UNIFIED THEORY OF MODEL REDUCTION

A UNIFIED THEORY OF MODEL REDUCTION FOR LINEAR TIME INVARIANT DYNAMICAL SYSTEMS

by

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ABSTRACT

The approximation of linear, time-invariant, dynamical systems by similar systems having fewer state variables is investigated. A class of reduced-order approximants called nonminimal partial realizations is introduced which includes many published methods as special cases, and thus represents a unification of the theory of model reduction. Since the concept of linear state variable feedback is central to many of the design procedures of modern control theory, the behaviour of the approximated system to such feedback laws derived from analysis of the approximating system is studied. The specific results derived give a credibility heretofore nonexistant to the class of reduced models called minimal partial realizations by virtue of the fact that they form a subclass of the nonminimal partial realizations. The use of canonical form state equations is advocated as a means of simplifying the computational procedure for an important class of reduced models termed aggregated partial realizations. Such realizations are shown to be useful for designing suboptimal linear quadratic servomechanism compensators, since guaranteed stability of the large-scale system is possible.

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CHAPTER 1

INTRODUCTION

1.1 What is Model Reduction?

Model reduction is essentially the practice of approximating the behaviour of a complex mathematical model of a physical system by a model of reduced complexity. Such a definition, however, is largely unenlightening, as the meaning of the term "complexity" has not been precisely defined. For instance, many physical systems are inherently nonlinear and of a distributed parameter nature, which is to say that the process mechanics are modelled by nonlinear partial differential equations. Such models may certainly be called complex as a quantitative mathematical treatment of them may be difficult or impossible. In another context, such models may be deemed relatively simple from the point of view of the number of parameters used in the equations. The mathematics of linear systems, although detailed and vast in scope, is certainly easier to manage than that required for the nonlinear case. In this case, complexity enters in the form of an excessive number of parameters which inhibits numerical calculations by taxing the abilities of the largest and fastest digital computers currently available. The most obvious example of this phenomenon is the linear quadratic regulator problem of modern control theory whose solution is elegantly stated in terms of a matrix Ricatti equation (c.f., Chapter 5). For systems of modest size, of perhaps twenty state variables, the problem tends to get out of hand either in computational effort or in memory requirements, depending on the algorithm chosen to solve the Ricatti equation. Since the memory requirements grow as n^2 and the computation time as n^3, where n is the number of states, the trade off of optimality for running time or memory restrictions is a well justified procedure.

The linear quadratic regulator problem is not the sole area where application of model reduction techniques is profitable, however, or the effort currently being spent in this area of research would not be justified. Indeed, many design techniques in the classical control theoretic setting are iterative in nature and the use of a reduced model in such a setting is commonplace. As an example, consider the feedback stabilization/compensation problem associated with a multistage transistor amplifier using the root locus technique. Here the assumption that the dominant poles of the amplifier are the only ones to be appreciably affected by variation of the feedback network is equivalent to the use of a class of reduced models termed aggregated partial realizations. This class is extensively treated in this thesis.

Having given some justification for the idea and application of reduced models in general, the class of complex models to which this thesis is directed is now defined. It is the class of linear, time-invariant, irreducible dynamical systems, having a large number of states, which may be modelled by differential equations of the form $\dot{x} = Ax + Bu$, $y = Cx + Du$. Here x is a state vector, u an input vector, y an output vector, and A, B, C, D are matrices having entries in the field of real numbers.† The direct transmission map D: u → y actually may be taken as zero without loss in generality for the purposes of developing reduction techniques, so attention will mainly be focussed on the matrices A, B, and C, which are assumed to be known exactly. The model reduction problem may therefore be loosely stated as: given $\dot{x} = Ax + Bu$, $y = Cx$, find a model $\dot{z} = Fz + Gu$, $\hat{y} = Hz$ which approximates the given system in some specified manner, where z has fewer components than x.

1.2 An Overview of Reduction Theory

In the past decade, much literature has appeared on the subject of model reduction, most of which deals with the reduction of order (number of state variables) of linear time-invariant dynamical systems. It is therefore of interest to examine briefly the history of model reduction. The beginnings of the subject may be traced to a stability test called the root locus technique (Evans, 1950), which has become a widely used tool in the field of classical control. The root locus method allows one to plot the locus of the poles of a dynamic system as a parameter is varied. The observation that the poles relatively close to the imaginary axis of the complex plane are appreciably affected by parameter variation, while those far into the left half plane do not change too much, led directly to the concepts of dominant poles, and reduced-order models which retained only the dominant poles. Classical

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† This restriction is not necessary. Indeed fields of arbitrary characteristic may be used, but the finite fields (having characteristic > 0) do not often occur in control theoretic settings.

control theory was developed to treat input-output (transfer function) relationships of single-input single-output systems. Early attempts at generalizing the application to multi-input multi-output (multivariable) systems failed until the introduction of the state space (matrix differential equation) description in the early 1960's. By this time, the dominant pole idea had become so entrenched in classical control that the first attempts at reduction of a multivariable system (Nicholson, 1964, Davison, 1966) used precisely this idea, and with good success. The computational effort, however, was much greater, for now calculations equivalent to determining the eigenvalues and eigenvectors of a matrix were required, whereas in the single variable case, only the factorization of a scalar polynomial was needed. The fact that a numerically stable eigenanalysis algorithm for real assymetric matrices did not appear until 1970 (Martin, et al., 1970) precluded the use of the dominant eigenvalue technique for any system having more than twenty state variables, although several researchers developed different eigenvalue preservation schemes during this time (Marshall, 1966, Mitra, 1967, Aoki, 1968, Chidambara, 1969). Such reduction techniques are also termed projective or aggregation methods. The term aggregation comes from the fact that the state vector of the reduced model is the image of a linear map (or aggregate) of the original state vector. This map may be factored as the product of another map with a projection, thus giving the term projective (c.f., Chapter 3). Mainly to reduce the large computational effort associated with the dominant eigenvalue technique, the continued fraction approach was introduced in 1968 (Chen and Shieh, 1968). This method consisted of expressing the transfer function as a continued fraction. Different order reduced models could then be generated by simply ignoring the last quotients and reversing the continuation process. Knowledge of the quotients also allowed the easy formulation of state equations for the reduced model. Zakian (1973) has proved that the reduced models of a single variable system obtained by Chen's method are in fact Padé approximants of the original transfer function. It is well known that the Padé algorithm is the most computationally efficient method for the approximation of a scalar function, so that Chen's method is the fastest reduction technique for scalar systems. The multivariable extension of Chen's algorithm yields a Padé type approximant as well, but is restricted to systems having an equal number of inputs and outputs, and polynomial matrix inversion is required. To circumvent these restrictions Shamash (1973) has proposed a direct extension of the Padé approximation algorithm to the multivariable case. Although this technique is certainly computationally more efficient than Chen's, a very serious drawback is that the order of a state variable realization of the reduced model may actually equal or exceed that of the original system. For example, consider

$$G(s) = \begin{bmatrix} 1/(s+2)(s+3) & 1/(s+2)^2 \\ 1/(s+3) & 1/(s+2) \end{bmatrix}$$

whose characteristic polynomial is $\Delta(s) = (s+2)^2(s+3)$ having degree 3. Using the method of Shamash one obtains a "reduced" model

$$G(s) = \begin{bmatrix} 1/(s+6) & 1/(s+4) \\ 1/(s+3) & 1/(s+2) \end{bmatrix}$$

whose characteristic polynomial is $\Delta(s) = (s+2)(s+3)(s+4)(s+6)$ having degree $4 > 3$!

Chen's method is a special case of the technique of minimal partial realization developed in this thesis. Intuitively this means the matching of a maximum number of terms of the Taylor series of the original transfer function with a reduced model having a minimal order characteristic polynomial. Although Chen showed that such reduced models are useful in the design of dynamic compensators for the original system, no straightforward method exists to allow the design of static (i.e., linear state variable feedback) compensation. The aggregation technique, however, is ideal for the design of static compensation, although this realization was long overdue, and, at that, only proven for the special case of Davison's reduction technique (Lamba and Rao, 1972 and 1975). The first breakthrough in the area of static compensator design came with the method of singular perturbations (Sanutti and Kokotović, 1969). This method is also called two time scale control, since reduction and compensator design are achieved by assuming that the fast dynamics of the system are infinitely fast compared with the slower dynamics (the derivative of the fast dynamics is normalized by dividing by a small parameter which, when set to zero, produces a reduced model and a singular derivative). Moreover this reduction technique is applicable to nonlinear systems as well as linear ones. Good success was reported for its application to the suboptimal regulator problem.

The major goals of this thesis are to develop a method of model reduction which includes existing reduction techniques as special cases and also allows the design of static compensation based on the reduced model. The proposed method combines the ideas of aggregation and minimal partial realization and is termed nonminimal partial realization. This unification is largely achieved as only the singular perturbation technique in the nonlinear case and the minimax approach (Genesio and Milanese, 1976) cannot be viewed as special cases.

1.3 Synopsis

The thesis is organized as follows. Chapter 2 reviews four different types of reduction which have appeared in the literature, including the eigenvalue preservation and minimal partial realization algorithms. Later chapters give a detailed treatment of the techniques.

In Chapter 3, the theory of model reduction by aggregation is studied. This method was pioneered by Aoki (1968), and held great promise, as a linear relationship $z = Kx$ related the state of the reduced model to that of the larger system. This method was largely forgotten, however, as Aoki gave no method for the determination of the aggregation matrix K. The remainder of the chapter is devoted to solving this problem, specifying the reaction of the large system to control laws based on the reduced model (the aggregation matrix K plays a central role here), and the derivation of some sensitivity formulae for the reduced model based on eigenvalue/eigenvector sensitivities of the A matrix. An important consequence of the formulae for determining the aggregation matrix is that all eigenvalue preservation methods are cases of aggregation and vice versa. This is the first step in the unification of model reduction theory.

Chapter 4 dwells mainly on the state variable formulation of the transfer function reduction methods introduced in Chapter 2. Here the concept of a partial realization is shown to follow naturally from that of a minimal realization generated from the Markov parameters of the transfer function. A new minimal/partial realization algorithm is developed which has some special iterative properties making the derivation of various partial realizations a very efficient process. This extension of the theory of minimal realization allows the unification of many of the existing reduction methods based on time moments or Markov parameters. Indeed, the algorithm is a multivariable generalization of the process of Padé approximation of proper rational functions. The emphasis then shifts from (minimal) partial realizations to nonminimal partial realization. The extra degrees of freedom gained by using a reduced state vector of greater length than necessary for minimal partial realization allow the development of reduced models which are both aggregated and partial realizations. Thus the class of nonminimal partial realizations is seen to be very large, and the unification of reduction theory is complete. Efficient methods of determining such aggregated partial realizations are given. The analytical techniques of state aggregation are generalized to apply to nonminimal (and minimal) partial realizations. It is precisely these results which make nonminimal partial realizations such good reduced order models of systems. Throughout Chapter 4 the use of canonical form state equations is stressed. Such forms are equally useful for proving otherwise difficult theorems and also afford an economy for simulation, both in memory and running time requirements.

Chapter 5 is devoted to the suboptimal linear quadratic regulator problem. The merits of various reduction schemes for determining suboptimal controls are investigated. The suboptimal tracking (servomechanism) problem is treated as a special form of the regulator. It is found that aggregated partial realizations are the most suitable reduced models when the signals to be tracked are polynomial time functions.

Until this point in the thesis, attention has focussed exclusively on dynamical systems of the continuous type, i.e., representable as $\dot{x} = Ax + Bu$. An extremely important class of dynamical systems is the discrete type, having state equations of the form $x(k+1) = \Phi x(k) + \Theta u(k)$. Chapter 6 discusses the reduction of such systems using the notion of nonminimal partial realizations as developed for the continuous time case. Differences between the behaviour of such reduced models from their continuous time counterparts are investigated.

Finally, in Chapter 7, the technique of aggregated partial realization is applied to the design and implementation of a suboptimal feedback law for the linear quadratic servomechanism problem for a dynamical system having thirty three states.

1.4 Contributions of the Thesis

Claimed as original contributions of this thesis are:

1. The unification of a large number of model reduction techniques through the use of nonminimal partial realization.
2. The development of an analytical framework which allows description of the reaction of the large system to linear state variable feedback laws derived from consideration of the reduced

model.

3. The use of aggregated partial realizations for the determination of suboptimal controls in the linear quadratic regulator problem guarantees that the original system may be stabilized. Matching of time moments then guarantees that the steady state tracking error will be zero for the servomechanism problem when the target is a polynomial function of time.
4. The use of minimal realization algorithms to iteratively determine Padé approximants of strictly proper rational functions. This iterative procedure allows the generation of the [r', r+1] or [r'+1, r] approximant given the [r', r] approximant.
5. The proof that aggregated models always preserve eigenvalues.
6. The proof that an aggregated reduced model may be used to place a subset of the poles of the large system arbitrarily.

CHAPTER 2

A REVIEW OF EXISTING METHODS OF MODEL REDUCTION

2.1 Introduction

This chapter provides a review of existing work in four major areas of the field. The two central methods, eigenvalue preservation, and partial realization, are discussed in detail, while the methods of singular perturbations and minimax approximation are given somewhat less attention. This bias in coverage is not intended to disclaim the latter areas of research, but reflects the success of the method developed in Chapter 4 in unifying the topic of model reduction.

2.2 Four Philosophies of Model Reduction

This section introduces four major areas of endeavour in the field of model reduction. The eigenvalue preservation methods of Davison (1966) and Mitra (1967), the moment matching methods of Chen and Shieh (1968) and Shamash (1973a), the singular perturbations method of Kokotović (1972), and the minimax approximation method of Milanese (1971), are discussed.

2.2.1 Eigenvalue Preservation Methods

Perhaps one of the oldest methods of simplifying linear systems is the retention of dominant modes. Certainly the first attempts at reducing multivariable linear systems were the techniques of Nicholson (1964) and Davison (1966), wherein the dominant eigenvalues and their corresponding eigenvectors (in the sense that the ratios of the elements of the reduced eigenvectors are the same as the ratios of corresponding elements of the original eigenvectors) were retained. The system to be reduced was represented by

$$\dot{x} = Ax + Bu, \; x \in R^n \tag{2.1}$$

where u(t) was a scalar unit step function. If the modal matrix (Porter and Crossley, 1972) of A and its inverse are given respectively by

$$V = [v_1 \; v_2 \; \ldots \; v_n] \tag{2.2a}$$

$$V^{-1} = \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_n^T \end{bmatrix} \tag{2.2b}$$

where

$$AV = V\Lambda = V \,\mathrm{diag}\, (\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{2.3}$$

and it is desired to retain the first r eigenvalues of A in a reduced model, then A, V, and V^{-1} are partitioned as follows

$$A = \begin{array}{c} \begin{matrix} \underbrace{r}_{} & \quad \underbrace{n-r}_{} \end{matrix} \\ \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \end{array} \begin{array}{l} (r \\ (n-r \end{array} \tag{2.4a}$$

$$V = \begin{array}{cc} \underbrace{r} & \underbrace{n-r} \\ \multicolumn{2}{c}{\left[\begin{matrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{matrix}\right]} \end{array} \begin{array}{l} (r \\ (n-r \end{array} \qquad (2.4b)$$

$$W = V^{-1} = \begin{array}{cc} \underbrace{r} & \underbrace{n-r} \\ \multicolumn{2}{c}{\left[\begin{matrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{matrix}\right]} \end{array} \begin{array}{l} (r \\ (n-r \end{array} \qquad (2.4c)$$

The dynamical equations of the reduced model of (2.1) are now given by

$$\dot{z} = Fz + Gu \qquad (2.5a)$$

$$F = A_{11} + A_{12} V_{21} V_{11}^{-1} \qquad (2.5b)$$

$$G = V_{11} [W_{11} \quad W_{12}]B \qquad (2.5c)$$

In Chapter 3 it is shown that the state variables x and z are linked by $z = V_{11} [W_{11} \quad W_{12}]x$. This important result was unnoticed by Davison but was discovered by Lamba and Rao (1972).

Davison's pioneering work treated the approximation of the state trajectories for a step input and did not consider approximation of an output map. This was an acceptable method providing the outputs of the system were identical to some of the original states, or that the reduced model was solely intended for the calculation of a suboptimal control for a linear quadratic regulator problem as investigated by Lamba and Rao (1972). One major defect was nonpreservation of the DC steady state.

In many instances, however, one is interested in closely approximating the output of the large-scale system. The addition of an output equation to the reduced model adds a new dimension to the problem, for now the choice of eigenvalues to retain may become critical (Hickin and Sinha, 1976a). In his work, Mitra (1967, 1969) proposed a method of optimal reduction in the sense of optimizing a linear quadratic error functional, for some fixed input u(t). To retain the first r eigenvalues using the following functional (for which u(t) is an impulse function) calculate

$$W = \int_0^\infty \exp(A\tau)\ BDB^T \exp(A^T\tau)\ d\tau,\ D > 0^+ \tag{2.6}$$

The following projection matrix is then calculated

$$P = I_n - \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} \left\{ [V^*_{12} \quad V^*_{22}]W^{-1} \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} \right\}^{-1} [V^*_{12} \quad V^*_{22}]W^{-1} \tag{2.7}$$

It is easily shown that PA = $\hat{A}$P so that the decontrolled trajectory $\hat{x}$ = Px satisfies the following differential equation

$$\dot{\hat{x}} = PAx + PBu = \hat{A}\hat{x} + \hat{B}u,\ x \in R^n \tag{2.8}$$

The reduction is now possible by virtue of the fact that (2.8) is uncontrollable, for it easily follows that $\hat{U}$ = PU where U and $\hat{U}$ are the controllability matrices of (2.1) and (2.8) respectively. Since P is calculated by (2.7), and W by (2.6), the subspace of R^n in which the

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$^+$ D > 0, the square matrix D is positive definite.

error $x-\hat{x}$ lies is orthogonal to the subspace in which $\hat{x}$ lies, when $u(t) = D^{1/2}\,\delta(t)$, a weighted set of unit impulses. Removal of the uncontrollable modes is now accomplished by choosing the r columns of a matrix E to span the controllable subspace of (2.8), and partitioning $\hat{A}$ as in (2.4a), and $\hat{B}$ and E as follows

$$\hat{B} = \begin{array}{cl} \begin{bmatrix} \hat{B}_{11} \\ \\ \hat{B}_{21} \end{bmatrix} & \begin{array}{l} (r \\ \\ (n-r \end{array} \end{array} \tag{2.9a}$$

$$E = \begin{array}{cl} \begin{bmatrix} E_{11} \\ \\ E_{21} \end{bmatrix} & \begin{array}{l} (r \\ \\ (n-r \end{array} \end{array} \tag{2.9b}$$

and by calculation of

$$F = E_{11}^{-1}\,\hat{A}_{11}\,E_{11} + E_{11}^{-1}\,\hat{A}_{12}\,E_{21} \tag{2.10a}$$

$$G = E_{11}^{-1}\,\hat{B}_{11} \tag{2.10b}$$

$$H = CE \tag{2.10c}$$

The reduced model would then be given by

$$\dot{z} = Fz + Gu, \quad z \in R^r \tag{2.11a}$$

$$\hat{y} = Hz \tag{2.11b}$$

The similarity of (2.5) and (2.10) should be quite evident. In Chapter 3 a general formula for such reduced models is derived.

To optimally reduce the system (2.4) it is necessary to calculate P and E matrices corresponding to each of the $\binom{n}{r}$ ways in which r eigenvalues may be selected from the n original eigenvalues. When n is large and r is small the ensuing combinatorial search will be so time consuming as to throw serious doubts as to the utility of the method. A common compromise is to retain the dominant eigenvalues only.

Davison's and Mitra's methods each suffer from the drawback that an eigenvalue/eigenvector calculation must be accomplished for the A matrix, which will generally be assymmetric. After transformation to Hessenberg form, several iterations of the QR algorithm, requiring $4n^2$ operations per iteration, may be needed to determine each eigenvalue (Dahlquist and Björck, 1974). Calculation of the eigenvectors will take at least as many operations and thus the computer time involved may become prohibitive for a large matrix ($n>50$). In the following section two computationally efficient methods of model reduction are presented.

2.2.2 Moment Matching and Padé Approximation

To overcome the computational difficulties of eigenanalysis, Chen and Shieh (1968) proposed the expansion of the transfer function $G(s) = C(sI-A)^{-1}B$ into a continued fraction. Generation of reduced-order models is then easily accomplished by ignoring some of the quotients and inverting the truncated continued fraction. Examination of the continued fraction expansion also allowed the easy writing of state equations of the transfer function and all reduced models. For scalar systems (single-input, single-output) the expansion of $g(s) = c^T(sI-A)^{-1}b$ proceeds as

$$g(s) = \cfrac{1}{h_1 + \cfrac{s}{h_2 + \cfrac{s}{h_3 + \cfrac{s}{h_4 + \cfrac{s}{\ddots}}}}} \qquad (2.12)$$

Although this method is very simple, and gives reduced models which are often very good approximations to the original, it suffers from two serious drawbacks. First, a reduced model of a stable (resp. unstable) G(s) may be unstable (resp. stable), and second, the method loses its computational appeal in the multivariable case where polynomial matrix inversion becomes necessary, and the number of inputs must equal the number of outputs. In Chapter 4 each of these difficulties is removed.

Zakian (1973) has noted that the method of Chen and Shieh yields an [r-1,r] Padé approximant[†] to g(s), and thus that the first 2r time moments of g(s) and its reduced model $\hat{g}(s)$ are equal. The steady state error between the responses of the two models will then be zero for step, ramp, etc., inputs if both systems are asympotically stable. Shamash (1973a) has proposed the use of more general [r',r] Padé approximants for model reduction. This usually enables a reduced model of prescribed order r having suitable stability properties to be found

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[†] If g(s) is analytic at s=0, then $\hat{g}(s)$ is an [m,n] Padé approximant to g(s) if $\hat{g}(s)$ is an [m,n] rational function and the first m+n+1 terms of the Taylor series of $g(s)-\hat{g}(s)$ are zero.

by choosing $r' \leq r-1$. Unfortunately, the choice of r' is governed by a trial and error technique. The method may be readily extended to multivariable systems, however, without the need for any matrix inversion. A serious deficiency of the multivariable algorithm of Shamash is the introduction of multiple poles into the "reduced" system which may then have a characteristic polynomial whose degree is greater than that of the original system! Other moments methods are discussed in the survey by Bosley and Lees (1972) and may be found in the supplementary bibliography.

2.2.3 Singular Perturbations

Each technique of model reduction mentioned so far has been explicitly formulated for linear time-invariant dynamical systems. The technique of singular perturbations (Kokotović, 1972) was formulated to reduce general nonlinear systems. Assuming the dynamical equations of the large-scale system to be in the following form

$$\dot{x}_1 = f(x_1, x_2, \mu, u, t) \qquad (2.13a)$$

$$\mu\dot{x}_2 = g(x_1, x_2, \mu, u, t) \qquad (2.13b)$$

where μ is a small, generally positive, parameter, reduction is accomplished by setting $\mu = 0$ and solving the now algebraic equation (2.13) to yield

$$\bar{x}_2 = h(\bar{x}_1, \bar{u}, t) \qquad (2.14a)$$

$$\dot{\bar{x}}_1 = f(\bar{x}_1, \bar{x}_2, 0, \bar{u}, t) = \bar{f}(\bar{x}_1, \bar{u}, t) \qquad (2.14b)$$

Under certain conditions, the states $\overline{x}_1$ will closely approximate the states x_1 after an initial transient where the magnitude of the error may be quite large. The nature of the reduction method also allows a boundary layer approximation to the behaviour of the neglected dynamics of the system when $\mu \neq 0$ as

$$\frac{dx_2}{d\tau} = g(x_1(t),\ x_2(\tau),\ 0,\ u(0),\ \tau),\ \tau = t/\mu \qquad (2.15)$$

The analysis and application becomes much simplified when dealing with linear time-invariant systems. A serious disadvantage, however, is that identification of the state variable transformation which is required to identify x_1, x_2, and μ may not be simple.†

The method of singular perturbations has been successfully applied to optimal control theory (for the linear quadratic loss case) where a boundary layer system was used to perturb the Ricatti equation (Sannuti and Kokotovic, 1969). At this time, no other branch of model reduction allows such compensation for the lost information of the neglected states.

2.2.4 Minimax Approximation

A greatly different philosophy to the problem of model reduction is the computation of an input-output independent measure of the error norm between a system and its approximant in the time domain. The subsequent minimization of this error norm is then used to generate a

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† This was subsequently removed, c.f., Chow and Kokotović (1976).

reduced model. Different choices of the norm will lead to different models. A popular choice for the input and output spaces are the so-called L_p spaces ($p>1$) (Naylor and Sell, 1971), while the error between the system responses is measured with the L_∞ or minimax (uniform) norm.

Advantages of this method include the possibility of identifying a reduced-order model from noisy input-output data, its application to nonlinear distributed parameter systems, and the possibility of guaranteed cost control (Genesio and Milanese, 1976). A serious disadvantage of the method is the great amount of data processing that may be necessary, for instance in the minimization of the error functional of the form

$$E = \sup_{u \in U} \frac{||y-\hat{y}||}{||u||} \tag{2.16}$$

In the above equation U represents a class of admissible inputs and $y-\hat{y}$ denotes the output error between the system and its approximating model. Because of the worst case nature of the approximation, the minimum error bound E^* may be too pessimistic to be of practical use.

2.3 Summary

A review of four major methods of model reduction has been presented. The eigenvalue preserving methods require much computer time to calculate a reduced model, particularly the optimal reduction method of Mitra. The moment matching methods of Chen and Shieh and Shamash require the least amount of computation but suffer from the serious drawback that the stability of the reduced model cannot be guaranteed. While the method of singular perturbations suffers from the disadvantage that identification of the proper form of the large-scale dynamical equations may be very difficult, it remains the only method which allows the partial recoverey of the information lost upon passage to the reduced model. The computational requirements of minimax approximation are variable, but can become very large when the set of admissible types of input is large. The worst case nature of the reduced model, often stated as a great advantage of the method, can sometimes give too pessimistic an error bound, particularly if the input set is not suitably restricted.

