POLICY-ORIENTED MODELS OF INTERREGIONAL DEVELOPMENT

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by

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ABSTRACT

In this dissertation government influence is an endogenous component of interregional development. The purpose of the study is to determine some of the implications of economic and/or population goals, in terms of the types of government actions that will be required to meet these goals, and in turn, the effect that these actions will have on regional income and population levels over time. These planning issues are treated as optimal control problems, which are solved using Pontryagin's Maximum Principle. This study represents one of the first applications of Pontryagin's Maximum Principle to geographic systems, and was developed independent of other related studies.

Four models are analyzed. All are simple, and serve as precursors of more comprehensive models. The first deals with the magnitude of government intervention when economic projections do not concur with established goals. The second model focuses specifically on per capita regional government spending as a policy instrument. The third model emphasizes the feedback between production levels and migration. The fourth is particularly sensitive to regional disparity, as well as to regional growth.

Results of the models indicate that in planned regional economies, levels of per capita income and population will

probably exhibit steady trends, without sharp upturns or downturns in either economic or population growth. This is true even when there are only constraints on the income and population levels at the end of the planning interval, but none on the trajectories during that interval. Underlying this scenario of steady trends, however, is the possibility of extremely drastic fiscal, monetary and immigration policies, which are required to maintain such trends. The severity of these policies will depend on the regions' economic and population goals, the magnitude of these goals relative to those of other regions, the propensity of migrants to move to the more prosperous regions, and the labour intensiveness of regions' economies.

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CHAPTER 1

INTRODUCTION

1.1 POLICY-ORIENTED MODELS OF INTERREGIONAL DEVELOPMENT

Governments throughout the world have recognized the importance of regional development to balanced national economic growth. Indeed, the public sector has taken a dominant rôle in guiding the development of regional economies. This is of course true in socialist countries, whose number has increased significantly in the last several decades. But even within capitalist countries, the growing levels of government spending have led to mixed economies, in which the public sector, as well as the private sector, has become a powerful force within the national economy.

Since administrative and political organization is based on regional divisions, national economic planning reflects regional concerns. In Canada, the Department of Regional Economic Expansion (DREE) was established to stimulate economic growth in the Maritime Provinces, eastern Quebec, and other designated lagging regions. In the United States, several Regional Action Planning Commissions have been created under Title V of the Public Works and Economic Development Act of 1965. A regional perspective is useful since, to some degree, regional patterns mirror economic and social class distinctions. Thus,problems of regional disparity are also issues of social concern. Notwithstanding, a nation's economic growth and social development depend on the equitable growth and development of its individual regions, not only in terms of social justice and an equitable distribution of wealth and well-being - though these are reasons enough - but also because national economies are characterized by a complex network of interregional flows of goods and services. The demise of a regional economy may well signal a similar fate for that of the nation, and indeed for its international partners in trade.

The increasing importance of the public role in guiding the national economy, the interregional makeup of such an economy, and concern for issues of social justice and regional disparity, all point to the need for and relevance of policy-oriented models of interregional development. This study analyzes several policy-oriented models of regional income growth. Each model embraces three major themes.

The first is that the models are interregional. In turn, interregional variations are represented four ways. The first of these is locational, but only implicitly: the interaction amongst the regions is reflected in the relative magnitude of the parameters. In Chapters 3-6, the target income levels that are established for one region, for example, implicitly take into account those of the other regions. But since these parameters are determined exogenously, locational relationships are only implicit. A second type of formulation is explicitly locational: the magnitudes of region-specific variables are interdependent and are determined endogenously. This is the case in Chapter 6, in which the income levels are determined as a function of their relative magnitudes. A third situation is implicitly spatial: the relative locations of regions are implied in the given parameter values. In Chapters 3 and 4, the net exports of each region can be related to an entropy-maximizing spatial interaction commodity flow model; the commodity flows are functions of distance. The fourth case is explicitly spatial: the variables are functions of interregional distances. This characterization is found in Chapters 3 and 5, in which migration and population distribution are functions of the relative locations of the regions.

The second major theme is the dynamic aspect of the models. The notion of economic development clearly implies change over time; and in each of the models, the variables are expressed as time-dependent[1] trajectories. In this manner, time is treated analytically, rather than in terms of simulation or a quasi-dynamic cross-sectional approach.

The third major theme is that the models are policy-oriented. Of course, this does not necessarily mean that they are value-laden political arguments. Rather, within a broader interpretation, a policy-oriented model is a representation of reality, whose analysis lends insights into the nature of

1 The term "time-dependent" indicates that the variables are functions of time. It should be made clear, however, that these models are termed "time-invariant" in the control theory literature because the parameters do not vary over time.

preferred policies, given a description of the system within which they operate. A model may be deemed to be policy-oriented in three ways. First, the results of the model can suggest policies, even though a formal policy instrument is not embedded within the analysis. For example, the derived profile of regional income trajectories will have significant political repercussions, although these may remain unspecified within the model itself. Second, policy implications are indicated in a more formal manner by means of sensitivity analysis. In this instance, analytical or numerical variation in the parameter values leads to qualitative and quantitative impacts on system behaviour. Some of the parameters defined in subsequent chapters are influenced by the relative political clout of individual regions, by personal income tax rates, by interest rates, and by other governmental policies. If the parameter values change, then it is likely a result of changes in government policies. The third type of policy implication is one in which variables representing policy instruments or strategies are treated endogenously. The optimal trajectories of these control variables, and of the system's state variables that they affect, are determined within the model.

This study is at the heart of geographical research in several respects. It is concerned with economic and demographic aspects of the landscape, their variations from place to place, and their evolution over time. It also combines the strength of deductive reasoning with the relevant issues of equity, planning and policy. The study is exploratory, and the models are simple and naive. But their methodology forms a basis for a generation of models in geography.

1.2 METHODOLOGY

The problem of finding a suitable policy implies one of determining a preferred or best alternative. This is ultimately a subjective decision, since it rests on the value-laden choice of an appropriate criterion. However, viewed as a heuristic device, a space-time optimization approach can still be illuminating in terms of policy implications. The relevance of optimization simply reflects the notion that the decision-making body or government wishes to find the optimal or best policy, and hence transcends political peculiarity. Of course, within an actual planning context, the solution of such a problem should not be the unqualified basis of a particular decision. Any model is simply a scenario that reflects a certain economic, political and social environment, and provides information that could be used in a decision-making process.

Model identification and parametric estimation are important issues, but are not of prime concern in this study. For instance, consumer behaviour (as measured by the marginal propensity to consume), interregional commodity flows, and the relationship between capital stock and regional output, are all determined exogenously. Given such information, the subsequent analysis is axiomatic: deductive reasoning is used to draw conclusions (results) from one or more premises (assumptions). The utility of this strategy lies in the discovery of new logical connections, the possible prediction of yet unobserved phenomena, and the efficiency in achieving a comparatively great number of deductions from a parsimonious set of initial propositions.

The analysis utilizes a powerful mathematical tool - Pontryagin's Maximum Principle, which gives the necessary conditions for the optimal control and state trajectories. The state variables are system attributes, such as regional income; and the control variable represents either implicitly or explicitly, a set of policy instruments which affect regional growth and which are manipulated by the government. The control variable is endogenous in that it rests with the model to determine its optimal values over time, given a set of hypothetical goals. These goals are expressed in two forms. First, a set of equations defines the regional (income) targets to be attained at the end of the planning horizon. Second, a time-dependent objective functional represents the cost of implementing the appropriate government policies. Past trends and other prior information provide estimates for a set of differential equations that represent the time-dependent behaviour of the system, and for a set of equations that summarize other relevant information. These components define a space-time optimization problem in which the objective functional is minimized (or maximized) subject to the system of differential equations and the other constraints. The problem is solved using Pontryagin's Maximum Principle, which is outlined in Appendix 1.

1.3 STRUCTURE OF THE STUDY

This research is motivated by the need for analytical models that encompass the locational, temporal, as well as policy-oriented aspects of interregional development. The salient property of these models is the endogenous characterization of control or policy variables within a dynamic interregional context. Too often, the distinction between goals and policies is blurred, with policy recommendations being mere paraphrased restatements of goals. The distinction is made clear in this study however. Goals, which are formulated within a broad forum of consumers, entrepreneurs and politicians, are stated as assumptions. The deduced propositions that ensure that these assumptions are met are then the policies.

An appropriate background for this study is presented in a brief review of related literature (Chapter 2). Then, different models of interregional development are presented in each of the next four chapters. In each case, goals are specified relating to the regional income levels and to the costs of the programs necessary to ensure that these targets are attained. Income growth is a function of a control or policy variable, and the analysis uncovers its optimal trajectory. Since there is no generally accepted model of interregional development, various models are entertained in this study. In each instance, the modelling strategy is to focus on a parsimonious set of attributes and to determine logical implications of this fundamental system, rather than to pursue a more detailed disaggregate approach. That step would follow only after more complete analysis of the simple, fundamental models.

1.4 NOTATION AND OTHER CONVENTIONS

The symbols are uniquely defined within any given chapter (except for Chapter 2 when they are redefined as required), though the definitions of symbols may differ from chapter to chapter. Parameters are positive real numbers unless specified otherwise, are given exogenously, are stationary with respect to time, and are indicated by small Greek letters, such as α_j. An asterisk denotes a constant - for example, $y_j(\tau)^*$. Variables are endogenous and are indicated by English letters - for example $y_j(t)$. Space is discrete: $j = 1,\ldots, J$; and time is continuous, with the initial time set to be 0 and the planning horizon set to be τ: $t\varepsilon[0,\tau]$. The control variable, $u_j(t)$, is a real number and is an element of a fixed closed set for all t. Other variables are positive real numbers and are continuous in t. Subscripts are used to identify region-specific elements as well as to distinguish between different elements. In the latter case, parentheses are used: $\zeta_{(1)}$. Differentiation is denoted by both $\frac{'d'}{d}$ and '.' notation, so that dy_j/dt is equivalent to $\dot{y}_j(t)$.

Each chapter is divided into sections; for example, Section 4.1 is the first section in Chapter 4. All equations are numbered consecutively within each chapter as they are introduced, so that Equation (3.2) is the second equation stated in Chapter 3, for example. Unless specified otherwise, equations involving subscripted terms are assumed to hold for all elements identified with the subscript. For instance,

$$r_j(t) = dq_j(t)/dt \qquad (1.1)$$

holds for all j, although '$\forall j$' is omitted. The assumptions and results are numbered consecutively within each chapter as well. The assumptions are the initial premises which represent the known dynamics of the system and the presumed goals of the government. The results are the deduced propositions which describe the properties of the system. No effort is made to differentiate among lemmas, theorems, corollaries and remarks. The proofs of all results are outlined in the Appendices.

CHAPTER 2

A REVIEW OF RELATED MODELS OF INTERREGIONAL DEVELOPMENT

2.1 INTRODUCTION

A missing ingredient in almost all geographical models is a policy-related component that can be analyzed within a deductive framework. In this study, such a component consists of a control variable that represents government influences on economic and/or demographic development. The magnitude of the control variable over time is determined by using Pontryagin's Maximum Principle (Pontryagin et al., 1962), a major theorem in optimal control theory (see Appendix 1). We should make it clear that the use of the word "control" does not necessarily imply strict government controls on price and wage levels, nor on immigration; but encompasses any fiscal, monetary, or population policy in general.

A policy component is important because any geographical system evolves within a government-regulated reality; and governments are showing more and more concern (if we measure this by per capita government expenditures) about their nation's and their individual regions' economic, demographic, and social structures. We treat the policy aspects within a deductive framework because its rigor and logic are appealing. At the same time, we do not disparage the utility of inductive or even intuitive approaches: all are important. But since we are concentrating on a deductive approach, we will not be dealing with problems of model indentification and estimation, nor with the social issues that underlie public decisions on goals and policies. On the other hand, in any serious applications to planning problems, it would be sheer folly to neglect such issues.

The study deals with geographical systems, in which regions exist, are different one from another, and interact with each other. We are not primarily concerned with economic systems per se, and with the details of consumer demand, unemployment, residential construction, production functions, prices and wage rates, corporate profits, money supply, and the like. Similarly, we are not principally concerned with demographic systems, age-specific death rates, or age and sex cohort structures. Construction of an integrated inter-regional scheme, with both demographic and economic components, would be welcomed from a theoretical as well as planning point of view. However, most regional econometric models (Klein and Glickman, 1977) do not develop a multiregional framework in which any region's economy is affected by the affairs of other regions; are weak in their representation of labour migration and population growth; and ignore the feed-back between regional structure and the spatial interaction of commodities, services, capital and labour. On the other hand, most of the interregional demographic models (Rogers, 1975; Wilson and Rees, 1974) do not relate their parameters to economic factors. We only see this in the highly aggregated migration models (Greenwood, 1973).

Because current dynamic models of urban and regional systems have significant deficiencies, there is no pressing need to consider a disaggregate econometric or demographic framework. Instead, we choose to explore the general usefulness of control theory in simple, aggregate models which focus on a limited number of locational attributes. In this chapter, we review two major groups of models that apply optimal control theory to interregional development; and in particular, relate their approaches to those in our study.

2.2 OPTIMAL INTERREGIONAL ECONOMIC GROWTH

If we talk about economic growth, policy, and planning, then we talk about society's welfare in the future; we talk about forecasting, and changes over time. For a policy-oriented model to be useful and relevant, it should be dynamic - that is, time-dependent. Inspecting the current dynamic spatial models, however, we observe that almost all of them are only implicitly policy-oriented. That is, we can estimate the sensitivity of regional income, say, to variations in the parameters; but we do not have an endogenous policy component that is sensitive to alternative sets of goals.

This led us to develop the models in Chapter 3 and 4, which adopt Jutila's (1971, 1972) fundamental model and extend it into a control theory context. Our models do not consider the more complex, and interesting, interregional export-import mechanisms that were proposed by Jutila (1971, 1972) and Sheppard (1976); but it is a more than reasonable conjecture that these, as well as other dynamic regional systems, can be accommodated within a control theory format.

This view is supported by evidence in the economics literature in which descriptive growth models have been expanded into a normative framework (Arrow, 1967; Arrow and Kurz, 1970; Athans and Kendrick, 1974; Dobell, 1968; Hadley and Kemp, 1971). As an example, Advani and Mukundan (1970) reformulated Solow's (1956) descriptive model into an optimal control problem through the introduction of a performance index:

$$\text{Max.} \quad I = \int_{t=0}^{\tau} c(t)\, g^{*}(t)\, dt \tag{2.1}$$

with respect to u(t),

subject to

$$\dot{q}(t) = u(t)\, q(t)^{\omega} - \phi\, q(t) \,, \tag{2.2}$$

where

- I is the performance index
- t denotes time
- τ is the finite planning horizon
- $c(t)$ is the per capita consumption in the country
- $g^{*}(t)$ is a given discount factor
- $q(t)$ is per capita capital

u(t) is the control variable, the fraction of output saved

ϕ is the population growth rate: $\dot{x}(t)=\phi x(t)$; and national output, $x(t)y(t)$, is described by a Cobb-Douglas relation $x(t)y(t)=[x(t)q(t)]^{\omega}[x(t)]^{(1-\omega)}$.

An interactive algorithm obtained numerical solutions using the Runge-Kutta method to integrate the canonical equations.

The formulation defined by Equations (2.1) and (2.2) is a basic version of the gamut of optimal economic growth models that were inspired by Ramsey's (1928) original work. A more detailed two-sector model with capital freely transferable between industries is given by Hadley and Kemp (1971, p. 327-347). There, the problem facing the government planning authority was to find the path of commodity prices which maximized the total utility over time. Sector 1 produced the consumption good; Sector 2, the capital good. Utility was a function of consumption. Capital accumulation was a function of current capital stock, prices of capital and consumption goods, and the labour force growth rate. In addition, there were several static constraints that defined the production function, full employment of factors, and other competitive conditions. The necessary conditions for optimality were used to construct phase diagrams that described general characteristics of the levels of prices and capital. Such multi-sector models can lead to ideas on how to construct multiregional models if we re-interpret the sectors as regions. But as always, blind analogies will be dangerous.

The Hadley and Kemp (1971) model provides a contrast to our models along several fronts. First, in Hadley and Kemp, the control variable is commodity prices; in our models the control variable is usually not so explicitly defined. While we think it unlikely that governments will have strict controls over such prices, we concede that a more specific definition of the controls affords more information and a better specification of the model. This feeling is elaborated in subsequent chapters, in which we criticize our use of extremely aggregate policy instruments, and suggest refinements to be considered in later studies.

Second, in Hadley and Kemp, the performance criterion is some function of total consumption. This appears to be typical of these types of models, and presents a rather narrow view of social welfare. In our models the regional income distribution (also population distribution in Chapter 5) reflects the societal goals. In general, however, it would be desirable to segment the population into several socio-economic groups, and to include such attributes as concentration of pollutants, and any other aspects of quality of life for which quantitative measures are feasible.

Third, most economic growth models incorporate a production function in which output depends on both capital and labour inputs, and different types of output are identified. In our models, we have not emphasized labour's role in the production of goods, and have either focused on capital accumulation (in Chapters 3 and 4), or have implied labour's role only indirectly (in Chapter 5).

Fourth, in our models, we have stressed the interregional aspects in which a country is concerned about the interregional distribution of wealth. The Hadley and Kemp study is typical of economic growth models in that it neglects these interregional concerns.

An exception to this latter observation is Domazlicki's (1977) model in which the key element is a transportation cost associated with interregional transfer of capital. The performance index that is maximized is the total capital in the two regions at the end of the planning horizon:

$$\text{Max.} \quad I = \rho_1 q_1(\tau) + \rho_2 q_2(\tau) \tag{2.3}$$

with respect to $u_j(t)$,

subject to

$$\begin{aligned} \dot{q}_1(t) &= (1-u_1(t))\ (1-\mu_1)\ y_1(t) + (1-\beta)\ u_2(t) \\ &\quad (1-\mu_2)\ y_2(t) \\ &= (1-u_1(t)\ \rho_1\ q_1(t) + (1-\beta)\ u_2(t) \\ &\quad \rho_2\ q_2(t) \quad , \end{aligned} \tag{2.4}$$

$$\begin{aligned} \dot{q}_2(t) &= (1-\beta)\ u_1(t)\ \rho_1\ q_1(t) + (1-u_2(t)) \\ &\quad \rho_2\ q_2(t) \end{aligned} \tag{2.5}$$

where $q_j(t)$ is the stock of capital in region j at time t

$u_j(t)$ is the proportion of capital in region j which is transferred to region i at time t

$y_j(t)$ is the output in region j at time t

μ_j is the constant consumption ratio in region j, so that $(1-\mu_j)$ is the savings ratio

β is the proportion of transferred capital that is used in transporting it from one region to another

ρ_j is the output-capital ratio in region j

τ is the planning horizon.

Equations (2.4) and (2.5) are linear in $u_j(t)$, which results in the bang-bang control situation. At any time the proportion of capital that is transported will be one of $\{u_1=1, u_2=0\}$, $\{u_1=0, u_2=1\}$, or $\{u_1=u_2=0\}$; and there will either be two, one, or no switches from one case to another.

The Domazlicki (1977) model has a rather questionable objective functional in that the planning agency only cares about the total amount of output at the end of the planning horizon - not output levels over time, nor their interregional distribution, nor the costs of achieving this maximum capital accumulation. It does, however, have the interesting mechanism that allows for the interregional transfer of capital, which would probably prove more insightful if one were to consider a multi-sector economy.

Thus far, the models that we have outlined all involve applications of Pontryagin's Maximum Principle, which is the method used in this study. Other methods do exist for solving certain types of optimal control problems. They are the calculus of variations which dates back to the ancient Greeks and which was developed more substantially by Lagrange, Euler, Legendre, Jacobi, Weierstrass, Bernoulli, Newton and L'Hospital (Gelfand and Fomin, 1963; Kirk, 1970); dynamic programming which was developed by Bellman (1957); and linear programming which was originated by Kantorovich (1939) (we can also include nonlinear programming (Zangwill, 1969) in this latter group of approaches). All can be used in control problems, though of course some are more suited than others for particular situations. In the remainder of this section, we outline some of the previous studies that have employed these other, related techniques.

The classical calculus of variations is generally regarded as a special case of optimal control theory, which was developed by Pontryagin et al. (1962). The necessary conditions associated with each approach are related and can be used to solve similar types of problems. The major restriction in the calculus of variations is that the control variable, $u_j(t) \equiv dy_j(t)/dt$, belongs to the space of differentiable functions; whereas in optimal control theory, the admissible controls are some closed and (possibly) bounded region of the space of piecewise continuous functions. This allows for inequalities in the constraints. Although variational analysis involving such constraints goes back to a paper by

Valentine (1937), it was Pontryagin and his associates who developed a radically different and more general approach to optimal control.

Isard and Liossatos (1975) have developed a series of related models that have utilized the calculus of variations. One of these considers the maximization of a welfare index which is a function of consumption and employment; the dynamic system consists of an output identity, capital flow, population growth, and migration:

$$\text{Max.} \quad I = \int_{t=0}^{\tau} \int_{j=0}^{\infty} F\,(C,x)\,dj\,dt \tag{2.6}$$

with respect to C ,

subject to

$$\dot{Q} = f(Q,x) - C - \overset{j}{E} \tag{2.7}$$

$$\ell_{(1)}\dot{E} = \xi_{(1)}Q - \sigma_{(1)}E \tag{2.8}$$

$$\dot{x} = \phi x - \overset{j}{N} \tag{2.9}$$

$$\ell_{(2)}\dot{N} = -\xi_{(2)}\overset{j}{x} - \sigma_{(2)}N \quad , \tag{2.10}$$

and appropriate initial and boundary conditions,

where t is time , $0 \leq t \leq \tau$

j is location, $0 \leq j \leq +\infty$

F is the welfare function

$C(j,t)$ is consumption at location j at time t

$x(j,t)$ is the labour force

$Q(j,t)$ is the stock of capital, so that $\dot{Q} \equiv \partial Q(j,t)/\partial t$ is investment

$f(Q,x)$ is the production function

$\overset{j}{E} \equiv \partial E(j,t)/\partial j$ is net exports

$\ell_{(1)}\dot{E}$ is the inertial force resisting change in the flow of capital E

$-\xi_{(1)}\overset{j}{Q}$ is the force for capital movement

$\sigma_{(1)}E$ are the transportation costs for moving capital E

ϕ is the rate of net natural increase in population

$\overset{j}{N}$ is net out-migration of labour

$\ell_{(2)}\dot{N}, -\xi_{(2)}\overset{j}{x}, \sigma_{(2)}N$ are analogous to $\ell_{(1)}\dot{E}, -\xi_{(1)}\overset{j}{Q}$, and $\sigma_{(1)}E$ respectively.

The model is analyzed using the calculus of variations. Explicit solutions are not obtained. However, the Euler-Lagrange equations (necessary conditions) are used to derive some relationships that are interpreted as investment, transportation, labour migration, and labour allocation principles. The major general criticism of the Isard and Liossatos models is that they hinge too much on previously derived results in general relativity, so that they have forced and artificial interpretations of the terms and results.

In general, analytical solutions to optimal control problems will not always be possible using either the calculus of variations or Pontryagin's Maximum Principle. Under such circumstances, and especially for problems of small dimension, dynamic programming can be a more efficient numerical procedure. In the discrete time case, the Principle of Optimality leads to a functional recurrence relation; while in a continuous process, the Hamilton-Jacobi-Bellman partial differential equation results. The major and well-known nemesis of dynamic programming is the "curse of dimensionality" in which computer storage constraints limit the size of the system that can be analyzed.

We illustrate the use of dynamic programming by summarizing Chow's (1976) approach, which is cast within a general macroeconomic framework. The task for the government is to keep the state of the system "on track", as close as possible to a predefined trajectory. A penalty is imposed for deviations from this target trajectory, and is defined by a quadratic loss function:

$$\text{Min.} \quad I = \sum_{t=1}^{\tau} (\underline{y}_t - \underline{a}_t^*)' \; \underline{\underline{\zeta}}_t \; (\underline{y}_t - \underline{a}_t^*) \tag{2.11}$$

with respect to $\underline{u}_t$,

subject to

$$\underline{y}_t = \underline{\underline{\nu}}_t \, \underline{y}_{t-1} + \underline{\underline{C}}_t^* \underline{u}_t + \underline{b}_t^* + \underline{e}_t^* \tag{2.12}$$

where a single underline denotes a vector, a double underline denotes a matrix, and ' denotes transpose

$\underline{y}_t$ is a vector of the state variables at time t (discrete)

$\underline{a}_t^*$ is a vector of the target trajectories, given exogenously

$\underline{\underline{\nu}}_t, \underline{\underline{C}}_t^*$ are matrices of parameters which are unknown, but have given probability distributions

$\underline{\zeta}_t$ is a given matrix of weights or coefficients

$\underline{u}_t$ is a vector of controls

$\underline{b}_t^*$ is a vector of variables not affected by $\underline{u}_t$

$\underline{e}_t^*$ is a random disturbance term with 0 mean and given covariance matrix

τ is the planning horizon (t=1 is the initial time).

The solution procedure begins by substituting Equation (2.12) into (2.11) for $t=\tau$, and taking its expected value. This defines a loss function for $t=\tau$, which is an expectation conditioned on the data at $t=(\tau-1)$. This loss function is minimized to obtain an expression for the optimal value of $\underline{u}_\tau$, which when substituted into the $t=\tau$ loss function, gives the $t=(\tau-1)$ loss function. Recalling the Principle of Optimality, we repeat this process until we arrive at t=1.

A stochastic control approach, such as that just outlined, is generally preferred to a deterministic one, since it concedes that we live in a world of uncertainty. Another stochastic control model is that sketched by Curry (1976), in which the government strives to achieve a particular employment trajectory. Employment, without government intervention, is viewed as a random walk; and the government has at its disposal policies of varying severity, which are implemented depending on the level of employment. If the objective is to minimize the expected average cost of the policies, then the sequence of government actions follows a

Markov process. Neither this nor the Chow (1976) model are interregional; but the stochastic element is appealing, and provides some avenues for extending the deterministic models that we will be studying in Chapters 3-6.

Linear programming is characterized by a linear objective function and constraints. In some cases, an optimal control problem can be converted into a linear programming problem. For example, Mickle et.al. (1975) combined a dynamic input-output system (Leontief, 1951) with a similar occupational mobility system:

$$\text{Max.} \quad I = \sum_{t=0}^{\tau-1} \sum_{j=1}^{n} \sum_{i=1}^{m} \psi_{jj}\omega_{jj}\rho_{ij}u_{ij}(t) \tag{2.13}$$

with respect to $u_{ij}(t)$,

subject to

$$\sum_{j=i}^{n} \beta_{ij}\, u_{ij}(t)\, [D_j(t)-S_j(t)] \leq M_i(t) \tag{2.14}$$

$$\sum_{i=1}^{m} \beta_{ij}\, u_{ij}(t)\, [D_j(t)-S_j(t)] \leq V_j(t) \tag{2.15}$$

$$y_{ij}(t) \leq 0 \quad , \tag{2.16}$$

where t is time (discrete)

τ is the time horizon

j denotes industry type, $j=1,n$

i denotes occupation, $i=1,m$

ψ_{jj} is productivity (goods/man-hour)

ω_{jj} is the length of work day (work-hours/day)

ρ_{ij} is the efficiency of a worker of occupation i working in industry j, $0 \leq \rho_{ij} \leq 1$

$u_{ij}(t)$ is the number of occupation i workers assigned to produce a unit of good j at time t

$D_j(t)$ is the demand for commodity j

$S_j(t)$ is the supply of commodity j

β_{ij} is the portion of the differential $(D_j(t)-S_j(t))$ that is to be reduced by type i workers

$M_i(t)$ is the supply of type i workers

$V_j(t)$ is the employment capacity of industry j.

It is relatively straightforward to extend this into an inter-regional framework as in input-output models (Isard, 1960).

2.3 POPULATION POLICY MODELS

Population is a relatively minor aspect in most optimal economic growth models. An exception is Pitchford's (1974) study in which population is an endogenous state variable (instead of being exogenous and increasing exponentially). The welfare criterion was per capita consumption; so that the objective was to minimize the differential between the actual consumption levels and the maximum possible. This maximum level is allowed if per capita production is maximized through a suitable allocation of capital and labour. However, these factors are generally not available in suitable amounts. Capital is made available through investment; and growth in the labour force is from net natural increase without any government intervention, is retarded by any reduction in births as a result of public policy, and is augmented by immigration. Formally, the problem is stated as:

$$\text{Min.} \quad I \int_{t=0}^{T} (\hat{c}-c(t))\, dt \tag{2.17}$$

with respect to $c(t)$, $R_{(1)}(t)$, $R_{(2)}(t)$, $R_{(3)}(t)$, and T, subject to

$$\dot{Q}(t) = R_{(1)}(t); \quad Q(0)=Q(0)^{*}, \quad Q(T)=\hat{Q} \tag{2.18}$$

$$\dot{x}(t) = \phi\, x(t) + R_{(2)}(t)/\sigma_{(2)} - R_{(3)}(t)/\sigma_{(1)}; \quad x(0)=x(0)^{*}, \quad x(T)=\hat{x} \tag{2.19}$$

$$Y(t) \geq (c(t)-\bar{c})\, x(t) + R_{(1)}(t) + R_{(2)}(t) + R_{(3)}(t) \tag{2.20}$$

$$c(t) \geq \bar{c}; \quad R_{(1)}(t), R_{(2)}(t), R_{(3)}(t) \geq 0, \tag{2.21}$$

where

t is time

T is the end of the planning horizon and is variable

$\hat{c}$ is the maximum level of per capita consumption, which results when $\hat{Q}$ and $\hat{x}$ are used in the production process

$Q(t)$ is the capital

$x(t)$ is the labour force

$\hat{Q}, \hat{x}$ are the levels of capital and labour respectively that maximize per capita output

$R_{(1)}(t)$ is investment

$R_{(2)}(t)$ is the amount of national income used to attract immigrants

$R_{(3)}(t)$ is the amount of national income spent on preventing births

ϕ is the rate of labour force increase when there is no government intervention

$\sigma_{(2)}$ is the number of immigrants attracted by one dollar

$\sigma_{(1)}$ is the number of births prevented by one dollar (one dollar of government expenditure on covering costs of contraception, for example)

$Y(t)$ is output (national income)

$\bar{c}$ is the subsistence level of consumption.

Output is always allocated to investment, population control expenditures, or consumption, so that Equation (2.20) is an equality. This reduces the number of possibilities (combinations of control variables) in which the optimal values of the control variables either equal the boundary value or take some interior point. The necessary conditions are then used to construct phase diagrams between Q(t) and x(t) for each of the possible combinations of control variables.

Although an immigration factor exists in Pitchford (1974) there is no interregional interaction. The models presented in Chapters 3 and 5 do include such interaction by focusing on the effects of income differentials and relative location on regions' population levels. Another model that represents such interaction is that of Osayimwese (1974). He considers two regions, urban and rural, in which increases in urban employment are due to rural-urban migration. These migrants are attracted to the urban area because of its population size (a surrogate for amenities), positive income differential, and rate of change in this differential (which reflects the expectation of future increases in income).

In an informal research memorandum Mehra (1975) considered a demographic control model that was suggested by MacKinnon (1974). Unlike the models discussed previously, this model concentrated on demographic relationships, and neglected the effects of the economy. The objective was to minimize the deviation of the multiregional population distribution from the preferred distribution. Population growth is from net natural increase and immigration. For any given year, the total number of immigrants is given. Also, each immigrant results in certain government expenditures, first for attracting him/her to the region, and then for initial support. The amount varies from region to region and time to time, but the total expenditures cannot exceed a predetermined budget constraint. The setup of the control problem is then:

$$\text{Min.} \quad I = \tfrac{1}{2} \sum_{t=1}^{\tau} (\underline{x}(t) - \underline{g}(t)^{*})' \, \underline{\underline{S}} \, (\underline{x}(t) - \underline{g}(t)^{*}) \tag{2.22}$$

with respect to $\underline{u}(t)$,

subject to

$$\underline{x}(t+1) = \underline{\underline{\phi}} \, \underline{x}(t) + \underline{u}(t) \tag{2.23}$$

$$\sum_{t=0}^{\tau-1} \underline{u}(t)' \, \underline{\alpha}(t) \leq u^{*} \tag{2.24}$$

$$\sum_j u_j(t) = \hat{u}(t) \qquad (2.25)$$

$$\underline{u}(t) \geq \underline{0} \qquad (2.26)$$

where

- τ is the planning horizon
- $\underline{x}(t)$ is the population distribution vector
- $\underline{g}(t)^*$ is the desired distribution of population at time t
- $\underline{\zeta}$ is a matrix of penalties for the deviation from $\underline{g}(t)^*$
- $\underline{u}(t)$ is the vector of controls - immigration
- $\underline{\phi}$ denotes net natural increase (a diagonal matrix)
- $\underline{\alpha}(t)$ is the cost of locating an immigrant
- $\hat{u}(t)$ is the total number of immigrants that are allowed into the country in year t.

A branch and bound technique was used to indicate whether the optimal trajectory of $\underline{u}(t)$ is given by the boundary constraint, or is an interior point in the space of admissible controls.

2.4 CONCLUDING COMMENTS

We have collected a set of studies that tackle much the same sorts of problems that we will be pursuing ourselves. These studies not only form a backdrop for viewing our own endeavours, they also provide a reason for them. We have not launched into a critical tirade of these other models, however, for that would have been a performance in banality. It is too much to expect that these dynamic control models will induce in us feelings of contentedness, when dynamic models in themselves have far to go.

Notwithstanding, it is evident that the regional economic models have virtually ignored the interregional interactions which are so much a part of any region's development. It is hardly surprising then, that they also fail to consider the distributional issues of regional disparity, which is such a major concern today. On the other hand, multiregional demographic models pay scant attention to the economic bases for migration and population growth.

To be sure, this dissertation does not really come to grips with the task of developing an accurate model of interaction and feedback between regional economic structure, production, commodity flows, migration, the growth of population, and the influence of government instruments on these various components. Instead, our strategy is to formulate a set of models, each of which highlights a different facet of the problem that has previously been under-emphasized.

In reviewing the previous studies, we had hoped to summarize the major, relevant thrusts. It was not our intent to provide a bibliographic compendium of studies: the references that have been cited provide an adequate springboard for diving into the literature. However, we would be negligent if we did not at least mention Ralston's (1976) introduction of change agents into diffusion theory, the normative models that have begun to mushroom in urban economics (Papageorgiou, 1976), and Nijkamp and Paelinck's (1973) work on environmental management.

CHAPTER 3

REGIONAL ECONOMIC DEVELOPMENT, GOVERNMENT POLICIES, AND POPULATION DISTRIBUTION

3.1 INTRODUCTION

This chapter introduces a model of regional development that encompasses several economic sectors - output, consumption, investment, interregional trade, and government spending within each region. The regions can be viewed either as large-scale entities, such as the Atlantic Region, Ontario, Quebec, the Prairies and British Columbia in Canada; or as smaller areal units more reflective of the interurban system, such as the 173 economic areas established by the United States Bureau of Economic Analysis. The model provides a suitable starting point for this investigation, since it is simple and manageable, yet sufficiently rich to discover meaningful insights.

Section 3.2 focuses on per capita regional income levels, reducing the macroeconomic system so that income is expressed solely in terms of time and of known parameter values. These income profiles would evolve if current technology, consumer preferences and government policies remain unaltered.

Unfortunately, the previous scenario can lead to imbalances in regional growth. Section 3.3 augments the analysis by introducing a control variable. Under these circumstances, the rate of change in regional income is expressed as a function of two additive terms: one, the reduced form derived in Section 3.2, which reflects the extension of past trends; and two, a control variable which represents the effects of newly-created government policies.

In Section 3.4, interregional migration responds to regional income differentials. Two migration projections are given, corresponding to the results of Sections 3.2 and 3.3. Population forecasts are then determined from the migration estimates, together with information on birth, death and immigration rates.

The numerical examples in Section 3.5 illustrate the differences between the income trajectories of Sections 3.2 and 3.3, as well as their respective impacts on population distribution. Section 3.5 also examines the sensitivity of the models to variations in the parameters.

3.2 REDUCED FORM OF A DYNAMIC REGIONAL ECONOMIC SYSTEM

This section derives an expression for per capita regional income levels over time. Assumption 3.1 encapsulates hypothetical, observed regularities in the macroeconomic system. If these past trends provide a reasonable indication of future developments, then Result 3.1 describes the anticipated per capita regional income trajectory. Assumption 3.2 is a special case that leads to the income trajectory given by Result 3.2; and a comparative dynamics analysis of this expression is summarized in Result 3.3.

<u>Assumption 3.1</u> A country consists of j=1,...,J regions; the regional economies are represented by the following relationships, true for all j:

$$y_{(1)j}(t) = c_j(t) + o_j(t) + e_j^*(t) - m_j^*(t) + g_j^*(t) \qquad (3.1)$$

$$r_j(t) = dq_j(t)/dt \qquad (3.2)$$

$$dw_j(t)/dt = \alpha_j \; y_{(1)j}(t) \qquad (3.3)$$

$$y_{(1)j}(t) = \rho_j \quad (q_j(t)-w_j(t)) \qquad (3.4)$$

$$r_j(t) = \int_{z=0}^{t} o_j(z) \; k_{(r)}^*(t-z) \; dz \qquad (3.5)$$

$$c_j(t) = \mu_j \int_{z=0}^{t} y_{(1)j}(z) \; k_{(c)}^*(t-z) \; dz \qquad (3.6)$$

where $y_{(1)j}(t)$ is the per capita gross regional product of region j per unit time at time t, assuming that present government policies are unaltered; the subscript (1) distinguishes this income trajectory from that which arises when a control variable is introduced;

$c_j(t)$ is the per capita private consumption in region j per unit time at time t;

$o_j(t)$ is the per capita product saved from consumption as an outlay for investment in region j per unit time at time t;

$e_j^*(t)$ is the per capita export of regional product from region j to all other regions per unit time at time t;

$m_j^*(t)$ is the per capita imports into region j per unit time at time t;

$g_j^*(t)$ is the per capita government spending in region j per unit time at time t;

$q_j(t)$ is the per capita stock of capital that would have accumulated in region j at time t had there been no consumption of capital;

$w_j(t)$ is the per capita stock of depleted capital in region j at time t;

$r_j(t)$ is the per capita investment in region j per unit time at time t;

α_j is the capital consumption rate in region j;

ρ_j is the rate of return in region j per unit time on the net available capital;

μ_j is the marginal propensity to consume available output in region j per unit time;

$k^*_{(r)}, k^*_{(c)}$ are kernels - functions representing time lags in investment and consumption, respectively

Assumption 3.1 draws upon the work of Jutila (1971, 1972, 1973). It is a terribly simplistic representation of the regional economies, yet allows us to deduce several interesting properties. Though a major advantage of the more comprehensive econometric models is their high level of disaggregation and more detailed representation of market conditions (Evans and Klein, 1968; Fromm and Tauban, 1968; Klein, 1971), the accuracy of simpler economic models is in some respects comparable to that of econometric models (Gordon, 1971; Klein, 1971, p. 137).

Equation (3.1) is an accounting identity; as is customary in such accounts, per capita regional income is equivalent to per capita gross regional product. This is the money value of all final goods and services produced in the region in any given time period. These goods and services are either consumed, set aside as outlays for investment, exported, or take the form of government expenditures.

In this model, per capita exports and imports are treated exogenously, as indicated by the asterisk notation. These flows include trade abroad, as well as interregional trade. Although the mix of commodities can vary, the total money value of regional exports and imports are thought to be sufficiently predictable over time.

Regional government spending includes that from local and regional sources, as well as federal funds in the region. Public aid and public works projects are anticipated to the extent that total government expenditures are exogenous - though given the uncertain political and economic climate, any planning applications of this type of model would consider a range of government spending alternatives. Of course, the model cannot forecast exogenous events - it does not unscramble the political manoeuvering responsible for a particular decision, nor does it evaluate its social ramifications.

