# CONTINUITY OF POSITIVE FUNCTIONALS AND REPRESENTATIONS

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AND REPRESENTATIONS

By

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## ABSTRACT

This dissertation is mainly concerned with the study of the continuity of positive functionals on topological *-algebras and of the representations of topological *-algebras by operators on a Hilbert space. In each case, both the locally convex and the non-locally convex topological *-algebras are considered. First we examine the general situation; then we discuss the results for the various known classes of topological *-algebras as special cases of our general considerations. These algebras include bounded algebras, MQ*-algebras, uniformly A-convex algebras and Banach algebras for the locally convex case, and F-algebras and p-normed algebras for the non-locally convex case. Meanwhile we relax the condition for the requirement of an identity element and the condition on the continuity of the involution map in the algebra. In this way we partially generalize some previous results, and the known results in some cases follow from ours. Those that are not particular cases of our results will also be discussed.

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*To my parents*

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## INTRODUCTION

In this dissertation we are mainly concerned with the study of the continuity of positive functionals on topological *-algebras and of the representations of topological *-algebras by operators on a Hilbert space. In each case we shall first look at the general situation, from which the corresponding results for known classes of topological *-algebras will be derived as special cases.

An important aspect of the study of positive functionals is its use in the study of representation theory. A remarkable fact is that all positive functionals on a B*-algebra with identity are continuous. Other classes of *-algebras more general than B*-algebras were subsequently studied and the above known result generalized. The wide range of study includes MQ*-algebras by Husain and Rigelhof [14], BP*-algebras by Husain and Warsi [17], GB*-algebras by Dixon [11], F-algebras by Zelazko [31] and p-normed algebras by Xia Dao-Xing [29].

A common feature in the study of the various kinds of *-algebras mentioned in the last paragraph is that they all require the identity element and the continuity of the involution map. The first condition was relaxed and the second relinquished for Banach *-algebras by Murphy [20] who recently proved that every positive functional on a commutative Banach *-algebra A such that $A^2 = A$ is continuous. Our aim, here, is to carry on the study of the subject in this direction. But instead of restricting our attention to a particular class of *-algebras, we shall give a systematic study for both the locally convex and the non-locally convex topological *-algebras in the most general situation possible. After that, we come back to some known classes of *-algebras as special cases of our general considerations. In this way we partially generalize some previous results, and the known results in some cases follow from ours. Those that are not particular cases of our results will also be discussed.

We prove the continuity of positive functionals on general topological *-algebras with the condition for the requirement of an identity being relaxed. While the condition on the continuity of the involution map cannot be dropped entirely beyond the Banach *-algebra case, we are able to relax it somewhat. A similar approach is adopted for the study of representation theory for more general *-algebras to the same degree of success as in the case of positive functionals, since a positive functional determines a *-representation of the given *-algebra and vice versa. Here again we generalize some well-known results.

Chapter 1 contains the definitions and known results from topological vector spaces and topological algebras which will be used in the sequel. Proofs are in general not included; however, proper references are given.

In Chapter 2 we investigate the continuity of positive functionals on both locally convex and non-locally convex topological *-algebras. In the former case the results on certain special classes like bounded algebras (introduced by Allan [1]), MQ*-algebras (Husain and Rigelhof [14]), uniformly A-convex algebras (introduced by Cochran [8]) and Banach algebras are discussed. In the latter case we compare our results with those of Zelazko [31] on F-algebras and Xia Dao-Xing [29] on p-normed algebras. After a short digression on C-symmetric algebras, we give a brief account of the continuity of multiplicative linear functionals as well.

The subject of Chapter 3 is on representation theory. Throughout the chapter both the locally convex and non-locally convex topological *-algebras are discussed. First we establish the *-representation of topological *-algebras by operators on a Hilbert space. Then we concern ourselves with the problem of studying those topological *-algebras so that every *-representation of which on a Hilbert space is necessarily continuous. It is well-known that each *-representation of a B*-algebra on a Hilbert space is continuous (e.g. see [32]). We show that this is true for a large class of *-algebras more general than B*-algebras, including certain non-locally convex *-algebras. The last section of this chapter deals with the conditions under which a positive functional on a topological *-algebra is representable.

Finally, the last chapter deals briefly with the algebras that look like Segal algebras. We shall apply the results established in Chapter 2 and 3 to such algebras. We study the continuity of positive functionals and representation theory on these algebras.

# CHAPTER 1

## Preliminaries

The purpose of the preliminaries is to present the basic definitions and theorems which will be recalled in subsequent chapters. To avoid tedium, proofs are in general omitted as they can be found in the given references.

### §1 Basic concepts

We shall assume throughout that all the vector spaces and algebras considered here are over the complex field $\mathbb{C}$.

Definition 1.1 A topological vector space is a vector space endowed with a Hausdorff linear topology with respect to which addition and scalar multiplication are continuous maps in both variables.

Since we are mainly concerned with algebras, the elementary properties and relevant definitions in a topological vector space are assumed. For reference, see [25].

Definition 1.2 A topological algebra is a topological vector space which is also an algebra such that multiplication is separately continuous.

Definition 1.3 An F-algebra is a complete metrizable topological algebra.

Since a locally convex topological algebra A has its topology generated by a non-empty family of seminorms $\{p_\alpha\}_{\alpha\varepsilon\Gamma}$, for brevity we shall denote a locally convex topological algebra by $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$. Otherwise we do not assume local convexity. For example, $(A, d)$ merely means a metrizable topological algebra with metric d.

<u>Definition 1.4</u> $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ is a locally m-convex algebra if each $p_\alpha$ satisfies the following submultiplicative condition:

$$p_\alpha(xy) \leq p_\alpha(x)\ p_\alpha(y) \qquad (x,y\varepsilon A).$$

Equivalently, A has a basic neighbourhood system $\{U_\alpha\}_{\alpha\varepsilon\Gamma}$ of 0 such that $U_\alpha U_\alpha \subset U_\alpha$ $(\alpha\varepsilon\Gamma)$ and each $U_\alpha$ is convex and circled.

If A is an algebra, we define $\tilde{A} = A\oplus\mathbb{C} = \{(x,\alpha):\ x\varepsilon A,\ \alpha\varepsilon\mathbb{C}\}$ with pointwise addition and scalar multiplication, and $(x_1,\alpha_1)(x_2,\alpha_2) = (x_1x_2 + \alpha_2x_1 + \alpha_1x_2,\ \alpha_1\alpha_2)$. Then $\tilde{A}$ has the element $(0,1)$ as an identity, and is called the algebra with an identity adjoined to A. If $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ is a topological algebra, by defining $\tilde{p}_\alpha((x,\alpha)) = p_\alpha(x) + |\alpha|$, $[\tilde{A}, \{\tilde{p}_\alpha\}_{\alpha\varepsilon\Gamma}]$ is also a topological algebra whose topology is compatible with the product topology on $A\oplus\mathbb{C}$.

<u>Definition 1.5</u> $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ has a left (resp. right) approximate identity $\{e_\lambda:\ \lambda\varepsilon\Lambda\}$ if $\{e_\lambda:\ \lambda\varepsilon\Lambda\}$ is a net in A such that for each $x\varepsilon A$, $\lim\limits_\lambda e_\lambda x = x$ (resp. $\lim\limits_\lambda x\, e_\lambda = x$). $\{e_\lambda:\ \lambda\varepsilon\Lambda\}$ is an approximate identity when it is both a left and right approximate identity.

The left approximate identity $\{e_\lambda: \lambda\varepsilon\Lambda\}$ is said to be bounded if for each $\alpha\varepsilon\Gamma$ there exists $K_\alpha > 0$ such that $p_\alpha(e_\lambda) \leq K_\alpha$ $(\lambda\varepsilon\Lambda)$. It is uniformly bounded if it is bounded and $\sup_\alpha K_\alpha < \infty$.

Definition 1.6 Let A be an algebra. The circle operation $\circ$ on A is defined by $x\circ y = x + y - xy$ $(x,y\varepsilon A)$.

This operation is clearly associative and has the zero element of A as an identity. If A has an identity e, then the relationship between the circle operation and multiplication is given by

$$(e - x)(e - y) = e - (x\circ y). \tag{1}$$

Definition 1.7 An element x of an algebra A is said to be left (resp. right) quasi-regular if there is an element $y\varepsilon A$ such that $y\circ x = 0$ (resp. $x\circ y = 0$). In this case, y is called the left (resp. right) quasi-inverse of x. x is quasi-regular if it is both left and right quasi-regular. It is (left, right) quasi-singular if it is not (left, right) quasi-regular.

We note that if A has an identity e, then by (1) it is clear that an element $x\varepsilon A$ is (left, right) quasi-regular iff $e - x$ is (left, right) regular.

Definition 1.8 Let A be an algebra and $x\varepsilon A$.

(a) The spectrum of x in A is the set $Sp_A(x) = \{\lambda \varepsilon \mathbb{C}: \lambda \neq 0, \lambda^{-1}x$ is quasi-singular in A$\}$, plus 0 if x is singular, where all elements are considered as singular if A has no identity.

(b) The spectral radius of x in A is the extended real number

$$\sigma_A(x) = \sup \{|\lambda|: \lambda \varepsilon Sp_A(x)\}.$$

Definition 1.9 A topological algebra A is a Q-algebra if the set of all quasi-regular elements of A is open in A.

Lemma 1.10 Let A be a topological algebra. The following are equivalent:

(i) A is a Q-algebra.

(ii) There is a neighbourhood of 0 consisting of quasi-regular elements.

(iii) The set of quasi-regular elements has an interior.

Proof (i) $\Longrightarrow$ (ii), (ii) $\Longrightarrow$ (iii) are obvious.
(iii) $\Longrightarrow$ (i) [19] pp. 80.

Proposition 1.11 Let A be a locally convex topological algebra. The following are equivalent:

(i) A is a Q-algebra.

(ii) The set $S = \{x \varepsilon A: \sigma_A(x) \leq 1\}$ is a neighbourhood of 0.

(iii) $S$ has an interior.

Proof [19] pp. 58.

Proposition 1.12 Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a locally m-convex algebra which is either complete or a Q-algebra. Then

$$\sigma_A(x) = \sup_\alpha \lim_{n\to\infty} [p_\alpha(x^n)]^{1/n} (x\varepsilon A) .$$

Proof [19] pp. 22 and pp. 24.

The following proposition is found in [19], pp. 80.

Proposition 1.13 Every element of a Q-algebra has a compact spectrum.

Since in a Banach algebra $(A, \|\cdot\|)$, $\sigma_A(x) \leq \|x\|$ $(x\varepsilon A)$, Proposition 1.11 says that a Banach algebra is a Q-algebra.

Definition 1.14 (a) A locally convex space is called a barrelled space if every closed, convex, balanced and absorbing subset is a neighbourhood of 0.

(b) A topological space X is a Baire space if X cannot be written as a countable union of nowhere dense subsets.

Theorem 1.15 (a) Every locally convex space which is a Baire space is barrelled.

(b) Every complete metrizable space is a Baire space.

Proof (a) See [26] pp. 60.

(b) See [18] pp. 28.

<u>Definition 1.16</u> A topological vector space is said to be locally bounded if there is a bounded neighbourhood of 0.

<u>Definition 1.17</u> Let A be an algebra. A mapping $x \to x^*$ of A onto itself is called an involution if the following conditions are satisfied:

(i) $(x^*)^* = x$

(ii) $(x+y)^* = x^* + y^*$

(iii) $(xy)^* = y^* x^*$

(iv) $(\alpha x)^* = \bar{\alpha} x^*$, $\alpha \in \mathbb{C}$.

An algebra with an involution is called a *-algebra. A subalgebra $\mathcal{C}$ of a *-algebra A is called a *-subalgebra of A provided $\mathcal{C}^* = \mathcal{C}$.

We shall call a topological algebra with an involution a topological *-algebra. We shall not assume the continuity of the involution map unless otherwise stated.

<u>Definition 1.18</u> Let A be a *-algebra.

(a) An element $h \in A$ such that $h^* = h$ is called hermitian. The set of all hermitian elements in A is denoted by $\mathcal{H}_A$.

(b) An element $x \in A$ such that $x x^* = x^* x$ is called normal.

(c) A subset $\mathcal{E}$ of A is called normal if the set $\mathcal{E} \cup \mathcal{E}^*$ is commutative, where $\mathcal{E}^* = \{x^* : x \in \mathcal{E}\}$.

If $\mathcal{E}$ is normal, then by Zorn's lemma, it is easy to see that there exists a maximal normal subset of A which contains $\mathcal{E}$.

If A is a $*$-algebra, let $\tilde{A}$ be the algebra with an identity adjoined. By defining $(x, \alpha)^* = (x^*, \bar{\alpha})$, the map $(x,\alpha) \to (x,\alpha)^*$ is then an involution in $\tilde{A}$.

<u>Proposition 1.19</u> Let A be a $*$-algebra and c a quasi-regular element with its quasi-inverse denoted by $c^\circ$. Then the following are true:

(a) $(c^*)^\circ = (c^\circ)^*$.

(b) If $x \in A$ satisfies $cx = xc$, then $c^\circ x = xc^\circ$.

(c) If c is normal, so is $c^\circ$.

<u>Proof</u> (a) $c^* \circ (c^\circ)^* = c^* + (c^\circ)^* - c^*(c^\circ)^* = (c+c^\circ - c^\circ c)^* = 0$. Similarly $(c^\circ)^* \circ c^* = 0$.

(b) $cx = xc$ implies that $c \circ x = x \circ c$. By associativity of the operation $\circ$, we have $c^\circ \circ x = c^\circ \circ x \circ c \circ c^\circ = c^\circ \circ c \circ x \circ c^\circ = x \circ c^\circ$, and so $c^\circ x = xc^\circ$.

(c) $cc^* = c^*c$ implies that $c \circ c^* = c^* \circ c$. Thus $(c^\circ)^* \circ c = (c^\circ)^* \circ c \circ c^* \circ (c^*)^\circ = (c^\circ)^* \circ c^* \circ c \circ (c^*)^\circ = (c^*)^\circ \circ c^* \circ c \circ (c^\circ)^* = c \circ (c^\circ)^*$, where the second last equality follows from (a). This in turn implies that $(c^\circ)^* c = c(c^\circ)^*$. From (b), we have $(c^\circ)^* c^\circ = c^\circ (c^\circ)^*$.

<u>Definition 1.20</u> A $*$-algebra is symmetric if every element of the form $-x^*x$ is quasi-regular.

The proof of the following proposition is straightforward and therefore omitted.

Proposition 1.21 Let A be a *-algebra. Then, for each $x \in A$,

(i) $Sp_A(x^*) = \overline{Sp_A(x)} = \{\bar{\lambda}: \lambda \in Sp_A(x)\}$.

(ii) $\sigma_A(x^*) = \sigma_A(x)$.

Proposition 1.22. Let A be an algebra. If A has no identity and $\tilde{A}$ is the algebra obtained by adjunction of an identity to A, then $Sp_A(x) = Sp_{\tilde{A}}(x)$ $(x \in A)$.

A proof of the above proposition may be found in [22], pp. 32.

§2 Linear maps

Proposition 1.23 Let f be a linear functional on a topological vector space. Then the following are equivalent:

(a) f is continuous.

(b) f is bounded in some neighbourhood of 0.

A proof may be found in [25], pp. 14.

Theorem 1.24 Let X, Y be locally convex spaces such that X is barrelled, or let X, Y be topological vector spaces such that X is a Baire space. Suppose $\Lambda$ is a collection of continuous linear maps from X into Y such that for each $x \in X$, the set $\Lambda(x) = \{f(x): f \in \Lambda\}$ is bounded in Y. Then $\Lambda$ is equicontinuous.