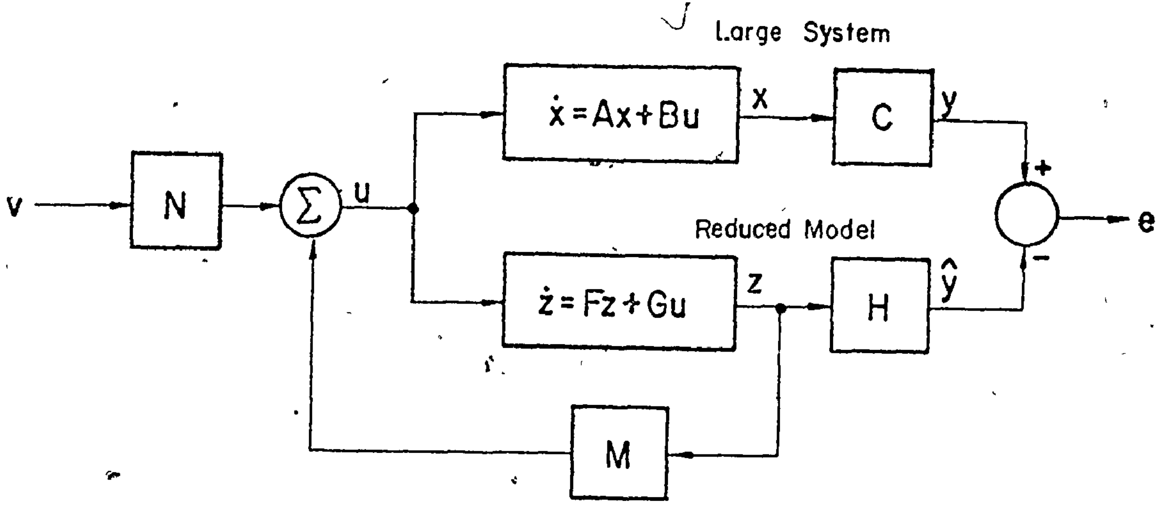
In Chapters 3 and 4, certain invariance properties of eigenvalue preserving and moment matching methods with respect to linear state variable feedback will be proven. These properties, not always shared by singularly perturbed models, and never shared by minimax models, give a new credibility to the earlier methods of model reduction. This is not surprising, as the latter methods of model reduction are not restricted to linear time-invariant dynamical systems. It is quite true that the more restrictive the statement of a problem is, the more specific are the results of the analysis.

CHAPTER 3

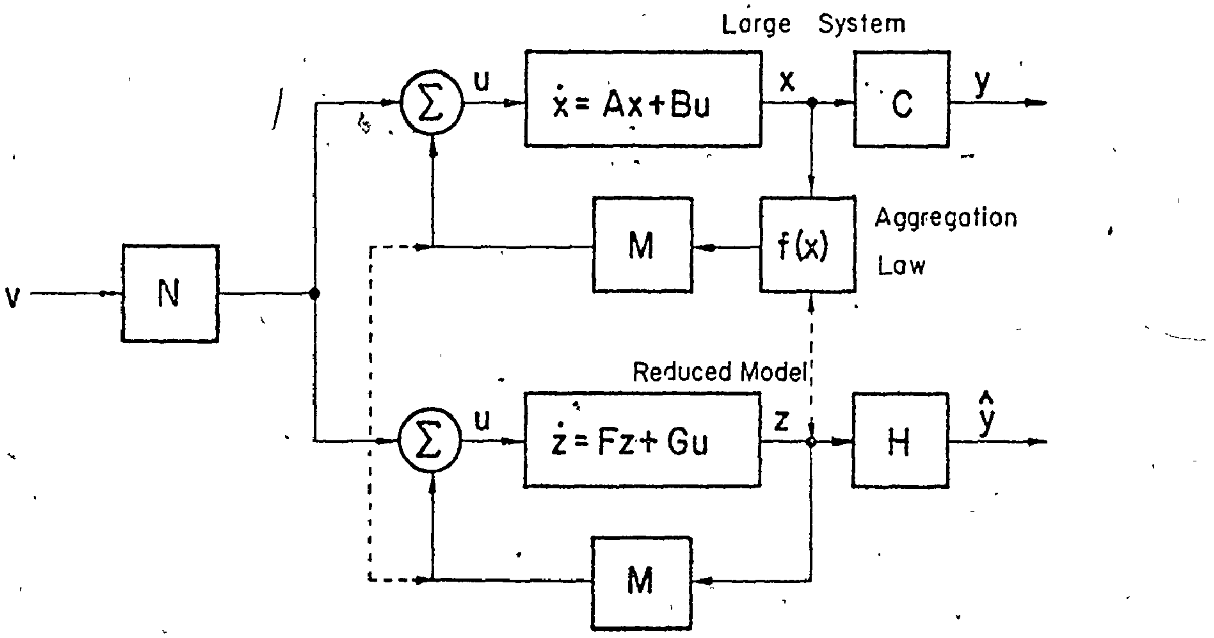
THE THEORY OF STATE AGGREGATION

3.1 Introduction

The early efforts at model reduction were primarily concerned with approximating the output of the large system for a certain specified input. The fact that no relation between the large-scale and reduced-order systems existed made the theories unsuitable for the design of static control laws based on the reduced model, as the resulting system was bound to be of an open loop nature (Figure 3.1a). The introduction of non-zero initial conditions was also impossible for the same reason. To construct a state variable feedback law for the large system based on such a law for the reduced model obviously requires a relationship z=f(x) between the state vectors (Figure 3.1b). The case of a linear relationship, z=Kx, is especially useful and was introduced to control engineers by Aoki (1968). It is of interest to determine if the large system and its model form an aggregated pair after the introduction of linear state variable feedback laws. Although Aoki did not show this to be true, therein lies the success of the method. Rao and Lamba (1975) proved this for the case of Davison's reduction method and the result was extended to aggregated reduced-order models in general by Hickin and Sinha (1975a). The fact that Lamba and Rao (1972) proved Davison's model to be aggregated provided the impetus



a) Compensation by general reduced model is dynamic.



b) Compensation by aggregated reduced model is static by virtue of the aggregation law. Dashed lines depict equivalent signals.

Figure 3.1 Illustrating the use of reduced models for compensation of a large system.

for Hickin and Sinha (1975b) to prove that all eigenvalue preserving reduction methods were special cases of aggregation. Hickin and Sinha (1975c) and Michailesco, et al. (1975) then gave general formulae for determining the aggregation matrix and the reduced model. The invariance property of aggregated models under linear state variable feedback is the foundation of the utility of the method and a justification of the rather large amount of computation involved (in the form of an eigenvalue/vector calculation) to reduce a system.

3.1.1 Aoki's Theory of Aggregation

It is supposed that the dynamical equations of the large-scale system are given as

$$\dot{x} = Ax + Bu \;, \quad x \in R^n,\; u \in R^p \tag{3.1a}$$

$$y = Cx \;, \quad y \in R^q \tag{3.1b}$$

and that it is wished to aggregate the state vector x into a state vector z of lower dimension, i.e., z should be a linearly related to x as

$$z = Kx \;, \quad z \in R^r,\; \rho(K) = r < n \tag{3.2}$$

K is termed the aggregation matrix. If z is to be useful, it should also satisfy a differential equation such as

$$\dot{z} = Fz + Gu \;, \quad z \in R^r,\; u \in R^p \tag{3.3a}$$

$$\hat{y} = Hz \;, \quad y \in R^q \tag{3.3b}$$

Substituting (3.2) and (3.3) into (3.1) give the following equations to be satisfied by K

$$FK = KA \tag{3.4a}$$

$$G = KB \tag{3.4b}$$

$$HK = C \tag{3.4c}$$

In general, (3.4c) is only approximately satisfied, but it is shown that (3.4c) can be exact if and only if there are some uncontrollable modes (Chen, 1970) in (3.1).

Assuming that K and A satisfy

$$KA = KAK^{+}K \tag{3.5}$$

where K^{+} denotes the Moore-Penrose pseudoinverse of K (Albert, 1972 and Boullion and Odell, 1969), Aoki derived F as

$$F = KAK^{+} \tag{3.6}$$

The output equation (3.4c) was not explicitly considered, but one choice for the H matrix could have been (in the spirit of (3.6))

$$H = CK^{+} \tag{3.7}$$

Under the condition (3.5) it can be determined that $|sI_r-F|$ divides $|sI_n-A|$. Hence the eigenvlaue preservation method is a special case of aggregation. It is shown in the next section that any aggregated model must preserve eigenvalues. Aggregated models are seen to be useful by virtue of the aggregation law (3.2). Since Aoki gave no explicit formula for the aggregation matrix K, the utility of the method was questionable.

3.2 A General Formula for the Aggregation Matrix

A subtle change in viewing (3.4) provides a general formula for the aggregation matrix. Instead of treating F as unknown, we address ourselves to the following problem. Given two square matrices A and F, under what conditions does (3.4a) have a nontrivial solution for K? The answer, as suspected, is given in the following theorem.

Theorem 3.1. Let the Jordan decomposition of A as a direct sum of Jordan blocks be

$$A \sim \sum_{i=1}^{\alpha}{}_{d} \sum_{j=1}^{\beta(i)}{}_{d} N_{n(ij)}(\lambda_i)$$

where $\{\lambda_i | i \epsilon \underline{\alpha}\}^{\dagger}$ denotes the set of distinct eigenvalues of A, $N_k(\lambda)$ is the Jordan block of dimension k and eigenvalue λ, and $\Sigma n(ij) = n$. Then a full rank aggregation law $z = Kx$ exists if and only if the decomposition of F is given by

$$F \sim \sum_{i=1}^{\hat{\alpha}}{}_{d} \sum_{j=1}^{\hat{\beta}(i)}{}_{d} N_{n(ij)}(\lambda_i)$$

where $\hat{\alpha} \leq \alpha$, $\hat{\beta}(i) \leq \beta(i)$ for $i \epsilon \underline{\hat{\alpha}}$. Then the rows of K may be selected from the rows of V^{-1}, where V is the modal matrix of generalized eigenvectors of A.

Proof The proof is a direct generalization of that of Gantmacher (1959) for the solution of the equation $AX = XB$ where A and B are the same size. Letting the modal matrices of A and F be V and W respectively we have

---

† For any positive integer α, $\underline{\alpha} = \{1, 2, \ldots, \alpha\}$

$$\tilde{A} = V^{-1} A V = \sum_{i=1}^{\alpha}{}^{d} \sum_{j=1}^{\beta(i)}{}^{d} N_{n(ij)}(\lambda_i) \tag{3.8a}$$

$$\tilde{F} = \sum_{i=1}^{\hat{\alpha}}{}^{d} \sum_{j=1}^{\hat{\beta}(i)}{}^{d} N_{\hat{n}(ij)}(\hat{\lambda}_i) \tag{3.8b}$$

where $\hat{\alpha} \leq \alpha$, $\hat{\beta}(i) \leq \beta(i)$ for $i \in \underline{\alpha}$, $\Sigma\, \hat{n}(ij) = r$, and $\sigma(\tilde{F}) = \sigma(F) = \{\hat{\lambda}_i | e \in \underline{\hat{\alpha}}\}$. The equation FK = KA now becomes

$$\tilde{F}\tilde{K} = \tilde{K}\tilde{A} \tag{3.9a}$$

$$\tilde{K} = W^{-1}KV \tag{3.9b}$$

It is seen that for a nontrivial solution $\tilde{K}$ to exist we must have $\hat{\lambda}_i = \lambda_i$ for at least one $\hat{\lambda}_i \in \sigma(F)$. Since it is wished that $\tilde{K}$ have full rank, we must have $\hat{\lambda}_i = \lambda_i$ for each $i \in \underline{\hat{\alpha}}$. There are then many solutions for K, one being $K = [I_r \mid 0]$ where

$$r = \sum_{i=1}^{\hat{\alpha}} \sum_{j=1}^{\hat{\beta}(i)} n(ij)$$

is the order of $\tilde{F}$.

<u>Corollary 3.1</u>. Let the state equations of a large scale plant be given by (3.1). Then if V is the modal matrix of A the following system constitutes an aggregated reduced-order model of (3.1) retaining the eigenvalues $\{\lambda_i\}$, $1 \leq i \leq r$.

$$K = [T \quad 0]V^{-1},\ |T| \neq 0 \text{ (aggregation matrix)} \tag{3.10a}$$

$$F = KAK^{n} \tag{3.10b}$$

$$G = KB \tag{3.10c}$$

$$H = CK^n \tag{3.10d}$$

where K^n is any right inverse of K, i.e., $KK^n = I_r$. The notation K^n is suggested by Boullion and Odell (1969). T is any nonsingular r x r matrix. If $T = I_r$, then F is diagonal. To avoid handling complex quantities, the following algorithm is useful for determing K.

(1) Perform eigenanalysis on A^T, i.e. find W such that $A^T W = W \text{diag} (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Then $V^{-1} = W^T$ and the rows of V^{-1} are the columns of W (Porter and Crossley, 1972).

(2) Suppose λ_i is real and $\lambda_j = \lambda_{j+1}$ is complex for some i,j. Then the i^{th} row of K is taken as the i^{th} column of W and the j^{th} and $j+1^{st}$ rows of K are taken as the real and imaginary parts of the j^{th} column of W. Also $f_{ii} = \lambda_i$ and $f_{jj} = f_{j+1,j+1} = \text{Re} (\lambda_j)$, $f_{j,j+1} = 1$ $f_{j+1,j} = - \text{Im} (\lambda_j)$. It may be noted that among all right inverses K^n of K, the one most easily calculated is the pseudoinverse given by $K^+ = K^T (KK^T)^{-1}$ which involves roughly $r^2(4/3r+n)$ operations. A slightly different approach to determining the aggregation matrix was given by Michailesco, et al. (1975).

3.3 Invariance of Aggregated Models Under Linear State Variable Feedback

If the aggregated model is to be of any use, it must remain aggregated after the introduction of linear state variable feedback. Suppose that the large-scale system is given by (3.1) and an aggregated model matching the first r eigenvalues of A is given by (3.10). If the control law u = MKx + Nv is now substituted in (3.1) there follows

$$\dot{x} = (A + BMK)x + BNv = A_f x + BNv \tag{3.11}$$

From (3.4) it is now evident that

$$K\dot{x} = KA_f x + KBNv = (F + GM)Kx + GNv \tag{3.12}$$

Setting $u = Mz + Nv$ in (3.10) gives

$$\dot{z} = (F + GM)z + GNv = F_f z + GNv \tag{3.13}$$

It follows that $F_f K = KA_f$ and that the reduced model continues to be an aggregated model with the original matrix K as the aggregation matrix. This is expressed in the following theorem.

<u>Theorem 3.2</u>. The introduction of a linear state variable feedback law $u = Mz + Nv = MKx + Nv$ does not affect the aggregation matrix. Furthermore, the spectrum of $F_f = F + GM$ is entirely contained in the spectrum of $A_f = A + BKM$, and those eigenvalues of A excluded from F appear without change in A_f. Thus $A_f \backsim F_f \oplus \bar{A}$, where $\bar{A}$ denotes the invariant eigenvalues.

<u>Proof</u>. (3.11) to (3.13) prove the first assertion. That $\sigma(F_f) \subset \sigma(A_f)$ ($\sigma(\cdot)$ denotes the spectrum) follows from $F_f K = KA_f$ and Theorem 3.1. Now let $Av_i = \lambda_i v_i$ where $\lambda_i \notin \sigma(F)$. Then $KAv_i = (FK)v_i = F(Kv_i) = \lambda_i(Kv_i)$. But $\lambda_i \notin \sigma(F)$ so $Kv_i = 0$. Hence $A_f v_i = (A + BMK)v_i = Av_i = \lambda_i v_i$ so $\lambda_i \in \sigma(A_f)$.

Theorem 3.2 was proven for the case of Davison's reduced model by Rao and Lamba (1975) and was subsequently generalized to apply to all aggregated models by Hickin and Sinha (1975a). The above proof is new and considerably simpler in comparison with that of the latter reference.

3.4 Other Properties of Aggregated Models

The following theorem is particularly interesting.

Theorem 3.3. Let $|A| \neq 0$ and let $f(\lambda) = \Sigma\, \alpha_i \lambda^i$ converge uniformly on some annulus $D = \{(\rho,\theta): \rho \in [\rho_1,\rho_2]\}$ of the complex plane. Let $||\cdot||$ denote the spectral norm of a matrix (Naylor and Sell, 1971), and let $f(A)$ converge uniformly. Then

$$K\, f(A) = f(F)\, K \tag{3.14a}$$

$$f(F) = K\, f(A)\, K^n \tag{3.14b}$$

where K^n is any right inverse of K, and FK = KA.

Proof. Since FK = KA it is easily established by induction that

$$KA^i = F^i K \qquad , \; i = 0, \pm 1, \ldots \tag{3.15a}$$

$$F^i = F^i(KK^n) = (F^i K)K^n = KA^i K^n, \; i = 0, \pm 1, \ldots \tag{3.15b}$$

For any square matrix T, $||T|| = \max\{|\lambda| : \lambda \in \sigma(T)\}$. Hence, as $\sigma(F) \subset \sigma(A)$, $||F|| \leq ||A||$ and $||F^{-1}|| \geq ||A^{-1}||$. But f(A) converges uniformly so $||A|| \in [\rho_1,\rho_2]$, $||A^{-1}|| \in [\rho_1,\rho_2]$. Hence $||F||$, $||F^{-1}|| \in [\rho_1,\rho_2]$ and f(F) converges uniformly. With the aid of (3.15), equations (3.14) now follow at once.

Corollary 3.2. Let $\langle A|\beta\rangle$ be the controllable subspace of (A,B) where β = Im(B) (the image under the map B (Wohnam, 1975)). Then if (F,G) is an aggregated model with aggregation matrix K, and $\mathcal{G}$ = Im(G) then

$$K\, \langle A|\beta\rangle = \langle F|\mathcal{G}\rangle \tag{3.16}$$

<u>Proof</u> $K\langle A|\beta\rangle = K(\beta + A\beta + \ldots + A^{n-1}\beta) = (K\beta) + F(K\beta) + \ldots$

$$+ F^{n-1}(K\beta)$$

$$= \langle F|K\beta\rangle = \langle F|\mathcal{G}\rangle$$

A consequence of this is the fact that (F,G) is controllable if (A,B) is controllable. If (A,B) is uncontrollable, then state aggregation may be used to remove all uncontrollable modes. In this case, there exists a right inverse K^n of K such that $HK = CK^nK = C$, i.e., (3.4c) is then exact.

3.5 The Models of Davison and Mitra Revisited

In the previous section, it was shown that eigenvalue preservation reduction methods were cases of aggregation. The aggregation matrices for Davison's (1966) and Mitra's (1967, 1969) models will now be given. The notion used in this section conforms with that of Section 2.2.1. Since the equation $G = KB$ relates the input to state matrices of the reduced and large-scale systems, the aggregation matrix K is relatively easy to isolate. In a similar manner, the form of the right inverse K^n of K may be determined by inspection of the equation $H = CK^n$ relating the state to output matrices. A more complete collection of aggregation matrices for different reduction schemata may be found in Michailesco, et al. (1976).

3.5.1 Davison's Model

Inspection of (2.5) determines K and K^n to be given by

$$K = V_{11}\,[W_{11}\ W_{12}] \qquad (3.17a)$$

$$K^n = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} V_{11}^{-1} = \begin{bmatrix} I_r \\ V_{21}\ V_{11}^{-1} \end{bmatrix} \qquad (3.17b)$$

3.5.2 Mitra's Model

Examination of (2.7), (2.10) gives

$$K = [E_{11}^{-1} \quad 0]P \qquad (3.18a)$$

$$K^n = E \qquad (3.18b)$$

In the calculation of $F = KAK^n$, the result $PE = E$, which follows from the way in which E is chosen, is needed (c.f., Hickin and Sinha, 1975b).

3.6 Using Aggregated Models for Control System Design

Aoki (1968) proposed the use of aggregated models for the calculation of suboptimal controls. Given the state equations (3.1) and (3.3) and a quadratic performance index

$$J = \int_0^\tau (x^TQx + u^TR\ u)\ dt,\ Q \geq 0,\ R > 0 \qquad (3.19)$$

the minimization of J yields a feedback solution in the form of a Ricatti equation (Kalman, 1960).

$$u = -R^{-1}\ B^T\ P(t)x,\ P(t) > 0 \qquad (3.20a)$$

$$-\dot{P} = A^T\ P + PA - PBR^{-1}\ B^TP + Q,\ P(\tau) = 0 \qquad (3.20b)$$

In order to reduce the computational effort at a sacrifice of optimality, the problem was reposed in terms of the reduced model as

$$\text{minimize} \quad \hat{J} = \int_{0}^{\tau} (z^T\hat{Q}z + u^TRu)dt \tag{3.21a}$$

$$\hat{Q} = (K^+)^T \; Q \; K^+ \tag{3.21b}$$

whose solution was

$$u = -R^{-1}G^T \; \hat{P}(t) \; z \tag{3.22a}$$

$$-\dot{\hat{P}} = F^T \; \hat{P} + \hat{P}F - \hat{P}GR^{-1} \; G^T \; \hat{P} + \hat{Q}, \; \hat{P}(\tau) = 0 \tag{3.22b}$$

Aoki considered the special case $\tau \to \infty$ using the steady state solutions to the Ricatti equations to implement the control laws. Although some success was reported with this method (Lamba and Rao, 1972), poor performance could result if the technique was applied to the optimal infinite time tracking problem, where the signal to be tracked had a nonzero steady state. In Chapter 5 this problem is overcome by suitable choice of the matrix K^n used to specify $H = CK^n$. Aoki's choice, the pseudoinverse K^+, almost never gives a satisfactory result.

Another type of control problem is that of eigenvalue placement. Such a problem is ideally suited to aggregated models by virtue of Theorem 3.2, which guarantees that a control law $u = Lz$ to shift the eigenvalues of the aggregated model to specified locations may be implemented on the large-scale system as $u = LKx$.

3.6.1 Sensitivity Considerations

It is evident that the concept of aggregation is similar to the modal control approach of Porter and Crossley (1972). Thus their sensitivity methods may be used to advantage. Recently Zein El-Din, et al. (1977) have extended the results of Porter and Crossley and these are used. It should be noted that the results are applicable only to the case of distinct eigenvalues in the A matrix.

Let the modal matrix of A and its inverse be taken as in Section 2.2, and let A be a function of the parameters ξ, η. Then set

$$p_{ij} = (\lambda_j-\lambda_i)^{-1} w_i^T \frac{\partial A}{\partial \xi} v_j , \; i \neq j \tag{3.23a}$$

$$q_{ij} = (\lambda_j-\lambda_i)^{-1} w_i^T \frac{\partial A}{\partial \eta} v_j , \; i \neq j \tag{3.23b}$$

The scalars p_{ii} and q_{ii} are arbitrary but it is advantageous to take them as zero. The first order eigenvalue and eigenvector sensitivities are then given by

$$\frac{\partial \lambda_i}{\partial \xi} = w_i^T \frac{\partial A}{\partial \xi} v_j \tag{3.24a}$$

$$\frac{\partial v_i}{\partial \xi} = \sum_{j=1}^{n} v_j \, p_{ji} \quad i = 1, 2, \ldots, n \tag{3.24b}$$

The last equation may now be written as

$$\frac{\partial V}{\partial \xi} = VP \tag{3.25a}$$

$$P = [p_{ij}], \; p_{ii} = 0 \tag{3.25b}$$

There then follows

$$\frac{\partial V^{-1}}{\partial \xi} = - V^{-1} \frac{\partial V}{\partial \xi} V^{-1} = -PV^{-1} \qquad (3.26)$$

The second order eigenvalue sensitivities are given by

$$\frac{\partial^2 \lambda_i}{\partial \xi \partial \eta} = w_i^T \{\frac{\partial A}{\partial \xi} Vq_i + \frac{\partial A}{\partial \eta} Vp_i + \frac{\partial^2 A}{\partial \xi \partial \eta} v_i\} \qquad (3.27)$$

where $Q = [q_{ij}]$ and p_i, q_i are the i^{th} columns of P, Q, respectively.