In Equation (3.2), per capita investment is defined as per capita capital formation: the rate of change in the per capita capital stock, not including the effects of depreciation. The variable $w_j(t)$ accounts for capital consumption. The rate of change in this stock is a fraction, α_j, of total production. Greater output means that more capital will be consumed in the production process. The parameter α_j reflects the industrial structure of the region, including the age and technology of the capital stock; more mature economies are likely to have greater α_j values.

The parameter ρ_j is the output-capital ratio. In effect, Equation (3.4) is an aggregate regional production function, in which there are constant returns to scale, the marginal product of capital is positive, the law of diminishing returns does not hold, and the elasticity of production with respect to capital is unity. The available capital stock equals the total capital that has existed at some time, minus the total stock of depreciated capital; and the available capital stock divided by the population, $(q_j(t)-w_j(t))$, equals the capital-labour ratio times the participation rate. The fact that there is only one explicit factor of production precludes the possibility of substitution between labour and capital; but this matter is not of immediate concern in this discussion.

A fraction of total output is apportioned to investment; and the conversion of these outlays to capital stock is lagged over time. These lags are incorporated in the kernel function, $k^*_{(r)}(t-z)$, which states what fraction of past outlays, $o_j(z)$, is converted into private investment, the fraction depending on how far in the past, $(t-z)$, the outlay was made. Consumption is lagged in a similar fashion; and the parameter μ_j is the marginal propensity to consume available output, some of which has been produced in previous periods.

A significant shortcoming of both Equations (3.5) and (3.6) is the implication that regional population levels are constant over time. Otherwise, the per capita levels $o_j(z)$ and $y_{(1)j}(z)$ would not be comparable to $o_j(z')$ and $y_{(1)j}(z')$ respectively. For example, at time z, the population of j could be very small, so that although total outlays are not that great, per capita outlays are still large; but at time z', with a much expanded population, the outlays at time z relative to population at time z' would be extremely small. Assuming that population is increasing over time, there will be a biased estimate of investment, with outlays from distant past time periods having a misleadingly great impact. There are two recourses. One is to assume known population trends, and to adjust the kernel appropriately. The other alternative, particularly if population is treated endogenously, is to have all outlays immediately take the form of investments, with no time lag. This latter strategy is employed in Section 3.4.

The first result of the analysis gives the reduced form for per capita income, expressed as a function of time and of known parameters or functions.

<u>Result 3.1</u> If Assumption 3.1 holds, then the per capita gross regional product (or equivalently, the per capita income) of region j at time t is:

$$y_{(1)j}(t) = \mathcal{L}^{-1}\Big[\{y_{(1)j}(0)^{*} - \rho_j\, \mathcal{L}[k^{*}_{(r)}(t)]\ (\mathcal{L}[e^{*}_j(t)] - \mathcal{L}[m^{*}_j(t)] + \mathcal{L}[g^{*}_j(t)])\}\ \{s - \rho_j\ (\mathcal{L}[k^{*}_{(r)}(t)] - \alpha_j - \mu_j\, \mathcal{L}[k^{*}_{(r)}(t)]\ \mathcal{L}[k^{*}_{(c)}(t)])\}^{-1}\Big],$$

where

$\mathcal{L}[\cdot]$ denotes the Laplace transform with respect to t, so that $\mathcal{L}^{-1}[\cdot]$ is the inverse Laplace transform (see Appendix 2);

$y_{(1)j}(0)^*$ is the initial per capita income level, given a priori;

s is the complex variable corresponding to the Laplace transform with respect to t.

Result 3.1 relates regional income levels to given trends in regional exports, imports, government spending, and investment and consumption lags. Assumption 3.2 specifies some hypothetical functional forms for each of these values.

<u>Assumption 3.2</u> Let

$$e_j^*(t) - m_j^*(t) = \varepsilon_j \quad , \varepsilon_j \text{ is a real number} \tag{3.8}$$

$$\mathcal{L}[k_{(r)}^*(t)], \mathcal{L}[k_{(c)}^*(t)] = 1 \tag{3.9}$$

$$g_j^*(t) = g_j^* \quad , \text{a constant.} \tag{3.10}$$

Equation (3.8) sets the per capita regional trade balance to be constant over time. This approximation is useful if there are no drastic changes in the commodities produced and exported by the region as well as those demanded and imported. It should be noticed that trade balances with individual partners do not necessarily remain constant - only the aggregate trade balance remains constant. In actual circumstances, the value of ε_j is determined in part by those of $\varepsilon_{i \neq k}$. In fact, a characterization of commodity flows in which $\sum_j \varepsilon_j = 0$, for example, creates an explicitly interregional structure.

Equation (3.9) states that there are no time lags in the conversion of outlays to capital, nor in the consumption of goods and services produced during any given time period. This assumption is not at all forbidding; it supposes that no money is physically set aside. Money can be saved, but it is assumed to be deposited in financial institutions, which in turn make loans to be used for consumption or investment.

Equation (3.10) is a heuristic device; and suggests a highly stable environment in which governments maintain constant levels of spending.

Assumptions 3.1 and 3.2 lead to the ensuing result.

<u>Result 3.2</u> If Assumptions 3.1 and 3.2 hold, then for each region j:

$$y_{(1)j}(t) = y_{(1)j}(0)^* \exp(\rho_j \delta_j t) + (\varepsilon_j + g_j^*)/\delta_j \, (1-\exp(\rho_j \delta_j t)) \quad (3.11)$$

where $\delta_j = 1-\mu_j-\alpha_j$ (3.12)

and $y_{(1)j}(0)^*$ is the known initial per capita income level.

Result 3.2 specifies how regional income levels are a function of productivity characteristics (as reflected in ρ_j), capital depreciation rates (α_j), and population's propensity to consume (μ_j), interregional trade (ε_j), and government spending (g_j^*). Parameter estimates then allow a plot of the income trajectory, as in Section 3.5. Those numerical results complement the analytical ones derived in the following sensitivity analysis.

<u>Result 3.3</u> If Assumptions 3.1 and 3.2 hold, then the effect of a unit parameter increase on region j's income trajectory, holding other parameters constant, is given in column one of Table 3.1 on page 49.

Result 3.3(a) indicates that income levels will always increase over time, unless α_j or ε_j is very large. That is, in this model recessions can only occur when either capital consumption rates or exports are extremely high, thereby negating the effects of capital accumulation; and since the parameters are time-invariant, this situation would eventually lead to the total collapse of the region. This becomes clearer if we recall that $y_{(1)j}(0)^* = \mu_j y_{(1)j}(0) + o_j(t) + \varepsilon_j + g_j^*(t)$.

This same observation proves useful when we examine the other results in Table 3.1. Hence, except under the most extraordinary conditions, rapid capital depreciation rates or high marginal propensity to consume will lead to lower production levels; and increasing the rate of return on capital will increase output.

The effect of an increase in net exports on output is identical to that of a similar increase in government spending, assuming that all other parameter values remain unchanged. In both instances, output invariably declines. This is surprising in the case of exports, and exposes a major flaw in the model. Usually, regions aim to increase exports while restraining their demands for imports. In the model, however, no real benefit accrues from exporting goods; there is no mechanism by which the money from exports can be reinvested in producers' plant and equipment. Instead, increasing exports simply means that less is available for consumption and investment within the region, thus reducing overall output. This is made clear in the following result.

<u>Result 3.4</u> If Assumptions 3.1 and 3.2 hold, then

(a)
$$\frac{\partial c_j(t)}{\partial \varepsilon_j} = \mu_j[1-\exp(\rho_j\delta_j t)]/\delta_j \qquad (3.13)$$
$$< 0$$

(b)
$$\frac{\partial r_j(t)}{\partial \varepsilon_j} = [\alpha_j - (1-\mu_j)\exp(\rho_j\delta_j t)]/\delta_j \qquad (3.14)$$
$$\lesssim 0$$

A parallel situation exists in the case of government spending. No mechanism exists for public investment to stimulate growth; and funds channeled into the public sector at the same time reduce capital outlays in the private domain. Although an increase in government spending reduces output over each subsequent time period the extent of the impact diminishes over time.

This latter observation is in contrast to the effect of the initial income level on subsequent levels, which from Result 3.3(e), increases over time. Hence, other things being equal, the more prosperous regions not only retain, but extend their relative advantages.

3.3 CHANGING GOVERNMENT POLICIES TO ACHIEVE REGIONAL INCOME TARGETS

The exponential nature of the income trajectory in Result 3.2 indicates the likelihood of increasing divergence among regional income levels. Social and political demands may dictate that government policies be altered in order to achieve a more desirable income distribution. The model in this section recognizes this possibility, and determines the optimal timing and magnitude of policy changes which ensure that income targets are met. It is not, however, the purpose of the model to evaluate the fairness of these targets. This is a value judgment that can be resolved only through a wider exchange among the people of the country. The question of whether income targets can ever become a political reality is a moot issue. Although discrete finite targets are impractical, it is reasonable, if not desirable, to have a target area or range of values, which would give some quantitative indication of what one means by 'reducing regional disparity'. Any set of income targets represents only one possible scenario, and it is an easy task to evaluate the sensitivity of the optimal solution to a suitable range of alternatives. Assumption 3.3 conforms to this strategy, because the boundary conditions are left in a general form, rather than defined as specific numerical targets.

Assumption 3.3 Without loss of generality, the initial time period is set to be t=0. A planning horizon, τ, is specified; and the per capita income levels at time τ are given to be

$$y_j(\tau) = y_j(\tau)^* \quad . \tag{3.15}$$

In Equation (3.15), the income variable does not have the subscript '(1)'; this indicates that it is affected by endogenous government policies. The income trajectory derived in the previous section does not generally satisfy the levels specified in Assumption 3.3. Consequently, government policies will have to be altered in order to induce the proper economic environment. Changes in government policy which either expand or restrict economic growth are represented by a control varialbe, $u_j(t)$. The resulting growth in per capita income emerges as the sum of two terms. One is the growth rate corresponding to prior fiscal and monetary policies; the other term is the control variable, which represents the perturbations arising from policy changes. There is no loss of information in using $\dot{y}_{(1)j}(t)$ rather than $y_{(1)j}(t)$, since $y_{(1)j}(0)^*$ is given. By the fundamental existence and uniqueness theorem for first order initial value problems (Boyce and DiPrima, 1965), $\dot{y}_{(1)j}(t)$ and $y_{(1)j}(0)^*$ admit $y_{(1)j}(t)$.

Assumption 3.4 If Assumptions 3.1 and 3.2 hold, and if government policies are subsequently modified, then the rate of change in per capita income levels is:

$$\dot{y}_j(t) = \dot{y}_{(1)j}(t) + u_j(t) , \qquad (3.16)$$

where $y_{(1)j}(t)$ is given by Equations (3.11) and (3.12), and

$$y_{(1)j}(0)^* \equiv y_j(0)^*$$

$u_j(t)$ is the control variable.

Even when government policies are changed, the fundamental trajectory trends maybe unaffected. In that case, $y_{(1)j}(t)$ remains as a pivotal element in the income trajecgory, so that $y_j(t) \simeq y_{(1)j}(t)$. In general, policy changes do alter at least one of the parameter values. For example, the federal government could eliminate the double tax on corporate profits and dividends. If the tax on dividends were eliminated, then investors would be encouraged to reduce consumption and to invest in the stock market. This would reduce

μ_j, the marginal propensity to consume; it might also reduce g_j^*, the level of government spending, assuming that the government is striving for a balanced budget.

In addition to satisfying the income targets, it is also desirable to minimize the costs of such a programme. This latter objective is represented in Assumption 3.5.

Assumption 3.5 The total cost of implementing the new government programmes is to be minimized. This cost is defined as:

$$I = \int_{t=0}^{\tau} \sum_j \zeta_j \, u_j(t)^2 \, dt \qquad , \qquad (3.17)$$

where ζ_j is a cost parameter.

The objective functional in Assumption 3.5 indicates that costs are associated with both expansionary (positive $u_j(t)$) and restrictive (negative $u_j(t)$) policies. Extreme policy changes (very large $|u_j(t)|$) are discouraged through the use of $u_j(t)^2$, instead of a $|u_j(t)|$ term. This action simply reflects the general desire to maintain a degree of continuity in government programmes - to cushion political repercussions, as well as to reduce management and organizational costs, that are associated with any abrupt change in government policy.

Increased funding for public works projects may act as a stimulant that increases productivity (ρ_j). However, a negative side effect is the likelihood of increased taxation and reduced disposable income for the consumer. In the objective functional, the effect of the stimulants would be represented by $u_j(t)$; and the unit cost ζ_j is multiplied by $u_j(t)^2$.

As an example of a restrictive policy, severe corporate taxes that are region-specific may tend to retard the 'natural' growth of a region. Although such actions may be consistent with national aims to reduce regional disparities, a costly burden is imposed on those regions whose growth is curtailed in the national interest. In this instance, $u_j(t)$ would be negative and would represent a containment on income growth - the decrease in $\dot{y}_j(t)$.

Assumptions 3.3-3.5 define an optimal control problem in which Equation (3.17) is minimized subject to Equations (3.15) and (3.16); the solution is determined using Pontryagin's Maximum Principle.

<u>Result 3.5</u> If Assumptions 3.3-3.5 hold, then the optimal trajectories are:

$$u_j(t) = [y_j(\tau)^* - y_{(1)j}(\tau)] \,/\, \tau \quad , \qquad (3.18)$$

$$y_j(t) = y_{(1)j}(t) + [y_j(\tau)^* - y_{(1)j}(\tau)] \; t/\tau, \qquad (3.19)$$

Surprisingly, neither $u_j(t)$ nor $y_j(t)$ are functions of ζ_j. This means that the optimal trajectories of the control instrument and of per capita income are not at all affected by the cost of implementing the control, as long as there is some positive-valued cost involved. The latter proviso is necessary to ensure that the Hessian is negative definite (refer to Appendix 3).

<u>Result 3.6</u> If $y_{(1)j}(t)$ is concave (convex), then so is $y_j(t)$.

Although $y_{(1)j}(t)$ refers to the income trajectory arising from Assumptions 3.1 and 3.2, the relationships given in Result 3.5 are true for any differentiable $y_{(1)j}(t)$. Consequently, Result 3.5 may be used to analyze other economic systems.

The constant $u_j(t)$ implies the appropriateness of some more permanent fiscal or monetary policy that promotes the desired year to year effects, rather than a one-shot stimulus which is unlikely to have any sustained, longer term impact. Not surprisingly, Equation (3.19) makes it evident that the new trajectory $y_j(t)$ increasingly diverges from $y_{(1)j}(t)$ as t increases.

Substituting Result 3.2 into Result 3.5 gives:

<u>Result 3.7</u> If Assumptions 3.1-3.5 hold, then the optimal trajectories of the control and income variables are:

$$u_j(t) = u_j = [y_j(\tau)^* - \kappa_{(1)j} - \kappa_{(2)j}\exp(\rho_j\delta_j\tau)]/\tau \qquad (3.20)$$

$$y_j(t) = \kappa_{(1)j} + \kappa_{(2)j}\exp(\rho_j\delta_j t) + u_j t, \quad ; \qquad (3.21)$$

where $\delta_j = 1-\mu_j-\alpha_j$, as before;

$$\kappa_{(1)j} = (\varepsilon_j+g_j^*)/\delta_j \qquad (3.22)$$

$$\kappa_{(2)j} = y_j(0)^* - \kappa_{(1)j} \qquad (3.23)$$

u_j is given by Equation (3.20).

The structure of the control and income trajectories are both fairly simple: $u_j(t)$ is constant as expected; and the income trajectory, $y_j(t)$, is the sum of an exponential and a linear term. Results 3.8 and 3.9 evaluate the sensitivity of $u_j(t)$ and $y_j(t)$ to variations in the parameters.

<u>Result 3.8</u> If Assumptions 3.1-3.5 hold, then the effect of a unit parameter increase on the control variable $u_j(t)$, holding other parameters constant, is given in Table 3.1 on page 49.

<u>Result 3.9</u> If Assumptions 3.1-3.5 hold, then the effect of a unit parameter increase on the income trajectory, holding other parameters constant, is given in Table 3.1 on page 49.

A rather dramatic qualitative property that emerges from Results 3.3, 3.8 and 3.9 is that $y_j(t)$ and $u_j(t)$ tend to react similarly to variations in the parameters, whereas $y_{(1)j}(t)$ reacts in an opposite manner. This observation is somewhat astounding for it implies the following possibility. Consider two regions with identical economies save for one parameter value, so that although the initial income levels are equal, one region will become richer than the other. Then assume that this is undesirable, that equal target income levels are specified for each region, and that the optimal policy (as specified in the previous discussion) is pursued. What happens is that under this new regime, the

TABLE 3.1: SUMMARY OF COMPARATIVE DYNAMICS ANALYSIS ON INCOME AND CONTROL TRAJECTORIES, $\partial(\cdot)/\partial(\cdot)$ (For example: $\partial y_{(1)j}(t)/\partial t$.)

Parameter		Trajectory $y_{(1)j}(t)$	$y_j(t)$	$u_j(t)$
(a)	t	$\rho_j\delta_j\kappa_{(2)j}\exp(\rho_j\delta_j t)$	$\partial y_{(1)j}(t)/\partial t + u_j$	0
		> 0, if $\delta_j>0$ and $\kappa_{(2)j}>0$		
(b)	α_j or μ_j	$\kappa_{(2)j}\rho_j\exp(\rho_j\delta_j\tau)t+$ $\kappa_{(1)j}[1-\exp(\rho_j\delta_j t)/\delta_j]$	$\kappa_{(2)j}\rho_j t[\exp(\rho_j\delta_j\tau)-$ $\exp(\rho_j\delta_j t)] + \kappa_{(1)j}/\delta_j$ $[1-\exp(\rho_j\delta_j t)-t/\tau(1-$ $\exp(\rho_j\delta_j\tau))]$	$\kappa_{(2)j}\rho_j\exp(\rho_j\delta_j\tau)$ $-\kappa_{(1)j}(1-\exp(\rho_j\delta_j\tau))/$ $(\delta_j\tau)$
		< 0, if $(\varepsilon_j+g_j^*)$, δ_j, $\kappa_{(2)j}>0$	> 0, if $(\varepsilon_j+g_j^*)$, δ_j, $\kappa_{(2)j}>0$	> 0, if $(\varepsilon_j+g_j^*)$, δ_j, $\kappa_{(2)j}>0$

TABLE 3.1 (continued)

Parameter	Trajectory $y_{(1)j}(t)$	$y_j(t)$	$u_j(t)$
(c) ρ_j	$[y_{(1)j}(0)^*\delta_j-\varepsilon_j-g_j^*]$ $\exp(\rho_j\delta_j t)t$ $\gtrless 0$, if $y_{(1)j}(0)^*\delta_j \gtrless (\varepsilon_j+g_j^*)$	$[y_j(0)^*\delta_j-\varepsilon_j-g_j^*]$ $t\,[\exp(\rho_j\delta_j t)-\exp(\rho_j\delta_j\tau)]$ $\lessgtr 0$, if $y_j(0)^*\delta_j \gtrless (\varepsilon_j+g_j^*)$ and $\delta_j>0$; <0, if $\delta_j<0$	$[\varepsilon_j+g_j^*-y_j(0)^*\delta_j]$ $\exp(\rho_j\delta_j\tau)$ $\lessgtr 0$, if $y_j(0)^*\delta_j \gtrless (\varepsilon_j+g_j^*)$
(d) ε_j or g_j^*	$(1-\exp(\rho_j\delta_j t))/\delta_j$ < 0	$[1-\exp(\rho_j\delta_j t)-t/\tau+\exp(\rho_j\delta_j\tau)t/\tau]/\delta_j$	$(\exp(\rho_j\delta_j\tau)-1)\delta_j/\tau$ > 0
(e) $y_j(0)^*$	$\exp(\rho_j\delta_j t)$ > 0	$\exp(\rho_j\delta_j t)-\exp(\rho_j\delta_j\tau)t/\tau$	$-\exp(\rho_j\delta_j\tau)/\tau$ < 0
(f) $y_j(\tau)^*$	--	t/τ $0\leq t/\tau\leq 1$	$1/\tau$ $0\leq 1/\tau\leq 1$

TABLE 3.1 (continued)

Parameter	Trajectory		
	$y_{(1)j}(t)$	$y_j(t)$	$u_j(t)$
(g) τ	--	$\partial u_j(t)/\partial \tau \; t$	$[\kappa_{(2)j}(1-\rho_j\delta_j\tau)\exp(\rho_j\delta_j\tau) + (\kappa_{(1)j} - y_j(\tau)^*)] / \tau^2$

where

$$\kappa_{(1)j} = (\varepsilon_j + g_j^*)/\delta_j$$

$$\kappa_{(2)j} = y_j(0)^* - \kappa_{(1)j}$$

$$u_j = [y_j(t)^* - y_j(0)^* \exp(\rho_j\delta_j\tau) - (\varepsilon_j + g_j^*)/\delta_j (1 - \exp(\rho_j\delta_j\tau))]/\tau \; .$$

region that would have become richer under the old scheme now experiences lower income levels, with the two regional income levels converging only at the end of the planning horizon.

Of course, the question of how this dilemma can be resolved remains open. Equity does not necessarily imply that relative output levels be maintained. In some ways, the optimal policy penalizes the region that showed more resourcefulness in developing a more efficient industrial mix. On the other hand, there is nothing to say that the initial economic conditions were at all fair. Perhaps the federal revenue-sharing formulae made it particularly advantageous for that region.

Results 3.10 and 3.11 give further evidence of the differences between $y_{(1)j}(t)$ and $y_j(t)$.

Result 3.10 If Assumptions 3.1-3.5 hold,

$$y_j(t) - y_{(1)j}(t) = u_j t , \qquad (3.24)$$

where u_j is given in Equation (3.20).

Result 3.11 If Assumptions 3.1-3.5 hold, then the effect of a unit parameter increase on the difference between income trajectories, $(y_j(t)-y_{(1)j}(t))$, holding other parameters constant, equals t times the corresponding change in $u_j(t)$.

The effect of any parameter changes on $(y_j(t)-y_{(1)j}(t))$ is magnified over time. Cases in which the original net exports, original government spending, or target income level are larger, are those in which $y_j(t)$ is larger relative to $y_{(1)j}(t)$. On the other hand, if the region's initial per capita income is great, then $y_j(t)$ tends to be smaller relative to $y_{(1)j}(t)$. Finally, the effects of the original capital depreciation rate, consumption propensity, productivity, and length of planning horizon, are not so unequivocal, but tend to be opposite that in the uncontrolled case.

3.4 IMPACT ON POPULATION PATTERNS

Numerous studies have estimated how interregional migration flows and population growth are affected by the relative prosperity of the different regions (Greenwood, 1973; Muth, 1968, 1971; Shaw, 1975). Its remarkably simple hypothesis is that migrants tend to move out of poorer regions into richer ones - the more prosperous regions hold promise of more and better paying job opportunities, as well as the amenities that develop hand in hand with economic growth. However, most of these migration models are static in that only cross-sectional relationships were estimated. Without predictions of regional economic growth, there can be no forecasts of migration, nor of population distribution. This observation presents a sharp contrast with many stochastic models in which forecasts are provided, but with scant theoretical explanation (as noted in Ginsberg, 1972).

In this section, the regional development models analyzed in previous sections are integrated with an income differential migration model. This conceptualization is incorporated in Assumption 3.6, providing an economic underpinning for predicting migration patterns over time. Assumption 3.7 combines this information with that on birth, death and immigration rates to define an equation for population growth.

<u>Assumption 3.6</u> Interregional migration flows respond to interregional income differentials in the following manner:

$$n_{ij}(t) = \sigma + \eta\ (y_j(t) - y_i(t)) - \beta\ f^*(d_{ij}),\ i \neq j \quad (3.25)$$

where $n_{ij}(t)$ is the number of people who move from region i to region j per unit time at time t, divided by the population of i

$y_j(t)$ is the regional per capita income level per unit time in region j at time t

$f^*(d_{ij})$ is a function of the distance separating i and j, d_{ij}; $f^*(d_{ij})>0$

σ, η, β are parameters (>0)

<u>Assumption 3.7</u> The rate of change in the population of region j is given by

$$\dot{x}_j(t) = \phi_j\ x_j(t) + \sum_{i \neq j} (\gamma_i\ n_{ij}(t) - \gamma_j\ n_{ji}(t)) \quad (3.26)$$

where $x_j(t)$ is the population of region j at time t

ϕ_j, γ_j are parameters (>0)

There are many possible surrogates for regional prosperity, including median family income, average wage rate, average disposable income and per capita income. Although not equivalent measures, there is usually a high degree of correlation among these. Per capita income is used here since this enables us to apply the results in the previous sections.

The parameters in Equation (3.25) are estimated exogenously, and are such that for reasonable values of $y_j(t)$ and d_{ij}, migration flows $n_{ij}(t) \geq 0$. The distance term suggests that migration is less between regions that are further apart. Migration between pairs of regions also depends on their relative per capita income levels, though it would be more realistic if these were measured against those of other regions as well.

Notwithstanding, Assumption 3.6 is a reasonable first approximation, being similar to a model estimated in Stone (1969, p. 182). In that study, migration was measured in terms of net interchange of males; income, by personal income per worker; and the distance function by highway mileage between the principal urban centers of the two regions. The regression coefficients were significant at the .005 level; the coefficient of determination was 0.61.

Assumption 3.6 implies that spatial influences, as well as economic factors, are responsible for the magnitude of net migration between regions. For example, set $f^*(d_{ij})=d_{ij}/\sum_k d_{ik}$, $f^*(d_{ji}) = d_{ji}/\sum_k d_{jk}$ in Equation (3.26). Then Result 3.12 shows how net migration flows between two regions may go to the poorer one rather than a distant, richer region.

<u>Result 3.12</u> If Assumption 3.6 holds, then

$$n_{ij}(t)-n_{ji}(t) = 2\eta\,(y_j(t)-y_i(t)) - \beta\,(f^*(d_{ij})-f^*(d_{ji})), \quad i\neq j. \tag{3.27}$$

Implications from Assumption 3.6 point to several weaknesses in the migration model. Migrants are affected by regional income differentials, but regional output is minimally sensitive to changes in the labour force. Total production does change, but only to the extent that per capita output (as well as investment, consumption, government spending, and exports) follows the previously determined path, $y_j(t)$. In addition, this adjustment is instantaneous. When the migrant arrives in a region, he/she immediately takes up the consumption habits of the population within that region. The migrant either becomes employed, or production expands so that per capita output and investment levels follow the previously determined trajectories, $y_j(t)$ and $r_j(t)$. Since neither the production function nor the parameters change, this implies an instantaneous creation of not only a job, but of capital as well. A much more reasonable situation is one with feedback between production, consumption and investment on the one hand, and migration and population growth on the other. Chapter 5 is an initial formulation of such a system, though it suffers from a weaker economic foundation.

In Assumption 3.7, net natural increase and net migration lead to population growth. The parameter ϕ_j equals the birth rate minus the death rate plus the net immigration rate. The parameter γ_j is an index of the number of potential out-migrants. It would be more appealing to replace γ_j by $x_j(t)$, but as a first approximation γ_j can be set equal to $x_j(0)^*$, the initial population.

Equations (3.25) and (3.26) are empirical relationships rather than identities, and are hence approximations of reality. In terms of detail, they fall far short of the accounting-based demographic models formulated by Rogers (1975) and Wilson and Rees (1974). They also fail to represent the interaction between population and output growth as a feedback process, as attempted in Chapter 5. But whereas analytic statements of population distribution are harder to come by in that chapter, the migration-population growth model in this chapter leads to the following results.

<u>Result 3.13</u> If Assumptions 3.1, 3.2, 3.6, and 3.7 hold, then for all j:

$$x_{(1)j}(t) = x_j(0)^* \exp(\phi_j t) + \sum_{i \neq j} [A^*_{(1)ij} (\exp(\phi_j t) - 1) + A^*_{(2)ij} (\exp(\rho_j \delta_j t) - \exp(\phi_j t)) - A^*_{(3)ij} (\exp(\rho_i \delta_i t) - \exp(\phi_j t))], \tag{3.28}$$

where $A^*_{(1)ij} = [\eta\ (\gamma_i + \gamma_j)\ (\kappa_{(1)j} - \kappa_{(1)i}) -$

$$\sigma\ (\gamma_i - \gamma_j) - \beta\ (\gamma_i f^*(d_{ij}) - \gamma_j f^*(d_{ji}))] \quad (3.29)$$

$$/\ \phi_j$$

$$A^*_{(2)ij} = \eta\ (\gamma_i + \gamma_j)\ \kappa_{(2)j}\ /\ (\rho_j \delta_j - \phi_j) \quad (3.30)$$

$$A^*_{(3)ij} = \eta\ (\gamma_i + \gamma_j)\ \kappa_{(2)i} / (\rho_i \delta_i - \phi_j) \quad (3.31)$$

$\kappa_{(1)j}$, $\kappa_{(2)j}$ are defined in Result 3.7

Result 3.14 If Assumptions 3.1-3.7 hold, then for all j:

$$x_j(t) = x_{(1)j}(t) - \sum_{i \neq j} A^*_{(4)ij}\ (1 + \phi_j t - \exp(\phi_j t)), \quad (3.32)$$

where $A^*_{(4)ij} = \eta\ (\gamma_i + \gamma_j)\ (u_j - u_i)\ /\ \phi_j^2$ (3.33)

and u_j is defined by Equation (3.20).

The variable $x_{(1)j}(t)$ is the population pattern that corresponds to $y_{(1)j}(t)$, the income trajectory without endogenous changes in government policy. (Note that $x_j(0)^* \equiv x_{(1)j}(0)^*$.) The trajectory $x_j(t)$ corresponds to $y_j(t)$, which describes income levels if endogenous policies are introduced.

The differences between $y_{(1)j}(t)$ and $y_j(t)$, and $x_{(1)j}(t)$ and $x_j(t)$ are illustrated in the following numerical results.

3.5 NUMERICAL EXAMPLES

In this section, numerical examples are presented for a two region case. All parameters are assigned hypothetical values which are substituted into the appropriate equations. For several of the parameters, the values are varied so as to illustrate qualitative properties of the models. Many of these properties were given in Results 3.8, 3.9, and 3.11. However, these numerical experiments augment the previous results by indicating quantitative attributes as well.

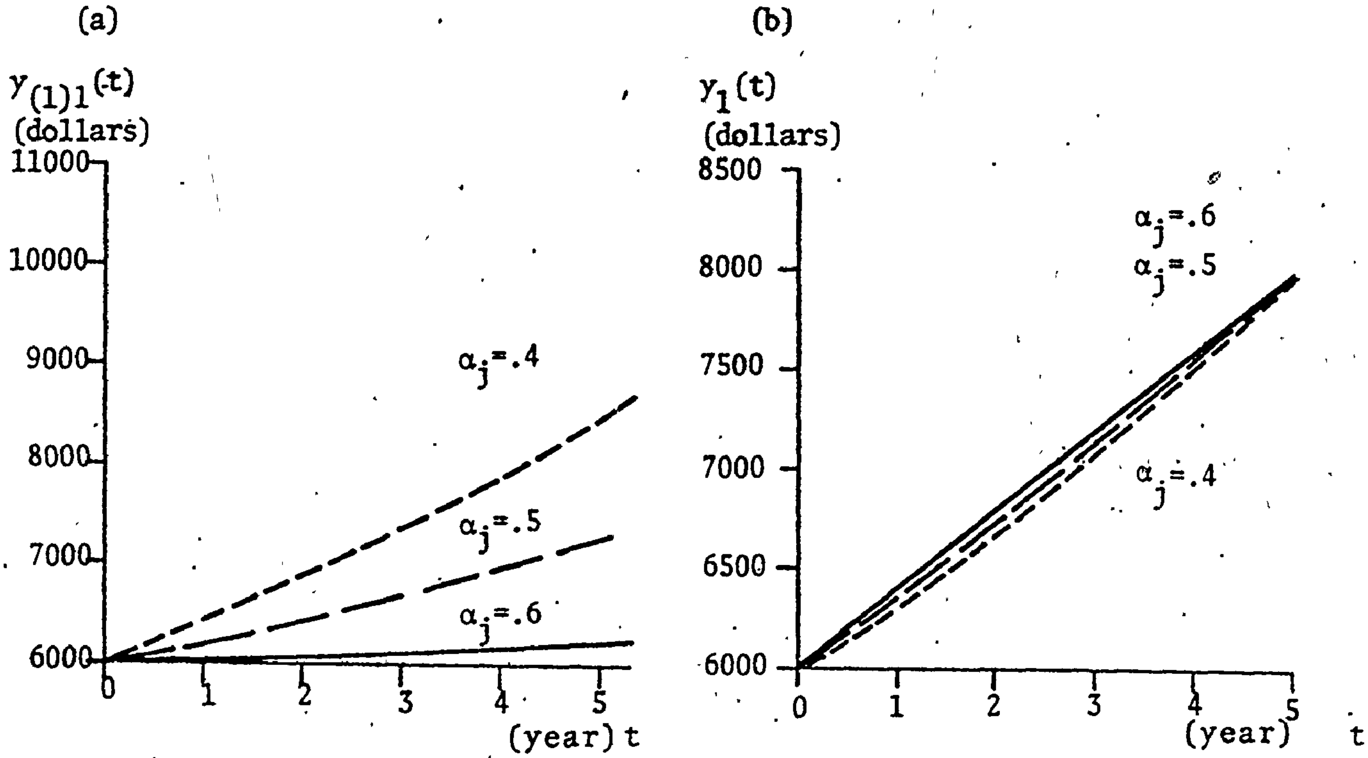
The input is listed in Table 3.2. These values are substituted into Equation (3.11) to obtain the income trajectory that would result if there were no new government policies, and into Equation (3.19) to obtain the income trajectory when new optimal policies are instituted. In addition, the values of the exogenously estimated capital depreciation rate, productivity, and marginal propensity to consume are varied so as to determine the impacts of these parameters on income trajectories and population patterns.

Several salient characteristics are evident in the income trajectories. As expected (see page 49), without the control variable, rapid capital depreciation rates or high consumption propensities lead to lower production/income levels, other things being equal. Figures 3.1(a), 3.2(a), 3.5(a), and 3.6(a) not only show this to be true, but also reveal how the effect becomes increasingly distinct over time. In certain instances, the per capita production/income levels actually decline (when $\alpha_2+\mu_2>0.8$ in Figures 3.2(a), 3.4(a), and 3.6(a)).

TABLE 3.2

INPUT FOR NUMERICAL EXAMPLES

Parameter	Value(s)
J	2.
τ	5.
$y_j(0)^*$	6000.
$y_1(\tau)^*$	8000.
$y_2(\tau)^*$	6000.
α_j	.4, .5, or .6
μ_j	.4, .3, or .2
ρ_j	.4, .3, or .2
ε_1	-500
ε_2	500
g_j^*	1000.
η	1.6×10^{-5}
β	3.0×10^{-6}
σ	.04
$x_1(0)^*$	20,000,000.
$x_2(0)^*$	10,000,000.
ϕ_j	.04
γ_1	20,000,000
γ_2	10,000,000
d_{ij}	1000



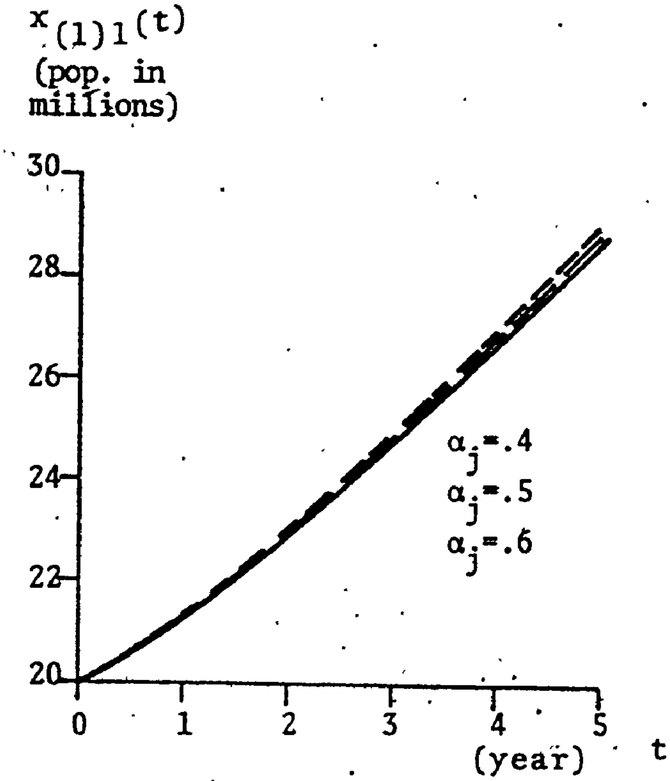
t	$y_{(1)1}(t)$		
	$\alpha_j=.4$	$\alpha_j=.5$	$\alpha_j=.6$
1	6408.	6216.	6030.
2	6855.	6446.	6062.
3	7343.	6690.	6094.
4	7878.	6949.	6127.
5	8463.	7225.	6162

t	$y_1(t)$		
	$\alpha_j=.4$	$\alpha_j=.5$	$\alpha_j=.6$
1	6312.	6372.	6398.
2	6670.	6756.	6797.
3	7066.	7156.	7197
4	7508.	7570.	7598.
5	8000.	8000.	8000.

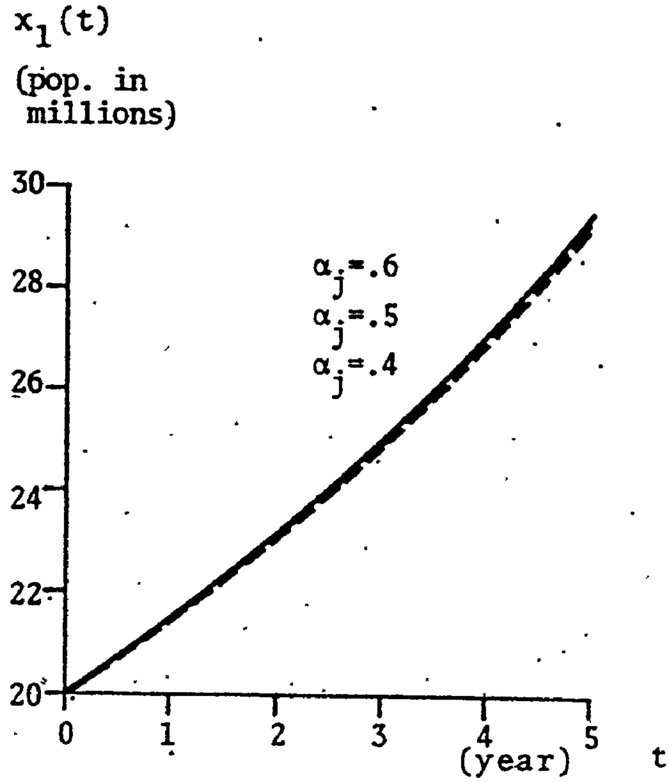
α_j	$u_1(t)$
.4	-93.
.5	155.
.6	368.

Figure 3.1(a),(b): Income Trajectories in Region 1 for Various Values of α_j, with $\mu_j=.3$, $\rho_j=.3$.

