UNIVERSAL ALGEBRA IN TOPOI

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ABSTRACT

This thesis represents an attempt to marry two distinct directions of research in Modern Algebra. On the one hand we have the theory of topoi which has been undergoing vigorous development within the last three years. This area grew out of careful consideration of the tools of Modern Algebraic Geometry as promoted by the French School under the leadership of Alexandre Grothendieck. Lawvere and Tierney have shown how the theory of topoi may be axiomatized conveniently and have established that much of Mathematics may be carried out in the environment of a topos, in which scheme the topos of definition replaces the category of sets. The second direction referred to is what is called Universal Algebra. By this we mean internal Universal Algebra, that is to say the study of equations and their solutions in mathematical structures. In the present work we undertake the study of the behaviour of universal algebras modelled in a topos.

In fact the topoi we choose to study are those arising as categories of sheaves of sets on a suitable parametrizing object (Grothendieck site). In this framework we introduce the notion of equation and solution, now for sheaves of algebras. We establish as major results that a compact Hausdorff sheaf of algebras is equationally (algebraically) compact and that for sheaves of modules, homological purity is equivalent to a form of equational purity. On the path towards the proof of these results we establish new facts about certain topoi, and some new facts regarding

"external" universal algebra, that is, the purely categorical aspect of universal algebra. For example we characterize the points of the category of double negation sheaves on a topological space, we give a useful characterization of those continuous maps whose inverse image functor on the associated sheaf categories is cotripleable, we establish a Birkhoff-type subdirect representation theorem for sheaves of algebras and we exhibit sufficient conditions for injectivity to be well-behaved in the category of sheaves of universal algebras associated with a specified theory.

What we have carried out here represents only a beginning in terms of the possibilities inherent in studying universal algebras modelled in topoi. The old theorems of universal algebra take on a more dynamic and geometrical flavour in their new environment, which we hope will lead to their application in Combinatorics and Algebraic Geometry.

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INTRODUCTION

The purpose of this chapter is to give some historical background and to sketch the line of development within the thesis. Comments on possible future directions of this research will be made in chapter VI.

Grothendieck was among the first to recognize the usefulness of categorical algebra as a tool in modern Algebraic Geometry. He developed an extensive theory of sheaves which subsumed the older theory of sheaves on a topological space. The parametrizing objects become small categories equipped with "topologies" and sheaves are diagrams (of sets, to begin) which are coherent in a sense made explicit using the "topology" on the site, as it is called. The developement of these concepts and their use in Algebraic Geometry, especially with regard to (co) homology theories may be followed in [EGA], [SGA].

The category of sheaves on a site $\mathfrak{C}$, denoted $\tilde{\mathfrak{C}}$, was recognized to be the fundamental invariant of the site. Such categories were named topoi. Recently Lawvere and Tierney have extracted the essence of these categories and produced an axiomatization whose models are called elementary topoi. Every category of the form $\tilde{\mathfrak{C}}$ is in fact an elementary topos although the converse is not true. With regard to the converse, Barr and Freyd have, within the past two years, developed many powerful theorems which provide for embeddings of (parts of) elementary topi into topoi of the form $\tilde{\mathfrak{C}}$.

This thesis is motivated by the work of Lawvere and Tierney, for their approach made it seem possible that a great deal of universal algebra could, and should, be carried out in an arbitrary topos. We have chosen to work with topoi of the form $\tilde{\mathbb{C}}$ for reasons explained in chapter I, 4. Certainly much of the work, if not all, could be carried out in an elementary topos under certain completeness and smallness hypotheses. Another approach in this vein would be to establish that results about universal algebra in an elementary topos $\mathbb{E}$ survive passage to the internally constructed sheaves $\mathbb{E}_j$ associated with a "Grothendieck modal operator" j [QS]. Indeed we often use this technique implicitly, proving results first for presheaves $\hat{\mathbb{C}}$ and then invoking exactness properties of the associated sheaf functor to conclude their validity in $\tilde{\mathbb{C}}$.

The tools we use come from the work of the French mathematicians on the topoi $\tilde{\mathbb{C}}$, from the work of Lawvere and Tierney on elementary topoi, from categorical algebra in general and from classical universal algebra. Most of what we use with respect to the first area can be found in [SGA]. Notable among the techniques found helpful is that of localizing a topos $\tilde{\mathbb{C}}$ at $\mathbb{C}/_C$ for $C \in |\mathbb{C}|$. This corresponds to restriction to an open set for sheaves on a topological space. For the theory of elementary topoi we refer the reader to [QS], [AST], [AT], [SB], [ET]. For categorical algebra the standard reference remains Mitchell [TC], complemented recently by MacLane [CWM]. For concepts of classical universal algebra the reader is referred to Grätzer [UA]. In chapter V we find it necessary to assume a good deal of knowledge about the category of modules over a sheaf of rings. The relevant information may be found for example in Artin [GT], MacDonald [AG] or Mumford [IAG].

This research represents an attempt to address the problem of understanding the internal structure of sheaves of universal algebras, objects which have been found useful in various branches of mathematics, notably geometry. We prefer to view a sheaf of algebras as a collection of algebras coherently parametrized by the site of definition, believing that this interpretation provides the correct insight into such objects, leading as it does to the notion of equation in a sheaf of algebras through which the internal structure of the algebra (which may be thought of as developing along parameters such as space, time, etc. to paraphrase Lawvere [QS]) can be described. The results of this are theorems such as those of chapter V establishing satisfying connections between topological and algebraic compactness on the one hand and between homological and equational purity on the other.

The theorems of chapter V are deceptively simple to state. They constitute a dynamic and geometrical reformulation of facts well known from classical universal algebra. As such their proofs are substantially more difficult and much careful preparation in necessary. This is undertaken in chapter I - III, during which several new results concerning the tools are presented, notably theorems characterizing the points of the category of double negation sheaves on a topological space (theorem 3.2.1, chapter II) and those continuous maps of topological spaces whose associated inverse image functors on the sheaf categories are cotripleable (theorem 4.3.4, chapter II). Proofs are also given for theorems characterizing (a) the points of the category of sheaves on a topological space, (b) the points of the category of sheaves associated in a certain way with a compact measure space and (c) the categories of sheaves on a topological space among the categories $\widetilde{\mathfrak{C}}$ of sheaves on an arbitrary Grothendieck site. These results were announced without proof in [SGA] and the arguments presented here may or may not represent improvements upon those envisaged by the inventors; the author does not know, for he has not seen their proofs.

Chapter IV provides an answer to the question of how the behaviour with respect to injectivity of an equational class is reflected in the result of passage to sheaves. Here it is established that if the site of definition "has enough points" (a fairly common situation including sheaves on a topological space and full functor categories for example) then the properties of the base equational class crucial with respect to injectivity are inherited by the sheaves with values in that class.

The organization of the material is straightforward, although it should be pointed out that a numbered reference which does not explicitly name a chapter is understood to refer to that number within the chapter currently being developed. Bibliographic references use mnemonics rather than numbers. For example [QS] refers to Lawvere's paper "Quantifiers and Sheaves". Entries in the bibliography which have been included for completeness, in that they bear upon the present work, but which are not explicitly referenced within the text, have no associated mnemonic.

Chapter I: Topologies, Sheaves and Topoi

This chapter introduces the environment in which algebra is to be studied in the sequel, namely those categories called topoi. Also introduced are seven fairly representative examples of the sort of topos we will be working with. The material in this chapter is all well-known, at least for sheaves on a topological space. We have chosen to study Grothendieck sites because this notion has greater generality than that of a topological space insofar as parametrizing of mathematical objects is concerned. In fact, the major result of chapter III regards a fact true for algebras modelled in any category of sheaves on a topological space, but apparently false in general for algebras modelled in categories of sheaves on a Grothendieck site. For detailed information on Grothendieck sites the reader is referred to [SGA].

1 Grothendieck Sites

1.1 Throughout $\mathbb{C}$ will denote a small category, fixed for the discussion. Its objects, collectively denoted by $|\mathbb{C}|$, will be called $A, B, C, \ldots$ and its maps $f, g, h, \ldots$. The functor category $\mathrm{Sets}^{\mathbb{C}^*}$ is denoted $\widehat{\mathbb{C}}$ and the full embedding $\mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ of the (contravariant) representable functors will be considered an inclusion. That is, the object $\mathbb{C}(-, C)$ of $\widehat{\mathbb{C}}$ will also be called C, and for $B \xrightarrow{f} C$, the induced map $\mathbb{C}(-, f)$: $\mathbb{C}(-,B)$ —— $\mathbb{C}(-, C)$ will be called simply f.

Of fundamental importance is the Yoneda lemma: the following correspondence is bijective and natural in C,F, for $C \in |\mathbb{C}|$ and $F \in |\hat{\mathbb{C}}|$

$$\begin{array}{ll} C \longrightarrow F & (\text{in } \hat{\mathbb{C}}) \\ \hline 1 \longrightarrow F(C) & (\text{in Sets}) \end{array}$$

Maps $1 \longrightarrow F(C)$ in sets are of course in 1-1 correspondence with elements $\sigma \in F(C)$. The two maps $C \longrightarrow F$ and $1 \longrightarrow F(C)$ will also be called "σ", hopefully with no confusion caused. The general rule of thumb is that, in studying a particular $F \in |\hat{\mathbb{C}}|$, we think of "$\sigma \in F(C)$" as an aid to intuition (i.e. "sections of a sheaf F over C"), but we use $\sigma : C \longrightarrow F$ for carrying out algebraic manipulations. The fact that the Yoneda correspondence is natural in C is summed up in the following useful diagram

$$\begin{tikzcd} C \arrow[r, "\sigma"] & F \\ C' \arrow[u, "f"] \arrow[ur, "\sigma'"'] & \end{tikzcd}$$

where σ' corresponds to $F(f)(\sigma) \in F(C')$

That is, $\sigma f = F(f)(\sigma)$. Note that this operation $(\sigma, f) \longmapsto \sigma f$ has been the basis of viewing functors as "generalized" M-sets where M is a "generalized" monoid, i.e. a category. For whenever the equation makes sense we must have $(\sigma f)f' = \sigma(ff')$ expressing the functoriality of F. Mitchell has exploited this vigourously, in the additive case, in [RSO].

Note For a presheaf F, and later for sheaves F, we will speak of the elements $\sigma \in F(C)$ as sections of F over C. This leads to the idea of "sectionwise" behaviour of a presheaf. For example a morphism $\varphi : F \longrightarrow G$ in C is mono iff it is "sectionwise" a mono, i.e. $\phi_C : F(C) \longrightarrow G(C)$ is mono for each $C \in |\hat{\mathbb{C}}|$. The more traditional way of saying this would use the phrase "pointwise mono", but in the theory of topoi "point" has a very special meaning. The term "section" is borrowed from the theory of sheaves on a topological space, and its use in this more general setting has been popularized in recent years by the developers of the elementary theory of topoi.

1.2 Grothendieck topologies For $C \in |\mathbb{C}|$ a sieve in C (French: "crible") is a subfunctor of $C \in |\hat{\mathbb{C}}|$. A topology on $\mathbb{C}$ is a collection $\{\mathrm{Cov}(C) \mid C \in |\mathbb{C}|\}$ where each $\mathrm{Cov}(C)$ is a collection of sieves in C, subject to:

(T1) Given $f : C \longrightarrow C'$ and $R \in \mathrm{Cov}(C')$, $f^{-1}(R) \in \mathrm{Cov}(C)$ i.e. $R \times_{C'} C$ $\mathrm{Cov}(C)$ where $R \times_{C'} C$ is defined by the following pullback in C

$$\begin{tikzcd}
R \times_{C'} C \arrow[r] \arrow[d] & C \arrow[d, "f"] \\
R \arrow[r] & C'
\end{tikzcd}$$

(T2) For R a sieve in C, $R \in \mathrm{Cov}(C)$ if $\exists R' \in \mathrm{Cov}(C)$ such that $\forall B \in |\mathbb{C}|$, $\forall B \longrightarrow R$, $R' \times_C B \in \mathrm{Cov}(B)$

$$\begin{tikzcd}
R' \times_C B \arrow[r] \arrow[dd] & B \arrow[d] \\
 & R \arrow[d, tail] \\
R' \arrow[r] & C
\end{tikzcd}$$

pullback in $\hat{\mathbb{C}}$.

(T3) $C \in \mathrm{Cov}(C)$.

Let us rephrase these concepts and conditions in a more "elementwise" fashion. Sieves are in 1-1 correspondence with ideals in the monoid-theoretic sense. That is, a sieve in C is a collection of maps each of whose codomain is C and which is closed under composition from the left. Such a collection, if it is in $\mathrm{Cov}(C)$, will be called a covering family, or a "covering of C". Then T1 - T3 above will be easily seen to be equivalent to the following four conditions

(i) if $R \in \mathrm{Cov}(C)$, $B \xrightarrow{f} C$ then $\left\{A \longrightarrow B \mid A \longrightarrow B \xrightarrow{f} C \in R\right\}$ $\in \mathrm{Cov}(B)$

(ii) an ideal which contains a covering family is itself a covering family.

(iii) $\left\{C_i \xrightarrow{f_i} C\right\}_{i \in I} \in \mathrm{Cov}(C)$, $\left\{C_{ij} \xrightarrow{g_{ij}} C_i\right\}_{j \in J_i} \in \mathrm{Cov}(C_i)$ all $i \in I$ $\left\{C_{ij} \xrightarrow{f_i g_{ij}} C \mid i \in I,\ j \in J_i\right\} \in \mathrm{Cov}(C)$

(iv) $\left\{B \longrightarrow C \mid B \in |\mathbb{C}|\right.$, all maps $\left.B \longrightarrow C\right\} \in \mathrm{Cov}(C)$

The topologies on $\mathbb{C}$, ordered by inclusion, form a complete lattice, for they are closed under intersection. The smallest topology is that for which $\mathrm{Cov}(C) = \{C\}$, all $C \in |\mathbb{C}|$. The largest is that for which $\mathrm{Cov}(C) = \{R \mid R \subseteq C \text{ in } \hat{\mathbb{C}}\}$ = the set of all sieves in C. It follows that any collection of sieves generates a smallest topology for which these are covering. In case the collection of sieves satisfies (i), (ii) and (iv) immediately above, one need only add all sieves larger than the given ones to obtain the generated topology.

1.3 *Simplifications induced by the existence of fibre products* If there are sufficient fibre products (pullbacks) available in $\mathbb{C}$, a different characterization of the structure of topology may be given. This will allow considerable simplification when we consider sheaves on a category equipped with a topology. To proceed, a pretopology on $\mathbb{C}$ is given by specifying for each $C \in |\mathbb{C}|$ a set $\mathrm{PCov}(C)$ of collections of maps with codomain C satisfying

(PT0) $\forall C \in |\mathbb{C}|$, $\forall B \longrightarrow C$ in $\mathbb{C}$, $\forall A \longrightarrow C \in R \in \mathrm{PCov}(C)$,

the fibre product exists

$$\begin{tikzcd} B \times_C A \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

"all maps in any covering are squarable"

(PT1) $\forall \{C_i \longrightarrow C\}_{i \in I} \in \mathrm{PCov}(C)$, $\forall f : B \longrightarrow C$ in $\mathbb{C}$,

the family $\{C_i \times_C B \longrightarrow B\}_{i \in I}$ obtained by pulling back along f is in $\mathrm{PCov}(B)$.

(PT2) Composition of coverings, as in T3.

(PT3) $\forall C$ $\{C \xrightarrow{1} C\}$ is in $\mathrm{PCov}(C)$.

Given a pretopology the sieves generated by the elements of the families $\mathrm{PCov}(C)$ are cofinal in the generated topology, so one need only add the "super-sieves" to obtain this topology. Clearly, if $\mathbb{C}$ has pullbacks, any topology is already a protopology.

Finally, a site, more often called a Grothendieck site in the literature, is a category equipped with a topology (not necessarily generated by a pretopology).

2 Sheaves on a Site

2.1. Having defined the structure of topology on a category it would seen incumbent to write down what it means for a functor to preserve the structure. This leads to the notions of continuous and cocontinuous functors. However the relevant definitions are difficult to understand without first studying the notion of sheaf, and therefore we shall defer them for the time being.

A sheaf cannot be construed in a natural way as a structure-preserving map between mathematical objects of the same species, although it is possible to view the category of sets as a site in a canonical way, and sheaves of sets are in particular functors from the given site into Sets. The situation may be usefully compared to that of Moore-Smith limits in point-set topology. Here we have on the one hand the idea of a directed set and the indispensable accompanying notion of morphism of directed sets. On the other hand, the raison d'etre of directed sets insofar as point-set Topology is concerned is their use in parametrizing elements of a topological space. So it is with sites and sheaves. For we view a sheaf (of sets, say) as a collection of sets coherently parametrized by a site. The word "parametrized" indicates of course that a sheaf is to be a functor. The requirement that the parametrization be coherent is more subtle to express. The precise definitions follow.

Definitions 2.2 A presheaf of sets on a category $\mathbb{C}$ is a functor $F\colon \mathbb{C}^* \longrightarrow \mathrm{Sets}$. The category of presheaves, whose morphisms are natural transformations of functors, is denoted $\widehat{\mathbb{C}}$.

F is said to be separated if $\forall C \in |\mathbb{C}|$, $\forall R \in \mathrm{Cov}(C)$, the map $\hat{\mathbb{C}}(C,F) \longrightarrow \hat{\mathbb{C}}(R,F)$, induced by $R \subseteq C$, is 1-1. F is a sheaf if these maps are always 1-1 and onto. This will be called the patching condition, for reasons to become evident shortly. Note that the notion of presheaf is defined for any category $\mathbb{C}$, but the notions of presheaf and sheaf presuppose a site structure on $\mathbb{C}$.

2.3 To illuminate these definitions somewhat, let us analyze further the maps in question. First of all, what is a morphism $R \longrightarrow F$ in $\hat{\mathbb{C}}$? Viewing R as a sieve and indexing the maps in it by $\{f_i : C_i \longrightarrow C \mid i \in I\}$, R is determined by elements $\sigma_i \in F(C_i)$ satisfying the compatibility condition:

$$\forall g_i : C_i \longrightarrow C_j \text{ in } \mathbb{C},\ F(g)(\sigma_j) = \sigma_i$$

(Recall F is contravariant)

The Yoneda lemma gives an isomorphism $F(C) \xrightarrow{\simeq} \hat{\mathbb{C}}(C,F)$. Then the map $F(C) \xrightarrow{\simeq} \hat{\mathbb{C}}(C,F) \longrightarrow \hat{\mathbb{C}}(R,F)$ takes an element $\sigma \in F(C)$ to the family $\{F(f_i)(\sigma) \mid i \in I\}$. Let us denote $F(f_i)(\sigma)$ by $\sigma | C_i$, read " σ restricted to C_i". The condition that F be separated now reads: if two elements σ, $\sigma' \in F(C)$ have the same restrictions over a cover $R \subseteq C$ then they are equal. The condition that F be a sheaf reads: every compatible (wrt R) family $\{\sigma_i \mid \sigma_i \in F(C_i), i \in I\}$ arises as the restrictions (over R) of a uniquely determined element of $F(C)$.

$$\begin{tikzcd}
 & & C & & \\
{} \arrow[urr] & & & & {} \arrow[ull] \\
\cdots & C_i \arrow[uur, "f_i"] \arrow[rr] & & C_j \arrow[uul] & \cdots
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \sigma \in F(C) \arrow[dll] \arrow[dl, "F(f_i)"'] \arrow[dr] \arrow[drr] & & \\
{} & & & & {} \\
\cdots & \sigma_i \in F(C_i) & & F(C_j) \arrow[ll] & \cdots \\
 & & & \sigma_j \in &
\end{tikzcd}$$

Proceeding further with the analysis we have

$$F(C) \xrightarrow{\sim} \widehat{\mathbb{C}}(C,F) \xrightarrow{\sim} \widehat{\mathbb{C}}(R,F) \xrightarrow{\sim} \widehat{\mathbb{C}}(\varinjlim_{(B,b)\in \mathbb{C}/R} B,F) \xrightarrow{\sim} \cdots$$

$$\varinjlim_{(B,b)\in \mathbb{C}/R} \widehat{\mathbb{C}}(B,F) \xrightarrow{\sim} \varinjlim_{(B,b)\in \mathbb{C}/R} F(B)$$

Here the category C/R has as objects maps $B \xrightarrow{b} R$ in $\widehat{\mathbb{C}}$ and as morphisms

$$\begin{tikzcd}
B \arrow[rr, "h"] \arrow[dr] & & B' \arrow[dl] \\
 & R &
\end{tikzcd}$$

rendering the diagram commutative. That is, the objects are $b \in R(B) \subseteq \mathbb{C}(B,A)$ and maps are $B \xrightarrow{h} B'$ with $R(h)(b') = b$, i.e. $b'h = b$. It is a well-known fact that any $R \in |\widehat{\mathbb{C}}|$ (sieve or not) is a colimit of generators $B \in |\mathbb{C}|$ over the associated diagram $\mathbb{C}/R$. For example see [SGA], expose I, 3.4. Using the basic construction of a projective limit via equalizers and products we are demanding, for F to be a sheaf, that the following diagram be an equalizer (the condition that F be a separated presheaf is that the first map be mono):

$$\begin{tikzcd}
F(C) \arrow[r, "g"] & \prod\limits_{i \in I} F(C_i) \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & \prod\limits_{i \in I} \prod\limits_{C_j \xrightarrow{h} C_i} F(C_j)
\end{tikzcd}$$

Diagrammatically the maps are defined by their projections as follows:

$$\begin{tikzcd}
F(C) \arrow[r, "g"] \arrow[dr, "F(f_i)"'] & \prod F(C_i) \arrow[d, "pr_i"] \\
& F(C_i)
\end{tikzcd}$$

$$\begin{tikzcd}
\prod F(C_i) \arrow[r, "g_1"] \arrow[ddr, "pr_j"'] & \prod\prod F(C_j) \arrow[d, "pr_i"] \\
& \prod F(C_j) \arrow[d, "pr_h"] \\
& F(C_j)
\end{tikzcd}$$

$$\begin{tikzcd}
\prod F(C_i) \arrow[rr, "g_2"] \arrow[dr, "pr_i"'] & & \prod\prod F(C_j) \arrow[d, "pr_i"] \\
& F(C_i) \arrow[dr, "F(h)"'] & \prod F(C_j) \arrow[d, "pr_h"] \\
& & F(C_j)
\end{tikzcd}$$

2.4 The above characterization in terms of $\varprojlim$'s shows that we can define a sheaf with values in any category. The fact that covariant hom functors preserve $\varprojlim$'s and collectively reflect them shows that $\mathbb{C}^* \xrightarrow{F} \mathbb{E}$ is a sheaf iff $\mathbb{C}^* \xrightarrow{F} \mathbb{E} \xrightarrow{(E,-)}$ Sets is a sheaf of sets for every $E \in \mathbb{E}$. Provided $\mathbb{E}$ actually has all of the $\varprojlim$'s being tested for, it is sufficient that $(E,-)F$ be a sheaf for all $E \in \Upsilon$ where $\Upsilon \leq |\mathbb{E}|$ is a strongly generating family. Recall Υ is a generating family if its associated hom functors are collectively faithful, and is strongly generating if it is generating and its associated hom functors collectively reflect isomorphisms, i.e. $\forall\ E' \xrightarrow{f} E''$, (E,f) iso all $E \in \Upsilon \implies f$ iso. If Υ is generating, it is sufficient to quantify over monomorphisms f, and in fact this is the definition of generator used in [SGA]. Grothendieck has popularized the interpretation of maps $E \longrightarrow E'$ in an arbitrary category $\mathbb{E}$ as "elements of E' defined over E". In this language we can say $F\colon \mathbb{C}^* \longrightarrow \mathbb{E}$ is a sheaf iff it is "pointwise" a sheaf of sets.

For example if $\mathbb{E}$ is a variety of universal algebras, $W(1)$ is a strong generator where $W(1)$ is the free algebra on a one-element set. Hence F is a sheaf iff $\mathbb{E}(W(1), F(\ ))$ is a sheaf of sets. But $\mathbb{E}(W(1),F(\ )) \cong UF$ where U is the forgetful functor from the variety to sets (because W is left adjoint to U). That is, a sheaf of algebras is a functor whose underlying presheaf of sets is a sheaf of sets.

2.5 One proves (see [SGA]) that given a collection $\{K(C): C \in |\mathbb{C}|\}$ where $K(C)$ is a collection of families of maps each of whose codomain is C, and supposing the collection is stable under "change of base" (i.e. satisfies T1 of I.1.2):

A functor $F:\mathbb{C}^* \longrightarrow$ Sets is a sheaf for the topology generated by the $K(C)$ iff it satisfies the patching condition for families in the $K(C)$.

If the topology is defined by a pretopology (I.1.3), the patching condition is easier to write down. It is equivalent to the following being an equalizer for each family $\{C_i \longrightarrow C \mid i \in I\}$ in $PCov(C)$, and for all $C \in |\mathbb{C}|$:

$$F(C) \longrightarrow \prod_{i \in I} F(C_i) \rightrightarrows \prod_{i,j} F(C_i \times_C C_j)$$

The maps are defined in the obvious fashion (cf I.2.3). This condition is the one which is presented in the literature whenever sheaves on a topological space are defined using the "coverings-restrictions" technique as opposed to the "espace étalé" technique.

2.6 Given a collection of presheaves we can always find a finest topology for which they are sheaves. Recall that "finer topology" "more patching conditions" "fewer sheaves". This sets up a Galois correspondence between subcollections of $|\hat{\mathbb{C}}|$ and sets of patching conditions on $\mathbb{C}$ (collections $\{K(C) \mid C \in |\mathbb{C}|\}$ where each $K(C)$ is a family of maps each of whose codomain is C). The closed elements in the first case are subtopoi of $\hat{\mathbb{C}}$, that is, reflective subcategories where the reflection preserves finite $\underleftarrow{\lim}$'s . This is essentially a theorem due to J. Giraud ([SG4], expose IV, 1.2). In the second case the closed elements are topologies. The proof of this fact is fairly technical and will be omitted here since it has no real bearing on the subject of this work.

It is sometimes quite useful to know that the representable functors on $\mathbb{C}^*$ are sheaves. For this reason the finest topology for which they are sheaves has been baptized the <u>canonical</u> topology. It can be shown that the covering families for this topology are the universal strict epimorphic ones. A strict epimorphic family is a collection of maps $\{f_i : C_i \longrightarrow C \mid i \in I\}$ with the property that for any family $\{g_i : C_i \longrightarrow B \mid i \in I\}$:

$$(\forall\, i,j \;\; \forall\, h_i : A \longrightarrow C_i,\; h_j : A \longrightarrow C_j,\; (f_i h_i = f_j h_j \Longrightarrow g_i h_i = g_j h_j))$$

$$\Longrightarrow \exists !\; g : C \longrightarrow B \text{ with } gf_i = g_i \text{ all } i \in I.$$

A family $\{f_i\}$ as above is universal strict epimorphic if $\forall\, B \longrightarrow C$, $\{C_i \longrightarrow B \mid i \in I,\; C_i \longrightarrow B \longrightarrow C = f_i\}$ is strict epimorphic. If $\mathbb{C}$ has pullbacks this is just the family obtained by pulling back the f_i along $B \longrightarrow C$.

3. The Associated Sheaf Functor

3.1 Denote by $\tilde{\mathfrak{C}}$ the full subcategory of $\hat{\mathfrak{C}}$ determined by the sheaves. $\hat{\mathfrak{C}}$, being a functor category, is relatively straightforward in terms of its basic categorical properties. It has all $\varprojlim$'s and $\varinjlim$'s, and the evaluation functors preserve them ("limits are computed sectionwise"). The representable functors constitute a generating family. Epimorphisms are sectionwise onto (and vice versa) and so on. Clearly an investigation of $\tilde{\mathfrak{C}}$ must begin by studying its relationship with $\hat{\mathfrak{C}}$. The fundamental fact here is that $\tilde{\mathfrak{C}}$ is a reflective subcategory of $\hat{\mathfrak{C}}$, and the reflection preserves finite left limits.

3.2 The construction of the associated sheaf is straightforward. To make the best possible sheaf out of a presheaf P we must clearly add to each P(C) new elements corresponding to compatible families over coverings of C. Then we must identify any which are bound to lead to the same "patched element", i.e. which agree on a common subcover. To be precise define

$$(aP)(C) = \varinjlim_{R \in \mathrm{Cov}(C)} \hat{\mathfrak{C}}(R,P)$$

The action on maps is obvious, as in the reflection map $\varsigma : P \longrightarrow aP$ (arising from the fact that $C \in \mathrm{Cov}(C)$). The verification that aP is a sheaf and that a is left adjoint to the inclusion map may be seen in [SGA], for example. Now the indexing category of this $\varinjlim$ is down-directed (Cov(C) is a filter) and therefore the diagram itself is up-directed. This has the consequence that a preserves finite projective limits, or is left exact as one says. The fact is absolutely crucial to the study of algebra in categories $\widetilde{\mathfrak{C}}$. For a proof that in Sets finite $\varprojlim$'s commute with colimits over up-directed sets see MacLane [CWM], page 211. For a discussion of this fact in the more general setting of regular categories see Grillet [RC]. We might point out that this type of commutation holds in any category $\widetilde{\mathfrak{C}}$, as follows from Grillet's work for example.

3.3 There are other methods of constructing the associated sheaf. We shall mention three of them. First of all for presheaves on a topological space one may build, using the "stalks" of a presheaf its associated étalé space over the space given. This is endowed with a suitable topology and the associated sheaf is obtained by taking continuous sections over this "fibred" object. Another method, used by Van Osdal, exhibits sheaves on a topological space T as coalgebras over a certain cotriple in $\mathrm{Sets}^{|T|}$, the functor category whose exponent is the discrete category on the underlying set of the space T. The fact that sheaves are cotripleable over $\mathrm{Sets}^{|T|}$ is a special case of a far more general result which we discuss briefly in chapter II, 4.3. The third method of constructing the associated sheaf is the internal one developed by Lawvere and Tierney. In this setting a topos is defined by elementary axioms (see section I.4) and a topology on such a topos $\mathbb{E}$ is an endomorphism of a special object in the category, which satisfies certain properties. Using this "Grothendieck modal operator" as Lawvere calls it, a notion of dense subobject is developed, and sheaves are defined by the way they behave with respect to dense subobjects. The category of sheaves is a subcategory of $\mathbb{E}$. Applying this to our case, we would start with the topos $\hat{\mathbb{C}}$, and the notion of a topology on $\mathbb{C}$ would become an internal property of $\hat{\mathbb{C}}$ (in fact an endomorphism of $\Omega \in |\hat{\mathbb{C}}|$). The notion of sheaf and the construction of associated sheaves would then arise with no further reference to $\mathbb{C}$.

4 The Two Notions of a Topos

4.1 Categories of the form $\tilde{\mathbb{C}}$ are called topoi (singular topos). They arose originally in the work of French algebraic geometers. Recently Lawvere and Tierney have developed an axiomatic theory of topoi. Their notion of topos, which is called an "elementary topos" is more general than that of categories $\tilde{\mathbb{C}}$. In fact the role played by the topoi $\tilde{\mathbb{C}}$ among the elementary topoi is analogous to that played by categories of modules among abelian categories.

4.2 An elementary topos ([QS], [AST], [AT], etc.) is a category $\mathbb{E}$ with the following properties:

(i) $\mathbb{E}$ has all finite limits and colimits including an initial object 0 and a terminal object 1.

(ii) $\mathbb{E}$ is cartesian closed.

(ii) $\mathbb{E}$ has a subobject classifier Ω , that is to say represents the functor $\mathbb{E}^* \longrightarrow$ Sets which takes $F \in |\mathbb{E}|$ to the set of subobjects of F and acts on maps by pulling back.

The resulting theory is powerful and breathtakingly elegant. Indeed Freyd has called this axiomatization and theory "the most important event in the history of categorical algebra since its creation" [AT].

4.3 We have chosen to study algebra in categories $\tilde{\mathfrak{C}}$ rather than in arbitrary elementary topoi. The main reason is that our applications require a certain measure of cocompleteness, so that free algebras are available, and a set of generators, so that equations may be more easily discussed. But the existence of arbitrary colimits and a generating set is already enough to characterize the $\tilde{\mathfrak{C}}$ among the elementary topoi. This is an easy consequence of theorem I.2, exposé IV of [SGA], together with the basic exactness properties of elementary topoi. In spite of the fact that we do not work in the setting of elementary topoi in their full generality, we will freely use many of the basic facts about elementary topoi in our proofs. Although direct proofs of these facts could be given for the topoi $\tilde{\mathfrak{C}}$, the most elegant arguments use the tools of the Lawvere-Tierney theory. The reader is referred to the literature for details, for example, [QS], [AST], [AT], [SB], [ET].

5. Examples of Sites and Sheaves

5.1 Relatively Pseudo-complemented Lattices Recall a lattice $\mathbb{L}$ is relatively pseudo-complemented iff it is cartesian closed qua category. That is, $\forall$ a,b $\in \mathbb{L}$ $\{x \in \mathbb{L} \mid a \wedge x \leq b\}$ has a largest element, denoted $a \Longrightarrow b$. For the existence of $a \Longrightarrow b$ means $\forall x \in \mathbb{L}$ we have a bijection of hom sets $\mathbb{L}(a \wedge x, b) \cong \mathbb{L}(x, a \Longrightarrow b)$ showing that

$a \wedge (\ )$ is left adjoint to $a \Longrightarrow (\ )$. It is a well-known result of elementary lattice theory that a complete lattice is relatively pseudo-complemented iff $\forall$ family $\{b_i\}_{i \in I}$ in $\mathbb{L}$, $\forall a \in \mathbb{L}$, $a \wedge \bigvee_i b_i = \bigvee_i a \wedge b_i$. This property makes the canonical topology on such an L easy to describe. For we can take as the covering families $\{a_i \longrightarrow a\}$ those collections $\{a_i\}_{i \in I}$ with $\bigvee_i a_i = a$. The above distributivity condition shows this is a pretopology. Any family $\{a_i\}$ with $a_i \leq a$ all i is epimorphic when considered as representing a family of maps with codomain a. It is easy to check that such a family is strict epimorphic iff $a = \bigvee_i a_i$. Finally, the distributivity condition mentioned above shows that every strict epimorphic family is universally strict epimorphic. Thus the pretopology defined above, in the way natural for a complete lattice, is in fact the canonical topology (cf. 2.5).

A sheaf $F \in |\tilde{\mathbb{L}}|$ is a presheaf $\mathbb{L}^* \longrightarrow$ Sets satisfying the pretopology patching condition. That is, for any situation $a = \bigvee_i a_i$ the following is an equalizer:

$$F(a) \longrightarrow \prod_{i \in I} F(a_i) \rightrightarrows \prod_{i,j \in I} F(a_i \wedge a_j)$$

5.2 <u>Sheaves on a Topological Space</u> Historically there were the first sites to be studied. From the point of view of site structure, this example is a subcase of 5.1 above, where we take for $\mathbb{L}$ the lattice of open sets of a topological space T. We denote this lattice by Open (T). We will briefly point out two characteristic features of the category of sheaves of sets on a topological space T, denoted $\tilde{T}$ rather than Open $(T)^{\sim}$.

First of all the representable functors $U: \text{Open }(T)^* \longrightarrow \text{Sets}$ defined by $U(V) = \begin{cases} 1 & V \subseteq U \\ \emptyset & V \nsubseteq U \end{cases}$ are sheaves, for the topology on Open (T) is the canonical one. Moreover these U generate $\tilde{T}$. In fact for any site $\mathbb{C}$ the collection of sheaves $a(C)$ where $C \in |\mathbb{C}|$ generate $\tilde{\mathbb{C}}$ (left adjoints take generating families to generating families). But in this case, in fact in the general situation of 5.1, the representable functors are clearly subobjects of the sheaf represented by $T \in \text{Open }(T)$. The latter is the sheaf constant at 1 and is the terminal object of $\tilde{T}$, also denoted by 1. To sum up:

In $\tilde{T}$ the subobjects of 1 generate.

