NEUTRON STARS AND BLACK HOLES IN SCALAR-TENSOR GRAVITY

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By

MICHAEL W. HORBATSCH, M.Sc., B.Sc.

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AUTHOR: Michael W. Horbatsch, M.Sc. (McMaster University), B.Sc. (University of Toronto)

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Abstract

The properties of neutron stars and black holes are investigated within a class of alternative theories of gravity known as Scalar-Tensor theories, which extend General Relativity by introducing additional light scalar fields to mediate the gravitational interaction.

It has been known since 1993 that neutron stars in certain Scalar-Tensor theories may undergo 'scalarization' phase transitions. The *Weak Central Coupling* (WCC) expansion is introduced for the purpose of describing scalarization in a perturbative manner, and the leading-order WCC coefficients are calculated analytically for constant-density stars. Such stars are found to scalarize, and the critical value of the quadratic scalar-matter coupling parameter $\beta_s = -4.329$ for the phase transition is found to be similar to that of more realistic neutron star models.

The influence of cosmological and galactic effects on the structure of an otherwise isolated black hole in Scalar-Tensor gravity may be described by incorporating the *Miracle Hair Growth Formula* discovered by Jacobson in 1999, a perturbative black hole solution with scalar hair induced by time-dependent boundary conditions at spatial infinity. It is found that a double-black-hole binary (DBHB) subject to these boundary conditions is inadequately described by the Eardley Lagrangian and emits scalar dipole radiation.

Combining this result with the absence of observable dipole radiation from quasar OJ287 (whose quasi-periodic 'outbursts' are consistent with the predictions of a general-relativistic DBHB model at the 6% level) yields the bound $|\dot{\phi}/M_{\rm pl}| \lesssim (16\,{\rm days})^{-1}$ on the cosmological time variation of canonically-normalized light ($m \lesssim 10^{-23}\,{\rm eV}$) scalar fields at redshift $z \sim 0.3$.

Acknowledgements

Contents

1   Outline

Scalar-Tensor theories are alternatives to General Relativity which modify the widely-accepted theory by introducing additional light scalar particles to mediate the gravitational interaction while retaining the usual graviton. The principal motivation for considering these alternatives stems from theoretical attempts to unify gravity with quantum mechanics at high energies, such as string theory, whose low-energy limit has recently been shown to contain scalars sufficiently light to have interesting astrophysical consequences [9]. Scalar-Tensor theories also play an important role in classical gravitational physics by providing a concrete framework for interpreting tests of relativistic gravity in the strong-field regime, where a phenomenological theory-independent interpretation is often problematic.

The subject of this thesis is the structure of highly-relativistic compact objects, namely, neutron stars and black holes, within Scalar-Tensor theories of gravity. Two novel theoretical results are obtained — the *Weak Central Coupling* (WCC) formalism for the perturbative description of scalarization phase transitions in neutron stars, developed in section 3, and *Miracle Hair Growth* in a double-black-hole binary system induced by cosmological and/or galactic effects, which is the subject of section 5.

The thesis begins with a comprehensive discussion of the basic properties of Scalar-Tensor gravity in section 2. In 2.1, the motivation for this theory stemming from classical gravitational physics on the one hand, and quantum high-energy physics on the other, is reviewed, and the historical development of the theory is summarized. A survey of more recent work is also provided, in order to set the context for the novel theoretical results developed in later sections. In 2.2 and 2.3, the action and field equations of Scalar-Tensor gravity are presented in two different mathematical formulations commonly referred to as the Einstein and Jordan frames, respectively. Conservation laws are discussed in 2.4, while the weak-field limit is taken in 2.5 in order to identify the 'physical' Newton constant $\tilde{G}$, among other things. This section contains no new results, and experts should feel free to skip it.

The purpose of section 3 is to develop the *Weak Central Coupling*

(WCC) formalism for the description of neutron star scalarization. The Landau approach to phase transitions is followed [113], and the coefficients therein are expressed in terms of perturbative solutions to the equations of stellar structure. In 3.1, the phenomenon of spontaneous scalarization is explained in some detail, and the key ideas behind the WCC expansion are introduced. The field equations for a static spherically-symmetric perfect fluid are obtained in 3.2, and the external vacuum solution is described in 3.3. Subsequently, the equations of stellar structure are derived in 3.4, and the formulas for matching their solution to an exterior metric are obtained in 3.5. With all the required tools in hand, the WCC perturbative expansions are then introduced in 3.6, and the leading-order coefficients are expressed in terms of perturbative solutions to the equations of stellar structure. This formalism is finally applied to constant-density stars in 3.7, and the complete leading-order solution is expressed analytically in terms of Heun functions, enabling one to describe the scalarization of such stars without numerically solving the equations of stellar structure.

Section 4 concerns the formalism and analytical approximation methods employed in the description of a binary inspiral with small size-to-separation ratio $\epsilon \sim R/D$, as a prerequisite for the *Miracle Hair Growth* described in section 5. The non-relativistic two-body problem is briefly reviewed in 4.1, and the complications arising at the relativistic level are first introduced in 4.2. The formalism of dividing the two-body problem into the 'interior' and 'exterior' regions is explained in 4.3, and in particular it is demonstrated how the boundary conditions simplify in the 'point-particle' limit $\epsilon \to 0$. Subsequently, the work of Damour and Esposito-Farèse [50] on binary systems in Scalar-Tensor gravity and radiation therefrom is summarized in 4.4 and 4.5, respectively, and the foundations upon which this work was built are explained. The results of a recent Jordan-frame calculation by Alsing and collaborators [6] of the radiation flux in *massive* Scalar-Tensor theory are summarized in 4.6, and converted to the Einstein frame, confirming in particular that the radiated power is a frame-independent quantity up to re-scaling of units. Finally, the Effective Field Theory (EFT) approach to the two-body problem initi-

ated by Goldberger and Rothstein [78, 80] is motivated and introduced in 4.7. Omitting the details of Feynman diagrams and power counting, the discussion focuses on the fundamental role played by the point-particle Lagrangian, its derivative expansion, and the relation thereof to an expansion in powers of the size-to-separation ratio $\epsilon \sim R/D$. As a prerequisite for the material in section 5, the derivative expansion in Scalar-Tensor gravity is explicitly written down to second order.

Building heavily on the material of section 4, section 5 discusses the influence of cosmological and galactic effects on the orbital dynamics of a double-black-hole binary inspiral in Scalar-Tensor gravity, and the radiation emitted therefrom. Following a brief introduction in 5.1, it is explained in 5.2 how Jacobson's *Miracle Hair Growth Formula* implies that such effects cause isolated black holes to grow scalar hair. Subsequently, Jacobson's arguments are applied to a double-black-hole binary in 5.3, and it is shown that the truncation of the point-particle Lagrangian to zeroth order in the derivative expansion, namely, the Eardley Lagrangian, is inadequate to describe the orbital corrections induced by galactic and cosmological effects. While the difficult problem of calculating these corrections is left for future work, the leading-order scalar dipole radiation flux is calculated in a relatively straightforward manner. Finally, in 5.4, an application to quasar OJ287 (which has been successfully modelled as a DBHB within GR) is discussed, and a bound on the cosmological time evolution of light scalar fields at its redshift is obtained.

Finally, the key results are summarized in section 6, and prospects for future work are discussed.

Mathematical preliminaries and explanations of notation, units, and conventions, have all been relegated to appendix A.

2   Scalar-Tensor Gravity

2.1   Introduction

2.1.1   Motivation from Classical Gravitational Physics

General Relativity, the current widely-accepted standard theory of gravitation, arose out of the attempt to unify the Newtonian theory of gravity with the special theory of relativity. Shortly after being formulated by Einstein in 1916, General Relativity was tested in the solar system, and was found to correctly describe the bending of starlight in the gravitational field of the sun, as measured by Eddington during a total solar eclipse in 1919. Moreover, General Relativity was able to explain the anomalous precession of Mercury's orbit, which had been a long-standing unsolved problem in celestial mechanics.

Subsequent solar-system tests have become much more precise, and have confirmed the validity of General Relativity to a high degree of accuracy [198, 196, 199]. For instance, Very Long Baseline Interferometry (VLBI) [155] and the Very Long Baseline Array (VLBA) [73] have been used to measure the deflection of radio waves from distant sources in the sun's gravitation field, and have confirmed the validity of General Relativity to a few parts in 10^4. According to General Relativity, electromagnetic signals propagating through a gravitational field experience a time delay [154] in addition to being deflected. This delay has been measured for radio signals transmitted from Earth to the Cassini spacecraft, and found to agree with the general-relativistic prediction to a few parts in 10^5 [18].

While the accuracy of these solar-system tests is impressive, it is important to keep in mind that such tests only probe the *weak-field*, or *Post-Newtonian* limit of General Relativity, where relativistic effects are merely small corrections to Newtonian gravity.

In 1975, a remarkable new testing ground for relativistic gravity was discovered — the binary pulsar PSR B1913+16 [92, 93, 169], for which Hulse and Taylor were awarded the Nobel Prize. This system consists of two neutron stars in a binary orbit, one of which is a pulsar, rotating with a period on the order of milliseconds, and having a magnetic field whose axis is not aligned

with the axis of rotation. This creates a 'lighthouse effect', where an observer on Earth sees radio signals emitted from the pulsar at regular intervals. These signals are measured by radio telescopes, and the Doppler shift is used to reconstruct the properties of the orbit. It was found that the reconstructed orbit agrees beautifully with the predictions of General Relativity [168]. In particular, a secular decrease of the orbital period due to the emission of gravitational radiation has been observed. Recent observations of this system continue to be consistent with the predictions of General Relativity [170, 193, 192].

Subsequently, many other binary pulsars have been discovered, and their reconstructed orbits all spectacularly agree with the predictions of General Relativity [164, 108]. Of particular interest is the double pulsar PSR J0737-3039 [110, 105], a binary system consisting of *two* pulsars. Although one of the pulsars has a slow rotational period on the order of seconds, its timing gives access to an orbital parameter that can not be measured in single-pulsar binary systems.

These binary pulsars typically have orbital periods on the order of hours, and light-crossing times on the order of seconds, and thus orbital velocities on the order of $v/c \sim 0.0001$, implying that the orbital motion may be described as a perturbation of the non-relativistic Kepler problem. However, the gravitational fields inside the neutron stars are very strong, and thus the binary pulsar tests probe some limited aspects of strong-field General Relativity.

Although the measurement of the decrease of the orbital period of binary pulsars provides indirect evidence for the existence of gravitational waves, a major experimental research program is currently being undertaken in order to directly measure these waves on Earth with interferometric detectors [128, 151]. The 'first generation' ground-based detectors (LIGO, VIRGO, GEO 600, and TAMA 300) have taken data, and set upper limits on the amplitudes of gravitational waves. Since a signal above the background noise has not yet been detected, the method of matched filtering is employed to look for signals buried in noise. After carrying out six 'science runs', LIGO is currently being

upgraded to Advanced LIGO [85], which is planned to become operational in 2014. This upgrade will increase the sensitivity ten-fold, which is expected to be sufficient to detect a gravitational wave signal above the noise. Similar 'second generation' upgrades are planned for the other ground-based detectors.

Since the frequency range of any ground-based detector is fundamentally limited by seismic noise, there exist propoals for 'third-generation' detectors, which will eliminate this noise and be sensitive to lower frequencies. The Einstein Telescope [142] will be cryogenically cooled and located in an underground facility, whereas eLISA/NGO [7] will be space-based and consist of three satellites.

One of the most promising sources of gravitational waves are the inspiral, merger, and ringdown of a binary system consisting of neutron stars and/or black holes. During the merger, orbital velocities may approach the speed of light, and thus, direct gravitational wave detection will test a regime of General Relativity which has not yet been tested by binary pulsar timing.

It is useful to interpret the results of experiments and observations in a framework which does not assume General Relativity, or any other particular theory of gravity, to be correct. For instance, in the weak-field regime it is possible to express the predictions of a broad class of relativistic theories of gravity in terms of the Parametrized Post-Newtonian (PPN) phenomenological parameters [196]. The result of any experiment or observation in a weak-field setting may then be phrased as a bound on some combination of the PPN parameters.

For the timing of binary pulsars whose relative orbital velocity is much slower than the speed of light, i.e. $v/c \ll 1$, the Parametrized Post-Keplerian (PPK) formalism may be used to describe the properties of the orbits, and relate the times-of-emission of pulses to their times-of-arrival on Earth in a broad class of relativistic theories of gravity [196, 54].

In order to describe the gravitational waveform emitted by a binary inspiral in a theory-independent manner, Yunes and Pretorius have developed the Parametrized Post-Einsteinian (PPE) framework [209, 41]. Following an investigation by Arun [8], this framework was extended to include amplitude

corrections due to dipole radiation [39]. However, recent work has revealed that the PPE framework is not sufficiently general to describe the waveform in massive Brans-Dicke theory [17].

In regimes where the gravitational fields are strong *everywhere*, such as binary systems with v/c approaching unity, it is usually very difficult, if not downright impossible, to express the predictions of a broad class of relativistic theories of gravity in terms of a finite set of phenomenological parameters. Thus, the only practical approach is the explicit construction of well-motivated alternative theories of gravity, which depend on a finite set of parameters, and reduce to General Relativity for a specific choice of parameters. Then the result of any experiment or observation may be phrased as a bound on some combination of these parameters.

In addition to their crucial role in the interpretation of strong-field observations, alternative theories of gravity are useful for gaining insight into the special properties of General Relativity. For example, the Strong Equivalence Principle (SEP), which holds in General Relativity, states that extended bodies with internal self-gravity respond to an external gravitational field in the same manner as microscopic test particles [196]. Nordtvedt [119, 120, 118] was the first to point out that the SEP may be violated in alternative theories of gravity, and suggest an experimental test by Lunar Laser Ranging (LLR). The results of this test have confirmed the validity of the SEP to a very high degree of accuracy [58, 204].

Another special property of General Relativity which breaks down in most alternative theories is the effacement principle, which states that in a binary system with $v/c \ll 1$, the internal structure of the two bodies affects the orbital motion at a much higher order in v/c than one would have *a priori* expected [44]. Thus, the successful general-relativistic description of the binary pulsar orbits in terms of two neutron star masses, and no other 'internal' parameters, is an excellent test of the effacement principle.

2.1.2   Motivation from High-Energy Physics

Being a classical field theory, General Relativity is expected to break down when quantum-gravitational effects become important, that is, when energies become comparable to the Planck energy $M_{\text{pl}}c^2$, or when distance scales become comparable to the Planck length ℓ_{pl}, where

$$M_{\text{pl}} \;=\; \sqrt{\frac{\hbar c}{G}} \sim 10^{19}\, \frac{\text{GeV}}{c^2}\,, \tag{2.1}$$

$$\ell_{\text{pl}} \;=\; \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-35}\, \text{m}\,. \tag{2.2}$$

Although such high energies are far outside the range of present-day and any foreseeable future experimental technologies, the problem of unifying gravity with quantum mechanics is of great theoretical interest, and has received much attention over the past few decades. The principal obstacle to the straight-forward quantization of General Relativity, and most alternative theories of gravity, is that G has mass dimension -2 in relativistic units, and thus the resulting quantum field theory of gravity is non-renormalizable, in the sense that an infinite number of counterterms are required to cancel ultraviolet divergences.

String theory [13], the most popular candidate for a quantum theory of gravity, solves this problem by postulating that matter is composed of extended objects with length scale ℓ_p called strings rather than point particles at the fundamental level. This length scale eliminates ultraviolet divergences by providing a natural high-energy and short-distance cut-off.

In order to be consistent at the quantum level, string theory forces the number of space-time dimensions to be equal to ten. One way to reconcile this requirement with the observed four space-time dimensions is to postulate that space-time is the product of a 'large' four-dimensional manifold and an 'internal' six-dimensional manifold. In the low-energy limit, an effective four-dimensional theory is obtained, and the information about the geometry of the 'internal' manifold is encoded in four-dimensional scalar fields. This idea of 'compactification' goes back to Kaluza [100] and Klein [103].

Moreover, string theory incorporates supersymmetry, which predicts,

among other things, that the gravitational interaction is mediated not only by the rank-2 symmetric tensor field of General Relativity, but by an entire supergravity multiplet. Although its precise structure depends on the type of supersymmetry considered, this multiplet generically contains a spin-3/2 'gravitino' field, and a scalar 'dilaton' field.

Thus, the low-energy limit of string theory is not pure General Relativity, but a more complicated theory of gravity which contains scalar fields. In order for these scalar fields to mediate long-range forces which are relevant in gravitational physics, they must be either massless, or very light. Although quantum corrections generically give large masses to scalars, it has been recently argued that many of the scalar fields predicted by string theory remain sufficiently light to have interesting cosmological and astrophysical consequences [9].

2.1.3 Partial Literature Survey

Relativistic theories of gravity which contain scalar fields in addition to the usual metric tensor are generically called Scalar-Tensor, or Tensor-Scalar theories [50, 25, 158, 75, 40]. Motivated by Mach's principle and building on earlier work by Jordan [98, 97] and Fierz [71], Brans and Dicke [24, 26, 57] constructed the first Scalar-Tensor theory in 1962, now referred to as Brans-Dicke (BD) gravity, or Jordan-Fierz-Brans-Dicke (JFBD) gravity. It contains one massless scalar field, and one free parameter ω which determines the strength of interactions between this scalar field and other matter fields. In the limit $\omega \to \infty$, General Relativity is recovered. The tests of relativistic gravity within the solar system described in section 2.1.1 place the bound $\omega \gtrsim 40000$. Consequently, all the predictions of Brans-Dicke gravity, in both the strong-field and weak-field regimes, are very close to those of General Relativity. For instance, in 1967 Salmona [150] investigated how the properties of neutron stars in Brans-Dicke theory differ from those in General Relativity, and found small corrections which vanish as $\omega \to \infty$.

Thus, in order to construct a Scalar-Tensor theory whose strong-field predictions deviate from General Relativity in an interesting manner, while

the weak-field limit passes the stringent solar-system tests, it is necessary to go beyond Brans-Dicke gravity. In the years 1968-1970, Bergmann [15], Nordtvedt [121], and Wagoner [188], have taken steps in this direction, and generalized the work of Brans and Dicke by replacing the parameter ω with a scalar-matter coupling function $\omega(\phi)$.

Following the discovery of the binary pulsar PSR B1913+16 in 1975, the predictions of several alternative theories of gravity were compared with the reconstructed orbit from pulsar timing, and a bound on the Brans-Dicke parameter ω was obtained [62, 203, 202], while the alternative bimetric theory of Rosen which contains no free parameters was ruled out [200].

In 1992, Damour and Esposito-Farèse generalized Scalar-Tensor gravity to include multiple scalar fields, and worked out the orbital Lagrangian and radiative energy loss for the N-body problem in these so-called Tensor-Multi-Scalar theories. They explicitly constructed a two-parameter family of models $T(\beta', \beta'')$, which contains two scalar fields, and passes solar system tests while predicting deviations from General Relativity in the strong-field regime. Binary pulsar timing data was used to obtain constraints on β' and β'' [50].

In deriving their theoretical results, Damour and Esposito-Farèse employed the 'Einstein frame' formulation of Scalar-Tensor gravity, which is mathematically simpler than the 'Jordan frame' formulation used by Clifford Will and others. There has been a great deal of confusion and controversy in the literature surrounding the issue of the physical equivalence of the Einstein and Jordan frames, or lack thereof [19, 35, 72, 69, 38]. The point of view taken here is that the distinction between these two frames is no different than the distinction between Cartesian and polar coordinates — all physical predictions, when worked out carefully, should be independent of the choice of conformal frame [72, 38]. In fact, in section 4.6, it is explicitly demonstrated that a recent Jordan-frame calculation of the radiative energy loss [6] in *massive* Brans-Dicke theory is consistent with the results of Damour and Esposito-Farèse.

In 1993, Damour and Esposito-Farèse investigated stellar structure in

the quasi-Brans-Dicke, or 'quadratic' model, which contains a single scalar field whose couplings to matter are determined by the Einstein-frame scalar-matter coupling function $\alpha(\varphi) = \alpha_s + \beta_s\varphi$. They found a phase-transition phenomenon which is analogous to the spontaneous magnetization of ferro-magnets at low temperatures, and has been called spontaneous scalarization [51]. The quantity playing a role analogous to the applied magnetic field is the 'applied scalar-matter coupling' α_∞ asymptotically far away from the star, while the analogue of the magnetization is the scalar-charge-to-mass ratio Q/M of the star. For non-relativistic stars it may be shown that $Q/M = \alpha_\infty$ which vanishes in the limit $\alpha_\infty \to 0$, and thus scalarization is not possible. On the other hand, for a relativistic star, if $\beta_s \lesssim -4$, then scalarization *is* possible — there exists a critical mass whose value depends on β_s, and relativistic stars heavier than this critical mass may have finite Q/M in the limit $\alpha_\infty \to 0$.

The discovery of scalarization was an important milestone for the study of stellar structure in Scalar-Tensor gravity, which has motivated much of the subsequent research. In 1998, the properties of scalarization were further investigated by Salgado and Sudarsky [148], while Harada employed catastrophe theory to quasi-analytically verify that scalarized stars are stable against perturbations, while non-scalarized stars heavier than the critical scalarization mass are unstable [83, 84]. These stability properties were also verified by Novak by means of a direct numerical simulation [122]. Other numerical simulations of scalarization phenomena have been carried out in [123, 124] and [2].

In further research, the spontaneous scalarization of boson stars [194], as well as a possible relation between scalarization and violation of the Weak Energy Condition (WEC) [195, 149] were investigated.

Subsequently, oscillations of scalarized stars were investigated by Sotani and Kokkotas [161, 160] and DeDeo and Psaltis [55, 56, 141]. The latter authors used observed redshifts of spectral lines from neutron stars to obtain the bound $\beta_s \gtrsim -9$ on the quasi-Brans-Dicke model. Psaltis has also written a more general review on the subject of testing strong-field gravity with observations in the electromagnetic spectrum [140].

It was found that the orbital dynamics of a binary system of scalarized stars differs substantially from that of non-scalarized stars, and consequently, the spectacular agreement between the pulsar timing data and General Relativity places strong constraints on the quasi-Brans-Dicke model. A detailed analysis yields the bound $\beta_s \gtrsim -5$ [68, 66, 67, 65, 64, 53, 63, 52, 49, 20, 106, 74, 89], ruling out a large portion of the theory-space in which scalarization is possible.

In 2010, an argument claiming that string theory predicts light scalars with masses in the range $10^{-33}\,\mathrm{eV}/c^2 \ldots 10^{-10}\,\mathrm{eV}/c^2$ [9] has led to an interest in massive Scalar-Tensor theories of gravity. For a massive scalar, the superradiance mechanism enables a Kerr black hole to develop a non-trivial scalar field profile at the expense of its rotational energy, leading to potentially observable phenomena [10, 104, 61]. In the context of a binary system with at least one rotating black hole, superradiance enables gravitational radiation to be emitted at the expense of the rotational energy of the black hole rather than the orbital binding energy, leading to the notion of a *floating orbit* which radiates without shrinking [36, 207]. Recently, Alsing and collaborators [6] have calculated the power loss formulas for circular orbits in massive Brans-Dicke theory, and in further work [17] these results were combined with a Fisher matrix analysis to obtain prospective bounds on massive Brans-Dicke theory from future gravitational wave detection, along lines similar to the earlier work of Will, Scharre, and Yunes [197, 152, 201], Damour and Esposito-Farèse [53], and Yagi and Tanaka [206].

In recent years there has been interest in stars [125], black holes [211, 208, 210], and binary inspirals [205, 166, 159, 127] in extended versions of Scalar-Tensor gravity motivated by string theory, where the scalar is coupled to quadratic curvature invariants. The two cases of special interest are Dynamical Chern Simons (DCS) gravity [94, 3, 4] and Einstein-Dilaton-Gauss-Bonnet (EDGB) gravity [116, 126], which are distinguished by the fact that the scalar is coupled to a topological invariant, giving rise to second-order field equations (which generically are fourth order).