(c)



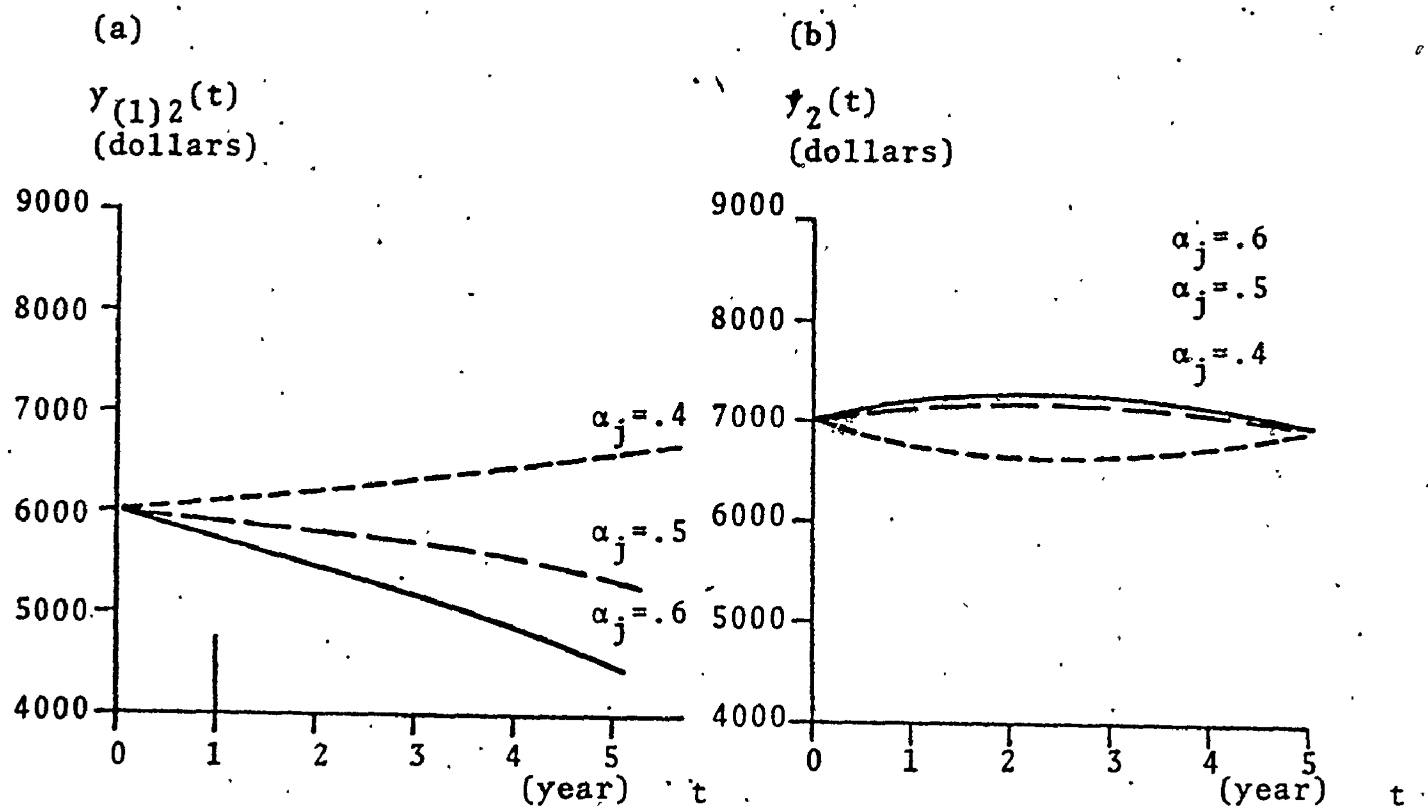
(d)



t	$x_{(1)1}(t)$ (in millions) $\alpha_j=.4$	$\alpha_j=.5$	$\alpha_j=.6$
1	21.33	21.33	21.33
2	22.88	22.87	22.86
3	24.78	24.64	24.62
4	26.77	26.65	26.60
5	29.18	28.94	28.83

t	$x_1(t)$ (in millions) $\alpha_j=.4$	$\alpha_j=.5$	$\alpha_j=.6$
1	21.33	21.34	21.35
2	22.87	22.92	22.94
3	24.66	24.75	24.79
4	26.72	26.86	26.91
5	29.11	29.26	29.32

Figure 3.1 (c),(d): Population Trajectories in Region 1 for Various Values of α_j, with $\mu_j=.3$, $\rho_j=.3$.

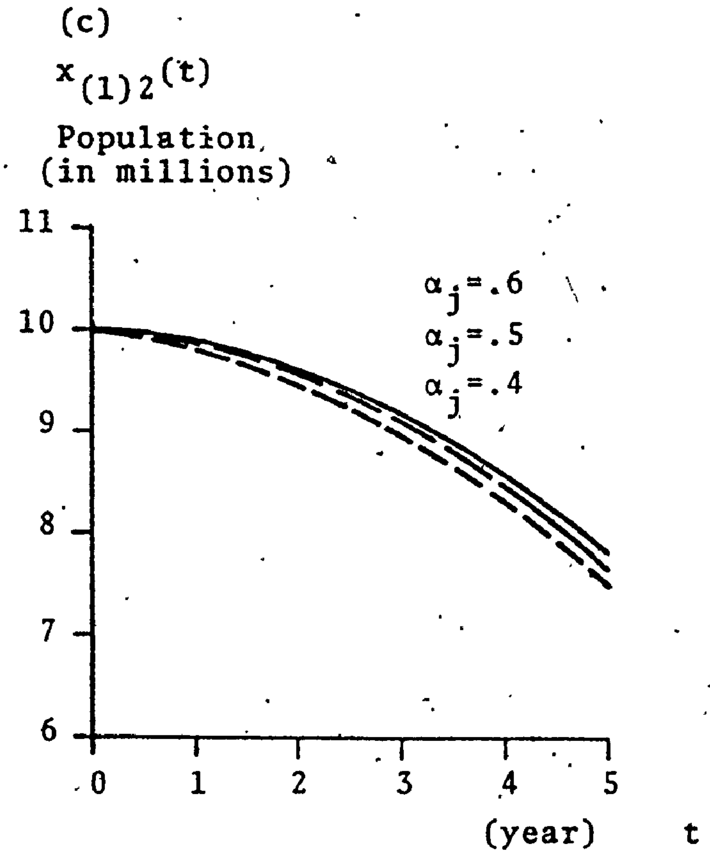


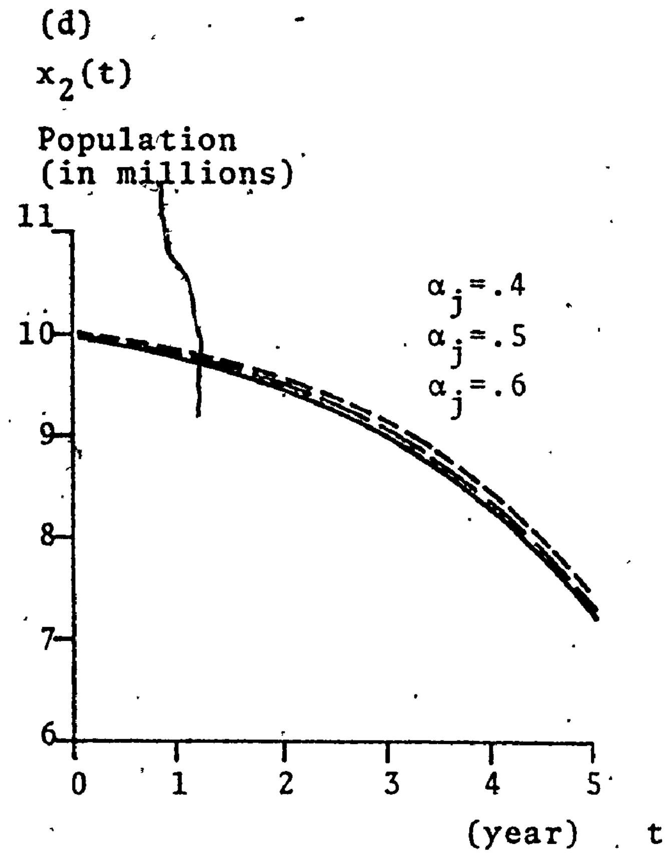
t	$y_{(1)2}(t)$ $\alpha_j=.4$	$\alpha_j=.5$	$\alpha_j=.6$
1	6094.	5907.	5726.
2	6197.	5809.	5443.
3	6310.	5704.	5152.
4	6433.	5593.	4853.
5	6568.	5475.	4543.

t	$y_2(t)$ $\alpha_j=.4$	$\alpha_j=.5$	$\alpha_j=.6$
1	5981.	6012.	6017.
2	5970.	6019.	6026.
3	5969.	6019.	6026.
4	5979.	6013.	6018.
5	6000.	6000.	6000.

α_j	$u_2(t)$
.4	-114.
.5	105.
.6	291.

Figure 3.2 (a),(b): Income Trajectories in Region 2 for Various Values of α_j, with $\mu_j=.3$, $\rho_j=.3$.

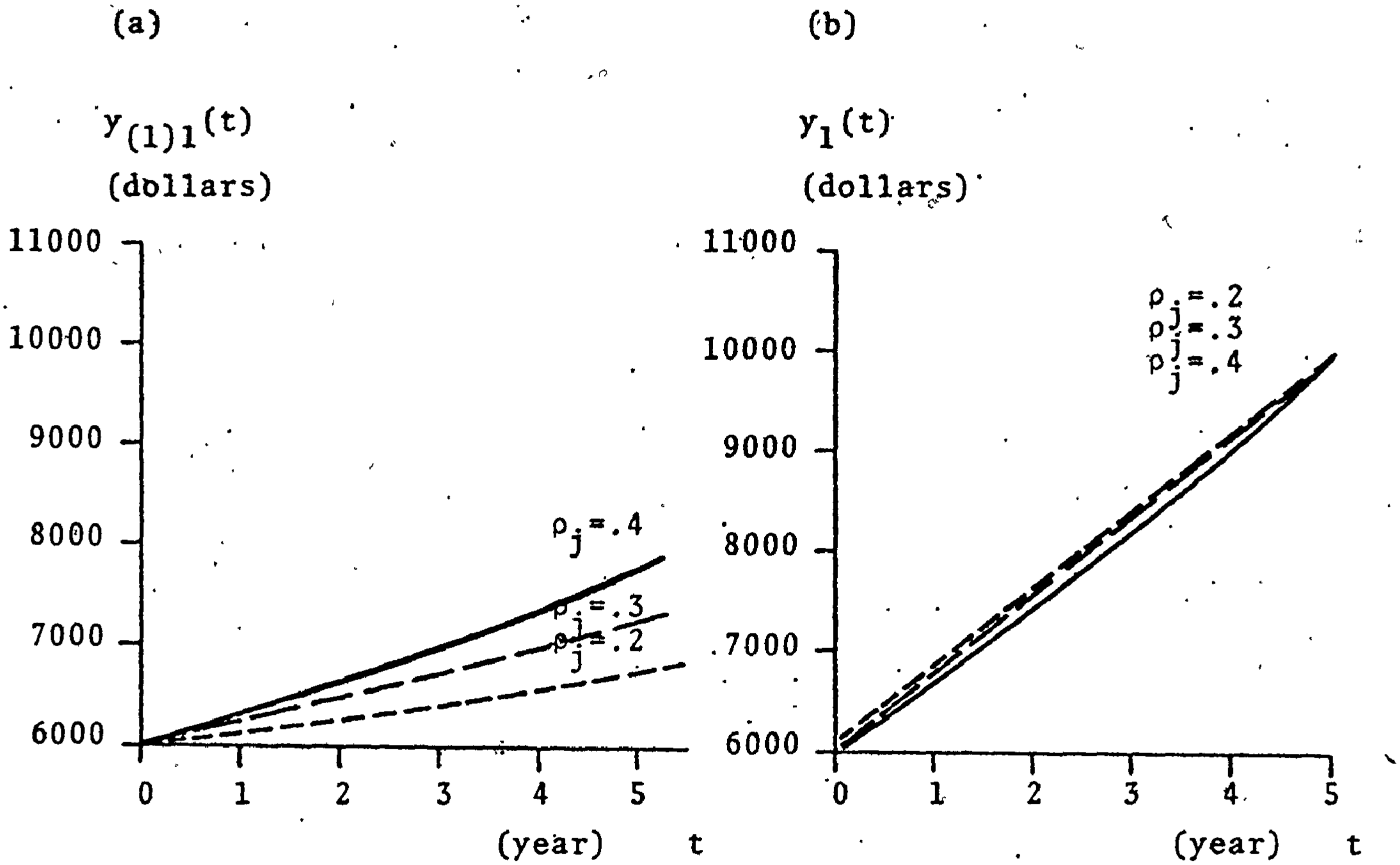




t	$x_{(1)2}(t)$ (in millions)		
	$\alpha_j=.4$	$\alpha_j=.5$	$\alpha_j=.6$
1	9.89	9.89	9.90
2	9.62	9.63	9.64
3	9.17	9.19	9.21
4	8.50	8.55	8.60
5	7.59	7.70	7.81

t	$x_2(t)$ (in millions)		
	$\alpha_j=.4$	$\alpha_j=.5$	$\alpha_j-.6$
1	9.89	9.88	9.88
2	9.60	9.58	9.56
3	9.12	9.07	9.04
4	8.41	8.35	8.30
5	7.45	7.38	7.32

Figure 3.2 (c), (d): Population Trajectories in Region 2 for Various Values of α_j, with $\mu_j=.3$, $\rho_j=.3$.



t	$y_{(1)1}(t)$		
	$\rho_j=.2$	$\rho_j=.3$	$\rho_j=.4$
1	6143.	6216.	6291.
2	6292.	6446.	6607.
3	6446.	6690.	6949.
4	6607.	6949.	7320.
5	6775.	7225.	7721.

t	$y_1(t)$		
	$\rho_j=.2$	$\rho_j=.3$	$\rho_j=.4$
1	6388.	6372.	6347.
2	6782.	6756.	6719.
3	7181.	7156.	7117.
4	7587.	7570.	7543.
5	8000.	8000.	8000.

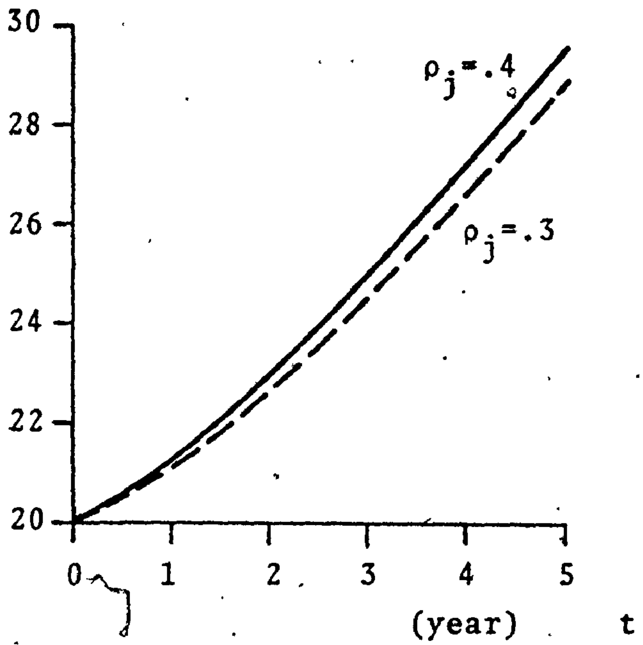
ρ_j	$u_1(t)$
.2	245.
.3	155.
.4	56.

Figure 3.3 (a),(b): Income Trajectories in Region 1 for Various Values of ρ_j, with $\mu_j=.3$, $\alpha_j=.5$.

(c)

$x_{(1)1}(t)$

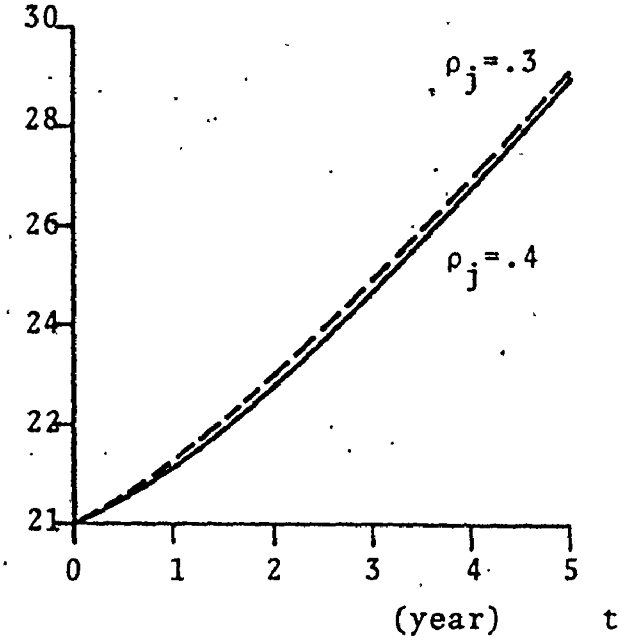
(pop. in millions)



(d)

$x_1(t)$

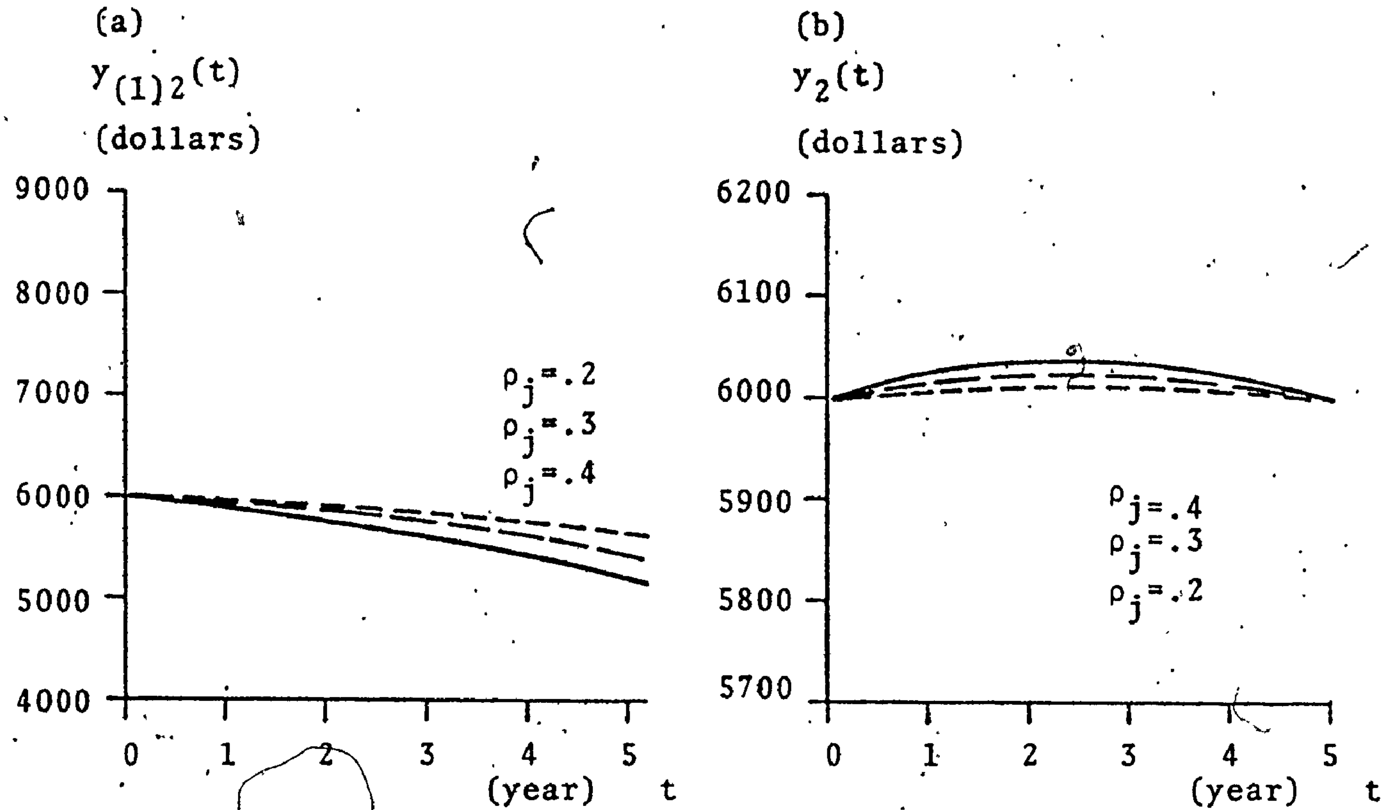
(pop. in millions)



t	$x_{(1)1}(t)$ (in millions) $\rho_j=.3$	$\rho_j=.4$
1	21.33	21.35
2	22.87	22.98
3	24.64	24.90
4	26.65	27.14
5	28.94	29.75

t	$x_1(t)$ (in millions) $\rho_j=.3$	$\rho_j=.4$
1	21.34	21.33
2	22.92	22.89
3	24.75	24.69
4	26.86	26.77
5	29.26	29.16

Figure 3.3 (c),(d): Population Trajectories in Region 1 for Various Values of ρ_j, with $\mu_j=.3$, $\alpha_j=.5$.



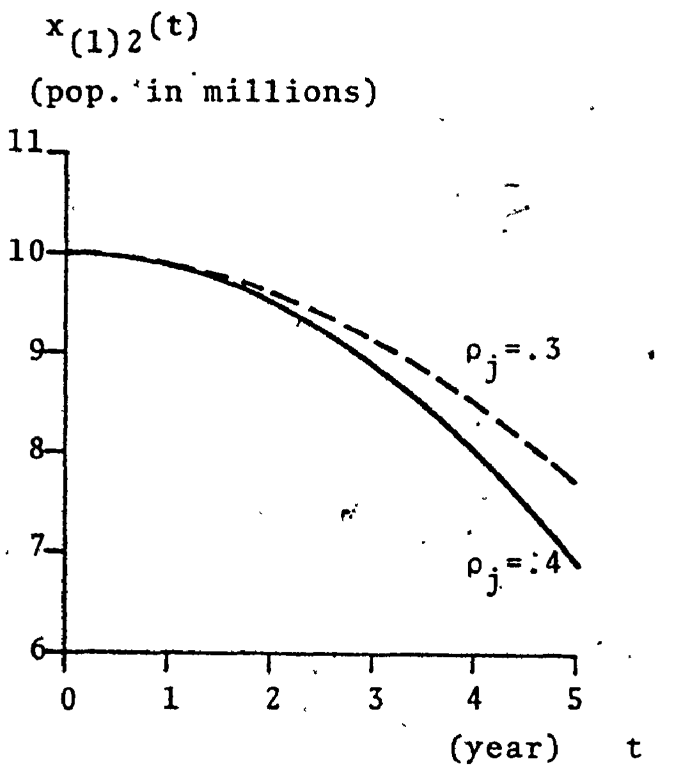
t	$y_{(1)2}(t)$		
	ρ_j=.2	ρ_j=.3	ρ_j=.4
1	5939.	5907.	5895.
2	5875.	5809.	5740.
3	5809.	5704.	5593.
4	5740.	5593.	5434.
5	5668.	5475.	5262.

t	$y_2(t)$		
	ρ_j=.2	ρ_j=.3	ρ_j=.4
1	6005.	6012	6023.
2	6008.	6019.	6035.
3	6008.	6019.	6036.
4	6005.	6013.	6024.
5	6000.	6000.	6000.

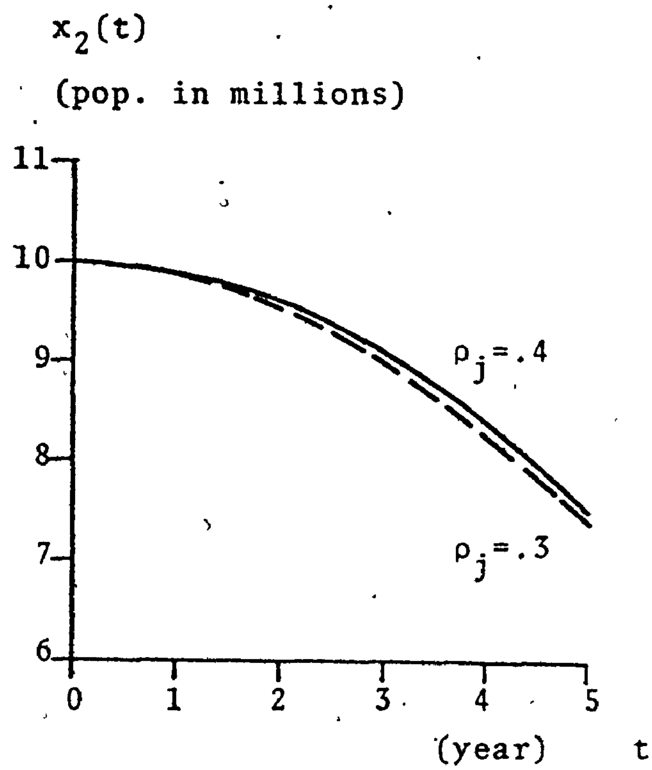
ρ_j	$u_2(t)$
.2	66.
.3	105.
.4	148.

Figure 3.4 (a),(b): Income Trajectories in Region 2 for Various Values of ρ_j, with μ_j=.3, α_j=.5.

(c)



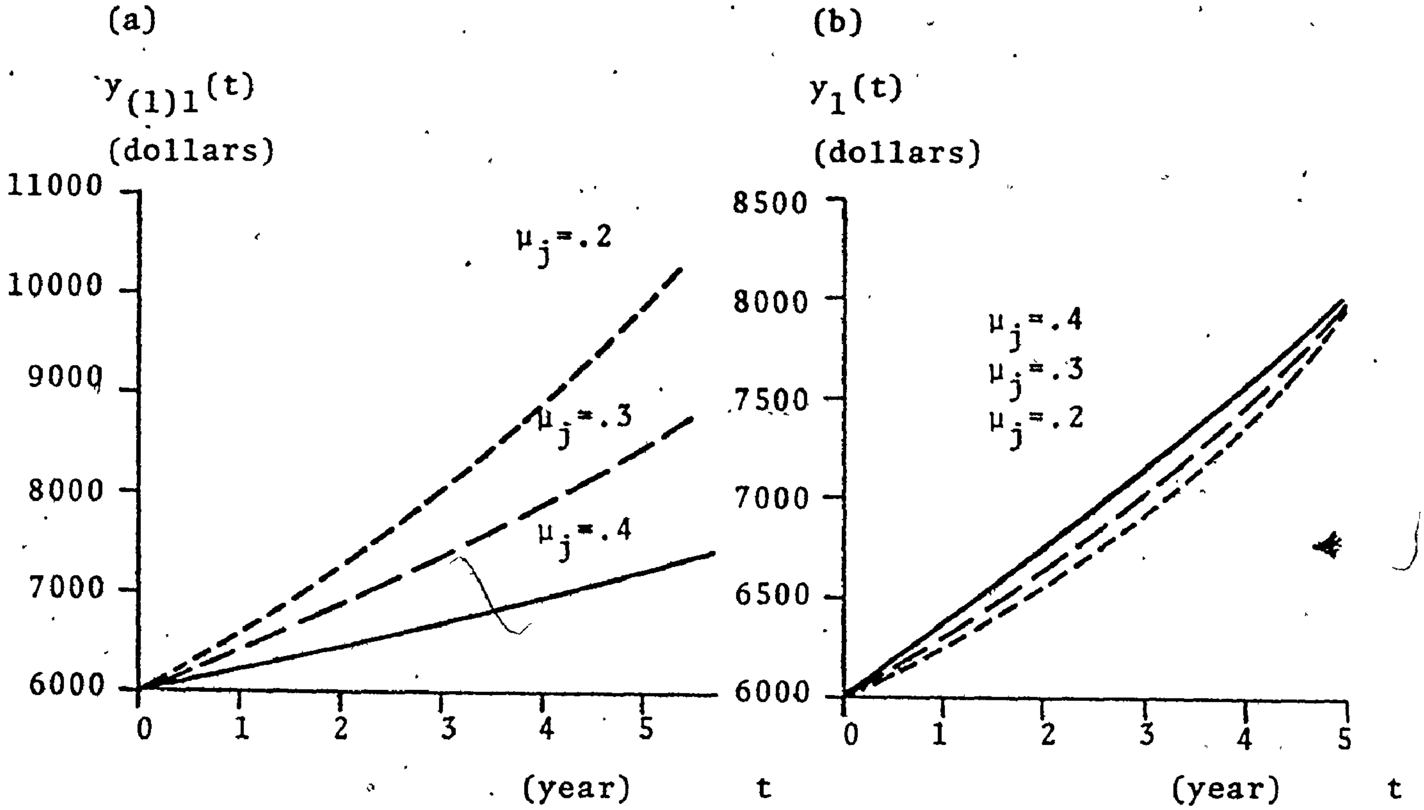
(d)



t	$x_{(1)2}(t)$ (in millions)	
	$\rho_j=.3$	$\rho_j=.4$
1	9.89	9.87
2	9.63	9.52
3	9.19	8.93
4	8.55	8.06
5	7.70	6.89

t	$x_2(t)$ (in millions)	
	$\rho_j=.3$	$\rho_j=.4$
1	9.88	9.89
2	9.58	9.61
3	9.07	9.14
4	8.35	8.44
5	7.38	7.48

Figure 3.4 (c),(d): Population Trajectories in Region 2 for Various Values of ρ_j, with $\mu_j=.3$, $\alpha_j=.5$.

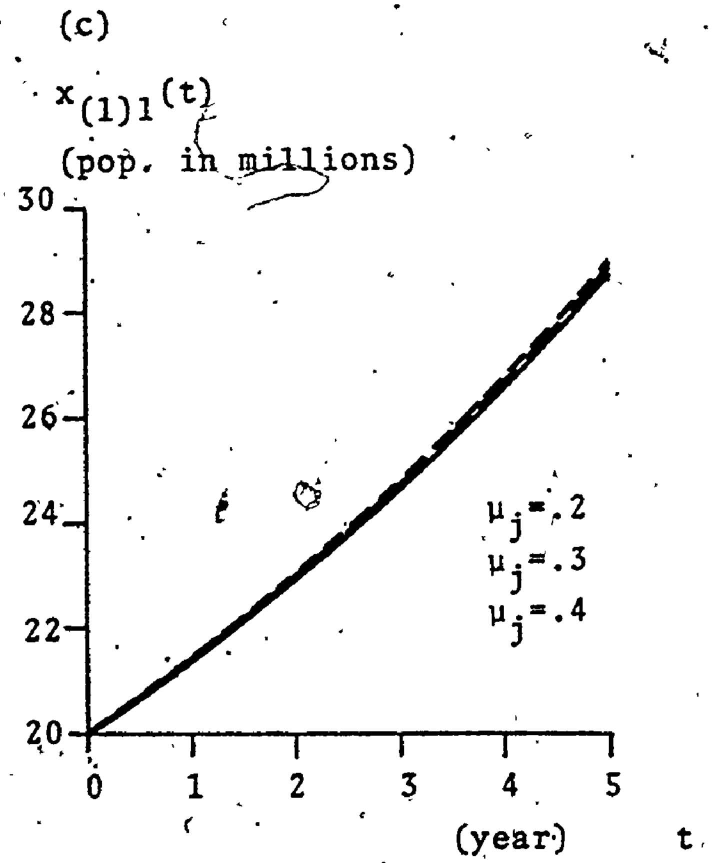


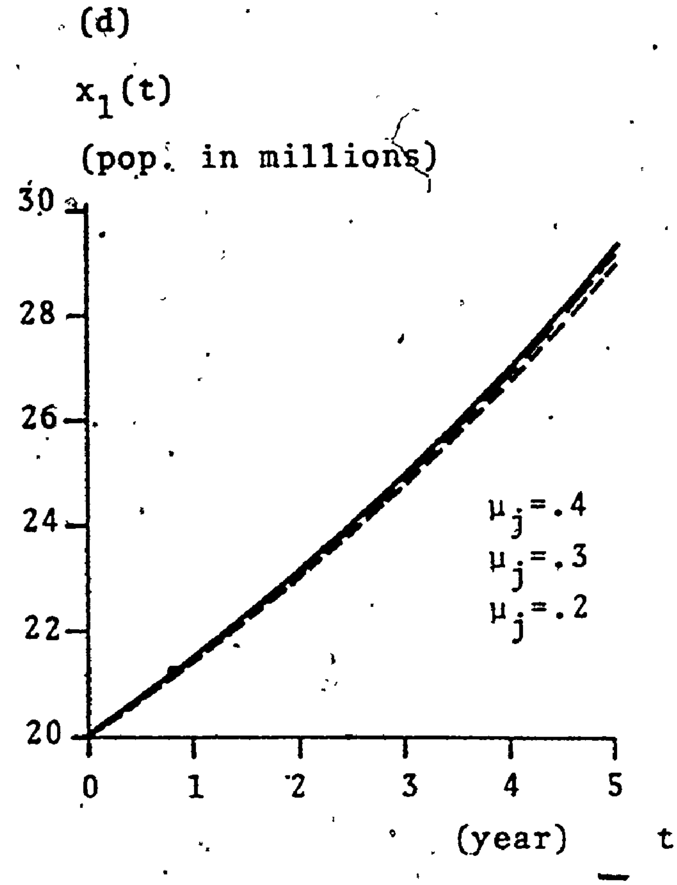
t	$y_{(1)1}(t)$		
	μ_j=.2	μ_j=.3	μ_j=.4
1	6066.	6408.	6216.
2	7288.	6855.	6446.
3	8058.	7343.	6690.
4	8926.	7878.	6949.
5	9905.	8463.	7225.

t	$y_1(t)$		
	μ_j=.2	μ_j=.3	μ_j=.4
1	6225.	6316.	6372.
2	6526.	6670.	6756.
3	6915.	7066.	7156.
4	7402.	7508.	7570.
5	8000.	8000.	8000.

μ_j	$u_1(t)$
.2	-381.
.3	-93.
.4	155.

Figure 3.5 (a),(b): Income Trajectories in Region 1 for Various Values of μ_j, with ρ_j=.3, α_j=.4.

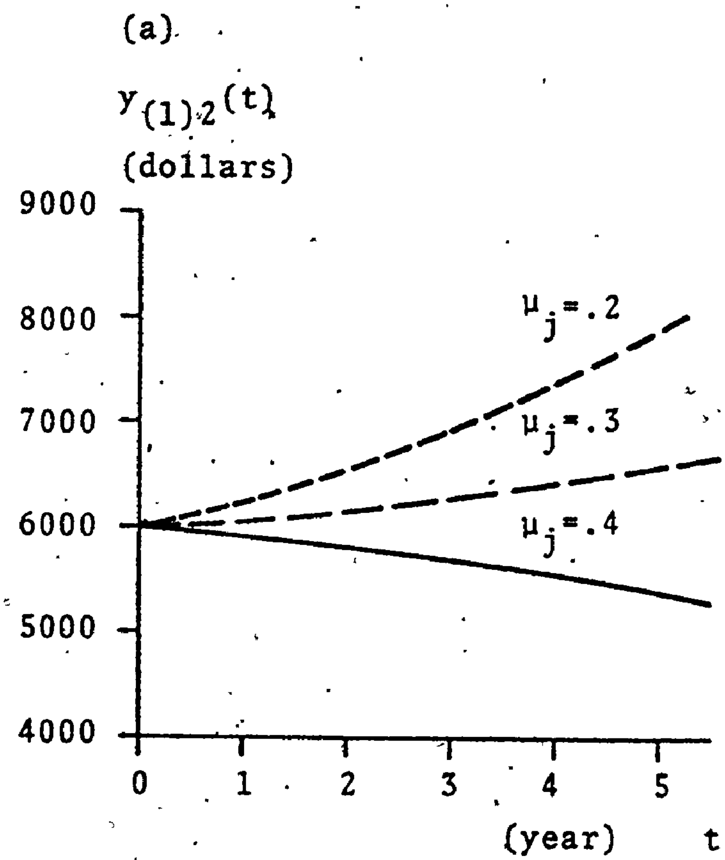


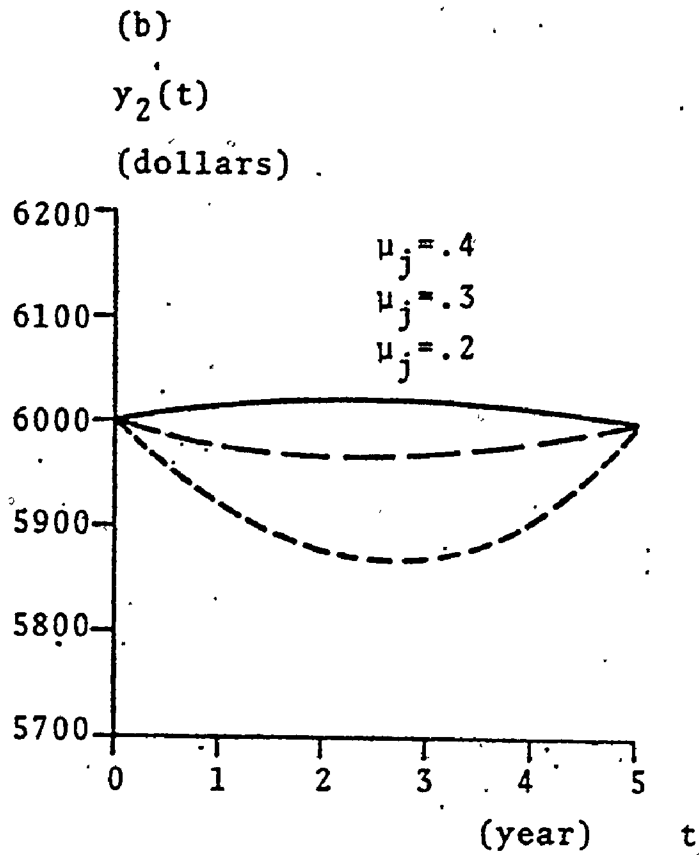


t	$x_{(1)1}(t)$ (in millions)		
	$\mu_j=.2$	$\mu_j=.3$	$\mu_j=.4$
1	21.33	21.33	21.33
2	22.88	22.88	22.87
3	24.68	24.66	24.64
4	26.77	26.71	26.65
5	29.18	29.05	28.94

t	$x_1(t)$ (in millions)		
	$\mu_j=.2$	$\mu_j=.3$	$\mu_j=.4$
1	21.33	21.36	21.34
2	22.87	22.90	22.92
3	24.66	24.71	27.75
4	26.72	26.79	26:86
5	29.11	29.19	29.26

Figure 3.5 (c),(d): Population Trajectories in Region 1 for Various Values of μ_j, with $\rho_j=.3$, $\alpha_j=.4$.





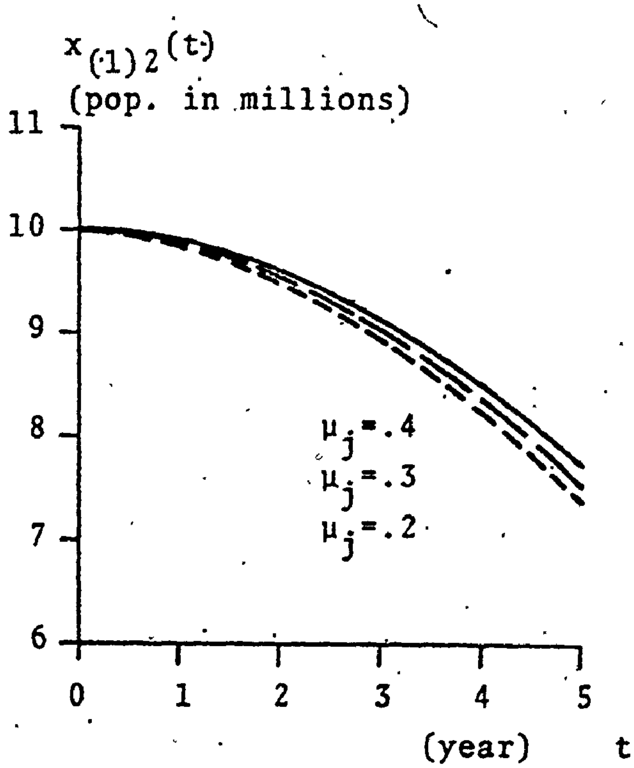
t	$y_{(1)2}(t)$ $\mu_j=.2$	$\mu_j=.3$	$\mu_j=.4$
1	6287.	6094.	5907.
2	6610.	6192.	5809.
3	6975.	6310.	5704.
4	7386.	6433.	5593.
5	7850.	6568.	5475.

t	$y_2(t)$ $\mu_j=.2$	$\mu_j=.3$	$\mu_j=.4$
1	5917.	5981.	6012.
2	5870.	5970.	6019.
3	5865.	5959.	6019.
4	5906.	5979.	6013.
5	6000.	6000.	6000.