A proof may be found in [26], pp. 83.

Definition 1.25 A linear functional f on an algebra A is multiplicative if $f(xy) = f(x)f(y)$ $(x, y \in A)$.

Definition 1.26 Let A be a *-algebra and f a linear functional on A.

(a) f is hermitian if $f(x^*) = \overline{f(x)}$ $(x \in A)$.

(b) f is positive if $f(x^*x) \geq 0$ for each $x \in A$.

Proposition 1.27 Let A be a *-algebra.

(a) If f is a linear functional on A such that $f(x^*x)$ is real for each $x \in A$, then f has the hermitian property: $f(y^*x) = \overline{f(x^*y)}$ $(x, y \in A)$. Thus, if A has identity, then $f(x^*) = \overline{f(x)}$.

(b) (Cauchy-Schwarz inequality) If f is a positive functional on A, then $|f(y^*x)|^2 \leq f(y^*y)\ f(x^*x)$ $(x, y \in A)$. Thus, if A has identity e, then $|f(x)|^2 \leq f(e)\ f(x^*x)$.

Proof [22] pp. 212.

## §3 Representations

Definition 1.28 Let A be an algebra and $\mathfrak{X}$ a vector space. A representation T of A on $\mathfrak{X}$ is a homomorphism $x \to T_x$ of A into the algebra of all linear operators on $\mathfrak{X}$. That is, for each $x \in A$, $T_x$ is an operator on $\mathfrak{X}$, and $T_{xy} = T_x T_y$, $T_{x+y} = T_x + T_y$, $T_{\alpha x} = \alpha T_x$ $(x, y \in A, \alpha \in \mathbb{C})$.

Definition 1.29 Let A be a *-algebra and H a Hilbert space. A *-representation T of A on H is a representation of A by bounded linear operators on H such that $T_{x^*} = (T_x)^*$ $(x \in A)$, where $(T_x)^*$ is the adjoint operator of $T_x$.

<u>Definition 1.30</u> Let A be a topological algebra, H a Hilbert space and $\mathcal{B}(H)$ the collection of all bounded linear operators on H. A continuous representation of A on H is a representation of A in $\mathcal{B}(H)$ which is continuous with respect to the topology on A and the norm topology on $\mathcal{B}(H)$.

<u>Definition 1.31</u> A representation T of an algebra A by operators on a vector space $\mathcal{X}$ is said to be strictly cyclic if there is a vector $\xi \varepsilon \mathcal{X}$ such that $\{T_x \xi : x \varepsilon A\} = \mathcal{X}$. In this case, $\xi$ is called a strictly cyclic vector.

<u>Definition 1.32</u> A representation T of an algebra A by operators on a topological vector space X is said to be topologically cyclic if there is a vector $\xi \varepsilon X$ such that the subspace $\{T_x \xi : x \varepsilon A\}$ is dense in X. $\xi$ is then called a topologically cyclic vector.

<u>Definition 1.33</u> Let f be a linear functional on the *-algebra A and let $x \to T_x$ be a *-representation of A on a Hilbert space H. Then f is said to be representable by $x \to T_x$ provided there exists a topologically cyclic vector $\xi \varepsilon H$ such that $f(x) = \langle T_x \xi, \xi \rangle$ $(x \varepsilon A)$ where $\langle , \rangle$ denotes the inner product on H.

CHAPTER 2

# Continuity of Positive and Multiplicative Linear Functionals

Ever since it was proved that every positive functional on a B*-algebra with identity is continuous, the attempt to generalize this result to more general topological *-algebras has been extensive. Among others, Husain and Rigelhof [14] studied for MQ*-algebras, and Zelazko [31] F-algebras. We shall present a systematic study of this subject for general locally convex and non-locally convex topological *-algebras which need not possess an identity. The results for certain known classes evolve later as special cases.

## §1 Continuity of positive functionals on locally convex *-algebras

We first prove a fundamental theorem (Theorem 2.3) which asserts the continuity of positive functionals on a topological *-algebra which is either locally convex or otherwise. We then concentrate on the locally convex case.

Definition 2.1 Given positive functionals f and g on a *-algebra, we say that f dominates g if f-g is positive.

Definition 2.2 An algebra A satisfies the relation $A^2 = A$ if for each $x \in A$, there are elements $a_i$, $b_i$ $(i=1,\ldots,n;\ n \in \mathbb{N})$ in

A such that $x = \sum_{i=1}^{n} a_i b_i$.

Theorem 2.3 Let A be a topological *-algebra such that A is either a Baire space or a barrelled space. Suppose $A^2 = A$ and every non-zero positive functional on A dominates a non-zero continuous positive functional. Then every positive functional on A is continuous.

Proof Clearly a zero positive functional is continuous. Let f be a non-zero positive functional on A. Since $A^2 = A$, every $x \in A$ is expressible as $\sum_{i=1}^{m} a_i b_i$ and therefore as $\sum_{i=1}^{n} \alpha_i x_i^* x_i$ where $\alpha_i \in \mathbb{C}$ by the identity:

$$4ab = (b+a^*)^* (b+a^*) - (b-a^*)^* (b-a^*)$$
$$+ i(b+ia^*)^* (b+ia^*) - i(b-ia^*)^* (b-ia^*).$$

Hence, linear functionals that agree on all elements $x^*x$ are identical.

Let $\mathcal{F} = \{g \neq 0:\ g$ a continuous positive functional on A dominated by f$\}$. By hypothesis $\mathcal{F} \neq \emptyset$. Define a relation "$>$" on $\mathcal{F}$ by "$g>h$" iff g dominates h. "$>$" is then a partial order on $\mathcal{F}$.

If $\mathcal{F}_0$ is a totally ordered subset of $\mathcal{F}$, then $\mathcal{F}_0$ is directed by $>$. For each $x \in A$ with $x = \sum_{i=1}^{n} \alpha_i x_i^* x_i$, we have

$$|g(x)| = |g(\sum_{i=1}^{n} \alpha_i x_i^* x_i)| \le \sum_{i=1}^{n} |\alpha_i|\ g(x_i^* x_i) \le \sum_{i=1}^{n} |\alpha_i|\ f(x_i^* x_i)$$

$(g \varepsilon \mathscr{F}_0)$. Since by hypothesis A is either Baire or barrelled, by Theorem 1.24 $\mathscr{F}_0$ is equicontinuous.

For each $y \varepsilon A$, $\lim_{> \mathscr{F}_0} g(y^*y)$ $(g \varepsilon \mathscr{F}_0)$ exists, as $g(y^*y) \leq f(y^*y)$ $\forall\, g \varepsilon \mathscr{F}_0$. Since each $x \varepsilon A$ can be expressed as $x = \sum_{i=1}^{n} \alpha_i x_i^* x_i$ $(\alpha_i \varepsilon \mathbb{C},\ x_i \varepsilon A)$, we can define, $\phi(x) = \lim_{g \stackrel{>}{\varepsilon} \mathscr{F}_0} g(x)$ $(x \varepsilon A)$.

$\phi$ is clearly a non-zero positive functional on A dominated by f. Further, for each $\varepsilon > 0$, since $\mathscr{F}_0$ is equicontinuous, there is a neighbourhood V of 0 in A such that $|g(x)| < \varepsilon$ $(x \varepsilon V,\ g \varepsilon \mathscr{F}_0)$. Thus $|\phi(x)| \leq \varepsilon$ $(x \varepsilon V)$ and so $\phi$ is continuous. Hence $\phi \varepsilon \mathscr{F}$ and $\phi$ is an upper bound for $\mathscr{F}_0$. By Zorn's lemma, $\mathscr{F}$ has a maximal element, say, $g_0$.

If $f - g_0 \neq 0$, by hypothesis there is a non-zero continuous positive functional $g_1$ such that $f - g_0 - g_1$ is positive. But clearly $g_0 + g_1 \ \varepsilon\ \mathscr{F}$ and $g_0 + g_1 > g_0$. This contradicts the maximality of $g_0$. Therefore $f - g_0 = 0$ or $f = g_0$ and so f is continuous.

<u>Remark</u> If A has an identity, then trivially $A^2 = A$. But there are non-trivial cases. Rudin [23] showed that every function in $L^1(\mathbb{R})$ is the convolution of two other functions in $L^1(\mathbb{R})$; a fact which he later generalized to $L^1(G)$ where G is either the additive group of Euclidean n-space, or the n-dimensional torus [24]. Cohen [9] proved further that in

any Banach algebra with a bounded left approximate identity each element can be factorised (written as the product of two elements). Recently Craw [10] proved this factorization theorem for a locally m-convex F-algebra with a uniformly bounded left approximate identity.

Next we prove a few lemmas which will lead to the main theorem.

<u>Lemma 2.4</u> Every maximal normal subset $\mathcal{C}$ of a topological $*$-algebra A is a closed maximal commutative $*$-subalgebra of A such that $Sp_A(c) = Sp_{\mathcal{C}}(c)$ $(c \in \mathcal{C})$. If A has identity e, then $e \in \mathcal{C}$.

<u>Proof</u> If $\mathcal{E}$ is any normal subset of A then it is obvious from the definition that $\mathcal{E} \cup \mathcal{E}^*$ is normal. Since $\mathcal{C}$ is maximal normal, $\mathcal{C} = \mathcal{C} \cup \mathcal{C}^*$ or $\mathcal{C} = \mathcal{C}^*$. Hence in order to prove that an element $x \in A$ belongs to $\mathcal{C}$, we have only to show that

(i) x is normal;

(ii) x commutes with every element of $\mathcal{C}$.

For in this case, $\mathcal{C} \cup \{x\}$ is normal so that $x \in \mathcal{C}$ by maximality of $\mathcal{C}$. By commutativity of $\mathcal{C} = \mathcal{C} \cup \mathcal{C}^*$, it is easy to verify that if $x, y \in \mathcal{C}$, $\alpha \in \mathbb{C}$, then $x+y$, $\alpha x$, $xy$ satisfy (i) and (ii). Thus $\mathcal{C}$ is a subalgebra and hence a maximal commutative $*$-sub-algebra. That $\mathcal{C}$ contains the identity element, if it exists, is trivial.

Let $\{c_\alpha\}$ be a net in $C$ which converges to $x \varepsilon A$. Since $c_\alpha c = cc_\alpha$ for every $c \varepsilon C$, we have $xc = \lim_\alpha (c_\alpha c) = \lim_\alpha (cc_\alpha) = cx$ $(c \varepsilon C)$. Since $C$ is a *-algebra, $x^*$ also commutes with every element of $C$. In particular $c_\alpha x^* = x^* c_\alpha \ \forall \alpha$. Thus $x\, x^* = x^* x$ so that $x \varepsilon C$. Therefore $C$ is closed. Finally, it is obvious that $Sp_A(c) \subset Sp_C(c)$. Now let $c \varepsilon C$ which is quasi-regular in A. Since c is normal, so is its quasi-inverse $c^\circ$, and $c^\circ$ commutes with $C$ because c does so (Proposition 1.19). Consequently, $c^\circ \varepsilon C$. Thus $\mathbb{C} \setminus Sp_A(c) \subset \mathbb{C} \setminus Sp_C(c)$ or $Sp_C(c) \subset Sp_A(c)$.

<u>Definition 2.5</u> A topological *-algebra A is said to have locally separately continuous (resp. locally pseudo-continuous) involution if in every maximal commutative *-subalgebra $C$ of A, the map $x \to x^*y$ is continuous for each fixed $y \varepsilon C$ (resp. the map $x \to x^*x$ is continuous). The word 'locally' will be omitted if we mean that the map $x \to x^*y$ for fixed $y \varepsilon A$ (resp. $x \to x^*x$) is continuous on A.

The above definition is a generalised notion of the so called 'locally continuous involution', namely, $x \to x^*$ is continuous on each maximal commutative *-subalgebra. It is clear that in a topological *-algebra A, 'locally continuous involution' implies 'locally separately continuous involution' and the two are equivalent if A has identity. If, moreover, A has jointly continuous multiplication, then 'locally continuous involution' implies 'locally pseudocontinuous involution'.

<u>Lemma 2.6</u> Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be sequentially complete. Let $h\varepsilon A$ with $\overline{\lim_{n\to\infty}} [p_\alpha(h^n)]^{1/n} < 1$ for each $\alpha\varepsilon\Gamma$. Then there exists an element $k\varepsilon A$ such that $2k - k^2 = h$. If, in addition, $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ is a topological *-algebra with locally separately continuous involution, and $h\varepsilon \mathcal{H}_A$, then $k\varepsilon \mathcal{H}_A$.

<u>Proof</u> Let $k_n = -\sum_{j=1}^{n} \binom{\frac{1}{2}}{j} \left(-h\right)^j$. Since $\left|\binom{\frac{1}{2}}{j}\right| \leq 1$ $\forall j$ and $\overline{\lim_{n\to\infty}} [p_\alpha(h^n)]^{1/n} < 1$ the series $\sum_{j=1}^{\infty} \binom{\frac{1}{2}}{j} p_\alpha \left(h^j\right)$ is absolutely convergent for each $\alpha\varepsilon\Gamma$ and so $\{k_n\}$ is Cauchy. Thus there exists $k\varepsilon A$ such that $k_n \to k$. Since the series $f(\zeta) = -\sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} (-\zeta)^n$ is absolutely convergent for $|\zeta| < 1$ and $2f(\zeta) - (f(\zeta))^2 = \zeta$, it follows that $2k - k^2 = h$.

If A is a topological *-algebra and $h\varepsilon \mathcal{H}_A$, let $C$ be a maximal commutative *-subalgebra of A which contains h. This is possible by Zorn's lemma since the set $\{h\}$ is normal. By Lemma 2.4, $C$ is a closed maximal commutative *-subalgebra and therefore contains k. Now $k_n \to k$ implies $k_n k \to k^2$ and $k_n k = k_n{*}k \to k{*}k$ since involution is locally separately continuous. Thus $k^2 = k{*}k$ and so $2k - k{*}k = 2k - k^2 = h$. But then $2k^* - k{*}k = (2k - k{*}k)^* = h^* = h$ and so $k = k^*$.

<u>Remark</u> If $[A \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ is a sequentially complete topological *-algebra with jointly continuous multiplication and locally pseudocontinuous involution, let $h\varepsilon \mathcal{H}_A$ such that

$\overline{\lim\limits_{n\to\infty}} [p_\alpha(h^n)]^{1/n} < 1$ for each $\alpha\varepsilon\Gamma$. Then there exists $k\varepsilon \mathcal{H}_A$ such that $2k - k^2 = h$. For, as in the proof of Lemma 2.6, in this case $k_n \to k$ implies $k_n{}^2 \to k^2$ and $k_n{}^2 = k_n{*}k_n \to k{*}k$.

<u>Lemma 2.7</u> Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a sequentially complete topological *-algebra with locally separately continuous involution. Let f be a positive functional on A. If $h\varepsilon \mathcal{H}_A$ satisfies $\sup\limits_\alpha \overline{\lim\limits_{n\to\infty}} [p_\alpha(h^n)]^{1/n} < \infty$, then

$$|f(u^*hu)| \leq M_h \; f(u^*u) \qquad (u\varepsilon A)$$

for some $M_h > 0$.

