LOCALIZATION IN NON-NOETHERIAN RINGS

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by

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A Thesis

Submitted to the School of Graduate Studies

in Partial Fulfilment of the Requirements

for the Degree

Doctor of Philosophy

McMaster University

April 1978

DOCTOR OF PHILOSOPHY (1978)
(Mathematics)

McMASTER UNIVERSITY
Hamilton, Ontario

TITLE: Localization in Non-Noetherian Rings

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SUPERVISOR: Professor B. J. Mueller

NUMBER OF PAGES: vi, 83

ABSTRACT

P. Gabriel constructed rings of quotients by inverting elements of multiplicative sets which satisfy the Ore and the reversibility conditions. We employ this technique in our study of localizations of non-noetherian rings at Goldie semiprime ideals. The three types of clans developed in this thesis enable us to decompose in a unique fashion (weakly) classical sets of prime ideals into (weak) clans which, in essence, are minimal localizable sets of prime ideals, satisfying certain properties. We further show that these (weak) clans are mutually disjoint sets. The different types of rings, brought into consideration, exhibit many interesting properties in the context of our localization theory.

We characterize the AR-property for the Jacobson radical of a semilocal ring by considering finitely generated modules. In the study of rings which are module-finite over their centres, we describe expressly the injective hull of the semilocal ring modulo its Jacobson radical. These two facts enable us to establish an interrelationship between the (strongly) classical semiprime ideals of the ring and those of its central subring. Furthermore, we show that under certain conditions the Q-sets are precisely all the minimal localizable sets of prime ideals of the ring. In the case of group rings, the flatness condition can be lifted without jeopardizing the validity of the assertion.

Lastly, we apply localization technique to characterize the group theoretic notion of q-nilpotency.

ACKNOWLEDGEMENTS

The author wishes to express his heartfelt gratitude to his supervisor Dr. B. J. Mueller whose unfailing guidance and encouragement have been of inestimable value during the preparation of this thesis. Special thanks also go to the author's friend J. Royle for his many helpful comments and suggestions.

The financial support provided by McMaster University is also gratefully acknowledged.

TABLE OF CONTENTS

INTRODUCTION

In the study of commutative rings, localization at multiplicative sets has been a well-understood and useful technique since the foundation of the theory. Similar techniques have been developed recently from several different standpoints to handle non-commutative rings. As a consequence of these generalizations, various concepts have evolved, for instance, localizing subcategory, torsion theory, Ore condition, etc. The scope of this dissertation covers only one aspect of localization in non-commutative rings. The approach, we have adopted here, was initiated by P. Gabriel who, in his thesis [6], discussed the Ore and the reversibility conditions on arbitrary multiplicative sets. The main advantage of his technique lies not only in the fact that it closely resembles the usual commutative ring localization but also that it provides a certain structure for the ring of quotients in which every element is explicitly expressible in terms of the elements of the original ring. Moreover, the resulting torsion theory is perfect, hence rendering an explicit way of describing the quotient functor. (See [22].)

In recent years, Gabriel's technique has been employed in studying localizability of semiprime ideals of non-commutative noetherian rings. J. Lambek and G. Michler ([16], [17]), A. V. Jategaonkar ([11], [12]) and B. J. Mueller ([23], [24]) are among those who have been working along this line of investigation. Our prime objective is to extract information from what has been known about noetherian ring localizations and to apply this to localizations of rings not necessarily noetherian. A few elementary results in the same direction have been obtained by J. Beachy and W. Blair [2] and overlap partly some work in [22].

The first chapter begins with some foundational work for our later undertaking. Here we make an introductory comment on the Ore and the reversibility conditions, and show by way of counterexamples that these two conditions are independent of one another. At the same time, we indicate certain classes of non-noetherian rings in which the reversibility condition can be deduced from the Ore condition.

Even at the initial stage of these developments, Goldie's Theorem serves as a key technique in our investigations. The definition of a localizable semiprime ideal entails Goldie's Theorem, the Ore and the reversibility conditions. The indispensability of Goldie's Theorem in this definition is elucidated by the endomorphism ring of an infinite dimensional vector space. Moreover, the localization R_S of a ring R at a localizable semiprime ideal S is a semilocal ring with SR_S as its Jacobson radical. (See [2] or [22]) This observation facilitates our further considerations.

The concept of a clan of prime ideals, introduced in [24] for noetherian rings, proves useful in the localization theory of non-noetherian rings as well. However, we find it necessary to formulate two definitions of clans which are termed "clans" and "strong clans".

At the present moment we have no example justifying this distinction. On the other hand, neither can we provide a proof to ascertain the equivalence of these two concepts in general. We, however, do have examples of certain classes of non-noetherian rings where these two concepts merge together. As the name implies, strong clans are always clans. The disjointness of clans, the unique decomposition of a classical set of prime ideals into clans and the building-up of a classical set from clans are all assured just as in the noetherian situation. The application of (strong) clans to the class of perfect rings is intended merely as an illustration of our theory and is by no means an exhaustive treatment.

In the second chapter, we introduce a variant concept of the theory developed in the first chapter. The incentive for doing this is derived from the notion of FP-injective modules which was studied by B. Stenström [34]. The concept of a weak clan, introduced here, extends the concept of a clan. Indeed, it is shown that clans are always weak clans. The two concepts coincide when the ring under consideration is noetherian. Although both of them share in common the properties indicated in the preceding paragraph, they are two distinct concepts; we include an example to substantiate this. In other words, prime ideals which constitute a clan remain together to form a weak clan. At the same time, under this new definition, more prime ideals may belong to weak clans even if they fail to belong to clans. The class of valuation rings is brought in for investigation: we find that all three types of clans coincide here and that the localization at a classical prime ideal gives rise to a noetherian local ring.

Rings which are finitely generated as modules over their centres constitute a rather important class of rings. This is the topic under study in the third chapter. A comprehensive localization theory has been formulated by B. J. Mueller [24] and P. F. Smith [33] in this area within the noetherian framework. Here we explore the interrelationship between the (strongly) classical semiprime ideals of the ring and those of its central subring by looking at the so-called Q-set. When Q ranges over all the prime ideals of the centre of the ring, we see that under certain conditions the Q-sets completely characterize the minimal localizable sets of prime ideals. It would be interesting to know if these constraints can be lifted. For group rings, we are able to safely remove one of the constraints.

Our pursuit in the third chapter also leads to an external characterization of the AR-property for the Jacobson radical J(R) of the semilocal ring R as well as to an explicit description of the injective hull of the R-module R/J(R).

Finally, group rings of finite groups over commutative rings provide substantial examples for this class of rings. Here we have patterned our arguments after [24] by employing block ideals in our deliberations. With the help of some group representation theory, we establish a characterization of a group theoretic property, namely, the concept of q-nilpotency, in terms of the localizability of a certain Q-augmentation ideal. Examples are listed to serve as an illustration.

CHAPTER I

AN APPROACH TO LOCALIZATION

Throughout this thesis all rings will have identity elements and all modules will be unitary right modules unless indicated otherwise. For any ring R, J(R) stands for the Jacobson radical of the ring. An ideal of R is always understood to be two-sided unless specified by a qualifier such as left or right. The same connotation extends to other concepts like noetherian, artinian, perfect, localizable, classical, etc. A regular element of R is a non-zero divisor. A standard notation for the injective hull of an R-module M is $E_R(M)$; when no confusion arises, we simply write E(M).

Just as for commutative rings, our localizations arise from suitable multiplicative subsets of the ring, which will be studied in the following section.

§1 THE ORE AND THE REVERSIBILITY CONDITIONS

Definitions. A multiplicative subset X of a ring R is a right Ore set if for any $r \in R$ and $s \in X$, there exist $r' \in R$ and $s' \in X$ such that $rs' = sr'$. It is called right reversible if $sr = 0$ for $s \in X$, $r \in R$ implies $rs' = 0$ for some $s' \in X$.

The left analogue is similarly defined. It should be noted that these two concepts are independent of each other. There are right reversible sets which are not right Ore, for instance, the set of all regular elements of a left Ore domain which is not right Ore. Conversely, a right Ore set is not necessarily right reversible as illustrated by the following example.

Example. Let R be the ring of endomorphisms of an $\aleph_0$-dimensional vector space V over a field K with basis $\{e_i \mid i \in \mathbb{N}\}$. Let $f : V \to V$ be the K-endomorphism given by $f(e_{2n}) = e_n$ and $f(e_{2n-1}) = 0$ for all $n \in \mathbb{N}$. Clearly, f is surjective and hence is left regular (that is, $hf = 0$ implies $h = 0$) in R, since V is a projective K-module. It is not right regular because it is not an automorphism. Let $X = \{1, f, f^2, \ldots\}$. A straightforward checking will verify that X is a right Ore but not a right reversible set.

However, there are rings in which the right Ore condition implies the right reversibility condition. This is obviously true for any domains. Another class of rings with this property consists of all those rings which satisfy the ascending chain condition on right annihilators of the form $\operatorname{ann}_R(c) \subset \operatorname{ann}_R(c^2) \subset \operatorname{ann}_R(c^3) \subset \ldots$, where $\operatorname{ann}_R(A) = \{r \in R \mid Ar=0\}$ denotes the right annihilator of a non-empty subset of the ring R. (cf. [35], Chapter II, Proposition 1.5.) Right perfect rings are members of this class. This is because any right perfect ring has the descending chain condition on principal left ideals (see [1]) and hence satisfies the ascending chain condition on right annihilators of the form prescribed above. In particular, semiprimary rings are examples of such rings. These are perfect rings with nilpotent Jacobson radicals.

Another type of rings which also belong to the aforesaid class is found in [13]. These are rings R with Krull dimension [8] such that Kd(I) = Kd(R) for all non-zero right ideals I of R. Here Kd(M) denotes the Krull dimension of an R-module M if it exists. It was shown that these rings satisfy the ascending chain condition on right annihilators at large. ([13], Theorem 7.)

Rings which can be embedded in rings with the ascending chain condition on right annihilators certainly inherit this property. Indeed, this is the situation where C. Procesi [28] proved that if R is an affine algebra over a commutative noetherian ring C, and if R can be embedded in a C-algebra S which is module-finite over its centre, then R has the ascending chain condition on right as well as on left annihilators. The crux of the proof of this statement lies essentially in the embedding of R in a noetherian subring of S.

We shall call a right Ore and right reversible set <u>right localizable</u>. P. Gabriel [6] has the following characterization for right localizable sets.

<u>Proposition 1.1</u>. For a multiplicative subset X of a ring R, the following conditions are equivalent:

(1) X is a right localizable set.

(2) There exists a classical right quotient ring for X.

Such classical right quotient ring is usually denoted by R_X. It is well-known that if X is a localizable set, then the classical right quotient ring for X coincides with the classical left quotient ring for X.

§2 RIGHT GOLDIE SEMIPRIME IDEALS

Definition. A ring R is a right Goldie ring if it has the following properties:

(i) R is a (semi)prime ring,

(ii) R has finite Goldie dimension, and

(iii) R satisfies the ascending chain condition on right annihilators.

A right Goldie ring is precisely the one which has a (semi)simple artinian classical right quotient ring for the set of all regular elements. This fact is generally known as Goldie's Theorem. ([7])

For a semiprime ideal S of a ring R, we define a multiplicative set $C(S) = \{c \in R \mid c \text{ is regular modulo } S\}$. S is called right Goldie if R/S is a right Goldie ring. In this case, C(S) coincides with the set $\{c \in R \mid cx \in S \text{ implies } x \in S\}$.

The purpose of this section is to investigate some of the basic properties of right Goldie semiprime ideals. In [12], the right Ore condition of C(S) is characterized in terms of $E_R(R/S)$ for a semiprime ideal S of a right noetherian ring R. We want to show that this characterization is also true for non-noetherian rings at large.

Notation. For any ring R, let mod-R be the category of all R-modules, and S a right Goldie semiprime ideal of R. Then the S-torsion theory is the one determined by $C(S)$, or equivalently cogenerated by $E(R/S)$. We shall denote this torsion theory by $(T_S, F_S, \rho_S, \theta_S)$ where T_S is the torsion class, F_S is the torsion-free class, ρ_S is the torsion radical and θ_S is the Gabriel filter.

For any R-module M, $m \in M$ and a submodule N of M, let $m^{-1}N = \{r \in R \mid mr \in N\}$. The closure of N in M with respect to the S-torsion theory is $\{m \in M \mid m^{-1}N \in \theta_S\}$. In short, it will be called the S-closure of N in M. For any right ideal I of R, we shall simply speak of the S-closure of I with the understanding that it is taken in R.

Proposition 1.2. Let S be a right Goldie semiprime ideal of a ring R. Then $C(S)$ is right Ore if and only if every element of $C(S)$ operates regularly on $E(R/S)$. (That is, for any $e \in E(R/S)$, $c \in C(S)$, $ec = 0$ implies $e = 0$.)

Proof. Suppose $C(S)$ is a right Ore set and there exist non-zero $e \in E(R/S)$ and $c \in C(S)$ with $ec = 0$. By essentiality of $E(R/S)$, there exists $r \in R$ such that $0 \neq er \in R/S$. Moreover, the right Ore condition of $C(S)$ implies $rc' = cr'$ for some $r' \in R$, $c' \in C(S)$, and so $erc' = ecr' = 0$, forcing $er\overline{c'} = 0$ in R/S. But $\overline{c'}$ is a regular element of R/S. Hence $er = 0$, a contradiction.

Conversely, assume that every element of $C(S)$ operates regularly on $E(R/S)$. Our first claim is that $R/cR \in T_S$ for any $c \in C(S)$. Suppose on the contrary that there exists some $c \in C(S)$ with $R/cR \notin T_S$.

Then for such element c, there must exist a non-zero R-homomorphism $f : R/cR \to E(R/S)$. Let $e = f(\bar{1})$ which is obviously non-zero. However, $ec = f(\bar{1})c = f(\bar{c}) = 0$, contradicting the assumption. This proves our claim. That means $cR \in \theta_S$ for all $c \in C(S)$. Hence, $D = \{x \in R \mid rx \in cR\} \in \theta_S$ for any given $r \in R$, and so $rD \subset cR$. Pick an element $c' \in D \cap C(S)$. Then $rc' = cr'$ for some $r' \in R$. ||

Given an S-torsion theory, its quotient ring will be denoted by R_S. When $C(S)$ is right localizable, R_S is actually the classical right quotient ring for $C(S)$. Henceforth, we will call a semiprime ideal S <u>right localizable</u> if it is right Goldie and $C(S)$ is a right localizable subset of R. One further point to be noted is that in any ring, a right Goldie semiprime ideal S is uniquely expressible as a finite irredundant intersection of prime ideals. Each of these prime ideals is right Goldie, and they account for all the minimal prime ideals over S. ([22])

<u>Proposition 1.3</u>. Let $S = \bigcap_{i=1}^{n} P_i$ be a right localizable semiprime ideal of a ring R and $T = \bigcap_{i\in I} P_i$ for some subset I of $\{1,\ldots,n\}$. Then $C(T)$ is right Ore (respectively, right reversible) in R if and only if $C(TR_S)$ is right Ore (respectively, right reversible) in R_S.

<u>Proof</u>. (1) First we claim that $C(TR_S) = \{cs^{-1} \in R_S \mid c \in C(T)\}$. Let $c \in C(T)$. It suffices to show that $c1^{-1} \in C(TR_S)$. Suppose $c1^{-1}at^{-1} \in TR_S$, that is, $cat^{-1} \in TR_S$. Because T is S-closed, $ca \in T$ which then implies $a \in T$ as $c \in C(T)$. Hence $at^{-1} \in TR_S$ and so $c1^{-1} \in C(TR_S)$.

Conversely, suppose $cs^{-1} \varepsilon C(TR_S)$. Let $cx \varepsilon T$ for some $x \varepsilon R$. Then $cs^{-1}sx1^{-1} \varepsilon TR_S$ implies $sx1^{-1} \varepsilon TR_S$, from which it follows that $x1^{-1} \varepsilon TR_S$ since s is invertible in R_S. Hence $x \varepsilon T$. This proves our claim.

(2) Next we want to show that both $C(T)$ and $C(TR_S)$ are right Ore if either one is. Observe that $E_R(R/T)$ takes on an R_S-module structure and $E_R(R/T) = \bigoplus_{i\varepsilon I} E_R(R/P_i) \simeq \bigoplus_{i\varepsilon I} E_{R_S}(R_S/P_iR_S) = E_{R_S}(R_S/TR_S)$ as R_S-modules. Proposition 1.2 and (1) above then complete the proof.

(3) Finally it remains to show that both $C(T)$ and $C(TR_S)$ are right reversible if either one is. First, we assume the right reversibility for $C(T)$ and let $cs^{-1} \varepsilon C(TR_S)$, $at^{-1} \varepsilon R_S$ with $cs^{-1}at^{-1} = 0$. Then $s^{-1}a = bd^{-1}$ for some $bd^{-1} \varepsilon R_S$. So $cs^{-1}at^{-1} = cbd^{-1}t^{-1} = (cb)(td)^{-1} = 0$ which means $cbx = 0$ for some $x \varepsilon C(S)$. By assumption and (1) above, there exists $c' \varepsilon C(T)$ with $bxc' = 0$. Now we have $(at^{-1})(tdxc')1^{-1} = (adxc')1^{-1} = (sbxc')1^{-1} = 0$. This establishes the right reversibility of $C(TR_S)$ as $tdxc' \varepsilon C(T)$.

Conversely, suppose $C(TR_S)$ is right reversible. Let $c \varepsilon C(T)$ and $r \varepsilon R$ with $cr = 0$. Then $(c1^{-1})(r1^{-1}) = 0$ in R_S. By assumption, there exists $st^{-1} \varepsilon C(TR_S)$ with $(r1^{-1})(st^{-1}) = 0$, from which we have $rsd = 0$ for some $d \varepsilon C(S)$. This proves the right reversibility of $C(T)$ as $sd \varepsilon C(T)$. ||

Remark. While right localizable semiprime ideals of a ring are right Goldie by definition, the converse is false. (See [24], Lemma 12, for instance.) There are non-noetherian rings where none of the prime ideals is Goldie; below is an example.

