo-isomorphic to the following elementary Λ-module:*

$$M \sim \bigoplus_{i=1}^{m} \Lambda/\mathfrak{p}_i^{n_i} \bigoplus \Lambda^r.$$

Here the $\mathfrak{p}_i$'s are primes of height one. Moreover, the $\mathfrak{p}_i$'s, n_i's and r are uniquely determined by M.

With the notations of the theorem above we have the following definitions:

- $\sum_{i=1}^{m} n_i \mathfrak{p}_i$ is the **divisor** of M.
- $\prod_{i=1}^{m} \mathfrak{p}_i{}^{n_i}$ is the **characteristic ideal** of M.
- This characteristic ideal is principal and generated by a polynomial of the form $\pi^{\mu} \cdot f^*(T) \in \mathcal{O}[T]$ for a distinguished polynomial $f^*(T)$ of degree λ by the Weierstrass Preparation Theorem (cf. Theorem 1.1.3). The **characteristic polynomial** of M is defined to be $char_{\Lambda}(M) := f^*(T)$, and μ and λ are called the Iwasawa $\boldsymbol{\mu}$**-invariant** and the Iwasawa $\boldsymbol{\lambda}$**-invariant** of M, respectively.

To explain why $char_{\Lambda}(M)$ is called the characteristic polynomial of M, we consider a finitely generated Λ-torsion module M, and we note that $V := M \otimes_{\mathcal{O}} \bar{\mathbb{Q}}_p$ is a vector space of dimension λ over $\bar{\mathbb{Q}}_p$ and $F(T)$ is the characteristic polynomial of the endomorphism of V defined by multiplication by $\gamma - 1$.

We have the following important lemma about the Γ-invariants and Γ-coinvariants of Λ-modules (see [22] for a proof).

Lemma 1.1.5. *Let M be a torsion Λ-module with characteristic polynomial $F(T)$. The following are equivalent:*

1. *M^{Γ} is finite.*
2. *M_{Γ} is finite.*
3. *$F(T) \neq 0$*

If these conditions hold, then

$$\frac{|M^{\Gamma}|}{|M_{\Gamma}|} = |char_{\Lambda}(M)(0)|_v = p^{-f \cdot v(F(0))},$$

where v is the normalized valuation, i.e. $v(\pi) = 1$, and f is the residue degree of π over p.

One can construct Iwasawa modules by equipping certain Galois groups with Λ-module structures. Let F_{∞} be a fixed $\mathbb{Z}_p$-extension of F with Galois group Γ, and let K_{∞} be an abelian pro-p-extension of F_{∞} so that K_{∞}/F is Galois. We obtain the extension of groups

$$0 \to Gal(K_{\infty}/F_{\infty}) \to Gal(K_{\infty}/F) \to \Gamma \to 0,$$

and Γ acts on $Gal(K_\infty/F_\infty)$ by inner automorphisms as follows: if $x \in Gal(K_\infty/F_\infty)$ and γ is a topological generator of Γ, then we define $x^\gamma := \tilde{\gamma} x \tilde{\gamma}^{-1}$ where $\tilde{\gamma}$ is a lift of γ to $Gal(K_\infty/F)$. This yields a Λ-module structure on $X := Gal(K_\infty/F_\infty)$.

Example 1.1.6. *Let F_∞/F be a fixed $\mathbb{Z}_p$-extension with intermediate fields F_n so that $Gal(F_n/F) \simeq \mathbb{Z}/p^n\mathbb{Z}$ and let S_p be the set of primes above p in F. Let L_∞ and L'_∞ be the maximal abelian unramified pro-p-extension of F_∞ and the maximal abelian unramified pro-p-extension of F_∞, in which all primes above p split, respectively. Let $X_\infty := Gal(L_\infty/F_\infty)$ and $X'_\infty := Gal(L'_\infty/F_\infty)$. Let H_n and H'_n be the p-parts of the Hilbert class field of F_n and the Hilbert S_p-class field of F_n, i.e. the maximal abelian unramified p-extension, in which all primes above p split, respectively. Because of the maximality, the extensions L_∞/F and L'_∞/F are Galois. Let A_n and A'_n denote the p-parts of the class group and the S_p-class group of F_n, respectively. Then we obtain by class field theory that $Gal(H_n/F_n) \simeq A_n$ and $Gal(H'_n/F_n) \simeq A'_n$. Consequently, by taking the projective limit with respect to the restriction maps on the left and the norm maps on the right hand side, we obtain*

$$X_\infty \simeq \varprojlim_n A_n$$

$$X'_\infty \simeq \varprojlim_n A'_n.$$

One can show that X_∞ and X'_∞ are finitely generated Λ-torsion modules (cf. [14], Theorem 5). By a conjecture of Iwasawa the μ-invariant of X_∞ vanishes, i.e. X_∞ is a finitely generated $\mathbb{Z}_p$-module. This was shown in the case that $F/\mathbb{Q}$ is abelian by Ferrero and Washington [9]. Moreover, by a conjecture of Greenberg, if F is a real number field, then X_∞ is supposed to be finite, i.e. both the Iwasawa μ-invariant and the Iwasawa λ-invariant vanish.

Let S be finite set of primes of F which contains the primes above p. For a place v let $\hat{U_F^S}$ and $\hat{F}_v$ be the p-adification of the S-unit group of F and the local field F_v, respectively. Then class field theory (see [16]) relates the Galois group of M_F^S , the maximal abelian pro-p-extension of F, which is unramified outside the primes in S, over F to the Galois group of the p-part of the Hilbert S-class field H_F^S over F by the following exact sequence:

$$0 \to D_F \to \hat{U_F^S} \to \prod_{v \in S} \hat{F}_v \to Gal(M_F^S/F) \to Gal(H_F^S/F) \to 0, \tag{1.2}$$

where the kernel D_F of the first map is a finitely generated $\mathbb{Z}_p$-module, whose rank is given by the Leopoldt defect δ_F.

Example 1.1.7. *Let F_∞/F be a fixed $\mathbb{Z}_p$-extension with intermediate fields F_n and let S be a finite set of primes in F containing the primes above p. Let M_∞^S be the maximal abelian pro-p-extension of F_∞, which is unramified outside the primes above S. Then M_∞^S/F_∞ is a Galois extension, whose Galois group is denoted by $\mathfrak{X}_\infty^S$. From the exact sequence above one can see that $\mathfrak{X}_\infty^S$ is a finitely generated Λ-module. However, $\mathfrak{X}_\infty^S$ is not torsion in general.*

Let S_f denote the set of finite primes in S, and let $\mathfrak{X}_\infty^f := \mathfrak{X}_\infty^{S_f}$. Since infinite primes are unramified in an odd degree extension, $\mathfrak{X}_\infty^S = \mathfrak{X}_\infty^f$ for any odd prime p. At this point we make the following convention, which holds in the whole thesis. The assumption $\mu = 0$ refers to the assumption of the following statement:

$$\boldsymbol{\mu = 0:} \qquad \textbf{The } \boldsymbol{\mu}\textbf{-invariant of } \boldsymbol{\mathfrak{X}_\infty^f} \textbf{ is zero, i.e. } \boldsymbol{\mathfrak{X}_\infty^f} \textbf{ is a finitely generated } \mathbb{Z}_p\textbf{-module.} \tag{1.3}$$

1.1.2 Galois groups as Λ-modules

From now on, let S be a finite set of primes of F containing the primes above p and the infinite primes. Let $\mathfrak{X}_\infty^S$ be the Galois group introduced in Example 1.1.7. One can show that

$$rank_\Lambda(\mathfrak{X}_\infty^S) \geq r_2(F) \tag{1.4}$$

Proposition 1.1.8. *(cf. [29], Theorem 10.3.22). Let G_∞^S be the Galois group of the maximal algebraic pro-p-extension of F_∞, which is unramified outside all primes in S. The following are equivalent:*

1. $rank_\Lambda(\mathfrak{X}_\infty^S) = r_2(F)$
2. *The Leopoldt defect δ_n of the field F_n is bounded independent of $n \geq 0$.*
3. $H^2(G_\infty^S, \mathbb{Q}_p/\mathbb{Z}_p) = 0$
4. $H_2(G_\infty^S, \mathbb{Z}_p) = 0$.

As a definition, F_∞/F is said to satisfy **the weak Leopoldt conjecture**, if these equivalent conditions hold. It is easy to show that the Leopoldt conjecture implies the weak Leopoldt conjecture. An example of such an extension satisfying the weak Leopoldt conjecture is given by the following (see Theorem 10.3.25 in [29] for a proof):

Proposition 1.1.9. *For any cyclotomic $\mathbb{Z}_p$-extension, the weak Leopoldt conjecture holds.*

We also have the following proposition (see Theorem 11.3.2 in [29] and Proposition 7 in [10] for a proof):

Proposition 1.1.10. *If the weak Leopoldt conjecture holds for a $\mathbb{Z}_p$-extension F_∞/F, then $\mathfrak{X}_\infty^S$ is a finitely generated Λ-module of rank $r_2(F)$ with no non-trivial finite submodules.*

Corollary 1.1.11. *Let F be a number field with no real prime. Then for the cyclotomic $\mathbb{Z}_p$-extension of F, $\mathfrak{X}_\infty^S$ is a finitely generated Λ-torsion module with no non-trivial finite submodules.*

We recall that $\mathfrak{X}_\infty^S = \mathfrak{X}_\infty^f$ for any odd prime p. For $p = 2$ we have the following proposition:

Proposition 1.1.12. *Let F be a number field with $r_1(F)$ real primes. Assume that the $\mathbb{Z}_2$-extension F_∞/F satisfies the weak Leopoldt conjecture. Then we have the following exact sequence of Λ-modules:*

$$0 \to (\Lambda/2)^{r_1(F)} \to \mathfrak{X}_\infty^S \to \mathfrak{X}_\infty^f \to 0.$$

If we further assume that F/k is an abelian extension of totally real number fields with Galois group G, then

$$0 \to (\Lambda[G]/2)^{r_1(k)} \to \mathfrak{X}_\infty^S \to \mathfrak{X}_\infty^f \to 0$$

is an exact sequence of $\Lambda[G]$-modules.

Proof. We have the following commutative diagram by (1.2) for the finite sets S and S_f:

$$\begin{array}{ccccccccc} 0 & \to & D_F & \to & \hat{U_F^S} & \to & \prod_{v\in S} \hat{F}_v & \to & Gal(M_F^S/H_F^S) \to 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & D'_F & \to & \hat{U_F^{S_f}} & \to & \prod_{v\in S_f} \hat{F}_v & \to & Gal(M_F^{S_f}/H_F^{S_f}) \to 0, \end{array}$$

where D_F and D'_F are the kernels of the corresponding maps and are bounded by the Leopoldt defect δ_F. Since $\hat{F}_v$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ for a real prime and trivial for complex primes, we obtain the following exact sequence:

$$0 \to D_F \to D'_F \to \prod_{\text{real } w\in S\setminus S_f} \mathbb{Z}/2\mathbb{Z} \to Gal(M_F^S/M_F^{S_f}) \to 0. \tag{1.5}$$

Since the real primes are unramified in F_∞/F, we can write the exact sequence above for the unique intermediate field F_n of F_∞/F with $Gal(F_n/F) \simeq \mathbb{Z}/2^n\mathbb{Z}$ for any $n \geq 0$, as follows:

$$0 \to D_{F_n} \to D'_{F_n} \to (\mathbb{Z}/2\mathbb{Z})^{2^n r_1(F)} \to Gal(M^S_{F_n}/M^{S_f}_{F_n}) \to 0.$$

Now the claim is the following:

$$\varprojlim (\mathbb{Z}/2\mathbb{Z})^{2^n r_1(F)} \simeq (\Lambda/2\Lambda)^{r_1(F)}$$

Since $\Lambda/2\Lambda \simeq \varprojlim \mathbb{Z}/2\mathbb{Z}[T]/T^{2^n}$, it suffices to show for a fixed real prime w of E that the inverse limits of $\{\prod_{w_n|w} \mathbb{Z}/2\mathbb{Z}\}$ and $\{\mathbb{Z}/2\mathbb{Z}[T]/T^{2^n}\}$ are isomorphic. For this we inductively define an isomorphism

$$f_n : \prod_{w_n|w} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}[T]/T^{2^n}$$

compatible with the norm maps as follow: Let f_0 be the identity and assume we have defined the isomorphisms f_m compatible with the norm maps for all $m \leq n$. Let w_{n+1} and w'_{n+1} be the extensions of w_n to F_{n+1}. We define

$$f_{n+1} : \prod_{w_{n+1}} \prod_{w'_{n+1}} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}[T]/T^{2^{n+1}}$$

as follows:

$$f_{n+1}(a_1, \cdots, a_{2n}, b_1, \cdots, b_{2n}) = f_n(a_1 + b_1, \cdots, a_{2n} + b_{2n}) + T^{2n} f_n(a_1, , a_{2n}).$$

Now we have the commutative diagram

$$\begin{array}{ccc} \prod_{w_n|w} \mathbb{Z}/2\mathbb{Z} & \xrightarrow{f_n} & \mathbb{Z}/2\mathbb{Z}[T]/T^{2^n} \\ \downarrow & & \downarrow \\ \prod_{w_{n+1}|w} \mathbb{Z}/2\mathbb{Z} & \xrightarrow{f_{n+1}} & \mathbb{Z}/2\mathbb{Z}[T]/T^{2^{n+1}} \end{array}$$

for any $n \geq 0$ and hence

$$\varprojlim_n \prod_{w_n|w} \mathbb{Z}/2\mathbb{Z} \simeq (\Lambda/2\Lambda).$$

This completes the proof of the claim. Since

$$0 \to Gal(M^S_F/M^{S_f}_F) \to \mathfrak{X}^S_\infty \to \mathfrak{X}^f_\infty \to 0$$

is exact we obtain the first exact sequence in the proposition. For the second part we note that under the assumption of the weak Leopoldt conjecture for F_∞/F, the cokernels of $D_{F_n} \to D'_{F_n}$, for all $n \geq 0$ are finite elementary 2-groups of order bounded

independent of n (cf. Proposition 1.1.8). This completes the proof of the first part of the proposition since $\Lambda/2\Lambda$ has no non-trivial finite submodules.

To prove of the second part of the proposition it is enough to consider all groups as $\Lambda[G]$-modules. We note that the term $(\mathbb{Z}/2\mathbb{Z})^{r_1(F)} = \prod_{w\in S\setminus S_f} \mathbb{Z}/2\mathbb{Z}$ in the sequence (1.5) is isomorphic to $(\mathbb{Z}[G]/2)^{r_1(k)}$, since G acts transitively on the set of primes w in F lying above a real place v of k, i.e.

$$\prod_{w|v} \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}[G]/2.$$

□

As a consequence of this proposition the μ-invariant of $\mathfrak{X}_\infty^S$ equals $r_1(F)$ under the assumption $\mu = 0$ (cf. (1.3)). This coincides with the statement of Proposition 8 in [10], which says that the μ-invariant of $\mathfrak{X}_\infty^S$ equals $[F : \mathbb{Q}]$ under the assumption of the vanishing of the μ-invariant of X_∞ (cf. Example 1.1.6), for any totally real number field F.

Now we recall the definition of the Tate-twist. Let ζ_{2p} be a $2p$-th root of unity and let $E := F(\zeta_{2p})$ and $\Delta := Gal(E/F)$. Let $E_\infty = E(\mu_{p^\infty})$ be the cyclotomic $\mathbb{Z}_p$-extension of E with $\tilde{\Gamma} := Gal(E_\infty/F)$. The exact sequence

$$0 \to \Gamma \to \tilde{\Gamma} \to \Delta \to 0$$

splits for any odd prime p. For $p = 2$ the situation is slightly different. Let $k \geq 2$ be the smallest integer such that $\zeta_{2^k} \in E$. The non-trivial element of $Gal(E/F)$ acts on E either by $\zeta_{2^k} \to -\zeta_{2^k}$ or $\zeta_{2^k} \to -\zeta_{2^k}^{-1} = \zeta_{2^k}^{-1+2^{k-1}}$. In the former case $Gal(E_\infty/F) \simeq \mathbb{Z}/2\mathbb{Z} \times \Gamma$, and in the latter case $k \geq 3$ and $Gal(E_\infty/F) \simeq \mathbb{Z}_2$. This can be summarized in the following lemma:

Lemma 1.1.13. *If $p = 2$, then $\tilde{\Gamma} = \mathbb{Z}_2$ if and only if $\zeta_4 \in F$ or $\zeta_{2^k} - \zeta_{2^k}^{-1} \in F$ for some $k \geq 3$.*

As a corollary, the exact sequence above splits for $p = 2$ if F is a totally real number field.

Since $\mu_{p^\infty} \subseteq E_\infty$, $\tilde{\Gamma}$ acts on μ_{p^∞}. For $\sigma \in \tilde{\Gamma}$ and $\zeta \in \mu_{p^\infty}$ the **cyclotomic character**

$$\rho : \tilde{\Gamma} \to \mathbb{Z}_p^*$$

is defined by the following relation:

$$\sigma \cdot \zeta = \zeta^{\rho(\sigma)}$$

Assuming $\tilde{\Gamma} \simeq \Delta \times \Gamma$, the restriction of ρ to Δ, which is denoted by ω, is called the **Teichmüller character**. The restriction of ρ to Γ is denoted by κ.

Now let M be a $\mathbb{Z}_p$-module with a $\tilde{\Gamma}$-action. $M(n)$, the ***n*-th Tate twist** of M, is defined to be the same underlying $\mathbb{Z}_p$-module M with a new $\tilde{\Gamma}$-action as follows:

$$\sigma *_n m := \rho(\sigma)^n m^\sigma$$

for $\sigma \in \tilde{\Gamma}$ and $m \in M$. As an example, $\mathcal{T} := \varprojlim_n \mu_{p^n}$ is the first Tate twist of $\mathbb{Z}_p$, which is called the Tate module. It is easy to see that $M(n) = M \otimes \mathbb{Z}_p(n)$, and

$$\mathbb{Z}_p(n) = \begin{cases} \mathcal{T}^{\otimes n} & \text{if } n > 0 \\ \mathbb{Z}_p & \text{if } n = 0 \\ Hom_{\mathbb{Z}_p}(\mathcal{T}^{-n}, \mathbb{Z}_p) & \text{if } n < 0 \end{cases}$$

where $\mathcal{T}^{\otimes n}$ is endowed with the diagonal $\tilde{\Gamma}$-action, and $Hom_{\mathbb{Z}_p}(\mathcal{T}^{-n}, \mathbb{Z}_p)$ is endowed with the contravariant $\tilde{\Gamma}$-action defined in general as follows: For M and N, two $\mathbb{Z}_p[[\tilde{\Gamma}]$-modules, $Hom_{\mathbb{Z}_p}(M, N)$ is equipped with a $\mathbb{Z}_p[[\tilde{\Gamma}]$-module structure with either a covariant action, i.e. $g \cdot f(x) = f(x^g)^{g^{-1}}$, or a contravariant action, i.e. $g \cdot f(x) = f(x^{g^{-1}})^g$, where $f \in Hom_{\mathbb{Z}_p}(M, N)$, $m \in M$ and $g \in \tilde{\Gamma}$.

Lemma 1.1.14. *([22]). Let M be a $\mathbb{Z}_p$-module with a $\tilde{\Gamma}$-action. For any integer n we have the following canonical isomorphisms:*

$$Hom_{\mathbb{Z}_p}(M(n), \mathbb{Q}_p/\mathbb{Z}_p) \simeq Hom_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p(-n)) \simeq Hom_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)(-n)$$

We also define for a Λ-module M, the **inverse** module $M^\#$ with the same underlying $\mathbb{Z}_p$-module as M but the inverse Γ-action given by $\gamma \cdot m := m^{\gamma^{-1}}$, for any $m \in M$. Since the characteristic polynomial of any Λ-module is determined by the action of $1 + T$, a change of variable leads to the following lemma:

Lemma 1.1.15. *([22]) Let M be a Λ-module with characteristic polynomial $f(T)$, where the base field F is a number field which contains a $2p$-th primitive root of unity ζ_{2p}. Then*

1. $char_\Lambda(M(n)) = f(\kappa(\gamma)^{-n}(1 + T) - 1)$
2. $char_\Lambda(M^\#) = f((1 + T)^{-1} - 1)$

Iwasawa modules usually appear with an extra group action. The eigenspaces with respect to this extra group action are defined as follows: Let M be a $\mathbb{Z}_p[[T]]$-module with G-action for a finite group G. For a character χ of G the χ-eigenspace of M is defined to be the following $\mathcal{O}_\chi[[T]]$-module:

$$M^\chi := \left\{ x \in M \otimes_{\mathbb{Z}_p} \mathcal{O}_\chi \mid \sigma(x) = \chi(\sigma)x \text{ for all } \sigma \in G \right\}$$

where $\mathcal{O}_\chi$ is the ring obtained by adjoining all χ-character values to $\mathbb{Z}_p$.

It is worth to highlight the isomorphism

$$(M_\Gamma)^* \simeq (M^*)^\Gamma, \tag{1.6}$$

where M^* is the Pontryagin dual $Hom_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$. We also have the following lemma (cf. [12], Lemma 5.16):

Lemma 1.1.16. *Let M be a $\mathbb{Z}_p[[H \times \Gamma]]$-module, where H is a finite group. Assume that M is free of finite rank as a $\mathbb{Z}_p$-module and M_Γ is finite. Then we have the following isomorphism of $\mathbb{Z}_p[H]$-modules:*

$$(M^\vee)_\Gamma \simeq (M_\Gamma)^*,$$

where $M^\vee$ denotes the dual $Hom_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ and both dual modules are viewed with either the covariant or the contravariant $(H \times \Gamma)$-actions.

1.1.3 Adjoints

Let M be a finitely generated Λ-torsion module. We have the following natural map:

$$\psi_M : M \to \prod_{\mathfrak{p} \in P^{(1)}} M_\mathfrak{p} = \oplus_{\mathfrak{p}|div(M)} M_\mathfrak{p},$$

where $P^{(1)}$ is the set of all primes of height one in Λ. Since pseudo-zero Λ-modules are the finite ones for the ring Λ, the kernel of this map is the maximal finite submodule of M. The cokernel of this map is called the **co-adjoint** of M and denoted by

$$\beta(M) := coker(\psi_M).$$

The co-adjoint is a right exact covariant functor on the category of finitely generated Λ-torsion modules. Moreover, since $ker(\psi_M)$ is the maximal finite submodule of M, it is also pseudo-left exact, i.e. if $M \mapsto N$ is an injective morphism of finitely generated Λ-torsion modules, then the kernel of $\beta(M) \to \beta(N)$ is finite.

Now for any finitely generated Λ-torsion module M, the **adjoint** $\alpha(M)$ of M is defined as the Pontryagin dual of $\beta(M)$:

$$\alpha(M) := Hom_{\mathbb{Z}_p}(\beta(M), \mathbb{Q}_p/\mathbb{Z}_p)$$

with covariant Λ-action

$$(\gamma \cdot f)(m) = f(m^\gamma),$$

where γ is a topological generator of Γ, $f \in \alpha(M)$ and $m \in \beta(M)$. The adjoint is then a left-exact contravariant functor on the category of finitely generated Λ-torsion modules. Another description of co-adjoints and adjoints is as follows:

Let M be a finitely generated Λ-torsion module and let $\mathfrak{m}$ be the maximal ideal of Λ. A sequence $\{\pi_n\}_{n\geq 0}$ of non-zero elements of Λ, which are all disjoint from $div(M)$, is called M**-admissible** if $\pi_0 \in \mathfrak{m}$ and $\pi_{n+1} \in \pi_n\mathfrak{m}$ for all $n \geq 0$. For example, $\{p^{n+1}\}_n$ is M-admissible, if the μ-invariant of M is trivial, and $\{T^{n+1}\}_n$ is M-admissible, if the λ-invariant of M is trivial.

Proposition 1.1.17. *(cf. [14], 1.3). Let M be a finitely generated Λ-torsion module. For any M-admissible sequence $\{\pi_n\}$ we have*

$$\beta(M) \simeq \varinjlim_n M/\pi_n M$$

$$\alpha(M) \simeq \varprojlim_n Hom_{\mathbb{Z}_p}(M/\pi_n M, \mathbb{Q}_p/\mathbb{Z}_p).$$

As an example, applying this proposition to $Y_\infty = Gal(L_\infty/H_{n_0})$ with an appropriate M-admissible sequence $\{v_{n_0,n}\}_n$ yields the following (cf. [14], Theorem 11):

$$\beta(Y_\infty) \simeq \varprojlim_n A_n = X_\infty.$$

We also have the following property of adjoints due to Iwasawa (cf. [14], 1.3):

Proposition 1.1.18. *If E is an elementary Λ-module, then $\alpha(E) \simeq E$ as Λ-modules.*

As a corollary, since the adjoint of any finite module is trivial, we have:

Corollary 1.1.19. *For any finitely generated Λ-module M, $\alpha(M)$ is pseudo-isomorphic to M and has no finite non-trivial Λ-submodule.*

1.2 L-functions

1.2.1 Introduction

Let E/F be a finite Galois extension of number fields with Galois group G. For a finite place w in E sitting above the place v in F, let G_w denote the decomposition

group, let I_w denote the inertia group and let σ_w be a Frobenius automorphism attached to w, i.e. σ_w is a generator of G_w/I_w. Now let V be a finite representation of G, i.e. a finite $\mathbb{C}$-dimensional $\mathbb{C}[G]$-module, with associated character χ_V.

The Artin L-function is defined as a product of Euler factors $L_v(s, V)$ for all finite places v in F as follows:

$$L(s, V) := \prod_v L_v(s, V) := \prod_v \frac{1}{det(1 - \sigma_w Nm(\mathfrak{p}_v)^{-s} | V^{I_w})},$$

where $Nm(\mathfrak{p}_v)$ is the norm of the prime $\mathfrak{p}_v$ associated to the place v, V^{I_w} is the fixed points of V under the action of the inertia group I_w and $det(1 - \sigma_w Nm(\mathfrak{p}_v)^{-s} | V^{I_w})$ is indeed the characteristic polynomial of σ_w acting on V^{I_w} evaluated at $T = Nm(\mathfrak{p}_v)^{-s}$. It is easy to see that these characteristic polynomials are independent of the choice of w and σ_w.

This product converges absolutely and defines an analytic function in the half plane $Re(s) > 1$. In fact it has a meromorphic continuation to the whole complex plane and by a deep conjecture of Artin, the analytic continuation gives an entire function if V is not the trivial representation. Some basic properties are as follows (cf. [1]):

1. $L(s, V \oplus V') = L(s, V) \cdot L(s, V')$ for any representations V and V'. This implies that it suffices to study Artin L-functions for irreducible representations.

2. For finite extensions of number fields $F \subseteq L \subseteq E$ and a representation V on $Gal(L/F)$, $L(s, V) = L(s, inf(V))$ where $inf(V)$ denotes the inflated representation of V on $Gal(E/F)$.

3. For finite extensions of number fields $F \subseteq L \subseteq E$ and a representation V on $Gal(E/L)$, $L(s, V) = L(s, ind(V))$ where $ind(V)$ denotes the induced representation of V on $Gal(E/F)$.

4. (Functional equation). Let r_1^+ and r_1^- denote the number of unramified real primes and ramified real primes in F, respectively. If we let

$$\Lambda(s, \chi_V) := \Gamma(\frac{s}{2})^a \cdot \Gamma(\frac{s+1}{2})^b \cdot L(s, V)$$

for $a := (r_2 + r_r^+)dim(V)$ and $b := (r_2 + r_1^-)dim(V)$, then for constant numbers $C_{\chi_V} \neq 0$ and $B_{\chi_V} > 0$ we have

$$\Lambda(1 - s, \chi_V) = C_{\chi_V} \cdot B_{\chi_V}^s \Lambda(s, \bar{\chi}_V).$$

The Artin L-function $L(s,V)$ is also denoted by $L(s,\chi_V)$. We note that for finite 1-dimensional respresentations, the Artin L-functions coincide with the Dirichlet L-functions via the following Artin reciprocity map:

$$\begin{cases} I_E^S \to Gal(E/F) \\ \\ w \to \sigma_w \end{cases}$$

Here S is a finite set of primes in E containing the ramified primes, I_E^S is the set of all fractional ideals in E which are relatively prime to the primes in S and σ_w is the Frobenius automorphism for any place $w \notin S$.