<u>Proof</u> If $\overline{\lim\limits_{n\to\infty}} [p_\alpha(h^n)]^{1/n} < 1$ for each $\alpha\varepsilon\Gamma$, by Lemma 2.6 there exist elements r, $s\varepsilon \mathcal{H}_A$ such that $2r - r^2 = h$ and $2s - s^2 = -h$. If $u\varepsilon A$, set $v = u - ru$, $w = u - su$ then $v^*v = (u^* - u^*r)(u - ru) = u^*u - u^*ru - u^*ru + u^*r^2u = u^*u - u^*\,(2r - r^2)u = u^*u - u^*hu$ and $w^*w = u^*u + u^*hu$. Since f is positive, $f(u^*u - u^*hu) = f(v^*v) \geq 0$,

$$f(u^*u + u^*hu) = f(w^*w) \geq 0 \text{ and so } |f(u^*hu)| \leq f(u^*u). \qquad (1)$$

Finally let $\delta_h = \sup\limits_\alpha \overline{\lim\limits_{n\to\infty}} [p_\alpha(h^n)]^{1/n}$. If $\delta_h = 0$, we obtain (1). Therefore assume $\delta_h > 0$. Let $\varepsilon>0$ be given. Then for each $\alpha\varepsilon\Gamma$, $\overline{\lim\limits_{n\to\infty}} \; p_\alpha\left[\left(\left\{\frac{h}{\delta_h+\varepsilon}\right\}^n\right)\right]^{1/n} =$

$$\overline{\lim_{n\to\infty}} \left[\frac{p_\alpha(h^n)}{(\delta_h+\varepsilon)^n}\right]^{1/n} = \frac{\overline{\lim\limits_{n\to\infty}} [p_\alpha(h^n)]^{1/n}}{\delta_h+\varepsilon} < 1$$

and so

$$\left| f\left(u^* \frac{h}{\delta_h+\varepsilon} u\right)\right| \le f(u^*u) \quad (u\varepsilon A)$$

i.e. $|f(u^*hu)| \le (\delta_h+\varepsilon)f(u^*u) \quad (u\varepsilon A)$.

Since $\varepsilon$ is arbitrary, we have

$$|f(u^*hu)| \le \delta_h \; f(u^*u) \qquad (u\varepsilon A).$$

Definition 2.8 Let f be a linear functional on the *-algebra A and for a fixed element $u\varepsilon A$, we define

$$f_u(x) = f(u^*xu) \qquad (x\varepsilon A).$$

Then $f_u$ is a linear functional and is positive if f is positive.

Following Tiller [28], we give the definition below:

Definition 2.9 Let A be a *-algebra. If f is a positive functional on A, let $I_f = \{x\varepsilon A: \; f(x^*x) = 0\}$. Set $P = \bigcap I_f$ where the intersection is taken over all positive functionals f on A. Then A is called P-commutative if $xy - yx\varepsilon P$ for all $x, y\varepsilon A$.

Clearly every commutative *-algebra is P-commutative. The following example shows a P-commutative *-algebra which is noncommutative.

Example 2.10 (W. Tiller [28]) Let A' be a noncommutative algebra with involution $x \to x'$. Let $A = \{(x,y): \; x,y\varepsilon A'\}$ with pointwise algebraic operations and involution

$(x,y)^* = (y',x')$. Then A is noncommutative. Let f be a positive functional on A, let x be an arbitrary element in A', and set $a = (x,0)$. Then $a^*a = 0$ which implies $f(a^*a) = 0$; hence $a \in I_f$ and since f is arbitrary, $a \in P$. Similarly, every element of A of the form $(0,y)$ is in P. Thus, by an application of the Cauchy-Schwarz inequality, it can be shown that $(x,y) = (x,0) + (0,y) \in P$ for every $x,y \in A'$; i.e. $A = P$. Therefore, A is P-commutative. If, moreover, A' has its linear topology generated by a family of seminorms $\{p'_\alpha\}_{\alpha\in\Gamma}$, then the family of seminorms $\{p_\alpha\}_{\alpha\in\Gamma}$, where $p_\alpha((x,y)) = p'_\alpha(x) + p'_\alpha(y)$, defines a linear topology on A. $(A, \{p_\alpha\}_{\alpha\in\Gamma})$ is (sequentially) complete if $(A', \{p'_\alpha\}_{\alpha\in\Gamma})$ is (sequentially) complete.

Tiller also proved that a Banach *-algebra with an approximate identity is P-commutative if $\sigma_A(x^*x) \leq \sigma_A(x)^2$ $(x \in A)$.

<u>Theorem 2.11</u> Let $[A, \{p_\alpha\}_{\alpha\in\Gamma}]$ be a sequentially complete, barrelled, topological *-algebra with locally separately continuous involution. Suppose A is P-commutative and $A^2 = A$. If there is a neighbourhood V of 0 in A and an $M>0$ such that for every $x \in V$, $\sup_\alpha \overline{\lim_{n\to\infty}} [p_\alpha\{(x^*x)^n\}]^{1/n} \leq M$, then every positive functional on A is continuous.

<u>Proof</u> The theorem is obvious for a zero positive functional. So let f be a non-zero positive functional on A. For a fixed $u \in A$, by the Cauchy-Schwarz inequality, we have

$|f_u(x)|^2 = |f(u^*xu)|^2 \leq f(u^*u)\ f(u^*x^*xu) \qquad (x\varepsilon A).$

If $x\varepsilon V$, then, as $x^*x\varepsilon \mathcal{H}_A$, by Lemma 2.7, $f(u^*x^*xu) \leq M_1 f(u^*u)$ where $M_1 = \max(1,M)$ and so $|f_u(x)|^2 \leq M_1 f(u^*u)^2$. Thus $f_u$ is bounded on V, hence continuous by Proposition 1.23.

If $f_u = 0$ for every $u\varepsilon A$, then again by Cauchy-Schwarz inequality, $|f(u^*x^*y)|^2 \leq f(u^*x^*xu)\ f(y^*y) = f_u(x^*x)\ f(y^*y) = 0$ $(u,x,y\varepsilon A)$. Hence $f(A^3) = \{0\}$, which is false, since $A^3 = A$ and $f \neq 0$. Therefore choose $u\varepsilon A$ such that $f_u \neq 0$. We may suppose $\overline{\lim_{n\to\infty}}\ [p_\alpha\{(u^*u)^n\}]^{1/n} < 1$ for each $\alpha\varepsilon\Gamma$. For otherwise, choose $\lambda>0$ so that $\sqrt{\lambda}u\varepsilon V$. Then $\sup_\alpha \overline{\lim_{n\to\infty}}\ [p_\alpha\{(\lambda u^*u)^n\}]^{1/n} \leq M$ by hypothesis and so $\overline{\lim_{n\to\infty}}\ [p_\alpha\{(\frac{\lambda}{M+1}\ u^*u)^n\}]^{1/n} < 1$ for each $\alpha\varepsilon\Gamma$. Since $f_{\alpha u} = |\alpha|^2\ f_u$ $(\alpha\varepsilon\mathbb{C})$, the linearity, positivity and continuity of $f_u$ is not affected if we replace u by a scalar multiple of u. We then replace $\sqrt{\frac{\lambda}{M+1}}$ u by u to obtain $\overline{\lim_{n\to\infty}}\ [p_\alpha\{(u^*u)^n\}]^{1/n} < 1$ for each $\alpha\varepsilon\Gamma$. Since $u^*u\varepsilon \mathcal{H}_A$, as seen in the proof of Lemma 2.7, there is $v_x\varepsilon A$ such that $v_x{}^*v_x = x^*x - x^*u^*ux$ for each $x\varepsilon A$.

Next we show that $f_u(x^*x) = f_x(u^*u)$ for every $x\varepsilon A$. First we observe that $I_f$ is a (proper) left ideal in A. Also the linear space $A/I_f$, with the inner product defined by $\langle x + I_f,\ y + I_f\rangle = f(y^*x)$, is an inner product space. Since A is P-commutative, we have $ux + I_f = xu + I_f \quad (x\varepsilon A)$. Hence $f_u(x^*x) = f(u^*x^*xu) = \langle xu + I_f,\ xu + I_f\rangle = \langle ux + I_f,\ ux + I_f\rangle$ $= f(x^*u^*ux) = f_x(u^*u)$. But then $(f - f_u)\ (x^*x) = f(x^*x) -$ $f_u(x^*x) = f(x^*x) - f_x(u^*u) = f(x^*x - x^*u^*ux) = f(v_x^*\ v_x) \geq 0$.

Therefore f dominates a non-zero continuous positive functional $f_u$ on A. Now Theorem 2.3 applies and so every positive functional on A is continuous.

Henceforth, we shall say that a topological *-algebra $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ has property $\mathfrak{Q}$ if the following condition $\mathfrak{Q}$ is satisfied: $\mathfrak{Q}$: There is a neighbourhood V of 0 in A and an $M>0$ such that $\sup_\alpha \overline{\lim_{n\to\infty}} [p_\alpha\{(x^*x)^n\}]^{1/n} \leq M \quad (x\varepsilon V)$.

§2 Some classes of locally convex topological *-algebras

After establishing Theorem 2.11 for a general locally convex topological *-algebra, we now provide concrete examples that satisfy condition $\mathfrak{Q}$.

Example 2.12 A locally m-convex Q-algebra $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ such that the involution is pseudocontinuous at the origin has property $\mathfrak{Q}$. For, in this case, by a theorem of Michael ([19], pp. 24), $\sigma_A(x) = \sup_\alpha \overline{\lim_{n\to\infty}} [p_\alpha(x^n)]^{1/n}$ $(x\varepsilon A)$. Since A is a Q-algebra, by Proposition 1.11 the set
$S = \{x\varepsilon A: \sup_\alpha \overline{\lim_{n\to\infty}} [p_\alpha(x^n)]^{1/n} \leq 1\} = \{x\varepsilon A: \sigma_A(x) \leq 1\}$ is a neighbourhood of 0. By continuity of $x \to x^*x$ at 0, there is a neighbourhood V of 0 such that $x\varepsilon V$ implies that $x^*x\varepsilon S$. Thus $\sup_\alpha \overline{\lim_{n\to\infty}} [p_\alpha\{(x^*x)^n\}]^{1/n} \leq 1$. $(x\varepsilon V)$.

Remark In view of the remark after Lemma 2.6, it is clear that the conclusions of Lemma 2.6 and Lemma 2.7 hold if $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ is a sequentially complete topological *-algebra with jointly continuous multiplication and locally pseudocontinuous involution. Therefore the conclusion of Theorem 2.11 holds if A is, in addition, P-commutative, $A^2 = A$, barrelled and has property $\mathfrak{Q}$. Since every *-subalgebra contains 0, locally pseudocontinuous involution implies pseudocontinuous involution at 0. We thus arrive at the following corollaries:

<u>Corollary 2.13</u> Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a sequentially complete, barrelled, locally m-convex Q-algebra with locally pseudocontinuous involution. Suppose A is P-commutative and $A^2 = A$. Then every positive functional on A is continuous.

<u>Corollary 2.14</u> Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a complete metrizable locally m-convex Q-algebra with locally pseudocontinuous involution. Suppose A is P-commutative and $A^2 = A$. Then every positive functional on A is continuous.

As seen in the first part of the proof of Theorem 2.11, we established the continuity of $f_u$ without requiring A to be P-commutative and barrelled since we did not make use of Theorem 2.3. Now if A has an identity e, since $f_e(x) = f(x)$ $(x\varepsilon A)$, we get a sharpened form of Theorem 2.11.

<u>Theorem 2.15</u> Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a sequentially complete topological *-algebra with identity and has property $\mathfrak{Q}$. Suppose A has either (i) locally (separately) continuous involution or (ii) jointly continuous multiplication and locally pseudocontinuous involution. Then every positive functional

on A is continuous.

As a particular case of Theorem 2.15, we deduce the following corollary, which is a slight generalization of Theorem 2 of Husain and Rigelhof [14].

Corollary 2.16 Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a sequentially complete locally m-convex Q *-algebra with identity and locally pseudo-continuous involution. Then every positive functional on A is continuous.

Following Allan [1] we say that an element x in a locally convex topological algebra $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ is bounded if for some non-zero $\lambda\varepsilon\mathbb{C}$, the set $\{(\lambda x)^n: \ n=1,2,\ldots\}$ is a bounded subset of A. The set of all bounded elements of A is denoted by $A_0$. We say that A is bounded if $A = A_0$.

Clearly every normed algebra is bounded. Also $[L^\infty[0,1], \{\|\cdot\|_p\}_{p=1}^\infty]$ i.e. $L^\infty$ with the induced $L^\omega$-topology is bounded. For, since $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$ ([2] pp. 932), we have, for each p,

$$\overline{\lim_{n\to\infty}} (\|(\bar{f}f)^n\|_p)^{1/n} = \overline{\lim_{n\to\infty}} (\|f^{2n}\|_p)^{1/n} = \overline{\lim_{n\to\infty}} \|f^2\|_{np} = \|f^2\|_\infty \leq$$

$(\|f\|_\infty)^2$ and so $\sup_p \overline{\lim_{n\to\infty}} (\|\bar{f}\ f)^n\|_p)^{1/n} \leq (\|f\|_\infty)^2 < \infty$. Thus by Propositions 2.14 and 2.18 [1], A is bounded. We shall further encounter other known classes in the ensuing discussion.

Proposition 2.17 Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a commutative, pseudo-complete ([1] Definition 2.5), bounded Q*-algebra. Then A has property $\mathfrak{Q}$.

Proof Suppose first A possesses an identity. By Proposition 2.14, Corollary 2.16 and Theorem 2.18 [1], $\beta(x) = \sup_{\alpha} \overline{\lim_{n\to\infty}} [p_\alpha(x^n)]^{1/n}$ is a submultiplicative seminorm on A. By Definition 3.1 and Theorem 3.12 [1], we see that $\beta(x) = \sigma_A(x)$ $(x\varepsilon A)$ and so $\sup_{\alpha} \overline{\lim_{n\to\infty}} [p_\alpha\{(x^*x)^n\}]^{1/n} = \beta(x^*x)$ $\leq \beta(x^*)\beta(x) = \sigma_A(x^*)\sigma_A(x) = (\sigma_A(x))^2$ $(x\varepsilon A)$.

Now suppose $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ possesses no identity. Then $[\tilde{A}, \{\tilde{p}_\alpha\}_{\alpha\varepsilon\Gamma}]$ fulfils all the conditions in the last paragraph. The fact that $\tilde{A}$ is bounded and pseudocomplete is proved respectively in Proposition 4.3 [17] and Proposition 2.8 [1]. Hence we have $\sup_{\alpha} \overline{\lim_{n\to\infty}} [\tilde{p}_\alpha\{(x^*x)^n\}]^{1/n} \leq (\sigma_{\tilde{A}}(x))^2$ $(x\varepsilon\tilde{A})$. If $x\varepsilon A$, then by Proposition 1.22

$\sup_{\alpha} \overline{\lim_{n\to\infty}} [p_\alpha\{(x^*x)^n\}]^{1/n} = \sup_{\alpha} \overline{\lim_{n\to\infty}} [\tilde{p}_\alpha\{(x^*x)^n\}]^{1/n} \leq (\sigma_{\tilde{A}}(x))^2 = (\sigma_A(x))^2$.

Since A is a Q-algebra, the set $\{x\varepsilon A\colon\ \sigma_A(x) \leq 1\}$ is then a neighbourhood of 0. Thus we conclude that $(A,\{p_\alpha\}_{\alpha\varepsilon\Gamma})$ has property $\mathfrak{Q}$.

Remark In particular a BP* Q-algebra([17]) has property $\mathfrak{Q}$.

The following is now an immediate corollary of Theorem 2.11 and Theorem 2.15.

Theorem 2.18 (a) Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a commutative, sequentially complete, barrelled, bounded Q-algebra with locally separately continuous involution and $A^2 = A$. Then every positive functional on A is continuous.

(b) Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a commutative, sequentially complete, bounded Q-algebra with identity and locally separately continuous involution. Then every positive functional on A is continuous.