Example. Consider again the example in §1. Let I be the set of all endomorphisms of finite rank. Then I is an ideal of R. In fact, 0 and I are the only prime ideals of R and are not Goldie. We will provide proofs of these facts for the convenience of the reader.

Claim 1. 0 and I are the only prime ideals of R.

Proof. Note that I is the only non-zero ideal of R, so it is maximal, hence prime. To show that 0 is prime, suppose $\phi R\psi = 0$ and $\psi \neq 0$. Let $\mathcal{B}$ be a basis for $\operatorname{im}\psi$ and complete it to a basis $\mathcal{D}$ of V. $\phi\psi = 0$ implies $\operatorname{im}\psi \subset \ker\phi$ from which $\phi(w) = 0$ for all $w \in \mathcal{B}$. Take any $y \in \mathcal{D} - \mathcal{B}$ and $w \in \mathcal{B}$. Define an endomorphism $f : V \to V$ by

$$f(x) = \begin{cases} y & \text{if } x = w \\ 0 & \text{if } x \in \mathcal{D} - \{w\} \end{cases}$$

Let $z \in V$ with $\psi(z) = w$. By assumption, $\phi f\psi = 0$, that is, $0 = \phi f\psi(z) = \phi f(w) = \phi(y)$. Hence $\phi = 0$. ||

Claim 2. Both 0 and I are not Goldie.

Proof. Take a basis $\{v_i \mid i \in \mathbb{N}\}$ of V. For every prime number p, define an endomorphism $f_p : V \to V$ by

$$f_p(v_i) = \begin{cases} v_i & \text{if } i \text{ is a power of } p \\ 0 & \text{otherwise} \end{cases}$$

Note that $(f_p)^2 = f_p$ and $f_p f_q = 0$ if $p \neq q$. So these endomorphisms produce an infinite direct sum in R. Therefore, R does not have finite Goldie dimension. In other words, 0 is not right Goldie.

Clearly, all $f_p \notin I$ and so $\bar{f}_p \neq 0$ in R/I. By the same token, R/I does not have finite Goldie dimension. Hence I is not right Goldie. Likewise, 0 and I are also not left Goldie. ||

The above example further illustrates the following fact. Observe that $\mathcal{C}(0)$ consists of all automorphisms of V, and hence satisfies the Ore and the reversibility conditions. However, the classical quotient ring of R for $\mathcal{C}(0)$ is R itself, which is not a Goldie ring by Claim 2. This observation clearly indicates that in our definition of a right localizable semiprime ideal S, the right localizability of C(S) alone is insufficient to make S right Goldie.

On the other hand, there are rings where every prime ideal is right Goldie, such as commutative rings, left or right perfect rings, PI rings and rings with Krull dimension. Prime ideals of the first three types of rings are even left Goldie. The reason for being so varies in each case. For commutative rings, every prime ideal is completely prime, hence Goldie. For one-sided perfect rings R, every prime ideal P contains J(R) and so R/P is simple artinian. In the case of PI rings, Posner's Theorem [28] accounts for this fact. Finally, it has been shown in [8] that a semiprime ring with Krull dimension is right Goldie.

§3 CLASSICAL SEMIPRIME IDEALS AND PERFECT RINGS

The notion of classical semiprime ideals has been studied mainly in noetherian rings, for instance, in [12], [17], [24] and [32]. Here we adopt this notion for the study of non-noetherian rings. As a preliminary attempt, we will investigate it in the context of perfect rings.

<u>Definitions</u>. (a) An ideal I of a ring R is said to have the <u>right AR-property</u> if for every right ideal A of R, there exists an integer $n > 0$ such that $A \cap I^n \subset AI$.

(b) A right localizable semiprime ideal S of a ring R is called <u>right classical</u> if $E(R/S) = \bigcup_{n=1}^{\infty} \operatorname{ann}_{E(R/S)} S^n$. It is called <u>strongly right classical</u> if $J(R_S)$ has the right AR-property.

It can be easily verified that a strongly right classical semiprime ideal is always right classical. (cf. [17], Proposition 4.3.) For noetherian rings, there is no distinction between these two definitions. Whether this will be so for non-noetherian rings in general is yet to be settled. However, there are quite a few kinds of non-noetherian rings where such a distinction also disappears. Such examples will be given in §5.

The proof of the next proposition is adapted from [24] for semiprimary rings. We include the proof here as we will need it later for right perfect rings.

<u>Proposition 1.4</u>. Let S be a semiprime ideal of a semiprimary ring R. Then the following conditions are equivalent:

(1) S is strongly right classical.

(2) S is right localizable.

(3) S has the right AR-property.

(4) There exists an idempotent element $e \in R$ such that $eR(1-e) = 0$ and $S = Re + J(R) = eR + J(R)$.

In this situation, R_S is a semiprimary ring.

Proof. (1) implies (2) trivially. Given (2), let $K = \rho_S(R)$. First, we claim that $C(S) \subset C(K)$ and $K \subset S$. Let $c \in C(S)$ and suppose $cx \in K$ for some $x \in R$. Then there exists $y \in C(S)$ with $cxy = 0$. Since $C(S)$ has the right reversibility condition, $xyc' = 0$ for some $c' \in C(S)$, which implies $x \in K$. Thus $C(S) \subset C(K)$. That $K \subset S$ is obvious from the definitions of K and of $C(S)$. This proves our claim.

Now we put $S = \bigcap_{i=1}^{n} P_i$ which is the unique representation of S as a finite irredundant intersection of prime ideals. Our next claim is that the P_i are the only prime ideals containing K. Suppose this is not the case. Then let Q be a prime ideal containing K but different from all the P_i. Since prime ideals of R are maximal, $Q + P_i = R$ for all $i = 1,\ldots,n$. This implies $C(P_i) \subset Q + P_i$ and thus $C(P_i) \cap Q \neq \emptyset$ for all i. For each i, pick an element $c_i \in C(P_i) \cap Q$. Then there exist $x_i \in R$ such that $c = \sum_{i=1}^{n} c_i x_i \in C(S)$. So $c \in C(K) \cap Q$. That means c is right regular, hence invertible in R/K. We then have $cx = 1 + k$ for some $x \in R$ and $k \in K$, from which follows $1 = cx - k \in Q$, a contradiction. This proves our second claim. Therefore, $J(R/K) = \bigcap_{i=1}^{n} P_i/K = S/K$. By nilpotency of the Jacobson radical, there exists an integer $m > 0$ with $S^m \subset K$.

To verify the right AR-property, take any right ideal A of R. We now claim that $A \cap S^m \subset AS$. Let $r \in A \cap S^m$. Then $rd = 0$ for some $d \in C(S)$ because $r \in K$. Note that d is invertible modulo S, and so $dz = 1 + s$ for some $z \in R$ and $s \in S$. Therefore, $rdz = r + rs = 0$, or simply $r = -rs \in AS$, thus proving (3).

Assume (3). Since R is a semiperfect ring, by lifting idempotent modulo $J(R)$, there exists an idempotent $e \in R$, unique and central modulo $J(R)$, with $S = eR + J(R) = Re + J(R)$. It remains to show that $eRf = 0$ where $f = 1-e$. Applying the right AR-property to the ideal $A = RfR$, we obtain an integer $n > 0$ with $A \cap S^n \subset AS$. Observe that $e \in S^n$. Therefore, $eRf \subset A \cap S^n \subset AS = RfR(Re + J(R)) = RfRe + RfJ(R)$, which leads to $eRf \subset eRfJ(R)f = eRfJ(fRf)$. This implies $eRf = 0$ since fRf is a perfect ring and $eRfJ(fRf)$ is small in eRf as a right fRf-submodule.

Finally, the implication of (1) from (4) proceeds as follows: we first identify the ring R with the matrix ring

$$\begin{pmatrix} eRe & 0 \\ fRe & fRf \end{pmatrix}.$$

Then S is the ideal

$$\begin{pmatrix} e(Re + J(R))e & 0 \\ f(Re + J(R))e & f(Re + J(R))f \end{pmatrix} = \begin{pmatrix} eRe & 0 \\ fRe & fJ(R)f \end{pmatrix}$$

and $C(S)$ is the multiplicative set

$$\left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \;\middle|\; c \text{ is invertible in } fRf \right\}.$$

S is evidently a Goldie semiprime ideal. To show that it is right localizable, take any elements

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in C(S) \quad \text{and} \quad \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \in R.$$

A direct checking will verify that

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & c^{-1}z \end{pmatrix} = \begin{pmatrix} x & 0 \\ y & z \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence $C(S)$ has the right Ore condition, from which follows the right reversibility condition since R is semiprimary. Moreover,

$$\rho_S(R) = \begin{pmatrix} eRe & 0 \\ fRe & 0 \end{pmatrix}.$$

Hence, $R_S \simeq R/\rho_S(R) \simeq fRf$, that is, R_S is a semiprimary ring, and $J(R_S)$, being nilpotent, obviously has the AR-property. ||

<u>Remark</u>. If the ring R in the preceding proposition is merely a right perfect ring, we still retain the equivalence of (2) and (4); the proof will be given below.

<u>Notation</u>. For any right perfect ring R, let I^α denote the right transfinite powers of an ideal I of R, defined inductively as follows: $I^\alpha = I^\beta I$ for $\alpha = \beta + 1$; $I^\alpha = \bigcap_{\beta<\alpha} I^\beta$ if α is a limit ordinal. It is easy to check that $J(R)^\alpha = 0$ for some ordinal α. In the same manner, we define the left transfinite powers of an ideal of a left perfect ring.

<u>Proposition 1.5</u>. Let S be a semiprime ideal of a right perfect ring R. Then the following conditions are equivalent:

(1) S is right localizable.

(2) There exists an idempotent element $e \in R$ such that $eR(1-e) = 0$ and $S = Re + J(R) = eR + J(R)$.

In this situation, R_S is a right perfect ring.

<u>Proof</u>. We go over the proof of the implication of (4) from (2) via (3) in Proposition 1.4, replacing $J(R/K)^m$ with an appropriate right transfinite power. Then we obtain the implication of (2) from (1). The proof of the converse implication is identical with that of (4) implying (1) in the preceding proposition, except that we do not get the right

AR-property for $J(R_S)$. That R_S is right perfect is also evident. ||

The left analogue of Proposition 1.5 can be easily formulated for left perfect rings. With this we obtain immediately the following corollary.

Corollary 1.6. If R is a ring-directly indecomposable perfect ring, then its Jacobson radical is the only localizable semiprime ideal.

Proof. The assertion follows directly from (2) of Proposition 1.5 and its left analogue. ||

Proposition 1.7. Let S be a right localizable semiprime ideal of a left perfect ring R. Then S is strongly right classical if and only if it has the right AR-property. Moreover, R_S is semiprimary.

Proof. Suppose S is strongly right classical. First we want to show that R_S is semiprimary. Let A be the right R_S-socle of $J(R_S)$. Then there exists an integer $n > 0$ such that $A \cap J(R_S)^n \subset AJ(R_S)$. But $AJ(R_S) = 0$, and so $A \cap J(R_S)^n = 0$. Therefore $J(R_S)^n = 0$ by essentiality of A. This proves that R_S is semiprimary. Since $J(R_S) = SR_S$, we have $S^nR_S = 0$, implying $S^n \subset \rho_S(R)$. A direct checking verifies $I \cap S^n \subset IS$ for any right ideal I of R.

Conversely, assume the right AR-property for S. We need to prove that $J(R_S)$ also has the right AR-property. Let A be any right ideal of R_S and $I = \varepsilon_S^{-1}(A)$ where $\varepsilon_S : R \to R_S$ is the localization map. Then there exists an integer $n > 0$ such that $I \cap S^n \subset IS$, which then yields

$IR_S \cap S^n R_S \subset IR_S SR_S$ by flatness of R_S as a left R-module. ||

Corollary 1.8. The Jacobson radical of a left perfect ring R has the right AR-property if and only if R is semiprimary.

Remark. For a semiprime ideal S of a right perfect ring R, the right AR-property for S is sufficient to make S strongly right classical.

§4 THE STRUCTURE OF CLANS

This section studies the structure of classical set of prime ideals. First and foremost, given such a set, we will partition it into mutually disjoint non-empty subsets in a certain way that each subset is a clan. Secondly, we will prove that no two distinct clans contain a common prime ideal, and that a classical set of prime ideals can be constructed from clans.

To begin with, a few remarks on notation and terminology are necessary. Two prime ideals are incomparable if neither one of them is a subset of the other. A non-empty finite set $\{P_1,\ldots,P_n\}$ of pairwise incomparable prime ideals of a ring R is a (strongly) classical set if the associated semiprime ideal $S = \bigcap_{i=1}^{n} P_i$ is (strongly) classical. Such a set is a (strong) clan if no proper non-empty subset of it is (strongly) classical. In general, we shall also speak of a localizable set of prime ideals when its associated semiprime ideal is localizable.

Recall that a (semi)local ring R is a ring such that $R/J(R)$ is (semi)simple artinian. Given such a ring R, we denote by $\hat{R}$ the completion of R with respect to the $J(R)$-adic topology on R. For brevity, it is usually called the $J(R)$-adic completion. When $J(R)^{\omega} = \bigcap_{n=1}^{\infty} J(R)^n = 0$, we may identify R with a subring of its Hausdorff completion $\hat{R}$. Moreover $\hat{R}$ is a semiperfect ring with $J(\hat{R}) = J(R)^{\wedge}$, the closure of $J(R)$ in $\hat{R}$. In general, $I^{\wedge}$ shall denote the closure in $\hat{R}$ of any right, left or two-sided ideal I of R. Note that $\hat{R}/(J(R)^n)^{\wedge} \simeq R/J(R)^n$ for all $n > 0$. The reader may consult [15] and [36] for more details of $J(R)$-adic topology and completion.

Lemma 1.9. If a semiprime ideal S of a ring R is right classical in R, so is $J(R_S)$ in R_S.

Proof. The assertion follows obviously from the fact that $E_R(R/S)$ takes on an R_S-module structure and is indeed the injective hull of $R_S/J(R_S)$ in mod-R_S. ||

Lemma 1.10. If R is a semilocal ring with a right classical Jacobson radical, then the $J(R)$-adic topology on R is Hausdorff.

Proof. (1) Let $e \in E(R/J(R))$. Then $eJ(R)^n = 0$ for some $n > 0$, and a fortiori $eJ(R)^{\omega} = 0$. Hence $E(R/J(R))J(R)^{\omega} = 0$.

(2) Let X be an R-module with an essential socle U. Then $U = \bigoplus_{i \in I} U_i$ where each U_i is a simple submodule of X. Since R is semilocal, each U_i is isomorphic to some submodule of $R/J(R)$. Let $\delta : U \to \prod_{i \in I} E(R/J(R))$ be the composite of two canonical R-monomorphisms $U \to \prod_{i \in I} R/J(R)$ and $\prod_{i \in I} R/J(R) \to \prod_{i \in I} E(R/J(R))$. Then δ extends to a map $\delta' : X \to \prod_{i \in I} E(R/J(R))$ which is an R-monomorphism by essentiality of U in

X. It follows from (1) that $XJ(R)^{\omega} = 0$.

(3) For a given right ideal A of R, consider the composite $\xi : A \to E(A/AJ(R))$ of two canonical homomorphisms $A \to A/AJ(R)$ and $A/AJ(R) \to E(A/AJ(R))$. Then there exists an element $e \in E(A/AJ(R))$ such that $\xi(x) = ex$ for all $x \in A$. Note that $A/AJ(R)$ is an essential socle of $E(A/AJ(R))$. Hence $E(A/AJ(R))J(R)^{\omega} = 0$ by (2). In particular, $eJ(R)^{\omega} = 0$. That means $A \cap J(R)^{\omega} \subset A \cap \ker \xi = AJ(R)$.

(4) Now let $x \in J(R)^{\omega}$ and $A = xR$. By (3), $A \cap J(R)^{\omega} \subset AJ(R)$. That is, $A = AJ(R)$. Thus $x = 0$ by Nakayama's Lemma. ||

<u>Lemma 1.11</u>. Let $S = \bigcap_{i=1}^{m} P_i$ be a right Goldie semiprime ideal of a ring R such that $E(R/S) = \bigcup_{n<\omega} \operatorname{ann}_{E(R/S)} S^n$. Suppose $T = \bigcap_{i \in I} P_i$ with $I \subset \{1,\dots,m\}$ is such that T/S^n is right localizable in R/S^n for all $n > 0$. Then T is right classical in R.

<u>Proof</u>. (1) First we want to show that $C(T)$ is a right Ore set. Suppose on the contrary that there is some non-zero element $e \in E(R/T)$ such that $ec = 0$ for some $c \in C(T)$. Since $E(R/T) \subset E(R/S)$, $eS^{n} = 0$ for some integer $n > 0$. So $e \in \operatorname{ann}_{E(R/T)} S^n = E_{R/S^n}(R/T)$. By assumption, the elements of $C(T/S^n)$ operate regularly on $E_{R/S^n}(R/T)$ as $R/T \simeq (R/S^n)/(T/S^n)$. However, $C(T/S^n) = C(T) + S^n/S^n$. Therefore $e\bar{c} = ec = 0$, a contradiction.

(2) Next we claim that $C(T)$ is a right reversible set. Take any $c \in C(T)$ and $r \in R$ with $cr = 0$. Pick an arbitrary element $e \in E(R/T)$.