In order to understand gravitational physics in regimes where analyti-

cal methods break down, such as the merger or collision of compact objects, it is necessary to resort to numerical relativity. This mature subfield of gravitational physics is well-developed, and numerous textbooks have been written on the subject (for example, see [12]). In contrast, numerical simulation in alternative theories of gravity is relatively uncharted territory. Although theoretical work has been carried out to cast the field equations of Scalar-Tensor gravity into a form suitable for numerical integration [147, 146] four years ago, it was only very recently that the first results of a binary inspiral simulation were reported [87]. It was found that whenever the scalar field is given an initial non-trivial profile, dipole radiation is emitted, and the dynamics differs very strongly from that of a double-black-hole binary inspiral in General Relativity. On the other hand, if the initial scalar field profile is turned off, the dynamics is indistinguishable from that of General Relativity, and dipole radiation is absent. Future numerical work in Scalar-Tensor gravity, as well as other alternative theories, has great potential to give insights into the properties of these theories which are inaccessible by analytical means.

In addition to Brans-Dicke theory and its natural generalizations, which are the principal subject of this thesis, there exist numerous other approaches to incorporating light scalar fields in a relativistic theory of gravity. For example, the Chameleon mechanism developed by Khoury and Weltman [102] by building on earlier work of Mota and Barrow [115] employs a matter-dependent scalar mass to satisfy solar system constraints, whereas the Galileon mechanism developed by Nicolis, Rattazzi, and Trincherini [117] satisfies these constraints by incorporating the Vainshtein effect [173] to decouple the scalar field from matter in gravitationally bound systems. A comprehensive survey of such alternative approaches is beyond the scope of this thesis.

Another subject beyond the scope of this thesis is cosmology. Although the effects of cosmology on an double-black-hole binary are investigated in section 5, this is implemented in a phenomenological manner, independent of the details of any particular cosmological model. For a comprehensive introduction to cosmology in Scalar-Tensor gravity, and motivation therefor, the reader is referred to the monographs [70] and [16].

2.2   Action and Field Equations in the Einstein Frame

The class of Scalar-Tensor theories considered in this thesis is defined by the following action in the so-called Einstein frame:

$$
\begin{aligned}
S \;=\;& \frac{c^4}{4\pi G} \int \frac{d^dx}{c} \sqrt{-g} \left(\frac{R}{4} - \frac{1}{2}\gamma_{ab}(\varphi)g^{\mu\nu}\nabla_\mu\varphi^a\nabla_\nu\varphi^b - B(\varphi) \right) \\
& + S_{\mathrm{m}}[\Psi; A^2(\varphi)g_{\mu\nu}]\,,
\end{aligned}
\tag{2.3}
$$

where the units, conventions, and geometrical quantities have been defined in appendices A.1 and A.2. The φ^a are scalar fields, and a is an index used to denote coordinates on an N-dimensional target-space with metric $\gamma_{ab}(\varphi)$. The notation used is similar, but not identical, to that of Damour and Esposito-Farèse [50]. In particular, the 'star' subscripts used by these authors to denote Einstein-frame quantities are dropped for brevity. It is always assumed that boundary terms in actions may be dropped, in other words, one may 'integrate by parts'.

The first line of (2.3) is the gravitational action, consisting of kinetic terms for both the tensor and scalar fields, as well as a scalar potential $B(\varphi)$. It is the most general coordinate-space-covariant and target-space-covariant functional of a metric and N scalar fields, expanded in numbers of coordinate-space derivatives to second order.[1] The motivation for such an expansion comes from effective field theory, which is described in section 4.7.

The second line of (2.3) is the matter action, which functionally depends on a set of matter fields, denoted collectively by Ψ. The choice of coupling these matter fields only to the combination $A^2(\varphi)g_{\mu\nu}$ is motivated by the well-tested Weak Equivalence Principle (WEP) [196], which states that trajectories of test particles follow the geodesics of a metric. Moreover, couplings of this form arise in the low-energy limit of several higher-dimensional models [99, 1, 47, 48].

The tensor and scalar field equations are derived from the requirement that the action (2.3) be stationary under variations of the metric and scalar fields, respectively. Using equations (A.46) and (A.42) to find the variations

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[1]A priori, arbitrary functions of φ may appear in the kinetic terms. However, these functions may be eliminated by field redefinitions. Such a redefinition is explicitly carried out in section 2.3 to put the action into Jordan-frame form.

of R and $\sqrt{-g}$, respectively, and using equation (A.41) to write $\delta g_{\mu\nu}$ in terms of $\delta g^{\mu\nu}$, it is found that the tensor field equation is given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - 2\gamma_{ab}(\varphi)\left(\nabla_\mu\varphi^a\nabla_\nu\varphi^b - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\varphi^a\nabla_\sigma\varphi^b\right)$$
$$+2B(\varphi)g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}\,, \qquad (2.4)$$

where the Einstein-frame energy-momentum tensor is defined by

$$T^{\mu\nu} = \frac{2c}{\sqrt{-g}}\frac{\delta S_{\mathrm{m}}[\Psi; A^2(\varphi)g_{\rho\sigma}]}{\delta g_{\mu\nu}}\,, \qquad (2.5)$$

and with the help of equation (A.41), its lowered-index form may be written as

$$T_{\mu\nu} = -\frac{2c}{\sqrt{-g}}\frac{\delta S_{\mathrm{m}}[\Psi; A^2(\varphi)g_{\rho\sigma}]}{\delta g^{\mu\nu}}\,. \qquad (2.6)$$

Contracting the indices in the tensor equation (2.4) yields

$$R = 2\gamma_{ab}(\varphi)g^{\mu\nu}\nabla_\mu\varphi^a\nabla_\nu\varphi^b + \frac{4dB(\varphi)}{d-2} - \frac{16\pi GT}{(d-2)c^4}\,, \qquad (2.7)$$

where T is the trace of the energy-momentum tensor (2.5), and substituting this result back in to (2.4) yields the 'trace-reverse' form of the tensor field equation:

$$R_{\mu\nu} - 2\gamma_{ab}(\varphi)\nabla_\mu\varphi^a\nabla_\nu\varphi^b - \frac{4}{d-2}B(\varphi)g_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{d-2}Tg_{\mu\nu}\right)\,. \quad (2.8)$$

Varying the action (2.3) with respect to φ^a yields the scalar field equation

$$\Box\varphi^a + \gamma^a{}_{bc}(\varphi)g^{\mu\nu}\nabla_\mu\varphi^b\nabla_\nu\varphi^c - B^a(\varphi) = -\frac{4\pi G}{c^4}\alpha^a(\varphi)T\,, \qquad (2.9)$$

where

$$\gamma^a{}_{bc}(\varphi) = \frac{1}{2}\gamma^{ad}(\varphi)\left(\frac{\partial\gamma_{cd}(\varphi)}{\partial\varphi^b} + \frac{\partial\gamma_{bd}(\varphi)}{\partial\varphi^c} - \frac{\partial\gamma_{bc}(\varphi)}{\partial\varphi^d}\right) \qquad (2.10)$$

are the target-space Christoffel symbols, and the derivatives of the functions $A(\varphi)$ and $B(\varphi)$ are denoted by

$$\alpha_a(\varphi) = \frac{d\log A(\varphi)}{d\varphi^a}\,, \qquad B_a(\varphi) = \frac{dB(\varphi)}{d\varphi^a}\,. \qquad (2.11)$$

Note that target-space indices are lowered and raised with the metric $\gamma_{ab}(\varphi)$ and its inverse $\gamma^{ab}(\varphi)$, respectively. For example, $\alpha^a(\varphi) = \gamma^{ab}(\varphi)\alpha_b(\varphi)$.

2.3 Transformation to the Jordan Frame

The essential feature of the Einstein-frame formulation of Scalar-Tensor gravity presented in section 2.2 is the absence of 'mixed' kinetic terms of the form $F(\varphi)R$ in (2.3), which implies that the last two terms in (A.46) may be neglected when varying the action. Consequently, the Einstein-frame field equations (2.8) and (2.9) have a very simple mathematical form. However, the downside is that matter couples to a combination of $g_{\mu\nu}$ and φ, complicating the physical interpretation of a given solution to the field equations.

The idea behind the Jordan-frame formulation of Scalar-Tensor gravity is to write the gravitational action (2.3) in terms of the *Jordan-frame metric* $\tilde{g}_{\mu\nu} = A^2(\varphi)g_{\mu\nu}$, to which matter is universally coupled. This simplifies the physical interpretation of solutions to the field equations, but the price paid is that the field equations themselves become substantially more complicated.

A *Jordan-frame scalar field* ϕ is introduced, and defined by the requirement that the Ricci term of (2.3) take the canonical form $c^3\sqrt{-\tilde{g}}\phi\tilde{R}/16\pi$ when written in terms of Jordan-frame variables[2], where $\tilde{g}$ and $\tilde{R}$ are the determinant and Ricci scalar, respectively, built out of the Jordan-frame metric $\tilde{g}_{\mu\nu}$. This requirement implies that

$$\phi = \frac{1}{GA^{d-2}(\varphi)} \ . \tag{2.12}$$

Whereas the scalar fields φ^a are all on equal footing in the Einstein-frame gravitational action, the field ϕ plays a distinguished role in the Jordan-frame gravitational action, which complicates the Jordan-frame description of Tensor-Multi-Scalar theories. For simplicity, only the case of a single scalar field ($N = 1$) will be considered.

The results of section A.5 (with $\Omega = A(\varphi)$) may be used to calculate the various geometric objects built out of the Jordan-frame metric $\tilde{g}_{\mu\nu}$. In particular, equation (A.56), along with (2.12) may be used to write the action

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[2]Note that with this definition of ϕ, the Jordan-frame quantity ϕ^{-1} may be interpreted as a spacetime-dependent Newton constant. This interpretation has been historically important in the development of Brans-Dicke theory [24, 26, 57].

(2.3) in the Jordan-frame form

$$S = \frac{c^4}{16\pi} \int \frac{d^d x}{c} \sqrt{-\tilde{g}} \left(\phi \tilde{R} - \frac{\omega(\phi)}{\phi} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \phi \tilde{\nabla}_\nu \phi - V(\phi) \right) + S_{\rm m}[\Psi; \tilde{g}_{\mu\nu}] , \quad (2.13)$$

where

$$\omega(\phi) = \frac{2}{(d-2)^2 \alpha^2(\varphi)} - \frac{d-1}{d-2} \quad (2.14)$$

is the Jordan-frame scalar-matter coupling function, and

$$V(\phi) = \frac{4B(\varphi)}{GA^d(\varphi)} \quad (2.15)$$

is the Jordan-frame scalar potential.

The method described in the section 2.2 may be used to derive the the Jordan-frame scalar field equation[3]

$$\tilde{R} + 2\frac{\omega(\phi)}{\phi} \tilde{\Box} \phi + \left(\frac{\omega'(\phi)}{\phi} - \frac{\omega(\phi)}{\phi^2} \right) \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \phi \tilde{\nabla}_\nu \phi - V'(\phi) = 0 , \quad (2.16)$$

and tensor field equation

$$\phi \left(\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{R} \tilde{g}_{\mu\nu} \right) - \frac{\omega(\phi)}{\phi} \left(\tilde{\nabla}_\mu \phi \tilde{\nabla}_\nu \phi - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{g}^{\rho\sigma} \tilde{\nabla}_\rho \phi \tilde{\nabla}_\sigma \phi \right)$$
$$+ \tilde{g}_{\mu\nu} \tilde{\Box} \phi - \tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi + \frac{1}{2} V(\phi) \tilde{g}_{\mu\nu} = \frac{8\pi}{c^4} \tilde{T}_{\mu\nu} , \quad (2.17)$$

where the Jordan-frame energy-momentum tensor is defined by

$$\tilde{T}^{\mu\nu} \equiv \frac{2c}{\sqrt{-\tilde{g}}} \frac{\delta S_{\rm m}[\Psi; \tilde{g}_{\rho\sigma}]}{\delta \tilde{g}_{\mu\nu}} = \frac{T^{\mu\nu}}{A^{d+2}(\varphi)} , \quad (2.18)$$

and its indices and lowered and raised by $\tilde{g}_{\mu\nu}$ and $\tilde{g}^{\mu\nu}$, respectively, so that

$$\tilde{T}_{\mu\nu} \equiv \tilde{g}_{\mu\rho} \tilde{g}_{\nu\sigma} \tilde{T}^{\rho\sigma} = -\frac{2c}{\sqrt{-\tilde{g}}} \frac{\delta S_{\rm m}[\Psi; \tilde{g}_{\rho\sigma}]}{\delta \tilde{g}^{\mu\nu}} = \frac{T_{\mu\nu}}{A^{d-2}(\varphi)} , \quad (2.19)$$

and

$$\tilde{T}^\mu_{\ \nu} \equiv \tilde{g}_{\nu\sigma} \tilde{T}^{\mu\sigma} = \frac{T^\mu_{\ \nu}}{A^d(\varphi)} . \quad (2.20)$$

---

[3]Alternatively, the Jordan-frame field equations (2.16) and (2.17) may be derived directly from the Einstein-frame field equations (2.9) and (2.4) by means of the conformal transformation formulas given in section A.5.

Contracting the tensor equation (2.17) yields

$$\phi\tilde{R} = \frac{\omega(\phi)}{\phi}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi + \frac{2(d-1)}{d-2}\tilde{\Box}\phi + \frac{dV(\phi)}{d-2} - \frac{16\pi\tilde{T}}{(d-2)c^4}\,, \qquad (2.21)$$

and susbstituting this result back into (2.17) yields the 'trace-reverse' form of the tensor field equation

$$\phi\tilde{R}_{\mu\nu} - \frac{\tilde{g}_{\mu\nu}\tilde{\Box}\phi}{d-2} - \frac{V(\phi)\tilde{g}_{\mu\nu}}{d-2} - \frac{\omega(\phi)}{\phi}\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi - \tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi$$
$$= \frac{8\pi}{c^4}\left(\tilde{T}_{\mu\nu} - \frac{\tilde{g}_{\mu\nu}\tilde{T}}{d-2}\right). \qquad (2.22)$$

Equation (2.21) may also be used to exchange the curvature term in the scalar field equation (2.16) for a 'matter' term :

$$\left(2\omega(\phi) + \frac{2(d-1)}{d-2}\right)\tilde{\Box}\phi + \omega'(\phi)\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi + \frac{d}{d-2}V(\phi) - \phi V'(\phi)$$
$$= \frac{16\pi\tilde{T}}{(d-2)c^4}. \quad (2.23)$$

2.4   Conservation Laws

In relativistic theories of gravity, the equations describing the local flow of energy and momentum are intricately tied to the contracted form of the second Bianchi identity, equation (A.19). For the Scalar-Tensor theories constructed in sections 2.2 and 2.3, the simplest method of deriving these equations involves working with (A.19) in the Einstein frame, and subsequently transforming the result to the Jordan frame.

Applying ∇^μ to the Einstein-frame tensor field equation (2.4) and using the Bianchi identity (A.19) yields the Einstein-frame conservation law

$$\nabla^\mu\left(T_{\mu\nu} + T_{\mu\nu}^{(\varphi)}\right) = 0\,, \qquad (2.24)$$

where the second term in the brackets denotes an effective energy-momentum tensor for the scalar fields, given by

$$T_{\mu\nu}^{(\varphi)} = \frac{c^4}{4\pi G}\left(\left[\nabla_\mu\varphi^a\nabla_\nu\varphi^b - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\varphi^a\nabla_\sigma\varphi^b\right]\gamma_{ab}(\varphi) - B(\varphi)g_{\mu\nu}\right).$$
$$(2.25)$$

Applying ∇^μ to equation (2.25) and using the scalar field equation (2.9) yields a more explicit form of the conservation law (2.24):

$$\nabla^\mu T_{\mu\nu} = -\nabla^\mu T^{(\varphi)}_{\mu\nu} = \alpha_a(\varphi)T\nabla_\nu\varphi^a\,. \tag{2.26}$$

Since matter does not couple to the scalar field ϕ in the Jordan frame, it is expected that the Jordan-frame energy-momentum tensor (2.18) is covariantly conserved, in other words, the Jordan-frame version of equation (2.24) should not have a 'scalar energy-momentum tensor'. In order to demonstrate this, the covariant derivative of (2.18) is written in terms of Einstein-frame quantities by means of equations (A.8) and (A.48):

$$\begin{aligned}\tilde{\nabla}_\mu\tilde{T}^{\mu\nu} &= \partial_\mu\tilde{T}^{\mu\nu} + \tilde{\Gamma}^\mu_{\ \mu\lambda}\tilde{T}^{\lambda\nu} + \tilde{\Gamma}^\nu_{\ \mu\lambda}\tilde{T}^{\mu\lambda} \\ &= \nabla_\mu\tilde{T}^{\mu\nu} + C^\mu_{\ \mu\lambda}\tilde{T}^{\lambda\nu} + C^\nu_{\ \mu\lambda}\tilde{T}^{\mu\lambda}\,,\end{aligned} \tag{2.27}$$

where in the second line, $C^\lambda_{\ \mu\nu}$ denotes the terms proportional to $\log\Omega$ on the right-hand side of equation (A.48), which are given by

$$C^\lambda_{\ \mu\nu} = \alpha(\varphi)\left(\delta^\lambda_\mu\partial_\nu\varphi + \delta^\lambda_\nu\partial_\mu\varphi - g_{\mu\nu}g^{\lambda\rho}\partial_\rho\varphi\right)\,. \tag{2.28}$$

Using equation (2.18) to write $\tilde{T}^{\mu\nu}$ in terms of $T^{\mu\nu}$, and also using the Einstein-frame conservation law (2.26) finally yields the desired result, namely,

$$\tilde{\nabla}_\mu\tilde{T}^{\mu\nu} = 0\,. \tag{2.29}$$

2.5   The Weak-Field Limit

In order to connect Scalar-Tensor gravity with Newtonian gravity and identify the physical Newton constant $\tilde{G}$ which is the quantity measured by Cavendish experiments, it is necessary to take the weak-field limit.

To this end, specialize the construction of sections 2.2 and 2.3 to the case of a single scalar field ($N = 1$) with no potential ($B(\varphi) = V(\phi) = 0$). Consider a 'weakly-gravitating' isolated system whose matter distribution in a globally-defined Minkowskian coordinate system is given by $\tilde{T}_{\mu\nu}$, and assume that the solutions to the Scalar-Tensor field equations admit perturbative expansions of the form

$$\begin{aligned}g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(h^2)\,, \tag{2.30}\\ \varphi &= \varphi_\infty + \psi + \mathcal{O}(\psi^2) \tag{2.31}\end{aligned}$$

in the Einstein frame, and

$$\tilde{g}_{\mu\nu} \;=\; \tilde{\eta}_{\mu\nu} + \tilde{h}_{\mu\nu} + \mathcal{O}(\tilde{h}^2) \,, \tag{2.32}$$

$$\phi \;=\; \phi_\infty + \chi + \mathcal{O}(\chi^2) \tag{2.33}$$

in the Jordan frame, where φ_∞ and ϕ_∞ are constants, and

$$\tilde{\eta}_{\mu\nu} := A^2(\varphi_\infty)\eta_{\mu\nu} \,, \qquad \tilde{\eta}^{\mu\nu} := A^{-2}(\varphi_\infty)\eta^{\mu\nu} \tag{2.34}$$

is the 'Jordan-frame flat metric'.

Assume that all the quadratic terms on the right-hand sides of (2.30)-(2.33), as well as cross-terms of $\mathcal{O}(h\psi)$ and $\mathcal{O}(\tilde{h}\chi)$ may be neglected. It follows from (2.30) that $g^{\mu\nu} = \eta^{\mu\nu} + \mathcal{O}(h)$, and therefore the indices on $h_{\mu\nu}$ may be lowered and raised with the flat metric $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$, respectively. For example,

$$h_\mu{}^\nu \;:=\; g^{\nu\lambda}h_{\mu\lambda} = [\eta^{\nu\lambda} + \mathcal{O}(h)]h_{\mu\lambda} = \eta^{\nu\lambda}h_{\mu\lambda} \,. \tag{2.35}$$

The identity $g_{\mu\nu}g^{\nu\lambda} = \delta_\mu^\lambda$ then implies that

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \,. \tag{2.36}$$

In the Jordan frame, the indices on $\tilde{h}_{\mu\nu}$ are lowered and raised with $\tilde{g}_{\mu\nu}$ and $\tilde{g}^{\mu\nu}$ *by definition*, and it follows from the linearization approximation that one may instead use the flat metric $\tilde{\eta}_{\mu\nu}$ and its inverse $\tilde{\eta}^{\mu\nu}$ to lower and raise these indices. Moreover, the identity $\tilde{g}_{\mu\nu}\tilde{g}^{\nu\lambda} = \delta_\mu^\lambda$ implies that

$$\tilde{g}^{\mu\nu} = \tilde{\eta}^{\mu\nu} - \tilde{h}^{\mu\nu} \,. \tag{2.37}$$

The Einstein-frame (2.30)-(2.31) and Jordan-frame (2.32)-(2.33) perturbative expansions are not independent, but rather are related by $\tilde{g}_{\mu\nu} = A^2(\varphi)g_{\mu\nu}$ and equation (2.12). Linearizing these relations yields

$$\tilde{h}_{\mu\nu} = A^2(\varphi_\infty)h_{\mu\nu} + 2\alpha(\varphi_\infty)\psi\tilde{\eta}_{\mu\nu} \,, \tag{2.38}$$

and

$$\phi_\infty \;=\; \frac{1}{GA^{d-2}(\varphi_\infty)} \,, \tag{2.39}$$

$$\frac{\chi}{\phi_\infty} \;=\; -(d-2)\alpha(\varphi_\infty)\psi \,, \tag{2.40}$$

respectively. Furthermore, linearizing the relation between the Jordan-frame and Einstein-frame scalar-matter coupling functions (2.14), and using equations (2.39)-(2.40) yields

$$\omega(\phi_\infty) = \frac{2}{(d-2)^2\alpha^2(\varphi_\infty)} - \frac{d-1}{d-2}, \tag{2.41}$$

$$\omega'(\phi_\infty) = \frac{4GA^{d-2}(\varphi_\infty)\alpha'(\varphi_\infty)}{(d-2)^3\alpha^4(\varphi_\infty)} = \frac{4\alpha'(\varphi_\infty)}{(d-2)^3\phi_\infty\alpha^4(\varphi_\infty)}, \tag{2.42}$$

where $\omega'(\phi) := d\omega(\phi)/d\phi$, and $\alpha'(\varphi) := d\alpha(\varphi)/d\varphi$.

In order to write down the linearized field equations in a simple form that is readily solved, it is convenient to employ harmonic coordinates, which are defined by the condition (A.22), or equivalently, (A.23). Linearizing the latter equation yields

$$\partial_\mu \bar{h}^{\mu\nu} = 0, \tag{2.43}$$

where

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}, \tag{2.44}$$

is the so-called trace-reverse[4] of $h_{\mu\nu}$, and $h := \eta^{\mu\nu}h_{\mu\nu}$ is the trace of $h_{\mu\nu}$. Linearizing the expression for the Ricci tensor in harmonic coordinates (A.24) yields

$$R_{\mu\nu} = -\frac{1}{2}\Box_\eta h_{\mu\nu}, \tag{2.45}$$

and therefore, the so-called Einstein tensor takes the form

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{1}{2}\Box_\eta \bar{h}_{\mu\nu}, \tag{2.46}$$

where $\Box_\eta = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the d'Alembertian operator of the flat Minkowski metric $\eta_{\mu\nu}$. With equation (2.46) in hand, it is now a straightforward matter to linearize the Einstein-frame field equations (2.4) and (2.9):

$$\Box_\eta \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}, \tag{2.47}$$

$$\Box_\eta \psi = -\frac{4\pi G}{c^4}\alpha(\varphi_\infty)T. \tag{2.48}$$

---

[4]This nomenclature derives from the identity $\bar{h} := \eta^{\mu\nu}\bar{h}_{\mu\nu} = -\eta^{\mu\nu}h_{\mu\nu} = -h$ which holds when $d = 4$.