The sensitivities of an aggregated model (F,G,H) of (A,B,C) are now easily written. The aggregation matrix K and any right inverse K^n are

$$K = [T \;\; 0]V^{-1} \qquad (3.28a)$$

$$K^n = V \left[\begin{matrix} T^{-1} \\ S \end{matrix}\right] \qquad (3.28b)$$

where S is an arbitrary constant matrix. Hence

$$\frac{\partial F}{\partial \xi} = [T \;\; 0] \frac{\partial}{\partial \xi} (V^{-1}AV)\left[\begin{matrix} T^{-1} \\ S \end{matrix}\right] = T \operatorname{diag} (\frac{\partial \lambda_1}{\partial \xi}, \frac{\partial \lambda_2}{\partial \xi}, \ldots, \frac{\partial \lambda_r}{\partial \xi})T^{-1} \qquad (3.29a)$$

$$\frac{\partial^2 F}{\partial \xi \partial \eta} = T \operatorname{diag} (\frac{\partial^2 \lambda_1}{\partial \xi \partial \eta}, \frac{\partial^2 \lambda_2}{\partial \xi \partial \eta}, \ldots, \frac{\partial^2 \lambda_r}{\partial \xi \partial \eta}) \qquad (3.29b)$$

$$\frac{\partial G}{\partial \xi} = [T \;\; 0] \frac{\partial}{\partial \xi} (V^{-1}B) = [T \;\; 0] \{V^{-1} \frac{\partial B}{\partial \xi} - PV^{-1}B\} \qquad (3.29c)$$

$$\frac{\partial H}{\partial \xi} = \{\frac{\partial C}{\partial \xi} V + CVP\} \left[\begin{matrix} T^{-1} \\ S \end{matrix}\right] \qquad (3.29d)$$

3.7 A Simple Example

It is required to derive a second order aggregated model for the following third order system:

$$\dot{x} = \begin{bmatrix} 0 & 0 & -4 \\ 1 & 0 & -6 \\ 0 & 1 & -4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u = Ax + Bu$$

$$y = [0 \quad 1 \quad 3]\, x = c^T x$$

The eigenvalues are the roots of $s^3 + 4s^2 + 6s + 4 = 0$ or $s = -2, -1 \pm j$. The only second-order aggregated model allowed is that which retains the conjugate roots. The modal matrix of A^T is given by (Chen, 1970)

$$W = \begin{bmatrix} 1 & 1 & 1 \\ -1+j & -1-j & -2 \\ (-1+j)^2 & (-1-j)^2 & (-2)^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1+j & -1-j & -2 \\ -j2 & +j2 & 4 \end{bmatrix}$$

and the aggregation matrix K is then given by

$$K = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

F is written down by inspection as

$$F = \begin{bmatrix} \mathrm{Re}(-1+j) & -\mathrm{Im}(-1+j) \\ -\mathrm{Im}(-1+j) & \mathrm{Re}(-1+j) \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

and g follows from

$$G = Kb = [1 \quad 0]^T$$

It is easily verified that

$$FK = \begin{bmatrix} -1 & 0 & 2 \\ 0 & -2 & 2 \end{bmatrix} = KA$$

Calculation of the output matrix h^T may be done in a number of ways. Among all possible matrices $h^T = c^T K^n$, that given by $h^T = c^T K^+$ will be the least squares solution, as the equation $h^T K = c^T$ is overdetermined (Albert, 1972). With this choice

$$h^T = c^T K^+ = [2/9 \quad 13/9]$$

Another choice for h^T is that which minimizes

$$I(u,h^T) = \int_0^\infty e^2\,(u(t),t)\,dt$$

where u(t) is taken as a step input, and $e(t) = c^T x(t) - h^T z(t)$. The analysis gives

$$h^T_{opt} = [1/5 \quad 3/10].$$

The control law $u = -[2 \; 0]z = -\ell^T z$ gives

$$\dot{z} = \begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix} z = F_f z, \quad |sI_2 - F_f| = (s+2)^2$$

setting $u = -\ell^T Kx = [-2 \quad 2 \quad 0]x$ gives

$$\dot{x} = \begin{pmatrix} -2 & 2 & -4 \\ 1 & 0 & -6 \\ 0 & 1 & -4 \end{pmatrix} x = A_f x, \quad |sI_3 - A_f| = (s+2)^3$$

As expected

$$F_f K = \begin{pmatrix} -3 & 2 & 2 \\ 1 & -2 & 2 \end{pmatrix} = KA_f$$

If the 3,3 element of the A matrix is perturbed by -0.2 the sensitivity formulae give

$$\lambda_1 \approx -0.9 + j\,0.9$$

$$\lambda_2 \approx -0.9 - j\,0.9$$

$$\lambda_3 \approx -2.4$$

and

$$K' \approx \begin{pmatrix} 0.95 & -1.1 & 0.3 \\ -0.15 & 0.8 & -1.7 \end{pmatrix}$$

$$F \approx \begin{pmatrix} -0.9 & -0.9 \\ 0.9 & -0.9 \end{pmatrix} \quad g \approx \begin{pmatrix} 0.95 \\ -0.15 \end{pmatrix}$$

The acutal values to two significant figures are

$$K' = \begin{pmatrix} 1 & -0.91 & -0.02 \\ 0 & 0.92 & -1.67 \end{pmatrix}$$

$$F = \begin{pmatrix} -0.91 & -0.92 \\ 0.92 & -0.91 \end{pmatrix} \quad g = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$\lambda_1 = -0.91 + j0.92 \qquad \lambda_2 = -0.91 - j0.92 \qquad \lambda_3 = -2.39$$

3.8 Conclusions

The concept of model reduction by state aggregation as introduced by Aoki was discussed. His work, however, gave no help in the determination of the aggregation matrix K. Instead, he considered K as known and the reduced-order model was calculated via equations (3.4b) and (3.6). This difficulty was removed in Section 3.2 by temporarily considering the reduced model to be known, and then deriving conditions under which the resulting solution for the aggregation matrix was nontrivial. Theorem 3.1 shows that all eigenvalue preservation methods are cases of aggregation and vice versa and thus provides a unifying framework for such reduction techniques. This is to be contrasted with earlier approaches wherein the matching of eigenvalues was assumed a priori (Michailesco, et al., 1975). Theorem 3.1 and its corollary also specified the calculation of the aggregation matrix in terms of the inverse modal matrix of A. This formula has also been independently discovered by Michailesco, et al. (1975). An algorithm for avoiding complex matrices (which occur when retaining complex eigenvalues) has also been given. Theorem 3.2 of Section 3.3, which asserts the invariance of the aggregation matrix under linear state variable feedback, is probably the most important property of aggregated models. This theorem was first proven for the special case of Davison's model (Rao and Lamba, 1975) and was subsequently generalized to apply to all aggregated models by Hickin and Sinha (1975a). The proof given here is new and more easily followed than that of the latter reference. In Section 3.4 other mathematical properties of aggregated models were developed. Theorem 3.3 states general conditions under which the equation FK = KA may be generalized to f(F)K = Kf(A) and its corollary deals with the controllable subspace of the pair (A,B). Section 3.5 gave the aggregation matrices for the reduced models of Davison and Mitra as promised in Chapter 2. Sensitivity considerations were investigated in Section 3.6 and an illustrative example was given. Sensitivity calculations are useful in the selection of eigenvalues to be retained in the reduced model, e.g., by retaining the eigenvalues having low sensitivities, control laws may be derived which are relatively insensitive to parameter variation.

CHAPTER 4

PARTIAL AND AGGREGATED PARTIAL REALIZATIONS OF LINEAR DYNAMIC SYSTEMS

4.1 Introduction

The goal of this chapter is to combine the method of aggregation, discussed in Chapter 3, with that of moments matching, or Padé approximation, discussed in Chapter 2, in such a way as to secure the separate advantages of each technique while simultaneously removing their disadvantages. In fact, a generalization of moments matching, called partial realization, is considered. The existance of reduced models which are both aggregated and partial realizations of a large scale system represents a unification of the theory of model reduction, since a good number of existing methods may be classified as special cases of what is termed nonminimal partial realization. This is depicted in Figure 4.1 where an arrow → connecting two boxes represents proper inclusion and an arrow ↔ represents a non-null intersection. Thus, the mixed method (Chuang, 1970), Routh approximation (Hutton, 1975), aggregation (Aoki, 1968), moments (Chen and Shieh, 1968), Padé approximation (Shamash, 1973a), and singular perturbations (Koktović, 1972), may all be viewed as special cases.

Methods of partial realization of linear dynamic systems are closely allied with the minimal realization problem via the Hankel matrix of Markov parameters. Accordingly, Section 4.2 presents a

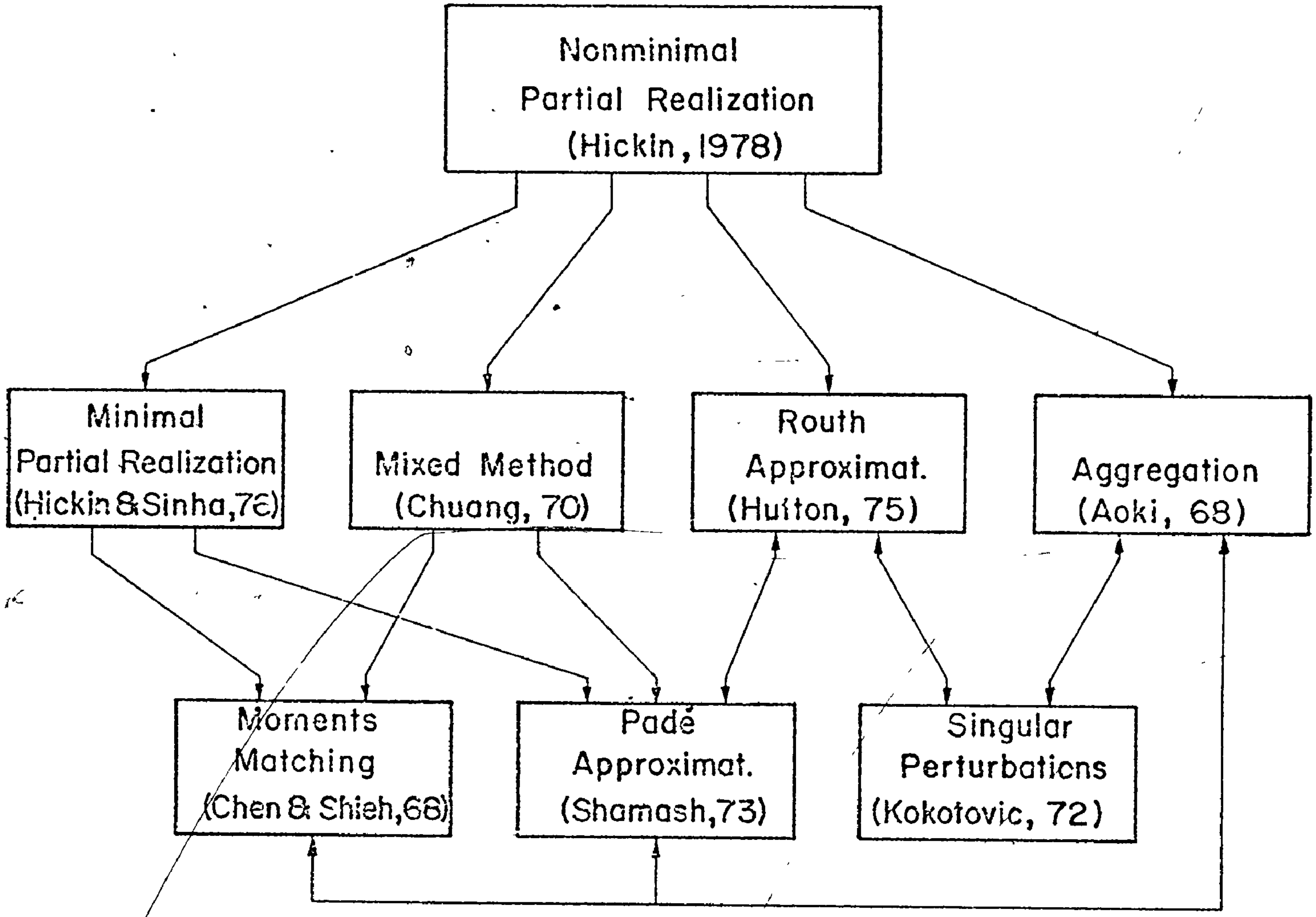


Figure 4.1 Relationships between nonminimal partial realization and other reduction methods

detailed discussion of this topic and its generalization (Hickin and Sinha, 1976a). Connections between the theory of Padé approximation and partial realization are explored in Section 4.3. Following this, the idea of nonminimal partial realization is introduced in Section 4.4. A central result describes the properties of such realizations under the application of linear state variable feedback laws. The main result of the thesis is given in Section 4.5, where an algorithm for generating aggregated partial realizations is presented. The final link of Figure 4.1 is completed in Section 4.6 where aggregation and singular perturbations are compared.

4.2 The Minimal Realization Problem

Given a pxq strictly proper transfer function matrix G(s), it is desirable to determine a state variable description (A, B, C), of minimal order n, which is zero state equivalent to G(s) (Chen, 1970). This problem, called minimal realization, has received much attention in the literature (Ho and Kalman, 1965, Tether, 1970, Rissanen, 1971, Silverman, 1971, Ackerman and Bucy, 1971, Budin, 1971, Chen and Mital, 1972, Mital and Chen, 1973, Dickenson, et al., 1974, Rissanen, 1974, Rosenbrock, 1970, Rózsa and Sinha, 1974, 1975, Hickin and Sinha, 1976a). Many algorithms obtain a minimal realization from the Hankel matrix of Markov parameters (Kalman, 1965). The most efficient methods of minimal realization perform a basis factorization of the Hankel matrix and extract a realization in a Luenberger canonical form (Silverman, 1971, Rissanen, 1974, Rózsa and Sinha, 1974, 1975). An extension of the Rózsa-Sinha algorithm to more general Hankel matrices was given in

Hickin and Sinha (1976a). Before this algorithm is presented, a discussion of reducing a matrix to Hermite normal form (MacDuffee, 1943) is given.

4.2.1 Reduction of a Matrix to Hermite Normal Form

A real general matrix is said to be in the Hermite normal form (MacDuffee, 1943) if any column is either a unit vector, or linearly dependent on unit vectors among the columns to its left. Evidently the rank of the matrix is given by the number of (distinct) unit vectors. For example, the following 5 x 7 rank 4 matrix is in Hermite normal form

$$X = \begin{pmatrix} 1 & 0 & x & 0 & x & 0 & x \\ 0 & 1 & x & 0 & x & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 1 & x \end{pmatrix}$$

where the x's denote elements of X having arbitrary values. The utility of such a matrix form is that every real general matrix may be brought into Hermite normal form by a nonsingular transformation.

<u>Theorem 4.1</u> (MacDuffee, 1943). Every real general matrix X has a Hermite normal form X_H, obtainable from X by elementary row and column operations. The following algorithm may be used to transform X to its Hermite normal form.

Theorem 4.2 (Rózsa and Sinha, 1974). Let X be a real m x n matrix. Then X can be brought into Hermite normal form in at most $\rho(X)+1 \leq \min(m,n)+1$ steps.

Proof Let x_{*i} and x_{j*} denote the i^{th} column and j^{th} row of X respectively. Inductively suppose that k distinct unit vectors $e_{\sigma\underline{k}}$ have already been generated,[†] and that column j of X (where $j \geq i$, of course), is next to be examined. Let $x_{\ell * j}$ denote any nonzero element of x_{*j} for which $\ell \notin \sigma_{\underline{k}}$ (it is computationally advantageous to select the largest magnitude such element). If no such element exists, x_{*j} is linearly dependent on the $x_{*\underline{j-1}}$, and proceed to examine x_{*j+1} in a similar manner. If, however, such an element exists, perform the calculation $X \leftarrow X - x_{\ell * j}^{-1} (x_{*j} - e_{\ell}) x_{\ell *}$. It is easily verified that the jth column of X is now e_{ℓ}, that the previous j-1 columns are unaffected, and that the transformation is nonsingular (having a determinant of unity). Thus after k+1 steps, there are k+1 distinct unit column vectors in X. This procedure now continues until no suitable pivot element can be found, which in view of Theorem 4.1, takes $\rho(X)$ steps. The final step is rearranging the rows so that the unit column vectors appear in their natural order. For minimal realization purposes, however, this step is not needed.

[†] Following Wohnam (1974), $\underline{k} = \{1, 2, \ldots, k\}$ if $k>0$, $\underline{0}=\emptyset$. $\sigma_{\underline{k}} = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$.

Example

$$X = \begin{pmatrix} 1 & -2 & 5 & -8 \\ -2 & 4 & -8 & 16 \end{pmatrix}$$

$$X^{(1)} = X - \frac{1}{1}\begin{pmatrix} 1-1 \\ -2 \end{pmatrix}[1 \quad -2 \quad 5 \quad -8] = \begin{pmatrix} 1 & -2 & 5 & -8 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

$$X^{(2)} = X^{(1)} - \frac{1}{2}\begin{pmatrix} 5 \\ 2-1 \end{pmatrix}[0 \quad 0 \quad 2 \quad 0] = \begin{pmatrix} 1 & -2 & 0 & -8 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Applications of the algorithm include calculation of the rank of a matrix and matrix inversion. If $|X| \neq 0$ for some square matrix X, then $X^{-1}Y$ may be found by reducing the compound matrix [X Y] to Hermite normal form and rearranging the rows of the reduced matrix so that the unit vectors appear in their natural order. If $|X| = 0$, then reduction of [X Y] also gives $X^{-1}Y$ where X^{-1} is now interpreted as the inverse image† of the map $X:R^n \rightarrow R^m$. This idea may be used to calculate maximal (A, B) invariant subspaces (Wohnam, 1974). The use of the Hermite normal form in minimal canonical realization of linear time invariant dynamic systems is now given.

4.2.2 Minimal Realization of Transfer Function Matrices in Canonical Forms

Consider an n^{th} order, p input, q output irreducible system

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† Let $f:A \rightarrow B$. Then $f^{-1}(X) = \{a \in A \mid f(a) \in X \cap B\}$ is the inverse image of X under f.

$$\dot{x} = Ax + Bu \qquad , x \in R^n \tag{4.1a}$$

$$y = Cx \tag{4.1b}$$

It is well known (Chen, 1970) that the transfer function matrix associated with (4.1) is given by

$$G(s) = C(sI_n - A)^{-1}B \tag{4.2}$$

If G(s) is formally expanded in a Taylor or Laurent series then

$$G(s) = -J_{-1} - J_{-2}s - J_{-3}s^2 - \dots \tag{4.3a}$$

$$G(s) = J_0s^{-1} + J_1s^{-2} + J_2s^{-3} + \dots \tag{4.3b}$$

where

$$J_i = CA^iB, \quad i \in Z \tag{4.4}$$

It may be noted that (4.3a) is possible if and only if G(s) has no poles at the origin and hence A^{-1} exists. The J_i for arbitrary i will be called generalized Markov parameters. When $i \geq 0$, the J_i will be termed Markov parameters, while if $i < 0$, they will be called time moments (this is a slight abuse of the strict definition of a time moment). A generalized Hankel matrix H_{ijk}, of order (i,j) and index k is written as

$$H_{ijk} = \begin{bmatrix} J_k & J_{k+1} & \cdots & J_{k+j-1} \\ J_{k+1} & J_{k+2} & \cdots & J_{k+j} \\ \vdots & \vdots & & \vdots \\ J_{k+i-1} & J_{k+i} & \cdots & J_{k+i+j-2} \end{bmatrix} \tag{4.5}$$

Because of (4.4), (4.5) may be factored as a block outer product

$$H_{ijk} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{i-1} \end{bmatrix} A^k[B \quad AB \quad A^2B \quad \dots \quad A^{j-1}B] = V_i \, A^k \, U_j \qquad (4.6)$$

V_n and U_n will be the observability and controllability matrices (Chen, 1970) of (4.1) respectively. Given G(s), the J_i may be readily calculated by synthetic division. Equation (4.6) may be used to derive a minimal canonical realization of G(s) as follows. Take $i \geq \alpha$, $j \geq \beta$ and $k \leq 0$, where α and β are the observability and controllability indices (Chen, 1970) of the system. Of course, $k < 0$ can be chosen if and only if G(s) has no poles at the origin. Construct H_{ijk} and reduce it to Hermite normal form. There will be n unit vectors, where $n = \rho(V_i) = \rho(U_j)$ (Kalman, 1965). Deletion of all zero rows now leaves U_j, from which B and A may be easily identified because of the unit vectors. C may then be derived from H_{ijk}, starting at row $-k+1$.

Example

$g(s) = (s+3)/(s+1)(s+2) = s^{-1}+0s^{-2}+(-2)s^3+ \dots = -(-3/2) - 7/2\ s - \dots$

Then

$$H_{23(-1)} = \begin{bmatrix} -3/2 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$H^{(1)} = H - \begin{bmatrix} -3/2 \\ 0 \end{bmatrix} [1 \quad 0 \quad -2] = \begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix}$$

$$H^{(2)} = H^{(1)} - \begin{pmatrix}0\\0\end{pmatrix}[0 \quad 1 \quad -3] = H^{(1)} = [e_2 \quad e_1 \quad \alpha]$$

As there are no zero rows, $H^{(2)} = U_3 = [b \quad Ab \quad A^2b]$ so that

$$b = \begin{pmatrix}0\\1\end{pmatrix} \qquad A = \begin{pmatrix}-3 & 1\\-2 & 0\end{pmatrix}$$

The output vector c^T is now taken from row 2, columns 2 and 1 of $H_{23(-1)}$, viz.,

$$c^T = [0 \quad 1]$$

In the multivariable case, the realization so obtained is called the input identifiable form. The generic case for a p input, q output system would be a realization of the form $B = [e_1 \quad e_2 \quad \cdots \quad e_p]$, $A = [e_{p+1} \quad e_{p+2} \quad \cdots \quad e_n \quad \alpha_1 \quad \alpha_2 \cdots \alpha_p]$, $C = [J_0 \quad J_1 \cdots J_{\sigma-1} \quad \bar{J}_\sigma]$, where $\sigma = \underline{|n/p}^{\dagger}$ and $\bar{J}_\sigma$ consists of the first $n-p\sigma$ columns of J_σ. The α_i would be obtained from columns $n+i$ of the Hermite normal form of H (after deletion of all zero rows). Some thought will show that the input identifiable form (including all non-generic cases, where the first n columns of H are not all linearly independent) is easily transformed into the controllable canonical form. The details may be found in Hickin and Sinha (1977b).

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† $\underline{|x}$ denotes the integer part of x.

Example

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} -2 & 0 & 1 \\ -3 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \qquad C = [4 \quad 5 \quad 6]$$

The Knonnecker invariants are obviously $n_1 = 1$, $n_2 = 2$. Take

$$X = [q_1^T \quad q_2^T \quad A^T q_2^T \quad A^T q_1^T \quad (A^T)^2 q_2^T \quad C^T] = \begin{bmatrix} 1 & 0 & 0 & -2 & -3 & 4 \\ 0 & 0 & 1 & 0 & 3 & 5 \\ 0 & 1 & 3 & 1 & 11 & 6 \end{bmatrix}$$

Reduction to Hermite normal form takes one step

$$X^{(1)} = X - \frac{1}{1} \begin{pmatrix} 0 \\ 1-1 \\ 3 \end{pmatrix} [0 \quad 0 \quad 1 \quad 0 \quad 3 \quad 5] = \begin{bmatrix} 1 & 0 & 0 & -2 & -3 & 4 \\ 0 & 0 & 1 & 0 & 3 & 5 \\ 0 & 1 & 0 & 1 & 2 & -9 \end{bmatrix}$$

Hence the controllable canonical form (A, B, C) is given by

$$\bar{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \bar{A} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & 2 & 3 \end{bmatrix}, \qquad \bar{C} = [4 \quad -9 \quad 5]$$

If the elements of the Hankel matrix are rearranged so that one block occurs for each input/output pair, the Hermite normal form algorithm may be used to realize a system in the column companion form (Rózsa and Sinha, 1975). This canonical form, and its generalization, are useful in model reduction (c.f., Section 4.5.4). An obvious corollary to realization of transfer function matrices in canonical forms is the transformation of state equations to canonical forms. This can be efficiently done by the Hermite normal form algorithm (c.f., Hickin and Sinha, 1977b).

4.2.3 Partial Realization of Transfer Function Matrices in Canonical Forms

A partial realization of a q x p strictly proper transfer function matrix G(s) is a triple (F, G, H) such that

$$HF^{i}G = CA^{i}B \quad , \ i = k,k+1,\dots,k+\sigma-1, \ k\leq 0 \qquad (4.7a)$$

$$HF^{i}G \neq CA^{i}B \quad , \ i = k+\sigma,k+\sigma+1, \ \dots \qquad (4.7b)$$

where (A, B, C) is any n^{th} order minimal realization of G(s) and $\sigma<\alpha+\beta$ where α and β are the controllability and observability indices of (A, B, C). The restriction (4.7b) insures that (F, G, H) is not equivalent to (A, B, C).

The minimal realization algorithm of Section 4.2.2 may be used to obtain partial realizations of a system G(s). The procedure is to construct a generalized Hankel matrix H_{ijk} and proceed with its reduction to Hermite normal form. After $r<n=\rho(H_{ijk})$ steps, r unit vectors have been generated, and an r^{th} order canonical partial realization (F, G, H) may be identified in the same manner as that of Section 4.2.2. It is obvious that (F, G) is controllable. (F, H) is genericaly observable, that is to say it is almost certainly observable. Hence, such systems are termed minimal partial realizations. A minimal partial realization matches a given number of generalized Markov parameters with a minimum number of states. In contrast, the term nonminimal partial realization refers to a partial realization matching a given number of generalized Markov parameters with more than the minimum number of state variables. Such realizations are generically irreducible (that is, minimal in the sense of Kalman), so that care must

be observed in interpreting the terms minimal and nonminimal as applied to partial realizations.

4.3 Partial Realization and Padé Approximation

It is evident that partial realization of strictly proper linear systems is closely allied to the problem of Padé approximation. It is shown that partial realization is a generalization of Padé approximation when the approximating functions to be derived are constrained to be strictly proper rational functions. An iterative algorithm for solving the Padé approximation problem under these conditions is given.

4.3.1 The Padé Approximation Problem

In his thesis, H. Padé (1892) investigated a method of approximating analytic functions by rational polynomial functions. Given a scalar function g(s) which is analytic at $s=\alpha$, a rational polynomial in $(s-\alpha)$ is sought, the Taylor series of which agrees with that of g(s) for the first few terms. Specifically, let

$$g(s) = c_0 + c_1(s-\alpha) + c_2(s-\alpha)^2 + \dots \qquad (4.8)$$

An [m,n] Padé approximant to g(s) about $s=\alpha$ is a rational function

$$\frac{p_m(s)}{q_n(s)} = \frac{a_0+a_1(s-\alpha)+\dots+a_m(s-\alpha)^m}{b_0+b_1(s-\alpha)+\dots+b_n(s-\alpha)^n} \qquad (4.9)$$

whose Taylor series agrees with (4.8) up to and including the term s^{m+n}. For control theoretic purposes, it is assumed that $m<n$, and that either

b_o or b_n may be taken as unity. Cross multiplication and equating of like powers in $(s-\alpha)$ gives the following m+n+1 equations

$$a_i = \sum_{k=0}^{i} b_k c_{m-k} \quad , \; i = 0, 1, \ldots, m \tag{4.10a}$$

$$0 = \sum_{k=0}^{n} b_k c_{m-k+j} \quad , \; j = 1, 2, \ldots, n \tag{4.10b}$$

from which the a_i and b_i may be calculated. Padé's method is one of the most computationally efficient methods of rational function approximation. The value of the approximant at a given s_o is also easily evaluated by synthetic division.

Although the method is straightforward, certain pathological cases arise in which an [m,n] approximant may match fewer (or more) than the theoretical n+m+1 terms of the Taylor series expansion of g(s). The first case occurs because of common factors in $p_m(s)$ and $q_n(s)$, while the second case may be resolved by appealing to the theory of continued fractions (Wall, 1948).

Padé approximation about $s=\infty$ may also be accomplished. In this case, the series (4.8) is replaced by a Laurent series and the approximant is written $p_m(1/s)/q_n(1/s)$. Equations for the coefficients of p_m and q_n are similar to (4.10).

Baker (1965) has extended the theory to include the idea of Padé approximation about more than one point. Evidently partial realization corresponds to Padé approximation about $s=0$ and $s=\infty$, and the Hermite normal form algorithm may be used to derive such approximants in state variable form by assuming the function g(s) to represent a linear system transfer function. If the coefficients $-c_i$ in (4.8) are placed in a Hankel matrix, which is then reduced to Hermite normal form, an [r-1,r] approximant results (viz., $g(s) = h^T(sI_r-F)^{-1}g$ where (F,g,h^T) are the matrices comprising the partial realization). More generally, an $[r^*,r]$ approximant, where $r^* \leq r-1$, may be derived by padding the sequence of time moments $-c_i$ with an extension sequence of $r-r^*-1$ zeros considered as Markov parameters.