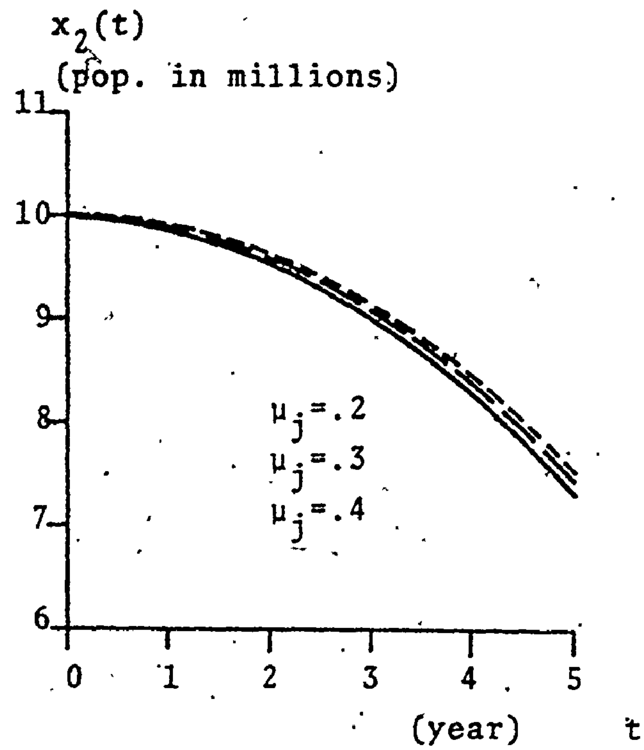
μ_j	$u_2(t)$
.2	-370.
.3	-114.
.4	105.

Figure 3.6 (a), (b): Income Trajectories in Region 2 for Various Values of μ_j, with $\rho_j=.3$, $\alpha_j=.4$.

(c)



(d)



t	$x_{(1)2}(t)$ (in millions)		
	$\mu_j = .2$	$\mu_j = .3$	$\mu_j = .4$
1	9.89	9.89	9.89
2	9.62	9.62	9.63
3	9.14	9.17	9.19
4	8.44	8.50	8.55
5	7.47	7.59	7.70

t	$x_2(t)$ (in millions)		
	$\mu_j = .2$	$\mu_j = .3$	$\mu_j = .4$
1	9.90	9.89	9.88
2	9.63	9.60	9.58
3	9.17	9.12	9.08
4	8.48	8.41	8.35
5	7.45	7.45	7.38

Figure 3.6 (c),(d): Population Trajectories in Region 2 for Various Values of μ_j, with $\rho_j = .3$, $\alpha_j = .4$.

Trajectories are concave, so that any regional disparities would be magnified over the long run.

The only explanation for the differences between Figures 3.1(a) and 3.2(a) is that the per capita net exports in Region 1 is $-500, while Region 2 has a positive trade balance of $500. From this, it is tempting to suggest that it is preferable to have trade deficits; and that these are signs of a vigorous economy, expanding so rapidly that it is unable to provide its own energy and other resource needs. However, a more sober perspective points out that the model offers no mechanism through which export revenues are used to finance subsequent capital formation.

In these experiments, productivity has relatively little impact on production/income levels (Figures 3.3(a) and 3.4(a)). However, lowering government spending or increasing imports magnifies the effect of greater productivity (as measured by the incremental output-capital ratio) on output. This conclusion is apparent from the greater divergence in the trajectories of Figure 3.3(a) than those of Figure 3.4(a).

Thusfar, we have discussed only the situation without a control variable. With the introduction of regional income targets and new, region-specific policies, we get the results illustrated in Figures 3.1(b)-3.6(b). For a given region and income target, there is little difference among the per capita production/income trajectories.

In Region 1, the trajectories are all convex, but are almost linear. This is rather comforting because we have a steady, sustained economic expansion, and not short-term business cycles with mini-recessions, which would have been indicated by highly convex or concave trajectories. A stable economy is desirable socially since the lower income groups tend to suffer more during recessionary times, and to lag in their recovery.

In Region 2 as well, there is little difference among the income trajectories when we specify an income target. It is surprising, however, that even though the target is set equal to the initial level, the optimal income trajectory, $y_2(t)$, does not remain constant. Rather, it is concave (convex) as $y_{(1)2}(t)$ is concave (convex). This means that for cases in which $y_{(1)2}(t) \leq \$6000$. (as for $\mu_2=0.4$ in Figure 3.6(a)), $y_2(t) \geq \$6000$. (as in Figure 3.6(b)). At first sight, it seems that the optimal solution "over-compensates", and that production is stimulated more than necessary, at excess cost. But the analysis tells us that it would be more costly to maintain a constant level of per capita production, or any other level except that defined in Equation (3.19).

We now discuss the variations in population patterns. For a given region, the population trajectories are similar, not only for the various controlled situations (Figures 3.1(d), 3.3(d), 3.5(d) and 3.2(d), 3.4(d), 3.6(d)), but for the uncontrolled cases as well (Figures 3.1(c), 3.3(c), 3.5(c) .

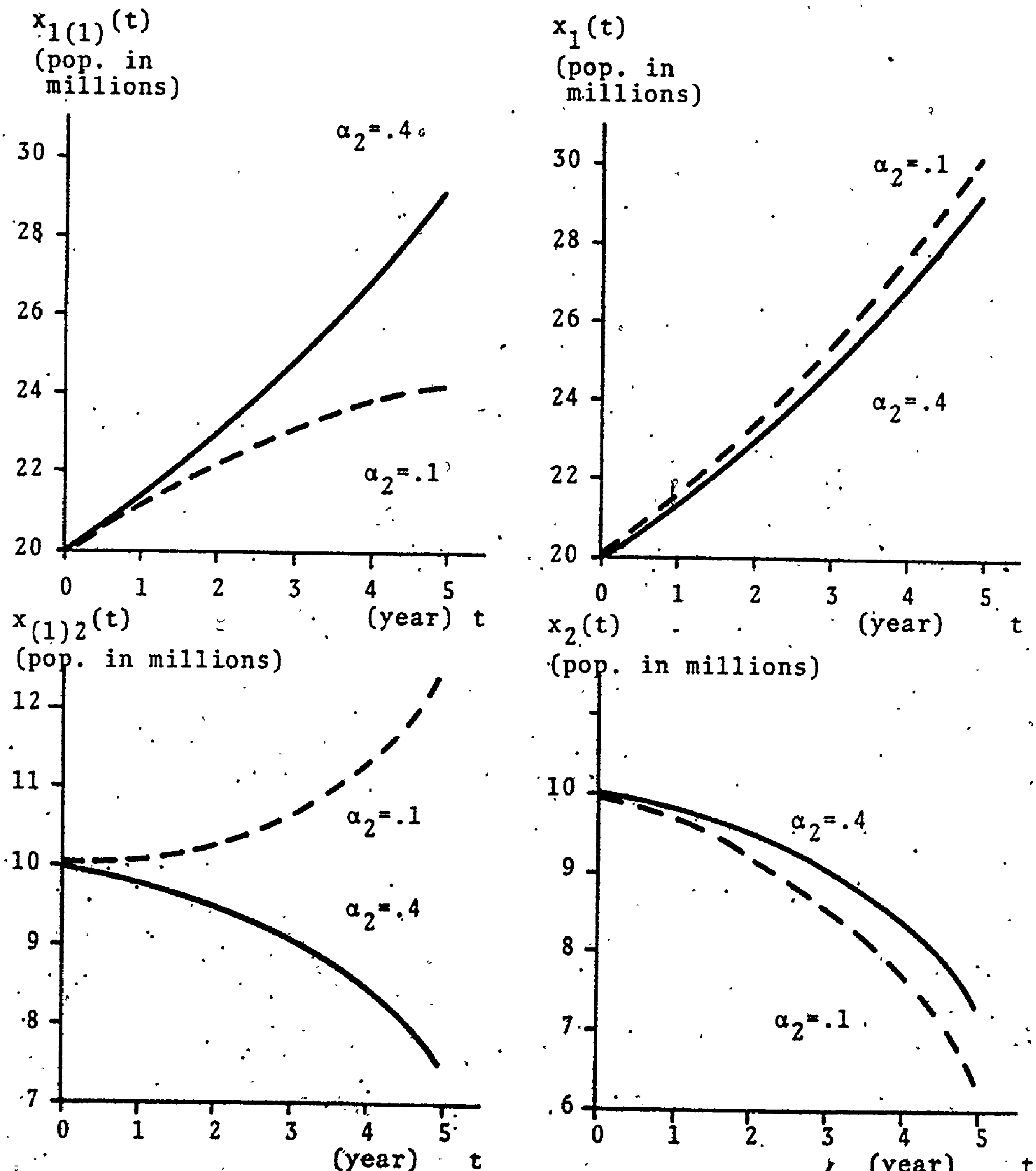


Figure 3.7: Population Trajectories for Regions 1 and 2 With ρ_j=.3, μ_j=.3, α_1=.4, and α_2=.4 or α_2=.1.

the given numerical inputs, population levels are fairly predictable, being relatively insensitive to past characteristics of the regional economy and to government attempts to redistribute wealth. This reduces the uncertainty in planning for housing construction, transportation facilities, and social services.

Under a different set of circumstances, however, radically different population patterns may emerge. For example, Figure 3.7 depicts the population trajectories for the $\alpha_2=0.4$ case given in Figures 3.1(c),(d) and 3.2(c),(d); and compares these to the case in which $\alpha_2=0.1$.

3.6 CONCLUDING COMMENTS

In this chapter, we have taken a dynamic regional economic system and determined its future state, assuming that past trends, including government policies, continue into the future. Next, we augmented this structure to allow for future government intervention aimed at effecting a more desirable geographical distribution of wealth. Then, we derived population patterns based on known birth-death rates; and on an interregional migration model in which migrants' decisions to move depend on the regions' relative attractiveness, and on the distance between potential destinations and the current residence.

The model is policy-oriented in that policy implications are treated endogenously. Such a claim must be taken in its proper perspective, however. The model's task is not to unravel social or political complexities, but to give insights which may have some bearing on policy decision-making. The control variable, $u_j(t)$, represents changes in the growth rate of regional output/income which are a consequence of newly-initiated government policies. We do not know what specific policies actually achieve the optimal levels of $u_j(t)$ - what capital gains tax, what incentives for private economic expansion, what allocation of grants-in-aid, and what other fiscal and monetary policies. But at least we know $u_j(t)$, which from Result 3.5 is constant over time. This suggests a fairly stable government posture, in which policies are consistently maintained over the duration of the planning period, rather than initiated in short bursts. In addition, the magnitude of $u_j(t)$ indicates the severity of new government actions. If $|u_j(t)| \simeq 0$, then little change is needed from past policies. But if $|u_j(t)|$ is relatively large, then the governments are faced with a much tougher task; and policies must be selected more judiciously, often in the face of political unpopularity.

The model in this chapter is an initial step towards more comprehensive models of relevance to regional economic planning. In large part, its simplicity and drawbacks were intentional. Since it is a first exploratory enquiry, we were not concerned with more realistic details; and such aspects as stochastic control, distributed parameters, perturbation analysis, and more comprehensive economic representations are the subject of ongoing study.

CHAPTER 4

REGIONAL GOVERNMENT EXPENDITURE AS A POLICY VARIABLE

4.1 INTRODUCTION

The control variable in Section 3.3 represents the impact of hypothetical economic policies on regional income growth. However, it does not represent any specific economic instrument. Although that model is sufficiently general to incorporate both monetary and fiscal considerations, it far from indicates which particular policies to follow. The model in this chapter takes one step in that direction by considering only the fiscal aspect, though at an extremely aggregate level.

Fiscal measures have two major components: taxation and government spending. For purposes of exposition, per capita regional taxation is set exogenously in the remainder of the chapter; and spending is endogenous. The converse situation, with endogenous taxation and exogenous spending, is not given, but can be studied in an analogous manner.

In the model of Section 3.3, spending and taxation are not explicit. There, it is assumed that, historically, per capita government spending is a constant, g_j^*; but the model does not reveal spending changes associated with the control variable $u_j(t)$. Also, taxation is subsumed within a deflated marginal propensity to consume, so that any tax implications associated with $u_j(t)$ are reflected in a new marginal propensity to consume. But this new value is unknown; only the old value, μ_j, is known.

In this chapter, as in the previous, a normative stance is taken: regional production targets are realized at the end of the planning horizon. The control variable is per capita regional government spending, which is one component of a regional macroeconomic system. In the next section, this system is defined, and then reduced to an expression for per capita gross regional product. For any given region, this trajectory is a function of per capita government spending. The task is to determine how spending should vary so as to achieve the regional production target, while minimizing the costs of the associated programmes.

4.2 REGIONAL ECONOMIC DEVELOPMENT

This section derives an expression for per capita gross regional product (or per capita income) as a function of time. The economic variables in the system are measured in terms of dollar values relative to some base year. This accounts for inflation, but does not present any model for its explanation.

Regional government spending is the control variable, and is defined as the expenditures by the local and regional governments, as well as federal spending in the region; taxation is defined in a similar fashion. An important component of the control variable is the regional pattern of federal expenditures. The 1969-71 allocations of federal expenditures in the United States were such that 34% were direct payments to individuals and 14% were grants-in-aid, whose dissemination is based on region-specific attributes. But since spending is also effected by local and regional governments, the model indicates the need for significant federal-regional cooperation. Interestingly, some recently-initiated research within the private sector is directed to understanding the dynamics of federal-regional economic relations (The First National Bank of Boston, 1976).

A formal representation of the macroeconomic system is stated in Assumption 4.1. As indicated in Chapter 1, the control variable, $u_j(t)$, is an element of a fixed closed set for all t. The other variables are positive and continuous in t; the parameters are all positive real numbers unless otherwise indicated. The proofs of all results are given in Appendix 4.

<u>Assumption 4.1</u> A country consists of j=1,...,J regions; and the regional economic system is represented by the following relationships, which hold for all j:

$$y_j(t) = c_j(t) + o_j(t) + e_j^*(t) - m_j^*(t) + u_j(t) \quad (4.1)$$

$$r_j(t) = dq_j(t)/dt \quad (4.2)$$

$$dw_j(t)/dt = \alpha_j \, y_j(t) \quad (4.3)$$

$$y_j(t) = \rho_j \ (q_j(t) - w_j(t)) \tag{4.4}$$

$$r_j(t) = \int_{z=0}^{t} o_j(z) \ k^*_{(r)}(t-z)dz \tag{4.5}$$

$$c_j(t) = \mu_j \int_{z=0}^{t} (y_j(z)-T^*_j(z)) \ k^*_{(c)}(t-z) \ dz \tag{4.6}$$

where

$y_j(t)$ is the per capita gross regional product of region j per unit time at time t;

$c_j(t)$ is the per capita private consumption in region j per unit time at time t;

$o_j(t)$ is the per capita product saved from consumption as an outlay for investment in region j per unit time at time t;

$e^*_j(t)$ is the per capita exports of regional product from region j to all other regions per unit time at time t;

$m^*_j(t)$ is the per capita imports into region j per unit time at time t;

$u_j(t)$ is the control variable: per capita government spending in region j per unit time at time t;

$q_j(t)$ is the per capita stock of capital that would have accumulated in region j at time t had there been no consumption of capital;

$w_j(t)$ is the per capita stock of depleted capital in region j at time t;

$r_j(t)$ is the per capita investment in region j per unit time at time t;

$T_j^*(t)$ is the per capita taxation in region j per unit time at time t, including local, regional and federal taxes;

α_j is the capital depletion rate in region j;

ρ_j is the rate of return in region j per unit time on the net available capital;

μ_j is the marginal propensity to consume disposable income in region j per unit time;

$k^*_{(r)}, k^*_{(c)}$ are kernels - functions representing time lags in investment and consumption, respectively.

Assumption 4.1 extends the work of Jutila (1971, 1972, 1973) by introducing taxation and endogenous government spending. The model sacrifices realism in terms of disaggregation and a more accurate specification of the economic dynamics, in order to concentrate on major aggregate sectors of the regional economies. It differs from the models in Chapter 3 in the following respect. There, Assumption 3.1 refers to a system without endogenous policy instruments; government spending is fixed, and taxation only implicit in the marginal propensity to consume. This is subsequently extended in Section 3.3, but the impacts of the newly-introduced control variable on spending and taxation are unclear. Whereas in this chapter, the control variable is per capita government spending itself.

The interpretation of the relationships in Assumption 4.1 is virtually identical to that in Assumption 3.1, except that consumption is related to disposable income $(y_j(t)-T_j^*(t))$ rather than total income; and government spending is endogenous. The reduced form of Assumption 4.1 is given by the following result.

<u>Result 4.1</u> If Assumption 4.1 holds, then the per capita gross regional product per unit time (or equivalently, the per capita income) of region j at time t is:

$$y_j(t) = L^{-1} [\{y_j(0)^* - \rho_j L[k^*_{(r)}(t)] (L[e^*_j(t)] - L[m^*_j(t)] + L[u_j(t)] - \mu_j L [k^*_{(c)}(t)] L[T^*_j(t)])\} \{s - \rho_j (L[k^*_{(r)}(t)] - \alpha_j - \mu_j L[k^*_{(r)}(t)] L[k^*_{(c)}(t)])\}^{-1}] \quad , \qquad (4.7)$$

where, as in Chapter 3:

$L[.]$ denotes the Laplace transform with respect to t;

$y_j(0)^*$ is the initial per capita income level, given a priori;

s   is the complex variable corresponding to the Laplace transform with respect to t.

If $L[k^*_{(r)}(t)]$, $L[k^*_{(c)}(t)]$, $L[e^*_j(t)]$, $L[m^*_j(t)]$, $L[T^*_j(t)]$ and $L[u_j(t)]$ are given, then $y_j(t)$ is determined explicitly. The first five are specified in Assumption 4.2. Government spending, $u_j(t)$, is determined from the optimal control problem analyzed in Section 4.3.

<u>Assumption 4.2</u> Let

$$e^*_j(t) - m^*_j(t) = \varepsilon_j \, , \quad \varepsilon_j \text{ is a real number} \tag{4.8}$$

$$L[k^*_{(r)}(t)], L[k^*_{(c)}(t)] = 1 \tag{4.9}$$

$$T^*_j(t) = \chi_j \exp(\psi_j t) \tag{4.10}$$

Assumption 3.2 specifies a fixed level of government spending, and implies tax rates which maintain a constant marginal propensity to consume. Assumption 4.2, on the other hand, states that per capita taxation will grow exponentially (since $\psi_j > 0$), and imposes no suppositions on government expenditures. The latter case assumes that local, regional, and federal governments anticipate the taxes they expect to levy. This, and other decision-making criteria (such as production goals), determine the optimal levels of government expenditures over time. Incidentally, if the optimal solution indicates inadequate support for education, health, and other services, then the production goals or taxation levels must be altered.

The per capita trade balances in each region are constant (Equation 4.8), so that the discussion on page 38 is appropriate here, as well. This simple export-import structure means that the interregional aspects of the model are rather weak, though they can be enriched along several fronts.

One possibility is to incorporate Jutila's (1971, 1972, 1973) growth pole - diffusion system, in which one region's exports to an adjacent region stimulate its development; and the interaction then continues from that region to others. This provides an explicitly interregional format in which location and trade play a key role in regional economic development.

Another possibility is to relate ε_j to an entropy-maximizing commodity flow model (Wilson, 1970). Actually, for our purposes, it is unnecessary to consider individual commodities - only the dollar value of the flows. The idea is to use past information to estimate pertinent relationships, and then to obtain forecasts by extending these relationships. We determine the least-biased estimate of commodity flows by maximizing the modeller's uncertainty:

Maximize

$$S = -\sum_{ij}\sum N_{ij} \log N_{ij} \qquad (4.11)$$

with respect to N_{ij}

subject to:

$$\sum_i N_{ij} = N_{\cdot j}^{*} \tag{4.12}$$

$$\sum_j N_{ij} = N_{i \cdot}^{*} \tag{4.13}$$

$$\sum_i \sum_j . F^{*}(d_{ij})\, N_{ij} = F^{*} \tag{4.14}$$

where S is the entropy, a measure of the modeller's uncertainty about the pattern of commodity flows (Webber, 1977);

N_{ij} is the dollar value of all commodity flows (not per capita) from region i to j in one year;

$N_{\cdot j}^{*}$ is the given value of all import demands in one year by region j, a reflection of its industrial structure, consumers' demands, and government import policies;

$N_{i \cdot}^{*}$ is the given value of all exports supplied in one year by region i, also a reflection of industrial structure and consumer demands;

$F^{*}(d_{ij})$ is the cost of transporting one dollar of commodity from i to j, which is an average rate based on the distance between i and j(d_{ij}), on the typical export products from i, and on the transportation modes.

F^* is the total cost spent on transporting all commodities between all regions.

The entropy-maximizing problem can be solved by the usual means (Tribus, 1969) to give commodity flows as a function of the distances between regions:

$$N_{ij} = \lambda_{(1)i} N^*_{i\cdot} \lambda_{(2)j} N^*_{\cdot j} \exp(-\beta F^*(d_{ij})) \quad , \tag{4.15}$$

where $\lambda_{(1)i}$, $\lambda_{(2)j}$, β are Lagrangian multipliers.

If N_{ij} data are available, then this spatial interaction model can be calibrated (Batty, 1976). If the parameters and distance function are stationary over time, Equation (4.15) becomes:

$$N_{ij}(t) = \lambda_{(1)i} N^*_{i\cdot}(t) \lambda_{(2)j} N^*_{\cdot j}(t) \exp(-\beta F^*(d_{ij})). \tag{4.16}$$

Assume that $e^*_j(t)$ and $m^*_j(t)$ are known (and satisfy Assumption 4.2). Assume too that population growth depends on interregional migration flows, which are sensitive to regional income differentials in a manner similar to that in Section 3.4 - so that regional population levels $x_j(t)$ are predictable. Note that

$$N^*_{i\cdot}(t) \equiv e^*_i(t) \; x_i(t) \tag{4.17}$$

and $$N^*_{\cdot j}(t) \equiv m^*_j(t) \; x_j(t) \tag{4.18}$$

These can be substituted into Equation (4.16) to estimate the distance-dependent interregional commodity flows over time. Although ε_j is constant, $N_{ij}(t)$ can vary over time.

We have a system in which regional economic development affects population growth which feeds back to affect commodity flow patterns.

Assumption 4.2 requires constant per capita trade balances in each region, no time lags in investment and consumption, and exponentially increasing per capita taxation. Given this, we are able to derive the reduced form for income as a function of government spending and time.

<u>Result 4.2</u> If Assumptions 4.1 and 4.2 hold, then:

$$y_j(t) = y_j(0)^* \exp(\rho_j\delta_j t) + \varepsilon_j/\delta_j \ (1-\exp(\rho_j\delta_j t))$$

$$- \rho_j \int_{z=0}^{t} (u_j(z) - \mu_j\chi_j\exp(\psi_j z)) \qquad (4.19)$$

$$\exp(\rho_j\delta_j(t-z))\ dz \quad ,$$

$$\delta_j = 1-\mu_j-\alpha_j \qquad (4.20)$$

<u>Result 4.3</u> Result 4.2 can be re-expressed as:

$$\dot{y}_j(t) = \rho_j\delta_j (y_j(0)^* - \varepsilon_j/\delta_j) \exp(\rho_j\delta_j\ t)$$

$$- \rho_j^2\delta_j \int_{z=0}^{t} (u_j(z)-\mu_j\chi_j\exp(\psi_j z)) \qquad (4.21)$$

$$\exp(\rho_j\delta_j(t-z))\ dz$$

$$- \rho_j\ (u_j(t)-\mu_j\chi_j\exp(\psi_j t)) \quad ,$$

where $y_j(0) = y_j(0)^*$ is given.

If $u_j(t)$ were specified exogenously, then

$$\frac{\partial y_j(t)}{\partial \varepsilon_j} < 0 \quad , \qquad (4.22)$$

as in Section 3.3. However $u_j(t)$ is now an endogenous variable; and in the control problem posed in the next section $u_j(t)$ is found to be a function of ε_j. Hence, Equation (4.22) is not necessarily true.

Also, from Equations (4.10) and (4.19) it is tempting to conclude that

$$\frac{\partial y_j(t)}{\partial T_j^*(t)} > 0 \qquad (4.23)$$

However, increasing $T_j^*(t)$ implies increasing χ_j or ψ_j, which in turn alters $u_j(t)$ and hence $y_j(t)$. Increasing χ_j or ψ_j may indeed increase $y_j(t)$; but this does not necessarily imply an increase in disposable income, $(y_j(t)-T_j^*(t))$. It appears that the validity of Equation (4.22) can only be established numerically for a particular situation.

4.3 GOVERNMENT SPENDING AS A POLICY INSTRUMENT

Regional income levels are a function of government spending, which can be specified exogenously as in Section 3.2, or determined endogenously according to certain policy objectives. The latter approach is adopted in this section, for the policy objectives stated in Assumptions 4.3 and 4.4.

Assumption 4.3 Denote the initial time by $t=0$, and the end of the planning horizon by $t=\tau$. Regional government expenditures are such that the cost associated with total government spending over all regions is minimized. This cost is defined as

$$I = \int_{t=0}^{\tau} \sum_j \zeta_j u_j(t)\, dt \quad , \tag{4.24}$$

where

ζ_j is the cost associated with one dollar of government spending in region j.

Assumption 4.4 The income levels for each region at the end of the planning period, τ, are prespecified:

$$y_j(\tau) = y_j(\tau)^* \quad . \tag{4.25}$$

Assumption 4.3 suggests that federal and regional governments cooperate in economic relations. Otherwise, the assumption is unrealistic, for the costs of policies depend on government decisions at local, regional, as well as federal levels. The federal government, however, has significant influence, since it constructs revenue-sharing formulae based on regional characteristics. Hence, at the federal decision-making level in particular, Assumption 4.3 is reasonable. In essence, it represents the undesirability of runaway government expenditures; the value of ζ_j indicates the weight attached to the undesirability. (The parameter ζ_j also reflects regional size differences, since $u_j(t)$ is measured per capita.) Popular opinion suggests that escalating levels of government spending are at the root of the recent, widespread wage-price spiral that climaxed in 1974-75. A large positive value of ζ_j commits a restrictive fiscal policy, that curbs inflationary pressures arising from uncontrolled public spending. On the other hand, a stagnant regional economy must be stimulated. In this instance, increasing the money supply, for instance, reduces the value of ζ_j so as to encourage expansionary pursuits.

After government expenditures are determined, they can be compared to tax revenues. Then we can see for which regions and at what times $u_j(t)$ is much greater (or less) than $T_j^*(t)$. This allows an a posteriori evaluation of the extent of surplus or deficit spending in regional budgets.

A more direct approach, however, is to adopt the objective functional:

$$I = \int_{t=0}^{\tau} \sum_j \zeta_j (u_j(t) - T_j^*(t))^2 \, dt \quad ; \qquad (4.26)$$

or to include a constraint such as:

$$\sum_j \gamma_j \, (u_j(t) - T_j^*(t))^2 \leq M^* \qquad (4.27)$$

where γ_j is a surrogate for population

M^* is a positive real number representing some limit on surpluses or deficits.

Both suggest a concern for balanced regional budgets. Not only does this lead to balanced national budgets, but it also reduces instances of disproportionate spending, in which a significant portion of one region's taxes are spent elsewhere. Unfortunately, using Equation (4.26) instead of Equation (4.24) as the objective functional leads to an intractable nonlinear system. The notion embodied in Equations (4.26) and (4.27) is an area requiring further study.

No generally accepted indicator of welfare exists, but any appropriate indicator must include measures of both economic growth and distributional equity. Assumption 4.4 is sensitive to both, since it can reflect growth objectives, as well as demands for reduced regional income disparities.

The assumptions define an optimal control problem. Assumption 4.3 is the objective functional; Assumption 4.4 states the boundary conditions; and previously-derived Result 4.3 gives the dynamic system. Pontryagin's Maximum Principle yields a solution for $u_j(t)$ and $y_j(t)$, stated in Result 4.4.

<u>Result 4.4</u> If Assumptions 4.1 to 4.4 hold, then the optimal trajectories of per capita regional government spending and per capita regional income are:

$$u_j(t) = u_j(0)^* \left[\frac{G_j^* \rho_j + \zeta_j}{\zeta_j} - \frac{G_j^* \rho_j}{\zeta_j} \exp\left[\frac{\rho_j \delta_j \zeta_j}{G_j^* \rho_j + \zeta_j} t \right] \right] \quad (4.28)$$

$$y_j(t) = y_j(0)^* \exp(\rho_j \delta_j t) + \frac{\varepsilon_j}{\delta_j}(1-\exp(\rho_j \delta_j t))$$

$$+ \frac{\chi_j \rho_j \mu_j}{\psi_j - \rho_j \delta_j}(\exp(\psi_j t) - \exp(\rho_j \delta_j t)) \tag{4.29}$$

$$+ \frac{u_j(0)^*}{\delta_j}(1-\exp(\rho_j \delta_j t))$$

$$\left[\frac{G_j^* \rho_j + \zeta_j}{\zeta_j} - \frac{G_j^* \rho_j}{\zeta_j} \exp\left[\frac{\rho_j \delta_j \zeta_j}{G_j^* \rho_j + \zeta_j} t\right]\right] ,$$

if $G_j^* > 0$ or (4.30)

$G_j^* \rho_j + \zeta_j < 0$, (4.31)

where G_j^* is a constant of integration.

The constants of integration (G_j^*,j=1,---,J) are determined from the boundary conditions. Setting t=τ in Equation (4.29) results in a transcendental expression with an exponential term in the unknown, G_j^*. The solution of this problem must be determined numerically. One of Equations (4.30) or (4.31) must be true for Result 4.4 to hold, so that only after a numerical evaluation do we know whether Equations (4.28) and (4.29) represent the optimal trajectories. Surprisingly, if neither of the conditions are met, then the trajectories represent those corresponding to a maximum, rather than a minimum, in *I*. In that case, the original problem of minimizing *I* would remain unsolved. Inequality constraints involving the control variable would have to be imposed. Then we would be in the realm of bang-bang and similar types of controls, in which optimal levels of government expenditure would be at the extreme, permissable limits.

If G_j^* satisfies either of Equations (4.30) or (4.31), then it is substituted into Equations (4.28) and (4.29) to give explicit solutions for $u_j(t)$ and $y_j(t)$. An interesting qualitative property follows:

Result 4.5 If Assumptions 4.1 and 4.4 hold and if either Equations (4.30) or (4.31) are true, then the optimal trajectory of per capita regional government spending is constant, if and only if

$$1-\mu_j-\alpha_j=0 \quad \text{(which is unlikely to occur).} \qquad (4.32)$$

Otherwise the trajectory is monotonic.

Whereas the optimal time path of per capita government spending is generally monotonic, the per capita income trajectory may not be. This is clear from Equation (4:29), in which there may be both positive and negative exponential terms. This equation also reaffirms that the effects of increased taxation are inconclusive without resorting to a numerical analysis.

4.4 NUMERICAL EXAMPLES

In the previous section the optimal levels of government spending and income were functions of a constant of integration (see Result 4.4). Only after we determine its value are we assured that the Hamiltonian is maximized, and the optimal solution found. Also, since we could not express the explicit functional form of the constant; we could not determine how it is affected by variations in the parameters. Consequently, general sensitivity properties of the model were not apparent.

In this section we summarize some numerical examples of the results derived in the previous section. This allows us to inspect the nature of particular optimal trajectories, and to observe how they are affected by variations in the parameter values.

The given input is listed in Table 4.1. We adopt a two-region situation in which each region has the same population. The initial income levels are identical; but at the end of the five year planning horizon, Region 1's per capita income level increases one third, while Region 2's remains unchanged. Interregional trade is only between the two regions, so that $\varepsilon_1 = -\varepsilon_2$. At year 0, government spending and taxation amount to \$1,000 per person; and per capita taxes increase exponentially over time.

The problem is to determine the optimal levels of per capita government spending over time. Since government spending affects income levels, these are derived as well. The solution is obtained by substituting the numerical input into Equation (4.29), setting $t=\tau=5$, solving for G_j^* [1], and then substituting this value into Equations (4.28) and (4.29). Results for different values of μ_j, ρ_j, and α_j are stated in Tables 4.2 and 4.3, and are graphed in Figures 4.1 and 4.2.

In Region 1, the income profiles are strikingly similar, (see Figure 4.1(b)), even with significant variations in consumption, productivity, and capital depreciation rates. Indeed, the trajectories here are very similar to those in Chapter 3, which were generated by a different model. This suggests that optimally planned economies can have very similar income trajectories, even though they may differ in their economic or political makeup.

Although the income levels remain relatively stable from case to case, government expenditures vary significantly. Situations that have either lower personal consumption or lower capital depreciation rates require higher levels of government spending, and results in higher income levels. On the other hand, with a lower output-capital ratio, there is less government spending; and the corresponding income level is relatively lower to begin with, but eventually

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[1] In these examples, we used a Taylor series expansion to obtain a polynomial in G_j^*. As a result, our solutions are approximate; but they appear to be fairly accurate since $y_j(\tau) \approx y_j(\tau)$ as required.

TABLE 4.1. INPUT FOR NUMERICAL EXAMPLES

PARAMETER	VALUE
τ	5.
$y_1(0)^*$ $y_2(0)^*$	6000.
$y_1(\tau)^*$	8000.
$y_2(\tau)^*$	6000.
ε_1	-500.
ε_2	500.
$u_1(0)^*, u_2(0)^*$	1000.
χ_1, χ_2	1000.
ψ_1, ψ_2	0.01
ζ_1, ζ_2	1.
μ_j, ρ_j, α_j	Varied. See Tables 4.2 and 4.3

TABLE 4.2. REGION 1 VALUES OF PER CAPITA GOVERNMENT SPENDING, $u_1(t)$; PER CAPITA INCOME, $y_1(t)$; AND GOVERNMENT SPENDING AS A PERCENTAGE OF INCOME, $u_1(t)/y_1(t)$. VALUES OF THE MARGINAL PROPENSITY TO CONSUME, μ_1, OUTPUT-CAPITAL RATIO, ρ_1, AND CAPITAL CONSUMPTION RATE, α_1, ARE VARIED.

μ_1	ρ_1	α_1	t	$u_1(t)$	$y_1(t)$	$u_1(t)/y_1(t)$ x 100%
.4	.4	.5	1	913	6240	14.6
			2	831	6560	12.7
			3	753	6961	10.8
			4	678	7440	9.1
			5	606	8000	7.6
.4	.3	.5	1	790	6217	12.7
			2	619	6548	9.5
			3	478	6968	6.9
			4	364	7459	4.9
			5	270	8006	3.4
.3	.3	.5	1	975	6317	15.4
			2	948	6672	14.2
			3	921	7068	13.0
			4	893	7509	11.9
			5	864	8000	10.8
.4	.3	.4	1	994	6343	15.7
			2	988	6712	14.7
			3	981	7109	13.8
			4	974	7538	12.9
			5	966	8000	12.1

TABLE 4.3. REGION 2 VALUES OF PER CAPITA GOVERNMENT SPENDING, $u_2(t)$; PER CAPITA INCOME, $y_2(t)$; AND GOVERNMENT SPENDING AS A PERCENTAGE OF INCOME, $u_2(t)/y_2(t)$. VALUES OF THE MARGINAL PROPENSITY TO CONSUME, μ_2, OUTPUT-CAPITAL RATIO, ρ_2, AND CAPITAL CONSUMPTION RATE, α_2, ARE VARIED.

μ_2	ρ_2	α_2	t	$u_2(t)$	$y_2(t)$	$u_2(t)/y_2(t)$ x 100%
.4	.4	.5	1	886	5843	15.2
			2	780	5770	13.5
			3	683	5774	11.8
			4	593	5852	10.1
			5	510	6000	8.5
.4	.3	.5	1	883	5884	15.0
			2	776	5832	13.3
			3	679	5837	11.6
			4	590	5895	10.0
			5	510	6001	8.5

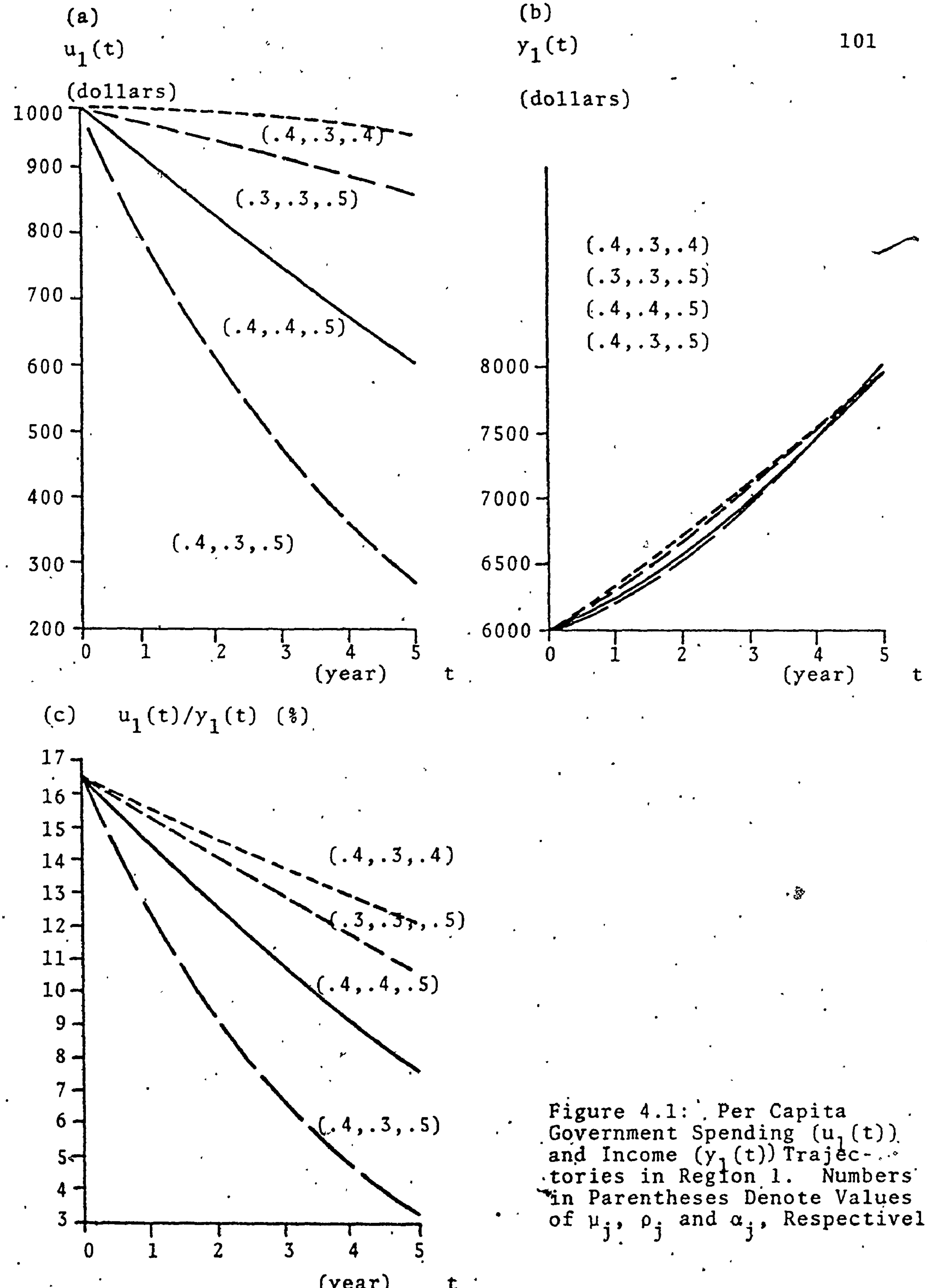

Figure 4.1: Per Capita Government Spending ($u_1(t)$) and Income ($y_1(t)$) Trajectories in Region 1. Numbers in Parentheses Denote Values of μ_j, ρ_j and α_j, Respectively.