Remark The corresponding results of Theorem 2.18 for BP*-algebras may be found in Theorem 5.7 and Theorem 5.14 [17]. Although a BP*-algebra requires a weaker condition than sequential completeness, namely pseudo-completeness, its definition (3.1 [17]) requires a lot more than a commutative bounded algebra. Moreover, the condition on continuous involution and identity is indispensable.

Now let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a locally m-convex Q-algebra. Then we have $\sup_\alpha \overline{\lim_{n\to\infty}} [p_\alpha(x^n)]^{1/n} = \sigma_A(x) < \infty$ $(x\varepsilon A)$ (Propositions 1.12 and 1.13). Thus, as argued before, $A = A_0$, i.e., A is bounded. If, in addition, A is a commutative pseudo-complete *-algebra, then by Proposition 2.17 $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ has property $\mathfrak{Q}$. Therefore, from Theorem 2.18 we deduce the following as a complement to Corollary 2.13 and Corollary 2.16:

Corollary 2.19 Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a commutative, sequentially complete, locally m-convex Q-algebra with locally separately continuous involution such that either

(i) A is barrelled and $A^2 = A$ or

(ii) A has identity.

Then every positive functional on A is continuous.

Example 2.20. Let $C^{\infty}(I)$ be the algebra of all complex-valued functions on the closed interval $I = [0,1]$ which have derivatives of all orders, where one-sided derivatives are taken at the end-points of $[0,1]$. Endow $C^{\infty}(I)$ with the topology which has a neighbourhood system of 0 consisting of the sets

$$U_n = \{f \in C^{\infty}(I): \ |f^{(k)}(t)| < 2^{-n} \text{ for } k \leq n \text{ and } t \in [0,1]\}$$
$$(n = 0,1,2,\ldots).$$

Then $C^{\infty}(I)$ is a commutative, complete, metrizable, locally m-convex algebra (see [19], pp. 11 and 15) with identity.

If $f = u + iv$, where $u,v$ are real functions, then $f$ has derivatives of all orders iff both $u$ and $v$ have derivatives of all orders, and $f^{(k)} = u^{(k)} + iv^{(k)}$ $(k = 1,2,\ldots)$. Thus $f \to \bar{f}$ is a continuous involution.

Since $U_0 = \{f \in C^{\infty}(I): \ |f(t)| < 1 \text{ for } t \in [0,1]\}$ is a neighbourhood of 0 such that each $f \in U_0$ has a quasi-inverse $-\frac{f}{1-f}$, by Lemma 1.10, $C^{\infty}(I)$ is a Q-algebra.

Thus $C^{\infty}(I)$ satisfies the hypotheses of Corollary 2.14, in particular, those of Corollary 2.13 and Corollary 2.19, so that every positive functional on $C^{\infty}(I)$ is continuous.

We note that the algebra $[L^{\infty}[0,1], \{\|\cdot\|_p\}_{p=1}^{\infty}]$ furnishes an example of a bounded algebra which is not a Q-algebra ([17] pp. 13).

The following definition is due to Cochran [8].

<u>Definition 2.21</u> $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ is uniformly A-convex if for each $x\varepsilon A$, there are positive constants $M_x$, $N_x$ such that

$$p_\alpha(xy) \leq M_x p_\alpha(y),$$
$$p_\alpha(yx) \leq N_x p_\alpha(y)$$

for each $\alpha\varepsilon\Gamma$ and each $y\varepsilon A$.

From the definition, it is clear that a uniformly A-convex algebra is a topological algebra.

It is easy to verify that $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ is uniformly A-convex iff A has a basic neighbourhood system $\{U_\alpha\}$ of 0 such that for each $x\varepsilon A$, $xU_\alpha \subset M_x U_\alpha$ and $U_\alpha x \subset M_x U_\alpha$, for each $\alpha\varepsilon\Gamma$.

We give an example of a complete uniformly A-convex algebra which is not locally m-convex.

<u>Example 2.22</u> Let $C_b(\mathbb{R})$ denote the algebra of bounded continuous complex-valued functions on $\mathbb{R}$ and $C_0^+(\mathbb{R})$ the set of positive continuous functions on $\mathbb{R}$ which vanish at infinity.

The family of seminorms $\{P_\phi : \phi\varepsilon C_o^+(\mathbb{R})\}$ determines a locally convex linear topology $\beta$ on $C_b(\mathbb{R})$, where

$$P_\phi(f) = \sup_{x\varepsilon\mathbb{R}} |f(x)\phi(x)| \quad (f\varepsilon C_b(\mathbb{R})).$$

The space $(C_b(\mathbb{R}), \beta)$ is commutative uniformly A-convex, since for each $f\varepsilon C_b(\mathbb{R})$, $P_\phi(fg) \leq M(f)\, P_\phi(g)$ $(\phi\varepsilon C_o^+(\mathbb{R}),\ g\varepsilon C_b(\mathbb{R}))$ where $M(f) = \sup_{x\varepsilon\mathbb{R}} |f(x)|$. The fact that it is not locally m-convex is given in [7]. Completeness follows from Theorem 3.6 [27].

<u>Proposition 2.23</u> Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a commutative, pseudo-complete, uniformly A-convex Q*-algebra. Then $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ has property $\mathfrak{Q}$.

<u>Proof</u> For each $x\varepsilon A$, $\alpha\varepsilon\Gamma$, $p_\alpha(x^n) \leq M_x^{\,n-1}\, p_\alpha(x)$ $(n\varepsilon\mathbb{N})$. Thus $\overline{\lim_{n\to\infty}}\, [p_\alpha(x^n)]^{1/n} \leq \overline{\lim_{n\to\infty}}\, M_x^{\frac{n-1}{n}}\ \overline{\lim_{n\to\infty}}\, [p_\alpha(x)]^{1/n} \leq M_x$ and so

$\sup_\alpha \overline{\lim_{n\to\infty}}\, [p_\alpha(x^n)]^{1/n} \leq M_x < \infty$ $(x\varepsilon A)$, i.e., $A_0 = A$. The result now follows from Proposition 2.17.

If $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ is a barrelled uniformly A-convex algebra, then by Proposition 1 [15], A is locally m-convex. Thus the study of positive functionals on such algebras is contained in Corollary 2.13 and Corollary 2.19. If A is also complete, then we have $\sigma_A(x) = \sup_\alpha \overline{\lim_{n\to\infty}}\, [p_\alpha(x^n)]^{1/n} < \infty$ $(x\varepsilon A)$. Thus by Theorem 1 [15], A is a Q-algebra.

As a corollary to Theorem 2.18 (b), we have:

Corollary 2.24 Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a commutative, sequentially complete, uniformly A-convex Q-algebra with identity and locally separately continuous involution. Then every positive functional on A is continuous.

The following example shows that when certain conditions are dropped, not every positive functional is continuous.

Example 2.25 Consider $L^\infty[0,1]$ with the induced $L^\omega$-topology. The positive function F on $L^\infty[0,1]$ defined by

$$F(f) = \int_0^1 \frac{f(t)\,dt}{t(\log t - 1)^2}$$

is not continuous ([11], pp. 714).

By Theorem 1 [2], $[L^\infty[0,1], \{\|\cdot\|_p\}_{p=1}^\infty]$ is not complete; and, as we mentioned before, though it is bounded, it is not a Q-algebra. Thus not all the conditions in Theorem 2.18 (b) are satisfied.

We note that in the following lemma, no topological property of the involution map is invoked.

Lemma 2.26 Let $(A, \|\cdot\|)$ be a Banach $*$-algebra and f a positive functional on A. Then for each $h\varepsilon\mathcal{H}_A$, $|f(u^*hu)| \leq M_h\, f(u^*u)$ $(u\varepsilon A)$ for some $M_h > 0$.

<u>Proof</u> Let $\tilde{A}$ be the algebra with an identity adjoined to A. Then $(\tilde{A}, \|\cdot\|_1)$ is a Banach *-algebra where $\|(x,\alpha)\|_1 = \|x\| + |\alpha|$.

First suppose that $h \in \mathcal{H}_A$ satisfies $\sigma_A(h) < 1$. But then $1 > \sigma_A(h) = \lim_{n\to\infty} \|h^n\|^{1/n} = \lim_{n\to\infty} \|[e - (e-h)]^n\|_1^{1/n}$

$$= \lim_{n\to\infty} \|[e - (e+h)]^n\|_1^{1/n}$$

i.e., $\sigma_{\tilde{A}}(e - (e-h)) = \sigma_{\tilde{A}}(e - (e+h)) < 1$. As $e - h$, $e + h \in \mathcal{H}_{\tilde{A}}$, by Lemma ([12]), there exist $r, s \in \mathcal{H}_{\tilde{A}}$ with $r^2 = e - h$, $s^2 = e + h$. Now if $u \in A$, set $v = ru$, $w = isu$ so that $v^* = u^*r$, $w^* = -iu^*s$. Then $v^*v = u^*r^2u = u^*(e-h)u = u^*u - u^*hu, w^*w = u^*s^2u = u^*(e+h)u = u^*u + u^*hu$. But as A is an ideal of $\tilde{A}$, $u \in A$ implies that $v, w \in A$. Hence $f(u^*u) - f(u^*hu) = f(u^*u - u^*hu) = f(v^*v) \geq 0$, and $f(u^*u) + f(u^*hu) = f(u^*u + u^*hu) = f(w^*w) \geq 0$. Therefore $|f(u^*hu)| \leq f(u^*u)$.

If $\sigma_A(h) \geq 1$, then for $\varepsilon > 0$, since $\sigma_A(x) < \infty$ $(x \in A)$ because A is a Banach algebra,

$$v = \frac{h}{\sigma_A(h)+\varepsilon} \qquad \text{satisfies } \sigma_A(v) < 1.$$

Hence $|f(u^*hu)| \leq (\sigma_A(h)+\varepsilon)\ f(u^*u)$. Letting $\varepsilon \to 0$, we get $|f(u^*hu)| \leq \sigma_A(h)\ f(u^*u)$.

<u>Remark</u> If A is a Banach *-algebra, we may then replace Lemma 2.7 by Lemma 2.26 whenever the former was used in the proof of Theorem 2.11. Thus we can remove the condition on the topological property of the involution map. As a Banach algebra

is a barrelled, locally m-convex Q-algebra, it has property $\mathcal{Q}$. Thus, the following theorem, which may be found in Tiller [28], falls as a corollary to Theorem 2.11.

Corollary 2.27 Every positive functional on a P-commutative Banach *-algebra A satisfying $A^2 = A$ is continuous.

## §3 Continuity of positive functionals on non-locally convex *-algebras

After a study of the continuity of positive functionals on locally convex *-algebras, we choose to give a brief survey of the same topic for non-locally convex case because there are interesting examples of non-locally convex space, e.g., the Lebesgue spaces $L^p[0,1]$ $(0<p<1)$ are already well known. It is also part of our aim to have a parallel result of Corollary 2.14 for non-locally convex, or at least non-locally m-convex, F-algebras, as Arens [2] has given an example of a locally convex F-algebra which is not locally m-convex.

In the discussion below we assume throughout that A is a complete topological algebra (not necessarily locally convex) which has jointly continuous multiplication and on which is defined a non-negative function $\|x\|$ satisfying the following conditions:

(I) $\|0\| = 0$

(II) $\|x+y\| \leq \|x\| + \|y\|$

(III) $\|\alpha x\| \leq \|x\|$ if $|\alpha| \leq 1$

(IV) for each $x \in A$, $\|x^n\| \leq \beta \|x\|^n$ $(n \in \mathbb{N})$ for some $\beta = \beta(x) > 0$.

(V) the sets $\{x \in A: \|x\| < \varepsilon\}$ for positive $\varepsilon$ are basic neighbourhoods of 0.

Lemma 2.28 Suppose A has locally pseudocontinuous involution. If $h \in \mathcal{H}_A$ satisfies $|h| < 1$, then there exists an element $k \in \mathcal{H}_A$ such that $2k - k^2 = h$.

Proof Let $k_n = -\sum_{j=1}^{n} \binom{\frac{1}{2}}{j} (-h)^j$. It suffices to show that $\{k_n\}$ is Cauchy as the rest follows exactly as in the proof of Lemma 2.6. Given $\varepsilon > 0$, since $\sum_{j=1}^{\infty} \| - \binom{\frac{1}{2}}{j} (-h)^j \| \le \sum_{j=1}^{\infty} \|h^j\| \le \beta \sum_{j=1}^{\infty} \|h\|^j < \infty$, choose a positive integer N such that $\sum_{j=n}^{\infty} \| - \binom{\frac{1}{2}}{j} (-h)^j \| < \varepsilon$ for $n > N$. Now if $n > m > N$, then $\|k_n - k_m\| \le \sum_{j=m+1}^{n} \| - \binom{\frac{1}{2}}{j} (-h)^j \| < \varepsilon$.

Lemma 2.29 Suppose A has locally pseudocontinuous involution. Let f be a positive functional on A. If $h \in \mathcal{H}_A$ satisfies $\|h\| < 1$, then $|f(u^*hu)| \le f(u^*u)$ $(u \in A)$.

Proof This follows exactly as in the first part of the proof of Lemma 2.7.

Theorem 2.30 Let A be Baire with locally pseudocontinuous involution. Suppose A is P-commutative and $A^2 = A$. Then every positive functional f on A is continuous.

Proof This follows almost exactly as in the proof of Theorem 2.11. We shall only indicate the slight modification.

Choose a neighbourhood V of 0 in A such that $\|x^*x\| < 1$ $(x\epsilon V)$. Then for each fixed $u\epsilon A$, $|f_u(x)|^2 = |f(u^*xu)|^2 \leq f(u^*u)\ f(u^*x^*xu) \leq f(u^*u))^2$ $(x\epsilon V)$ by Lemma 2.29. Thus $f_u$ is continuous.

Choose $u\epsilon A$ such that $f_u \neq 0$. We may assume $\|u^*u\| < 1$. For otherwise, choose $\lambda>0$ so that $\lambda u\epsilon V$. Then $\|(\lambda u)^*(\lambda u)\| < 1$ and we may replace $\lambda u$ by u.

<u>Corollary 2.31</u> Suppose A has identity and locally pseudo-continuous involution. Then every positive functional on A is continuous.

Let (A, d) be an F-algebra. Then multiplication is jointly continuous in A (Arens [3] pp. 629). Moreover, there is defined on A an F-norm $\|x\| = d(x,0)$ with the following properties (Köthe [18], pp. 163):

(FI) $\|x\| \geq 0$

(FII) $x = 0$ if $\|x\| = 0$

(FIII) $\|\lambda x\| \leq \|x\|$ if $|\lambda| \leq 1$

(FIV) $\|x+y\| \leq \|x\| + \|y\|$

(FV) $\|\lambda x_n\| \to 0$ if $\|x_n\| \to 0$

(FVI) $\|\lambda_n x\| \to 0$ if $\lambda_n \to 0$

If we require the metric d to satisfy the condition: for each $x\epsilon A$, $d(x^n,0) \leq \beta(d(x,0))^n$ $(n\epsilon\mathbb{N})$ for some $\beta = \beta(x) > 0$, we call such an F-algebra (A,d) pseudomultiplicative. Then the F-norm satisfies properties (I) to (V).

Remark Obviously a submultiplicative metric d, i.e., $d(xy,0) \leq d(x,0)\, d(y,0)$, is pseudomultiplicative.

The following are now immediate.

Theorem 2.32 Let (A, d) be a P-commutative pseudomultiplicative F-algebra with locally pseudo-continuous involution and $A^2 = A$. Then every positive functional on A is continuous.

Corollary 2.33 Let (A, d) be a pseudomultiplicative F-algebra with identity and locally pseudocontinuous involution. Then every positive functional on A is continuous.