Alternatively, one may linearize (2.8) instead of (2.4) to obtain an equation for $h_{\mu\nu}$ rather than $\bar{h}_{\mu\nu}$:

$$\Box_\eta h_{\mu\nu} = -\frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{T\eta_{\mu\nu}}{d-2}\right).$$
(2.49)

Note that applying ∂^μ to equation (2.47) and using the linearized harmonic coordinate condition (2.43) yields the linearized conservation law

$$\partial^\mu T_{\mu\nu} = 0\,,$$
(2.50)

which is consistent with the exact nonlinear conservation law (2.26) because $T = \mathcal{O}(\psi)$.[5] The fact that $T = \mathcal{O}(\psi)$ also implies that indices on $T_{\mu\nu}$ may be lowered and raised with the flat metric $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$, and that the linearized version of the relations between the Jordan-frame and Einstein-frame energy-momentum tensors (2.18)-(2.20) is simply

$$\tilde{T}_{\mu\nu} = \frac{T_{\mu\nu}}{A^{d-2}(\varphi_\infty)}\,, \qquad \tilde{T}^{\mu\nu} = \frac{T^{\mu\nu}}{A^{d+2}(\varphi_\infty)}\,, \qquad \tilde{T}^\mu{}_\nu = \frac{T^\mu{}_\nu}{A^d(\varphi_\infty)}\,.$$
(2.51)

Equation (2.51) implies that $\tilde{T} = \mathcal{O}(\psi)$ and therefore, indices on $\tilde{T}_{\mu\nu}$ may be lowered and raised with the flat metric $\tilde{\eta}_{\mu\nu}$ and its inverse $\tilde{\eta}^{\mu\nu}$, respectively.

Equations (2.47)-(2.48) are wave equations in d-dimensional Minkowski space-time, which may be formally solved by the method of Green's functions, as explained in section A.6. In the case $d = 4$ which is of physical interest, the Green's function has the simple form given by equation (A.61), which yields the following solution of the field equations:

$$\bar{h}_{\mu\nu}(x^0, \vec{x}) = \frac{4G}{c^4}\int d^3\vec{y}\,\frac{T_{\mu\nu}(x^0 - |\vec{x}-\vec{y}|, \vec{y})}{|\vec{x}-\vec{y}|}\,,$$
(2.52)

$$\psi(x^0, \vec{x}) = \frac{G}{c^4}\alpha(\varphi_\infty)\int d^3\vec{y}\,\frac{T(x^0 - |\vec{x}-\vec{y}|, \vec{y})}{|\vec{x}-\vec{y}|}\,.$$
(2.53)

If $T_{\mu\nu}$ is time-independent, then the solution to the field equations may be taken to also be time-independent, and thus the d'Alembertian wave operator $\Box_\eta$ in d-dimensional Minkowski space-time reduces to the Laplacian Δ in $(d-1)$-dimensional Euclidean space. The Green's function for this operator

---

[5]This follows from the linearized scalar field equation (2.48).

is given by equation (A.62) (with $N = d - 1$), and therefore the solution to the field equations is given by

$$\bar{h}_{\mu\nu}(\vec{x}) = \frac{16\pi G}{(d-3)\Omega_{d-2}c^4} \int d^{d-1}\vec{y}\, \frac{T_{\mu\nu}(\vec{y})}{|\vec{x} - \vec{y}|^{d-3}}\,, \tag{2.54}$$

$$\psi(\vec{x}) = \frac{4\pi G\alpha(\varphi_\infty)}{(d-3)\Omega_{d-2}c^4} \int d^{d-1}\vec{y}\, \frac{T(\vec{y})}{|\vec{x} - \vec{y}|^{d-3}}\,, \tag{2.55}$$

where Ω_{d-2} is the area of the sphere S^{d-2}, and is given by equation (A.64).

If the matter distribution is further specialized to pressure-less dust with mass density ρ, then the form of the energy-momentum becomes particularly simple — $\tilde{T}_{00} = \rho c^2 A^2(\varphi_\infty)$, and all other components vanish. Using the relations (2.51), one finds that $T_{00} = -T = \rho c^2 A^d(\varphi_\infty)$, which in conjunction with the solution (2.54)-(2.55) yields

$$h_{00}(\vec{x}) = \frac{16\pi G A^d(\varphi_\infty)}{(d-2)\Omega_{d-2}c^2} \int d^{d-1}\vec{y} \frac{\rho(\vec{y})}{|\vec{x} - \vec{y}|^{d-3}}\,, \tag{2.56}$$

$$h_{0i}(\vec{x}) = 0\,, \tag{2.57}$$

$$h_{ij}(\vec{x}) = \delta_{ij} \cdot \frac{16\pi G A^d(\varphi_\infty)}{(d-2)(d-3)\Omega_{d-2}c^2} \int d^{d-1}\vec{y} \frac{\rho(\vec{y})}{|\vec{x} - \vec{y}|^{d-3}}\,, \tag{2.58}$$

$$\psi(\vec{x}) = -\frac{4\pi G\alpha(\varphi_\infty)A^d(\varphi_\infty)}{(d-3)\Omega_{d-2}c^2} \int d^{d-1}\vec{y} \frac{\rho(\vec{y})}{|\vec{x} - \vec{y}|^{d-3}}\,. \tag{2.59}$$

Equations (2.56)-(2.59) express the solution of the linearized field equations in a form that is suitable for comparison to Newtonian gravity.

In order to complete the connection with Newtonian gravity, it is necessary to investigate the trajectory of a test particle in the presence of the fields (2.56)-(2.59). The universal coupling of matter to the Jordan-frame metric implies that such a trajectory $z^\mu(t)$ is a geodesic of the Jordan-frame metric, and thus satisfies

$$\frac{d^2 z^\lambda}{d\tau^2} + \tilde{\Gamma}^\lambda{}_{\mu\nu} \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} = 0\,, \tag{2.60}$$

where τ is the proper time. The first step in evaluating equation (2.60) involves the calculation of the Jordan-frame Christoffel symbols $\tilde{\Gamma}^\lambda{}_{\mu\nu}$, whose independent non-vanishing components (in the weak and static field limit) are found

using equation (A.9):

$$\tilde{\Gamma}^0_{0i} = \tilde{\Gamma}^i_{00} = -\frac{1}{2A^2(\varphi_\infty)}\partial_i\tilde{h}_{00}\,, \tag{2.61}$$

$$\tilde{\Gamma}^i_{jk} = \frac{1}{2A^2(\varphi_\infty)}\left(\partial_j\tilde{h}_{ki} + \partial_k\tilde{h}_{ji} - \partial_i\tilde{h}_{jk}\right)\,. \tag{2.62}$$

The temporal and spatial components of the geodesic equation (2.60) may then be written as

$$\frac{d^2t/d\tau^2}{dt/d\tau} = -\frac{2}{c^2}\frac{d\vec{z}}{d\tau}\cdot\vec{\nabla}U^{\text{Newt}}\,, \tag{2.63}$$

$$\frac{d^2z^i}{d\tau^2} = -\partial_iU^{\text{Newt}}\cdot\left(\frac{dt}{d\tau}\right)^2 - \tilde{\Gamma}^i_{jk}\frac{dz^j}{d\tau}\frac{dz^k}{d\tau}\,, \tag{2.64}$$

where the Newtonian potential is identified as

$$U^{\text{Newt}}(\vec{x}) = -\frac{c^2\tilde{h}_{00}(\vec{x})}{2A^2(\varphi_\infty)}\,. \tag{2.65}$$

Up to this point, all calculations in this section have employed coordinates x^μ in which the Einstein-frame metric asymptotically tends to $\eta_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$, while the Jordan-frame metric asymptotically tends to $\tilde{\eta}_{\mu\nu} = A^2(\varphi_\infty)\eta_{\mu\nu}$ (see equations (2.30) and (2.32)). However, in order to identify the 'physical' Jordan-frame Newton constant $\tilde{G}$, it is useful to introduce new coordinates $\hat{x}^\mu := A(\varphi_\infty)x^\mu$, in which the Jordan-frame metric asymptotically tends to $\tilde{\eta}_{\hat{\mu}\hat{\nu}} = \text{diag}(-1, 1, \ldots, 1)$. Let $\hat{U}^{\text{Newt}}$ and $\hat{\rho}$ denote the Newtonian potential and mass density in these new coordinates, so that $\hat{\rho}(\hat{x}) := \rho(\hat{x}/A(\varphi_\infty))$, and likewise for $\hat{U}^{\text{Newt}}$. Combining equations (2.65), (2.38), (2.56), and (2.59) yields an explicit integral formula for the Newtonian potential, which in hatted coordinates reads

$$\hat{U}^{\text{Newt}}(\hat{x}) = -\tilde{G}\int d^{d-1}\hat{y}\,\frac{\hat{\rho}(\hat{y})}{|\hat{x} - \hat{y}|^{d-3}}\,, \tag{2.66}$$

where

$$\tilde{G} = \frac{4\pi G A^{d-2}(\varphi_\infty)[2(d-3) + (d-2)\alpha^2(\varphi_\infty)]}{(d-2)(d-3)\Omega_{d-2}} \tag{2.67}$$

is the 'physical' Jordan-frame Newton constant. This expression simplifies to

$$\tilde{G} = A^2(\varphi_\infty)(1 + \alpha^2(\varphi_\infty)) \tag{2.68}$$

in the case of physical interest, $d = 4$. Note that the Newtonian limit of the geodesic equation (2.63)-(2.64), expressed in 'hatted' coordinates, has the canonical form

$$\frac{d^2 \hat{z}^i}{d\hat{t}^2} = -\hat{\partial}_i \hat{U}^{\text{Newt}} \,. \tag{2.69}$$

3 The Weak-Central-Coupling Formalism

3.1 Introduction

The purpose of this introductory section is to explain the phenomenon of spontaneous scalarization and outline the key ideas behind the Weak Central Coupling (WCC) formalism. The starting point is the definition of the mass and scalar charge of a possibly strongly-gravitating star, which is motivated by the asymptotic properties of the gravitational fields of a 'weakly-gravitating' object found in section 2.5

To this end, the hyperspherical coordinates described in section A.3 are employed, and a multipole expansion of equations (2.56)-(2.59) is carried out, demonstrating that the behaviour of the fields asymptotically far away from such a body is given by

$$
g_{00} \;=\; -1 + \frac{16\pi G M}{(d-2)\Omega_{d-2}c^2 r^{d-3}} + \mathcal{O}\left(\frac{1}{r^{d-2}}\right), \tag{3.1}
$$

$$
g_{0i} \;=\; 0, \tag{3.2}
$$

$$
g_{ij} \;=\; \delta_{ij}\left[1 + \frac{16\pi G M}{(d-2)(d-3)\Omega_{d-2}c^2 r^{d-3}}\right] + \mathcal{O}\left(\frac{1}{r^{d-2}}\right), \tag{3.3}
$$

$$
\varphi \;=\; \varphi_\infty - \frac{4\pi G Q}{(d-3)\Omega_{d-2}c^2 r^{d-3}} + \mathcal{O}\left(\frac{1}{r^{d-2}}\right), \tag{3.4}
$$

where

$$
M := A(\varphi_\infty)\tilde{M} = A^d(\varphi_\infty)\int d^{d-1}\vec{y}\,\rho(\vec{y}) \tag{3.5}
$$

is defined to be the Einstein-frame mass of the body,

$$
\tilde{M} := \int d^{d-1}\hat{y}\,\hat{\rho}(\hat{y}) \tag{3.6}
$$

is the 'physical' Jordan-frame mass, and

$$
Q := \frac{c^2}{4\pi G}\int_{S^{d-2}} \partial_\mu\varphi\,n^\mu d\Omega_{d-2} = \alpha(\varphi_\infty)M \tag{3.7}
$$

is the Einstein-frame 'scalar charge', where n^μ and $d\Omega^\mu_{d-2}$ denote the outward-pointing normal vector and canonical volume element on S^{d-2}, respectively.

The important lesson to take away from this analysis is that in the slow-motion weak-field regime of Scalar-Tensor gravity, the magnitude of scalar effects relative to tensor effects is determined by the asymptotic value of the

scalar-matter coupling, $\alpha(\varphi_\infty)$. The purpose of this section is to explore how things change in the strong-field regime, while restricting attention to static spherically-symmetric isolated bodies for simplicity. It turns out that an asymptotic expansion of the form (3.1)-(3.4) continues to hold[6], however, the constants M, Q, and φ_∞ characterizing the asymptotic fields are no longer given by equations (3.5)-(3.7). In fact, it turns out that in certain Scalar-Tensor theories, it is possible for the ratio Q/M to be non-zero, and even of order unity, when $\alpha(\varphi_\infty) = 0$. This phenomenon, called spontaneous scalarization, was discovered in 1993 by Damour and Esposito-Farèse [51]. The principal goal of this section is to develop a formalism for describing scalarization perturbatively.

It is possible to expand Q/M in a series of the form [51]

$$\frac{Q}{M} = \alpha(\varphi_\infty) \left(1 + a_1 s + a_2 s^2 + \ldots \right) , \tag{3.8}$$

where $s := GM/R^{d-3}c^2$, the *compactness* of the star, is a dimensionless measure of the gravitational field strength in the stellar interior, where R denotes the stellar radius. For instance, Schwarzschild black holes in $d = 4$ have $s = \frac{1}{2}$, while neutron stars typically have $s \sim 0.1$. It is important to note that all the properties of a star depend on the scalar boundary condition φ_∞ imposed at infinity, and thus, the right-hand side of (3.8) depends on φ_∞ not only explicitly, but also implicitly through s. Thus, it may happen that Q/M remains finite in the limit $\alpha(\varphi_\infty) \to 0$, provided that the series in s diverges sufficiently rapidly. It is in this sense that spontaneous scalarization has been called a 'non-perturbative' phenomenon [51].

The approach taken in the WCC formalism begins with the observation that the difficulties in analytically understanding scalarization stem from the multi-valuedness of (3.8) as a function of φ_∞. The crucial insight is that an expansion of Q/M in powers of the central values of the scalar field φ_0 and pressure P_0 is guaranteed to define a single-valued function. Indeed, for a given choice of these central values (φ_0, P_0), it is possible to integrate the equations of stellar structure, match the solution to the exterior metric, and

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[6]This will be demonstrated in section 3.3

thus uniquely determine all properties of the star. Therefore, the so-called Weak-Central-Coupling (WCC) series expansions are employed:

$$\frac{GM}{R^{d-3}c^2} \equiv s = c_1 + c_2\alpha_0^2 + c_3\alpha_0^4 + \mathcal{O}(\alpha_0^6)\,, \tag{3.9}$$

$$\alpha_\infty = d_1\alpha_0 + d_2\alpha_0^3 + \mathcal{O}(\alpha_0^5)\,, \tag{3.10}$$

$$Q/M = e_1\alpha_0 + e_2\alpha_0^3 + \mathcal{O}(\alpha_0^5)\,, \tag{3.11}$$

where $\alpha_0 := \alpha(\varphi_0)$, $\alpha_\infty := \alpha(\varphi_\infty)$, and the coefficients c_i, d_i and e_i are functions of P_0. When the cubic terms are neglected, it follows from (3.10)-(3.11) that $Q/M \propto \alpha_\infty$, and thus scalarization is impossible. However, when the cubic terms are included, equation (3.10) may be inverted to write α_0 in terms of α_∞, d_1, and d_2 by means of Cardano's formula [90]. In the limit $\alpha_\infty \to 0$, the expression for α_0 simplifies to

$$\alpha_0 = \pm\sqrt{-d_1/d_2}\,. \tag{3.12}$$

Therefore, scalarization is possible whenever $d_1/d_2 < 0$, and the criterion for the onset of scalarization is given by $d_1 = 0$.

Arguments of this type for describing phase transitions go back to Landau [113], and have been previously put forward in the literature on scalarization. For instance, the ansatz

$$M(Q, \varphi_\infty) = \mu(Q) - Q\varphi_\infty\,, \tag{3.13}$$

$$\mu(Q) = \frac{1}{2}a(\bar{M}_{\text{cr}} - \bar{M})Q^2 + \frac{1}{4}bQ^4\,, \tag{3.14}$$

was postulated in [52], however, the coefficients a and b were not calculated, either numerically or analytically. The quantity $\bar{M}$ denotes the baryonic mass[7] of the star, and $\bar{M}_{\text{cr}}$ is the critical value thereof for the onset of scalarization.

The novelty of the present approach is the development of a formalism for the systematic calculation of the coefficients c_i, d_i and e_i. This formalism is applied to constant-density stars, for which the $i = 1$ coefficients are expressed in terms of Heun functions, and used to demonstrate that such stars exhibit scalarization.

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[7]The baryonic mass is defined to be the total mass of the baryons making up the star when their gravitational interaction is 'turned off'. This is a useful quantity because it is often conserved.

3.2   Field Equations

There exist two distinct classes of astrophysical objects with strong internal gravitational fields — black holes and neutron stars. The former are described by vacuum solutions of the field equations, whereas the latter may be modelled by a *perfect fluid* at zero temperature, characterized by the mass-energy density ρ and pressure P, as well as a functional relation between these two quantities, called an *equation of state*. The purpose of this section is to specialize the field equations of Scalar-Tensor gravity to such a perfect fluid source, and write them out explicitly in a suitable coordinate system.

The starting point is the perfect fluid energy-momentum tensor [37], which has the form

$$\tilde{T}^{\mu\nu} = (\rho + P/c^2)\tilde{u}^\mu\tilde{u}^\nu + P\tilde{g}^{\mu\nu} \, , \tag{3.15}$$

where $\tilde{u}^\mu$ is the velocity four-vector of the fluid. For a static spherically-symmetric fluid, the requirement that $\tilde{u}^\mu$ must point in the positive time direction, together with the normalization condition

$$\tilde{g}_{\mu\nu}\tilde{u}^\mu\tilde{u}^\nu = -c^2 \, , \tag{3.16}$$

uniquely determines

$$\tilde{u}^0 \;\; = \;\; \frac{c}{A(\varphi)\sqrt{f}} \, , \tag{3.17}$$

and

$$\tilde{u}_0 \;\; := \;\; \tilde{g}_{00}\tilde{u}^0 = -cA(\varphi)\sqrt{f} \, , \tag{3.18}$$

where the Einstein-frame metric is taken to have the form (A.25).

In order to write down the Einstein-frame field equations (2.8) and (2.9), it is necessary to calculate the lower-index components and trace of the Einstein-frame energy momentum tensor, which may be accomplished with the help of equations (2.19) and (2.20):

$$T_{00} \;\; = \;\; A^d(\varphi)f\rho c^2 \, , \tag{3.19}$$

$$T_{rr} \;\; = \;\; A^d(\varphi)hP \, , \tag{3.20}$$

$$T_{\theta_i\theta_i} \;\; = \;\; A^d(\varphi)Pg_{\theta_i\theta_i} \, , \tag{3.21}$$

$$T \;\; = \;\; A^d(\varphi)[(d-1)P - \rho c^2] \, . \tag{3.22}$$

It is thus found that the independent non-trivial components of the Einstein-frame tensor field equation (2.8) are given by

$$R_{00} + \frac{4}{d-2}B(\varphi)f = \frac{8\pi G}{c^4(d-2)}A^d(\varphi)f[(d-3)\rho c^2 + (d-1)P]\,, \quad (3.23)$$

$$R_{rr} - 2\varphi'^2 - \frac{4}{d-2}B(\varphi)h = \frac{8\pi G}{c^4(d-2)}A^d(\varphi)h(\rho c^2 - P)\,, \quad (3.24)$$

$$R_{\theta_1\theta_1} - \frac{4}{d-2}B(\varphi)k = \frac{8\pi G}{c^4(d-2)}kA^d(\varphi)(\rho c^2 - P)\,, \quad (3.25)$$

while (A.11) may be used to write the Einstein-frame scalar field equation (2.9) in the form

$$\frac{1}{\sqrt{fh}k^{(d-2)/2}}\left(\sqrt{\frac{f}{h}}k^{(d-2)/2}\varphi'\right)' - B'(\varphi)$$

$$= \frac{4\pi G}{c^4}\alpha(\varphi)A^d(\varphi)(\rho c^2 - (d-1)P)\,. \quad (3.26)$$

Substituting into (3.23)-(3.25) the components of the Ricci tensor $R_{\mu\nu}$ given by equations (A.38)-(A.40) yields

$$\frac{f''}{2f} - \frac{f'^2}{4f^2} - \frac{f'h'}{4fh} + \frac{(d-2)f'k'}{4fk} + \frac{4}{d-2}B(\varphi)h =$$

$$\frac{8\pi G}{c^4(d-2)}A^d(\varphi)h[(d-3)\rho c^2 + (d-1)P]\,, \quad (3.27)$$

$$-\frac{f''}{2f} + \frac{f'^2}{4f^2} + \frac{f'h'}{4fh} + (d-2)\left(-\frac{k''}{2k} + \frac{k'^2}{4k^2} + \frac{k'h'}{4kh}\right)$$

$$-2\varphi'^2 - \frac{4}{d-2}B(\varphi)h = \frac{8\pi G}{c^4(d-2)}A^d(\varphi)h(\rho c^2 - P)\,, \quad (3.28)$$

$$(d-3)\frac{h}{k} - \frac{k''}{2k} - \frac{(d-4)k'^2}{4k^2} + \frac{k'h'}{4kh} - \frac{k'f'}{4kf} - \frac{4}{d-2}B(\varphi)h =$$

$$\frac{8\pi G}{c^4(d-2)}A^d(\varphi)h(\rho c^2 - P)\,. \quad (3.29)$$

Thus, the field equations for a static spherically-symmetric perfect fluid boil down to (3.26)-(3.29), a system of four coupled radial differential equations for the dependent variables $f = -g_{00}$, $h = g_{rr}$, $k = g_{\theta_1\theta_1}$, and φ. Note that the freedom to re-define the radial coordinate allows one to impose a constraint involving f, h, k, φ, and r. For the interior of a star, which is treated in section 3.4, it is convenient to make the choice $k = r^2$. On the other hand, the

solution to the vacuum equations with $P = \rho = 0$ (described in section 3.3), which is relevant for describing a black hole or the exterior of a star, is known analytically in a coordinate system where $f = h^q$ for some constant exponent q. The coordinate transformation required to match the interior and exterior solutions at a stellar boundary is the subject of section 3.5.

3.3 Vacuum

The field equations (3.26)-(3.29) simplify considerably in the absence of matter, that is, when $P = \rho = 0$. To simplify things even further and enable an analytical solution, it will be assumed that the scalar potential $B(\varphi)$ vanishes.

Under these assumptions, the scalar field equation (3.26) may be immediately integrated, yielding

$$\frac{f}{h}k^{d-2}\varphi'^2 = \text{const}. \tag{3.30}$$

After multiplying through by f/f', one finds that the (tt) equation (3.27) may *also* be immediately integrated, yielding

$$\frac{f'^2}{fh}k^{d-2} = \text{const}. \tag{3.31}$$

These two 'conservation laws' imply that

$$\varphi' = \text{const} \times \frac{f'}{f}. \tag{3.32}$$

When $d = 4$, it is not difficult to find the most general solution to the remaining equations (3.28) and (3.29) by means of standard techniques. However, for arbitrary d, it appears that there exists no straightforward method of arriving at the most general solution without making any additional assumptions. One way to understand why the case $d \neq 4$ is challenging, is to notice that the k'^2 term drops out of (3.29) when $d = 4$, considerably simplifying this equation.