Then $eS^n = 0$ for some integer $n > 0$. By assumption, $C(T/S^n)$ is right reversible in R/S^n. So $\bar{c}\bar{r} = 0$ in R/S^n implies the existence of an element $b \in C(T)$ with $rb = h \in S^n$. Thus $erb = eh = 0$ which then implies $er = 0$ by (1). As e is an abitrary element, we have $r \in \text{ann}_R E(R/T) = \rho_T(R)$. Hence there exists $c' \in C(T)$ such that $rc' = 0$ since $C(T)$ is right Ore.

(3) It remains to show that $E(R/T) = \bigcup_{n=1}^{\infty} \text{ann}_{E(R/T)} T^n$. Since $P_j R_T = R_T$ for all $j \notin I$, $SR_T = TR_T$. Take any $e \in E(R/T)$. Then $eS^n = 0$ for some integer $n > 0$. Therefore $0 = eS^n R_T = e(SR_T)^n = eT^n R_T$ as both S and T are Goldie. Hence $eT^n = 0$, given the fact that $E(R/T)$ is also an R_T-module. ||

We now come to the first structure theorem for classical semiprime ideals.

Theorem 1.12. Let R be a ring with a (strongly) classical semiprime ideal $S = \bigcap_{i=1}^{m} P_i$. Then there is a one-to-one correspondence between the central idempotents of $\hat{R}_S$ and the localizable subsets of $\{P_1, \ldots, P_m\}$. Such subsets are also (strongly) classical.

Proof. (1) First consider the given S as a classical semiprime ideal. We want to associate a given localizable subset of $\{P_1, \ldots, P_m\}$ with a central idempotent of $\hat{R}_S$. Let $\{P_i \mid i \in I\}$ be a localizable subset and put $T = \bigcap_{i \in I} P_i$. By Proposition 1.3, TR_S is localizable in R_S. Then $\overline{TR_S}$ is localizable in $\bar{R}_S = R_S/J(R_S)^n$ for every $n > 0$ since $\bar{R}_S$ is semiprimary. By Proposition 1.4, there exists a unique central idempotent $\bar{e}_n$ $\bar{R}_S$ with $\overline{TR_S} = \bar{e}_n\bar{R}_S + \overline{J(R_S)}$ for each n. Let e_n be a representative of the coset $\bar{e}_n$ modulo $J(R_S)^n$. We claim that (e_n) form a Cauchy sequence in R_S. Observe also $TR_S = e_nR_S + J(R_S)$ for all n. Then for any pair of integers k and n with $k > n$, we have $\bar{e}_n\bar{R}_S + \overline{J(R_S)} = \bar{e}_k\bar{R}_S + \overline{J(R_S)}$ in $\bar{R}_S = R_S/J(R_S)^n$. The uniqueness of the central idempotent ensures $\bar{e}_n = \bar{e}_k$. That is, $e_n - e_k \in J(R_S)^n$ for any $k > n$. This proves the claim. Hence there exists uniquely an element $e \in \hat{R}_S$ with $\lim e_n = e$. Since $\hat{R}_S = \varprojlim R_S/J(R_S)^n$, the element e is a central idempotent of $\hat{R}_S$. We associate such e with the given localizable subset.

(2) Conversely, we want to associate a given central idempotent of $\hat{R}_S$ with a localizable subset. Let e be a central idempotent of $\hat{R}_S$, and let $\alpha : R \to \hat{R}_S$ be the composite of the localization map ε_S of R and the completion map of R_S. We claim that $T = \alpha^{-1}(e\hat{R}_S + J(\hat{R}_S))$ is a semiprime ideal of R. Put $T^* = (e\hat{R}_S + J(\hat{R}_S)) \cap R_S$. Then T^* is a semiprime ideal because $J(R_S) \subset T^*$ and R_S is semilocal. Moreover, the map $\psi : R/T \to R_S/T^*$, given by $\psi(\bar{r}) = \overline{\varepsilon_S(r)}$, is a well-defined R-monomorphism. Thus $\psi(J(R/T)) \subset J(R_S/T^*) = 0$ implies $J(R/T) = 0$, from which we conclude that T is semiprime and our claim is proved.

Notice $TR_S = (e\hat{R}_S + J(\hat{R}_S)) \cap R_S$ and $\hat{R}_S/(J(R_S)^n)^\wedge \simeq R_S/J(R_S)^n$ for all $n > 0$. We deduce from this observation that $e\hat{R}_S + J(\hat{R}_S)/(J(R_S)^n)^\wedge \simeq TR_S/J(R_S)^n$. But $e\hat{R}_S + J(\hat{R}_S)/(J(R_S)^n)^\wedge$ is localizable in $\hat{R}_S/(J(R_S)^n)^\wedge$ by Proposition 1.4 since $\bar{e}$ is a central idempotent of the semiprimary ring $\hat{R}_S/(J(R_S)^n)^\wedge$. Hence $\overline{TR}_S$ is localizable in $\bar{R}_S = R_S/J(R_S)^n$. Notice also $J(R_S)$ is classical in R_S by Lemma 1.9. In order to conclude that TR_S is classical in R_S by invoking Lemma 1.11, we need to show that $TR_S = \bigcap_{i\varepsilon I} P_iR_S$ for some subset I of $\{1,\ldots,m\}$. This is equivalent to showing that $T = \bigcap_{i\varepsilon I} P_i$. This, in turn, will imply the localizability of T in R according to Proposition 1.3. We associate $\{P_i \mid i \varepsilon I\}$ with the given e.

(3) Continuing from (2), we now show that $T = \bigcap_{i\varepsilon I} P_i$ for some subset I of $\{1,\ldots,m\}$. Being a semiprime ideal, $e\hat{R}_S + J(\hat{R}_S) = \bigcap_{\mu} Q_\mu$ where Q_μ are prime ideals of $\hat{R}_S$. For each μ, we claim that $\alpha^{-1}(Q_\mu) = P_{i_\mu}$ for some $i_\mu \varepsilon \{1,\ldots,m\}$. Consider $Q_\mu^* = Q_\mu \cap R_S$ which will be shown to be prime as follows: let $a, b \varepsilon R_S$ with $aR_Sb \subset Q_\mu^*$. Then $a\hat{R}_Sb \subset (aR_Sb)^\wedge \subset Q_\mu$ since Q_μ is closed in $\hat{R}_S$. Therefore $a \varepsilon Q_\mu$ or $b \varepsilon Q_\mu$, that is, $a \varepsilon Q_\mu^*$ or $b \varepsilon Q_\mu^*$. So Q_μ^* is a prime ideal. On the other hand, $J(R_S) \subset Q_\mu^*$, thus $P_1R_S \ldots P_mR_S \subset Q_\mu^*$, implying $P_{i_\mu}R_S \subset Q_\mu^*$ for some i_μ. But each P_iR_S is a maximal ideal, hence $P_{i_\mu}R_S = Q_\mu^*$ which leads to $\alpha^{-1}(Q_\mu) = \varepsilon_S^{-1}(Q_\mu^*) = P_{i_\mu}$. This completes the proof of our claim. Hence $T = \alpha^{-1}(e\hat{R}_S + J(\hat{R}_S)) = \alpha^{-1}(\bigcap_\mu Q_\mu) = \bigcap_\mu \alpha^{-1}(Q_\mu) = \bigcap_\mu P_{i_\mu}$.

(4) Next we will show that the associations in (1) and (2) are inverse of each other. Suppose $\{P_i \mid i \varepsilon I\}$ and $\{P_i \mid i \varepsilon I'\}$ are two localizable subsets which associate with the same central idempotent $e \varepsilon \hat{R}_S$ via (1). Let $T = \bigcap_{i\varepsilon I} P_i$ and $T' = \bigcap_{i\varepsilon I'} P_i$. Then by (1), there are two Cauchy sequences (e_n) and (e_n') such that $TR_S = e_nR_S + J(R_S)$ and $T'R_S = e_n'R_S + J(R_S)$. Since $e = (\bar{e}_n) = (\bar{e}_n')$, it follows that

$e_n - e_n' \in J(R_S)^n$. Thus $TR_S = e_nR_S + J(R_S) = e_n'R_S + J(R_S) = T'R_S$ which then implies $T = T'$ as both T and T' are S-closed. Therefore $\{P_i \mid i \in I\} = \{P_i \mid i \in I'\}$.

Now take any central idempotent e of $\hat{R}_S$. Then $T = \alpha^{-1}(e\hat{R}_S + J(\hat{R}_S))$ is a semiprime ideal of R and $T = \bigcap_{i\in I} P_i$ for some localizable subset $\{P_i \mid i \in I\}$. By (1), there exists a Cauchy sequence (e_n) in R_S with $TR_S = e_nR_S + J(R_S)$ for all n. We claim that e is the limit point of (e_n). Let $\kappa_n : \hat{R}_S/(J(R_S)^n)\hat{} \to R_S/J(R_S)^n$ be the natural isomorphisms. Then $\kappa_n(e\hat{R}_S + J(\hat{R}_S)/(J(R_S)^n)\hat{}\,) = TR_S/J(R_S)^n$, that is, $\kappa_n(\bar{e})\bar{R}_S + \overline{J(R_S)} = \overline{TR}_S = \bar{e}_n\bar{R}_S + \overline{J(R_S)}$ in $\bar{R}_S = R_S/J(R_S)^n$. By uniqueness, $\kappa_n(\bar{e}) = \bar{e}_n$. But $\kappa_n(\bar{e}) = \phi_n(e)$ where $\phi_n : \hat{R}_S \to R_S/J(R_S)^n$ are the projection maps. Hence $\bar{e}_n = \phi_n(e)$, that is, $e = \lim e_n$.

(5) Let $\{P_i \mid i \in I\}$ be a localizable subset and put $T = \bigcap_{i\in I} P_i$. TR_S is localizable in R_S by Proposition 1.3 and hence classical by Lemma 1.11. It follows that T is classical since $E_R(R/T) = E_{R_S}(R_S/TR_S)$.

(6) Finally, we assume that S is strongly classical. Since strongly classical semiprime ideals are always classical, it suffices to show that every localizable subset is strongly classical. Let $\{P_i \mid i \in I\}$ be a localizable subset and put $T = \bigcap_{i\in I} P_i$. Take any right ideal B of R_T. Then $B = AR_T$ where A is the inverse image of B under the localization map $\varepsilon_T : R \to R_T$. Since $J(R_S)$ has the right AR-property, there exists an integer $k > 0$ such that $AR_S \cap S^kR_S \subset ASR_S$. Moreover, $SR_T = TR_T$ and R_T is a flat left R_S-module. Thus we have

$$(AR_S \cap S^kR_S) \otimes_{R_S} R_T \cong (AR_S \otimes_{R_S} R_T) \cap (S^kR_S \otimes_{R_S} R_T) = AR_T \cap S^kR_T = AR_T \cap T^kR_T .$$

Likewise $ASR_S \underset{R_S}{\otimes} R_T \simeq ASR_T = AR_T TR_T$. Thus $AR_T \cap T^k R_T \subset AR_T TR_T$. The left AR-property is similarly verified. ||

Corollary 1.13. Every strong clan is a clan. The latter is also a minimal localizable set.

Before making an attempt on the second main result, we need a lemma which is supposedly well-known. Nevertheless we provide the proof here for the convenience of the reader. (cf. [22])

Lemma 1.14. Let $S = \bigcap_{i=1}^{m} P_i$ be a right localizable semiprime ideal of a ring R. Then any right Goldie prime ideal P contained in $\bigcup_{i=1}^{m} P_i$ is S-closed. Moreover, PR_S is a prime ideal of R_S.

Proof. Let P be a right Goldie prime ideal which is contained in $\bigcup_{i=1}^{m} P_i$. Take any $x \in c\ell(P)$, the S-closure of P. Then $xc \in P$ for some $c \in C(S)$, hence $\bar{x}\bar{c} = 0$ in $\bar{R} = R/P$. To ensure $x \in P$, it suffices to show that c is regular modulo P.

By Goldie's Theorem, $\bar{R}$ has a simple artinian classical right quotient ring Q. For any element $z \in Q$, denote the left and the right annihilators of z in Q by $\ell(z)$ and $r(z)$ respectively. We apply the left and the right maximum conditions on Q to the sets $\{\ell(\bar{t}) \mid t \in C(S), \ell(\bar{c}) \subset \ell(\bar{t})\}$ and $\{r(\bar{t}) \mid t \in C(S), r(\bar{c}) \subset r(\bar{t})\}$ to get an element $t \in C(S)$ such that $\ell(\bar{t})$ and $r(\bar{t})$ are maximal in their respective sets. Suppose $\bar{t}ab^{-1} = 0$ for some non-zero $ab^{-1} \in Q$ where $a, b \in \bar{R}$. Then $\bar{t}a = 0$. On the one hand, $\bar{R}$, being prime, ensures the existence of an element $d \in \bar{R}$ with $ad\bar{t}^2 \neq 0$. On the other hand, the right Ore condition of $C(S)$ yields elements $u \in C(S)$ and $w \in \bar{R}$ with $ad\bar{t}^2\bar{u} = \bar{t}w$. Therefore $0 = \bar{t}ad\bar{t}^2\bar{u} = \bar{t}^2 w$, implying $w \in r(\bar{t}^2) = r(\bar{t})$, thus $\bar{t}w = 0$. That is, $ad\bar{t}^2\bar{u} = 0$ or simply $ad\bar{t} \in \ell(\bar{t}\bar{u})$ which leads to $\ell(\bar{t}) \subsetneq \ell(\bar{t}\bar{u})$, contradicting the maximality of $\ell(\bar{t})$. Hence $\bar{t}$ is right regular in Q and so is $\bar{c}$. Since Q is artinian, $\bar{c}$ is invertible in Q and is certainly regular in $\bar{R}$. Because P is S-closed, PR_S is prime, given the observation that it is an ideal of R_S. ||

Theorem 1.15. Every prime ideal of a ring R belongs to at most one clan.

Proof. Consider two clans $\{P_1,\ldots,P_n\}$ and $\{Q_1,\ldots,Q_m\}$. Put $S = \bigcap_{i=1}^{n} P_i$ and $T = \bigcap_{j=1}^{m} Q_j$. Assume $P_1 \subset Q_1$ and let $P_1,\ldots,P_s$ be exactly all the P_i which are contained in $\bigcup_{j=1}^{m} Q_j$. For any $i \in \{1,\ldots,s\}$, P_iR_T is a prime ideal of R_T by Lemma 1.14. For any $i > s$, $P_iR_T = R_T$. Thus $SR_T = AR_T$ where $A = \bigcap_{i=1}^{s} P_i$.

First we want to show that $C(A)$ is an Ore set. Suppose there are elements $e \in E(R/A)$ and $c \in C(A)$ with $ec = 0$. Then there exists an integer $k > 0$ such that $eS^k = 0$. Since A is T-closed by Lemma 1.14, we may view $E(R/A)$ as an R_T-module. Hence $eS^kR_T = 0$, implying $eA^kR_T = 0$. That is, $eA^k = 0$.

For any $b \in C(A)$, $b1^{-1} \in C(AR_S)$ and is therefore regular, even invertible modulo AR_S since R_S/AR_S, being a factor ring of R_S/SR_S, is semisimple artinian. Let $\bar{R}_S = R_S/A^kR_S$. Then $J(\bar{R}_S) = \overline{AR_S}$. All these imply $\overline{b1^{-1}}$ is invertible in $\bar{R}_S$ for any $b \in C(A)$. Moreover,

$E(R/A)$ is an R_S-module because A is S-closed. Since $eA^k = 0$, we may consider eR_S as an $\bar{R}_S$-module. Thus $e\overline{cl^{-1}} = ecl^{-1} = 0$ implies $e = 0$ as $\overline{cl^{-1}}$ is invertible in $\bar{R}_S$. Proposition 1.2 then completes the proof of the Ore condition for $\mathcal{C}(A)$.

By Proposition 1.3, $\mathcal{C}(AR_S)$ is an Ore set in R_S and so $\overline{\mathcal{C}(AR_S)}$ is localizable in the semiprimary ring R_S/S^nR_S for every integer $n > 0$. By Lemmas 1.9 and 1.11, AR_S is classical in R_S from which we deduce that A is classical in R by virtue of Proposition 1.3 and Theorem 1.12. Hence $A = S$ as $\{P_1,\ldots,P_n\}$ is a clan. That means $\bigcup_{i=1}^{n} P_i \subset \bigcup_{j=1}^{m} Q_j$, implying each P_i is contained in some Q_j. In particular, if $P_1 = Q_1$, then by symmetry the two clans coincide. ||

The corollary below is a consequence of Thearems 1.12 and 1.15. It describes how to partition a classical set into clans in a unique fashion. To accomplish this, a partial ordering is necessary to facilitate our proof. For a ring R, we define a partial ordering on the set B of all the central idempotents of R as follows: given two central idempotents e and f, we say that $e \leq f$ if and only if the following equivalent conditions are satisfied:

(1) $ef = e$.

(2) $eR + J(R) \subset fR + J(R)$.

(3) $eR \subset fR$.

(4) $f = e + e'$ for some $e' \in B$ such that $ee' = 0$.

Remark. With this partial ordering defined on the set of all the central idempotents of $\hat{R}_S$ in Theorem 1.12, it can be shown that for any two localizable subsets L_1 and L_2 together with their respective central idempotents e_1 and e_2, $e_1 \leq e_2$ if and only if $L_1 \supset L_2$.

Corollary 1.16. Let $\{P_1,\ldots,P_m\}$ be a classical set of prime ideals of a ring R. Then this set is the disjoint union of clans in a unique fashion. A subset is localizable if and only if it is the union of some of these clans.

Proof. Let $S = \bigcap_{i=1}^{m} P_i$. Then $\hat{R}_S$ is a semiperfect ring. Let $1 = e_1 + \ldots + e_n$ where all the e_i are non-zero mutually orthogonal centrally-indecomposable central idempotents of $\hat{R}_S$. Let $f_i = 1 - e_i$ for $i = 1,\ldots,n$.