Zelazko [31] reported that every positive functional on an F-algebra with identity and continuous involution is continuous. Theorem 2.32, while imposing an extra condition, namely pseudomultiplicativity, extends Zelazko's result to an F-algebra without identity. In Corollary 2.33 this extra condition is compensated by relaxing the continuity of involution.

Definition 2.34 Fix $p\varepsilon(0,1]$. An algebra A is called a p-normed algebra if there is defined on A a function $\|x\|_p$, called a p-norm, satisfying

(PI) $|x|_p \geq 0$; $|x|_p = 0$ iff $x = 0$

(PII) $\|\alpha x\|_p = |\alpha|^p\, |x|_p$ $(\alpha\varepsilon\mathbb{C})$

(PIII) $|x+y|_p \leq |x|_p + |y|_p$

A is then metrizable with the metric d given by $d(x,y) = \|x-y\|_p$. Thus a complete p-normed algebra is an F-algebra. We call a p-normed algebra A pseudomultiplicative if for each $x \in A$, $\|x^n\|_p \leq \beta(\|x\|_p)^n$ $(n \in \mathbb{N})$ for some $\beta = \beta(x) > 0$.

<u>Theorem 2.35</u> Let A be a complete, P-commutative, pseudomultiplicative p-normed algebra with locally pseudocontinuous involution and $A^2 = A$. Then every positive functional on A is continuous.

<u>Proof</u> This follows directly from Theorem 2.32.

<u>Corollary 2.36</u> Let A be a complete p-normed algebra with identity and locally pseudocontinuous involution. Then every positive functional on A is continuous.

<u>Proof</u> It is easy to see that a p-normed algebra is locally bounded. Since A has identity, by Theorem 1 [30], there is an equivalent submultiplicative p-norm on A. The result then follows from Corollary 2.33.

In Theorem 6 [29], Xia proved that every positive functional on a complete p-normed algebra with identity and continuous involution is continuous. Thus Theorem 2.35 extends this result to the case without identity, whereas Corollary 2.36 is a slight extension.

Let A be a locally bounded algebra. Then, as shown in [18] pp. 161, A is p-normable for a suitable $p \in (0,1]$.

If, in addition, A has identity and is complete, then the P-norm may be chosen to satisfy $\|xy\|_p \leq \|x\|_p \|y\|_p$ (Zelazkȯ [30], Theorem 1). Thus we have

<u>Corollary 2.37</u> Let A be a complete locally bounded algebra with identity and locally pseudocontinuous involution. Then every positive functional on A is continuous.

<u>Example 2.38</u> Let $W_p$, $0<p\leq 1$, be the collection of all holomorphic functions on the unit disc, $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$, for which $\|\phi\|_p = \sum_{n=0}^{\infty} |a_n|^p < \infty$, with the pointwise multiplication.

$W_p$ has the constant function 1 as identity. It is easy to check that $\|\phi\|_p$ is a complete p-norm on $W_p$ with $\|\phi\psi\|_p \leq \|\phi\|_p \|\psi\|_p$. Moreover, the map $\phi \to \phi^*$ defined by $\phi^*(z) = \sum_{n=0}^{\infty} \bar{a}_n z^n$ is a continuous involution.

<u>Example 2.39</u> Let $A = \mathbb{C} \times L^p[0,1]$, where $0 < p \leq 1$.

With pointwise addition and scalar multiplication and multiplication defined by $(\alpha,f)(\beta,g) = (\alpha\beta, \alpha g + \beta f)$, A is an algebra with identity (1,0). The function $\|\ \|_p$ defined on A by $\|(\alpha,f)\|_p = |\alpha|^p + \int_0^1 |f|^p$ is a complete submultiplicative p-norm.

Further the map $(\alpha,f) \to (\alpha,f)^*$ given by $(\alpha,f)^* = (\bar{\alpha},\bar{f})$ is a continuous involution.

## §4 C-symmetric algebras and the continuity of multiplicative linear functionals.

We give a short account of C-symmetric algebras which are slight generalization of symmetric algebras. Then we proceed to consider the continuity of multiplicative linear functionals on C-symmetric topological algebras.

Definition 2.40 A *-algebra A is said to be C-symmetric if $-x^*x$ is quasi-regular for every normal x.

The above definition is thus a generalized notion of symmetric algebras. The following proposition justifies the name of C-symmetric algebras.

Proposition 2.41 A *-algebra A is C-symmetric iff every maximal commutative *-subalgebra of A is symmetric.

Proof Suppose A is C-symmetric. Let $C$ be a maximal commutative *-subalgebra of A. Then for each $c \in C$, $cc^* = c^*c$. By hypothesis, $-c^*c$ has a quasi-inverse $y \in A$. Since $-c^*c$ is normal (in fact hermitian), so is y (Proposition 1.19 (c)). By commutativity of $C$, $(-c^*c)x = x(-c^*c)$ $(x \in C)$. Hence $yx = xy$ $(x \in C)$ by Proposition 1.19 (b). $C$ being a commutative *-subalgebra is a normal subset of A. It is maximal normal. For, if $C \subset \mathcal{E}$ where $\mathcal{E}$ is a normal subset of A, then $\mathcal{E} \subset \mathfrak{M}$ for some maximal normal subset $\mathfrak{M}$. By Lemma 2.4, $\mathfrak{M}$ is a commutative *-subalgebra. Hence $C = \mathfrak{M} = \mathcal{E}$. Since y is normal and commutes with $C$, it follows that $y \in C$ as shown in the proof of Lemma 2.4. So $-c^*c$ is quasi-regular in $\mathcal{C}$ for each $c \varepsilon \mathcal{C}$.

Conversely, let $x \varepsilon A$ be normal. Then $x$ is contained in a maximal normal subset $\mathcal{C}$ of A. $\mathcal{C}$ is a maximal commutative $*$-subalgebra of A such that $Sp_{\mathcal{C}}(c) = Sp_A(c)$ $(c \varepsilon \mathcal{C})$ by Lemma 2.4. By hypothesis, as $-x^*x \varepsilon \mathcal{C}$, $-x^*x$ has a quasi-inverse in $\mathcal{C}$. Hence $1 \notin Sp_{\mathcal{C}}(-x^*x) = Sp_A(-x^*x)$, which implies that $-x^*x$ is quasi-regular in A.

It is well known ([19], pp. 26) that if A is a symmetric algebra, then the following holds: if $x = x^*$, then $Sp_A(x)$ is real; if $x = -x^*$, then $Sp_A(x)$ is imaginary. We shall prove that the same conclusion holds for C-symmetric algebras.

<u>Proposition 2.42</u> Let A be a C-symmetric algebra. If $x = x^*$, then $Sp_A(x)$ is real; if $x = -x^*$, then $Sp_A(x)$ is imaginary.

<u>Proof</u> Either $x = x^*$ or $x = -x^*$ implies that x is normal. Now any normal element x is contained in a maximal commutative $*$-subalgebra $\mathcal{C}$ such that $Sp_A(x) = Sp_{\mathcal{C}}(x)$. By hypothesis, $\mathcal{C}$ is symmetric. Therefore $Sp_A(x) = Sp_{\mathcal{C}}(x)$ is real if $x = x^*$ and $Sp_A(x) = Sp_{\mathcal{C}}(x)$ is imaginary if $x = -x^*$.

<u>Proposition 2.43</u> Let f be a multiplicative linear functional on an algebra A. Then $f(x) \neq 1$ if x is quasi-regular.

<u>Proof</u> Suppose $x+y - xy = 0$ for some y. Then $f(x) + f(y) - f(x)\, f(y) = 0$. Hence $f(x) \neq 1$.

Proposition 2.44 Let f be a non-zero multiplicative linear functional on an algebra A. Then $f(x)\varepsilon Sp_A(x)$ $(x\varepsilon A)$.

Proof Fix $x\varepsilon A$ and let $\lambda = f(x)$. If $\lambda = 0$, then certainly x cannot have an inverse and so $\lambda\varepsilon Sp_A(x)$. (We recall Definition 1.8, where every element is considered as singular if A has no identity). If $\lambda \neq 0$, then $\lambda^{-1}x$ cannot be quasi-regular; for if it were so, then by Proposition 2.43, $f(\lambda^{-1}x) \neq 1$. But $f(\lambda^{-1}x) = \lambda^{-1}f(x) = 1$.

Proposition 2.45 A multiplicative linear functional f on a C-symmetric algebra A is hermitian, i.e. $f(x^*) = \overline{f(x)}$ $(x\varepsilon A)$.

Proof We need only to consider $f \neq 0$. Using Proposition 2.42 and Proposition 2.44, since $x+x^* = (x+x^*)^*$ and $x-x^* = -(x-x^*)^*$, we have $f(x) + f(x^*) = f(x+x^*)\varepsilon Sp_A(x+x^*)$ is real and $f(x) - f(x^*) = f(x-x^*)\varepsilon Sp_A(x-x^*)$ is imaginary. Thus $f(x^*) = \overline{f(x)}$.

The following are now immediate since a hermitian multiplicative linear functional is positive.

Theorem 2.46 Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a sequentially complete barrelled topological *-algebra with locally separately continuous involution. Suppose A is P-commutative, $A^2 = A$, C-symmetric and has property $\mathfrak{Q}$. Then every multiplicative linear functional on A is continuous.

Proof By Theorem 2.11 and Proposition 2.45.

Corollary 2.47 Let $[A, \{p_\alpha\}_{\alpha\epsilon\Gamma}]$ be a sequentially complete topological *-algebra with locally separately continuous involution. Suppose A has identity and property $\mathcal{Q}$ and is C-symmetric. Then every multiplicative linear functional on A is continuous.

Proof By Theorem 2.15 and Proposition 2.45.

Remarks

(i) Theorem 2.46 holds for a sequentially complete C-symmetric barrelled Q-algebra with $A^2 = A$ which is either (a) commutative, bounded (in particular, locally m-convex) and has locally separately continuous involution or (b) P-commutative locally m-convex with locally pseudocontinuous involution.

(ii) Corollary 2.47 holds for a sequentially complete C-symmetric Q-algebra with identity which is either (a) commutative, bounded (in particular, locally m-convex and uniformly A-convex) and has locally separately continuous involution or (b) P-commutative locally m-convex with locally pseudocontinuous involution.

Theorem 2.48 Let A be a C-symmetric pseudomultiplicative F-algebra with locally pseudocontinuous involution. Suppose A is either (i) P-commutative and $A^2 = A$ or (ii) possesses an identity. Then every multiplicative linear functional on A is continuous.

Proof By Theorem 2.32, Corollary 2.33, and Proposition 2.45.

Theorem 2.49 Let A be a complete C-symmetric p-normed algebra with locally pseudocontinuous involution. Suppose A is either (i) P-commutative, pseudomultiplicative and $A^2 = A$ or (ii) possesses an identity. Then every multiplicative linear functional on A is continuous.

Proof By Theorem 2.35, Corollary 2.36, and Proposition 2.45.

Corollary 2.50 Let A be a complete C-symmetric locally bounded algebra with identity and locally pseudocontinuous involution. Then every multiplicative linear functional on A is continuous.

CHAPTER 3

## Representations

In this chapter, we make use of positive functionals for the study of representations of topological *-algebras. The results established in Chapter 2 will be frequently applied here. Again both the locally convex and non-locally convex algebras will be discussed. Conditions on representability of positive functionals will also be investigated.

### §1 Representations of *-algebras

Let A be a *-algebra and f a positive functional on A. Set $I_f = \{x \in A: f(x^*x) = 0\}$. Then $X_f = A/I_f$ is an inner product space with inner product defined by $\langle x+I_f, y+I_f \rangle = f(y^*x)$. We denote by $H_f$ the completion of $X_f$, $\mathcal{L}(X_f)$ the vector space of all linear operators on $X_f$ and by $\mathcal{B}(H_f)$ the vector space of all bounded linear operators on $H_f$.

We are interested in the *-representation of a topological *-algebra A in $\mathcal{B}(H_f)$. Of interest also is when this *-representation can be made continuous.

Theorem 3.1 Let $[A, \{p_\alpha\}_{\alpha\in\Gamma}]$ be a sequentially complete topological *-algebra with either (i) locally separately continuous involution, or (ii) jointly continuous multiplication and locally pseudocontinuous involution. Let f be a positive functional on A. Then f induces a representation $x \to T_x$ of A in $\mathcal{L}^*(X_f)$ such that

(a) if x satisfies $\sup_\alpha \overline{\lim_{n\to\infty}} \{p_\alpha[(x^*x)^n]\}^{1/n} < \infty$ then $T_x$ is continuous on $X_f$, hence extendable to an operator (again denoted by $T_x$) in $\mathcal{B}(H_f)$; in this case $(T_x)^* = T_{x^*}$.

(b) if there exists a neighbourhood V of 0 in A and an M>0 such that $\sup_\alpha \overline{\lim_{n\to\infty}} \{p_\alpha[(x^*x)^n]\}^{1/n} \leq M$ $(x\varepsilon V)$, then the conclusion in (a) holds for all $x\varepsilon A$, i.e., f induces a *-representation of A in $\mathcal{B}(H_f)$. Moreover, the representation $x \to T_x$ is continuous.

(c) if A has identity, then the representation is strictly cyclic with a cyclic vector $a\varepsilon X_f$ such that

$$f(x) = \langle T_x a, a\rangle \quad (x\varepsilon A).$$

<u>Proof</u> The map $x \to T_x$ of A into $\mathcal{L}(X_f)$ defined by

$$T_x(y + I_f) = xy + I_f \quad (y\varepsilon A)$$

is a representation. It is well defined because $I_f$ is a left ideal of A.

(a) If x satisfies $\sup_\alpha \overline{\lim_{n\to\infty}} \{p_\alpha[(x^*x)^n]\}^{1/n} < \infty$, then by Lemma 2.7,

$$f(y^*x^*xy) \leq M_x f(y^*y) \qquad (y\varepsilon A)$$

for some $M_x > 0$. Thus

$$\begin{aligned}\|T_x(y + I_f)\|^2 &= \|xy + I_f\|^2 \\ &= \langle xy + I_f,\ xy + I_f\rangle \\ &= f(y^*x^*xy) \\ &\leq M_x\ f(y^*y) \\ &= M_x\ \|y + I_f\|^2 \quad (y\varepsilon A)\end{aligned} \tag{1}$$

from which it follows that $T_x$ is continuous on $X_f$, hence extendable to an operator in $\mathcal{B}(H_f)$ which we again denote by $T_x$.

Since $T_x \varepsilon \mathcal{B}(H_f)$, $(T_x)^*$ exists as an operator in $\mathcal{B}(H_f)$, and to show that $(T_x)^* = T_{x^*}$, it suffices to show that $(T_x)^*(y+I_f) = T_{x^*}(y+I_f)$ for all $y\varepsilon A$. But this follows from the following:

$$\begin{aligned}\langle y + I_f,\ (T_x)^*\ (z + I_f)\rangle &= \langle T_x(y + I_f),\ z + I_f\rangle \\ &= f(z^*xy) = f((x^*z)^*y) \\ &= \langle y + I_f,\ x^*z + I_f\rangle \\ &= \langle y + I_f,\ T_{x^*}(z + I_f)\rangle \quad (y,z\varepsilon A).\end{aligned}$$

(b) If there is a neighbourhood V of 0 in A and an $M>0$ such that $\sup_\alpha \overline{\lim_{n\to\infty}} \{p_\alpha[(x^*x)^n]\}^{1/n} \leq M$ $(x\varepsilon V)$, then as seen in the proof of Lemma 3.7,

$$f(y^*x^*xy) \leq M_1\ f(y^*y) \quad (x\varepsilon V,\ y\varepsilon A), \tag{2}$$

where $M_1 = \max(1, M)$. Now if $x\varepsilon A$, choose $\delta_x > 0$ such that

$\delta_x\, x \in V$ ($\delta_x$ exists because V is absorbing). Thus

$$f(y^*x^*xy) \le \frac{M_1}{\delta_x^2}\; f\,(y^*y)$$

and so from (1), $T_x$ is continuous on $X_f$. Therefore the conclusion in (a) holds for each $x\epsilon A$, i.e., there is a *-representation of A in $\mathcal{B}(H_f)$.