By means of an ansatz, the following three-parameter class of solutions

has been obtained for arbitrary d by Bagchi and Kalyana Rama [11]:

$$f(\xi) = (1 - (\xi_0/\xi)^{d-3})^{1-(d-3)q}, \tag{3.33}$$

$$h(\xi) = (1 - (\xi_0/\xi)^{d-3})^{q-1}, \tag{3.34}$$

$$k(\xi) = \xi^2(1 - (\xi_0/\xi)^{d-3})^q, \tag{3.35}$$

$$\varphi(\xi) = \varphi_\infty + \frac{p}{2}\log(1 - (r_0/r)^{d-3}), \tag{3.36}$$

where the constants p and q are related by

$$2p^2 = (d-2)(2 - (d-3)q)q, \tag{3.37}$$

and the radial coordinate has been renamed to ξ to distinguish it from the coordinate used in the interior of a star. Note that it follows from equation (3.37) that

$$0 \le q \le \frac{2}{(d-3)} \equiv q_{\text{max}}. \tag{3.38}$$

While the solution (3.33)-(3.36) for arbitrary d is relatively recent, the special case of $d = 4$ has been known for a long time. For references to these earlier works, the reader is referred to the bibliography of [11].

The asymptotic expansions

$$\begin{aligned}
g_{00} &= -1 + (1 - (d-3)q)(\xi_0/\xi)^{d-3} + \ldots \\
&\equiv -1 + \frac{G\hat{M}}{\xi^{d-3}c^2} + \ldots, \tag{3.39}
\end{aligned}$$

$$\begin{aligned}
\varphi &= \varphi_\infty - \frac{p}{2}(\xi_0/\xi)^{d-3} + \ldots \\
&\equiv \varphi_\infty + \frac{G\hat{Q}}{\xi^{d-3}c^2} + \ldots, \tag{3.40}
\end{aligned}$$

agree with those derived earlier in the weak-field context, namely, equations (3.1) and (3.4). The 'hatted' mass $\hat{M}$ and scalar charge $\hat{Q}$ have been introduced for simplicity of notation, and are related to the corresponding Einstein-frame quantities by

$$\hat{M} = \frac{16\pi M}{(d-2)\Omega_{d-2}}, \tag{3.41}$$

$$\hat{Q} = -\frac{4\pi Q}{(d-3)\Omega_{d-2}}. \tag{3.42}$$

For the later purpose of matching to a stellar interior (which is the subject of section 3.5), it is useful to express the constants $(\hat{M}, \hat{Q}, \varphi_\infty)$ characterizing the vacuum solution in the form

$$\frac{G\hat{M}}{c^2} = \frac{k^{d-3}}{(d-3)\xi^{d-4}}\frac{df}{d\xi}, \tag{3.43}$$

$$\frac{\hat{Q}}{\hat{M}} = -\frac{d\varphi/d\xi}{d\log f/d\xi}, \tag{3.44}$$

$$\varphi_\infty = \varphi - \log f\frac{d\varphi/d\xi}{d\log f/d\xi}. \tag{3.45}$$

In order to investigate whether the solution (3.33)-(3.36) may be employed to describe a black hole, it is necessary to study its behaviour in the vicinity of $\xi = \xi_0$, which involves the computation of scalar curvature invariants. In General Relativity, where the Ricci scalar vanishes by the vacuum field equations, one must compute 'quadratic' curvature scalars such as $R_{\rho\sigma\mu\nu}R^{\rho\sigma\mu\nu}$. However, in Scalar-Tensor gravity, the Ricci scalar may be written in terms of $(\partial\varphi)^2$ by means of (2.7), yielding

$$R = \frac{p^2(d-3)^2\xi_0^{2d-6}}{2\xi^{2d-4}(1-(\xi_0/\xi)^{d-3})^{1+q}}, \tag{3.46}$$

which is singular at $\xi = \xi_0$ unless $q = 0$. Thus, the solution (3.33)-(3.36) may describe a black hole only in the case $q = 0$, which is nothing other than the well-known d-dimensional generalization of the Schwazschild black hole. Statements about vanishing scalar charges of black holes are usually referred to as 'no-hair theorems' [86], and have been recently extended to a broad class of Scalar-Tensor theories [162]. These theorems generically need to assume some form of time-independence, as explicitly illustrated by the *Miracle Hair Growth Formula* [96] discussed in section 5.

3.4   Interior

In this section, the Einstein-frame field equations for a static spherically-symmetric perfect fluid (3.26)-(3.29) are simplified. Schwarzschild-like coordinates with $k = r^2$ are employed, and the dependent variables are transformed from (f, h, k, φ) to (μ, ν, P, φ), where $\nu = \log f = \log(-g_{00})$, and

$\mu = (1 - h^{-1})/2 = (1 - g^{rr})/2$. It turns out that ν may be eliminated, in the sense that the μ, P, and φ equations form a closed system.

In order to carry out these simplifications, one starts by adding the (tt) (3.27) and (rr) (3.28) equations to obtain

$$-\frac{k''}{2k} + \frac{k'^2}{4k^2} + \frac{f'k'}{4fk} + \frac{h'k'}{4hk} - \frac{2}{d-2}\varphi'^2 = \frac{8\pi G}{c^4(d-2)}A^d(\varphi)h(\rho c^2 + P). \quad (3.47)$$

Adding the $(\theta_1\theta_1)$ (3.29) equation to this result yields an equation which does not depend on P,

$$-\frac{k''}{k} - \frac{(d-5)k'^2}{4k^2} + \frac{k'h'}{2kh} - \frac{2}{d-2}\varphi'^2 + (d-3)\frac{h}{k} - \frac{4}{d-2}B(\varphi)h$$
$$= \frac{16\pi G}{c^2(d-2)}A^d(\varphi)h\rho, \quad (3.48)$$

whereas subtracting the $(\theta_1\theta_1)$ (3.29) equation from (3.47) yields an equation which allows $\nu' = f'/f$ to be eliminated,

$$\nu' = \frac{2k}{k'}\left((d-3)\left(\frac{h}{k} - \frac{k'^2}{4k^2}\right)\right.$$
$$\left. + \frac{2\varphi'^2 - 4B(\varphi)h + 16\pi GA^d(\varphi)hP/c^4}{d-2}\right). \quad (3.49)$$

Writing out the r component of the energy conservation equation (2.26) quickly yields

$$P' = -(\rho c^2 + P)\left(\frac{\nu'}{2} + \alpha(\varphi)\varphi'\right), \quad (3.50)$$

a simple result which may also be derived the hard way by starting from (3.26)-(3.29). Finally, writing out the scalar equation (3.26) and using (3.49) to eliminate $\nu' = f'/f$ yields

$$\varphi'' + \left(-\frac{k''}{k'} + \frac{k'}{k} - \frac{8khB(\varphi)}{(d-2)k'} + 2(d-3)\frac{h}{k'}\right)\varphi' - B'(\varphi)h =$$
$$\frac{4\pi GA^d(\varphi)h}{c^4}\left(\alpha(\varphi)(\rho c^2 - (d-1)P) + \frac{4\varphi'k}{(d-2)k'}(\rho c^2 - P)\right). \quad (3.51)$$

This new system of ODEs (3.48)-(3.51) in which the dependent variables are (h, ν, P, φ) and k is regarded as being fixed by a constraint is equivalent to the original system (3.26)-(3.29).

In order to further simplify (3.48)-(3.51), define the radial coordinate r by the condition $k(r) = r^2$, and define

$$h(r) = \frac{1}{1 - 2\mu(r)}, \qquad \mu(r) = \frac{Gm(r)}{c^2 r^{d-3}}. \tag{3.52}$$

Then the h equation (3.48) becomes

$$r\mu' + (d-3)\mu = \frac{1}{d-2}(1-2\mu)r^2\varphi'^2 + \frac{2}{d-2}r^2 B(\varphi) + \frac{8\pi G A^d(\varphi)r^2\rho}{c^2(d-2)}, \tag{3.53}$$

the ν equation (3.49) becomes

$$\nu' = \frac{2(d-3)\mu}{r(1-2\mu)} + \frac{2r\varphi'^2}{d-2} - \frac{4r B(\varphi)}{(d-2)(1-2\mu)} + \frac{16\pi G A^d(\varphi)rP}{c^4(1-2\mu)(d-2)}, \tag{3.54}$$

the P equation (3.50) remains unchanged, and the scalar equation (3.51) becomes

$$\varphi'' + \varphi'\left(\frac{(d-2) - 2\mu}{r(1-2\mu)} - \frac{4r B(\varphi)}{(d-2)(1-2\mu)}\right) - \frac{B'(\varphi)}{1-2\mu} =$$
$$\frac{4\pi G A^d(\varphi)}{c^4(1-2\mu)}\left(\alpha(\varphi)(\rho c^2 - (d-1)P) + \frac{2r\varphi'(\rho c^2 - P)}{d-2}\right). \tag{3.55}$$

For a solution regular at the stellar centre, the appropriate initial conditions are found to be

$$\nu(r = 0) = \nu_0, \qquad \mu(r = 0) = 0, \qquad P(r = 0) = P_0,$$
$$\varphi(r = 0) = \varphi_0, \qquad \varphi'(r = 0) = 0. \tag{3.56}$$

The final simplification involves casting the ODEs (3.53),(3.54), (3.50), and (3.55) into dimensionless form, which is useful for both analytical and numerical work. To this end, write the dimensionful variables as $\rho = \rho_0\bar{\rho}$, $P = \rho_0 c^2\bar{P}$, $B(\varphi) = \lambda\zeta\bar{B}(\varphi)$, and $r = \sqrt{u/\zeta}$, where $\rho_0 := \rho(r = 0)$ is the central mass density, and $\zeta := 8\pi G\rho_0 A^d(\varphi_0)/c^2$ has units of inverse length squared, so that $\zeta^{-1/2}$ sets the scale for the radius of the star. The conformal factor is written as $A(\varphi) = A(\varphi_0)\bar{A}(\varphi)$.

Under this transformation, the μ equation (3.53) becomes

$$\dot{\mu} + \frac{d-3}{2u}\mu = 2u\frac{1-2\mu}{d-2}\dot{\varphi}^2 + \frac{\lambda\bar{B}(\varphi)}{d-2} + \frac{\bar{A}^d(\varphi)\bar{\rho}}{2(d-2)}, \tag{3.57}$$

where dots denote derivatives with respect to u. The P equation (3.50) becomes

$$\dot{\bar{P}} = -(\bar{\rho} + \bar{P})\left(\frac{\dot{\nu}}{2} + \alpha(\varphi)\dot{\varphi}\right),$$ (3.58)

the ν equation (3.54) becomes

$$\dot{\nu} = \frac{(d-3)\mu}{u(1-2\mu)} + \frac{4u\dot{\varphi}^2}{d-2} - \frac{2\lambda\bar{B}(\varphi)}{(d-2)(1-2\mu)} + \frac{\bar{A}^d(\varphi)\bar{P}}{(1-2\mu)(d-2)},$$ (3.59)

and finally, the φ equation (3.55) becomes

$$\ddot{\varphi} + \frac{d-1-4\mu}{2u(1-2\mu)}\dot{\varphi} - \frac{2\lambda\bar{B}(\varphi)\dot{\varphi}}{(d-2)(1-2\mu)} - \frac{\lambda\bar{B}'(\varphi)}{4u(1-2\mu)}$$
$$= \frac{\bar{A}^d(\varphi)}{8u(1-2\mu)}\left(\alpha(\varphi)(\bar{\rho} - (d-1)\bar{P}) + \frac{4u\dot{\varphi}}{d-2}(\bar{\rho} - \bar{P})\right).$$ (3.60)

The system of ODEs (3.57)-(3.60) is the final simplified dimensionless form of the Einstein-frame field equations for a static spherically-symmetric perfect fluid. The appropriate initial conditions for the scalar gradient are given by

$$\dot{\varphi}(u=0) = \frac{2\lambda\bar{B}'(\varphi_0) + (1-(d-1)\bar{P}_0)\alpha(\varphi_0)}{4(d-1)},$$ (3.61)

where $\bar{P}_0 := \bar{P}(u=0)$.

3.5 Matching Conditions

At the surface of a star, it is necessary to match the interior solution described in section 3.4 to the exterior solution described in section 3.3. This matching procedure is complicated by the fact that the interior and exterior problems are formulated using different radial coordinates — the interior coordinate, denoted by r, is defined by the condition $g_{\theta_1\theta_1} = r^2$, whereas the exterior coordinate, denoted by ξ, is defined by the condition (3.35), in which $k = g_{\theta_1\theta_1}$. Therefore, the relation between r and ξ is given explicitly by

$$r^2 = (1 - (\xi_0/\xi)^{d-3})^q \xi^2.$$ (3.62)

Equating the other components of the metric and the scalar field yields the following matching conditions to be satisfied at the stellar boundary:

$$-g_{00} = e^{\nu} = (1 - (\xi_0/\xi)^{d-3})^{1-(d-3)q}, \tag{3.63}$$

$$g_{rr} = \frac{1}{1 - 2\mu} = (1 - (\xi_0/\xi)^{d-3})^{-(1-q)} \left(\frac{d\xi}{dr}\right)^2, \tag{3.64}$$

$$\varphi = \varphi_{\infty} + \frac{p}{2} \log(1 - (\xi_0/\xi)^{d-3}). \tag{3.65}$$

Differentiating equation (3.62) with respect to ξ and combining the result with equation (3.64) yields

$$\frac{d\xi}{dr} = \frac{(1 - (\xi_0/\xi)^{d-3})^{1-q/2}}{1 - (1 - q/q_{\max})(\xi_0/\xi)^{d-3}} \geq 0, \tag{3.66}$$

$$\sqrt{1 - 2\mu} = \frac{1 - (1 - q/q_{\max})(\xi_0/\xi)^{d-3}}{\sqrt{1 - (\xi_0/\xi)^{d-3}}}, \tag{3.67}$$

where $q_{\max}$ has been defined in equation (3.38).

The ultimate goal of the matching procedure is to express the constants which characterize the external solution $(\hat{M}, \hat{Q}, \varphi_{\infty})$ in terms of the boundary values of the internal solution $(\mu(R), \nu(R), \varphi(R), \varphi'(R))$, where R denotes the value of the coordinate r at the stellar boundary, (not to be confused with the Ricci scalar), which is defined by the condition of vanishing pressure — $P(R) = 0$.

To achieve this, equations (3.43)-(3.45) are evaluted at the stellar boundary, and then re-written in terms of 'internal' quantities. This is a triviality for the latter two of these equation, since the derivatives with respect to ξ may be replaced by derivatives with respect to r. On the other hand, more work is required to express (3.43) in terms of internal quantities, and the desired result may be obtained by the use of equations (3.66), (3.67), (3.63), and (3.62). It is found that

$$\frac{G\hat{M}}{c^2} = \sqrt{1 - 2\mu} \frac{R^{d-2}}{d-3} \frac{d\nu}{dr} e^{\nu/2}, \tag{3.68}$$

$$\frac{\hat{Q}}{\hat{M}} = -\frac{d\varphi/dr}{d\nu/dr}, \tag{3.69}$$

$$\varphi_{\infty} = \varphi - \nu \cdot \frac{d\varphi/dr}{d\nu/dr}. \tag{3.70}$$

In principle, the set of equations (3.68)-(3.70) is the complete solution to the matching problem. Given an interior solution, the right-hand sides of these equations may be evaluated at the stellar boundary, and used to determine the parameters $\hat{M}$, $\hat{Q}$, and φ_∞ which characterize the external gravitational and scalar fields.

However, it turns out that the quantities which appear on the right-hand sides of equations (3.68)-(3.70) are not all independent, and in particular, $\nu(R)$ and $\nu'(R)$ may be expressed in terms of the other quantities. It is useful to eliminate ν and its derivative, because the interior ODEs for μ, P, and φ, namely, equations (3.57), (3.58), and (3.60), form a closed system.

While it is fairly straightforward to express $\nu'(R)$ in terms of other quantities by evaluating equation (3.54) at the stellar boundary,

$$\nu'(R) = \frac{2(d-3)\mu(R)}{R(1-2\mu(R))} + \frac{2R\varphi'^2(R)}{d-2} \, , \tag{3.71}$$

it takes much more work to carry this out for $\nu(R)$. Since the gradients $(e^\nu)' = -g_{00}'$ and φ' are required to be continuous at the stellar boundary, equations (3.63) and (3.65) may be differentiated and then combined with (3.66) to yield

$$\varphi' = \frac{p(d-3)}{2\xi} \cdot \frac{(1-(\xi_0/\xi)^{d-3})^{-q/2}(\xi_0/\xi)^{d-3}}{1-(1-q/q_{max})(\xi_0/\xi)^{d-3}} \, , \tag{3.72}$$

$$\nu' = \frac{(1-(d-3)q)(d-3)}{\xi} \cdot \frac{(1-(\xi_0/\xi)^{d-3})^{-q/2}(\xi_0/\xi)^{d-3}}{1-(1-q/q_{max})(\xi_0/\xi)^{d-3}} \, . \tag{3.73}$$

With these two expressions in hand, it is now a matter of algebra to verify that one may write

$$\nu = -\frac{2\nu'}{\sqrt{\nu'^2 + 8(d-3)\varphi'^2/(d-2)}}$$
$$\times \operatorname{arctanh}\left(\frac{\sqrt{\nu'^2 + 8(d-3)\varphi'^2/(d-2)}}{\nu' + 2(d-3)/R}\right) \, . \tag{3.74}$$

Finally, substituting the expressions (3.74) for $\nu(R)$ and (3.71) for $\nu'(R)$ into

the matching formulas (3.68)-(3.70) and simplifying yields

$$\hat{s} := \frac{G\hat{M}}{R^{d-3}c^2} = \frac{2\mathcal{K}}{(d-2)(d-3)\sqrt{1-2\mu}} \exp\left(-\frac{\mathcal{K}}{\mathcal{H}}\text{arctanh}\frac{\mathcal{H}}{\mathcal{J}}\right) , \quad (3.75)$$

$$\frac{\hat{Q}}{\hat{M}} = -\frac{(d-2)(1-2\mu)R\varphi'}{2\mathcal{K}} , \quad (3.76)$$

$$\varphi_\infty = \varphi(R) + \frac{(d-2)(1-2\mu)R\varphi'}{\mathcal{H}}\text{arctanh}\left(\frac{\mathcal{H}}{\mathcal{J}}\right) , \quad (3.77)$$

where the following auxilliary quantities have been defined:

$$\mathcal{K} = (d-2)(d-3)\mu + (1-2\mu)R^2\varphi'^2 , \quad (3.78)$$

$$\mathcal{H} = \sqrt{\mathcal{K}^2 + 2(d-2)(d-3)(1-2\mu)^2 R^2\varphi'^2} , \quad (3.79)$$

$$\mathcal{J} = (d-2)(d-3)(1-\mu) + R^2\varphi'^2(1-2\mu) . \quad (3.80)$$

From the matching conditions in their final form, (3.75)-(3.80), one deduces that the quantities $\hat{s}$, $\hat{Q}/\hat{M}$, and $\varphi_\infty - \varphi(R)$ depend only on $\mu(R)$ and $R\varphi'(R)$.

3.6   The Weak-Central-Coupling (WCC) Expansion

In this section, it is demonstrated that the quantities characterizing the external gravitational and scalar fields of a star, namely, $s = GM/R^{d-3}c^2$, Q/M, and $\alpha_\infty := \alpha(\varphi_\infty)$, admit formal Weak-Central-Coupling (WCC) series expansions in powers of $\alpha_0 := \alpha(\varphi_0)$ of the type (3.9)-(3.11), and explicit formulas for the first WCC coefficients c_1, d_1, and e_1 are obtained. For simplicity, attention is restricted to the 'quadratic', or quasi-Brans-Dicke model with vanishing scalar potential, that is, $B(\varphi) = 0$, and conformal factor

$$A(\varphi) = \exp\left(\alpha_s(\varphi - \varphi_\star) + \frac{1}{2}\beta_s(\varphi - \varphi_\star)^2\right) , \quad (3.81)$$

which leads to the Einstein-frame scalar-matter coupling

$$\alpha(\varphi) = \alpha_s + \beta_s(\varphi - \varphi_\star) , \quad (3.82)$$

where $\varphi_\star$ is an arbitrary constant[8].

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[8]This constant is related to the choice of units and drops out of all physical predictions.

The first step in the calculation of the WCC coefficients involves an expansion of the solution to the equations of stellar structure (3.57)-(3.60) in powers of $\alpha_0 := \alpha(\varphi_0)$. In order to write down such a series explicitly, it is useful to 'move' the quantity φ_0 'from the initial conditions to the differential equations', by shifting and scaling the Einstein-frame scalar field φ. To this end, introduce $\psi := (\varphi - \varphi_0)/\alpha(\varphi_0)$, which is well-defined as long as $\alpha(\varphi_0)$ does not vanish[9]. When written in terms of ψ, the equations of stellar structure (3.57)-(3.60) take the form

$$\dot{\mu} = -\frac{d-3}{2u}\mu + \frac{2\alpha_0^2 u(1-2\mu)\dot{\psi}^2}{d-2} + \frac{\bar{\rho}e^{d\alpha_0^2\psi(1+\beta_s\psi/2)}}{2(d-2)}, \tag{3.83}$$

$$\dot{P} = -(\bar{\rho}+\bar{P})\left(\frac{\dot{\nu}}{2} + \alpha_0^2(1+\beta_s\psi)\dot{\psi}\right), \tag{3.84}$$

$$\dot{\nu} = \frac{(d-3)\mu}{u(1-2\mu)} + \frac{4\alpha_0^2 u\dot{\psi}^2}{d-2} + \frac{\bar{P}e^{d\alpha_0^2\psi(1+\beta_s\psi/2)}}{(d-2)(1-2\mu)}, \tag{3.85}$$

$$\ddot{\psi} = -\frac{d-1-4\mu}{2u(1-2\mu)}\dot{\psi}$$
$$+\frac{e^{d\alpha_0^2\psi(1+\beta_s\psi/2)}}{8u(1-2\mu)}\left(\frac{4u}{d-2}\dot{\psi}(\bar{\rho}-\bar{P})\right.$$
$$\left. + (1+\beta_s\psi)(\bar{\rho}-(d-1)\bar{P})\right), \tag{3.86}$$

which is readily suited for expansion in powers of α_0^2. It follows from equations (3.56) and (3.61) that the initial conditions for a solution regular at the stellar centre are given by

$$\nu(0) = \nu_0, \qquad \mu(0) = 0, \qquad \bar{P}(0) = \bar{P}_0,$$
$$\psi(0) = 0, \qquad \dot{\psi}(0) = \frac{1-(d-1)\bar{P}_0}{4(d-1)}, \tag{3.87}$$

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[9]If $\alpha(\varphi_0)$ vanishes, then $\varphi \equiv \varphi_0$ solves the scalar equation (3.60), and the remaining equations reduce to those of General Relativity. On the other hand, if $\alpha(\varphi_\infty)$ vanishes, it does *not* follow that the equations of stellar structure reduce to those of GR, which is the lesson of scalarization, and the whole motivation behind working with φ_0 rather than φ_∞.

and therefore, the coefficients of the series

$$\mu = \mu^{(0)} + \alpha_0^2 \mu^{(1)} + \alpha_0^4 \mu^{(2)} + \dots, \tag{3.88}$$

$$\bar{P} = \bar{P}^{(0)} + \alpha_0^2 \bar{P}^{(1)} + \alpha_0^4 \bar{P}^{(2)} + \dots, \tag{3.89}$$

$$\psi = \psi^{(0)} + \alpha_0^2 \psi^{(1)} + \alpha_0^4 \psi^{(2)} + \dots, \tag{3.90}$$

depend only on the parameters β_s, $\bar{P}_0$, and d.[10] The stellar boundary, located at $u = U$ and defined by the condition $\bar{P}(u = U) = 0$, may also be expanded in powers of α_0^2,

$$U = U^{(0)} + \alpha_0^2 U^{(1)} + \alpha_0^4 U^{(2)} + \dots, \tag{3.91}$$

where the coefficients in (3.91) are related to those in (3.89) by expansion of $\bar{P}(U) = 0$. For instance,

$$\bar{P}^{(0)}(U^{(0)}) = 0, \qquad U^{(1)} = -\frac{\bar{P}^{(1)}(U^{(0)})}{\dot{\bar{P}}^{(0)}(U^{(0)})}. \tag{3.92}$$

The final step involves an expansion of the matching conditions (3.75)-(3.80) in powers of α_0^2. One may use

$$R\varphi'(R) = 2U\dot{\varphi}(U) = 2\alpha_0 U\dot{\psi}(U) \tag{3.93}$$

to re-write these equations in terms of ψ and the dimensionless radial variable u introduced in section 3.4, and subsequently conclude that the quantities $\hat{s}$, $\hat{Q}/(\hat{M}\alpha_0)$, and

$$\mathcal{F} := \frac{\varphi_\infty - \varphi_0}{\alpha_0} = \psi(U) + \frac{2(d-2)(1-2\mu)U\dot{\psi}(U)}{\mathcal{H}}\text{arctanh}\left(\frac{\mathcal{H}}{\mathcal{J}}\right) \tag{3.94}$$

all admit series expansions in powers of α_0^2, with coefficients depending on β_s, $\bar{P}_0$, and d. Finally, converting from 'hatted' to 'physical' quantities by means of (3.41)-(3.42) and writing $\alpha_\infty = \alpha_0(1 + \beta_s\mathcal{F})$ yields the desired WCC expansions (3.9)-(3.11).