(1) First we claim that each clan of the classical set corresponds to some f_i in the sense of Theorem 1.12. Let L be a clan together with its corresponding central idempotent e of $\hat{R}_S$. Then $e = e_{i_1} + \ldots + e_{i_k}$ for some subset $\{i_1,\ldots,i_k\} \subset \{1,\ldots,n\}$. Clearly $k \leq n-1$, otherwise $e = 1$ in which case $\cap L = R$. Suppose $k < n-1$. Then there exists a non-zero central idempotent $g \in \hat{R}_S$ with $e + g = 1 - e_j = f_j$ for some $j \in \{1,\ldots,n\}$, implying $e \leq f_j$. Therefore $L_j \subset L$ where L_j is the localizable subset corresponding to f_j. Hence $L_j = L$ as L is a clan. By Theorem 1.12, $e = f_j$ which contradicts $k < n-1$. Hence $k = n-1$ as required.

(2). Conversely, we want to show that each f_j corresponds to a clan. Let L_j be the localizable subset which corresponds to f_j and let L be a clan contained in L_j. By (1), L corresponds to some f_k. So

$f_j \leq f_k$, that is, $1-e_j \leq 1-e_k$. So there is a central idempotent h with $1-e_k = (1-e_j) + h$ and $(1-e_j)h = 0$ from which we get $e_j = e_k$ or equivalently $f_j = f_k$.

In view of (1) and (2) there are exactly n clans and according to Theorem 1.15, they are mutually disjoint. Therefore $\{P_1,\ldots,P_m\}$ is expressible uniquely as the disjoint union of these clans.

(3) Now take r distinct clans $L_1,\ldots,L_r$ together with their corresponding central idempotents $f_{i_1},\ldots,f_{i_r}$. Let $e = 1 - \sum_{k=1}^{r} e_{i_k}$ and L be its corresponding localizable subset. We claim that $L = \bigcup_{k=1}^{r} L_k$. Notice that $e_{i_k} e = 0$ for any e_{i_k}. That means $f_{i_k} e = e$, hence $e \leq f_{i_k}$ or equivalently, $L_k \subset L$ for each k. Therefore $\bigcup_{k=1}^{r} L_k \subset L$. To reverse the inclusion, take any $P \in L$. Then P belongs to some clan $L_j \subset L$. Let f_j be the central idempotent corresponding to L_j. Suppose $f_j \neq f_{i_k}$ for any k. Then $e_j e = e_j$. On the other hand, $L_j \subset L$ implies $e \leq f_j$, that is, $ef_j = e$. So $ee_j = 0$, a contradiction. This establishes $L = \bigcup_{k=1}^{r} L_k$ as required.

A repetition of the arguments in (1) and (2) will yield the fact that every localizable subset is the union of some of the clans. ||

Corollary 1.17. The localization R_S of a ring R at the semiprime ideal S associated with a clan is ring-directly indecomposable.

Proof. This follows trivially from Corollary 1.16. ||

Remark. The building-up of a localizable set from clans in Corollary 1.16 is done within a given classical set. The question whether the same can be done from clans which do not necessarily come from a fixed classical set has an affirmative answer to certain extent.

Proposition 1.18. Let U be the union of a finite collection of (strong) clans of a ring R. Suppose no two prime ideals from U are comparable. Then U is (strongly) classical.

Proof. Let $S_1,\ldots,S_m$ be the semiprime ideals associated with the given clans and $U = \{P_1,\ldots,P_n\}$, the union of all the given clans such that no two P_i are comparable. Then $S = \bigcap_{j=1}^{m} S_j = \bigcap_{i=1}^{n} P_i$, $C(S) = \bigcap_{j=1}^{m} C(S_j) = \bigcap_{i=1}^{n} C(P_i)$ and $E(R/S) = \bigoplus_{j=1}^{m} E(R/S_j)$. From these follows immediately the Ore condition of $C(S)$ via Proposition 1.2. For the reversibility condition of $C(S)$, suppose $cr = 0$ for some $c \in C(S)$ and $r \in R$. Then for each $j = 1,\ldots,m$, there exists $c_j \in C(S_j)$ such that $rc_j = 0$. Since $C(S)$ is an Ore set, there exist $x_j \in R$ with $c' = \sum_{j=1}^{m} c_j x_j \in C(S)$. Thus $rc' = 0$. The left reversibility condition is similarly verified.

To show $E(R/S) = \bigcup_{k=1}^{\infty} \mathrm{ann}_{E(R/S)}\, S^k$, take any element $e \in E(R/S)$. Then $e = (e_1,\ldots,e_m)$ for some $e_j \in E(R/S_j)$. For each j, there exists an integer $k(j) > 0$ such that $e_j(S_j)^{k(j)} = 0$. By taking k = the maximum integer among all the k(j), we see that $e_j S^k = 0$ for all j and hence $eS^k = 0$. Therefore S is classical.

Now suppose all the above S_j are strongly classical. Clearly S is localizable by the above reasoning. Therefore it remains to show that $J(R_S)$ has the AR-property. Take any right ideal I of R_S and let $A = \varepsilon_S^{-1}(I)$ where $\varepsilon_S : R \to R_S$ is the localization map. For each j, there is an integer $k(j) > 0$ such that $AR_{S_j} \cap (S_jR_{S_j})^{k(j)} \subset AR_{S_j}S_jR_{S_j}$.

Note that $SR_{S_j} = S_jR_{S_j}$ for every j. Let $k = \max\{k(j) \mid j = 1,\ldots,m\}$. Then we have $(A \cap S^k)R_{S_j} \subset ASR_{S_j}$, which implies $A \cap S^k \subset \bigcap_{j=1}^{m} S_j\text{-}c\ell(AS) = S\text{-}c\ell(AS)$. Therefore $AR_S \cap S^kR_S = (A \cap S^k)R_S \subset ASR_S = AR_SSR_S$. Likewise we also have the left AR-property. ||

§5 EXAMPLES AND COUNTEREXAMPLES

In this section we list a few examples and counterexamples pertinent to this chapter.

(A) Rings in which every localizable semiprime ideal is strongly classical:

a) Semiprimary rings are of this type. This is obvious from Proposition 1.4.

b) A ring R is a fully left (respectively right) idempotent ring if $I = I^2$ for every left (respectively right) ideal I of R. All such rings R have $J(R) = 0$; the class of these rings is closed under localization at any localizable set. In fact the localization at a localizable semiprime ideal is semisimple artinian. Examples of such rings are von Neumann regular rings and left (respectively right) V-rings.

c) In [29] appears the following ring. Consider the commutative polynomial ring $K[x,y]$ in two indeterminates x and y over a field K. Let

$$R = \left\{ \frac{f}{g} \;\middle|\; f, g \in K[x,y] \text{ with } g(0,y) \neq 0 \text{ and } \frac{f(0,y)}{g(0,y)} \in K \right\}$$

Then R is a commutative non-noetherian local ring with

$$J(R) = \left\{ \frac{f}{g} \in R \;\middle|\; f(0,y) = 0 \right\}$$

For any non-zero ideal I of R, there exists an integer $n > 0$ such that $J(R)^n \subset I$. Hence $J(R)$ is the only non-zero prime ideal of R and has the AR-property.

(B) Rings in which every right classical semiprime ideal is also strongly right classical:

This class of rings obviously includes all the rings mentioned in (A). Another kind of rings which belong to this class is the right FGS rings. These are rings over which every cyclic module has a finitely generated socle. One of the characterizations of right FGS rings R is the fact that every finitely generated R-module has finite Goldie dimension. (See [14]) Examples of such rings include right valuation rings and rings with Krull dimension [8].

To see why a right classical semiprime ideal S of a right FGS ring R is strongly right classical, we can imitate the proof of Theorem 3.5 in [23], bearing in mind the key step to be observed in this proof is the fact that every cyclic R_S-module has a finitely generated socle. We shall demonstrate this observation in the case of a right FGS ring.

Proposition 1.19. Let S be a right localizable semiprime ideal of a right FGS ring R. Then R_S is also a right FGS ring.

Proof. By Proposition 2.2 in [14], it suffices to show that every cyclic R_S-module has finite Goldie dimension. Take any $M = eR_S$ and put $N = eR$. Then by hypothesis N has finite Goldie dimension, say n. Suppose on the contrary that M has no finite Goldie dimension. Then there must exist $e_1, \ldots, e_{n+1} \varepsilon M$, forming a direct sum $\bigoplus_{i=1}^{n+1} e_i R_S$ in M. For each i, $e_i = er_i c_i^{-1}$ for some $r_i c_i^{-1} \varepsilon R_S$. By finding a common right denominator, we may as well assume $c_i^{-1} = c^{-1}$ for all i. Thus $e_i c = er_i$. Evidently $\sum_{i=1}^{n+1} e_i cR$ cannot be a direct sum in N. Without loss of generality, we may assume there is a non-zero element $x \varepsilon \sum_{i=1}^{n} e_i cR \cap e_{n+1} cR$. That is, $x = e_{n+1} cb = \sum_{i=1}^{n} e_i cd_i$ for some $b, d_i \varepsilon R$. This implies $cb1^{-1} \neq 0$ in R_S. Therefore $e_{n+1} R_S \cap \sum_{i=1}^{n} e_i R_S \neq 0$, a contradiction. ||

(C) Rings in which some right localizable semiprime ideals are not (strongly) right classical:

a) Let $R = C^{\infty}_{\mathbb{R}}(\mathbb{R})$ and $M = \{f \varepsilon R \mid f(0) = 0\}$ which is a maximal ideal of R. Then $J(R_M)$ is not classical since the $J(R_M)$-adic topology on R_M is not even Hausdorff.

b) Let R be a left perfect but not semiprimary ring. Then $J(R)$ is localizable but not strongly right classical according to Corollary 1.8. However, the question remains open as to whether there exists a left perfect but not semiprimary ring with a right classical Jacobson radical.

CHAPTER II

A VARIATION IN THE THEORY OF CLANS

This chapter is devoted to a further generalization of our localization theory developed in Chapter I. Just as the module theoretic concept of FP-injectivity extends that of injectivity, we introduce here a more generalized concept of a clan. Consequently, many of the previous results will find their respective analogues here. This new development proves useful at least in the case of coherent rings where some of these rings reveal the limitation of our earlier theory. Suffice it to say in the meantime that our effort in formulating this new theory calls for the help of the FP-injective modules.

Definition. Let R be a ring. An R-module M is called finitely presented if there exists a short exact sequence

$$0 \to K \to P \to M \to 0$$

where P is a finitely generated projective R-module and K is a finitely generated R-module.

A ring R is right coherent if every finitely generated right ideal of R is finitely presented. A coherent ring is a ring which is both right and left coherent. Right noetherian rings and right semi-hereditary rings are right coherent. So are right valuation domains and von Neumann regular rings. We will examine later some specific examples of right coherent rings.

§1 FP-INJECTIVE MODULES

The notion of FP-injective modules was introduced in [34] as an extension of the notion of injective modules. For any ring R, an R-module M is called FP-injective if it satisfies the following equivalent conditions:

(1) $\mathrm{Ext}^1_R(F,M) = 0$ for every finitely presented R-module F.

(2) For every short exact sequence $0 \to A \xrightarrow{\alpha} B \to F \to 0$ with F finitely presented and any homomorphism $f : A \to M$, there exists a homomorphism $f' : B \to M$ such that $f'\alpha = f$.

The verification of the equivalence of these two conditions is just a matter of straightforward checking and hence is omitted.

For a while our ring R will remain arbitrary until we further confine our attention to specific types of rings. Recall that given a multiplicative subset X of R, it determines a torsion theory $(T_X, F_X, \rho_X, \theta_X)$, called the X-torsion theory. A monomorphism is called X-dense if its cokernel is X-torsion. An R-module M is called X-divisible if for every X-dense monomorphism $f : A \to B$ and any homomorphism $h : A \to M$, there exists a homomorphism $h' : B \to M$ such that $h'f = h$. Denote by A the quotient category of mod-R determined by the X-torsion theory, and let Q denote the corresponding quotient functor. For any R-module M, D(M) denotes the divisible hull of M with respect to the X-torsion theory, or simply the divisible hull of M when the torsion theory under consideration is unambiguous. Explicitly, $D(M) = \kappa^{-1}(\rho_X(E(M)/M))$ where $\kappa : E(M) \to E(M)/M$ is the canonical epimorphism. It is the smallest X-divisible submodule of $E(M)$ containing M.

Proposition 2.1. Let X be a right Ore subset of a ring R and E an X-torsionfree FP-injective R-module. Then E is X-divisible.

Proof. Take any right ideal $I \in \theta_X$. Then $I \cap X \neq \emptyset$. Pick an element $c \in I \cap X$. Then the short exact sequence

$$0 \to I/cR \overset{\alpha}{\to} R/cR \overset{\beta}{\to} R/I \to 0$$

in which α and β are natural maps induces a long exact sequence

$$\mathrm{Hom}_R(I/cR,E) \xrightarrow{\theta_1} \mathrm{Ext}_R^1(R/I,E) \xrightarrow{\beta_1^*} \mathrm{Ext}_R^1(R/cR,E) \longrightarrow \cdots$$

Since X is right Ore, I/cR is X-torsion. So $\mathrm{Hom}_R(I/cR,E) = 0$. On the other hand, the FP-injectivity of E renders $\mathrm{Ext}_R^1(R/cR,E) = 0$. Hence $\mathrm{Ext}_R^1(R/I,E) = \ker \beta_1^* = \mathrm{im}\ \theta_1 = 0$ from which it follows that E is X-divisible. ||

Corollary 2.2. In the above situation, E has an R_X-module structure.

Proof. $Q(E) = D(E/\rho_X(E)) = D(E) = E \in A$ by Proposition 2.1. Then by Theorem 2.8 in [22], $E \approx \mathrm{Hom}_R(R_X,E) \in \text{mod-}R_X$. ||

Definition. A Silver right localization of a ring R is a ring epimorphism $f : R \to S$ such that S is a flat left R-module.

Proposition 2.3. Under the same hypotheses as in Proposition 2.1, if in addition X is right localizable in R, then E is an FP-injective R_X-module.

Proof. Corollary 2.2 asserts that E is an R_X-module. Thus we only need to show that it is FP-injective as an R_X-module. Take any finitely presented R_X-module F and consider a short exact sequence in mod-R_X

$$0 \to K \to (R_X)^n \to F \to 0 .$$

Then $K = \sum_{j=1}^{m} e_j R_X$ for some $e_j \in (R_X)^n$. Multiplying the components of each e_j by a common denominator, we may write $e_j = (a_{j1}1^{-1},\ldots,a_{jn}1^{-1})$ where all $a_{jk} \in R$. Let $e_j' = (a_{j1},\ldots,a_{jn})$ and $K' = \sum_{j=1}^{m} e_j' R$. Then consider the short exact sequence in mod-R

$$0 \to K' \overset{\alpha}{\to} R^n \to F' \to 0$$

where α is the inclusion map and $F' = \operatorname{coker} \alpha$. Since the right localization map $\varepsilon_X : R \to R_X$ is Silver, we still have, after tensoring with R_X, a short exact sequence

$$0 \to K' \otimes_R R_X \to R^n \otimes_R R_X \to F' \otimes_R R_X \to 0$$

which gives rise to the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & K' \otimes_R R_X & \to & R^n \otimes_R R_X & \to & F' \otimes_R R_X & \to & 0 \\ & & \downarrow \mu_1 & & \downarrow \mu_2 & & & & \\ 0 & \to & K & \to & (R_X)^n & \to & F & \to & 0 \end{array}$$

where μ_1 and μ_2 are defined by multiplication in a natural way. But both μ_1 and μ_2 are isomorphisms. Hence the induced map between the cokernels, making the second square commutative, is also an isomorphism. Then by Proposition 4.1.3 ([4], Chapter 6, §4), we have

$$\operatorname{Ext}_R^1(F',E) \simeq \operatorname{Ext}_{R_X}^1(F' \otimes_R R_X, E) = \operatorname{Ext}_{R_X}^1(F,E)$$

which implies $\operatorname{Ext}_{R_X}^1(F,E) = 0$ as desired. ||

Given below is a characterization of the right Ore condition in terms of FP-injective modules. This result generalizes Proposition 1.2 in Chapter I.

Proposition 2.4. Let S be a right Goldie semiprime ideal of a ring R and E any FP-injective submodule of E(R/S) with $R/S \subset E$. Then $C(S)$ is right Ore if and only if every element of $C(S)$ operates regularly on E.

Proof. If $C(S)$ is right Ore, then every element of $C(S)$ operates regularly on E(R/S) by Proposition 1.2 and even more so on E. Conversely, suppose every element of $C(S)$ operates regularly on E. We claim that for any $c \in C(S)$, $R/cR \in T_S$. Suppose on the contrary there exists an element $c \in C(S)$ with $R/cR \notin T_S$. Then there must exist a non-zero R-homomorphism $f : R/cR \to E(R/S)$. Let $e = f(\bar{1})$ which is non-zero. By essentiality of E(R/S) over E, there exists $x \in R$ with $0 \neq ex \in E$. Let g be the restriction of f to $xR + cR/cR$. Then g is a non-zero homomorphism from $xR + cR/cR$ to E. Consider now the short exact sequence

$$0 \to xR + cR/cR \to R/cR \to R/xR + cR \to 0$$

where the maps are defined canonically. Obviously $R/xR + cR$ is finitely presented. Since E is FP-injective, g is then extended to a map $g' : R/cR \to E$. Therefore $z = g'(\bar{1}) \neq 0$. However, $zc = 0$ which then implies $z = 0$ as c operates regularly on E. This contradicts the fact that g is non-zero, hence asserting the claim. That is, $cR \in \theta_S$ for all $c \in C(S)$ and hence follows the right Ore condition for C(S) as required. ||

§2 WEAKLY CLASSICAL SEMIPRIME IDEALS

Definition. A semiprime ideal $S = \bigcap_{i=1}^{n} P_i$ of a ring R is called weakly right classical if S is right localizable and if there exist FP-injective R-modules E_i with $R/P_i \subset E_i \subset E(R/P_i)$ for $i = 1,\ldots,n$ such that $E_S = \bigoplus_{i=1}^{n} E_i = \bigcup_{k=1}^{\infty} \mathrm{ann}_{E_S} S^k$.