Example

$$g(s) = (s+1)/(s+2)(s^2+2s+2) = 1/4 - 1/8s - 1/16s^2 + 5/32s^3 - 9/64s^4 + ..$$

A [1,2] approximant to g(s) about s=0 is derived by forming the generalized Hankel matrix

$$H = \begin{pmatrix} -5/32 & 1/16 & 1/8 \\ 1/16 & 1/8 & -1/4 \end{pmatrix}$$

Reduction of H yields

$$\overline{H} = \begin{pmatrix} 0 & 1 & -4/3 \\ 1 & 0 & -4/3 \end{pmatrix}$$

so that

$$g = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ F = \begin{pmatrix} -4/3 & 1 \\ -4/3 & 0 \end{pmatrix}, \ h^T = [1/8 \quad -1/4]F^2 = [1/6 \quad 1/9]$$

$$h^T(sI_2-F)^{-1}g = (1/6)(s+2)/(s^2+4/3s+4/3) = 1/4-1/8s-1/16s^2+5/32s^3-7/56s^4+ \ldots$$

which agrees with the series for g(s) in 4 terms as expected. To derive a [0,2] approximant, start with

$$\underset{\sim}{H} = \begin{bmatrix} 1/16 & 1/8 & -1/4 \\ 1/8 & -1/4 & 0 \end{bmatrix}$$

Reduction now leads to

$$\bar{H} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix}$$

so that

$$g = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; F = \begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix}, \; h^T = [-1/4 \quad 0]F = [0 \quad 1/2] \quad ,$$

$$h^T(sI_2-F)^{-1}g = (1/2)/(s^2+s+2) = 1/4 - 1/8s - 1/16s^2 + 3/32s^3 + \ldots$$

4.3.2 Minimal Partial Realizations as Reduced-Order Models

It was shown that minimal partial realization and Padé approximation (about s=0 and s=∞) are equivalent under certain circumstances valid in a control theoretic setting. Thus, minimal partial realizations may be used to derive reduced-order models of large-scale systems. The method inherits all the advantages and disadvantages of Padé approximation. The main difference is that minimal partial realization gives the reduced models in state variable form rather than transfer function form. For single variable systems, the pathological case of pole-zero cancellation which afflicts Padé approximation, occurs in the form of an unobservable (but controllable) "minimal" partial realization (or vice versa). Although this occurance is rare, the observability of such reduced models should always be checked.

More serious from the control theoretic point of view is the possibility of generating unstable (resp. stable) reduced models of a stable (resp. unstable) system. While an unobservable model may at least be used for simulation and control (provided no observer is needed), an unstable model of a stable system (or vice versa) seems to be of no practical use whatsoever. To overcome this problem, either the specification for the desired order of the reduced model, the set of generalized Markov parameters to be matched, or both, must be changed. The iterative nature of the Hermite normal form algorithm allows this to be accomplished without starting the partial realization process from the beginning. It is obvious that an $r+1^{th}$ order model may be derived from an r^{th} order one by performing one more reduction step. Ignoring the unit vectors in the first p columns of the reduced Hankel matrix (where p is the number of inputs), and performing enough iterations of the Hermite normal form algorithm to restore the number of unit vectors to r, allows the determination of an r^{th} order partial realization matching $\{J_{k+1}, J_{k+2}, \ldots\}$ instead of $\{J_k, J_{k+1}, \ldots\}$, where $k < 0$.

Example

$$g(s) = (s+1)/(s+2)(s^2+2s+2)$$

An approximant to g(s) matching $\{j_{-4}, j_{-3}, j_{-2}, j_{-1}\}$ was derived earlier. An approximant matching $\{j_{-3}, j_{-2}, j_{-1}, j_0\}$ could have been obtained had an extra column been carried in the Hankel matrix, viz.,

$$H = \begin{bmatrix} -5/32 & 1/16 & 1/8 & -1/4 \\ 1/16 & 1/8 & -1/4 & 0 \end{bmatrix}$$

Two steps of the reduction process yield

$$H^{(2)} = \begin{bmatrix} 0 & 1 & -4/3 & -2/3 \\ 1 & 0 & -4/3 & 4/3 \end{bmatrix}$$

Ignoring the first column and choosing the 23 element as a point gives

$$\overline{H}^{(2)} = \begin{bmatrix} - & 1 & 0 & -2 \\ - & 0 & 1 & -1 \end{bmatrix}$$

The partial realization obtained from $\overline{H}^{(2)}$ is

$$F = \begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix}, \ g = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ h^T = [-1/4 \quad 0]F = [0 \quad 1/2]$$

$$h^T(sI_2-F)^{-1}g = (1/2)/(s^2+s+2)$$

which was obtained before, and is known to match $\{j_{-3}, j_{-2}, j_{-1}, j_0\}$.

4.4 Nonminimal Partial Realization

A nonminimal partial realization of a strictly proper transfer function matrix G(s) may be envisaged as failing to match the theoretical number of generalized Markov parameters. (One must be careful, however, to refrain from including the pathological cases of minimal partial realization in this definition). Such nonminimal realizations may be derived by changing some of the calculated generalized Markov parameters of G(s) and reducing the generalized Hankel matrix to Hermite normal form. The method of Section 4.3.1 for generating $[r^*,r]$ Padé approximants about s=0 for a scalar transfer function g(s) is thus an example of nonminimal partial realization.

Roman and Bullock (1975) have proved that the characteristic polynomial of a nonminimal partial realization may be arbitrarily assigned by a suitable choice of the extension sequence (the perturbed generalized Markov parameters). Choosing an extension sequence to give a desired set of eigenvalues, or simply to assure stability, however, is no easy task. Ledwich and Moore (1976) give a decision theoretic approach to test for the existance of a stable partial realization, which could be applied to test a given extension sequence, but it is doubtful that the method could be extended to generate one.

The class of nonminimal partial realizations of a given system G(s) is very large, and provides a unifying framework to the reduction of large-scale systems. Specifically, it is obvious that the methods of Chen and Shieh (1968), Shamash (1973a), and Chuang (1970) may be viewed as special cases. The result of Roman and Bullock (1975) shows the existance of aggregated partial realizations, which occur when the eigenvalues to be matched are eigenvalues of the original system. The method of singular perturbations yields partial realizations, along with the Routh approximation method of Hutton (1975).

In Chapter 3 it was shown that the link between the large-scale system and its reduced-order aggregated model provided a rich analytical framework for the design of state variable feedback laws. The following section derives similar results for (nonminimal) partial realizations thereby giving such reduced models a new credibility.

4.4.1 Properties of Partial Realizations Under Linear State Variable Feedback

Theorem 3.2 stated that aggregated models remained aggregated after the introduction of certain linear state variable feedback laws. It is now shown that partial realizations are invariant (in a weak sense) under any linear state variable feedback law.

Theorem 4.3 Let any realization of a strictly proper transfer function matrix G(s) be

$$\dot{x} = Ax + Bu \ , \ x \in R^n, \ u \in R^p \tag{4.11a}$$

$$y = Cx \ , \ y \in R^q \tag{4.11b}$$

Furthermore suppose that the generalized Markov parameters $J_i = CA^iB$ are identically zero for $i \in I = \{-i_0, -i_0+1, \ldots, -1, 0, \cdots, i_1\}$. Then $J_{if} = C(A+BK)^iBL = 0$ for $i \in I$ and any conformable matrices K and L.

Proof Consider the expansion of $(A+BK)^i$ for $i \geq 0$. It shall be shown by induction that

$$(A+BK)^i = A^i + A^{i-1}BK + A^{i-2}BK(A+BK) + \ldots + ABK(A+BK)^{i-2} + BK(A+BK)^{i-1}, \ i \geq 0 \tag{4.12}$$

where it is understood that the expansion is terminated when any indicated power of a matrix is negative. (4.12) is trivially true for i=0,1. Suppose by induction that (4.12) holds for some $j \geq 0$ and consider

$$\begin{aligned}(A+BK)^{j+1} &= (A+BK)^j(A+BK)\\ &= \{A^j + A^{j-1}BK + A^{j-2}BK(A+BK) + \ldots + BK(A+BK)^{j-1}\}(A+BK)\\ &= A^{j+1} + A^jBK + A^{j-1}BK(A+BK) + A^{j-2}(A+BK)^2 + \ldots + BK(A+BK)^j\end{aligned}$$

Hence (4.12) is valid for $j+1$ and by induction for all integers $i \geq 0$. When $i<0$, set $i' = -i$. The matrix inversion lemma (Rózsa, 1974) gives

$$(A+BK)^{i} = [(A+BK)^{-1}]^{i'} = [A^{-1}-A^{-1}B(I_p+KA^{-1}B)^{-1}KA^{-1}]^{i'} \tag{4.13}$$

as $i' > 0$, (4.12) may now be used to give

$$\begin{aligned}(A+BK)^{i} = {} & A^{i}-A^{i}B(I_p+KA^{-1}B)^{-1}KA^{-1}-A^{i+1}B(I_p+KA^{-1}B)^{-1}KA^{-1}(A+BK)^{-1} \\ & - \dots - A^{-1}B(I_p+KA^{-1}B)^{-1}KA^{-1}(A+BK)^{i+1}, \ i<0\end{aligned} \tag{4.14}$$

Premultiplication by C and postmultiplication by BL now give

$$J_{if} = C(A+BK)^{i}BL = \begin{cases} J_iL+J_{i-1}KL+J_{i-2}K(A+BK)L+\dots+J_0K(A+BK)^{i-1}L, & i\geq 0 \\ J_iL+J_{i+1}\Gamma(A+BK)^{-1}L+\dots+J_{-1}\Gamma(A+BK)^{i+1}L, & i<0 \end{cases} \tag{4.15a}$$

$$\Gamma = (I_p+KA^{-1}B)^{-1}KA^{-1} \tag{4.15b}$$

where the identity $I_p - (I_p+KA^{-1}B)^{-1}KA^{-1}B = (I_p+KA^{-1}B)^{-1}$ has been used. Since $J_i = 0$ for $i \in I = \{-i_0, -i_0+1, \dots, -1\} \cup \{0, 1, \dots, i_1\}$, the application of (4.15) now gives $J_{if} = 0$ for $i \in I$.

<u>Theorem 4.4</u> Let a large-scale system and any partial realization be given by

$$\dot{x} = Ax + Bu \quad , \ y = Cx$$

$$\dot{z} = Fz + Gu \quad , \ \hat{y} = Hz$$

and suppose that the generalized Markov parameters of the two systems agree on the set I of Theorem 4.3. Then the generalized Markov parameters $J_{if}^{(e)}$ of the error system $y-\hat{y}$ are zero for $i \in I$, and any linear state variable feedback law of the form $u = Kx + Lz + Mv$.

Proof The generalized Markov parameters $J_i^{(e)}$ of the error system

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} B \\ G \end{pmatrix} u$$

$$e = [C \quad -H] \begin{pmatrix} x \\ z \end{pmatrix}$$

are obviously zero for $i \in I$. Application of Theorem 4.3 now gives $J_{if}^{(e)} = 0$ for $i \in I$ where $u = [K \quad L] \begin{pmatrix} x \\ z \end{pmatrix} + Mv = Kx + Lz + Mv$.

A useful application of Theorem 4.4 is the case of polynomial input functions $u(t) = \Sigma_{i=0}^{k-1} \alpha_i t^i/i!$. If the original and reduced-order models have only open left half plane poles, and the generalized Markov parameters $\{J_{-1}, J_{-2}, \ldots, J_{-k}\}$ are matched, then the steady state error (and its first k-1 derivatives) are zero regardless of any feedback control law introduced. Hence if $u = Lz + Mv$ is applied to (F, G, H) to track some specified polynomial function, this control law applied to (A, B, C) will achieve the same resulting steady state.

4.5 Canonical Aggregated Partial Realizations

Although the work of Roman and Bullock (1975) proves the existance of aggregated partial realizations, it is not of much use in their determination, as the specification of an extension sequence to give a desired set of eigenvalues is by no means a simple matter.

A more direct approach is the use of certain canonical forms of state equations where the elements may be easily chosen to give a desired characteristic polynomial. The input and output matrices are then chosen to match some generalized Markov parameters. The problem is first solved for single-input single-output systems and this solution is then generalized to the multivariable case. The canonical forms which are most convenient to use are the column companion and controllable canonical forms (Luenberger, 1967). A generalization of the column companion form is now given.

4.5.1 A Generalized Column Companion Form for Single Variable Systems

Let (A,b,c^T) denote a controllable, n^{th} order, single-input, single-output system. If a transformation matrix is chosen as (Luenberger, 1967)

$$P = [b \quad Ab \; \ldots \; A^{n-1}b] \tag{4.16}$$

then the controllability of (A,b) insures that P is nonsingular and

$$\overline{A} = P^{-1}AP = \begin{bmatrix} 0 & 0 & \ldots & 0 & -\alpha_n \\ 1 & 0 & \ldots & 0 & -\alpha_{n-1} \\ 1 & 0 & \ldots & 0 & -\alpha_{n-2} \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \ldots & 1 & -\alpha_1 \end{bmatrix}, \quad \overline{b} = e_1 \tag{4.17a}$$

$$\overline{c}^T = c^TP = [j_0 \quad j_1 \; \ldots \; j_{n-2} \quad j_{n-1}] \tag{4.17b}$$

where the j_i are the Markov parameters of the system. It is well known (Chen, 1970) that the characteristic polynomial of A is

$$|\lambda I_n - A| = \lambda^n + \alpha_1\lambda^{n-1} + \ldots + \alpha_{n-1}\lambda + \alpha_n \tag{4.18}$$

Theorem 4.5

Let $g(s) = (\beta_1 s^{n-1} + \beta_2 s^{n-2} + \ldots + \beta_{n-1} s + \beta_n)/(s^n + \alpha_1 s^{n-1} + \ldots + \alpha_{n-1} s + \alpha_n)$.

Then if g(s) has no common factors, a minimal realization is given by

$$\dot{x} = \bar{A}x + e_k u = \bar{A}x + \bar{b}u \qquad , k \in \underline{n} \tag{4.19a}$$

$$y = [j_{-k+1} \quad j_{-k+2} \quad \cdots \quad j_{-k+n}]x = \bar{c}^T x \tag{4.19b}$$

where $\bar{A}$ is a companion matrix as in (4.17).

Proof The special case of k=1 is valid from consideration of (4.17). If k>1, (4.19b) requires the computation of time moments, so that g(s) must have no pole at the origin. In this case $|A| \neq 0$, and A^{-1} exists. Writing A in columns as

$$\bar{A} = [e_2 \quad e_3 \quad \cdots \quad e_n \quad \alpha] \tag{4.20}$$

it is then easily established that

$$\bar{A}^i = [e_{i+1} \quad e_{i+2} \quad \cdots \quad e_n \quad \alpha \quad \bar{A}\alpha \quad \cdots \quad \bar{A}^{i-1}\alpha], \ i \geq 0 \tag{4.21}$$

Since $\bar{A}^{-1}\bar{A} = I_n$, then for some vector $\alpha^{(-1)}$ we have

$$\bar{A}^{-1} = [\alpha^{(-1)} \quad e_1 \quad e_2 \quad \cdots \quad e_{n-1}] \tag{4.22a}$$

$$\bar{A}^{-i} = [\bar{A}^{-i+1}\alpha^{(-1)} \quad \bar{A}^{-i+2}\alpha^{(-1)} \quad \cdots \quad \alpha^{(-1)} \quad e_1 \quad e_2 \quad \cdots \quad e_{n-i}], \ i>0 \tag{4.22b}$$

If equation (4.17) is now transformed according to $P = A^{-k+1}$ then

$$\tilde{A} = P^{-1}\bar{A}\,P = \bar{A} \tag{4.23a}$$

$$\tilde{b} = P^{-1}e_1 = e_k \tag{4.23b}$$

$$\tilde{c}^T = \tilde{c}^T I_n = \tilde{c}^T[\tilde{A}^{-k+1}e_k \ \tilde{A}^{-k+2}e_k \ \dots \ \tilde{A}^{-k+n}e_k] = [j_{-k+1} \ \ j_{-k+2} \ \cdots \ j_{-k+n}] \tag{4.23c}$$

where the fact that the generalized Markov parameters are invariant under a nonsingular transformation has been used. Since g(s) was assumed to have no common factors, (4.17) and hence (4.19) are minimal realizations.

Corollary 4.1 The state variable representation

$$\dot{x} = \bar{A}x + \bar{A}e_n u \tag{4.24a}$$

$$y = [j_{-n} \ \ j_{-n+1} \ \cdots \ j_{-1}]x \tag{4.24b}$$

is equivalent to (4.19).

This type of realization is useful in casting the Routh Approximation Method into state variable form, for it is well known (Hutton and Friedland, 1975) that the r^{th} order Routh approximant of a scalar system matches the first r time moments of the original system.

4.5.2 Canonically Aggregated Partial Realizations For Single Variable Systems

As the characteristic polynomial of a companion matrix is easily determined, such matrices play an important role in the derivation of aggregated reduced-order models of single-input single-output systems. The remaining problem is the choice of the input and output matrices (b and c^T) which allow the easy determination of the aggregation matrix and easy matching of generalized Markov parameters.

The generalized column companion form of Section 4.5.1 is obviously useful for matching the generalized Markov parameters. It is shown that the aggregation matrix is also easily derived. The controllable canonical form is also suitable for generating aggregated partial realizations; the matching of generalized Markov parameters requiring marginally more computation than the case of the column companion form.

Theorem 4.6 Let g(s) be as in Theorem 4.5 and let the poles of g(s) be $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then an r^{th} order aggregated partial realization of g(s) is given by

$$\dot{z} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\delta_r \\ 1 & 0 & \cdots & 0 & -\delta_{r-1} \\ 0 & 1 & \cdots & 0 & -\delta_{r-2} \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & -\delta_1 \end{bmatrix} z + e_m u = \bar{F}z + \bar{g}u, \quad m \in \underline{r} \qquad (4.25a)$$

$$y = [j_{-m+1} \quad j_{-m+2} \quad \cdots \quad j_{-m-1+r} \quad j_{-m+r}]z = \bar{h}^T z \qquad (4.25b)$$

where $s^r + \delta_1 s^{r-1} + \ldots + \delta_{r-1}s + \delta_r = (s-\lambda_1)(s-\lambda_2)\ldots(s-\lambda_r)$. Furthermore, if a realization of g(s) is taken as in (4.19), then the aggregation matrix is

$$K = [\bar{F}^{-k+1}\bar{g} \quad \bar{F}^{-k+2}\bar{g} \quad \ldots \quad \bar{F}^{-k+n}\bar{g}] \qquad (4.26)$$

Proof By Theorem 4.5 and the choice of the δ_i, (4.25) obviously represents an aggregated partial realization of g(s). The formula for the aggregation matrix follows from (4.21), (4.22) and $K = KI_n = K[\bar{A}^{-k+1}\bar{b} \quad \bar{A}^{-k+2}\bar{b} \quad \ldots \quad \bar{A}^{-k+n}\bar{b}]$.

It is expedient to choose $m \geq k$ (if possible), for then the aggregation matrix K given in (4.25) will contain a maximal number (r) of unit vectors.

Theorem 4.7 Let g(s) be realized in the controllable canonical form (Chen, 1970)

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1 \end{bmatrix} x + e_n u = Ax + bu$$

$$y = [\beta_n \quad \beta_{n-1} \quad \beta_{n-2} \quad \cdots \quad \beta_1]x = c^T x$$

If the eigenvalues of A are $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ then an aggregated partial realization matching $\{j_{-k+1}, j_{-k+2}, \ldots, j_{-k+r}\}$ where k e r is given by

$$z = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\delta_r & -\delta_{r-1} & -\delta_{r-2} & \cdots & -\delta_1 \end{bmatrix} z + e_r u = Fz + gu \qquad (4.26a)$$

$$y = [j_{-k+1} \; j_{-k+2} \; j_{-k+3} \cdots j_{-k+r}] \begin{bmatrix} \delta_{r-1} & \delta_{r-2} & \cdots & \delta_1 & 1 \\ \delta_{r-2} & \vdots & \iddots & 1 & 0 \\ \vdots & \delta_1 & \iddots & 0 & 0 \\ \delta_1 & 1 & & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} F^{k-1} z = h^T z \qquad (4.26b)$$

and the aggregation matrix is given by

$$K = \begin{pmatrix} \varepsilon_{n-r} & \varepsilon_{n-r-1} & \varepsilon_{n-r-2} & \cdots & \varepsilon_2 & \varepsilon_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \varepsilon_{n-r} & \varepsilon_{n-r-1} & \cdots & \varepsilon_3 & \varepsilon_2 & \varepsilon_1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \varepsilon_r & \varepsilon_{r-1} & \varepsilon_{r-2} & \cdots & & & 1 & 0 \\ 0 & 0 & 0 & \cdots & \varepsilon_{r+1} & \varepsilon_r & \varepsilon_{r-1} & \cdots & & \cdots & 0 & 1 \end{pmatrix} \tag{4.27}$$

where $s^r + \delta_1 s^{r-1} + \ldots + \delta_{r-1} s + \delta_r = (s-\lambda_1)(s-\lambda_2)\ldots(s-\lambda_r)$ and

$$s^{n-r} + \varepsilon_1 s^{n-r-1} + \ldots + \varepsilon_{n-r-1} s + \varepsilon_{n-r} = (s-\lambda_{r+1})(s-\lambda_{r+2})\ldots(s-\lambda_n).$$

Proof Since

$$(s^r+\delta_1 s^{r-1} + \ldots + \delta_r)(s^{n-r}+\varepsilon_1 s^{n-r-1} + \ldots + \varepsilon_{n-r}) = s^n+\alpha_1 s^{n-1} + \ldots + \alpha_n ,$$

then $FK = KA$ and $g = Kb$ may be directly verified for K given by (4.27). Hence (F,b) is an aggregated model of (A,b).

The generalized Markov parameters of (F,g,h^T) may be calculated from

$$[j_{-k+1} \quad j_{-k+2} \ \cdots \ j_{-k+r}] = h^T F^{-k+1}[g \quad Fg \ \ldots \ F^{r-1}g] = h^T F^{-k+1} U.$$

where U is the controllability matrix, which is always invertible. Hence $h^T = [j_{-k+1} \quad j_{-k+2} \ \cdots \ j_{-k+r}]U^{-1}F^{k-1}$. Use of the representation of U^{-1} as given in Chen (1970) gives (4.26b) immediately.

Example

$$g(s) = (s+1)/(s+2)(s^2+2s+2)$$

A minimal realization of g(s) in the generalized column companion form is

$$\dot{x} = \begin{pmatrix} 0 & 0 & -4 \\ 1 & 0 & -6 \\ 0 & 1 & -4 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u = Ax + bu$$

$$y = [1/8 \quad -1/4 \quad 0]\ x = c^T x$$

An aggregated model retaining the eigenvalues -1±j and matching $\{j_{-1}, j_0\}$ is given by

$$\dot{z} = \begin{pmatrix} 0 & -2 \\ 1 & -2 \end{pmatrix} z + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = Fz + gu$$

$$\hat{y} = [-1/4 \quad 0]\ z = h^T z$$

$$K = [F^{-2}g \quad F^{-1}g \quad g] = \begin{pmatrix} -1 & 1 & 0 \\ -1/2 & 0 & 1 \end{pmatrix}$$

If the controllable canonical form is to be used in the realizations, then

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -6 & -4 \end{pmatrix} \cdot x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u = Ax + bu$$

$$y = [1 \quad 1 \quad 0]\ x = c^T x$$

$$\dot{z} = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} z + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$\hat{y} = [-1/4 \quad 0] \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} z = [1/2 \quad 0]\ z = h^T z$$

$$K = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

In each case it is easily verified that $FK = KA$ and $g = Kb$ as required. The reduced order transfer function is evidently $g(s) = h^T(sI-F)^{-1}g = (1/2)/(s^2+2s+2)$.

4.5.3 A Generalized Column Companion Form for Multivariable Systems

Let (A,B,C) denote an n^{th} order, p input, q output, controllable realization of some strictly proper transfer function matrix G(s). The transformation to the column companion form is achieved by taking (Luenberger, 1967)

$$P = [b_1 \quad Ab_1 \; \dots \; A^{n_1-1} b_1 \quad b_2 \quad Ab_2 \; \dots \; A^{n_2-1} b_2 \; \dots \; b_p \quad Ab_p \; \dots \; A^{n_p-1} b_p] \tag{4.28}$$

The transformed system matrices are given by

$$\bar{A} = P^{-1}AP = [e_2 \quad e_3 \; \dots \; e_{n_1} \quad \alpha_1 \quad e_{n_1+2} \quad e_{n_1+3} \; \dots \; e_{n_1+n_2} \quad \alpha_2 \; \dots \quad e_{n-n_p+2} \quad e_{n-n_p+3} \; \dots \; e_n \quad \alpha_p] \tag{4.29a}$$

$$\bar{B} = P^{-1}B = [e_1 \quad e_{n_1+1} \quad e_{n_1+n_2+1} \; \dots \; e_{n-n_p+1}] \tag{4.29b}$$

$$\bar{C} = CP = [j_{01} \; j_{11} \dots j_{n_1-1,1} \quad j_{02} \; j_{12} \dots j_{n_2-1,2} \; \dots \; j_{0p} \; j_{1p} \; \dots \; j_{n_p-1,p}] \tag{4.29c}$$

Theorem 4.8 Let $\sigma + 1 \in \underline{\check{n}}$ where $\check{n} = \min\{n_i, i \in \underline{p}\}$. Assume that G(s) has no poles at the origin, and let $P_1 = A^{-\sigma} P$ for P given by (4.28). Then

$$\tilde{A} = P_1^{-1}AP_1 = P^{-1}AP = \overline{A} \tag{4.30a}$$

$$\tilde{B} = P_1^{-1}B = P^{-1}A^{\sigma}B = [e_{\sigma+1} \quad e_{\sigma+n_1+1} \quad e_{\sigma+n_1+n_2+1} \cdots e_{\sigma+n-n_p+1}] \tag{4.30b}$$

$$\tilde{C} = CP_1 = CA^{-\sigma}P = [j_{-\sigma,1} \; j_{1-\sigma,1} \cdots j_{n_1-1-\sigma,1} \quad j_{-\sigma,2} \; j_{1-\sigma,2} \cdots j_{n_2-1-\sigma,2} \cdots$$

$$\cdots \quad j_{-\sigma,p} \quad j_{1-\sigma,p} \cdots j_{n_p-1-\sigma,p}] \tag{4.30c}$$

<u>Proof</u> (4.30a) and (4.30c) follows from (4.28) and (4.29) for any integer σ. The restriction $\sigma+1 \epsilon \underline{n}$ is imposed to insure, that B will contain only unit vectors as indicated. Since $P^{-1}P = I_n$ it follows that $P^{-1}A^{i}b_j = e_k$ where $j \epsilon \underline{p}$, $i+1 \epsilon \underline{n}_j$ and $k = n_1+n_2+\ldots+n_{j-1}+i+1$. Hence for $\sigma+1 \epsilon \underline{n}_j$ it follows that

$$P^{-1}A^{\sigma}b_j = e_{n_1+n_2+\ldots+n_{j-1}+\sigma+1}$$

and (4.30b) is verified.