In summary, it has been demonstrated that the coefficients c_i, d_i, and e_i in the WCC series (3.9)-(3.11) may be explicitly calculated in terms of the

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[10]These coefficients also depend on the functional form of the equation of state. An expansion of ν is not considered because this quantity does not enter into the matching conditions (3.75)-(3.80) derived in section 3.5.

perturbative solutions to the equations of stellar structure (3.83)-(3.86). For example,

$$c_1 = \frac{(d-2)\Omega_{d-2}}{8\pi}\mu^{(0)} , \tag{3.95}$$

$$d_1 = 1 + \beta_s \left(\psi^{(0)} - \frac{U^{(0)}\dot{\psi}^{(0)}}{(d-3)\mu^{(0)}}(1 - 2\mu^{(0)})\log(1 - 2\mu^{(0)}) \right) , \tag{3.96}$$

$$e_1 = \frac{4(1 - 2\mu^{(0)})U^{(0)}\dot{\psi}^{(0)}}{(d-2)\mu^{(0)}} , \tag{3.97}$$

where the right-hand sides of the above equations are all to be evaluated at the zeroth-order stellar boundary $u = U^{(0)}$.

3.7   Constant-Density Stars

The purpose of this section is to put some 'flesh' on the skeletal framework developed in the preceeding sections 3.1 - 3.6 by solving the equations of stellar structure (3.83)-(3.86) and calculating the coefficients of the WCC series (3.9)-(3.11) to leading order for an explicit model. As in section 3.6, the scalar-matter coupling is taken to have the 'quasi-Brans-Dicke' form (3.82), while the stellar model employed is that of an 'incompressible', or constant-density star, defined by $\rho(r) \equiv \rho_0$ rather than a functional relation between P and ρ.

The leading-order terms in the series (3.88)-(3.90) may be found by solving the equations of stellar structure (3.83)-(3.86) in the limit $\alpha_0^2 = 0$, namely,

$$\dot{\mu}^{(0)} = -\frac{d-3}{2u}\mu^{(0)} + \frac{1}{2(d-2)} , \tag{3.98}$$

$$\dot{\bar{P}}^{(0)} = -\frac{(1 + \bar{P}^{(0)})[(d-3)(d-2)\mu^{(0)} + u\bar{P}^{(0)}]}{2u(d-2)(1 - 2\mu^{(0)})} , \tag{3.99}$$

$$\ddot{\psi}^{(0)} = -\frac{d - 1 - 4\mu^{(0)}}{2u(1 - 2\mu^{(0)})}\dot{\psi}^{(0)}$$
$$+ \frac{1}{8u(1 - 2\mu^{(0)})}\left(\frac{4u}{d-2}\dot{\psi}^{(0)}(1 - \bar{P}^{(0)}) \right.$$
$$\left. + (1 + \beta_s\psi^{(0)})(1 - (d-1)\bar{P}^{(0)}) \right) , \tag{3.100}$$

with initial conditions

$$\mu^{(0)}(0) = 0 \,, \qquad \bar{P}^{(0)}(0) = \bar{P}_0 \,, \qquad \psi^{(0)}(0) = 0 \,, \tag{3.101}$$

and

$$\dot{\psi}^{(0)}(0) = \frac{1 - (d-1)\bar{P}_0}{4(d-1)} \tag{3.102}$$

at the stellar centre. Equation (3.98), which is self-contained, may be immediately solved to yield

$$\mu^{(0)}(u) = \frac{u}{(d-1)(d-2)} \,. \tag{3.103}$$

This result is substituted into equation (3.99), whose solution may then be found by standard techniques to be

$$\bar{P}^{(0)} = \frac{((d-1)\bar{P}_0 + d - 3)\sqrt{1 - 2\mu^{(0)}} - (d-3)(1 + \bar{P}_0)}{(d-1)(1 + \bar{P}_0) - ((d-1)\bar{P}_0 + d - 3)\sqrt{1 - 2\mu^{(0)}}} \,, \tag{3.104}$$

which vanishes at

$$U^{(0)} = \frac{2(d-1)(d-2)\bar{P}_0(d - 3 + (d-2)\bar{P}_0)}{(d - 3 + (d-1)\bar{P}_0)^2} \,. \tag{3.105}$$

Equations (3.103)-(3.104) describe the famous constant-density stellar solution in General Relativity which saturates the Buchdahl inequality[11] and was first discovered by Schwarzschild in 1916 [153]. The leading-order 'scalar corrections' to this general-relativistic solution are contained in the remaining equation (3.100), whose solution is considerably more complicated than that of (3.98)-(3.99)

In the special case of Brans-Dicke theory, namely, $\beta_s = 0$, equation (3.100) reduces to a linear first-order equation for $\dot{\psi}^{(0)}$, whose solution may be written in terms of the hypergeometric function (A.67),

$$\dot{\psi}^{(0)} = \frac{(1 + \bar{P}_0)(d-2)\,{}_2F_1(\frac{1}{2}, \frac{d-1}{2}; \frac{d+1}{2}; 2\mu^{(0)}) - d(\frac{d-3}{d-1} + \bar{P}_0)}{4(d-1)\sqrt{1 - 2\mu^{(0)}}[(1 + \bar{P}_0) - (\frac{d-3}{d-1} + \bar{P}_0)\sqrt{1 - 2\mu^{(0)}}]} \,. \tag{3.106}$$

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[11]The general-relativistic Buchdahl inequality, an upper bound on the mass-to-radius ratio of a star, was initially derived in [28], and its generalization to Scalar-Tensor gravity was investigated in [172].

On the other hand, for $\beta_s \neq 0$ the change of variables

$$y = 1 + \beta_s \psi^{(0)}, \qquad z = \frac{1}{2}\left(1 - \sqrt{1 - \frac{2u}{(d-1)(d-2)}}\right) \tag{3.107}$$

reduces (3.100) to the Heun equation (A.69) with parameters

$$a_H = -\frac{1}{d-3+(d-1)\bar{P}_0}, \qquad q_H = -\frac{(d-1)(d-2)[1-(d-1)\bar{P}_0]\beta_s}{4[d-3+(d-1)\bar{P}_0]}, \tag{3.108}$$

$$\alpha_H = \frac{d-1}{2}\left(1 + \sqrt{1 - \frac{d(d-2)}{d-1}\beta_s}\right), \quad \beta_H = \frac{d-1}{2}\left(1 - \sqrt{1 - \frac{d(d-2)}{d-1}\beta_s}\right), \tag{3.109}$$

and

$$\gamma_H = \delta_H = \frac{d-1}{2}, \qquad \epsilon_H = 1. \tag{3.110}$$

In the latter case, the leading-order WCC coefficients (3.95)-(3.97) are found to be

$$c_1 = \frac{(d-2)\Omega_{d-2}\bar{P}_0(d-3+(d-2)\bar{P}_0)}{4\pi[d-3+(d-1)\bar{P}_0]^2}, \tag{3.111}$$

$$d_1 = \text{HeunG} - \frac{(1+\bar{P}_0)\text{HeunG}'}{d-3+(d-1)\bar{P}_0}\log\left(\frac{(d-3)(1+\bar{P}_0)}{d-3+(d-1)\bar{P}_0}\right), \tag{3.112}$$

$$e_1 = \frac{2(d-3)(1+\bar{P}_0)\text{HeunG}'(a_H, q_H; \alpha_H, \beta_H, \gamma_H, \delta_H; Z)}{(d-2)\beta_s[d-3+(d-1)\bar{P}_0]}, \tag{3.113}$$

where in (3.112), the arguments of the Heun functions are the same as those in (3.113) and have been suppressed for brevity. The derivative of the Heun function (A.71) with respect to its argument is denoted by HeunG', and

$$Z = \frac{\bar{P}_0}{d-3+(d-1)\bar{P}_0} \tag{3.114}$$

is the value of the radial variable z at the stellar boundary.

Recalling from section 3.1 that the onset of scalarization takes place when $d_1 = 0$, and evaluating (3.112) in Maple for the case of physical interest, namely, $d = 4$, reveals the critical value of β_s for this transition to be -4.329 [90], which is similar to that of more realistic neutron star models [51]. The ability to calculate this critical value by means of special functions without having to numerically integrate the equations of stellar structure and employ the

shooting method[12] is the most remarkable achievement of the Weak-Central-Coupling formalism[13]. For other applications, the reader is referred to [90].

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[12]In the numerical integration of the equations of stellar structure, initial conditions are specified at the stellar centre. In order to obtain a solution with a desired boundary condition at infinity, it is necessary to vary the initial conditions until the required boundary condition is obtained to the desired level of accuracy. This procedure is often referred to as the shooting method.

[13]Note that heuristic analytical methods of estimating the critical value of β_s, based on minimizing an approximate energy functional, or neglecting certain terms in the field equations, were presented in [51, 52].

4 Binary Systems

4.1 The Non-Relativistic Problem

The gravitational two-body problem has a long history, going back to Kepler and Newton in the 1600s, who obtained the complete solution at the non-relativistic level [88]. In this limit, the 'relative' dynamics decouples from that of the centre of mass, and is described by the equations of motion

$$\ddot{\vec{r}} = -\frac{GM\vec{r}}{r^3} \,, \tag{4.1}$$

which may be derived from the Lagrangian

$$\mathcal{L} = \frac{1}{2}\mu v^2 + \frac{GM\mu}{r} \,, \tag{4.2}$$

where $\vec{r}$ is the relative position vector, and $\vec{v} = \dot{\vec{r}}$ is its time derivative. The masses of the two bodies are denoted by M_1 and M_2, the total mass is denoted by $M = M_1 + M_2$, and the reduced mass is denoted by $\mu = M_1 M_2 / M$. On account of the time-translational and rotational symmetries of the Lagrangian (4.2), the total mechanical energy,

$$E = \frac{1}{2}\mu v^2 - \frac{GM\mu}{r} \,, \tag{4.3}$$

and the angular momentum

$$\vec{L} = \mu \vec{r} \times \vec{v} \,, \tag{4.4}$$

are conserved — $\dot{E} = \dot{\vec{L}} = 0$. The famous result for which Kepler is well-known, is that every bound orbit with $E < 0$ is a non-precessing[14] ellipse with one of the foci at $r = 0$, described mathematically by

$$r = \frac{a(1 - e^2)}{1 + e \cos\theta} \,, \tag{4.5}$$

where

$$a = \frac{GM\mu}{2|E|} \,, \qquad e = \sqrt{1 - \frac{2|E|L^2}{G^2 M^2 \mu^3}} \,, \tag{4.6}$$

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[14]This lack of precession is related to an abstract symmetry of (4.2), and the conservation of the so-called Laplace-Runge-Lenz vector which points in the direction of the periastron.

are the semi-major axis and eccentricity, respectively, and θ is the angle between the periastron and $\vec{r}$, as measured in the plane of the motion. Less well-known is Kepler's equation describing the motion in time[15],

$$\omega(t - t_0) = \psi - e \sin \psi \,, \tag{4.7}$$

where

$$\omega = \sqrt{\frac{GM}{a^3}} = \sqrt{\frac{8|E|^3}{G^2 M^2 \mu^3}} \tag{4.8}$$

is the orbital (angular) frequency, and ψ is an auxilliary variable called the *eccentric anomaly*, which is related to r and θ by

$$r = a(1 - e \cos \psi) \,, \qquad \tan \frac{\theta}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{\psi}{2} \,. \tag{4.9}$$

4.2   Introduction to The Relativistic Problem

The feature of General Relativity (or alternatives) absent in Newtonian gravity which substantially complicates the two-body problem is the finite propagation speed of gravity, which leads to gravitational radiation, rendering the orbits unstable. Due to this instability, the relativistic two-body problem consists of three successive stages — the *inspiral*, when the separation between the bodies greatly exceeds their radii, the *merger*, when the separation is of the same order as the radii, and the *ringdown*, when the two bodies have coalesced into a single oscillating body which gradually 'rings' down to the final state. A variety of approximation methods have been developed for the purpose of describing the inspiral and ringdown analytically, whereas the merger is not amenable to analytical methods and must be treated numerically.

The purpose of this section is to provide a brief overview of the approximation methods employed in the analytical description of a binary inspiral in Scalar-Tensor gravity. The Einstein-frame approach of Damour and Esposito-Farèse [50] which draws heavily on the methods of [42, 22] is followed, and it is explicitly checked that the power loss formulas obtained are consistent with the

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[15]This equation may be inverted by means of a Fourier series, the nth coefficient being proportional to the Bessel function $J_n(ne)$ [190].

recent Jordan-frame calculation in *massive* Scalar-Tensor gravity carried out by Alsing and collaborators [6], which draws heavily on the methods described in the book by Clifford Will [196].

4.3   Formalism

The natural dimensionless expansion parameter in a binary inspiral is the size-to-separation ratio $\epsilon \sim R/D$. If the masses of the two bodies have equal order of magnitude[16], i.e. $M_1 \sim M_2 \sim M$, then the assumption $\epsilon \ll 1$ implies $GM/Dc^2 \ll GM/Rc^2 \lesssim 1$, that is to say, the gravitational fields exterior to the bodies are weak, whereas the interior fields may be strong, as in the case of neutron stars and black holes. The obvious way to proceed, is to break up space-time into three distinct regions — the interiors of the two bodies, in which the full non-linear field equations are to be solved, and the exterior, in which a weak-field approximation similar to that described in section 2.5 may be employed. To formalize this idea, introduce the 'matching' length scale $L \sim \sqrt{RD}$ so that $R \ll L \ll D$, construct regions of space-time $\mathcal{D}_1$ and $\mathcal{D}_2$ of radius L enclosing the world-tubes of bodies 1 and 2, respectively, and let $\mathcal{D}_{\text{ext}}$ denote the complement of $\mathcal{D}_1 \cup \mathcal{D}_2$. While the problem of solving the field equations, both in $\mathcal{D}_A$ and $\mathcal{D}_{\text{ext}}$, is a difficult one[17], due to the necessity of imposing appropriate boundary conditions, a substantial simplification takes place in the limit $\epsilon \to 0$.

Indeed, for a dimensionless coordinate system covering $\mathcal{D}_A$ in which the size of the Ath body is of order unity, the boundary conditions are imposed at a radius of order $L/R \sim \epsilon^{-1/2}$, which tends to infinity as $\epsilon \to 0$. Thus, in this limit, the problem to be solved in the interior of $\mathcal{D}_A$ reduces to that of an isolated body subject to boundary conditions $(g_{\mu\nu}^{\text{ext}}, \varphi^{\text{ext}})$ at infinity. Moreover, the boundary condition for the metric may be reduced to asymptotic flatness by means of a coordinate transformation.

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[16]Recall that $M := M_1 + M_2$ is defined to be the total mass. An *extreme mass ratio inspiral* (EMRI) with $M_1 \ll M_2$ is treated by perturbing about geodesics, employing methods beyond the scope of this thesis.

[17]The letter A labels the bodies and runs over $1, 2$.

Similarly, for a dimensionless coordinate system covering $\mathcal{D}_{\text{ext}}$ in which the separation between the bodies is of order unity, the boundary conditions are imposed on tubes of size $L/D \sim \epsilon^{1/2}$, which vanishes in the limit $\epsilon \to 0$. Thus, in this limit, the tubes shrink down to the worldlines of the two bodies.

In General Relativity, it may be shown [42] that imposing these 'near-worldline' boundary conditions is equivalent to introducing the following fictitious source term into the field equations:

$$T_{(a)}^{\mu\nu}(x) = \sum_A M_A \int ds_A \, Z_{(a)}(x - z_A(s_A)) \frac{u^\mu u^\nu}{\sqrt{-g}\sqrt{-g_{\rho\sigma}u^\rho u^\sigma/c^2}} \,, \qquad (4.10)$$

where $z_A^\mu(s_A)$ is the arclength-parametrized worldline of body A, and $u_A^\mu = dz_A^\mu/ds_A$ is its velocity. The function $Z_{(a)}$ which depends on a complex parameter a is called the Riesz kernel, and is defined to be

$$Z_{(a)}(x) = \frac{(-\eta_{\mu\nu}x^\mu x^\nu)^{(a-d)/2}}{\pi^{(d-2)/2}2^{a-1}\Gamma(a/2)\Gamma((a+2-d)/2)} \qquad (4.11)$$

for future-directed timelike vectors x, and zero otherwise[18]. It reduces to the delta function in the limit $a \to 0$, and serves to regularize divergences[19]. If these divergences are disregarded and Z_Λ is replaced by $\delta^{(4)}$, then the fictitious energy-momentum tensor (4.10) may be formally derived[20] from the 'point particle' matter action

$$S_m = -\sum_A c \, M_A \int_{\Gamma_A} ds_A \,, \qquad (4.12)$$

---

[18]Equations (4.10) and (4.11) are written in a 'Minkowskian' coordinate system x^μ globally defined on $\mathcal{D}_{\text{ext}}$. In these coordinates the metric $g_{\mu\nu}$ is asymptotically flat, and satisfies the Kirchoff 'no-incoming-radiation' boundary condition at past null infinity.

[19]Taking the limit $\epsilon \sim R/D \to 0$ essentially amounts to approximating extended bodies by point particles, and it is well known that a field theory coupled to point sources generically contains divergences due to the lack of a short-distance cutoff. In equations (4.10) and (4.11), the parameter a essentially plays the role of such a cutoff. It is taken to zero at the end of the calculation, and all physical quantities, when calculated carefully, should be independent of it.

[20]The one-dimensional integral over the worldline Γ_A is written as a four-dimensional integral over the entire space-time, with a delta function restricting the integration variables to the worldline. This expression is functionally differentiated with respect to $g_{\mu\nu}$, and equation (2.5) is used to calculate $T^{\mu\nu}$.

where Γ_A denotes the worldline of body A.

In Scalar-Tensor gravity, the situation is more complicated. Although one may introduce fictitious source terms to impose the correct 'near-worldline' boundary conditions as in General Relativity, it turns out that these sources may *not* be formally derived from a matter action of the form considered in (2.3) where matter couples solely to the Jordan-frame metric. Rather, the appropriate action, first written down by Eardley [62] in 1975, has the form

$$S_m = -\sum_A c \int_{\Gamma_A} M_A(\varphi) ds_A \,, \tag{4.13}$$

where $M_A(\varphi^{\text{ext}})$ is the Einstein-frame mass[21] of body A, found by solving the field equations in the interior of $\mathcal{D}_A$. This action may also be expressed in terms of Jordan-frame quantities as

$$S_m = -\sum_A c \int_{\Gamma_A} \tilde{M}_A(\varphi) d\tilde{s}_A \,, \tag{4.14}$$

where $\tilde{M}_A(\varphi^{\text{ext}}) := M_A(\varphi^{\text{ext}})/A(\varphi^{\text{ext}})$ is the Jordan-frame mass of body A, and $d\tilde{s}_A = A(\varphi) ds_A$ is the differential arclength along Γ_A measured by the Jordan-frame metric. In the special case of a 'weakly-gravitating' body, the results of sections 2.5 and 3.1 may be applied in the interior of $\mathcal{D}_A$, and thus the Jordan-frame mass of body A is given by equation (3.6), which is independent of φ^{ext}. It then follows that the matter action for body A *does* couple solely to the Jordan-frame metric, and is of the form considered in (2.3). However, this need not be the case for strongly gravitating bodies, indicating a violation of the Strong Equivalence Principle (SEP).

4.4   Solution of the External Field Equations

With the source terms in hand, one turns to the problem of solving the field equations in $\mathcal{D}_{\text{ext}}$. To this end, approximation schemes are employed which are similar in spirit to those described in section 2.5, but technically more sophisticated. Whereas the field equations were linearized and all quadratic

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[21]For a strongly-gravitating body, the Einstein-frame mass is defined by the asymptotic expansion (3.1).

terms were dropped in section 2.5, the method employed here may be used to systematically calculate higher-order corrections.

The starting point is the definition of harmonic coordinates (A.22), in which the Ricci tensor takes the form (A.24) of a nonlinear wave operator acting on the metric. Since (A.22) may be written as $\partial_\mu \mathfrak{g}^{\mu\nu} = 0$, where

$$\mathfrak{g}_{\mu\nu} \equiv \frac{g_{\mu\nu}}{\sqrt{-g}}\,, \qquad \mathfrak{g}^{\mu\nu} \equiv \sqrt{-g}\, g^{\mu\nu}\,, \tag{4.15}$$

is the 'gothic' metric, it is useful to work with a perturbative expansion of $\mathfrak{g}^{\mu\nu}$ (rather than $g_{\mu\nu}$, as was done in section 2.5), so that the harmonic coordinate condition retains its simple form at all orders. The *Post-Minkowskian* (PM) series, written as

$$\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + G h_{(1)}^{\mu\nu} + G^2 h_{(2)}^{\mu\nu} + \dots\,, \tag{4.16}$$

$$\varphi = \varphi_\infty + G \varphi_{(1)} + G^2 \varphi_{(2)} + \dots\,, \tag{4.17}$$

is an expansion in the 'nonlinearity' of gravity, where φ_∞ is the constant 'background' value of the scalar field asymptotically far away from the binary system. At i^{th} order in G, the harmonic coordinate condition simply reads $\partial_\mu h_{(i)}^{\mu\nu} = 0$, while the Einstein-frame field equations (2.8)-(2.9) take the form

$$\Box_\eta h_{(i)}^{\mu\nu} = F^{\mu\nu}[h_{(1)}, \dots, h_{(i-1)}, \varphi_{(1)}, \dots, \varphi_{(i-1)}; m_A, \Gamma_A]\,, \tag{4.18}$$

$$\Box_\eta \varphi_{(i)} = F[h_{(1)}, \dots, h_{(i-1)}, \varphi_{(1)}, \dots, \varphi_{(i-1)}; m_A, \Gamma_A]\,, \tag{4.19}$$

where $\Box_\eta = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the d'Alembertian operator of the flat Minkowski metric, and the right-hand sides depend on the lower-order solutions, as well as the mass functions $m_A(\varphi)$ and the worldlines Γ_A. Equations (4.18)-(4.19) may be formally solved by means of the Green's function (A.60), and in principle the procedure may be iterated to arbitrarily high order in G.

The harmonic coordinate condition at i^{th} order, $\partial_\mu h_{(i)}^{\mu\nu} = 0$, is a set of four constraints involving the worldlines, the mass functions, and the solution up to order $i - 1$. These constraints may be regarded in some sense as the $\mathcal{O}(G^{i-1})$ 'equations of motion' that the worldlines must satisfy. In General Relativity, this has been carefully worked out [42] for $i = 4$, and moreover, these constraints have been further expanded in a *Post-Newtonian* series, that

is, an expansion in powers of v/c, to yield the relativistic corrections to the equations of motion (4.1), up to and including G^3/c^5, which is the order at which non-conservative effects first arise [44].