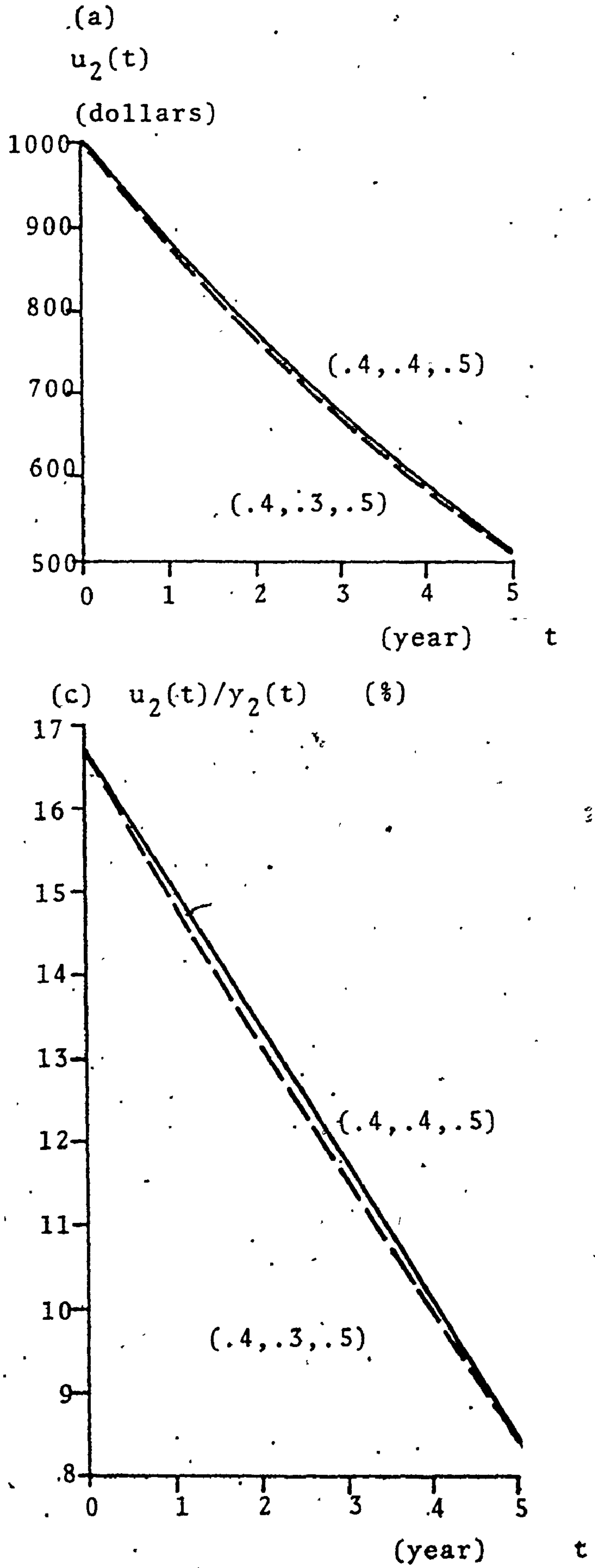

Figure 4.2: Per Capita Government Spending ($u_2(t)$) and Income ($y_2(t)$) Trajectories in Region 2. Numbers in Parentheses Denote Values of μ_j, ρ_j and α_j, Respectively.

exceeds that of the higher output-capital ratio case.

In the more unusual no-growth scenario, represented by Region 2, the income trajectories are convex (Figure 4.2(b)). Income levels dip below the initial level before recovering at the end of the planning interval. Concave trajectories, such as those in Chapter 3, do not arise here; but in general we cannot rule out the possibility of their occurrence.

Just as in Region 1, if Region 2 has a lower output-capital ratio, then less government spending is required. But in this case, the corresponding income levels are greater! Thus, there is no straightforward relationship between government spending and regional income levels. This not only makes the model more interesting, but realistic as well.

In several cases (that is, for certain values of μ_j, ρ_j, and α_j) optimal solutions could not be found since neither Equations (4.30) nor (4.31) were satisfied (see Table 4.4). This suggests that if these hypothetical scenarios are at all realistic, then there will be many instances in which optimal solutions do not exist. When faced with such a situation, governments can introduce additional constraints into the model, which alter the space of feasible solutions and which then allow an optimal solution.

For example, we can set limits on government spending: a lower limit so as to ensure a minimum level of public services, and an upper limit so as to alleviate inflationary pressures. Such a strategy may not only be necessary to ensure an optimal solution, it is also likely desirable. Indeed, in our numerical examples, the optimal per capita government spending trajectories all exhibit sharp declines to what would normally be unacceptable levels.

TABLE 4.4. LISTING OF CASES IN WHICH OPTIMAL SOLUTION FOUND, AND THOSE IN WHICH OPTIMAL SOLUTION DOES NOT EXIST.

μ_j	ρ_j	α_j	Solution Exists for Region 1	Solution Exists for Region 2
.4	.4	.4	No	No
.3	.4	.4	No	No
.4	.3	.4	Yes	No
.3	.3	.4	No	No
.4	.4	.5	Yes	Yes
.3	.4	.5	No	No
.4	.3	.5	Yes	Yes
.3	.3	.5	Yes	No

In some instances, optimal policies exist for one region, but not for the other. Solutions for only two of the four cases were derived for Region 2. Yet, in the model, this does not undermine Region 1's policies, since its economy does not explicitly interact with that of Region 2 - aspects such as interregional trade are exogenous and fixed. This underscores the weakness in models that are not explicitly interregional; and a very desirable extension of our model would have one region's economy and government policies explicitly related to those of other regions.

4.5 CONCLUDING COMMENTS

In the previous chapter governments were free to choose any combination of policies that had the desired impact on income growth. This impact on income growth was the control variable, $u_j(t)$, defined as the rate of change in income growth due to new or altered government policies. The value of $u_j(t)$ was a function of the consumption, productivity and other characteristics of the economy at the beginning of the planning period; and although not necessary, these parameters likely changed as government policies were introduced. However, we did not know whether these policies reflected a more liberal position with increased public spending, a more conservative stance, or simply a re-allocation of resources.

In this chapter the control variable is per capita regional government spending. This is more explicit than the control variable of Chapter 3. There, we found the extent to which regional economies had to be stimulated or possibly controlled, but we had no inkling of what policies would lead to such changes. The use of government expenditures still has a significant drawback, however, in that all spending is assumed to have the same effect on a region's economy, regardless of whether it is spent on health, education, welfare, highways, or other infrastructure. A more comprehensive version of the present model should certainly have a disaggregate public sector. However, that must wait until we have estimates of the impacts of different types of government spending.

Disaggregation along other lines is desirable as well; and would include output by industry group, wages, prices, retail sales, labour, and monetary factors. With regard to these more detailed studies, the results of our analysis lead us to expect that yearly industrial production can be predicted with reasonable accuracy, so long as governments are commited to reaching certain production targets at the end of the planning horizon. However, such a goal will mean that governments will have to be extremely judicious. This is because our numerical examples indicate that optimal government policies will be extremely sensitive to personal consumption habits, capital depreciation, and productivity. In addition, because solutions to certain problems (that is, models) may not exist, governments must be ready to be more explicit in specifying goals and in identifying feasible or acceptable solutions.

CHAPTER 5

GUIDING AN INTERREGIONAL SYSTEM IN WHICH INCOME AND POPULATION ARE INTERDEPENDENT

5.1 INTRODUCTION

The population patterns in Chapter 3 arose from a situation in which the labour force was sensitive to regional income differentials. Migrants were attracted to prosperous regions that had higher per capita income levels. But although in-migration was a consequence of economic growth, it did not affect that growth. This is a limited view of the interaction between population growth and economic development, because it ignores the economic stimulus provided by an expanding labour force. Some of the in-migrants will be highly-skilled and/or professional workers, who will contribute to the educational and technological development of the region. To some extent, the larger labour force results in a more competitive labour market, so that more qualified people fill the jobs and productivity is increased. This does not necessarily lead to greater unemployment (and hence reduced per capita income levels) since the multiplier concept indicates the development of support services and industries, as well as an expanded consumer market.

In this chapter, we explore the use of government policies to guide a system of regions in which economic growth and population are interdependent. Having this interdependence is very much in the spirit of Greenwood (1973), Muth (1968, 1971), Okun (1968), Olvey (1970), and Persky and Kain (1970), who have argued that a feedback process exists between migration and income growth. However, our model goes beyond this previous work by introducing an endogenous policy-related component. This component represents the influence that governments can have on income and population growth through the creation of new or altered policies.

In the next section, we formulate a simple model that depicts the interdependence between economic and population growth. Next, numerical results are presented for a two-region example. The chapter ends with some concluding commments.

5.2 CONSTRUCTION OF OPTIMAL INCOME AND POPULATION DISTRIBUTIONS

In this section, we give a construction for determining the optimal income and population distributions, and the magnitude of government intervention necessary to achieve these distributions. Of course, the notion of optimality is meaningless without the specification of goals, as we have noted before.

As in previous chapters we assume a finite planning horizon, at the end of which we have a certain interregional distribution of income. Governments may wish to specify income targets in order to reduce regional disparities within a country. The public sector represents the people of the country, and has a responsibility to ensure that all people and all regions have a fair share of the nation's wealth. Increasing equity may, in the long run, increase efficiency as well. Funds channelled to poorer regions to support short term development may help to sustain longer term growth. In that case, the long run tax burden on the more prosperous regions may be reduced, and lead to a more vibrant business environment.

In addition to economic concerns per se, we also introduce population targets. These can represent attempts to decentralize and/or to ensure that there exists a labour force compatible with the anticipated industrial structures of the regions. Both the population and income targets are not meant to be inflexible objectives, to be reached at all costs. In reality, no such specific goals are likely to be stated. Rather, the practical usefulness of these types of models lies in their evaluation of alternative scenarios. We can consider a range of possible income and population targets, some of which may be more desirable than others. Then, for each set of targets, we can determine the optimal solution. This gives a distribution of solutions, one for each scenario; and under conditions of political uncertainty or risk aversion, we can select that optimal trajectory which is the weighted mean or the mode of all the trajectories corresponding to the different scenarios.

In addition to these targets, governments would also be concerned about the costs of the policies that were designed to achieve these objectives. The costs associated with population programmes would include the tremendous administrative expenses, the costs of an advertising campaign, and possibly state-subsidized medical expenses for contraceptives or surgery. The costs associated with economic policies include not only the administrative costs, but the possible negative repercussions of these programmes. Policies that stimulate regional growth, such as an extensive job-creation programme or urban revitalization projects, tend to encourage inflationary trends. On the other hand, restrictive policies, that contain inflation but allow only moderate economic growth, eat away at both consumer and business confidence. For example, a restrictive monetary policy with high interest rates will reduce consumer borrowing to purchase goods and business borrowing for capital investment. As a result, the region may experience stagnation in the long run - a policy designed to moderate growth in the short term may be a leash on longer term development.

Thus far, we have outlined two government objectives: attaining income and population targets, and minimizing the costs and negative impacts of government programmes. These are stated formally in Assumptions 5.1 and 5.2, respectively.

<u>Assumption 5.1</u> A country consists of a finite number of regions $j=1,\ldots,J$. An initial time period, $t=0$, and a planning horizon, $t=\tau$, are specified. The population and per capita income of each region attain the following target levels at $t=\tau$:

$$x_j(\tau) = x_j(\tau)^* \tag{5.1}$$

$$y_j(\tau) = y_j(\tau)^* \quad , \tag{5.2}$$

where $x_j(t)$ is the population of region j at time t

$y_j(t)$ is the per capita income per unit time of region j at time t.

<u>Assumption 5.2</u> Government policies are selected such that their total costs and negative impacts are minimized. The total costs and negative impacts are represented by:

$$I = \int_0^\tau \sum_j [\zeta_{(1)j} u_j(t)^2 + \zeta_{(2)j} v_j(t)^2] \, dt, \tag{5.3}$$

where $u_j(t)$ is the change in region j's per capita income growth rate at time t resulting from new government policies (see Assumption 5.3)

$v_j(t)$ is the change in population growth rate at time t resulting from new government policies

$\zeta_{(1)j}$, $\zeta_{(2)j}$ are unit cost parameters.

In Assumption 5.1, $x_j(\tau)^*$ and $y_j(\tau)^*$ are the exogenous population and income targets. In Assumption 5.2, $u_j(t)$ and $v_j(t)$ are termed the control variables. These are not specific government instruments themselves, but rather reprethe impacts of such instruments on income and population growth rates, respectively. If $u_j(t)$ or $v_j(t)$ have either large positive or negative values, then this discloses the need for intense political reform. New legislation will involve high administrative costs, and can have negative impacts, such as inflationary pressures, reduced allocation of funds to other public services, or a subdued business environment. This is not to say that present policies do not carry any such costs or risks. Rather, this reflects an inclination to perpetuate existing policies, and a concession to practical limitations in effecting vast changes. Any legislative proposals that will involve large changes in $\dot{x}_j(t)$ or $\dot{y}_j(t)$ (that is large positive or negative values of $v_j(t)$ or $u_j(t)$) will be scrutinized through arduous and prolonged debate - for often, they challenge policies that have already been submitted to such scrutiny, and that have passed that test. These are the reasons why $u_j(t)^2$ and $v_j(t)^2$ are used in computing the total costs, rather than absolute values. Since $u_j(t)^2$ and $v_j(t)^2$ have different units of measurement, $\zeta_{(1)j}$ and $\zeta_{(2)j}$ standardize for units, and allow different priorities among regions. Total negative impacts are then simply the sum of those related to each region, over the whole time horizon.

Having defined the government's objectives in Assumptions 5.1 and 5.2, we now specify the interaction among regional per capita income, interregional migration, and population distribution; and the effect of the control variables on income and population growth.

Assumption 5.3 The control variables $u_j(t)$ and $v_j(t)$ are defined as the changes in $\dot{y}_j(t)$ and $\dot{x}_j(t)$, respectively, that are a result of new government policies. Past trends in $\dot{y}_j(t)$ and $\dot{x}_j(t)$ have been estimated exogenously, and the parameters are assumed to be stationary over the time horizon. From this the temporal interrelationships among population, income, migration, and the control variables take the following form:

$$\dot{x}_j(t) = \sigma_{(1)j} + \phi_j x_j(t) + \gamma_j \sum_{i \neq j} [\Gamma_i^* n_{ij}(t) - \Gamma_j^* n_{ji}(t)] + v_j(t) \quad (5.4)$$

$$\dot{y}_j(t) = \sigma_{(2)j} + \nu_j y_j(t) + \pi_j \sum_{i \neq j} [\Gamma_i^* n_{ij}(t)/\Gamma_j^* - n_{ji}(t)] + u_j(t) \quad (5.5)$$

$$n_{ij}(t) = \sigma_{(3)} + \eta\, [y_j(t) - y_i(t)] - \beta \cdot f^*(d_{ij}), \quad i \neq j \quad (5.6)$$

where $n_{ij}(t)$ is the flow of migrants from region i to region j, per unit time, at time t, divided by the population of i at time t

$f^*(d_{ij})$ is the distance decay function, d_{ij} being the distance between regions i and j

Γ_j^* is a given constant

$\sigma_{(1)j}$, $\sigma_{(2)j}$, $\sigma_{(3)}$, ϕ_j, γ_j, ν_j, π_j, β are parameters.

Unlike the model in Chapter 3, regional demographic and economic attributes are treated concomitantly here. The rate of population change, $\dot{x}_j(t)$, is a function of population size, migration from other regions, and a control variable; and we assume that the parameters will remain relatively stable over the planning interval.

We are primarily concerned with a closed interregional system, so that we have omitted a net immigration term in Equation (5.4). However, its effects on population growth are reflected in the values of $\sigma_{(1)j}$, ϕ_j, and γ_j.

Net natural increase is reflected in ϕ_j, so that population increases tend to be greater in more populated regions.

The constant Γ_i^* is given. If we let $\Gamma_i^*=x_i(0)^*$, then $\Gamma_i^* n_{ij}(t)$ is an approximation of $x_i(t)n_{ij}(t)$, the number of migrants from i to j at time t; and it follows that $\sum_{i\neq j} [\Gamma_i^* n_{ij}(t)-\Gamma_j^* n_{ji}(t)]$ is an approximation of the net migration into region j from all other regions (within the country) at time t. For small values of t, the approximation will be reasonable if the estimate of γ_j has a small standard error.

We have been assuming that past trends will give a reasonable forecast of future population levels; and of course, it is also assumed that we know no other information that would lead us to believe otherwise. Often, these forecasts indicate a population distribution in which some regions are overcrowded, with ghetto areas, traffic congestion and air pollution; and other areas have low population. In the latter case, these regions will not have a sufficient size market to support tertiary activities, nor an adequate pool of skilled labour to attract new industry. As a result, we have assumed the existence of population targets in Assumption 5.1; and to ensure that these are met, the government can institute various programmes. These might include state funding for contraceptive devices or for surgery, advertising campaigns or financial inducements for new immigrants, or even stringent controls on where new immigrants can locate. The optimal trajectory of $v_j(t)$ indicates the extent to which the government will have to initiate such programmes in order to meet its population objectives.

Thusfar, we have discussed the population growth equation, Equation (5.4). In that discussion, we alluded to the role of recent arrivals in the labour force helping to stimulate a regional economy. There are exceptions. For example, Friedlander (1965) concluded that emigration (most of which was out-migration to other regions within the United States) was beneficial to Puerto Rico's economy. Usually however, and especially in developed economies, one expects that in-migration would have a positive influence on a region's economy. This hypothesis is supported by Okun and Richardson (1961), Okun (1968), and Greenwood (1973), who estimated that regional in-migration tended to increase regional economic growth rates within the United States.

Equation (5.5) reflects these findings. The rate of change in regional per capita income, $\dot{y}_j(t)$, is a function of the net regional in-migration rate, $\sum_{i=j} [\Gamma_i^* n_{ij}(t)/\Gamma_j^* - n_{ji}(t)]$, where we set $\Gamma_i^* = x_i(0)^*$. As before, this is an approximation; and the discussion in the context of Equation (5.4) is pertinent here as well.

The assumption that $\dot{y}_j(t)$ is a function of $y_j(t)$ means that the rich regions become richer, other things being equal. This is similar to Myrdal's (1957) cumulative causation effect. The industrial structure of a regional economy largely determines its growth characteristics. Hence, a region's industrial makeup is symbolized by the magnitude of ν_j.

Providing a basis for our projections of future income levels are the parameters $\sigma_{(2)j}$, ν_j, and π_j. If indications are that the targets, $y_j(\tau)^*$, will not be achieved, then the government will launch new policies to ensure the attainment of these goals. How extensive the policy changes must be will be determined by the value of $u_j(t)$. A large (positive or negative) value of $u_j(t)$ means that past policies are not at all consistent with current goals; and suggests a re-allocation of transfer payments from the federal to regional governments, extraordinary grants to particular industries or firms, selective taxing schemes which would serve as incentives/disincentives for future industrial development, local public works grants, and/or public employment programmes. Hopefully, such policies will have a significant impact on employment, gross regional product, and average income levels.

Equation (5.6) makes clear the circularity between the demographic and the economic aspects. Just as in-migrants tend to encourage regional growth (Equation 5.5), a region's prosperity is a significant factor in migrants' decisions to move. Equation (5.6) depicts a situation in which out-migration rates are greater from poorer to richer regions. Indeed, net migration favours the richer region as the per capita income differential increases:

$$\partial[x_i(t)n_{ij}(t)-x_j(t)n_{ji}(t)]/\partial[y_j(t)-y_i(t)] \qquad (5.7)$$

$$= \eta\ (x_i(t)+x_j(t)) \text{ , from Equation (5.6).}$$

However, Equation (5.7) also has the peculiar property that the poorer of two regions can in fact experience the positive net migration. This is because:

$$x_i(t)n_{ij}(t) - x_j(t)n_{ji}(t)$$

$$= (x_i(t)-x_j(t))\ (\sigma_{(3)} - f^*(d_{ij})) \qquad (5.8)$$

$$+ (x_i(t)+x_j(t))\ \eta\ (y_j(t)-y_i(t)) \text{ ,}$$

assuming $d_{ij}=d_{ji}$, for simplicity.

In Equation (5.8), $[x_i(t)n_{ij}(t)-x_j(t)n_{ji}(t)]$ can be positive even though $y_j(t)>y_i(t)$; but this will only occur when $x_i(t)$ is extremely small relative to $x_j(t)$. Other contributing factors in this case would be the relative magnitudes of $\sigma_{(3)}$, β and η; and even though Equation (5.8) allows the possibility of net in-migration into poorer regions, we think this unlikely to happen with reasonable values of $\sigma_{(3)}$, β, η, $x_i(t)$ and $x_j(t)$. In that case, Equation (5.8) would be consistent with empirical evidence which shows that regions with higher per capita income experience greater net migration (Stone, 1969; Lowry, 1966; Okun, 1968; Greenwood, 1973).

The other significant factor affecting the migration rate is the distance separating the two regions. This effect is represented by the friction of distance parameter β, and distance function $f^*(d_{ij})$. The distance function allows a more flexible interpretation of distance - perceived distance, moving costs, distance between largest cities in each region, distance between the population centroids, or some other distance measure.

Since we are concerned with the government's influences on economic development and population growth, it would be advantageous to include a control variable in the regional migration equation. This would allow the government to encourage moves from one region to another, by offering tax breaks to certain interregional migrants. As it stands in the model, the government's influence on regional population patterns appears limited to immigration policies.

A second major drawback in the model is that income growth is stimulated by regional in-migrants, but not by immigrants. Immigrants, as well as those already in the country, help satisfy the labour demands of industry, and also become part of the consumer market. Extensions of the present model should incorporate these factors; but for now, we limit ourselves to an analysis based on Assumptions 5.1-5.3.

These assumptions form an optimal control problem with quadratic performance index and linear dynamic system. This is one of the forms that has received considerable attention in the control literature; and we now outline a construction for the solution of this problem:

$$\underset{\underline{u}(t)}{\text{Min}} \quad I = \int_0^\tau [\sum_j \zeta_{(1)j} u_j(t)^2 + \zeta_{(2)j} v_j(t)^2]\, dt, \tag{5.3}$$

$$\text{s.t.} \quad \dot{x}_j(t) = \sigma_{(1)j} + \phi_j x_j(t) + \gamma_j \sum_{i \neq j} [\Gamma_i^* n_{ij}(t) - \Gamma_j^* n_{ji}(t)] + v_j(t) \tag{5.4}$$

$$\dot{y}_j(t) = \sigma_{(2)j} + \nu_j y_j(t) + \pi_j \sum_{i \neq j} [\Gamma_i^* n_{ij}(t) / \Gamma_j^* - n_{ji}(t)] + u_j(t) \tag{5.5}$$

$$n_{ij}(t) = \sigma_{(3)} + \eta\,(y_j(t) - y_i(t)) - \beta f^*(d_{ij}), \quad i \neq j \tag{5.6}$$

$$x_j(0) = x_j(0)^*, \quad y_j(0) = y_j(0)^* \text{ are known,} \tag{5.7}$$

$$x_j(\tau) = x_j(\tau)^*, \quad y_j(\tau) = y_j(\tau)^* \text{ are given.} \tag{5.8}$$

When Equation (5.6) is substituted into Equations (5.4) and (5.5), we get, respectively:

$$\dot{x}_j(t) = \kappa_{(1)j} + \phi_j x_j(t) + \gamma_j \sum_{i \neq j} [(\Gamma_i^* + \Gamma_j^*) \ln (y_j(t) - y_i(t))] + v_j(t) \tag{5.9}$$

$$\dot{y}_j(t) = \kappa_{(2)j} + \kappa_{(3)j} y_j(t) - \pi_j \sum_{i \neq j} [(\Gamma_i^* / \Gamma_j^* + 1) \ln y_i(t)] + u_j(t) \ , \tag{5.10}$$

where $\kappa_{(1)j} = \sigma_{(1)j} + \gamma_j \sum_{i \neq j} [(\Gamma_i^* - \Gamma_j^*) (\sigma_{(3)} - \beta f^*(d_{ij}))]$ $\quad i \neq j \quad$ (5.11)

$$\kappa_{(2)j} = \sigma_{(2)j} + \pi_j \sum_{i \neq j} [(\Gamma_i^* / \Gamma_j^* - 1) (\sigma_{(3)} - \beta f^*(d_{ij}))], \quad i \neq j \tag{5.12}$$

$$\kappa_{(3)j} = \nu_j + \pi_j \sum_{i \neq j} (\Gamma_i^* / \Gamma_j^* + 1) \ln \ldots \tag{5.13}$$

Use Equations (5.3), (5.9) and (5.10) to form the Hamiltonian:

$$H = - \sum_j [\zeta_{(1)j} u_j(t)^2 + \zeta_{(2)j} v_j(t)^2] + \sum_j p_{(1)j}(t) [\kappa_{(1)j} + \phi_j x_j(t) + \gamma_j \sum_{i \neq j} [(\Gamma_i^* + \Gamma_j^*) \ln y_j(t)] - \gamma_j \sum_{i \neq j} [(\Gamma_i^* + \Gamma_j^*) \ln y_i(t)] + v_j(t)]$$

$$+ \sum_j p_{(2)j}(t) \; [\kappa_{(2)j} + \kappa_{(3)j} \, y_j(t) - \pi_j$$

$$\sum_{i \neq j} [(\Gamma_i^*/\Gamma_j^*+1) \; \eta \; y_i(t)] + u_j(t)] \; . \qquad (5.14)$$

One set of necessary conditions for I to be a minimum is that:

$$- \frac{\partial H}{\partial x_j(t)} = - p_{(1)j}(t) \; \phi_j = \dot{p}_{(1)j}(t) \qquad (5.15)$$

and

$$- \frac{\partial H}{\partial y_j(t)} = - p_{(1)j}(t) \; \gamma_j \sum_{i \neq j} (\Gamma_i^* + \Gamma_j^*) \; \eta \; +$$

$$\sum_{i \neq j} [p_{(1)i}(t) \; \gamma_i \; (\Gamma_j^* + \Gamma_i^*)] \; \eta \; -$$

$$p_{(2)j}(t) \; \kappa_{(3)j} + \sum_{i \neq j} [p_{(2)i}(t) \; \pi_i$$

$$(\Gamma_j^*/\Gamma_i^*+1)] \; \eta \qquad (5.16)$$

$$= \dot{p}_{(2)j}(t)$$

The other set of necessary conditions for I to be minimized is that H is maximized:

$$\frac{\partial H}{\partial u_j(t)} = - 2\zeta_{(1)j} u_j(t) + p_{(2)j}(t) = 0 \qquad (5.17)$$

$$\frac{\partial H}{\partial v_j(t)} = - 2\zeta_{(2)j} v_j(t) + p_{(1)j}(t) = 0 \; , \qquad (5.18)$$

which imply that

$$u_j(t) = p_{(2)j}(t) \;/\; 2\zeta_{(1)j} \tag{5.19}$$

and $$v_j(t) = p_{(1)j}(t) \;/\; 2\zeta_{(2)j} \tag{5.20}$$

Equations (5.15) and (5.16) give a system of linear first order differential equations for $p_{(1)j}(t)$ and $p_{(2)j}(t)$, the solution of which has 2J constants of integration. The expressions for $p_{(1)j}(t)$ and $p_{(2)j}(t)$ can then be substituted into Equations (5.19) and (5.20), which in turn are substituted into Equations (5.9) and (5.10). Solving this resulting system of differential equations for $x_j(t)$ and $y_j(t)$ results in 4J constants of integration (2J from before). The initial values, $x_j(0)^*$ and $y_j(0)^*$, and the boundary conditions, $x_j(\tau)^*$ and $y_j(\tau)^*$, can then be utilized to determine the constants of integration. Substituting these values where appropriate produces explicit expressions for $x_j(t)$, $y_j(t)$, $v_j(t)$ and $u_j(t)$. A two-region example is given in the following section.

5.3 TWO-REGION EXAMPLE

Assumptions 5.1-5.3 have defined a problem in which we must determine the magnitude of public policies that will be necessary to achieve the stated economic and demographic goals of the government. In this section, we present the solution for a hypothetical two-region system, defined in Assumption 5.4.

<u>Assumption 5.4</u> The country consists of two regions, and the constants and parameters have the following values:

$$\zeta_{(1)j} = 1x10^6 \ , \ \zeta_{(2)j} = 1 \qquad j=1,2 \qquad (5.21)$$

$$\sigma_{(1)j}, \sigma_{(2)j} = 0 \ , \qquad j=1,2 \qquad (5.22)$$

$$\phi_j = 2x10^{-2} \ , \qquad j=1,2 \qquad (5.23)$$

$$\gamma_j = 1 \ , \qquad j=1,2 \qquad (5.24)$$

$$\nu_j = 5x10^{-2} \ , \qquad j=1,2 \qquad (5.25)$$

$$x_1(0)^* = \Gamma_1^* = 2x10^7 ; \ x_2(0)^* = \Gamma_2^* = 1x10^7 \quad \text{(people)} \qquad (5.26)$$

$$\tau=5 \qquad \text{(years)} \qquad (5.27)$$

$$x_1(\tau)^* = 2.5x10^7 ; \ x_2(\tau)^* = 1.5x10^7 \quad \text{(people)} \qquad (5.28)$$

$$y_1(0)^* = y_2(0)^* = 6x10^3 \qquad \text{(dollars)} \qquad (5.29)$$

$$y_1(\tau)^* = 8x10^3 ; \ y_2(\tau)^* = 6x10^3 \quad \text{(dollars)} \qquad (5.30)$$

$$\sigma_{(3)} = 8x10^{-3} \qquad (5.31)$$

$$\eta = 4x10^{-6} \qquad (5.32)$$

$$\beta = 1.5\text{x}10^{-6} \tag{5.33}$$

$$\pi_j = 1\text{x}10^2 \tag{5.34}$$

$$f^*(d_{ij}) = d_{ij} = 1\text{x}10^3 \qquad \text{(miles)} \quad . \tag{5.35}$$

When this input is used in the construction outlined in the previous section, we obtain the optimal trajectories for population control, income control, population, and per capita income.

<u>Result 5.1</u> If Assumptions 5.1-5.4 are true, then (see Appendix 5 for proof):

$$\begin{aligned} x_1(t) = 1\text{x}10^3 [&3.24998\text{x}10^3 + 3.284717\text{x}10^4 \exp(.02t) \\ &+ 1.71 \exp(.0518t) - 1.61\text{x}10^4 \\ &\exp(-.02t) + 2.8\exp(-.0518t) \\ &- 4.4\text{x}10^{-1}\exp(-.0836t) - 1.2 \\ &\exp(-.05t)] \end{aligned} \tag{5.36}$$

$$\begin{aligned} x_2(t) = 1\text{x}10^3 [&-3.25002\text{x}10^3 + 3.115283\text{x}10^4 \\ &\exp(.02t) - 1.71\text{x}10^3\exp(.0518t) \\ &- 1.79\text{x}10^4\exp(-.02t) - 2.8 \\ &\exp(-.0518t) + 4.4\text{x}10^{-1}\exp(-.0836t) \\ &+ 1.2\exp(-.05t)] \end{aligned} \tag{5.37}$$

$$y_1(t) = 1\text{x}10^3 \, [1.0\text{x}10^{-2} + 5.30\exp(.05t)$$
$$+ 1.51\exp(.0518t) + 1.97\exp(-.0836t) \qquad (5.38)$$
$$- 8.21\exp(-.0518t) + 5.43\exp(-.05t)]$$

$$y_2(t) = 1\text{x}10^3 \, [-1.\text{x}10^{-2} + 5.30\exp(.05t)$$
$$- 3.02\exp(.0518t) - 1.96\exp(-.0836t)$$
$$+ 8.52\exp(-.0518t) - 2.85\exp(-.05t)]$$

$$v_1(t) = 6.45\text{x}10^5 \, \exp(-.02t) \qquad (5.40)$$

$$v_2(t) = 7.15\text{x}10^5 \, \exp(-.02t) \qquad (5.41)$$

$$u_1(t) = 1\text{x}10^2 \, [-2.6418\exp(-.0836t)$$
$$+ 8.6568\exp(-.0518t) - 5.35 \qquad (5.42)$$
$$\exp(-.05t)]$$

$$u_2(t) = 1\text{x}10^2 \, [2.6418\exp(-.0836t)$$
$$- 8.6568 \exp(-.0518t) + \qquad (5.43)$$
$$2.675\exp(-.05t)]$$

The values of $x_j(t)$, $y_j(t)$, $v_j(t)$ and $u_j(t)$ are summarized in Table 5.1 for different values of t, and are graphed in Figures 5.1 and 5.2. Inspecting these results we see that the optimal income trajectories for both regions are very similar to those in Chapters 3 and 4 (compare Figures 5.1(a), 3.1(b), and 4.1(b); and Figures 5.2(a), 3.2(b), and 4.2(b));

TABLE 5.1 TRAJECTORIES FOR PER CAPITA INCOME, $y_j(t)$; POPULATION, $x_j(t)$; INCOME CONTROL, $u_j(t)$; AND POPULATION CONTROL, $v_j(t)$.

(a) Region 1

t	$y_1(t)$	$x_1(t)$	$u_1(t)$	$v_1(t)$
1	6354.	20980000.	-797.	632000.
2	6720.	21970000.	-797.	620000.
3	7110.	22970000.	-801.	607000.
4	7523.	23970000.	-806.	595000.
5	7961.*	24990000.*	-815.	584000.

(b) Region 2

t	$y_2(t)$	$x_2(t)$	$u_2(t)$	$v_2(t)$
1	5957.	10980000.	-325.	701000.
2	5942.	11970000.	-315.	687000.
3	5935.	12970000.	-305.	673000.
4	5937.	13970000.	-296.	660000.
5	5948.*	14980000.*	-286.	647000.

*Rounding errors in computation.

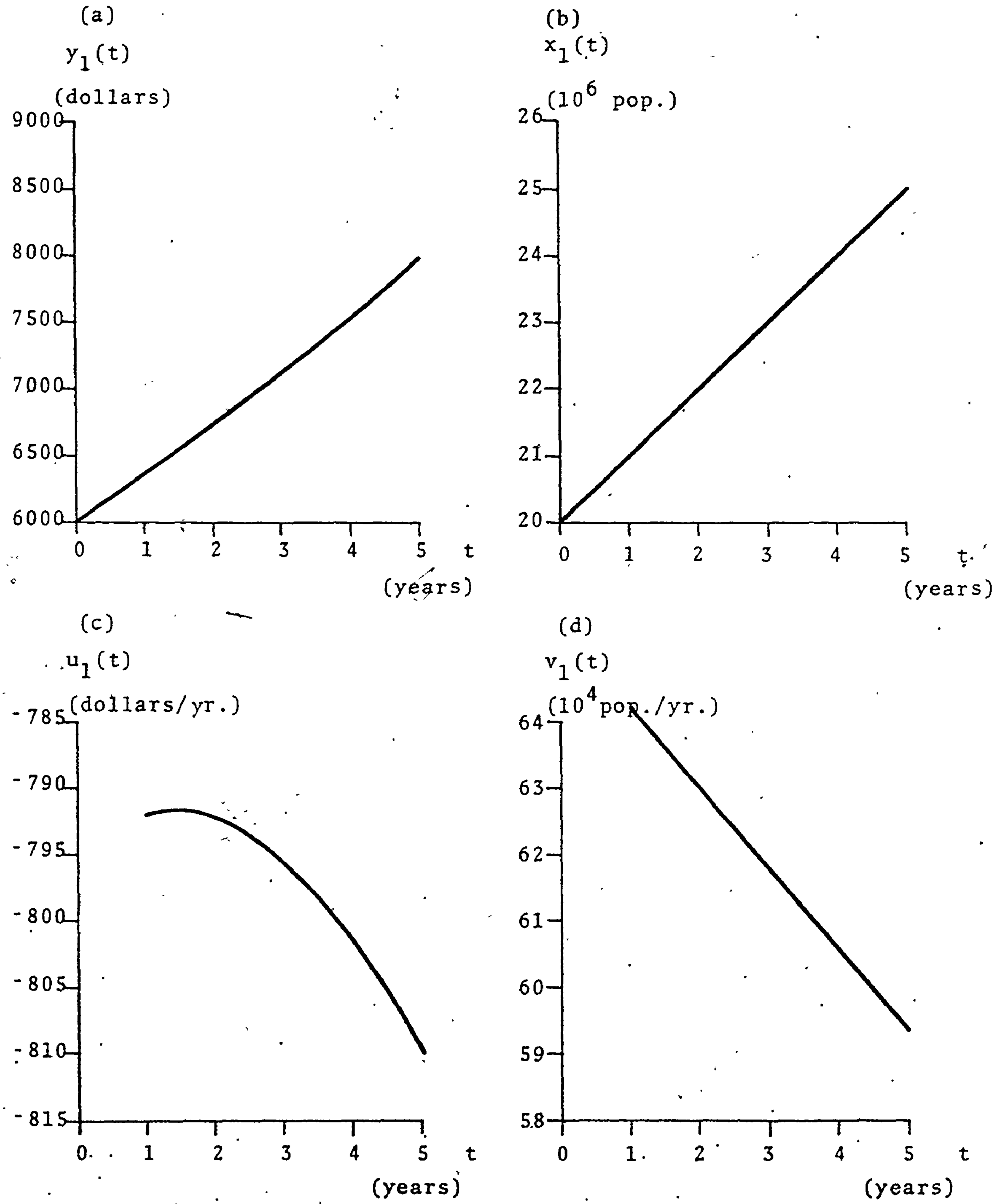

Figure 5.1: Plots of Per Capita, Income Population, Income Control, and Population Control for Region 1.

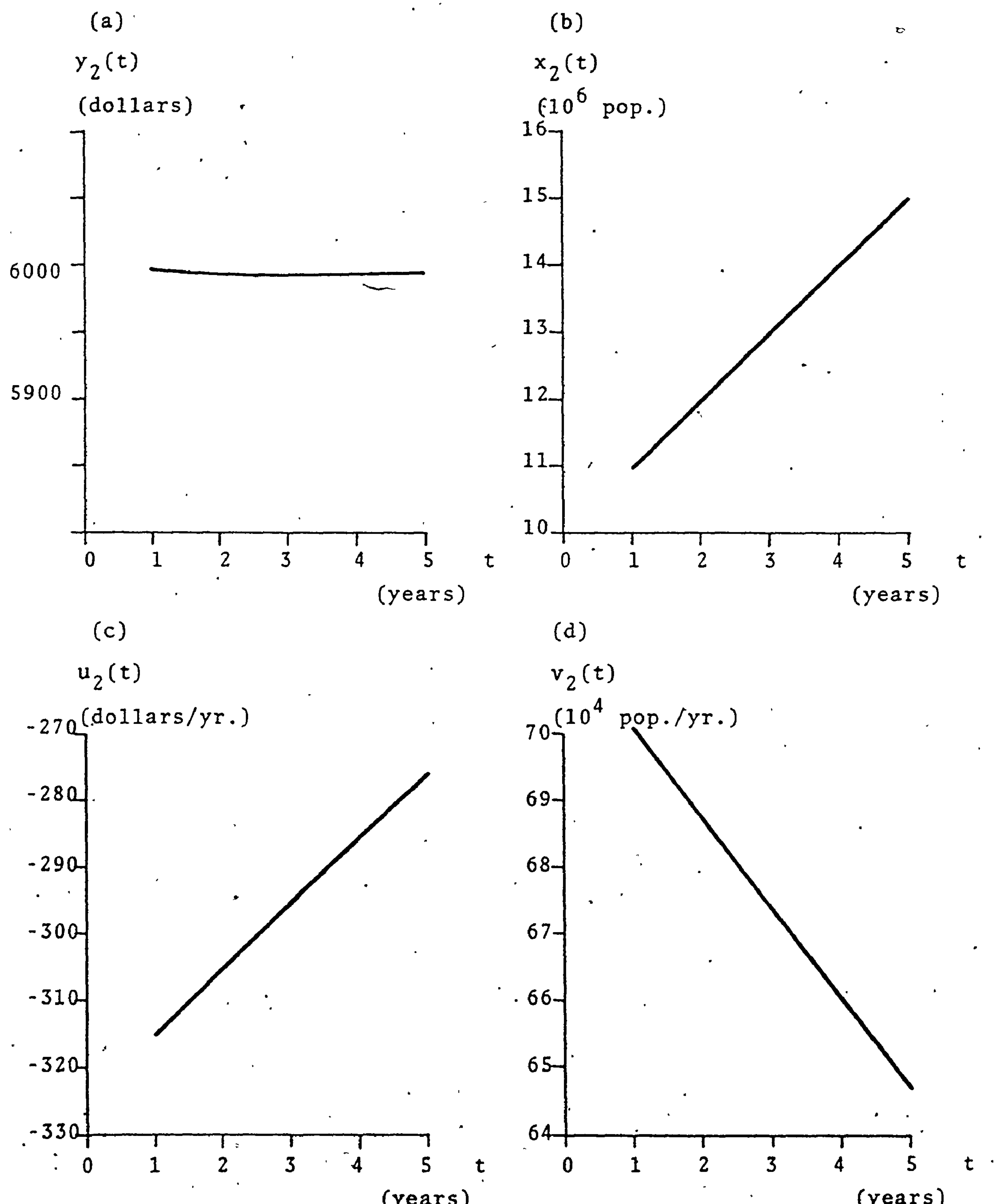

Figure 5.2: Plots of Per Capita Income, Population, Income Control, and Population Control for Region 2.

but the population trajectories differ significantly from those in Chapter 3 (contrast Figures 5.1(b) and 3.1(d), and 5.2(b) and 3.2(d).