Moreover, given $\varepsilon>0$, by (2) $f(y^*x^*xy) \le \varepsilon^2 f(y^*y)$ $(x\epsilon \frac{\varepsilon}{\sqrt{M_1}} V,\ y\epsilon A)$. Hence from (1), $\|T_x(y + I_f)\|^2 \le \varepsilon^2 \|y + I_f\|^2$ $(x\epsilon \frac{\varepsilon}{\sqrt{M_1}}\ V,\ y\epsilon A)$, i.e., $\|T_x\| \le \varepsilon$ if $x\epsilon \frac{\varepsilon}{\sqrt{M_1}} V$. Therefore the map $x \to T_x$ is continuous.

(c) If A has identity e, let $a = e + I_f$. Since the set $\{T_x a:\ x\epsilon A\}$ is exactly $X_f$, a is strictly cyclic. Further, $\langle T_x\, a, a\rangle = \langle x + I_f,\ e + I_f\rangle = f(e^*x) = f(x)\quad (x\epsilon A)$.

<u>Corollary 3.2</u> Let $[A, \{p_\alpha\}_{\alpha\epsilon\Gamma}]$ be a sequentially complete topological *-algebra with either (i) locally separately continuous involution, or (ii) jointly continuous multiplication and locally pseudocontinuous involution. Let f be a positive functional on A. Then we have the following:

(a) if each $x\epsilon A$ satisfies $\sup_\alpha \overline{\lim_{n\to\infty}} \{p_\alpha[(x^*x)^n]\}^{1/n} < \infty$, then f induces a *-representation $x \to T_x$ of A on a Hilbert space H. Further, if A has identity, then the representation is topologically cyclic with a topologically cyclic vector $a\epsilon H$ such that $f(x) = \langle T_x\, a, a\rangle \quad (x\epsilon A)$.

(b) If A has property $\mathfrak{Q}$, then the *-representation in (a) is continuous.

<u>Proof</u> If follows immediately from Theorem 3.1.

<u>Corollary 3.3</u> Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a sequentially complete locally m-convex Q-algebra with locally pseudocontinuous involution. Then each positive functional f on A induces a continuous *-representation $x \to T_x$ of A on a Hilbert space H. Further, if A has identity, then the representation is topologically cyclic with a topologically cyclic vector $a \varepsilon H$ such that

$$f(x) = \langle T_x a, a\rangle \qquad (x \varepsilon A).$$

<u>Proof</u> This follows directly from Corollary 3.2 since the hypothesis implies that A has property $\mathfrak{Q}$.

<u>Remark</u>. Brooks ([4] Theorem 6.1) proved the conclusions in Corollary 3.3 for a complete locally m-convex algebra with identity and continuous involution such that f is a continuous positive functional on A. While we impose an extra condition that A be a Q-algebra, we relax other conditions considerably; particularly that f need not be continuous.

<u>Theorem 3.4</u> Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a sequentially complete bounded algebra with locally separately continuous involution. Then each positive functional f on A induces a *-representation $x \to T_x$ of A on a Hilbert space H. If A has identity, then the

representation is topologically cyclic with a topologically cyclic vector $a \in H$ such that $f(x) = \langle T_x a, a \rangle$ $(x \in A)$. If $A$ is a commutative Q-algebra, then the *-representation is continuous.

Proof The first and second parts follow from Corollary 3.2 (a) and [1] Proposition 2.14 and Proposition 2.18. The last part follows from Corollary 3.2 (b).

Remark The corresponding results of Theorem 3.4 for BP*-algebras may be found in [16] Theorem 4.2.

As a corollary to Theorem 3.4 and a supplement to Corollary 3.3, we have

Corollary 3.5 Let $[A, \{p_\alpha\}_{\alpha \in \Gamma}]$ be a commutative sequentially complete locally m-convex Q-algebra with locally separately continuous involution. Then each positive functional $f$ on $A$ induces a continuous *-representation $x \to T_x$ of $A$ on a Hilbert space $H$. If $A$ has identity, then the representation is topologically cyclic with a topologically cyclic vector $a \in H$ such that

$$f(x) = \langle T_x a, a \rangle \qquad (x \in A).$$

Proof As seen before, a locally m-convex Q-algebra is bounded, and so it follows from Theorm 3.4.

Corollary 3.6 Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a sequentially complete uniformly A-convex algebra with locally separately continuous involution and f a positive functional on A. Then f induces a *-representation $x \to T_x$ of A on a Hilbert space H. If A has identity, then the representation is topologically cyclic with a topologically cyclic vector $a\varepsilon H$ such that $f(x) = \langle T_x a, a\rangle$ $(x\varepsilon A)$. If A is a commutative Q-algebra, then the *-representation is continuous.

Proof This follows immediately from Theorem 3.4 since a uniformly A-convex algebra is bounded, and has property $\mathcal{Q}$ if, in addition, it is a commutative sequentially complete Q*-algebra.

Example 3.7 As seen in Example 2.24 $[L^\infty[0,1]\ \{\|\cdot\|_p\}_{p=1}^\infty]$ is bounded, but neither complete nor a Q-algebra. We show that the positive functional

$$F(f) = \int_0^1 \frac{f(t)\ dt}{t\ (\log t - 1)^2}$$

does not induce a continuous *-representation. Thus Theorem 3.4 could fail when certain conditions are dropped.

If $F(\bar{f}f) = 0$, then $f = 0$ a.e. in $[0,1]$, so that $I_F = \{0\}$ and $X_F = L^\infty[0,1]$. The map $f \to T_f$ of $L^\infty[0,1]$ into $\mathcal{L}(X_F) = \mathcal{L}(L^\infty[0,1])$ defined by $T_f g = fg$ $(g\varepsilon L^\infty[0,1])$ is a representation. We show that each $T_f$ is continuous on $L^\infty[0,1]$ with respect to the inner product topology, where

$$\langle g, h\rangle = F(\bar{h}g) = \int_0^1 \frac{\bar{h}(t)\, g(t)}{t\,(\log t - 1)^2}\, dt.$$

Let $\{g_n\}_{n=1}^{\infty}$ be a sequence in $L^{\infty}[0,1]$ such that $\langle g_n, g_n\rangle \to 0$ as $n \to \infty$. Then

$$\langle T_f g_n, T_f g_n\rangle = \langle fg_n, fg_n\rangle = \int_0^1 \frac{|f(t)|^2\, |g_n(t)|^2}{t(\log t - 1)^2}\, dt.$$

$$\leq (\|f\|_{\infty})^2 \int_0^1 \frac{|g_n(t)|^2}{t\,(\log t - 1)^2}\, dt = (\|f\|_{\infty})^2 \langle g_n, g_n\rangle \to 0$$

as $n \to \infty$ where $\|f\|_{\infty}$ denotes the $L^{\infty}$-norm.

Therefore each $T_f$ is extendable to an operator in $\mathscr{B}(H_F)$, which we again denote by $T_f$. That $T_{\bar{f}} = (T_f)^*$ follows from the following:

$$\langle h, T_{\bar{f}}g\rangle = \int_0^1 \frac{f(t)\, \bar{g}(t)\, h(t)}{t\,(\log t - 1)^2}\, dt$$

$$= \langle T_f h, g\rangle = \langle h, (T_f)^* g\rangle$$

$$(g, h \in L^{\infty}[0,1]).$$

The representation $f \to T_f$ is therefore a $*$-representation of $L^{\infty}[0,1]$ in $\mathscr{B}(H_F)$. Since $L^{\infty}[0,1]$ has identity 1, we have $F(f) = \langle T_f 1, 1\rangle$ $(f \in L^{\infty}[0,1])$.

We claim that the $*$-representation $f \to T_f$ is not continuous. For, otherwise $f_n \to f$ in $[L^{\infty}[0,1], \{\|\cdot\|_p\}_{p=1}^{\infty}]$ implies that

$T_{f_n}1 \to T_f 1$ in $H_F$ and so $F(f_n) = \langle T_{f_n}1, 1\rangle \to \langle T_f 1, 1\rangle = F(f)$. This implies that F is continuous on $[L^\infty[0,1], \{\|\cdot\|_p\}_{p=1}^\infty]$, a contradiction to Example 2.25.

Remark The algebra $C^\infty(I)$ given in Example 2.20 satisfies the hypotheses of Corollary 3.3 and Corollary 3.5. Therefore, every positive functional F on $C^\infty(I)$ induces a continuous *-representation $f \to T_f$ of $C^\infty(I)$ on a Hilbert space H. Moreover, the representation is topologically cyclic with a topologically cyclic vector $a \in H$ so that $F(f) = \langle T_f a, a\rangle$ $(f \in C^\infty(I))$.

Theorem 3.8 Let (A,d) be a pseudomultiplicative F-algebra with locally pseudocontinuous involution. Then every positive functional f on A induces a continuous *-representation $x \to T_x$ of A on a Hilbert space H. If A has identity, then the representation is topologically cyclic with a topologically cyclic vector $a \in H$ such that $f(x) = \langle T_x a, a\rangle$ $(x \in A)$.

Proof As in the proof of Theorem 3.1, we define $x \to T_x$ of A into $\mathcal{L}(X_f)$ by $T_x(y + I_f) = xy + I_f$ $(y \in A)$.

Choose a neighbourhood V of 0 in A such that $d(x^*x, 0) < 1$ $(x \in V)$. Then by Lemma 2.29, $f(y^*x^*xy) \le f(y^*y)$ $(x \in V, y \in A)$. That $x \to T_x$ is a continuous *-representation follows now exactly as in the proof of Theorem 3.1 (b).

If A has identity e, since each $T_x \in \mathcal{B}(H_f)$ and $\{T_x(e+I_f): x \in A\} = X_f$ is dense in $H_f$, the *-representation is topologically cyclic with $a=e+I_f$ as the topologically cyclic vector.

Corollary 3.9 Let A be a complete p-normed algebra with identity and locally pseudocontinuous involution. Then every positive functional f on A induces a continuous *-representation $x \to T_x$ of A on a Hilbert space H. The representation is topologically cyclic with a topologically cyclic vector $a \in H$ such that

$$f(x) = \langle T_x a, a\rangle \quad (x \in A).$$

Proof This follows from Theorem 3.8 since the p-norm may be assumed to be submultiplicative when A has identity.

Remark (i) Since a locally bounded algebra A is p-normable for some $p \in (0,1]$, the same conclusion in Corollary 3.9 holds if A is complete and has identity and locally pseudocontinuous involution.

(ii) The algebra $W_p$ given in Example 2.38 satisfies the hypotheses of Corollary 3.9.

## §2 Continuity of *-representations

Our aim now is to investigate those topological *-algebras A which have the property that every *-representation of A on a Hilbert space is necessarily continuous.

Definition 3.10 A positive functional f on a *-algebra A is said to be extendable if f is hermitian and if there exists $\mu > 0$ such that

$$|f(x)|^2 \leq \mu f(x^*x) \quad (x \varepsilon A). \tag{1}$$

Let f be a positive functional on a *-algebra A. Then a necessary and sufficient condition for the existence of a positive functional $\tilde{f}$ on $\tilde{A}$ such that $\tilde{f}|_A = f$ is that f be extendable. Moreover, if (1) holds, $\tilde{f}(e)$ may be taken to be equal to $\mu$ (Hewitt and Ross [13], pp. 317). Further, it can be seen from the construction of $\tilde{f}$ that if A is a topological *-algebra, then f is continuous on A iff $\tilde{f}$ is continuous on $\tilde{A}$.

<u>Lemma 3.11</u> If A is a topological *-algebra with locally continuous involution, then $\tilde{A}$ has locally continuous involution.

<u>proof</u> Let $\mathfrak{M}$ be a maximal commutative *-subalgebra of $\tilde{A}$. Suppose $\{(x_\nu, \zeta_\nu)\}$ is a net in $\mathfrak{M}$ converging to $(x,\zeta) \varepsilon \mathfrak{M}$. Let $\mathfrak{M}_0 = \{z: (z,\mu) \varepsilon \mathfrak{M}\}$. Then $\mathfrak{M}_0$ is a maximal commutative *-subalgebra of A as is easy to see. Since $x_\nu \to x$ in $\mathfrak{M}_0$ and $\zeta_\nu \to \zeta$ in $\mathbb{C}$, we have $x_\nu^* \to x^*$ and so

$$(x_\nu, \zeta_\nu)^* = (x^*, \bar{\zeta}) \to (x^*, \bar{\zeta}) = (x,\zeta)^*.$$

<u>Theorem 3.12</u> Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a sequentially complete topological *-algebra which has property $\mathfrak{A}$ and locally continuous involution. Let T be a *-representation of A on a Hilbert space H. Then T is continuous.

<u>Proof</u> Let $\zeta$ be a nonzero element of H and put $f(x) = \langle T_x\zeta, \zeta\rangle (x\varepsilon A)$.

Then for any $x \epsilon A$, $0 \leq \|T_x \zeta\|^2 = f(x^*x)$ so that f is a positive functional on A. Since $f(x^*) = \langle T_{x^*}\zeta, \zeta\rangle = \langle \zeta, T_x\zeta\rangle = \overline{f(x)}$ and

$$|f(x)|^2 = |\langle T_x\zeta,\zeta\rangle|^2 \leq \|\zeta\|^2 \|T_x\zeta\|^2 = \|\zeta\|^2 f(x^*x) \tag{2}$$

f is extendable. Let $\tilde{f}$ be the extension of f to $\tilde{A}$. Since $[\tilde{A}, \{\tilde{p}_\alpha\}_{\alpha\epsilon\Gamma}]$ satisfies the hypothesis of Lemma 2.7 and $\sup_\alpha \overline{\lim_{n\to\infty}} [\tilde{p}_\alpha\{(x^*x)^n\}]^{1/n} = \sup_\alpha \overline{\lim_{n\to\infty}} [p_\alpha\{(x^*x)^n\}]^{1/n} \leq M$ $(x\epsilon V)$, we have $f(x^*x) = \tilde{f}(x^*x) \leq M_1\tilde{f}(e) = M_1\|\zeta\|^2$ $(x\epsilon V)$ where $M_1 = \max(1,M)$ and the last equality follows from (2). Let $\varepsilon>0$ be given. Then $\|T_x\zeta\|^2 = f(x^*x) \leq \varepsilon^2\|\zeta\|^2$ if $x\epsilon \frac{\varepsilon}{\sqrt{M_1}} V$, or $\|T_x\zeta\| \leq \varepsilon\|\zeta\|$ $(x\epsilon\frac{\varepsilon}{\sqrt{M_1}} V)$. Since $\zeta$ is an arbitrary non-zero element of H, we have $\|T_x\| \leq \varepsilon$ $(x\epsilon\frac{\varepsilon}{\sqrt{M_1}} V)$.

As an immediate corollary to Theorem 3.12, the following corollary also slightly generalises Theorem 3 [14] with continuous involution there replaced by locally continuous involution.

<u>Corollary 3.13</u> Let $[A, \{p_\alpha\}_{\alpha\epsilon\Gamma}]$ be a sequentially complete locally m-convex Q-algebra with locally continuous involution. Then every *-representation of A on a Hilbert space is continuous.

<u>Theorem 3.14</u> Let $[A, \{p_\alpha\}_{\alpha\epsilon\Gamma}]$ be a commutative sequentially complete bounded Q-algebra with locally continuous involution. Then every *-representation of A on a Hilbert space is continuous.