The second feature we wish to point out is the existence of stalks. For each $x \in T$ we have a functor $\tilde{T} \longrightarrow \text{Sets}$ denoted by $F \longmapsto F_x$, $f \longmapsto f_x$ and computed by

$$F_x = \varinjlim_{U \ni x} F(U)$$

These stalk functors have the property that they are left exact and have right adjoints (equivalently by the special adjoint functor theorem and the presence of a generating family for $\tilde{T}$, they are left exact and preseve all colimits). Such a functor on a sheaf category $\tilde{\mathbb{C}}$ is called a stalk of $\mathbb{C}$ and its right adjoint is called a point of $\mathbb{C}$. In general $\tilde{T}$ has more points than those represented by the above formula for $x \in T$. Later we will describe completely the points of $\tilde{T}$. However these stalk functors collectively reflect isomorphisms. When a sheaf category $\tilde{\mathbb{C}}$ has enough stalk functors to reflect isomorphisms, we say that $\mathbb{C}$ has enough points.

It is a fact ([SGA], exposé IV, 7.1.9) that a topos $\widetilde{\mathbb{C}}$ is equivalent to one of the form $\widetilde{T}$ iff

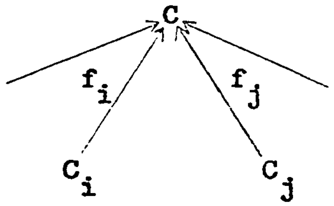
(a) the subobjects of 1 generate and

(b) it has enough points.

In section 4.2 of chapter II we will indicate how one proves this.

5.3 <u>The topos $\widehat{\mathbb{C}}$</u> Every small category $\mathbb{C}$, has a smallest topology on it, for which $\mathrm{Cov}(C)$ consists of one sieve, C itself, or in the language of ideals, the ideal of all maps with codomain C. For this topology it is clear that every presheaf is a sheaf, i.e. $\widetilde{\mathbb{C}} \subseteq \widehat{\mathbb{C}}$ is an isomorphism of categories. At the other end of the lattice of topologies is the discrete topology for which every sieve, including the empty one, is covering. Because $0 \rightarrowtail C$ is covering $\forall C \in |\mathbb{C}|$ we must have for any sheaf F, that $\widehat{\mathbb{C}}(C,F) \longrightarrow \widehat{\mathbb{C}}(0,F) \cong 1$ is an isomorphism. This means $F(C) = 1$. Thus $\widetilde{\mathbb{C}}$ reduces to the zero topos where $1 = 0$ and there is, up to isomorphism, only one object.

5.4 <u>Fan Sites</u> The simplest sort of non-trivial site one can imagine is clearly a "fan" $\mathbb{C}$, looking as follows:



where $\mathrm{Cov}(C_i) = \{1_{C_i}\}$, $\mathrm{Cov}(C) = \{f_i \mid i \in I\} \cup \{1_C\}$.

A little reflection shows that the sheaves on such a fan are not particularly interesting. For obviously a sheaf on $\mathbb{C}$ consists of a family of maps $F(C) \longrightarrow F(C_i)$ constituting a product diagram (this is the patching condition). The fact that maps of sheaves are natural transformations means $\tilde{\mathbb{C}}$ is equivalent to the category whose objects are I-indexed families of sets and whose maps are I-indexed families of maps with the obvious connections between domains and codomains of the data involved.

i.e. $$\tilde{\mathbb{C}} \cong \underbrace{\text{Sets} \times \text{Sets} \times \dots}_{\text{"I times"}} \cong \prod_{i \in I} \text{Sets} \cong \text{Sets}^{|I|} \cong \tilde{T} \cong \widetilde{P(I)}$$

where $|I|$ denotes the discrete category whose objects are the elements of I and whose only morphisms are identities. T denotes the discrete topological space on the set I. $P(I)$ is the Boolean algebra of all subsets of I, equipped with its canonical topology (see 5.1).

Let us point out that although such sites are simpleminded from a sheaf-theoretic point of view, they are indispensible for the study of injectivity.

5.5 <u>Regular Epimorphism Topology on a Regular Category</u> Let $\mathbb{C}$ be a regular category. That is, every map may be factored as a regular epi (= coequalizer) followed by a mono, and regular epimorphisms are universal which is to say the pullback of any regular epimorphism along any map is again a regular epi. Clearly for a regular category $\mathbb{C}$ the regular epimorphisms are covering families (one element coverings) for a pretopology. A sheaf then is a contravariant set valued functor which inverts the regular epimorphisms (i.e. carries them to bijective maps). This example has been exploited extensively by Michael Barr in proving non-abelian embedding theorems [EC].

5.6 *A Measure-theoretical Example* Let K be a compact topological space equipped with a complete regular Borel measure μ. From this data we shall construct (in three ways) a topos reminiscent of $\widetilde{K}$, the sheaves on the space K. In fact in the topos we construct the subobjects of 1 generate but we shall see later that in general there will be very few "points" (in the case of Lebesgue measure on [0,1], no points !)

This example is due to P. Déligne and the construction is indicated briefly in [SGA] ("démonstration au lecteur").

Define a category $\mathbb{A}$, for "almost measurable", as follows: The objects of $\mathbb{A}$ are subsets $A \subseteq K$ which are almost measurable. A is said to be almost measurable if there is a set $A' \subseteq K$ of measure 0 with $A \cup A'$ measurable. Equivalently there exists a measurable set E with $\mu(E \Delta A) = 0$. Here Δ is symmetric difference. The morphisms of $\mathbb{A}$ are inclusions, that is, we view $\mathbb{A}$ as a poset ordered by inclusion. Elementary measure-theoretic arguments show that $\mathbb{A}$ is a σ-algebra containing the σ-algebra of measurable subsets of K. A family $\{A_i \subseteq A: i \in I\}$ will be a covering of A iff I is countable and $\mu(A - \bigcup_i A_i) = 0$. Observe that this topology (in fact pretopology) is not comparable with the canonical topology on $\mathbb{A}$, for each topology has covers which are not covers in the other. In particular the representable functors on $\mathbb{A}$ will not in general be sheaves.

Let $\mathbb{M}$ be the site defined as $\mathcal{A}$ above but taking only measurable subsets of K as objects. Define $\mathbb{M}/_{\equiv}$ as follows. For measurable $A,B \subseteq K$ put $A \equiv B$ iff $\mu(A \triangle B) = 0$. This is an equivalence relation on $\mathbb{M}$. Define $[A]_{\equiv} \leq [B]_{\equiv}$ iff $\mu(B-A) = 0$. This is a well defined partial order. It turns out that $[A]_{\equiv} \leq [B]_{\equiv}$ iff $\exists\ A' \equiv A,\ B' \equiv B$ with $A' \subseteq B'$. $\mathbb{M}/_{\equiv}$ is a σ-algebra and the canonical $\mathbb{M} \xrightarrow{\nu} \mathbb{M}/_{\equiv}$ is a morphism of σ-algebras. Define a pretopology on $\mathbb{M}/_{\equiv}$ by taking countable suprema as coverings. Clearly if we define a relation $\sim$ on $\mathcal{A}$ by $A \sim B$ iff $\mu(A \triangle B) = 0$ we obtain a quotient $\mathcal{A}/_{\sim} \cong \mathbb{M}/_{\equiv}$ and a commutative diagram

$$\begin{tikzcd}
\mathbb{M} \arrow[rr, "\sigma"] \arrow[dr, "\nu"'] & & \mathcal{A} \arrow[dl, "\xi"] \\
& \mathbb{M}/_{\equiv} &
\end{tikzcd}$$

σ is the canonical injection

These three functors induce functors on the sheaf categories

$$\begin{tikzcd}
\tilde{\mathbb{M}} & & \tilde{\mathcal{A}} \arrow[ll, "\sigma_s"'] \\
& \widetilde{\mathbb{M}}/_{\equiv} \arrow[ul, "\nu_s"] \arrow[ur, "\xi_s"'] &
\end{tikzcd}$$

This will follow from a later section on morphisms of topoi (Chapter II, 1.2). The important thing here is that $\sigma_s \xi_s = \nu_s$ and all three functors are equivalences of categories. One can easily convince oneself, without the general theory of morphisms of sites, that $\mathcal{A}$, $\mathbb{M}$ and $\mathbb{M}/\equiv$ serve to define the same sheaves by observing that if B, A are (almost) measurable sets with $B \subseteq A$ and $\mu(A-B) = 0$ then B covers A and any sheaf must have the property that restriction $F(A) \longrightarrow F(B)$ is an isomorphism. Similarly a sheaf cannot distinguish between parameters whose symmetric difference is 0.

I have included $\mathcal{A}$ because this is the way [SGA] introduces the example. I have included $\mathbb{M}$ because it seems to me less cumbersome to work with than $\mathcal{A}$ or $\mathbb{M}/\equiv$ $(\mathcal{A}/\sim)$. I have included $\mathbb{M}/\equiv$ since it exhibits the sheaves as sheaves on a site whose topology is coarser than the canonical one (not the case in the representation as sheaves on $\mathcal{A}$ or $\mathbb{M}$). At any rate any one of these parametrizations is as good as the others in terms of the resulting topos.

Let us illustrate the techniques used in studying $\widetilde{\mathbb{M}}$ by briefly discussing the situation with generators.

First of all the representable functors on $\mathbb{M}$ will rarely be sheaves. For take $A \in |\mathbb{M}|$ and suppose we have $B \geq A$, $B \neq A$, $\mu(B-A) = 0$. Then $A(B) = 0$, $A(A) = 1$ but if A were a sheaf it would follow that $A(A) = A(B)$ since $B \geq A$ is a covering. However we can easily describe the sheaf associated to A. $a(A)(B) = \begin{cases} 1 & \mu(B-A) = 0 \\ \emptyset & \text{otherwise} \end{cases}$

If $B \subseteq A$ then $B-A = \emptyset$ so $\mu(B-A) = 0$. This shows that $A(B) = 1 \Longrightarrow a(A)(B) = 1$ so we have a natural map $A \longrightarrow a(A)$. $a(A)$ is evidently a contravariant functor. Let us show that it is a sheaf.

Let $\{B_i \subseteq B : i \in I\}$ be a covering. We must show the following is an equalizer:

$$a(A)(B) \longrightarrow \prod_i a(A)(B_i) \rightrightarrows \prod_{i,j} a(A)(B_i \cap B_j)$$

If $a(A)(B_i) = \emptyset$ some i then necessarily $a(A)(B) = \emptyset$, $\prod_i a(A)(B_i) = \emptyset$ and the last two arrows are equal and the first is an isomorphism. Hence the diagram is trivially an equalizer.

If $a(A)(B_i) = 1$ all $i \in I$ then $a(A)(B_i \cap B_j) = 1$ all i,j simply by functoriality of $a(A)$. Thus the last two arrows are the identity on 1 and we are finished if we know $a(A)(B) = 1$. But $\mu(B_i - A) = 0$ all $i \in I$ and $\bigcup_i B_i - A = \bigcup_i (B_i - A)$ so $\mu(\bigcup B_i - A) = 0$ (recall I is countable). But since the B_i cover B $\mu(B - \bigcup_i B_i) = 0$. Moreover

$$\begin{aligned} B - A &= (B-A) \cap ((\bigcup_i B_i)^c \cup \bigcup_i B_i) \\ &= (B \cap A^c \cap (\bigcup B_i)^c) \cup (B \cap A^c \cap \bigcup_i B_i) \\ &= (B - (A \cup \bigcup_i B_i)) \cup (\bigcup_i B_i - A) \\ &\subseteq (B - \bigcup_i B_i) \cup (\bigcup_i B_i - A) \end{aligned}$$

Since these last two sets have measure 0 so does $B - A$. Finally, $a(A)$ is the reflection of A in the category of sheaves.

$$\begin{tikzcd}
A \arrow[d] \arrow[rrd, "\sigma"] & & \\
a(A) \arrow[rr, dashed, "\bar{\sigma}"'] & & F
\end{tikzcd}$$

Given $\sigma : A \longrightarrow F$, which corresponds to a section $\sigma \in F(A)$ we construct $\bar{\sigma}$ as follows:

If $a(A)(B) = \emptyset$ there is no choice for $\bar{\sigma}_B$.

If $a(A)(B) = 1$, $\mu(B-A) = 0$ and $B - A \subseteq B$ is a covering. Thus $F(B) \longrightarrow F(B-A)$ is an isomorphism. Then $\bar{\sigma}_B$ maps the unique element of 1 to the element of $F(B)$ obtained by restricting $\sigma \in F(A)$ to $F(B-A)$ and applying the inverse of $F(B) \longrightarrow F(B-A)$.

This evidently gives a factorization of σ through $A \longrightarrow a(A)$ and the reader may esily show the factorization is unique.

As mentioned above we will return to this example when we study points of a topos.

5.7 Double Negation Topology on Open(T) Take a topology space T and call a family $U_i \subseteq U$ of open sets covering if $U \subseteq \text{Int}\,\Gamma\,(\bigcup_{i \in I} U_i)$. Here "Int" and "$\Gamma$" stand for interior and closure respectively. This defines a pretopology, for take such a covering and any $V \subseteq U$. Then $V = V \cap U = V \cap \text{Int}\,\Gamma(\bigcup_i U_i) \subseteq \text{Int}\,\Gamma(V) \cap \text{Int}\,\Gamma(\bigcup_i U_i) = \text{Int}\,\Gamma(V \cap \bigcup_i U_i)$ $= \text{Int}\,\Gamma(\bigcup_i V \cap U_i)$. i.e. $\{V \cap U_i : i \in I\}$ is a covering of V. Note that "Int Γ" commutes with intersections when the arguments are open sets ([BA], page 15).

The topology generated by this pretopology is called the double negation topology on Open(T). The theory of elementary topoi allows one to associate with any topos a subtopos consisting of its double negation sheaves. What we have described above is the category of double negation sheaves associated with $\tilde{T}$ (or $\hat{T}$ - they have the same double negation sheaves, technically because the double negation topology on Open(T) is finer than the canonical topology). By $\text{Open}(T)^{\perp\perp}$ we denote the category of open sets of T with the double negation topology and by $\tilde{T}^{\perp\perp}$ the associated category of sheaves. The notation "$\perp\perp$" is borrowed from [BA] where it is used to represent $\text{Int}\,\Gamma$. One may easily see that the sheaf associated to $U \in |\text{Open}(T)^{\perp\perp}|$ is the sheaf represented by $U^{\perp\perp}$. Thus the representable functors on Open(T) are sheaves iff the open set representing them is regular. Indeed $\tilde{T}^{\perp\perp}$ can be described equally well as the sheaves on the site whose category is the poset of regular open sets of T and whose topology is the canonical one.

Later we will show that $\tilde{T}^{\perp\perp}$ is a topos which may be very poor in points. In fact for T a T_2 space $\tilde{T}^{\perp\perp}$ often has no points at all. This in spite of the fact that $\tilde{T}^{\perp\perp}$ is a Boolean topos in which the subobjects of 1 generate. We recall that in the theory of elementary topoi a topos $\mathbb{E}$ is said to be Boolean if the canonical Heyting algebra structure on Ω is Boolean. Equivalently, for every $F \in |\mathbb{E}|$ and $F' \rightarrowtail F,\ \exists\, F'' \rightarrowtail F$ with $F' \perp\!\!\perp F'' \xrightarrow{\sim} F$. In an arbitrary topos one has pseudocomplements available, but in general not complements.

Chapter II Morphisms of Topoi

1. Morphisms of Topoi and Sites

1.1 Definition The experience of the associated sheaf functor and of the direct and inverse image under continuous maps of sheaves on a topological space leads to the following definition. A (geometric) morphism of topoi is a functor $f: \mathbb{E} \longrightarrow \mathbb{E}'$ equipped with an exact left adjoint f^*. "exact" means f^* preserves finite limits and finite colimits (the latter will be preserved by virtue of the fact that f^* is a left adjoint). The inclusion $\tilde{\mathbb{C}} \subseteq \hat{\mathbb{C}}$ of sheaves into presheaves is a geometric morphism of topoi. Given a continuous map $f:X \longrightarrow Y$ of topological spaces a geometric morphism $f_*:\tilde{X} \longrightarrow \tilde{Y}$ is induced via the formula $f_*(F)(V) = F(f^{-1}(V))$, the "direct image" functor [TF]. We shall study this example in more detail later.

There is another notion of morphism of topoi, that of logical morphism. This is a functor which preserves the structure of the topos as an elementary topos, i.e. a functor which preserves finite $\varprojlim$'s and $\varinjlim$'s, exponentiation and Ω. A geometric morphism whose left adjoint part is a logical morphism is called a local homeomorphism. As is to be expected a geometric morphism induced by a local homeomorphism of topological spaces is logical and thus a local homeomorphism in the topos-theoretical sense.

1.2 *Morphisms of Sites, Continuous Functors* As is to be expected, the proper notion of a morphism $u:\mathbb{C}\longrightarrow\mathbb{C}'$ of sites will induce a morphism of the associated topoi.

First of all any functor $u:\mathbb{C}\longrightarrow\mathbb{C}'$ induces a functor $\hat{\mathbb{C}}'\longrightarrow\hat{\mathbb{C}}$ which has left and right adjoints ("Kan extension", cf [CWM], page 229). These three functors are denoted by u^*, $u_!$, u_* . Here and in what follows we follow the complex notation of [SGA] for want of a better system. Thus u_* is a geometric morphism of topoi - u^* preserves all limits and colimits since it is both a left and right adjoint. u^* will be a geometric morphism whenever $u_!$ is left exact, for example when $\mathbb{C}$ has finite left limits and u preserves them (SGA 4, I.5.4).

We say u is continuous if u^* carries sheaves to sheaves. The restriction to sheaves is denoted by u_s. Equivalent conditions are that $u_!$ takes bidense morphisms to bidense morphisms, or that for any covering sieve $R \subseteq C$ in $\hat{\mathbb{C}}$, $u_!(R)\longrightarrow u_!(C)$ is bidense. Recall that a morphism is bidense if application of the associated sheaf functor turns it into an isomorphism. A monomorphism is called dense if it is transformed by the associated sheaf into an epimorphism (and automatically an isomorphism). Hence by exactness properties of "a" a morphism is bidense iff

(1) The diagonal is dense in its kernel relation

(2) The image (at the presheaf level) is dense in the codomain.

Note that for any $C \in |\mathbb{C}'|$, $u_!(C) = u(C) \in |\mathbb{C}|$. That is, the following diagram commutes

$$\begin{tikzcd}
\hat{\mathbb{C}}' & \hat{\mathbb{C}} \arrow[l, "u_!"'] \\
\mathbb{C}' \arrow[u, "h'"] & \mathbb{C} \arrow[l, "u"] \arrow[u, "h"']
\end{tikzcd}$$

h, h' are the Yoneda embeddings.

The composite $\tilde{\mathbb{C}} \subseteq \hat{\mathbb{C}} \xrightarrow{u_!} \hat{\mathbb{C}}' \xrightarrow{a} \tilde{\mathbb{C}}'$ provides a left adjoint, denoted u^s to u_s. If $u_!$ is left exact, so is u^s and thus u_s is a geometric morphism of topoi. This will be the case, as pointed out above, if $\mathbb{C}$ has finite left limits and u preserves them. Of course if $u_!$ is left exact it is clearly necessary that u preserve whatever finite left limits exist in $\mathbb{C}$, because of the equation $u_! h = h' u$ in the diagram above.

Any continuous functor must carry covering families to covering families. In case the topology of $\mathbb{C}$ is defined by a pretopology and u commutes with fibre products, this condition is also sufficient for u to be continuous. In fact it is sufficient that u carry coverings of the pretopology to coverings in $\mathbb{C}'$.

It follows that a continuous mapping $f: X \longrightarrow Y$ of topological spaces induces a continuous functor f^{-1}: Open$(Y) \longrightarrow$ Open(X). Thus we obtain a pair of functors

$$\begin{tikzcd}
\tilde{X} \arrow[r, shift left, "(f^{-1})_s"] & \tilde{Y} \arrow[l, shift left, "(f^{-1})^s"]
\end{tikzcd}$$

f^{-1} preserves finite left limits since they amount to intersections hence $(f^{-1})^s$ is left exact and $(f^{-1})_s$ is a geometric morphism of topoi, as promised sometime previously. To add to the notational confusion, it is traditional to denote $(f^{-1})_s$ by f_* and $(f^{-1})^s$ by f^*.

$f_*(E)$ is called the direct image of the sheaf F and $f^*(G)$ is called the inverse image of G.

1.3 Cocontinuous Functors Between Sites The notion of continuity of a functor $u:\mathbb{C} \longrightarrow \mathbb{C}'$ has lead us to an adjoint pair

$$\begin{tikzcd} \tilde{\mathbb{C}}' \arrow[r, shift left, "u_s"] & \tilde{\mathbb{C}} \arrow[l, shift left, "u_s"] \end{tikzcd}$$

$u^s \dashv u_s$. The additional assumption of cocontinuity provides a right adjoint to u_s. We say u is cocontinuous if it satisfies one of the two following equivalent conditions:

(1) u_* carries sheaves to sheaves

(2) $\forall\, C \in |\mathbb{C}|$, $\forall\, R \in Cov(u(C))\ \exists\, S \in Cov(C)$ with $u(S) \subseteq R$.

This last inequality is to be understood in the language of ideals: $u(S) = \left\{ u(f) \,\middle|\, A' \xrightarrow{f} A \in S \right\}$.

In this case the restriction of u_* to sheaves is also denoted u_* . When there is possibility of confusion we write $\hat{u}_*$ for the functor at the level of presheaves.

If u is cocontinuous (not necessarily continuous) the composite $\tilde{\mathbb{C}}' \subseteq \hat{\mathbb{C}}' \xrightarrow{u^*} \hat{\mathbb{C}} \xrightarrow{a} \tilde{\mathbb{C}}$ provides a left adjoint to u_*, and in fact a left exact left adjoint, so that u_* is a geometric morphism of topoi. If u is also continuous this composite is just u_s so we have $u^s \dashv u_s \dashv u_*$.

The important example of a functor which is continuous and cocontinuous is the inclusion $Open(U) \xrightarrow{v} Open(T)$ where $U \subseteq T$ is open. Note that this is not a functor induced by a continuous map but the functor Open(T) $\longrightarrow$ Open(U) induced by $i : U \subseteq T$ is in fact right adjoint to the functor v. There is a theorem about such situations, viz: for $\mathbb{C} \underset{v}{\overset{u}{\rightleftarrows}} \mathbb{C}'$ with $v \dashv u$, u is continuous iff v is cocontinuous. In this case $v_* \cong u_s$, $v^* \cong u^s$.

1.4 Summary Let us attempt to clear away at least some confusion by summing all this up in a few diagrams.

Presheaves

$$\begin{tikzcd}
\hat{\mathbb{C}}' \arrow[r, "u^*"] & \hat{\mathbb{C}} \arrow[l, bend right=50, "u_*"'] \arrow[l, bend left=50, "u_!"]
\end{tikzcd}$$

$u_! \dashv u^* \dashv u_*$

$u^*(F)(C) = F(u(C))$

$u_!(G)(C') = \varinjlim_{u(C) \to C'} G(C)$

$u_*(G)(C') = \varprojlim_{C' \to u(C)} G(C)$

Presheaves in the presence of an adjoint pair $\mathbb{C} \underset{v}{\overset{u}{\rightleftarrows}} \mathbb{C}'$

$v \dashv u$

$$\begin{tikzcd}
\hat{\mathbb{C}}' \arrow[r, shift left, "u^* = v_*"] \arrow[r, bend right=60, "v_!"'] & \hat{\mathbb{C}} \arrow[l, shift left, "v^* = u_!"] \arrow[l, bend right=60, "u_*"']
\end{tikzcd}$$

$v_! \dashv (v^* = u_!) \dashv (v_* = u^*) \dashv u_*$

Sheaves-continuity

definition

$$\begin{tikzcd}
\hat{\mathbb{C}}' \arrow[r, "u^*"] & \hat{\mathbb{C}} \\
\tilde{\mathbb{C}}' \arrow[u, "i'"] \arrow[r, dashed, "\exists u_s"'] & \tilde{\mathbb{C}} \arrow[u, "i"']
\end{tikzcd}$$

$$\begin{tikzcd}
\tilde{\mathbb{C}}' \arrow[r, "u_s"] & \tilde{\mathbb{C}} \arrow[l, bend left=50, "u^s"]
\end{tikzcd}$$

$u^s \dashv u_s$

$u^s = a\, u_!\, i$

u_s a geometric morphism if C has and u preserves finite $\varprojlim$'s

Sheaves-cocontinuity

definition
$$\begin{tikzcd}
\widehat{\mathbb{C}}' & \widehat{\mathbb{C}} \arrow[l, "\hat{u}_*"'] \\
\widetilde{\mathbb{C}}' \arrow[u] & \widetilde{\mathbb{C}} \arrow[u] \arrow[l, dashed, "\exists\, u_*"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{\mathbb{C}}' \arrow[r, shift right, "au^*i"'] & \widetilde{\mathbb{C}} \arrow[l, shift right, "u_*"']
\end{tikzcd}
\qquad au^*i \dashv u_*$$

u_* a geometric morphism

Sheaves-Continuity and Cocontinuity

$$\begin{tikzcd}
\widetilde{\mathbb{C}}' \arrow[r, "u_s"] & \widetilde{\mathbb{C}} \arrow[l, bend right=50, "u_*"'] \arrow[l, bend left=50, "u^s"]
\end{tikzcd}
\qquad u^s \dashv u_s \dashv u_*$$

Sheaves in the presence of an adjoint pair

$$\begin{tikzcd}
\mathbb{C} \arrow[r, shift left, "u"] & \mathbb{C}' \arrow[l, shift left, "v"]
\end{tikzcd}
\qquad u \dashv v \qquad u \text{ continuous} \Longleftrightarrow v \text{ cocontinuous}$$

If u continuous
$$\begin{tikzcd}
\widetilde{\mathbb{C}}' \arrow[r, shift left, "u^s = v_*"] & \widetilde{\mathbb{C}} \arrow[l, shift left, "u_s = v^*"]
\end{tikzcd}
\qquad u^s \text{ is a geometric morphism}$$

Sheaves: Continuous mapping $X \xrightarrow{f} Y$ of Topological Spaces

$$\begin{tikzcd}
\widetilde{X} \arrow[r, shift left, "f_*"] & \widetilde{Y} \arrow[l, shift left, "f^*"]
\end{tikzcd}
\qquad f^* \dashv f_* \qquad f_* \text{ a geometric morphism}$$

$$f_*(F)(V) = F(f^{-1}(V))$$

Sheaves - Inclusion of an Open Subspace $U \subseteq T$

v: Open(U) $\longrightarrow$ Open(T)

$i: U \longrightarrow T$ yields i^{-1}: Open(T) $\longrightarrow$ Open(U)

$v \dashv i^{-1}$ v is continuous and cocontinuous

$$\begin{tikzcd}
\widetilde{U} \arrow[r, shift left, "i_* = (i^{-1})_s = v_*"] \arrow[r, bend right=60, "v^s"'] & \widetilde{T} \arrow[l, shift left, "v_s = (i^{-1})^s = i^*"]
\end{tikzcd}$$

$$v_*(F)(W) = F(U \cap W)$$

$$v^s(F)(W) = \begin{cases} F(W) & W \subseteq U \\ \emptyset & W \nsubseteq U \end{cases}$$

$$v_s(G)(V) = G(V) \quad \forall\, V \subseteq U$$

2. Points of a Topos

2.1 Preliminaries

The category of sets, which we have denoted by Sets, is a topos. For example it may be represented as the functor category $\text{Sets}^{\mathbb{1}}$ where $\mathbb{1}$ is the terminal object in the category of categories. Now suppose we have an elementary topos $\mathbb{E}$ and a geometric morphism $\mathbb{E} \xrightarrow{f} \text{Sets}$. Then f^* is left exact and preserves right limits. By left exactness $f^*(1) = 1$, the terminal object in $\mathbb{E}$. By the preservation of left limits it follows that $f^*(n) = \coprod_n 1$ for any $n \in |\text{Sets}|$. Thus there can be at most one such geometric morphism and if it exists its right adjoint part is represented by $1 \in |\mathbb{E}|$. The topoi $\tilde{\mathbb{C}}$ are always complete so $\mathbb{E}(1,-)$ is a geometric morphism, for f^* defined by the above formula is left exact, a fact which follows from the exactness properties of an elementary topos. To sum up, Sets is the terminal object in the category of topoi of the form $\tilde{\mathbb{C}}$ (as opposed to the more general elementary topoi).

Now whenever we stumble upon a terminal object in a category, it is natural to attempt to assess how close it come to being a generator. In our case we wish to study geometric morphisms $p: \text{Sets} \longrightarrow \tilde{\mathbb{C}}$. These are, for now obvious reasons, called points of $\tilde{\mathbb{C}}$ (or of $\mathbb{C}$). The category of topoi inherits a 2-structure from the category of categories. In particular the class of points of a topos $\tilde{\mathbb{C}}$ is a category which we denote Point $(\mathbb{C})$. The left exact left adjoint p^* of a point p is called a stalk or fibre of $\mathbb{C}$ and the category of these is denoted Fib $(\mathbb{C})$. A natural transformation $p \longrightarrow p'$ induces a natural transformation

$(p')^* \longrightarrow p^*$ by a well known proposition about adjoint functors. This mediates an isomorphism $\mathrm{Fib}(\mathbb{C})^* \xrightarrow{\sim} \mathrm{Point}(\mathbb{C})$. If $F \in |\widetilde{\mathbb{C}}|$ and p is a point we often write $p^*(F) = Fp$.

2.2 How Points Arise as Direct Limits Observe that every $F \in |\widetilde{\mathbb{C}}|$ is the colimit of a diagram of generators and since p^* is colimit preserving, p^* is determined by its values on any generating set and in particular by its values on the sheaves associated to the representables

$$F = \varinjlim_{\mathbb{C}/F} C$$

Hence $F = aF = a(\varinjlim_{\mathbb{C}/F} C) = \varinjlim_{\mathbb{C}/F} a(C) \Longrightarrow p^*(F) = \varinjlim_{\mathbb{C}/F} p^*(a(C))$

$\mathbb{C}/F$ is the comma category whose objects are maps $C \longrightarrow F$.

Since p^* is determined by the composite $\mathbb{C} \longrightarrow \widetilde{\mathbb{C}} \xrightarrow{p^*} \mathrm{Sets}$ we ask what the properties of this composite are. One knows ([SGA 4], IV.6.3) that the composite, which we from time to time also denote by p^*,

(1) is left exact

(2) carries covering families to covering families (i.e. epimorphic ones)

Moreover, provided $\mathbb{C}$ has finite left limits, these properties characterize the stalks on $\mathbb{C}$.

We can pin down the stalks a little further. Following [SGA] define a neighbourhood of a point p to be an object $F \in |\widetilde{\mathbb{C}}|$ together with an element $u \in F_p = p^*(F)$. In $\mathrm{Sets}^{\widetilde{\mathbb{C}}}$ this is a morphism $F \longrightarrow p^*$, identifying F with the representable functor $\tilde{\mathbb{C}}(F,-)$. Defining a morphism of neighbourhoods in the obvious way we have the category $N(p) = (\tilde{\mathbb{C}}, p^*) = \tilde{\mathbb{C}}/p^*$

$$\text{and} \quad p^* = \varinjlim_{F \in N(p)^*} F$$

Since this limit is computed pointwise in $\mathrm{Sets}^{\tilde{\mathbb{C}}}$ we have

$$p^*(G) = \varinjlim_{F \in N(p)^*} F(G) = \varinjlim_{F \in N(p)^*} \tilde{\mathbb{C}}(F,G)$$

viewing $N(p)$ as a comma category it is clear that it has finite limits (in fact it's complete and cocomplete). Therefore $N(p)^*$ is filtered.

[Recall a category $\mathbb{K}$ is filtered if $\forall$

$$\begin{tikzcd} & K' \\ K \arrow[ur] \arrow[dr] & \\ & K'' \end{tikzcd} \quad \exists \quad \begin{tikzcd} K' \arrow[dr] & \\ & K''' \\ K'' \arrow[ur] & \end{tikzcd}$$

making the square commute, and $\forall$ $K \rightrightarrows K'$ $\exists$ $K' \longrightarrow K''$ such that the composites are equal.] Moreover if we consider the full subcategory of $N(p)$ whose objects are those $F \longrightarrow p^*$ where F runs through a generating family closed under finite left limits, we obtain a cofinal (c.f. [FE]) subcategory of $N(p)^*$. Suppose then that $\mathbb{C}$ has finite $\varprojlim$'s. Then the subcategory of $N(p)$ determined by $a(C) \longrightarrow p^*$, $C \in |\mathbb{C}|$ constitutes a filtered, cofinal subcategory. Denote it by $N'(p)$.

$$\text{Then} \quad p^*(G) = \varinjlim_{N(p)^*} \tilde{\mathbb{C}}(F,G)$$

$$= \varinjlim_{N'(p)^*} \tilde{\mathbb{C}}(a(C),G)$$

$$= \varinjlim_{N'(p)^*} \hat{\mathbb{C}}(C,G)$$

$$= \varinjlim_{N'(p)^*} G(C)$$

The point of this discussion is that every stalk functor is obtained from a filtered diagram in $\mathbb{C}$ by taking direct limits. It is not however the case that every filtered diagram in $\mathbb{C}$ gives rise to a stalk by the above formula. Conditions (1) and (2) for stalks (see above) must always be checked. Neither one follows from the nature of the formula suggested.

2.3 A topos $\tilde{\mathbb{C}}$ <u>has enough points</u> if $\forall \phi : F \longrightarrow G$ in $\tilde{\mathbb{C}}$, ϕ_p iso all points $p \Longrightarrow \phi$ iso. i.e. the stalks collectively reflect isomorphisms. It follows that the stalks collectively reflect whatever limits and colimits they happen to preserve, namely finite left limits and all right limits. This is well-known in the case of sheaves on a topological space. i.e. a diagram of sheaves is a pullback, coequalizer etc. iff it is "stalkwise".

If a topos has enough points it admits a <u>set</u> of points whose stalks collectively reflect isomorphisms ([SGA 4], I.7.7, IV.6.5).

Finally, if a set of fibres collectively reflects isomorphisms, its members are collectively faithful. For suppose $F \underset{\psi}{\overset{\phi}{\rightrightarrows}} G$ are maps in $\tilde{\mathbb{C}}$ with $\phi_p = \psi_p$ all p of the set envisaged. Let $E \xrightarrow{\mu} F$ be the equalizer of ϕ and ψ. Then μ_p is the equalizer of ϕ_p, ψ_p all $p \Longrightarrow \mu_p$ is an isomorphism, all $p \Longrightarrow \mu$ is an isomorphism $\Longrightarrow \phi = \psi$.

<u>Example</u> 2.3.1 $\hat{\mathbb{C}}$ has enough points for any $\mathbb{C}$, since the evaluations $e_C : \hat{\mathbb{C}} \longrightarrow Sets$ via $F \longmapsto F(C)$ are left exact left adjoints which collectively reflect isomorphisms. Hence the oft-repeated

"everything works pointwise in a functor category". Note that for $\widetilde{\mathfrak{C}}$ evaluations are almost never left adjoints since they normally fail to preserve right limits.

<u>Example</u> 2.3.2 For a topological space T, the stalks of $\widetilde{T}$ associated with the set-theoretical points of the space (cf. I.5.2) collectively reflect isomorphisms.

<u>Note</u> 2.3.3 Categories $\widetilde{\mathfrak{C}}$ need not have enough points, as we shall see shortly.

2.4 <u>Open Sieves</u> We note in passing the interplay between subobjects of 1 in $\widetilde{\mathfrak{C}}$, which we call open sieves, and the points of $\mathfrak{C}$. Applying the fibre p^* to $R \subseteq 1$ we obtain $R_p \subseteq 1_p = 1$. Thus $R_p = 1$ or 0 and the full subcategory of $N(p)$ whose objects are open sieves is isomorphic to the category whose objects are $R \subseteq 1$ with $R_p = 1$ and whose maps are inclusions $R \subseteq R'$. The set of points of $\mathfrak{C}$ may be endowed with a topology by taking as open sets $R^s = \left\{p\colon R_p = 1\right\}$. It follows that $(\bigcup R_i)^s = \bigcup R_i^s$, $(R \cap S)^s = R^s \cap S^s$, $0^s = 0$ and $1^s =$ set of all points.