From now on, let E/F be a finite abelian Galois extension with Galois group G, let $\chi \in \hat{G}$ be a $\mathbb{C}$-valued character of G and let S be a finite set of primes in F which contains the primes ramified in E and the infinite primes. The corresponding 1-dimensional Artin L-function is as follows:

$$L_{E/F}(s,\chi) = \prod_{\mathfrak{p}} \frac{1}{1-\chi(\sigma_{\mathfrak{p}})Nm(\mathfrak{p})^{-s}}$$

where the product runs over all prime ideals of F. We know that $L_{E/F}(s,\chi)$ is analytic for $Re(s) > 1$ and has a meromorphic continuation to the whole complex plane. Moreover, its analytic continuation is an entire function if the character χ is not trivial and otherwise it has only one simple pole at $s = 1$. As a consequence of the functional equation of Artin L-functions, $L_{E/F}(1-n,\chi)$ is non-zero for $n \geq 2$ precisely in the following situations:

- E is a totally real number field and n is even.
- E is a CM field, i.e. a totally complex field which is a quadratic extension of its maximal real subfield, and n is odd.

By removing the Euler factors corresponding to the primes in S, one can define a S-incomplete L-function as

$$L_{E/F}^S(s,\chi) := \prod_{\mathfrak{p} \notin S} \frac{1}{1-\chi(\sigma_{\mathfrak{p}})Nm(\mathfrak{p})^{-s}}.$$

Now by taking the direct sum over all characters of G we construct the G-equivariant L-function attached to E/F. For this let

$$e_\chi := \frac{1}{\mid G \mid} \sum_{\sigma \in G} \chi(\sigma)\sigma^{-1}$$

be the idempotent of $\mathbb{C}[G]$ attached to any character χ of the group G. The G-equivariant L-function associated to E/F is defined as

$$\Theta_{E/F}(s) := \sum_{\chi \in \hat{G}} L_{E/F}(s, \chi^{-1}) \cdot e_\chi,$$

where the sum runs over all characters of G. Similarly removing the Euler factors corresponding to the primes in S leads to the following definition of the G-equivariant S-incomplete L-function associated to E/F:

$$\Theta^S_{E/F}(s) := \sum_{\chi \in \hat{G}} L^S_{E/F}(s, \chi^{-1}) \cdot e_\chi.$$

So $\Theta^S_{E/F} \in \mathbb{C}[G]$ is the unique element of the complex group ring $\mathbb{C}[G]$ such that for any character χ of G we have

$$\chi(\Theta^S_{E/F}(s)) = L^S_{E/F}(s, \chi^{-1}).$$

Theorem 1.2.1. *(Klingen-Siegel, cf. [42]). For any integer $n \geq 1$,*

$$\Theta^S_{E/F}(1-n) \in \mathbb{Q}[G].$$

Moreover, by the following theorem, a suitable multiple of $\Theta^S_{E/F}(s)$ is in fact in the integral group ring $\mathbb{Z}[G]$:

Theorem 1.2.2. *(Deligne-Ribet, cf. [7]). For any integer $n \geq 1$,*

$$Ann_{\mathbb{Z}[G]}(H^0(E, \mathbb{Q}/\mathbb{Z}(n))) \cdot \Theta^S_{E/F}(1-n) \subset \mathbb{Z}[G].$$

For simplicity we will drop the index E/F in the notations above, if there is no confusion about the extension.

1.2.2 p-adic L-functions

Let ψ be a 1-dimensional p-adic Artin character for the totally real number field F of finite order, i.e. a group homomorphism

$$\psi : Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}) \to \bar{\mathbb{Q}}_p^*$$

of finite order and let $\mathcal{O}_\psi$ denote the ring obtained by adjoining all character values of ψ to $\mathbb{Z}_p$. We first mention the terminology of Greenberg (cf. [11]) for the different types of the characters ψ: the character ψ is of **type S** if

$$F_\psi \cap F_\infty = F,$$

and of **type W** if

$$F_\psi \subseteq F_\infty.$$

So the trivial character is the only character of both types. By fixing an embedding $\mathbb{C} \to \mathbb{C}_p$ we identify the groups of the complex and the p-adic characters.

For a 1-dimensional p-adic Artin character ψ for $F = \mathbb{Q}$ of finite order, Kubota and Leopoldt have defined the p-adic L-function $L_p(s, \psi)$ as the unique continuous function of p-adic numbers, which interpolates the special values of the Artin L-function, with the Euler factor at p removed as

$$L_p(1-n, \psi) = (1 - \psi\omega^{-n}(p)p^{n-1})L(1-n, \psi\omega^{-n})$$

for any integer $n \geq 1$ and the Teichmüller character ω on $\mathbb{Q}$. Deligne and Ribet have extended the construction to an arbitrary number field F. They have shown that for any 1-dimensional p-adic valued Artin character ψ for F of finite order there is a p-adic L-function $L_p(s, \psi)$ which interpolates the special values of the Artin L-function with the Euler factors above p removed as

$$L_p(1-n, \psi) = \prod_{\mathfrak{p}|p}(1 - \psi\omega^{-n}(\mathfrak{p})Nm(\mathfrak{p})^{n-1}) \cdot L(1-n, \psi\omega^{-n}),$$

where $n \geq 1$ is an integer and ω is the Teichmüller character over F. $L_p(s, \psi)$ turns out to be p-adically analytic everywhere if ψ is not trivial and with at most one simple pole at $s = 1$ otherwise.

Let S_p denote the set of all primes of F sitting above p and let S be a finite set of primes in F containing S_p. By removing the Euler factors at the primes in $S \setminus S_p$ one can construct the S-incomplete p-adic L-function which satisfies

$$L_p^S(1-n, \psi) = \prod_{\mathfrak{p}|p}(1 - \psi\omega^{-n}(\mathfrak{p})\mathfrak{p}^{n-1}) \cdot L^S(1-n, \psi\omega^{-n}).$$

Deligne and Ribet have also proved that for any such character ψ for F, there is a power series $G_{\psi,S}$ in $\mathcal{O}_\psi[[T]]$ such that

$$L_p^S(1-s, \psi) = \frac{G_{\psi,S}(u^s - 1)}{H_\psi(u^s - 1)},$$

where $u = \kappa(\gamma) \in 1 + p\mathbb{Z}_p$ and $H_\psi(T) = \psi(\gamma)(1+T) - 1$ if ψ is of type W and 1 otherwise. Therefore, for any integer $n \geq 1$ and S-incomplete L-function $L^S(1-n, \psi\omega^{-n})$, we have

$$L^S(1-n, \psi\omega^{-n}) \sim_p \frac{G_{\psi,S}(u^n - 1)}{H_\psi(u^n - 1)}, \tag{1.7}$$

where $\sim_p$ denotes the equality of the p-adic valuations of both sides.

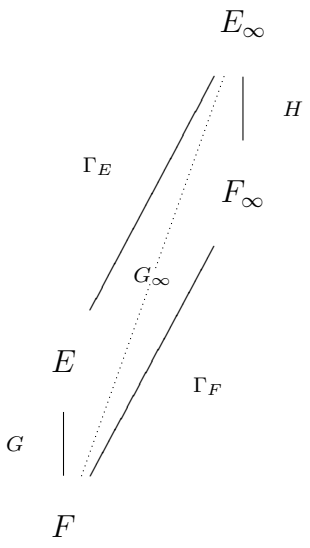
Remark 1.2.3. *By the functional equation of Artin L-functions (cf. Section 1.2.1) we see that $G_{\psi,S}$ vanishes if ψ is an odd character, i.e. $\psi(-1) = -1$.*

1.2.3 p-adic pseudo-measures

Let p be a fixed prime number. Let E/F be a finite abelian extension of totally real number fields with Galois group G. Let E_∞ (resp. F_∞) be the cyclotomic $\mathbb{Z}_p$-extension of E (resp. of F), with Galois group Γ_E (resp. Γ_F). We denote by H the Galois group of E_∞/F_∞, and by G_∞ the Galois group of E_∞/F. Since Γ_F is topologically generated by one element, the exact sequence

$$0 \to H \to G_\infty \leftrightarrows \Gamma_F \to 0 \tag{1.8}$$

splits. We denote by Γ the image of Γ_F under this splitting map, and by γ a topological generator of Γ. Assume that G_∞ is abelian. Then $G_\infty \simeq H \times \Gamma$. The following diagram illustrates the situation:



We set $\mathbb{A} := \mathbb{Z}_p[[G_\infty]]$ and denote by $Q(\mathbb{A})$ the quotient ring of $\mathbb{A}$. We freely use the identification

$$\mathbb{A} \simeq \mathbb{Z}_p[H][[T]],$$

where $\gamma - 1$ maps to T (see identification (1.1)).

Let S be a finite set of primes in F containing the primes above p, the primes ramified in E_∞ and the infinite primes, and let S_f denote the set of finite primes in S. We define equivariant versions of $G_{\psi,S}$ and H_ψ as follows (cf. [32], Proposition 5.4): For a character ψ of G_∞, let $G_{\psi,S}(T), H_\psi \in \mathcal{O}_\psi[[T]]$ be the power series defined

in Section 1.2.2. Let

$$G_S := \sum_{\psi \in \hat{H}} G_{\psi,S}(\gamma - 1) \cdot e_\psi \in \frac{1}{|H|}\mathcal{O}[H][[\Gamma]]$$
$$H_S := \sum_{\psi \in \hat{H}} H_{\psi}(\gamma - 1) \cdot e_\psi \in \frac{1}{|H|}\mathcal{O}[H][[\Gamma]] \tag{1.9}$$

be the equivariant versions of $G_{\psi,S}$ and H_ψ. For any character χ of G_∞ they satisfy the following:

$$\chi(G_S) = G_{\chi,S}(0) \quad , \quad \chi(H_S) = H_\chi(0).$$

We recall that by the Weierstrass Preparation Theorem (cf. Theorem 1.1.3),

$$G_{\psi,S}(T) = \pi^{\mu(G_{\psi,S})} \cdot G^*_{\psi,S}(T)$$

for $G^*_{\psi,S}(T) := g^*_{\psi,S}(T) \cdot u_{\psi,S}(T)$, where $g^*_{\psi,S}(T) \in \mathcal{O}_\psi[T]$ is a distinguished polynomial, $u_{\psi,S}(T) \in \mathcal{O}_\psi[[T]]$ is a unit, and π is a fixed uniformizer in $\mathcal{O}_\psi$. The modified equivariant L-function G^*_S is now defined as follows:

$$G^*_S := \sum_{\psi \in \hat{H}} G^*_{\psi,S}(\gamma - 1) \cdot e_\psi \in \frac{1}{|H|}\mathcal{O}[H][[\Gamma]]. \tag{1.10}$$

The following lemma relates G_S and G^*_S, assuming $\mu = 0$ (cf. (1.3)):

Lemma 1.2.4. *Under the assumption $\mu = 0$ we have the following equalities:*

1. *$G_S = G^*_S$ for any odd prime p.*
2. *$G_S = 2^{r_1(F)} G^*_S$ for $p = 2$.*

Proof. By a result of Wiles [47], $\mu(G_{\psi,S})$ is the same as the Iwasawa μ-invariant of the Λ-module $\mathfrak{X}_\infty^{S_f}$, which is zero by the assumption, for all odd primes p (see Section 1.5). In the case of the prime 2, the μ-invariant of $\mathfrak{X}_\infty^{S}$ differs from the μ-invariant of $\mathfrak{X}_\infty^{S_f}$ by the factor $2^{r_1(F)}$, by Proposition 1.1.12. Therefore the μ-invariant of $\mathfrak{X}_\infty^{S}$ is $2^{r_1(F)}$. Hence $\pi^{\mu(G_{\psi,S})} = 2^{r_1(F)}$ for any character ψ of G (see [11], pages 82 and 87). □

In particular, we have both equalities in Lemma 1.2.4 for any abelian number field F, since the assumption of the lemma holds (cf. [9]).

To see the next lemma we briefly review the definition of a p-adic pseudo-measure of a certain Galois group and its relation to the p-adic L-function. For more properties one can consult [40]. For a commutative profinite group $\mathcal{G}$ the element $\lambda_S \in Q(\mathbb{Z}_p[[\mathcal{G}]])$ is called a pseudo-measure on $\mathcal{G}$ if $(g-1)\lambda_S$ is a measure, i.e. $(g-1)\lambda_S \in \mathbb{Z}_p[[\mathcal{G}]]$, for any $g \in \mathcal{G}$, where $Q(\mathbb{Z}_p[[\mathcal{G}]])$ denotes the quotient ring of $\mathbb{Z}_p[[\mathcal{G}]]$. Let $\mathfrak{X} := \mathfrak{X}_F^{S_f}$ denote the Galois group of the maximal abelian pro-p-extension of F which is unramified outside the primes in S_f, over F. By a result of Deligne and Ribet in [7] there is a unique pseudo-measure on $\mathfrak{X}$ denoted by $\lambda_S \in Q(\mathbb{Z}_p[[\mathfrak{X}]])$, which satisfies the following relation for any finite order character χ of $\mathfrak{X}$:

$$L_{p,S}(1-s,\chi) =< \chi\kappa^s, \lambda_S > .$$

Equivalently if we let $\varepsilon : \mathfrak{X} \to \mathbb{Z}_p$ be the locally constant function defined by $\varepsilon(g) = 1$ if g has image 1 in H, and zero otherwise, then

$$\zeta_p^S(\varepsilon_h, 1-n) =< \varepsilon_h\rho^n, \lambda_S > .$$

Here ρ is the cyclotomic character, ε_h is the locally constant function satisfying $\varepsilon_h(x) = \varepsilon(hx)$ and $\zeta_p^S(\varepsilon_h, s)$ is the S-incomplete p-adic partial zeta function associated to ε_h. The image of λ_S under the natural surjection $\pi : \mathfrak{X} \twoheadrightarrow G_\infty$ is a p-adic pseudo-measure on G_∞ which is denoted by $\theta_S \in Q(\mathbb{A})$. So if $\hat{\gamma} \in \mathfrak{X}$ denotes a pre-image of $\gamma \in G_\infty$ under the surjection above, then

$$Z_S := \pi((\hat{\gamma}-1)\lambda_S) \in \mathbb{A}.$$

In fact $\theta_S = (\gamma-1)^{-1}Z_S \in Q(\mathbb{A})$. Now with notations as above we have the following:

Lemma 1.2.5. *Let d_∞ and c_∞ in $Q(\mathbb{A})$ satisfy $d_\infty = c_\infty((\gamma-1)e + (1-e)) \in \Delta G_\infty$, where e is the idempotent associated to the trivial character of H and ΔG_∞ denotes the augmentation ideal in $\mathbb{A}$. Then*

$$c_\infty G_S = d_\infty \theta_S \in \mathbb{A}$$

Proof. By a calculation in Proposition 12 in [33] for any character χ of G_∞ satisfying $\chi(\gamma) = 1$, we have the following:

$$\frac{G_{\chi,S}(T)}{T^{<\chi,1>}} = \sum_{h\in H} \chi(h)\frac{Z_S(h,T)}{T}$$

Here $Z_S(h,T)$ is given by the relation $Z_S = \sum_{h\in H} Z_S(h,\gamma-1)h \in \mathbb{A}$. As a result,

$$G_S/H_S = \sum_{\chi\in\hat{H}} \frac{G_{\chi,S}(T)}{T^{<\chi,1>}} e_\chi = \theta_S.$$

Since $H_S = (\gamma-1)e + (1-e)$, we obtain $c_\infty G_S = d_\infty \theta_S$. Therefore, for the p-adic pseudo-measure θ_S on G_∞ we have $c_\infty G_S = d_\infty \theta_S \in \mathbb{A}$. □

Let $\tilde{E} = E(\zeta_{2p})$ be the field obtaining by adjoining a primitive $2p$-th root of unity ζ_{2p} to E, and let $\tilde{E}_\infty := E_\infty(\zeta_{2p}) = E(\mu_{p^\infty})$ be the cyclotomic $\mathbb{Z}_p$-extension of $\tilde{E}$. We denote by $\tilde{G}_\infty$ the Galois group of $\tilde{E}_\infty/F$. Since $\tilde{E}_\infty$ contains all p-power roots of unity, we have the cyclotomic character

$$\rho : \tilde{G}_\infty \to \mathbb{Z}_p^* = Aut(\mu_{p^\infty})$$

of $\tilde{G}_\infty$. We extend the definitions of a Tate twisted module and an inverse module of Section 1.1.2 to the following:

- Let t_n be the unique continuous isomorphism of $\mathcal{O}$-algebras

$$t_n : \mathcal{O}[[\tilde{G}_\infty]] \to \mathcal{O}[[\tilde{G}_\infty]], \tag{1.11}$$

  which satisfies $t_n(g) = \rho(g)^n \cdot g$ for all $g \in \tilde{G}_\infty$ and $n \in \mathbb{Z}$. For a $\mathcal{O}[[\tilde{G}_\infty]]$-module M let the Tate twisted module $M(n)$ be the same underlying group M with a new $\mathcal{O}[[\tilde{G}_\infty]]$-action given by $\sigma *_n m := t_n(\sigma)m^\sigma$ for $\sigma \in \mathcal{O}[[\tilde{G}_\infty]]$ and $m \in M$.

- Let ι be the unique continuous isomorphism of $\mathcal{O}$-algebras

$$\iota : \mathcal{O}[[G_\infty]] \to \mathcal{O}[[G_\infty]]^{op}, \tag{1.12}$$

  which satisfies $\iota(g) = g^{-1}$ for all $g \in G_\infty$. For a $\mathcal{O}[[G_\infty]]$-module M, let the inverse module $M^\#$ be the same underlying group M with a new $\mathcal{O}[[G_\infty]]$-action given by $\sigma * m := \iota(\sigma)m^\sigma$ for $\sigma \in \mathcal{O}[[G_\infty]]$ and $m \in M$.

By using Lemma 1.1.15 we obtain the following lemma:

Lemma 1.2.6. *With notations as above we have the following:*

$$(\iota \circ t_n)(G_S) = \sum_{\psi \in \hat{H}} G_{\psi^{-1}\omega^n, S}(u^n(\gamma)^{-1} - 1) \cdot e_\psi$$

$$(\iota \circ t_n)(H_S) = \sum_{\psi \in \hat{H}} H_{\psi^{-1}\omega^n, S}(u^n(\gamma)^{-1} - 1) \cdot e_\psi$$

This lemma yields the following equality:

$$(\pi \circ \iota \circ t_n)G_S/H_S = \sum_{\chi \in \hat{G}} \frac{G_{\chi^{-1}\omega^n, S}(u^n - 1)}{H_{\chi^{-1}\omega^n, S}(u^n - 1)} \cdot e_\chi,$$

where $\pi : \mathbb{\Lambda} \to \mathbb{Z}_p[G]$ is the projection map sending $\gamma - 1$ to zero, and $u = \kappa(\gamma)$. Therefore we obtain:

Corollary 1.2.7.

$$(\pi \circ \iota \circ t_n)G_S/H_S = \Theta^S_{E/F}(1-n)$$

Remark 1.2.8. *We note that in all the dyadic and the non-dyadic L-functions we have defined, the set S can be replaced by S_f, since infinite primes have no influence on the definitions.*

1.3 Fitting ideals

1.3.1 Introduction

Let R be a commutative ring with identity and let M be a finitely generated R-module. We assume that M is a finitely presented R-module, i.e. that the module of relations for M is a finitely generated R-module. So we have the following presentation of M:

$$R^a \xrightarrow{h} R^b \to M \to 0$$

Let A be a matrix associated to h. The (initial) **Fitting ideal** of M is defined to be the ideal of R generated by all b-minors of A if $a \geq b$, and R otherwise. It is denoted by $Fitt_R(M)$, and if there is no confusion about the ring, we simply write $Fitt(M)$. One can see as a corollary of Schanuel's Lemma that this definition is independent of the choice of the finite presentation. Now we list some basic properties of Fitting ideals. For the proof one can consult [28].

1. $Fitt(M)$ is a finitely generated ideal of R satisfying

   $$(Ann_R(M))^b \subseteq Fitt_R(M) \subseteq Ann_R(M),$$

   where $Ann_R(M)$ is the annihilator ideal of M and b is an integer so that M can be generated by b elements as a R-module.

2. If $M \twoheadrightarrow M'$ is a surjective map of finitely presented R-modules, then

   $$Fitt_R(M) \subseteq Fitt_R(M').$$

3. If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of finitely presented R-modules, then

   $$Fitt_R(M') \cdot Fitt_R(M'') \subseteq Fitt_R(M).$$

   Moreover, we have equality if the exact sequence splits, i.e.

   $$Fitt_R(M' \oplus M'') = Fitt_R(M') \cdot Fitt_R(M'').$$

4. If $M \simeq R/\mathfrak{a}$ is a cyclic module, then

$$Fitt_R(M) = Ann_R(M) = \mathfrak{a}.$$

More generally, applying the previous property to $M \simeq R/\mathfrak{a}_1 \oplus R/\mathfrak{a}_2 \oplus \cdots \oplus R/\mathfrak{a}_n$, a direct sum of n cyclic R-modules, leads to

$$Fitt_R(M) = \mathfrak{a}_1\mathfrak{a}_2 \cdots \mathfrak{a}_n.$$

5. As a consequence of properties 3 and 4, the Fitting ideal of a finite module M over a PID R is principal and generated by the cardinality $|M|$ of M:

$$Fitt_R(M) =< |M| > .$$

6. If $I \subseteq R$ is any ideal and $\overline{Fitt_R(M)}$ is the image of $Fitt_R(M)$ in R/I, then

$$Fitt_{R/I}(M/I \cdot M) = \overline{Fitt_R(M)}.$$

One can also see that if $I \subseteq R$ is finitely generated, then

$$Fitt_R(M/I \cdot M) \subseteq (Fitt_R(M), I).$$

7. If M is a cyclic R-module, then

$$Fitt_R(M^*) = Ann_R(M^*) \simeq Ann_R(M) = Fitt_R(M),$$

where $M^* = Hom(M, \mathbb{Q}_p/\mathbb{Z}_p)$ is the Pontryagin dual.

8. Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a direct sum of some rings. Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ be a R-module, where M_i is a R_i-module for each i. Then

$$Fitt_R(M) = (Fitt_{R_1}(M_1), Fitt_{R_2}(M_2), \cdots, Fitt_{R_n}(M_n)),$$

as an ideal in the ring $R = R_1 \times R_2 \times \cdots \times R_n$.

We can say more about the relation between the Fitting ideals in an exact sequence in the following situation (cf. [28]):

Proposition 1.3.1. *Let R be a principal ideal domain and let*

$$0 \to E \to N \to M \to 0$$

be a short exact sequence of finitely generated R-torsion modules. Then

$$Fitt_R(N) = Fitt_R(M) \cdot Fitt_R(E).$$

Proof. Since one side of this equality holds for any short exact sequence (see property 3), it suffices to show $Fitt(N) \subseteq Fitt(M) \cdot Fitt(E)$. Let $\{m_1, m_2, \cdots, m_t\}$ and $\{e_1, e_2, \cdots, e_r\}$ generate M and E as R-modules, respectively. Letting n_i be a pre-image of m_i, then N is generated by $\{n_1, n_2, \cdots, n_t, e_1, e_2, \cdots, e_r\}$. Since M is a torsion R-module and R is a principal ideal domain, we can form a $(t \times t)$-matrix A, whose rows are some relations of $\{m_1, m_2, \cdots, m_t\}$ which generate all other relations. For any relation $(a_1, a_2, \cdots, a_t)$ of $\{m_1, m_2, \cdots, m_t\}$ the element $a_1n_1 + a_2n_2 + \cdots + a_tn_t \in N$ lies in E, and so generated by $\{e_1, e_2, \cdots, e_r\}$. Therefore, there exist b_j's so that $(a_1, \cdots, a_t, b_1, \cdots, b_r)$ is a relation of $\{n_1, \cdots, n_t, e_1, \cdots, e_r\}$. Hence the $(t \times t)$-matrix A can be extended to the $(t \times t + r)$-matrix $(A|B)$, in which the rows are relations of $\{n_1, \cdots, n_t, e_1, \cdots, e_r\}$. We note that if $(a'_1, \cdots, a'_t, b'_1, \cdots, b'_r)$ is an arbitrary relation of $\{n_1, \cdots, n_t, e_1, \cdots, e_r\}$, then by the way we chose the matrix A, the n-tuples $(b'_1, \cdots, b'_r)$ can be written as a linear combination of rows of B. Now let D be a matrix with $t + r$ columns so that each row is a relation of $\{n_1, \cdots, n_t, e_1, \cdots, e_r\}$. By adding t rows we get $D' = (\frac{A|B}{D})$, whose rows are still relations of $\{n_1, \cdots, n_t, e_1, \cdots, e_r\}$. By elementary row operations we change D' to its equivalent matrix $D'' = \left(\begin{smallmatrix} A & B \\ 0 & B' \end{smallmatrix}\right)$. We observe that any $(t + r)$-minor of D is also a $(t + r)$-minor of D' and consequently, a $(t + r)$-minor of D''. But, any $(t + r)$-minor of D'' vanishes unless it can be written as the product of a r-minor of B' and $det(A)$. Therefore, since Fitting ideals are generated by the minors, the statement of the proposition follows. □

Since any submodule of a finitely generated torsion module over a principal ideal domain is still finitely generated and torsion, we can state more generally the following for a long exact sequence:

Corollary 1.3.2. *Let $M_0, M_1, \cdots, M_n$ be finitely generated R-torsion modules where R is a principal ideal domain, which are related by the following exact sequence*

$$0 \to M_n \to M_{n-1} \to \cdots \to M_0 \to 0.$$

Then

$$Fitt(M_0) \cdot Fitt(M_1)^{-1} \cdots Fitt(M_n)^{(-1)^n} = 1,$$

where Fitting ideals are viewd as fractional ideals of R.

Now let R be a local ring and let M be a R-module of projective dimension at most one (see Section 1.4.1 for the definition of the projective dimension). Then the R-module M has a finite presentation

$$0 \to R^b \xrightarrow{h} R^b \to M \to 0.$$

As a result, the Fitting ideal of the R-module M is principal, and generated by the determinant of h. In the following subsection we describe the Fitting ideal of an important module of projective dimension at most one over some local ring.

1.3.2 Fitting ideals of certain Iwasawa modules

For this part let $\mathcal{O}$ be the integral closure of $\mathbb{Z}_p$ in some finite extension of $\mathbb{Q}_p$, and let M be a torsion $\mathcal{O}[[T]]$-module with no non-trivial finite submodules.