<u>Proof</u> Immediate from Theorem 3.12 as the hypothesis implies that A has property $\mathfrak{Q}$.

Corollary 3.15 Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a commutative sequentially complete uniformly A-convex Q-algebra with locally continuous involution. Then every *-representation of A on a Hilbert space is continuous.

Proof Immediate from Theorem 3.14.

Remark (i) Example 3.7 shows that when certain conditions are dropped from Theorem 3.14, there exists a discontinuous *-representation of A on a Hilbert space.

(ii) The algebra $C^\infty(I)$ of Example 2.20 satisfies the hypotheses of Corollary 3.13, so that every *-representation of $C^\infty(I)$ on a Hilbert space is continuous.

Theorem 3.16 Let (A, d) be a pseudomultiplicative F-algebra with identity and locally pseudocontinuous involution. Then every *-representation T of A on a Hilbert space H is continuous.

Proof The proof follows closely that of Theorem 3.12. Let $f(x) = \langle T_x\zeta, \zeta\rangle$ $(x\varepsilon A)$, where $\zeta$ is a nonzero element of H. Then f is a positive functional on A. Choose a neighbourhood V of 0 in A such that $d(x^*x, 0) < 1$ $(x\varepsilon V)$. By Lemma 2.29, $f(x^*x) \leq f(e)$ $(x\varepsilon V)$. Let $\varepsilon>0$ be given. Then

$$\|T_x\zeta\|^2 = f(x^*x) \leq \varepsilon^2 f(e) = \varepsilon^2\langle T_e\zeta, \zeta\rangle = \varepsilon^2\langle\zeta, \zeta\rangle = \varepsilon^2\|\zeta\|^2$$

$(x\varepsilon\varepsilon V)$. Hence $\|T_x\| \leq \varepsilon$ $(x\varepsilon\varepsilon V)$.

The result in Theorem 3.16 can be improved in the case of a p-normed algebra as we see below.

Theorem 3.17 Let $(A, \|\cdot\|_p)$ be a complete p-normed algebra with locally continuous involution. Then every *-representation T of A on a Hilbert space H is continuous.

Proof Let $f(x) = \langle T_x\zeta, \zeta\rangle$ where $0 \neq \zeta\varepsilon H$. As seen in the proof of Theorem 3.12, f is an extendable positive functional on A. Let $\tilde{f}$ be the extension of f to $\tilde{A}$. We note that $\tilde{A}$ is also a complete p-normed algebra with locally continuous involution if we define $\|(x, \alpha)\|_p' = \|x\|_p + |\alpha|^p$. Since $\tilde{A}$ has identity, we may assume without loss of generality that $\|\cdot\|_p'$ is submultiplicative ([30], Theorem 1).

Since A has jointly continuous multiplication and locally continuous involution, it has locally pseudocontinuous involution. Thus choose a neighbourhood V of 0 such that $\|x^*x\|_p < 1$ $(x\varepsilon V)$. Since $(A, \|\cdot\|_p')$ satisfies the hypothesis of Lemma 2.29 and $\|x^*x\|_p' = \|x^*x\|_p < 1$ $(x\varepsilon V)$, we have $f(x^*x) = \tilde{f}(x^*x) \leq \tilde{f}(e) = \|\zeta\|^2$ $(x\varepsilon V)$. The rest follows exactly as in the proof of Theorem 3.12.

Remark (i) Theorem 3.17 holds for a complete locally bounded algebra with locally continuous involution.

(ii) The algebra $W_p$ of Example 2.38 satisfies the hypotheses of Theorem 3.17.

## §3 Representability of positive functionals

In this section we shall discuss conditions for a positive functional to be representable.

Lemma 3.18 Let $x \to T_x$ be any *-representation of a *-algebra $A$ on a Hilbert space $H$. Let $a$ be a nonzero vector in $H$ and define

$$f(x) = \langle T_x a, a\rangle \quad (x \in A).$$

Then there exists a closed invariant subspace $H_0$ of $H$ such that $f$ is representable by the restriction of $x \to T_x$ to $H_0$.

Proof [22] pp. 216, Lemma 4.5.6.

Theorem 3.19 Let $[A, \{p_\alpha\}_{\alpha\in\Gamma}]$ be a sequentially complete topological *-algebra with locally continuous involution such that $\sup_\alpha \overline{\lim_{n\to\infty}} \{p_\alpha[(x^*x)^n]\}^{1/n} < \infty$ for each $x\in A$. Let $f$ be a hermitian functional on $A$. Then $f$ is representable iff $|f(x)|^2 \leq \mu f(x^*x)$ $(x\in A)$ where $\mu$ is a positive constant independent of $x$.

Proof Suppose $x \to T_x$ is a *-representation of $A$ on a Hilbert space $H$ such that $f(x) = \langle T_x a, a\rangle$ $(x\in A)$ for some topologically cyclic vector $a\in H$. Then

$$|f(x)|^2 = |\langle T_x a,a\rangle|^2 \leq \|T_x a\|^2\|a\|^2 = \|a\|^2 f(x^*x) \quad (x\in A).$$

Conversely, let $[\tilde{A}, \{\tilde{p}_\alpha\}_{\alpha\epsilon\Gamma}]$ be the *-algebra with an identity adjoined. Extend f to a linear functional $\tilde{f}$ on $\tilde{A}$ by defining

$$\tilde{f}((x,\alpha)) = f(x) + \mu\alpha, \text{ for } (x,\alpha)\epsilon\tilde{A}. \quad \text{Since}$$

$$\begin{aligned}
\tilde{f}((x,\alpha)^*(x,\alpha)) &= \tilde{f}(x^*x + \bar{\alpha}x + \alpha x^*, |\alpha|^2) \\
&= f(x^*x) + \bar{\alpha}f(x) + \alpha f(x^*) + \mu|\alpha|^2 \\
&= f(x^*x) + \bar{\alpha}f(x) + \alpha\overline{f(x)} + \mu|\alpha|^2 \\
&\geq f(x^*x) - 2|\alpha|\;|f(x)| + \mu|\alpha|^2 \\
&\geq f(x^*x) - 2|\alpha|\mu^{1/2}\;(f(x^*x))^{1/2} + \mu|\alpha|^2 \\
&= (f(x^*x) - |\alpha|\mu^{1/2})^2 \geq 0,
\end{aligned}$$

$\tilde{f}$ is a positive functional on $\tilde{A}$.

We may assume $f \neq 0$ so that $\tilde{f} \neq 0$. By Theorem 3.1, $\tilde{f}$ induces a representation $T\colon\ x \to T_x$ of $\tilde{A}$ in $\mathcal{L}(\tilde{A}/I_{\tilde{f}})$ such that $\tilde{f}(x) = \langle T_x a, a\rangle$ $(x\epsilon\tilde{A})$ for some nonzero vector $a\epsilon\tilde{A}/I_{\tilde{f}} \subset H_{\tilde{f}}$ = completion of $\tilde{A}/I_{\tilde{f}}$.

Since $\sup_\alpha \overline{\lim_{n\to\infty}} \{\tilde{p}_\alpha[(x^*x)^n]\}^{1/n} = \sup_\alpha \overline{\lim_{n\to\infty}} \{p_\alpha[(x^*x)^n]\}^{1/n} < \infty$ $(x\epsilon A)$, by Theorem 3.1(a), the restriction of T on A is a *-representation of A in $\mathcal{B}(H_{\tilde{f}})$. As $f(x) = \tilde{f}(x) = \langle T_x a, a\rangle$ $(x\epsilon A)$, the result follows from Lemma 3.18.

It is clear from the proof of Theorem 3.19 that a representable positive functional is extendable. As an immediate consequence of Theorem 3.19, we have

Corollary 3.20 Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a sequentially complete topological *-algebra with locally continuous involution and such that $\sup_\alpha \overline{\lim_{n\to\infty}} \{p_\alpha[(x^*x)^n]\}^{1/n} < \infty$ $(x\varepsilon A)$. Let f be a positive functional on A. Then f is representable iff f is extendable.

The following corollaries are now immediate from Corollary 3.20. The first is a slight generalization of Theorem 1 [14].

Corollary 3.21 Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a locally m-convex *-algebra with locally continuous involution such that A is either (i) complete and $\sigma_A(x^*x) < \infty$ $(x\varepsilon A)$ or (ii) a sequentially complete Q-algebra. Then a positive functional on A is representable iff it is extendable.

Corollary 3.22 Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a sequentially complete bounded algebra with locally continuous involution. Then a positive functional on A is representable iff it is extendable.

Corollary 3.23 Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a sequentially complete uniformly A-convex *-algebra with locally continuous involution, and f a positive functional on A. Then f is representable iff f is extendable.

So far in our discussion in this section, we have not assumed the existence of an identity in the algebra. The following corollary gives sufficient conditions for the representability of a positive functional when the algebra possesses an identity.

Corollary 3.24 Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a sequentially complete topological *-algebra with identity such that $\sup_\alpha \overline{\lim_{n\to\infty}} \{p_\alpha[(x^*x)^n]\}^{1/n} < \infty$ $(x\varepsilon A)$. Suppose A has either (i) locally separately continuous involution, or (ii) jointly continuous multiplication and locally pseudocontinuous involution. Then every positive functional on A is representable.

Proof This follows directly from Corollary 3.2 (a).

Thus the conclusion of Corollary 3.24 holds for a sequentially complete *-algebra A with identity which has either (i) locally separately continuous involution, or (ii) jointly continuous multiplication and locally pseudocontinuous involution; and which is either (a) bounded, in particular a locally m-convex Q-algebra and a uniformly A-convex algebra or (b) complete locally m-convex such that $\sigma_A(x^*x) < \infty$ $(x\varepsilon A)$.

Theorem 3.25 Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a sequentially complete topological *-algebra with continuous involution such that $\sup_\alpha \overline{\lim_{n\to\infty}} \{p_\alpha[(x^*x)^n]\}^{1/n} < \infty$ $(x\varepsilon A)$. Suppose A has a right approximate identity $\{e_\lambda\}$ such that the set $\{e_\lambda^* e_\lambda\}$ is bounded. Then every continuous positive functional on A is representable.

Proof Since the involution map is continuous,
$\lim_\lambda e_\lambda^* x = \lim_\lambda (x^* e_\lambda)^* = x^{**} = x.$

The positivity of f implies that $f(x^*e_\lambda) = \overline{f(e_\lambda^* x)}$ for each $\lambda$. Since f is continuous, we have $f(x^*) = \overline{f(x)}$, so that f is hermitian.

By continuity of f, there exists a neighbourhood U of 0 such that $|f(x)| \leq 1$ $(x \varepsilon U)$. Choose $\kappa > 0$ such that $\{e_\lambda * e_\lambda\} \subset \kappa U$. Then $f(e_\lambda * e_\lambda) \leq \kappa$ for all $\lambda$.

By the Cauchy-Schwarz inequality, we have $|f(x^* e_\lambda)|^2 \leq f(x^* x)\, f(e_\lambda * e_\lambda) \leq \kappa\, f(x^* x)$ $(x \varepsilon A)$, where $\kappa > 0$ is independent of $\lambda$ and x. Passing to the limit, we have $|f(x)|^2 = |f(x^*)|^2 \leq \kappa\, f(x^* x)$ $(x \varepsilon A)$. By Theorem 3.19, f is representable.

<u>Remarks</u> (i) If a topological *-algebra $[A, \{p_\alpha\}_{\alpha \varepsilon \Gamma}]$ satisfies $p_\alpha(x^* x) = [p_\alpha(x)]^2$ $(x \varepsilon A,\ \alpha \varepsilon \Gamma)$, then boundedness of $\{e_\lambda\}$ is equivalent to boundedness of $\{e_\lambda * e_\lambda\}$.

(ii) As seen in the proof of Theorem 3.25, if $\{e_\lambda\}$ is an approximate identity consisting of hermitian elements, then the condition on "continuous involution" is not required.

(iii) If $\{e_\lambda\}$ is bounded and involution is pseudocontinuous at 0, then the set $\{e_\lambda * e_\lambda\}$ is bounded. For, given any neighbourhood U of 0, choose a neighbourhood V of 0 such that $x^* x \varepsilon U$ $(x \varepsilon V)$. Choose $\alpha > 0$ so that $\{e_\lambda\} \subset \alpha V$. Then $\{e_\lambda * e_\lambda\} \subset \alpha^2 U$.

(iv) A topological *-algebra $[A, \{p_\alpha\}_{\alpha \varepsilon \Gamma}]$ satisfying $p_\alpha(x^* x) \leq \kappa_\alpha [p_\alpha(x)]^2$ for some $\kappa_\alpha > 0$ $(x \varepsilon A,\ \alpha \varepsilon \Gamma)$ has the property that involution is pseudocontinuous at the origin.

<u>Corollary 3.26</u> Let $[A, \{p_\alpha\}_{\alpha \varepsilon \Gamma}]$ be a locally m-convex *-algebra with continuous involution and a bounded right approximate identity. Suppose A is either (i) complete and $\sigma_A(x^* x) < \infty$ $(x \varepsilon A)$ or (ii) a sequentially complete Q-algebra.

Then every continuous positive functional on A is representable.

<u>Proof</u> This follows from Theorem 3.25, the fact that jointly continuous multiplication and continuous involution implies pseudocontinuous involution, and Remark (iii) after Theorem 3.25.

<u>Corollary 3.27</u> Let $[A, \{p_\alpha\}_{\alpha\epsilon\Gamma}]$ be a sequentially complete topological *-algebra with identity such that $\sup_\alpha \overline{\lim_{n\to\infty}} \{p_\alpha[(x^*x)^n]\}^{1/n} < \infty$ $(x\epsilon A)$. Then every continuous positive functional on A is representable.

<u>Proof</u> Follows easily from Theorem 3.25 and Remark (ii) after the theorem.

It suffices to say that the conclusion of Corollary 3.27 holds for a sequentially complete *-algebra A with identity which is either (i) bounded, in particular a locally m-convex Q-algebra and a uniformly A-convex algebra or (ii) complete locally m-convex such that $\sigma_A(x^*x) < \infty$ $(x\epsilon A)$.

We now turn to a brief discussion of the representability of positive functionals on non-locally convex *-algebras.

<u>Corollary 3.28</u> Let A be a pseudomultiplicative F-algebra (or a complete p-normed algebra) with identity and locally pseudocontinuous involution. Then every positive functional on A is representable.

<u>Proof</u> By Theorem 3.8 and Corollary 3.9.

For a p-normed algebra, not necessarily possessing an identity, we have parallels of Theorem 3.19 and Corollary 3.20.

Theorem 3.29 Let A be a complete p-normed algebra with locally continuous involution. Then a hermitian functional f on A is representable iff $|f(x)|^2 \leq \mu f(x^*x)$ $(x \varepsilon A)$ for some $\mu > 0$ independent of x. Thus a positive functional is representable iff it is extendable.

Proof We follow the proof of Theorem 3.19. The necessity emerges trivially. For sufficiency, we define $\tilde{f}((x,\alpha)) = f(x) + \mu\alpha$ on $\tilde{A}$. Then $\tilde{f}$ is a positive functional on $\tilde{A}$. By Corollary 3.9, $\tilde{f}(x) = \langle T_x a, a \rangle$ $(x \varepsilon \tilde{A})$ for some topologically cyclic vector a in a Hilbert space H, where $x \to T_x$ is a *-representation of $\tilde{A}$ on H induced by $\tilde{f}$. The result now follows from Lemma 3.18.

Remark Theorem 3.29 holds for a complete locally bounded algebra with locally continuous involution.