In the per capita income trajectories the initial and target levels were identical to those in Chapters 3 and 4. But the income models were different, and in this chapter, regional per capita income levels were affected by regional migration of the labour force and their dependents. Previously, they were not. The similarity in the income trajectories among the different models was rather surprising then, especially in light of the fact that the population trajectories differed so radically. Part of the explanation for the divergence between the population trajectories is due to the different numerical inputs. However, the main reason is the population targets that were set in this chapter.

For Region 1, the income control variable remained fairly constant over the first two years, and then declined more steeply towards the end of the planning interval. For Region 2, it increased (became less negative) almost linearly over time. In Chapter 3, the income control trajectories were always constant; were of the same order of magnitude as in this chapter; but the values were usually greater (less negative). That is, the income controls in this chapter indicate the need for more restrictive economic policies when population targets are also of concern. When a government prescribes both income and population targets, then it is faced with the prospect of instituting rather extreme measures to ensure that both trajectories remain "on track".

Also, we notice that the government must attempt to increase population growth in both regions. This suggests a very "open" immigration policy, so as to encourage immigrants to settle in both regions.

5.4 CONCLUDING COMMENTS

In this chapter we have outlined a simple model in which:

(a) population growth rates are a function of population size, regional migration rates, a population control variable, and parameter estimates that reflect immigration and net natural increase;

(b) per capita income growth is a function of the región's prosperity, regional migration rates, an income control variable, and parameter estimates that reflect the regions' industrial structures and past fiscal and monetary policies;

(c) migration rates are sensitive to regional income differentials, and the distances separating regions; and

(d) the government specifies population and income targets, and attempts to minimize the costs and negative impacts of its policies.

Section 5.2 gave a construction for obtaining the optimal trajectories of the control variables. Numerical results for a two-region case indicated that the optimal per capita income paths had little curvature, and were very similar to those obtained in the previous chapters. However, the population paths differed significantly, since the model in this chapter imposed population targets. Inspecting the magnitudes of the control variables (which affect the rates of change in population and per capita income), and comparing them to the rates of change in the optimal population and income trajectories, we observe that the control variables are much greater (in absolute value). One ramification is clear - and that is that in order to have any chance of realizing its goals, the government will have to shift drastically its former economic and immigration policies. Whether any such changes will be successful in reality, or even politically feasible, is another question.

It is easy to criticize the simplicity of the economic aspects in the model. However, such criticisms will be buffered by precise parameter estimates. Then, simplicity will be an advantage, because of the parsimonious explanation of previously observed variations in the data.

Although we have expressed our attention to regional equity in terms of income targets, we should also be sensitive to regional income distributions as they develop during the planning period. This in fact is one of the major concerns in the next chapter.

CHAPTER 6

TRADE-OFFS AMONG CONFLICTING GOALS

6.1 ECONOMIC GROWTH AND REGIONAL DISPARITY

This chapter takes up one of the issues encountered in previous chapters. Specifically, it takes a direct approach to the problem of trade-offs among conflicting goals within an interregional system.

Rapid economic growth is a prime objective, not only at the national scale, but in regional economies as well. In previous chapters, economic growth objectives were represented by boundary constraints that define per capita regional output/income targets. In that situation, satisfactory economic growth is achieved if prespecified levels of per capita gross regional product are attained at the end of the planning horizon. One problem which could arise is that although final targets are met, production levels before the end of the planning time period are too low. For example, at the end of a five year time span regional output might have increased by 25%; but production for the first four years could have been extremely low, with a large spurt in production in the fifth and final year. This would be inferior to the situation in which the region experienced steady growth throughout; or better yet, the case where production levels are increased by

25% in the first year and maintained in subsequent years. In fact, if these output levels were measured using an appropriate price deflator, then there would also be less overall unemployment in these latter cases. Increased labour demands would stem from increased total production over the five year interval.

Although the goal of full employment is usually compatible with that of economic growth, a possibly conflicting goal is that of distributional equity in regional income levels. Interestingly, Myrdal (1957) and Williamson (1965), though differing in their long run assessments (Myrdal arguing that income levels will continue to diverge, Williamson claiming eventual convergence), are in agreement as to their predictions of short run regional growth imbalances. In any event, given the unpredictability of even short run economic developments, reassurances of long run convergence in regional income levels are but a trite consolation. Galbraith (1958) has argued that general economic growth does relatively little for those at the lower end of the social-economic ladder. From a geographical perspective, an appropriate corollary is that economic growth does not necessarily imply interregional income convergence. This feeling is evident in Canadian fiscal policy, as carried out by the Department of Regional Economic Expansion, in the form of transfer payments and other fiscal incentives in the Atlantic provinces and other less developed regions. Similarly, in the United States, revenue-sharing formulae have been constructed so as to promote growth in Appalachia and in the southern Sunbelt States; and federally constituted regional development commissions have been established to promote economic development in low income-high unemployment regions.

Previous chapters have been sensitive to the issue of regional disparities primarily through the boundary conditions, $y_j(\tau)^*$ - the same constraints which were thought to represent acceptable growth targets. Hence, these parameters mirror implicitly the dual goals of economic growth and distributional equity. As a result, the trade-offs between these goals and the resolution of any conflicts are determined exogenously. This is not to underplay the importance of such evaluations. Rather, it displays the normative stance taken by these models, and concedes that issues of equity and conflict resolution involve subjective, value-laden decisions, which cannot be accommodated within a deductive analysis. On the other hand, a deductive analysis uncovers the implications which follow logically from value-based assumptions.

Still, just as the use of only boundary conditions fails to consider income levels between the initial year and the planning horizon, it also fails to measure regional disparities within that time span. In addition, although the magnitudes of the regional income targets are based on both growth and equity considerations, it is impossible to distinguish completely the relative weights commanded by each. To eliminate these shortcomings, terms representing regional income levels and a measure of regional disparity are now introduced into the objective functional. In this way, levels of regional disparity and income growth at all time periods are as important as the final target levels. Further, separating income growth and distributional equity into two separate terms within a multi-objective expression allows the trade-offs between these objectives to be evaluated with a finer degree of resolution.

The purpose of the model is not to impart such an evaluation per se, but rather to uncover implications arising from alternative value judgments on the relative importance of different objectives. The model assumes that decisions emanating from value judgments can be depicted deterministically; that is, the parameters have particular values. Although in reality we may not be able to identify the relative importance of growth vis-a-vis equity, the utility of the model hinges on the supposition that we can at least entertain some crude ordering or range of parameter values representing the relative importance of each goal. For example, although it may be unrealistic to state that equity, as defined by a particular statistic, is 0.8 times as important as the value of all per capita gross regional products, it might be reasonable to state that it is somewhere between 0.5 and 1.4 times as important. Under such circumstances we can determine the sensitivity of the model to a range of alternative assumptions.

In order to realize regional income targets, it will generally be necessary to effect expansionary, or possibly even restrictive, fiscal policies; and options exist for both regional and national governments. These can take the form of federal transfer payments or grants-in-aid that are targeted for specific regions, regional business and personal income tax incentives, support for venture capital accumulation, or backing regional bonds. However, costs are associated with such programmes. For instance, funding one programme means concomitant cutbacks in others, reduced social services, or increased taxes - so that although gross regional products may rise, certain regions may not experience corresponding increases in personal disposable income. To model this situation, a term representing the costs related to expansionary or restrictive programmes is introduced into the objective functional; and the importance of minimizing these costs is weighted against the growth and equity goals.

It is clear that the multi-goal objective functional described above is still incomplete as an all-encompassing welfare criterion. Economic goals such as a national balance of payments, balance between expenditures and receipts at both the regional and national levels, and price stability are not at all explicit. Issues of social welfare, justice and equity are subsumed within economic surrogates. Environmental concerns are reflected, at best, only in the costs associated with expansionary policies: increasing economic growth may at the same time decrease environmental quality. Nevertheless, this new multi-goal objective functional, and the control problem it defines, mark a distinct extension of previous approaches.

6.2 A MODEL ORIENTED TOWARDS REGIONAL ECONOMIC GROWTH AND EQUITY

The model formulated in this section considers the trade-offs among competing goals more directly than the models studied in previous chapters. Whereas the previous models included only a cost term within the objective functional, this model introduces three terms measuring economic growth, regional disparity, and costs of government programmes associated with planning goals. However, the model is extremely simplistic in its economic structure, and lacks even the detail of the aggregate macroeconomic system of Chapter 4. In addition, it falls short of the interaction between economic and demographic aspects achieved in Chapter 5. The assumptions of the model follow.

<u>Assumption 6.1</u> There exists a finite set of regions $j=1,\ldots,J$; and for each region the rate of change in per capita gross regional product (per capita income) is:

$$\dot{y}_j(t) = \nu_j\, y_j(t) + u_j(t) \quad , \qquad (6.1)$$

where $y_j(t)$ is the per capita gross regional product (per capita income) of region j per unit time at t

$u_j(t)$ is the control variable, which reflects the impact of government policies on the income growth rate of region j at time t

ν_j is a parameter representing the growth rate, without the influence of controls.

This is an extremely simple description of the regional income trajectory. If there is a steady growth rate ($\nu_j>0$) and if there is no intervention so that $u_j(t)=0$, then income levels will increase exponentially. Although simplistic, the qualitative properties of this trajectory are a reasonable representation of observed trends in the intermediate and long run.

Assumption 6.1 can be related to Myrdal's (1957) cumulative causation hypothesis. Even if the growth rates, ν_j, are the same for two regions, the richer region with a larger $y_j(0)^*$ will grow more than the poorer, so that the income difference increases over time. But in spite of the fact that $y_j(0)^*$ might be large, a small ν_j portrays a recessionary situation. In this case, unlikely in Myrdal's (1957) scheme, a poorer region may quickly overtake a slow growth region.

As in Chapters 3 and 5, the specific government policy represented by the control variable $u_j(t)$ is not identified. It is a construct that represents the effectiveness of a particular policy. Thus, it represents the impact on income growth of such policies as selected tax cuts, provision of jobs through public funding, or lowering of interest rates. But the model avoids the problem of determining what type of government policy is most efficient. Instead, this is a separate issue to be tackled after we determine the appropriate magnitude of government action - that is, after $u_j(t)$ is solved.

<u>Assumption 6.2</u> Denote the initial time by t=0. Then, either

(i) $$y_j(0) = y_j(0)^* \quad , \text{ and} \tag{6.2}$$

$$\left.\frac{dy_j(t)}{dt}\right|_{t=0} = \dot{y}_j(0)^* \tag{6.3}$$

are given;

or

(ii) a planning horizon, τ, is specified, and

$$y_j(0) = y_j(0)^* \quad , \quad \text{and}$$

$$y_j(\tau) = y_j(\tau)^* \tag{6.4}$$

are given.

As developed later, the mathematical analysis results in a system of first order differential equations, whose solution requires either the initial conditions stated in Assumption 6.2 (i) or the boundary conditions of Assumption 6.2 (ii).

Assumption 6.2 (i) states that both the initial per capita regional income levels and initial rates of change in these levels are known. No planning horizon is specified. Instead of achieving distributional equity by means of boundary constraints, an equity measure is introduced into the objective functional (see Assumption 6.3).

On the other hand, Assumption 6.2 (ii) adopts the boundary constraint employed in previous chapters. In addition, the equity measure in the objective functional is maintained, since the boundary constraint reflects distributional considerations only at $t=\tau$, whereas the equity measure is based on distributional considerations over the whole planning time span.

Assumption 6.3. The optimal government action is such that the objective functional, I, is minimized:

$$I = \int_0^\tau \sum_j \left[\zeta_{(1)} \frac{y_j(t)}{\sum_k y_k(t)} \log \frac{y_j(t)}{\sum_k y_k(t)} + \zeta_{(2)} u_j(t)^2 - \zeta_{(3)} y_j(t)\right] dt \quad , \qquad (6.5)$$

where

$\zeta_{(1)}$ is the cost associated with the inequity statistic (see Equation (6.6)),

$\zeta_{(2)}$ is the cost associated with the magnitude $u_j(t)^2$, where $u_j(t)$ represents the impacts of government policy on the rate of change in per capita regional income,

$\zeta_{(3)}$ is the benefit derived from a unit increase in the per capita regional income level,

and $\zeta_{(1)}, \zeta_{(2)}, \zeta_{(3)} > 0$

Equation (6.5) is a mathematical statement reflecting much of the discussion in Section 6.1. There are three terms, denoting inequity, costs associated with government policy, and income growth. Trade-offs among the three separate goals are reflected in the values of $\zeta_{(1)}$, $\zeta_{(2)}$ and $\zeta_{(3)}$. For example, if distributional equity is extremely important then $\zeta_{(1)}$ would be relatively large.

If $y_j(t)$ is cost indexed but unadjusted for inflation, then Assumptions 6.1 and 6.3 indicate that economic growth unadjusted for inflation and minimizing government activity are conflicting goals. Their relative importance is reflected in the values of $\zeta_{(2)}$ and $\zeta_{(3)}$. Even if $y_j(t)$ is adjusted for inflation, increasing government activity $u_j(t)$, can increase $y_j(t)$, yet decrease disposable income after taxes.

An interesting methodological innovation is the introduction of the inequity term:

$$N = \int_0^{\tau} \sum_j \frac{y_j(t)}{\sum_k y_k(t)} \log \frac{y_j(t)}{\sum_k y_k(t)} \, dt \tag{6.6}$$

This is based on the entropy statistic that is used to measure the uniformity in a distribution:

$$S = - \sum_j \frac{y_j}{\sum_k y_k} \log \frac{y_j}{\sum_k y_k} \tag{6.7}$$

For instance, Hodge and Gatrell (1976) used S to measure the distribution of benefits derived from the location of a public facility. If all of the y_j are equal, then S is at a maximum (and N would be at a minimum). On the other hand, if production was just in one region such that $y_j = \sum_k y_k$ and $y_{i \neq j} = 0$, then S is at a minimum. In general, the more even the distribution of per capita regional income at each time, the smaller the value of N - the smaller the inequity. The inequity is summed up over all time periods to obtain a total measure. The use of such a measure implies that equality is used as a surrogate for equity. Of course, whether this is proper depends on one's political philosophy.

The optimal control problem defined in Assumptions 6.1-6.3 is distinct because of the nonlinear inequity term. Consequently, most of the results in control theory that have been applied in previous studies (Pindyck, 1973; Fujita, 1976; Willikens, 1976) are inapplicable. Because of the nonlinearity, analytic solutions are not possible. Result 6.1 gives an expression for per capita regional incomes in terms of a system of second order nonlinear autonomous ordinary differential equations; and it is straightforward to reduce this to a set of first order differential equations.

<u>Result 6.1</u> If Assumptions 6.1 and 6.3 hold, then an expression for the optimal trajectory of per capita regional income levels is given by:

$$\ddot{y}_j(t) = \frac{1}{2\zeta_{(2)}} \left[\zeta_{(1)} \left[\frac{1}{(\sum_k y_k)^2} \sum_k y_k \log \frac{y_j}{y_k} \right] - \zeta_{(3)} + 2\dot{\zeta}_{(2)} \nu_j{}^2 y_j(t) \right] . \tag{6.8}$$

Result 6.1 cannot be solved analytically, though invoking either Assumption 6.2(i) or 6.2(ii) allows a numerical solution. The use of Assumption 6.2(i) admits a system of first order differential equations with given initial conditions, so that a unique solution exists (Boyce and DiPrima, 1969, Theorem 7.1), and numerical integration is relatively straightforward (Kirk, 1970, p. 127). Assumption 6.2(ii) leads to a nonlinear boundary value problem which can be treated by steepest descent, variation of extremals, or quasilinearization techniques.

Since there is no $\dot{y}_j(t)$ term in Equation (6.8), neither the theorem of Lienard nor that due to Levinson and Smith (Davis 1962, p. 304, 307) are satisfied. Hence, we are unable to show that the optimal trajectory of $y_j(t)$ is periodic;

and a reasonable conjecture is that the per capita regional income trajectories are not periodic. Further, because of the qualitative similarity between business cycles and periodic trajectories, we add the conjecture that business cycles represent a sub-optimal profile.

6.3 A TWO-REGION CASE

Result 6.1 conveys limited information since it implies few general qualitative properties. A special two-region case allows additional qualitative aspects to be investigated. One of the two regions has a constant, exogenously-given level of per capita income; and no controls are imposed on this region. The problem is to determine qualitative properties of the income and control trajectories of the other region. Assumptions 6.1 to 6.3 are redefined for this special case by Assumptions 6.4 to 6.6. These lead to Result 6.2.

Assumption 6.4 (a) The rate of change in the per capita gross regional product of Region 1 is:

$$\dot{y}_1(t) = \nu_1 y_1(t) + u_1(t) \quad ; \qquad (6.9)$$

(b) The per capita gross regional product of Region 2 is a given constant over time:

$$y_2(t) = y_2^* \qquad (6.10)$$

Assumption 6.5 Denote the initial time by t=0.
Then, either

(i) $y_1(0) = y_1(0)^*$, and (6.11)

$$\left.\frac{dy_1(t)}{dt}\right|_{t=0} = \dot{y}_1(0)^* \tag{6.12}$$

are given;

or

(ii) a planning horizon, τ, is specified, and

$$y_1(0) = y_1(0)^* \quad \text{and} \tag{6.13}$$

$$y_1(\tau) = y_1(\tau)^* \tag{6.14}$$

are given.

Assumption 6.6 The optimal government action is such that the objective functional, I, is minimized:

$$I = \int_{t=0}^{\tau} [\zeta_{(1)} \frac{y_1(t)}{y_1(t)+y_2^*} \log \frac{y_1(t)}{y_1(t)+y_2^*} + \frac{y_2^*}{y_1(t)+y_2^*} \tag{6.15}$$

$$\log \frac{y_2^*}{y_1(t)+y_2} + \zeta_{(2)}\, u_1(t)^2 - \zeta_{(3)}\, (y_1(t)+y_2^*)]\, dt$$

Result 6.2 If Assumptions 6.4 to 6.6 hold, then the optimal level of per capita income in Region 1 satisfies the following differential equation:

$$\ddot{y}_1(t) = \frac{\zeta_{(1)}}{2\zeta_{(2)}(y_1(t)+y_2^*)^2}\,(y_2^*\log y_1(t) - y_2^*\log y_2^*) \quad (6.16)$$

$$- \frac{\zeta_{(3)}}{2\zeta_{(2)}} + \nu_1^2 y_1(t) \quad .$$

With change in notation, this can be re-expressed as:

$$v^2 = - \frac{\zeta_{(1)}}{\zeta_{(2)}}\, y_2^* \left[\frac{\log x}{x+y_2^*} + \frac{1}{y_2^*}\log \frac{x+y_2^*}{x}\right] \quad (6.17)$$

$$+ \frac{\zeta_{(1)} y_2^* \log y_2^*}{\zeta_{(2)}(x+y_2^*)} - \frac{\zeta_{(3)}}{\zeta_{(2)}}\, x + \nu_1^2 x^2 + G \quad ,$$

where

$v = \dot{y}_1(t)$ (6.18)

$x = y_1(t)$ and (6.19)

G is a constant of integration.

This nonlinear autonomous second order differential equation cannot be solved analytically, so that a qualitative analysis is indicated. A common method of analyzing qualitative properties of differential equations is through the use of phase diagrams (Boyce and DiPrima, 1969; Davis, 1962; Gandolfo, 1971; McDonald, 1950). This involves a plot of $dy_1(t)/dt$ versus $y_1(t)$, so that Equation (6.16) must be re-expressed in a more suitable form, Equation (6.17). Both of these equations are stated in Result 6.3, though only the latter is utilized in the subsequent analysis.

In order to illustrate the form of the phase diagram indicated by Result 6.3, a numerical example is considered in the rest of this section. The values assigned to the parameters ν_1 and y_2^* represent realistic orders of magnitude. Let

$$\zeta_{(1)}, \zeta_{(2)}, \zeta_{(3)} = 1 \tag{6.20}$$

$$y_2^* = 6000 \tag{6.21}$$

$$\nu_1 = 0.05 \tag{6.22}$$

Equation (6.21) states that the per capita income level of Region 2 is $6,000 per year. Equation (6.22) states that, without any additional government programmes, Region 1 experiences a 5% annual growth rate in income. The values assigned to $\zeta_{(1)}$, $\zeta_{(2)}$, and $\zeta_{(3)}$ were arbitrarily selected.

For a given value of x, $(v^2 - G)$ can be computed after substituting Equations (6.20)-(6.22) into (6.17). Table 6.1 lists some of these calculations (column two), used to sketch Figure 6.1.

Inspecting Figure 6.1, one obtains the phase diagram, Figure 6.2, consisting of a family of curves corresponding to different values of G. The interpretation of Figure 6.2 is straightforward. The G values reflect various possible initial conditions. Assume that the initial conditions are (x_o, v_o), as indicated in Figure 6.2. Then the top branch (above the x axis) of the $C_{(1)}$ curve is the appropriate x-v trajectory. At the initial point x_o, v_o is positive; hence, x will increase, at which point v is still positive; so that x will continue to increase. As a result,

TABLE 6.1

VALUES OF $(v^2\text{-}G)$ AND $(.0025x^2\text{-}x)$ FOR SELECTED VALUES OF x

x	$(v^2\text{-}G)$	$(.0025x^2\text{-}x)$
.001	-1.02×10^{-3}	-1.00×10^{-3}
.01	-1.00×10^{-2}	-1.00×10^{-2}
10	-9.76	-9.75
100	-7.51×10	-7.50×10
200	-9.67×10	-1.00×10^{2}
300	-7.52×10	-7.50×10
400	2.47	0.00
500	1.25×10^{2}	1.25×10^{2}
1000	1.50×10^{3}	1.50×10^{3}
2000	8.00×10^{3}	8.00×10^{3}
5000	5.75×10^{4}	5.75×10^{4}
6000	8.40×10^{4}	8.40×10^{4}
7000	1.15×10^{5}	1.16×10^{5}
8000	1.52×10^{5}	1.52×10^{5}
9000	1.93×10^{5}	1.94×10^{5}
10000	2.40×10^{5}	2.40×10^{5}

the path of the trajectory along the top $C_{(1)}$ branch is in the direction of the arrows. In this situation, the system is unstable, for x increases without bound. If interest is only in a finite time horizon, then long run equilibrium properties are not of overbearing concern. Yet we should still recognize that the end of the present time period becomes the beginning of the next, and that perpetuation of the policy represented by $C_{(1)}$ would lead to explosive conditions.

If one assumes that the initial value of x (income) is 6000, then the different trajectory possibilities are illustrated in Figure 6.3. There are four possibilities:

(i) If $v>0$ at $t=0$, then the system is unstable and follows the trajectory $\ell_{(1)}$. That is, if Region 1 is experiencing growth in production at $t=0$, then optimal production levels will continue to increase monotonically. Another way of interpreting this case is that if $y_1(\tau)^* > y_1(0)^*$ then the region will have continually increasing production levels until $y_1(\tau)^*$ is attained.

(ii) If $v=0$ at $t=0$, then the system is stable and income levels will remain constant at x_0. This is the interpretation within the context of Assumption 6.2 (i). Within the context of Assumption 6.2 (ii), the interpretation is that if $y_1(\tau)^* = y_1(0)^*$, then the optimal income trajectory is a constant: $y_1(t) = y_1(0)^* = y_1(\tau)^*$.

(iii) If $v<0$ at $t=0$, then the system may be stable and converge monotonically to the equilibrium point x^1, which is less than the initial income level. This can only occur when the region is experiencing a slight economic recession at $t=0$; in a situation in which a more serious economic depression is evident, Case (iv) would be more likely. As in the previous cases, a dual interpretation of the trajectory behaviour is possible. That is: if a target income level is less than the present level, then the income trajectory decreases monotonically over time until the target is achieved. In contrast to Case (i), however, once this target is achieved, income levels will not have any tendency to deviate from this equilibrium level.

(iv) Again, if $v<0$ at $t=0$, then the system may follow $\ell_{(4)}$ and, in the long run, approach (but never attain) $x=0$. This possibility would only occur when the region is experiencing a rapid decline in production at $t=0$. Note that $v_{(4)}<v_{(3)}$ in Figure 6.3; this will always be the case. The dual interpretation of this case is that if the target income level is very low, then there will be no solution. This is not mathematically inconsistent, since there is no guarantee that a solution exists for the two-point boundary value problem. In practice, target

income levels will always be higher than present levels, so that this situation would not arise.

Inspection of Figure 6.3 with v versus x implies that 1/v versus x takes on the form illustrated in Figure 6.4. Then, since

$$t = \int \frac{1}{v}\, dx \qquad \text{(McDonald, 1950)}, \tag{6.23}$$

Figure 6.4 indicates that t versus x takes the form given in Figure 6.5, whence one obtains a plot of income over time, Figure 6.6. Taking the slope of x versus t gives v versus t (Figure 6.7); and then, recalling Equation (6.1), these two graphs can be combined to produce a plot of u_1 versus t (Figure 6.8). As indicated previously, these graphs are not accurate, but reflect qualitative properties of the various trajectories.

Interestingly, for the given parameter values, $(v^2\text{-}G)$ can be approximated by its last two terms: $.0025x^2-x$ (see Table 6.1). As a check on certain qualitative attributes revealed in Figures 6.6-6.8, the rate of change in Region 1's income levels is set to be:

$$v = (.0025x^2-x)^{1/2} \quad , \tag{6.24}$$

which, when substituted into Equation (6.23), admits:

$$t = \frac{1}{(.0025)^{1/2}} \log \left. \frac{(.0025x' - 1 + \sqrt{.0025x'(.0025x'-1)}\,)^2}{(.0025x'-1)} \right|_{x_0}^{x} \tag{6.25}$$

From Equations (6.9) and (6.24) the control variable trajectory is:

$$u_1 = (.0025x^2 - x)^{1/2} - .05x \quad . \tag{6.26}$$

Table 6.2 tabulates values of x, v and u_1, assuming $x_0 \equiv y_1(0)^*$ = 6000; these are plotted in Figures 6.9-6.11, which can be compared with the $\ell_{(1)}$ curves in Figures 6.6-6.8.

TABLE 6.2

VALUES OF x, v AND u_1 AS FUNCTIONS OF t
(ASSUMING x_0=6000 AND v^2= $.0025x^2$-x)

x	t	v	u_1
6,500	1.65	3.15×10^2	-10.16
7,000	3.18	3.40×10^2	-10.15
7,500	4.60	3.65×10^2	-10.14
8,000	5.93	3.90×10^2	-10.13
9,000	8.34	4.40×10^2	-10.11
10,000	10.49	4.90×10^2	-10.10

The numerical approximation is instructive. It suggests that the optimal levels of additional government control are such that $u_1(t)$ may be almost constant over time. This may or may not mean constant government fiscal and monetary policy per se. It does mean that these policies are such that normal income growth (equal to $\nu_1 y_1$) should be decreased by a small constant magnitude at each time (note that the absolute value

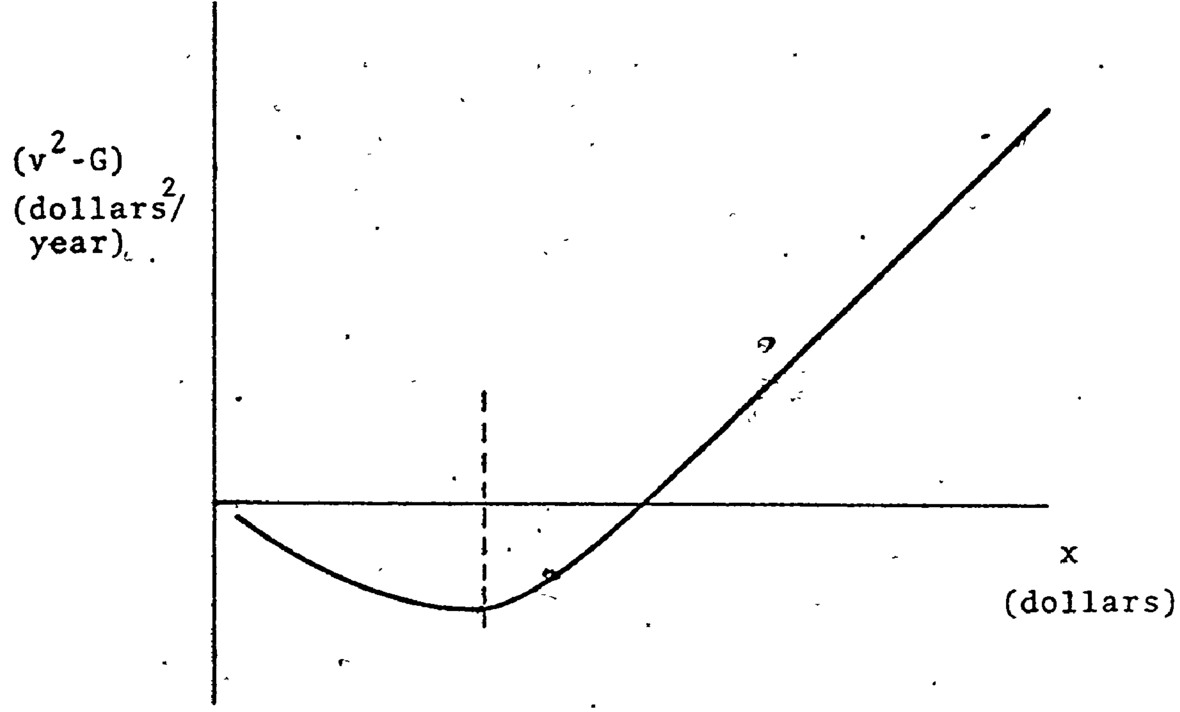

Figure 6.1: Plot of (v^2-G) versus x (not to scale).

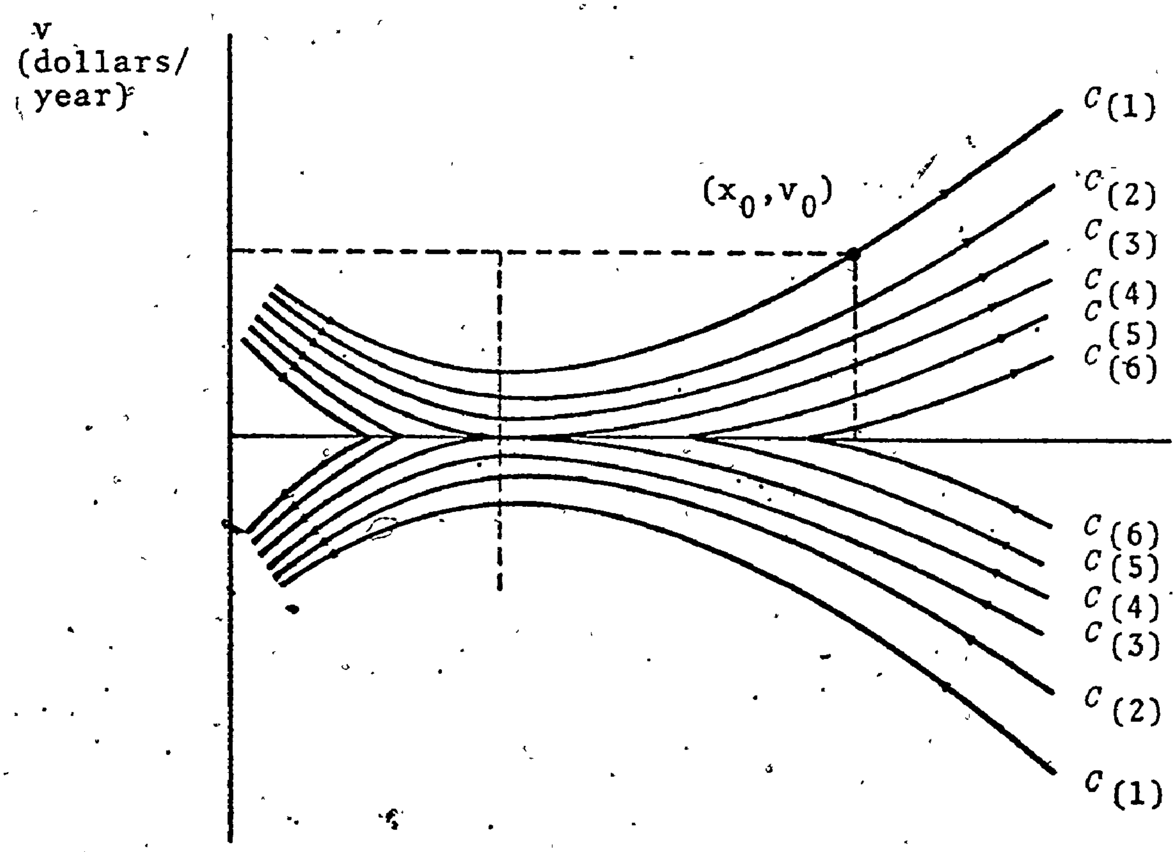

Figure 6.2: Plot of v versus x (Not to scale).

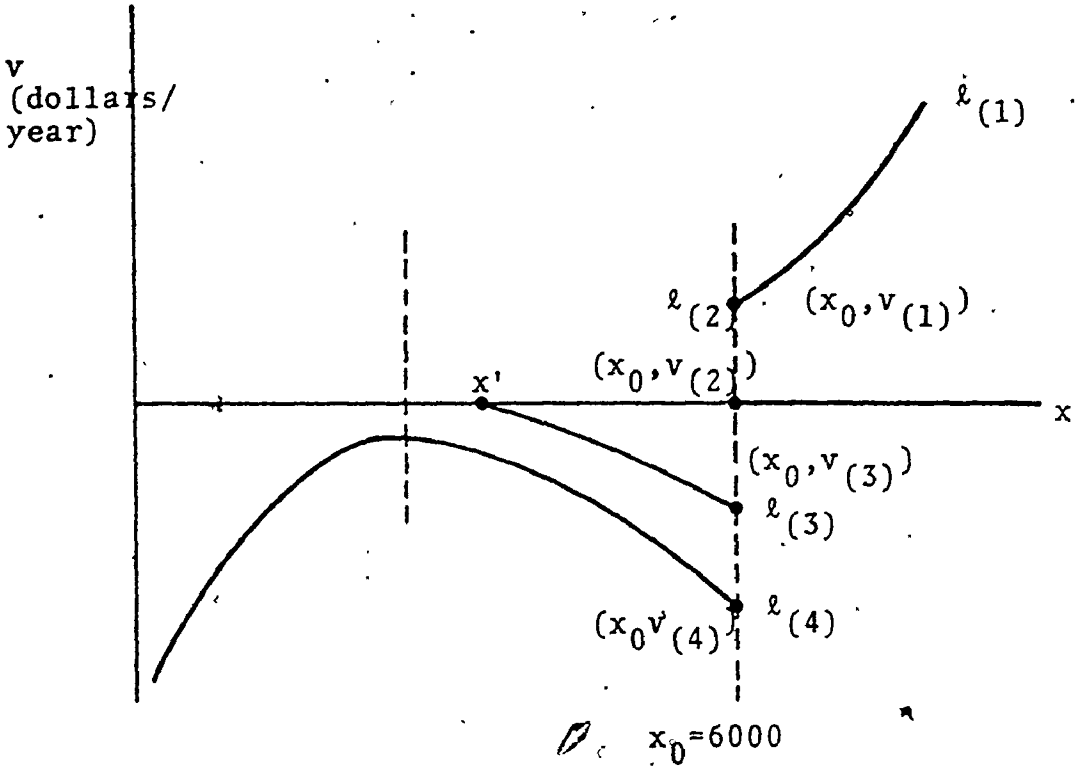

Figure 6.3: Plot of v versus x, Assuming x_0=6000 (Not to Scale).

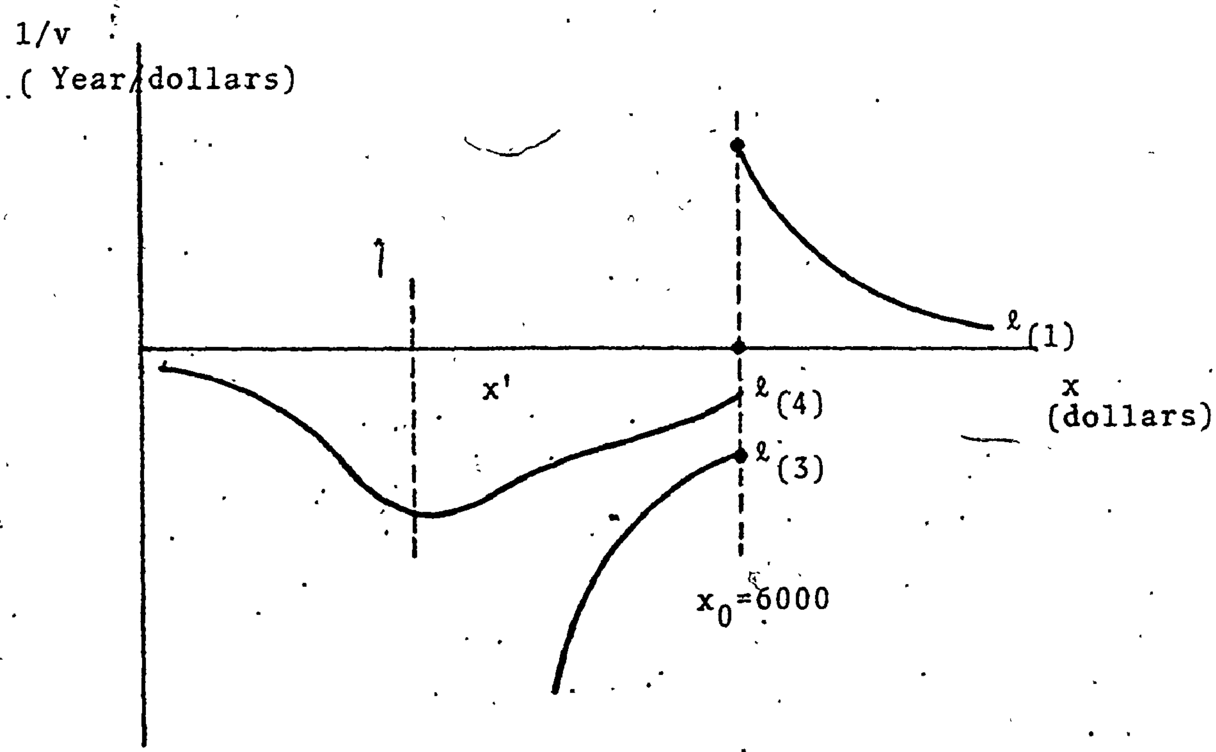

Figure 6.4: Plot of 1/v versus x, Assuming x_0=6000 (Not to Scale).