4.5.4 Canonically Aggregated Partial Realizations for Multivariable Systems

The generalized column companion form of the previous section seems to be a good canonical form to use when an aggregated partial realization of a multivariable system is needed. It turns out that this is the only suitable canonical form for the multivariable case, as the single variable system results for the controllable canonical form cannot be extended. The main reason for this is the fact that the characteristic polynomial of a multivariable controllable canonical realization cannot be determined by inspection, except in the case where the matrix is block triangular. For this reason, the use of the input identifiable canonical form is ruled out as well.

The column companion canonical realization and its generalizations, however, may always be taken in the block upper triangular form, with the diagonal blocks being companion matrices. The characteristic polynomial is readily seen to be a product of characteristic polynomials of the companion blocks. An aggregated model may now be constructed in an iterative manner by first separately aggregating the companion blocks as in the scalar case and then determining the coupling between these aggregated blocks. Because of the block structure present, special care must be exercised when multiple eigenvalues are present. To gain insight into these special problems, a simple example is examined.

Let

$$\dot{x} = \begin{pmatrix} 0 & 0 & 0 & -6 & 0 & 1 \\ 1 & 0 & 0 & -17 & 0 & 2 \\ 0 & 1 & 0 & -17 & 0 & 3 \\ 0 & 0 & 1 & -7 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & -5 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} u = Ax + Bu$$

$$y = \begin{pmatrix} 0 & 1 & -2 & -4 & 1 & 0 \\ 0 & -1 & 1 & -1 & 1 & 1 \end{pmatrix} x = Cx$$

It is easily verified that $|sI_6-A| = \{(s+1)^2(s+2)(s+3)\}\{(s+1)(s+4)\}$. Suppose that a third order aggregated model is to be constructed retaining the eigenvalues $s=-1$ (three times). In this case, the F and G matrices are chosen as

$$F = \begin{bmatrix} 0 & -1 & f_1 \\ 1 & -2 & f_2 \\ 0 & 0 & -1 \end{bmatrix} \qquad G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

where f_1 aand f_2 are to be determined. The aggregation matrix is written by inspection as

$$K = KI_6 = K[b_1\ Ab_1\ A^2b_1\ A^3b_1\ \ b_2\ Ab_2] = [g_1\ Fg_1\ F^2g_1\ F^3g_1\ \ g_2\ Fg_2]$$

$$= \begin{bmatrix} 1 & 0 & -1 & 2 & 0 & f_1 \\ 0 & 1 & -2 & 3 & 0 & f_2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Now since F, K, and A must satisfy $FK = KA$ there follows $Fk_6 = Ka_6$ which gives

$$\begin{bmatrix} -f_1-f_2 \\ f_1-3f_2 \end{bmatrix} = \begin{bmatrix} 6-5f_1 \\ 8-5f_2 \end{bmatrix}$$

giving $f_1 = 20/9$ and $f_2 = 26/9$. The output matrix H may be chosen as

$$H = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

which matches j_{01}, j_{11}, and j_{10} of the sixth order model. Suppose that a model ratining $s=-1$ (twice) and $s=-4$ is desired. The F and K matrices become

$$F = \begin{bmatrix} 0 & -1 & f_1 \\ 1 & -2 & f_2 \\ 0 & 0 & -4 \end{bmatrix} \qquad K = \begin{bmatrix} 1 & 0 & -1 & 2 & 0 & f_1 \\ 0 & 1 & -2 & 3 & 0 & f_2 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{bmatrix}$$

The condition $Fk_6 = Ka_6$ imposed as before gives

$$\begin{bmatrix} -4f_1-f_2 \\ f_1-6f_2 \end{bmatrix} = \begin{bmatrix} 6-5f_1 \\ 8-5f_2 \end{bmatrix}$$

which is an inconsistant set of equations. If the (1,6) element of A were changed to 3 (instead of 1), the underdetermined set $f_1-f_2 = 8$ results. Thus, when multiple eigenvalues are present, no aggregated model or an infinite number of such models may exist.

The underdetermined case warrants further scrutiny. Putting $a_{16} = \alpha$, the A matrix is now a function of α, denoted $A(\alpha)$, which may be partitioned as

$$A(\alpha) = \begin{array}{cc} \begin{array}{cc} \quad 4 & \qquad 2 \end{array} & \\ \begin{bmatrix} A_{11} & A_{12}(\alpha) \\ 0 & A_{22} \end{bmatrix} & \begin{array}{l} (4 \\ (2 \end{array} \end{array}$$

Hence the classical adjoint of $sI_6-A(\alpha)$ is

$$\operatorname{adj}(sI_6-A(\alpha)) = \begin{bmatrix} \Delta_{22}(s)\ \operatorname{adj}(sI_4-A_{11}) & X(s,\alpha) \\ 0 & \Delta_{11}(s)\ \operatorname{adj}(sI_2-A_{22}) \end{bmatrix}$$

where $\Delta_{11}(s) = (s+1)^2(s+2)(s+3)$, $\Delta_{22}(s) = (s+1)(s+4)$, and

$$X(s,\alpha) = \begin{bmatrix} \alpha s^3 + (7\alpha-24)s^2 + (17\alpha-18)s + (17\alpha-12) \\ 2s^3 + (\alpha-54)s^2 + (7\alpha-41)s + (17\alpha-18) \\ 3s^3 - 45s^2 + (\alpha-54)s + (7\alpha-24) \\ 4s^3 + 3s^2 + 2s + \alpha \end{bmatrix} [1 \quad s]$$

Since $\Delta_{11}(s)$ and $\Delta_{22}(s)$ share a common factor, namely s+1, then choosing $\underline{q}$ such that $X(s,\alpha) = (s+1)\,\bar{X}(s,\alpha)$ will cause the minimal polynomial of A to differ from its characteristic polynomial. For $\alpha=3$ it is easily verified that

$$X(s,\alpha) = (s+1) \begin{bmatrix} s^2 - 6s + 39 \\ 2s^2 - 53s + 33 \\ 3s^2 - 48s - 1 \\ 4s^2 - s - 3 \end{bmatrix} [1 \quad s]$$

In this case the eigenvalue $s=-1$ is assoicated with two Jordan blocks instead of one, and each block may be considered separately in the process of aggregation.

A algorithm for constructing an aggregated model in the column companion form is now given. Consider the equations (4.29). Because of the selection procedure for the columns of the transformation matrix P it is evident that $\bar{A}$ is a block upper triangular matrix whose diagonal blocks are companion matrices and whose superdiagonal blocks are zero excepting the final column. Thus, dropping the bar for notational simplicity

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1p} \\ 0 & A_{22} & A_{23} & \cdots & A_{2p} \\ 0 & 0 & A_{33} & \cdots & A_{3p} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & A_{pp} \end{bmatrix}, \qquad (4.31a)$$

$$A_{ii} = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_1(ii) \\ 1 & 0 & \cdots & 0 & a_2(ii) \\ 0 & 1 & \cdots & 0 & a_3(ii) \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{n_i}(ii) \end{bmatrix}, \ (n_i x n_i) \qquad (4.31b)$$

$$A_{ij} = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_1(ij) \\ 0 & 0 & \cdots & 0 & a_2(ij) \\ 0 & 0 & \cdots & 0 & a_3(ij) \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{n_i}(ii) \end{bmatrix}, \ (n_i x n_j), \ i \neq j \qquad (4.31c)$$

It is seen that

$$\sigma(A) = \bigcup_{i=1}^{p} \sigma(A_{ii})$$

To determine an aggregated model F, having the same structure, it is obvious that $\sigma(F_{ii}) \subset \sigma(A_{ii})$ is a necessary condition. Thus F_{ii} is an aggregated model of A_{ii}. To find K such that FK = KA, it is seen that K has the same block structure as A and F except that the main diagonal blocks K_{ii} will not be square, since they obviously satisfy $F_{ii}K_{ii} = K_{ii}A_{ii}$.

It remains to determine the F_{ij} and K_{ij} where $i \neq j$. This can always be done providing one is careful when multiple eigenvalues are present. The algorithm takes advantage of the canonical structure. Putting

$$F = [e_2 \;\; e_3 \; \dots \; e_{r_1} \;\; \beta_1 \;\; e_{r_1+2} \;\; e_{r_1+3} \; \dots \; e_{r_1+r_2} \;\; \beta_2 \; \dots$$

$$\dots \; e_{r-r_p+2} \;\; e_{r-r_p+3} \; \dots \; e_r \;\; \beta_p]$$

$$G = [e_1 \;\; e_{r_1+1} \;\; e_{r_1+r_2+1} \; \dots \; e_{r-r_p+1}]$$

Then

$$K = KI_n = K[b_1 \;\; Ab_1 \; \dots \; A^{n_1-1} b_1 \;\; b_2 \;\; Ab_2 \; \dots \; A^{n_2-1} b_2 \; \dots$$

$$\dots \; b_p \;\; Ab_p \; \dots \; A^{n_p-1} b_p]$$

$$= [g_1 \;\; Fg_1 \; \dots \; F^{n_1-1} g_1 \;\; g_2 \;\; Fg_2 \; \dots \; F^{n_2-1} g_2 \; \dots \; g_p \;\; Fg_p \; \dots \; F^{n_p-1} g_p]$$

$$= [e_1 \;\; e_2 \; \dots \; e_{r_1} \;\; \beta_1 \;\; F\beta_1 \; \dots \; F^{n_1-r_1-1} \beta_1 \;\; e_{r_1+1} \;\; e_{r_1+2} \; \dots$$

$$\dots \; e_{r_1+r_2} \;\; \beta_2 \;\; F\beta_2 \; \dots \; F^{n_2-r_2-1} \beta_2 \; \dots \; e_{r-r_p+1} \;\; e_{r-r_p+2} \; \dots$$

$$\dots \; e_r \;\; \beta_p \;\; F\beta_p \; \dots \; F^{n_p-r_p-1} \beta_p]$$

The columns β_i of F have the following form

$$\beta_i^T = [\beta_i(1) \ldots \beta_i(m_i) \quad \beta_i(m_i+1) \ldots \beta_i(m_i+r_i) \quad 0 \ldots 0]$$

$$= [\tilde{\beta}_i^T \quad \bar{\beta}_i^T \quad 0^T]$$

where $m_i = r_1+r_2+\ldots+r_{i-1}, m_1 \equiv 0$. The first m_i entries of β_i are unknown, the next r_i are determined by $\sigma(F_{ii})$, and the last are zero. The columns α_i of A have the same form, where we replace the r_j by n_j, except that these vectors are completely known. The equation FK = KA (c.f. Chapter 3) then gives rise to the following system of equations

$$[F^{n_1-r_1}\beta_1 \quad F^{n_2-r_2}\beta_2 \ldots F^{n_p-r_p}\beta_p] = [K\alpha_1 \quad K\alpha_2 \ldots K\alpha_p]$$

Because of the form of α_i and β_i, one may derive an equation of the form

$$T_i(\beta_1, \beta_2, \ldots, \beta_{i-1})\tilde{\beta}_i = u(\beta_1, \beta_2, \ldots, \beta_{i-1}) \qquad (4.32)$$

where $\tilde{\beta}_i$ is a vector of the r_i unknown entries of β_i. If $\sigma(F_{ii})$ was properly chosen, then the above equation has at least one solution for $\tilde{\beta}_i$. To quantify the statement "properly chosen" suppose that a decomposition of A as a direct sum of Jordan blocks is

$$A \sim \Sigma_{d\,\substack{i\in\underline{u}\\ j\in\underline{v}}} N_{\ell_{ij}}(\lambda_i) \qquad (4.33)$$

where $N_k(\lambda_j)$ is a Jordan block of size k with eigenvalue λ_j and Σ_d denotes the direct sum. From Chapter 3, for a full-rank aggregation matrix K to exist we must have*.

$$F \sim \Sigma_{d\,\substack{i\in U_1\\ j\in V_1}} N_{t_{ij}}(\lambda_i) \qquad (4.34)$$

where $U_1 \subset \underline{u}$ and $V_1 \subset \underline{y}$. If the eigenvalues of A are distinct there is obviously no problem. When this is not the case, however, (4.32) may very well have no solution, since the decomposition (4.33) is unknown. To insure a solution the safest procedure is to retain or discard all A-invariant subspaces associated with a given eigenvalue λ. That is, if $|sI_n-A| = (s-\lambda)^m\phi(s)$, then take $|sI_r-F| = (s-\lambda)^\delta\xi(s)$ where $\delta \in \{0,m\}$. In this case, (4.32) will in fact have a unique solution for each β_i as $|T_i| \neq 0$.

From the discussion in Section 4.5.3 it is evident that the output matrix H of the aggregated model may be chosen to match the generalized Markov parameters of the original system (A, B, C), and the existance of aggregated partial realizations of multivariable systems is proven. This is stated as

<u>Theorem 4.9</u> Let (A, B, C) be a controllable realization. Then it is always possible to find an aggregated partial realization (F, G, H) which is controllable.

<u>Proof</u> Since (A, B, C) is a controllable, it has a column companion form (A, B, C) as in (4.29). By the above discussion there is an aggregated partial realization (F, G, H). Since (F, G) is itself in column companion form, it is controllable.

<u>Theorem 4.10</u> Let (F, G, H) be an aggregated partial realization of (A, B, C) where the generalized Markov parameters agree on the set $I = (-\underline{i}_0) \cup \underline{i}_1$. If K is the aggregation matrix and $u = Lz+Mv = LKx+Mv$ is any linear state variable feedback law, then $H(F+GL)^iGM = C(A+BLK)^iBM$ for

$i \in I$.

Proof Since $FK = KA$ it is easily shown by induction that $K(A+BLK)^i = (F+GL)^iK$ for all integers i. Adapting equation (4.15) to the current setting gives

$$J_{if} = C(A+BLK)^iBM = \begin{cases} J_iM+J_{i-1}LKM+J_{i-2}LK(A+BLK)M+\ldots+J_0LK(A+BK)^{i-1}M, & i\geq 0 \\ \\ J_iM+J_{i+1}\Gamma(A+BLK)^{-1}M+\ldots+J_{-1}\Gamma(A+BK)M & , i<0 \end{cases}$$

where $\Gamma = (I_p+LKA^{-1}B)^{-1}KA^{-1} = (I_p+LF^{-1}G)^{-1}F^{-1}K = \Omega K$, p being the number of inputs. The theorem now follows since $\bar{J}_i = J_i$ when $i \in I$ where $\bar{J}_i$ is the i^{th} generalized Markov parameter HF^iG.

This result is far stronger than that of Theorem 4.4 since it states that the generalized Markov parameters of the reduced feedback system agree with those of the original system on the set I whereas before only the generalized Markov parameters of the error system were zero on the set I. For single-input single-output systems Theorem 4.10 may be made stronger. The result is based on the fact that the system zeros are feedback invariants in the single variable case.

4.5.5 Advantages and Disadvantages of Aggregated Partial Realization

The following advantages of the method of aggregated partial realization are evident.

1. It retains all the good features of the method of state aggregation, e.g.,
   - a relationship between the states of the original and reduced model.
   - the invariance property under LSVF.
   - a stable (unstable) reduced model of a stable (unstable) system.
2. The use of canonical forms permits considerable simplification of the calculation of F, G, and K and also allows easy matching of the generalized Markov parameters.
3. The invariance property of matched moments under LSVF is retained.

The disadvantages of the method are

1. The reduced-order model may not be observable.
2. The tendency of many real systems to be controllable through the first input results in a trade-off of modelling accuracy for the remaining inputs to increase the accuracy with respect to the first input.

The first disadvantage cannot be cured. Observability of the reduced model should always be checked. Many practical systems are controllable through each input, but usually there is also an input which affects the system more than any other, or is especially suited to manipulation for control purposes. By exchanging rows of G(s) this special input can always be made the first one. Now the second disadvantage may actually be a great advantage since the modelling accuracy with respect to this input is very good. If this is not acceptable, i.e., equal accuracy with respect to all inputs and outputs is desired, the aggregation process may be carried out by calculation of the left eigenvectors of A as in Chapter 3. The aggregated model may then be transformed to the input identifiable form (Hickin and Sinha, 1977b), for which the output matrix H consists of the (generalized) Markov parameters of all outputs. Since the order of the reduced model is generally much smaller than that of the original system, this transformation may be carried out with minimal effort.

4.6 Aggregation and Singular Perturbations

In this section a connection between a certain class of aggregated models and the method of singular perturbations is established. In particular, it is shown that a slight modification of Davison's algorithm results in a singularly perturbed model. Let the state equations of a large-scale stable system be as usual

$$\dot{x} = Ax + Bu \quad , x \in R^n, \; u \in R^p \tag{4.35a}$$

$$y = Cx \quad , y \in R^q \tag{4.35b}$$

Let the eigenvalues of A, ranked by increasing magnitude of the real part, be $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and let V be the corresponding modal matrix. Now consider a transformation of variables

$$x = V \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix} w \;, \; |T_{11}| \neq 0, \; |T_{22}| \neq 0 \tag{4.36}$$

so that (4.41) becomes

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} = \begin{pmatrix} F_{11} & 0 \\ 0 & F_{22} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} G_{11} \\ G_{21} \end{pmatrix} u, \; w_1 \in R^r, \; w_2 \in R^{n-r} \tag{4.37a}$$

$$y = [H_{11} \quad H_{12}] \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \tag{4.37b}$$

where

$$\begin{aligned} F_{11} &= K_{11} A K^n_{11} \\ G_{11} &= K_{11} B \\ F_{22} &= K_{22} A K^n_{22} \\ G_{22} &= K_{22} B \\ K_{11} &= [T_{11} \quad 0] V^{-1} \\ K_{22} &= [0 \quad T_{22}] V^{-1} \\ K^n_{11} &= V[T^{-1}_{11} \quad 0] \\ K^n_{22} &= V[0 \quad T^{-1}_{22}] \end{aligned} \tag{4.38}$$

It is clear that $\sigma(F_{11}) = \{\lambda_1, \ldots, \lambda_r\}$ and $\sigma(F_{22}) = \{\lambda_{r+1}, \ldots, \lambda_n\}$, and that the substates w_1 and w_2 of w are each aggregated states of x. Now set

$$\mu = \mathrm{Re}(\lambda_1)/\mathrm{Re}(\lambda_{r+1}) \tag{4.39a}$$

$$w_2 = \mu \hat{w} \qquad (4.39b)$$

Equations (4.36) now become

$$\dot{w}_1 = F_{11}w_1 + G_{11}u \qquad (4.40a)$$

$$\mu\dot{\hat{w}} = \hat{F}_{11}\hat{w} + G_{21}u \qquad (4.40b)$$

$$y = H_{11}w_1 + \hat{H}_{12}\hat{w} \qquad (4.40c)$$

where $\hat{F}_{22} = \mu F_{22}$ and $\hat{H}_{12} = \mu H_{12}$. Because of the ranking of the eigenvalues, μ will be a small (positive) parameter. Setting $\mu = 0$ in (4.40b) now gives a singularly perturbed model

$$\dot{\bar{w}} = F_{11}\bar{w} + G_{11}\bar{u} \qquad (4.41a)$$

$$\bar{\hat{w}} = -\hat{F}_{22}^{-1} G_{21}\bar{u} \qquad (4.41b)$$

$$\bar{y} = H_{11}\bar{w}_1 - \hat{H}_{12}\hat{F}_{22}^{-1}G_{21}\bar{u} = H_{11}\bar{w}_1 - H_{12}F_{22}^{-1}G_{21}\bar{u} \qquad (4.41c)$$

It is clear that this model is also aggregated, and stable by the assumed stability of the original system. In fact, reference to Section 3.5.1 shows that Davison's model is a special case of (4.41a). Since F_{11} is stable, it follows that $w_1(t) = w(t) + O(\mu)$[†]. Intuitively this means that after an initial transient period, during which $w_1 - w$ may be quite large, approximating function w comes and remains within a

---

[†] $w_1(t) = w(t) + O(\mu) \Rightarrow \exists\ \alpha > 0,\ \alpha^* \in [0,\alpha],\ t^* < \infty \ni |w_1(t) - w(t)| < \alpha^*\mu \ \forall t > t^*$. See, for example Kokotović (1972).

hypercylinder traced about the trajectory w_1. Now all the tools of singular perturbation theory may be brought to bear on any control design. The boundary layer system is evidently governed by

$$\dot{\hat{\bar{w}}}(\tau) = F_{11}\hat{\bar{w}}(\tau) + G_{21}\bar{u}(0), \quad \tau = t/\mu \qquad (4.42)$$

so that

$$\hat{\bar{w}}(\tau) = \exp(\hat{F}_{22}\tau)\hat{\bar{w}}(0)+\hat{F}_{22}^{-1}\{\exp(\hat{F}_{22}\tau)-I\}G_{21}\bar{u}(0) \qquad (4.43)$$

It is known that Davison's model provides a good transient response, but a poor DC steady-state response. The presence of the term $-H_{12}F_{22}^{-1}G_{21}\bar{u}$ ensures that the DC steady-state error will be zero, but in so doing, upsets the good transient tracking, and thus gives an intuitive insight into this characteristic behaviour of singularly perturbed models.

As an example consider g(s) as given in Section (3.7). Calculations give

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} u$$

$$y = [1/2 \quad 1/2 \quad -1/2] \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Now $\mu = (-1/-2) = 1/2$ and setting $w = \mu w_2$ the singularly perturbed model ($\mu = 0$) becomes

$$\dot{\bar{w}}_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} w_1 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

$$\bar{\hat{w}} = \bar{u}$$

$$\bar{y} = [1/2 \quad 1/2]\ \bar{w} = (1/4)\ \bar{u}$$

and the boundary layer system is given by

$$\dot{\bar{\hat{w}}}(\tau) = -\bar{\hat{w}}(\tau) + \bar{u}(0)\ , \ \tau = t/2$$

4.7 Conclusions

The theory of model reduction by nonminimal partial realizations has been proposed. This class of reduced models is seen to be very large. Specifically, several existing methods of model reduction are seen to be special cases of nonminimal partial realization. Thus the claim of a unified theory is justified.

The behaviour of such models under linear state variable feedback is examined. This is an important contribution, for it justifies the use of such models as a means of designing feedback compensators for large-scale systems.

Of particular interest are the aggregated partial realizations whose existance is shown in Section 4.5. Such reduced models are powerful tools since they combine the excellent qualities of aggregation and partial realization. The use of canonical realizations allows the aggregation procedure to be accomplished without an eigenvector calculation (c.f., Chapter 3), and facilitates the choice of the output matrix for matching generalized Markov parameters.

Model reduction by minimal partial realization is presented, along with a new algorithm for the determination of state variable representations of proper rational transfer functions. Several existing methods such as Padé approximation and continued fraction expansion are special cases. The proposed method has the advantage of determining reduced models directly in state variable form and its iterative nature is appreciated when a reduction of a given order fails to be useful, e.g., unobservable or unstable (assuming the original system was observable and stable).

CHAPTER 5

THE SUBOPTIMAL CONTROL PROBLEM

5.1 Introduction

The linear quadratic regulator problem and its solution via a time varying linear state variable feedback law is one of the most fundamental problems of modern control theory (Kalman, 1960). For an n^{th} order system, the crux of the problem is the solution of a matrix differential equation in 1/2 n(n+1) unknowns, which is further complicated by the presence of a terminal boundary value rather than an initial one. The solution of the optimal trajectory then requires integration of the autonomous system once the feedback gain matrix is known. This total solution is then termed the two point boundary value problem and has been the object of intensive research since 1960. All known algorithms, including the so-called generalized X-Y or Chandesekhar methods, take a prohibitive amount of computer time to solve even for systems of modest order, and it is doubtful whether such algorithms may ever be implemented in real time.

In order to save computational effort, the steady-state value of the feedback gain matrix is often used, reducing the calculation to the solution of an algebraic matrix equation in n(n+1)/2 unknowns. Even the best of the algorithms for solving this problem (Kleinmann, 1968) may require too much computer time for large systems ($n > 50$).

An alternative approach is to derive a reduced-order model of the large-scale system, repose the optimal control problem for the reduced model, solve this easier problem, and apply the resulting control signal to the large-scale system and hope for the best. If $n/r \geq 5$ the potential savings in computation are enormous, being roughly proportional to the 3rd power of this reduction ratio. Two methods of attacking this sub-optimal control problem are those of Aoki (1968) and Kokotović (1972). The next section presents a generalization of Aoki's method.

5.2 The Linear Quadratic Regulator

The linear quadratic regulator problem and its solution are well documented in the literature. This section states the problem in a somewhat more general manner than the original work (Kalman, 1960) but is nevertheless well known. Thus let

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \quad x \in R^n, \quad u \in R^p \tag{5.1a}$$

$$y = Cx \qquad\qquad , \quad y \in R^q \tag{5.1b}$$

The linear quadratic regulator problem is

$$\text{minimize PI} = \int_0^\tau (y^T y + u^T Ru)dt, \quad R > 0 \tag{5.2}$$

Theorem 5.1 If (A, B) is stabilizible and (A, C) is detectable then the optimal control $u^*(t)$ is given by

$$u^*(t) = -R^{-1}B^TP(t)x^*(t) \tag{5.3a}$$

$$-\dot{P}(t) = P(t)A+A^TP(t)-P(t)BR^{-1}B^TP(t)+C^TC, \quad P(\tau) = 0 \tag{5.3b}$$

furthermore the minimal value PI^* is given by

$$PI^* = x_o^T \; P(0)x_o \tag{5.4}$$

The proof of Theorem (5.1) is possible under slightly more general circumstances than have been stated, and may be found in Kwakernaak and Sivan (1972). The following corollory is also provable.

Corollary 5.1 For $\tau \to \infty$, $\lim\limits_{t\to\infty} P(t) = P \not< 0$. P is the unique positive definite solution of the algebraic Ricatti equation

$$0 = PA + A^TP - PBR^{-1}B^TP + C^TC \tag{5.5}$$

More recently, a different approach to the problem provides a relief in computational effort if $p + q < 1/2\ n$. This is the so-called generalized X-Y or Chandrasekhar method (Casti, 1974, and Kailath, 1972), and is stated below without proof.