Theorem 3.30 Let A be a complete p-normed algebra with continuous involution. Suppose A has a right approximate identity $\{e_\lambda\}$ such that the set $\{e_\lambda^* e_\lambda\}$ is bounded. Then every continuous positive functional on A is representable.

Proof This follows from Theorem 3.29 and the proof of Theorem 3.25.

<u>Remark</u> By Remark (iii) after Theorem 3.25, we note that the condition that $\{e_\lambda * e_\lambda\}$ be bounded in Theorem 3.30 is weaker than requiring $\{e_\lambda\}$ be bounded.

<u>Corollary 3.31</u> Let A be a complete p-normed algebra with identity. Then every continuous positive functional on A is representable.

<u>Proof</u> By Theorem 3.30 and Remark (ii) after Theorem 3.25.

Comparing respectively Corollary 3.24 and Corollary 3.27, and Corollary 3.28 and Corollary 3.31, we note the interesting fact that the continuity of involution and the continuity of positive functional are interchangeable.

## CHAPTER 4

## On Algebras of the Segal Type

### §1 Introduction

Reiter ([21] pp. 127) introduced the following notion of Segal algebras:

Let G be a locally compact abelian group. A Segal algebra S is a subalgebra of $L^1(G)$ satisfying the following conditions:

(S1) S is dense in $L^1(G)$ and is translation invariant (i.e., $f \varepsilon S \Rightarrow L_a f \varepsilon S$ for all $a \varepsilon G$, where $(L_a f)(x) = f(xa)$).

(S2) S is a Banach algebra under some norm $\|\cdot\|_S$ so that $\|L_a f\|_S = \|f\|_S$ for all $f \varepsilon S$, $a \varepsilon G$.

(S3) The map $a \to L_a f$ is continuous from G to $(S, \|\cdot\|_S)$.

On the basis of these assumptions the following properties may be deduced ([21], pp. 128):

(S4) There exists $M>0$ such that $\|f\|_{L^1} \leq M\|f\|_S$ $(f \varepsilon S)$

(S5) S is an ideal in $L^1(G)$ and

$$\|f * g\|_S \leq \|f\|_{L^1} \|g\|_S \quad (f \varepsilon L^1(G),\ g \varepsilon S).$$

Properties (S4) and (S5) prompted Cigler [6] to generalize the notion of Segal algebras to normed ideals. Accordingly, an ideal N in $L^1(G)$ is a normed ideal if the following conditions hold:

(N1) N is dense in $L^1(G)$.

(N2) N is a Banach space under some norm $\|\cdot\|_N$ such that $\|f\|_{L^1} \leq \|f\|_N$ $(f \in N)$.

(N3) $\|f*g\|_N \leq \|f\|_{L^1} \|g\|_N$ $(f \in L^1(G),\ g \in N)$.

The notion of normed ideals was further generalized by Burnham [5] to that of an abstract Segal algebra. Thus $(B, \|\cdot\|_B)$ is an abstract Segal algebra with respect to a Banach algebra $(A, \|\cdot\|_A)$ if the following axioms hold:

(1) B is a dense left ideal in A and B is a Banach algebra with respect to the norm $\|\cdot\|_B$.

(2) There exists $M>0$ such that

$$\|f\|_A \leq M\|f\|_B \quad (f \in B).$$

(3) There exists $C>0$ such that

$$\|fg\|_B \leq C\|f\|_A \|g\|_B \ (f \in A,\ g \in B).$$

In this chapter we give a brief study of the continuity of positive functionals and representations on *-algebras that have similar structure as that of abstract Segal algebras.

## §2 Continuity of positive functionals

We shall apply our study in Chapter 2 on the continuity of positive functionals to those algebras considered in this chapter.

Proposition 4.1 Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a locally m-convex *-algebra with continuous involution. Let B be a P-commutative *-subalgebra of A with $B^2 = B$. Suppose $\{q_\beta\}_{\beta\varepsilon\Lambda}$ is a family of seminorms defined on B such that $[B, \{q_\beta\}_{\beta\varepsilon\Lambda}]$ is a sequentially complete, barrelled, topological algebra, and

(i) for each $\alpha\varepsilon\Gamma$, there exist $M_\alpha>0$, $\beta\varepsilon\Lambda$ so that $p_\alpha(x) \le M_\alpha q_\beta(x)$ $(x\varepsilon B)$.

(ii) there exists $\alpha_0\varepsilon\Gamma$ so that for each $\beta\varepsilon\Lambda$, there exist $C_\beta>0$, $\nu\varepsilon\Lambda$ with $q_\beta(xy) \le C_\beta\, p_{\alpha_0}(x)\, q_\nu(y)$ $(x,y\varepsilon B)$.

Then every positive functional on $[B, \{q_\beta\}_{\beta\varepsilon\Lambda}]$ is continuous.

Proof The restriction of the map $x \to x^*$ on B is continuous with respect to the induced $\{p_\alpha\}$-topology. By (i) this topology is coarser than the $\{q_\beta\}$-topology. Hence $[B, \{q_\beta\}_{\beta\varepsilon\Lambda}]$ has continuous involution.

$$\text{By (ii), } q_\beta\{(x^*x)^n\} \le C_\beta p_{\alpha_0}\{(x^*x)^{n-1}\}q_\nu(x^*x) \le C_\beta\{p_{\alpha_0}(x^*x)\}^{n-1} q_\nu(x^*x).$$

Hence $\overline{\lim_{n\to\infty}} \{q_\beta(x^*x)^n\}^{1/n} \le p_{\alpha_0}(x^*x)$ $(\beta\varepsilon\Lambda)$ and so

$$\sup_\beta \overline{\lim_{n\to\infty}} \{q_\beta(x^*x)^n\}^{1/n} \le p_{\alpha_0}(x^*x) \quad (x\varepsilon B). \tag{1}$$

We show that the condition Ⓑ is satisfied by $[B, \{q_\beta\}_{\beta\varepsilon\Lambda}]$.

Observe that the joint continuity of multiplication and continuous involution imply that $x \to x^*x$ is continuous on $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$, hence continuous on B with respect to the induced $\{p_\alpha\}$-topology. Hence choose a $\{p_\alpha\}$-neighbourhood V of 0, which is also a $\{q_\beta\}$-neighbourhood of 0, in B such that $p_{\alpha_0}(x^*x) \leq 1$ whenever $x\varepsilon V$. Thus, by (1),

$\sup_\beta \overline{\lim_{n\to\infty}} [q_\beta\{(x^*x)^n\}]^{1/n} \leq 1$ $(x\varepsilon V)$. The result now follows from Theorem 2.11.

<u>Proposition 4.2</u> Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a locally m-convex *-algebra, and B a P-commutative *-subalgebra of A with $B^2 = B$. Suppose $\{q_\beta\}_{\beta\varepsilon\Lambda}$ is a family of seminorms defined on B such that $[B, \{q_\beta\}_{\beta\varepsilon\Lambda}]$ is a sequentially complete, barrelled, topological algebra, and (i) for each $\alpha\varepsilon\Gamma$, there exists $M_\alpha>0$, $\beta\varepsilon\Lambda$ so that $p_\alpha(x) \leq M_\alpha q_\beta(x)$ $(x\varepsilon B)$.

(ii) there exists $\alpha_0\varepsilon\Gamma$ so that for each $\beta\varepsilon\Lambda$ there exists $C_\beta > 0$ with $q_\beta(x^*y) \leq C_\beta p_{\alpha_0}(y)\, q_\beta(x)$ $(x,y\varepsilon B)$.

Then every positive functional on $[B, \{q_\beta\}_{\beta\varepsilon\Lambda}]$ is continuous.

<u>Proof</u> (ii) implies that $[B, \{q_\beta\}_{\beta\varepsilon\Lambda}]$ has separately continuous involution. For each $\beta\varepsilon\Lambda$, $q_\beta\{(x^*x)^n\} = q_\beta\{(x^*x)^*(x^*x)^{n-1}\}$ $\leq C_\beta p_{\alpha_0}\{(x^*x)^{n-1}\}q_\beta(x^*x) \leq C_\beta\{p_{\alpha_0}(x^*x)\}^{n-1} q_\beta(x^*x)$ so that

$$\sup_\beta \overline{\lim_{n\to\infty}} [q_\beta\{(x^*x)^n\}]^{1/n} \leq p_{\alpha_0}(x^*x) \qquad (x\varepsilon B). \tag{2}$$

Now (i) and (ii) imply that the map $x \to x^*x$ in $[B, \{q_\beta\}_{\beta\varepsilon\Lambda}]$ is continuous at the origin. Since the induced $\{p_\alpha\}$-topology in B is coarser than the $\{q_\beta\}$-topology, we may choose a neighbourhood V of 0 in $[B, \{q_\beta\}_{\beta\varepsilon\Lambda}]$ so that $p_{\alpha_0}(x^*x) \leq 1$ $(x\varepsilon V)$. Therefore, by (2), $[B, \{q_\beta\}_{\beta\varepsilon\Lambda}]$ has property $\mathfrak{Q}$. The result now follows from Theorem 2.11.

Proposition 4.3 Let $(A,d)$ be a metrizable *-algebra and $B$ a P-commutative *-subalgebra of $A$ with $B^2 = B$. Suppose $(B,\rho)$ is an F-algebra with locally pseudocontinuous involution, and

(i) there exists $M>0$ such that

$$d(x,0) \leq M\,\rho(x,0) \qquad (x \in B)$$

(ii) there exists $C>0$ such that

$$\rho(xy,\,0) \leq C\,d(x,0)\ \ \rho(y,0) \qquad (x,y \in B).$$

Then every positive functional on $(B,\rho)$ is continuous.

Proof (i) and (ii) imply that

$$\rho(xy,\,0) \leq CM\rho(x,0)\ \rho(y,0) \qquad (x,y \in B).$$

Replacing $\rho$ by the equivalent metric $\rho'$ given by $\rho'(x,y) = CM\,\rho(x,y)$, we see that $\rho'$ satisfies $\rho'(xy,0) \leq \rho'(x,0)\rho'(y,0)$ $(x,y \in B)$, hence is pseudomultiplicative. The result now follows from Theorem 2.32.

Proposition 4.4 Let $(A,d)$ be a metrizable *-algebra and $B$ a P-commutative *-subalgebra of $A$ with $B^2 = B$. Suppose $(B,\rho)$ is a pseudomultiplicative F-algebra, and

(i) there exists $M>0$ such that

$$d(x,0) \leq M\rho(x,0) \qquad (x \in B)$$

(ii) there exists $C>0$ such that

$$\rho(x^*y, 0) \leq Cd(x,0)\ \rho(y,0) \qquad (x,y\varepsilon B).$$

Then every positive functional on $(B,\rho)$ is continuous.

<u>Proof</u> Fix $x_0 \varepsilon B$ and let $\{x_n\}$ be a sequence converging to $x_0$ in $(B,\rho)$.

Then $\rho(x_n{}^*x_n - x_0{}^*x_0,\ 0)$

$$\leq \rho(x_n{}^*(x_n - x_0),\ 0) + \rho((x_n - x_0)^*x_0,\ 0)$$

$$\leq Cd(x_n,0)\rho(x_n - x_0, 0) + Cd(x_n - x_0, 0)\rho(x_0, 0)$$

$$\leq CM\rho(x_n,0)\rho(x_n - x_0, 0) + CM\rho(x_n - x_0, 0)\rho(x_0, 0)$$

$\to 0$ as $n \to \infty$ since $\{\rho(x_n,0)\}_{n=1}^{\infty}$ is bounded.

Therefore $(B,\rho)$ has pseudocontinuous involution. The result now follows from Theorem 2.32.

## §3 <u>Representations</u>

Our study in Chapter 3 on representations will now be applied to those algebras considered in this chapter.

<u>Proposition 4.5</u> Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a locally m-convex *-algebra, and B a *-subalgebra of A. Suppose $\{q_\beta\}_{\beta\varepsilon\Lambda}$ is a family of seminorms defined on B such that $[B, \{q_\beta\}_{\beta\varepsilon\Lambda}]$ is a sequentially complete topological algebra satisfying the following condition:

(i) there exists $\alpha_0 \varepsilon\Gamma$ so that for each $\beta\varepsilon\Lambda$, there exists $C_\beta > 0$ with $\quad q_\beta(x^*y) \leq C_\beta\ p_{\alpha_0}(y)\ q_\beta(x) \qquad (x,y\varepsilon B).$

Then every positive functional on B induces a *-representation of B on a Hilbert space H. If B has identity, then the *-representation is topologically cyclic with a topologically cyclic vector $a \varepsilon H$ such that

$$f(x) = \langle T_x a, a\rangle \quad (x \varepsilon B).$$

If, in addition to (i), we also have

(ii) for each $\alpha\varepsilon\Gamma$, there exists $M_\alpha > 0$, $\beta\varepsilon\Lambda$ so that $p_\alpha(x) \leq M_\alpha q_\beta(x)$ $(x\varepsilon B)$,

then the *-representation is continuous.

<u>Proof</u> Since condition (i) implies that the involution is separately continuous, the first part follows from Corollary 3.2 (a) and (2) in the proof of Proposition 4.2.

As seen in the proof of Proposition 4.2, (i) and (ii) imply that $[B, \{q_\beta\}_{\beta\varepsilon\Lambda}]$ has property $\mathcal{Q}$, so that the second part follows immediately from Theorem 3.2 (b).

<u>Proposition 4.6</u> Let $[A, \{p_\alpha\}_{\alpha\varepsilon\Gamma}]$ be a locally m-convex *-algebra with continuous involution, and B a *-subalgebra of A. Suppose $\{q_\beta\}_{\beta\varepsilon\Lambda}$ is a family of seminorms defined on B such that $[B, \{q_\beta\}_{\beta\varepsilon\Lambda}]$ is a sequentially complete topological algebra satisfying the following conditions:

(i) for each $\alpha\varepsilon\Gamma$, there exists $M_\alpha > 0$, $\beta\varepsilon\Lambda$ so that $p_\alpha(x) \leq M_\alpha q_\beta(x)$ $(x\varepsilon B)$.

(ii) there exists $\alpha_0 \varepsilon\Gamma$ so that for each $\beta\varepsilon\Lambda$, there exist $C_\beta > 0$, $\nu\varepsilon\Lambda$ with

$$q_\beta(xy) \leq C_\beta p_{\alpha_0}(x)\, q_\nu(y) \qquad (x,y\varepsilon B).$$

Then every *-representation of $[B, \{q_\beta\}_{\beta\varepsilon\Lambda}]$ on a Hilbert space is continuous. Moreover, a positive functional on B is representable iff it is extendable.

Proof From the proof of Proposition 4.1, $[B, \{q_\beta\}_{\beta\varepsilon\Lambda}]$ has property $\mathfrak{Q}$ and continuous involution. The results now follow from Theorem 3.12 and Corollary 3.20.

Proposition 4.7 Let (A,d) be a metrizable *-algebra and B a *-subalgebra of A. Suppose $(B,\rho)$ is an F-algebra with locally pseudocontinuous involution and

(i) there exists $M>0$ so that

$$d(x,0) \leq M\rho(x,0) \qquad (x\varepsilon B).$$

(ii) there exists $C>0$ so that

$$\rho(xy,0) \leq C\, d(x,0)\, \rho(y,0) \qquad (x,y\varepsilon B).$$

Then every positive functional on $(B,\rho)$ induces a continuous *-representation of $(B,\rho)$ on a Hilbert space H. If B has identity, then the representation is topologically cyclic with a topologically cyclic vector $a\varepsilon H$ such that $f(x) = \langle T_x a,a\rangle$ $(x\varepsilon B)$. Moreover, every *-representation of $(B,\rho)$ on a Hilbert space is continuous.

Proof By Theorem 3.8, Theorem 3.16 and the proof of Proposition 4.3.

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