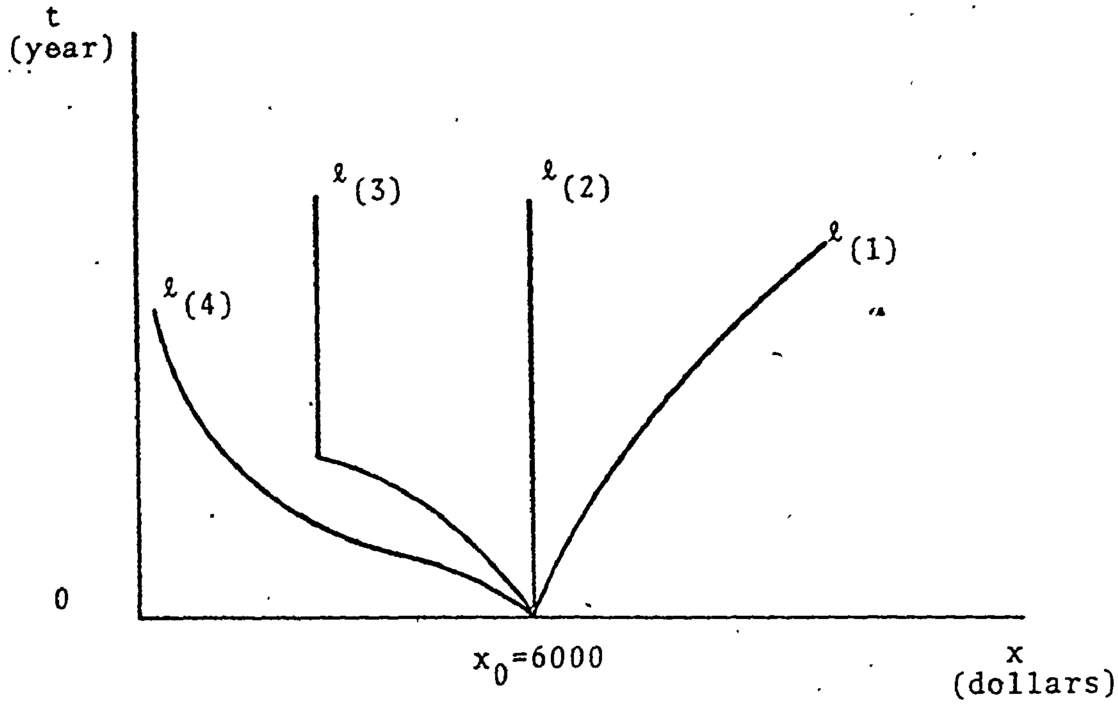

Figure 6.5: Plot of t versus x, Assuming $x_0=6000$ (Not to Scale).

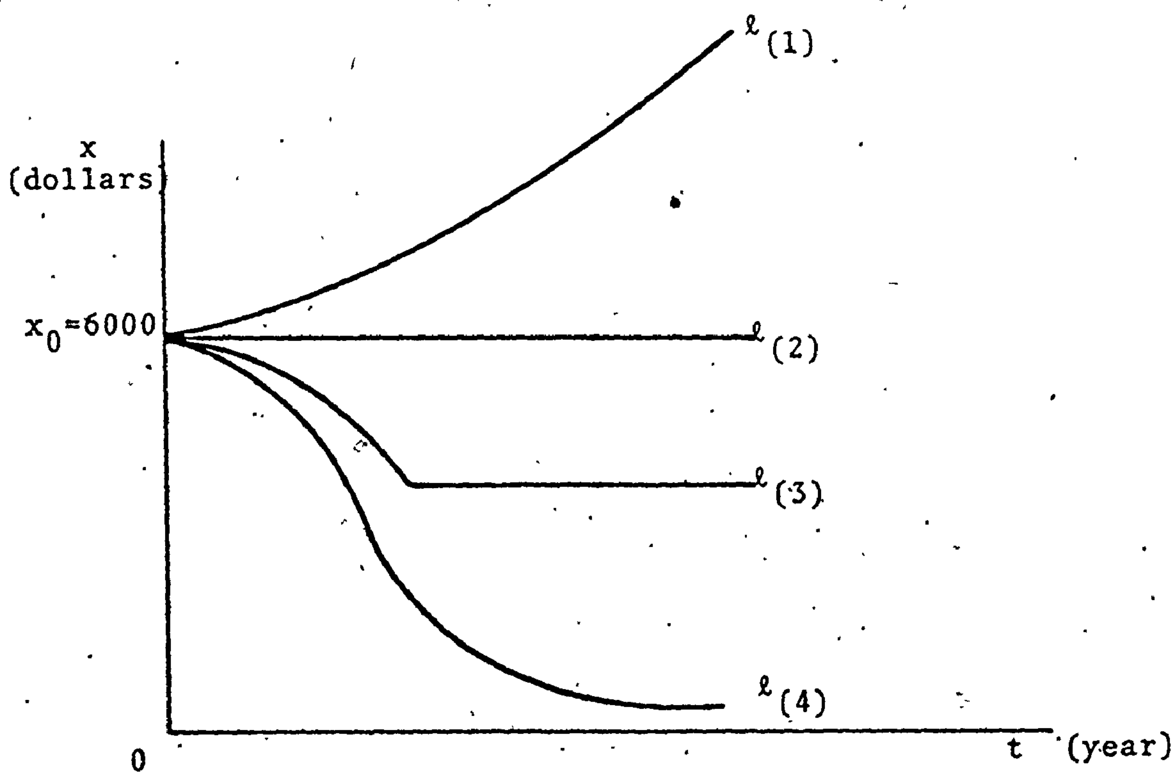

Figure 6.6: Plot of x versus t, Assuming $x_0=6000$ (Not to Scale).

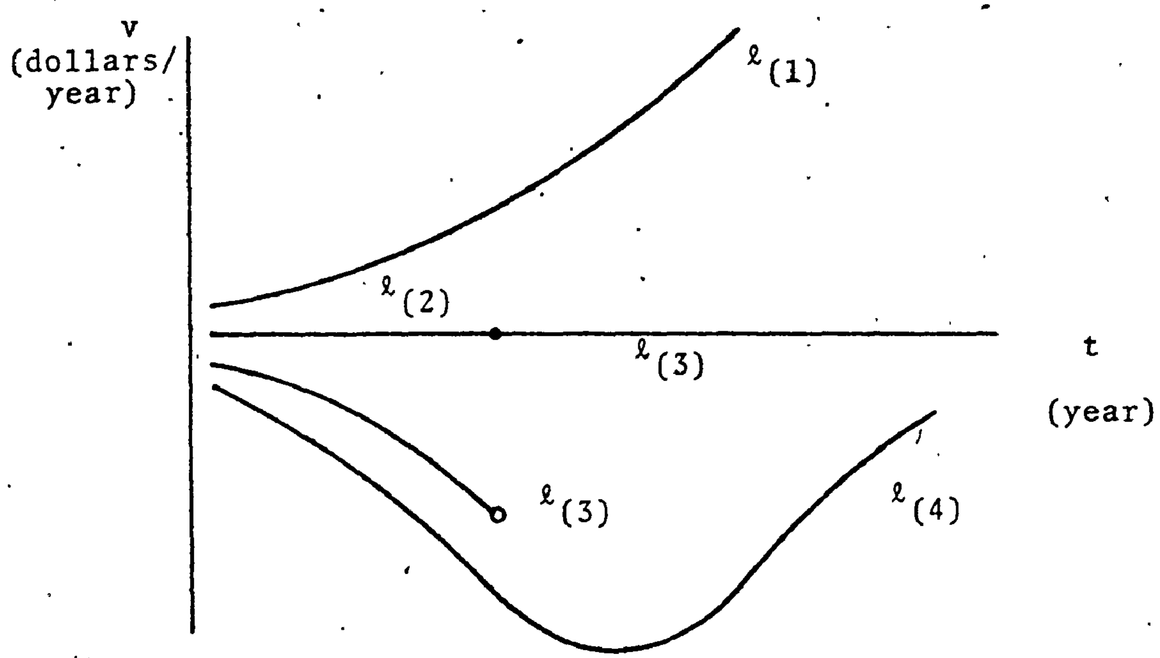

Figure 6.7: Plot of v versus t. Assuming x_0=6000 (Not to Scale).

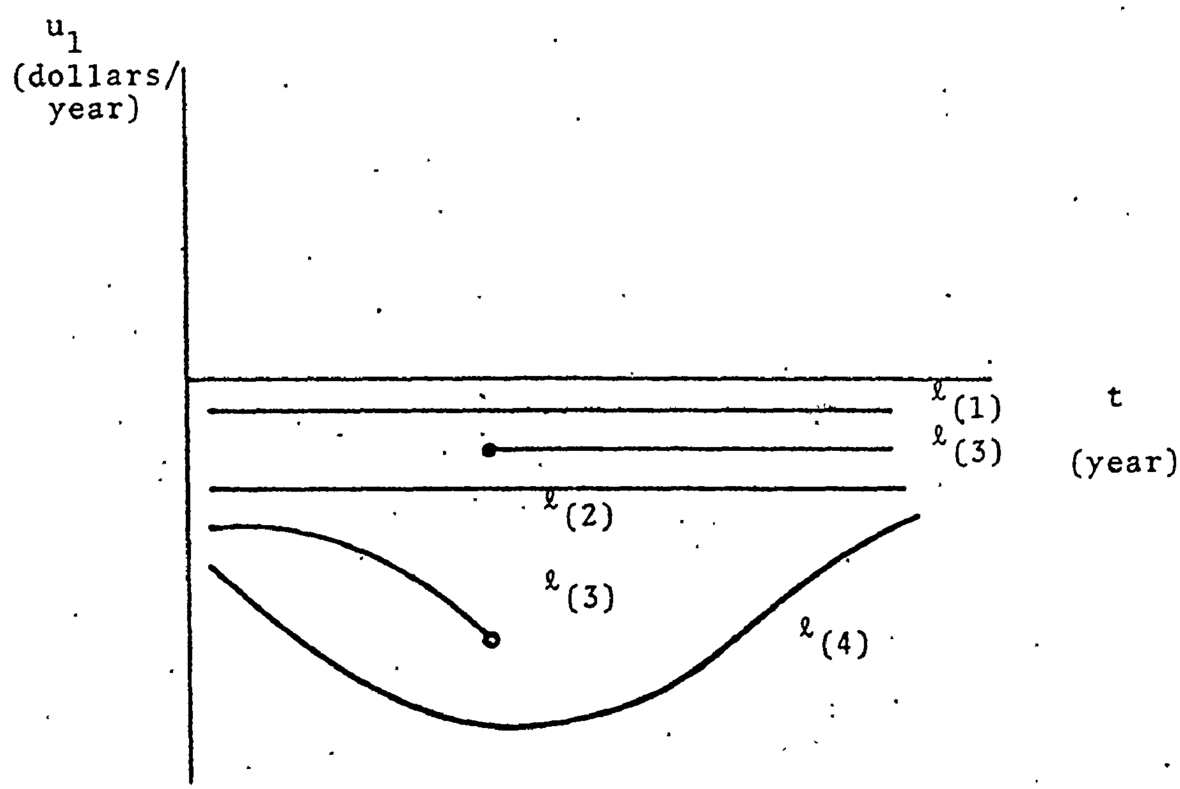

Figure 6.8: Hypothetical Plots of u_1 versus t, Assuming x_0=6000 (Not to Scale).

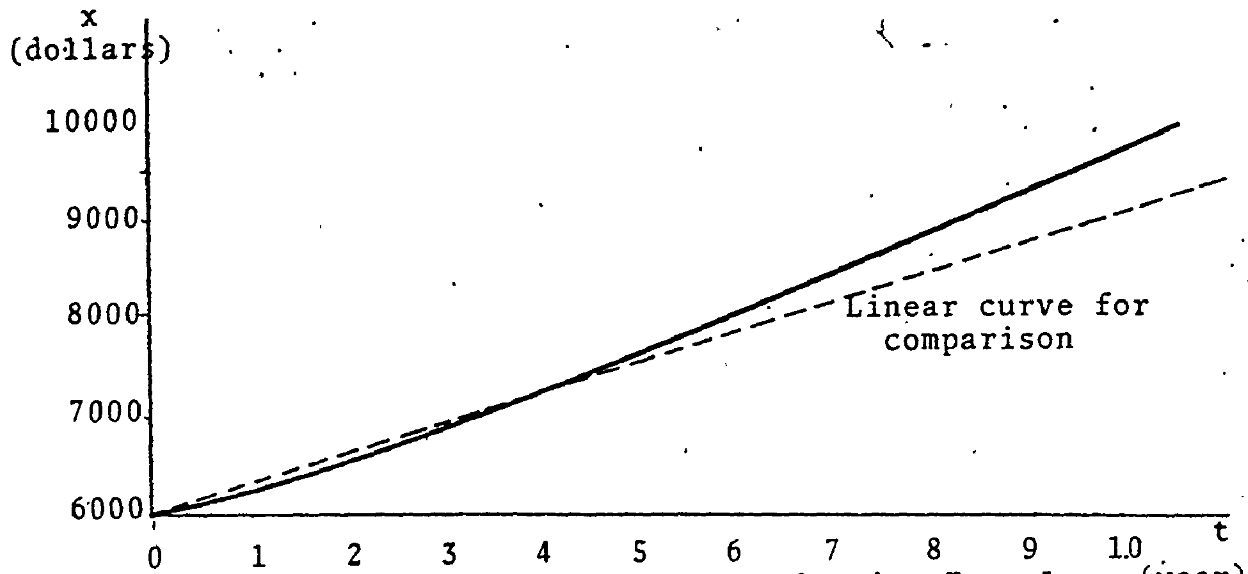

Figure 6.9: Plot of x versus t in Approximation Example.

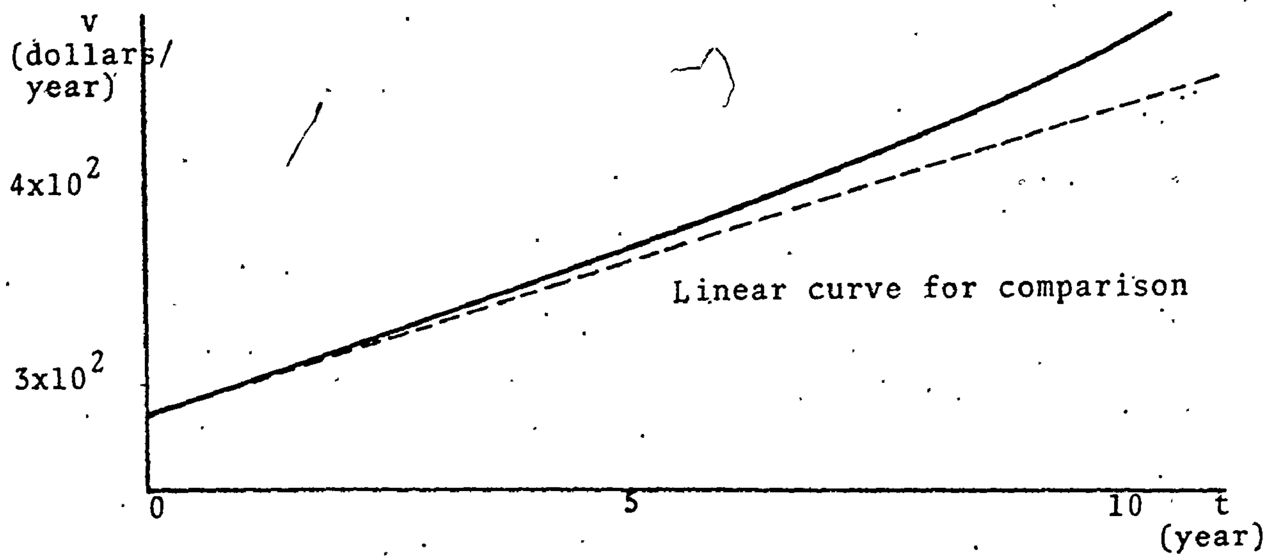

Figure 6.10: Plot of v versus t in Approximation Example.

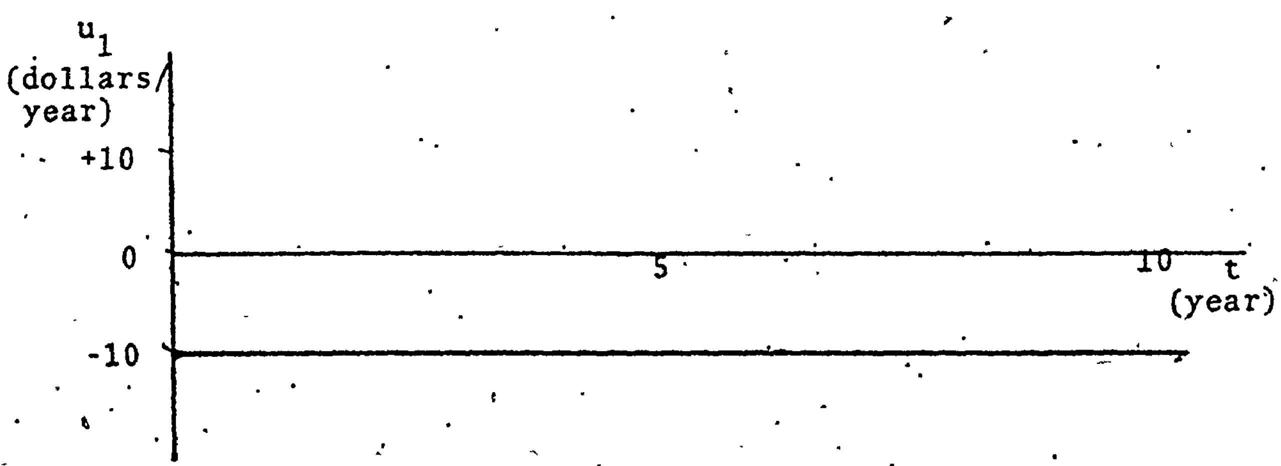

Figure 6.11: Plot of u_1 versus t in Approximation Example.

of u_1 is small relative to v). Although such an impact may possibly be realized through a one-shot economic package, it is more intuitively appealing to suggest a continuous programme of economic stimuli stretching over the planning interval. The qualitative structure of the income, rate of income growth, and control trajectories is encouraging. The optimal policy suggests an economy with sustained, yet controlled, growth. The relative stability of the control variable suggests much greater political and practical ease in implementation than for example a bang-bang type of trajectory.

In addition, the example reveals that if the goal of equity is to be important in determining the optimal decision, then $\zeta_{(1)}$ must be very large relative to $\zeta_{(3)}$ and particularly $\zeta_{(2)}$. In the numerical example above, the terms representing the impact of the equity factor in Equation (6.17) are insignificant: $v^2 - G \approx v_1{}^2 x^2 - \frac{\zeta_{(3)}}{\zeta_{(2)}} x$

Finally, as observed in previous chapters, a comparative dynamics (sensitivity) analysis is difficult, since the magnitude of G is in turn a function of the parameters.

6.4 CONCLUDING COMMENTS

In this chapter we assumed that per capita regional income levels increase exponentially, unless they are altered by government instruments designed to meet certain growth, cost, and equity considerations. The system was explicitly inter-regional in that one region's economy depended on those of other regions. The analysis did not focus on details of the regional economies, but rather on general qualitative properties that emerge from a multi-objective scheme.

Figures 6.6 and 6.8 encapsulate the results of a two-region numerical experiment. If the target per capita income level is greater than the present level, then the income trajectory is convex and the control variable is almost constant (the $\ell_{(1)}$ curves in Figures 6.6 and 6.8). Qualitatively, we can see how similar these results are to those of Chapter 3. There, the income trajectories were also always convex for Region 1 (see Figures 3.1(b)-3.6(b)) and the control variable was exactly constant. Comparing the $\ell_{(2)}$ curves in Figures 6.6 and 6.8 to Region 2's trajectories in Chapter 3 (Figures 3.2(b)-3.6(b)), we observe that the income trajectory was not constant, as it is here-though the two sets are not too unlike.

In contrasting the results of Chapters 3 and 6, we should realize that $y_{(1)1}(t)$ in Chapter 3 is comparable to $y_1(0)^* \exp(\nu_1 t)$ in this chapter. In both cases, the control variable is absent. Since the two trajectories differ, however, we can expect the corresponding trajectories arising out of the control problem to differ as well. This is especially so because of the different objective functionals in the two cases. Yet, we still observe qualitative similarities in both the income and control trajectories.

Additional numerical experimentation (not detailed here) indicates that even when regional equality becomes an extremely important concern ($\zeta_{(1)}$ is very large), the qualitative properties of the four cases illustrated in Figures 6.6-6.8 are unchanged. Hence, the analysis suggests that the introduction of regional equity and growth criteria, in addition to cost minimization, and income targets, does not significantly alter many of the general, qualitative characteristics of the regional income and optimal control trajectories.

CHAPTER 7

SOME FINAL THOUGHTS

While it is ennobling to surmise that a human landscape will evolve naturally to a state of bliss, such speculation is but little more than sheer surrealism, for it can only be a product of homo sapiens' penchant for optimism. But in the end it is a trite issue. That governments exist and live in our world is a verifiable fact. Their rôle is to act so as to maximize the welfare of their societies; and of late, they have been responding more and more to this task, at least if it is to be measured in terms of their expenditures.

Modelling the ability of a government to influence the evolution of a geographical landscape is the key element in this dissertation. The structure of the models which we have studied is remarkably simple. First, we have a set of goals which are established by those in power, be it the people, big business, labour, politicians, the military, a dictator, the church, or some combination of these. Second, we describe the relationships among the economic and/or demographic attributes of the landscape, and specify how government instruments can influence its evolution. Third, we determine the magnitude of the government policies that are necessary in order for our goals to be met.

Several caveats are worth noting. We realize that goals are not so well-defined in reality. The interplay between economics, morality, and political forces creates an array of attitudes and visions. Here, any set of goals is not meant to be a unanimous opinion, but rather a consensus view, or possibly a risk-aversion statement of priorities, or possibly a statement of popular opinion, or even a minority position. True, judgements are value-laden, but a quantitative representation of these judgements is reasonable. How else can we justify arguing about something, if we don't know what we want? The purpose of the models is not to impart the judgements, but to aid the decision-maker in evaluating the consequences of his/her judgements.

We have eschewed the problems of model identification and estimation, but have assumed implicitly that our initial premises are based on past and present information - there is no other - and that they are a reasonable representation of the system under study. We concede that a "reasonable representation" is not necessarily the most accurate, though its accuracy can only be ascertained after the fact. We also acknowledge that we live in a world of uncertainty; and indeed, suggest that stochastic control theory presents a fruitful line of enquiry. But we stand on the basis of the analytic viability of deterministic models, and on the belief that they provide some illuminating insights for investigating alternative scenarios.

After reviewing the scenarios presented in past studies, we decided that the economic models were limited in their treatment of migration, population growth, and interregional interaction (autarchies do not exist in reality). Whereas, the demographic models suppressed the influence of regional variations in economic prosperity, so much a factor, direct or otherwise, in individuals' decisions to move.

Our own models directed varying degrees of attention at each of these neglected areas. In Chapters 3 and 4, the economic apparatuses were only implicitly interregional. Their saving grace is the fact that they are easily linked to explicit, interregional export structures that already exist in the literature (although not in a control context). Chapters 5 and 6, however, do project strong interregional frameworks: the locations of the regions, their income levels, interregional migration, and population growth interact simultaneously (Chapter 5); and regional disparity is paired with concerns for stimulating economic growth and minimizing the costs of government programmes (Chapter 6). The details are given in the concluding comments in each chapter and in the Results.

If there are still calls for constructing a super-model of the geography and political economy of the human landscape, then we have certainly not even tried to answer that challenge. Instead, our energies were channelled towards exploring some very fundamental models - constructing much-needed skeletons, before filling in any flesh. In time, our models will either be discarded outright and forgotten, extended with many modifications, or used as a basis for developing other, possibly very different, models. If it is either of the latter two, then this study will have been worthwhile.

APPENDIX 1

PONTRYAGIN'S MAXIMUM PRINCIPLE

Pontryagin's Maximum Principle gives the necessary conditions for the solution of a dynamic optimization problem. Proofs of the theorem and well known "type problems" (such as the linear-quadratic, linear regulator, minimum-time, and minimum fuel problems) are given in Bryson and Ho (1969), Connors and Teichroew (1967), Hadley and Kemp (1971), Kirk (1970), Noton (1972), and Pontryagin et al. (1962).

The interregional systems that we are considering consist of state variables, such as income and population, which we shall denote by $x_j(t)$ in this discussion. In addition, there are control variables $u_i(t)$ (sometimes called decision or instrument variables) that are determined by the decision-maker, which is the government.

The state of the system changes over time, and is described by a set of differential equations which is called the dynamic system (or transition equations):

$$\dot{x}_j(t) = f_j(\underline{x}(t), \underline{u}(t), t), \tag{A1.1}$$

where the underline denotes a vector. The dynamic system together with the initial conditions $\underline{x}(0)^*$, and boundary conditions $\underline{x}(\tau)^*$, act as constraints. The initial conditions describe the state of the system at the beginning of the planning interval; the boundary conditions are the targets which the government has set as its goals. There may also be other constraints which are stated as (in)equalities involving the state and/or control variables.

In addition to the final targets for the state variables, the government will also want to achieve some other goal(s) over the whole planning interval. This is expressed in the objective functional (or performance index), which is to be minimized, by convention:

$$\underset{\underline{u}(t)}{\text{Min.}} \quad I = \int_{t=0}^{T} F[\underline{x}(t),\underline{u}(t),t]\, dt \qquad \text{(A1.2)}$$

I can represent the total welfare over the planning interval (if I is maximized) or the total costs and negative impacts of the government programmes, or some other performance criterion.

We shall assume that all functions satisfy sufficiently strong smoothness conditions.[1] Then, Pontryagin's Maximum Principle can be stated:

If $\underline{u}^{\dagger}(t)$ is an optimal control vector that minimizes Equation (A1.2) subject to (A1.1) and to the initial and boundary conditions, then there exist co-state variables (or auxiliary variables) $\underline{p}(t)$, one for each state variable, such that:

(a) $\dot{p}_j(t) = -\partial H/\partial x_j(t)$, (A1.3)

where the Hamiltonian

$$H = -F(\underline{x}(t),\underline{u}(t),t) + \sum_j p_j(t)\, f_j(\underline{x}(t),\underline{u}(t),t) \qquad \text{(A1.4)}$$

[1] I, $\underline{x}$, continuous; $\underline{u}$, F, $\underline{f}$ piece-wise continuous.

(b) the Hamiltonian achieves a <u>maximum</u> with respect to the admissible controls $\underline{u}(t)$ at $\underline{u}^{\dagger}(t)$.

Equation (A1.3) together with (A1.1) are often called the <u>canonical</u> <u>equations</u>.

Pontryagin's Maximum Principle gives the necessary conditions for the optimal solution, and thus provides candidates for the optimal control and state trajectories. If the necessary conditions yield many candidates, then a sufficiency theorem such as that due to Leitmann and Stalford (1971) can be used.

APPENDIX 2

THE LAPLACE TRANSFORM

Laplace transforms are especially useful for solving linear differential equations with constant coefficients. In essence, the use of the transform reduces the consideration of integral-differential equations to algebraic manipulations. Some references include Boyce and DiPrima (1969), Doetsch (1971), Fodor (1965), Holl et al. (1959), McCollum and Brown (1965), Sneddon (1972), Strum and Ward (1970), Takahashi (1966), and Thomson (1960). Fairly exhaustive tables are given in Erdelyi et al. (1954), Oberhettinger and Badii (1973), and Roberts and Kaufmann (1966).

The Laplace transform of a function, g(t), is defined as:

$$L\,[g(t)] = \int_0^\infty \exp(-st)\,g(t)\,dt \quad , \qquad (A2.1)$$

where s is complex.

The inverse Laplace transform is denoted by L^{-1}, where

$$L^{-1}[L[g(t)]] = g(t) \quad . \qquad (A2.2)$$

[1]

The following properties are stated without proof.

$$L[g(t)*f(t)] = L[g(t)]\;L[f(t)] \quad , \qquad (A2.3)$$

where $g(t)*f(t) \equiv \int_0^t g(t-z)\, f(z)\, dz \quad . \qquad (A2.4)$

$g(t), f(t)$ are functions of t

1. The inverse Laplace transform can be expressed as an integral equation, but analytical difficulties result since integration is over the complex plane. Hence, other approaches, such as the method of are to the im

$$L[dg(t)/dt] = s\ L[g(t)] - g(0) \tag{A2.5}$$

where g(0) is the value of g(t) at t=0.

$$L[\alpha_1 \exp(\alpha_2 t)] = \frac{\alpha_1}{s-\alpha_2} , \tag{A2.6}$$

where α_1, α_2 are constants.

APPENDIX 3

PROOFS OF RESULTS IN CHAPTER 3

<u>Result 3.1</u> Laplace transforms are a suitable method of treating the time-dependent systems encountered in Chapters 3 and 4. Appendix 2 lists the essential properties of Laplace transforms which are used in the analysis. Result 3.1 is obtained by using the relationships in Assumption 3.1 to obtain expressions for the Laplace transforms of $c_j(t)$ and $o_j(t)$, which in turn are substituted into the Laplace transform of the accounting equation (Equation 3.1).

First we take the Laplace transform of Equation (3.1):

$$L[y_{(1)j}(t)] = \mathcal{L}[c_j(t)] + \mathcal{L}[o_j(t)] + \mathcal{L}[e_j^*(t)] - \mathcal{L}[m_j^*(t)] + \mathcal{L}[g_j^*(t)] \qquad \text{(A3.1)}$$

Taking the derivative of Equation (3.4) and combining it with Equation (3.2) gives:

$$r_j(t) = \dot{y}_{(1)j}(t) / \rho_j + \dot{w}_j(t) \qquad \text{(A3.2)}$$

$$= y_{(1)j}(t) / \rho_j + \alpha_j \, y_{(1)j}(t), \text{ using Equation (3.3).} \qquad \text{(A3.3)}$$

This is an expression solely in terms of $y_{(1)j}(t)$, having eliminated the other variable $w_j(t)$.

Taking the Laplace transform of this equation gives:

$$\mathcal{L}[r_j(t)] = [s \, \mathcal{L}[y_{(1)j}(t)] - y_{(1)j}(0^*)] / \rho_j + \alpha_j \, \mathcal{L}[y_{(1)j}(t)]. \qquad \text{(A3.4)}$$

Also, the Laplace transforms of Equations(3.5) and (3.6) are:

$$L[r_j(t)] = L[k^*_{(r)}(t)]\ L[o_j(t)] \tag{A3.5}$$

$$L[c_j(t)] = \mu_j\ L[k^*_{(c)}(t)]\ L[y_{(1)j}(t)] \tag{A3.6}$$

Substituting Equation (A3.5) into (A3.4) leads to an expression for the Laplace tranform of $o_j(t)$:

$$L[o_j(t)] = \frac{1}{\rho_j}\left[\frac{s\ L[y_{(1)j}(t)] - y_{(1)j}(0)^*}{L[k^*_{(r)}(t)]}\right] + \alpha_j\left[\frac{L[y_{(1)j}(t)]}{L[k^*_{(r)}(t)]}\right] \tag{A3.7}$$

Then, substituting Equations (A3.6) and A3.7) into (A3.1) gives an expression solely with exogenous terms:

$$L[y_{(1)j}(t)] - \mu_j\ L[k^*_{(c)}(t)]\ L[y_{(1)j}(t)] - \frac{s\ L[y_{(1)j}(t)]}{\rho_j L[k^*_{(r)}(t)]} - \frac{\alpha_j\ L[y_{(1)j}(t)]}{L[k^*_{(r)}(t)]} \tag{A3.8}$$

$$= -\frac{y_j(0)^*}{\rho_j\ L[k^*_{(r)}(t)]} + L[e^*_j(t)] - L[m^*_j(t)] + L[g^*_j(t)]$$

Rearranging Equation (A3.8) so that $L[y_{(1)j}(t)]$ is on the left hand side of the equation, and then taking the inverse Laplace transform, gives Equation (3.7).

<u>Result 3.2</u> Apply Assumption 3.2 to Result 3.1, and evaluate the Laplace transforms:

$$y_{(1)j}(t) = L^{-1}\left[\frac{y_{(1)j}(0)^* - \rho_j(\varepsilon_j+g_j^*) / s}{s - \rho_j(1-\mu_j-\alpha_j}\right] \qquad \text{(A3.9)}$$

$$= L^{-1}\left[\frac{y_{(1)j}(0)^* \; s - \rho_j(\varepsilon_j+g_j^*)}{s\;[s - \rho_j(1-\mu_j-\alpha_j)]}\right] \qquad \text{(A3.10)}$$

$$= L^{-1}\left[\frac{y_{(1)j}(0)^*}{s[s-\rho_j(1-\mu_j-\alpha_j)]} - \frac{\rho_j(\varepsilon_j+y_j^*)}{s[s-\rho_j(1-\mu_j-\alpha_j)]}\right] \text{(A3.11)}$$

$$= y_{(1)j}(0)^* \exp(\rho_j(1-\alpha_j-\mu_j)t) \qquad \text{(3.11)}$$

$$- \frac{\varepsilon_j+g_j^*}{1-\mu_j-\alpha_j} \; [1-\exp(\rho_j(1-\mu_j-\alpha_j)t)] .$$

<u>Result 3.3</u> This result is obtained by taking the partial derivative of Equation (3.11) with respect to the appropriate parameters. For example,

$$\frac{\partial y_{(1)j}(t)}{\partial \mu_j} = y_{(1)j}(0)^* \exp\;(\rho_j(1-\alpha_j)t)(-\rho_j t)$$
$$\exp(-\rho_j\mu_j t) + (\varepsilon_j+g_j^*)/\delta_j^2 \qquad \text{(A3.12)}$$
$$\cdot(1-\exp(\rho_j\delta_j t)) + (\varepsilon_j+g_j^*)/\delta_j$$
$$[-\exp(\rho_j(1-\alpha_j)t)\;(-\rho_j t)\;\exp(-\rho_j\mu_j t)].$$

<u>Result 3.4(a)</u> Since $[k^*_{(c)}(t)]=1$ (Assumption 3.2), per capita consumption is:

$$c_j(t) = \mu_j \; y_{(1)j}(t), \text{ using Equation (3.6).} \qquad \text{(A3.15)}$$

Substituting Equation (3.11) into (A3.13), and then differentiating with respect to ε_j gives Equation (3.13).

(b) Since $L[k^*_{(r)}(t)]=1$ (Assumption 3.2), Equation (3.5) means that

$$r_j(t) = o_j(t) \tag{A3.14}$$

Substituting Equations (A3.14), (3.11), (A3.13), (3.8) and (3.10), into (3.1), and then differentiating with respect to ε_j gives:

$$\frac{\partial r_j(t)}{\partial \varepsilon_j} = \frac{\partial}{\partial \varepsilon_j} [y_{(1)j}(t) - c_j(t) - \varepsilon_j - g_j^*]$$

$$= (1-\mu_j) \frac{1-\exp(\rho_j \delta_j t)}{\delta_j} - 1$$

$$= [(1-\mu_j) (1-\exp(\rho_j \delta_j t) - (1-\mu_j(-\alpha_j))] / \delta_j$$

$$= [(1-\mu_j) (1-\exp(\rho_j \delta_j t)-1) + \alpha_j] / \delta_j$$

$$= [\alpha_j - (1-\mu_j) \exp(\rho_j \delta_j t)] / \delta_j \tag{3.14}$$

<u>Result 3.5</u> The result is obtained using Pontryagin's Maximum Principle. Define $Y_j(t)=\dot{y}_{(1)j}(t)$. From Assumptions 3.3-3.5, the optimal control problem is:

$$\text{Minimize} \quad I = \int_{t=0}^{\tau} \sum_j \zeta_j u_j(t)^2 \, dt \tag{3.17}$$

$$\text{subject to} \quad \dot{y}_j(t) = Y_j(t) + u_j(t), \tag{3.16}$$

$$y_j(\tau) = y_j(\tau)^* \tag{3.15}$$

Construct the Hamiltonian:

$$H = -\sum_j \zeta_j u_j(t)^2 + \sum_j p_j(t) [Y_j(t)+u_j(t)], \tag{A3.20}$$

where $p_j(t)$ is the costate variable.

To obtain the adjoint equations, set

$$-\frac{\partial H}{\partial y_j(t)} = 0 = \dot{p}_j(t), \tag{A3.21}$$

so that $p_j(t) = G_{(1)j}$, (A3.22)

where $G_{(1)j}^*$ is a constant of integration.

In order to solve the minimization problem, it is necessary to maximize the Hamiltonian with respect to $u_j(t)$:

$$\frac{\partial H}{\partial u_j(t)} = -2\zeta_j u_j(t) + p_j(t) = 0 \tag{A3.23}$$

which gives an expression for the control variable:

$$u_j(t) = p_j(t) / 2\zeta_j$$

$$= G_{(1)j}^* / 2\zeta_j \tag{3.24}$$

Equations (A3.22) and (A3.24) are necessary conditions for the solution of the optimization problem. Substituting them into Equation (3.16) gives a differential equation which leads to:

$$y_j(t) = \int^t Y_j(t') \, dt' + G_{(1)j}^* \, t / 2\zeta + G_{(2)j}^*$$

$$= y_{(1)j}(t) + G_{(1)j}^* \, t / 2\zeta_j + G_{(2)j}^* \quad , \tag{A3.25}$$

where $G_{(2)j}^*$ is another constant of integration.

Recall that we know the present income level, $y_{(1)j}(0)^*$, and the taget, $y_j(\tau)^*$. These are termed boundary conditions, which can be utilized by letting t=0 and then letting t=τ in Equation (A3.25); this leads to the following two equations:

$$G^*_{(2)j} = 0 \tag{A3.26}$$

$$G^*_{(1)j} \; \tau / 2\zeta_j = y_j(\tau)^* - y_{(1)j}(\tau) - G^*_{(2)j}, \tag{A3.27}$$

from which

$$G^*_{(1)j} = [y_j(\tau)^* - y_{(1)j}(\tau)] \; 2\zeta_j / \tau \; . \tag{A3.28}$$

Substituting Equation (A3.28) into (A3.24) and (A3.25) results in,

$$u_j(t) = [y_j(\tau)^* - y_{(1)j}(\tau)] / \tau \tag{3.18}$$

$$y_j(t) = y_{(1)j}(t) + [y_j(\tau)^* - y_{(1)j}(\tau)] \; t/\tau \; . \tag{3.19}$$

To show that these trajectories correspond to a maximum in the Hamiltonian, it is only necessary to note that

$$\frac{\partial^2 H}{\partial u_j(t) \partial u_i(t)} = 0 \qquad , \; i \neq j, \tag{A3.29}$$

and

$$\frac{\partial^2 H}{\partial u_j(t)^2} = 2\zeta_j \tag{A3.30}$$

Hence, the Hessian matrix, consisting of the partial derivatives $\partial^2 H / \partial u_j(t) \partial u_k(t)$, is diagonal, with negative elements (recall that $\zeta_j > 0$). In a diagonal matrix, the elements along the diagonal are the eigenvalues. Since the eigenvalues are all negative here, it follows that the Hessian is negative definite, in which case we have a maximum for H. Since the extremal given by Equation (3.18) is the only candidate for an optimal trajectory, it is just that.

Result 3.6 Since $t[y_j(\tau)^* - y_{(1)j}(\tau)]/\tau$ is linear, it follows from Result 3.5 that $y_j(t)$ is concave (convex) if $y_{(1)j}(t)$ is concave (convex) (Lancaster, 1968, p. 331).

Result 3.7 Substitute Result 3.2 into Result 3.5.