Theorem 5.2 Let (A,B) be stabilizible and (A,C) detectable. Then the optimal control $u^*(t)$ is given by

$$u^*(t) = BN(t) \tag{5.6a}$$

$$-\dot{M}(t) = (A - BN(t))^TM(t), \quad M(\tau) = C^T \tag{5.6b}$$

$$-N(t) = R^{-1}B^TM(t)M^T(t) \quad , \; N(\tau) = 0 \tag{5.6c}$$

In addition, the solution P(t) of (5.3b) is expressible as

$$P(t) = \int_t^\tau M(\sigma)M^T(\sigma)d\sigma \tag{5.7}$$

Note The resulting autonomous system $x^* = (A-BR^{-1}B^TP(t))x^*$, $x(0) = x_0$ is always stable if the conditions of the theorems are met.

5.3 Suboptimal Control via Reduced-Order Models

An intuitively appealing method for reducing the calculation in the linear quadratic regulator problem is to proceed as follows.

1. Derive a reduced-order model of the large-scale system (5.1).
2. Substitute the output $\hat{y}$ of the reduced model for y in the performance index PI (5.2).
3. Calculate the optimal control u^* by Theorem (5.1) or (5.2) and use it as a suboptimal control for the system (5.1).

Thus let

$$\dot{z} = Fz + Gu,\ z(0) = z_o,\ z \in R^r,\ u \in R^p \tag{5.8a}$$

$$\hat{y} = Hz \tag{5.8b}$$

be a reduced-order model of (5.1). To derive a suboptimal control, the following quadratic functional is minimized by theorems (5.1) or (5.2)

$$\hat{PI} = \int_0^\tau (\hat{y}^T\hat{y} + u^TRu)dt \tag{5.9}$$

Via Ricatti equations there follows

$$u^* = -R^{-1}G^T\hat{P}(t)z^*(t) \tag{5.10a}$$

$$-\dot{\hat{P}}(t) = \hat{P}(t)F+F^T\hat{P}(t)-\hat{P}(t)GR^{-1}G^T\hat{P}(t)+H^TH, \ \hat{P}(\tau) = 0 \tag{5.10b}$$

and via X-Y equations

$$-\dot{\hat{M}}(t) = (F - G\hat{N}(t))^T\hat{M}(t), \ \hat{M}(\tau) = H^T \tag{5.11a}$$

$$-\dot{\hat{N}}(t) = R^{-1}G^T\hat{M}(t)M^T(t) \quad , \ \hat{N}(\tau) = 0 \tag{5.11b}$$

The merits of aggregation, partial realizaton, and aggregated partial realization for the reduced-order model (F,G,H) are discussed in the following three sections.

5.3.1 Suboptimal Control via Aggregation

Let (F,G,H) be an aggregated reduced-order model of (A,B,C). Then z(t) = Kx(t) where K is the aggregation matrix. As an immediate consequence, the initial condition z_o in (5.8a) may be specified as Kx_o since x_o was specified in the initial problem (5.1, 5.2). Suppose, for the moment, that $C = I_n$, i.e., a state regulator problem is given. The choice $H = CK^+ = K^+$ allows the functional (5.9) to be written

$$\hat{PI} = \int_0^\tau \{x^T(K^+)^TKx + u^TRu\} \, dt$$

which is the form advocated by Aoki (1968).

The aggregation law $z = Kx$, and the relationships $FK = KA$, $G = KB$ allow equations (5.10) and (5.11) to be rewritten as

$$u(t) = -R^{-1}G^TK^T\hat{P}(t)K\ x(t) \tag{5.12a}$$

$$-K^T\dot{\hat{P}}(t)K = (K^T\hat{P}K)A+A^T(K^T\hat{P}K)-(K^T\hat{P}K)BR^{-1}B^T(K^T\hat{P}K)+(HK)^THK,$$

$$K^T\hat{P}(\tau)K = 0 \tag{5.12b}$$

$$-K^T\dot{\hat{M}}(t) = (A - B\hat{N}K)^TK^T\hat{M}\ ,\ K^T\hat{M}_r(\tau) = (HK)^T \tag{5.13a}$$

$$-\dot{\hat{N}}(t)K = R^{-1}B^T(K^T\hat{M})(K^T\hat{M})^T,\ \hat{N}_r(\tau)K = 0 \tag{5.13b}$$

Comparison of the above with equations (5.3) and (5.5) allows the following correspondences

$$P(t) \leftrightarrow K^T\hat{P}(t)K$$

$$M(t) \leftrightarrow K^T\hat{M}(t)$$

$$N(t) \leftrightarrow \hat{N}(t)K$$

which gives some apreciation for the usefulness of the suboptimal control. Specifically, a lower bound for J_* is

$$J(u_*,x_o) \geq x_o^TK^TPKx_o = z_o^TPz_o.$$

When the control $u_*(t) = -R^{-1}B^TK^T\hat{P}(t)Kx(t)$ is applied, it is easily determined that the autonomous system remains aggregated in spite of the time-varying component, viz.,

$$KA_f(t) = K\{A-BR^{-1}B^TK\hat{P}(t)K\} = \{FK-GR^{-1}G^T\hat{P}(t)K\}$$

$$= \{F-GR^{-1}G^T\hat{P}(t)\}K = F_f(t)K \qquad (5.14)$$

This allows the following theorem.

Theorem 5.3 Let $\mathrm{Re}\lambda \leq 0 \ \forall \ \lambda \in \sigma(A)\backslash\sigma(F)$.[†] Then the autonomous system

$$\dot{x} = A_f(t)x = \{A-BR^{-1}B^TK^TP(t)K\}x, \ x(0) = x_o$$

is stable in the sense of Lyapunov (ISL stable).

Proof By the assumed conditions of Theorem (5.1), the autonomous system

$$\dot{z} = F_f(t)z = \{F-GR^{-1}G^T\hat{P}(t)\}z, \ z(0) = Kx_o$$

is ISL stable. Hence the Jacobian matrix $\partial F_f(t)z/\partial z^T$, i.e., $F_f(t)$, has a stable spectrum for each $t \in [0, \tau]$. By (5.14) $\sigma(A_f(t)) \supset \sigma(F_f(t))$, and $\sigma(A_f(t))\backslash\sigma(F_f(t)) = \sigma(A)\backslash\sigma(F)$, which has left half plane eigenvlaues by assumption. Hence $\sigma(A_f)$ is ISL stable for each $t \in [0, \tau]$ and the theorem is proven.

[†] If $X \supset Y$, $X\backslash Y = \{x \in X \mid x \notin Y\}$, the set theoretic difference.

Corollary 5.2 $\sigma(A-BR^{-1}B^TK^T\hat{P}K)$ is ISL stable where $\hat{P} = \lim_{t\to\infty} \hat{P}(t)$.

There is more in Theorem 5.3 than is immediately apparent. Besides the stability consideration it was shown that the suboptimal control can be realized in closed-loop form.

In taking an aggregated model (F,G,H) of (A,B,C) no assumption on the nature of the H matrix was made. In general, $H = CK^n$, where K^n is some right inverse of the aggregation matrix K. The output $y^*(t)$ may tend to some finite, but nonzero limit, when the optimal control $u^*(t)$ is applied. To keep the quadratic functional finite when the suboptimal control $u_*(t)$ is applied, it may become necessary to choose the H matrix to match some time moments.

5.3.2 Suboptimal Control via Partial Realization

The linear optimal quadratic regulator is a special case of the more general linear optimal quadratic tracking problem where the performance functional takes the following form

$$PI = \int_0^\tau (e^Te + u^TRu)dt \tag{5.15a}$$

$$e(t) = y(t) - \bar{y}(t) \tag{5.15b}$$

where $\bar{y}(t)$ is some prescribed target function. If $\bar{y}(t) = 0$, it reverts to the regulator problem. Although the tracking problem may be solved by the direct application of Pontryagin's minimum principle to (5.15), it is possible to convert the problem into a regulator problem of higher dimension. Specifically, it is assumed that $\bar{y}(t)$ may be represented as the output of an autonomous linear system, viz.,

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}, \quad \bar{x}(0) = \bar{x}_0 \tag{5.16a}$$

$$\bar{y}(t) = \bar{C}\bar{x} \tag{5.16b}$$

Augmenting the state $x(t)$ of (5.1) by $\bar{x}(t)$ above allows the following respecification of the tracking problem

$$\begin{pmatrix} \dot{x} \\ \dot{\bar{x}} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} \begin{pmatrix} x \\ \bar{x} \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u, \quad \begin{pmatrix} x(0) \\ \bar{x}(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ \bar{x}_0 \end{pmatrix} \tag{5.17a}$$

$$e = [C \quad -\bar{C}] \begin{pmatrix} x \\ \bar{x} \end{pmatrix} \tag{5.17b}$$

Equations (5.17) plus the functional (5.15) are seen to constitute a regulator problem, the solution of which is possible via Theorems 5.1 or 5.2. Note that the state $\bar{x}$ is uncontrollable. This is necessary as the target $\bar{y}$ would be upset by the resulting feedback law.

The suboptimal tracking problem is now easily formulated by replacing (A, B, C) with a reduced model (F, G, H). The important case $\tau = \infty$ is now considered.

The specification $\tau = \infty$ in the functional (5.15) ensures that the steady state error is zero. If a reduced-order model is to be used, it must be such that the steady state error $\hat{e}(t) = \hat{y}(t) - \bar{y}(t)$ is also zero, or the resulting performance integral will be infinite. The special case where $\bar{y}(t)$ is a polynomial in t is common. In this case, by virtue of Theorem 4.2, a zero steady state error is possible if the generalized Markov parameters J_i (i<0) are matched between (A, B, C) and (F, G, H) prior to the solution of the Ricatti equation. The resulting feedback system will be of an open-loop nature, so the A matrix must be stable at the outset. This is summarized in the theorem below.

Theorem 5.4 Let (A, B, C) be a stable, minimal realization, and let (F, G, H) be a stabilizible-detectable partial realization of (A, B, C), such that the generalized Markov parameters J_i, $i \in \{-1, -2, \ldots, -k\}$ are matched. Then any sub-optimal control law derived on the basis of (F, G, H) will result in a zero steady state tracking error for target functions of the form

$$y(t) = \sum_{i=0}^{k-1} d_i t^i .$$

Proof Since (F, G, H) is stabilizible-detectable, it follows that the feedback law derived drives the output $\hat{y}(t)$ to the desired target $\overline{y}(t)$ as $t \rightarrow \infty$. By the assumed stability of the A matrix, the matched J_i, $i \in \{-1, -2, \ldots, -k\}$, and Theorem (4.2), it follows that the J_{if}, the generalized Markov parameters of the error system are zero for $i \in \{-1, -2, \ldots, -k\}$, and that $\lim_{t\rightarrow\infty} e(t) = 0$ for all targets

$$\overline{y}(t) = \sum_{i=0}^{k-1} d_i t^i,$$ the Laplace transform of which is

$$\overline{y}(s) = \sum_{i=0}^{k-1} d_i i! \; s^{-i-1}.$$

5.3.3 Suboptimal Control via Aggregated Partial Realization

The usefulness of an aggregated partial realization for the calculation of suboptimal controls should now be evident. Theorems 5.3 and 5.4 may be immediately combined to give

Theorem 5.5 Let (F, G, H) be a detectable aggregated partial realization of the stabilizible-detectable system (A, B, C), where the generalized Markov parameters J_i agree on the set {-1, -2, ..., -k}. Then equation (5.14) holds, the suboptimal control derived on the basis of (F, G, H) defines a stable system, and the steady-state tracking error $y - \bar{y}$ is zero for target functions

$$\bar{y}(t) = \sum_{i=0}^{k-1} d_i t^i .$$

5.4 Some Simple Examples

The first example is taken from Rao and Lamba (1974), which is cast in a slightly different form (addition of an output matrix C).

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1/2 & -28/5 & -61/10 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u = Ax + bu$$

$$y = \text{diag}\ (\sqrt{5}, \sqrt{4}, \sqrt{1})x = Cx$$

$$\text{minimize PI} = \int_0^\infty (y^T y + u^T u)dt$$

The characteristic polynomial is evidently $|sI_3 - A| = s^3 + 61/60s^2 + 28/5s + 1/2 = (s+1/10)(s+1)(s+5)$. The reduced model will have a characteristic polynomial $|sI_2 - F| = (s+1/10)(s+1) = s^2 + 11/10s + 1/10$ and the output matrix will be chosen to match J_{-1} and J_0, thus

$$\dot{z} = \begin{pmatrix} 0 & 1 \\ -1/10 & -11/10 \end{pmatrix} z + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = Fz + gu$$

$$\hat{y} = \begin{pmatrix} 1/\sqrt{5} & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} z = H_1 z$$

$$PI = \int_0^\infty (\hat{y}^T\hat{y} + u^T u)dt$$

$$K = \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix} \qquad \text{(aggregation matrix)}$$

The solution of the Ricatti equation

$$0 = \hat{P}F + F^T\hat{P} - \hat{P}gg^T\hat{P} + H^T H$$

and the suboptimal control law becomes

$$\hat{P}_1 = \begin{pmatrix} 0.649 & 0.358 \\ 0.358 & 0.611 \end{pmatrix}$$

$$u_{*1} = -g^T\hat{P}_1 z = -[0.358 \quad 0.611]z = -[1/791 \quad 3/412 \quad 0.611]x$$

where the law $z = Kx$ has been used in the last step. Rao and Lamba give the optimal control law as

$$u^* = -[1.791 \quad 2.087 \quad 0.410]x, \quad ||u^*-u_{*1}||/||u^*|| = 0.77$$

To demonstrate the effect of the choice for H, the problem, reworked for

$$H_2 = \begin{bmatrix} 1/\sqrt{5} & -\sqrt{5}/25 \\ 0 & 2/5 \\ 0 & 0 \end{bmatrix}$$

which provides matching of $\{J_{-2}, J_{-1}\}$, gives

$$\hat{P}_2 = \begin{bmatrix} 0.593 & 0.358 \\ 0.358 & 0.347 \end{bmatrix}$$

$$u_{*2} = -[1.791 \quad 2.094 \quad 0.347]x$$

$$||u^*-u_{*2}||/||u^*|| = 0.02$$

which is barely distinguishable from the optimal control. This second model is much better than the first because of the nonzero weight applied to the second output instead of the third. This illustrates the consequences of indiscriminant choice of the generalized Markov parameters to be matched.

A second example exhibits the suboptimal tracking problem in a multivariable setting. The plant is described by

$$\dot{x} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -6 & -11 & 0 & -6 & 0 \\ -120 & 0 & 0 & -74 & 0 & -15 \end{pmatrix} x + \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & -4 \\ 9 & -6 \\ -2 & 14 \\ -61 & 36 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} x$$

minimize $PI = \int_0^\infty \{100(y_1-1)^2+u_1^2+u_2^2\}dt$

Three second order reduced models were used to generate and compare suboptimal controls. These were

$$\dot{z} = \begin{pmatrix} -3/2 & 0 \\ 6 & -6 \end{pmatrix} z + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u$$

Model 1 Partial Realization Matching

$\{J_{-1}, J_0\}$

$$\hat{y} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} z$$

$$\dot{z} = \begin{pmatrix} -4 & 1 \\ 2 & -8 \end{pmatrix} z + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u$$

Model 2 Partial Realization Matching

$\{J_0, J_1\}$

$$\hat{y} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} z$$

$$\dot{z} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} z + \begin{pmatrix} 0.381 & 0 \\ 0.392 & -0.196 \end{pmatrix} u$$

Model 3 Aggregated Retaining

$\{-1, -2\}$ using pseudoinverse

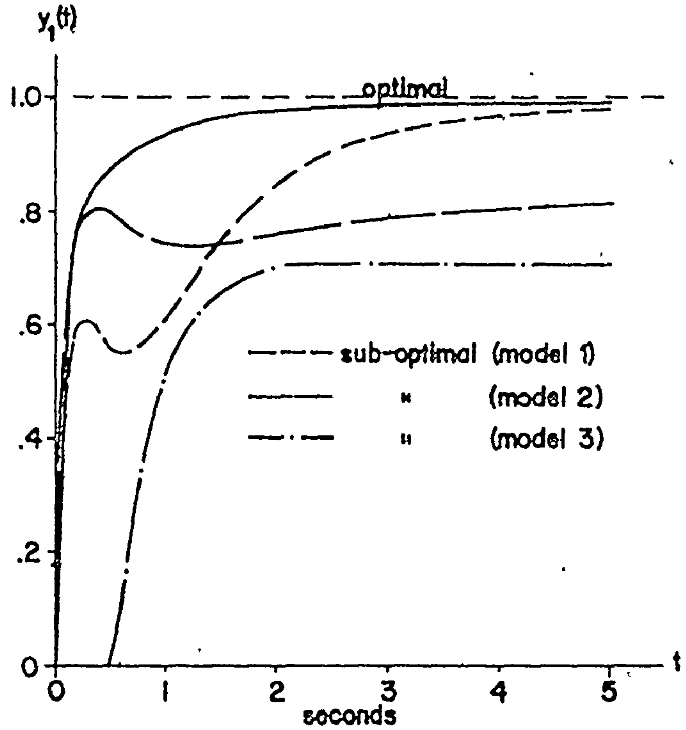
$$\hat{y} = \begin{pmatrix} 3.38 & -2.69 \\ 0 & 0 \end{pmatrix} z$$

Models 1 and 2 were partial realizations of the original system matching the generalized Markov parameters $\{J_{-1}, J_0\}$ and $\{J_0, J_1\}$ respectively. Model 3 was aggregated, retaining the dominant eigenvalues $\{-1, -2\}$ of the original system, and using the pseudoinverse to calculate the output matrix as $\hat{H} = CK^+$ (Aoki's method). The responses of the original system to the optimal and suboptimal controls is shown in Figure (5.1). As expected, model 1 provides the only suboptimal control which drives the original system to the specified target of $y_1 = 1$. Model 2 provides excellent transient tracking but fails in the steady state. The response to the control provided by model 3 provided neither acceptable transient nor steady-state tracking (the response $y_2(t)$ was omitted from Figure (5.1b) as it was always negative).

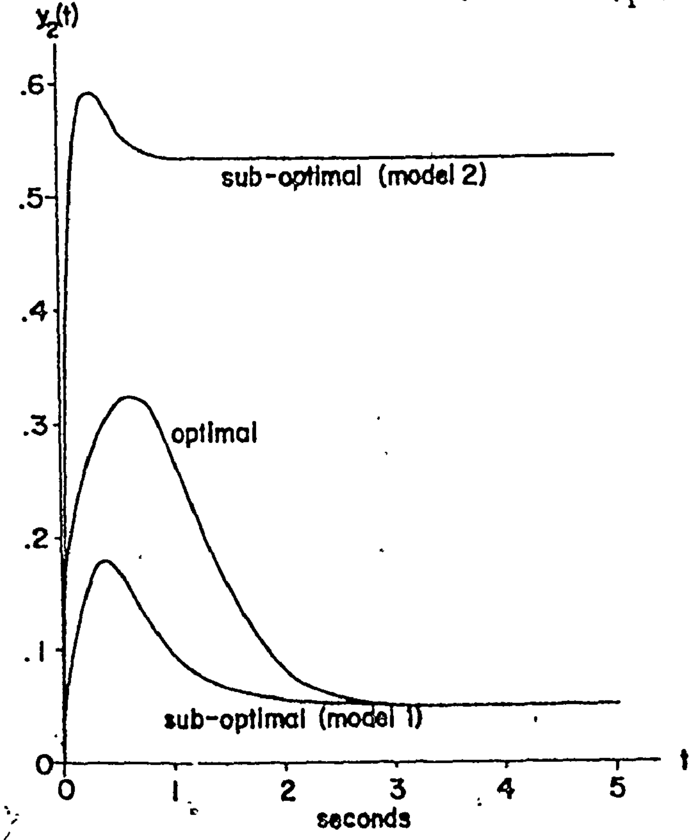
Three aggregated partial realizations, each matching J_{-1} but retaining different eigenvalues, were also derived. These second order reduced models were then used to generate suboptimal controls for linear quadratic regulator problem with weighting matrices $R = I_2$ and

$$S = \begin{pmatrix} 10 & 1 \\ 1 & 5 \end{pmatrix}.$$

The initial state was $x_o = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]^T$ and the optimal and suboptimal costs were calculated. The relevent data is given in Table 5.1. It is seen that the cost based on the dominant eigenvalue approach is not the best, although it is quite good. On the other hand, retaining an eigenvalue which is definitely nondominant results in a very poor value of the suboptimal cost. This clearly illustrates the difficulties that may face the designer if the eigenvalues of a system cannot be clearly split into a dominant and nondominant set.



Trajectories of $y_1(t)$ for different models



Trajectories of $y_2(t)$ for different models

Figure 5.1 System responses to optimal and suboptimal controls

Model #	F matrix	G matrix	H matrix	Suboptimal Cost	Suboptimal/ Optimal Cost
1	$\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$	$\begin{pmatrix} .381 & 0 \\ .392 & .196 \end{pmatrix}$	$\begin{pmatrix} .875 & 1.700 \\ -.875 & 1.700 \end{pmatrix}$	22.451	1.31
2	$\begin{pmatrix} -1 & 0 \\ 0 & -6 \end{pmatrix}$	$\begin{pmatrix} .381 & 0 \\ 0 & -.091 \end{pmatrix}$	$\begin{pmatrix} 1.750 & -10.977 \\ 0 & -10.977 \end{pmatrix}$	57.782	3.36
3	$\begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$	$\begin{pmatrix} .381 & 0 \\ .267 & 1.069 \end{pmatrix}$	$\begin{pmatrix} 1.640 & .468 \\ -.109 & .468 \end{pmatrix}$	19.844	1.15

Table 5.1 Various aggregated partial realizations and associated suboptimal costs. Optimal cost is 17.181.

5.5 Conclusions

The suboptimal linear quadratic regulator and tracking problems and their solution via reduced-order models have been discussed. Particular attention has been given to the use of aggregated models, whereby the suboptimal control law may be implemented in the classical feedback manner and stability guaranteed, and partially realized models, where stability cannot be induced, but if the original system is stable, a zero steady state tracking error for polynomial target functions can be arranged. The used of an aggregated partial realization obviously retains these good features of the separate methods.

CHAPTER 6

DISCRETE SYSTEMS

6.1 Introduction

To this point, the discussion has focussed exclusively on continuous systems. Discrete systems, however, are firmly established in linear systems theory, and thus deserve some consideration. Although simulation of such systems is considerably easier than simulation of continuous systems, the design of control laws is often more challenging (for instance, compare the discrete and continuous versions of the Kalman filter). It is well known that the frequency domain descriptions of continuous and discrete systems are essentially the same (excepting the incorporation of initial conditions), so that the techniques for reduction of continuous systems may be directly applied to the discrete case. The meaning of the complex frequency domain variable is different in each case, however, and this leads to differing behaviour of reduced models. The major differences in the application of the theory are presented.

6.2 Generalized Markov Parameters

A strictly proper discrete system is described in the time domain by

$$x_{k+1} = Ax_k + Bu_k, \quad x \in R^n, \ u \in R^p \tag{6.1a}$$

$$y_k = Cx_k \quad , \; y \in R^q \tag{6.1b}$$

where k denotes the k^{th} time instant ($k = 0, 1, \ldots$). A transfer function description of (6.1) is obtained by assuming zero initial conditions and Z-transforming to give

$$y(z) = G(z)u(z) = C(zI_n - A)^{-1}B\, u(z) \tag{6.2}$$

Evidently the formal expansions of G(z) and $(zI_n-A)^{-1}$ give

$$G(z) = \begin{cases} \sum_{i=0}^{\infty} J_i z^{-i-1} \\ \\ -\sum_{i=1}^{\infty} J_{-i} z^{i-1} \end{cases} \tag{6.3}$$

and the identification

$$J_i = CA^iB, \; i = 0, \pm 1, \ldots \tag{6.4}$$

It is thus evident that the computational techniques of Chapters 3 and 4 may be applied to (6.1) to produce a variety of reduced-order models.

The behaviour of partially realized discrete systems differs from that of their continuous counterparts by virtue of the fact that z^{-1} may be interpreted as a unit time delay operator whereas s^{-1} is essentially an integration operator. Matching the J_i ($i \geq 0$) provides reduced models which reproduce the output sequence exactly for the first few time instants, while matching the J_i ($i<0$) yields no special steady-state properties.

Theorem 6.1 Let

$$\phi_{k+1} = F\phi_k + Gu_k, \ \phi \in R^r, \ u \in R^p \qquad (6.5a)$$

$$\hat{y}_k = H\phi_k \qquad , \ y \in R^q \ . \qquad (6.5b)$$

be a partial realization of (6.1) matching the generalized Markov parameters J_i for $i \in \{0, 1, \ldots, \ell-1\}$. Then the output error $e_k = y_k - \hat{y}_k$ is identically zero for $k \in \{0, 1, \ldots, \ell\}$ and zero initial conditions.

Proof In the frequency domain let

$$u(z) = u_0 + u_1 z^{-1} + \ldots \qquad (6.6)$$

Now it is evident that

$$E(z) = \hat{G}(z)-G(z) = H(zI_r-F)^{-1}G-C(zI_n-A)^{-1}B = z^{-\ell-1}\tilde{G}(z) \qquad (6.7a)$$

$$\tilde{G}(z) = J_\ell + J_{\ell+1}z^{-1} + \ldots \qquad (6.7b)$$

Hence the output error is given by

$$e(z) = z^{-\ell-1}\{J_\ell u_0 + (J_\ell u_1 + J_{\ell+1}u_0)z^{-1} + \ldots\} \qquad (6.8)$$

and so $e_k = 0$ for $k \in \{0, 1, \ldots, \ell\}$.

It is fairly obvious that the DC steady state output error may be nulled by selecting $H(I_r-A)^{-1}G = C(I_n-A)^{-1}B$. This corresponds to the generalized Markov parameter J_{-1} only if the substitution $z \leftarrow z-1$ is made. Evidently the simultaneous matching of transient and steady state characteristics via reduction of a generalized Hankel matrix to Hermite Normal form is not possible.

6.3 Discretized Systems

Consider the continuous system described by

$$\dot{x} = Ax + Bu \qquad (6.9a)$$

$$y = Cx \qquad (6.9b)$$

If the input u(t) is sampled at time instants kτ, and followed by a zero order hold, then the system (6.9) may be converted into a discrete representation.

$$x(k\tau+\tau) = \Phi x(k\tau) + \theta\, u(k\tau) \qquad (6.10a)$$

$$y(k\tau) = Cx(k\tau) \qquad (6.10b)$$

where

$$\Phi = \exp(A\tau) \qquad (6.11a)$$

$$\theta = \int_0^\tau \exp(A\sigma)d\sigma B \qquad (6.11b)$$

Properties of reduced models of (6.10) induced by reduced models of (6.9) are now investigated.