We will later have occasion to refer to this topology on the points of $\mathfrak{C}$, when we characterize the topoi of the form $\widetilde{T}$.

3 <u>Examples of Points and Morphisms</u>

3.1 <u>Points of $\widetilde{T}$ for a Topological Space T</u>

3.1.1 Theorem Let T be a topological space. The category of points of $\tilde{T}$ is isomorphic to the poset of non-empty irreducible closed subsets of T, ordered by inclusion.

Recall that a closed $Z \subseteq T$ is irreducible iff $\forall F, F' \subseteq T$ closed, $Z = F \cup F' \Rightarrow F = \emptyset$ or $F' = \emptyset$. Equivalently $\forall U, U' \subseteq T$ open, $(U \cap Z \neq \emptyset \wedge V \cap Z \neq \emptyset) \Rightarrow U \cap V \cap Z \neq \emptyset$. It is clear from the statement of the theorem that irreducible closed subsets play an important role in classical sheaf theory. In a later section we will use them to "balance" topological spaces in the course of studying the functor which associates to a space its category of sheaves.

3.1.2 Proof Recall from 2.2 that the category of points of $\tilde{T}$ is dual to the category of stalks on $\tilde{T}$ which in turn is isomorphic to the category of set-valued functors on Open(T) which are left exact and which take covering families to covering families. Let p be a point of $\tilde{T}$ and $p^*: \text{Open}(T) \longrightarrow \text{Sets}$ the corresponding stalk restricted to Open(T). Since p^* is left exact and $T \in |\text{Open}(T)|$ is terminal, $p^*(T) = T_p = 1$. Every map $U \longrightarrow V$ in Open(T) is mono and left exact functors preserve monos, hence $U_p \longrightarrow T_p = 1$ is mono. Thus $U_p = \emptyset$ or 1, all $U \in |\text{Open}(T)|$. Hence we can consider p^* as taking values in the 2 element Boolean algebra. The condition that p^* be left exact means $(U \cap V)_p = U_p \cap V_p$ and the covering condition means $(\bigcup_{i \in I} U_i)_p = \bigvee_{i \in I} (U_i)_p$ where U, V, U_i are open sets and the join in the right hand side is taken in the Boolean algebra 2. Note that this implies $\emptyset_p = 0$. We have so far shown that the stalks of $\tilde{T}$ are no more or less than join-complete lattice homomorphisms from the Heyting algebra of open sets of T into the two-element Boolean algebra.

Let us consider the category $N'(p)$ of neighbourhoods of p (cf. 2.2). Its objects are maps $U \longrightarrow p^*$, i.e. elements of $p^*(U)$. Now $p^*(U)$ has at most one element so it is clear that $N'(p)$ is isomorphic to the subcategory of $\text{Open}(T)$ determined by those U for which $U_p = 1$. Hence, $\forall\, F \in |\tilde{T}|$, $F_p = \underset{U_p = 1}{\varinjlim} F(U)$.

Define $Z = \{x \in T\colon \forall \text{ open } U \subseteq T,\ x \in U \Longrightarrow U_p = 1\}$. That is, $x \in Z$ iff each of its topological neighbourhoods is a topos-theoretical neighbourhood of p. Z is closed, for take $x \in T$ with the property that every nbhd of x meets Z. Take any open $V \ni x$. $V \cap Z \neq \emptyset \Longrightarrow V_p = 1$. Since V was an arbitrary nbhd of x, $x \in Z$ and Z is closed. Z is irreducible, for suppose $U, V \subseteq T$ are open, $U \cap Z \neq \emptyset$, $V \cap Z \neq \emptyset$. If $U \cap V \cap Z = \emptyset$, then $\forall\ x \in U \cap V$, $\exists$ open $W_x \subseteq U \cap V$ with $x \in W_x$ and $(W_x)_p = 0$. $\bigcup\limits_x W_x = U \cap V$, hence

$$0 = \bigcup_x (W_x)_p = (\bigcup W_x)_p = (U \cap V)_p = U_p \cap V_p = 1 \quad \otimes$$

Now since $U_p = 1 \Longleftrightarrow U \cap Z = \emptyset$ we have shown that every point p of $\tilde{T}$ arises from an irreducible closed $Z \subseteq T$ via the formula (for p^*)

$$F_p = \underset{U \cap Z \neq \emptyset}{\varinjlim} F(U)$$

Conversely such a formula defines a point, for a non-empty irreducible closed Z. For clearly such a formula implies that $U_p = 1$ if $U \cap Z \neq \emptyset$, 0 otherwise. It is easy to check that this defines a join-complete lattice homomorphism from Open(T) to 2. Left exactness comes from the irreducibility and the fact that $Z \neq \emptyset$.

To this point we have exhibited a mapping from the irreducible closed subsets of T onto the stalks of $\tilde{T}$. To establish the 1-1 nature of the mapping, suppose Z, Z' are different irreducible closed subsets, say $x \in Z - Z'$. Then $\exists$ open $U \subseteq T$ with $x \in U$, $U \cap Z' = \emptyset$. But this means $U_Z = 1$, $U_{Z'} = 0$ where U_Z means $p^*(U)$ for the point p associated with Z. Thus $Z \neq Z' \Longrightarrow (\ )_Z \neq (\ )_{Z'}$. Finally we must show that for points p, p' induced by Z, Z'

$$\text{n.t.}(p, p') \cong \text{Hom}(Z,Z') = \begin{cases} 1 & Z \subseteq Z' \\ 0 & Z \nsubseteq Z' \end{cases}$$

But since Z, Z' are closed, $Z \subseteq Z' \Longleftrightarrow (\forall \text{ open } U,\ U \cap Z' = \emptyset \Longrightarrow U \cap Z = \emptyset)$. The latter condition is precisely what is needed for there to exist a natural transformation $p'^* \longrightarrow p^*$ (considered as 2-valued join-complete lattice homomorphisms on Open(T)). There can be at most one such natural transformation since p'^*, p^* take values in a poset.

3.1.3 Remarks Every element of the set underlying the space T yields a point of $\tilde{T}$ since $\forall x \in T$, $\Gamma\{x\}$ is an irreducible closed set and the corresponding stalk is computed at a sheaf F by $F_x = \varinjlim_{U \ni x} F(U)$ since the conditions $x \in U$ and $\Gamma\{x\} \cap U \neq \emptyset$

are equivalent. It is well-known that these particular points are sufficient to reflect isomorphisms, which fact is the basis of the "étalé space" approach to classical sheaf theory.

3.2 *Points of the Double Negation Sheaves for a Space T* Here we analyze further example 5.7 of chapter I. Using the terminology established previously, we have a morphism of sites (i.e. continuous functor) Open(T)$\longrightarrow$Open(T)$^{\perp\perp}$ given by the identity functor. For this functor is left exact and takes coverings to coverings. This yields a morphism of topoi $\tilde{T}^{\perp\perp}\longrightarrow\tilde{T}$ whose exact left adjoint carries generators $U \in |\text{Open}(T)|$ to $\text{Int}\,\Gamma(U)$. This latter functor induces by composition a map of the points of Open(T)$^{\perp\perp}$ into those of Open(T). If we look at the associated stalks, more precisely their restrictions to the generators, the mapping of the stalks of Open(T)$^{\perp\perp}$ to those of Open(T) is given by composition with the above mentioned identity functor Open(T)$\longrightarrow$Open(T)$^{\perp\perp}$ Hence every stalk of $\tilde{T}^{\perp\perp}$ is defined by a formula

$$F_p = \varinjlim_{U \cap Z \neq \emptyset} F(U) \qquad \text{some non-empty irreducible closed set } Z$$

For if p^* is a stalk on $\tilde{T}^{\perp\perp}$ $p^*(F) = \varinjlim_{p^*(U) = 1} F(U)$, and

$p^*(U) = 1 \Longleftrightarrow q^*(U) = 1$ where q is the point on Open(T) induced by p.

Thus $p^*(F) = \varinjlim_{q^*(U) = 1} F(U) = \varinjlim_{U \cap Z \neq \emptyset} F(U)$

It is clear now that the category of points of $\tilde{T}^{\perp\perp}$ is isomorphic to some subcategory of the category of non-empty irreducible closed subsets of T. This subcategory is identified by the following theorem.

3.2.1 Theorem A non-empty irreducible closed $Z \subseteq T$ defines a point of $\tilde{T}^{\perp\perp}$ iff it is regular i.e. $Z = \Gamma\,\mathrm{Int}(Z)$.

Note that for non-empty irreducible closed subsets, regularity is the same as having non-empty interior. For if $\mathrm{Int}(Z) \neq \emptyset$ take any $x \in Z$. $\forall$ nbhd $U \ni x$, $U \cap Z \neq \emptyset$ and by irreducibility $U \cap \mathrm{Int}\, Z \neq \emptyset$. Hence every nbhd of $x \in Z$ meets $\mathrm{Int}\, Z$, or $Z \subseteq \Gamma\,\mathrm{Int}\, Z$. On the other hand if $\mathrm{Int}\, Z = \emptyset$, Z cannot be regular and non-empty at the same time.

3.2.2. Proof of Theorem Let p be a point of T and Z its associated irreducible closed subset, i.e. $p^*(F) = \varinjlim_{U \cap Z \neq \emptyset} F(U)$. Consider the open set T-Z. Since $Z \cap (T-Z) = \emptyset$, $p^*(T-Z) = 0$. The inclusion $T-Z \subseteq \mathrm{Int}\,\Gamma(T-Z)$ is a cover in $\mathrm{Open}(T)^{\perp\perp}$ so $p^*(\mathrm{Int}\,\Gamma(T-Z)) = p^*(T-Z) = 0$. This in turn implies $Z \cap \mathrm{Int}\,\Gamma(T-Z) = \emptyset$, i.e. $T-Z \supseteq \mathrm{Int}\,\Gamma(T-Z)$. The other inclusion being trivial, we have $T-Z = \mathrm{Int}\,\Gamma(T-Z)$, equivalently $Z = \Gamma\,\mathrm{Int}(Z)$. Hence Z is regular. Conversely if Z is a non-empty regular irreducible closed set it must induce a point of $\tilde{T}^{\perp\perp}$. The formula $F_p = \varinjlim_{U \cap Z \neq \emptyset} F(U)$ certainly provides a left exact functor on $\mathrm{Open}(T)^{\perp\perp}$, as in 3.1.2, for this follows simply from the fact that Z is irreducible and non-empty. We need only check the covering condition. Suppose $U_i \subseteq U$, $i \in I$ is a covering of U. This means $U \subseteq \mathrm{Int}\,\Gamma(\bigcup_i U_i)$. We must show $p^*(\mathrm{Int}\,\Gamma(\bigcup_i U_i)) = 1 \Longrightarrow \exists\, i \in I$ with $p^*(U_i) = 1$. Since p^* is a fibre for the canonical topology we know $p^*(U_i) = 1$ some $i \in I \Longleftrightarrow p^*(\bigcup_i U_i) = 1$. Rephrasing, we must show $p^*(\mathrm{Int}\,\Gamma(\bigcup_i U_i)) = 1 \Longrightarrow p^*(\bigcup_i U_i) = 1$. Put $\bigcup_i U_i = V$. Suppose $p^*(\mathrm{Int}\,\Gamma(V)) = 1$ i.e. $Z \cap \mathrm{Int}\,\Gamma(V) \neq \emptyset$. By irreducibility of Z, $\mathrm{Int}\, Z \cap \mathrm{Int}\,\Gamma(V) \neq \emptyset$ (by regularity, it is impossible that $\mathrm{Int}\, Z = \emptyset$ since $Z \neq \emptyset$). Now $\mathrm{Int}\, Z \cap \mathrm{Int}\,\Gamma(V) = \mathrm{Int}\,\Gamma\ \mathrm{Int}\, Z \cap \mathrm{Int}\,\Gamma(V) = \mathrm{Int}\,\Gamma(\mathrm{Int}\, Z \cap V)$ ("$\mathrm{Int}\,\Gamma$" commutes with intersections, for open sets cf. 5.7 chapter I). Hence $\mathrm{Int}\,\Gamma(\mathrm{Int}\, Z \cap V) \neq \emptyset$, a fortiori $\mathrm{Int}\, Z \cap V \neq \emptyset$, so $Z \cap V \neq \emptyset$. Hence $p^*(V) = 1$.

3.2.3 Note that if T is, say, Hausdorff and has no isolated points, the only non-empty irreducible closed sets are the singletons, and these have empty interior. For such T then, 3.2.1 shows $\tilde{T}^{\perp\perp}$ has no points. Thus a very fundamental construction on the category of sheaves on a topological space (fundamental within the theory of elementary topoi, that is) may produce topoi which cannot be realized as sheaves on any topological space. As pointed out in 5.7 of chapter I, $\tilde{T}^{\perp\perp}$ is a Boolean topos. In fact the double negation sheaves associated with any elementary topos always form a Boolean topos. For example see [AT].

3.3 Points of $\tilde{M}$ Associated With a Measure Space (K,μ) As in 5.6, Chapter I, let (K,μ) be a compact topological space equipped with a complete regular Borel measure. The following result, announced in [SGA], answers the question of what the points of the associated Topos $\tilde{M}$ are.

3.3.1 Theorem The category of points of $\widetilde{\mathbb{M}}$ is isomorphic to the discrete category whose objects are those $x \in K$ with $\mu(\{x\}) \neq 0$.

3.3.2 Proof Let p be a point of $\widetilde{\mathbb{M}}$. For convenience we will denote the restriction of p^* to $\mathbb{M} \subseteq \widetilde{\mathbb{M}}$ by ϕ. Recall ϕ is left exact and takes covering families to covering families. The following facts are immediate and may be obtained in a manner quite similar to the proof of the analogous facts for $\widetilde{T}$ (see 3.1.2).

(F1) $\phi(K) = 1, \quad \phi(\emptyset) = 0, \quad \phi(A) = 0$ or $1 \quad \forall A \in |\mathbb{M}|$

(F2) $\phi(A \cap B) = \phi(A) \wedge \phi(B)$ and $\phi(A) = 1 \Longleftrightarrow \phi(K-A) = 0$.
For the latter, observe that $A, K-A$ cover K so since $\phi(K) = 1$ (F1), either $\phi(A) = 1$ or $\phi(K-A) = 1$. But not both, for $\phi(A) \wedge \phi(K-A) = \phi(A \cap (K-A)) = \phi(\emptyset) = 0$.

(F3) The covering condition says that for any countable family $A_i \subseteq A$ with $\mu(A - \bigcup A_i) = 0$, $\phi(A) = 1 \Longrightarrow \exists\, i$ with $\phi(A_i) = 1$.

(F4) $\forall A \subseteq K, \; \mu(A) = 0 \Longrightarrow \phi(A) = 0$ for $\mu(A) = 0 \Longrightarrow \emptyset \subseteq A$ is a covering family.

For $x \in K$ we shall write $\mu(x)$ instead of $\mu(\{x\})$. Suppose now we have $x \in K$ with $\mu(x) \neq 0$. Define $\phi : \mathbb{M} \longrightarrow \text{Sets}$ by

$$\phi(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Such a ϕ is in fact a point of $\mathbb{M}$. Clearly ϕ is left exact. To establish the covering condition (F3) let $A \supseteq \bigcup A_i$, $\mu(A - \bigcup A_i) = 0$

and $\phi(A) = 1$, i.e. $x \in A$. If $x \notin \bigcup A_i$ then $A - \bigcup A_i \supseteq \{x\}$ and $\mu(A - \bigcup A_i) \geq \mu(x) > 0$ $\otimes$. Hence $x \in \bigcup A_i$ and $x \in A_i$, some i. Thus $\phi(A) = 1 \Longrightarrow \exists i$ with $\phi(A_i) = 1$. Writing $\phi(x)$ instead of $\phi(\{x\})$ we have that $\phi(x) = 1$. Quite in general, for a stalk ϕ , if for some x $\phi(x) = 1$ then $\phi(x') = 0$ all $x' \neq x$. For $\phi(x) = 1 \Longrightarrow \phi(K - \{x\}) = 0$ (F2) and $x' \neq x \Longrightarrow \{x'\} \subseteq K - \{x\}$ so $\phi(x') = 0$.

Thus we have established a 1-1 map from $\{x \in K: \mu(x) \neq 0\}$ to the set of stalks of $\mathbb{M}$. Its image is $\{\phi : \phi$ a stalk of $\mathbb{M}$, $\exists\, x \in K \;\; \phi(x) = 1\}$. For suppose ϕ is a stalk of $\mathbb{M}$ and $\phi(x) = 1$, some $x \in K$. Then the neighbourhoods of the point associated with ϕ (cf. 2.2) are exactly those $A \subseteq |\mathbb{M}|$ with $x \in A$. Clearly $x \in A \Longrightarrow \phi(A) = 1$ since $\{x\} \subseteq A$. $x \notin A \Longrightarrow A \subseteq K - \{x\}$ so $\phi(K - \{x\}) = 0 \Longrightarrow \phi(A) = 0$. Therefore

$$\phi(A) = \varinjlim_{B \in N'(p)} A(B) = \varinjlim_{B \ni x} A(B) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} .$$

This establishes the bijection $\{x : \mu(x) \neq 0\} \xrightarrow{\sim} \{\phi : \phi$ a stalk, $\phi(x) = 1$, some $x\}$. Injectivity is clear for if ϕ , ϕ' correspond to x, x' then $x \neq x' \Longrightarrow \phi(x') = 0$, $\phi'(x') = 1$ i.e. $\phi \neq \phi'$.

Claim: this exhausts all of the stalks. That is, there are no stalks which vanish on every singleton. The proof involves fairly complicated but elementary considerations of a measure-theoretic and topological nature. We argue by reductio ad absurdum and work now with a fixed stalk ϕ which we suppose to vanish on every singleton.

The first lemma will allow us to reduce to the case in which all singletons have measure zero. (By F4 this is formally stronger than the condition on ϕ).

3.3.3 Lemma $B = \{x \in K: \ \mu(x) \neq 0\}$ has an open neighbourhood U (in the topological sense, of course) with $\phi(U) = 0$.

Proof Let $x \in B$. By regularity of μ we can find a decreasing sequence of open neighbourhoods of x, say V_n with $\lim_{n \to \infty} \mu(V_n) = \mu(x)$. Then $\mu(\bigcap V_n - \{x\}) = 0$. Thus $\{x\}$ covers $\bigcap V_n$ and since $\phi(x) = 0$, $\phi(\bigcap V_n) = 0$. It follows that $\phi(K - \bigcap V_n) = 1$ i.e. $\phi(\bigcup K - V_n) = 1$. Hence $\exists\, n_0$ with $\phi(K - V_{n_0}) = 1$ i.e. $\phi(V_{n_0}) = 0$. This shows every $x \in B$ has a neighbourhood V_x with $\phi(V_x) = 0$. Since $\mu(K) < \infty$ and $\mu(V_x) > 0$ all $x \in B$, B must be countable. Thus the V_x cover $\bigcup_{x \in B} V_x$ in the topology on $\mathbb{M}$ so $\phi(\bigcup V_x) = 0$ and $\bigcup_{x \in B} V_x$ is the sought-for neighbourhood of B.

Now we can make the announced reduction. If $K' \subseteq K$ is a compact set equipped with the restriction of μ and associated category of measurable subsets $\mathbb{M}'$ then the composite

$$\mathbb{M}' \hookrightarrow \mathbb{M} \xrightarrow{\phi} \text{Sets}$$

satisfies the covering condition and preserves non-empty intersections. All that is needed to be a stalk on $\mathbb{M}'$ is that the composite preserve the terminal object, and hence all finite left limits. In conclusion,

ϕ restricts to a stalk on $\mathbb{M}'$ iff $\phi(K') = 1$. In our case put $K' = K-U$. Since $\phi(U) = 0$, $\phi(K') = 1$. More importantly all singletons of K' have measure 0. If we can show that there are no stalks ϕ on $\mathbb{M}'$ with $\phi(x) = 0$ all $x \in K'$, we will have shown the same for $\mathbb{M}$, K. Thus we may now assume without loss of generality that all singletons of K have measure 0. With two more lemmas we will be ready to prove the result.

3.3.4 Lemma Suppose $A \subseteq K$ is measurable with $\phi(A) = 1$. Then there is a closed $F \subseteq A$ with $\phi(F) = 1$.

Proof $\phi(A) = 1 \Longrightarrow \mu(A) > 0$ (F4). By regularity construct an ascending sequence of closed sets $F_n \subseteq A$ with $\lim_{n \to \infty} \mu(F_n) = \mu(A)$. Then $\mu(A - \bigcup F_n) = 0$ so the F_n cover A. Since $\phi(A) = 1$ we must have $\phi(F_n) = 1$, some n.

3.3.5 Lemma Suppose $F \subseteq K$ is closed with $\phi(F) = 1$. Then there is a measurable $A \subseteq F$ with $\phi(A) = 1$ and $\mu(A) < \frac{1}{2}\mu(F)$.

Proof In fact we can choose A relatively open in F but this is of no consequence. For each $x \in F$ choose an open $U_x \subseteq K$, $x \in U_x$ with $\mu(U_x) < \frac{1}{2}\mu(F)$. This is possible since $\mu(F) > 0$, $\mu(x) = 0$ and regularity. Finitely many of the U_x will cover F, by compactness, say $F = (F_n \cap U_{x_1}) \cup \ldots \cup (F_n \cap U_{x_n})$. This is a cover in the sense of our topology on $\mathbb{M}$ so $\phi(F) = 1 \Longrightarrow \exists\, i$ with $\phi(F \cap U_{x_i}) = 1$. Put $A = F \cap U_{x_i}$.

3.3.6 Conclusion of the Proof of 3.3.2. Define sequences $\{A_n\}$, $\{F_n\}$ with (1) F_n are closed subsets of K.

(2) A_n are measurable subsets of K.

(3) $F_{n+1} \subseteq A_{n+1} \subseteq F_n \subseteq A_n$ all n

(4) $\phi(F_n) = \phi(A_n) = 1$ all n

(5) $\mu(F_n) < \frac{1}{2^n} \mu(K)$

Set $F_0 = K$ and select A_0 by 3.3.5 above. Having constructed F_n and A_n construct $A_{n+1} \subseteq F_n$ by 3.3.5 and $F_{n+1} \subseteq A_{n+1}$, by 3.3.4. The conditions (1)-(5) are clearly satisfied. Put $F = \bigcap F_n$. $\mu(F) = \lim_{n\to\infty} \mu(F_n) = 0$. By compactness $F \neq \emptyset$. Since $\mu(F) = 0$, $\phi(F) = 0$. $K-F = \bigcup_n (K-F_n)$ so the $K-F_n$ cover $K-F$. $\phi(F) = 0 \Longrightarrow \phi(K-F) = 1 \Longrightarrow$ for some n_0, $\phi(K-F_{n_0}) = 1 \Longrightarrow \phi(F_{n_0}) = 0$, contradicting the choice of the F_n.

This completes the proof that no stalk of $\mathbb{M}$ may vanish on all singletons and thus completes the proof of theorem 3.3.1.

3.3.7. Remark It follows that for Lebesgue measure on [0,1] for example, $\widetilde{\mathbb{M}}$ has no points. For this is a regular complete measure on a compact T_2 space for which all singletons have measure 0. As pointed out in 5.6, chapter I, this example of a topos with no points is due to P. Déligne.

4 Further Study of Sheaves on a Topological Space

4.1 On the Faithfulness of Top $\longrightarrow$ Topoi Given a continuous mapping $f: X \longrightarrow Y$ of topological spaces we have shown in section 1.2 of this chapter how to associate a geometric morphism of topoi $f_*: \tilde{X} \longrightarrow \tilde{Y}$. This in fact is easily seen to be functorial. We are interested here in the faithfulness of $f \longmapsto f_*$, i.e. when is it the case that $f: X \longrightarrow Y$ and $g: X \longrightarrow Y$ induce the same geometric morphism? The answer is given by theorem 4.1.6 of this section. The results here are announced without proof in [SGA]. The machinery built up in this section will be used in the next section to prove a theorem characterizing the topoi $\tilde{T}$ among the more general $\tilde{\mathfrak{C}}$.

We have previously noted (in section 3.1.1) that closed irreducible subsets of a space X play an important role in the study of $\tilde{X}$. We will now make this more precise. Call a topological space balanced (French: "sobre") if every non-empty closed irreducible subset has exactly one generic point. Recall $x \in Z$ is said to be generic if $\Gamma(\{x\}) = Z$. For example for an indiscrete space, the whole space is closed and irreducible but every point is generic. On the other hand the space whose elements are the natural numbers and whose open sets are the final segments $(n, \longrightarrow)$ has the property that the whole space is closed and irreducible but there are no generic points. These two examples are far from balanced. Any Hausdorff space is balanced for the only non-empty irreducible closed subsets are the singletons.

We construct now a best approximation of an arbitrary space X by a balanced one. This space, called the "balancing" of X, is denoted $b(X)$. The points of $b(X)$ are the non-empty closed irreducible $Z \subseteq X$.

The open sets of $b(X)$ are of the form

$$U^s = \{ Z \in b(X): \ Z \cap U \neq \emptyset \} \ , \ U \subseteq X \ \text{open.}$$

Clearly $(\bigcup U_i^s) = (\bigcup U_i)^s$ and $U_1^s \cap U_2^s = (U_1 \cap U_2)^s$, the latter by irreducibility of the elements of $b(X)$. $\emptyset^s = \emptyset \subseteq b(X)$ and $X^s = b(X)$. Thus we have a topology on $b(X)$. Moreover we have a continuous $\beta: X \longrightarrow b(X)$ by $\beta(x) = \Gamma(\{x\})$. Continuity comes from the fact that $\beta^{-1}(U^s) = \{x \in X: \Gamma(\{x\}) \cap U \neq \emptyset \ = \{x \in X: x \in U\} = U$. Incidently it is clear that β^{-1} is an isomorphism of the sites Open(b(X)) and Open(X). From this the following simple but important proposition follows.

4.1.1 <u>Proposition</u> $\beta_*: \tilde{X} \longrightarrow \widetilde{b(X)}$ is an isomorphism of categories.

4.1.2 <u>Proposition</u> For $U \subseteq X$ open, $b(X) - U^s$ is irreducible iff $X-U$ is irreducible. In this event, $X-U \in b(X)$ is the unique generic point of $b(X) - U^s$. In particular $b(X)$ is a balanced space.

<u>Proof</u> First of all observe that for any open $U, V \subseteq X$, $V \cap (X-U) \neq \emptyset \Longleftrightarrow V^s \cap (b(X) - U^s) \neq \emptyset$. For the members of $V^s \cap (b(X) - U^s)$ are the non-empty irreducible closed subsets of $X-U$ which meet V. If there is one such, a fortiori $X-U$ meets V. On the other hand if $V \cap (X-U) \neq \emptyset$, say $x \in V \cap (X-U)$, then $\Gamma(\{x\})$ is an irreducible closed subset of $X-U$ which meets V.

Now suppose $b(X) - U^s$ is irreducible. We shall show $X-U$ is irreducible. Say $V \cap (X-U) \neq \emptyset$, $W \cap (X-U) \neq \emptyset$. By the above remarks, $V^s \cap (b(X) - U^s) \neq \emptyset$,

$W^s \cap (b(X) - U^s) \neq \emptyset$ hence by irreducibility of $b(X) - U^s$, $V^s \cap W^s \cap (b(X) - U^s) \neq \emptyset$, i.e. $(V \cap W)^s \cap (b(X) - U^s) \neq \emptyset$ i.e. $(V \cap W) \cap (X-U) \neq \emptyset$, again by the above remark. Hence the irreducibility of $X-U$.

Now we show that if $X-U$ is irreducible, $\Gamma_{b(X)}(\{X-U\}) = b(X) - U^s$ which shows that the latter is irreducible and has at least one generic point. Clearly $X-U \in b(X) - U^s$. Take $Z \in b(X) - U^s$ and V^s a neighbourhood of Z i.e. $Z \cap U = \emptyset$, $Z \cap V \neq \emptyset$. It follows that $V \cap (X-U) \neq \emptyset$, i.e. $V^s \cap \{(X-U)\} \neq \emptyset$.

All that remains to be shown is that $X-U$ is the only generic point of $b(X) - U^s$. This boils down to showing that $X-U \neq X-V \Longrightarrow b(X) - V^s \neq b(X) - U^s$ i.e. $V^s \neq U^s$. But, as has already been pointed out the correspondence $U \longmapsto U^s$ is an isomorphism of sites (with inverse β^{-1}). This completes the proof of the proposition.

4.1.3 <u>Proposition</u> Any continuous map $f:X \longrightarrow Y$, with Y balanced, factors uniquely through $\beta:X \longrightarrow b(X)$ i.e. $b(X)$ is the reflection of X in the category of balanced topological spaces.

<u>Proof</u> First of all observe that for any continuous $f:X \longrightarrow Y$ where X,Y are arbitrary topological spaces, and for any closed irreducible $Z \subseteq X$, $\Gamma_Y f(Z)$ is irreducible in Y. For $U \cap \Gamma_Y f(Z) \neq \emptyset$, $V \cap \Gamma_Y f(Z) \neq \emptyset \Longrightarrow U \cap f(Z) \neq \emptyset$, $V \cap f(Z) \neq \emptyset \Longrightarrow f^{-1}(U) \cap Z \neq \emptyset$, $f^{-1}(V) \cap Z \neq \emptyset \Longrightarrow f^{-1}(U) \cap f^{-1}(V) \cap Z \neq \emptyset \Longrightarrow f^{-1}(U \cap V) \cap Z \neq \emptyset \Longrightarrow$ $U \cap V \cap f(Z) \neq \emptyset \Longrightarrow U \cap V \cap \Gamma_Y f(Z) \neq \emptyset$.

Now let us return to the situation envisaged in the proposition.

$$\begin{tikzcd} X \arrow[r, "\beta"] \arrow[d, "f"'] & b(X) \arrow[dl, "\bar{f}"] \\ Y & \end{tikzcd}$$

Define $\bar{f}$ as follows: $\bar{f}(Z)$ is the unique generic point of $\Gamma_Y f(Z)$. Then $\bar{f}\beta(x)$ is the unique generic point of $\Gamma_Y f\, \Gamma_X\{x\} = \Gamma_Y\{f(x)\}$. And the unique generic point of the latter is of course $f(x)$. Hence $\bar{f}\beta = f$.

Next let us show $\bar{f}$ is continuous.

$$\begin{aligned} \bar{f}^{-1}(U) &= \{Z \in b(X) : \bar{f}(Z) \in U\} \\ &= \{Z \in b(X) : \text{ the unique generic point of } \Gamma_Y f(Z) \text{ is in } U\} \\ &= \{Z \in b(X) : f(Z) \cap U \neq \emptyset\} \\ &= \{Z \in b(X) : Z \cap f^{-1}(U) \neq \emptyset\} \\ &= (f^{-1}(U))^s \end{aligned}$$

Finally we show uniqueness of the extension $\bar{f}$. First of all, for any $F \in b(X)$, $\Gamma_{b(X)}\beta(F) = b(X) - U^s$ where $U = X-F$. To show this take $x \in F$. Then $\Gamma_X\{x\} \subseteq X-U$, i.e. $\Gamma_X\{x\} \in b(X) - U^s$. Then $\beta(F) \subseteq b(X) - U^s$ so $\Gamma_{b(X)}\beta(F) \subseteq b(X) - U^s$. Now take any $Z \in b(X) - U^s$. Then $Z \cap U = \emptyset$ so $Z \subseteq F$ and if V^s is a neighbourhood of Z, $Z \cap V \neq \emptyset$ so $F \cap V \neq \emptyset \Longrightarrow V^s \cap \beta(F) \neq \emptyset$ i.e. $Z \in \Gamma_{b(X)}\, \beta(F)$. To show uniqueness of $\bar{f}$, let $g: b(X) \longrightarrow Y$ be any continuous map with Y balanced. Then $\Gamma_Y g(b(X) - U^s) = \Gamma_Y g\, \Gamma_{b(X)}\beta(F) = \Gamma_Y g\beta(F)$. But also $\Gamma_Y g(b(X) - U^s) = \Gamma_Y g \ \Gamma_{b(X)}\{F\} = \Gamma_Y\{g(F)\}$. i.e. $\Gamma_Y g\beta(F) = \Gamma_Y\{g(F)\}$, so $g(F)$ is the unique generic point of $\Gamma_Y g\beta(F)$ and hence g is determined by its values on the image of β.

4.1.4 Remark Proposition 4.1.3 show that b is a functor left adjoint to the inclusion of balanced spaces into all topological spaces. It is straightforward to show that if $f:X \longrightarrow Y$ is a continuous map then $b(f):b(X) \longrightarrow b(Y)$ maps a closed irreducible $F \subseteq X$ to $\Gamma_Y f(F)$ which, as pointed out before, is closed and irreducible.

4.1.5 Proposition If $f:X \longrightarrow Y$ and $g:X \longrightarrow Y$ are continuous maps with Y balanced, $f_* = g_* \Longrightarrow f = g$.

Proof Suppose $f \neq g$ and say $f(x) \neq g(x)$. By the uniqueness of generic points in Y, $\Gamma_Y\{f(x)\} \neq \Gamma_Y\{g(x)\}$. Assume without loss of generality that $\Gamma_Y\{g(x)\} \nsubseteq \Gamma_Y\{f(x)\}$. Put $U = X - \Gamma_X\{x\}$, $V = Y - \Gamma_Y\{f(x)\}$ and consider the representable sheaf $U \in |\tilde{X}|$. Claim $f_*(U) \neq g_*(U)$ and that in particular $f_*(U)(V) \neq g_*(U)(V)$. Now

$$f_*(U)(V) = U(f^{-1}(V)) = \begin{cases} 1 & f^{-1}(V) \subseteq U \\ 0 & \text{otherwise} \end{cases}$$

Similarly $$g_*(U)(V) = \begin{cases} g^{-1}(V) \subseteq U \\ 0 \ \text{otherwise} \end{cases}$$

Since $f\,\Gamma_X\{x\} \subseteq \Gamma_Y\{f(x)\}$, $V = Y - \Gamma_Y\{f(x)\} \subseteq Y - f\,\Gamma_X\{x\}$. Hence $f^{-1}(V) \subseteq f^{-1}(Y - f\,\Gamma_X\{x\}) = X - f^{-1}f\,\Gamma_X\{x\} \subseteq X - \Gamma_X\{x\} = U$. Thus $f^*(U)(V) = 1$. By assumption $\Gamma_Y\{g(x)\} \nsubseteq \Gamma_Y\{f(x)\}$, hence $g\,\Gamma_X\{x\} \nsubseteq \Gamma_Y\{f(x)\}$ (since $\Gamma_Y\{g(x)\} = \Gamma_Y g\,\Gamma_X\{x\}$). Equivalently, $\Gamma_X\{x\} \nsubseteq g^{-1}\,\Gamma_Y\{f(x)\}$, i.e. $g^{-1}(Y - \Gamma_Y\{f(x)\}) \nsubseteq X - \Gamma_X\{x\}$.

i.e. $g^{-1}(V) \nsubseteq U$, so $g_*(U)(V) = 0$. This establishes that $f \neq g \Longrightarrow f_* \neq g_*$, as required.

4.1.6 Theorem Suppose $f:X \longrightarrow Y$ and $g:X \longrightarrow Y$ are continuous mappings of topological spaces. Then $f_* = g_*$ iff $b(f) = b(g)$.

Proof We have the following diagrams

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "\beta"'] & Y \arrow[d, "\beta"] \\
b(X) \arrow[r, shift left, "b(f)"] \arrow[r, shift right, "b(g)"'] & b(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{X} \arrow[r, shift left, "f_*"] \arrow[r, shift right, "g_*"'] \arrow[d, "\beta_*"', "\wr"] & \widetilde{Y} \arrow[d, "\beta_*", "\wr"'] \\
\widetilde{b(X)} \arrow[r, shift left, "b(f)_*"] \arrow[r, shift right, "b(g)_*"'] & \widetilde{b(Y)}
\end{tikzcd}$$

By naturality of β, and by the fact that β_* is an isomorphism of categories, $f_* = g_* \Longleftrightarrow b(f)_* = b(g)_*$. But since $b(Y)$ is balanced, $b(f)_* = b(g)_* \Longleftrightarrow b(f) = b(g)$ (4.1.5).