Lemma 1.3.3. *For a finitely generated $\mathcal{O}[[T]]$-torsion module M with no non-trivial finite submodules, the projective dimension of M is at most one:*

$$pd_{\mathcal{O}[[T]]}(M) \leqslant 1.$$

Proof. By the Structure Theorem for $\mathcal{O}[[T]]$-modules (cf. Theorem 1.1.4), there is an embedding of M into an elementary Iwasawa module $\bigoplus_i \mathcal{O}[[T]]/\mathfrak{p}_i^{a_i}$ with finite cokernel B, where the $\mathfrak{p}_i$'s are some primes of height one in $\mathcal{O}[[T]]$. Let $\lambda \in \mathcal{O}[[T]]$ be a relatively prime element to the characteristic polynomial $char_{\mathcal{O}[[T]]}(M)$. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 & \to & M & \to & \bigoplus_{\mathfrak{p}_i} \mathcal{O}[[T]]/\mathfrak{p}_i^{a_i} & \to & B & \to & 0 \\ & & \downarrow{\scriptstyle \cdot\lambda} & & \downarrow{\scriptstyle \cdot\lambda} & & & & \\ & & M & \to & \bigoplus_{\mathfrak{p}_i} \mathcal{O}[[T]]/\mathfrak{p}_i^{a_i}, & & & & \end{array}$$

where the vertical maps are given by scalar multiplication by λ. Here the injectivity of the second vertical map follows from the way we have chosen λ to be prime to the characteristic polynomial. Therefore the first vertical map is also injective, and so the depth of the $\mathcal{O}[[T]]$-module M is at least one. Now applying the Auslander-Buchsbaum formula (cf. [2]) to the regular local ring $\mathcal{O}[[T]]$, i.e.

$$pd_{\mathcal{O}[[T]]}(M) + depth_{\mathcal{O}[[T]]}(M) = dim(\mathcal{O}[[T]]) = 2,$$

yields $pd_{\mathcal{O}[[T]]}(M) \leq 1$. □

As a corollary, the Fitting ideal of M is a principal ideal. This has the following explicit description:

Proposition 1.3.4. *For a finitely generated $\mathcal{O}[[T]]$-torsion module M with no non-trivial finite submodules we have the following equality of ideals in $\mathcal{O}[[T]]$:*

$$Fitt_{\mathcal{O}[[T]]}(M) = (char_{\mathcal{O}[[T]]}(M)).$$

Proof. First we note that the Fitting ideal is generated by the characteristic ideal for any elementary Iwasawa module by property 4 of Fitting ideals in 1.3.1:

$$Fitt_{\mathcal{O}[[T]]}(\bigoplus_i \mathcal{O}[[T]]/\mathfrak{p}_i^{a_i}) = \prod_i \mathfrak{p}_i^{a_i} = (char_{\mathcal{O}[[T]]}(\bigoplus_i \mathcal{O}[[T]]/\mathfrak{p}_i^{a_i})).$$

Since $pd_{\mathcal{O}[[T]]}(M) \leq 1$ (cf. Lemma 1.3.3), the Fitting ideal of M is principal, and generated by some element $f(T) \in \mathcal{O}[[T]]$. For the $\mathcal{O}[[T]]$-torsion module M with no non-trivial finite submodules, there are some primes $\mathfrak{p}_i \in \mathcal{O}[[T]]$ of height one and a finite $\mathcal{O}[[T]]$-module B, so that the sequence

$$0 \to M \to \bigoplus_i \mathcal{O}[[T]]/\mathfrak{p}_i^{a_i} \to B \to 0$$

is exact (cf. Theorem 1.1.4). Now let $\mathfrak{p} \in \mathcal{O}[[T]]$ be an arbitrary prime. Localizing the exact sequence above at the prime $\mathfrak{p}$ yields the isomorphism

$$M_{\mathfrak{p}} \simeq \bigoplus_i (\mathcal{O}[[T]]/\mathfrak{p}_i^{a_i})_{\mathfrak{p}}.$$

Since the Fitting ideal coincide with the ideal generated by the characteristic polynomial for the right hand side of the isomorphism above, the same holds for $M_{\mathfrak{p}}$. Now it is enough to vary the primes $\mathfrak{p}$ in the unique factorization domain $\mathcal{O}[[T]]$ to complete the proof. □

1.3.3 Complements

In this part we list some propositions that we need for the next chapters. First we describe the relation between Fitting ideals and determinantal ideals in a certain situation. For this we review the definition of the determinantal ideal, which plays a role similar to that of the characteristic ideal for some Λ-modules with an extra group action.

For a commutative ring R with identity, a finitely generated projective R-module P and $f \in End_R(P)$, the determinant of f is defined as

$$det_R(f \mid P) := det_R(f \oplus id_Q \mid P \oplus Q),$$

where Q is a complement of P, i.e. $P \oplus Q$ is free. One can check that the definition is independent of Q by using Schanuel's lemma. By the same strategy, since $P \otimes_R R[X]$ is a finitely generated projective $R[X]$-module, the monic polynomial $det_R(X - f \mid P) \in R[X]$ is defined to be

$$det_R(X - f \mid P) := det_R(id_P \otimes X - f \otimes 1 \mid P \otimes_R R[X])$$

for any projective R-module P. One can see that these definitions are well-behaved under base-change, i.e.

$$\begin{aligned} det_R(f \mid P) &= det_{R'}(f \otimes id_{R'} \mid P \otimes R') \\ det_R(X - f \mid P) &= det_{R'}(X - (f \otimes id_{R'}) \mid P \otimes R'), \end{aligned} \tag{1.13}$$

where R' is any R-algebra. We have the following proposition:

Proposition 1.3.5. *(cf. [13], Proposition 7.2). Let R be a commutative, semi-local, compact topological ring and let Γ be a pro-cyclic group with topological generator γ. Let M be a topological $R[[\Gamma]]$-module, which is projective and finitely generated as an R-module. Let*

$$F(X) := det_R(X - m_\gamma \mid M),$$

where m_γ is the $R[[\Gamma]]$-module automorphism of M given by multiplication by γ. Then the following holds:

1. *M is finitely presented as an $R[[\Gamma]]$-module. If we let $F(\gamma)$ be the image of $F(X)$ via the R-algebra morphism $R[X] \to R[[\Gamma]]$ sending X to γ, then we have an equality of $R[[\Gamma]]$-ideals*
$$Fitt_{R[[\Gamma]]}(M) = (F(\gamma)).$$
2. *If we view $M_R^\vee = Hom_R(M, R)$ as a topological $R[[\Gamma]]$-module with covariant Γ-action, then*
$$Fitt_{R[[\Gamma]]}(M) = Fitt_{R[[\Gamma]]}(M_R^\vee).$$
3. *If we view $M^\vee = Hom_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ as a topological $R[[\Gamma]]$-module with covariant Γ-action, where $R = \mathbb{Z}_p[G]$ and G is a finite abelian group, then*
$$Fitt_{R[[\Gamma]]}(M) = Fitt_{R[[\Gamma]]}(M^\vee).$$

We also quote Lemma 5 in [3], which relates the Fitting ideals of the modules of a 4-term exact sequence under some assumptions.

Proposition 1.3.6. *Let $R := \mathbb{Z}_p[G]$, for a finite abelian group G and a prime number p. Assume that we have an exact sequence of finite R-modules*

$$0 \to A \to P \to P' \to A' \to 0.$$

Further, assume that $pd_{\mathbb{Z}_p[G]}P \leq 1$ and $pd_{\mathbb{Z}_p[G]}P' \leq 1$. Then we have

$$Fitt_R(A^*) \cdot Fitt_R(P') = Fitt_R(A') \cdot Fitt_R(P),$$

where the Pontryagin dual $A^* := Hom(A, \mathbb{Q}_p/\mathbb{Z}_p)$ is endowed with the covariant G-action.

Finally, we have the following lemma describing the Fitting ideal, the annihilator ideal and the projective dimension under the Tate-twist action. Again, let $\mathcal{O}$ be the integral closure of $\mathbb{Z}_p$ in some finite extension of $\mathbb{Q}_p$ for a prime p. Let $\tilde{G}_\infty$ be the Galois group of $E_\infty(\zeta_{2p})/F$ defined in Section 1.2.3, and let t_n be the homomorphisms defined by (1.11) for all $n \in \mathbb{Z}$. We have the following lemma (see for example Lemma 7.4 in [13] for a proof):

Lemma 1.3.7. *Assume that $\tilde{G}_\infty$ is an abelian group and M is a finitely presented $\mathcal{O}[[\tilde{G}_\infty]]$-module. Then for all $n \in \mathbb{Z}$, the following holds:*

1. $Ann_{\mathcal{O}[[\tilde{G}_\infty]]}(M(n)) = t_{-n}(Ann_{\mathcal{O}[[\tilde{G}_\infty]]}(M))$
2. $Fitt_{\mathcal{O}[[\tilde{G}_\infty]]}(M(n)) = t_{-n}(Fitt_{\mathcal{O}[[\tilde{G}_\infty]]}(M))$
3. $pd_{\mathcal{O}[[\tilde{G}_\infty]]}(M) = pd_{\mathcal{O}[[\tilde{G}_\infty]]}(M(n))$.

1.4 Cohomology

1.4.1 Group cohomology

Let G be a group and let M be a G-module. We define the group of i-th cochains $C^i(G, M)$ of G to be the set of functions from G^i to M

$$C^i(G, M) := \{f : G^i \to M\},$$

and the i-th differentials $d^i : C^i(G, M) \to C^{i+1}(G, M)$ to be the following maps :

$$d^i(f)(g_0, g_1, \cdots, g_i) = g_0 f(g_1, g_2, \cdots, g_i)$$

$$+ \sum_{j=0}^{i-1} (-1)^j f(g_0, \cdots, g_{i-2}, g_{j-1}g_{j+1}, g_{j+2}, \cdots, g_i) + (-1)^{i+1} f(g_0, g_1, \cdots, g_{i-1}).$$

Since for any $i \geq 0$, $d^{i+1} \circ d^i = 0$ we obtain a cochain complex

$$\cdots \overset{d^{i-2}}{\to} C^{i-1}(G, M) \overset{d^{i-1}}{\to} C^i(G, M) \overset{d^i}{\to} C^{i+1}(G, M) \overset{d^{i+1}}{\to} \cdots,$$

which defines the i-th cohomology group of G with coefficients in M as follows:

$$H^n(G, M) = \frac{Ker(d^n)}{Im(d^{n-1})}.$$

Let G be a profinite group, i.e. a topological group given by a projective limit

$$G = \varprojlim G/U,$$

where U runs over open subgroups of G of finite index. Let M be a discrete G-module, i.e. $M = \cup M^U$, where the union is over open subgroups U of G. From now on, a G-module means a discrete G-module. We define the n-th cohomology group of G with coefficients in M in the same way with continuous cochains, i.e. $C^i(G, M)$ is the set of continuous functions from G^n to M. Here we list some basic properties of these groups:

- $H^0(G, M) = M^G$.
- For an injective G-module I, $H^n(G, I) = 0$, for any $n \geq 1$.
- For a short exact sequence $0 \to L \to M \to N \to 0$ of G-modules we obtain a long exact sequence in a functorial way as follows:

$$0 \to L^G \to M^G \to N^G \to H^1(G, L) \to H^1(G, M) \to H^1(G, N) \to H^2(G, L) \to \cdots .$$

- Let M be an induced module, i.e. $M = \mathbb{Z}[G] \otimes_{\mathbb{Z}} M_0$ for an abelian group M_0 with G-action given by $g \cdot (a \otimes b) = (ga) \otimes b$. Then $H^n(G, M) = 0$ for any $n \geq 1$.
- For a G-module M with trivial G-action we have $H^1(G, M) = Hom(G, M)$.

Now let H be a closed subgroup of a profinite group G and let M be a continuous G-module. By changing the group we have the following maps:

- For any $i \geq 0$, there is a restriction map on cochains $C^i(G, M) \to C^i(H, M)$, which induces the restriction map $res : H^i(G, M) \to H^i(H, M)$.
- If the index of H in G is finite, there is a norm map $M^H \to M^G$ given by $m \to \sum_g gm$, where g runs over a set of left coset representatives of G/H. One can show that this map extends to the corestriction map $cor : H^i(H, M) \to H^i(G, M)$ for every $i \geq 0$.
- For a closed normal subgroup H in G, M^H is a G/H-module. There is an inflation map on cochains $inf : C^i(G/H, M^H) \to C^i(G, M)$, which induces the inflation map $inf : H^i(G/H, M^H) \to H^i(G, M)$, for any $i \geq 0$.

It is not hard to see that for a closed finite subgroup H of finite index in G, the composition map $res \circ cor : H^i(G, M) \to H^i(G, M)$ is given by multiplication by $|G/H|$. In particular, for a finite group G the cohomology group $H^i(G, M)$, for $i \geq 0$, is annihilated by $|G|$. If we assume that H is also normal in G we have a five term exact sequence

$$0 \to H^1(G/H, M^H) \overset{inf}{\to} H^1(G, M) \overset{res}{\to} H^1(H, M) \overset{tr}{\to} H^2(G/H, M^H) \overset{inf}{\to} H^2(G, M), \tag{1.14}$$

where tr is the transfer map (cf. [29], Proposition 1.6.6). One can also check that for a profinite group G and a G-module M,

$$H^n(G, M) = \varinjlim H^n(G/U, M^U),$$

where the injective limit is with respect to the inflation maps and U runs over open subgroups of G of finite index in G (cf. [29], Proposition 1.2.6).

We have the following definitions:

- The **cohomological dimension** $cd(G)$ of the group G is defined to be the smallest integer n such that $H^q(G, M) = 0$ for all $q > n$ and all torsion G-module M, and ∞ if no such integer exists. For a prime p the **cohomological p-dimension** $cd_p(G)$ of the group G is also defined to be the smallest integer n, such that the p-primary part $H^q(G, M)(p) = 0$ for all $q > n$ and all torsion G-module M, and ∞ if no such integer exists. For a pro-p group G one can show that $cd_p(G) \leq n$ if and only if $H^{n+1}(G, \mathbb{Z}/p\mathbb{Z}) = 0$ (cf. [29], Proposition 3.3.2).

- Let $n \geq 0$ and let G be a pro-p group such that the groups $H^i(G, \mathbb{Z}/p\mathbb{Z})$ are finite for all i with $0 \leq i \leq n$. Then the **n-th partial Euler-Poincaré characteristic** of G is defined to be

$$\chi_n(G) = \sum_{i=0}^{n} dim_{\mathbb{Z}/p\mathbb{Z}}(H^i(G, \mathbb{Z}/p\mathbb{Z})).$$

- A G-module M is defined to be **cohomologically trivial** if $H^n(H, M) = 0$ for all $n \geq 0$ and all closed subgroups H of G.

Let p be a prime number and let F be a number field with $r_2(F)$ pairs of complex embeddings. Let S be a finite set of primes in F containing the primes above p and the infinite primes, let S_f denote the set of finite primes in S and let G_F^S (resp. $G_F^{S_f}$) be the Galois group of the maximal algebraic pro-p-extension of F unramified outside the primes in S (resp. in S_f).We need the following important lemma later on. For a proof we refer to Proposition 8.3.17 and Corollary 8.6.16 in [29] in the case p is odd, and Theorem 1 in [39] for the case $p = 2$.

Lemma 1.4.1. *The cohomological dimension of $G_F^{S_f}$ is at most 2:*

$$cd(G_F^{S_f}) \leq 2,$$

and the second partial Euler-Poincaré characteristic of the group $G_F^{S_f}$ is given by:

$$\chi_2(G_F^{S_f}) = -r_2(F).$$

Remark 1.4.2. *For any odd prime p we observe that $G_F^S = G_F^{S_f}$, since the infinite primes are unramified in any p-extension.*

Now we want to obtain another description of group cohomology in terms of projective resolution. First we remark that $\mathbb{Z}[G^i]$ with G-action defined by diagonal

left multiplication is $\mathbb{Z}[G]$-free and hence projective. We define the standard projective resolution of $\mathbb{Z}$ by G-modules as follows:

$$\cdots \xrightarrow{d_i} \mathbb{Z}[G^{i+1}] \xrightarrow{d_{i-1}} \mathbb{Z}[G^i] \to \cdots \xrightarrow{d_0} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where ϵ is the augmentation map and d_i, for any $i \geq 0$ is given by

$$d_i(g_0, g_1, \cdots, g_{i+1}) = \sum_{j=1}^{i} (-1)^j (g_0, \cdots, g_{j-2}, g_{j-1}g_{j+1}, g_{j+2}, \cdots, g_{i+1}).$$

One can show that the standard projective resolution is exact. Now for a G-module, M we obtain a complex

$$0 \to Hom_{\mathbb{Z}[G]}(\mathbb{Z}[G], M) \xrightarrow{D^0} \cdots \to Hom_{\mathbb{Z}[G]}(\mathbb{Z}[G^{i+1}], M)$$

$$\xrightarrow{D^i} Hom_{\mathbb{Z}[G]}(\mathbb{Z}[G^{i+2}], M) \to \cdots,$$

where $D^i = f \circ d_i$ for any $i \geq 0$. One can show that the map

$$\psi^i : Hom_{\mathbb{Z}[G]}(\mathbb{Z}[G^{i+1}], M) \to C^i(G, M)$$

defined by

$$\psi^i(f)(g_1, g_2, \cdots, g_i) = f(1, g_1, g_1g_2, \cdots, g_1g_2 \cdots g_i)$$

is a natural isomorphism in the category of G-modules for any $i \geq 0$. This provides an isomorphism of complexes and so induces the same cohomology groups. Since the standard resolution is in fact a projective resolution of $\mathbb{Z}$ we obtain

$$H^i(G, M) \simeq Ext^i_{\mathbb{Z}[G]}(\mathbb{Z}, M), \tag{1.15}$$

where $Ext^i_{\mathbb{Z}[G]}(\mathbb{Z}, -)$ is the right derived functor of the functor $Hom_{\mathbb{Z}[G]}(\mathbb{Z}, -)$.

We recall that for a ring R and a R-module M, the **projective dimension** $pd_R(M)$ of M is defined to be the minimal number n such that there exists a projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots P_0 \to M \to 0$$

for M of length n. If there is no such resolution we let $pd_R(M) = \infty$.

Let $\mathcal{O}$ be a finite integral extension of $\mathbb{Z}_p$, and let G be a profinite group. We remark that (1.15) still holds for any $\mathcal{O}[[G]]$-module M by a similar argument. We have the following proposition (see [41], Chapt. IX, §5, Theorem 8):

Proposition 1.4.3. *Let M be a $\mathcal{O}[G]$-module for a finite group G. Then M is cohomologically trivial if and only if the projective dimension of M is finite, if and only if the projective dimension of M is at most one:*

$$pd_{\mathcal{O}[G]}M \leq 1.$$

If we further assume that M is $\mathbb{Z}$-free, then M is cohomologically trivial if and only if M is projective as a $\mathcal{O}[G]$-module.

Remark 1.4.4. *Serre proved this proposition for any $\mathbb{Z}[G]$-module M, where G is finite. However, one can easily generalize the proof to any $R[G]$-module M, where R is a principal ideal domain and G is a finite group.*

We have the following lemma (cf. [29], Proposition 5.2.11 and Corollary 5.2.13):

Lemma 1.4.5. *Let M be an $\mathcal{O}[[G]]$-module. Then*

1. *$pd_{\mathcal{O}[[G]]}M \leq n$ if and only if $Ext^{n+1}_{\mathcal{O}[[G]]}(M, N) = 0$ for any simple $\mathcal{O}[[G]]$-module N.*
2. *$pd_{\mathcal{O}[[G]]}\mathcal{O} = cd_p G$.*

Now back to Iwasawa theory. Let $\Lambda = \mathcal{O}[[\Gamma]]$ be a completed group ring, where $\Gamma \simeq \mathbb{Z}_p$. We have the following lemma (for a proof see Proposition 2.2 and Lemma 2.3 in [30]):

Lemma 1.4.6. *Let M be a finitely generated $\Lambda[H]$module, where H is a finite group. Then $pd_{\Lambda[H]}M \leq 1$ if and only if the following conditions are satisfied:*

1. *$pd_{\mathbb{Z}_p[H]}M \leq 1$.*
2. *M has no non-trivial finite Λ-submodules (i.e., $pd_\Lambda(M) \leq 1$).*

If we consider $\mathbb{Z}[G^i]$ as a right G-module under the right diagonal multiplication, then for a G-module M the standard resolution of $\mathbb{Z}$ induces a complex

$$\cdots \to \mathbb{Z}[G^3] \otimes_{\mathbb{Z}[G]} M \xrightarrow{d_1} \mathbb{Z}[G^2] \otimes_{\mathbb{Z}[G]} M \xrightarrow{d_0} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} M \to 0,$$

whose i-th homology groups define the i-th homology groups $H_i(G, M)$ of G with coefficients in M for all $i \geq 0$. Here are some basic properties:

- $H_0(G, M) = M_G$.

- For a projective G-module P, $H_n(G,P)=0$ for any $n\geq 1$.
- For a short exact sequence $0\to L\to M\to N$ of G-modules we obtain a long exact sequence in a functorial way as follows:

$$\cdots\to H_2(G,N)\to H_1(G,L)\to H_1(G,M)\to H_1(G,N)\to L_G\to M_G\to N_G\to 0.$$

The following proposition describes the relation between cohomology and homology groups (cf. [29] Theorem 2.2.9):

Proposition 1.4.7. *Let G be a profinite group and let M be a compact G-module. Then there are functorial isomorphisms for all $i\geq 0$*

$$H_i(G,M)^*\simeq H^i(G,M^*),$$

where $*$ denotes the Pontryagin dual.

1.4.2 A useful commutative diagram

Let $\mathfrak{c}$ be a class of finite groups containing $\mathbb{Z}/p\mathbb{Z}$ and closed under taking subgroups, homomorphic images and group extensions, where p is a fixed prime number. Let

$$1\to\mathcal{H}\to\mathcal{G}\to G\to 1$$

be an exact sequence of pro-$\mathfrak{c}$ groups, i.e. an exact sequence of groups which are projective limits of groups in $\mathfrak{c}$. Let $\Delta\mathcal{H}$ denote the augmentation ideal of $\mathbb{Z}_p[\mathcal{H}]$, and let $I=\Delta\mathcal{H}\mathbb{Z}_p[[\mathcal{G}]]$. Then the sequence

$$0\to I\to\mathbb{Z}_p[[\mathcal{G}]]\to\mathbb{Z}_p[[G]]\to 0 \tag{1.16}$$

is exact. We apply the functor $H_0(\mathcal{H},-)=Tor_0^{\mathbb{Z}_p[[\mathcal{G}]]}(\mathbb{Z}_p[[G]],-)$ to the exact sequence

$$0\to\Delta\mathcal{G}\to\mathbb{Z}_p[[\mathcal{G}]]\to\mathbb{Z}_p\to 0,$$

where $\mathbb{Z}_p[[\mathcal{G}]]$ is homologically trivial as an induced $\mathcal{H}$-module. Therefore we obtain the following exact sequence:

$$0\to\Delta\mathcal{H}\mathbb{Z}_p[[\mathcal{G}]]/\Delta\mathcal{H}\Delta\mathcal{G}\to\Delta\mathcal{G}/\Delta\mathcal{H}\Delta\mathcal{G}\to\mathbb{Z}_p[[\mathcal{G}]]/\Delta\mathcal{H}\mathbb{Z}_p[[\mathcal{G}]]\to\mathbb{Z}_p\to 0.$$

We have the isomorphism $\mathbb{Z}_p[[G]]\simeq\mathbb{Z}_p[[\mathcal{G}]]/I$ by (1.16), and the map

$$f: I/I^2\to\Delta\mathcal{H}\mathbb{Z}_p[[\mathcal{G}]]/\Delta\mathcal{H}\Delta\mathcal{G},$$

using the inclusion $(\Delta\mathcal{H}\mathbb{Z}_p[[\mathcal{G}]])^2 \subseteq \Delta\mathcal{H}\Delta\mathcal{G}$. Consequently, the exact sequence

$$I/I^2 \xrightarrow{f} \mathbb{Z}_p[[G]] \otimes_{\mathbb{Z}_p[[\mathcal{G}]]} \Delta\mathcal{G} \to \Delta G \to 0 \tag{1.17}$$

is obtained.

Let G be a finitely generated pro-$\mathfrak{c}$ group with presentation

$$1 \to R \to F_d \to G \to 1, \tag{1.18}$$

where F_d is a free group of rank d. Writing the exact sequence (1.17) for the short exact sequence above, and noting that ΔF_d is a free $\mathbb{Z}_p[[F_d]]$-module (cf. [29], Propositions 5.6.3 and 5.6.4) lead to the following exact sequence, which is a profinite analogue of a theorem of Lyndon for discrete groups (cf. [26], Proposition 1.1):

Proposition 1.4.8. *There is a canonical exact sequence:*

$$0 \to R^{ab}(p) \to \mathbb{Z}_p[[G]]^d \to \mathbb{Z}_p[[G]] \to \mathbb{Z}_p \to 0$$

In particular, if $cd_p(G) \leq 2$, then $R^{ab}(p)$ is a projective $\mathbb{Z}_p[[G]]$-module.

We note that the abelian pro-p group $R^{ab}(p)$ is naturally a $\mathbb{Z}_p[[G]]$-module, which is called the p-relation module of the group G with respect to the given presentation.

Now let G be a finitely generated pro-$\mathfrak{c}$ group with presentation

$$1 \to R \to F \to G \to 1, \tag{1.19}$$

where F is a free pro-$\mathfrak{c}$-group of rank d, and let

$$1 \to \mathcal{H} \to \mathcal{G} \to G \to 1$$

be an exact sequence of pro-$\mathfrak{c}$ groups. We want to find a description of the $\mathbb{Z}_p$-module $X := \mathcal{H}^{ab}(p)$. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 1 & = & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & N & = & N & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \to & R & \to & F & \to & G & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \to & \mathcal{H} & \to & \mathcal{G} & \to & G & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1, & & & &
\end{array}$$

where N is the kernel of the corresponding maps. For

$$X := \mathcal{H}^{ab}(p) \quad \text{and} \quad Y := \Delta\mathcal{G}/\Delta\mathcal{H}\Delta\mathcal{G},$$

we have the following proposition (cf. [26], Proposition 1.7 and [15], Lemma 4.3):

Proposition 1.4.9. *With the notations as above, the diagram*

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & 0 \\
 & & & & & & \downarrow & & \downarrow \\
0 & \to & H_2(\mathcal{H},\mathbb{Z}_p) & \to & H_0(\mathcal{H},N^{ab}(p)) & \to & R^{ab}(p) & \to & X & \to 0 \\
 & & || & & || & & \downarrow & & \downarrow & \\
0 & \to & H_2(\mathcal{H},\mathbb{Z}_p) & \to & H_0(\mathcal{H},N^{ab}(p)) & \to & \mathbb{Z}_p[[G]]^d & \to & Y & \to 0 \\
 & & & & & & \downarrow & & \downarrow & \\
 & & & & & & \Delta G & = & \Delta G & \\
 & & & & & & \downarrow & & \downarrow & \\
 & & & & & & 0 & & 0 &
\end{array}$$

is commutative.

Proof. Writing the homological form of the five term exact sequence (1.14) (cf. Proposition 1.4.7) for the group extension

$$1 \to N \to R \to \mathcal{H} \to 1$$

and the module $\mathbb{Z}_p$, and noting that $cd_p(R) \leq cd_p(F) = 1$ lead to the first row of the diagram. We note that $H_1(\mathcal{H}, \Delta\mathcal{G}) \simeq H_2(\mathcal{H}, \mathbb{Z}_p)$. To obtain the second row we first apply Proposition 1.4.8 to the exact sequence $1 \to N \to F \to \mathcal{G} \to 1$ and obtain

$$0 \to N^{ab}(p) \to \mathbb{Z}_p[[\mathcal{G}]]^d \to \Delta\mathcal{G} \to 0.$$

Now taking $\mathcal{H}$-homology of the exact sequence above and noting that $\mathbb{Z}_p[[\mathcal{G}]]$ is homologically trivial as an $\mathcal{H}$-induced module leads to the second row of the diagram. Similarly we obtain the second column by taking the $\mathcal{H}$-homology of the sequence $0 \to \Delta\mathcal{G} \to \mathbb{Z}_p[[\mathcal{G}]] \to \mathbb{Z}_p \to 0$. The first column is also given by Proposition 1.4.8. Therefore, for all these natural maps and compatible identifications we have the commutative diagram of the proposition. □

1.4.3 Étale cohomology and number fields

Let F be a number field, let $\mathcal{O}_F$ be its ring of integers and let p be a prime number. For μ_{p^m}, the group of p^m-th roots of unity, and $n \geq 1$ let $\mu_{p^m}^{\otimes n}$ denote the n-fold tensor product. Let $\Omega_F^{(p)}$ denote the maximal algebraic extension of F which is unramified outside the primes above p, and let $G_F^{(p)} := Gal(\Omega_F^{(p)}/F)$. The étale cohomology groups $H^*_{\acute{e}t}(Spec(\mathcal{O}_F[1/p]), \mu_{p^m}^{\otimes n})$ of the scheme $Spec(\mathcal{O}_F[1/p])$ with values in the étale sheaf $\mu_{p^m}^{\otimes n}$ as defined by Grothendieck (cf. [25]), can be identified with the Galois cohomology groups $H^*_{\acute{e}t}(G_F^{(p)}, \mu_{p^m}^{\otimes n})$. To simplify notations we will write $H^*_{\acute{e}t}(\mathcal{O}'_F, \mathbb{Z}/p^m(n))$ instead of $H^*_{\acute{e}t}(Spec(\mathcal{O}_F[1/p]), \mu_{p^m}^{\otimes n})$, where $\mathcal{O}'_F := \mathcal{O}_F[1/p]$.