Notice that $R/S \subset E_S \subset E(R/S)$ and E_S is again an FP-injective R-module by Corollary 2.4 in [34]. With this definition we proceed to establish below several lemmas which lay the groundwork for the main results in the next section.

Lemma 2.5. Let $S = \bigcap_{i=1}^{n} P_i$ be a right localizable semiprime ideal of a ring R. Then E_S is an FP-injective R_S-module with embeddings $R_S/J(R_S) \rightarrowtail E_S \rightarrowtail E_{R_S}(R_S/J(R_S))$ as R_S-modules.

Proof. Let $E_S = \bigoplus_{i=1}^{n} E_i$. Then each E_i is an FP-injective R_S-module by Proposition 2.3, hence so is E_S. Moreover, since $E_R(R/S) \simeq E_{R_S}(R_S/J(R_S))$ as R_S-modules, there is a natural embedding of E_S into $E_{R_S}(R_S/J(R_S))$. For each i, $R/P_i \subset E_i$. Therefore tensoring with R_S, we get $R_S/P_iR_S = R/P_i \otimes_R R_S \rightarrowtail E_i \otimes_R R_S = E_i$ which yields an R_S-monomorphism $R_S/J(R_S) \rightarrowtail \bigoplus_{i=1}^{n} E_i = E_S$ since R_S is a flat left R-module. ||

Corollary 2.6. If S is a weakly right classical semiprime ideal of a ring R, so is $J(R_S)$ in R_S.

Proof. The assertion is an immediate consequence of Lemma 2.5. ||

Lemma 2.7. Let $S = \bigcap_{i=1}^{n} P_i$ be a right Goldie semiprime ideal of a ring R and E_S, as defined previously, be such that $E_S = \bigcup_{k=1}^{\infty} \text{ann}_{E_S} S^k$. Suppose $T = \bigcap_{i \in I} P_i$, with $I \subset \{1,\ldots,n\}$, has the property that T/S^k is right localizable in R/S^k for every integer $k > 0$. Then T is weakly right classical in R.

Proof. Let $E_S = \bigoplus_{i=1}^{n} E_i$ and put $E_T = \bigoplus_{i \in I} E_i$. By imitating the argument used in the proof of Lemma 1.11 with E(R/T) being replaced by E_T, we obtain almost the entire proof of our assertion except for the right reversibility condition of $\mathcal{C}(T)$. To this end, it suffices to prove $\text{ann}_R E_T = \text{ann}_R E(R/T)$. Obviously $\text{ann}_R E(R/T) \subset \text{ann}_R E_T$.

For the reverse inclusion, take any $z \in \text{ann}_R E_T$. By FP-injectivity of E_T, every R-homomorphism $f : zR \to E_T$ can be extended to a map $f' : R \to E_T$. That means there is an element $e \in E_T$ such that $f(zr) = ezr$ for all $r \in R$. But $ez = 0$. Hence $\text{Hom}_R(zR,E_T) = 0$. We now claim that $\text{Hom}_R(zR,E(R/T)) = 0$. Suppose on the contrary there is a non-zero homomorphism $h : zR \to E(R/T)$. Then there exists a non-zero element $w \in E(R/T)$ such that $h(zr) = wzr$ for all $r \in R$. Since E_T is an essential submodule of E(R/T), there exists $b \in R$ with $0 \neq wzb \in E_T$. Let g be the restriction of h to zbR. Then g is an R-homomorphism $zbR \to E_T$ and is non-zero because $wzb \neq 0$. This contradicts $\text{Hom}_R(zbR,E_T) = 0$ proven above. Therefore $\text{Hom}_R(zR,E(R/T)) = 0$ as claimed. This further implies $z \in \text{ann}_R E(R/T)$. ||

Lemma 2.8. Let R be a semilocal ring with a weakly right classical Jacobson radical J(R). Then the J(R)-adic topology on R is

Hausdorff.

Proof. Let $E_{J(R)}$ be the FP-injective R-module associated with the weakly right classical J(R). Since $\mathrm{ann}_R\, E_{J(R)} = \mathrm{ann}_R\, E(R/J(R))$ as indicated in the proof of Lemma 2.7, $E_{J(R)}J(R)^{\omega} = 0$ implies $E(R/J(R))J(R)^{\omega} = 0$. The rest of the proof proceeds as in that of Lemma 1.10. ||

§ 3 THE STRUCTURE OF WEAK CLANS

The observations made in the preceding section enable us to formulate statements parallel to Theorems 1.12, 1.15, Proposition 1.18 and some of their corollaries. Their proofs can be carried over mutatis mutandis. For this reason we simply state these analogues without proofs.

Theorem 2.9. Let R be a ring with a weakly classical semiprime ideal $S = \bigcap_{i=1}^{m} P_i$. Then there is a one-to-one correspondence between the central idempotents of $\hat{R}_S$ and the localizable subsets of $\{P_1,\ldots,P_m\}$. Such subsets are also weakly classical.

Here we take the liberty of calling a localizable set of prime ideals weakly classical when the associated semiprime ideal is weakly classical. Furthermore, Theorem 2.9 gives rise to the following concept.

Definition. A weakly classical set of prime ideals is called a weak clan if no proper non-empty subset of it is weakly classical.

Remark. It can be deduced immediately from Theorem 2.9 that a weak clan is actually a minimal localizable set of prime ideals.

Theorem 2.10. Every prime ideal of a ring R belongs to at most one weak clan.

Corollary 2.11. Every weakly classical set $\{P_1,\ldots,P_m\}$ of prime ideals of a ring R is expressible uniquely as a disjoint union of weak clans. Moreover, a subset of $\{P_1,\ldots,P_m\}$ is localizable if and only if it is the union of some of these weak clans.

Corollary 2.12. Every clan of a ring is also a weak clan.

The assertion of Corollary 2.12 follows trivially from Corollaries 1.16 and 2.11. We now see that this weaker notion of clans does not alter the structure of clans as defined in Chapter I. At the same time, prime ideals which fail to belong to clans may now belong to weak clans.

Corollary 2.13. The localization of a ring at a weak clan is ring-directly indecomposable.

Proposition 2.14. Let U be the union of a finite collection of weak clans of a ring. Suppose no two prime ideals from U are comparable. Then U is weakly classical.

Remark. In the above proposition we may take $E_S = \bigoplus_{i\in\Omega} E_i$ where $S = \cap U$ and the E_i are the FP-injective R-modules associated with the corresponding weak clans.

For noetherian rings a weakly classical semiprime ideal is also classical. This is because FP-injective modules over a noetherian ring are actually injective. Hence the E_S coincides with E(R/S) for any semiprime ideal S. Apart from noetherian rings, a weakly classical semiprime ideal is no longer classical in general whereas the converse of it is always true. The distinction between these two concepts will be elucidated by an example of a coherent ring in §5.

§4 RIGHT VALUATION RINGS

In this section we confine our attention to the class of right valuation rings. These are rings for which the lattice of all right ideals is linearly ordered by inclusion. They need not be domains in contrast to a more conventional definition of valuation rings. (cf. [31])

Proposition 2.15. Let P be any prime ideal of a right valuation ring R. Then

(1) C(P) is a right Ore set if P is right Goldie.

(2) P is right Goldie if and only if P is completely prime.

Proof. Let $r \in R$ and $c \in C(P)$. Then either $cR \subset rcR$ or $rcR \subset cR$. Suppose $cR \subset rcR$. Then $c = rct$ for some $t \in R$. Note that this element t belongs to C(P). Thus $ct \in C(P)$. On the other hand, $rcR \subset cR$ implies $rc = cr'$ for some $r' \in R$. In either case C(P) is right Ore, hence proving (1).

Suppose now P is right Goldie. Then R/P has a simple artinian classical right quotient ring Q for C(P) modulo P. Q, being a right valuation ring also, must therefore be a division ring. Hence R/P is a domain. That is, P is completely prime.

Conversely, suppose P is completely prime. Then R/P is a domain. This renders all elements of $C(P) = R - P$ regular modulo P, and $\overline{C(P)}$ is then right localizable in R/P via the same argument used in proving (1). Therefore the classical right quotient ring of R/P for $\overline{C(P)}$ is a division ring. This shows that P is right Goldie. ||

Corollary 2.16. If R is a right valuation domain, then a prime ideal is right localizable if and only if it is completely prime.

Proof. This follows directly from the two assertions of Proposition 2.15. ||

Remark. We do not know whether there exists a right valuation ring with a prime but not completely prime ideal, or equivalently, a right valuation prime ring which is not a domain.

However, H. H. Brungs and G. Törner have settled this problem affirmatively under a rather specialized setting. In [3], they studied right valuation rings R of the following types:

(i) J(R) is the only prime ideal of R

(ii) J(R) and 0 are the only prime ideals of R

subject to

(iii) char R ≠ char R/J(R)

Then in this setting they proved

(1) Every right valuation ring with (iii) of type (i) or (ii) is a right duo ring.

From (1) follows

(2) Every right valuation ring satisfying (ii) and (iii) is a domain.

A right duo ring is a ring in which every right ideal is two-sided. All the prime ideals of a right duo ring are completely prime. Hence the types of right valuation rings described in (1) must have all the prime ideals right Goldie according to Proposition 2.15.

Proposition 2.17. Let P be a right localizable non-zero prime ideal of a right valuation ring R. Then P is strongly right classical if and only if the $J(R_P)$-adic topology on R_P is Hausdorff.

In this case, R_P is a principal right ideal ring with $J(R_P)$ as the only non-zero prime ideal. If $J(R_P) = 0$, then R_P is a division ring.

Proof. For simplicity we set $J = J(R_P)$ and $S = R_P$. If P is strongly right classical, then $J^{\omega} = 0$ by Lemmas 1.9 and 1.10. Conversely, suppose $J^{\omega} = 0$. Then for any non-zero ideal I of S, there exists a smallest integer $n > 0$ such that $I \not\subset J^n$. This implies $J^n \subset I$ since R_P is a right valuation ring. Hence J has the right AR-property.

The above consideration further shows that J is the only non-zero prime ideal of S. If $J = 0$, then S is a division ring since it is both a simple artinian and a right valuation ring. So it remains to prove that S is a principal right ideal ring in general.

Assume $J \neq 0$. Take any integer $n > 0$ and suppose $J^n/J^{n+1} \neq 0$. Then J^n/J^{n+1} is a semisimple S-module. Actually it is a simple S-module due to the fact that S is a right valuation ring. This leads to $J^n = xS$ for some $x \in S$. On the other hand, if $J^n/J^{n+1} = 0$, then J is nilpotent for the J-adic topology is Hausdorff. In this case, take n to be the index of nilpotency of J. Then J^{n-1} is a simple S-module by the same token and is therefore a principal right ideal. This shows that all non-zero powers of J are principal right ideals. Furthermore, they account for all the non-zero right ideals of S because for any non-zero right ideal A of S, there exists an integer $n > 0$ with $J^{n+1} \subset A \subset J^n$. A repetition of the above argument will yield either $A = J^{n+1}$ or $A = J^n$. Hence S is a principal right ideal ring. ||

Corollary 2.18. Suppose R is a domain in addition to the hypotheses in Proposition 2.17. Then the height of such P is one.

Proof. The Hausdorff property of $J(R_P)$ renders $\bigcap_{n=1}^{\infty} P^n = 0$ via the canonical ring monomorphism $R \to R_P$. Then the assertion follows trivially since R is a right valuation ring. ||

Remarks. (*a*) Proposition 2.17 indicates that there is no distinction between weak clans and strong clans as far as valuation rings are concerned. Hence the three definitions of classical semiprime ideals coincide here.

(*b*) When the ring R is a right valuation domain, Corollary 2.18 assures that any right classical non-zero prime ideal is indeed a minimal prime. The converse, however, is false. Such is the case for instance when we consider the commutative ring of fractional power series

$R = K[[x^{(\frac{1}{2})^n} \mid n = 0,1,2,\ldots]]$ over a field K with addition and multiplication defined as usual. The Jacobson radical J(R) is the only non-zero prime ideal of R, is idempotent and therefore is not classical. On the other hand, the next example, extracted from [8], is a commutative domain whose minimal prime is classical.

Let A be a discrete valuation domain with maximal ideal xA and let $B = A[y]_{(y)}$, the localization of the polynomial ring A[y] at (y). Then the commutative domain $R = A + yB$ is a non-noetherian rank 2 valuation ring. The prime spectrum of R consists of xR, yB and 0. The minimal prime yB is classical.

(c) The question whether the assertion of Corollary 2.18 remains valid for right valuation rings other than domains seems open. However, we do have a partial affirmative for right valuation right duo rings. We shall demonstrate this fact in the following.

Proposition 2.19. Let R be a right valuation right duo ring and P a right classical non-zero prime ideal of R. Then the height of P is at most one.

Proof. Suppose the height of P is greater than one. Then there must exist prime ideals Q_1 and Q_2 such that the inclusions $Q_1 \subset Q_2 \subset P$ are proper. Since every prime ideal of a right duo ring is completely prime, both Q_1 and Q_2 are right Goldie by Proposition 2.15. Hence both Q_1R_P and Q_2R_P are prime ideals of R_P by Lemma 1.14. We now have two cases to consider, namely, Case 1: If $Q_2R_P = 0$, then $Q_1R_P = Q_2R_P = 0$ which implies $Q_1 = Q_2$ since both Q_i are P-closed. This contradicts the proper inclusion $Q_1 \subsetneqq Q_2$.

Case 2: If $Q_2R_P \neq 0$, then $Q_2R_P = PR_P$ by Proposition 2.17. This yields $Q_2 = P$ and hence contradicts the proper inclusion $Q_2 \subsetneqq P$. ||

As a matter of fact, Proposition 2.17 can be equivalently formulated as follows: a right localizable prime ideal P of a right valuation ring is right classical if and only if for $P^{(n)}$ = P-closure of P^n, $\bigcap_{n=1}^{\infty} P^{(n)} = \{r \in R \mid rc = 0 \text{ for some } c \in R - P\}$. Moreover, if $P^{(n)} = P^{(n+1)}$ for some n, then $P^{(n)} = P^{(n+k)}$ for all $k > 0$. In this situation, R_P is a right artinian right valuation ring with only a finite number of right ideals. For domains there is another interesting aspect, namely,

<u>Proposition 2.20</u>. Let P be a minimal prime ideal of a right valuation domain R. Then either P is idempotent or $\bigcap_{n=1}^{\infty} P^n = 0$.

<u>Proof</u>. Suppose $I = \bigcap_{n=1}^{\infty} P^n \neq 0$. Our aim is to show that I is a prime ideal of R. Consider $aRb \subset I$ with both $a, b \notin I$. Then there must exist an integer $k > 0$ with $a, b \notin P^k$. So both aR and bR properly contain P^k. From this we obtain $aRP^k \subsetneqq aRbR$ which leads to $P^{2k} \subsetneqq aRbR$. Hence $I \subsetneqq aRbR$, an obvious contradiction. This proves that I is a non-zero prime ideal and therefore must coincide with P, since the latter is a minimal prime. That is, P is idempotent. ||

§5 A COUNTEREXAMPLE

The following example is a commutative coherent ring having a prime ideal which constitutes a weak clan but fails to be classical.

Let $R = K[x_i \mid i \in \mathbb{N}]$, the commutative polynomial ring in countably many indeterminates x_i over a field K. It is a coherent ring and the ideal M, generated by $\{x_i \mid i \in \mathbb{N}\}$, is a maximal ideal of R. Since $K \simeq R/M$, we may endow K with an R-module structure via the natural map $R \to R/M$. Henceforth, K, when viewed as an R-module, is always understood in this context. Now let $T = K[[x_i^{-1} \mid i \in \mathbb{N}]]$, the ring of formal power series in countably many indeterminates x_i^{-1} over K where the expansion of each element of T could involve an infinite number of the x_i^{-1}. For $i, n \in \mathbb{N}$, we write $x_i^{-n} = x_i^{-1} \dots x_i^{-1}$ (n factors). Let T carry an R-module structure via the defining relations

$$(\alpha x_1^{-\nu_1} \dots x_q^{-\nu_q})(\beta x_1^{\mu_1} \dots x_q^{\mu_q}) = \alpha\beta x_1^{-(\nu_1-\mu_1)} \dots x_q^{-(\nu_q-\mu_q)} \quad \text{if } \mu_i \leqq \nu_i \text{ for each } i$$

$$= 0 \quad \text{otherwise}$$

where $\alpha, \beta \in K$ and $\nu_i, \mu_i \in \mathbb{N}$.

For each positive integer n, let $R_n = K[x_1, \dots, x_n]$ and $V_n = K[x_1^{-1}, \dots, x_n^{-1}]$, each being a polynomial ring in n indeterminates over K. By restricting the above defining relations to $x_1, \dots, x_n$, we can turn V_m into an R_n-module for every $m \geqq n$. Moreover, for $m \geqq n$, R_m is a flat R_n-module (in fact, it is even a free R_n-module) and so R, being the up-directed union of R_i, is a flat module over every R_n.

Claim 1. For $m \geq n$, V_m is an injective R_n-module.