4.2 On the Characterization of Topoi $\widetilde{T}$

4.2.1 Theorem A topos $\widetilde{\mathfrak{C}}$ is equivalent to $\widetilde{T}$ for a topological space T iff (1) the subobjects of 1 generate

(2) there are enough points.

Proof (1) and (2) are necessary by 5.1 of chapter I. To show sufficiency we utilize the topology induced on the points by the open sieves (cf 2.4). Let T be a set of points whose associated stalks collectively reflect isomorphisms. The open sets of T are of the form $U^s = \{p \in T : U_p = 1\}$ for $U \subseteq 1$ in $\tilde{\mathfrak{C}}$. Let us define a new site $\mathfrak{S}$ whose underlying category is (a skeleton of) the poset of subobjects of 1 in $\tilde{\mathfrak{C}}$, and whose covering families are exactly the families $U_i \subseteq U$ which are epimorphic in $\tilde{\mathfrak{C}}$.

Consider the functor $\mathfrak{S} \longrightarrow \mathrm{Open}(T)$ which takes U to U^s. It is clearly onto on objects (and maps for that matter). It is automatically faithful since $\mathfrak{S}$ and Open(T) are posets. Finally it is an isomorphism of categories since it is 1-1 on objects. To show this suppose $U^s = V^s$. Then $U_p = V_p$ all $p \in T$. Consider the inclusion $i : U \cap V \longrightarrow U$. $(U \cap V)_p = U_p \cap V_p = U_p$ all $p \in T$, i.e. i_p is iso all $p \in T \Longrightarrow i$ is iso, i.e. $U \cap V = U$. Symmetrically $U \cap V = V$ and hence $U = V$.

Next we show that $U \longmapsto U^s$ is not only an isomorphism of categories, it is in fact an isomorphism of sites, showing $\tilde{\mathfrak{S}} \cong \tilde{T}$. To establish this we must show $\{U_i \subseteq U : i \in I\}$ is epimorphic in $\tilde{\mathfrak{C}}$ iff $U^s = \bigcup_i U_i^s$. Now stalks preserve and reflect epimorphic families, so we have

$$\begin{aligned} & \{U_i \subseteq U : i \in I\} \quad \text{is epimorphic in } \tilde{\mathfrak{C}} \\ \Longleftrightarrow\ & \{(U_i)_p \subseteq U_p : i \in I\} \quad \text{is epimorphic in Sets, all } p \in T \\ \Longleftrightarrow\ & \forall\, p \in T,\ U_p = 1 \Longrightarrow \exists\, i \in I \text{ with } (U_i)_p = 1 \\ \Longleftrightarrow\ & U^s \subseteq \bigcup_i U_i^s \\ \Longleftrightarrow\ & U^s = \bigcup_i U_i^s \end{aligned}$$

Having now the isomorphism $\tilde{\mathfrak{S}} \cong \tilde{T}$ we are finished if we know that $\tilde{\mathfrak{S}}$ is equivalent to $\tilde{\mathfrak{C}}$. For this we will merely quote a result due to Giraud. For details of the proof see [SGA], exposé IV, 4.1.2.1.

4.2.2 Theorem Let $\mathfrak{C}$ be a site and $\mathfrak{S} \subseteq \tilde{\mathfrak{C}}$ a full subcategory. Define a functor $\tilde{\mathfrak{C}} \longrightarrow \tilde{\mathfrak{S}}$ by $F \longmapsto (S \longmapsto \tilde{\mathfrak{C}}(S,F))$. This functor is an equivalence of categories iff $\mathfrak{S}$ is a generating family for $\tilde{\mathfrak{C}}$. $\mathfrak{S}$ is assumed to carry the topology whose covering families are exactly the epimorphic ones.

4.3 On the Cotripleability of f^*

4.3.1 As pointed out in chapter I, 3.3, Van Osdol has shown in [SRC] that the functor $S:\tilde{T} \longrightarrow \mathrm{Sets}^{|T|}$, which sends a sheaf to its family of stalks, is cotripleable. Denoting the discrete space on $|T|$ by $|T|$ as well, we have that $|\tilde{T}| = \mathrm{Sets}^{|T|}$. The functor S, considered as a functor from $\tilde{T}$ to $|T|^{\sim}$ is just f^* where $f:|T| \longrightarrow T$ is the identity map, which is continuous. With this approach a more general problem is suggested by Van Osdol's result, namely, for which continuous maps $f:X \longrightarrow Y$ of topological spaces is f^* cotripleable? The answer is given in theorem 4.3.4 below. First a definition and some preliminaries.

4.3.2 Definition A subspace A of a topological space Y is said to be super-dense iff $\forall$ open $U,V \subseteq Y$, $U \cap A = V \cap A \Longrightarrow U = V$. The following are some immediate consequences of the definition.

(1) For $A \subseteq A' \subseteq Y$, A superdense $\Longrightarrow A'$ superdense.

(2) Superdense $\Longrightarrow$ dense. For if $A \subseteq Y$ is not dense, $\exists\ U \subseteq Y-A,\ U \neq \emptyset$. Then $U \cap A = \emptyset \cap A$ but $U \neq \emptyset \Longrightarrow A$ not superdense.

(3) A superdense $\Longrightarrow Y-A$ contains no non-empty closed set.

For if $Z \subseteq Y-A$ is non-empty, closed then $Y \cap A = (Y-Z) \cap A$ but $Y-Z \neq Y$. In particular a T_1 space has no proper superdense sets.

(4) If $A \subseteq Y$ is superdense then $\forall\, U,V \subseteq Y$ open, $U \cap A \subseteq V \cap A \Longrightarrow U \subseteq V$. For $U \cap A \subseteq V \cap A \Longrightarrow U \cap A = U \cap V \cap A$ $\Longrightarrow U = U \cap V \Longrightarrow U \subseteq V$.

(5) Every point of an indiscrete space is superdense. Any topological space is superdense in its Alexandrov compactification, in fact in any extension by a single point.

(6) (Banaschewski) $A \subseteq Y$ superdense, A,Y T_0 $\Longrightarrow A \subseteq Y$ is an essential extension in the category of T_0 spaces (cf [EETS]).

4.3.3 <u>Lemma</u> Suppose A is a subspace of a topological space Y, with inclusion mapping $i:A \longrightarrow Y$. Then the following are equivalent:

(1) The set of stalks on $\tilde{Y}$ corresponding to those $y \in A$ is sufficient to reflect isomorphisms. i.e. A already provides "enough points" for $\tilde{Y}$.

(2) i^* reflects isomorphisms.

(3) A is superdense in Y.

<u>Proof</u> (1) and (2) are equivalent simply because $\forall\, F \in |\tilde{Y}|$, and any $y \in A$, $i^*(F)_y = F_{i(y)} = F_y$.

(2) $\Longrightarrow$ (3) One can easily show that in general for a continuous $f:X \longrightarrow Y$ and sheaf $F \in |\tilde{Y}|$, $f^*(F)$ may be computed by taking the sheaf associated to the presheaf $f^0(F)$ defined by $f^0(F)(U) = \underrightarrow{\lim}_{f(U) \subseteq V} F(V)$.

In the case at hand this formula shows that $i^*(U) = U \cap A$, that is, i^* takes generators to generators in the obvious way.

Now if A is not superdense in Y we have open sets $U, V \subseteq Y$ with $U \cap A = V \cap A$ but $U \neq V$. Also $U \cap V \cap A = U \cap A = V \cap A$. Since $U \neq V$ we cannot have both $U \cap V = U$ and $U \cap V = V$. Say $U \cap V \neq U$, i.e. $U \cap V \subsetneqq U$. This is a mapping in $\tilde{Y}$ which is not an isomorphism, but whose image under i^* is an isomorphism. Thus A not superdense $\Longrightarrow i^*$ does not reflect isomorphisms, showing $(2) \Longrightarrow (3)$.

$(3) \Longrightarrow (2)$ Hypothesis (3) implies that i^* restricted to the generators $U \in |\tilde{Y}|$, $U \subseteq Y$ open, is an isomorphism onto its image. First of all note that this means i^* is faithful on hom sets of the form $\tilde{Y}(U,F)$. For suppose σ, τ are two maps $U \longrightarrow F$ and $i^*(\sigma) = i^*(\tau)$. The equalizer of σ, τ is a subobject of U, i.e. an open set $V \subseteq U$. Since i^* preserves equalizers we have $i^*(V) \longrightarrow i^*(U) \rightrightarrows i^*(F)$ is an equalizer. Since the two maps $i^*(\sigma)$, $i^*(\tau)$ are supposed to be equal, $i^*(V) \longrightarrow i^*(U)$ is an isomorphism and hence $V = U$, implying $\sigma = \tau$. Next, take a map $f : F \longrightarrow G$ in $\tilde{Y}$ and suppose $i^*(f)$ is an isomorphism. First of all f must be mono, for if not $\exists\ \sigma, \tau : U \longrightarrow F$ with $\sigma \neq \tau$ but $f\sigma = f\tau$. Then $i^*(f)\, i^*(\sigma) = i^*(f)\, i^*(\tau)$ which implies $i^*(\sigma) = i^*(\tau)$, which in turn by the above means $\sigma = \tau$ $\otimes$ To show f is epi it is sufficient to show every $\sigma : U \longrightarrow G$ factors through f. Now $i^*(\sigma)$ factors through $i^*(f)$.

$$\begin{tikzcd}
i^*(F) \arrow[r, "i^*(f)"] & i^*(G) \\
i^*(U) \arrow[u, dashed, "\xi"] \arrow[ur, "i^*(\sigma)"'] &
\end{tikzcd}$$

Since $i^*(U) = U \cap A$, ξ represents a section of $i^*(F)$ over $U \cap A$. Such a section must come from a section over $U \cap A$ of the presheaf $i^0(F)$. $i^0(F)(U \cap A) = \underset{i(U \cap A) \subseteq V}{\underrightarrow{\lim}} F(V)$.

An element of this last set may be represented by $\overline{\xi} \in F(V)$ for some V with $U \cap A \subseteq V$. But then $U \cap A \subseteq V \cap A$ so $U \subseteq V$ and $\overline{\xi}$ may be restricted to U to obtain $\xi' \in F(U)$, i.e. $\xi' : U \longrightarrow F$. It is then routine to check that $i^*(\xi') = \xi$ and thus $i^*(\sigma) = i^*(f)\, i^*(\xi') = i^*(f\xi') \Longrightarrow \sigma = f\xi'$. Thus we have shown an arbitrary $\sigma : U \longrightarrow G$ factors through f. Thus f is epi, and since every topos is balanced, f is iso.

4.3.4 Theorem For $f : X \longrightarrow Y$ a continuous mapping of topological spaces, the following are equivalent:

(1) f^* is cotripleable.

(2) f^* reflects isomorphisms.

(3) The set-theoretical image of f in Y is superdense.

Proof (1) $\Longrightarrow$ (2) Beck's "Crude Tripleability Theorem" [TAC] applies here. Dualized it reads that the conjunction of the following conditions is sufficient for a left adjoint to be cotripleable: the domain has equalizers, the functor preserves them, the functor reflects isomorphisms. In our case $\tilde{Y}$ always has equalizers and the functors f^*, being left exact, always preserve them. Since any cotripleable functor must necessarily reflect isomorphisms, the equivalence of (1) and (2) is clear.

(2) $\Leftrightarrow$ (3) Denote by A the image f(X) in Y. Denoting the inclusion $A \longrightarrow Y$ by i and the natural map $X \longrightarrow A$ by $\bar{f}$ we have

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\bar{f}"'] & & Y \\
& A \arrow[ur, "i"'] &
\end{tikzcd}$$ commutes

hence

$$\begin{tikzcd}
\tilde{X} \arrow[rr, "f_*"] \arrow[dr, "\bar{f}_*"'] & & \tilde{Y} \\
& \tilde{A} \arrow[ur, "i_*"'] &
\end{tikzcd}$$ commutes

hence

$$\begin{tikzcd}
\tilde{X} & & \tilde{Y} \arrow[ll, "f^*"'] \arrow[dl, "i^*"] \\
& \tilde{A} \arrow[ul, "\bar{f}^*"] &
\end{tikzcd}$$ commutes

Now $\bar{f}^*$ reflects isomorphisms because $\bar{f}$ is onto. For suppose $\bar{f}^*(\alpha)$ is an isomorphism, some α a morphism in $\tilde{A}$. Then $f^*(\alpha)_x = \alpha_{\bar{f}(x)}$ is iso, all $x \in X$. But as x runs through X, $\bar{f}(x)$ runs through A, so α_y is iso all $y \in A$ which implies α is iso in $\tilde{A}$. Since $\bar{f}^*$ reflects isomorphisms and since any functor preserves them, clearly f^* reflects isomorphisms iff i^* reflects them. By lemma 4.3.3 we are finished.

Chapter III: Sheaves of Algebras

1 Introduction

1.1 For the rest of this work we shall concern ourselves with the interplay between topoi and classical universal algebra. There are two points of view available. One is to study algebras of a given species modelled in a topos, for example "group objects" in the category of sheaves on the unit interval qua topological space. The second is to study sheaves whose values are algebras of a given species, that is, algebras coherently parametrized by a site. In flavour these directions are obviously algebraic in the first case and geometric in the second. One of the main reasons for the richness of the subject is that they are in fact the same. We will make extensive use of both points of view, choosing whichever is more convenient for the task at hand. In general we adopt the "algebras in topoi" point of view for intuition but we are often forced to the "sheaf of algebras" approach in arguments.

Let us briefly recall why we have the two alternatives. The ultimate reason is that phenomenon which one author has dubbed "the grandaddy of all adjointness relations", namely that which expresses the cartesian closed structure of the category of all categories.

$$(\mathbb{A}, \mathbb{B}^{\mathbb{C}}) \cong (\mathbb{A} \times \mathbb{C}, \mathbb{B})$$

By symmetry of the cartesian product it follows that iteration of functor categories may be carried out in any order i.e.

$$(\mathbb{A}^{\mathbb{B}})^{\mathbb{C}} \cong (\mathbb{A}^{\mathbb{C}})^{\mathbb{B}}$$

Suppose we select a set of cones in $\mathbb{A}$ and $\mathbb{B}$ and denote by square brackets around the exponent the full subcategory of the full functor category determined by those functors transforming the specified cones into limits. Then the isomorphism still holds i.e.

$$(\mathbb{A}^{[\mathbb{B}]})^{[\mathbb{C}]} \cong (\mathbb{A}^{[\mathbb{C}]})^{[\mathbb{B}]}$$

Essentially this is because limits in functor categories are computed pointwise. In our case we have

$$(\mathrm{Sets}^{[\mathbb{C}^*]})^{[\mathbb{H}]} = (\mathrm{Sets}^{[\mathbb{H}]})^{[\mathbb{C}^*]}$$

where $\mathbb{C}$ is a site, $\mathbb{H}$ an algebraic theory, and the relevant diagrams to be turned into limits are the obvious ones. Symbolically $M(\mathbb{H},\tilde{\mathbb{C}}) \cong Sh(\mathbb{C},M(\mathbb{H}, \mathrm{Sets}))$. In words (and defining once and for all "M", "Sh") we have that the models of $\mathbb{H}$ in $\tilde{\mathbb{C}}$ are the same as the sheaves on $\mathbb{C}$ with values in the equational class (variety) determined by $\mathbb{H}$, i.e. the models of $\mathbb{H}$ in Sets.

Observe that an object of $Sh(\mathbb{C}, \mathbb{E})$ must satisfy the patching condition in $\mathbb{E}$ (chapter I, 2.4). In the case of $\mathbb{E}$ an equational class this is obscured by the fact that a patching condition holds in an equational class iff it holds at the level of sets, technically because the underlying set functor in such instances preserves and reflects limits. If for example one finds it necessary to discuss sheaves with values in Top, the category of topological spaces and continuous maps, as we shall do, one must be very careful with regard to patching conditions.

1.2 From this point on $\mathbb{H}$ will denote a fixed but otherwise arbitrary finitary theory. The examples we have in mind are abelian groups, modules over an arbitrary ring, rings, lattices and so on. We will try to adhere to the following conventions:

(1) Upper case roman letters for sheaves of sets

(2) Script capitals for sheaves of algebras $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ etc.

(3) The underlying sheaf of sets of an algebra will be denoted by the corresponding roman letter, i.e. A for $\mathcal{A}$, B for $\mathcal{B}$ etc.

1.3 <u>Sectionwise Behaviour</u> Let $\mathcal{A}$ be a sheaf of algebras with carrier A, and F an arbitrary sheaf of sets. We are interested here in $\widetilde{\mathfrak{C}}(F,A)$, the "points of A defined over F", or as we prefer to say, the sections of A over F. Now $\widetilde{\mathfrak{C}}(F,-)$ is a product-preserving functor, hence carries algebra objects in $\widetilde{\mathfrak{C}}$ to algebras in Sets, and homomorphisms of algebra objects to homomorphisms of set-valued algebras. Furthermore for a fixed algebra $\mathcal{A}$, the mappings on the hom sets induced by varying F along homomorphisms are also homomorphisms of set-valued algebras. The converse of these observations is true (cf. [CF], chapter 4, for example). Namely, A has an algebra structure on it iff the hom functor $\tilde{\mathfrak{C}}(-,A):\tilde{\mathfrak{C}}^* \longrightarrow \text{Sets}$ factors through the forgetful functor $M(\mathbf{H},\text{Sets}) \longrightarrow \text{Sets}$. In fact, bringing in the homomorphisms, $M(\mathbf{H},\tilde{\mathfrak{C}})$ is equivalent to the following pullback in Cat

$$\begin{tikzcd}
\mathbf{P} \arrow[r] \arrow[d] & M(\mathbf{H},\text{Sets})^{\tilde{\mathfrak{C}}^*} \arrow[d] \\
\tilde{\mathfrak{C}} \arrow[r] & \text{Sets}^{\tilde{\mathfrak{C}}^*}
\end{tikzcd}$$

The right-hand map is induced by the forgetful functor on the base equational class. The bottom map is the Yoneda embedding.

These remarks imply in particular that for an algebra $\mathcal{A}$ and $C \in |\mathfrak{C}|$, the two sets $\tilde{\mathfrak{C}}(C,A)$ and $A(C)$ are equipped with algebra structures. In fact the Yoneda isomorphism $\tilde{\mathfrak{C}}(C,A) \cong A(C)$ is an isomorphism of algebras. Note that a sheaf of sets A carries an algebra structure as soon as each $\tilde{\mathfrak{C}}(C,A)$ does, for all $C \in |\mathfrak{C}|$. This is an instance of the more general fact that the factorization discussed above need only be known for a generating subcategory, then it automatically extends to the whole category.

These observations are useful in manipulating "internal powers" or "exponentials" of algebras. For $X \in \tilde{\mathfrak{C}}$, the functor $F \longmapsto F^X$ preserves products (it is a right adjoint), hence takes algebra objects to algebra objects. If $\mathcal{A}$ is an algebra, so is $\mathcal{A}^X$ and at the level of sections over $F \in \tilde{\mathfrak{C}}$, $\tilde{\mathfrak{C}}(F,\mathcal{A}^X) \cong \tilde{\mathfrak{C}}(F \times X,\mathcal{A})$ is an isomorphism of algebras. As an example of the sectionwise technique of studying algebras in $\tilde{\mathfrak{C}}$, let $f:X \longrightarrow Y$ be a morphism of sheaves of sets. I claim the induced map $\mathfrak{a}^Y \longrightarrow \mathfrak{a}^X$ is a homomorphism of algebras. By the above it is sufficient to show $\tilde{\mathfrak{C}}(F,A^Y) \longrightarrow \tilde{\mathfrak{C}}(F,A^X)$ is a homomorphism of set-valued algebras. Consider the following:

$$\begin{tikzcd}
\tilde{\mathfrak{C}}(F \times Y, A) \arrow[r, "\sim"] \arrow[d] & \tilde{\mathfrak{C}}(F, A^Y) \arrow[d] \\
\tilde{\mathfrak{C}}(F \times X, A) \arrow[r, "\sim"] & \tilde{\mathfrak{C}}(F, A^X)
\end{tikzcd}$$

The left-hand arrow is induced by $F \times f: F \times X \longrightarrow F \times Y$ and is a homomorphism of set-valued algebras, by the general considerations above. The two horizontal arrows are the exponential adjointness isomorphisms and are actually isomorphisms of algebras, as pointed out previously. The diagram commutes by naturality of the adjointness transformations, hence the right-hand arrow is a homomorphism of algebras, and we are finished.

2 Free Algebras

2.1 We can do no universal algebra without free algebras. Let us convince ourselves that they are available. There are several theorems around which give the existence of left adjoints in our case, but we will construct them directly since it is a simple matter and the description thus obtained is vital for what follows.

Constructing free algebras means, of course, exhibiting a left adjoint for $U : M(\mathbb{H},\tilde{\mathbb{C}}) \longrightarrow \tilde{\mathbb{C}}$. What we do, simply, is to assign to a sheaf G of sets the presheaf built by forming the free $\mathbb{H}$-algebra on G "sectionwise", then taking the associated sheaf of the latter.

$$\tilde{\mathbb{C}} = \mathrm{Sh}(\mathbb{C},\mathrm{Sets}) \longrightarrow \mathrm{Psh}(\mathbb{C},\mathrm{Sets}) \longrightarrow \mathrm{Psh}(\mathbb{C},M(\mathbb{H},\mathrm{Sets})) \longrightarrow \mathrm{Sh}(\mathbb{C},M(\mathbb{H},\mathrm{Sets}))$$

Denote this composite by W. The second arrow above is induced by composition with the free algebra fuctor $|W|: \mathrm{Sets} \longrightarrow M(\mathbb{H},\mathrm{Sets})$ familiar from universal algebra. The third arrow is the associated sheaf at the level of algebras. Why is it there? Simply because the basic associated sheaf functor $\hat{\mathbb{C}} \longrightarrow \tilde{\mathbb{C}}$ preserves finite left limits and therefore survives the passage to finitary algebra objects, i.e.

$$\left[M(\mathbb{H},\hat{\mathbb{C}}) \longrightarrow M(\mathbb{H},\tilde{\mathbb{C}})\right] \cong \left[\mathrm{Psh}(\mathbb{C},M(\mathbb{H},\mathrm{Sets}) \longrightarrow \mathrm{Sh}(\mathbb{C},M(\mathbb{H},\mathrm{Sets}))\right]$$

At this point we should attempt to describe explicitly what the sections of such a free sheaf of algebras look like. Let us denote the "forgetful" and "free" functors for the basic equational class $M(\mathbb{H},\mathrm{Sets})$ by $|U|$ and $|W|$ respectively. Then for $G \in |\tilde{\mathbb{C}}|$ and $C \in |\mathbb{C}|$ we have

$$W(G)(C) = \varinjlim_{R \in \mathrm{Cov}(C)} \hat{\mathbb{C}}(R,\hat{W}(G))$$

$\hat{W}(G)$ is the presheaf defined by $\hat{W}(G)(C) = |W|(G(C))$, i.e. the free algebra on the sections over C of G. Thus an element of $W(G)(C)$ is represented, up to the refinement equivalence relation, by a covering $\left\{C_i \longrightarrow C : i \in I\right\}$ of C together with elements

$\sigma_i \in |W|(G(C_i))$ which are compatible with respect to the covering. To sum up: the sections of $W(G)$ over C are equivalence classes, modulo refinement, of compatible families of polynomials parametrized by coverings of C. To repeat: a word of $W(G)$ over C is ultimately a family of "real" polynomials associated with the theory $\mathbb{H}$, and the variables from which the polynomials are built are determined by coverings of C with respect to the site structure.

Functors considered to this point:

$$\begin{tikzcd}
M(\mathbb{H},\tilde{\mathbb{C}}) \arrow[rr, "i"'] \arrow[dr, "\Gamma_C"] \arrow[ddd, "U"] & & M(\mathbb{H},\hat{\mathbb{C}}) \arrow[ll, dashed, bend right=10, "a"'] \arrow[dl, "\hat{\Gamma}_C"'] \arrow[ddd, "\hat{U}"] \\
& M(\mathbb{H},\mathrm{Sets}) \arrow[ul, dashed, bend left=10] \arrow[ur, dashed, bend right=10] \arrow[d, "|U|"] & \\
& \mathrm{Sets} \arrow[u, dashed, bend left=20, "|W|"] \arrow[dl, dashed, bend right=10] \arrow[dr, dashed, bend left=10] & \\
\tilde{\mathbb{C}} \arrow[uuu, dashed, bend left=10, "W"] \arrow[ur, "\Gamma_C"'] \arrow[rr, "i"'] & & \hat{\mathbb{C}} \arrow[uuu, dashed, bend right=10, "\hat{W}"'] \arrow[ul, "\hat{\Gamma}_C"] \arrow[ll, dashed, bend right=10, "a"']
\end{tikzcd}$$

Γ_C = sections over $C \in |\mathbb{C}|$ (sheaves)

= $\hat{\mathbb{C}}(C,-)$

$\hat{\Gamma}_C$ = sections over $C \in |\mathbb{C}|$ (presheaves)

i = inclusion of sheaves in presheaves

a = associated sheaf

Notes:

(1) All solid arrows are right adjoints in the pairs in which they occur.

(2) All square diagrams of right adjoints (there are 3) commute. Hence the corresponding diagrams of left adjoints commute.

(3) $\hat{\Gamma}_C \hat{W} = |W| \hat{\Gamma}_C$

(4) $W = a\hat{W}i$

(5) Since $a\hat{W} = Wa$ (from (2)), if we wish to build a free algebra in $\tilde{\mathbb{C}}$ on a <u>presheaf</u> it makes no difference whether or not we "sheafify" it first.

(6) The vertical adjoint pairs are all tripleable. Of special interest is the fact that $\tilde{\mathbb{C}}$ is regular and the triple induced on it by the pair $W \dashv U$ is regular (cf [TM]) hence one knows from general principles that U creates regular coimage factorizations and in particular preserves and reflects the property of being a coequalizer.

2.2 <u>Geometric Morphisms</u> Although this chapter is discussing free algebras, this is a convenient place to discuss geometric morphisms of topoi and algebras. The fundamental fact is that geometric morphisms survive the passage to finitary algebra objects. Thus if we have a pair of functors $f: \tilde{\mathbb{C}} \longrightarrow \tilde{\mathbb{D}}$, $f^*: \tilde{\mathbb{D}} \longrightarrow \tilde{\mathbb{C}}$ with $f^* \dashv f$, f^* left exact, we obtain a similar situation for the algebras, and a commutative diagram

$$\begin{tikzcd}
M(\mathbb{H}, \tilde{\mathbb{C}}) \arrow[r, shift right, "f^{\mathbb{H}}"'] \arrow[d, "U"'] & M(\mathbb{H}, \tilde{\mathbb{D}}) \arrow[l, shift right, "(f^*)^{\mathbb{H}}"'] \arrow[d, "U"] \\
\tilde{\mathbb{C}} \arrow[r, shift right, "f"'] & \tilde{\mathbb{D}} \arrow[l, shift right, "f^*"']
\end{tikzcd}$$

Usually we write $f^{\mathbb{H}} = f$, $(f^*)^{\mathbb{H}} = f^*$ where no confusion can arise.

The previously mentioned phenomenon of the associated sheaf construction being available for sheaves of algebras is merely a special case of this more general fact about geometric morphisms. An important consequence of the lifting of geometric morphisms to sheaves of algebras is that any points which $\mathbb{C}$ may have will lift to "points" of the sheaves of algebras, over the base equational class. That is to say, the stalks of a sheaf of algebras will be algebras (in Sets). Because the underlying set functor for equational classes preserves and reflects limits and isomorphisms, if $\mathbb{C}$ has enough points, $M(\mathbb{H}, \tilde{\mathbb{C}})$ has enough algebra-valued points and all of the exactness properties mentioned in chapter II, 2.3 will hold for sheaves of algebras.

2.3 Properties of the Free Algebra Functor Besides the fundamental colimit preservation property which belongs to any left adjoint, the functor W constructed in 2.1 has additional features reminiscent of the word algebras encountered in classical universal algebra.

2.3.1 Proposition The sheaves $W(C)$, $C \in |\mathbb{C}|$ form a generating family in $M(\mathbb{H}, \tilde{\mathbb{C}})$. If the topology on $\mathbb{C}$ is not coarser than the canonical one, i.e. if the $C \in |\mathbb{C}|$ are not necessarily sheaves, then by $W(C)$ we mean $W(a(C))$, or what is the same thing, $a\hat{W}(C)$.

Proof Any left adjoint carries generating families to generating families.

The reason we single out this elementary fact as a proposition is that the $W(C)$ are the most important example of finitely generated sheaves of algebras (soon to be defined). Observe that sheaves of algebras differ from classical equational classes in that in general no single algebra will generate the category.

2.3.2 <u>Proposition</u> W is faithful and preserves monomorphisms.

<u>Proof</u> That W preserves monomorphisms is clear since $W = a\hat{W}i$ and $a, \hat{W}, i$ preserve monos. $\hat{W}$ preserves monos since $|W|$ does, and $\hat{W}$ is sectionwise just W.

To establish that W is faithful requires a bit more work. It will suffice to show that the front adjunction $\eta : I \longrightarrow UW$ is pointwise a monomorphism. Now $\eta_F : F \longrightarrow UW(F)$ is mono iff it is 1-1 at each $C \in |\mathbb{C}|$. We must show the following map is mono

$$F(C) \longrightarrow UW(F)(C) = W(F)(C)$$

This map takes $\sigma \in F(C)$ to the element of $W(F)(C)$ defined by the identity cover of C, which consists of the single map $1 : C \longrightarrow C$, together with the "compatible" family consisting of σ itself, considered now as a generator of $\hat{W}(F)(C) = |W|(F(C))$. Suppose $\sigma, \tau \in F(C)$ get mapped to the same thing. That means the families $\{\sigma\}, \{\tau\}$ agree on a common refinement of their defining covers, which happen to be the same. To be more precise, this means that the section $\sigma, \tau \in \hat{W}F(C)$ agree on a cover of C, say $\{C_i \longrightarrow C\}$. Symbolically $\sigma|C_i = \tau|C_i$ with respect to restriction in the presheaf $\hat{W}F$.

But $\sigma|C_i$ and $\tau|C_i$ are generators of the free $\mathbb{H}$-algebra $(\hat{W}F)(C_i)$, so $\sigma|C_i = \tau|C_i$ in the sheaf F. Here we have used the fact that $\hat{W}$ is faithful, i.e. that $F \longrightarrow \hat{U}\hat{W}F$ is mono. This follows from a pointwise application of the analogous result for universal algebras. Finally $\sigma|C_i = \tau|C_i$ in $F \Rightarrow \sigma = \tau$, since F is a sheaf.

3 Finiteness Conditions

3.1 Previously we have emphasized the connection between the finitary nature of $\mathbb{H}$ and the exactness properties of geometric morphisms, in particular the associated sheaf functor. We shall now attempt to exploit this interplay further by introducing and studying the concept of a finitely generated object of $M(\mathbb{H},\tilde{\mathfrak{C}})$. Of course it is clear that once we have defined the notion of a finite sheaf of sets, we will inevitably be forced to declare as finitely generated precisely those algebras which are quotients of algebras free on a finite sheaf of sets. In this connection recall that a morphism of algebras is a coequalizer (quotient map) in $M(\mathbb{H},\tilde{\mathfrak{C}})$ iff it is one in $\tilde{\mathfrak{C}}$ (2.1 Note (6)). To define the notion of a finite sheaf we find it convenient to begin with a brief study within the framework of elementary topoi of a concept introduced by Mitchell for abelian categories.

3.2 Definition Let $\mathbb{E}$ be a category and $\mathfrak{G}$ a collection of objects of $\mathbb{E}$. Then $X \in \mathbb{E}$ is said to be finitely generated with respect to $\mathfrak{G}$ if $\exists$ $S_1,\ldots,S_n \in \mathfrak{G}$ and an epimorphism $\coprod_{i=1}^{n} S_i \twoheadrightarrow X$. The collection of objects finitely generated with respect to $\mathfrak{G}$ is denoted $\bar{\mathfrak{G}}$.

3.2.1 Proposition Using the notation of definition 3.2, and if $\mathbb{E}$ is an elementary topos, the following facts obtain:

(1) $\overline{\mathfrak{G}}$ is closed under finite coproducts and epimorphic images.

(2) If $\mathfrak{G}$ is closed under finite non-empty products, so is $\overline{\mathfrak{G}}$. Hence if $1 \in \mathfrak{G}$ then $\overline{\mathfrak{G}}$ is closed under all finite products.

(3) If $\mathfrak{G}$ is closed under subobjects, so is $\overline{\mathfrak{G}}$.

Proof (1) is immediate. (2) and (3) follow from the exactness properties of topoi, specifically that pulling back preserves colimits, and the formal consequences that (a) the product of epimorphisms is an epimorphism and (b) products distribute over sums as in arithmetic. Suppose then that $\coprod_{i=1}^{n_j} S_{i_j} \twoheadrightarrow X_j$ $j = 1,\ldots,m$ are epimorphisms representing the X_j as finitely generated. We obtain an epimorphism $\prod_{j=1}^{m} \coprod_{i=1}^{m_j} S_{i_j} \twoheadrightarrow \prod_{j=1}^{m} X_j$. But $\prod_{j=1}^{m} \coprod_{i=1}^{n_j} S_{i_j} = \coprod \prod_{j=1}^{m} S_{i_j;j}$ where the coproduct is taken over all sequences $(i_1,\ldots,i_m) \in n_1 \times \ldots \times n_m$. The products involved here are by hypothesis in $\mathfrak{G}$, so $\prod_{j=1}^{m} X_j$ is in $\mathfrak{G}$. To establish (3) suppose $\coprod_{i=1}^{n} S_i \twoheadrightarrow X$ is an epimorphism and $A \rightarrowtail X$ a monomorphism. Pulling back we obtain

$$\begin{tikzcd}
(\coprod S_i) \times_X A \arrow[r, twoheadrightarrow] \arrow[d, tail] & A \arrow[d, tail] \\
\coprod S_i \arrow[r, twoheadrightarrow] & X
\end{tikzcd}$$

Again from the exactness properties of topoi the top arrow is an epimorphism and $(\coprod S_i) \times_X A \cong \coprod (S_i \times_X A)$. These summands are subobjects of S_i and hence are in $\mathfrak{G}$.

3.2.3 Definition A sheaf $F \in |\tilde{\mathbb{C}}|$ is finite iff it is finitely generated with respect to the set $\mathfrak{G} = \{a(C) \mid C \in |\mathbb{C}|\}$, i.e. the sheaves associated to the representable ones.

This definition is chosen for several reasons. The defining set $\mathfrak{G}$ of "basic" finite objects constitutes a generating family, which is useful. Although the members of $\mathfrak{G}$ are not in general projective, they are as close to being projective as we can expect of a non-trivial (i.e. non-0) sheaf. Considered as presheaves, they are of course projective. If $\mathbb{C}$ has finite left limits the conditions of 3.2.1 are satisfied, yielding a good calculus of finite objects in $\tilde{\mathbb{C}}$. Finally, and most important, the definition seems to be the right notion to describe the phenomena of chapter V of this work.

3.3 Small Objects of $\tilde{\mathbb{C}}$ There is a concept of "small" object in a category which was introduced by Mitchell [TC] and shown to have an intimate relationship with "finitely generated" in the case of abelian categories. In this section we shall prove two results of a similar type, for topoi. Recall that in any category an object is called small if any map from it to a coproduct necessarily factors through a finite sub-coproduct.

3.3.1 Proposition In $\tilde{\mathbb{C}}$ (in fact, in any elementary topos) the induced mapping of a sub-coproduct into a coproduct is a monomorphism.

<u>Proof</u> If we have a family $\{F_i : i \in I\}$ of objects of $\tilde{\mathbb{C}}$ and a subset $J \subseteq I$ then the map $\coprod_{i \in J} F_i \longrightarrow \coprod_{i \in I} F_i$ is one injection of the coproduct diagram

$$\begin{tikzcd}
\coprod_{i \in J} F_i \arrow[dr] & \\
 & \coprod_{i \in I} F_i \\
\coprod_{i \in I-J} F_i \arrow[ur] &
\end{tikzcd}$$

But it is well-known that in any elementary topos the injections of a coproduct are monomorphisms (eg. see [SB]).