Similarly for S, a finite set of primes of F containing the primes above p, we obtain the étale cohomology groups $H^*_{ét}(\mathcal{O}_F^S, \mathbb{Z}/p^m(n))$ as Galois cohomology groups, where we replace the extension $\Omega_F^{(p)}$ by Ω_F^S, the maximal algebraic extension of F, which is unramified outside the primes in S. We will also denote by:

$$H^i_{ét}(\mathcal{O}'_F, \mathbb{Z}_p(n)) = \varprojlim_m H^i_{ét}(\mathcal{O}'_F, \mu_{p^m}^{\otimes n})$$

the p-adic cohomology groups and set

$$H^i_{ét}(\mathcal{O}'_F, \mathbb{Q}_p/\mathbb{Z}_p(n)) = \varinjlim_m H^i_{ét}(\mathcal{O}'_F, \mu_{p^m}^{\otimes n}).$$

The long exact sequence of étale cohomology coming from the short exact sequence

$$0 \to \mathbb{Z}_p(n) \to \mathbb{Q}_p(n) \to \mathbb{Q}_p/\mathbb{Z}_p(n) \to 0$$

gives the following boundary maps:

$$\delta_i : H^{i-1}_{ét}(\mathcal{O}'_F, \mathbb{Q}_p/\mathbb{Z}_p(n)) \to H^i_{ét}(\mathcal{O}'_F, \mathbb{Z}_p(n))$$

The kernel and cokernel of these boundary maps were determined by Tate as follows: the kernel of δ_i is the maximal divisible subgroup of $H^{i-1}_{ét}(\mathcal{O}'_F, \mathbb{Q}_p/\mathbb{Z}_p(n))$ and the image of δ_i is the torsion subgroup of $H^i_{ét}(\mathcal{O}'_F, \mathbb{Q}_p/\mathbb{Z}_p(n))$ (cf. [44], Proposition 2.3). In particular, we have the following isomorphism:

$$\delta_1 : H^0_{ét}(\mathcal{O}'_F, \mathbb{Q}_p/\mathbb{Z}_p(n)) \xrightarrow{\sim} H^1_{ét}(\mathcal{O}'_F, \mathbb{Z}_p(n))_{tors}.$$

We have the following proposition about the p-adic cohomology groups:

Proposition 1.4.10. *(cf. [43]). For any number field F and $k \geq 3$,*

1. $H^0_{ét}(\mathcal{O}'_F, \mathbb{Z}_p(n)) = 0$ *if* $n \neq 0$
2. $H^k_{ét}(\mathcal{O}'_F, \mathbb{Z}_p(n)) = 0$ *if p is odd*
3. $H^k_{ét}(\mathcal{O}'_F, \mathbb{Z}_2(n)) \simeq \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{r_1(F)} & \text{if } k+n \text{ is even} \\ 0 & \text{otherwise,} \end{cases}$

   where $r_1(F)$ is the number of real embeddings of F.

We see that for a number field F the first and the second p-adic étale cohomology groups are of special interest. The structure can be described by using Borel's structure Theorem for algebraic K-groups together with Soulé's Chern character as

Proposition 1.4.11. *(cf. [19], corollary 2.5). For $n \geq 2$,*

1. $rank_{\mathbb{Z}_p}(H^1_{\acute{e}t}(\mathcal{O}'_F, \mathbb{Z}_p(n))) = \begin{cases} r_1(F) + r_2(F) & \text{if } n \text{ is odd} \\ r_2(F) & \text{if } n \text{ is even} \end{cases}$

2. $H^2_{\acute{e}t}(\mathcal{O}'_F, \mathbb{Z}_p(n))$ *is finite and trivial for almost all p.*

As a corollary we obtain that $H^1_{\acute{e}t}(\mathcal{O}'_F, \mathbb{Z}_p(n))$ is torsion for $n \geq 2$ if and only if F is totally real and n is even. For $n = 1$ we have

$$H^1_{\acute{e}t}(\mathcal{O}'_F, \mathbb{Z}_p(1)) \simeq \mathbb{Z}_p \otimes U'_F \quad \text{and} \quad H^2_{\acute{e}t}(\mathcal{O}'_F, \mathbb{Z}_p(1)) \simeq A'_F \oplus \mathbb{Z}_p^{s_p - 1},$$

where U'_F is the unit group of $\mathcal{O}'_F$, A'_F is the p-part of the class group of $\mathcal{O}'_F$ and s_p is the number of primes of F sitting above p (cf. [19], Chap. 2).

The relation between the étale cohomology groups of $spec(\mathcal{O}'_F)$ and of $spec(F)$ is given by Soulé's exact localisation sequence as follows:

Proposition 1.4.12. *(cf. [43]). Let $n \geq 2$ be an integer and let p be an odd prime. Then the following sequence is exact:*

$$0 \to H^1_{\acute{e}t}(\mathcal{O}'_F, \mu_{p^m}^{\otimes n}) \to H^1_{\acute{e}t}(F, \mu_{p^m}^{\otimes n}) \to \oplus_v H^0_{\acute{e}t}(k_v, \mu_{p^m}^{\otimes n-1}) \to H^2_{\acute{e}t}(\mathcal{O}'_F, \mu_{p^m}^{\otimes n}) \to \cdots,$$

where v runs over all finite places of F not dividing p and k_v is the residue field of F at v.

Taking inverse limits in the exact sequence of Proposition 1.4.12 yields

$$H^1_{\acute{e}t}(\mathcal{O}'_F, \mathbb{Z}_p(n)) \simeq H^1_{\acute{e}t}(F, \mathbb{Z}_p(n)),$$

since $H^0_{\acute{e}t}(k_v, \mathbb{Z}_p(n-1)) = 0$ for $n \neq 1$. Now we take the long exact sequence of étale cohomology of the exact sequence

$$0 \to \mathbb{Z}_p(n) \xrightarrow{\cdot p^m} \mathbb{Z}_p(n) \to \mathbb{Z}/p^m(n) \to 0$$

to obtain another useful sequence called the Bockstein sequence:

$$0 \to H^{i-1}_{\acute{e}t}(\mathcal{O}'_F, \mathbb{Z}_p(n))/p^m \to H^{i-1}_{\acute{e}t}(\mathcal{O}'_F, \mathbb{Z}/p^m\mathbb{Z}(n)) \to \quad {}_{p^m}H^i_{\acute{e}t}(\mathcal{O}'_F, \mathbb{Z}_p(n)) \to 0$$

for $i \geq 1$. Here ${}_{p^m}H^i_{\acute{e}t}(\mathcal{O}'_F, \mathbb{Z}_p(n))$ denotes the p^m-torsion part of $H^i_{\acute{e}t}(\mathcal{O}'_F, \mathbb{Z}_p(n))$.

In order to see the relation between algebraic K-theory and p-adic étale cohomology we need to consider étale Chern characters defined by Soulé:

$$ch^{(p)}_{i,n} : K_{2n-i}(\mathcal{O}_F) \otimes \mathbb{Z}_p \to H^i_{\acute{e}t}(\mathcal{O}'_F, \mathbb{Z}_p(n))$$

for $i = 1, 2$ and $n \geq 2$. He constructed these homomorphisms for odd p in [43] and showed the surjectivity. Dwyer and Friedlander in [6] used another approach which includes the prime 2 as well by using étale K-theory. The **Quillen-Lichtenbaum Conjecture** predicts that for a number field F the étale Chern characters above are isomorphisms for $i = 1, 2$, $n \geq 2$, unless $p = 2$ and F has a real embedding. In general, one can see that this conjecture is a consequence of the Bloch-Kato conjecture (cf. [19], chap. 2) which has recently been proven by Voevodsky (cf. [45]). The kernel and cokernel of the Chern characters are also described for $p = 2$ explicitly as follows:

Theorem 1.4.13. *(cf. [38]). The 2-primary Chern character*

$$ch^{(2)}_{i,n} : K_{2n-i}(\mathcal{O}_F) \otimes \mathbb{Z}_2 \to H^i_{\acute{e}t}(\mathcal{O}'_F, \mathbb{Z}_2(n))$$

for $i = 1, 2$ and $n \geq 2$ is

$$\begin{cases} \textit{an isomorphism} & \textit{if } 2n - i \equiv 0,1,2,7 \textit{ mod } 8, \\ \textit{surjective with kernel} \simeq (\mathbb{Z}/2\mathbb{Z})^{r_1} & \textit{if } 2n - i \equiv 3 \textit{ mod } 8, \\ \textit{injective with cokernel} \simeq (\mathbb{Z}/2\mathbb{Z})^{r_1} & \textit{if } 2n - i \equiv 6 \textit{ mod } 8. \end{cases}$$

For $n \equiv 3$ mod 4, there is an exact sequence

$$0 \to K_{2n-1}(\mathcal{O}_F) \otimes \mathbb{Z}_2 \to H^1_{\acute{e}t}(\mathcal{O}'_F, \mathbb{Z}_2(n)) \to (\mathbb{Z}/2\mathbb{Z})^{r_1}$$
$$\to K_{2n-2}(\mathcal{O}_F) \otimes \mathbb{Z}_2 \to H^2_{\acute{e}t}(\mathcal{O}'_F, \mathbb{Z}_2(n)) \to 0.$$

Now we turn our attention to the relation between algebraic K-groups and motivic cohomology groups. For a smooth scheme X over a base scheme B and an integer $n \geq 2$, we define the motivic cohomology groups $H^i_{\mathcal{M}}(X, \mathbb{Z}(n)$ to be the hypercohomology groups of Bloch's cycle complexes $\mathbb{Z}(n)$ $(n \geq 0)$ in the Zariski topology [23]. If $B = SpecF$, then the groups $H^i_{\mathcal{M}}(X, \mathbb{Z}(n))$ agree with Bloch's higher Chow groups and with the motivic cohomology groups defined by Levine and Voevodsky [23]. For details one can consult [17]. We use Chern classes and characters constructed by Pushin in [31] to relate K-theory to motivic cohomology. For the number field F he constructed Chern characters

$$ch^{\mathcal{M}}_{i,n} : K_{2n-i}(F) \to H^i_{\mathcal{M}}(F, \mathbb{Z}(n))$$

for $i = 1, 2$ and $n \geq 2$. This induces the étale Chern character after tensoring by $\mathbb{Z}_p$. Therefore the Quillen-Lichtenbaum conjecture shows that these Chern characters are isomorphisms up to 2-torsion parts. The 2-primary information is the same as in the theorem above.

Let E/F be a finite Galois extension of number fields with Galois group G. Let p be a prime number and let S be a finite set of primes of F containing the primes

above p and the primes ramified in E. So $Spec(\mathcal{O}_E^S) \to Spec(\mathcal{O}_F^S)$ is étale. Studying the Galois descent and the Galois co-descent of étale cohomology is the goal of this part. First we note that the p-adic version of the Hochschild-Serre spectral sequence is as follows:

$$E_2^{ij} := H^i(G, H^j_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_p(n))) \Rightarrow E^{i+j} := H^{i+j}_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_p(n)).$$

By Proposition 1.4.10, the E_2-terms are zero if $j = 0$ or if $j \geq 3$ and p is odd, and we obtain the following Galois descent relations for odd primes:

$$H^1_{\acute{e}t}(E, \mathbb{Z}_p(n))^G \simeq H^1_{\acute{e}t}(F, \mathbb{Z}_p(n)),$$

$$0 \to H^1(G, H^1_{\acute{e}t}(E, \mathbb{Z}_p(n))) \to H^2_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_p(n)) \to$$

$$H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_p(n))^G \to H^2(G, H^1_{\acute{e}t}(E, \mathbb{Z}_p(n))) \to 0.$$

Similarly, the étale version of the Tate spectral sequence, i.e.

$$E_2^{ij} := H_{-i}(G, H^j_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_p(n))) \Rightarrow E^{i+j} := H^{i+j}_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_p(n))$$

for $i \leq 0$ and $j \geq 0$ leads to the Galois co-descent relation for odd primes as follows:

$$H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_p(n))_G \simeq H^2_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_p(n)),$$

where the isomorphism is induced by the corestriction map. Since the higher étale cohomology groups do not vanish for the prime 2, the study of the Galois descent and the Galois codescent for the prime 2 is more complicated.

In this part we let E/F be an arbitrary quadratic extension of number fields with Galois group G generated by σ, $n \geq 2$ an odd integer, and r_∞ the number of infinite primes of F ramified in E. Furthermore, we denote by S a finite set of primes in F containing the primes above 2, the primes ramified in E and the infinite primes. The Galois codescent relations for $H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))$ are described by the following proposition (see Proposition 2.1 in [18] for a proof).

Proposition 1.4.14. *(cf. [18], Proposition 2.1). If there is a ramified infinite prime, i.e. if $r_\infty \neq 0$, then*

$$0 \to (\mathbb{Z}/2\mathbb{Z})^{r_\infty - 1} \to H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))_G \to H^1_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Q}_2/\mathbb{Z}_2(n)) \to 0$$

is exact, and otherwise if $r_\infty = 0$, then

$$H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))_G \simeq H^1_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Q}_2/\mathbb{Z}_2(n)).$$

Next we consider the Galois descent relations for $H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))$ in the following proposition (see Proposition 2.3 in [18] for a proof).

Proposition 1.4.15. *If there is a ramified infinite prime, i.e. if $r_\infty \neq 0$, then*

$$0 \to H^1(G, \mu(E)) \to H^1_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Q}_2/\mathbb{Z}_2(n)) \to H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^G \to 0$$

is exact, and if $r_\infty = 0$, then

$$0 \to H^1(G, \mu(E)) \to H^1_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Q}_2/\mathbb{Z}_2(n)) \to H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^G \to H^2(G, \mu(E)) \to 0$$

is exact.

We define $H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^-$ to be the kernel of the surjective corestriction map

$$H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n)) \overset{cor}{\to} H^1_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Q}_2/\mathbb{Z}_2(n)),$$

and $H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^{(-1)}$ to be the kernel of the cohomological norm

$$H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n)) \overset{1+\sigma}{\to} H^1_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Q}_2/\mathbb{Z}_2(n)).$$

As a consequence of Propositions 1.4.14 and 1.4.15 a diagram chase in the commutative diagram

$$\begin{array}{ccccc} H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^- & \rightarrowtail & H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n)) & \overset{cor}{\twoheadrightarrow} & H^1_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Q}_2/\mathbb{Z}_2(n)) \\ \downarrow & & || & & \downarrow \\ H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^{(-1)} & \rightarrowtail & H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n)) & \overset{1+\sigma}{\to} & H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^G, \end{array}$$

yields the following lemma:

Lemma 1.4.16. *There is an exact sequence*

$$0 \to H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^- \to H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^{(-1)} \to H^1(G, \mu(E)(2)) \to 0.$$

We also have the Galois descent and codescent relations for $H^2_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_2(n))$ as follows (see Proposition 2.8 in [18] for a proof):

Proposition 1.4.17. *If s_∞ denotes the 2-rank of the cokernel of the signature map $H^1_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_2(n)) \to (\mathbb{Z}/2\mathbb{Z})^{r_1(F)}$, then we have the following exact sequences:*

1. $$0 \to (\mathbb{Z}/2\mathbb{Z})^{s_\infty} \to H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))_G \to H^2_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_2(n)) \to 0,$$

   and

2.

$$0 \to H^1(G, H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))) \to H^2_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_2(n)) \to H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))^G$$

$$\to H^2(G, H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))) \to (\mathbb{Z}/2\mathbb{Z})^{r_\infty - s_\infty} \to 0.$$

Again, we define $H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))^-$ to be the kernel of the surjective corestriction map

$$H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n)) \overset{cor}{\to} H^2_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_2(n)),$$

and $H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))^{(-1)}$ to be the kernel of the cohomological norm

$$H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n)) \overset{1+\sigma}{\to} H^2_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_2(n)).$$

By a diagram chase in the commutative diagram

$$\begin{array}{ccccc} H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))^- & \rightarrowtail & H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n)) & \twoheadrightarrow & H^2_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_2(n)) \\ \downarrow & & || & & \downarrow \\ H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))^{(-1)} & \rightarrowtail & H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n)) & \to & H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))^G, \end{array}$$

and using Proposition 1.4.17 we obtain:

Lemma 1.4.18. *The following sequence is exact:*

$$0 \to H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))^- \to H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))^{(-1)} \to H^1(G, H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))) \to 0.$$

At the end of this section we discuss the relation between the Fitting ideals of $H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^-$ and of $H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))^-$, where E is a CM-extension of a totally real number field F, and $n \geq 2$ is odd.

First let E/F be quadratic. We note that $H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))$ and $H^1_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_2(n))$ are of the same rank by a consequence of Proposition 1.4.11. Therefore, the n-th Q-index

$$Q_n := [H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n)) : H^1_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_2(n))H^0(E, \mathbb{Q}_2/\mathbb{Z}_2)]$$

is finite. Since the image of $1-\sigma$ acting on $H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))$ is torsion, we have the map

$$H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n)) \overset{1-\sigma}{\to} H^0(E, \mathbb{Q}_2/\mathbb{Z}_2)/2,$$

whose kernel equals to $H^1_{\acute{e}t}(\mathcal{O}_F^S, \mathbb{Z}_2(n)) \cdot H^0(E, \mathbb{Q}_2/\mathbb{Z}_2)$. As a consequence, the n-th Q-index is either 1 or 2. Moreover, since the image of the above map lies in $H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))^-$, the n-th Q-index is equal to 2 if and only if $H^1(G, H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n)))$ is trivial (cf. [18], Section 3).

If Div_E is the maximal 2-divisible subgroup of $H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_2/\mathbb{Z}_2(n))$, then by an observation of Tate we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & Div_E^- & \to & H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_2/\mathbb{Z}_2(n))^- & \to & H^2_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Z}_2(n))^- & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Div_E & \to & H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_2/\mathbb{Z}_2(n)) & \to & H^2_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Z}_2(n)) & \to & 0 \\
 & & \downarrow \alpha & & \downarrow & & \downarrow & & \\
0 & \to & Div_F & \to & H^1_{\acute{e}t}(\mathcal{O}^S_F, \mathbb{Q}_2/\mathbb{Z}_2(n)) & \to & H^2_{\acute{e}t}(\mathcal{O}^S_F, \mathbb{Z}_2(n)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0, & &
\end{array} \tag{1.20}$$

in which the minus parts of the first and the second étale cohomology groups of $\mathcal{O}'_E$ with $\mathbb{Q}_2/\mathbb{Z}_2$-coefficients are compared.

Proposition 1.4.19. *Let $H^2_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Z}_2(n))^-$ and $H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_2/\mathbb{Z}_2(n))^-$ be the kernels of the corresponding corestriction maps. Then*

$$Fitt_{\mathbb{Z}_2}(H^2_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Z}_2(n))^-) = (2^{-r_1(F)}Q_n) \cdot Fitt_{\mathbb{Z}_2}(H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_2/\mathbb{Z}_2(n))^-),$$

where $r_1(F)$ is the number of real primes of F.

Proof. Since the kernels of the vertical maps in diagram (1.20) are all finite modules over the PID $\mathbb{Z}_p$, by Corollary 1.3.2 it suffices to show

$$Fitt_{\mathbb{Z}_2}(ker(\alpha)) = (2^{r_1(F)})(Q_n^{-1}).$$

Furthermore, since Div_E and Div_F are both of co-rank 2^{r_1}, by a diagram chase it is enough to show that the Fitting ideal of $ker(Div_F \overset{res}{\to} Div_F)$ is generated by (Q_n). Now we see that this holds by looking at the following commutative diagram (cf. [18], Proposition 3.1):

$$\begin{array}{ccccccccc}
 & & & & H^1(G, \mu(E)(2)) & \to & H^1(G, H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Z}_2(n))) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & Div_F & \to & H^1_{\acute{e}t}(\mathcal{O}^S_F, \mathbb{Q}_2/\mathbb{Z}_2(n)) & \to & H^2_{\acute{e}t}(\mathcal{O}^S_F, \mathbb{Z}_2(n)) & \to & 0 \\
 & & \downarrow res & & \downarrow & & \downarrow & & \\
0 & \to & Div_E & \to & H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_2/\mathbb{Z}_2(n)) & \to & H^2_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Z}_2(n)) & \to & 0 \\
 & & \downarrow & & & & & & \\
 & & 0. & & & & & &
\end{array}$$

□

We now let E/F be a finite abelian extension of number fields of order $2m$ for m odd, where E is CM and F is totally real. Let $G = Gal(E/F)$, $H = Gal(E/E^+)$ and $N = Gal(E^+/F)$, where E^+ denotes the maximal real subfield of E. We can consider

diagram (1.20) as a diagram of $\mathbb{Z}_2[N]$-modules. We note that for the group N of odd order, $\mathbb{Z}_2[N]$ is the maximal order in $\mathbb{Q}_2[N]$, and so

$$\mathbb{Z}_2[N] = \prod_{\psi \in \hat{N}} \mathcal{O}_\psi,$$

where ψ runs over the irreducible (real) $\mathbb{Q}_2$-characters of N.

We use the commutative diagram (1.20) for the quadratic extension E/E^+ and take ψ-eigenspaces of the $\mathbb{Z}_2[N]$-modules, which is a maximal order, for any character ψ of N. We obtain the exact sequence

$$0 \to (Div_E^-)^\psi \to (H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^-)^\psi \to (H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))^-)^\psi \to 0$$

of $\mathcal{O}_\psi$-modules. Since $\mathcal{O}_\psi$ is a PID, the same argument as in the proof of Proposition 1.4.19 shows that

$$Fitt_{\mathcal{O}_\psi}(H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))^\psi) = (2^{-r_1(E^+)}Q_n) \cdot Fitt_{\mathcal{O}_\psi}(H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^\psi)$$

for any character ψ of N. Now taking the direct sum over all characters of N and using property 8 of Fitting ideals in Section 1.3.1 yields

$$Fitt_{\mathbb{Z}_2[N]}(H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))^-) = (2^{-r_1(E^+)}Q_n) \cdot Fitt_{\mathbb{Z}_2[N]}(H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^-).$$

Consequently, since the action of the non-trivial character χ of $Gal(E/E^+)$ on the above modules is given by multiplication by -1, we obtain:

Proposition 1.4.20. *Let E/F be an abelian extension of number fields of order $2m$, for odd m, with Galois group G, where E is CM and F is totally real. If we consider the kernels $H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))^-$ and $H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^-$ of the corresponding corestriction maps as $\mathbb{Z}_p[G]$-modules, then*

$$Fitt_{\mathbb{Z}_2[G]}(H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))^-) = (2^{-r_2(E)}Q_n) \cdot Fitt_{\mathbb{Z}_2[G]}(H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^-),$$

where $r_2(E)$ is the number of pairs of complex embeddings in E.

1.5 The Main Conjecture in Iwasawa theory

Let F be a totally real number field, let F_∞ be the cyclotomic $\mathbb{Z}_p$-extension of F and let $\Gamma = Gal(F_\infty/F)$ be topologically generated by γ. Let S be a finite set of primes of F containing the primes above p and the infinite primes, and let S_f denote the set of finite primes in S. Let ψ be a 1-dimensional Artin character for F, i.e.

$$\psi : Gal(\bar{\mathbb{Q}}_p/F) \to \bar{\mathbb{Q}}_p^*$$

with finite order, let F_ψ be the associated field of ψ, i.e. the largest extension of F such that

$$\psi : Gal(F_\psi/F) \to \bar{\mathbb{Q}}_p^*$$

is a faithful character, and let $\mathcal{O}_\psi := \mathbb{Z}_p[\psi]$ be the ring obtained by adjoining all character values $\psi(g)$ to $\mathbb{Z}_p$ for $g \in Gal(F_\psi/F)$, with a fixed uniformizer π.

We fix an embedding $\mathbb{C} \to \mathbb{C}_p$, which identifies the groups of the complex and the p-adic characters. We assume that ψ is an even character, i.e. $\psi(-1) = 1$, and that the character ψ is of *type* S, i.e.

$$F_\psi \cap F_\infty = F.$$

Let L be a totally real finite abelian extension of F containing F_ψ, such that $L \cap F_\infty = F$, and let $L_\infty := LF_\infty$ denote the cyclotomic $\mathbb{Z}_p$-extension of L. Let $\mathfrak{X}_\infty^S$ be the Galois group of the maximal abelian pro-p-extension of L_∞, which is unramified outside the primes in S, over L. We denote by $F_{\psi,S} \in \mathcal{O}_\psi[T]$ the characteristic polynomial of the ψ-component

$$\mathfrak{X}_\infty^{S,\psi} := \left\{ x \in \mathfrak{X}_\infty^S \otimes_{\mathbb{Z}_p} \mathcal{O}_\psi | \sigma(x) = \psi(\sigma)x \quad \text{for all } \sigma \in Gal(L/F) \right\}$$

of $\mathfrak{X}_\infty^S$. This polynomial can be uniquely written as

$$F_{\psi,S}(T) = \pi^{\mu(F_{\psi,S})} \cdot f_{\psi,S}^*(T)$$

for a distinguished polynomial $f_{\psi,S}^* \in \mathcal{O}_\psi[T]$.

Remark 1.5.1. *By Proposition 1 in [11], $F_{\psi,S}$ is independent of the choice of the field L. So one can simply take $L = F_\psi$ for any even character ψ of type S.*

On the other hand, by the result of Deligne and Ribet there is a unique power series $G_{\psi,S}$ in $\mathcal{O}_\psi[[T]]$, such that

$$L_p^S(1-s,\psi) = \frac{G_{\psi,S}(\kappa(\gamma)^s - 1)}{H_\psi(\kappa(\gamma)^s - 1)},$$

where $H_\psi(T) = T$ if ψ is the trivial character, and $H_\psi(T) = 1$ otherwise (cf. Section 1.2.2). By the Weierstrass Preparation Theorem (cf. Theorem 1.1.3) the power series $G_{\psi,S}(T) \in \mathcal{O}_\psi[[T]]$ can be uniquely written as

$$G_{\psi,S}(T) = \pi^{\mu(G_{\psi,S})} \cdot g_{\psi,S}^*(T) \cdot u_{\psi,S}(T)$$

for the distinguished polynomial $g_{\psi,S}^* \in \mathcal{O}_\psi[T]$ and the unit $u_{\psi,S} \in \mathcal{O}_\psi[[T]]$. Now the classical Main Conjecture is as follows (cf. [47]):

Theorem 1.5.2. *(Wiles). For a totally real number field F, an even 1-dimensional Artin character ψ for F which is of type S, and any odd prime p,*

$$f^*_{\psi,S} = g^*_{\psi,S}.$$

If F is also assumed to be abelian, the equality still holds for $p = 2$.

Wiles also proved for odd primes that the algebraic μ-invariant $\mu(F_{\psi,S})$ and the analytic μ-invariant $\mu(G_{\psi,S})$ are the same provided that the character ψ is of order prime to p. However, the same equality is deduced for odd primes and an arbitrary character ψ, by using the following definition of μ-invariants of $\mathfrak{X}^{S,\psi}_\infty$ due to Greenberg (cf. [29], Chap. XI, §6): First we write the Galois group of the finite abelian extension F_ψ/F as $Gal(F_\psi/F) = Q \times P$, where P is the cyclic p-Sylow subgroup of $Gal(F_\psi/F)$ with generator h of order p^n. This gives the decomposition $\psi = \phi \cdot \phi'$, where ϕ is of order prime to p and ϕ' is of p-power order. Let $e_\phi \in \mathcal{O}_\psi[Q]$ be the idempotent associated to ϕ. Then μ_ψ of $\mathfrak{X}^S_\infty$ is defined to be the μ-invariant of the following $\mathcal{O}_\psi[[\Gamma]]$-module:

$$\mu_\psi = \begin{cases} \mu(\mathfrak{X}^{S,\psi}_\infty) = \mu(F_{\psi,S}) & \text{if } \psi \text{ is of order prime to } p \\ \mu((h^{p^{n-1}} - 1)e_\phi(\mathfrak{X}^S_\infty \otimes_{\mathbb{Z}_p} \mathcal{O}_\psi)) & \text{otherwise.} \end{cases}$$

With this definition we have the equality of the algebraic and the analytic μ-invariants $\mu_\psi = \mu(G_{\psi,S})$ for odd primes p. However, conjecturally both sides of this equality are zero for odd primes p, which was shown for abelian fields F, i.e. for the case that $F/\mathbb{Q}$ is abelian.