Proof. Take any R_n-module X. It suffices to show that $\mathrm{Ext}^1_{R_n}(X,V_m) = 0$. Since R_m is a flat R_n-module, $\mathrm{Ext}^1_{R_n}(X,V_m)$ is then isomorphic to $\mathrm{Ext}^1_{R_m}(X \otimes_{R_n} R_m, V_m)$ according to Proposition 4.1.3 ([4], Chapter 6, §4). However, V_m is an injective R_m-module by Theorem 2 of [26]. Hence $\mathrm{Ext}^1_{R_m}(X \otimes_{R_n} R_m, V_m) = 0$. That means $\mathrm{Ext}^1_{R_n}(X,V_m) = 0$ as required. ||

Claim 2. $V = \vec{\bigcup} V_n$ is an injective R_n-module for every n.

Proof. For any $n > 0$, it is clear that $V = \vec{\bigcup}_{m \geq n} V_m$. By Claim 1, each V_m is an injective R_n-module. This implies that V is an injective R_n-module as R_n is a noetherian ring. (See [34]) ||

Claim 3. V is an FP-injective R-module.

Proof. V takes on an R-module structure via the same set of defining relations. Let F be a finitely presented R-module. Since R is the up-directed union of the R_n, by virtue of Lemma 2.15 in [18], there exist an integer $n > 0$ and a finitely presented R_n-module F_n such that $F_n \otimes_{R_n} R \cong F$. Therefore, by applying Proposition 4.1.3 ([4], Chapter 6, §4), we obtain $\mathrm{Ext}^1_{R_n}(F_n,V) \cong \mathrm{Ext}^1_R(F_n \otimes_{R_n} R, V) \cong \mathrm{Ext}^1_R(F,V)$. Claim 2 forces $\mathrm{Ext}^1_{R_n}(F_n,V) = 0$. That is, $\mathrm{Ext}^1_R(F,V) = 0$. ||

Claim 4. $V = \bigcup_{n=1}^{\infty} \mathrm{ann}_V\, M^n$.

Proof. Let $z \in V$. Then there exists an integer $n > 0$ with

$z \in V_n$. Hence $zM^k = 0$ for some sufficiently large integer k. ||

Claim 5. As an R-module, V is essential over K but is not the injective hull of K.

Proof. Let z be a non-zero element of V. Then $z \in V_n$ for some n. Select a non-zero term $\beta x_1^{-\mu_1} \dots x_n^{-\mu_n}$ from z so that the sum $\mu_1 + \dots + \mu_n$ is as large as possible. If $\gamma x_1^{-\lambda_1} \dots x_n^{-\lambda_n}$ is any other non-zero term in z, then $\lambda_1 + \dots + \lambda_n \leq \mu_1 + \dots + \mu_n$, and so $\lambda_i < \mu_i$ for at least one i. This leads to $(\gamma x_1^{-\lambda_1} \dots x_n^{-\lambda_n})(x_1^{\mu_1} \dots x_n^{\mu_n}) = 0$. Therefore $0 \neq z(x_1^{\mu_1} \dots x_n^{\mu_n}) = \beta \in K$. We conclude that V is essential over K as an R-module.

For the second assertion, it suffices to prove that V is not isomorphic to $E_R(K)$. Consider the element $e = x_1^{-1} + \dots + x_n^{-n} + \dots$ of T. For any integer $n > 0$, $ex_n^n = 1 \in K$, implying $K \subset eR$. Take any $s \in R$ and suppose $es \neq 0$. Then es is a formal power series of the form $\beta_1 x_{i_1}^{-\nu_1} + \dots + \beta_n x_{i_n}^{-\nu_n}$ where each $\beta_j \neq 0$. Without loss of generality, we may assume $\nu_1 = \max \{\nu_1, \dots, \nu_n\}$. Then $esx_{i_1}^{\nu_1} = \beta_1 \in K$. That is, eR is essential over K. Hence we may identify eR with a submodule of $E_R(K)$. Moreover, $eM^n \neq 0$ for any integer $n > 0$ because $ex_n^n = 1$. Therefore Claim 4 indicates that V cannot be isomorphic to $E_R(K)$. ||

The proof of Claim 5 also demonstrates that $E_R(K) \neq \bigcup_{n=1}^{\infty} \operatorname{ann}_{E_R(K)} M^n$. This together with Claim 4 establishes the assertion that the ideal M forms a weak clan but not a clan.

CHAPTER III

RINGS MODULE-FINITE OVER THEIR CENTRES

Rings which are finitely generated as modules (or in short, module-finite) over their centres constitute an interesting class for study. In this chapter we concern ourselves with the application of the localization theory, developed in Chapter I, to this particular class of rings. More specifically, we are going to examine the relationship between a localizable semiprime ideal of such a ring R and its counterpart in the centre of R. The latter gives rise to the usual localizations at prime ideals in commutative ring theory. We begin our study by simply assuming that

(*I*) the given ring is module-finite over a subring of its centre.

Further on, our assumptions will be more restrictive.

§1 CENTRAL LOCALIZATION

Let A be a central subring of a ring R satisfying (*I*). Take any prime ideal Q of A. Then the set $X = A - Q$ is evidently an Ore and reversible multiplicative subset of R. Hence there is a localization of R at X which will be denoted by R_Q and will be called the central localization of R at Q. Denote the canonical localization map by $\varepsilon_Q : R \to R_Q$. The set $\{P \subset \mathrm{Spec}(R) \mid P \cap A = Q\}$ is called the Q-set. With this set-up, we list below a few basic observations.

Proposition 3.1. Let Q, A and R be as abovementioned. Then

(1) R_Q is module-finite over A_Q, and R_Q/QR_Q is an artinian ring.

(2) $J(R_Q)$ contains QR_Q and hence R_Q is semilocal.

(3) The Q-set is finite and localizable in R. Moreover, $R_S \simeq R_Q$ where $S = \bigcap$Q-set.

(4) There exists an integer $k > 0$ such that $S^k R_Q \subset QR_Q$.

Proof. The module-finiteness of R_Q over A_Q is a direct consequence of that of R over A. Likewise, R_Q/QR_Q is also module-finite over A_Q/QA_Q. The latter being a field makes R_Q/QR_Q a finite dimensional A_Q/QA_Q-vector space. Hence R_Q/QR_Q is artinian. This proves (1).

R_Q being module-finite, hence integral over A_Q implies $J(A_Q) = J(R_Q) \cap A_Q$ by a result in [10]. Therefore $QA_Q \subset J(R_Q)$ which further yields $QR_Q \subset J(R_Q)$. Now $R_Q/J(R_Q) \simeq (R_Q/QR_Q)/(J(R_Q)/QR_Q)$ indicates that R_Q is semilocal, thus confirming (2).

Let $X = A - Q$. Note that all the prime ideals from the Q-set are X-closed and, upon passing to R_Q, account for all maximal ideals of R_Q since R_Q/QR_Q is artinian. Moreover, the fact that R_Q is semilocal establishes the finiteness of the Q-set.

Denote the Q-set by $\{P_1,\ldots,P_n\}$ and put $S = \bigcap_{i=1}^{n} P_i$. Clearly all the P_i are pairwise incomparable as they are X-closed. Furthermore, R is a PI ring since it satisfies a standard identity $s_m(x_1,\ldots,x_m)$ for some suitable m. Therefore all prime ideals of R are Goldie. So it suffices to prove that C(S) is localizable. First, observe that

$SR_Q \simeq (\bigcap_{i=1}^{n} P_i) \otimes_R R_Q = \bigcap_{i=1}^{n} (P_i \otimes_R R_Q) \simeq \bigcap_{i=1}^{n} (P_i R_Q) = J(R_Q)$. Also $R_Q/J(R_Q) = R_Q/SR_Q \simeq (R/S)_X$. Now take any $t \in C(S)$. Then $\bar{t}$ is regular in R/S and hence is also regular in $(R/S)_X$ since $X \subset C(S)$. In fact, $\bar{t}$ is invertible in $(R/S)_X$ for $(R/S)_X$ is a semisimple artinian ring. Via the ring isomorphism, $\varepsilon_Q(t)$ becomes invertible modulo $J(R_Q)$ and therefore is invertible in R_Q. By Proposition 1.1, $C(S)$ is localizable and so $R_S \simeq R_Q$. This proves (3).

Statement (4) results trivially from the nilpotency of $J(R_Q/QR_Q) = SR_Q/QR_Q$. ||

<u>Remarks</u>. (*a*) The torsion theory determined by X, that is, by taking $\{I \mid I \text{ is a right ideal of } R \text{ with } I \cap X \neq \emptyset\}$ as the Gabriel filter, coincides with the S-torsion theory. This is because both torsion theories are perfect and correspond to the same Silver localization as asserted by (3). (See [22], Corollary 2.10.)

(*b*) When Q ranges over Spec(A), the Q-sets are then in a one-to-one correspondence with the prime ideals of A. In fact, they induce an equivalence relation on Spec(R) in which the equivalence classes are precisely all the Q-sets. Assertion (3) ensures the localizability of these equivalence classes. It is then natural to ask when they will become minimal localizable sets or better still (strong) clans. We shall undertake the study of this problem in the next two sections.

§2 MINIMALITY OF LOCALIZABLE SETS

Let R be a ring which satisfies the following two assumptions:

(*II*) R is module-finite over its centre C.

(*III*) For any prime ideal Q of C, the $J(C_Q)$-adic completion $\hat{C}_Q$ is a flat C_Q-module.

We want to show that these two conditions are sufficient for the minimality of the Q-sets among the localizable sets. But first we need two lemmas.

Lemma 3.2. If $u : A \to B$ is a flat homomorphism of commutative rings and D is a ring which is module-finite over A, then $C \otimes_A B$ is the centre of $D \otimes_A B$ where C is the centre of D.

The above lemma is due to P. Gabriel ([6], p.432). Applying it to the flat homomorphisms $C \to C_Q$ and $C_Q \to \hat{C}_Q$, we may identify C_Q and $\hat{C}_Q$ with the centres of R_Q and $\hat{R}_Q$ respectively. Here $\hat{R}_Q$ is the $J(R_Q)$-adic completion of R_Q. Because $\hat{C}_Q$ is a local ring, $\hat{R}_Q$ is ring-directly indecomposable.

Lemma 3.3. There exists an integer $k > 0$ such that $R_Q/J(R_Q)^k$ has no non-trivial central idempotents.

Proof. We proceed to prove the lemma by contradiction. Then for each integer $n > 0$, the set B_n, consisting of all non-trivial central idempotents of $R_Q/J(R_Q)^n$, is by assumption a non-empty finite set. Denote by δ_n the canonical map $R_Q/J(R_Q)^{n+1} \to R_Q/J(R_Q)^n$. Obviously $\delta_n(B_{n+1}) \subset B_n$. By König's Graph Theorem, there exists a sequence (e_n) such that $e_n \in B_n$ and $\delta_n(e_{n+1}) = e_n$. By definition of δ_n, (e_n) is a central idempotent of $\hat{R}_Q$. But $\hat{R}_Q$ is ring-directly indecomposable. Thus $(e_n) = 0$ or $(e_n) = 1$, a contradiction. This completes the proof of the lemma. ||

We are now ready to prove

Proposition 3.4. Every Q-set is a minimal localizable set.

Proof. Let $\{P_1,\ldots,P_n\}$ be a Q-set. Then $\{P_1R_Q,\ldots,P_nR_Q\}$ is a localizable set in R_Q since $R_Q \simeq R_S$ with $S = \cap$Q-set. By Lemma 3.3, we may pick an integer $k > 0$ such that $R_Q/J(R_Q)^k$ has no non-trivial central idempotents. Since $\overline{R}_Q = R_Q/J(R_Q)^k$ is a semiprimary ring, $\{\overline{P_1R_Q},\ldots,\overline{P_nR_Q}\}$ is localizable, hence strongly classical in $\overline{R}_Q$ by virtue of Proposition 1.4. In fact, it is a strong clan by Corollary 1.6, hence a fortiori a minimal localizable set. From this follows the minimality of the Q-set in view of Proposition 1.3. ||

Observe that given a finite collection of Q-sets such that all the prime ideals in the union U of these Q-sets are pairwise incomparable. Then U is localizable in R. The proof of this observation is identical with that of Proposition 1.18, except that we do not have the semiprime ideal, associated with U, to be classical.

Concerning the converse implication of Proposition 3.4, we do not know whether it holds in general. Nonetheless, we do have an affirmative answer in a more specialized situation, especially if we further impose

(IV) All the prime ideals of R are maximal.

This condition is equivalent to having all the prime ideals of C maximal. (See [9])

So now the ring R satisfies assumptions (*II*), (*III*) and (*IV*), and for such a ring R, we propose to show that the Q-sets give a complete description of the minimal localizable sets of prime ideals in R. We commence our pursuit with a few lemmas.

<u>Notation</u>. Let $\{P_1,\ldots,P_q\}$ be a minimal localizable set of prime ideals of R. Let $S = \bigcap_{i=1}^{q} P_i$ and $Q_i = P_i \cap C$ for $i = 1,\ldots,q$. We may assume, without loss of generality, the first t Q_i are exactly all the distinct prime ideals among the Q_i. Clearly they are pairwise incomparable because of condition (*IV*). Let $X = \bigcap_{i=1}^{t} (C - Q_i)$ which is an Ore and reversible subset of R. Through an abuse of notation, we write R_Q instead of R_X where $Q = \bigcap_{i=1}^{t} Q_i$.

<u>Lemma 3.5</u>. The maximal ideals of R_Q are precisely those PR_Q where P belongs to the union of all the Q_i-sets for $i = 1,\ldots,t$.

<u>Proof</u>. Observe that C_Q is a semilocal ring with maximal ideals Q_iC_Q for $i = 1,\ldots,t$ and R_Q is module-finite over C_Q. With this observation, a direct verification will establish the lemma. ||

<u>Lemma 3.6</u>. $\hat{C}_Q$ is a flat C_Q-module and $\hat{C}_Q \simeq \hat{C}_{Q_1} \times \ldots \times \hat{C}_{Q_t}$ as rings.

<u>Proof</u>. Consider the localization maps

$$\begin{array}{ccc} C & \overset{\alpha}{\rightarrow} & C_Q \\ \beta\downarrow & & \\ C_{Q_i} & & \end{array}$$

Then the induced map $\gamma : C_Q \to C_{Q_i}$ is a flat ring homomorphism, making the diagram

$$\begin{array}{ccc} C & \overset{\alpha}{\to} & C_Q \\ \beta\downarrow & \swarrow\gamma & \\ C_{Q_i} & & \end{array}$$

commute.

By (*III*), the completion map $\delta : C_{Q_i} \to \hat{C}_{Q_i}$ is flat. Hence the composite map $\delta\gamma : C_Q \to \hat{C}_{Q_i}$ is flat. This gives the flatness of $\hat{C}_{Q_1} \times \ldots \times \hat{C}_{Q_t}$ as a C_Q-module. It remains to show that $\hat{C}_Q \simeq \hat{C}_{Q_1} \times \ldots \times \hat{C}_{Q_t}$ as rings.

Assume now $rc^{-1} \quad r'c^{-1}$ (Mod Q^nC_Q) for $rc^{-1}, r'c^{-1} \in C_Q$. Then there exists $z \in X$ with $(r-r')z \in Q^n \subset Q_i^n$ for all $i = 1,\ldots,t$. This leads to $rc^{-1} \equiv r'c^{-1}$ (Mod $Q_i^nC_{Q_i}$) since $X \subset C - Q_i$ for all i. Therefore the diagonal map $\phi_n : C_Q/Q^nC_Q \to \prod_{i=1}^{t} C_{Q_i}/Q_i^nC_{Q_i}$ is a well-defined ring homomorphism.

To show ϕ_n is one-to-one, consider $rc^{-1} \in C_Q$ such that $rc^{-1} \in Q_i^nC_{Q_i}$ for all i. Then for each i, there exists $c_i \in C - Q_i$ with $rc_i \in Q_i^n$. Since by (*IV*) $Q_1^n, \ldots, Q_t^n$ are pairwise relatively prime, we may invoke the Chinese Remainder Theorem to get an element c' of C with $c' \equiv c_i$ (Mod Q_i^n) for all i. Obviously, $c' \in X$, and so $rc' \equiv rc_i$ (Mod Q_i^n), implying $rc' \in \bigcap_{i=1}^{t} Q_i^n = Q^n$. Thus $rc^{-1} \in Q^nC_Q$.

As for surjectivity of ϕ_n, consider $r_i c_i^{-1} \in C_{Q_i}$ for $i = 1,\ldots,t$. Again the Chinese Remainder Theorem yields elements $r \in C$ and $c \in X$ with $r \equiv r_i$ and $c \equiv c_i$ (Mod Q_i^n). From this follows $c^{-1} \equiv c_i^{-1}$ (Mod $Q_i^n C_{Q_i}$), and so $rc^{-1} \equiv r_i c_i^{-1}$ (Mod $Q_i^n C_{Q_i}$) for $1 = 1,\ldots,t$.

Therefore ϕ_n is a ring isomorphism for every n. Furthermore, they induce a ring isomorphism between $\hat{C}_Q$ and $\hat{C}_{Q_1} \times \ldots \times \hat{C}_{Q_t}$. ||

The above lemma with the help of Lemma 3.2 identifies the centre of $\hat{R}_Q$ with $\hat{C}_{Q_1} \times \ldots \times \hat{C}_{Q_t}$. Thus $\hat{R}_Q$ has exactly t centrally indecomposable central idempotents. Let $1 = e_1 + \ldots + e_t$ where each e_i is a centrally indecomposable central idempotent of $\hat{R}_Q$, and let ν_n be the canonical maps $\hat{R}_Q \to R_Q/J(R_Q)^n$. For any integers $n, i > 0$, $\nu_n(e_i)$ is a central idempotent of $R_Q/J(R_Q)^n$.

Lemma 3.7. For each i, there exists an integer $n_i > 0$ such that $\nu_{n_i}(e_i)$ is centrally indecomposable.