3.3.2 <u>Proposition</u> For $C \in |\mathbb{C}|$, the sheaf $a(C)$ is small if every sieve $R \in \mathrm{Cov}(C)$ is small in $\hat{\mathbb{C}}$ (considered now as a subfunctor $R \subseteq C$ rather than as an ideal of maps in $\mathbb{C}$). If no $C' \in |\mathbb{C}|$ is covered by the empty sieve, which from the $\hat{\mathbb{C}}$ point of view is the initial object and from the "ideal" point of view is the empty ideal, this condition is also necessary.

<u>Proof</u> To show the condition is sufficient, take a family $\{F_i : i \in I\}$ in $|\tilde{\mathbb{C}}|$ and a mapping $a(C) \xrightarrow{\sigma} \coprod_I F_i$. Now $\coprod_I F_i$ is just $a(\hat{\coprod_I} F_i)$ where the circumflex indicates that the coproduct is formed in $\hat{\mathbb{C}}$. The mapping σ may be looked upon as a section of $\coprod_I F_i$ over C and as such is locally a section of $\hat{\coprod_I} F_i$ that is, over a cover $R \subseteq C$. This is expressed by the diagram

$$\begin{tikzcd}
R \arrow[r, dashed, "\bar{\sigma}"] \arrow[d, tail, "i"'] & \widehat{\coprod_I} F_i \arrow[dd, "\varsigma"] \\
C \arrow[d, "\varsigma"'] & \\
a(C) \arrow[r, "\sigma"'] & \coprod_I F_i
\end{tikzcd}$$

By hypothesis R is small in $\widehat{\mathfrak{C}}$ so we have a factorization of $\bar{\sigma}$ into $R \longrightarrow \widehat{\coprod_J} F_i \subseteq \widehat{\coprod_I} F_i$ with J a finite subset of I. Applying the associated sheaf functor, the reflection maps ς collapse, as does i and the coproducts $\widehat{\coprod}$ are transformed into coproducts in $\widetilde{\mathfrak{C}}$, leaving us with the factorization

$$\begin{tikzcd}
a(C) \arrow[rr, "\sigma"] \arrow[dr] & & \coprod_I F_i \\
& \coprod_J F_i \arrow[ur] &
\end{tikzcd}$$

Hence $a(C)$ is small in $\widetilde{\mathfrak{C}}$.

To show that the condition is necessary, under the additional hypothesis, suppose some $R \in \mathrm{Cov}(C)$ is not small. That is, we have a map $R \xrightarrow{\sigma} \widehat{\coprod} F_i$ which does not factor through a finite sub-coproduct. Exactness properties of topoi imply that a subobject of a coproduct is a coproduct of subobjects, hence we may as well assume that the image of σ is $\widehat{\coprod} F_i$, i.e. σ is an epimorphism. We may also assume $F_i \neq 0$ all i, for 0 summands contribute nothing to a coproduct. Applying the associated sheaf we obtain an epimorphism $a(C) \twoheadrightarrow \coprod a(F_i)$. Now each F_i is non-zero hence has a section $C_i' \longrightarrow F_i$ over C_i'. Then we have a map $a(C_i') \longrightarrow a(F_i)$ and since each $a(C_i') \neq 0$, $a(F_i) \neq 0$

(in an elementary topos any map with codomain 0 must be an isomorphism). This shows that the coproduct $\coprod a(F_i)$ is properly infinite and since $a(\sigma)$ is epi, it cannot factor through any proper subobject of the coproduct, let alone a finite subcoproduct. We have used the easily established fact that $\forall\ C \in |\mathbb{C}|$, the empty sieve covers C iff $a(C) = 0$.

3.3.3 Proposition In the setting of proposition 3.2.1, if $\mathfrak{G}$ consists of small objects, so does $\overline{\mathfrak{G}}$.

Proof Clearly the initial object is small, for the usual pathological reasons. Let us show that the coproduct of two small objects is small, thereby ensuring that any finite coproduct of small objects is small. Say $F, G \in |\mathbb{E}|$ are small and we have a map $F+G \longrightarrow \coprod_I H_i$. Here we use "+" for binary coproduct. Then the induced maps on the summands F and G factor through finite sub-coproducts $\coprod_J H_i$ and $\coprod_K H_i$. Clearly by putting the factored maps together in the obvious way we obtain a factorization of the given map through $\coprod_{J \cup K} H_i$.

To show epimorphic images of small objects are small, let $F \xrightarrow{\sigma} G$ be an epimorphism with F small and suppose $G \xrightarrow{\tau} \coprod_I H_i$ is any map. Then $\tau\sigma$ factors through a finite sub-coproduct, by the smallness of F.

$$\begin{tikzcd}
F \arrow[r, "\sigma"] \arrow[dr, "\bar{\sigma}"'] & G \arrow[r, "\tau"] & \coprod_I H_i \\
 & \coprod_J H_i \arrow[ur, tail, "\mu"'] &
\end{tikzcd}$$

Now σ is the coequalizer of its kernel pair and $\bar{\sigma}$ clearly composes equally with the two maps of the kernel pair of σ, hence $\bar{\sigma}$ factors through σ, say $\bar{\sigma} = \phi\sigma$. Then we have $\mu\phi\sigma = \mu\bar{\sigma} = \tau\sigma$, so $\mu\phi = \tau$ since σ is epi. Thus we have, as required, factorized τ through a finite sub-coproduct.

3.3.4 Corollary If $\mathbb{C}$ is a Noetherian site (cf. 4.2.3), all finite sheaves are small. The most typical example of a Noetherian site is Open(T) with its canonical topology, where T is the spectrum of a commutative Noetherian ring.

3.3.5 Before leaving this introduction to finite sheaves let us point out that even for "nice" categories $\mathbb{C}$, for example finitely complete ones, the full subcategory of finite sheaves rarely constitutes a topos in its own right. The problem is that exponentials and the subobject classifier are often forced to be infinite. However, if $\mathbb{C}$ is a finite category, (i.e. the set of morphisms of $\mathbb{C}$ is finite), the finite objects of $\tilde{\mathbb{C}}$ do constitute a topos and in fact the finite sheaves on $\mathbb{C}$ in this case are just the sheaves with values in the category of finite sets (which is, by the way, a topos). But observe that for non-finite categories $\mathbb{C}$, being a finite sheaf is much stronger than merely being a sheaf with values in finite sets. And even if $\mathbb{C}$ has enough points, a sheaf all of whose fibres are finite need not be finite in our sense, although the converse is true.

4 Subalgebra Lattices in $M(\mathbb{H},\tilde{\mathfrak{C}})$

4.1 Basic Properties of Subalgebra Lattices

4.1.1 *Remarks* Recall the situation in classical universal algebra. We start with the category of sets, where subobject lattices are complete atomic Boolean algebras. In equational classes $M(\mathbb{H},\text{Sets})$, subalgebra lattices are complete and meet-continuous (i.e. meets distribute over up-directed joins). Furthermore the inaccessible elements are join dense in the lattice. The relationship between subalgebra lattices and the subobject lattices of the carriers of the algebras is fundamental and is usually summed up with the phrase "the union of a directed family of subalgebras is an algebra". For our purposes this is better expressed by: the forgetful functor $U:M(\mathbb{H},\text{Sets})\longrightarrow\text{Sets}$ preserves joins of up-directed families of subobjects. The following sequence of propositions establishes that these facts survive fairly well the passage to sheaves.

4.1.2 *Proposition* Subobject lattices in $\tilde{\mathfrak{C}}$ are complete Heyting algebras.

Proof This is a basic fact about elementary topoi. Joins of subobjects are computed first in $\hat{\mathfrak{C}}$ by taking the pointwise unions, and then "sheafified". Intersections are computed pointwise, as for presheaves. The Heyting algebra structure may be looked upon as a consequence of the fact that in an elementary topos pulling back preserves colimits, for this implies that in a subalgebra lattice meets distribute over arbitrary joins. For a complete lattice this last-mentioned fact is equivalent to the existence of pseudo-complements, which is the basis of the structure of a Heyting algebra.

4.1.3 <u>Proposition</u> The underlying object functor $U:M(\mathbb{H},\tilde{\mathfrak{C}})\longrightarrow\tilde{\mathfrak{C}}$ preserves joins of up-directed families of subalgebras.

<u>Proof</u> This relies heavily on the fact that the corresponding functor $\hat{U}$ (for presheaves) has the property. This of course in turn follows from the fact that the property is true for algebras in Sets, i.e. for $|U|$, and that everything in a functor category works "pointwise".

Recall the following diagram from section 2.3 of this chapter.

$$\begin{tikzcd}
M(\mathbb{H},\tilde{\mathfrak{C}}) \arrow[r, shift right, "i"'] \arrow[d, dash, "U"'] & M(\mathbb{H},\hat{\mathfrak{C}}) \arrow[l, shift right, "a"'] \arrow[d, dash, "U"] \\
\tilde{\mathfrak{C}} \arrow[r, shift right, "i"'] & \hat{\mathfrak{C}} \arrow[l, shift right, "a"']
\end{tikzcd}$$

As we have seen before, $Ua = a\hat{U}$, $iU = \hat{U}i$. We are concerned with diagrams $D:\Sigma\longrightarrow M(\mathbb{H},\tilde{\mathfrak{C}})$ where Σ is an up-directed poset and $D(\alpha)$ is mono for each morphism α in Σ. Then we have

$$\begin{aligned}
U\,\varinjlim D &= U\,\varinjlim aiD && \text{(since } ai = \text{identity)}\\
&= Ua\,\varinjlim iD && \text{(since } a \text{ preserves } \varinjlim\text{'s)}
\end{aligned}$$

$$= a\hat{U}\ \varinjlim iD$$
$$= a\ \varinjlim \hat{U}iD \qquad \text{(since } \hat{U} \text{ preserves these } \varinjlim\text{'s)}$$
$$= a\ \varinjlim iUD$$
$$= \varinjlim aiUD$$
$$= \varinjlim UD.$$

4.1.4 Proposition Subalgebra lattices in $M(\mathbb{H},\tilde{\mathbb{C}})$ are complete and meet-continuous.

Proof It is well-known that for any category $\mathbb{E}$ and theory $\mathbb{H}$, finitary or not, all left limits which exist in $\mathbb{E}$ exist in $M(\mathbb{H}, \mathbb{E})$, and the underlying object preserves and reflects them (in fact, even stronger it creates them, [TM]). Thus given a family of subalgebras of an algebra in $M(\mathbb{H},\tilde{\mathbb{C}})$, its intersection exists and is computed in $\tilde{\mathbb{C}}$. That is to say, the carrier of the intersections is the intersection of the carriers. Meets distribute over joins of chains since both constructions are made at the $\tilde{\mathbb{C}}$ level (4.1.3), and the distributivity holds there as pointed out in the proof of 4.1.2.

4.2 On the Algebraicity of Subalgebra Lattices in $M(\mathbb{H},\tilde{\mathbb{C}})$.

4.2.1 In classical universal algebra one calls a lattice algebraic if it is complete and every element is the join of the compact elements it exceeds [UA]. An element of a lattice is compact if, whenever it is exceeded by the join of a family of elements, it is already exceeded by the join of finitely many of those elements. One proves that a lattice is algebraic iff it is isomorphic to the lattice of subalgebras of some universal algebra. By passing to sheaves of algebras it might be expected that a wider variety of subalgebra lattices might appear. This is in fact true, as example 4.2.2 shows. Following this example we show that a certain finiteness condition on a site, introduced by Artin [GT] in connection with algebraic geometry, is sufficient to ensure that subalgebra lattices in $M(\mathbb{H},\tilde{\mathfrak{T}})$ are algebraic. To prove this result we will find it convenient to use another characterization of algebraic lattices. A lattice is algebraic if and only if it is complete, meet continuous, and the intranscessible elements are join dense (every element is the join of the intranscessable ones it exceeds). An element is intranscessable iff whenever it is exceeded by the join of a directed family of elements, it is already exceeded by one of them. For complete lattices it is sufficient that the condition hold for chains (it follows that it will hold for directed sets in general).

4.2.2 <u>Example</u> The subalgebra lattice of $1 \in \tilde{T}$ where T is the open unit interval (0,1) has only one compact element, namely $\emptyset$, and hence is not algebraic. Note that here we are taking $\mathbb{H}$ to be the trivial theory for which $U{:}M(\mathbb{H},\tilde{T}) \longrightarrow \tilde{T}$ is an isomorphism.

<u>Proof</u> The subalgebra under discussion is of course nothing more or less than the Heyting algebra of open subsets of (0,1). For an open set to be a compact element of this lattice is equivalent to being topologically compact. But in a Hausdorff space any compact subspace is closed. In (0,1), in fact in any connected topological space, the only open-closed subspaces are $\emptyset$ and the whole space. In this example even the whole space fails to be compact.

4.2.3 <u>Definitions</u> Recall that a topological space is said to be Noetherian if every open set is quasicompact. The spectrum of a commutative Noetherian ring is such a space. We also sometimes encounter in nature topological spaces which have at least a basis of quasicompact open sets. The spectrum of any commutative ring is such a space. We must extend these concepts to sites. Given a site $\mathbb{C}$, an object $C \in |\mathbb{C}|$ will be called quasicompact if every $\{C_i \longrightarrow C \mid i \in I\} \in \mathrm{Cov}(C)$ contains a finite subfamily which is in $\mathrm{Cov}(C)$. In the language of ideals this says every ideal which covers C is finitely generated. The site $\mathbb{C}$ is called Noetherian (Artin, [GT]) if every object is quasicompact, and is called locally Noetherian if every object can at least be covered by quasicompact objects.

4.2.4 <u>Remarks</u> Let $\mathbb{C}$ be a locally Noetherian site and denote by $\mathbb{K}$ the full subcategory determined by the compact objects. Then $\mathbb{K}$ in a certain vague sense generates $\mathbb{C}$. In fact if $i: \mathbb{K} \longrightarrow \mathbb{C}$ is the inclusion functor, i_* is a good candidate for an isomorphism. However certain exactness conditions must be imposed before this is necessarily true. For example if $\mathbb{C}$ is finitely left complete and $\mathbb{K}$ is closed under the formation of finite left limits in $\mathbb{C}$, i_* will in fact be an isomorphism [GT].

4.2.5 <u>Definition</u> In keeping with the definition of finitely generated algebras, we define an algebra $\mathfrak{A} \in |M(\mathbb{H}, \tilde{\mathbb{C}})|$ to be compactly generated if there is a quasicompact $C \in |\mathbb{C}|$ and a quotient map (= regular epi = coequalizer = epi in $\tilde{\mathbb{C}}$) $Wa(C) \longrightarrow\!\!\!\!\triangleright \mathfrak{A}$.

4.2.6 Proposition Let $\mathfrak{C}$ be any site. Then the compactly generated subalgebras of an algebra $\mathcal{A}$ are intranscessable elements of the subalgebra lattice of $\mathcal{A}$.

Proof Let $\mathcal{B} \subseteq \mathcal{A}$ be a compactly generated subalgebra and $\{\mathcal{A}_i \mid i \in I\}$ a chain of subalgebras of $\mathcal{A}$ with $\mathcal{B} \subseteq \bigvee_i \mathcal{A}_i$. As usual we write $U(\mathcal{A}) = A$, $U(\mathcal{B}) = B$ and so on for the carriers of the algebras. Say $Wa(C) \twoheadrightarrow \mathcal{B}$ is a quotient map in $M(\mathbb{H}, \tilde{\mathfrak{C}})$ with C quasicompact. Then the associated map $a(C) \longrightarrow \bigvee_i A_i$ in $\tilde{\mathfrak{C}}$ corresponds to a section $\sigma \in (\bigvee_i A_i)(C)$, that is, a cover $\{C_j \longrightarrow C : j \in J\}$ of C together with compatible sections $\sigma_j \in \bigcup_i A_i(C_j)$. Recall that this is because $\bigvee A_i$ is the sheaf associated to $\bigcup A_i$ (sectionwise union in $\hat{\mathfrak{C}}$). Further, since $\mathfrak{C}$ is quasicompact we may suppose J to be finite. Now for each $j \in J$ we have $\alpha(j) \in I$ with $\sigma_j \in A_{\alpha(j)}(C_j)$. The finite set $\{A_{\alpha(j)} : j \in J\}$ is linearly ordered since we started with a chain. Say $A_{\alpha(j')}$ is the largest. Then for each $j \in J$ $\sigma_j \in A_{\alpha(j)}(C_j) \subseteq A_{\alpha(j')}(C_j)$ so the σ_j are all sections of $A_{\alpha(j')}$ over the various C_j. They are clearly compatible with respect to $A_{\alpha(j')}$ (they were given as compatible with respect to $\bigvee A_i$) and hence determine a section of $A_{\alpha(j')}(C)$ which of course must be σ . Hence we have a factorization

$$\begin{tikzcd}
a(C) \arrow[rr] \arrow[dr] & & \bigvee A_i \\
& A_{\alpha(j')} \arrow[ur, tail] &
\end{tikzcd}$$

Note that the inclusion of summands into a join is a monomorphism in a topos. Lifting the above diagram by the adjointness of W and U we obtain

$$\begin{tikzcd}
Wa(C) \arrow[r, -{Triangle[open]}] \arrow[dr] & \mathfrak{B} \arrow[r, tail] & \bigvee\mathfrak{A}_i \\
& \mathfrak{A}_{\alpha(j')} \arrow[ur, tail] &
\end{tikzcd}$$

The top two arrows constitute a regular image factorization in $M(\mathbb{H},\tilde{\mathbb{C}})$ and hence we have an induced morphism of algebras $\mathfrak{B} \longrightarrow \mathfrak{A}_{\alpha(j')}$ making the two triangles commute. In particular, this shows $\mathfrak{B} \rightarrowtail \bigvee\mathfrak{A}_i$ factors through $\mathfrak{A}_{\alpha(j')}$ i.e. $\mathfrak{B} \subseteq \mathfrak{A}_{\alpha(j')}$. Thus $\mathfrak{B}$ is intranscessable.

4.2.7 Proposition Let $\mathbb{K} \subseteq \mathbb{C}$ be a full subcategory, $\mathbb{C}$ a site. Suppose every $C \in |\mathbb{C}|$ has a cover $\{K_i \longrightarrow C : i \in I\}$ with all $K_i \in |\mathbb{K}|$ Then in any subalgebra lattice the subalgebras generated by the $K \in |\mathbb{K}|$ are join dense.

Proof Evidently it suffices to show that any algebra is the join of the $\mathbb{K}$-generated subalgebras it contains. To this end let $\mathfrak{A}$ be an algebra and $\{\mathfrak{A}_i : i \in I\}$ the set of its $\mathbb{K}$-generated subalgebras. i.e. for each $i \in I$ there is a $K \in |\mathbb{K}|$ and a quotient map $Wa(K) \longrightarrow \mathfrak{A}_i$. Then $\bigvee_i \mathfrak{A}_i \subseteq \mathfrak{A}$ and we must establish equality. Let $\sigma : C \longrightarrow A$ be any section of the carrier of $\mathfrak{A}$ over an arbitrary $C \in |\mathbb{C}|$. Let $\{K_j \longrightarrow C : j \in J\}$ be a covering of C by objects of K, and $\mathfrak{B}_j$ the images in $\mathbb{M}(\mathbb{H},\tilde{\mathbb{C}})$ of the composite

$Wa(K_j) \longrightarrow Wa(C) \longrightarrow \mathcal{A}$ where the latter map is the free extension of σ . Then the B_j are $\mathbb{K}$-generated. Claim $\sigma \in (\bigvee_J \mathcal{B}_j)(C)$. But the restriction of σ to K_j clearly factors through B_j:

$$\begin{tikzcd}
K_j \arrow[r] \arrow[dr] & C \arrow[r, "\sigma"] & A \\
& B_j \arrow[ur] &
\end{tikzcd}$$

Thus $\sigma | K_j \in (\bigcup_J B_j)(K_j)$ where the union is taken in $\hat{\mathfrak{C}}$. This implies that $\sigma \in (\bigvee_J B_j)(C)$ where $\bigvee_J B_j$ is the sheaf associated to $\bigcup_J B_j$. But $\bigvee_J B_j \subseteq U(\bigvee_J \mathcal{B}_j)$, possibly properly of course, since $\mathcal{B}_j \subseteq \bigvee_J \mathcal{B}_j \Longrightarrow U(\mathcal{B}_j) \subseteq U(\bigvee \mathcal{B}_j)$ i.e. $B_j \subseteq U(\bigvee_J \mathcal{B}_j)$ all $j \in J$. The upshot of all this is that in $\tilde{\mathfrak{C}}$ we have the factorizations

$$\begin{tikzcd}
C \arrow[r, "\sigma"] \arrow[dr] & \bigvee_I A_i \arrow[r, tail] & U(\bigvee_I \mathcal{A}_i) \\
& \bigvee_J \mathcal{B}_j \arrow[u, tail] \arrow[r, tail] & U(\bigvee_J \mathcal{B}_j) \arrow[u, tail]
\end{tikzcd}$$

In particular σ is a section over C of the carrier of $\bigvee_J \mathcal{B}_j$ which is a subalgebra of $\bigvee_I \mathcal{A}_i$. This shows that every section of $\mathcal{A}$ is already a section of $\bigvee_I \mathcal{A}_i$, i.e. $\bigvee_I \mathcal{A}_i = \mathcal{A}$, as required.

4.2.8 Theorem If $\mathfrak{C}$ is a locally Noetherian, subalgebra lattices in $M(\mathbb{H},\tilde{\mathfrak{C}})$ are algebraic.

Proof By 4.2.7 the compactly generated elements of the subalgebra lattice are join dense. Thus by 4.2.6 the intranscessable elements of the lattice are join dense. These lattices are complete and meet-continuous by 4.1.4, whether $\mathbb{C}$ is locally Noetherian or not. As pointed out in 4.2.1, this ensures that the subalgebra lattices are algebraic.

5 Congruences and Subdirectly Irreducible Algebras

5.1 Congruences in $\mathbb{E}'(\mathbb{H},\tilde{\mathbb{C}})$

5.1.1 Definitions Recall that an equivalence relation on an object F of a category $\mathbb{E}$ is a subobject $R \subseteq F \times F$ such that

(1) (symmetry) there is a factorization of the diagonal Δ through R.

$$\begin{tikzcd} F \arrow[rr, "\Delta"] \arrow[dr, dashed] & & F \times F \\ & R \arrow[ur] & \end{tikzcd}$$

(2) (reflexivity) the twisting map $\tau = (pr_2, pr_1)$ preserves R.

$$\begin{tikzcd} F \times F \arrow[r, dash, "\tau"] & F \times F \\ R \arrow[u, tail] \arrow[r, dashed] & R \arrow[u, tail] \end{tikzcd}$$

(3) (transitivity) consider the pullback

$$\begin{tikzcd} P \arrow[r, "f_2"] \arrow[d, "f_1"'] & R \arrow[d, "pr_1"] \\ R \arrow[r, "pr_2"'] & F \end{tikzcd}$$

Then there is a map $P \xrightarrow{\gamma} R$ ("composition") with

$pr_1\gamma = pr_2f_1$, $pr_2\gamma = pr_1f_2$.

Equivalent to the above is that the hom functor $\mathbb{E}(-,F)\colon \mathbb{E}^* \longrightarrow \mathrm{Sets}$ factors through the forgetful functor $\mathrm{Rel} \longrightarrow \mathrm{Sets}$ where Rel is the category of sets equipped with relations, and relation-preserving maps. In fact it is sufficient that the factorization hold for $\mathbb{E}(-,F)$ restricted a full subcategory whose objects generate $\mathbb{E}$. We define a congruence on an object $\mathfrak{A} \in |M(\mathbb{H},\tilde{\mathbb{C}})|$ to be an equivalence relation on $\mathfrak{A}$ in this categorical sense.

5.1.2 Facts We list here without proof some basic facts about congruences. They are all either standard results from the theory of triples (theories) or else are easily obtained using techniques already employed in this chapter.

(1) $R \rightarrowtail A \times A$ defines a congruence on $\mathfrak{A}$ iff it is a subalgebra of $\mathfrak{A} \times \mathfrak{A}$ and is an equivalence relation in $\tilde{\mathbb{C}}$.

(2) $R \rightarrowtail A \times A$ defines a congruence on $\mathfrak{A}$ iff $R(C) \rightarrowtail A(C) \times A(C)$ is a congruence on $\mathfrak{A}(C)$ for each $C \in |\mathbb{C}|$, and restriction preserves the relational structure.

(3) For any point p of the site $\mathbb{C}$ and any congruence $\mathfrak{R} \rightarrowtail \mathfrak{A} \times \mathfrak{A}$, $\mathfrak{R}_p \rightarrowtail \mathfrak{A}_p \times \mathfrak{A}_p$ is a congruence on $\mathfrak{A}_p$ in $M(\mathbb{H},\mathrm{Sets})$.

(4) The coequalizer $\mathfrak{R} \rightrightarrows \mathfrak{A} \longrightarrow \mathfrak{A}/\mathfrak{R}$ of the projections may be computed in $\tilde{\mathbb{C}}$. That is, if we apply U to the diagram we obtain $R \rightrightarrows A \longrightarrow A/R$, the coequalizer in $\tilde{\mathbb{C}}$. Note that this relies heavily on the fact that $\mathbb{H}$ is finitary.

(5) Meets of congruences in the subalgebra lattice of a sheaf of algebras $\mathcal{A} \times \mathcal{A}$, which may of course be computed in the subobject lattice of $A \times A$, are again congruences. Joins of directed families of congruences on $\mathcal{A}$ are congruences. Hence the lattice of congruences on $\mathcal{A}$ is complete and meet-continuous. Congruences generated, as congruences, by sections of $\mathcal{A} \times \mathcal{A}$ over quasicompact objects of $\mathbb{C}$ are intranscessable and, if $\mathbb{C}$ is locally Noetherian, join dense. Hence for locally Noetherian sites $\mathbb{C}$, congruence lattices in $M(\mathbb{H},\tilde{\mathbb{C}})$ are algebraic.

(6) The standard isomorphism theorems of universal algebra hold:

(i) Every epimorphic image of an algebra $\mathcal{A}$, where the epimorphism is an epimorphism in $\tilde{\mathbb{C}}$, is obtained by factoring out the kernel relation ("kernel pair"), which is a congruence. More precisely, given any map $\mathcal{A} \longrightarrow \mathcal{B}$ in $M(\mathbb{H},\tilde{\mathbb{C}})$, with kernel relation $\mathcal{R}$ (in $\tilde{\mathbb{C}}$), $\mathcal{R}$ is a congruence and $\mathcal{A} \longrightarrow \mathcal{A}/_{\mathcal{R}} \rightarrowtail \mathcal{B}$ is a regular image factorization (i.e. coequalizer followed by a mono with the standard factorization property of categorical images [TC]). The kernel relation of $\mathcal{A} \longrightarrow \mathcal{A}/_{\mathcal{R}}$ is $\mathcal{R}$.

(ii) $\mathcal{B} \rightarrowtail \mathcal{A}$ a subalgebra, $\mathcal{R}$ a congruence on $\mathcal{A}$. Then $\mathcal{R}$ restricts to a congruence $\mathcal{R}|\mathcal{B}$ on $\mathcal{B}$ and we obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{B} \arrow[r, tail] \arrow[d] & \mathcal{A} \arrow[d] \\
\mathcal{B}/_{\mathcal{R}|\mathcal{B}} \arrow[r, tail] & \mathcal{A}/_{\mathcal{R}}
\end{tikzcd}$$

(iii) $\mathcal{R}$ a congruence on $\mathcal{A}$. Then the lattice of congruences on $\mathcal{A}$ containing $\mathcal{R}$ is isomorphic to the lattice of congruences on $\mathcal{A}/\mathcal{R}$. The isomorphisms associates to $\mathcal{Q} \supseteq \mathcal{R}$ a congruence $\mathcal{Q}/\mathcal{R}$ on $\mathcal{A}/\mathcal{R}$ with

$$\mathcal{A}/\mathcal{Q} \cong \mathcal{A}/\mathcal{R} \big/ \mathcal{Q}/\mathcal{R}$$

(7) A morphism in $M(\mathbb{H},\tilde{\mathbb{C}})$ is a coequalizer iff it is a quotient map (i.e. of the form $\mathcal{A} \longrightarrow \mathcal{A}/\mathcal{R}$) iff its underlying map in $\tilde{\mathbb{C}}$ is epi in $\tilde{\mathbb{C}}$.

5.2 Subdirectly Irreducible Algebras

5.2.1 The basic result of this section is that if $\mathbb{C}$ is a locally Noetherian site, the Birkhoff subdirect representation theorem of universal algebra holds in $M(\mathbb{H},\tilde{\mathbb{C}})$. Once one knows the facts of 5.1.2 the proof of this assertion can almost be copied from any treatise on classical universal algebra. We shall go into a little detail, however, in order to give an indication of how one manipulates congruences in $M(\mathbb{H},\tilde{\mathbb{C}})$.

5.2.2 Definition $\mathcal{A} \in |M(\mathbb{H},\tilde{\mathbb{C}})|$ is subdirectly irreducible iff whenever we have a monomorphism $\mu: \mathcal{A} \longrightarrow \prod_{i \in I} \mathcal{B}_i$ into a product of algebras, then for some projection pr_i, $pr_i\mu$ is mono.

5.2.3 Proposition Let $\mathbb{E}$ be any category with kernel pairs and suppose we have a product $\prod_{i \in I} B_i$ and a map $f: A \longrightarrow \prod_{i \in I} B_i$.

Denoting $pr_i f = f_i$ and $R_i \subseteq A \times A$, $R \subseteq A \times A$ the kernel relations of f_i, f respectively, then $R = \bigcap_{i \in I} R_i$.

Proof We must show that with respect to the natural ordering on subobjects of $A \times A$, if $S \subseteq R_i \subseteq A \times A$ all $i \in I$, then $S \subseteq R$. But $S \subseteq R_i \Longleftrightarrow S \rightarrowtail A \times A \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} A \xrightarrow{f_i} B_i$ compose equally, by the universality property which R_i has as the kernel pair of f_i (i.e. pullback of f_i with itself). Thus $S \subseteq R_i$ all $i \Longrightarrow$

$S \rightarrowtail A \times A \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} A \xrightarrow{f} \prod B_i \xrightarrow{pr_i} B_i$ compose equally, all $i \Longrightarrow$

$S \rightarrowtail A \quad A \rightrightarrows A \longrightarrow B$ compose equally (projections of a product are jointly monomorphic) $\Longrightarrow S \rightarrowtail A \times A$ factors through $R \rightarrowtail A \times A$ by the universality property of R as the kernel relation of f.

5.2.4 Proposition $\mathcal{A} \in |M(\mathbf{H},\tilde{\mathcal{C}})|$ is subdirectly irreducible iff $\bigcap \{\mathcal{R} : \mathcal{R} \text{ congruence on } \mathcal{A}, \mathcal{R} \neq \Delta\} \neq \Delta$ where Δ is the diagonal relation, i.e. $(pr_1, pr_2) : \mathcal{A} \rightarrowtail \mathcal{A} \times \mathcal{A}$

Proof To show the condition is necessary, suppose that the set of congruences in the proposition statement is denoted by I and $\bigcap I = \Delta$. Consider $\mathcal{A} \longrightarrow \prod_I \mathcal{A}/_{\mathcal{R}}$ where the $\mathcal{R}$th projection is the canonical quotient map. By 5.2.3 the kernel relation of this map is $\bigcap_I \ker(\mathcal{A} \longrightarrow \mathcal{A}/_{\mathcal{R}}) = \bigcap I = \Delta$, applying 5.2.1 fact (i). Hence $\mathcal{A} \longrightarrow \prod \mathcal{A}/_{\mathcal{R}}$ is mono but no $\mathcal{A} \longrightarrow \mathcal{A}/_{\mathcal{R}}$ is mono ($\mathcal{R} \neq \Delta$, all $\mathcal{R} \in I$), showing $\mathcal{A}$ is not subdirectly irreducible. To show the condition is sufficient suppose $(f_j):\mathcal{A} \longrightarrow \prod_{j \in J} \mathcal{B}_j$ is a mono, where $\ker(f_j) = \mathcal{R}_j$. Then if f_j ,is not mono for all j, $\bigcap_{j \in J} \mathcal{R}_j \geq \bigcap I \supsetneq \Delta$. i.e. $\ker((f_j)) \neq \Delta$ which is a contradiction. Note that throughout this proof we have made considerable use of the fact, true in any category, and completely elementary to establish, that a map $f:X \longrightarrow Y$ is mono iff it has a kernel relation, and that kernel relation is $\Delta:X \longrightarrow X \times X$. Equivalently, the following is a pullback diagram

$$\begin{tikzcd}
X \arrow[r, "1"] \arrow[d, "1"'] & X \arrow[d, "f"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

5.2.5 <u>Proposition</u> $\mathcal{A} \in |M(\mathbb{H},\tilde{\mathbb{C}})|$, $\mathcal{R}$ a congruence on $\mathcal{A}$. Then $\mathcal{A}/_{\mathcal{R}}$ is subdirectly irreducible iff $\mathcal{R}$ is completely meet-irreducible in the lattice of congruences on $\mathcal{A}$.

<u>Proof</u> 5.2.4 and 5.2.1 fact 6 (iii) (the "second isomorphism theorem" of universal algebra).

5.2.6 <u>Theorem</u> If $\mathbb{C}$ is a locally Noetherian site then the Birkhoff subdirect representation theorem holds in $M(\mathbb{H},\tilde{\mathbb{C}})$. That is, every algebra may be mapped monomorphically into a product of subdirectly irreducible algebras.

<u>Proof</u> By 5.1.2 (6) the congruence lattice on $a \in |M(\mathbb{H},\tilde{\mathfrak{C}})|$ is algebraic, so the McCoy-Fuchs theorem may be applied, which says that in algebraic lattices every element is the meet of completely meet-irreducible ones. Hence for such an a, $\Delta = \bigcap\{\mathcal{R} \mid \mathcal{R}$ a completely meet-irreducible congruence$\}$. This gives a monomorphism $a \longrightarrow \prod a/_{\mathcal{R}}$ where the product is taken over all completely meet-irreducible $\mathcal{R}$. But for such $\mathcal{R}$, $a/_{\mathcal{R}}$ is subdirectly irreducible by 5.2.5.

Chapter IV Injectivity in $M(\mathbb{H},\tilde{\mathbb{C}})$

1 Existence of Injectives

1.1 When $\mathbb{H}$ is Trivial

1.1.1 Here we treat the case for which $\mathbb{H}$ is trivial in the sense of having "no operations or equations". Technically, as a category, $\mathbb{H}$ is just the category of finite ordinals and $U:M(\mathbb{H},\tilde{\mathbb{C}}) \longrightarrow \tilde{\mathbb{C}}$ is an equivalence of categories. Categories $\tilde{\mathbb{C}}$ always have enough injectives. In fact any elementary topos has enough injectives. For it is the case that the subobject classifier Ω of an elementary topos $\mathbb{E}$ is injective. In fact we have canonical extensions of maps into Ω. Say $F' \rightarrowtail F$ is mono and $\phi : F' \longrightarrow \Omega$ is any map. Then ϕ is the "characteristic function" of $F'' \rightarrowtail F'$. By composing $F'' \rightarrowtail F' \rightarrowtail F$, F'' is exhibited as a subobject of F and thus has a characteristic function $\psi : F \longrightarrow \Omega$. It is then a simple matter to show that ψ extends ϕ. One then can show all "power sets" Ω^F, $F \in |\mathbb{E}|$, are injective. But every $F \in |\mathbb{E}|$ may be mapped monomorphically into its associated "power set" as follows ("embedding of singletons"): the characteristic function of the diagonal $\Delta : F \longrightarrow F \times F$ is a map $F \times F \longrightarrow \Omega$. By exponential adjointness one obtains the desired $F \longrightarrow \Omega^F$ which, with a little work, can be shown to be a monomorphism (note that it is not necessarily a retraction however). This embedds every F into an injective, Ω^F. In fact this construction may be refined somewhat. There is a factorization

$$\begin{tikzcd} F \arrow[rr, tail] \arrow[dr, tail] & & \Omega^{F} \\ & \tilde{F} \arrow[ur, tail] & \end{tikzcd}$$

where $\tilde{F}$ is the object which classifies partial maps with domain F. $\tilde{F}$ is injective. Indeed in some topoi every injective is of the form $\tilde{F}$ for some F, e.g. G-sets for a group G.