Remark 1.5.3. *The assumption $\mu = 0$ (cf. (1.3)) implies the vanishing of the μ-invariants of $\mathfrak{X}^{f,\psi}_\infty$ for all even characters ψ of G, since the Galois group of the maximal abelian pro-p-extension of E_∞ maps to the Galois group $\mathfrak{X}^{S_f}_\infty$ with a finite cokernel.*

By using the identification (1.1), which maps γ to $T + 1$ and the remark above, we have also the following formulation of the classical Main Conjecture for all primes p under the assumption $\mu = 0$:

$$(G^*_{\psi,S}(T)) = (det_{Q(\mathcal{O}_\psi)}((T + 1) - m_\gamma \mid \mathfrak{X}^{f,\psi}_\infty \otimes Q(\mathcal{O}_\psi)) \tag{1.21}$$

for $G^*_{\psi,S}(T) = g^*_{\psi,S}(T)u_{\psi,S}(T)$, where m_γ denote the automorphism of $\mathfrak{X}^{f,\psi}_\infty \otimes Q(\mathcal{O}_\psi)$ given by multiplication by γ. Here we recall that $Q(\mathcal{O}_\psi)$ is the quotient ring of $\mathcal{O}_\psi$, and $\mathfrak{X}^S_\infty = \mathfrak{X}^f_\infty$ for odd primes p. Finally, we obtain yet another formulation of the classical Main Conjecture for all primes p under the assumption $\mu = 0$ by using Proposition 1.3.4 as follows:

$$Fitt_{\mathcal{O}_\psi[[T]]}(\mathfrak{X}^{f,\psi}_\infty) = (G^*_{\psi,S}(T)). \tag{1.22}$$

Chapter 2

An Equivariant Main Conjecture in Iwasawa Theory

2.1 An Equivariant Main Conjecture

2.1.1 Algebraic construction and formulation

Let E/F be a finite abelian extension of totally real number fields with Galois group G, and let p be an arbitrary prime. Let E_∞ (resp. F_∞) be the cyclotomic $\mathbb{Z}_p$-extension of E (resp. F). We denote the multiplicative group $Gal(E_\infty/E)$ (resp. $Gal(F_\infty/F)$) by Γ_E (resp. Γ_F). Let H denote the Galois group of the finite abelian extension E_∞/F_∞. Hence $G_\infty := Gal(E_\infty/F)$ is **abelian**. We let S denote a finite set of primes in F, which contains the primes above p, the primes ramified in E_∞ and the infinite primes. The set of finite primes in S is also denoted by S_f. We use the same notations for the set of primes above the primes in S and S_f, respectively, in any intermediate field of E_∞/F. Since Γ_F is topologically generated by one element, the exact sequence

$$0 \to H \to G_\infty \leftrightarrows \Gamma_F \to 0 \tag{2.1}$$

splits. We denote by $\Gamma \leq G_\infty$ the image of Γ_F, so that $G_\infty \simeq H \times \Gamma$, and by Λ the group ring $\mathbb{Z}_p[[\Gamma]]$. Let E' be the fixed field of E_∞ under the action of the closed subgroup Γ. Then $E' \cap F_\infty = F$, $E_\infty = E' \cdot F_\infty$, $Gal(E'/F) \simeq H$ and E_∞/E' is also a cyclotomic $\mathbb{Z}_p$-extension.

Let M_∞^S and $M_\infty^{S_f}$ be the maximal abelian pro-p-extensions of E_∞ unramified outside the primes in S and S_f, respectively, with Galois groups $\mathfrak{X}_\infty := \mathfrak{X}_\infty^S$ and $\mathfrak{X}_\infty^f := \mathfrak{X}_\infty^{S_f}$, over E_∞, respectively. Since E is totally real, $\mathfrak{X}$ is a Λ-torsion module with no non-trivial finite submodule by Propositions 1.1.9 and 1.1.10. The Λ-module

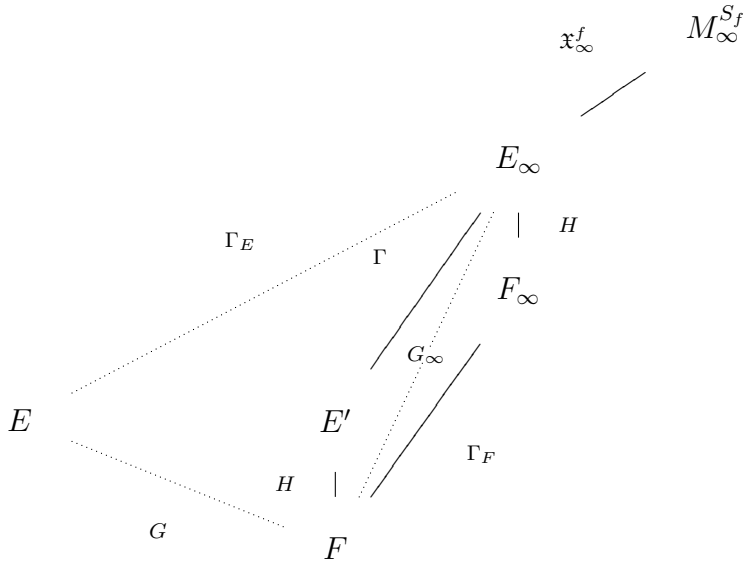
$\mathfrak{X}_\infty^f$, which is a quotient of $\mathfrak{X}_\infty$, is also torsion. Finally, we set $\mathbb{A} := \mathbb{Z}_p[[G_\infty]]$ and we freely use the identification

$$\mathbb{A} \simeq \mathbb{Z}_p[H][[T]], \tag{2.2}$$

which is given by mapping γ to $1+T$ (cf. (1.1)).

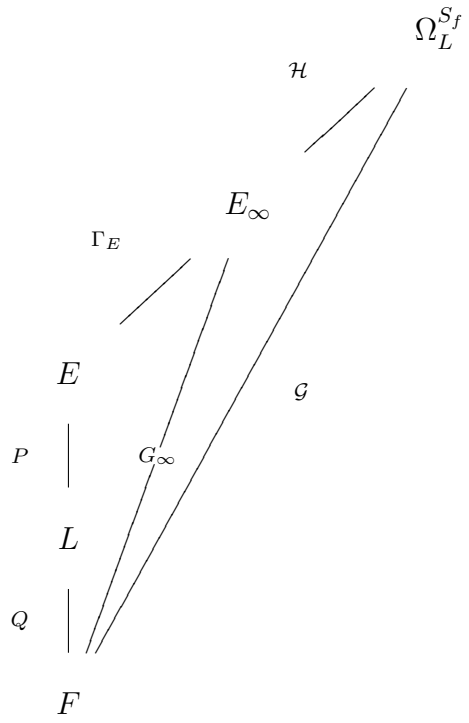
Remark 2.1.1. *For an odd prime p, infinite primes of F are unramified in a pro-p-extension. Hence M_∞^S and $M_\infty^{S_f}$ coincide. Consequently, $\mathfrak{X}_\infty = \mathfrak{X}_\infty^f$ for odd primes.*

The following diagram illustrates the situation:



We recall that the classical Main Conjecture in Iwasawa theory can be written in the form of equality (1.22). In this formulation the Fitting ideal of the (finitely generated) $\mathcal{O}_\psi[[\Gamma]]$-torsion module $\mathfrak{X}^{S_f,\psi}$, whose projective dimension is at most 1, is principal, and generated by the L-function associated to ψ, where ψ is a character of G. Hence, to formulate an Equivariant Main Conjecture we construct an appropriate (finitely generated) $\mathbb{A}$-torsion module of projective dimension at most 1, so that the Fitting ideal of that module is principal, and generated by an equivariant L-function. The strategy of this part is as follows: Since, the $\mathbb{A}$-torsion module $\mathfrak{X}_\infty^f$ is not necessarily of projective dimension at most one, in the first step the so-called envelope $\mathcal{Y}_\infty^f$ is constructed, whose projective dimension is at most one. However, this module is not necessarily torsion. In the next step, by taking a proper quotient of the envelope, the $\mathbb{A}$-module $\mathcal{Z}_\infty^f$ is constructed. We will see that $\mathcal{Z}_\infty^f$ is a (finitely generated) $\mathbb{A}$-torsion module of projective dimension at most 1, whose principal Fitting ideal is generated by an equivariant L-function.

Let P be the p-Sylow subgroup of G and let L be the fixed field of E under the action of P with Galois group Q, over F. Let $\Omega_L^{S_f}$ be the maximal algebraic pro-p-extension of L, which is unramified outside the primes in S_f. We denote by $\mathcal{H}$ the Galois group of $\Omega_L^{S_f}$ over E_∞, and by $\mathcal{G}$ the Galois group of $\Omega_L^{S_f}$ over F. The finitely generated group $\mathcal{G}$ has a presentation of the form $\mathcal{G} \simeq \mathcal{F}/\mathcal{W}$, where $\mathcal{F}$ is an appropriate free profinite group of rank d and $\mathcal{W}$ is a relation subgroup of $\mathcal{F}$ of rank r. For a certain relation subgroup $\mathcal{R}$ of $\mathcal{F}$ we then have an isomorphism $G_\infty \simeq \mathcal{F}/\mathcal{R}$. The following diagram illustrates the situation:



We apply Proposition 1.4.9 to the profinite groups in the commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{W} & = & \mathcal{W} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \to & \mathcal{R} & \to & \mathcal{F} & \to & G_\infty & \to & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \to & \mathcal{H} & \to & \mathcal{G} & \to & G_\infty & \to & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1, & & & &
\end{array}$$

and obtain a commutative diagram:

$$\begin{array}{ccccccccccc}
 & & & & & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \\
0 & \to & H_2(\mathcal{H},\mathbb{Z}_p) & \to & H_0(\mathcal{H},\mathcal{W}^{ab}) & \to & \mathcal{R}^{ab} & \to & \mathfrak{X}_\infty^f & \to & 0 \\
 & & || & & || & & \downarrow & & \downarrow & & \\
0 & \to & H_2(\mathcal{H},\mathbb{Z}_p) & \to & H_0(\mathcal{H},\mathcal{W}^{ab}) & \to & \mathbb{A}^d & \to & \mathcal{Y}_\infty^f & \to & 0 \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & \Delta G_\infty & = & \Delta G_\infty & & \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0, & &
\end{array} \tag{2.3}$$

where ΔG_∞ denotes the augmentation ideal of $\mathbb{A}$, and $\mathcal{Y}_\infty^f = H_0(\mathcal{H}, \Delta\mathcal{G})$ is the $\mathcal{H}$-coinvariant of the augmentation ideal $\Delta\mathcal{G}$ of $\mathbb{Z}_p[[\mathcal{G}]]$.

Remark 2.1.2. *One can form diagram* (2.3) *more generally for any intermediate field of* $\Omega_L^{S_f}/L$ *with a similar construction.*

The cyclotomic $\mathbb{Z}_p$-extension E_∞/E satisfies the weak Leopoldt conjecture by Proposition 1.1.9, and therefore in diagram (2.3) the group $H_2(\mathcal{H},\mathbb{Z}_p)$ vanishes (cf. Proposition 1.1.8). We also have the following proposition:

Proposition 2.1.3. *The* $\mathbb{A}$*-module* $H_0(\mathcal{H},\mathcal{W}^{ab})$ *is projective.*

Proof. Since $|Q|$ is prime to p, we have the equality of cohomological dimensions

$$cd_p(\mathcal{G}) = cd_p(Gal(\Omega_L^{S_f}/L)).$$

The p-cohomological dimension of the pro-p group $Gal(\Omega_L^{S_f}/L)$ is at most 2 (cf. Lemma 1.4.1), and so

$$cd_p(\mathcal{G}) \leq 2$$

for any prime p. By Lemma 1.4.5 we have

$$pd_{\mathbb{Z}_p[[\mathcal{G}]]}(\mathbb{Z}_p) = cd_p(\mathcal{G}).$$

This together with the exact sequence

$$0 \to \mathcal{W}^{ab} \to \mathbb{Z}_p[[\mathcal{G}]]^d \to \mathbb{Z}_p[[\mathcal{G}]] \to \mathbb{Z}_p \to 0,$$

given in Theorem 1.4.8, implies the projectivity of $\mathcal{W}^{ab}$ as a $\mathbb{Z}_p[[\mathcal{G}]]$-module. Now it suffices to take $\mathcal{H}$-coinvariant to conclude the projectivity of $H_0(\mathcal{H},\mathcal{W}^{ab})$ as a $\mathbb{Z}_p[[G_\infty]]$-module. □

Let χ be a $\mathbb{C}_p$-valued character of the group Q, let

$$e_\chi := \frac{1}{\mid Q \mid} \sum_{\sigma \in Q} \chi(\sigma)\sigma^{-1}$$

be the idempotent of Q attached to the character χ, and let $\mathbb{A}_\chi := \mathcal{O}_\chi[[G_\infty(p)]]$, where $\mathcal{O}_\chi$ is the ring obtained by adjoining all character values of χ to $\mathbb{Z}_p$. Since $G_\infty(p) \simeq P \times \Gamma$ is a pro-p group, $\mathbb{A}_\chi$ is a local ring and therefore $e_\chi H_0(\mathcal{H}, \mathcal{W}^{ab})$ is a free $\mathbb{A}_\chi$-module of rank r_χ:

$$e_\chi H_0(\mathcal{H}, \mathcal{W}^{ab}) \simeq \mathbb{A}_\chi^{r_\chi}.$$

Now by applying e_χ to the exact sequence in the second row of diagram (2.3) we obtain:

$$0 \to \mathbb{A}_\chi^{r_\chi} \to \mathbb{A}_\chi^{d} \to e_\chi \mathcal{Y}_\infty^f \to 0.$$

This implies that $\chi_2(Gal(\Omega_L^{S_f}/L)(p)) = -1 + d - r_\chi$, where $\chi_2(-)$ is the second partial Euler-Poincaré characteristic (cf. Section 1.4.3). Since L is totally real, $\chi_2(Gal(\Omega_L^{S_f}/L)(p))$ is zero by Lemma 1.4.1. Now by taking the direct sum over all characters of Q we obtain:

$$H_0(\mathcal{H}, \mathcal{W}^{ab}) \simeq \mathbb{A}^r$$

for $r = d - 1$. Therefore diagram (2.3) can be rewritten as

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{A}^r & \xrightarrow{f} & \mathcal{R}^{ab} & \to & \mathfrak{X}_\infty^f & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{A}^r & \xrightarrow{\phi} & \mathbb{A}^{r+1} & \to & \mathcal{Y}_\infty^f & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Delta G_\infty & = & \Delta G_\infty & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array} \tag{2.4}$$

So far we have constructed the envelope $\mathcal{Y}_\infty^f$ of $\mathfrak{X}_\infty^f$, which fits into the exact sequence

$$0 \to \mathfrak{X}_\infty^f \to \mathcal{Y}_\infty^f \to \Delta G_\infty \to 0. \tag{2.5}$$

The second row in diagram (2.4) implies that the $\mathbb{A}$-module $\mathcal{Y}_\infty^f$ is of projective dimension at most one. To make it into a torsion $\mathbb{A}$-module we now take a quotient of $\mathcal{Y}_\infty^f$ by a certain submodule as follows:

Let $d_\infty \in \Delta G_\infty$ be a non-zero divisor in the augmentation ideal of $\mathbb{A}$ and let c_∞ be an invertible element in $Q(\mathbb{A})$ such that

$$d_\infty = c_\infty((\gamma - 1)e + (1 - e)), \tag{2.6}$$

where γ is the fixed (topological) generator of $\Gamma \leq G_\infty$ and $e = \frac{1}{|H|}\sum_{h\in H} h$ is the idempotent in $Q(\mathbb{Z}_p[H])$ attached to the trivial character of H. Here we note that $\gamma - 1$ and $1 - e$ generate the augmentation ideal ΔG_∞, and $\gamma - 1$ and $1 - e$ can be written in the form (2.6) as follows:

$$\gamma - 1 = (e + (\gamma - 1)(1 - e))((\gamma - 1)e + (1 - e)),$$

$$1 - e = (1 - e)((\gamma - 1)e + (1 - e)).$$

Let $y_\infty \in \mathcal{Y}_\infty^f$ be a pre-image of d_∞ in diagram (2.4). We have the following diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathbb{A} & = & \mathbb{A} & & \\ & & & & \downarrow \Psi & & \downarrow \psi & & \\ 0 & \to & \mathfrak{X}_\infty^f & \to & \mathcal{Y}_\infty^f & \to & \Delta G_\infty & \to & 0 \\ & & || & & \downarrow & & \downarrow & & \\ 0 & \to & \mathfrak{X}_\infty^f & \to & \mathcal{Z}_\infty^f & \to & z_\infty^f & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0, & & \end{array} \tag{2.7}$$

where Ψ and ψ are defined by mapping $1 \in \mathbb{A}$ to y_∞ and to d_∞, respectively, and $\mathcal{Z}_\infty^f$ and z_∞^f are the quotients of $\mathcal{Y}_\infty^f$ and ΔG_∞ by the images of Ψ and ψ, respectively. We note that the vertical maps are injective, since $d_\infty \in \mathbb{A}$ is a non-zero divisor. By a diagram chase in the diagram

$$\begin{array}{ccccccc} & & \mathbb{A} & = & \mathbb{A} & & \\ & & \downarrow \psi & & \downarrow \psi & & \\ 0 & \to & \Delta G_\infty & \to & \mathbb{A} & \to \mathbb{Z}_p \to 0, & \end{array}$$

we obtain:

Lemma 2.1.4. *The exact sequence*

$$0 \to z_\infty^f \to \mathbb{A}/d_\infty\mathbb{A} \to \mathbb{Z}_p \to 0 \tag{2.8}$$

is exact, where the middle term is of projective dimension one and

$$Fitt_{\mathbb{A}}(\mathbb{A}/d_\infty\mathbb{A}) = (d_\infty).$$

By using the middle column of diagram (2.7) and the first row of diagram (2.4) we also obtain the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbb{A}^r & \to & \mathbb{A}^{r+1} & \to & \mathbb{A} & \to & 0 \\ & & \downarrow f & & \downarrow \Phi & & \downarrow \psi & & \\ 0 & \to & \mathcal{R}_\infty & \to & \mathbb{A}^{r+1} & \to & \Delta G_\infty & \to & 0 \\ & & \downarrow & & \downarrow & & || & & \\ 0 & \to & \mathfrak{X}_\infty^f & \to & \mathcal{Z}_\infty^f & \to & z_\infty^f & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0, & & \end{array} \tag{2.9}$$

which implies the following proposition:

Proposition 2.1.5. *$\mathcal{Z}_\infty^f$ is a finitely generated $\mathbb{A}$-torsion module of projective dimension at most one:*

$$pd_{\mathbb{A}}(\mathcal{Z}_\infty^f) \leq 1.$$

Now we formulate the Equivariant Main Conjecture for all primes. We recall that multiplying $c_\infty \in Q(\mathbb{A})$ to the modified equivariant L-function G_S^* for the field F (cf. (2.10) for the definition) yields an element in $\mathbb{A} \simeq \mathbb{Z}_p[H][[T]]$ by Lemma 1.2.5:

$$c_\infty G_S^* := c_\infty \sum_{\psi \in \hat{H}} G_{\psi,S}^*(\gamma - 1) \cdot e_\psi \in \mathbb{A}, \tag{2.10}$$

where S is a finite set of primes in F containing the primes above p, the primes ramified in E_∞, and the infinite primes, and e_ψ is the idempotent associated to the character ψ of H. Here we recall that for the abelian extension E/F of totally real number fields with Galois group G, and the extension E_∞/F_∞ of the cyclotomic $\mathbb{Z}_p$-extensions of E and F with Galois group H, the abelian Galois group $G_\infty = Gal(E_\infty/F)$ is of the form $H \times \Gamma$, where $\Gamma =< \gamma >$ is isomorphic to $\mathbb{Z}_p$.

Conjecture 2.1.6. *(The Equivariant Main Conjecture). For the abelian extension E_∞/F with Galois group G_∞, and a non-zero divisor $d_\infty = c_\infty((\gamma - 1)e + (1 - e))$ in the augmentation ideal of $\mathbb{A} = \mathbb{Z}_p[[G_\infty]]$ so that $\mathbb{A}/d_\infty\mathbb{A}$ is a finitely generated $\mathbb{Z}_p$-free module, e.g. $d_\infty = \gamma - 1$, we have the following equality of ideals in $\mathbb{A}$:*

$$Fitt_{\mathbb{A}}(\mathcal{Z}_\infty^f) = (c_\infty G_S^*).$$

Remark 2.1.7. *For any odd prime p, the formulation of the Equivariant Main Conjecture 2.1.6 coincides with the formulation of Ritter-Weiss in [33] (see Lemma 1.2.4 and Remark 2.1.1).*

2.1.2 Proof under the assumption $\mu = 0$

In this part we prove conjecture 2.1.6 based on the classical Main Conjecture in Iwasawa theory under the assumption $\mu = 0$. For some technical reasons we need to apply the contravariant functors $E^i(-) := Ext^i_{\mathbb{A}}(-, \mathbb{A})$ to $\mathcal{Z}^f_\infty$, for $i = 0, 1$. We will see that $E^1(\mathcal{Z}^f_\infty)$ is a finitely generated $\mathbb{A}$-torsion module of projective dimension at most one, whose Fitting ideal is in terms of the modified equivariant L-function.

We first note that $E^0(M) = Hom_{\mathbb{A}}(M, \mathbb{A})$ is a left exact functor for any $\mathbb{A}$-module M, and that the contravariant functor $E^i(-)$ turns any left $\mathbb{A}$-module M into a right $\mathbb{A}$-module. So for any left $\mathbb{A}$-module M we still have $E^i(M)^\#$ as a left $\mathbb{A}$-module. We recall that $\#$ denotes the inverse action defined as (1.12). To make this section more precise we see some general properties of the functors $E^i(M)$ in the following lemma. For a proof see Propositions 5.4.17, 5.5.6 and Corollary 5.5.7 in [29]. One can see [15] for more properties.

Lemma 2.1.8. *Let M be an $\mathbb{A}$-module, let $\alpha(M)$ denote the adjoint of M and let $M^\vee = Hom_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ be the dual with contravariant G_∞-action. Then*

1. *$E^i(M) = Ext^i_\Lambda(M, \Lambda)$ as Λ-modules for any $\mathbb{A}$-module M and $i \geq 0$.*
2. *$E^1(M) \simeq \alpha(M)$ as Λ-modules, provided M is a finitely generated Λ-torsion module.*
3. *$E^1(M)^\# \simeq M^\vee$ as Λ-modules, provided M is a Λ-torsion module with trivial μ-invariant, i.e. M is a finitely generated $\mathbb{Z}_p$-module.*

We list some results obtained by applying the contravariant functors $E^i(-)$ for $i = 1, 2$, to the commutative diagrams above. We first remark that $E^i(\mathbb{A}) = 0$ for $i \geq 1$ since $pd_{\mathbb{A}}(\mathbb{A}) = 0$ by Lemma 1.4.5.

Lemma 2.1.9. *The $\mathbb{A}$-module $E^1(\mathcal{Z}^f_\infty)^\#$ is of projective dimension at most one, and*

$$Fitt_{\mathbb{A}}(E^1(\mathcal{Z}^f_\infty)^\#) = Fitt_{\mathbb{A}}(\mathcal{Z}^f_\infty)^\#.$$

Proof. We first apply $E^i(-)$ to the last column of diagram (2.7). We observe that $Hom_{\mathbb{A}}(z^f_\infty, \mathbb{A})$ is the set of all morphisms in $Hom_{\mathbb{A}}(\Delta G_\infty, \mathbb{A})$, whose restriction to $d_\infty \mathbb{A}$ vanishes. This observation and the choice of $d_\infty \in \Delta G_\infty$ as a non-zero divisor imply that $E^0(z^f_\infty) = 0$. By part 1 of Lemma 2.1.8 $E^0(\mathfrak{X}^f_\infty)$ is also trivial for the Λ-torsion module $\mathfrak{X}^f_\infty$. Hence, by applying the contravariant functor $E^i(-)$ to the last row of diagram (2.7), we obtain $E^0(\mathcal{Z}^f_\infty) = 0$. On the other hand $E^i(\mathbb{A})$ is trivial

for $i \geq 1$ as we mentioned before. Therefore applying $E^i(-)$ to the middle column of diagram (2.9) leads to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & E^0(\mathbb{A})^{r+1} & \overset{E^0(\Phi)}{\to} & E^0(\mathbb{A})^{r+1} & \to & E^1(\mathcal{Z}_\infty^f) & \to & 0 \\ & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ 0 & \to & \mathbb{A}^{r+1} & \to & \mathbb{A}^{r+1} & \to & E^1(\mathcal{Z}_\infty^f)^\# & \to & 0, \end{array}$$

which shows that the projective dimension of $E^1(\mathcal{Z}_\infty^f)^\#$ is at most one. As another consequence of the diagram above, the Fitting ideal of the right $\mathbb{A}$-module $E^1(\mathcal{Z}_\infty^f)$ is given by the determinant of $E^0(\Phi)$. The Fitting ideal of $\mathcal{Z}_\infty^f$ is given by the determinant of Φ defined in diagram (2.9) and this completes the proof. □

Lemma 2.1.10. *We have the following exact sequence of finitely generated $\mathbb{A}$-torsion modules:*

$$0 \to \mathbb{Z}_p \to \mathbb{A}/d_\infty\mathbb{A} \to E^1(z_\infty^f)^\# \to 0.$$

Proof. We saw in the proof of Lemma 2.1.9 that $E^0(z_\infty^f)$ is trivial. Since $pd_\Lambda(\mathbb{Z}_p) = 1$ by a consequence of Proposition 1.4.5, $E^2(\mathbb{Z}_p)$ is also trivial. So by applying $E^i(-)$ to the exact sequence (2.8) we obtain

$$\begin{array}{ccccccccc} 0 & \to & E^1(\mathbb{Z}_p) & \to & E^1(\mathbb{A}/d_\infty\mathbb{A}) & \to & E^1(z_\infty^f) & \to & 0 \\ & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ 0 & \to & \mathbb{Z}_p & \to & \mathbb{A}/d_\infty\mathbb{A} & \to & E^1(z_\infty^f)^\# & \to & 0, \end{array}$$

where the first vertical isomorphism follows from part 2 of Lemma 2.1.8 together with Proposition 1.1.18, and the second vertical isomorphism is a consequence of $pd_\mathbb{A}(\mathbb{A}) = 0$. □

Lemma 2.1.11. *We have the following exact sequence of finitely generated $\mathbb{A}$-torsion modules:*

$$0 \to E^1(z_\infty^f) \to E^1(\mathcal{Z}_\infty^f) \to E^1(\mathfrak{X}_\infty^f) \to 0.$$

Proof. First we observe that $E^2(z_\infty^f)$ is trivial by applying $E^i(-)$ to the exact sequence (2.8) and by noting that the projective dimensions of $\mathbb{Z}_p$ and $\mathbb{A}/d_\infty\mathbb{A}$ are both one. Now we apply $E^i(-)$ to the last row of diagram (2.7) to obtain the exact sequence above. We note that the surjectivity of the last map in the diagram follows from the observation that $E^2(z_\infty^f) = 0$, and that the injectivity of the first map in the diagram is a consequence of the observation that $E^0(\mathfrak{X}_\infty^f) = 0$ in the proof of Lemma (2.1.9). □

We combine 2.1.10 and 2.1.11 to obtain the following theorem:

Theorem 2.1.12. *We have the following exact sequence of finitely generated $\mathbb{A}$-torsion modules:*

$$0 \to \mathbb{Z}_p \to \mathbb{A}/d_\infty\mathbb{A} \to E^1(\mathcal{Z}_\infty^f)^\# \to E^1(\mathfrak{X}_\infty^f)^\# \to 0,$$

in which

$$pd_{\mathbb{A}}(\mathbb{A}/d_\infty\mathbb{A}) \leq 1 \quad \textit{and} \quad pd_{\mathbb{A}}(E^1(\mathcal{Z}_\infty^f)^\#) \leq 1.$$