Proof. Fix an integer i and suppose the assertion is false. Then $\nu_n(e_i)$ must be centrally decomposable for every $n > 0$. Let A_n denote the set of all central idempotents of $R_Q/J(R_Q)^n$ and put $B_n = \{r \in A_n \mid \nu_n(e_i) = r + y$ for some non-zero $y \in A_n$ with $ry = 0\}$. By assumption, $B_n \neq \emptyset$ for all $n > 0$. Denote by δ_n the canonical maps $R_Q/J(R_Q)^{n+1} \to R_Q/J(R_Q)^n$. Then the König's Graph Theorem yields a sequence (b_n) with $b_n \in B_n$ and $\delta_n(b_{n+1}) = b_n$. Note that (b_n) is a central idempotent of $\hat{R}_Q$, and for each n, there exists $c_n \in B_n$ with $\nu_n(e_i) = b_n + c_n$. Obviously (c_n) is also a central idempotent of $\hat{R}_Q$.

Therefore $e_i = (b_n) + (c_n)$, a contradiction. ||

Corollary 3.8. There exists an integer $n > 0$ such that $\nu_n(e_i)$ are centrally indecomposable for all $i = 1,\ldots,t$.

Proof. The result follows readily from Lemma 3.7 by taking $n = \max\{n_1,\ldots,n_t\}$. ||

We finally come to our main result:

Theorem 3.9. $\{P_1,\ldots,P_q\}$ is a Q_i-set for some i.

Proof. It is clear that $\{P_1R_Q,\ldots,P_qR_Q\}$ is a localizable set in R_Q. Also, Corollary 3.8 yields an integer $n > 0$ such that $\nu_n(e_i)$ are centrally indecomposable for all $i = 1,\ldots,t$. Since $\bar{1} = \nu_n(e_1) + \ldots + \nu_n(e_t)$ in $\bar{R}_Q = R_Q/J(R_Q)^n$, that means $\bar{R}_Q$ has exactly t centrally indecomposable central idempotents.

On the other hand, each Q_i-set, upon passing to $\bar{R}_Q$, becomes a strong clan in $\bar{R}_Q$ since they constitute t mutually disjoint strongly classical sets in the semiprimary ring $\bar{R}_Q$. Hence $\{\overline{P_1R_Q},\ldots,\overline{P_qR_Q}\}$ is a union of some of these strong clans. This implies $\{P_1,\ldots,P_q\}$ must contain some Q_i-set as subset, and thus must coincide with that Q_i-set by minimality. ||

§3 CLASSICAL SEMIPRIME IDEALS OF THE RING AND CLANS OF ITS CENTRE

This section consists of two parts. The first part deals with the relationship between classical semiprime ideals of the ring R and those of a central subring A over which R is module-finite. We are aiming to establish here two results. First and foremost, every clan is contained in some Q-set where $Q \in \mathrm{Spec}(A)$. Secondly, a Q-set is classical if and only if Q itself constitutes a clan in A. In the same vein, we proceed to analyze strongly classical semiprime ideals for the second part of this section.

To begin with, let R be a ring which is module-finite over a central subring A. This setting will be assumed throughout the entire section.

<u>Proposition 3.10</u>. Every clan in R is a subset of some Q-set with $Q \in \mathrm{Spec}(A)$.

<u>Proof</u>. Let $\{P_1,\ldots,P_n\}$ be a clan in R. Put $S = \bigcap_{i=1}^{n} P_i$. Without loss of generality, we may assume $Q = P_1 \cap A$ is a minimal prime among all the $P_i \cap A$. Also, by re-indexing the prime ideals in the clan, we may assume that $P_1,\ldots,P_t$ are all the prime ideals from the clan such that $P_i \cap A = Q$. Put $T = \bigcap_{i=1}^{t} P_i$. Then a repetition of the argument used in the proof of Theorem 1.15 will confirm $T = S$, and so $t = n$. ||

In [27], J. Osterburg proved the following result:

<u>Proposition 3.11</u>. Let R be a ring which is module-finite over a semilocal noetherian central subring A. Then $E_R(R/J(R)) \simeq \mathrm{Hom}_A(R,V)$ where $V = E_A(A/J(A))$.

Our first main theorem below requires in its proof a generalization of the above proposition. Therefore our primary concern is to show that the noetherian assumption in Proposition 3.11 is really superfluous.

Lemma 3.12. Let R be a ring which is module-finite over a semilocal central subring A. Then $E_R(R/J(R)) \simeq \mathrm{Hom}_A(R,V)$ where $V = E_A(A/J(A))$.

Proof. It is a well-known fact that $H = \mathrm{Hom}_A(R,V)$ is an injective R-module. For brevity, we write $J = J(R)$ and let $M = \{f \in H \mid J \subset \ker f\}$. Then as an R-module, $M = \mathrm{ann}_H J$. Therefore, M is semisimple both as an R-module and as an R/J-module, since R is semilocal. This implies $M = R\text{-socle}(H)$.

From the definition of M follows $M \simeq \mathrm{Hom}_A(R/J,V)$ as R-modules. Furthermore, $\mathrm{Hom}_A(R/J,V) \simeq \mathrm{Hom}_{A/J(A)}(R/J,\mathrm{ann}_V J(A)) = \mathrm{Hom}_{A/J(A)}(R/J,A/J(A))$, and by Proposition 3.11, $\mathrm{Hom}_{A/J(A)}(R/J,A/J(A)) \simeq R/J$ as R-modules. Thus we may regard R/J as an R-submodule of H. To complete the proof, it suffices to prove that M is essential in H.

Let $V_n = \mathrm{ann}_V J(A)^n$ and observe that $\bigcup_{n=1}^{\infty} V_n$ is essential in V. Then we make the following claim: if $Y = v_1A + \dots + v_sA$ is a non-zero finitely generated A-submodule of V, then there is an element $a \in A$ with the property that all $v_ia \in \bigcup_{n=1}^{\infty} V_n$ and at least one $v_ia \neq 0$. We proceed to prove this claim by induction on s. For $s = 1$, the claim is trivial. So take $s > 1$. By the inductive hypothesis, there exists an element $a \in A$ such that for $i = 1,\dots,s-1$, all $v_ia \in \bigcup_{n=1}^{\infty} V_n$ and at least one $v_i a \neq 0$. If $v_s a = 0$, then a is the desired element. If, on the other hand, $v_s a \neq 0$, then there exists an element $b \in A$ with $0 \neq v_s ab \in \bigcup_{n=1}^{\infty} V_n$ by essentiality, and so ab is the desired element. This completes the inductive proof of the claim.

Now take any non-zero A-homomorphism $f : R \to V$. Then $\operatorname{im} f$ is a finitely generated A-submodule of V. So we may let $\operatorname{im} f = \sum_{i=1}^{s} v_i A$. By the above claim, there exists an element $a \in A$ such that all $v_i a \in V_n$ for some n, and at least one $v_i a \neq 0$. Thus $f(ar) \in V_n$ for all $r \in R$. This implies $fa : R \to V_n$ is a non-zero A-homomorphism. Hence $\operatorname{Hom}_A(R,V)$ is an essential R-module over $\bigcup_{n=1}^{\infty} \operatorname{Hom}_A(R,V_n)$.

Consider, next, any non-zero A-homomorphism $g : R \to V_n$. That is, $\operatorname{im} g$ is annihilated by $J(A)^n$. We may assume n to be the smallest such integer. If $n = 1$, then $g \in \operatorname{Hom}_A(R,V_1)$. Suppose $n > 1$. Since $(\operatorname{im} g)J(A)^{n-1} \neq 0$, there must exist an element $j \in J(A)^{n-1}$ with $(\operatorname{im} g)j \neq 0$. However, $(\operatorname{im} g)jJ(A) = 0$. Hence $gj \in \operatorname{Hom}_A(R,V_1)$. This demonstrates the essentiality of $\bigcup_{n=1}^{\infty} \operatorname{Hom}_A(R,V_n)$ over $\operatorname{Hom}_A(R,V_1)$.

As $J(A) \subset J(R)$ by a result in [10], $M \subset \operatorname{Hom}_A(R,V_1)$. Note that $R/J(A)R$ is an artinian ring with Jacobson radical $J(R)/J(A)R$ due to its module-finiteness over the commutative artinian ring $A/J(A)$. (See [5]) Therefore $J(R)^k \subset J(A)R$ for some integer $k > 0$. This leads to $J(R)^k \subset \ker f$ for any $f \in \operatorname{Hom}_A(R,V_1)$, hence proving the essentiality of $\operatorname{Hom}_A(R,V_1)$ over M. Piecing together all the above observations, we see that M is essential in H. ||

Theorem 3.13. Let $Q \in \mathrm{Spec}(A)$. Then $\{Q\}$ is a clan in A if and only if the Q-set is classical in R.

Proof. Suppose Q constitutes a clan in A. That is, $V = E_A(A/Q) = \bigcup_{n=1}^{\infty} V_n$ where $V_n = \mathrm{ann}_V\, Q^n$. Since R_Q is module-finite over A_Q, $\mathrm{Hom}_{A_Q}(R_Q,V) = \bigcup_{n=1}^{\infty} \mathrm{Hom}_{A_Q}(R_Q,V_n)$, given the observation that V is the injective hull of A_Q/QA_Q as an A_Q-module. By Lemma 3.12, we have $\mathrm{Hom}_{A_Q}(R_Q,V) = E_{R_Q}(R_Q/J(R_Q)) = H$. Let $S = \bigcap Q$-set. Then by Proposition 3.1, there exists an integer $m > 0$ such that $S^m R_Q \subset QR_Q$. Now take any $f \in \mathrm{Hom}_{A_Q}(R_Q,V_n)$ for some arbitrary n. Then $fQ^n R_Q = 0$, which implies $fS^{mn}R_Q = 0$, and thus $f \in \mathrm{ann}_H\, S^{mn}R_Q$. Hence $H = \bigcup_{n=1}^{\infty} \mathrm{ann}_H\, S^n R_Q$. In other words, S is classical since $R_Q \simeq R_S$ by Proposition 3.1.

Conversely, suppose S is classical. Then by Lemma 3.12, we get
$$H = E_R(R/S) = \bigcup_{n=1}^{\infty} \mathrm{ann}_H\, S^n = \bigcup_{n=1}^{\infty} \mathrm{ann}_H\, Q^n = \mathrm{Hom}_{A_Q}(R_Q,V) = \bigcup_{n=1}^{\infty} \mathrm{Hom}_{A_Q}(R_Q,V_n).$$
(Note: The last equality is obtained by repeating the preceding one.) Now take any non-zero $v \in V$ and define an A_Q-homomorphism $g : A_Q \to V$ by $g(x) = vx$ for all $x \in A_Q$. Then the injectivity of V extends g to an A_Q-homomorphism $h : R_Q \to V$. But then h is a map with image contained in V_n for some n, as indicated above. Hence $h(1) = g(1) = v \in V_n$. This shows $V = \bigcup_{n=1}^{\infty} V_n$. ||

Remarks. Let C be the centre of R. In the context of Theorem 3.13, the Q-set can be partitioned into pairwise disjoint clans by virtue of Corollary 1.16. The number of prime ideals of C lying over Q is at most equal to the number of clans in the partition. All these prime ideals are also classical in C in view of Theorems 1.12 and 3.13 since R is evidently module-finite over C. In particular, if the Q-set is a clan, then there is only one prime ideal of C over Q.

Some groundwork is needed for the second part of this section on strongly classical semiprime ideals. Our objective is to establish an analogue of Theorem 3.13. First, we notice that in [17] the right AR-property for the Jacobson radical of a semilocal noetherian ring has been characterized internally as well as externally. As a matter of fact, one of these characterizations remains valid for semilocal non-noetherian rings. We will look at this observation again later when we give another criterion for the right AR-property. The latter characterization will be used subsequently in achieving our objective.

Lemma 3.14. Let R be a semilocal ring whose Jacobson radical J(R) has the right AR-property. Then for any semisimple R-module M, $E(M) = \bigcup_{n=1}^{\infty} \operatorname{ann}_{E(M)} J(R)^n$.

Proof. The proof for the implication of (b) from (a) in Proposition 4.3 of [17] can be carried over here verbatim. ||

Lemma 3.15. Let R be a semilocal ring whose Jacobson radical J(R) has the right AR-property. Then for any finitely generated R-module M, there exists an integer $n > 0$ such that socle $(M) \cap MJ(R)^n = 0$.

Proof. We are confronted with two cases: First, suppose that socle (M) is essential in M. Then $M \subset E_R(\text{socle } (M))$. By Lemma 3.14, $MJ(R)^n = 0$ for some n since M is finitely generated. Hence socle $(M) \cap MJ(R)^n = 0$ holds trivially.

Next, suppose that socle (M) is not essential in M. In this case, the set $\{I \mid I$ is a non-zero submodule of M with $I \cap$ socle $(M) = 0\}$ is non-empty. The Zorn's Lemma then yields a maximal member, say L, in this set. We claim that socle $(M) \oplus L$ is essential in M. Take any non-zero $m \in M$ such that $m \notin$ socle $(M) \oplus L$. Then socle $(M) \cap (L + mR) \neq 0$ by maximality of L. So there exists a non-zero $x = z + mr \in$ socle (M) for some $z \in L$ and $r \in R$. If $z = 0$, then socle $(M) \cap mR \neq 0$, and a fortiori, (socle $(M) \oplus L) \cap mR \neq 0$. On the other hand, if $z \neq 0$, then $zs \neq 0$ for some $s \in J(R)$. However, $(z + mr)s = 0$. This leads to $zs = m(-rs)$ which implies $L \cap mR \neq 0$, and a fortiori, (socle $(M) \oplus L) \cap mR \neq 0$. Hence the claim is proved.

Therefore, $M \subset E(\text{socle } (M)) \oplus E(L)$. By Lemma 3.14, we get $W = E(\text{socle } (M)) = \bigcup_{n=1}^{\infty} E_n$ where $E_n = \text{ann}_W\, J(R)^n$. Let M be generated by $m_1, \ldots, m_k$. Then for each i; $m_i = x_i + y_i$ for some $x_i \in W$ and $y_i \in E(L)$. Since each x_i annihilates some power of $J(R)$, there exists an integer $n > 0$ such that $x_i J(R)^n = 0$ for all i. Thus $m_i J(R)^n = y_i J(R)^n$ for all i. This implies $MJ(R)^n \subset E(L)$, and so socle $(MJ(R)^n) = 0$. From this follows socle $(M) \cap MJ(R)^n = 0$. ||

<u>Proposition 3.16</u>. Let R be a semilocal ring with Jacobson radical $J(R)$. Then the following conditions are equivalent:

(1) $J(R)$ has the right AR-property.

(2) For any finitely generated R-module M and any submodule N of M, there exists an integer $n > 0$ such that $N \cap MJ(R)^n \subset NJ(R)$.

(3) Every right ideal of R is closed in the J(R)-adic topology.

(4) For every semisimple R-module M, $E(M) = \bigcup_{n=1}^{\infty} \text{ann}_{E(M)} J(R)^n$.

Proof. Assume (1). Let N be a submodule of a finitely generated R-module M. Apply Lemma 3.15 to $\bar{M} = M/NJ(R)$ to get an integer $n > 0$ such that $\text{socle}(\bar{M}) \cap \bar{M}J(R)^n = 0$. Observe that $\bar{N} = N/NJ(R) \subset \text{socle}(\bar{M})$. Thus $\bar{N} \cap \bar{M}J(R)^n = 0$. That is, $N \cap MJ(R)^n \subset NJ(R)$, hence proving (2).

Given (2), let I be a right ideal of R. Put $\bar{R} = R/I$. By (2), the J(R)-adic topology on $\bar{R}$ is Hausdorff. Thus $\bigcap_{n=1}^{\infty} (I + J(R)^n) = I$, yielding (3).

The proof of (d) implying (a) in Proposition 4.3 of [17] can be used to establish the implication of (1) from (3).

The implication of (4) from (1) is actually the assertion of Lemma 3.14. Conversely, suppose (4) is given. Let I be a right ideal of R. Then $I/IJ(R)$ is a semisimple R-module since R is semilocal. By (4), $E = E(I/IJ(R)) = \bigcup_{n=1}^{\infty} \text{ann}_E J(R)^n$. Let $f : I \to E$ be the composite of the canonical maps $I \to I/IJ(R) \to E$. Then there exists an element $e \in E$ such that $f(x) = ex$ for all $x \in I$. Since $eJ(R)^n = 0$ for some n, it follows that $I \cap J(R)^n \subset \ker f = IJ(R)$. This proves (1). ||

Condition (3) in the above proposition appears in [17]. Condition (4) completes the converse implication of Lemma 3.14, and Condition (2) is the one to be used in the proof of the following theorem.

Theorem 3.17. Let A be a commutative semilocal ring. Then J(A) has the AR-property if and only if for every ring R which is module-finite over A, J(R) has the AR-property.

Proof. Suppose J(A) has the AR-property. Let R be a ring which is module-finite over A. Observe that $J(A)R \subset J(R)$ and $R/J(A)R$ is an artinian ring with Jacobson radical $J(R)/J(A)R$. Thus there exists an integer $k > 0$ such that $J(R)^k \subset J(A)R$. Take any right ideal I of R. Clearly I is an A-submodule of R. By Proposition 3.16, $I \cap RJ(A)^n \subset IJ(A)$ for some integer n. This, together with the above observation, yields $I \cap J(R)^{kn} \subset IJ(R)$, hence demonstrating the right AR-property. The left AR-property is similarly verified.

The converse implication is trivial. ||

Our second main result now becomes a corollary of the preceding theorem.

Corollary 3.18. Let A be a commutative ring and $Q \in \mathrm{Spec}(A)$. Then Q is strongly classical in A if and only if for every ring R which is module-finite over A, the Q-set is strongly classical in R.