In suitable coordinates, the $\mathcal{O}(v^4/c^4)$ relativistic corrections to the equations of motion may be derived from a two-body Lagrangian[22], whose symmetries may be employed to decouple the centre-of-mass and relative problems, and furthermore define the energy and angular momentum, i.e., the relativistic generalizations of (4.3)-(4.4). Although the mathematical description of trajectories at this order involves complicated hyperelliptic functions [43], Damour and Deruelle [45] have shown that shifting the radial coordinate and introducing 'relativistic eccentricities' allows the $\mathcal{O}(v^2/c^2)$ trajectories to be described by means of the standard Keplerian functions (4.7) and (4.9), providing the foundation upon which the Parametrized Post Keplerian (PPK) framework and pulsar timing formula [46, 54] have been built.

At order v^5/c^5, the equations of motion contain a 'radiation reaction' force, and thus may not be derived from a Lagrangian. The rate of energy loss due to this force has been found to be [44]

$$\frac{dE}{dt} = -\frac{8G^3M^2\mu^2}{15r^4c^5}[12v^2 - 11(\vec{v}\cdot\hat{r})^2]. \tag{4.20}$$

Moreover, the relative dynamics no longer decouples from that of the centre of mass, since a system emitting non-symmetric radiation experiences a recoil, as first pointed out by Bekenstein [14]. In recent years it has been found that a double-black-hole binary with anti-aligned spins in the orbital plane experiences a particularly large recoil, which has been called a 'superkick' [32, 27, 81, 33, 139, 101, 109, 82, 163].

4.5   Radiation at Infinity

The conventional interpretation of equation (4.20) is that mechanical energy is converted into 'gravitational-wave energy', which then gets 'radiated out to infinity'. The purpose of the present section is to make this intuition precise,

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[22]Although this Lagrangian generically depends on positions, velocities, and *accelerations*, a special coordinate choice eliminates the acceleration-dependence.

and demonstrate that the energy flux through a large sphere at infinity is given by (4.20). In addition to providing a consistency check of the general-relativistic result, this method enables one to calculate the power loss in Scalar-Tensor gravity without the use of equations of motion[23].

The principal obstacle to the description of gravitational radiation in terms of the coordinates x^μ and fields $(g_{\mu\nu}, \varphi)$ employed in the previous section, is the non-analyticity of the metric $g_{\mu\nu}$ at future null infinity — in an expansion of the form

$$g_{\mu\nu}(x^0, \vec{x}) = \eta_{\mu\nu} + \frac{k_{\mu\nu}(x^0 - r, \, \vec{x}/r)}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \qquad (4.21)$$

where $r = |\vec{x}|$, terms logarithmic in r appear at high Post-Minkowskian orders, because the light-cones of $g_{\mu\nu}$ along which the waves propagate differ from those of $\eta_{\mu\nu}$ [23]. However, it is possible to construct a 'radiative' coordinate system (which is *not* harmonic) adapted to the light cones of $g_{\mu\nu}$, in which the metric *is* analytic at future null infinity [21].

Therefore, it is necessary to break up space-time yet again. A region $\mathcal{D}_{\text{near}}$ (the *near zone*) which encloses the binary system is constructed, and $\mathcal{D}_{\text{far}}$ (the *far wave zone*) is defined to be its complement. In the latter region, the radiative coordinates constructed in [21] are employed and denoted by $X^\mu = (X^0, \vec{X})$, while the metric and scalar fields are denoted by $G_{\mu\nu}$ and Φ, respectively. The solution to the vacuum field equations in $\mathcal{D}_{\text{far}}$, subject to past-stationarity and no incoming radiation at past null infinity, has been obtained in [23] to all orders[24] in the Post-Minkowskian expansion. The resulting metric depends functionally on an infinite set of multipole moments, which are functions of the retarded time $U = (X^0 - |\vec{X}|)/c$, and are conventionally denoted by $M_{i_1 \cdots i_l}(U)$ and $S_{i_1 \cdots i_l}(U)$.[25] Similarly, the scalar field Φ depends on an infinite set of multipole moments denoted by $\Psi_{i_1 \cdots i_l}(U)$. Matching the

---

[23]These equations of motion are only known to order v^2/c^2, and calculating them to higher order is extremely difficult.

[24]This is made possible by the simplified structure of the Post-Minkowskian field equations (4.18)-(4.19) in the absence of matter. The solution is first obtained in harmonic coordinates, and then transformed to radiative coordinates.

[25]Note that these time-dependent multipole moments [34] differ from the frequency-dependent moments often used in electrodynamics [95]. For a comprehensive introduction

radiative fields $(G_{\mu\nu}, \Phi)$ to the near fields $(g_{\mu\nu}, \varphi)$ at the boundary leads to expressions for the multipole moments in terms of integrals over the sources in $\mathcal{D}_{\text{near}}$, the first few of which are given by

$$M_{ij}(U = t) = \int_{\mathcal{D}_{\text{near}}} d^3\vec{x} \left(x_i x_j - \tfrac{1}{3}\delta_{ij}|\vec{x}|^2\right) \rho(t, \vec{x}) + \mathcal{O}\left(\frac{1}{c^2}\right), \quad (4.22)$$

$$\psi(U = t) = \int_{\mathcal{D}_{\text{near}}} d^3\vec{x}\, j(t, \vec{x}) + \mathcal{O}\left(\frac{1}{c^2}\right), \quad (4.23)$$

$$\psi_i(U = t) = \int_{\mathcal{D}_{\text{near}}} d^3\vec{x}\, x_i\, j(t, \vec{x}) + \mathcal{O}\left(\frac{1}{c^2}\right), \quad (4.24)$$

$$\psi_{ij}(U = t) = \int_{\mathcal{D}_{\text{near}}} d^3\vec{x} \left(x_i x_j - \tfrac{1}{3}\delta_{ij}|\vec{x}|^2\right) j(t, \vec{x}) + \mathcal{O}\left(\frac{1}{c^2}\right), \quad (4.25)$$

in the non-relativistic limit, where $\rho = T^{00}/c^2 + \mathcal{O}(1/c^2)$ is the mass density, while j is the source term[26] in the scalar field equation

$$\sqrt{-g}\,\Box\varphi = -\frac{4\pi G}{c^2} j. \quad (4.26)$$

In the particular case of the matter action (4.13), it is found that

$$\rho(t, \vec{x}) = \sum_A M_A(\varphi_\infty)\delta^{(3)}(\vec{x} - \vec{z}_A(t)) + \mathcal{O}\left(\frac{1}{c^2}\right), \quad (4.27)$$

$$j(t, \vec{x}) = -\sum_A M_A'(\varphi_\infty)\delta^{(3)}(\vec{x} - \vec{z}_A(t)) + \mathcal{O}\left(\frac{1}{c^2}\right), \quad (4.28)$$

where $M_A'(\varphi) := dM_A(\varphi)/d\varphi$. By considering the variation of a Tolman-like integral expression for M_A, it may be shown [50] for a static body that $M_A'(\varphi) = Q_A(\varphi)$, where Q_A is defined by the integral in equation (3.7), or equivalently by the asymptotic expansion (3.4).

With expressions for the multipole moments in hand, it remains to express the radiative fields in terms of these moments, and calculate the energy flow through a large sphere at infinity. The leading-order term in the multipole expansion of the metric is given by

$$G_{ij}(U, \vec{X}) = \delta_{ij} + \frac{1}{R}\left[\frac{2G}{c^4}\ddot{M}_{ij}(U) + \mathcal{O}\left(\frac{1}{c^5}\right)\right] + \mathcal{O}\left(\frac{1}{R^2}\right), \quad (4.29)$$

---

to multipole expansions in relativistic gravity, the reader is referred to the review of Thorne [171].

[26]Note that j is generically *not* equal to $\sqrt{-g}\alpha(\varphi)T$ on account of the violation of the Strong Equivalence Principle (SEP).

where dots denote derivatives with respect to U, $R = |\vec{X}|$, and the time-time and time-space components of the metric vanish in the Transverse Traceless (TT) gauge. A similar expansion for the scalar field Φ has the form

$$
\begin{aligned}
\Phi \;=\; & \varphi_\infty + \frac{1}{R}\left[\frac{G}{c^2}\psi(U) + \frac{G}{c^3}N^i\dot{\psi}_i(U)\right. \\
& \left. + \frac{G}{2c^4}N^iN^j\ddot{\psi}_{ij}(U) + \mathcal{O}\left(\frac{1}{c^5}\right)\right] + \mathcal{O}\left(\frac{1}{R^2}\right), \quad (4.30)
\end{aligned}
$$

where $\vec{N} = \vec{X}/R$. The scalar energy flux through a large sphere of radius $R_\star$ may be calculated by means of the energy-momentum tensor defined in (2.25),

$$
\begin{aligned}
F_\varphi(U) \;=\; & \int_{S^2_{R_\star}} T^{0i}_{(\varphi)}(U,\vec{X})N_i dA \\
\;=\; & \frac{G}{c}\dot{\psi}^2(U) + \frac{G}{3c^3}\sum_{i=1}^{3}[\ddot{\psi}_i(U)]^2 \\
& + \frac{G}{30c^5}\sum_{i,j=1}^{3}[\dddot{\psi}_{ij}(U)]^2 + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (4.31)
\end{aligned}
$$

where the first three terms describe monopole, dipole, and quadrupole scalar radiation, respectively, and the limit $R_\star \to \infty$ has been taken so that the $1/R^2$ terms in (4.30) do not contribute.

The notion of a gravitational energy flux may only be defined in special circumstances. In particular, given a decomposition of the metric into a 'background' and 'perturbation', as in (4.29), the quadratic corrections to equation (2.46) may be interpreted as an effective gravitational energy-momentum tensor [189, 37]. Employing such a construction and calculating the corresponding flux, one finds the leading-order contribution

$$
F_g = \frac{G}{5c^5}\sum_{i,j=1}^{3}[\dddot{M}_{ij}(U)]^2 + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (4.32)
$$

which is the famous quadrupole formula. The fluxes (4.31)-(4.32) may be explicitly calculated for a binary system by means of equations (4.22)-(4.25) and (4.27)-(4.28), and in General Relativity it is found that $F_g = -dE/dt$, where the right-hand side is given by (4.20), confirming that the mechanical energy loss is balanced by the radiated power, as promised.

In Scalar-Tensor gravity, the tensor flux picks up an additional factor,

$$F_g^{\text{quad}} = \frac{8G}{15c^5} \left(\frac{G_{12}M\mu}{r^2} \right)^2 \left[12v^2 - 11(\vec{v} \cdot \hat{r})^2 \right], \tag{4.33}$$

where $G_{12} = G(1+\alpha_1\alpha_2)$ is the total (gravity plus scalar) coupling between the two bodies, and the charge-to-mass ratios $\alpha_A = Q_A/M_A$ may be interpreted as effective scalar-matter couplings. The scalar flux $F_\varphi = F_\varphi^{\text{mon}} + F_\varphi^{\text{dip}} + F_\varphi^{\text{quad}}$ contains three contributions given by

$$F_\varphi^{\text{mon}} = \mathcal{O}\left(\frac{1}{c^5} \right), \tag{4.34}$$

$$F_\varphi^{\text{dip}} = \frac{G}{3c^3} \left(\frac{G_{12}M\mu}{r^2} \right)^2 [\alpha_1 - \alpha_2]^2 + \mathcal{O}\left(\frac{1}{c^5} \right), \tag{4.35}$$

$$F_\varphi^{\text{quad}} = \frac{G}{30c^5} \left(\frac{G_{12}M\mu}{r^2} \right)^2 \left[32v^2 - \frac{88}{3}(\hat{r} \cdot \vec{v})^2 \right] (\alpha_1 X_2 + \alpha_2 X_1)^2$$
$$+ \mathcal{O}\left(\frac{1}{c^7} \right), \tag{4.36}$$

where $X_A = M_A/M$ are mass ratios. In equations (4.33)-(4.36), the masses and charges are all to be evaluated at φ_∞, the value of the scalar field asymptotically far away from the binary system. Note that the leading contribution to the monopole flux vanishes on account of the constancy of the bodies' scalar charges. The first non-trivial contribution to this flux depends on the second derivatives $M_A''(\varphi_\infty)$, and is given in [50].

4.6 Radiation in Massive Brans-Dicke Theory

Motivated in part by the recent argument that string theory predicts a broad range of scalar masses with potentially observable astrophysical consequences [9], Alsing and collaborators [6] have recently turned to the problem of finding the energy flux in *massive* Scalar-Tensor theories, whose solution is complicated by the unwieldy form of the Green's function for the massive wave equation in four-dimensional Minkowski space-time[27], and requires sophisticated contour integration methods [6, 190]. The most prominent feature of the obtained solution is the presence of Heaviside step functions cutting off

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[27]This Green's function may be written in terms of Bessel functions [6, 190]

the radiation when the scalar mass gets larger than (a constant multiple of) the orbital frequency, the existence of which has been intuitively anticipated in [31]. In order to simplify the calculation, Alsing and collaborators restricted attention to the special case of circular orbits in Brans-Dicke theory[28] and obtained the Jordan-frame scalar fluxes[29]

$$\tilde{F}_\phi^{\text{dip}} = \frac{2\tilde{G}^3\mathcal{G}^2\tilde{M}^2\tilde{\mu}^2\xi}{3\hat{r}^4c^3}(s_1 - s_2)^2\left(1 - \frac{m_s^2c^4}{\hbar^2\Omega^2}\right)\Theta(\hbar\Omega - m_sc^2),\quad (4.37)$$

$$\tilde{F}_\phi^{\text{quad}} = \frac{8\tilde{G}^3\mathcal{G}^2\tilde{M}^2\tilde{\mu}^2\xi\Gamma^2v^2}{15\hat{r}^4c^5}\left(1 - \frac{m_s^2c^4}{4\hbar^2\Omega^2}\right)\Theta(2\hbar\Omega - m_sc^2),\quad (4.38)$$

and the tensor flux

$$\tilde{F}_{\tilde{g}}^{\text{quad}} = \frac{8\tilde{G}^3\mathcal{G}^2\tilde{M}^2\tilde{\mu}^2v^2}{15\hat{r}^4c^5}(12 - 6\xi).\quad (4.39)$$

Here

$$m_s^2 = \frac{\hbar^2}{c^2}B''(\varphi_\infty) = \frac{\hbar^2}{c^2}\cdot\frac{V''(\phi_\infty)\phi_\infty}{2\omega_\infty + 3}\quad (4.40)$$

is the mass of the scalar field, Ω is the orbital angular frequency, and $\tilde{G}$ is the physical Newton constant given by equation (2.68). The hat on r indicates that units are used in which the Jordan-frame metric is asymptotically-Minkowski[30], and the tildes on the total and reduced masses indicate that these quantities are calculated using the Jordan-frame point-particle action (4.14).

The sensitivities of the two bodies are defined by

$$s_A \equiv \frac{d\log\tilde{M}_A(\phi)}{d\log\phi}\bigg|_{\phi=\phi_\infty},\quad (4.41)$$

and may be expressed in terms of Einstein-frame quantities by means of equation (2.12). It is found that

$$\begin{aligned}\frac{d\log\tilde{M}_A(\phi)}{d\log\phi} &= \frac{1}{2}\left(1 - \frac{d\log M_A(\varphi)}{d\log A(\varphi)}\right) \\ &= \frac{1}{2}\left(1 - \frac{\alpha_A(\varphi)}{\alpha(\varphi)}\right),\end{aligned}\quad (4.42)$$

---

[28]Work is in progress to relax both of these restrictions [6].

[29]The scalar monopole flux vanishes for a circular orbit.

[30]This notation was first introduced in equation (2.66)

which implies

$$1 - 2s_A = \frac{\alpha_A}{\alpha_\infty} \,. \tag{4.43}$$

With this result in hand, the constants ξ, $\mathcal{G}$, and Γ appearing in equations (4.37)-(4.39) may be transformed into the Einstein frame:

$$\xi \;:=\; \frac{1}{\omega_\infty + 2} = \frac{2\alpha_\infty^2}{1 + \alpha_\infty^2} \,, \tag{4.44}$$

$$\mathcal{G} \;:=\; 1 - \xi(s_1 + s_2 - 2s_1 s_2) = \frac{1 + \alpha_1 \alpha_2}{1 + \alpha_\infty^2} \,, \tag{4.45}$$

$$\Gamma \;:=\; 1 - 2(s_1 X_2 + s_2 X_1) = \frac{\alpha_1 X_2 + \alpha_2 X_1}{\alpha_\infty} \,. \tag{4.46}$$

Finally, upon putting everything together, it is found that[31]

$$\tilde{F}_\phi^{\text{dip}} \;=\; \frac{G A_\infty^{-2}}{3c^3} \left(\frac{G_{12} M \mu}{r^2}\right)^2 (\alpha_1 - \alpha_2)^2 \times$$
$$\left(1 - \frac{m_s^2 c^4}{\hbar^2 \Omega^2}\right) \Theta(\hbar\Omega - m_s c^2) \,, \tag{4.47}$$

$$\tilde{F}_\varphi^{\text{quad}} \;=\; \frac{16 G A_\infty^{-2}}{15 c^5} \left(\frac{G_{12} M \mu v}{r^2}\right)^2 (\alpha_1 X_2 + \alpha_2 X_1)^2 \times$$
$$\left(1 - \frac{m_s^2 c^4}{4\hbar^2 \Omega^2}\right) \Theta(2\hbar\Omega - m_s c^2) \,, \tag{4.48}$$

$$\tilde{F}_{\tilde{g}}^{\text{quad}} \;=\; \frac{96 G A_\infty^{-2}}{15 c^5} \left(\frac{G_{12} M \mu v}{r^2}\right)^2 \,, \tag{4.49}$$

which is consistent with equations (4.33)-(4.36) in the limit of circular orbits and a massless scalar, that is, $\hat{r} \cdot \vec{v} \to 0$ and $m_s \to 0$, explicitly confirming the equivalence of the Einstein-frame and Jordan-frame formulations of Scalar-Tensor gravity.

4.7 Effective Field Theory

4.7.1 Motivation

In General Relativity, the relativistic corrections to the equations of motion described in section 4.4 were calculated in a mathematically well-defined manner to order v^5/c^5, by employing the Riesz kernel (4.11) to regularize divergences

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[31]The notation A_∞ is short-hand for $A(\varphi_\infty)$.

arising in the point-particle limit [44, 42]. Two independent groups have attempted to extend this calculation to order v^6/c^6, and both found that the final equations of motion depend on an undetermined parameter arising in the regularization procedure. Although initially conjectured that this ambiguity is related to the physical incompleteness of the point-particle model, it was later understood how the unknown parameter may be uniquely determined by means of dimensional regularization. Subsequently, the results obtained by these two groups were shown to be physically equivalent [22].

Motivated by the desire to obtain an intuitive physical understanding of these complicated calculations, Goldberger and Rothstein [80, 78, 76] applied the formalism of Effective Field Theory (EFT) [30] to the gravitational two-body problem. This approach employs the tools of Quantum Field Theory (QFT) [112] to handle divergences, and explains in very simple terms why the traditional Post-Newtonian calculations run into difficulty at order v^6/c^6. Moreover, the equations of motion, radiative multipole moments, and power loss formulas may be calculated by means of Feynman diagrams and power counting, which drastically reduces the amount of work required to obtain results at a given Post-Newtonian order [77, 145, 80, 78, 76]. These methods are particularly useful for incorporating the effects of spin [129, 134, 133, 132, 137, 138, 130, 136, 135] and disspiation [79, 131].

4.7.2   Introduction

Motivated by the separation of scales[32] ubiquitous in nature, Effective Field Theory (EFT) [30] is a framework built upon the axiom that it is always possible to describe physics at energies $E \ll \Lambda$ (or length scales $r \gg r_0$) using only degrees of freedom accessible at those energies (or length scales). The high-energy (or short-distance) degrees of freedom are 'integrated out', and their effects are encoded into a set of low-energy (or long-distance) phenomenological parameters. For instance, hydrodynamics is an effective field theory in

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[32]For example, it is possible to understand the macroscopic properties of fluids without understanding atomic physics (QED), and the latter may be understood without knowledge of QCD.

which molecular length scales have been integrated out, and the viscosity is an example of a long-distance phenomenological parameter encoding information about the short-distance molecular interactions. Another example more relevant to the subject of this thesis is General Relativity itself, which is theoretically consistent and observationally well-tested at energies[33] $E \ll M_{\mathrm{pl}}c^2$ [29, 60, 59].

In the context of a binary inspiral, the effective field theory framework may be applied on two different levels [80]. For the purpose of obtaining the equations of motion, it is useful to integrate out length scales shorter than the sizes of the bodies, and encode their effects into a 'point particle' Lagrangian such as (4.12) or (4.13), which enters as a source into the field equations describing the orbital dynamics. On the other hand, for the purpose of obtaining the energy flux at infinity, length scales shorter than the orbital separation are integrated out, and the orbital dynamics is encoded into multipole moments such as (4.22)-(4.25), in terms of which the flux is expressed.

Of principal interest here is the former application, namely, the point-particle description of orbital dynamics, which will be relevant in section 5 where a double-black-hole binary system embedded in a cosmological background (or slowly moving through a scalar gradient) is considered, and found to be inadequately described by the Eardley Lagrangian (4.13).

4.7.3   Formalism

The point-particle description of a binary inspiral (in General Relativity or alternative theories) begins with a matter action of the form

$$S_m = \sum_A \int_{\Gamma_A} \mathcal{L}_A ds_A \,, \tag{4.50}$$

where the *point-particle Lagrangian* $\mathcal{L}_A$ functionally depends on the metric, worldline, and any other fields in the theory[34], and is invariant under coordinate transformations and worldline reparametrizations. In Scalar-Tensor

---

[33]The Planck mass M_{pl} is defined in equation (2.1).

[34]In some instances it is necessary to introduce additional degrees of freedom [129, 79, 131].

gravity, it has the explicit functional form

$$
\begin{aligned}
\mathcal{L}_A \;=\;\; & \mathcal{L}_A[g_{\mu\nu} \;;\; \epsilon_{\mu\nu\rho\sigma} \;;\; R_{\rho\sigma\mu\nu} \;;\; \nabla_{\lambda_1}\!\cdots\!\nabla_{\lambda_i} R_{\rho\sigma\mu\nu} \;; \\
& \varphi \;;\; \nabla_{\lambda_1}\!\cdots\!\nabla_{\lambda_i}\varphi \;;\; u_A^\lambda \;;\; D^i u_A^\lambda / ds_A^i\,],
\end{aligned}
\tag{4.51}
$$

where $z_A^\lambda(s_A)$ is an arclength parametrization of the worldline Γ_A, $u_A^\lambda :=$ dz_A^λ/ds_A is its velocity, and $D/ds_A := u_A^\lambda \nabla_\lambda$ is the covariant derivative along it.

The point-particle Lagrangian plays two distinct roles. On one hand, its functional derivatives appear as sources in the field equations, and on the other, its variation with respect to the worldline yields the equations of motion for body A in the test-particle limit. In the case of a 'true' point particle with mass M_A in General Relativity, it is well-known that

$$
\mathcal{L}_A = -M_A c,
\tag{4.52}
$$

while the point-particle Lagrangian of an extended body generically contains infinitely many correction terms describing the internal structure. Equation (4.51) suggests a natural way of organizing these terms — expanding in *numbers of derivatives*, which is related[35] to an expansion in powers of the size-to-separation ratio $\epsilon \sim R_A/D$ for the two-body problem, since the coefficient of a term with k derivatives[36] has dimension $ML^{k+1}T^{-1}$. The terms in $\mathcal{L}_A$ with this scaling will be denoted by $\mathcal{L}_A^{(k)}$, so that

$$
\mathcal{L}_A = \sum_{k=0}^{\infty} \mathcal{L}_A^{(k)}.
\tag{4.53}
$$

Carrying out this expansion in General Relativity, one finds that $\mathcal{L}_A^{(0)}$ is given by[37] (4.52), $\mathcal{L}_A^{(1)}$ vanishes[38], and

$$
\mathcal{L}_A^{(2)} = a_1 R + a_2 R_{\mu\nu} u^\mu u^\nu + a_3 \dot{u}^\mu \dot{u}^\nu g_{\mu\nu},
\tag{4.54}
$$

---

[35]This relation is not always a simple one, particularly when the orbital dynamics depends on a length scale other than D. Such a situation will be considered in section 5.