Theorem 6.2 Let (F, G) be an aggregated model of (A, B) with aggregation matrix K. Then $(K\,\Phi\,K^n, K\theta)$ is an aggregated model of (Φ, θ), where K^n is any right inverse of K.

Proof By Theorem 3.3, it follows that $K\exp(At)K^n = \exp(Ft)\ \forall t$. Hence $K\,\Phi\,K^n = \exp(F\tau)$ and

$$K\theta = \int_0^{\tau} \exp(F\sigma)d\sigma G.$$

$(K\,\Phi\,K^n, K\theta)$ is said to be induced by (F, G).

6.4 Conclusions

It is shown that the ideas of model reduction of continuous time systems via nonminimal partial realization may be directly applied to systems of the discrete type. In this case, however, the generalized Markov parameters $J_i = CA^iB$, where $i < 0$, do not have any connection with the time moments. Hence the matrix oriented methods of Chapter 4 may not be used to simultaneously match Markov parameters and time moments. This of course does not imply that such reduced-order models do not exist for discrete systems, since Padé approximation about $z = \infty$ (Markov parameters) and $z = 1$ (time moments) is obviously possible by direct manipulation of a scalar transfer function g(z). In many applications, however, the steady state step (ramp, etc.) responses are more important to preserve than the transient response. When multivariable systems are reduced by Padé approximation of the transfer function about more than one point it may very well happen that the number of states in a realization of the approximant $\hat{G}(z)$ may be larger than the number of states in the original system $G(z)$. Thus it seems that the disadvantage of the proposed algorithm is only apparent in the scalar case.

CHAPTER 7

APPLICATION OF THE THEORY TO A LARGE SYSTEM

7.1 Introduction

An application of the theory of model reduction by aggregated partial realization to a control system of order thirty three, having three inputs and three outputs, is presented. The reduced model, of order eight, is used in the design of a suboptimal output tracking control law. The performance is compared to the optimal control.

7.2 Specifications of the System

The system chosen to illustrate the salient points of the theory has thirty three states, three inputs, and three outputs, and will be denoted by (A, B, C). The realization is given in the column companion form, the main diagonal blocks of A having sizes of twelve, eleven, and ten, respectively. The numerical values of the matrices are given in Figure 7.1 while the eigenvalues of A are listed in Table 7.1. Note that there are four sets of repeated eigenvalues. The system is obviously controllable, and the observability was checked by calculating $\rho[C^T\ A^TC^T\ A^{2T}\ \ldots\ A^{30T}] = 33$. Since the column companion form is used, the Markov parameters are easily identified by inspection of the columns of C. For the reduction process, however, the first time moment ($J_{-1} = CA^{-1}B$) is required and is given in Figure 7.2.

7.3 Derivation of the Reduced Model

An aggregated partial realization of (A, B, C) is derived using the techniques of Chapter 4. The aggregation process is described first, while the matching of generalized Markov parameters is deferred to Section 7.3.2.

7.3.1 Derivation of the Aggregated Model

The algorithm of Theorem 4.9 may be described as aggregating the diagonal blocks of A separately, followed by the calculation of the coupling columns to ensure that FK = KA, where F is the reduced state transition matrix and K is the aggregation matrix. An engineering rule of thumb states that any control system may be reasonably approximated by a system of third order, hence a reduced model of order nine, i.e., three subsystems of order three, is sought. Following another time honoured practice, the dominant poles of the system will first be examined for possible inclusion in the aggregated model (F, G). To insure the existance of such a model, any repeated pole will be entirely included or else excluded from the reduced model. Note that choosing each subsystem to have order three constrains each to have at least one real pole. When all the above constraints are considered, no set of nine eigenvalues can be found. Relaxing the subsystem order constraint allows the set of eight eigenvalues listed in Table 7.2 to qualify. The characteristic polynomial of each subsystem is readily calculated, thus determining the diagonal blocks of F, which is given in Figure 7.3. Since the coupling between the blocks is not yet known, these values have been denoted by β_{ij}. The structure of the aggregation matrix K is given in Figure 7.4 where β_{*i} denotes the column vector $[\beta_{1i} \;\; \beta_{2i} \; \cdots \; \beta_{ii} \; 0 \; \cdots \; 0]^T$. Application of the algorithm of Theorem 4.9 yields unique solutions for the β_{ij} which are tabulated in Table 7.3. The numerical accuracy of the procedure for determining the β_{ij} was of the order of 10^{-9} using single precision arithmetic on a CDC 6400 computer. This completes the aggregation procedure.

7.3.2 Selection of the Output Matrix

Selection of the output matrix H to match some of the generalized Markov parameters of (A, B, C) should be done with the intended application of the reduced model in mind. Since it is intended to design a suboptimal output tracking compensator, where the targets are step functions, it is expediant to match at least one time moment to provide a zero steady state step error. Since the Markov parameters are closely associated with the transient response of a system, as many as possible should be matched in order to track the transient closely. Hence, the columns of H are to be chosen from the set $\{J_{-1}, J_0, J_1\}$. This necessitates use of the generalized column companion form of Section 4.5.3. The F matrix is invariant under this transformation, and we have

$$\bar{F} = FFF^{-1} = F$$

$$\bar{G} = FG = [e_2 \;\; e_5 \;\; e_7]$$

$$\bar{K} = FK$$

$$\bar{H} = [j_{-11} \;\; j_{01} \;\; j_{11} \;\vdots\; j_{-12} \;\; j_{02} \;\vdots\; j_{-13} \;\; j_{03} \;\; j_{13}]$$

This completes the determination of the aggregated partial realization $(\overline{F}, \overline{G}, \overline{H})$ of (A, B, C). The zero state responses of (A, B, C) and $(\overline{F}, \overline{G}, \overline{H})$ to an input $u^T = [1 \quad 1 \quad 1]$ are shown in Figure 7.5.

7.4 Design of a Suboptimal Output Tracking Compensator

It is desired to design a compensator to allow the original system (A, B, C) to track step changes. Thus a performance integral

$$PI = \int_0^\infty (e^T Se + u^T Ru)dt$$

is to be minimized, where $e = y - \overline{y}$ is the deviation of the output from the target signal $\overline{y}$, which is a vector of step functions. The method of Section 5.3.3 is used to convert this problem into an equivalent quadratic regulator problem by appending three extra states to generate the target signals.

The use of aggregated partial realizations to implement suboptimal controls is straightforward. The procedure may also be used for design, however. One of the major difficulties associated with linear quadratic regulator theory is the choice of the weighting matrices to give an acceptable response. The usual cut and try approach can be very expensive and time consuming when dealing with a large system. If, however, a reduced model is available, it is relatively cheap to use the cut and try approach with the hope that the resulting weighting matrices will also provide an acceptable optimal control. This approach is taken here.

From the unit step responses of Figure 7.5 it is evident that y_1 and y_3 take 200 seconds to reach 98% of the steady state value while y_2 takes 140 seconds. The long settling time is mainly due to the double eigenvalues at $-.0479 \pm j0.0119$. The design objective will be to reduce all settling times to 100 seconds and having a steady state response of $y_{ss}^T = [0.5 \quad 0.02 \quad 4.0]$ for an input $u^T = [1 \quad 1 \quad 1]$. A class of feedback laws required to produce the desired steady state may be derived as solutions to the optimal linear servomechanism problem discussed above. It remains to choose the weighting matrices to give an acceptable settling time.

The input weighting matrix R was taken as the identity matrix and four runs were made with the different output weighting matrices listed in Table 7.4. The tracking error responses for the first three choices of S are plotted in Figure 7.6. It is apparent that a 2% settling time of less than 100 seconds for all three outputs is ill posed, since reducing the settling time of y_1 and y_3 increases that of y_2. Ignoring our specification for y_2, the fourth choice for S produces acceptable settling times for y_1 and y_3. The tracking error responses are plotted in Figure 7.7 along with the response of the original system to this control law. The similarity between the signals $y-\overline{y}$ and $\hat{y}-\overline{y}$, where $\overline{y}$ is the target, is evident, and the settling times are roughly 100 seconds. The cost of the control strategy for the reduced system was $PI^* = 2.56 \times 10^5$ while the cost for the large system was $PI = 4.64 \times 10^5$. The agreement between the costs is not exceptional and occurs because of the method of calculating the optimal control and is discussed in Section 7.5.

7.5 Some Notes on Computational Procedures

All calculations were executed to meet or exceed a tolerance of 10^{-7} (absolute). The continuous time Ricatti equation was solved by the diagonalization method of Potter (1966), the eigenanalysis being carried out by the QR algorithm (Martin, et al.) after reduction of the original matrix to almost triangular (Hessenberg) form. Simulations of dynamic systems were obtained through the discretization procedure mentioned in Chapter 6. The computation involves taking the exponential of the block matrix

$$\exp\{\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\tau\} \quad \text{which yields} \quad \begin{pmatrix} \Phi & \theta \\ 0 & I \end{pmatrix} \quad \text{where}$$

$$\Phi = \exp(A\tau), \ \theta = \int_0^\tau \exp(A\sigma)d\sigma B,$$

and τ is the integration step size. The matrix exponential routine is due to Källström (1973) which uses a scaled back-iterated Taylor series expansion to minimize error. The continuous Lyapunov equation, used to calculate the cost of the suboptimal feedback law, was solved using the recent algorithm of Hoskins, et al. (1977). Most of the above mentioned routines involve matrix multiplication. Rather than explicitly code the multiplication wherever needed, a subroutine has been used whose code is optimized to reduce the roundoff errors bound to occur. This code is due to Kahan (1971).

The solution of the servomechanism problem was effected by the method of Section 5.3.2, where the $\bar{A}$ matrix was taken as diag (-.0015,

-.002, -.0025). This choice generates target signals $\bar{y}$ which are slowly decaying exponentials and thus approximate unit steps. This was done to keep the system asymptotically stable. If $\bar{A}$ had been taken to be identically zero, unboundness in the P matrix would have occurred. Thus the costs of control given in Section 7.4 for the 33rd order example will approach ∞ as $\bar{A} \rightarrow 0$. They are evidently becoming unbounded at the same rate.

7.6 Conclusions

An aggregated partial realization of order eight has been derived for a thirty third order system. This reduced model has been successfully used to design a suboptimal static compensator for the linear quadratic servomechanism problem, where the choice of the weighting matrices is understood to be an integral part of the design. It is to be noted that an aggregated partial realization is the only type of reduced model other than singular perturbations which may be successfully used to solve this type of problem when the signal to be tracked is a polynomial function of time. It may be added that using singular perturbations, the equation for the approximated output $\hat{y}$ becomes $\hat{y} = Hz + Du$, resulting in a modified Ricatti equation, for which the solution is only positive semidefinite, instead of positive definite. The proposed method results in a suboptimal control $u = -R^{-1}B^T(K^TPK)x$, where P is positive definite. It is easily appreciated that the factor K^TPK is now only positive semidefinite, so that the two methods are quite similar.

$$\left[\begin{array}{ccccccccccccccccccccccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&-0.074481071&&&&&&&&&&&1.812&&&&&&&&&&1.404\\
1&0&0&0&0&0&0&0&0&0&0&-5.399374063&&&&&&&&&&&1.256&&&&&&&&&&4.719\\
0&1&0&0&0&0&0&0&0&0&0&-154.3861120&&&&&&&&&&&-2.761&&&&&&&&&&0.129\\
0&0&1&0&0&0&0&0&0&0&0&-2185.931765&&&&&&&&&&&2.342&&&&&&&&&&3.831\\
0&0&0&1&0&0&0&0&0&0&0&-15813.01737&&&&&&&&&&&-0.123&&&&&&&&&&4.929\\
0&0&0&0&1&0&0&0&0&0&0&-55514.51472&&&&&&&&&&&2.739&&&&&&&&&&-0.975\\
0&0&0&0&0&1&0&0&0&0&0&-86675.49320&&&&&&&&&&&-4.250&&&&&&&&&&2.573\\
0&0&0&0&0&0&1&0&0&0&0&-90220.51188&&&&&&&&&&&-2.071&&&&&&&&&&0.420\\
0&0&0&0&0&0&0&1&0&0&0&-30023.05951&&&&&&&&&&&-0.243&&&&&&&&&&3.404\\
0&0&0&0&0&0&0&0&1&0&0&-5017.443394&&&&&&&&&&&3.179&&&&&&&&&&-1.121\\
0&0&0&0&0&0&0&0&0&1&0&-465.8109327&&&&&&&&&&&4.191&&&&&&&&&&-2.829\\
0&0&0&0&0&0&0&0&0&0&1&-27.19770000&&&&&&&&&&&3.326&&&&&&&&&&4.147\\
&&&&&&&&&&&&0&0&0&0&0&0&0&0&0&0&-10.57034850&&&&&&&&&&-0.338\\
&&&&&&&&&&&&1&0&0&0&0&0&0&0&0&0&-303.3995079&&&&&&&&&&-2.214\\
&&&&&&&&&&&&0&1&0&0&0&0&0&0&0&0&-3399.177807&&&&&&&&&&-2.590\\
&&&&&&&&&&&&0&0&1&0&0&0&0&0&0&0&-18594.30974&&&&&&&&&&2.550\\
&&&&&&&&&&&&0&0&0&1&0&0&0&0&0&0&-50653.06932&&&&&&&&&&-2.906\\
&&&&&&&&&&&&0&0&0&0&1&0&0&0&0&0&-63157.70758&&&&&&&&&&-2.430\\
&&&&&&&&&&&&0&0&0&0&0&1&0&0&0&0&-33907.22121&&&&&&&&&&1.818\\
&&&&&&&&&&&&0&0&0&0&0&0&1&0&0&0&-11003.19089&&&&&&&&&&-1.956\\
&&&&&&&&&&&&0&0&0&0&0&0&0&1&0&0&-2267.074240&&&&&&&&&&4.452\\
&&&&&&&&&&&&0&0&0&0&0&0&0&0&1&0&-304.4672696&&&&&&&&&&3.974\\
&&&&&&&&&&&&0&0&0&0&0&0&0&0&0&1&-24.66800000&&&&&&&&&&-2.835\\
&&&&&&&&&&&&&&&&&&&&&&&0&0&0&0&0&0&0&0&0&-0.071675871\\
&&&&&&&&&&&&&&&&&&&&&&&1&0&0&0&0&0&0&0&0&-3.483734142\\
&&&&&&&&&&&&&&&&&&&&&&&0&1&0&0&0&0&0&0&0&-57.84973215\\
&&&&&&&&&&&&&&&&&&&&&&&0&0&1&0&0&0&0&0&0&-366.6131279\\
&&&&&&&&&&&&&&&&&&&&&&&0&0&0&1&0&0&0&0&0&-1102.057804\\
&&&&&&&&&&&&&&&&&&&&&&&0&0&0&0&1&0&0&0&0&-1885.253290\\
&&&&&&&&&&&&&&&&&&&&&&&0&0&0&0&0&1&0&0&0&-1919.444259\\
&&&&&&&&&&&&&&&&&&&&&&&0&0&0&0&0&0&1&0&0&-1002.770752\\
&&&&&&&&&&&&&&&&&&&&&&&0&0&0&0&0&0&0&1&0&-168.8338998\\
&&&&&&&&&&&&&&&&&&&&&&&0&0&0&0&0&0&0&0&1&-16.32680000
\end{array}\right]
\quad
\left[\begin{array}{ccc}
1&&\\
0&&\\
0&&\\
0&&\\
0&&\\
0&&\\
0&&\\
0&&\\
0&&\\
0&&\\
0&&\\
0&&\\
&1&\\
&0&\\
&0&\\
&0&\\
&0&\\
&0&\\
&0&\\
&0&\\
&0&\\
&0&\\
&0&\\
&&1\\
&&0\\
&&0\\
&&0\\
&&0\\
&&0\\
&&0\\
&&0\\
&&0\\
&&0
\end{array}\right]$$

A matrix

B matrix

Figure 7.5 The A, B, and C matrices

$$\left[\begin{array}{ccccccccccccccccccccccccccccccccc}
0&10^{-6}&10^{-6}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&10^{-6}&0&10^{-6}&0&0&0&0&0&0&0&0&0&0&0&0&10^{-6}\\
10^{-6}&0&0&0&0&0&0&0&0&0&0&10^{-6}&0&0&0&0&0&0&0&0&0&0&10^{-6}&0&0&0&0&0&0&10^{-6}&0&0&0\\
0&0&0&10^{-6}&0&0&0&0&0&0&0&0&10^{-6}&0&0&0&0&0&0&0&0&0&10^{-6}&0&0&0&0&0&0&0&0&10^{-6}&10^{-6}
\end{array}\right]$$

C matrix

B	1,2	-3.57±j12.58
L	3,4	-0.338±j0.602
O	5,6	-0.0479±j0.0119
C	7	-0.0619
K	8,9	-0.132±j0.00804
	10,11	-5.28±j3.657
1	12	-8.4
B	13,14	-0.132±j0.00804
L	15	-0.154
O	16	-0.212
C	17	-0.598
K	18,19	-2.68±j3.57
	20,21	-3.76±j4.85
2	22,23	-5.28±j3.657
B	24	-7.62
L	25,26	-3.62±j0.56
O	27,28	-0.0479±j0.0119
C	29	-0.224
K	30	-0.598
	31	-0.765
3	32,33	-3.15±j8.65

Table 7.1 Eigenvalues of the A matrix

$$J_{-1} = \begin{bmatrix} -0.3142_{10}{}^{-1} & -0.8809_{10}{}^{-2} & -0.5738 \\ -0.8584_{10}{}^{-4} & -0.1469_{10}{}^{-4} & -0.1554_{10}{}^{-1} \\ -0.2123 & -0.3642_{10}{}^{-1} & -0.3987_{10}{}^{+1} \end{bmatrix}$$

Figure 7.2 The first time moment $CA^{-1}B$ of (A, B, C)

BLOCK 1	1,2	-0.0479±j0.0119
	3	-0.0619
BLOCK 2	4	-0.154
	5	-0.212
BLOCK 3	6,7	-0.0479±j0.0119
	8	-0.224

Table 7.2 Eigenvalues of the reduced system

β_{12}	$-0.1709_{10}{}^{-1}$
β_{22}	0.4422
β_{32}	$0.8377_{10}{}^{+1}$
β_{13}	$0.1095_{10}{}^{-1}$
β_{23}	$-0.1185_{10}{}^{-1}$
β_{33}	$0.5548_{10}{}^{-1}$
β_{43}	$-0.3340_{10}{}^{-2}$
β_{53}	$-0.1427_{10}{}^{-1}$

Table 7.3 Values of the unknowns β_{ij} of Fig. 7.2

$$F = \left[\begin{array}{cccccccc}
0 & 0 & -0.1507_{10}-3 & & \beta_{12} & & & \beta_{13} \\
1 & 0 & -0.8366_{10}-2 & & \beta_{22} & & & \beta_{23} \\
0 & 1 & -0.1577 & & \beta_{32} & & & \beta_{33} \\
 & & & 0 & -0.3265_{10}-1 & & & \beta_{43} \\
 & & & 1 & -0.3660 & & & \beta_{63} \\
 & & & & & 0 & 0 & -0.54567_{10}-3 \\
 & & & & & 1 & 0 & -0.2390_{10}-1 \\
 & & & & & 0 & 1 & -0.3198
\end{array}\right]$$

Figure 7.3 Structure of F and G with unknown entries β_{ij}

$$K = [g_1 \; Fg_1 \; \ldots \; F^{11}g_1 \quad g_2 \; Fg_2 \; \ldots \; F^{10}g_2 \quad g_3 \; Fg_3 \; \ldots \; F^{8}g_3]$$

$$= [e_1 \; e_2 \; e_3 \; \beta_{*1} \; F\beta_{*1} \; \ldots \; F^{8}\beta_{*1} \; e_4 \; e_5 \; \beta_{*2} \; F\beta_{*2} \; \ldots \; F^{8}\beta_{*2} \; \ldots$$

$$\ldots \; e_6 \; e_7 \; e_8 \; \beta_{*3} \; F\beta_{*3} \; \ldots \; F^{6}\beta_{*3}]$$

Figure 7.4 Structure of the aggregation matrix K
(β_{*i} is the final column corresponding to block i of F)

Design #	Output Weighting Matrix S
1	diag(100, 10, 100)
2	diag(1000, 10, 1000)
3	diag(600, 100, 500)
4	diag(500, 1000, 1000)

Table 7.4 Various output weighting matrices used in the design problem

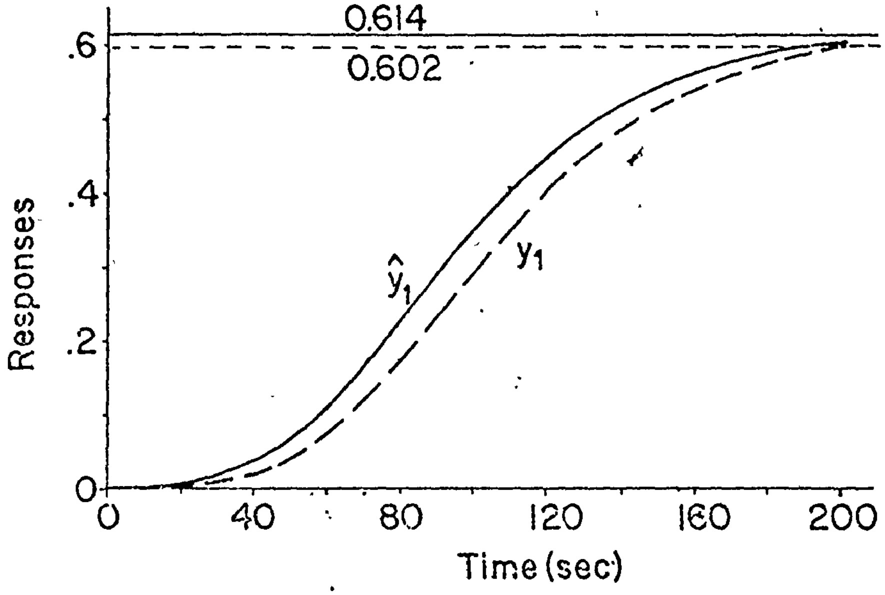


Fig. 7.5a

Figure 7.5 Step response of (A, B, C) and (F, G, H)

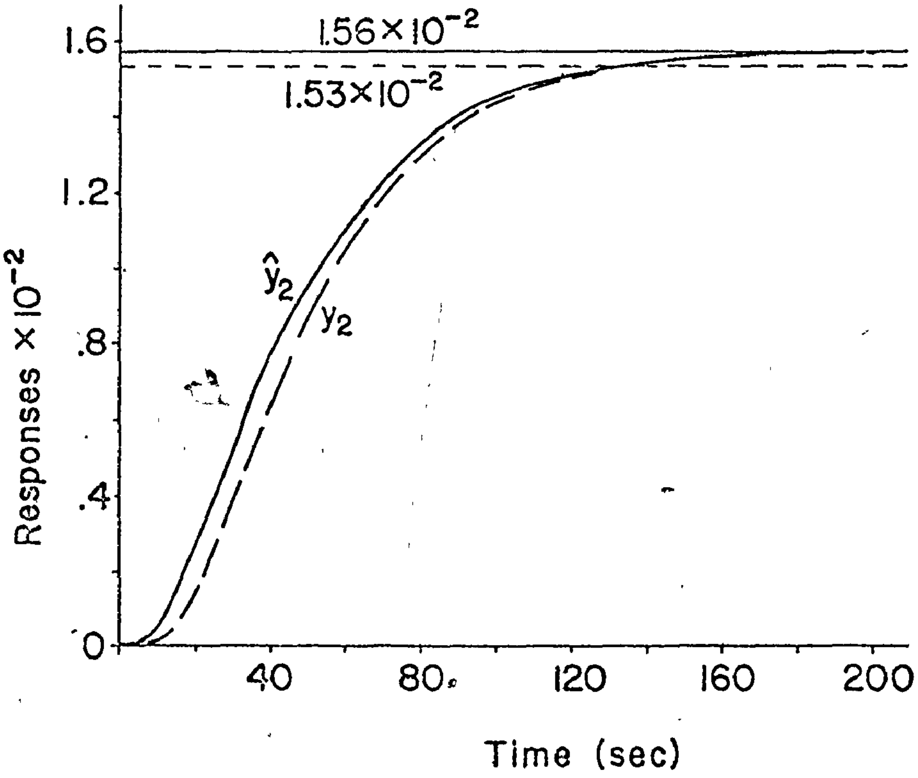

1.6
1.2
.8
.4
0
1.56×10^{-2}
1.53×10^{-2}
$\hat{y}_2$
y_2
Responses $\times10^{-2}$
40
80
120
160
200
Time (sec)