The Λ-module $E^1(\mathfrak{X}_\infty^f)^\#$ is the same as the adjoint of $\mathfrak{X}_\infty^f$ by part 1 of Lemma 2.1.8 and so it is a finitely generated $\mathbb{Z}_p$-free module under the assumption $\mu = 0$. Therefore, by the exact sequence in Theorem 2.1.12 we have the following proposition:

Proposition 2.1.13. *Let $d_\infty \in \Delta G_\infty$ be a non-zero divisor so that $\mathbb{A}/d_\infty\mathbb{A}$ is a finitely generated $\mathbb{Z}_p$-free module, e.g. $d_\infty = \gamma - 1$. Then under the assumption $\mu = 0$, the sequence*

$$0 \to \mathbb{Z}_p \to \mathbb{A}/d_\infty\mathbb{A} \to E^1(\mathcal{Z}_\infty^f)^\# \to E^1(\mathfrak{X}_\infty^f)^\# \to 0$$

is an exact sequence of finitely generated $\mathbb{Z}_p$-free modules. Moreover, if we consider this sequence as an exact sequence of $\mathbb{Z}_p[H]$-modules, then under the assumptions of $\mu = 0$ we have

$$pd_{\mathbb{Z}_p[H]}(\mathbb{A}/d_\infty\mathbb{A}) = 0 \quad \textit{and} \quad pd_{\mathbb{Z}_p[H]}(E^1(\mathcal{Z}_\infty^f)^\#) = 0.$$

Proof. We only need to prove the second part. We first remark that $\mathbb{A}/d_\infty\mathbb{A}$ and $E^1(\mathcal{Z}_\infty^f)$ are both H-cohomologically trivial by Lemma 1.4.6, since their projective dimensions are at most one as $\mathbb{A}$-modules. By a classical theorem of Nakayama $pd_{\mathbb{Z}_p[H]}(M) = 0$ if and only if M is $\mathbb{Z}_p$-free and H-cohomologically trivial (cf. Proposition 1.4.3). Therefore both modules are of projective dimension zero as $\mathbb{Z}_p[H]$-modules. □

By using Proposition 1.3.5 for the ring $R = \mathbb{Z}_p[H]$ and the finitely generated R-modules $M = E^1(\mathcal{Z}_\infty^f)^\#$ and $M = \mathbb{A}/d_\infty\mathbb{A}$, which are projective by Proposition 2.1.13, we obtain:

Lemma 2.1.14. *If m_γ denotes the $R[[\Gamma]]$-module automorphism of M given by multiplication by γ, then*

$$Fitt_{\mathbb{A}}(E^1(\mathcal{Z}_\infty^f)^\#) = (det_{\mathbb{Z}_p[H]}((T+1) - m_\gamma \mid E^1(\mathcal{Z}_\infty^f)^\#)),$$

$$Fitt_{\mathbb{A}}(\mathbb{A}/d_\infty\mathbb{A}) = (det_{\mathbb{Z}_p[H]}((T+1) - m_\gamma \mid \mathbb{A}/d_\infty\mathbb{A})).$$

Let $\mathcal{O}$ be the ring of integers obtained by adjoining all character values of the characters of H to $\mathbb{Z}_p$, let π be a fixed uniformizer in $\mathcal{O}$ and let $Q(\mathcal{O})$ denote the field of fraction of $\mathcal{O}$. We consider $\mathcal{O}$ and $Q(\mathcal{O})$ as $\mathbb{A}$-modules with trivial G_∞-action. We note that for the idempotent e attached to the trivial character of H we have $H_S(T) = T \cdot e + (1 - e)$ using the identification (2.2) (see (1.9) for the definition of $H_S(T)$). Therefore, using Lemma 2.1.14 we obtain the following lemma:

Lemma 2.1.15. *We have the following equalities of ideals in $Q(\mathcal{O})[H]$:*

$$(H_S(T)) = (det_{Q(\mathcal{O})[H]}((T+1) - m_\gamma \mid Q(\mathcal{O}))),$$

$$(d_\infty) = (det_{Q(\mathcal{O})[H]}((T+1) - m_\gamma \mid \mathbb{A}/d_\infty\mathbb{A})).$$

Remark 2.1.16. *Any character χ of H can be extended to a $Q(\mathcal{O})[X]$-algebra homomorphism, for a variable X, and to a $Q(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}[[\Gamma]]$-algebra homomorphism*

$$\chi : Q(\mathcal{O})[H][X] \to Q(\mathcal{O})[X],$$

$$\chi : Q(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}[[G_\infty]] \to Q(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}[[\Gamma]],$$

which map $h \to \chi(h)$ for $h \in H$, respectively.

Lemma 2.1.17. *We have the following equality of ideals in $\mathcal{O}[[\Gamma]]$:*

$$(det_{Q(\mathcal{O})}((T+1) - m_\gamma \mid e_\psi(\mathfrak{X}_\infty^f \otimes_{\mathbb{Z}_p} Q(\mathcal{O}))))^\# = (det_{Q(\mathcal{O})}((T+1) - m_\gamma \mid e_\psi(E^1(\mathfrak{X}_\infty^f)^\# \otimes_{\mathbb{Z}_p} Q(\mathcal{O}))))).$$

Proof. For simplicity let $M = \mathfrak{X}_\infty^f$ and $V = \mathfrak{X}_\infty^f \otimes_{\mathbb{Z}_p} Q(\mathcal{O})$. We use the argument in the proof of part 3 of Lemma 7.5 in [13] as follows: Let $\{x_\chi\}$ be a fixed $\mathcal{O}$-basis of $e_\chi \cdot M$, for any character χ of H. We note that x_χ is also a $Q(\mathcal{O})$-basis for $(e_\chi \cdot V)$. Let $A_{\gamma,\psi}$ denote the matrix of the automorphism m_γ restricted to $e_\psi \cdot V$ with respect to the basis x_ψ. We note that $A_{\gamma,\psi}$ is an invertible matrix of size m_ψ, where m_ψ is the dimension of $e_\psi \cdot V$ over $Q(\mathcal{O})$. By linear algebra we note that $(A^{-1}_{\gamma,\psi^{-1}})^t$ is the matrix of m_γ restricted to $e_\psi \cdot V^\vee = (e_{\psi^{-1}} \cdot V)^\vee$ with respect to the dual basis $x^*_{\psi^{-1}}$, where t stands for transposition. Now since $A_{\gamma,\psi^{-1}}$ is invertible and $\gamma \in \mathcal{O}[[\Gamma]]^\times$, we have

$$\begin{aligned} det_{Q(\mathcal{O})}((T+1) - m_\gamma \mid e_\psi \cdot V^\vee) &= det(\gamma \cdot I_m - (A^{-1}_{\gamma,\psi^{-1}})^t) \\ &\cong det(\gamma^{-1} \cdot I_m - A_{\gamma,\psi^{-1}}) \\ &= det_{Q(\mathcal{O})}((T+1) - m_\gamma \mid e_\psi \cdot V)^\#, \end{aligned}$$

where $\cong$ denotes the equality up to a unit in $\mathcal{O}[[\Gamma]]$. Therefore, we have the following equality of ideals in $\mathcal{O}[[\Gamma]]$:

$$(det_{Q(\mathcal{O})}((T+1)-m_\gamma \mid e_\psi(\mathfrak{X}_\infty^f \otimes_{\mathbb{Z}_p} Q(\mathcal{O}))))^\# \\ = (det_{Q(\mathcal{O})}((T+1)-m_\gamma \mid e_\psi(\mathfrak{X}_f^\vee \otimes_{\mathbb{Z}_p} Q(\mathcal{O})))),$$

where $\mathfrak{X}_f^\vee = Hom_{\mathcal{O}}(\mathfrak{X}_\infty^f, \mathcal{O})$ is endowed with the covariant action. Now part 3 of Lemma 2.1.8 completes the proof. □

From this lemma and the formulation (1.21) of the classical Main Conjecture in Iwasawa theory, i.e.

$$(G_{\psi,S}^*(T)) = (det_{Q(\mathcal{O})}((T+1)-m_\gamma \mid e_\psi(\mathfrak{X}_\infty^f \otimes_{\mathbb{Z}_p} Q(\mathcal{O})))),$$

we obtain the following equality of ideals in $\mathcal{O}[[T]]$:

$$(\psi(G_S^*(T)^\#)) = (G_{\psi,S}^*(T)^\#) \\ = ((det_{Q(\mathcal{O})}((T+1)-m_\gamma \mid e_\psi(E^1(\mathfrak{X}_\infty^f)^\# \otimes_{\mathbb{Z}_p} Q(\mathcal{O})))))$$

$$= (\psi(det_{Q(\mathcal{O}[H])}((T+1)-m_\gamma \mid E^1(\mathfrak{X}_\infty^f)^\# \otimes_{\mathbb{Z}_p} Q(\mathcal{O})))).$$

Therefore the exact sequence

$$0 \to \mathbb{Z}_p \to \mathbb{A}/d_\infty\mathbb{A} \to E^1(\mathcal{Z}_\infty^f)^\# \to E^1(\mathfrak{X}_\infty^f)^\# \to 0$$

in Theorem 2.1.12, and Lemma 2.1.14 together with the base-change property of determinants (cf. (1.13)) imply the following equality of ideals in $\mathcal{O}[[T]]$:

$$\psi((c_\infty G_S^*(T))^\#) = (\psi(det_{Q(\mathcal{O}[H])}((T+1)-m_\gamma \mid E^1(\mathcal{Z}_\infty^f)^\# \otimes_{\mathbb{Z}_p} Q(\mathcal{O})))). \qquad (2.11)$$

Now we complete the proof of the Equivariant Main Conjecture 2.1.6 under the assumption $\mu = 0$ using the classical main conjecture. We first recall that the μ-invariant $\mu(F)$ of a power series $F \in \mathcal{O}[[T]]$ is the largest exponent $\mu \geq 0$ such that $f \in (\pi^\mu)\mathcal{O}[[T]]$. For $F \in \mathbb{A}$ we define the μ-invariant of F to be zero if $\mu(\chi(F)) = 0$ for any p-adic valued character χ of H.

Let $F = det_{\mathbb{Z}_p[H]}((T+1)-m_\gamma \mid E^1(\mathcal{Z}_\infty^f)^\#)$ and $G := (c_\infty G_S^*(T))^\#$ in $\mathbb{A}$ (cf. Lemma 1.2.5). We have the following:

- $\boldsymbol{\mu(F) = 0}$, since the determinantal polynomial $F \in \mathbb{Z}_p[H][[T]]$ of the projective $\mathbb{Z}_p[H]$-module $E^1(\mathcal{Z}_\infty^f)^\#$ (cf. Proposition 2.1.13) is monic.

- $\boldsymbol{\mu(G) = 0}$, since for any p-adic character ψ of the group H, we have

$$\mu(\psi(G_S^*)) = \mu(g_{\psi,S}^* \cdot u_{\psi,S}) = 0 \quad , \quad \mu(\psi(H_S)) = 0 \quad , \quad \mu(\psi(d_\infty)) = 0.$$

  Here we note that the determinantal polynomials of the filed $Q(\mathcal{O})$ and the projective $\mathbb{Z}_p[H]$-module $\mathbb{A}/d_\infty\mathbb{A}$ (cf. Proposition 2.1.13), which are generated by $H_S \in \mathbb{A}$ and $d_\infty \in \mathbb{A}$, respectively (cf. Lemma 2.1.15), are monic.

- $\boldsymbol{(\psi(F)) = (\psi(G))}$, using equality (2.11).

In the terminology of [3], $R := \mathbb{Z}_p[H]$ is admissible for the abelian group H, i.e. R is a finite product of strictly admissible rings R_i, which means that each R_i is separated and complete in the $rad(R_i)$-adic topology and also $R_i/rad(R_i)$ is a skew field. Since the μ-invariants of $F, G \in R[[T]]$ are both zero, Proposition 2.1 in [3] as an equivariant Weierstrass Preparation Theorem implies the existence of unique distinguished polynomials $f^*, g^* \in R[T]$ and units $u, v \in (R[[T]])^\times$ such that

$$F = u \cdot f^* \quad \text{and} \quad G = v \cdot g^*.$$

We apply a p-adic character ψ of H to both sides, and note that $\psi(f^*)$ and $\psi(g^*)$ are both distinguished polynomials in $\mathcal{O}[T]$, and that $\psi(u), \psi(v) \in \mathcal{O}[[T]]^\times$ are units. Hence the equality $(\psi(F)) = (\psi(G))$ together with the uniqueness of the Weierstrass decomposition yields

$$\psi(f^*) = \psi(g^*)$$

for any p-adic character ψ of H. Therefore, $f^* = g^*$ and $F = uv^{-1}G$. The equality $(F) = (G)$ now implies the following:

$$\begin{aligned} (\iota(c_\infty G_S^*)) &= (det_{\mathbb{Z}_p[G]}((T+1) - m_\gamma \mid E^1(\mathcal{Z}_\infty^f)^\#)) \\ &= Fitt_{\mathbb{A}}(E^1(\mathcal{Z}_\infty^f)^\#) \qquad \text{by Lemma 2.1.14} \qquad (2.12) \\ &= Fitt_{\mathbb{A}}(\mathcal{Z}_\infty^f)^\# \qquad \text{by Lemma 2.1.9.} \end{aligned}$$

Consequently, the equality

$$Fitt_{\mathbb{A}}(\mathcal{Z}_\infty^f) = (c_\infty G_S^*)$$

holds, and this completes the proof of the Equivariant Main Conjecture 2.1.6 under the assumptions of the classical Main Conjecture and of $\mu = 0$.

Theorem 2.1.18. *The Equivariant Main Conjecture 2.1.6 follows from the classical Main Conjecture in Iwasawa theory under the assumption $\mu = 0$.*

Remark 2.1.19. *For any odd prime p, or for the prime 2 if F is an absolutely abelian number field, the classical Main Conjecture holds, and hence the Equivariant Main Conjecture 2.1.6 is verified under the assumption $\mu = 0$.*

We recall that the assumption $\mu = 0$ holds for any absolute abelian number field E, i.e. for any number field E whose Galois group over $\mathbb{Q}$ is abelian, by [9]. Hence

Corollary 2.1.20. *If E is an absolute abelian number field, the Equivariant Main Conjecture 2.1.6 holds.*

2.2 Application

The Equivariant Main Conjecture combined with Galois descent proves the Coates-Sinnott Conjecture for totally real abelian extensions. In this part we first describe the formulation of the Coates-Sinnott Conjecture, and then we will prove it for real extensions under the assumption $\mu = 0$.

2.2.1 Formulation of the Coates-Sinnott Conjecture

Let E/F be an abelian extension of number fields with Galois group G, let $n \geq 2$ be an integer, and let p be an arbitrary prime. Let S be a finite set of primes in F containing the primes above p, the primes ramified in E and the infinite primes, and let S_f denote the set of all finite primes in S. Let

$$\Theta^S_{E/F}(s) = \sum_{\chi \in \hat{G}} L^S_{E/F}(s, \chi^{-1}) \cdot e_\chi$$

be the G-equivariant S-incomplete L-function associated to E/F (cf. Section 1.2.1). We recall that for an integer $n \geq 1$ by Theorem 1.2.1

$$\Theta^S_{E/F}(1-n) \in \mathbb{Q}[G],$$

and furthermore, by Theorem 1.2.2,

$$Ann_{\mathbb{Z}[G]}(H^0(E, \mathbb{Q}/\mathbb{Z}(n))) \cdot \Theta^S_{E/F}(1-n) \subset \mathbb{Z}[G].$$

For $n \geq 1$, the n-th higher Stickelberger ideal is defined as follows:

$$Stick^S_{E/F}(n) := Ann_{\mathbb{Z}[G]}(H^0(E, \mathbb{Q}/\mathbb{Z}(n))) \cdot \Theta^S_{E/F}(1-n) \subset \mathbb{Z}[G].$$

Remark 2.2.1. *The classical Theorem of Stickelberger states that*

$$Stick^S_{E/\mathbb{Q}}(1) \subseteq Ann_{\mathbb{Z}[G]}(Cl(\mathcal{O}_E),$$

where $Cl(\mathcal{O}_E)$ denotes the class group of $\mathcal{O}_E$. Brumer conjectured that the same holds for any abelian extension E/F of number fields.

The formulation of the Coates-Sinnott Conjecture is as follows:

Conjecture 2.2.2. *(The Coates-Sinnott Conjecture, K-theoretic version). Let E/F be an abelian Galois extension of number fields with Galois group G, and let $n \geq 2$. Then*

$$Stick^S_{E/F}(n) \subseteq Ann_{\mathbb{Z}[G]}(K_{2n-2}(\mathcal{O}_E)).$$

There is another formulation of the Coates-Sinnott Conjecture in terms of motivic cohomology, which is closely related to the original conjecture:

Conjecture 2.2.3. *(The Coates-Sinnott Conjecture, motivic version). Let E/F be an abelian Galois extension of number fields with Galois group G, and let $n \geq 2$. Then*

$$Stick^S_{E/F}(n) \subseteq Ann_{\mathbb{Z}[G]}(H^2_{\mathcal{M}}(E, \mathbb{Z}(n))).$$

The motivic version is equivalent to the following p-adic version for all primes p:

Conjecture 2.2.4. *(The Coates-Sinnott Conjecture, p-adic version). Let E/F be an abelian Galois extension of number fields with Galois group G, let p be prime, and let $n \geq 2$. Then*

$$Ann_{\mathbb{Z}_p[G]}(H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))) \cdot \Theta^S_{E/F}(1-n) \subseteq Ann_{\mathbb{Z}_p[G]}(H^2_{\acute{e}t}(\mathcal{O}'_E, \mathbb{Z}_p(n))).$$

We remark that by the functional equation of L-functions (cf. Section 1.2.1) $L_{E/F}(s, \chi)$ vanishes at negative integers $1-n$ for $n \geq 2$, unless F is a totally real number field and $\chi(1) = (-1)^n$. Therefore the following situations are of interest:

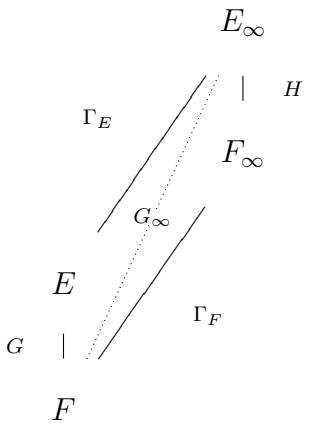
- E is a totally real number field and $n \geq 2$ is even.
- E is a CM field and $n \geq 2$ is odd.

Therefore, by Quillen-Lichtenbaum conjecture and Proposition 1.4.13, the motivic version is slightly stronger than the K-theoretic version of the Coates-Sinnott Conjecture. More precisely, if n is congruent (*mod* 4) to 1 or 2, then both versions are equivalent, and if n is congruent (*mod* 4) to 0 or 3, then the motivic version implies the K-theoretic version of the Coates-Sinnott Conjecture.

2.2.2 Proof for totally real extensions E/F assuming $\mu = 0$

We recall the set up from Section 2.1.1. Let E/F be an abelian Galois extension of totally real number fields with Galois group G, and let $n \geq 2$ be an even integer.

Let E_∞ (resp. F_∞) be the cyclotomic $\mathbb{Z}_p$-extension of E (resp. F) with Galois group Γ_E (resp. Γ_F), over E (resp. over F). We denote by G_∞ the Galois group of E_∞/F, by H the Galois group of E_∞/F_∞, and by $\Gamma =< \gamma >$ the image of Γ_F under the splitting map in (2.1).



Since G_∞ is abelian, $G_\infty = H \times \Gamma$ and the completed group ring $\mathbb{\Lambda} = \mathbb{Z}_p[[G_\infty]]$ is identified with $\mathbb{Z}_p[H][[T]]$ under the identification (2.2). We let $d_\infty \in \Delta G_\infty$ be a non-zero divisor so that $\mathbb{\Lambda}/d_\infty\mathbb{\Lambda}$ is a finitely generated $\mathbb{Z}_p$-free module, e.g. $d_\infty = \gamma - 1$. By Theorem 2.1.12 we obtain an exact sequence of finitely generated $\mathbb{\Lambda}$-torsion modules

$$0 \to \mathbb{Z}_p \to \mathbb{\Lambda}/d_\infty\mathbb{\Lambda} \to E^1(\mathcal{Z}^f_\infty)^\# \to E^1(\mathfrak{X}^f_\infty)^\# \to 0, \tag{2.13}$$

in which the middle terms are of projective dimensions at most one. By the first equality of (2.12) we also have

$$Fitt_{\mathbb{\Lambda}}(\mathbb{\Lambda}/d_\infty\mathbb{\Lambda}(n)) = (t_n(d_\infty)),$$

$$Fitt_{\mathbb{\Lambda}}(E^1(\mathcal{Z}^f_\infty)^\#(n)) = ((\iota \circ t_n)(c_\infty G^*_S)).$$

We note that the sequence (2.13) is also an exact sequence of finitely generated Λ-modules, where $\Lambda = \mathbb{Z}_p[[\Gamma]]$. Moreover, the exact sequence (2.13) is an exact sequence of finitely generated $\mathbb{Z}_p$-free modules since we have assumed $\mu = 0$.

Lemma 2.2.5. *Let $G_E^{S_f}$ be the Galois group of the maximal algebraic pro-p-extension of E unramified outside the primes above S_f, over E. Then*

1. *$H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n)) \simeq \mathbb{Z}_p(n)_{\Gamma_E}$ for $n \geq 2$.*
2. *$H^2(G_E^{S_f}, \mathbb{Z}_p(n)) \simeq (E^1(\mathfrak{X}^f_\infty)^\#(n))_{\Gamma_E}$ for even $n \geq 2$ (under the hypothesis $\mu = 0$).*

Proof. 1. It is enough to take the Γ_E-invariants and Γ_E-coinvariants of the exact sequence $0 \to \mathbb{Z}_p(n) \to \mathbb{Q}_p(n) \to \mathbb{Q}_p(n)/\mathbb{Z}_p(n) \to 0$ to get a 6 term exact sequence in which $\mathbb{Q}_p(n)_{\Gamma_E} = \mathbb{Q}_p(n)^{\Gamma_E} = 0$ for $n \geq 2$. We note that $H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n)) = \mathbb{Q}_p/\mathbb{Z}_p(n)^{\Gamma_E}$.

2. Let $\tilde{E} := E(\zeta_p)$ be the cyclotomic extension of E obtained by adjoining a primitive p-th root of unity ζ_p, let $\tilde{E}^+$ be the maximal real subfield of $\tilde{E}$ with Galois group $\Delta = Gal(\tilde{E}^+/E)$. We note that the order of Δ is prime to p, and that for the prime 2, $\tilde{E} = E$. Let E'^+_∞ be the cyclotomic $\mathbb{Z}_p$-extension of $\tilde{E}^+$, with Galois group isomorphic to Γ_E. As before we denote by $\Omega^{S_f}_{\tilde{E}^+}$ (resp. $\Omega^{S_f}_E$) the maximal algebraic pro-p-extension of $\tilde{E}^+$ (resp. E) unramified outside the primes above S_f. We have the following calculations:

$$
\begin{aligned}
E^1(\mathfrak{X}^f_\infty)^\#(n)_{\Gamma_E} &= Hom(\mathfrak{X}^f_\infty, \mathbb{Z}_p)(n)_{\Gamma_E} && \text{by part 3 in Lemma 2.1.8} \\
&\simeq Hom((\mathfrak{X}^f_\infty)_{\Gamma_E}, \mathbb{Q}_p/\mathbb{Z}_p(n)) && \text{by Lemma 1.1.16} \\
&\simeq Hom(\mathfrak{X}^f_\infty, \mathbb{Q}_p/\mathbb{Z}_p(n))^{\Gamma_E} && \text{by isomorphism (1.6)} \\
&\simeq Hom(Gal(\Omega^{S_f}_E/E_\infty), \mathbb{Q}_p/\mathbb{Z}_p(n))^{\Gamma_E} \\
&\simeq Hom(Gal(\Omega^{S_f}_{\tilde{E}^+}/E_\infty), \mathbb{Q}_p/\mathbb{Z}_p(n))^{\Delta\times\Gamma_E} \\
&\simeq H^1(Gal(\Omega^{S_f}_{\tilde{E}^+}/E_\infty), \mathbb{Q}_p/\mathbb{Z}_p(n))^{\Delta\times\Gamma_E} && \text{since } n \text{ is even and } \mu_p \subseteq \tilde{E} \\
&\simeq H^1(Gal(\Omega^{S_f}_E/E_\infty), \mathbb{Q}_p/\mathbb{Z}_p(n))^{\Gamma_E} && \text{since } |\Delta| \text{ is prime to } p \\
&\simeq H^1(Gal(\Omega^{S_f}_E/E), \mathbb{Q}_p/\mathbb{Z}_p(n)) && \text{since } cd_p(\Gamma_E) = 1 \\
&\simeq H^2(Gal(\Omega^{S_f}_E/E), \mathbb{Z}_p(n)),
\end{aligned}
$$

where the last isomorphism follows from the finiteness of groups $H^1(\mathcal{H}, \mathbb{Q}_p/\mathbb{Z}_p(n))$ and $H^2(\mathcal{H}), \mathbb{Z}_p(n))$ for even n and totally real field E (cf. Proposition 1.4.11). □

As a consequence of the lemma above together with Proposition 1.4.11 we see that $E^1(\mathfrak{X}^f_\infty)^\#(n)_{\Gamma_E}$ and $\mathbb{Z}_p(n)_{\Gamma_E}$ are both finite and so by Lemma 1.1.5 $(E^1(\mathfrak{X}^f_\infty)^\#)(n)^{\Gamma_E}$ and $\mathbb{Z}_p(n)^{\Gamma_E}$ are both trivial for even $n \geq 2$. We note that $\mathbb{Z}_p(n)$ and $(E^1(\mathfrak{X}^f_\infty)^\#)(n)$, which is equal to the adjoint of $\mathfrak{X}^f_\infty(n)$ as a Λ-module by Lemma 2.1.8, have no non-trivial finite submodules. On the other hand $(\mathbb{A}/d_\infty\mathbb{A})(n)^{\Gamma_E} = 0$ since $\kappa(\gamma)^n \neq 1$ for $n \geq 2$. Therefore $(\mathbb{A}/d_\infty\mathbb{A})(n)_{\Gamma_E}$ is again finite by Lemma 1.1.5. As a result, the

Γ_E-coinvariants of $E^1(\mathcal{Z}_\infty^f)^\#(n)$ are also finite and similarly $(E^1(\mathcal{Z}_\infty^f)^\#)(n)^{\Gamma_E} = 0$, for any even $n \geq 2$. Hence, by taking the Γ_E-coinvariants of the exact sequence (2.13), we obtain the following exact sequence of finite $\mathbb{Z}_p[G]$-modules for any even $n \geq 2$:

$$0 \to H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n)) \to (\mathbb{A}/d_\infty\mathbb{A})(n)_{\Gamma_E} \to (E^1(\mathcal{Z}_\infty^f)^\#)(n)_{\Gamma_E} \to H^2(G_E^{S_f}, \mathbb{Z}_p(n)) \to 0, \tag{2.14}$$

where the two middle $\mathbb{Z}_p[G]$-modules are of projective dimension at most one by Lemma 2.1.8. Furthermore, by Proposition 1.3.5 and by property 6 of Fitting ideals in Section 1.3.1, we have

$$Fitt_{\mathbb{Z}_p[G]}((\mathbb{A}/d_\infty\mathbb{A})(n)_{\Gamma_E}) = ((\pi \circ t_n)(d_\infty)),$$

$$Fitt_{\mathbb{Z}_p[G]}((E^1(\mathcal{Z}_\infty^f)^\#)(n)_{\Gamma_E}) = ((\pi \circ \iota \circ t_n)(c_\infty G_S^*),$$

where $\pi : \mathbb{A} \to \mathbb{Z}_p[G]$ is the projection mapping $\gamma - 1$ to 0.