[36]This term is built from the objects listed in the square brackets on the right-hand side of equation (4.51). The Riemann tensor is counted as having two derivatives, while the velocity is counted as having zero derivatives.

[37]Note that $u^\mu u^\nu g_{\mu\nu} = -1$.

[38]Note that $(Du^\mu/ds)u^\nu g_{\mu\nu} = \frac{1}{2}\frac{D}{ds}(u^\mu u^\nu g_{\mu\nu}) = 0$.

where $\dot{u}^\mu$ is short-hand notation for the covariant derivative Du^μ/ds, and the body labels A on u^μ, s, and a_i have been suppressed for brevity. Since the variation of the gravitational action $\int d^4x\sqrt{-g}R$ is a linear combination of $R_{\mu\nu}$ and R (see equation (A.46)), a re-definition of the metric may be employed to remove the a_1 and a_2 terms from $\mathcal{L}_A^{(2)}$, as explained in the appendix of [76]. A power-counting argument [78] demonstrates that the a_1 and a_2 terms may only appear in the equations of motion at orders v^6/c^6 and higher, explaining why the traditional approaches ran into difficulties at this order, while the fact that a_1 and a_2 may be removed explains why it was possible to surmount these difficulties[39].

Turning to the a_3 term quadratic in $\dot{u}$, one finds that it may be removed by redefining the dynamical variables z_A^λ, since the variation of $\int \mathcal{L}_A^{(0)} ds_A$ with respect to the worldline is proportional to $\dot{u}$ [37]. Continuing the derivative expansion to third order, one finds a plethora of terms involving R, ∇R, $\dot{u}$, $\ddot{u}$, and $\dddot{u}$, which the present author has not seen discussed anywhere in the literature. For instance, Goldberger and Rothstein [76, 79, 80] state that the first non-redundant terms arise at fourth order in the derivative expansion, and may be put into the canonical form

$$\mathcal{L}_A^{(4)} = c_E E_{\mu\nu} E^{\mu\nu} + c_B B_{\mu\nu} B^{\mu\nu} \,, \tag{4.55}$$

where $E_{\mu\nu} := C_{\mu\rho\nu\sigma} u^\rho u^\sigma$ and $B_{\mu\nu} := \tilde{C}_{\mu\rho\nu\sigma} u^\rho u^\sigma$ are the 'electric' and 'magnetic' parts of they Weyl tensor, respectively, where $\tilde{C}_{\mu\rho\nu\sigma} := \frac{1}{2}\epsilon_{\mu\rho\kappa\lambda} C^{\kappa\lambda}{}_{\nu\sigma}$ is the dual.

Turning to Scalar-Tensor gravity, it is found that $\mathcal{L}_A^{(0)}$ is given by the Eardley Lagrangian (4.13), whereas $\mathcal{L}_A^{(1)}$ is a total derivative,

$$\mathcal{L}_A^{(1)} = \frac{dF(\varphi)}{ds_A} \,, \tag{4.56}$$

contributing only a boundary term to the action. Although eight terms are *a*

---

[39]The fact that a_1 and a_2 may be removed is also an explicit illustration of the principle of effacement briefly mentioned in section 2.1.1.

priori possible at second order,

$$
\begin{aligned}
\mathcal{L}_A^{(2)} &= a_1(\varphi)R + a_2(\varphi)R_{\mu\nu}u^\mu u^\nu + a_3(\varphi)\dot{u}^\mu \dot{u}^\nu g_{\mu\nu} \\
&\quad + a_4(\varphi)g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi + a_5(\varphi)u^\mu u^\nu \nabla_\mu\varphi\nabla_\nu\varphi \\
&\quad + a_6(\varphi)g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi + a_7(\varphi)u^\mu u^\nu \nabla_\mu\nabla_\nu\varphi \\
&\quad + a_8(\varphi)\dot{u}^\mu \nabla_\mu\varphi\,,
\end{aligned}
\tag{4.57}
$$

the field redefinition procedure described above may be employed to remove the a_3, a_6, and a_8 terms, and absorb the a_1 and a_2 terms into the a_4 and a_5 terms, respectively [53]. In the latter reference, it was demonstrated that for bodies which are static and spherically-symmetric when unperturbed, the a_4 term is the dominant one in the Post-Newtonian limit.

In summary, the formalism of Effective Field Theory demonstrates that a derivative expansion of the point-particle Lagrangian is related to an expansion in powers of the size-to-separation ratio $\epsilon \sim R/D$ for a binary inspiral. The next section considers an astrophysically interesting situation where the Eardley Lagrangian (4.13) is inadequate, and it is necessary to go beyond the zeroth order in this derivative expansion to correctly describe the orbital dynamics.

5   Miracle Hair Growth

5.1   Introduction

The purpose of this section is to construct an astrophysically interesting example of a binary inspiral in which the Eardley Lagrangian (4.13) is inadequate to describe the orbital dynamics, and the derivative expansion explained in 4.7.3 must be taken to higher orders. This may be accomplished by introducing a new scale into the problem, and the natural candidate is the time scale $\mathfrak{T}$ over which the asymptotic background scalar φ_∞ evolves, which may physically arise either due to a cosmological embedding [70, 16], or a relative velocity to an external scalar gradient, such as that sourced by the dark matter distribution of a galaxy. In addition to the size-to-separation ratio $\epsilon \sim R/D$, the orbital dynamics will then depend on the ratios $T_{\mathrm{orb}}/\mathfrak{T}$ and $T_{\mathrm{int}}/\mathfrak{T}$, where T_{orb} is the orbital time scale[40], and T_{int} is the internal time scale[41] of the bodies. It turns out that $\mathcal{O}(T_{\mathrm{int}}/\mathfrak{T})$ effects cause black holes to grow scalar hair, which is the subject of the next section.

5.2   Jacobson's Formula

As discussed in section 3.3, black holes in Scalar-Tensor gravity often fail to have 'scalar hair', and reduce to those of General Relativity. Although it is not difficult to obtain hairy black-hole solutions, most of these constructions are irrelevant for astrophysics, since they employ higher dimensions, electromagnetic charge, scalar potentials, and combinations thereof[42]. An interesting exception is *Jacobson's Miracle Hair Growth Formula*, discovered back in 1999 [96], which states that a Schwarzschild[43] black hole subject to time-dependent boundary conditions for the scalar field at spatial infinity *grows hair*, in other words, acquires a non-zero scalar charge[44] Q_A. More precisely, if the asymp-

---

[40]This quantity may be related to other orbital variables by equation (4.8).

[41]For a black hole of mass M, $T_{\mathrm{int}} = GM/c^3$.

[42]For a few examples, see [165, 156, 5, 111].

[43]The result generalizes to a wide class of black holes in a fairly straightforward manner.

[44]Recall that Q_A is defined by the integral in equation (3.7), or equivalently by the asymptotic expansion (3.4).

totic value of the scalar φ_∞ evolves linearly in time with scale $\mathfrak{T}$, that is, $\varphi(t, r = \infty) = t/\mathfrak{T} + \text{const}$, and one works to linear order in the ratio[45] $\delta := GM_A/\mathfrak{T}c^3$, then the dimensionless quantity characterizing the amount of hair grown[46] is given by

$$\alpha_A := \frac{Q_A}{M_A} = 4\delta = \frac{4GM_A}{\mathfrak{T}c^3} \ . \tag{5.1}$$

On a technical level, the vacuum field equations to be solved boil down to the wave equation $\Box\varphi = 0$ in a Schwarzschild background upon linearization in δ, and the scalar field profile giving rise to the hair is the following globally-defined zero mode of the d'Alembertian operator:

$$\varphi(t, r) = \frac{t}{\mathfrak{T}} + 2\delta \log \left| 1 - \frac{2GM_A}{rc^2} \right| \ , \tag{5.2}$$

where both terms individually satisfy the linear wave equation, and coefficients are chosen so that the breakdown of the time coordinate at the horizon is precisely cancelled by the singularity in the second term, as explained in [96, 91].

5.3   Double-Black-Hole Binaries

In the context of a double-black-hole binary, Jacobson's formula implies that both black holes acquire scalar charges given by (5.1) at linear order in δ, and thus a scalar interaction between the black holes arises at order δ^2. To find the orbital corrections induced by this interaction, it is insufficient to use the Eardley Lagrangian (4.13), because the mass (or any other physical property) of a black hole in Scalar-Tensor gravity is invariant under shifts of the scalar field, implying that the source term $\delta S_m^{\text{Eardley}}/\delta\varphi$ in the scalar field equation vanishes.

It follows from the discussion in 4.7.3 that the appropriate point-particle Lagrangian from which the $\mathcal{O}(\delta^2)$ orbital corrections may be derived has the

---

[45]In taking this quantity to be small, one is assuming that the cosmological or galactic time scale $\mathfrak{T}$ associated with the evolution of φ_∞ is much longer than the light crossing time scale GM_A/c^3 of the black hole.

[46]This quantity, initially defined in section 4.5, enters into the power loss formulas (4.33)-(4.36).

general form

$$\mathcal{L}_A = -M_A c + (a_4 g^{\mu\nu} + a_5 u^\mu u^\nu)\nabla_\mu \varphi \nabla_\nu \varphi + a_7 u^\mu u^\nu \nabla_\mu \nabla_\nu \varphi, \quad (5.3)$$

where the coefficients a_i do *not* depend on φ on account of the scalar-shift symmetry of black holes, and may be determined by a matching calculation. This calculation is reserved for future work, and attention is restricted to the simpler problem of finding the dipole flux emitted by the accelerating scalar charges. To leading order in δ and leading Post-Newtonian order, this flux is given by the second term on the second line of equation (4.31), where the scalar dipole moment $\psi_i(U)$ is calculated by means of equation (4.24), in which the source j is given by (4.28), with $z_A(t)$ being the $\mathcal{O}(\delta^0)$ unperturbed trajectory, and $M'_A(\varphi_\infty)$ being replaced by $Q_A = \alpha_A M_A$ as given by the Jacobson formula (5.1). Putting everything together yields the final radiated scalar dipole power

$$F_\varphi^{\text{dip}} = \frac{G^3 M_1^2 M_2^2}{3r^4 c^3} \left[\alpha_1 - \alpha_2\right]^2 + \mathcal{O}\left(\frac{v^5}{c^5}\right) + \mathcal{O}(\delta^3), \quad (5.4)$$

for a double-black-hole binary in Scalar-Tensor gravity subject to the boundary conditions $\varphi(t, r = \infty) = t/\mathfrak{T} + \text{const.}$

5.4   The Quasar OJ287

The result (5.4) has a nice application to quasar OJ287, whose light intensity as a function of time is quasi-periodic, with two 'bursts' occurring every 12 years. In 1988, this system was modelled by Sillanpää and collaborators [157] as a supermassive black hole binary, with an outburst occurring whenever the orbit of the lighter black hole punctures the accretion disc of the heavier companion. Work on the model continued following the outbursts of 1994-5 [107, 176, 167, 178, 143], and the first outburst of 2007 [182, 179, 177, 174]. In particular, a prediction was made for the date of the second outburst, which was subsequently verified by observations to within 6% [185]. Further work on the model [183, 180, 184] has led to an estimate of the spin of the heavier black hole [187, 181, 175], and it is expected that observations of future outbursts will yield a test of the no-hair theorem [186].

The agreement of the general-relativistic quadrupolar flux F_g^{quad} (given by (4.20)) with the observations of OJ287 implies a bound on the ratio $F_\varphi^{\text{dip}}/F_g^{\text{quad}}$, which by means of equations (5.4), (5.1), and (4.20) may be translated into the bound $\mathfrak{T} \gtrsim 16$ days [91]. Although not impressive by cosmological or galactic standards, it is quite remarkable that such a bound is even possible, given that it is completely independent of any assumptions about how the scalar couples to matter.

6   Summary and Prospects

In this thesis, strong-field effects in Scalar-Tensor gravity were investigated, and two novel results have been presented — The *Weak Central Coupling* (WCC) framework for describing neutron star scalarization perturbatively, and the *Miracle Hair Growth* in double-black-hole binaries induced by cosmological and/or galactic effects.

The former has been worked out in the 'quadratic' model with conformal factor (3.81) and Einstein-frame scalar-matter coupling (3.82) for simplicity. However, since pulsar timing observations have already ruled out a large region of the quadratic-model theory-space in which scalarization is allowed [68, 66, 67, 65, 64, 53, 63, 52, 49, 20, 106, 74, 89], it would be worthwhile to extend the WCC formalism to more complicated coupling functions, as well as more general Scalar-Tensor theories such as Einstein-Dilaton-Gauss-Bonet (EDGB) gravity [116, 126], in which a detailed study of stellar structure has recently been undertaken [125].

A calculation of the scalar dipole flux emitted by a double-black-hole binary has led to a bound on galactic and cosmological effects in quasar OJ287 which is interesting in principle, but not very useful in practice. However, observations of future OJ287 outbursts, as well as direct gravitational wave observations by second-generation detectors may yield better bounds.

A calculation of theoretical interest which has not yet been undertaken, is the determination of the coefficients a_i in the effective point-particle Lagrangian (5.3) describing a 'Jacobson' black hole with 'miracle hair'.

A Mathematical Preliminaries

The purpose of this section is to establish conventions and notation, and list the mathematical formulae that are used throughout the thesis.

A.1 Units

The speed of light is denoted by c. In the Einstein-frame formulation of Scalar-Tensor gravity, the gravitational constant is denoted by G and scalar fields are denoted by φ, while in the Jordan-frame formulation, the gravitational constant (which is measured in a Cavendish experiment) is denoted by $\tilde{G}$ while scalar fields are denoted by ϕ. Geometrical units are *not* used, and all factors of G and c are explicitly written. Note that φ is dimensionless, while ϕ has units of $1/G$, where G has units of L^{d-1}/MT^2 in d dimensions.

A.2 Basic Differential Geometry

The number of space-time dimensions is denoted by d. Although the case $d = 4$ is of physical interest, results are derived for arbitrary d whenever possible. Space-time coordinates are denoted by $x^\mu = (x^0, x^i)$, where $x^0 = ct$ is the time coordinate, and x^i are spatial coordinates. Greek indices run from 0 to $d - 1$, and latin indices run from 1 to $d - 1$. The partial derivative with respect to x^μ is denoted by ∂_μ.

The space-time metric is denoted by $g_{\mu\nu}$, and its inverse is denoted by $g^{\mu\nu}$, so that $g_{\mu\nu}g^{\nu\lambda} = \delta^\lambda_\mu$. Differentiating this identity relates partial derivatives of the metric to partial derivatives of the inverse metric:

$$\partial_\lambda g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\partial_\lambda g_{\rho\sigma}\,, \qquad \partial_\lambda g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\partial_\lambda g^{\rho\sigma}\,. \tag{A.1}$$

Unless stated otherwise, indices are raised and lowered with $g^{\mu\nu}$ and $g_{\mu\nu}$, respectively. Signs are chosen so that time-like vectors have negative norm, and space-like vectors have positive norm. In other words, the metric is 'mostly plus'. MTW [114], Carroll [37], and Weinberg [191] all use this same choice of sign, but some other authors use opposite signs.

The flat Minkowski metric is denoted by $\eta_{\mu\nu}$. In canonical coordinates on Minkowski space-time, it takes the form $\eta_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$. The notation for symmetrization and antisymmetrization of tensors is given by

$$T_{(\mu_1 \cdots \mu_k)} = \frac{1}{k!} \sum_{\sigma \in S_k} T_{\mu_{\sigma(1)} \cdots \mu_{\sigma(k)}} , \tag{A.2}$$

$$T_{[\mu_1 \cdots \mu_k]} = \frac{1}{k!} \sum_{\sigma \in S_k} (\text{sgn}\sigma) T_{\mu_{\sigma(1)} \cdots \mu_{\sigma(k)}} . \tag{A.3}$$

Note that some authors may omit the factor of $k!$.

The determinant of the metric, which is negative-definite, is denoted by g. It appears in the canonical volume element of the space-time manifold,

$$dV = \sqrt{-g} dx^0 \cdots dx^{d-1} \equiv \sqrt{-g} d^d x , \tag{A.4}$$

and its derivative may be computed by using the matrix identity $\log \det = \text{tr} \log$ and equation (A.1):

$$\partial_\lambda \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial_\lambda g_{\mu\nu} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \partial_\lambda g^{\mu\nu} . \tag{A.5}$$

The completely antisymmetric Levi-Civita tensor is defined to be

$$\epsilon_{\mu_1 \cdots \mu_d} = \sqrt{-g} \cdot \text{sgn}(\mu_1 \cdots \mu_d) , \tag{A.6}$$

and its raised-index form is given by

$$\epsilon^{\mu_1 \cdots \mu_d} = -\frac{1}{\sqrt{-g}} \cdot \text{sgn}(\mu_1 \cdots \mu_d) \tag{A.7}$$

When the partial derivative operator ∂_μ acts on a tensor, the resulting object is generally *not* a tensor. However, it is possible to define a covariant derivative operator ∇_μ, which maps tensors to tensors, annihilates the metric, and reduces to the partial derivative operator when acting on scalars. Its action on single-index tensors is given by

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\lambda} V^\lambda , \qquad \nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda{}_{\mu\nu} V_\lambda , \tag{A.8}$$

and its action on multi-index tensors is given by straightforward generalizations of (A.8), with a $+\Gamma$ correction term for each upper index, and a $-\Gamma$ correction term for each lower index.

The $\Gamma^\lambda{}_{\mu\nu}$ are called Christoffel symbols. They are not tensors, and they are constructed to cancel the 'non-covariant' parts of ∂_μ. An explicit formula for them may be found by writing out the metric compatibility condition $\nabla_\lambda g_{\mu\nu} = 0$:

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right) . \tag{A.9}$$

Contracting equation (A.9) and combining the result with equation (A.5) yields

$$\Gamma^\mu{}_{\mu\lambda} = \frac{1}{2} g^{\mu\nu} \partial_\lambda g_{\mu\nu} = -\frac{1}{2} g_{\mu\nu} \partial_\lambda g^{\mu\nu} = \frac{\partial_\lambda \sqrt{-g}}{\sqrt{-g}} . \tag{A.10}$$

The d'Alembertian wave operator is defined by $\square \equiv \nabla_\mu \nabla^\mu$. By means of equation (A.10), the d'Alembertian of a scalar function f may be written explicitly in coordinates as

$$\begin{aligned} \square f &= \partial_\mu \partial^\mu f + \Gamma^\mu{}_{\mu\nu} \partial^\nu f \\ &= \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu f \right) . \end{aligned} \tag{A.11}$$

When acting on a scalar function f, covariant derivatives commute: $\nabla_\mu \nabla_\nu f = \nabla_\nu \nabla_\mu f$. However, this need not be the case when covariant derivatives act on tensors of higher rank. The failure of covariant derivatives to commute is quantified by the Riemann curvature tensor $R^\rho{}_{\sigma\mu\nu}$, so that for single-index tensors,

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma , \qquad [\nabla_\mu, \nabla_\nu] V_\rho = -R^\sigma{}_{\rho\mu\nu} V_\sigma , \tag{A.12}$$

and for multi-index tensors, the action of $[\nabla_\mu, \nabla_\nu]$ is given by a straightforward generalization of equation (A.12), with a $+R$ term for each upper index, and a $-R$ term for each lower index.

The curvature conventions used are those of MTW [114], Wald [189], and Carroll [37]. They differ from those of Weinberg [191] by a sign. The advantage of this choice is that the Ricci scalar curvature (A.18) is positive on spheres and negative on hyperboloids, in line with intuition. However, the disadvantage is that in an action, the gravitational kinetic term has the opposite sign of the kinetic terms of other fields.

Using (A.8) to evaluate the left-hand side of one of the equations in (A.12) yields the following expression for the Riemann tensor in coordinates:

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}{}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma}. \tag{A.13}$$

It may be shown that the Riemann tensor has the following symmetry properties under the interchange of two indices:

$$R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu} = -R_{\sigma\rho\mu\nu} = R_{\mu\nu\rho\sigma}. \tag{A.14}$$

The first equality in (A.14) follows immediately from the definition (A.12), whereas it takes more work to derive the other two equalities. The Riemann tensor also satisfies two Bianchi identities, which involve antisymmetrization over three indices. The first of these is given by

$$R_{\rho[\sigma\mu\nu]} = 0, \tag{A.15}$$

and the second is given by

$$\nabla_{[\lambda}R_{\rho\sigma]\mu\nu} = 0. \tag{A.16}$$

On account of the symmetries (A.14), there exists only one independent non-vanishing contraction of the Riemann tensor. It is called the Ricci tensor, and is defined by

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}. \tag{A.17}$$

It follows from (A.14) that the Ricci tensor is symmetric: $R_{\mu\nu} = R_{\nu\mu}$. It may be further contracted, to define the Ricci scalar:

$$R = R^{\mu}{}_{\mu}. \tag{A.18}$$

Writing out and contracting the second Bianchi identity (A.16) yields the following differential relation between the Ricci tensor and scalar:

$$\nabla^{\mu}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = 0. \tag{A.19}$$

The trace-free part of the Riemann tensor, with the Ricci tensor and scalar 'subtracted out', is called the Weyl tensor, is given by

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{d-2}\left(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}\right) + \frac{2}{(d-1)(d-2)}g_{\rho[\mu}g_{\nu]\sigma}R, \tag{A.20}$$

and has the same symmetry properties (A.14)-(A.15) as the Riemann tensor under interchange of indices.

By means of equations (A.13), (A.9), and (A.1), the Ricci tensor (A.17) may be written explicitly in terms of the metric and its partial derivatives as

$$
\begin{aligned}
R_{\mu\nu} &= \frac{1}{2} g^{\rho\sigma} (\partial_\nu \partial_\sigma g_{\mu\rho} - \partial_\rho \partial_\sigma g_{\mu\nu} + \partial_\mu \partial_\rho g_{\nu\sigma} - \partial_\mu \partial_\nu g_{\rho\sigma}) \\
&\quad + \frac{1}{4} g^{\kappa\lambda} g^{\rho\sigma} \big(2 \partial_\kappa g_{\mu\nu} \partial_\sigma g_{\rho\lambda} - \partial_\kappa g_{\mu\nu} \partial_\lambda g_{\rho\sigma} \\
&\quad\quad - 2 \partial_\nu g_{\mu\kappa} \partial_\sigma g_{\rho\lambda} + \partial_\nu g_{\mu\kappa} \partial_\lambda g_{\rho\sigma} \\
&\quad\quad - 2 \partial_\mu g_{\nu\kappa} \partial_\sigma g_{\rho\lambda} + \partial_\mu g_{\nu\kappa} \partial_\lambda g_{\rho\sigma} \\
&\quad\quad + 2 \partial_\sigma g_{\nu\lambda} \partial_\rho g_{\kappa\mu} - 2 \partial_\sigma g_{\nu\lambda} \partial_\kappa g_{\rho\mu} \\
&\quad\quad + \partial_\mu g_{\rho\lambda} \partial_\nu g_{\sigma\kappa} \big) .
\end{aligned}
\tag{A.21}
$$

In order to write the above expression as a non-linear wave operator acting on the metric, it is useful to introduce harmonic coordinates, defined by the condition

$$
0 = \Box x^\mu = \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} g^{\mu\nu}) .
\tag{A.22}
$$

It follows from equation (A.5) and (A.1) that the harmonic coordinate condition (A.22) is equivalent to

$$
\frac{1}{2} g^{\mu\nu} \partial_\rho g_{\mu\nu} = \partial^\sigma g_{\rho\sigma} .
\tag{A.23}
$$

Equation (A.23) may be used to simplify equation (A.21), and obtain the following expression for the Ricci tensor in harmonic coordinates:

$$
\begin{aligned}
R_{\mu\nu} &= -\frac{1}{2} g^{\rho\sigma} \partial_\rho \partial_\sigma g_{\mu\nu} \\
&\quad + \frac{1}{4} g^{\kappa\lambda} g^{\rho\sigma} \big(2 \partial_\nu g_{\rho\lambda} \partial_\sigma g_{\mu\kappa} + 2 \partial_\mu g_{\rho\lambda} \partial_\sigma g_{\nu\kappa} \\
&\quad\quad + 2 \partial_\sigma g_{\nu\lambda} \partial_\rho g_{\kappa\mu} - 2 \partial_\sigma g_{\nu\lambda} \partial_\kappa g_{\rho\mu} \\
&\quad\quad - \partial_\mu g_{\rho\lambda} \partial_\nu g_{\sigma\kappa} \big) .
\end{aligned}
\tag{A.24}
$$

A.3   Static Spherically-Symmetric Metrics

In this section, coordinates for describing static spherically-symmetric bodies are introduced, and the various curvature tensors are explicitly computed in these coordinates.