Proof. A direct application of Theorem 3.17 to R_Q which is module-finite over the local ring A_Q yields the desired result. ||

§4 EXAMPLES.

The following examples are rings in which every Q-set is a strong clan.

(A) Let R be a ring which is module-finite over its centre C, and suppose C is a von Neumann regular ring. Take any $Q \in \mathrm{Spec}(C)$. Then R_Q is a finite dimensional vector space over the field C_Q and hence is itself an artinian ring. Therefore both C_Q and R_Q coincide with $\hat{C}_Q$ and $\hat{R}_Q$ respectively. Lemma 3.2 shows that C_Q is the centre of R_Q. On account of Theorem 1.12, the Q-set is a strong clan.

Such ring R in general need not be von Neumann regular. For instance, take any commutative von Neumann regular ring C and let $R = \begin{pmatrix} C & C \\ 0 & C \end{pmatrix}$. Then $J(R) = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}$ shows that R is not von Neumann regular.

(B) There is a commutative ring C (due to M. Nagata [25]) which has infinitely many maximal ideals, and the localization C_M at every maximal ideal M is noetherian. This ring differs from the commutative ring described in above example at least for the reason that it is not coherent whereas the preceding one is. Now let R be a ring having C as its centre and being module-finite over it.

For every maximal ideal M of C, R_M is module-finite over C_M and hence is noetherian by [5]. Moreover, C_M is the centre of R_M. The M-set, upon passing to R_M, becomes the $J(C_M)$-set which is a strong clan in R_M on account of Theorem 7 of [24]. Then Proposition 1.3 shows that the M-set is actually a strong clan in R.

We now consider a non-maximal prime ideal Q of C. Then Q is contained in some maximal ideal M of C. In order to see that the Q-set is a strong clan in R by means of Theorem 7 of [24], it suffices to show R_Q is a noetherian ring. Since R_M is noetherian as indicated above, the proof will be completed by showing $R_Q \simeq (R_M)_{QC_M}$.

Let $R \overset{\alpha}{\to} R_M \overset{\beta}{\to} (R_M)_{QC_M}$ be canonical maps, and let $\gamma = \beta\alpha$. Our aim is to prove that γ is indeed the localization map for $X = C - Q$. For any $t \in X$, $\gamma(t)$ is obviously an invertible element of $(R_M)_{QC_M}$. Now let $ab^{-1} \in R_M$ and $cd^{-1} \in C_M - QC_M$. Then $ab^{-1} = \alpha(a)\alpha(b)^{-1}$ and so $\beta(ab^{-1}) = \beta(\alpha(a)\alpha(b)^{-1}) = \beta\alpha(a)\beta\alpha(b)^{-1} = \gamma(a)\gamma(b)^{-1}$. Similarly, $\beta(cd^{-1}) = \gamma(c)\gamma(d)^{-1}$, which implies $\beta(cd^{-1})^{-1} = \gamma(d)\gamma(c)^{-1}$. Thus $(ab^{-1})(cd^{-1})^{-1} = \gamma(ad)\gamma(cb)^{-1}$. Moreover, if $\gamma(a) = 0$ for $a \in R$, then $\alpha(a)\alpha(c) = 0$ for some $c \in X$. This further implies $acd = 0$ for some $d \in C - M \subset X$. Hence γ is the localization map for X as required.

§5 GROUP RINGS

Group rings of finite groups over commutative rings are examples of the kind of rings under discussion in this chapter. Given a group ring $R = AG$ where A is a commutative ring and G is a finite group, the centre C of R is given by $\{ \Sigma a_g g \mid a_g = a_h$ if g and h are conjugate$\}$. One interesting aspect of the group ring R in terms of localization is that all the Π-sets with $\Pi \in \mathrm{Spec}(C)$ are minimal localizable sets in R without having to impose the flatness condition (*III*) as seen earlier in §2. Hence, if all the prime ideals of A are maximal, it is then natural to expect that the Π-sets will completely characterize the minimal localizable sets in R in the same manner as asserted in Theorem 3.9 whose proof nevertheless requires condition (*III*).

Before delving in any further, we record below some elementary information concerning centrally indecomposable central idempotents. Their proofs are rather straightforward and are thus omitted. Henceforth, these facts will be used without mention.

(*a*) Any two distinct centrally indecomposable central idempotents of a ring R are mutually orthogonal.

(*b*) Let $1 = e_1 + \ldots + e_n$ where all the e_i are centrally indecomposable central idempotents of R. Then for every maximal ideal M of R, there exists uniquely one such e_i such that $e_i \notin M$.

(*c*) Let e be a centrally indecomposable central idempotent of R. Then for any maximal ideal M of R with $e \notin M$, eM is a maximal ideal of eR. Conversely, given a maximal ideal N of eR, $e^{-1}N = \{r \in R \mid er \in N\}$ is a maximal ideal of R with $e \notin e^{-1}N$. This defines a one-to-one correspondence between maximal ideals of R not containing e and maximal ideals of eR.

<u>Definition</u>. A non-zero ideal B of a ring R is called a <u>block ideal</u> (or <u>block</u>, in short) if there exists a centrally indecomposable central idempotent e such that B = eR. Such e is uniquely determined by B. The block B, per se, is a ring with identity e. The reader may consult [19] for more details of block ideals.

The following lemma generalizes a result in [24].

Lemma 3.19. Let $Q = \bigcap_{i=1}^{n} Q_i$ be an irredundant intersection of prime ideals of a commutative ring A, and let $K = A_Q/QA_Q$. Denote by C the centre of the group ring R = AG where G is a finite group. Then there is a one-to-one correspondece between all the prime ideals Π of C over all the Q_i and the blocks of KG. Moreover, the prime ideals of R over such Π correspond bijectively to the maximal ideals of the block.

Proof. (1) Since R_Q/QR_Q is artinian, the set of all maximal ideals of R_Q consists of exactly all PR_Q where $P \in \mathrm{Spec}(R)$ such that $P \cap A = Q_i$ for $i = 1,\ldots,n$. Hence the prime ideals of R over Q_i for $i = 1,\ldots,n$ correspond bijectively to the maximal ideals of R_Q/QR_Q. Moreover, we have $R_Q/QR_Q \simeq A_QG/QA_QG \simeq (A_Q/QA_Q)G = KG$. Therefore the prime ideals of R over Q_i for $i = 1,\ldots,n$ correspond bijectively to the maximal ideals of KG.

(2) Obviously, C_Q/QC_Q is an artinian ring as R_Q/QR_Q is module-finite over it. Since C is integral over A, it follows as in (1) above that the prime ideals of C over Q_i for $i = 1,\ldots,n$ correspond bijectively to the maximal ideals of C_Q/QC_Q. But the restriction of the natural map $R_Q \to KG$ to C_Q induces an isomorphism between C_Q/QC_Q and the centre Z(KG) of KG. Thus the prime ideals of C over Q_i for $i = 1,\ldots,n$ correspond bijectively to the maximal ideals of Z(KG) via this isomorphism, hence to the indecomposable idempotents of Z(KG), and then accordingly to the blocks of KG.

(3) Let $1 = e_1 + \ldots + e_m$ be the decomposition of 1 into centrally indecomposable central idempotents e_i in KG. Denote the blocks

by $B_i = e_iKG$ for $i = 1,\ldots,m$. Suppose Π is a prime ideal of C lying over some Q_j. Then by (2), Π corresponds to a unique block B_i. Our task now is to establish a one-to-one correspondence between the prime ideals of R over Π and the maximal ideals of B_i. Let P_1 and P_2 be prime ideals of R over Π. Then they are also prime ideals lying over Q_j. Hence by (1), each P_k corresponds to a maximal ideal M_k of KG uniquely. Therefore e_iM_1 and e_iM_2 are distinct maximal ideals of B_i because $e_i \notin M_1 \cup M_2$ by (2).

Conversely, if N is a maximal ideal of B_i, then $M = B_1 \oplus \ldots \oplus B_{i-1} \oplus N \oplus B_{i+1} \oplus \ldots \oplus B_m$ is a maximal ideal of KG and $e_i \notin M$. By (1), M corresponds to a unique prime ideal P of R over some Q_h. However, $P \cap C = \Pi$ since $(P \cap C)_Q/QC_Q \simeq M \cap Z(KG)$. Hence $h = j$ by (2). ||

Specializing Lemma 3.19 to the case where $Q = Q_1$, we have

<u>Proposition 3.20</u>. For a group ring $R = AG$ with centre C, the Π-set is a minimal localizable set in R for every $\Pi \in \mathrm{Spec}(C)$.

<u>Proof</u>. Let $\{P_1,\ldots,P_n\}$ be a Π-set in R, $Q = \Pi \cap A$ and K, the quotient field of A/Q which is also the residue field A_Q/QA_Q. We know by Proposition 3.1 that the Π-set is localizable. Let $M_1,\ldots,M_n$ be the maximal ideals of KG which correspond to $P_1,\ldots,P_n$ respectively via the canonical map $R \to R_Q \to R_Q/QR_Q \simeq KG$. Then $\{M_1,\ldots,M_n\}$ is localizable, hence strongly classical in KG.

If Π is the only prime ideal of C lying over Q, then by Lemma 3.19, 1 is centrally indecomposable in KG, and then Spec(KG) is a strong clan on account of Theorem 1.12. Thus $\{M_1,\ldots,M_n\}$ = Spec(KG) which in turn implies the Π-set contains no proper localizable subset and hence is minimal.

On the other hand, if there are more than one prime ideal of C lying over Q, then again Lemma 3.19 assures the existence of a centrally indecomposable central idempotent e of KG such that $e \notin \bigcup_{i=1}^{n} M_i$, since all M_i lie over the same maximal ideal of Z(KG) and that maximal ideal is isomorphic to $\Pi C_Q/QC_Q$. In this case, $eM_1,\ldots,eM_n$ are distinct maximal ideals of $B = eKG$. We claim now the set $\{eM_1,\ldots,eM_n\}$ is localizable in B. To this end, it suffices to show $ec \in \bigcap_{i=1}^{n} C_B(eM_i)$ if and only if $c \in \bigcap_{i=1}^{n} C_{KG}(M_i)$.

Let $c \in \bigcap_{i=1}^{n} C_{KG}(M_i)$ and suppose $(ec)(er) \in \bigcap_{i=1}^{n} eM_i$. That is, $cer \in \bigcap_{i=1}^{n} eM_i \subset \bigcap_{i=1}^{n} M_i$ which implies $er \in \bigcap_{i=1}^{n} M_i$, and so $er \in \bigcap_{i=1}^{n} eM_i$. Thus $ec \in \bigcap_{i=1}^{n} C_B(eM_i)$. Conversely, let $ec \in \bigcap_{i=1}^{n} C_B(eM_i)$ and suppose $cr \in \bigcap_{i=1}^{n} M_i$. This implies $er \in \bigcap_{i=1}^{n} eM_i$, or equivalently, $er \in \bigcap_{i=1}^{n} M_i$. Since e is central and does not belong to any M_i, it follows that $r \in \bigcap_{i=1}^{n} M_i$. This proves the claim. However, B has only one strong clan and this forces the Π-set to be minimal. ||

We now reinstate assumption (*IV*) from §2. With this condition added, we proceed to prove our second assertion (Proposition 3.21). In this case, all the prime ideals of A and of R are maximal. ([9])

Proposition 3.21. Assume the group ring $R = AG$ with centre C satisfies (IV). Then the Π-sets account for all the minimal localizable sets in R where Π ranges over all the prime ideals of C.

Proof. Let $\{P_1,\ldots,P_n\}$ be a minimal localizable set in R. Put $\Pi_i = P_i \cap C$ and $Q_i = \Pi_i \cap A$. Without loss of generality, we may assume that $\Pi_1,\ldots,\Pi_t$, with $t \leq n$, are all the distinct ones among the Π_i, and $Q_1,\ldots,Q_r$, with $r \leq t$, are all the distinct ones among the Q_i. Evidently, they are all pairwise incomparable by maximality. Let $Q = \bigcap_{i=1}^{r} Q_i$, $X = \bigcap_{i=1}^{r}(A - Q_i)$ and $K = A_Q/QA_Q$. If $t = 1$, then $\{P_1,\ldots,P_n\} = \Pi_1$-set by Proposition 3.20. So we assume $t > 1$.

Each P_i corresponds to a unique maximal ideal M_i of KG via the canonical map $R \to KG$. Then $\{M_1,\ldots,M_n\}$ is a localizable, hence strongly classical set in KG. By the same token, each Π_i-set, upon passing to KG, becomes a strongly classical set in KG. Since $t > 1$, Lemma 3.19 assures the existence of a centrally indecomposable central idempotent e_i of KG such that $e_i \notin \bigcup N_i$ where N_i denotes the set of images in KG of the Π_i-set under the canonical map. A repetition of the last part of the proof of Proposition 3.20 will confirm that each N_i is a strong clan in KG. Hence $\{M_1,\ldots,M_n\}$ must contain all these strong clans. In other words, $\{P_1,\ldots,P_n\}$ must contain all the Π_i-sets. By minimality, $\{P_1,\ldots,P_n\} = \Pi_i$-set for some i. ||

The following consideration requires some group representation theory. Let KG be the group algebra of a finite group G over a field K. Associated with any group representation Ψ of G is a K-vector space V

which carries a KG-module structure simultaneously. Such V is called the representation module of G belonging to Ψ. If V is a simple KG-module, then Ψ is said to be irreducible.

A simple KG-module W is said to belong to a block B of KG if $W \simeq B/M$ for some maximal right ideal M of B. Hence an irreducible representation is said to belong to a block B if its representation module belongs to B. A principal block is the one to which the trivial representation belongs.

Let q be a prime number. A finite group G is called q-nilpotent if there is a normal subgroup N whose order $|N|$ is not divisible by q but such that G/N is a q-group. In connection with this definition we record the following results.

Proposition 3.22. Let K be a field of characteristic $q > 0$ and G a finite group. Then we have:

(1) If G is q-nilpotent, then each block of KG has a unique simple module.

(2) The intersection of the kernels of the irreducible representations belonging to any block of KG is a q-nilpotent subgroup of G.

These two assertions can be found in [20] and [21] respectively. Our further discussion also necessitates the use of Maschke's Theorem [30] which states that given a division ring K, a group algebra KG is semi-simple artinian if and only if G is a finite group and char K does not divide the order of G.

For a prime ideal Q of a commutative ring A, the Q-augmentation ideal of the group ring AG is defined to be $\Delta_Q = \{\Sigma a_g g \mid \Sigma a_g \in Q\}$ which is a prime ideal of AG. Let K be the quotient field of A/Q and C be the centre of AG. Then according to Lemma 3.19, $\Delta_Q \cap C$ corresponds to a block of KG. This block is the principal block of KG.

Given below is a characterization of a q-nilpotent group in terms of localization.

Proposition 3.23. Let Q be a prime ideal of a commutative ring A and q be the characteristic of K, the quotient field of A/Q. Then for the group ring R = AG of a finite group G, the following conditions are equivalent:

(1) Every prime ideal P of R with $P \cap A = Q$ is localizable.

(2) Δ_Q is localizable.

(3) G is q-nilpotent.

Proof. If q does not divide the order |G| of G, then KG is a semisimple artinian ring by Maschke's Theorem. Hence each block of KG is a simple artinian ring. By Lemma 3.19, there lies only one prime ideal of R over any given prime ideal Π of C with $\Pi \cap A = Q$. The localizability of the Π-set which is a singleton set is assured by Proposition 3.20. Trivially, G is a q-nilpotent group.

So we now assume |G| to be divisible by q. Trivially (1) implies (2). Given (2), then $\{\Delta_Q\}$ is the $\Delta_Q \cap C$-set by Proposition 3.20. This implies the principal block has exactly one maximal ideal owing to Lemma 3.19.

Hence every irreducible representation belonging to the principal block has the same kernel as that of the trivial representation. The kernel of the latter is G. Thus by assertion (2) of Proposition 3.22, G is q-nilpotent.

Given (3), then it follows from assertion (1) of Proposition 3.22 that each block of KG has only one maximal ideal. Consequently, (1) can be immediately deduced from Lemma 3.19 and Proposition 3.20. ||

Corollary 3.24. All prime ideals of R = AG are localizable if and only if G is q-nilpotent for all prime numbers q which are not invertible in A.

Proof. Suppose G is q-nilpotent for all prime numbers q which are not invertible in A. Let P be a prime ideal of R and put $Q = P \cap A$. Let $q = \text{char } K$ where K is the quotient field of A/Q. Then $q \in Q$, implying it is not invertible in A. By assumption, G is q-nilpotent and so P is localizable by Proposition 3.23.

Conversely, suppose all the prime ideals of R are localizable. Let q be a prime number which is not invertible in A. Then $q \in Q$ for some prime ideal Q of A. Also, $q = \text{char } K$ where K is the quotient field of A/Q. From Proposition 3.23 follows then the q-nilpotency of G. ||

We conclude this section with two examples.

(A) Let G be a finite nilpotent group, for instance, the quaternion group, and let A be any commutative ring. Then all the prime ideals of AG are localizable on account of Corollary 3.24, since G is q-nilpotent for any prime number q.

(B) Proposition 3.23 enables us to construct minimal localizable sets consisting of more than one prime ideal. For example, take A to be a commutative ring of characteristic 5 and G to be the dihedral group D_5 with defining relations $a^5 = b^2 = e$ and $ab = ba^{-1}$ on its generators a and b. Let Q be any prime ideal of A. Then the characteristic of the quotient field of A/Q is also 5. However, G is q-nilpotent for any prime number $q \neq 5$ and is not 5-nilpotent. Hence Δ_Q is not localizable in AG due to Proposition 3.23. If C denotes the centre of AG, then $\Delta_Q \cap$ C-set contains other prime ideals besides Δ_Q in view of Proposition 3.20.

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