Fig. 7.5b

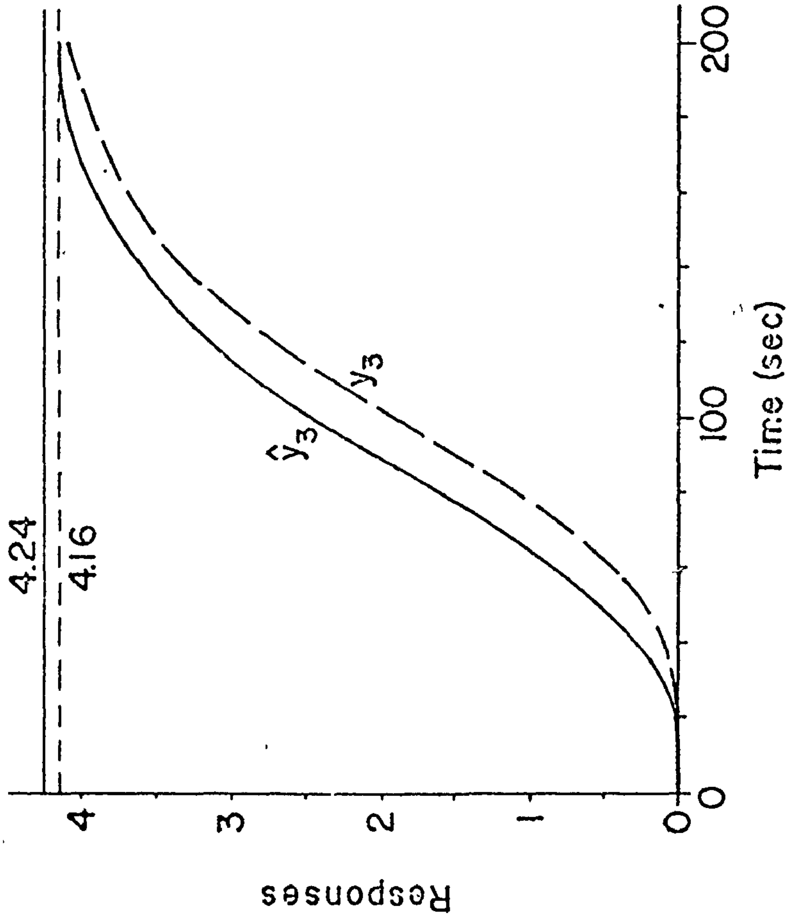


Fig. 7.5c

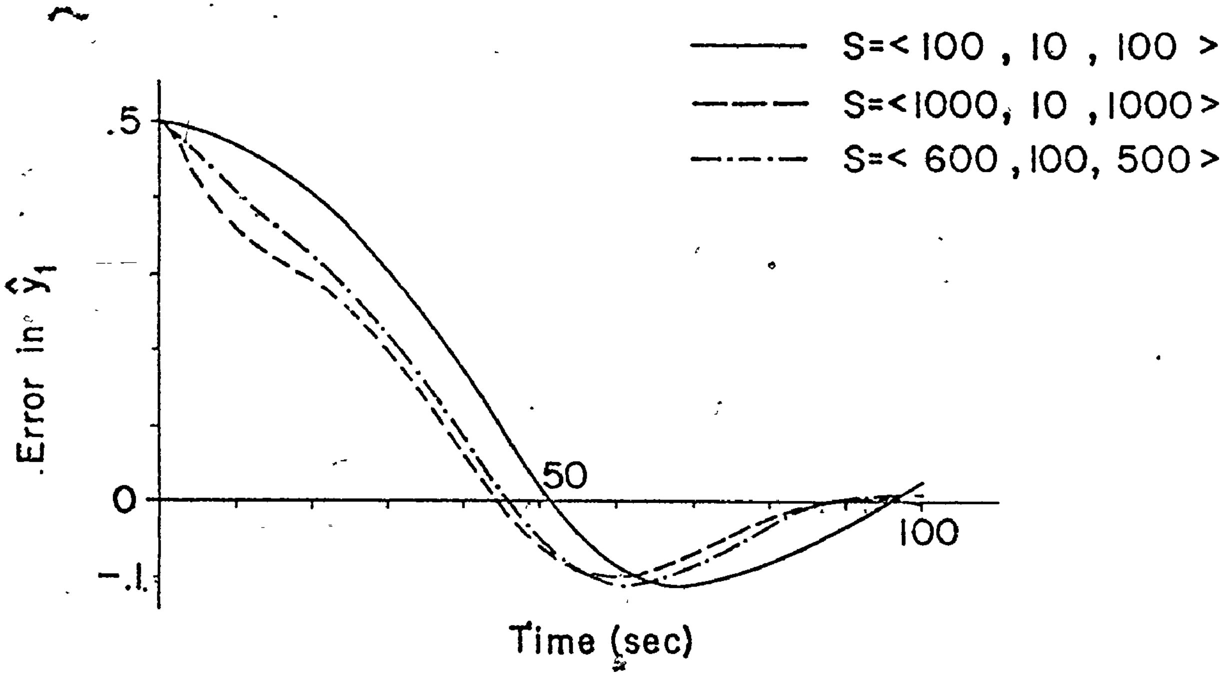


Fig. 7.6a

Figure 7.6 Optimal response of (F, G, H) to various output weighting matrices S

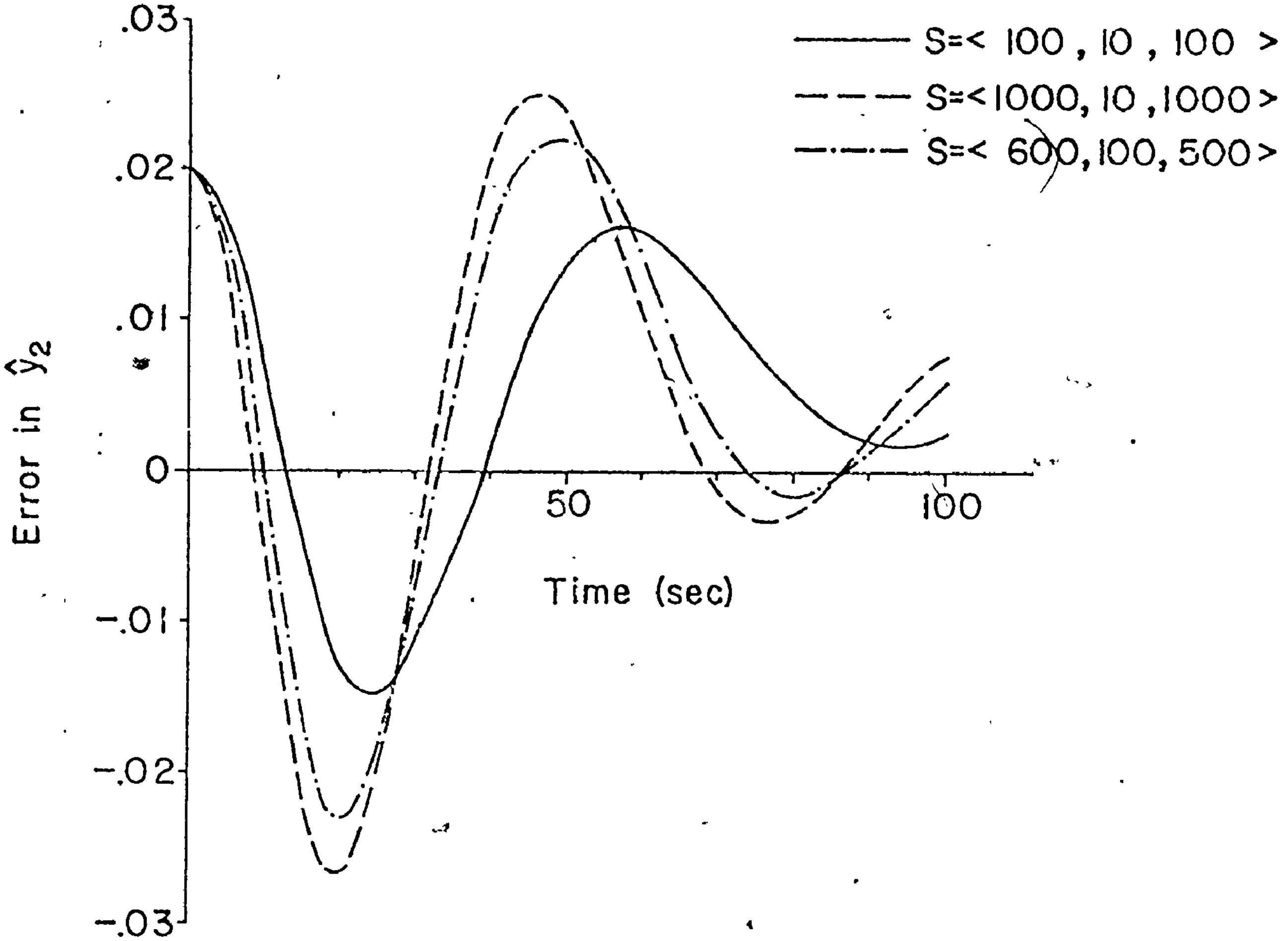

.03
.02
.01
0
-.01
-.02
-.03
Error in $\hat{y}_2$
50
100
Time (sec)
S=< 100 , 10 , 100 >
S=<1000, 10 ,1000>
S=< 600,100, 500>

Fig. 7.6b

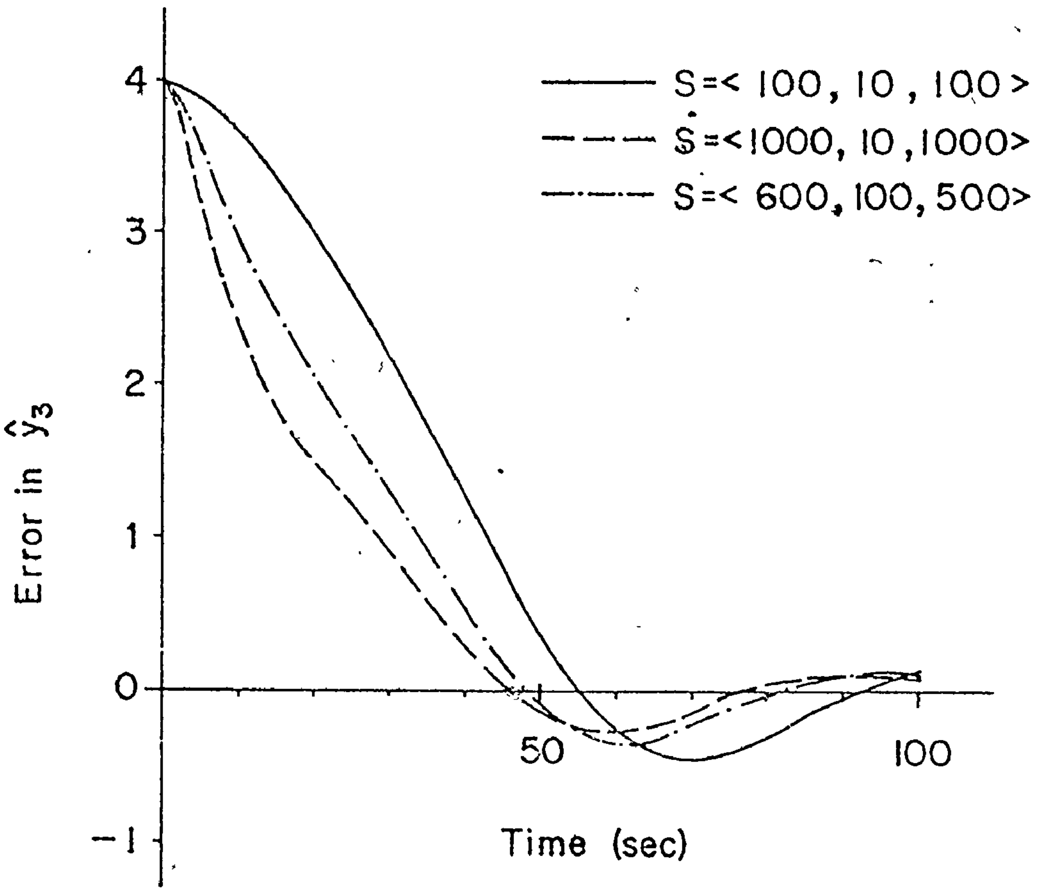

S=< 100, 10 , 100 >
S=<1000, 10 ,1000>
S=< 600, 100, 500>
4
3
2
1
0
−1
Error in $\hat{y}_3$
50
100
Time (sec)

Fig. 7.6c

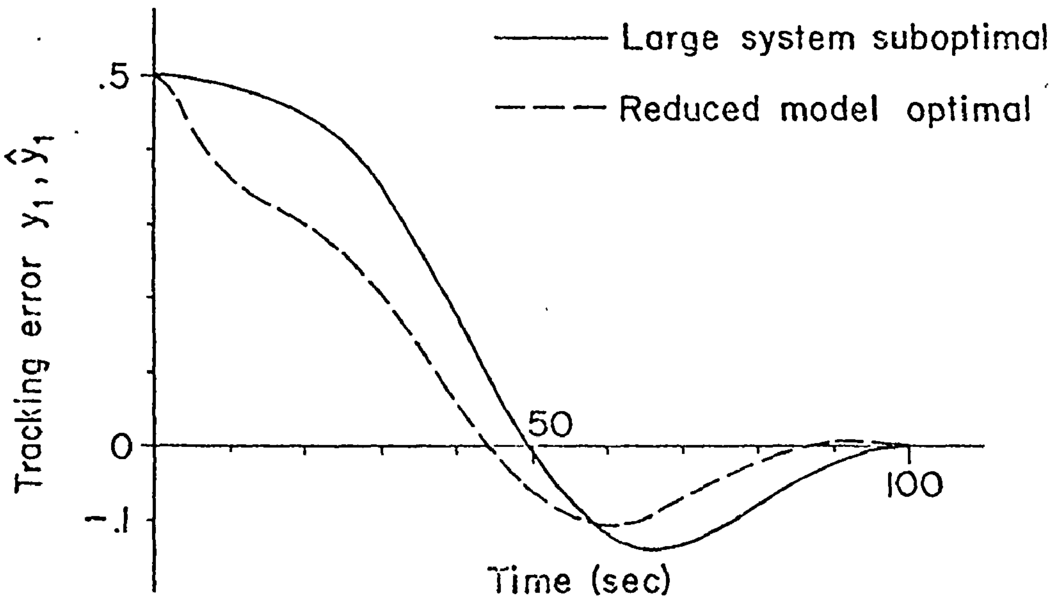


Fig. 7.7a

Figure 7.7 Optimal and suboptimal responses of (A, B, C)

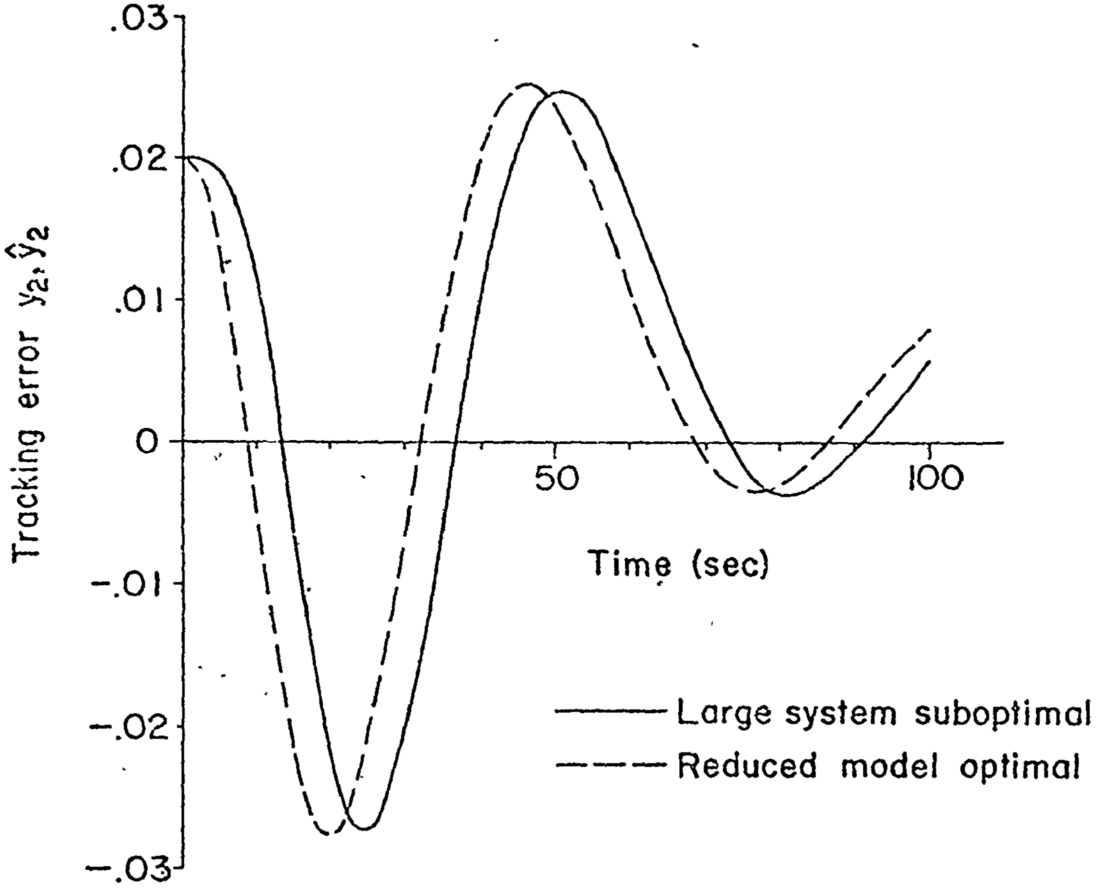

.03
.02
.01
0
-.01
-.02
-.03
Tracking error $y_2, \hat{y}_2$
50
100
Time (sec)
Large system suboptimal
Reduced model optimal

Fig. 7.7b

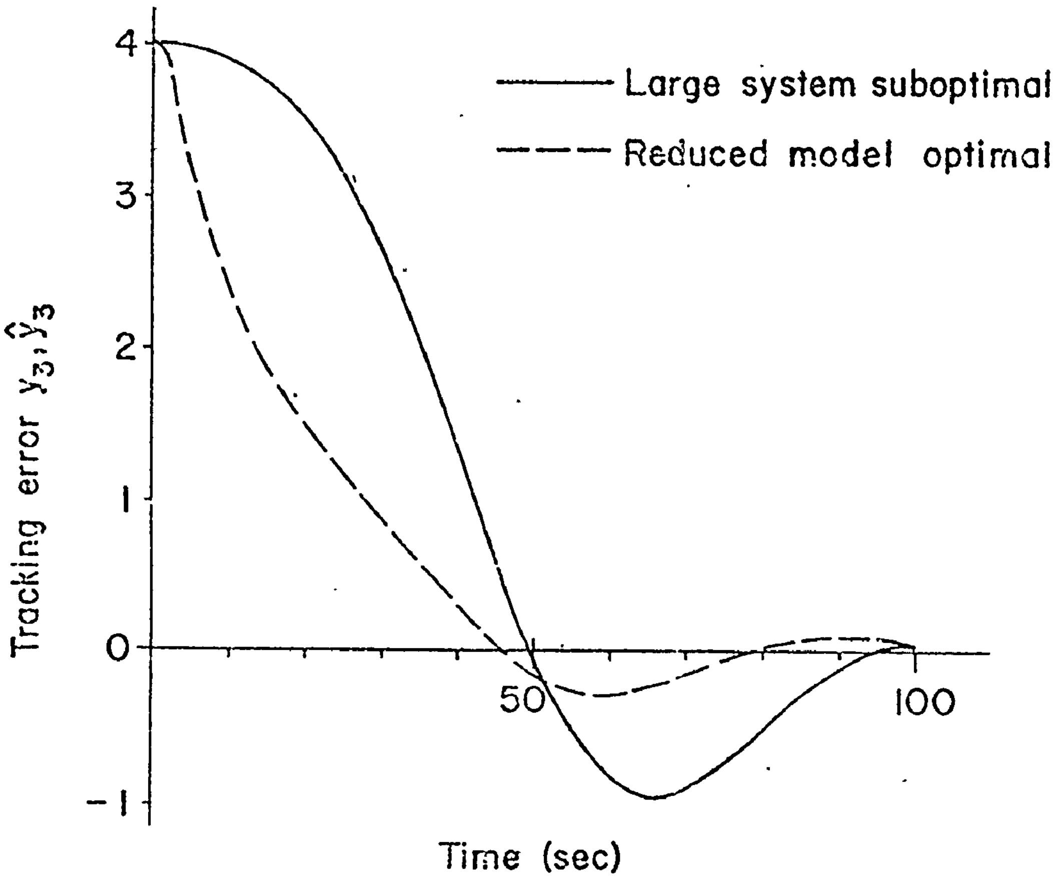

Large system suboptimal
Reduced model optimal
4
3
2
1
0
-1
50
100
Tracking error $y_3, \hat{y}_3$
Time (sec)

Fig. 7.7c

CHAPTER 8

CONCLUSIONS

The problem of approximating large-scale, linear, time-invariant dynamical systems has been investigated in detail. A close examination of many existing reduction methods has revealed that they may be classified in two main groups, those which preserve eigenvalues (aggregated models), and those which preserve part of the Taylor (or Laurent) series of the system transfer function (partial realizations). Furthermore, these two reduction methods may simultaneously be applied, and such approximants (aggregated partial realizations) are seen to be a special, but very important case of model reduction by nonminimal partial realization. Nonminimal partial realization is thus a unifying concept in the field of model reduction. The reaction of the large system to static compensation schemes derived from consideration of such reduced models was derived. Although some results were known for special cases of aggregated models, the extension to the case of partial realizations is new, and gives such reduced models a previously nonexistant credibility. By combining aggregation with partial realization it has been made possible to solve suboptimal control problems which could have not been successfully attacked before. Specifically, in the case of the suboptimal linear quadratic servomechanism problem, where the signal to be tracked is a polynomial function in time (for example, step functions), it is possible to ensure the stability of the large system and the convergence of the performance integral to a finite value.

The detailed theory of aggregation has resulted in a general formula for deriving aggregated reduced models and the aggregation matrix. While this result is not entirely new, the approach used proves that all aggregated reduced models preserve the eigenvalues of the original system. Previous work had started with this assumption a priori. An important result of this investigation is that the rows of the aggregation matrix are linear combinations of the rows of the inverse modal matrix of the large system. The use of the generalized column companion form was then considered in order to remove the necessity of calculating the inverse modal matrix. This canonical form also allows the easy matching of generalized Markov parameters.

Model reduction by partial realization was seen to be closely related to the Padé approximation problem. In fact, for scalar proper systems, the two theories are equivalent. This has directly led to a more general theory of the minimal realization problem and the introduction of generalized canonical forms. The iterative nature of the algorithm used to generate such realizations allows the designer great flexibility in choosing a reduced-order model of a large-scale system. Starting from the generalized Hankel matrix, models of increasing order may be generated without starting again all over from the beginning. This advantage is not shared by Chen's matrix continued fraction technique, which is widely accepted as being one of the most effective reduction techniques. It also represents an improvement in the fact that multivariable models having an unequal number of inputs and outputs may be treated, the reduced model order need not be a multiple of the number of inputs, and matrix polynomial inversion is not needed.

A nontrivial application of the theory to a system of order 33 is given. Here an eighth order reduced model is used in the design of a suboptimal linear quadratic servomechanism compensator. Selection of the weighting matrices for such problems is rather arbitrary, and most applications of the theory are reduced to trying different weighting schemes until a satisfactory response is achieved. For a large-scale system, this process will become very expensive, both in terms of the computer time required (several minutes per run) and the lengthening of the design process due to poor turn around time likely to be experienced. The alternative is to calculate an aggregated partially realized model of the system (which, admittedly, is likely to be expensive, but need only be done once) and design on the basis of the reduced model. The time saved per run will be of the order of the cube of the reduction ratio (number of states in the reduced model divided by the number of states in the large system), which for the problem considered is close to 97%. Since computer throughput will be greatly accelerated, it is plausible that the design process will be a matter of hours instead of days. Now on-line design of servomechanism compensators becomes worthwhile and will accelerate the process even more. There is, of course, a price to be paid, and the final design is suboptimal in the sense of the quadratic performance index. This seems of little import when a workable design has been achieved which otherwise might have never been attempted. In those cases where optimality is absolutely essential (which those with industrial experience will be quick to say are few or nonexistent) the derived suboptimal control law may serve as the starting approximation in an iterative solution scheme (c.f., Kleinmann, 1968).

It is hoped that the partial unification of model reduction of linear time-invariant dynamical systems given in this dissertation will encourage further work in their application to the design process. Some suggstions for further research are given in the next section.

8.1 Suggestions for Further Research

The discussion following Theorem 4.10 provides a useful clue for further research which should eventually lead to a class of approximants which include nonminimal partial realizations as a special case. In the proof of Theorem 4.4 the key result was that the set of generalized Markov parameters which are identically zero, $\{J_i = CA^iB = 0 \mid i = -i_0, -i_0+1, \ldots, 0, 1, \ldots, i_1\}$, is invariant under linear state variable feedback. Again, with the aggregation method described in Chapter 3, it is noticed that the eigenvalues of the large system which are not incorporated into the reduced model are linear state variable feedback invariants (c.f., Theorem 3.3). Recently the role of the transmission (or invariant) zeros (Rosenbrock, 1970) in model reduction has been examined (Shaked and Karcanias, 1976). The transmission zeros are also linear state variable feedback invariants. This evidence suggests that feedback invariants of the large system play an important part in the philosophy of model reduction. A deep study of this would probably be initiated by placing a module structure on the transfer function space of linear systems rather than the more usual vector space approach (c.f., Kalman, 1969). Here the possibility of tortion (zero divisors) could lead to a useful method of system reduction. The interested reader is referred to Jacobson (1974) for the definition of the various algebraic terms.

In this thesis, the use of the column companion form for determining aggregated partial realizations is proposed. An alternative approach would be the use of a real Schur form of the A and F matrices. In this case there would be no question as to the validity of a proposed F matrix but the calculation of the output matrix H to match generalized Markov parameters will become more complicated.

The matching of time moments is equivalent to matching the derivatives of the transfer function at $s = 0$. Perhaps the output matrix may also be chosen to duplicate some of the phase information as well (at a point $s = j\omega$ instead of $s = 0$). This is certainly possible in an ad hoc manner, for consider

$$g(s) = (5s+6)/(s+1)(s+2)(s+3)$$

$$\measuredangle g(j\omega) = \tan^{-1}(5\omega/6)-\tan^{-1}(\omega/3)-\tan^{-1}\{3\omega/(2-\omega^2)\}$$

If $\hat{g}(s)$ is taken as

$$\hat{g}(s) = (as+b)/(s+1)(s+2)$$

$$\measuredangle \hat{g}(j\omega) = \tan^{-1}(a\omega/b)-\tan^{-1}\{3\omega/(2-\omega^2)\}$$

then choosing

$$a = b\omega_0^{-1} \tan\{\tan^{-1}(5\omega_0/6)-\tan^{-1}(\omega_0/3)\}$$

will give

$$\measuredangle \hat{g}(j\omega_0) = \measuredangle g(j\omega_0)$$

Now b may be chosen to have $|\hat{g}(j\omega_0)| = |g(j\omega_0)|$. It is not clear how that technique may be extended to multivariable systems, but the transmission zeros should figure prominently.

In Chapter 5, some results on suboptimal control by aggregated partial realization were given which indicate that the method is promising. No analytic results were given, however, to indicate how good the suboptimal control actually is in comparison with the optimal control law. This contrasts sharply with the method of suboptimal control by singular perturbations (Chow and Kokotović, 1976), where $O(\mu)$ and $O(\mu^2)$ controls laws may be derived. In view of the similarity between the two methods, it should be possible to derive similar results. The main problem would be the explicit identification of the parameter μ.

No application of reduced-order models to filtering theory has been given in this dissertation. One line of research might be an investigation of filtering for systems with correlated input and output noise rather than the usual assumption of white Gaussian noise (more rigorously the derivative of a Wiener process). Here the dynamic equations of the system under consideration could be considered as an aggregated version of a larger system driven by white noise.

Alternatively, the use of partial realization to match the time moments of a large system might be useful in the case of estimation in the presence of a Wiener noise process.

Model reduction can also be used as a tool in system identification. In many cases, a high order model may be obtained by fitting gathered data. The real process, however, could be a low order system in cascade with a time delay so that the identified model would have high order due to the extra phase shift of the delay. Here, an aggregated partial realization, preserving the dominant poles and the DC steady state, in cascade with a suitable delay element could be effectively used.

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