1.2 $\mathbb{H}$ the Theory of Modules Over a Ring

1.2.1 Theorem (Grothendieck [Tokoku]). If $\mathbb{A}$ is an abelian category satisfying AB5 and possessing a generator, then every $A \in |\mathbb{A}|$ may be embedded in an injective.

Proof The proof is transfinite in nature and will not be reproduced here. The interested reader may consult [Tokoku] for details.

1.2.2 Remarks For $\mathbb{H}$ the theory of modules over a ring R, $M(\mathbb{H},\tilde{\mathbb{C}})$ is a category of the type envisioned in 1.2.1 and thus Grothendieck's theorem yields enough injectives in $M(\mathbb{H},\tilde{\mathbb{C}})$ for these particular theories $\mathbb{H}$. More generally his theorem applies to sheaves of modules over a sheaf of rings. To handle such categories with the techniques we have been discussing, we would have to expand our frame of reference to encompass "many-sorted theories" $\mathbb{H}$ [SADC]. In fact most of the results of this work which apply to $M(\mathbb{H},C)$ apply also "mutatis mutandis" to n-sorted theories $\mathbb{H}$ with $n < \aleph_0$. Although the details will not be reproduced in this thesis, the extension is straightforward, and the techniques of extension will be implicit in a discussion in chapter V of how some of the results appear in the case of sheaves of modules over a sheaf of rings.

Before leaving this theorem of Grothendieck's, it should be pointed out that Mitchell [TC] gives a constructive proof for a special case, namely sheaves of abelian groups over a topological space. His construction appears as a special case of the general existence theorem which follows.

1.3 $\mathbb{H}$ Arbitrary

1.3.1 Theorem If $\mathbb{C}$ is a site with enough points and $\mathbb{H}$ is a finitary theory such that $M(\mathbb{H},\text{Sets})$ has enough injectives, then $M(\mathbb{H},\tilde{\mathbb{C}})$ has enough injectives.

Proof Let P be a set of points p whose stalks collectively reflect isomorphisms. Consider the functor

$$S : M(\mathbb{H},\tilde{\mathbb{C}}) \longrightarrow M(\mathbb{H},\text{Sets})^{|P|}$$

which sends a sheaf of algebras $\mathfrak{a}$ to the family $\{\mathfrak{a}_p : p \in P\}$. Here $(\ \ )^{|P|}$ indicates the functor category whose exponent is the discrete category whose class of objects is the set P. S clearly preserves all right limits, and since $M(\mathbb{H},\tilde{\mathbb{C}})$ has a set of generators, the special adjoint functor theorem provides for the existence of a right adjoint Q, which will in fact be a geometric morphism of topoi. From exactness properties of the stalks it follows that S preserves (and reflects) monomorphisms, and is faithful. Since $M(\mathbb{H},\text{Sets})$ has enough injectives, so does $M(\mathbb{H},\mathrm{Sets})^{|P|}$, for "everything works pointwise", in particular any P-indexed family of injective algebras is injective in the functor category. Thus by well-known results about adjoint functors with the above exactness properties, every algebra in $M(\mathbb{H},\tilde{\mathbb{C}})$ may be embedded in an injective. Given an algebra $\mathfrak{a}$, embedd each stalk $\mathfrak{a}_p$ into an injective algebra (in Sets) I_p and apply Q. The resulting map $\mathfrak{a} \longrightarrow Q(\{I_p\}_{p \in P})$ is an embedding of $\mathfrak{a}$ into an injective.

1.3.1 Constructions The above theorem does not give us a very clear picture of any particular injective into which we can embedd $\mathfrak{a}$, although the adjoint functor theorem is constructive and thus Q may be theoretically computed. We now exhibit a functor $Q_0 : M(\mathbb{H},\mathrm{Sets}) \xrightarrow{|P|} M(\mathbb{H},\tilde{\mathbb{C}})$ which is a very simple construction and which in two important cases is equal to Q. It appears that in general the values of Q_0 need not be sheaves, although they are always separated presheaves. Furthermore, supposing that $\mathbb{C}$ is such that Q_0 does have sheaves of values, there seems to be no general technique for constructing a back adjunction $SQ_0 \longrightarrow 1$ although a front adjunction $1 \longrightarrow Q_0 S$ is available.

The formula for Q_0 is

$$Q_0(\{\mathfrak{a}_p\}_{p \in P})(C) = \prod_{p \in P} \mathfrak{a}_p^{C_p}$$

C_p is of course the stalk at p of the representable functor $C \in |\tilde{\mathbb{C}}|$ (assume for this discussion that the topology on $\mathbb{C}$ is coarser than the canonical, although this is no real restriction - in general just replace C by $a(C)$ where necessary). If $C \longrightarrow C'$ is any map in $\mathbb{C}$ we obtain, upon application of the stalk at p, $C_p \longrightarrow C'_p$, inducing for any algebra

$\mathcal{B}$ in Sets a homomorphism $\mathcal{B}^{C'_p} \longrightarrow \mathcal{B}^{C_p}$. Thus given a family of algebras $\{\mathcal{A}_p : p \in P\}$ in sets, we obtain a map $\prod_{p \in P} \mathcal{A}_p^{C_p} \longrightarrow \prod_{p \in P} \mathcal{A}_p^{C'_p}$ which is the restriction map from C to C' for $Q_0(\{\mathcal{A}_p\})$. This is clearly functorial, hence $Q_0(\{\mathcal{A}_p\})$ is a presheaf. In fact it is a separated presheaf, for let $\{f_i : C_i \longrightarrow C \mid i \in I\}$ be a covering in $\mathbb{C}$ and take any section $\alpha = (\alpha^p) \in Q_0(\{\mathcal{A}_p\})(C)$, i.e. $\alpha^p : C_p \longrightarrow \mathcal{A}_p$ in Sets. The restriction of α to C_i is the family $(\alpha^p(f_i)_p)$. If β is another such section, whose restrictions to the C_i are the same as those of α, we have $\alpha^p(f_i)_p = \beta^p(f_i)_p$. But since the stalk at p must carry covering families to epimorphic families, it follows that for each p, $\{(f_i)_p \mid i \in I\}$ is epimorphic and therefore $\alpha^p = \beta^p$. Hence $\alpha = \beta$, showing the presheaf is separated. Now if $\mathcal{A}$ is a sheaf and $\mathcal{A}_p$ are its stalks we always have a morphism of presheaves

$$\eta : \mathcal{A} \longrightarrow Q_0 S(\mathcal{A})$$

In particular cases this is the best candidate for a front adjunction At each $C \in |\mathbb{C}|$ we must exhibit a map of algebras $\mathcal{A}(C) \longrightarrow \prod_{p \in P} \mathcal{A}_p^{C_p}$

Given a section $\sigma \in \mathcal{A}(C)$, we can represent it as a morphism of sheaves of sets $\sigma : C \longrightarrow \mathcal{A}$. Applying the stalk at p we obtain $\sigma_p : C_p \longrightarrow \mathcal{A}_p$. The morphism η takes $\sigma \in \mathcal{A}(C)$ to the family $(\sigma_p) \in \prod_{p \in P} \mathcal{A}_p^{C_p}$. The fact that η is a morphism of presheaves then comes down to showing that $\forall$ $f : C' \longrightarrow C$ and any $\sigma \in \mathcal{A}(C)$, $(f(\sigma)_p) = (\sigma_p \cdot f_p)$. But $f(\sigma)$ considered a a morphism $C' \longrightarrow \mathcal{A}$ is just σf (cf. Chapter I,1.1) so $f(\sigma)_p = (\sigma f)_p = \sigma_p f_p$. To show that η

is natural in $\mathcal{A}$, let $\mathcal{B}$ be any sheaf of algebras and $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ a morphism in $M(\mathbb{H},\tilde{\mathbb{C}})$. For each $C \in |\mathbb{C}|$ we must establish commutativity of

$$\begin{tikzcd}
\mathcal{A}(C) \arrow[r] \arrow[d, "\phi_C"'] & \prod_p \mathcal{A}_p^{C_p} \arrow[d, "Q_0 S(\phi)"] \\
\mathcal{B}(C) \arrow[r] & \prod_p \mathcal{B}_p^{C_p}
\end{tikzcd}$$

Up to this point we have not defined the action of Q_0 on morphisms, but it is fairly obvious, in fact in some senses dual to the action of restriction on a particular $Q_0(\{\mathcal{A}_p\})$. Precisely, if $\{\mathcal{A}_p\}$ and $\{\mathcal{B}_p\}$ are now arbitrary families of algebras, and $\psi_p : \mathcal{A}_p \longrightarrow \mathcal{B}_p$ is an arbitrary family of algebra homomorphisms, $Q_0(\{\psi_p\})$ at C takes a family (α^p) to $(\psi_p \circ \alpha^p)$. Returning to the naturality of η in $\mathcal{A}$, we must show that $\forall\ \sigma \in \mathcal{A}(C)$, $(\phi_p \circ \sigma_p) = (\phi_C(\sigma)_p)$. But again by the Yoneda lemma, $\phi_C(\sigma)$ considered as a morphism $C \longrightarrow \mathcal{B}$ is just $\phi \circ \sigma$ (σ here considered as a morphism $C \longrightarrow \mathcal{A}$). Hence $\phi_C(\sigma)_p = (\phi\sigma)_p = \phi_p\, \sigma_p$.

This is about as far as we are able to carry the general case, at the time of this writing. We briefly mention two large classes of examples to which it applies.

First of all if $\mathbb{C}$ is the site Open(T) for a topological space T and P is the set of points associated with the singletons $x \in T$, the formula becomes

$$Q_0(\{\mathcal{A}_x\})(U) = \prod_{x \in T} \mathcal{A}_x^{U_x} = \prod_{x \in U} \mathcal{A}_x$$

The simplification in the formula arises from the fact that $U_x = 1$ if $x \in U$ and $U_x = 0$ if $x \notin U$. So for $x \notin U$, $a_x^{U_x} = a_x^0 = 1$, which makes no contribution to the product. In this case Q_0 always has sheaves as values, and in fact is right adjoint to S with front adjunction η. This construction may be seen for abelian groups in [TC].

The second example is for sites $\mathbb{C}$ carrying the coarsest topology. Here Cov(C) consists of only one sieve, viz. C itself, for each $C \in \mathbb{C}$. $\tilde{\mathbb{C}}$ is isomorphic to $\hat{\mathbb{C}}$ (in fact equal - every presheaf is a sheaf). For P we take the points whose stalks are the evaluations $\{e_C \mid C \in |\mathbb{C}|\}$ (cf. 2.3.1, chapter II). Note that these evaluations are just the covariant hom functors $(C,-):\mathbb{C} \longrightarrow \text{Sets}$. Then

$$Q_0(\{a_C\})(C') = \prod_{C \in |\mathbb{C}|} a_C^{\mathbb{C}(C,C')}$$

Now in this example the functor S may be viewed as the exponentiation of $i:|\mathbb{C}| \longrightarrow \mathbb{C}$ with base $M(\mathbb{H},\text{Sets})$. i is the canonical functor from the discrete category on the underlying set of objects of $\mathbb{C}$ to $\mathbb{C}$. Then the formula for Q_0 is easily recognized as that for the right Kan extension of i (for example see [CWM]). Thus in this case also, Q_0 gives the right adjoint to S.

It is to be hoped that in other cases of interest Q_0 will be right adjoint to S, or at least provide enough injective sheaves of algebras (note in this regard that η is always a monomorphism). Clearly further investigation along these lines is necessary.

2 The Behaviour of Injectivity in $M(\mathbb{H},\tilde{\mathbb{C}})$

2.1 Introduction

2.1.1 We shall now obtain results on the behaviour of injectivity in sheaves of algebras, applying the general framework established by B. Banaschewski and exposed in a recent series of papers by him. The most concise reference is [IE^{3}CA] and the reader is referred to this paper for any undefined concepts in what follows. We confine ourselves here to studying injectivity with respect to monomorphisms (the work of Banaschewski is more general in its applicability). Our main result is that if injectivity is well-behaved in the base variety $M(\mathbb{H},\mathrm{Sets})$ then it is well-behaved in $M(\mathbb{H},\tilde{\mathbb{C}})$ for any site $\mathbb{C}$. First of all we reproduce the basic results of [IE^{3}CA].

2.1.2 Injectivity (with respect to monomorphisms) in a category $\mathbb{E}$ is said to be well-behaved if the following three propositions are true.

2.1.2.1 <u>Proposition</u> For any $F \in |\mathbb{E}|$ TFAE

(i) F is injective.

(ii) F is an absolute retract.

(iii) Any essential extension of F is an isomorphism.

2.1.2.2 <u>Proposition</u> Every $F \in |\mathbb{E}|$ has an injective hull, unique up to isomorphism.

2.1.2.3 <u>Proposition</u> For any monomorphism $F \rightarrowtail G$ TFAE

(H1) G is an injective hull of F.

(H2) $F \rightarrowtail G$ is a maximal essential extension.

(H3) $F \rightarrowtail G$ is a minimal injective extension.

Note (H2) and (H3) are stated with respect to the usual order on subobjects in a category. The next proposition gives a set of conditions sufficient to ensure that injectivity is well-behaved in $\mathbb{E}$. The same numbering as that in [IE^{3}CA] has been retained, but the first two conditions there are always satisfied for injectivity with respect to monomorphisms and have not been listed.

2.1.3 <u>Proposition</u> The following four conditions on a category $\mathbb{E}$ ensure that injectivity is well-behaved in $\mathbb{E}$.

(E3) For any monomorphism $F \rightarrowtail G$ there exists a morphism $G \longrightarrow H$ such that the composite $F \rightarrowtail G \longrightarrow H$ is essential.

(E4) Given a monomorphism $F \rightarrowtail G$ and any morphism $f: F \longrightarrow H$ then f may be extended at least to some "super-object" of H, that is the diagram may be completed as follows, with the bottom line a monomorphism:

$$\begin{tikzcd}
F \arrow[r, tail] \arrow[d] & G \arrow[d, dashed] \\
H \arrow[r, tail, dashed] & K
\end{tikzcd}$$

(E5) Any well-ordered diagram in $\mathbb{E}$, whose transition maps are monomorphisms, has an upper bound whose injections are monomorphisms.

(E6) Each $F \in |\mathbb{E}|$ has, up to isomorphism, only a set of essential extensions.

2.1.4 Proposition For an equational class M($\mathbb{H}$,Sets) TFAE

(i) Injectivity is well-behaved.

(ii) There are enough injectives.

(iii) (E4) and (E6) hold.

(iv) Every subdirectly irreducible algebra has injective extensions.

Note: Equational classes M($\mathbb{H}$,Sets) always satisfy (E3), (E5).

2.2 Results for Sheaves of Algebras

2.2.1 Theorem For a category of sheaves of algebras M($\mathbb{H},\tilde{\mathbb{C}}$) conditions (i) - (iii) of the following are equivalent, with no assumptions on the nature of the site $\mathbb{C}$. If $\mathbb{C}$ happens to be a locally Noetherian site, conditions (i) - (iv) are all equivalent.

(i) Injectivity is well-behaved.

(ii) There are enough injectives.

(iii) (E4) and (E6) hold.

(iv) Every subdirectly irreducible sheaf of algebras has injective extensions.

Proof

(i) $\Rightarrow$ (ii) Definition of "well-behaved injectivity".

(ii) $\Rightarrow$ (iii) (E4) is always satisfied for any category $\mathbb{E}$ having enough injectives. One need only embedd H into an injective K and lift the composite $F \longrightarrow H \rightarrowtail K$ to G. (E6) is always satisfied for a well-powered $\mathbb{E}$ having enough injectives, for any essential extension of $F \in |\mathbb{E}|$ must be a subobject of any arbitrarily selected injective extension of F.

(iii) $\Rightarrow$ (i) We must show (E3) and (E5) hold in $M(\mathbb{H},\tilde{\mathbb{C}})$. It is well-known that they hold in $M(\mathbb{H},\text{Sets})$ for any finitary theory $\mathbb{H}$. To establish (E3) let $\mathcal{A} \rightarrowtail \mathcal{B}$ be any monomorphism in $M(\mathbb{H},\tilde{\mathbb{C}})$. Let Σ be the set of congruences $\mathcal{R}$ on $\mathcal{B}$ such that $\mathcal{R}|\mathcal{A} = \Delta$. Note that $\mathcal{R}|\mathcal{A} = \mathcal{R} \cap \mathcal{A}\times\mathcal{A}$. Take a chain $\{\mathcal{R}_\alpha : \alpha \in I\}$ in Σ. Then $\bigvee_\alpha \mathcal{R}_\alpha|\mathcal{A} = \mathcal{A}\times\mathcal{A} \cap \bigvee_\alpha \mathcal{R}_\alpha = \bigvee_\alpha (\mathcal{A}\times\mathcal{A} \cap \mathcal{R}_\alpha) = \bigvee_\alpha \Delta = \Delta$ (by meet continuity of the lattice of congruences). Hence Σ is inductive and any maximal $\mathcal{R}$ in Σ will have the property that $\mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{B}/_{\mathcal{R}}$ is essential. This follows from the fact that the standard "isomorphism theorems" of universal algebra hold in $M(\mathbb{H},\tilde{\mathbb{C}})$, as pointed out in chapter III. 5.1.2. To establish (E5) note that since colimits are available, this condition is equivalent to "injections of $\varinjlim$'s of well-ordered systems of monomorphisms are monomorphisms". This is well-known to be true in $M(\mathbb{H},\text{Sets})$, hence it is true in $M(\mathbb{H},\hat{\mathbb{C}})$ since $\varinjlim$'s and monos are computed pointwise there. Given a well-ordered system of monomorphisms in $M(\mathbb{H},\tilde{\mathbb{C}})$ we compute its $\varinjlim$ by first computing it in $M(\mathbb{H},\hat{\mathbb{C}})$, where the injections are monomorphisms, and then applying the associated sheaf functor which preserves monomorphisms.

(ii) $\Rightarrow$ (iv) Always.

(iv) $\Rightarrow$ (ii) (for $\mathbb{C}$ locally Noetherian). Given an algebra $\mathcal{A}$, embedd it in a product of subdirectly irreducibles (by chapter III, 5.2.6). This product may be embedded in a product of injectives, each of which is an injective extension of one of the subdirectly irreducibles. But any product of injectives is injective (any category $\mathbb{E}$, any sort of injectivity).

2.2.2 <u>Theorem</u> If injectivity is well-behaved in $M(\mathbb{H},\text{Sets})$, then it is well-behaved in $M(\mathbb{H},\tilde{\mathbb{C}})$, provided $\mathbb{C}$ has enough points.

<u>Proof</u> The hypothesis implies $M(\mathbb{H},\text{Sets})$ has enough injectives. By 1.3.1 of this chapter, $M(\mathbb{H},\tilde{\mathbb{C}})$ has enough injectives. By 2.2.1 above, injectivity is well-behaved.

2.2.3 <u>Remarks</u> By arguments similar to those used in 2.2.1 above, showing that (E5) holds, one can show that if (E4) holds in $M(\mathbb{H},\text{Sets})$ it must hold in $M(\mathbb{H},\tilde{\mathbb{C}})$ for any $\mathbb{C}$. However it is not clear to the author that the validity of (E6) in $M(\mathbb{H},\text{Sets})$ necessarily implies by itself that (E6) holds in the sheaves of algebras, even when $\mathbb{C}$ has enough points. The problem is the following: it is true that if a map $\mu\colon \mathcal{A} \rightarrowtail \mathcal{B}$ is essential at each $C \in |\mathbb{C}|$, that is to say $\mu_C\colon \mathcal{A}(C) \rightarrowtail \mathcal{B}(C)$ is essential in $M(\mathbb{H},\text{Sets})$, then μ is essential. Similarly if $\mathbb{C}$ has enough points and μ_p is essential for each point p, then μ is essential. However the converses of these statements are in general false, even when $\mathbb{H}$ is trivial. For example consider $\hat{\mathbf{2}}$ where $\mathbf{2}$ is the ordinal number 2 considered as a poset and hence as a category. Then for any sets $F \subsetneq G$, the morphism in $\hat{\mathbf{2}}$

$$\begin{pmatrix} F \\ \cap \\ G \end{pmatrix} \longrightarrow \begin{pmatrix} G \\ \| \\ G \end{pmatrix}$$

given by inclusions at each point (0 and 1), is essential, but it fails to be essential either at its "sections" or its points (which, because we are in a full functor category, we can take to be the same). Note, by the way, that the above represents an injective hull for $(F \subset G) \in \hat{\mathcal{P}}$. The general problem of what conditions on $\{\mu_C : C \in |\mathbb{C}|\}$ are imposed by assuming μ to be essential seems difficult and would be worthwhile pursuing. Hopefully one might be able to decide whether or not (E6) for $M(\mathbb{H},\tilde{\mathbb{C}})$ follows from (E6) for $M(\mathbb{H},\text{Sets})$.

2.2.4 We close off this section with a result which is useful in the search for injectives in particular categories $M(\mathbb{H},\tilde{\mathbb{C}})$.

Proposition The forgetful functor $M(\mathbb{H},\tilde{\mathbb{C}}) \longrightarrow \tilde{\mathbb{C}}$ carries injectives to injectives.

Proof Proposition 2.3.2 of chapter III stated that the functor in question has a left adjoint which is faithful and which preserves monomorphisms.

Chapter V Equations Parametrized By a Site

1 Equations in a Sheaf of Algebras

In this section we introduce the notion of a sheaf of equations with coefficients in a sheaf of algebras, and whose "unknowns" are represented by a sheaf of variables. Working with the definition used requires extensive manipulation of exponentiation in $\tilde{\mathfrak{C}}$. To facilitate this we use the technique of localization as presented in [SGA]. With this tool we can usually deduce results about behaviour of algebras at the level of sections over $C \in |\tilde{\mathfrak{C}}|$ from analogous results about global sections. This approach has the advantage that global sections of exponentials are much easier to deal with, both conceptually and technically. A summary of the theory of localization, together with some extensions to algebras, is presented in 1.2. This might logically have been a part of Chapter I, but it has been deferred to this chapter since it is not required in Chapter I-IV.

1.1 Definition of Equations and Solutions

1.1.1 Recall that in universal algebra a set of equations in variables $x \in X$ (X a set) is a subset of the square of the free algebra on X, which we have been denoting $|W|(X)$. More generally, if $\mathfrak{a}$ is an algebra we have the algebra of polynomials with coefficients in $\mathfrak{a}$, denoted $\mathcal{A}[X]$ and defined to be the coproduct $\mathcal{A} + |W|(X)$ in $M(\mathbb{T},\mathrm{Sets})$. A set of equations Σ in variables X with coefficients in $\mathcal{A}$ is a subset $\Sigma \subseteq \mathcal{A}[X]^2$. We can translate thus to the situation at hand, for the relevant free algebras are available.

1.1.2 <u>Definition</u> Let $\mathcal{A}$ be a sheaf of algebras and X a sheaf of sets. Define the algebra of polynomials with coefficients in $\mathcal{A}$, and variables in X, to be the coproduct $\mathcal{A}[X] = \mathcal{A} + W(X)$ in $M(\mathbb{T},\tilde{\mathbb{C}})$. A sheaf of equations Σ with coefficients in $\mathcal{A}$ and variables in X is a subsheaf $\Sigma \subseteq U(\mathcal{A}[X])^2$. We will abuse our notation somewhat by writing $U(\mathcal{A}[X]) = A[X]$. Since A is just a sheaf of sets (the carrier of $\mathcal{A}$), $A[X]$ might be interpreted also as the corresponding construction in $\tilde{\mathbb{C}}$ where A is considered as a $\mathbb{T}'$-algebra for $\mathbb{T}'$ the trivial theory. However no confusion should arise in this regard. Note that for $\mathcal{A} = W(0)$, the free algebra on the initial object of $\tilde{\mathbb{C}}$, we have $\mathcal{A}[X] = \mathcal{A} + W(X) = W(0) + W(X) = W(0+X) = W(X)$ so that anything general we say about $\mathcal{A}[X]$ will apply to the sheaf of polynomials (terms) in the variables X.

1.1.3 <u>Interpretation</u> This definition can be meaningfully interpreted in terms of classical universal algebra. For any sheaf is defined by its sections over the generators $C \in |\mathbb{C}|$. Thus a sheaf of equations consists of collections $\Sigma(C)$ of sections $C \xrightarrow{\sigma} A[X]^2$, i.e. pairs of elements $(\sigma_1, \sigma_2) \in A[X](C)^2$. Now observe that $\mathcal{A}[X] = a(\mathcal{A} + \hat{W}(X))$, the coproduct being taken in presheaves. But $(\mathcal{A} + \hat{W}(X))(C) = \mathcal{A}(C) + |W|(X(C)) = \mathcal{A}(C)[X(C)]$. From the representation of $\mathcal{A}[X]$

as an associated sheaf, and the standard way of computing associated sheaves, we see that a section $\tau \in \mathcal{Q}[X](C)$ is, over some covering $\{C_i \longrightarrow C : i \in I\}$ a collection of "real" polynomials in the $X(C_i)$ with coefficients in the $\mathcal{Q}(C_i)$. Thus for our typical equation (σ_1, σ_2) defined over C we can find coverings $\{C_i \longrightarrow C : i \in I\}$ and $\{C_j \longrightarrow C : j \in J\}$ such that $\sigma_1 | C_i \in \mathcal{Q}(C_i)[X(C_i)]$ and $\sigma_2 | C_j \in \mathcal{Q}(C_j)[X(C_j)]$. Taking a common refinement of the two covers we see that locally σ is just a pair of polynomials with coefficients in "real" algebras (i.e. algebras in sets). Summing up, an "equation" defined over a generator C is a map $C \longrightarrow A[X]^2$ which is nothing more or less than a family of equations in variables which are certain sections of X, in coefficients which are certain sections of $\mathcal{Q}$, this family being coherently parametrized by a cover of C with respect to the topology on $\mathfrak{C}$. Besides speaking of equations defined over generators, we may of course discuss equations defined over any object of $\tilde{\mathfrak{C}}$. In particular we shall at times speak of globally defined equations, which are merely equations defined over 1, the terminal object of $\tilde{\mathfrak{C}}$.

1.1.4 Remarks The morphisms $\mathcal{Q} \longrightarrow \mathcal{Q}[X]$ and $W(X) \longrightarrow \mathcal{Q}[X]$, arising from the definition of $\mathcal{Q}[X]$ as a coproduct, are monomorphisms. For the analogous result is true for equational classes in Sets. Hence it is true for algebras modelled in presheaves. The coproduct in $M(\mathbb{H}, \tilde{\mathfrak{C}})$ is obtained by applying the associated sheaf functor to the analogous coproduct in $M(\mathbb{H}, \hat{\mathfrak{C}})$, and since the associated sheaf functor preserves monomorphisms, we are finished.

1.1.5 Proposition If p is a point of $\mathbb{C}$, $(\mathcal{A}[X])_p = \mathcal{A}_p[X_p]$.

Proof First of all we need to know that the taking of stalks commutes in a sense with the formation of free algebras. Recalling the discussion of Chapter II, 2.2 observe that if p is a point of $\mathbb{C}$ with associated fibre p^* on $\tilde{\mathbb{C}}$ then ip is a point of $\hat{\mathbb{C}}$ with associated fibre p^*a (i,a are the inclusion of sheaves in presheaves and the associated sheaf functor, respectively). Moreover $N'(p) = N'(ip)$.

Hence we have
$$\begin{aligned} W(X)_p &= (a(\hat{W}(X)))_p \\ &= \hat{W}(X)_{ip} \\ &= \varinjlim_{C \in N'(ip)} \hat{W}(X)(C) \\ &= \varinjlim_{C \in N'(p)} \hat{W}(X)(C) \\ &= \varinjlim_{C \in N'(p)} |W|(X(C)) \\ &= |W|(\varinjlim_{C \in N'(p)} X(C)) \qquad \text{since } |W| \text{ preserves } \varinjlim\text{'s} \\ &= |W|(X_p) \end{aligned}$$

Now then $\mathcal{A}[X]_p = (\mathcal{A} + W(X))_p = \mathcal{A}_p + W(X)_p = \mathcal{A}_p + |W|(X_p) = \mathcal{A}_p[X_p]$.

1.1.6 The Sheaf of Solutions of a Sheaf of Equations A concept of equation must be accompanied by a notion of solution if it is to be useful. Let $\Sigma \subseteq A[X]^2$ be a sheaf of equations and $\mathcal{B}$ a sheaf of algebras with $\mathcal{A} \subseteq \mathcal{B}$. Borrowing on experience from universal algebra and logic, a solution to Σ, in $\mathcal{B}$, should be a valuation (morphism in $\tilde{\mathbb{C}}$) $v: X \longrightarrow B$ with a certain property. If $\bar{v}$ is the extension to a map

$\mathcal{A}[X] \longrightarrow \mathcal{B}$ (extend freely on the component $W(X)$ and on the component $\mathcal{A}$ just map via the inclusion $\mathcal{A} \subseteq \mathcal{B}$), that property is simply that $\Sigma \subseteq \ker(\bar{v})$. Otherwise put, $\bar{v}q_1 = \bar{v}q_2$ where q_1, q_2 are the projections of Σ . But this is just a first approximation, for a topos we have available the object B^X which, besides containing the valuations $X \longrightarrow B$, is rather more rich in structure. The valuations referred to are only its global sections. Its structure at $C \in |\mathbb{C}|$ will be discussed at some length in 1.2. Right now we need only its universality properties. We look for an "object of solutions" for Σ , which is to be a subsheaf of B^X and is defined as follows:

$$\begin{tikzcd}
\Upsilon_\Sigma \arrow[r, tail] & B^X \arrow[r, "\lambda"] & B^{A[X]} \arrow[r, shift left, "B^{q_1}"] \arrow[r, shift right, "B^{q_2}"'] & B^\Sigma
\end{tikzcd}$$

Υ_Σ , the object of solutions, is the equalizer of the maps $B^{q_1}\lambda$ and $B^{q_2}\lambda$. λ is the "lifting map" which acts in such a way as to carry global sections of B^X, maps $v:X \longrightarrow B$ to their extensions $\mathcal{A}[X] \longrightarrow \mathcal{B}$. λ is defined as follows:

By exponential adjointness λ corresponds to a map $A[X] \longrightarrow B^{B^X}$

$$\begin{array}{l}
B^X \xrightarrow{\lambda} B^A \\
\hline
B^X \times A[X] \xrightarrow{\check{\lambda}} B \\
\hline
A[X] \times B^X \longrightarrow B \\
\hline
A[X] \longrightarrow B^{B^X}
\end{array}$$

This last map is induced by the double dual map $X \longrightarrow B^{B^X}$ and the map $A \longrightarrow B^{B^X}$ which corresponds to the composite $A \times B^X \xrightarrow{pr_1} A \longrightarrow B$. Of course to deduce the map $A[X] \longrightarrow B^{B^X}$ we must know that $A \longrightarrow B^{B^X}$ is a homomorphism of algebras. We are using the algebra structure inherited by B^{B^X} from B by virtue of the fact that it is an internal power. To establish that we do indeed have a homomorphism of algebras, take any $F \in |\tilde{\mathfrak{C}}|$

$$\begin{tikzcd}
\tilde{\mathfrak{C}}(F,A) \arrow[r, equal] \arrow[d] & \tilde{\mathfrak{C}}(F,A) \arrow[d] \\
\tilde{\mathfrak{C}}(F,B^{B^X}) \arrow[r, "\sim"] & \tilde{\mathfrak{C}}(F \times B^X, B)
\end{tikzcd}$$

The map on the right is induced by $pr_1 : F \times B^X \longrightarrow F$ on the first argument and $A \rightarrowtail B$ on the second, hence is a homomorphism of set-valued algebras. The bottom arrow is a homomorphism, in fact isomorphism, of algebras by the way in which the algebra structure on B^{B^X} is induced from that of B. The diagram commutes by elementary manipulations of the exponential adjointness involved. Hence the left-hand arrow is a homomorphism of set-valued algebras, and since F was arbitrary, $A \longrightarrow B^{B^X}$ is in fact a homomorphism of sheaves of algebras.

Let us now show that λ acts on global sections as we have indicated. Take any $\check{\sigma} : 1 \longrightarrow B^X$ corresponding to $\sigma : X \longrightarrow B$. Then the composite $\lambda\check{\sigma}:1\longrightarrow B^{A[X]}$ corresponds to a map $A[X]\longrightarrow B$ obtained as follows.

$$A[X]\longrightarrow 1\times A[X]\xrightarrow{\check{\sigma}\times(\ )} B^{X}\times A[X]\xrightarrow{\check{\lambda}} B$$

To show this map is the lifting of σ by the universality property of $\mathfrak{a}[X]$ we need only check that (1) preceeded by $X\longrightarrow A[X]$ it is equal to σ and (2) preceeded by $A\longrightarrow A[X]$ it is equal to $A\rightarrowtail B$. The following commutative diagrams establish this.

$$\begin{tikzcd}
A[X] \arrow[r, "{(k_{\check{\sigma}},1)}"] & B^{X}\times A[X] \arrow[r, "\check{\lambda}"] & B \\
X \arrow[u] \arrow[r, "{(k_{\check{\sigma}},1)}"'] & B^{X}\times X \arrow[u] \arrow[ur, "\varepsilon"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
A[X] \arrow[r, "{(k_{\check{\sigma}},1)}"] & B^{X}\times A[X] \arrow[r, dash, "\check{\lambda}"] & B \\
A \arrow[u] \arrow[r, "{(k_{\check{\sigma}},1)}"'] & B^{X}\times A \arrow[u] \arrow[r, "pr_2"] & A \arrow[u]
\end{tikzcd}$$

In these diagrams $k_{\check{\sigma}}$ is the "constant map $?\longrightarrow 1\xrightarrow{\check{\sigma}} B^{X}$; we have used the same name no matter what the domain. ε is the evaluation, the back adjunction of the exponential adjointness. The diagrams commute by the very definition of λ.

To describe the behaviour of the morphism λ at a particular $C\in|\mathbb{C}|$, using the usual representation of $B^{X}(C)$ as $\tilde{\mathbb{C}}(a(C)\times X,B)$

is possible, but rather complicated and not particularly enlightening. Instead we will use the technique of localization (to be developed in 1.2) to exhibit the sectionwise behaviour of λ, at C say, as the global behaviour of the corresponding map $\lambda | C$ between the algebras localized at C. As pointed out in the introduction to this paragraph, the localization technique, although it requires some preparatory work, is extremely economical and understandable as a technique of proof. Before proceeding to the study of localization, let us make some comments on the idea of solution that has been introduced.

1.1.7 Remarks General properties of equalizers and exponentiation can easily be shown to yield that $\Upsilon_{\Sigma} = \bigcap \Upsilon_{\Sigma'}$ where Σ' runs over any epimorphic family of subobjects of Σ. Now the set of finite subsheaves of Σ is an epimorphic family. A direct proof is easy and of course ultimately depends on the fact that finiteness was defined in terms of a generating family. In fact the singly-generated subsheaves of Σ already constitute an epimorphic family. Furthermore, the collection $\{ \Upsilon_{\Sigma'} \mid \Sigma' \subseteq \Sigma \text{ is finite} \}$ is closed under finite intersections. For if Σ_1 and Σ_2 are two finite subsheaves of Σ, their join in the subsheaf lattice is finite (it is a quotient of the coproduct $\Sigma_1 + \Sigma_2$) and together they constitute an epimorphic family of subobjects of their join, hence $\Upsilon_{\Sigma_1} \cap \Upsilon_{\Sigma_2} = \Upsilon_{\Sigma_1 \vee \Sigma_2}$.