Now by applying Proposition 1.3.6 to the exact sequence (2.14) of finite $\mathbb{Z}_p[G]$-modules we get the following equality:

$$Fitt_{\mathbb{Z}_p[G]}(H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))^*) Fitt_{\mathbb{Z}_p[G]}((E^1(\mathcal{Z}_\infty^f)^\#)(n)_{\Gamma_E})$$

$$= Fitt_{\mathbb{Z}_p[G]}((\mathbb{A}/d_\infty\mathbb{A})(n)_{\Gamma_E}) Fitt_{\mathbb{Z}_p[G]}(H^2(G_E^{S_f}, \mathbb{Z}_p(n))).$$

The Fitting and the annihilator ideals of $H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))$ are the same as the Fitting ideal of the Pontryagin dual $H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))^*$ by property 7 of Fitting ideals in Section 1.3.1. Hence we have the following equality of fractional ideals in $\mathbb{Z}_p[G]$:

$$Fitt_{\mathbb{Z}_p[G]}(H^2(G_E^{S_f}, \mathbb{Z}_p(n))) = Ann_{\mathbb{Z}_p[G]}(H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n)))((\pi \circ \iota \circ t_n)(c_\infty/d_\infty \cdot G_S^*)).$$

Finally, since $d_\infty/c_\infty = (\gamma - 1)e + (1 - e)$ is identified with $H_S(T)$ by the identification (2.2), we obtain the following theorem:

Theorem 2.2.6. *We have the following equality of ideals of $\mathbb{Z}_p[G]$:*

$$Fitt_{\mathbb{Z}_p[G]}(H^2(G_E^{S_f}, \mathbb{Z}_p(n))) = Ann_{\mathbb{Z}_p[G]}(H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n)))((\pi \circ \iota \circ t_n)(\frac{G_S^*}{H_S})).$$

Now let p be an odd prime. We note that $H^2(G_E^S, \mathbb{Z}_p(n)) = H^2(G_E^{S_f}, \mathbb{Z}_p(n))$, where $G_E^{S_f}$ denotes the Galois group of the maximal algebraic pro-p-extension of E unramified outside the primes in S, over E. Moreover, $G_S^* = G_S$ under the assumption $\mu = 0$ (cf. Lemma 1.2.4). Therefore $(\pi \circ \iota \circ t_n)(c_\infty/d_\infty G_S^*)) = \Theta_{E/F}^S(1-n)$ by equality 1.2.7, and by Theorem 2.2.6 we obtain:

$$Fitt_{\mathbb{Z}_p[G]}(H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_p(n))) = Ann_{\mathbb{Z}_p[G]}(H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))) \cdot \Theta_{E/F}^S(1-n).$$

This implies the p-adic version of the Coates-Sinnott Conjecture for odd primes.

For $p = 2$ we use the following lemma:

Lemma 2.2.7. *For the extension E/F of totally real number fields with Galois group G, where $[F : \mathbb{Q}] = r_1(F)$, and for even integers $n \geq 2$, we have*

$$2^{r_1(F)} Fitt_{\mathbb{Z}_2[G]}(H^2(G_E^{S_f}, \mathbb{Z}_2(n))) \subseteq Fitt_{\mathbb{Z}_2[G]}(H^2(G_E^S, \mathbb{Z}_2(n))).$$

Proof. We first take Γ_E-coinvariants and then the Pontryagin dual of the second exact sequence in Proposition 1.1.12 for the extension E/F. The same calculation as in the proof of the second part of Lemma 2.2.5 leads to the exact sequence

$$0 \to H^2(G_E^{S_f}, \mathbb{Z}_2(n)) \to H^2(G_E^S, \mathbb{Z}_2(n)) \to (\mathbb{Z}[G]/2)^{r_1(F)}(n),$$

which completes the proof of this lemma by property 3 of Fitting ideals in Section 1.3.1. □

From this lemma we obtain the following inclusion:

$$2^{r_1(F)} Ann_{\mathbb{Z}_p[G]}(H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n)))((\pi \circ \iota \circ t_n)(\frac{G_S^*}{H_S}) \subseteq Ann_{\mathbb{Z}_2[G]}(H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))).$$

Moreover, $G_S) = (2^{r_1(F)} \cdot G_S^*$ for the prime 2 under the assumption $\mu = 0$ (cf. Lemma 1.2.4). Consequently, $\Theta^S_{E/F}(1-n) = 2^{r_1(F)}((\pi \circ \iota \circ t_n)(\frac{G_S^*}{H_S}))$ (cf. 1.2.7) for any integer $n \geq 2$ and as a result,

$$Ann_{\mathbb{Z}_2[G]}(H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(n))) \cdot \Theta^S_E(1-n) \subseteq Ann_{\mathbb{Z}_2[G]}(H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))).$$

Finally, we note that $H^2_{\acute{e}t}(\mathcal{O}'_E, \mathbb{Z}_2(n)) \subseteq H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))$. Hence

Theorem 2.2.8. *For an abelian extension E/F of totally real number fields with Galois group G and even $n \geq 2$, the Coates-Sinnott Conjecture 2.2.4 holds for all primes p, under the assumptions of $\mu = 0$ and the 2-primary part of the classical Main Conjecture in Iwasawa theory.*

Chapter 3

The p-primary part of the Coates-Sinnott Conjecture without assuming $\mu = 0$ for specific cases

Let E/F be a finite abelian extension of number fields with Galois group G and let p be an arbitrary prime number. In the two specific cases below the p-adic version of the Coates-Sinnott Conjecture 2.2.4 can be deduced from the classical Main Conjecture in Iwasawa theory, which does not assume $\mu = 0$.

- $n \geq 2$ is an even number and E/F is a finite abelian extension of number fields of order prime to p, where E is a totally real number field.
- $n \geq 2$ is an odd number and E/F is a finite abelian extension of number fields of degree $2m$, where m is not divisible by p odd, E is a CM-field and F is a totally real number field.

However, for the prime 2 we have to assume the Main Conjecture in Iwasawa theory if F is not abelian over $\mathbb{Q}$, and the equality of the algebraic and the analytic μ-invariants for all characters of G (cf. Section 1.5). For odd primes p, $n \geq 2$ even, and a finite abelian extension E/F of order prime to p the result has been shown in [30] and [20].

3.1 Abelian extensions E/F of totally real fields with $p \nmid [E : F]$

The goal of this section is to prove the p-adic version of the Coates-Sinnott Conjecture 2.2.4 in the following set-up without assuming $\mu = 0$:

- Let E/F be a finite abelian extension of totally real number fields with Galois group G where the order of G is prime to p.
- Let $n \geq 2$ be an even integer.
- Let S be a finite set of primes of F containing the primes above p, the primes ramified in E and the infinite primes.

In this situation we remark that $\mathbb{Z}_p[G]$ is the maximal order in $\mathbb{Q}_p[G]$. Let E_∞ be the cyclotomic $\mathbb{Z}_p$-extension of E with multiplicative Galois group $\Gamma =< \gamma >$. Since the order of $G = Gal(E/F)$ is prime to p, we have the following isomorphism of abelian groups:

$$Gal(E_\infty/F) \simeq G \times \Gamma.$$

Let M_∞ be the maximal abelian pro-p-extension of E_∞ which is unramified outside the primes in S with Galois group $\mathfrak{X} := \mathfrak{X}_\infty^S$, over E_∞. The Galois group G acts on $\mathfrak{X}$ by inner automorphism (cf. Section 1.1.2) and so $\mathfrak{X}$ is equipped with a $\mathbb{Z}_p[G][[\Gamma]]$-module structure. We recall that by (1.1) we have the identification

$$\mathbb{Z}_p[G][[\Gamma]] \simeq \mathbb{Z}_p[G][[T]],$$

which maps $\gamma - 1$ to T. Since the Galois group G of E/F is of order prime to p, we have the following decomposition:

$$\mathbb{Z}_p[G] \simeq \prod_\chi \mathcal{O}_\chi,$$

where the sum runs over all irreducible $\mathbb{Z}_p[G]$-characters and $\mathcal{O}_\chi$ is the ring generated over $\mathbb{Z}_p$ by the values of χ. Therefore, to approach the Coates-Sinnott Conjecture we study the conjecture characterwise for each eigenspace

$$H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_p(n))^\chi,$$

where χ is an (even) character of G.

Let χ be an (even) character of G, let $\Lambda = \mathcal{O}_\chi[[T]]$ and let $\psi := \chi\omega^n$ be the corresponding (real) character of F where ω is the Teichmüller character. We note that in our situation n and χ have the same parity and that $\mathfrak{X}^\psi(-n)$, the $(-n)$-Tate twist of the ψ-eigenspace of $\mathfrak{X}$, is a finitely generated Λ-torsion module with no non-trivial finite submodules (cf. Section 1.1.2). In order to verify the conjecture we describe the Fitting ideal of $\mathfrak{X}^\psi(-n)_\Gamma$ and then the Fitting ideal of $H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_p(n))^\chi$ in terms of the n-th higher p-adic Stickelberger ideal attached to E/F. First we remark that by Proposition 1.3.4 we have an explicit description of the Fitting ideal of the finitely generated Λ-torsion module $\mathfrak{X}^\psi(-n)$ as follows:

$$Fitt_\Lambda(\mathfrak{X}^\psi(-n)) = (char_\Lambda(\mathfrak{X}^\psi(-n))). \tag{3.1}$$

To describe the Fitting ideal of $H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_p(n))$ we need the following lemma:

Lemma 3.1.1. *For the character χ of G, an even integer $n \geq 2$, and $\psi = \chi\omega^n$, we have the following isomorphism $\mathcal{O}_\chi$-modules:*

$$(\mathfrak{X}^\psi(-n)_\Gamma)^* \simeq H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_p(n))^{\chi^{-1}}.$$

Proof. Let $\tilde{E} := E(\zeta_p)$ a cyclotomic extension of E by adjoining a primitive p-th root of unity, let $\tilde{E}^+$ denote the maximal real subfield of $\tilde{E}$, and let $\Delta := Gal(\tilde{E}^+/E)$. Let $\Omega^S_{\tilde{E}^+}$ (resp. Ω^S_E) be the maximal algebraic p-extension of $\tilde{E}^+$ (resp. E) unramified outside the primes in S. Similar to the calculations in the proof of Lemma 2.2.5 we have the following isomorphisms:

$$\begin{aligned}
(\mathfrak{X}^\psi(-n)_\Gamma)^* \simeq &((\mathfrak{X}^\psi(-n))^*)^\Gamma \simeq Hom(\mathfrak{X}^\psi(-n), \mathbb{Q}_p/\mathbb{Z}_p)^\Gamma \quad \text{by isomorphism 1.6}\\
\simeq &Hom(\mathfrak{X}(-n)^\chi, \mathbb{Q}_p/\mathbb{Z}_p)^\Gamma\\
\simeq &(Hom(\mathfrak{X}(-n), \mathbb{Q}_p/\mathbb{Z}_p)^{\chi^{-1}})^\Gamma\\
\simeq &(Hom(\mathfrak{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))^{\chi^{-1}})^\Gamma \qquad \text{by Lemma 1.1.14}\\
\simeq &(Hom(Gal(\Omega^S_E/E_\infty), \mathbb{Q}_p/\mathbb{Z}_p(n))^{\chi^{-1}})^\Gamma\\
\simeq &(Hom(Gal(\Omega^S_{\tilde{E}^+}/E), \mathbb{Q}_p/\mathbb{Z}_p(n))^{\chi^{-1}})^{\Delta\times\Gamma} \quad \text{since } |\Delta| \text{ is prime to } p\\
\simeq &(H^1_{\acute{e}t}(\mathcal{O}^S_{E'^+_\infty}, \mathbb{Q}_p/\mathbb{Z}_p(n))^{\chi^{-1}})^{\Delta\times\Gamma} \qquad \text{since } n \text{ is even and } \mu_p \subseteq \tilde{E}\\
\simeq &(H^1_{\acute{e}t}(\mathcal{O}^S_{\tilde{E}^+}, \mathbb{Q}_p/\mathbb{Z}_p(n))^{\chi^{-1}})^{\Delta} \qquad \text{since } cd_p(\Gamma) = 1\\
\simeq &H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_p/\mathbb{Z}_p(n))^{\chi^{-1}} \qquad \text{since } |\Delta| \text{ is prime to } p\\
\simeq &H^2_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Z}_p(n))^{\chi^{-1}}.
\end{aligned}$$

The last isomorphism holds, since the groups $H^2_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Z}_p(n))$ and $H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Z}_p(n))^{\chi^{-1}}$ are finite for totally real number field E and even integers $n \geq 2$ (cf. Section 1.4.3). □

We note that the evaluation of the generator of $Fitt_{\mathcal{O}_\chi[[T]]}(\mathfrak{X}^\psi(-n))$ at $T = 0$ is the same as the computation of $Fitt_{\mathcal{O}_\chi}(\mathfrak{X}^\psi(-n)_\Gamma)$ because of the identification

$$\mathcal{O}_\chi[[\Gamma]] \simeq \mathcal{O}_\chi[[T]],$$

which maps $\gamma - 1$ to T (cf. (1.1)). As a result, for any character χ of G the Fitting ideal of $H^2_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Z}_p(n))^{\chi^{-1}}$ is the same as the characteristic polynomial of $\mathfrak{X}^\psi(-n)$ evaluated at zero. Therefore, by (3.1) and Lemma 3.1.1 we obtain:

Corollary 3.1.2.

$$Fitt_{\mathcal{O}_\chi}(H^2_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Z}_p(n))^{\chi^{-1}}) = (char_\Lambda(\mathfrak{X}^\psi(-n))(0)).$$

We have to relate this result to the L-function part of the conjecture. If we let

$$F_{\psi,S}(T) = \pi^{\mu(F_{\psi,S})} f^*_{\psi,S}(T) \in \mathcal{O}_\psi[T]$$

be the characteristic polynomial of $\mathfrak{X}^\psi$, then by Lemma 1.1.15 we have the following equality:

$$char_\Lambda(\mathfrak{X}^\psi(-n))(T) = F_{\psi,S}(\kappa(\gamma)^n(T+1)-1). \tag{3.2}$$

Now the Main Conjecture in Iwasawa theory (see Section 1.5) yields

$$f^*_{\psi,S}(T) = g^*_{\psi,S}(T)$$

for the monic polynomial $g^*_{\psi,S}$ defined by $G_{\psi,S}(T) = \pi^{\mu(G_{\psi,S})} g^*_{\psi,S}(T) u_{\psi,S}(T) \in \mathcal{O}_\psi[[T]]$, where $G_{\psi,S}(T) \in \mathcal{O}_\psi[[T]]$ is given by the following:

$$L^S_p(F, \psi, 1-s) = \frac{G_{\psi,S}(\kappa(\gamma)^s - 1)}{H_\psi(\kappa(\gamma)^s - 1)}.$$

Moreover, for any odd prime p the analytic μ-invariant $\mu(G_{\psi,S})$ and the algebraic μ-invariant $\mu(F_{\psi,S})$ are equal and hence

$$F_{\psi,S}(T) \sim_p G_{\psi,S}(T),$$

where $\sim_p$ denotes the equality of the p-adic valuations of both sides. If we also assume the equality of these analytic and algebraic μ-invariants for the prime 2, then the above equality still holds for $p = 2$.

Now we observe the following two situations:

1. Assume $\psi \neq 1$. Since F_ψ, the fixed field of the kernel of ψ, is in E and p does not divide the order of the Galois group of E/F, ψ is of type S. Therefore $H_\psi = 1$, and

$$\begin{aligned} F_{\psi,S}(\kappa(\gamma)^n - 1) &\sim_p G_{\psi,S}(\kappa(\gamma)^n - 1) \\ &\sim_p L^S_p(1-n, \psi) \\ &\sim_p L^S_{E/F}(1-n, \chi). \end{aligned} \tag{3.3}$$

2. Assume $\psi = 1$. Obviously ψ is of type W and so $H_\psi(\kappa(\gamma)^n - 1) = \kappa(\gamma)^n - 1$. If we let $M = \mathbb{Z}_p$ as a $\mathbb{Z}_p[[T]]$-module, then $char_\Lambda(M(-n))(T) = \kappa(\gamma)^n(T+1) - 1$. Since this characteristic polynomial does not vanish at zero, we have

$$\begin{aligned}\kappa(\gamma)^n - 1 =&|(M^\psi(-n))_\Gamma| = |((M^\psi(-n))_\Gamma)^*| = |((M^\psi(-n))^*)^\Gamma| \\ =&|(Hom_\Lambda(\mathbb{Z}_p(-n), \mathbb{Q}_p/\mathbb{Z}_p)^{\omega^n})^\Gamma| \\ =&|(Hom_\Lambda(\mathbb{Z}_p, \mathbb{Q}_p/\mathbb{Z}_p(n))^{\omega^n})^\Gamma| \\ =&|((\mathbb{Q}_p/\mathbb{Z}_p(n))^{\omega^n})^\Gamma| \\ =&|H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))^{\omega^n}|.\end{aligned}$$

The group $H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))$ is cyclic, and as a consequence, $H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))^\chi$ is trivial if and only if χ is not the trivial character. Therefore,

$$\begin{aligned}F_{\psi,S}(\kappa(\gamma)^n - 1) \sim_p\ & G_{\psi,S}(\kappa(\gamma)^n - 1) \\ \sim_p\ & |H_\psi(\kappa(\gamma)^n - 1)|\, L_p^S(1-n, \psi) \\ \sim_p\ & |H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))| L_p^S(1-n, \psi) \\ \sim_p\ & |H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))| L_{E/F}^S(1-n, \chi).\end{aligned} \tag{3.4}$$

Hence by (3.2) we obtain

$$char_\Lambda(\mathfrak{X}^\psi(-n)(0)) = |H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))| L_{E/F}^S(1-n, \chi),$$

and consequently, by Corollary 3.1.2 we obtain the following equality of ideals:

$$\begin{aligned}Fitt_{\mathcal{O}_\chi}(H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_p(n))^{\chi^{-1}}) =&|H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))|(L_{E/F}^S(1-n, \chi)) \\ =&Ann_{\mathbb{Z}_p[G]}(H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))) \cdot L_{E/F}^S(1-n, \chi).\end{aligned}$$

Finally, we take the direct sum of the $\mathcal{O}_\chi$-modules $H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_p(n))^{\chi^{-1}}$ in the situation of this section, and use property 8 of Fitting ideals in Section 1.3.1) to obtain:

$$\begin{aligned}Ann_{\mathbb{Z}_p[G]}(H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))) \cdot \Theta_{E/F}^S(1-n) =&Fitt_{\mathbb{Z}_p[G]}(H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_p(n))) \\ \subseteq&Ann_{\mathbb{Z}_p[G]}(H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_p(n))) \\ \subseteq&Ann_{\mathbb{Z}_p[G]}(H^2_{\acute{e}t}(\mathcal{O}'_E, \mathbb{Z}_p(n))),\end{aligned}$$

where the last relation follows from the fact that $H^2_{\acute{e}t}(\mathcal{O}'_E, \mathbb{Z}_2(n)) \subseteq H^2_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Z}_2(n))$. Hence

Theorem 3.1.3. *The p-adic version of the Coates-Sinnott Conjecture 2.2.4 holds for any prime number p, an even integer $n \geq 2$, and an abelian extension E/F of totally real number fields, with Galois group G, whose order is prime to p, based on the classical Main Conjecture and the equality of the algebraic and the analytic μ-invariants for the prime 2.*

3.2 Abelian extensions of a CM field E over a totally real field F of degree $2m$ with $p \nmid m$

The goal of this section is to prove the p-adic version of the Coates-Sinnott Conjecture 2.2.4 in the following set-up without assuming $\mu = 0$:

- Let E/F be a finite abelian extension of number fields of degree $2m$ with Galois group G, where m is not divisible by p, E is CM and F is totally real,
- Let $n \geq 2$ be an odd integer,
- Let S be a finite set of primes of F containing the primes above p, the primes ramified in E and the infinite primes.

We denote by J the complex conjugation, by E^+ the maximal real subfield of E, and by $H =< J >$ the Galois group of E/E^+.

3.2.1 The case $p \neq 2$

We have the following definitions of the minus and the plus parts of a $\mathbb{Z}_p[G]$-module M for p odd:

$$M^- := (\frac{1-J}{2})M \quad \text{and} \quad M^+ := (\frac{1+J}{2})M,$$

which give rise to the decomposition $M = M^- \oplus M^-$. In particular,

$$\mathbb{Z}_p[G] = \mathbb{Z}_p[G]^- \oplus \mathbb{Z}_p[G]^+.$$

In this situation we remark that $\mathbb{Z}_p[G]$ is the maximal order in $\mathbb{Q}_p[G]$, and that the L-function attached to E^+/F vanishes at $s = 1 - n$, for odd $n \geq 2$, by the functional equation of L-functions (cf. Section 1.2.1). So we only concentrate on the Fitting ideal of the minus part $H^2_{\acute{e}t}(\mathcal{O}'_E, \mathbb{Z}_2(n))^-$ of

$$H^2_{\acute{e}t}(\mathcal{O}'_E, \mathbb{Z}_p(n)) = H^2_{\acute{e}t}(\mathcal{O}'_E, \mathbb{Z}_p(n))^- \oplus H^2_{\acute{e}t}(\mathcal{O}'_E, \mathbb{Z}_p(n))^+.$$

Furthermore, we can study the Coates-Sinnott Conjecture characterwise, for odd characters of G, i.e. the characters which map -1 to -1, since

$$H^2_{\acute{e}t}(\mathcal{O}'_E, \mathbb{Z}_p(n))^- = \bigoplus_{\chi \text{odd}} H^2_{\acute{e}t}(\mathcal{O}'_E, \mathbb{Z}_p(n))^\chi.$$

Without loss of generality, **we can assume that $\zeta_p \in E$ for p odd**, since for any odd character χ of G we have:

$$H^1_{\acute{e}t}(\mathcal{O}'_E, \mathbb{Q}_p/\mathbb{Z}_p(n))^\chi \simeq H^1_{\acute{e}t}(\mathcal{O}'_{E(\zeta_p)}, \mathbb{Q}_p/\mathbb{Z}_p(n))^\chi.$$

We also remark that by Proposition 1.4.11, the $\mathbb{Z}_p$-rank of

$$H^1_{\acute{e}t}(\mathcal{O}'_E, \mathbb{Q}_p/\mathbb{Z}_p(n))^+ \simeq H^1_{\acute{e}t}(\mathcal{O}'_{E^+}, \mathbb{Q}_p/\mathbb{Z}_p(n))$$

is the same as the $\mathbb{Z}_p$-rank of $H^1_{\acute{e}t}(\mathcal{O}'_E, \mathbb{Q}_p/\mathbb{Z}_p(n))$, since E is a CM field. Therefore the minus part $H^1_{\acute{e}t}(\mathcal{O}'_E, \mathbb{Q}_p/\mathbb{Z}_p(n))^-$ is finite.

Let χ be an odd character of G, let $\psi = \chi\omega^n$ be the corresponding even character, and let $\Lambda = \mathcal{O}_\chi[[T]]$. Let $M^S_{E_\infty}$ (resp. $M^S_{E^+_\infty}$) be the maximal abelian pro-p-extension of E_∞ (resp. E^+_∞), which is unramified outside the primes in S, and let $\mathfrak{X}_E := \mathfrak{X}^S_{E_\infty}$ (resp. $\mathfrak{X}_{E^+} := \mathfrak{X}^S_{E^+_\infty}$) be the Galois group of the extension $M^S_{E_\infty}/E_\infty$ (resp. $M^S_{E^+_\infty}/E^+_\infty$).

By Galois theory, since p is odd and $\zeta_p \in E$, we have the isomorphism $\mathfrak{X}_E/(1-J)\mathfrak{X}_E \simeq \mathfrak{X}_{E^+}$. As a consequence,

$$\mathfrak{X}^\psi_E \simeq \mathfrak{X}^\psi_{E^+} \tag{3.5}$$

for the even character ψ. So $\mathfrak{X}^\psi_E$ is a finitely generated Λ-torsion module with no non-trivial finite submodule (cf. Section 1.1.2). By using Proposition 1.3.4 we have the same equality as (3.1):

$$Fitt_\Lambda(\mathfrak{X}^\psi_E(-n)_\Gamma) = (char_\Lambda(\mathfrak{X}^\psi_E(-n)(0))). \tag{3.6}$$

Lemma 3.2.1. *For the odd character χ of the Galois group G and an odd integer $n \geq 2$, we have the following isomorphism of $\mathcal{O}_\chi$-modules:*

$$(\mathfrak{X}^\psi_E(-n)_\Gamma)^* \simeq H^2_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Z}_p(n))^{\chi^{-1}}.$$

Proof. As we mentioned we can assume that E contains the primitive p-th roots of unity for p odd. Let Ω^S_E be the maximal algebraic p-extension of E unramified outside the primes in S. The same calculation as in the proof of Lemma 3.1.1 yields the following:

$$(\mathfrak{X}^\psi_E(-n)_\Gamma)^* \simeq (Hom(Gal(\Omega^S_E/E_\infty), \mathbb{Q}_p/\mathbb{Z}_p(n))^{\chi^{-1}})^\Gamma \text{ see calculation in Lemma 3.1.1}$$

$$\simeq (H^1_{\acute{e}t}(\mathcal{O}^S_{E_\infty}, \mathbb{Q}_p/\mathbb{Z}_p(n))^{\chi^{-1}})^\Gamma \qquad \text{for } n \text{ odd and } \mu_{p^\infty} \in E_\infty$$

$$\simeq H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_p/\mathbb{Z}_p(n))^{\chi^{-1}} \qquad \text{since } cd_p(\Gamma) = 1$$

$$\simeq H^2_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Z}_p(n))^{\chi^{-1}},$$

where the last isomorphism follows from the finiteness of the group $H^2_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Z}_p(n))$ and the inclusion

$$H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_p/\mathbb{Z}_p(n))^{\chi^{-1}} \subseteq H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_p/\mathbb{Z}_p(n))^{-}.$$

□

The same argument as in the proof of Corollary 3.1.2 shows:

$$Fitt_{\mathcal{O}_\chi}(H^2_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Z}_p(n))^{\chi^{-1}}) = (char_\Lambda(\mathfrak{X}^\psi_E(-n))(0)). \tag{3.7}$$

Let $F_{\psi,S}(T) = \pi^{\mu(F_{\psi,S})} f^*_{\psi,S}(T) \in \mathcal{O}_\psi[T]$ be the characteristic polynomial of $\mathfrak{X}^\psi_E$. By Lemma 1.1.15 the characteristic polynomial of $\mathfrak{X}^\psi_E(-n)$ is as follows:

$$char_\Lambda(\mathfrak{X}^\psi_E(-n))(T) = F_{\psi,S}(\kappa(\gamma)^n(T+1) - 1).$$

Because of the isomorphism (3.5) we can use the Main Conjecture in Iwasawa theory (see Section 1.5). This yields

$$f^*_{\psi,S}(T) = g^*_{\psi,S}(T)$$

for the monic polynomial $g^*_{\psi,S}$ defined by $G_{\psi,S}(T) = \pi^{\mu(G_{\psi,S})} g^*_{\psi,S}(T) u_{\psi,S}(T) \in \mathcal{O}_\psi[[T]]$, where $G_{\psi,S}(T) \in \mathcal{O}_\psi[[T]]$ is given by the following:

$$L^S_p(1-s, \psi) = \frac{G_{\psi,S}(\kappa(\gamma)^s - 1)}{H_\psi(\kappa(\gamma)^s - 1)}.$$

Moreover, since the analytic and the algebraic μ-invariants are equal for p odd, we obtain the following relation:

$$F_{\psi,S}(T) \sim_p G_{\psi,S}(T).$$

where $\sim_p$ denotes the equality of the p-adic valuations of both sides. The same calculations in (3.3) and (3.4) together with 3.7 yield

$$\begin{aligned} Fitt_{\mathcal{O}_\chi}(H^2_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Z}_p(n))^{\chi^{-1}}) =& |H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))|(L^S_{E/F}(1-n, \chi)) \\ =& Ann_{\mathbb{Z}_p[G]}(H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))) \cdot L^S_{E/F}(1-n, \chi). \end{aligned}$$

Finally, we note that $H^2_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Z}_p(n))^-$ is the direct sums of $H^2_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Z}_p(n))^{\chi^{-1}}$, where χ runs over the odd characters of G, and that $L^S_{E/F}(1-n, \chi) = 0$ for any even

characters χ of G. Therefore, by property 8 of Fitting ideals in Section 1.3.1, we obtain:

$$\begin{aligned} Ann_{\mathbb{Z}_p[G]}(H^0(E,\mathbb{Q}_p/\mathbb{Z}_p(n)))\cdot\Theta^S_{E/F}(1-n) =& Fitt_{\mathbb{Z}_p[G]}(H^2_{\acute{e}t}(\mathcal{O}^S_E,\mathbb{Z}_p(n))) \\ \subseteq& Ann_{\mathbb{Z}_p[G]}(H^2_{\acute{e}t}(\mathcal{O}^S_E,\mathbb{Z}_p(n))) \\ \subseteq& Ann_{\mathbb{Z}_p[G]}(H^2_{\acute{e}t}(\mathcal{O}'_E,\mathbb{Z}_p(n))), \end{aligned}$$

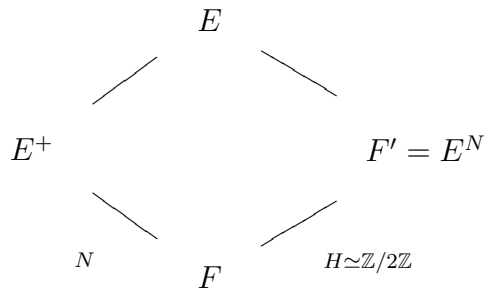
where the last relation follows from the inclusion $H^2_{\acute{e}t}(\mathcal{O}'_E,\mathbb{Z}_p(n)) \subseteq H^2_{\acute{e}t}(\mathcal{O}^S_E,\mathbb{Z}_p(n))$. Hence the p-adic version of the Coates-Sinnott Conjecture 2.2.4 holds for odd primes p in the situation of this part.