Result 3.8 Take the partial derivative of Equation (3.25) with respect to the appropriate parameter. For example,

$$\frac{\partial u_j(t)}{\partial \alpha_j} = \frac{1}{\tau} \{- y_j(0)^* \; (-\rho_j \;) \exp(\rho_j \delta_j \tau)$$

$$- [\frac{\varepsilon_j + g_j^*}{\delta_j} (-1) \; (-\rho_j \tau) \exp(\rho_j \delta_j \tau)$$

$$+ (1-\exp(\rho_j \delta_j \tau)) \; (-\varepsilon_j - g_j^*) / \delta_j^2]\}$$

$$= \frac{1}{\tau} [y_j(0)^* \; \rho_j \; \tau \exp(\rho_j \delta_j \tau) \tag{A3.31}$$

$$- \frac{(\varepsilon_j + g_j^*)}{\delta_j} \rho_j \; \tau \exp(\rho_j \delta_j \tau)$$

$$- \frac{(1-\exp(\rho_j \delta_j \tau)) \; (\varepsilon_j + g_j^*)}{\delta_j^2} \quad]$$

Result 3.9 Take the required partial derivatives of Equation (3.21). For example,

$$\frac{\partial y_j(t)}{\partial \rho_j} = \delta_j\ t\ \exp(\rho_j \delta_j t)\ [y_j(0)^* - (\varepsilon_j + g_j^*)]$$

$$+ \frac{t}{\tau}\ [-y_j(0)^* - (\varepsilon_j + g_j^*)/\delta_j]\ \tau\ \delta_j\ \exp(\rho_j \delta_j \tau)$$

$$= t\ \exp(\rho_j \delta_j t)\ [y_j(0)^*\ \delta_j - (\varepsilon_j + g_j^*)]$$

$$+ t\ \exp(\rho_j \delta_j \tau)\ [-y_j(0)^*\ \delta_j + (\varepsilon_j + g_j^*)]. \qquad (3.39)$$

Result 3.10 Recall Results 3.2 and 3.7, and subtract Equation (3.11) from (3.21).

Result 3.11 Follows from Result 3.10.

Result 3.12 Obvious.

Result 3.13 The proof is similar to that of Result 3.14, but with $u_j = 0$ (recall Result 3.10).

Result 3.14 Substituting Result 3.7 into Assumption 3.6, and then substituting this in turn into Assumption 3.7 gives a nonhomogeneous linear first order differential equation with constant coefficients, whose solution is given below:

$$x_j(t) = \exp(\phi_j t)\ \{ \int \exp(-\phi_j t) \sum_{i \neq j} [A^*_{(1)ij} + A^*_{(2)ij} \exp(\rho_j \delta_j t) - A^*_{(3)ij} \exp(\rho_i \delta_i t) + A^*_{(4)ij}\ t]\ dt + G^*_{(3)j} \} \qquad (3.32)$$

where $A^*_{(1)ij}$, $A^*_{(2)ij}$, $A^*_{(3)ij}$ and $A^*_{(4)ij}$ are defined by Equations (3.29), (3.30), (3.31), and (3.33), respectively;

and $G^*_{(3)j}$ is a constant of integration.

Evaluating the integral in Equation (A3.30) produces:

$$x_j(t) = \sum_{i\neq j}\left[\frac{A^*_{(1)ij}}{\phi_j} + \frac{A^*_{(2)ij}\exp(\rho_j\delta_j t)}{\rho_j\delta_j-\phi_j} - \frac{A^*_{(3)ij}\exp(\rho_j\delta_j t)}{\rho_i\delta_i-\phi_j} + \frac{A^*_{(4)ij}(-1-\phi_j t)}{\phi_j^2}\right] + G^*_{(3)j}\exp(\phi_j t). \qquad (A3.33)$$

We can take advantage of the fact that we know the present population. Let t=0 in Equation (A3.31), so that

$$G^*_{(3)j} = x_j(0)^* - \sum_{i\neq j}\left[-\frac{A^*_{(1)ij}}{\phi_j} + \frac{A^*_{(2)ij}}{\rho_j\delta_j-\phi_j} - \frac{A^*_{(3)ij}}{\rho_i\delta_i-\phi_j} - \frac{A^*_{(4)ij}}{\phi_j^{\,2}}\right] \qquad (A3.34)$$

Substitute Equation (A3.32) into (A3.31):

$$x_j(t) = x_j(0)^*\exp(\phi_j t) + \sum_{i\neq j}\left[\frac{A^*_{(1)ij}}{\phi_j}[\exp(\phi_j t)-1] - \frac{A^*_{(2)ij}}{\rho_j\delta_j-\phi_j}[\exp(\rho_j\delta_j t)-\exp(\phi_j t)] - \frac{A^*_{(3)ij}}{\rho_i\delta_i-\phi_i}[\exp(\rho_i\delta_i t)-\exp(\phi_j t)] - \frac{A^*_{(4)ij}}{\phi_j^2}[1+\phi_j t-\exp(\phi_j t)]\right] \qquad (A3.35)$$

APPENDIX 4

PROOFS OF RESULTS IN CHAPTER 4

Result 4.1 The proof is similar to that of Result 3.1. The Laplace transform of Equation (4.1) is:

$$L[y_j(t)] = L[c_j(t)] + L[o_j(t)] + L[e_j^*(t)] - L[m_j^*(t)] + L[u_j(t)] \quad \text{(A4.1)}$$

Differentiating Equation (4.4), substituting the resulting expression for $\dot{q}_j(t)$ into Equation (4.2), and then substituting the expression for $\dot{w}_j(t)$ from Equation (4.3) into that equation yields:

$$r_j(t) = \dot{y}_j(t)/\rho_j + \alpha_j y_j(t) , \quad \text{(A4.2)}$$

whose Laplace transform is

$$L[r_j(t)] = [s\ L[y_j(t)] - y_j(0)^*] / \rho_j + \alpha_j\ L[y_j(t)] . \quad \text{(A4.3)}$$

The Laplace transform of Equation (4.5) is

$$L[r_j(t)] = L[k^*_{(r)}(t)]\ L[o_j(t)], \quad \text{(A4.4)}$$

which when substituted into Equation (A4.3) gives:

$$L[o_j(t)] = \frac{1}{\rho_j}\left[\frac{s\ L[y_j(t)] - y_j(0)^*}{L[k^*_{(r)}(t)]}\right] + \alpha_j \frac{L[y_j(t)]}{L[k^*_{(r)}(t)]} \quad \text{(A4.5)}$$

The Laplace transform of Equation (4.5) is

$$L[r_j(t)] = L[k^*_{(r)}(t)]\ L[o_j(t)]\ , \qquad \text{(A4.4)}$$

which when substituted into Equation (A4.3) gives:

$$L[o_j(t)] = \frac{1}{\rho_j}\left[\frac{L[y_j(t)] - y_j(0)^*}{L[k^*_{(r)}(t)]}\right] + \alpha_j\ \frac{L[y_j(t)]}{L[k^*_{(r)}(t)]} \qquad \text{(A4.5)}$$

The Laplace transform of Equation (4.6) is

$$L[c_j(t)] = \mu_j\ L[k^*_{(c)}(t)]\ [L[y_j(t)] - L[T^*_j(t)]]. \qquad \text{(A4.6)}$$

Equations (A4.6) and (A4.5) denote the Laplace transform of $c_j(t)$ and $o_j(t)$ in terms of $y_j(t)$. Hence, we can obtain the reduced form for $y_j(t)$ by substituting Equations (A4.6) and (A4.5) into (A4.1), which results in:

$$L[y_j(t)] - \mu_j\ L[k^*_{(c)}(t)]\ L[y_j(t)] - \frac{s\ L[y_j(t)]}{\rho_j\ [k^*_{(r)}(t)]}$$

$$- \frac{\alpha_j\ L[y_j(t)]}{L[k^*_{(r)}(t)]}$$

$$= - \frac{y_j(0)^*}{\rho_j\ L[k^*_{(r)}(t)]} + L[e^*_j(t)] - L[m^*_j(t)]$$

$$+ L[u_j(t)] - \mu_j\ L[k^*_{(c)}(t)]\ L[T^*_j(t)]. \qquad \text{(A4.7)}$$

This expression can be rearranged so as to have only $L[y_j(t)]$ on the left side of the equation. Taking the inverse Laplace transform of this latter equation results in Equation (4.7).

<u>Result 4.2</u> Substitute Equations (4.8) and (4.9) into (A4.7),

using the fact that $L[\varepsilon_j]=\varepsilon_j/s$:

$$y_j(t) = L^{-1}\left[\frac{s\ y_j(0)^* - \rho_j\ [\varepsilon_j + s\ L[u_j(t)] - \mu_j\ s\ L[T_j^*(t)]]}{s\ [s - \rho_j(1-\mu_j-\alpha_j)]}\right]$$

$$= L^{-1}\left[\frac{y_j(0)^*}{s - \rho_j(1-\mu_j-\alpha_j)} - \frac{\rho_j\ \varepsilon_j}{s\ [s - \rho_j(1-\mu_j-\alpha_j)]}\right.$$

$$\left. - \frac{\rho_j\ [L[u_j(t)] - \mu_j\ L[T_j^*(t)]]}{s - \rho_j(1-\mu_j-\alpha_j)}\right] \qquad \text{(A4.8)}$$

$$= y_j(0)^*\ \exp(\rho_j(1-\mu_j-\alpha_j)t) - \rho_j \int_{z=0}^{t} u_j(z)$$

$$- \mu_j\ T_j^*(z)]\ \exp[\rho_j(1-\mu_j-\alpha_j)$$

$$(t-z)]\ dz \qquad \text{(A4.9)}$$

$$- \frac{\rho_j\ \varepsilon_j}{(-\rho_j)(1-\mu_j-\alpha_j)}\ [1 - \exp(\rho_j(1-\mu_j-\alpha_j)t]\ ,$$

which reduces to Equation (4.19) after Equation (4.10) is used.

Result 4.3   First recall the following result from Courant (1938, vol. 2, p. 220).

Let F(t) be a function:

$$F(t) = \int_{\Gamma_{(2)}(t)}^{\Gamma_{(1)}(t)} E\ (t,z)\ dz \qquad \text{(A4.10)}$$

where $\Gamma_{(1)}(t),\Gamma_{(2)}(t)$ are functions of t; E(t,z) is a function of t and z.

Then, $\dot{F}(t) = \int_{\Gamma_{(2)}(t)}^{\Gamma_{(1)}(t)} E_t(t,z)\ dz$

$$- \dot{\Gamma}_{(1)}(t)\ E(t,\Gamma_{(1)}(t)) \qquad \text{(A4.11)}$$

$$+ \dot{\Gamma}_{(2)}(t)\ E(t,\Gamma_{(2)}(t)) ,$$

where $\dot{F}(t) = dF(t)/dt$

$$E_t(t,z) = \partial E(t,z)/\partial t \quad .$$

Let

$$U_j(t) = u_j(t) - \mu_j\ T_j^*(t) \quad . \qquad (4.12)$$

Using Equation (A4.11),

$$\frac{d}{dt}[\int_0^t U_j(z)\ \exp(\rho_j\delta_j(t-z))\ dz\ ]$$

$$= \int_0^t U_j(z)\rho_j\delta_j\ \exp(\rho_j\delta_j(t-z))\ dz + U_j(t) \quad . \qquad \text{(A4.13)}$$

Differentiate Equation (4.19) and substitute Equations (A4.13) and (A4.12) into the resulting expression for $\dot{y}_j(t)$. This results in Equation (4.21), where it is assumed that $T_j^*(t)=\chi_j\exp(\psi_j t)$.

<u>Result 4.4</u> Pontryagin's Maximum Principle is used to solve the optimal control problem defined by Assumptions 4.3 and 4.4 and Result 4.3. First, a preliminary result is proved. Recall that

$$\frac{d}{dt}\ [\ \int_0^t\ (u_j(z) - \mu_j T_j^*(z))\ \exp(\rho_j\delta_j(t-z))\ dz]$$

$$= \int_0^t\ (u_j(z) - \mu_j\ T_j^*(z))\ \rho_j\delta_j\ \exp(\rho_j\delta_j(t-z))\ dz$$

$$+ (u_j(t) - \mu_j T_j^*(t)) \quad . \qquad \text{(A4.14)}$$

Then, after realizing that

$$\frac{dE(u_j(t),t)}{du_j(t)} = \frac{d}{dt}E(u_j(t),t)/\frac{du_j(t)}{dt} \quad , \qquad (A4.15)$$

where E is a function, we can use Equation (A4.14) to show that

$$\frac{d}{du_j(t)}[\int_0^t u_j(z)\exp(\rho_j\delta_j(t-z))\,dz] \qquad (A4.16)$$

$$= [\int_0^t u_j(z)\,\rho_j\delta_j\exp(\rho_j\delta_j(t-z))\,dz$$

$$+ u_j(t)] / \dot{u}_j(t) \quad .$$

Now turn to the optimal control problem:

$$\text{Minimize} \quad I = \int_{t=0}^{\tau} \sum_j \zeta_j u_j(t)\,dt \qquad (4.24)$$

$$\text{subject to } \dot{y}_j(t) = \rho_j\delta_j\,(y_j(0)^* - \varepsilon_j/\delta_j)\exp(\rho_j\delta_j t) \qquad (4.21)$$

$$- \rho_j^2\delta_j \int_0^t (u_j(z)-\mu_j T_j^*(z))\exp(\rho_j\delta_j$$

$$(t-z))\,dz - \rho_j\,(u_j(t)-\mu_j T_j^*(t))$$

$$y_j(\tau) = y_j(\tau)^* \qquad (4.25)$$

Construct the Hamiltonian:

$$H = -\sum_j \zeta_j u_j(t) + \sum_j p_j(t)\,[\rho_j\delta_j\,(y_j(0)^*-\varepsilon_j/\delta_j)\exp(\rho_j\delta_j t)$$

$$- \rho_j^2\delta_j \cdot \int_0^t (u_j(z)-\mu_j T_j^*(z))\exp(\rho_j\delta_j(t-z))\,dz$$

$$- \rho_j\,(u_j(t)-\mu_j T_j^*(t))] \quad , \qquad (A4.17)$$

where $p_j(t)$ is the costate variable.

The adjoint equations are obtained from:

$$\frac{\partial H}{\partial y_j(t)} = 0 = \dot{p}_j(t) \quad (A4.18)$$

whence

$$p_j(t) = G_j^* \text{ , a constant of integration.} \quad (A4.19)$$

In order to minimize I, it is necessary to maximize H:

$$\frac{\partial H}{\partial u_j(t)} = -\zeta_j - [p_j(t)\ \rho_j^3\delta_j^2 \int_0^t u_j(z) \exp(\rho_j\delta_j(t-z))\, dz$$

$$+ p_j(t)\ \rho_j^2\delta_j\ u_j(t)] / \dot{u}_j(t) - p_j(t)\rho_j \quad (A4.19)$$

$$= 0$$

The necessary condition given by Equation (A4.19) leads to an expression for $u_j(t)$. We first simplify the integro-differential equation by taking the Laplace transform:

$$L[u_j(t)] = \{ -(G_j^*\rho_j+\zeta_j)\ u_j(0)^*\ (s-\rho_j\delta_j)\}$$

$$\{-[G_j^*\rho_j+\zeta_j]\ s^2 + [-G_j^*\rho_j^2\delta_j$$

$$+ \rho_j\delta_j(G_j^*\rho_j+\zeta_j)]\ s\ \}^{-1} \quad (A4.20)$$

$$= \frac{u_j(0)^*\ (s-\rho_j\delta_j)}{s\ [s - \frac{\rho_j\delta_j\zeta_j}{G_j^*\rho_j+\zeta_j}]} \text{ , using Equation (A4.18).}$$

Taking the inverse Laplace transform of Equation (A4.20), and using the result given in Roberts and Kaufmann (1966, p. 181), yields:

$$u_j(t) = u_j(0)^* \left[\frac{\rho_j\delta_j}{\frac{\rho_j\delta_j\zeta_j}{G_j^*\rho_j+\zeta_j}} + \left[1 - \frac{\rho_j\delta_j}{\frac{\rho_j\delta_j\zeta_j}{G_j^*\rho_j+\zeta_j}} \right] \exp\left(\frac{\rho_j\delta_j\zeta_j}{G_j^*\rho_j+\zeta_j} t \right) \right], \tag{A4.21}$$

which can be rearranged to give Equation (4.28), the optimal trajectory of per capita government spending. This in turn is substituted into Equation (4.19):

$$\begin{aligned} y_j(t) &= \varepsilon_j/\delta_j + (y_j(0)^* - \varepsilon_j/\delta_j) \exp(\rho_j\delta_j t) \\ &+ \rho_j\mu_j \int_{z=o}^{t} T_j^*(z) \exp(\rho_j\delta_j(t-z))\, dz \\ &- \rho_j \int_{2=0}^{t} u_j(0)^* \left[\frac{G_j^*\rho_j}{\zeta_j} + 1 + \frac{G_j^*\rho_j}{\zeta_j} \exp(\rho_j\delta_j\zeta_j/(G_j^*\rho_j+\zeta_j)t) \right] \exp(\rho_j\delta_j(t-z))\, dz \qquad \text{(A4.22)} \\ &= \varepsilon_j/\delta_j + (y_j(0)^* - \varepsilon_j/\delta_j) \exp(\rho_j\delta_j t) \\ &+ \frac{\chi_j\rho_j\mu_j \exp(\rho_j\delta_j t)}{\psi_j - \rho_j\delta_j} [\exp(\psi_j - \rho_j\delta_j)t - 1] + \frac{u_j(0)^*}{\delta_j} \left[\frac{G_j^*\rho_j}{\zeta_j} + 1 + \frac{G_j^*\rho_j}{\zeta_j} \exp(\rho_j\delta_j\zeta_j/(G_j^*\rho_j+\zeta_j)t) \right] \end{aligned}$$

$$y_j(t) = \exp(\rho_j\delta_j t)\ [\exp(-\rho_j\delta_j t) - 1],$$

which can be rearranged to give Equation (4.29). These results are based on the necessary conditions for H to be a maximum (Equation (A4.19)). To prove sufficiency, note that:

$$\frac{\partial H^2}{\partial u_j(t)\partial u_i(t)} = 0 \qquad , i \neq j \tag{A4.23}$$

$$\frac{\partial^2 H}{\partial u_j(t)^2} = - \frac{1}{\dot{u}_j(t)} \{p_j(t)\rho_j{}^3\delta_j{}^2[\int_0^t u_j(z)\ \rho_j\delta_j \exp(\rho_j\delta_j(t-z))\ dz + u_j(t)] / u_j(t) + p_j(t)\rho_j{}^2\delta_j\}$$

$$= - \frac{G_j^*\rho_j{}^3\delta_j{}^2 \int_0^t u_j(z)\ \rho_j\delta_j \exp(\rho_j\delta_j(t-z))\ dz}{(\dot{u}_j(t))^2} - \frac{G_j^*\rho_j{}^3\delta_j{}^2 u_j(t)}{(\dot{u}_j(t))^2} - \frac{G_j^*\rho_j{}^2\delta_j}{\dot{u}_j(t)} \tag{A4.24}$$

To simplify this expression, we must eliminate all $u_j(t)$ terms using appropriate substitutions. Recalling Equation (4.28), we see that

$$\dot{u}_j(t) = - u_j(0)^* \frac{G_j^*\rho_j{}^2\delta_j}{(G_j^*\rho_j+\zeta_j)} \exp\left(\frac{\rho_j\delta_j\zeta_j}{G_j^*\rho_j+\zeta_j}\ t\right) \tag{A4.25}$$

Also,

$$\int_0^t u_j(z)\ \rho_j\delta_j\ \exp(\rho_j\delta_j(t-z))\ dz$$

$$= u_j(0)^*\ \frac{G_j^*\rho_j+\zeta_j}{\zeta_j}\left[\exp\left(\frac{\rho_j\delta_j\zeta_j}{G_j^*\rho_j+\zeta_j}t\right) - 1\right] . \qquad (A4.26)$$

Substituting Equations (4.28), (A4.25) and (A4.26) into (A4.24), we find that

$$\frac{\partial^2 H}{\partial u_j(t)^2} = -\ \frac{\zeta_j(G_j^*\rho_j+\zeta_j)}{u_j(0)^*\rho_j G_j^*}\ \exp\left(-\frac{\rho_j\delta_j\zeta_j}{G_j^*\rho_j+\zeta_j}\ t\right) . \qquad (A4.27)$$

If $\partial^2 H/\partial u_j(t)^2<0$, then the Hessian is negative definite, in which case H is maximized and *I* minimized. This happens when either

$$G_j^* > 0 \quad \text{or} \qquad (4.30)$$

$$G_j^*\rho_j+\zeta_j < 0 \quad . \qquad (4.31)$$

As indicated in Appendix 1, maximizing H is only a necessary condition for minimizing *I*. Yet it is unnecessary to invoke a formal sufficiency theorem, since all extremals (candidates for the optimal trajectory) have been identified; and in this case it is unique.

<u>Result 4.5</u> The result follows from

$$\frac{\partial u_j(t)}{\partial t} = -\ u_j(0)^*\ \frac{G_j^*\rho_j{}^2\delta_j}{G_j^*\rho_j+\zeta_j}\ \exp\left(\frac{\rho_j\delta_j\zeta_j}{G_j^*\rho_j+\zeta_j}\ t\right) , \qquad (A4.28)$$

where $u_j(t)$ is given by Equation (4.28).

APPENDIX 5

PROOF OF RESULT IN CHAPTER 5

<u>Result 5.1</u> We use the necessary conditions derived for the J region case, and the values given in Assumption 5.4. From Equation (5.15),

$$\begin{bmatrix} p_{(1)1}(t) \\ p_{(1)2}(t) \end{bmatrix} = \begin{bmatrix} G^*_{(1)1}\exp(-\phi_1 t) \\ G^*_{(1)2}\exp(-\phi_2 t) \end{bmatrix} = \begin{bmatrix} G^*_{(1)1}\exp(-.02t) \\ G^*_{(1)2}\exp(-.02t) \end{bmatrix}, \qquad \text{(A5.1)}$$

where $G^*_{(1)1}$, $G^*_{(1)2}$ are constants of integration. Substituting Equation (A5.1) into (5.16) results in a system of linear first order differential equations:

$$\begin{bmatrix} \dot{p}_{(2)1}(t) \\ \dot{p}_{(2)2}(t) \end{bmatrix} = \begin{bmatrix} -\nu_1 - \pi_1(\Gamma^*_2/\Gamma^*_1+1)\ \eta, & \pi_1(\Gamma^*_1/\Gamma^*_2+1)\ \eta \\ \pi_2\ (\Gamma^*_2/\Gamma^*_1+1)\ \eta, & \nu_2 - \pi_2(\Gamma^*_1/\Gamma^*_2+1)\ \eta \end{bmatrix} \begin{bmatrix} p_{(2)1}(t) \\ p_{(2)2}(t) \end{bmatrix} +$$

$$\begin{bmatrix} -G^*_{(1)1}\exp(-\phi_1 t)\ \gamma_1\ (\Gamma^*_2+\Gamma^*_1)\ \eta\ + \\ \qquad G^*_{(1)2}\exp(-\phi_2 t)\ \gamma_2(\Gamma^*_1+\Gamma^*_2)\ \eta \\ \\ -G^*_{(1)2}\exp(-\phi_2 t)\ \gamma_2\ (\Gamma^*_1+\Gamma^*_2)\ \eta\ + \\ \qquad G^*_{(1)1}\ \exp(-\phi_1 t)\ \gamma_1\ (\Gamma^*_2+\Gamma^*_1)\ \eta \end{bmatrix}$$

$$= \begin{bmatrix} -.0506 & .0012 \\ .0006 & -.0512 \end{bmatrix} \begin{bmatrix} p_{(2)1}(t) \\ p_{(2)2}(t) \end{bmatrix} \qquad (A5.2)$$

$$+ \begin{bmatrix} -120\exp(-.02t)\ G^*_{(1)1} + 120\exp(-.02t)\ G^*_{(1)2} \\ -120\exp(-.02t)\ G^*_{(1)2} + 120\exp(-.02t)\ G^*_{(1)1} \end{bmatrix},$$

whose homogeneous system is

$$\begin{bmatrix} \dot{p}_{(2)1}(t) \\ \dot{p}_{(1)1}(t) \end{bmatrix} = \begin{bmatrix} -.0506 & .0012 \\ .0006 & -.0512 \end{bmatrix} \begin{bmatrix} p_{(2)1}(t) \\ p_{(2)1}(t) \end{bmatrix} \qquad (A5.3)$$

The eigenvalues are:

-.0500, -.0518 ,

and the corresponding eigenvectors are:

$$\begin{bmatrix} -.8944 \\ .4472 \end{bmatrix} \qquad \begin{bmatrix} -.7454 \\ .7454 \end{bmatrix}$$

Hence, the fundamental matrix is:

$$\underline{\Psi}(t) = \begin{bmatrix} -.8944\exp(-.05t) & -.7454\exp(-.0518t) \\ .4472\exp(.05t) & .7454\exp(-.0518t) \end{bmatrix}, \quad \text{(A5.4)}$$

and

$$\underline{\Psi}^{-1}(t) = \begin{bmatrix} -2.2362\exp(.05t) & -2.2362\exp(.05t) \\ 1.3416\exp(.0518t) & 2.6832\exp(.0518t) \end{bmatrix} \quad \text{(A5.5)}$$

Using Equations (A5.4) and (A5.5) and recalling a result in Boyce and DiPrima (1969, page 318, equation 12), we get the solution of Equation (A5.2):

$$\begin{bmatrix} P_{(2)1}(t) \\ \\ P_{(2)2}(t) \end{bmatrix} = \begin{bmatrix} [2\exp(-.05t) - \exp(-.0518t)]\, G^*_{(2)1} + \\ [2\exp(-.05t) - 2\exp(-.0518t)]\, G^*_{(2)2} \\ [-\exp(-.05t) + \exp(-.0518t)]\, G^*_{(2)1} + \\ [-\exp(-.05t) + 2\exp(-.0518t)]\, G^*_{(2)2} \end{bmatrix} + \begin{bmatrix} 3.774\text{x}10^3\ (G^*_{(1)1}-G^*_{(1)2})\ [\exp(-.0836t) - \exp(-.0518t)] \\ -3.774\ (G^*_{(1)1}-G^*_{(1)2})\ [\exp(-.0836t) - \exp(-.0518t)] \end{bmatrix} \quad \text{(A5.6)}$$

Substituting the expressions for $u_j(t)$ and $v_j(t)$, obtained from Equations (5.17) and (5.18), into Equations (5.9) and (5.10), we get:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} = \begin{bmatrix} .02 & 0 & 1.2\text{x}10^2 & -1.2\text{x}10^2 \\ 0 & .02 & -1.2\text{x}10^2 & 1.2\text{x}10^2 \\ 0 & 0 & 5.06\text{x}10^{-2} & -6.0\text{x}10^{-4} \\ 0 & 0 & -1.2\text{x}10^{-3} & 5.12\text{x}10^{-2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ y_1(t) \\ y_2(t) \end{bmatrix}$$

$$+ \begin{bmatrix} -6.5\text{x}10^4 + p_{(1)1}(t)/2 \\ 6.5\text{x}10^4 + p_{(1)2}(t)/2 \\ -3.25\text{x}10^{-1} + p_{(2)1}(t)/2\text{x}10^6 \\ 6.5\text{x}10^{-1} + p_{(2)2}(t)/2\text{x}10^6 \end{bmatrix} . \qquad \text{(A5.7)}$$

The eigenvalues of this homogeneous system are:

.0200, .0200, .0500, .0518 ;

and the corresponding eigenvectors are:

$$\begin{bmatrix} 1.0000 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1.0000 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -.7071 \\ -.7071 \end{bmatrix}, \begin{bmatrix} .5337 \\ -.5337 \\ .4714 \\ -.9428 \end{bmatrix} .$$

Hence, the fundamental matrix here is:

$$\underline{\underline{\Phi}}(t) = \begin{bmatrix} \exp(.02t) & 0 & 0 & .5337\exp(.0518t) \\ 0 & \exp(.02t) & 0 & -.5337\exp(.0518t) \\ 0 & 0 & -.7071\exp(.05t) & .4714\exp(.0518t) \\ 0 & 0 & -7071\exp(.05t) & -.9428\exp(.0518t) \end{bmatrix}, \qquad \text{(A5.8)}$$

and

$$\underline{\underline{\Phi}}(t)^{-1} = \begin{bmatrix} \exp(-.02t) & 0 & -.3774\exp(-.02t) & .3774\exp(-.02t) \\ 0 & \exp(-.02t) & .3774\exp(-.02t) & -.3774\exp(-.02t) \\ 0 & 0 & -.9428\exp(-.05t) & -.4714\exp(-.05t) \\ 0 & 0 & .7071\exp(-.0518t) & -.7071\exp(-.0518t) \end{bmatrix} \tag{A5.9}$$

Again, we use the result in Boyce and DiPrima, together with Equations (A5.8), (A5.9), (A5.1), and (A5.6), to obtain the solution of Equation (A5.7):

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \exp(.02t) & 0 & 0 & .5337\exp(.0518t) \\ 0 & \exp(.02t) & 0 & -.5337\exp(.0518t) \\ 0 & 0 & -.7071\exp(.05t) & .4714\exp(.0518t) \\ 0 & 0 & -.7071\exp(.05t) & -.9428\exp(.0518t) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -.3774 & .3774 \\ 0 & 0 & .3774 & -.3774 \\ 0 & 0 & -.9428 & -.4714 \\ 0 & 0 & .7071 & -.7071 \end{bmatrix} \begin{bmatrix} 2x10^7 \\ 1x10^7 \\ 6x10^3 \\ 6x10^3 \end{bmatrix} +$$

$$\begin{bmatrix} \exp(.02t) & 0 & 0 & .5337\exp(.0518t) \\ 0 & \exp(.02t) & 0 & -.5337\exp(.0518t) \\ 0 & 0 & -.7071\exp(.05t) & .4714\exp(.0518t) \\ 0 & 0 & -.7071\exp(.05t) & -.9428\exp(.0518t) \end{bmatrix} \tag{A5.10}$$

$$\int_0^t \begin{bmatrix} \exp(-.02s) & 0 & -.3774\exp(-.02s) & .3774\exp(-.02s) \\ 0 & \exp(-.02s) & .3774\exp(-.02s) & -.3774\exp(-.02s) \\ 0 & 0 & -.9428\exp(-.05s) & -.4714\exp(-.05s) \\ 0 & 0 & .7071\exp(-.0518s) & -.7071\exp(-.0578s) \end{bmatrix}$$

$$\left[\begin{array}{l} -6.5\times10^{4} + G^{*}_{(1)1}\ \exp(-.02s)/2 \\ 6.5\times10^{4} + G^{*}_{(1)2}\ \exp(-.02s)/2 \\ -3.25\times10^{-1} + (2\times10^{6})^{-1}\{3.774\times10^{3}\ [\exp(-.0836s)-\exp(-.0518s)] \\ (G^{*}_{(1)1}-G^{*}_{(1)2}) + [2\exp(-.05s)-\exp(-.0518s)]\ G^{*}_{(2)1} + \\ [2\exp(-.05s)-2\exp(-.0518s)]\ G^{*}_{(2)2}\} \\ 6.5\times10^{-1} + (2\times10^{6})^{-1}\{-3.774\times10^{3}\ [\exp(-.0836s)-\exp(-.0518s)] \\ (G^{*}_{(1)1}-G^{*}_{(1)2}) + [-\exp(-.05s)+\exp(-.0518s)]\ G^{*}_{(2)1} + \\ [-\exp(-.05s)+2\exp(-.0518s)]\ G^{*}_{(2)2}\} \end{array}\right] ds$$

We can then set $t=\tau=5$, and use the boundary conditions to obtain the values of the constants of integration:

$$G^{*}_{(1)1} = 1.291\times10^{6} \quad (A5.11)$$

$$G^{*}_{(1)2} = -1.430\times10^{6} \quad (A5.12)$$

$$G^{*}_{(2)1} = 1.332\times10^{8} \quad (A5.13)$$

$$G^{*}_{(2)2} = -6.679\times10^{8} \quad ; \quad (A5.14)$$

which, when substituted into Equations (A5.10), (5.19), (5.20), (A5.6) and (A5.1), give Result 5.1.

Finally, it is well known that sufficiency conditions are satisfied in the linear-quadratic problem.

APPENDIX 6

PROOFS OF RESULTS IN CHAPTER 6

<u>Result 6.1</u> Assumptions 6.1 and 6.3 define an optimal control problem. To use Pontryagin's Maximum Principle, construct the Hamiltonian:

$$H = -\sum_j \zeta_{(1)} \frac{y_j(t)}{\sum_i y_i(t)} \log \frac{y_j(t)}{\sum_i y_i(t)} - \sum_j \zeta_{(2)} u_j(t)^2 \tag{A6.1}$$

$$+ \sum_j \zeta_{(3)} y_j(t) + \sum_j [p_j(t) \ (\nu_j y_j(t) + u_j(t))]$$

One set of necessary conditions for the optimal solution is that

$$-\frac{\partial H}{\partial y_j(t)} = \zeta_{(1)} A_j(t) - \zeta_{(3)} - p_j(t)\nu_j = \dot{p}_j(t), \tag{A6.2}$$

where

$$A_j(t) = \frac{\partial}{\partial y_j(t)} \sum_j \frac{y_j(t)}{\sum_i y_i(t)} \log \frac{y_j(t)}{\sum_i y_i(t)} \tag{A6.3}$$

$p_j(t)$ is the costate variable.

Equation (A6.2) gives an expression for the costate variable, with an undetermined constant of integration, G_j^*:

$$p_j(t) = \exp(-\nu_j t) \ [\int \exp(\nu_j t) \ (\zeta_{(1)} A_j(t)$$

$$-\zeta_{(3)}) \ dt + G_j^*] . \tag{A6.4}$$

Also, for H to be at a maximum:

$$\frac{\partial H}{\partial u_j(t)} = -\ 2\ \zeta_{(2)} u_j(t) + p_j(t) = 0 \quad , \tag{A6.5}$$

which implies that

$$u_j(t) = \frac{p_j(t)}{2\zeta_{(2)}} \quad . \tag{A6.6}$$

Following the proof of Result 3.5, we see that the Hessian is negative definite here as well, and that $u_j(t)$ given by Equation (A6.6) is indeed the optimal trajectory of the control variable. Substituting Equation (A6.4) into (A6.6), and then substituting the result into Equation (6.1) yields:

$$\dot{y}_j(t) = \nu_j y_j(t) + \frac{1}{2\zeta_{(2)}} \exp(-\nu_j t)$$

$$[\int \exp(\nu_j t)\ (\zeta_{(1)} A_j(t) - \zeta_{(3)})\ dt + G_j^*] \quad , \tag{A6.7}$$

whence

$$\ddot{y}_j(t) = \nu_j \dot{y}_j(t) + \frac{1}{2\zeta_{(2)}} \{\exp(-\nu_j t)\ [\exp(\nu_j t)$$

$$(\zeta_{(1)} A_j(t) - \zeta_{(3)})] + [\int \exp(\nu_j t)$$

$$(\zeta_{(1)} A_j(t) - \zeta_{(3)})\ dt + G_j^*]\ (-\nu_j) \tag{A6.8}$$

$$\exp(-\nu_j t)\} \quad .$$

To eliminate the constant of integration from the expression for $y_j(t)$, observe that Equation (A6.7) implies that:

$$\int \exp(-\nu_j t)\ (\zeta_{(1)} A_j(t) - \zeta_{(3)})\ dt + G_j^*$$

$$= (\dot{y}_j(t) - \nu_j y_j(t))\ \frac{2\zeta_{(2)}}{\exp(-\nu_j t)} \quad . \qquad (A6.9)$$

This expression can be used in Equation (A6.8) to give:

$$\ddot{y}_j(t) = \nu_j \dot{y}_j(t) + \frac{1}{2\zeta_{(2)}} [\zeta_{(1)} A_j(t) - \zeta_{(3)} +$$

$$(\dot{y}_j(t) - \nu_j y_j(t))\ 2\zeta_{(2)}\ \exp(\nu_j t)$$

$$(-\nu_j \exp(-\nu_j t))]$$

$$= \frac{1}{2\zeta_{(3)}} [\zeta_{(1)} A_j(t) - \zeta_{(3)} + 2\zeta_{(2)} \nu_j^2 \quad y_j(t)] . \qquad (A6.10)$$

Recall that $A_j(t)$ is defined as:

$$A_j(t) = \frac{\partial}{\partial y_j(t)} \left[\sum_j \frac{y_j(t)}{\sum_i y_i(t)} \log \frac{y_j(t)}{\sum_i y_i(t)} \right] . \qquad (A6.3)$$

This expression can be evaluated:

$$A_j(t) = \frac{\partial}{\partial y_j(t)} \left[\frac{y_1(t)}{y_1(t) + y_2(t) \ldots + y_j(t) + \ldots + y_J(t)} \right.$$

$$\log \frac{y_1(t)}{y_1(t) + y_2(t) + \ldots + y_j(t) + \ldots + y_J(t)}$$

$$A_j(t) = \frac{1}{(\sum_i y_i(t))^2} [- \sum_k y_k(t) \log y_k(t) + \sum_k y_k(t)$$

$$\log \sum_i y_i(t) + \sum_i y_i(t) \log y_j(t)$$

$$- \sum_i y_i(t) \log \sum_i y_i(t)]$$

$$= \frac{1}{(\sum_i y_i(t))^2} \sum_i y_i(t) (\log y_j(t) - \log y_i(t))$$

$$= \frac{1}{(\sum_i y_i(t))^2} \sum_i y_i(t) \log\left[\frac{y_j(t)}{y_i(t)}\right] . \qquad (A6.11)$$

Substituting Equation (A6.11) into (A6.10) results in Equation (6.8).

__Result 6.2__ Follows directly from Result 6.1. Although the general case defined by Assumptions 6.1 and 6.3 is modified with $y_2(t)=y_2^*$, Result 6.1 still holds for $y_1(t)$. This is easily verified by reconstructing its proof, substituting y_2^* for $y_2(t)$.

$$+\cdots+\frac{y_J(t)}{y_1(t)+y_2(t)+\cdots+y_j(t)+\cdots+y_J(t)}$$

$$\log\frac{y_J(t)}{y_1(t)+y_2(t)+\cdots+y_j(t)+\cdots+y_J(t)}\Bigg]$$

$$= \frac{y_1(t)}{\sum_i y_i(t)}\ \frac{\sum_i y_i(t)\cdot 0 - y_1(t)\cdot 1}{y_1(t)\ \sum_i y_i(t)} + \log\left[\frac{y_1(t)}{\sum_i y_i(t)}\right] \cdot \left[\frac{-y_1(t)}{(\sum_i y_i(t))^2}\right]+\cdots+$$

$$\frac{y_j(t)}{\sum_i y_i(t)} \cdot \frac{\sum_i y_i(t) - y_j(t)}{y_j(t)\sum_i y_i(t)} + \log\left[\frac{y_j(t)}{\sum_i y_i(t)}\right]\left(\frac{\sum_i y_i(t) - y_j(t)}{(\sum_i y_i(t))^2}\right)+\cdots$$

$$= \sum_{h\neq j}\left[-\frac{y_h(t)}{(\sum_i y_i(t))^2} - \frac{y_h(t)}{(\sum_i y_i(t))^2}\log\frac{y_h(t)}{\sum_i y_i(t)}\right]$$

$$+\frac{\sum_i y_i(t) - y_j(t)}{(\sum_i y_i(t))^2}\left(1+\log\frac{y_j(t)}{\sum_i y_i(t)}\right)$$

$$= \frac{1}{(\sum_i y_i(t))^2}\left\{\sum_k\left[-y_k(t)\left[1+\log\frac{y_k(t)}{\sum_i y_i(t)}\right]\right]\right.$$

$$\left.+ (\sum_i y_i(t))\left[1+\log\frac{y_j(t)}{\sum_i y_i(t)}\right]\right\}$$

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