As pointed out previously, a solution of Σ over an object $F \in |\tilde{\mathfrak{C}}|$ is a map $F \longrightarrow \Upsilon_{\Sigma}$ i.e. an element of the set $\tilde{\mathfrak{C}}(F, \Upsilon_{\Sigma})$. Σ is finitely solvable over such an F iff $\tilde{\mathfrak{C}}(F, \Upsilon_{\Sigma'}) \neq \emptyset$ all finite

$\Sigma' \subseteq \Sigma$. Consider the following three facts:

(1) $\widetilde{\mathfrak{C}}(F,-)$ commutes with intersections (since they are $\varprojlim$'s).

(2) $\{ \Upsilon_{\Sigma'} \mid \Sigma' \subseteq \Sigma , \Sigma' \text{ finite} \}$ is closed under finite intersections.

(3) $\widetilde{\mathfrak{C}}(F, \Upsilon_{\Sigma}) \rightarrowtail \widetilde{\mathfrak{C}}(F,\mathfrak{B}^X)$ is a monomorphism, and hence $\widetilde{\mathfrak{C}}(F, \Upsilon_{\Sigma})$ may be identified with a subset of $\widetilde{\mathfrak{C}}(F,\mathfrak{B}^X)$.

It follows that Σ is finitely solvable over F iff the collection $\{ \widetilde{\mathfrak{C}}(F, \Upsilon_{\Sigma'}) \mid \Sigma' \subseteq \Sigma , \Sigma' \text{ finite} \}$ has the finite intersection property. Moreover, since $\widetilde{\mathfrak{C}}(F,\Upsilon_{\Sigma}) = \bigcap \widetilde{\mathfrak{C}}(F, \Upsilon_{\Sigma'})$ where Σ' runs over all finite subsheaves of Σ , our goal is now clear: to search for a structure on the sheaf of algebras $\mathfrak{B}$ which will induce a compact T_2 topology on $\widetilde{\mathfrak{C}}(F,\mathfrak{B}^X)$ such that all $\widetilde{\mathfrak{C}}(F, \Upsilon_{\Sigma'})$ are closed subsets. Clearly such an algebra $\mathfrak{B}$ will be equationally compact in a very strong sense, with respect to equations with coefficients in $\mathfrak{A}$ (which we may take to be the free algebra on the initial object of $\widetilde{\mathfrak{C}}$, thus retrieving the general notion of equational compactness usually discussed in universal algebra). We shall prove in 2 that if $\mathfrak{B}$ is a topologically compact sheaf of algebras, it will satisfy these properties.

1.2 Localization

1.2.1 The material in this section comes mainly from [SGA], exposé III. The reader is referred there for details of the proofs, which we will not include here, for reasons of space. We will present a series of propositions, with comments designed to establish some geometric intuition into the situation.

1.2.2 Proposition Let $\mathbb{E}$ be an elementary topos and $f:F \longrightarrow G$ a morphism in $\mathbb{E}$. Then the functor $f^*: \mathbb{E}/_G \longrightarrow \mathbb{E}/_F$ which acts by "pulling back" has a left adjoint Σ_f and a right adjoint Π_f. Σ_f acts on objects of $\mathbb{E}/_F$ by composition with f. f^* is a logical morphism of topoi and hence Π_f is a local homeomorphism.

f^* is called the inverse image functor and Π_f the direct image functor. Of particular interest is the case in which $G = 1$ and $f = !:F \longrightarrow 1$, the unique map from F to the terminal object. Then $\mathbb{E}/_G \cong \mathbb{E}$, f^* sends an object X to the arrow $pr_2 : X \times F \longrightarrow F \in |\mathbb{E}/_F|$, Σ_f sends an arrow to its domain and Π_f sends an arrow to its "object of sections". We also denote f^* in this case by $\Delta : \mathbb{E} \longrightarrow \mathbb{E}/_F$. Often one interprets $\Delta(X)$ as the fibred object over F whose fibres are constant, with value X. In the present setting however, we prefer to view $\Delta(X)$ as the object X localized at F. This will become clearer in what follows.

1.2.3 We consider now the sheaf categories $\tilde{\mathbb{C}}$, and the effect of localizing them at $C \in |\mathbb{C}|$. First of all, we localize the site of definition. Denote by $j_C : \mathbb{C}/_C \longrightarrow \mathbb{C}$ the functor which acts on objects by sending an arrow to its domain. Equip $\mathbb{C}/_C$ with the finest topology for which j_C is continuous.

1.2.4 Proposition In the situation described immediately above,

(1) A family of maps in $\mathbb{C}/_C$ is covering iff its image under j_C is covering.

(2) j_C, besides being continuous, is also cocontinuous.

(3) The following three functors are defined on the sheaves of sets

$j_C^s : (\mathbb{C}/_C)^{\sim} \longrightarrow \tilde{\mathbb{C}}$ "extension by the empty set"

$j^*_C = j_{C,s} : \tilde{\mathbb{C}} \longrightarrow (\mathbb{C}/_C)^{\sim}$ "restriction to $\mathbb{C}/_C$"

$j_{C,*} : (\mathbb{C}/_C)^{\sim} \longrightarrow \tilde{\mathbb{C}}$ "direct image"

For (3), see chapter II, 1.4. We have $j_C^s \dashv j_{C,s} \dashv j_{C,*}$. Note that if $\mathbb{C} = \text{Open}(T)$ for a topological space T, and $C = U$, some open set in T, $\mathbb{C}/_C = \text{Open}(T)/_U$ is canonically isomorphic, as a site, to Open(U) and the three functors above are, modulo this isomorphism, nothing more than v^s, v_s, v_* described in chapter II, 1.4.

1.2.5 Proposition We have the following factorization of j_C^s, with e_C an equivalence of categories:

$$\begin{tikzcd}
(\mathbb{C}/_C)^{\sim} \arrow[rr, "j_C^s"] \arrow[dr, "e_C"'] & & \tilde{\mathbb{C}} \\
& \tilde{\mathbb{C}}/_{a(C)} \arrow[ur] &
\end{tikzcd}$$

The functor "restriction to $\mathbb{C}/_C$", composed with e_C is the functor Δ described in 1.2.2.

1.2.6 Remarks Propositions 1.2.2 and 1.2.5 show together that "restriction to $\mathbb{C}/_C$" is a logical morphism of topoi. Otherwise put,

$j_{C,*}$ is a local homeomorphism. Furthermore, j_C^* and $j_{C,*}$, because they are right adjoints, respect algebra structures and so the restriction and direct image constructions apply equally well to algebras. "extension by the empty set" may in certain cases be modified to "extension by the free algebra on the empty set" to provide a left adjoint to j_C^* at the level of algebras. This in in fact true for sheaves on a topological space.

Let us discuss exponentiation in sheaf categories using the tool of localization. We denote by $F|C$ the image of an object $F \in |\tilde{\mathbb{C}}|$ under restriction to $\mathbb{C}/_C$, and similarly $f|C$ for maps. We have the following adjoint correspondences:

$$\begin{array}{c}
C \longrightarrow F^X \\
\hline
j_C^{\varepsilon}(1) \longrightarrow F^X \\
\hline
1 \longrightarrow (F^X)|C \\
\hline
1 \longrightarrow F|C^{X|C} \\
\hline
X|C \longrightarrow F|C
\end{array}$$

Note that $(F^X)|C \cong F|C^{X|C}$ reflects the fact that restriction to $\mathbb{C}/_C$ is a logical morphism of topoi and hence in particular preserves exponentiation. The above adjoint correspondences yield an isomorphism of sets

$$F^X(C) \cong (\mathbb{C}/_C)^{\sim}(X|C \ , \ F|C)$$

This formula is familiar from the theory of sheaves on a topological space (see [TF], for example). The formula gives a very useful representation of the sections of an exponential. To extend this to maps, we need only observe that the following diagram commutes, for $C' \longrightarrow C$

$$\begin{tikzcd}
F^X(C) \arrow[d] \arrow[r, phantom, "\cong"] & (\mathbb{C}/_C)^\sim(X|C, F|C) \arrow[d] \\
F^X(C') \arrow[r, phantom, "\cong"] & (\mathbb{C}/_{C'})^\sim(X|C', F|C')
\end{tikzcd}$$

The right hand map is that induced by the functor $(\mathbb{C}/_C)^\sim \longrightarrow (\mathbb{C}/_{C'})^\sim$ which is in turn induced by the continuous functor $\mathbb{C}/_{C'} \longrightarrow \mathbb{C}/_C$ given by composition with $C' \longrightarrow C$.

1.2.7 <u>The Extension Map λ</u> We return briefly to a discussion of the map λ defined in 1.1.6 of this chapter. First note that the following diagrams commute:

$$\begin{tikzcd}
M(\mathbb{H},\widetilde{\mathbb{C}}) \arrow[r, "j_C^*"] \arrow[d, "U"'] & M(\mathbb{H},(\mathbb{C}/_C)^\sim) \arrow[d, "U"] \\
\widetilde{\mathbb{C}} \arrow[r, "j_C^*"'] & (\mathbb{C}/_C)^\sim
\end{tikzcd}$$

$$\begin{tikzcd}
M(\mathbb{H},\widetilde{\mathbb{C}}) \arrow[r, "j_{C,s}"] \arrow[d, "U"'] & M(\mathbb{H},(\mathbb{C}/_C)^\sim) \arrow[d, "U"] \\
\widetilde{\mathbb{C}} \arrow[r, "j_{C,s}"'] & (\mathbb{C}/_C)^\sim
\end{tikzcd}$$

This is merely a reflection of the fact, mentioned above, that restriction and direct image are constructions directly applicable to algebras. Now consider the second diagram. Each of the maps involved has a left adjoint, and it follows that the diagram of left adjoints must commute. Hence the following:

$$\begin{tikzcd}
M(\mathbb{H},\tilde{\mathfrak{C}}) \arrow[r, "j_C^*"] & M(\mathbb{H},(\mathfrak{C}/_C)^{\sim}) \\
\tilde{\mathfrak{C}} \arrow[u, "W"] \arrow[r, "j_C^*"'] & (\mathfrak{C}/_C)^{\sim} \arrow[u, "W"']
\end{tikzcd}$$

Using our special notation for restriction, this means that for any $X \in |\tilde{\mathfrak{C}}|$, $W(X)|C = W(X|C)$. Since restriction commutes with coproducts (it is a left adjoint) we have for any $X \in |\tilde{\mathfrak{C}}|$, any $\mathcal{A} \in |M(\mathbb{H},\tilde{\mathfrak{C}})|$,

$$\mathcal{A}[X]|C = (\mathcal{A} + W(X))|C = \mathcal{A}|C + W(X)|C = \mathcal{A}|C + W(X|C) = \mathcal{A}|C[X|C] .$$

Once we have noted this, and taking into account the fact that restriction is a logical morphism we can say that for

$$B^X \xrightarrow{\lambda} B^{A[X]}$$

restriction to C gives $B|C^{X|C} \xrightarrow{\lambda|C} B|C^{A|C[X|C]}$ and $\lambda|C$ *is the extension map* for the localized pair $\mathcal{A}|C \subseteq \mathcal{B}|C$ in $M(\mathbb{H},(\mathfrak{C}/C)^{\sim})$.

Thus we can describe, using localization, the action of $\lambda_C : B^X(C) \longrightarrow B^{A[X]}(C)$. For an element σ of $B^X(C)$ corresponds to map $X|C \longrightarrow B|C$, i.e. a *global* section of $B|C^{X|C}$. Applying $\lambda|C$ we get a morphism $A[X]|C \longrightarrow B|C$ which corresponds exactly to

$\lambda_C(\sigma) \in B^{A[X]}(C)$. The point of all this is that we know how to compute $\lambda | C$ on global sections (see 1.1.7). It acts according to the universal extension property of $\mathfrak{A}|C[X|C]$.

The technique here is clear. To study sections over an arbitrary $C \in |\mathbb{C}|$ of a complicated object, it is sometimes useful to localize at C, where these sections become transformed into global sections, and results about the latter may be applied in the localized category.

1.2.8 Localizing Solutions Recall the equalizer diagram used in 1.1.6 to define the concept of a solution sheaf.

$$\begin{tikzcd} \Upsilon_\Sigma \arrow[r] & B^X \arrow[r] & B^{A[X]} \arrow[r, shift left] \arrow[r, shift right] & B^\Sigma \end{tikzcd}$$

Since localization at $C \in |\mathbb{C}|$ preserves all of the structure involved we obtain

$$\begin{tikzcd} \Upsilon_\Sigma|C \arrow[r] & B|C^{X|C} \arrow[r] & B|C^{A|C[X|C]} \arrow[r, shift left] \arrow[r, shift right] & B|C^{\Sigma|C} \end{tikzcd}$$

This diagram is still an equalizer, hence $\Upsilon_\Sigma|C = \Upsilon_{\Sigma|C}$, that is $\Upsilon_\Sigma|C$ is the sheaf of solutions to the equations $\Sigma|C \subseteq A|C[X|C]^2$. It follows that a solution to Σ over C, which is merely a map $C \longrightarrow \Upsilon_\Sigma$ corresponds exactly to a global solution of the sheaf of equations $\Sigma|C \subseteq A|C[X|C]^2$.

2 Topological and Equational Compactness

2.1 Topologizing Sheaves of Algebras

In this section we introduce the notion of a compact Hausdorff (T_2) structure on a sheaf of algebras. In a theorem below several equivalent ways of specifying such a structure will be formulated. Two lemmas are required as preliminaries to the proofs.

2.1.1 *Lemma.* For any $F \in |\tilde{\mathfrak{C}}|$ the following is a coequalizer diagram in $\tilde{\mathfrak{C}}$:

$$\begin{tikzcd}
\coprod\limits_{(C,C')} \coprod\limits_{(C',F)} a(C') \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \coprod\limits_{(C,F)} a(C) \arrow[r, "h"] & F
\end{tikzcd}$$

The indexing sets of the coproducts are the hom sets in $\hat{\mathfrak{C}}$ and the three maps are defined as follows:

$$\begin{tikzcd}
\coprod \coprod a(C') \arrow[rr, "f"] & & \coprod a(C) \\
\coprod a(C') \arrow[u, "i_\alpha"] & a(C) \arrow[ur, "i_{\sigma\alpha}"'] & \\
a(C') \arrow[u, "i_\sigma"] \arrow[ur, "a(\alpha)"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\coprod\coprod a(C') \arrow[r, "g"] & \coprod a(C) \\
\coprod a(C') \arrow[u, "i_\alpha"] & \\
a(C') \arrow[u, "i_\sigma"] \arrow[uur, "i_\sigma"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\coprod a(C) \arrow[r, "h"] & F \\
a(C) \arrow[u, "i_\sigma"] \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

Note that in the above we have not drawn any notational distinction between $\sigma : C \longrightarrow F$ and $a(\sigma) : a(C) \longrightarrow F$. Recall that this is a 1-1 correspondence between such σ, $a(\sigma)$ and sections of F over C.

Proof We establish directly that h has the universal property of a coequalizer. Let $h' : \coprod_{(C,F)} a(C) \longrightarrow G$ be any morphism of sheaves with $h'f = h'g$. Considering h' preceeded by the injections

i_σ , $\sigma \in \hat{\mathfrak{C}}(C,F)$ we see that for each C and each $\sigma \in F(C)$, h' "picks out" an element, call it $\bar{h}_C(\sigma)$, of $G(C)$. Thus we have mappings $\bar{h}_C : F(C) \longrightarrow G(C)$. The fact that $h'f = h'g$ simply says $\bar{h}$ is natural in C, i.e. $\bar{h} : F \longrightarrow G$ is a morphism in $\tilde{\mathfrak{C}}$. By its definition, $\bar{h}\, h = h'$ and $\bar{h}$ is unique with this property since h is epi. For h is the map associated to $\coprod_{(C,F)} C \longrightarrow F$ in $\hat{\mathfrak{C}}$ which is clearly an epi in $\hat{\mathfrak{C}}$ (it is "onto" at the level of sections).

2.1.2 <u>Lemma</u> For any $X \in |\tilde{\mathfrak{C}}|$ and any n-ary operation f in the theory $\mathbb{H}$, there is a homomorphism of algebras $f_X : W(X) \longrightarrow \coprod_n W(X)$ such that for any algebra $\mathcal{A} \in |M(\mathbb{H},\tilde{\mathfrak{C}})|$, the composite

$$\tilde{\mathfrak{C}}(X,A)^n \longrightarrow (W(X),\mathcal{A})^n \longrightarrow (\coprod_n W(X),\mathcal{A}) \xrightarrow{(f_X,-)} (W(X),\mathcal{A}) \longrightarrow \tilde{\mathfrak{C}}(X,A)$$

is the n-ary operation, corresponding to f, on the algebra (in Sets) $\tilde{\mathfrak{C}}(X,A)$ whose structure is induced by $\mathcal{A}$ (Chapter III, 1.3).

<u>Proof</u> A discussion of this for algebras in Sets can be found in a paper by Freyd [AFG]. The reader should have no difficulty in extending this to presheaves of algebras and then sheaves of algebras, using the techniques developed to this point. Note that the operations described in the statement of the lemma expose the free algebras $W(X)$ as $\mathbb{H}$-coalgebras in $\tilde{\mathfrak{C}}$.

2.1.3 <u>Theorem</u> The following types of structure on $F \in |\tilde{\mathfrak{C}}|$ are equivalent in the sense that given one we may deduce either of the others, and the passages set up correspondence which are bijective.

(1) $F(C)$ carries a compact T_2 structure for each $C \in |\mathfrak{C}|$ and restriction is continuous.

(2) As in (1) with the added condition that the patching condition holds at the level of compact T_2 spaces, i.e. F is a sheaf with values in the category of compact T_2 spaces.

(3) There is a factorization

$$\begin{tikzcd}
\tilde{\mathfrak{C}} \arrow[rr, dashed] \arrow[dr, "\tilde{\mathfrak{C}}(-{,}F)"'] & & CH \arrow[dl, "U"] \\
& Sets &
\end{tikzcd}$$

where CH is the category of compact T_2 spaces and continuous mappings and U is the forgetful functor to Sets.

<u>Proof</u> Evidently (2) $\Longrightarrow$(1). Let us show (1) $\Longrightarrow$(2). The patching condition is the following equalizer (Chapter I, 2.3)

$$F \longrightarrow \prod F(C) \rightrightarrows \prod\prod F(C_i)$$

The maps involved are all continuous with respect to the product topologies. The diagram is an equalizer at the level of sets since F is a sheaf (of sets). Since the underlying set functor for compact T_2 spaces reflects equalizer diagrams (basically because any 1-1 continuous map between compact T_2 spaces is a homeomorphism onto its image) the above diagram must be an equalizer at the level of compact T_2 spaces

(1)$\Longrightarrow$(3) Take any $X \in |\tilde{\mathfrak{C}}|$ and exhibit X as a coequalizer as in lemma 2.1.1. Applying the hom functor $\tilde{\mathfrak{C}}(-,F)$ we obtain the following equalizer in sets

$$\tilde{\mathfrak{C}}(X,F) \longrightarrow \prod \tilde{\mathfrak{C}}(a(C),F) \rightrightarrows \prod \prod \tilde{\mathfrak{C}}(a(C),F)$$

Making use of the isomorphism $\tilde{\mathfrak{C}}(a(C),F) \cong F(C)$ we can write this as

$$\tilde{\mathfrak{C}}(X,F) \longrightarrow \prod_{(C,X)} F(C) \rightrightarrows \prod \prod F(C)$$

The latter two maps are constructed using only projections and restriction in the sheaf F. Thus the topologies on the $F(C)$ induce compact T_2 topologies on the product so that the two maps between the products are continuous. Since the diagram is an equalizer the image of the first map is closed in the product, so the topology transported from this subspace to $\tilde{\mathfrak{C}}(X,F)$ via the first map is compact T_2. Note that this is the initial topology determined by the first map, and since the product topology is the initial topology determined by the projections, we deduce that the topology on $\tilde{\mathfrak{C}}(X,F)$ is the initial topology determined by the maps $\tilde{\mathfrak{C}}(X,F) \longrightarrow F(C)$ which send $\phi: X \longrightarrow F$ to $\phi_C(\sigma)$ where C ranges over $|\mathfrak{C}|$ and σ ranges over $X(C)$. Thus the topology on $\tilde{\mathfrak{C}}(X,F)$ has as a basis sets of the form $V_{\{C_i,\ \sigma_i, G_i\}_{i=1}^{n}}$ where $C_i \in |\mathfrak{C}|$, $\sigma_i \in X(C_i)$, G_i is an open subset of $F(C_i)$ and

$$V_{\{C_i,\ \sigma_i, G_i\}} = \{\phi: X \longrightarrow F \mid \phi_{C_i}(\sigma_i) \in G_i,\ i = 1,\ldots,n\}$$

Now to complete the proof that (1)$\Longrightarrow$(3) we must show that as X is varied, $\tilde{\mathfrak{C}}(X,F)$ varies continuously. Take any $f: X \longrightarrow Y$ in $\tilde{\mathfrak{C}}$. Then

$$\tilde{\mathfrak{C}}(f,F): \tilde{\mathfrak{C}}(Y,F) \longrightarrow \tilde{\mathfrak{C}}(X,F)$$

and $\tilde{\mathfrak{C}}(f,F)^{-1}(V_{\{C_i,\ \sigma_i,G_i\}}) = \{\phi:Y\longrightarrow F \mid (\phi f)_{C_i}(\sigma_i) \in G_i \quad i = 1,\dots,n\}$

$$= \{\phi:Y\longrightarrow F \mid \phi_{C_i}(f_{C_i}(\sigma_i)) \in G_i \quad i = 1,\dots,n\}$$

$$= V_{\{C_i,f_{C_i}(\sigma_i),G_i\}}$$

Since the latter is open in $\tilde{\mathfrak{C}}(Y,F)$, this shows $\tilde{\mathfrak{C}}(f,F)$ is continuous.

(3) $\Longrightarrow$(1) Consider the following diagram

$$\begin{tikzcd}
\tilde{\mathfrak{C}}(a(C),F) \arrow[r] \arrow[d, "\tilde{\mathfrak{C}}(a(\alpha){,}F)"'] & F(C) \arrow[d, "F(\alpha)"] \\
\tilde{\mathfrak{C}}(a(C'),F) \arrow[r] & F(C')
\end{tikzcd}$$

where $\alpha:C'\longrightarrow C$ in $\mathfrak{C}$. By assumption (3) the two left hand sets have compact T_2 topologies on them and the map between them is continuous. Thus if we transport the topologies on the $\tilde{\mathfrak{C}}(a(C),F)$ to the $F(C)$ via the Yoneda isomorphism, we obtain a structure of the type envisioned in (1). Note that the restrictions $F(\alpha)$ are then continuous.

Finally we must show that the correspondences are 1-1. If we start with a structure as in (1) and pass to one of type (3), then back again, we are finished if we know that the topology on $\tilde{\mathfrak{C}}(a(C),F)$ defined as in (1)$\Longrightarrow$(3) (by the coequalizer technique) is the same (modulo the Yoneda isomorphism) as that of $F(C)$. The topology on $C(a(C),F)$ was defined via the equalizer

$$\tilde{\mathfrak{C}}(a(C),F) \longrightarrow \prod_{(C',C)} F(C') \rightrightarrows \prod_{(C'',C')} \prod_{(C',C)} F(C)$$

The first map followed by the projection corresponding to $1 \in (C,C)$ is the Yoneda isomorphism. It is a bijective continuous map of compact T_2 spaces, therefore a homeomorphism. Now if we start with a structure as in (3), pass to one of type (1) and back again, we return to the same topologies on the hom sets since $\tilde{\mathfrak{C}}(X,F)$ must be the equalizer in compact T_2 spaces:

$$\tilde{\mathfrak{C}}(X,F) \longrightarrow \prod F(C) \rightrightarrows \prod \prod F(C)$$

where the $F(C)$ carry the topology transported via the Yoneda isomorphism from $\tilde{\mathfrak{C}}(a(C),F)$. That is to say, the topologies on the $\tilde{\mathfrak{C}}(X,F)$ are completely determined by those on the hom sets $\tilde{\mathfrak{C}}(a(C),F)$ and for such hom sets the correspondence set up by the equivalence of (1) and (3) is clearly 1-1 (follows from the arguments at the beginning of this paragraph).

Note Analysis of the fact that the patching condition holds at the level of compact T_2 spaces (see $(1) \Longrightarrow (2)$ in the proof of 2.1.3) shows that for F a compact T_2 sheaf of sets we have the following condition:

$$\forall \text{ covering } \{C_i \longrightarrow C \mid i \in I\}, \ \forall \text{ open } G \subseteq F(C)$$
$$\exists \ i_1,\dots,i_n \text{ and open sets } G_{i_r} \subseteq F(C_{i_r}) \text{ such that } \forall \tau \in F(C),$$
$$\tau \mid C_{i_r} \in G_{i_r} \quad r = 1,\dots,n \text{ iff } \tau \in G.$$

Continuity of restriction would imply simply that we could ensure that restrictions of sections were close at finitely many places by ensuring that the sections were close. What we have here is stronger - we can guarantee that two sections are close by ensuring that their restrictions are close at finitely many places ("continuity of patching").

2.1.4 Definitions A sheaf F of sets with a structure of the type envisioned in 2.1.3 is called a compact T_2 sheaf of sets. A morphism $\phi : F \longrightarrow G$ between two such is said to be continuous if $\phi_C : F(C) \longrightarrow G(C)$ is continuous for each $C \in |\mathbb{C}|$.

2.1.5 Proposition Let F, G be compact T_2 sheaves of sets and $\phi : F \longrightarrow G$ a morphism of sheaves. Then ϕ is continuous iff $(X, \phi) : (X, F) \longrightarrow (X, G)$ is continuous for every $X \in |\tilde{\mathbb{C}}|$.

Proof Suppose (X, ϕ) is continuous for every X. Putting $X = a(C)$, $C \in |\mathbb{C}|$ we have the following commutative diagram

$$\begin{tikzcd}
(a(C), F) \arrow[r] \arrow[d, "{(a(C), \phi)}"'] & F(C) \arrow[d, "\phi_C"] \\
(a(C), G) \arrow[r] & G(C)
\end{tikzcd}$$

The top and bottom arrows are homeomorphisms. The left hand vertical arrow is continuous, hence ϕ_C is continuous.

Suppose now each ϕ_C is continuous and take any $X \in |\tilde{\mathbb{C}}|$, and open $V_{\{C_i, \sigma_i, G_i\}} \subseteq \tilde{\mathbb{C}}(X, G)$

$$\begin{aligned}
(X, \phi)^{-1}(V) &= \{\psi : X \longrightarrow F \mid (\phi\psi)_{C_i}(\sigma_i) \in G_i \quad i = 1, \ldots, n\} \\
&= \{\psi : X \longrightarrow F \mid \psi_{C_i}(\sigma_i) \in \phi^{-1}(G_i) \quad i = 1, \ldots, n\} \\
&= V_{\{C_i, \sigma_i, \phi_{C_i}^{-1}(G_i)\}}
\end{aligned}$$

Hence (X,ϕ) is continuous.

Note 2.1.3 and 2.1.5 combine to give an isomorphism between the categories of objects of $\tilde{\mathfrak{C}}$ equipped with compact T_2 topologies sectionwise and those equipped with such topologies by the lifting of hom functors.

2.1.7 Induced Topologies Projective limits, internal powers, and localizations (as in section 1.2) of compact T_2 sheaves of sets are again compact T_2 sheaves of sets. The structures are deduced from the following isomorphisms:

(1) $\tilde{\mathfrak{C}}(X,\varprojlim F_i) \cong \varprojlim \tilde{\mathfrak{C}}(X,F_i)$

(2) $\tilde{\mathfrak{C}}(X,F^G) \cong \tilde{\mathfrak{C}}(X \times G,F)$

(3) $(\mathfrak{C}/C)^{\sim}(Y,F|C) \cong \tilde{\mathfrak{C}}(j_C^s\ (Y),F)$

Note that it follows with respect to (2) and (3) that

$$\tilde{\mathfrak{C}}(1,F^G) \cong \tilde{\mathfrak{C}}(G,F) \quad \text{and}$$

$$\tilde{\mathfrak{C}}(C,F^G) \cong (\mathfrak{C}/C)^{\sim}(G|C,F|C)$$

are homeomorphisms.

2.1.8 Theorem The following types of structure on $\mathcal{A} \in |M(\mathbb{H},\tilde{\mathfrak{C}})|$ are equivalent in the sense that given one we may deduce any of the others and the passages set up correspondences with are bijective.

(1) $A \in |\tilde{\mathfrak{C}}|$ carries a compact T_2 topology so that is operations are continuous.

(2) $A \in |\tilde{\mathfrak{C}}|$ carries a compact T_2 topology so that for every

$X \in |\tilde{\mathfrak{C}}|$ the operations on the induced algebra $\tilde{\mathfrak{C}}(X,A)$ are continuous.

(3) $A \in |\tilde{\mathfrak{C}}|$ carries a compact T_2 topology so that for every $C \in |\mathfrak{C}|$ the operations on $A(C)$ are continuous.

(4) There is a factorization

$$\begin{tikzcd}
M(\mathbb{H},\tilde{\mathfrak{C}}) \arrow[rr, dashed] \arrow[dr, "(-{,}\mathfrak{a})"'] & & CH \arrow[dl, "U"] \\
& Sets &
\end{tikzcd}$$

Proof (1)$\Longleftrightarrow$(3) is just the definition of continuity for morphisms of sheaves.

To show (1)$\Longleftrightarrow$(2) let f be any n-ary operation of $\mathbb{H}$ and $f_A : A^n \longrightarrow A$ the corresponding operation of $\mathfrak{a}$. Let $f_X : \tilde{\mathfrak{C}}(X,A)^n \longrightarrow \tilde{\mathfrak{C}}(X,A)$ be the corresponding operation on the algebra $\tilde{\mathfrak{C}}(X,A)$. By the way in which the structure of the latter is deduced from that of A, the following commutes:

$$\begin{tikzcd}
\tilde{\mathfrak{C}}(X,A)^n \arrow[rr, "\sim"] \arrow[dr, "f_X"'] & & \tilde{\mathfrak{C}}(X,A^n) \arrow[dl, "\tilde{\mathfrak{C}}(X{,}f_A)"] \\
& \tilde{\mathfrak{C}}(X,A) &
\end{tikzcd}$$

The isomorphism is actually a homeomorphism, by 2.1.7. Thus $\tilde{\mathfrak{C}}(X,f_A)$ is continuous for all X iff f_X is continuous, all X. But in light of 2.1.5 this establishes the equivalence of (1) and (2).

(4)$\Longrightarrow$(1) The isomorphism $(W(X),\mathfrak{a}) = \tilde{\mathfrak{C}}(X,A)$ provides the topology on A and lemma 2.1.2 shows that the operations will be continuous.

(1) $\Longrightarrow$ (4) The following is an equalizer for sheaves of algebras $\mathfrak{B}$ and $\mathfrak{A}$

$$\begin{tikzcd} M(\mathbb{H},\tilde{\mathfrak{C}})(\mathfrak{B},\mathfrak{A}) \arrow[r] & \tilde{\mathfrak{C}}(B,A) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{n=0}^{\infty} \prod_{\mathbb{H}(n,1)} (B^n,A) \end{tikzcd}$$

p sends $\phi : B \longrightarrow A$ to the sequence whose entry at (n,f) for $f : n \longrightarrow 1$ in $\mathbb{H}$ is ϕf_B where f_B is the n-ary operation on B corresponding to f. q sends ϕ to the sequence whose entry is $f_A \phi^n$. Now a topology on A so that the operations are continuous induces topologies on $\tilde{\mathfrak{C}}(B,A)$ and $\tilde{\mathfrak{C}}(B^n,A)$ so that p and q are continuous. Thus $(\mathfrak{B},\mathfrak{A})$, being the equalizer of a pair of maps between compact T_2 spaces, inherits a compact T_2 structure as a subspace of $\tilde{\mathfrak{C}}(B,A)$. This is a weak topology determined by the evaluations $(\mathfrak{B},\mathfrak{A}) \longrightarrow (W(C),\mathfrak{A}) \cong A(C)$ which send ϕ to $\phi_C(\sigma)$ for $C \in |\mathfrak{C}|$, $\sigma \in B(C)$.

Thus the situation with algebras is analogous to that of 2.1.3 provided we replace C by $W(C)$. The remaining details of the proof of 2.1.8 proceed accordingly.

2.2 Topological Compactness Implies Equational Compactness

2.2.1 We now take up the discussion of 1.1.7. Recall the definition of solution given in 1.1.6.

$$\begin{tikzcd}\Upsilon_\Sigma \arrow[r] & B^X \arrow[r, "\lambda"] & B^{A[X]} \arrow[r, shift left] \arrow[r, shift right] & B^\Sigma\end{tikzcd}$$

For the purposes of this discussion, we assume $\mathcal{A} = W(0)$, i.e. $A[X] = UW(X)$. That is, we do not consider coefficients in our equations. This assumption is not necessary, but it is the usual framework in which equational compactness is discussed. We write $W(X) = UW(X)$ where no confusion can arise, using the same notation for the free algebra and its carrier. Rewriting the above equalizer diagram:

$$\begin{tikzcd}\Upsilon_\Sigma \arrow[r] & B^X \arrow[r, "\lambda"] & B^{W(X)} \arrow[r, shift left, "B^{q_1}"] \arrow[r, shift right, "B^{q_2}"'] & B^\Sigma\end{tikzcd}$$

If $\mathfrak{B}$ carries a compact T_2 topology, so do all the powers of B in the diagram. B^{q_1} and B^{q_2} are continuous (as can easily be seen by applying an arbitrary hom functor $\tilde{\mathfrak{C}}(T,-)$). If λ can be shown to be continuous, then the application of any hom functor $\tilde{\mathfrak{C}}(T,-)$ to the underlying diagram in $\tilde{\mathfrak{C}}$ will yield an equalizer diagram in compact Hausdorff spaces (not in compact Hausdorff algebras, for we are certainly not claiming λ is an algebra homomorphism - it is not). In particular $\tilde{\mathfrak{C}}(T, \Upsilon_\Sigma)$ will always be a closed subspace of $\tilde{\mathfrak{C}}(T,B^X)$ and we shall have proved the following:

2.2.2 <u>Theorem</u> Let $\mathfrak{B}$ be a sheaf of compact Hausdorff algebras. Then any sheaf of equations Σ in variables X which is finitely solvable over a particular $T \in |\tilde{\mathfrak{C}}|$ is solvable over T. In particular if the equations $\Sigma(C)$ are finitely solvable in $\mathfrak{B}(C)$, they are solvable there (taking $T = C \in |\tilde{\mathfrak{C}}|$).

Proof The idea of the proof has been outlined in 1.1.7 and 2.2.1 above. All that remains is to prove λ is continuous. From 2.1.1 it is enough to show λ is continuous at each $C \in |\tilde{\mathfrak{C}}|$. To establish this we show λ is continuous at the level of global sections and then appeal to localization techniques to obtain the statement for arbitrary $C \in |\tilde{\mathfrak{C}}|$.

We must show $\tilde{\mathfrak{C}}(1,\lambda)$ is continuous. We have

$$\begin{tikzcd}
\tilde{\mathfrak{C}}(1,B^X) \arrow[r] \arrow[d, "{(1,\lambda)}"'] & \tilde{\mathfrak{C}}(X,B) \arrow[d] \\
\tilde{\mathfrak{C}}(1,B^{W(X)}) \arrow[r] & \tilde{\mathfrak{C}}(W(X),B)
\end{tikzcd}$$

The diagram commutes. The horizontal arrows are homeomorphisms and the right-hand vertical arrow sends $X \longrightarrow B$ to the (underlying map of the) free extension to $W(X) \longrightarrow \mathfrak{B}$. (see 1.1.6). If we show the right-hand arrow is continuous, we are finished. To this end take $\phi : X \longrightarrow B$ and a basic neighbourhood $V_{\{C_i, \sigma_i, G_i\}}$ of the extension $\phi : W(X) \longrightarrow B$. We have then that $\sigma_i \in W(X)(C_i)$ and $\phi_{C_i}(\sigma_i) \in G_i$ $i = 1,\ldots,n$. Let us "take apart" the σ_i. For each $i = 1,\ldots,n$ there is a covering $\{C_{ij} \longrightarrow C_i \mid j \in J_i\}$ with $\sigma_i | C_{ij} = w_{ij} \in \hat{W}(X)(C_{ij}) = |W|(X(C_{ij}))$. i.e. w_{ij} is a "real" polynomial in a "real" free algebra. This comes from the way $W(X)$ is built as the associated sheaf to the presheaf $\widehat{W}(X)$ which in turn is just the "pointwise free algebra" functor. We can suppose w_{ij} depends on variables $x_1^{ij},\ldots x_{n_{ij}}^{ij} \in X(C_{ij})$. Next, by continuity of patching choose $G'_{ij} \subseteq B(C_{ij})$ so that $\forall \tau \in B(C_i)$, $\tau|C_{ij} \in G'_{ij}$ all $j \in J_i$ iff $\tau \in G_i$. Recall that this can be done so that for all $j \in J_i - F_i$ where $F_i \subseteq J_i$ is finite, $G'_{ij} = B(C_{ij})$.