3.2.2 The case $p = 2$

In the situation of this section, again the L-function attached to E^+/F vanishes at $s = 1-n$, for odd $n \geq 2$, by the functional equation of L-functions (cf. Section 1.2.1). Therefore we only concentrate again on the Fitting ideal of the minus part of $H^2_{\acute{e}t}(\mathcal{O}'_E,\mathbb{Z}_2(n))$. We recall that the minus part is defined as the kernel of the following surjective corestriction map:

$$cor : H^2_{\acute{e}t}(\mathcal{O}'_E,\mathbb{Z}_2(n)) \to H^2_{\acute{e}t}(\mathcal{O}'_F,\mathbb{Z}_2(n)).$$

We recall that E^+ denotes the maximal real subfield of E, and $H =< J >$ denotes the Galois group of the quadratic extension E/E^+. We also let N denote the Galois group of E^+/F, whose degree is not divisible by 2.



Since N is of odd order, we have the canonical isomorphism $G \simeq N \times H$. Therefore the group of characters $\hat{G}$ of G is also canonically isomorphic to the direct product

$$\hat{G} \simeq \hat{N} \times \hat{H}, \tag{3.8}$$

where $\hat{N}$ and $\hat{H}$ denote the group of characters of H and N, respectively. Since $\mathbb{Z}_2[N]$ is the maximal order in $\mathbb{Q}_2[N]$, we also have the decomposition

$$\Lambda[G] \simeq (\oplus_\psi \mathcal{O}_\psi \otimes_{\mathbb{Z}_2} \mathbb{Z}_2[H])[[T]],$$

where ψ runs over all (real) characters of N and $\mathcal{O}_\psi$ is defined as before.

Step 1: $i \in E$

In this step we assume that E contains the primitive 4-th root of unity $i = \zeta_4$. Let χ denote the non-trivial character of H, which is the same as the Teichmüller character ω in our situation, and let ψ be an arbitrary character of N. By the canonical isomorphism (3.8) every odd character of G is of the form $\psi\chi$ for an even character ψ of N.

Let F_∞ be the cyclotomic $\mathbb{Z}_2$-extension of F with Galois group Γ and let $\Lambda = \mathcal{O}_\psi[[\Gamma]]$ be the completed group ring. Let $E_\infty = E(\zeta_{2^\infty})$ (resp. E^+_∞) be the cyclotomic $\mathbb{Z}_2$-extension of E (resp. E^+), let $M^S_{E_\infty}$ (resp. $M^S_{E^+_\infty}$) be the maximal abelian pro-2-extension of E_∞ (resp. E^+_∞) which is unramified outside the primes in S with Galois group $\mathfrak{X}_E := \mathfrak{X}^S_{E_\infty}$ (resp. $\mathfrak{X}_{E^+}) := \mathfrak{X}^S_{E^+_\infty}$), over E_∞ (resp. over E^+_∞). Since ψ is an even character, $\mathfrak{X}^\psi_E$ and $\mathfrak{X}^\psi_{E^+}$ are both finitely generated Λ-torsion modules with no non-trivial finite submodules (cf. Propositions 1.1.9 and 1.1.10). The field $M^S_{E^+_\infty}$ is contained in $M^S_{E_\infty}$, since we allow infinite primes to be ramified, and furthermore, by Galois theory we have an exact sequence

$$0 \to \mathfrak{X}_E/(1-J)\mathfrak{X}_E \to \mathfrak{X}_{E^+} \to H \to 0, \tag{3.9}$$

where the complex conjugation J is the generator of H. The analogue of Lemma 3.1.1 in this step is as follows:

Lemma 3.2.2. *For odd $n \geq 2$ we have the following isomorphisms of finite groups:*

$$((\mathfrak{X}_E/(1-J)\mathfrak{X}_E)^\psi(-n)_\Gamma)^* \simeq H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_2/\mathbb{Z}_2(n))^{(\psi\chi)^{-1}}.$$

Proof. If we denote by Ω^S_E the maximal algebraic pro-2-extension of E unramified outside S, and by M the Λ-module $\mathfrak{X}_E/(1-J)\mathfrak{X}_E$, then we have the following isomorphisms:

$$\begin{aligned}
(M^\psi(-n)_\Gamma)^* \simeq & (M^\psi(-n)^*)^\Gamma \simeq Hom(M^\psi(-n), \mathbb{Q}_2/\mathbb{Z}_2)^\Gamma \text{ by isomorphism (1.6)} \\
\simeq & Hom(M(-n)^{\psi\chi}, \mathbb{Q}_2/\mathbb{Z}_2)^\Gamma \qquad \text{since } n \text{ is odd} \\
\simeq & (Hom(M(-n), \mathbb{Q}_2/\mathbb{Z}_2)^{(\psi\chi)^{-1}})^\Gamma
\end{aligned}$$

$$\begin{aligned}
&\simeq (Hom(M, \mathbb{Q}_2/\mathbb{Z}_2(n))^{(\psi\chi)^{-1}})^{\Gamma} && \text{by Lemma 1.1.14}\\
&\simeq (Hom(\mathfrak{X}_E, \mathbb{Q}_2/\mathbb{Z}_2(n))^{(\psi\chi)^{-1}})^{\Gamma} && \text{since } \psi \text{ is even}\\
&\simeq (Hom(Gal(\Omega_E^S/E_\infty), \mathbb{Q}_2/\mathbb{Z}_2(n))^{(\psi\chi)^{-1}})^{\Gamma}\\
&\simeq (H^1_{\acute{e}t}(\mathcal{O}^S_{E_\infty}, \mathbb{Q}_2/\mathbb{Z}_2(n))^{(\psi\chi)^{-1}})^{\Gamma} && \text{since } \mu_{2^\infty} \subseteq E_\infty\\
&\simeq H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_2/\mathbb{Z}_2(n))^{(\psi\chi)^{-1}} && \text{since } cd_p(\Gamma) = 1.
\end{aligned}$$

Finally, we note that $H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_2/\mathbb{Z}_2(n))$ and $H^1_{\acute{e}t}(\mathcal{O}^S_{E^+}, \mathbb{Q}_2/\mathbb{Z}_2(n))$ have the same rank for any odd integer n by Proposition 1.4.11. Therefore $H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_2/\mathbb{Z}_2(n))^-$ is finite, and by Lemma 1.4.16, the group $H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_2/\mathbb{Z}_2(n))^{(\psi\chi)^{-1}}$ is also finite for the odd character $\psi\chi$. □

On the other hand by (3.9), the Λ-modules $(\mathfrak{X}_E/(1-J)\mathfrak{X}_E)^{\psi}(-n)$ and $\mathfrak{X}^{\psi}_{E^+}(-n)$ have the same characteristic polynomials, which generate the same Fitting ideals by Proposition 1.3.4. Therefore, by evaluation at zero, we obtain the following equalities:

$$\begin{aligned}
Fitt_{\mathbb{Z}_2}(\mathfrak{X}^{\psi}_E(-n)_\Gamma) &= Fitt_{\mathbb{Z}_2}(\mathfrak{X}^{\psi}_{E^+}(-n)_\Gamma)\\
&= (char_\Lambda(\mathfrak{X}^{\psi}_{E^+}(-n))(0)),
\end{aligned}$$

under the identification (1.1). Consequently, by Lemma 3.2.2, we have

$$Fitt_{\mathbb{Z}_2}(H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_2/\mathbb{Z}_2(n))^{(\psi\chi)^{-1}}) = (char_\Lambda(\mathfrak{X}^{\psi}_{E^+}(-n))(0)). \tag{3.10}$$

Now we relate this Fitting ideal to the corresponding L-function by using the Main Conjecture in Iwasawa theory. If $F_{\psi,S}(T) = 2^{\mu(F_{\psi,S})} f^*_{\psi,S}(T) \in \mathcal{O}_\psi[T]$ is the characteristic polynomial of $\mathfrak{X}^{\psi}_{E^+}$, then by Lemma 1.1.15 the characteristic polynomial of $\mathfrak{X}^{\psi}_{E^+}(-n)$ is of the form

$$char_\Lambda(\mathfrak{X}^{\psi}(-n))(T) = F_{\psi,S}(\kappa(\gamma)^n(T+1) - 1).$$

The assumptions of the Main Conjecture in Iwasawa theory, and the equality of algebraic and the analytic μ-invariants for the prime 2 yield

$$F_{\psi,S}(T) \sim_2 G_{\psi,S}(T),$$

where $\sim_2$ denotes the equality of the 2-adic valuations, and $G_{\psi,S}(T) \in \mathcal{O}_\psi[[T]]$ is the numerator of the 2-adic L-function as follows (cf. Section 1.5):

$$L^S_2(1-n, \psi) = \frac{G_{\psi,S}(\kappa(\gamma)^s - 1)}{H_\psi(\kappa(\gamma)^s - 1)}.$$

This implies, in the case the S type character ψ is not the trivial one, that

$$G_{\psi,S}(\kappa(\gamma)^n - 1) \sim_p L_2^S(1-n,\psi).$$

The same argument as in (3.4) for the trivial character $\psi = 1$, which is of type W, yields

$$H_\psi(\kappa(\gamma)^n - 1) = |H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(n))^{\omega^n}|.$$

Therefore we obtain the following equalities similarly:

$$\begin{aligned} char_\Lambda(\mathfrak{X}_{E^+}^\psi(-n))(0) &= |H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(n))| L_2^S(1-n,\psi) \\ &= |H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(n))| L_{E/F}^S(1-n,\psi\chi). \end{aligned}$$

By equality (3.10) we can replace the ideal generated by the characteristic polynomial of $\mathfrak{X}_{E^+}^\psi(-n)$ evaluated at zero, by the Fitting ideal of $H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}/\mathbb{Z}_2(n))^{(\psi\chi)^{-1}}$ to obtain

$$Fitt_{\mathcal{O}_\psi}(H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^{(\psi\chi)^{-1}}) = (|H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(n))|) L_{E/F}^S(1-n,\psi\chi). \quad (3.11)$$

Since $\mathbb{Z}_2[N]$ is a maximal order for the group N of odd order, $H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^\chi$ has the following decomposition as a $\mathbb{Z}_2[N]$-module:

$$H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^\chi = \oplus_{\psi\in\hat{N}} H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^{(\psi\chi)^{-1}},$$

where ψ runs over all (real) characters of N. Now using (3.11) for each even character ψ and property 8 of Fitting ideals in Section 1.3.1 yields

$$Fitt_{\mathbb{Z}_2[N]}(H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^\chi) = (|H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(n))|) \sum_{\psi\in\hat{N}} L^S(1-n,\psi\chi) e_{\psi^{-1}}, \quad (3.12)$$

where $e_{\psi^{-1}}$ is the idempotent attached to ψ^{-1} for any character ψ of N. Since the action of H on $H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^\chi$ is via multiplication by -1, it is enough to multiply both sides of equality (3.12) by the idempotent $e_{\chi^{-1}} = e_\chi$ to obtain the Fitting ideal of the module $H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^\chi$ over the ring $\mathbb{Z}_2[G]$. This yields

$$\begin{aligned} Fitt_{\mathbb{Z}_2[G]}(H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^\chi) &= (|H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(n))|) \sum_{\text{even } \psi\in G} L^S(1-n,\psi\chi) e_{\psi^{-1}} e_{\chi^{-1}} \\ &= (|H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(n))|) \sum_{\text{even } \psi\in G} L^S(1-n,\psi\chi) e_{(\psi\chi)^{-1}}. \end{aligned}$$

By using Lemma 1.4.16 and Corollary 1.3.2 we have the following relation:

$$2^{-1}|H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(n))| \sum_{\text{even } \psi\in\hat{G}} L^S(1-n,\psi\chi) e_{(\psi\chi)^{-1}} \in Fitt_{\mathbb{Z}_2[G]}(H^1_{\acute{e}t}(\mathcal{O}_E^S, \mathbb{Q}_2/\mathbb{Z}_2(n))^-),$$

and hence Proposition 1.4.20 implies that

$$2^{-r_1-1}Q_n|H^0(E,\mathbb{Q}_2/\mathbb{Z}_2(n))|\sum_{\text{odd}\psi\chi\in G} L^S(1-n,\psi\chi)e_{(\psi\chi)^{-1}} \in Fitt_{\mathbb{Z}_2[G]}(H^2_{\acute{e}t}(\mathcal{O}^S_E,\mathbb{Z}_2(n))^-),$$

where r_1 is the number of real primes of E^+, and the sum is over all odd characters of G.

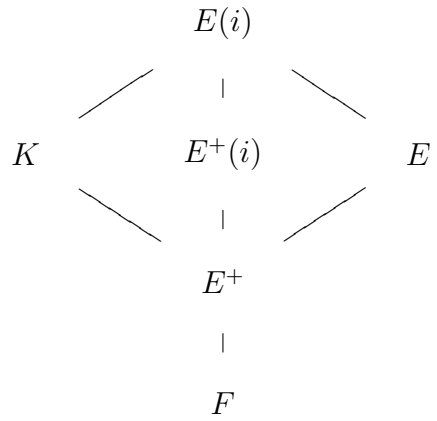
By the functional equation (cf. Section 1.2.1) the $L^S_{E/F}(1-n,\psi\chi)=0$ for any even character $\psi\chi$ of G, and an odd $n\geq 2$. We also note that the annihilator of the (cyclic) $\mathbb{Z}_2[G]$-module $H^0(E,\mathbb{Q}_2/\mathbb{Z}_2(n))$ is principal and generated by the order of the module. Hence

$$\begin{aligned} Ann_{\mathbb{Z}_2[G]}(H^0(E,\mathbb{Q}_2/\mathbb{Z}_2(n)))\cdot\Theta^S_{E/F}(1-n) \subseteq & Fitt_{\mathbb{Z}_2[G]}(H^2_{\acute{e}t}(\mathcal{O}^S_E,\mathbb{Z}_2(n))) \\ \subseteq & Ann_{\mathbb{Z}_2[G]}(H^2_{\acute{e}t}(\mathcal{O}^S_E,\mathbb{Z}_2(n))) \\ \subseteq & Ann_{\mathbb{Z}_2[G]}(H^2_{\acute{e}t}(\mathcal{O}'_E,\mathbb{Z}_2(n))), \end{aligned}$$

where the last relation follows from the inclusion $H^2_{\acute{e}t}(\mathcal{O}'_E,\mathbb{Z}_2(n))\subseteq H^2_{\acute{e}t}(\mathcal{O}^S_E,\mathbb{Z}_2(n))$. Hence, in the situation of step 1, the p-adic version of the Coates-Sinnott Conjecture 2.2.4 holds based on the classical Main Conjecture and the equality of μ-invariants for all characters of G.

Step 2: $i\notin E$

In this step, E does not contain the primitive 4-th root of unity $i=\zeta_4$. So the extension $E(i)/F$ is an abelian extension with Galois group isomorphic to $N\times\mathbb{Z}/2\times\mathbb{Z}/2$. Let K denote the maximal real subfield of $E(i)$. The following diagram illustrates the situation:



Both $E^+(i)/F$ and $E(i)/K$ are CM-extensions of type considered in step 1. We note that the non-trivial characters of $E^+(i)/E^+$ and $E(i)/K$ are the Teichmüller characters ω_{E^+} and ω_K, respectively. Let χ denote the non-trivial character of E/E^+, and let ψ be any (real) character of N. By (3.12) we have the following:

$$\begin{cases} Fitt_{\mathbb{Z}_2[N]}(H^1_{\acute{e}t}(\mathcal{O}^S_{E(i)}, \mathbb{Q}_2/\mathbb{Z}_2(n))^{\omega_K}) \\ \qquad\qquad = (|H^0(E(i), \mathbb{Q}_2/\mathbb{Z}_2(n))| \sum_{\psi\in\hat{N}} L^S_{E(i)/K}(1-n, \psi\omega_K)e_{\psi^{-1}}), \\ \\ Fitt_{\mathbb{Z}_2[N]}(H^1_{\acute{e}t}(\mathcal{O}^S_{E^+(i)}, \mathbb{Q}_2/\mathbb{Z}_2(n))^{\omega_{E^+}}) \\ \qquad\qquad = (|H^0(E^+(i), \mathbb{Q}_2/\mathbb{Z}_2(n))| \sum_{\psi\in\hat{N}} L^S_{E^+(i)/E^+}(1-n, \psi\omega_{E^+})e_{\psi^{-1}}), \end{cases}$$

where ψ runs over all (real) characters of N with the associated idempotent e_ψ.

The induced character $(\psi\omega_K)_*$ in the extension $E(i)/F$ is equal to $\psi\omega_{E^+} + \psi\chi$ by property 3 of Artin L-functions in Section 1.2.1. Therefore we have the following relation as well:

$$L^S_{E/F}(1-n, \psi\chi) = \frac{L^S_{E(i)/K}(\psi\omega_K, 1-n)}{L^S_{E^+(i)/F}(\psi\omega_{E^+}, 1-n)}.$$

We note that the quotient of $|H^0(E(i), \mathbb{Q}_2/\mathbb{Z}_2(n))|$ by $|H^0(F(i), \mathbb{Q}_2/\mathbb{Z}_2(n))|$ is 1 or 2, and that $|H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(n))| = 2$ for the field $E \not\ni i$. Hence

$$Fitt_{\mathbb{Z}_2[N]}(H^1_{\acute{e}t}(\mathcal{O}^S_{E(i)}, \mathbb{Q}_2/\mathbb{Z}_2(n))^{\omega_K}) \cdot Fitt_{\mathbb{Z}_2[N]}(H^1_{\acute{e}t}(\mathcal{O}^S_{E^+(i)}, \mathbb{Q}_2/\mathbb{Z}_2(n))^{\omega_{E^+}})^{-1} =$$

$$\begin{cases} (2^{-1}|H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(n))| \sum_{\psi\in\hat{N}} L^S_{E/F}(1-n, \psi\chi)e_{\psi-1}) & \text{if } \mu(E(i)(2)) \neq \mu(E^+(i)(2)), \\ \\ (|H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(n))| \sum_{\psi\in\hat{N}} L^S_{E/F}(1-n, \psi\chi)e_{\psi^{-1}}) & \text{if } \mu(E(i)(2)) = \mu(E^+(i)(2)), \end{cases}$$

where $\mu(E(i)(2))$ and $\mu(E^+(i)(2))$ denote the 2-primary parts of $\mu(E(i))$ and $\mu(E^+(i))$, respectively. So in order to verify the conjecture, it suffices to relate the left hand side of the equality above to the Fitting ideal of $H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_2/\mathbb{Z}_2(n))^\chi$, and then to the Fitting ideal of $H^2_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Z}_2(n))^-$. The following proposition is crucial:

Proposition 3.2.3. *We have the following relation of Fitting ideals:*

- *if $\mu(E(i)(2)) \neq \mu(F(i)(2))$, then*

  $$Fitt_{\mathbb{Z}_2[N]}(H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_2/\mathbb{Z}_2(n))^\chi)$$

  $$= Fitt_{\mathbb{Z}_2[N]}(H^1_{\acute{e}t}(\mathcal{O}^S_{E(i)}, \mathbb{Q}_2/\mathbb{Z}_2(n))^{\omega_K})(Fitt_{\mathbb{Z}_2[N]}(H^1_{\acute{e}t}(\mathcal{O}^S_{E^+(i)}, \mathbb{Q}_2/\mathbb{Z}_2(n))^{\omega_{E^+}})^{-1}.$$

- *if* $\mu(E(i)(2)) = \mu(F(i)(2))$, *then*

$$Fitt_{\mathbb{Z}_2[N]}(H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_2/\mathbb{Z}_2(n))^{\chi})$$

$$= 2^{-1} Fitt_{\mathbb{Z}_2[N]}(H^1_{\acute{e}t}(\mathcal{O}^S_{E(i)}, \mathbb{Q}_2/\mathbb{Z}_2(n))^{\omega_K})(Fitt_{\mathbb{Z}_2[N]}(H^1_{\acute{e}t}(\mathcal{O}^S_{E^+(i)}, \mathbb{Q}_2/\mathbb{Z}_2(n))^{\omega_{E^+}})^{-1}.$$

Proof. For simplicity, let A_L denote the group $H^1_{\acute{e}t}(\mathcal{O}^S_L, \mathbb{Q}_2/\mathbb{Z}_2(n))$ for any CM-field L, and let χ_L denote the non-trivial character of L/L^+. We note that $A_L^{\chi_L}$ is the same as the kernel $A_L^{(-1)}$ of the cohomological norm defined in Section 1.4.3. By Proposition 2.11 in [19] for the CM-extension E/E^+, where $i \notin E$, we have

$$|A_E^{(-1)}| = \begin{cases} \frac{|A^{(-1)}_{E(i)}|}{|A^{(-1)}_{E^+(i)}|} & \text{if } \mu(E(i)(2)) \neq \mu(E^+(i)(2)), \\ \\ 2\frac{|A^{(-1)}_{E(i)}|}{|A^{(-1)}_{E^+(i)}|} & \text{if } \mu(E(i)(2)) = \mu(E^+(i)(2)). \end{cases}$$

If one considers these groups as $\mathbb{Z}[N]$-modules and replaces the modules by their ψ-eigenspaces in the whole proof of Proposition 2.11 in [19] for an arbitrary (real) character ψ of N, we obtain the same result for ψ-eigenspaces as

$$|A_E^{\psi\chi}| = \begin{cases} |A^{\psi\omega_K}_{E(i)}||A^{\psi\omega_{E^+}}_{E^+(i)}|^{-1} & \text{if } \mu(E(i)(2)) \neq \mu(E^+(i)(2)), \\ \\ 2|A^{\psi\omega_K}_{E(i)}||A^{\psi\omega_{E^+}}_{E^+(i)}|^{-1} & \text{if } \mu(E(i)(2)) = \mu(E^+(i)(2)). \end{cases}$$

On the other hand since the groups above are $\mathcal{O}_\psi$-modules, their Fitting ideals are principal and generated by their orders. Therefore,

$$Fitt_{\mathcal{O}_\psi}(A_E^{\psi\chi}) = \begin{cases} Fitt_{\mathcal{O}_\psi}(A^{\psi\omega_K}_{E(i)})Fitt_{\mathcal{O}_\psi}(A^{\psi\omega_{E^+}}_{E^+(i)})^{-1} & \text{if } \mu(E(i)(2)) \neq \mu(E^+(i)(2)), \\ \\ (2)Fitt_{\mathcal{O}_\psi}(A^{\psi\omega_K}_{E(i)})Fitt_{\mathcal{O}_\psi}(A^{\psi\omega_{E^+}}_{E^+(i)})^{-1} & \text{if } \mu(E(i)(2)) = \mu(E^+(i)(2)). \end{cases}$$

Since $\mathbb{Z}_2[N]$ is a maximal order, it is enough to vary ψ in the group of characters of N to complete the proof. □

Now we can compute the Fitting ideal of the $\mathbb{Z}_2[N]$-module $H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_2/\mathbb{Z}_2(n))^{\chi}$ using Proposition 3.2.3:

$$Fitt_{\mathbb{Z}_2[N]}(H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_2/\mathbb{Z}_2(n))^{\chi}) = (2^{-1}|H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(n))| \sum_{\psi \in \hat{N}} L^S_{E/F}(1-n, \psi\chi)e_{\psi^{-1}}). \tag{3.13}$$

Again, to obtain the Fitting ideal of $H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_2/\mathbb{Z}_2(n))^\chi$ over $\mathbb{Z}_2[G]$, since the action of H on $H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_2/\mathbb{Z}_2(n))^\chi$ is via multiplication by -1, it is enough to multiply both sides of equality (3.13) by the idempotent $e_{\chi^{-1}}$. Consequently, we have

$$\begin{aligned} Fitt_{\mathbb{Z}_2[G]}(H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_2/\mathbb{Z}_2(n))^\chi) =& (2^{-1}|H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(n))|) \sum_{\psi \in \hat{N}} L^S_{E/F}(1-n, \psi\chi) e_{\psi^{-1}} e_{\chi^{-1}} \\ =& (2^{-1}|H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(n))|) \sum_{\psi \in \hat{N}} L^S_{E/F}(1-n, \psi\chi) e_{(\psi\chi)^{-1}}, \end{aligned}$$

which leads to the following relation by Lemma 1.4.16 and Corollary 1.3.2:

$$2^{-2}|H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(n))| \sum_{\text{odd } \psi\chi \in G} L^S_{E/F}(1-n, \psi\chi) e_{(\psi\chi)^{-1}} \in Fitt_{\mathbb{Z}_2[G]}(H^1_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Q}_2/\mathbb{Z}_2(n))^-).$$

Now by Proposition 1.4.20 we obtain

$$\begin{aligned} 2^{-r_1-2} Q_n |H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(n))| & \sum_{\text{odd } \psi\chi \in G} L^S_{E/F}(1-n, \psi\chi) e_{(\psi\chi)^{-1}} \\ & \in Fitt_{\mathbb{Z}_2[G]}(H^2_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Z}_2(n))^-), \end{aligned}$$

where r_1 is the number of real primes of E^+, and the sum is over all odd characters of G. Since by the functional equation of L-functions (cf. Section 1.2.1) we have $L^S_{E/F}(1-n, \psi\chi) = 0$ for any even characters $\psi\chi$ and n odd. Consequently, we obtain:

$$\begin{aligned} Ann_{\mathbb{Z}_2[G]}(H^0(E, \mathbb{Q}_2/\mathbb{Z}_2(n))) \cdot \Theta^S_{E/F}(1-n) &\subseteq Ann_{\mathbb{Z}_2[G]}(H^2_{\acute{e}t}(\mathcal{O}^S_E, \mathbb{Z}_2(n))) \\ &\subseteq Ann_{\mathbb{Z}_2[G]}(H^2_{\acute{e}t}(\mathcal{O}'_E, \mathbb{Z}_2(n))), \end{aligned}$$

where the last inclusion follows from $Spec(O^S_E) \subseteq Spec(O'_E)$. Hence the Coates-Sinnott Conjecture holds in this step as well. This together with the result from step 1, and the result from the case that p is odd, yields:

Theorem 3.2.4. *The p-adic version of the Coates-Sinnott Conjecture 2.2.4 holds for any prime number p, an odd integer $n \geq 2$, and an abelian extension E/F of a CM field E over a totally real number field F, with Galois group G, whose order is $2m$ for m not divisible by p, based on the classical Main Conjecture and the equality of the algebraic and the analytic μ-invariants for all real characters of G in the case $p = 2$.*

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