The polynomials $w_{ij} \in |W|(X(C_{ij}))$ determine mappings $\tilde{w}_{ij}: B(C_{ij})^{n_{ij}} \longrightarrow B(C_{ij})$ by evaluation. Since $B(C_{ij})$ is a topological algebra all of these mappings are continuous. Moreover $\tilde{w}_{ij}(\phi_{C_{ij}}(x_1^{ij}),\ldots, \phi_{C_{ij}}(x_{n_{ij}}^{ij})) = \bar{\Phi}_{C_{ij}}(w_{ij}) \in G'_{ij}$. The first equality comes from the way that $\bar{\Phi}$ acts - as the extension of ϕ to the free sheaf of algebras, locally it acts like a free extension in classical universal algebra. $\bar{\Phi}_{C_{ij}}(w_{ij}) \in G'_{ij}$ since $\bar{\Phi}_{C_{ij}}(w_{ij}) = \bar{\Phi}_{C_i}(\sigma_i)|C_{ij}$ and $\bar{\Phi}_{C_i}(\sigma_i) \in G_i$ (recall how the G'_{ij} were chosen).

Having established that $\tilde{w}_{ij}(\phi_{C_{ij}}(x_1^{ij}),\ldots) \in G'_{ij}$ use the continuity of $\tilde{w}_{ij}$ to select open sets $G''_{ijk} \subseteq B(C_{ij})$, $k = 1,\ldots,n_{ij}$ with $\phi_{C_{ij}}(x_k^{ij}) \in G''_{ijk}$ $k = 1,\ldots,n_{ij}$ and $\tilde{w}_{ij}(\prod_{k=1}^{n_{ij}} G''_{ijk}) \subseteq G'_{ij}$.

Claim: $V_{\{C_{ij},x_k^{ij},G''_{ijk}\}\, k=1,\ldots,n_{ij}}$ is a neighbourhood of ϕ which is mapped into $V_{\{C_i, \sigma_i, G_i\}}$ by the extension mapping $\tilde{\mathfrak{C}}(X,B)\longrightarrow\tilde{\mathfrak{C}}(W(X),B)$. Certainly ϕ is in $V_{\{C_{ij}, x_k^{ij}, G''_{ijk}\}_k}$ by the way the G''_{ijk} were chosen.

Take any ψ in this neighbourhood of ϕ. $\bar{\psi}_{C_i}(\sigma_i)|C_{ij} =$ $\bar{\psi}_{C_{ij}}(\sigma_i|C_{ij}) = \bar{\psi}_{C_{ij}}(w_{ij}) = \tilde{w}_{ij}(\psi_{C_{ij}}(x_1^{ij}),\ldots,\psi_{C_{ij}}(x_{n_{ij}}^{ij})) \in G'_{ij}$, since $\psi_{C_{ij}}(x_k^{ij}) \in G''_{ijk}$ because of the fact that $\psi \in V_{\{C_{ij}, x_k^{ij}, G''_{ijk}\}}$ From the way in which the G'_{ij} were chosen, it follows that $\bar{\psi}_{C_i}(\sigma_i) \in G_i$, hence $\bar{\psi} \in V_{\{C_i, \sigma_i, G_i\}}$, as was to be shown.

We have established now that $\tilde{\mathfrak{C}}(1,\lambda)$ is continuous. We must show $\tilde{\mathfrak{C}}(C,\lambda)$ is continuous for each $C \in |\mathfrak{C}|$. Consider the commutative diagram

$$\begin{tikzcd}
\tilde{\mathfrak{C}}(C,B^X) \arrow[r, "\sim"] \arrow[d, "{(C,\lambda)}"'] & (\mathfrak{C}/C)^\sim(1,B|C^{X|C}) \arrow[d, "{(1,\lambda|C)}"] \\
\tilde{\mathfrak{C}}(C,B^{W(X)}) \arrow[r, "\sim"] & (\mathfrak{C}/C)^\sim(1,B|C^{W(X|C)})
\end{tikzcd}$$

The horizontal maps are homeomorphisms. The map $\lambda|C$ is, as pointed out in 1.2.7, the extension map for the compact algebra $B|C$ in $M(\mathfrak{H},(\mathfrak{C}/C)^\sim)$. Thus by applying the above global result to the algebra localized at $\mathfrak{C}/C$ we see $(1,\lambda|C)$ is continuous and hence (C,λ) is continuous, completing the proof.

3 Homological and Equational Purity for Modules

3.1 Sheaves of Modules on a Ringed Space

3.1.1 A ringed space $(T,\mathcal{O})$ is a topological space T together with a sheaf of rings $\mathcal{O}$ defined on it. For example the spectrum of a commutative ring with unit has such a sheaf defined on it via (ring-theoretic) localization, and in fact these sheaves are the basis of modern Algebraic Geometry. We shall refer to a ringed space $(T,\mathcal{O})$ simply by $\mathcal{O}$ with the space of definition T understood. A (left) module over $\mathcal{O}$ is a sheaf $\mathcal{A}$ of abelian groups together with an action $\mathcal{O} \times \mathcal{A} \longrightarrow \mathcal{A}$ satisfying the usual module laws. Equivalently $\mathcal{O}(U) \times \mathcal{A}(U) \longrightarrow \mathcal{A}(U)$ is a module action for each $U \in |\mathrm{Open}(T)|$ and restriction (along the ring and group simultaneously) respects the module structure. The resulting category we denote $\mathcal{O}$-Mod. If $\mathcal{O}$ is the sheaf of rings associated to the presheaf constant at a fixed ring R, $\mathcal{O}$-Mod can be seen to be equivalent to the category of sheaves of R-modules, or what is the same thing, $M(\mathbb{H},\tilde{T})$ where $\mathbb{H}$ is the theory of R-modules treated as universal algebras (recall that this involves treating the ring action on the group as a family of unary operations indexed by R). However if $\mathcal{O}$ is more complicated than simply a "constant sheaf" the category $\mathcal{O}$-Mod need not be representable as $M(\mathbb{H},\tilde{T})$ for a finitary theory $\mathbb{H}$. The reason is that because the rings are allowed to vary, the theories corresponding to the various types of modules vary accordingly. To subsume the categories $\mathcal{O}$-Mod in the scheme developed to this point we would have to expand our treatment in one of two ways: (1) treat n-sorted theories instead of 1-sorted theories as we have done, or (2) study algebras defined over a sheaf of theories. It is not our intention to carry out either of these possible extensions here. Instead we shall merely point out that for a fixed ringed space (T,Θ), all of what we have said to this point in general about categories $M(\mathbb{H},\tilde{\mathfrak{T}})$ carries over mutatis mutandis to Θ-Mod.

3.1.2 <u>Facts</u>

(1) For $U \subseteq T$ open, the functor "restriction to U" on Θ-Mod has a left adjoint, "extension by 0". The formula for this is

$$j_U^s(\mathfrak{a})(V) = \begin{cases} 0 & V \nsubseteq U \\ \mathfrak{a}(V) & V \subseteq U \end{cases}$$

We will use the notation $\mathfrak{a}|U$ to refer to both the restriction of $\mathfrak{a} \in \Theta$-Mod to an algebra in $\Theta|U$-Mod, and to the restriction of $\mathfrak{a}$ to U, followed by the extension by 0. Thus $\mathfrak{a}|U$ refers to an algebra either in Θ-Mod or $\Theta|U$-Mod. In what follows, unless otherwise stated we assume that $\mathfrak{a}|U \in \Theta$-Mod, that is, $\mathfrak{a}|U$ is $\mathfrak{a}$ "cut down to U and extended by zero".

(2) For each $X \in |\tilde{T}|$ there is an Θ-module $W(X)$ freely generated by X. For $X = U$, $U \subseteq T$ open, we have $W(U) = \Theta|U$ i.e.

$$W(U)(V) = \begin{cases} 0 & V \nsubseteq U \\ \Theta(V) & V \subseteq U \end{cases}$$

(3) Θ-Mod is an AB5 abelian category.

(4) Every sheaf of modules is the colimit over a directed set of finitely presented ones. Recall finitely presented means cokernel of a map between finite sums of modules of the type $\Theta|U$.

(5) A tensor product on Θ^{op}-Mod × Θ-Mod is defined by forming, for any two arguments, their pointwise tensor product as presheaves, then taking the associated sheaf. For any right module $\mathcal{A}$, we have $\mathcal{A} \otimes \Theta|U = \mathcal{A}|U$.

(6) The additivity of Θ-Mod results in the fact that we can reduce equations to polynomials by "moving the variables to one side". That is, any system of equations $\Sigma \rightarrowtail \mathcal{A}[X]^2$ is equivalent to one of the form $\Sigma \rightarrowtail \mathcal{A}[X]^2$ where the first component is 0 and therefore is completely specified by a single map $\Sigma \longrightarrow \mathcal{A}[X]$. A global solution in an algebra $\mathcal{B} \supseteq \mathcal{A}$ is a map $X \longrightarrow B$ whose free extension to $\mathcal{A}[X]$ is 0 on Σ. A solution over $U \in |\mathrm{Open}(T)|$ is a map $X|U \longrightarrow B|U$ which is a solution to the restricted system $\Sigma|U \longrightarrow \mathcal{A}|U[X|U]$.

(7) If $X = U_1 + \ldots + U_n$, the coproduct of generators $U_i \in \tilde{T}$, $U_i \in \mathrm{Open}(T)$, $\mathcal{A}[X] = \mathcal{A} \oplus W(X) = \mathcal{A} \oplus W(U_1) \oplus \ldots \oplus W(U_n) = \mathcal{A} \oplus \Theta|U_1 \oplus \ldots \oplus \Theta|U_n$.

3.2 Equivalence of Homological and Equational Purity

3.2.1 The first order of business here is to prove a proposition which will lead to the notion of "affine purity". This concept will then be shown to be equivalent to homological purity in Θ-Mod. The equations we consider will have variables in a sheaf X which is restricted to be a finite coproduct of generators $X = U_1 + \ldots + U_n$ $U_i \subseteq T$ open. Since we deal always with finite sheaves of equations the finiteness of the coproduct is no real restriction. Requiring that X be a coproduct of generators does of course represent a very real restriction over the more general case of X an arbitrary (though finite, say) sheaf of sets. We will be interested in finite systems of equations (polynomials) $\Sigma \longrightarrow \mathcal{A}[X]$. Since Σ is a quotient $V_1 + \dots + V_m \twoheadrightarrow \Sigma$ we may at times consider the map $V_1 + \dots + V_m \longrightarrow \mathcal{A}[X]$ as representing the equations, and forget about speaking of its image (Σ) explicitly. We make the requirement that $V_1 = \dots = V_m = W$, say. That is, the equations in our system will be defined over the same open set W. Finally, if the equations in a system are defined over W, we look for solutions over W, and not over possibly larger open sets.

3.2.2 Definition Quite in general for $\mathcal{A} \rightarrowtail \mathcal{B}$ a subalgebra in $M(\mathbb{H},\tilde{\mathbb{C}})$ and $\Sigma \subseteq \mathcal{A}[X]^2$ a system of equations in variables X (any sheaf of sets) with coefficients in $\mathcal{A}$, we say Σ is locally solvable over $C \in |\mathbb{C}|$ if there is a covering $\{C_i \longrightarrow C \mid i \in I\}$ of C such that $\Sigma | C_i$ is globally solvable for each $i \in I$. Note that the solutions exhibited need not be in any sense compatible with respect to the cover. Also note that if Σ is solvable over C, it is automatically locally solvable over any cover of C (just restrict the solution to the objects of the cover).

Let us return now to the situation of 3.2.1.

3.2.3 Proposition Let $\mathcal{A} \rightarrowtail \mathcal{B}$ be a submodule ($\mathcal{A}$, $\mathcal{B}$ left Θ-modules). Then the following are equivalent:

(1) $\forall$ sheaf of variables X as in 3.2.1, $\forall$ open $W \subseteq T$,

$\forall$ system of equations $\coprod_{i=1}^{m} W \longrightarrow \mathcal{A}[X]$ with image $\Sigma \subseteq \mathcal{A}[X]$, [Σ solvable over W in $\mathcal{B} \Longrightarrow \Sigma$ is locally solvable over W in $\mathcal{A}$].

(2) Same hypotheses as (1), then [Σ locally solvable over W in $\mathcal{B} \Longrightarrow \Sigma$ locally solvable over W in $\mathcal{A}$].

(3) Hypotheses of (1) and $\forall$ $V \subseteq W$ open, then [Σ solvable over V in $\mathcal{B} \Longrightarrow \Sigma$ solvable locally over V in $\mathcal{A}$].

(4) Same hypotheses as (3), then [Σ locally solvable over V in $\mathcal{B} \Longrightarrow \Sigma$ locally solvable over V in $\mathcal{A}$].

Proof (4) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1) are obvious weakenings. (3) $\Rightarrow$ (2) Suppose W has a covering $W = \bigcup\{W_k | k \in K\}$ so that Σ is solvable in $\mathcal{A}$ over each W_k. Then applying (3) with $V = W_k$, we see there exists a covering $W_k = \bigcup\{W_{k\ell} | \ell \in L_k\}$ so that Σ is solvable in $\mathcal{A}$ over $W_{k\ell}$ for each $\ell \in L_k$. Then Σ is solvable in $\mathcal{A}$ over the composite covering $W = \bigcup\{W_{k\ell} | k \in K, \ell \in L_k\}$ and (2) is established. (1) $\Rightarrow$ (4) Given the primary hypothesis of (4), then to say Σ is solvable in $\mathcal{B}$ over V is equivalent to saying that the restriction of Σ to V is solvable in $\mathcal{B}$ over V. But this is now the situation of (1) with W = V and the result follows.

Note The proof that (1) $\Rightarrow$ (4) is somewhat deceptive in that it looks totally trivial. However, hidden within it is the fact, used implicitly, that the image of $(\amalg V \subseteq \amalg W \longrightarrow \mathcal{A}[X])$ is $\Sigma | V$ which comes from the fact that restriction to an open set preserves image factorizations. The reason for introducing (4), and its companion (3), is that solutions are guaranteed from hypothetical solutions defined over open sets V possibly smaller than W (over which the equations are defined).

3.2.4 Definition If any (and hence all) of the conditions (1) - (4) of 3.2.3 holds, we say $\mathcal{A}$ is an affine-pure submodule of $\mathcal{B}$.

The reason for the prefix "affine" is the third condition of the following theorem.

3.2.5 Theorem Let $\mathcal{A} \rightarrowtail \mathcal{B}$ be a monomorphism of left Θ-modules. Then the following are equivalent:

(1) $\mathcal{A}$ is an affine-pure submodule of $\mathcal{B}$.

(2) For any right Θ-module $\mathcal{C}$, $\mathcal{C} \otimes \mathcal{A} \longrightarrow \mathcal{C} \otimes \mathcal{B}$ is a monomorphism, i.e. $\mathcal{A}$ is a homologically pure submodule of $\mathcal{B}$.

(3) For any module homomorphism $\phi : \bigoplus_{i=1}^{n} \Theta|U_i \longrightarrow \bigoplus_{j=1}^{n} \Theta|V_j$ where all U_i, V_j are open sets in T, in the diagram

$$\begin{tikzcd}
\bigoplus_{i=1}^{n} \mathcal{A}|U_i \arrow[r, "\phi\otimes\mathcal{A}"] \arrow[d] & \bigoplus_{j=1}^{m} \mathcal{A}|V_j \arrow[d] \\
\bigoplus_{i=1}^{n} \mathcal{B}|U_i \arrow[r, "\phi\otimes\mathcal{B}"'] & \bigoplus_{j=1}^{m} \mathcal{B}|V_j
\end{tikzcd}$$

we have $\mathrm{Image}(\phi\otimes\mathcal{B}) \cap \bigoplus_{j=1}^{m} \mathcal{A}|V_j \subseteq \mathrm{Image}(\phi\otimes\mathcal{A})$

(4) As in (3) except $U_i = V_j = W$ all i,j.

Note in connection with (3) that $(\bigoplus_{i=1}^{n} \Theta|U_i)\otimes\mathfrak{a} = \bigoplus_{i=1}^{n}(\Theta|U_i\otimes\mathfrak{a}) = \bigoplus_{i=1}^{n}\mathfrak{a}|U_i$. $\phi_{ij}:\Theta|U_i \longrightarrow \Theta|V_j$ and hence is represented by an element of $(\Theta|V_j)(U_i)$, i.e. an element of the ring $\Theta(U_i)$, which element must be 0 if $U_i \nsubseteq V_j$. With this sort of identification, $\phi\otimes\mathfrak{a}$ may be described in terms of matrix multiplication with the matrix (ϕ_{ij}). This will become clearer in the proof of the theorem. Before proceeding with this proof, we need a lemma about abelian categories.

3.2.6 Lemma Let $\mathcal{A}$ be any abelian category and consider the following commutative diagram in $\mathcal{A}$:

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d, tail, "a"'] & B \arrow[r, "v"] \arrow[d, tail, "b"'] & C \arrow[d, "c"'] \\
A' \arrow[r, "\phi'"'] & B' \arrow[r, "v'"'] & C'
\end{tikzcd}$$

We suppose $v = \operatorname{coker}(\phi)$, $v' = \operatorname{coker}(\phi')$ and a and b are monomorphisms. Then c is a monomorphism iff $\operatorname{Image}(\phi') \cap B \subseteq \operatorname{Image}(\phi)$. Note that the reverse inclusion is always true simply by virtue of the commutativity of the left-hand square.

Proof Diagram chasing shows easily that the theorem is true in any category of modules, hence it is true in any abelian category.

3.2.7 Proof of Theorem 3.2.5 First of all (3) is equivalent to (2) for $\mathcal{C}$ a finitely presented module. For by definition such $\mathcal{C}$ are exactly the cokernels of such ϕ as appear in the statement of (3).

ϕ, $\mathcal{C}$ and $\mathcal{A} \rightarrowtail \mathcal{B}$ give rise to the following diagram, by tensoring the coequalizer diagram defining $\mathcal{C}$ with $\mathcal{A}$ and $\mathcal{B}$.

$$\begin{tikzcd}
\oplus \mathcal{A}|U_i \arrow[r, "\phi\otimes\mathcal{A}"] \arrow[d] & \oplus \mathcal{A}|V_j \arrow[r, "v\otimes\mathcal{A}"] \arrow[d] & \mathcal{C}\otimes\mathcal{A} \arrow[d] \\
\oplus \mathcal{B}|U_i \arrow[r, "\phi\otimes\mathcal{B}"] & \oplus \mathcal{B}|V_j \arrow[r, "v\otimes\mathcal{B}"] & \mathcal{C}\otimes\mathcal{B}
\end{tikzcd}$$

Assume for the moment no special properties of ϕ or the inclusion $\mathcal{A} \rightarrowtail \mathcal{B}$ The top and bottom rows are cokernels since $(\ )\otimes ?$ preserves cokernels (in fact all right limits). The left and middle vertical arrows are mono since $\mathcal{A} \rightarrowtail \mathcal{B}$ mono $\Rightarrow$ $\mathcal{A}|U_i \longrightarrow \mathcal{B}|U_i$ is mono and in an abelian category finite sums of monomorphisms are mono (since finite sums are products as well). Thus the fact that (3) is equivalent to (2) for finitely presented $\mathcal{C}$ is an immediate consequence of lemma 3.2.6.

At this point it is clear that (2) $\Rightarrow$ (3). On the other hand (3) $\Rightarrow$ (2) since every Θ-module $\mathcal{C}$ is the colimit over an up-directed set of finitely presented Θ-modules, say $\mathcal{C} = \varinjlim \mathcal{C}_i$. Since $\otimes$ preserves $\varinjlim$'s we have

$$\mathcal{C}\otimes\mathcal{A} \longrightarrow \mathcal{C}\otimes\mathcal{B} = (\varinjlim \mathcal{C}_i)\otimes\mathcal{A} \longrightarrow (\varinjlim \mathcal{C}_i)\otimes\mathcal{B}$$

$$= \varinjlim(\mathcal{C}_i\otimes\mathcal{A} \longrightarrow \mathcal{C}_i\otimes\mathcal{B})$$

By hypothesis (2) each of the maps $\mathcal{C}_i\otimes\mathcal{A} \longrightarrow \mathcal{C}_i\otimes\mathcal{B}$ is mono and since Θ-Mod is an AB5 category, $\varinjlim$'s over up-directed posets of monomorphisms are monomorphisms. Hence $\mathcal{C}\otimes\mathcal{A} \longrightarrow \mathcal{C}\otimes\mathcal{B}$ is mono.

Now $(3) \Rightarrow (4)$ is a weakening so at this point we have established $(2) \Leftrightarrow (3) \Rightarrow (4)$. We shall complete the chain of implications by showing $(4) \Rightarrow (1)$ and $(1) \Rightarrow (3)$.

$(4) \Rightarrow (1)$ Suppose we have a system of equations

$$\coprod_{j=1}^{m} W \xrightarrow{\phi} \mathfrak{A} \oplus \Theta|U_1 \oplus \dots \oplus \Theta|U_n = \mathfrak{A}[X]$$

with a solution in $\mathcal{B}$ over W, say $X|W = (\coprod U_i)|W \xrightarrow{f} \mathcal{B}|W$ with $\coprod_{j=1}^{m} W \longrightarrow \mathfrak{A}|W\ [X|W] \xrightarrow{\bar{f}} \mathcal{B}|W = 0.$

We can assume $W \subseteq U_i$ all $i = 1,\dots,n$ for if $W \nsubseteq U_i$ some i then $\phi_j : W \longrightarrow \mathfrak{A} \oplus \Theta|U_1 \oplus \dots \oplus \Theta|U_n$ has, when considered as a section over W of the codomain sheaf, 0 for its component at the "$\Theta|U_i$-place". Here ϕ_j is ϕ preceded by the jth injection of the coproduct $\coprod W$. The ϕ_j represent the individual equations of the system ϕ. It follows that $W \nsubseteq U_i \Rightarrow$ each ϕ_j does not depend on the "$\Theta|U_i$-place", that is, the assignment of the corresponding variable does not affect the polynomial since the variable is multiplied by 0. For the purposes of finding solutions we can ignore such cases, for given a solution to ϕ cut down to the components where not all ϕ_j are 0, we can expand the solution (by assigning 0 to the extra variables) to a solution for ϕ.

$W \subseteq U_i$ all $i \Rightarrow (\coprod_{i=1}^{n} U_i)|W = \coprod_{i=1}^{n} W$ and the condition that f be a solution in $\mathcal{B}$ may be written

$$\coprod_{j=1}^{m} W \longrightarrow \mathcal{A}|W \oplus \Theta|W \oplus \dots \oplus \Theta|W \xrightarrow{\bar{f}} \mathcal{B}|W = 0$$

$\bar{f}$ restricted to $\bigoplus_{i=1}^{n} \Theta | W$ corresponds to sections $b_i \in \mathcal{B}(W)$ since $\Theta|W$ are free modules on $W \in |\tilde{T}|$. $\phi_j : W \longrightarrow \mathcal{A}|W \oplus \Theta|W \oplus \dots \oplus \Theta|W$ represents a section over W of the codomain, say $(-\alpha_j, \phi_{1j}, \dots, \phi_{nj})$ with $\alpha_j \in \mathcal{A}(W)$ and $\phi_{ij} \in \Theta(W)$ $i = 1,\dots,n$. We use $-\alpha_j$ for convenience in formulas to be produced. The fact that f is a solution may be expressed by the equations

$$\alpha_j = \sum_{i=1}^{n} \phi_{ij}\, b_i \qquad \text{(equations in the module } \mathcal{B}(W))$$

Now the ϕ_{ij} define a map $\hat{\phi} : \bigoplus_{i=1}^{n} \Theta|W \longrightarrow \bigoplus_{j=1}^{m} \Theta|W$ and in fact $\hat{\phi}$ operates in such a way that $(\alpha_1,\dots,\alpha_m) = \hat{\phi} \otimes \mathcal{B}\,(b_1,\dots,b_n)$. Refer to the diagram of (3) with all $U_i = V_j = W$. It follows that $(\alpha_i,\dots,\alpha_m)$ is a section over W of $\text{Image}(\hat{\phi} \otimes \mathcal{B}) \cap \bigoplus_{j=1}^{m} \mathcal{A}|W$. By (4) we must have that $(\alpha_1,\dots,\alpha_m)$ is a section over W of $\text{Image}(\hat{\phi} \otimes \mathcal{A})$. That is, there is a covering $W = \bigcup\{W_k | k \in K\}$ and $\alpha^k = (\alpha_1^k,\dots,\alpha_n^k)$ sections over W_k of $\bigoplus_{i=1}^{n} \mathcal{A}|W$ such that $\alpha|W_k = \hat{\phi} \otimes \mathcal{A}(\alpha^k)$. But $\hat{\phi} \otimes \mathcal{A}(\alpha^k) = (\sum_{i=1}^{n} \phi_{i1}\, \alpha_i^k, \dots, \sum_{i=1}^{n} \phi_{im}\, \alpha_i^k)$, which says that the maps $\begin{pmatrix}\alpha_1^k \\ \vdots \\ \alpha_n^k\end{pmatrix} : \coprod_{i=1}^{n} W_k \longrightarrow \mathcal{A}$ provide solutions to ϕ in $\mathcal{A}$ over the W_k, as required.

(1) $\Rightarrow$ (3) Given the hypothesis of condition (3) we show the inclusion pointwise. Let $W \subseteq T$ be any open set and

$$\alpha = (\alpha_1,\ldots,\alpha_m) \in (\mathrm{Image}(\phi\otimes\mathcal{B}) \cap \bigoplus_{j=1}^{m} \mathcal{A}|V_j)(W)$$

$$= \mathrm{Image}(\phi\otimes\mathcal{B})(W) \cap \bigoplus_{j=1}^{m} \mathcal{A}|V_j(W)$$

Let us first assume that α is in the set-theoretical image of $\phi\otimes\mathcal{B}$. From this case we will later deduce the more general situation which is of course when α is locally (over W) in the set-theoretic image of $\phi\otimes\mathcal{B}$

The given morphism ϕ has components $\phi_{ij}:\mathcal{O}|U_i \longrightarrow \mathcal{O}|V_j$. We are restricting attention to the open set W. Now $(\mathcal{O}|U_i)(W) = 0$ unless $W \subseteq U_i$. ϕ_{ij} represents, by freeness of $\mathcal{O}|U_i$, a section of $\mathcal{O}|V_j$ over U_i and hence is 0 unless $U_i \subseteq V_j$. The point of these observations is that $(\phi_{ij})_W$, the component of the morphism ϕ_{ij} of sheaves of modules at the open set W, is 0 unless $W \subseteq U_i \subseteq V_j$. Without loss of generality we assume the indices of the U_i, V_j are ordered so that

$$W \subseteq U_1 \cap \ldots \cap U_{n'} \ , \ W \nsubseteq U_{n'+1},\ldots,U_n$$

$$W \subseteq V_1 \cap \ldots \cap V_{m'} \ , \ W \nsubseteq V_{m'+1},\ldots,V_m$$

Since $\alpha_{m'+1},\ldots,\alpha_m$ are elements of $(\mathcal{A}|V_{m'+1})(W)$,..., $(\mathcal{A}|V_m)(W)$ it follows that $\alpha_{m'+1} = \ldots = \alpha_m = 0$.

We are assuming α is in the set-theoretical image of $(\phi\otimes\mathcal{B})_W$, say $\alpha = \phi\otimes\mathcal{B}(b_1,\ldots,b_n)$ where $b_i \in (\mathcal{B}|U_i)(W)$. By the numbering of U_i, $(\mathcal{B}|U_i)(W) = 0$ for $i = n'+1,\ldots,n$ hence $b_{n'+1} = \ldots = b_n = 0$. Moreover $b_1,\ldots,b_{n'}$ are all elements of $\mathcal{B}(W)$ since $(\mathcal{B}|U_i)(W) = \mathcal{B}(W)$

for $W \subseteq U_i$. In the same way $\alpha_1,\ldots,\alpha_{m'}$, are all elements of $\mathcal{A}(W) \subseteq \mathcal{B}(W)$ and the relation $\alpha = \phi \otimes \mathcal{B}(b_1,\ldots,b_n)$ is equivalent to $\alpha_j = \sum_{i=1}^{n'} \psi_{ij}\, b_i \quad j = 1,\ldots,m'$ (equations in $\mathcal{B}(W)$) where $\psi_{ij} \in \theta(W)$ and $\psi_{ij} = \begin{cases} (\phi_{ij})_W & U_i \subseteq V_j \\ 0 & U_i \nsubseteq V_j \end{cases}$. When we say $\psi_{ij} = (\phi_{ij})_W$ we mean of course that ψ_{ij} is the element of $\theta(W)$ corresponding to the module map

$$(\phi_{ij})_W : (\theta|U_i)(W) \longrightarrow (\theta|V_j)(W) = \theta(W)$$

The ψ_{ij} yield a system of equations

$$\coprod_{j=1}^{m'} W \xrightarrow{\psi} \mathcal{A}|W \oplus \underbrace{\theta|W \oplus \ldots \oplus \theta|W}_{n' \text{ times}}$$

The component $\psi_j : W \longrightarrow \mathcal{A}|W \oplus \theta|W \oplus \ldots \oplus \theta|W$ corresponds to the section $(-\alpha_j, \psi_{1j}, \ldots, \psi_{n'j})$. Note that the sheaf of variables in which the system ψ is defined is $X' = \coprod_{i=1}^{n'} W$. The equations $\alpha_j = \sum_{i=1}^{n'} \psi_{ij}\, b_i$ say exactly that

$$\begin{pmatrix} b_1 \\ \vdots \\ b_{n'} \end{pmatrix} : \coprod_{i=1}^{n'} W \longrightarrow \mathcal{B}|W$$

is a solution for ψ over W. Our hypothesis of affine purity may now be invoked to yield a local solution in $\mathcal{A}$ to ψ, over some covering $W = \bigcup\{W_k | k \in K\}$. i.e. for each $k \in K$ we have $\alpha_1^k,\ldots,\alpha_{n'}^k \in \mathcal{A}(W_k)$ with $\alpha_j|W_k = \sum_{i=1}^{n'} \psi_{ij}|W_k . \alpha_i^k$ for $j = 1,\ldots,m'$.

Claim: $$\Big(\sum_{i=1}^{n'} \psi_{i1}|W_k.\alpha_i^k,\ldots,\sum_{i=1}^{n'} \psi_{im'}|W_k.\alpha_i^k,\underbrace{0,\ldots,0}_{m-m' \text{ times}}\Big)$$

$$= \phi\otimes\mathcal{a}(\alpha_1^k,\ldots,\alpha_{n'}^k,\underbrace{0,\ldots,0}_{n-n' \text{ times}})$$

First of all $\sum_{i=1}^{n} \phi_{ij}|W_k.\alpha_i^k = \sum_{i=1}^{n'} \phi_{ij}|W_k.\alpha_i^k$ assuming we have extended the definition of α_i^k so that $\alpha_i^k = 0$ for $n' < i \leq n$. Hence the equality claimed certainly holds in the first m' components. We must show that for $j > m'$

$$\sum_{i=1}^{n} \phi_{ij}|W_k.\alpha_i^k = 0$$

But if some $\phi_{ij}|W_k.\alpha_i^k \neq 0$ for $j > m'$,

(1) $i \leq n'$ since $\alpha_i^k = 0$ for $i > n'$

(2) $U_i \subseteq V_j$ since $\phi_{ij} \neq 0$

Now (1) $\Rightarrow W \subseteq U_i$ by our numbering of the U_i and with (2) we have $W \subseteq U_i \subseteq V_j$ i.e. $W \subseteq V_j$ which means $j \leq m'$, a contradiction. Thus we have shown that for $j > m'$, $\sum_{i=1}^{n} \phi_{ij}|W_k.\alpha_i^k = 0$, in fact we have shown each summand to be 0. This completes the proof of the claim, but the claim together with what immediately preceded it combine to give

$$(\alpha_1|W_k,\ldots,\alpha_{m'}|W_k,0,\ldots,0) = \phi\otimes\mathcal{a}(\alpha_1^k,\ldots,\alpha_{n'}^k,0,\ldots,0)$$

In vector notation $\alpha|W_k = \phi\otimes\mathcal{a}(\alpha^k)$. (Recall $\alpha_j = 0$ for $m' < j \leq m$)

We have shown that if α is in the set-theoretical image of $\phi\otimes\mathcal{B}$, then it is locally in the set-theoretical image of $\phi\otimes\mathcal{A}$ i.e. it is a section of the sheaf-theoretic image of $\phi\otimes\mathcal{A}$. Now let us consider the general case where α is a section of the sheaf-theoretic image of $\phi\otimes\mathcal{B}$, i.e. $\alpha \in \text{Image}(\phi\otimes\mathcal{B})(W) \cap \bigoplus_{j=1}^{m} (\mathcal{A}|V_j)(W)$.

Then there is a cover $W = \bigcup\{W_k | k \in K\}$ with $\alpha|W_k = \phi\otimes\mathcal{B}(b^k)$ (α is locally in the set-theoretic image of $\phi\otimes\mathcal{B}$). By what we have shown already, there is an open cover $W_k = \bigcup\{W_{k\ell} | \ell \in L_k\}$ with $\alpha|W_{k\ell} = \phi\otimes\mathcal{A}(\alpha^{k\ell})$ where $\alpha^{k\ell} \in \bigoplus_{i=1}^{n} (\mathcal{A}|U_i)(W_{k\ell})$. Combining the two levels of covering we obtain a covering of W with the property that α is, with respect to this covering, locally in the set theoretic image of $\phi\otimes\mathcal{A}$, i.e. $\alpha \in \text{Image}(\phi\otimes\mathcal{A})(W)$.

CHAPTER IV CONCLUSIONS

It is felt that the results of Chapter IV and V provide strong evidence that "internal" universal algebra may be effectively carried out in a topos. More precisely, by reformulating concepts such as "equation" in a more dynamic (i.e. sheaf-theoretic) manner insight may be obtained into the structure of algebras which are allowed to vary along certain parameters. Several things must be taken into account in the further development of these ideas.

With regard to the theorems of chapter IV and V several obvious questions remain open. For example it would be useful to establish that the condition that $\mathbb{C}$ have enough points cannot be removed in the theorem of chapter IV which says that under these circumstances injectivity in the sheaves of algebras is as well behaved as it is in the base equational class. In view of theorem 3.2.1 of chapter II the double negation sheaves on a suitably connected Hausdorff space would be a good topos to begin such an investigation. Attempts should also be made to find the correct relationships between the concepts of injectivity, equational compactness and purity in $M(\mathbb{H},\tilde{\mathbb{C}})$. It seems that, primarily due to the absence of the axiom of choice in the internal logic of an arbitary topos, the classical connections (for example equationally compact = pure injective) will not hold in general. Work in this direction must take into account the recent, and as yet unpublished, results of

Banaschewski which provide meaningful notions of purity and equational compactness in any category. For example in this scheme a pure extension appears as an up-directed colimit of split extensions.

Evidently much of classical universal algebra (including model theory) may be interpreted in a topos using the techniques developed in this research. We have given only a handful of examples of this, and the power of the extension to algebras varying along parameters awaits exploitation. It is felt for example that the theorem relating topological and equational compactness might prove useful in combinatorics, since it applies in particular to finite sets (or groups, rings etc.) parametrized by a graph.

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