The most general static spherically-symmetric metric may be written as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -f(r)c^2 dt^2 + h(r)dr^2 + k(r)d\Omega^2 \, , \tag{A.25}$$

where the coordinates are $x^\mu = (ct, r, \theta_1, \ldots, \theta_{d-2})$, and $d\Omega^2$ is the canonical line element on S^{d-2} in hyperspherical coordinates, namely,

$$
\begin{aligned}
d\Omega^2 &= d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \sin^2\theta_1 \sin^2\theta_2 d\theta_3^2 + \cdots \\
&\quad + (\sin^2\theta_1 \cdots \sin^2\theta_{d-3})d\theta_{d-2}^2 \, .
\end{aligned}
\tag{A.26}
$$

Note that the functions f and h are dimensionless, while the function k has units of length squared.

Given this choice of coordinates, it follows from equation (A.9) that the independent non-vanishing Christoffel symbols are given by

$$\Gamma^0{}_{0r} = \frac{f'}{2f} \, , \tag{A.27}$$

$$\Gamma^r{}_{00} = \frac{f'}{2h} \, , \tag{A.28}$$

$$\Gamma^r{}_{rr} = \frac{h'}{2h} \, , \tag{A.29}$$

$$\Gamma^r{}_{\theta_i\theta_i} = -\frac{k'}{2k} \cdot \frac{g_{\theta_i\theta_i}}{h} \, , \tag{A.30}$$

$$\Gamma^{\theta_i}{}_{r\theta_i} = \frac{k'}{2k} \, , \tag{A.31}$$

$$\Gamma^{\theta_j}{}_{\theta_i\theta_i} = \begin{cases} -\cot\theta_j g^{\theta_j\theta_j} g_{\theta_i\theta_i} & : j < i \\ 0 & : j \geq i \end{cases} \, , \tag{A.32}$$

$$\Gamma^{\theta_j}{}_{\theta_j\theta_i} = \begin{cases} \cot\theta_i & : i < j \\ 0 & : i \geq j \end{cases} \, , \tag{A.33}$$

where primes denote derivatives with respect to r, and it follows from equations (A.13) and (A.14) that the independent non-vanishing components of

the Riemann tensor are given by

$$R^0{}_{r0r} = -\frac{f''}{2f} + \frac{f'^2}{4f^2} + \frac{f'h'}{4fh}, \tag{A.34}$$

$$R^0{}_{\theta_i 0\theta_i} = -\frac{f'k'}{4fk}\frac{g_{\theta_i\theta_i}}{h}, \tag{A.35}$$

$$R^r{}_{\theta_i r\theta_i} = \left(-\frac{k''}{2k} + \frac{k'^2}{4k^2} + \frac{h'k'}{4hk}\right)\frac{g_{\theta_i\theta_i}}{h}, \tag{A.36}$$

$$R^{\theta_i}{}_{\theta_j\theta_i\theta_j} = \left(\frac{h}{k} - \frac{k'^2}{4k^2}\right)\frac{g_{\theta_j\theta_j}}{h} \qquad (i \neq j). \tag{A.37}$$

Carrying out the contraction (A.17) yields the following expressions for the non-vanishing independent components of the Ricci tensor:

$$R_{00} = \frac{f}{h}\left(\frac{f''}{2f} - \frac{f'^2}{4f^2} - \frac{f'h'}{4fh} + \frac{(d-2)f'k'}{4fk}\right), \tag{A.38}$$

$$R_{rr} = -\frac{f''}{2f} + \frac{f'^2}{4f^2} + \frac{f'h'}{4fh} + (d-2)\left(-\frac{k''}{2k} + \frac{k'^2}{4k^2} + \frac{k'h'}{4kh}\right), \tag{A.39}$$

$$R_{\theta_i\theta_i} = \left[d - 3 + \frac{k}{h}\left(-\frac{k''}{2k} - \frac{(d-4)k'^2}{4k^2} + \frac{k'h'}{4kh} - \frac{k'f'}{4kf}\right)\right]\frac{g_{\theta_i\theta_i}}{k}. \tag{A.40}$$

A.4   Variations of the Metric

In order to derive the field equations from the action in a relativistic theory of gravity, it is necessary to consider a variation of the metric $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, and compute how the various curvature tensors transform under such a variation. The variations of the metric and its inverse are related by

$$\delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}, \qquad \delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}, \tag{A.41}$$

which is analogous to equation (A.1). The variation of the determinant of the metric is given by

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}, \tag{A.42}$$

which is analogous to equation (A.5). It follows from equations (A.9), (A.8), and (A.41) that the variation of the Christoffel symbols is given by

$$\delta\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\nabla_\mu\delta g_{\nu\sigma} + \nabla_\nu\delta g_{\mu\sigma} - \nabla_\sigma\delta g_{\mu\nu}\right). \tag{A.43}$$

The variation of the Riemann curvature tensor is given by

$$
\begin{aligned}
\delta R^{\rho}{}_{\sigma\mu\nu} &= \nabla_{\mu}\delta\Gamma^{\rho}{}_{\nu\sigma} - \nabla_{\nu}\delta\Gamma^{\rho}{}_{\mu\sigma} \\
&= \frac{1}{2}g^{\rho\lambda}\big([\nabla_{\mu},\nabla_{\nu}]\delta g_{\sigma\lambda} + \nabla_{\mu}(\nabla_{\sigma}\delta g_{\nu\lambda} - \nabla_{\lambda}\delta g_{\nu\sigma}) \\
&\qquad\qquad - \nabla_{\nu}(\nabla_{\sigma}\delta g_{\mu\lambda} - \nabla_{\lambda}\delta g_{\mu\sigma})\big)\,, \quad \text{(A.44)}
\end{aligned}
$$

where the first equality follows equations (A.13) and (A.8), and the second equality follows from equation (A.43). Taking variations of the contractions of the Riemann tensor, one obtains

$$
\begin{aligned}
\delta R_{\mu\nu} &= \nabla_{\lambda}\delta\Gamma^{\lambda}{}_{\mu\nu} - \nabla_{\nu}\delta\Gamma^{\lambda}{}_{\lambda\mu} \\
&= \nabla^{\lambda}\nabla_{(\mu}\delta g_{\nu)\lambda} - \frac{1}{2}\Box\delta g_{\mu\nu} - \frac{1}{2}g^{\lambda\rho}\nabla_{\mu}\nabla_{\nu}\delta g_{\lambda\rho} \qquad \text{(A.45)}
\end{aligned}
$$

for the variation of the Ricci tensor, and

$$
\begin{aligned}
\delta R &= -(R^{\mu\nu} + g^{\mu\nu}\Box - \nabla^{\mu}\nabla^{\nu})\delta g_{\mu\nu} \\
&= (R_{\mu\nu} + g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu})\delta g^{\mu\nu} \qquad\qquad \text{(A.46)}
\end{aligned}
$$

for the variation of the Ricci scalar.

A.5   Conformal Transformations

Scalar-tensor theories of gravity have two different mathematical representations, called the Einstein and Jordan frames. In order to convert between them, it is necessary to know how the various curvature tensors transform when the metric is re-scaled by a conformal factor:

$$
\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}\,, \qquad \tilde{g}^{\mu\nu} = \Omega^{-2}g^{\mu\nu}\,, \qquad\qquad \text{(A.47)}
$$

where Ω is a function of space-time. Let $\tilde{\Gamma}^{\lambda}{}_{\mu\nu}$, $\tilde{\nabla}_{\mu}$, $\tilde{\Box}$, $\tilde{R}^{\rho}{}_{\sigma\mu\nu}$, $\tilde{R}_{\mu\nu}$, and $\tilde{R}$ be the Christoffel symbols, covariant derivative, d'Alembertian, Riemann tensor, Ricci tensor, and Ricci scalar, respectively, which are constructed using the re-scaled metric (A.47). In particular, this means that the indices on $\tilde{\Gamma}^{\lambda}{}_{\mu\nu}$, $\tilde{\nabla}_{\mu}$, $\tilde{R}^{\rho}{}_{\sigma\mu\nu}$, and $\tilde{R}_{\mu\nu}$ are raised and lowered with the re-scaled metric (A.47). For example, $\tilde{R} = \tilde{g}^{\mu\nu}\tilde{R}_{\mu\nu}$, and $\tilde{R}_{\rho\sigma\mu\nu} = \tilde{g}_{\rho\lambda}\tilde{R}^{\lambda}{}_{\sigma\mu\nu}$.

In order to obtain the transformation of the Christoffel symbols, one uses equations (A.9) and (A.47) to find that

$$
\begin{aligned}
\tilde{\Gamma}^{\lambda}{}_{\mu\nu} &= \frac{1}{2}\tilde{g}^{\lambda\sigma}\left(\partial_{\mu}\tilde{g}_{\nu\sigma} + \partial_{\nu}\tilde{g}_{\mu\sigma} - \partial_{\sigma}\tilde{g}_{\mu\nu}\right) \\
&= \Gamma^{\lambda}{}_{\mu\nu} + (\delta^{\lambda}_{\mu}\partial_{\nu} + \delta^{\lambda}_{\nu}\partial_{\mu} - g_{\mu\nu}g^{\lambda\rho}\partial_{\rho})\log\Omega\,.
\end{aligned} \tag{A.48}
$$

Now to find the transformation of the Riemann tensor, the above result is combined with equations (A.13) and (A.8), yielding

$$
\begin{aligned}
\tilde{R}^{\rho}{}_{\sigma\mu\nu} &= \partial_{\mu}\tilde{\Gamma}^{\rho}{}_{\nu\sigma} - \partial_{\nu}\tilde{\Gamma}^{\rho}{}_{\mu\sigma} + \tilde{\Gamma}^{\rho}{}_{\mu\lambda}\tilde{\Gamma}^{\lambda}{}_{\nu\sigma} - \tilde{\Gamma}^{\rho}{}_{\nu\lambda}\tilde{\Gamma}^{\lambda}{}_{\mu\sigma} \\
&= R^{\rho}{}_{\sigma\mu\nu} + 2(\delta^{\beta}_{\sigma}\delta^{\alpha}_{[\mu}\delta^{\rho}_{\nu]} - g^{\rho\beta}\delta^{\alpha}_{[\mu}g_{\nu]\sigma})\nabla_{\alpha}\nabla_{\beta}\log\Omega \\
&\quad + 2(\delta^{\rho}_{[\mu}\delta^{\alpha}_{\nu]}\delta^{\beta}_{\sigma} - g^{\rho\beta}g_{\sigma[\mu}\delta^{\alpha}_{\nu]} + g_{\sigma[\mu}\delta^{\rho}_{\nu]}g^{\alpha\beta})\nabla_{\alpha}\log\Omega\nabla_{\beta}\log\Omega \tag{A.49}
\end{aligned}
$$

Contracting equation (A.49) gives the transformation of the Ricci tensor,

$$
\begin{aligned}
\tilde{R}_{\mu\nu} &= R_{\mu\nu} - \left((d-2)\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu} + g^{\rho\sigma}g_{\mu\nu}\right)\nabla_{\rho}\nabla_{\sigma}\log\Omega \\
&\quad + (d-2)(\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu} - g^{\rho\sigma}g_{\mu\nu})\nabla_{\rho}\log\Omega\nabla_{\sigma}\log\Omega\,, \tag{A.50}
\end{aligned}
$$

and contracting again gives the transformation of the Ricci scalar,

$$
\tilde{R} = \Omega^{-2}\left[R - 2(d-1)\Box\log\Omega - (d-2)(d-1)g^{\mu\nu}\nabla_{\mu}\log\Omega\nabla_{\nu}\log\Omega\right]\,. \tag{A.51}
$$

Note that in equations (A.48)-(A.50), indices may *not* be freely moved up and down, because on the left-hand sides, the re-scaled metric (A.47) is used to raise and lower indices, whereas on the right-hand sides, the original metric is used to raise and lower indices.

Let $\tilde{g}$ be the determinant of the re-scaled metric $\tilde{g}_{\mu\nu}$. Its scaling is given by

$$
\sqrt{-\tilde{g}} = \Omega^{d}\sqrt{-g}\,, \tag{A.52}
$$

and therefore, the d'Alembertian operator acting on a scalar function, (A.11), transforms according to

$$
\begin{aligned}
\tilde{\Box}f &= \tilde{\nabla}_{\mu}\tilde{\nabla}^{\mu}f \\
&= \frac{1}{\sqrt{-\tilde{g}}}\partial_{\mu}\left(\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_{\nu}f\right) \\
&= \Omega^{-2}\left[\Box f + (d-2)g^{\mu\nu}\nabla_{\mu}\log\Omega\nabla_{\nu}f\right]\,. \tag{A.53}
\end{aligned}
$$

This relation may be inverted to write $\Box f$ in terms of re-scaled quantities:

$$\Box f = \Omega^2 \left[\tilde{\Box} f - (d-2) \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \log \Omega \tilde{\nabla}_\nu f \right] . \tag{A.54}$$

Now, equation (A.54) may be combined with equation (A.51) to write R in terms of re-scaled quantities:

$$R = \Omega^2 \left[\tilde{R} + 2(d-1) \tilde{\Box} \log \Omega - (d-1)(d-2) \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \log \Omega \tilde{\nabla}_\nu \log \Omega \right] . \tag{A.55}$$

It is also useful to write the relation between R and $\tilde{R}$ in the form

$$R - 2(d-1) \Box \log \Omega = \Omega^2 \left[\tilde{R} + (d-2)(d-1) \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \log \Omega \tilde{\nabla}_\nu \log \Omega \right] , \tag{A.56}$$

since the second term on the left-hand side is a total derivative.

A.6   Wave Equations and Green's Functions

A useful tool for solving wave equations is the Green's function $G(x, y)$, which is defined to be a solution of the wave equation with a point source:

$$\Box_x G(x, y) = \delta_y(x) , \tag{A.57}$$

where δ_y is the Dirac distribution centered on y, which is defined by the conditions $\delta_y(x) = 0$ for $x \neq y$, and

$$\int dV_x f(x) \delta_y(x) = f(y) \tag{A.58}$$

for all functions f. The subscripts x on $\Box$ and dV denote that the differentiation and integration is with respect to the coordinate x. Given a wave equation of the form $\Box f = j$, a solution may be written in terms of Green's function as

$$f(x) = \int dV_y G(x, y) j(y) . \tag{A.59}$$

Note that in order to uniquely determine G, it is necessary to supplement equation (A.57) with appropriate boundary conditions.

For a general manifold, the calculation of G is a difficult problem. Even in flat d-dimensional Minkowski space-time, there is no simple expression for

G. However, in the special case $d = 4$, the Green's function with 'retarded' boundary conditions is given by

$$G(x, y) = -\frac{1}{2\pi}\Theta(x^0 - y^0)\delta[(x^\mu - y^\mu)(x_\mu - y_\mu)]\,, \qquad (A.60)$$

where Θ is the Heaviside step function, which ensures that 'signals propagate forward in time', and δ is the one-dimensional Dirac distribution on $\mathbb{R}$, which ensures that 'signals propagate at the speed of light'.

Writing equation (A.60) explicitly in terms of the space and time components of the coordinates yields

$$G(x, y) = -\frac{1}{4\pi}\frac{\delta[y^0 - (x^0 - |\vec{x} - \vec{y}|)]}{|\vec{x} - \vec{y}|}\,. \qquad (A.61)$$

Another operator whose Green's function has a simple formula is the d'Alembertian constructed out of the flat metric on the N-dimensional Euclidean space $\mathbb{R}^N$, in other words, the Laplacian Δ. It may be shown that

$$G_N^\Delta(\vec{x}, \vec{y}) = -\frac{1}{(N - 2)\Omega_{N-1}|\vec{x} - \vec{y}|^{N-2}} \qquad (A.62)$$

is a Green's function for Δ, that is,

$$\Delta_x G_N^\Delta(\vec{x}, \vec{y}) = \delta^{(N)}(\vec{x} - \vec{y})\,. \qquad (A.63)$$

where $\delta^{(N)}$ is the Dirac distribution on $\mathbb{R}^N$. In equation (A.62), Ω_{N-1} denotes the area of the sphere S^{N-1}, and is given by

$$\Omega_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)}\,. \qquad (A.64)$$

A.7   Hypergeometric and Heun Functions

In this section, the hypergeometric and Heun differential equations are introduced. Their solutions are needed to describe constant-density stars in Scalar-Tensor gravity by means of perturbation theory.

Given a linear second-order differential equation with three regular singular points, it is possible to find a fractional linear transformation which maps

the singular points to 0, 1, and ∞, and thus brings the differential equation into the canonical form

$$\frac{d^2y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1}\right)\frac{dy}{dz} + \frac{\alpha\beta y}{z(z-1)} = 0\,, \tag{A.65}$$

where the coefficients are related by

$$\gamma + \delta = \alpha + \beta + 1\,. \tag{A.66}$$

Equation (A.65) is the famous hypergeometric equation, which shows up throughout theoretical and mathematical physics, and reduces to many well-known special functions for particular values of α, β, and γ. The regular solution about $z = 0$ has the series expansion

$$_2F_1(\alpha, \beta; \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{z^k}{k!}\,, \tag{A.67}$$

where

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = x(x+1)\cdots(x+k-1) \tag{A.68}$$

is the Pochhammer symbol.

Similarly, given a linear second-order differential equation with four regular singular points, it is possible to find a fractional linear transformation which maps the singular points to 0, 1, ∞, and a, and thus brings the differential equation into the canonical form

$$\frac{d^2y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right)\frac{dy}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)}y = 0\,, \tag{A.69}$$

where the parameters are related by

$$\gamma + \delta + \epsilon = \alpha + \beta + 1\,. \tag{A.70}$$

Equation (A.69) is called the Heun equation [144]. It was first studied by Karl Heun in 1889, and is not nearly as well-known as the hypergeometric equation. The regular solution about $z = 0$ has the series expansion

$$\text{HeunG}(a, q; \alpha, \beta, \gamma, \delta; z) = \sum_{r=0}^{\infty} c_r z^r\,, \tag{A.71}$$

where the coefficients c_r satisfy the recursion relations

$$-qc_0 + a\gamma c_1 = 0, \tag{A.72}$$

$$P_r c_{r-1} - (Q_r + q)c_r + R_r c_{r+1} = 0 \qquad (r \geq 1), \tag{A.73}$$

where

$$P_r = (r - 1 + \alpha)(r - 1 + \beta), \tag{A.74}$$

$$Q_r = r[(r - 1 + \gamma)(1 + a) + a\delta + \epsilon], \tag{A.75}$$

$$R_r = (r + 1)(r + \gamma)a. \tag{A.76}$$

References

[1] Andreas Albrecht, C.P. Burgess, F. Ravndal, and C. Skordis. Natural quintessence and large extra dimensions. *Phys.Rev.*, D65:123507, 2002, astro-ph/0107573.

[2] Miguel Alcubierre, Juan C. Degollado, Dario Nunez, Milton Ruiz, and Marcelo Salgado. Dynamic transition to spontaneous scalarization in boson stars. *Phys.Rev.*, D81:124018, 2010, 1003.4767.

[3] Stephon Alexander and Nicolas Yunes. Chern-Simons Modified Gravity as a Torsion Theory and its Interaction with Fermions. *Phys.Rev.*, D77:124040, 2008, 0804.1797.

[4] Stephon Alexander and Nicolas Yunes. Chern-Simons Modified General Relativity. *Phys.Rept.*, 480:1–55, 2009, 0907.2562.

[5] Masoud Allahverdizadeh and Ken Matsuno. Charged Rotating Kaluza-Klein Black Holes in Dilaton Gravity. *Phys.Rev.*, D81:044001, 2010, 0908.2484.

[6] Justin Alsing, Emanuele Berti, Clifford M. Will, and Helmut Zaglauer. Gravitational radiation from compact binary systems in the massive Brans-Dicke theory of gravity. *Phys.Rev.*, D85:064041, 2012, 1112.4903.

[7] Pau Amaro-Seoane, Sofiane Aoudia, Stanislav Babak, Pierre Binetruy, Emanuele Berti, et al. eLISA: Astrophysics and cosmology in the milli-hertz regime. 2012, 1201.3621.

[8] K.G. Arun. Generic bounds on dipolar gravitational radiation from inspi-ralling compact binaries. *Class.Quant.Grav.*, 29:075011, 2012, 1202.5911.

[9] Asimina Arvanitaki, Savas Dimopoulos, Sergei Dubovsky, Nemanja Kaloper, and John March-Russell. String Axiverse. *Phys.Rev.*, D81:123530, 2010, 0905.4720.

[10] Asimina Arvanitaki and Sergei Dubovsky. Exploring the String Axiverse with Precision Black Hole Physics. *Phys.Rev.*, D83:044026, 2011, 1004.3558.

[11] Arjun Bagchi and S. Kalyana Rama. Cosmology and static spherically symmetric solutions in D-dimensional scalar tensor theories: Some Novel features. *Phys.Rev.*, D70:104030, 2004, gr-qc/0408030.

[12] Thomas W. Baumgarte and Stuart L. Shapiro. *Numerical Relativity: Solving Einstein's Equations on the Computer.* Cambridge University Press, 2010.

[13] Katrin Becker, Melanie Becker, and John H. Schwarz. *String Theory and M-Theory: A Modern Introduction.* Cambridge University Press, 2007.

[14] Jacob D. Bekenstein. Gravitational-Radiation Recoil and Runaway Black Holes. *Astrophys.J.*, 183:657–664, 1973.

[15] Peter G. Bergmann. Comments on the scalar-tensor theory. *International Journal of Theoretical Physics*, 1(1):25–36, 1968.

[16] Marcelo Samuel Berman. *Introduction to General Relativistic and Scalar-tensor Cosmologies.* Nova Science Pub Inc, 2007.

[17] Emanuele Berti, Leonardo Gualtieri, Michael Horbatsch, and Justin Alsing. Light scalar field constraints from gravitational-wave observations of compact binaries. 2012, 1204.4340.

[18] B. Bertotti, L. Iess, and P. Tortora. A test of general relativity using radio links with the Cassini spacecraft. *Nature*, 425:374–376, 2003.

[19] A. Bhadra, K. Sarkar, D.P. Datta, and K.K. Nandi. Brans-Dicke theory: Jordan versus Einstein frame. *Mod.Phys.Lett.*, A22:367–376, 2007, gr-qc/0605109.

[20] N.D. Ramesh Bhat, Matthew Bailes, and Joris P.W. Verbiest. Gravitational-radiation losses from the pulsar-white-dwarf binary PSR J1141-6545. *Phys.Rev.*, D77:124017, 2008, 0804.0956.

[21] Luc Blanchet. Radiative gravitational fields in general relativity. 2. Asymptotic behaviour at future null infinity. *Proc.Roy.Soc.Lond.*, A409:383–399, 1987.

[22] Luc Blanchet. Gravitational radiation from post-Newtonian sources and inspiralling compact binaries. *Living Rev.Rel.*, 9:4, 2006.

[23] Luc Blanchet and Thibault Damour. Radiative gravitational fields in general relativity I. general structure of the field outside the source. *Phil.Trans.Roy.Soc.Lond.*, A320:379–430, 1986.

[24] C. Brans and R. H. Dicke. Mach's principle and a relativistic theory of gravitation. *Phys. Rev.*, 124(3):925–935, 1961.

[25] Carl H. Brans. The roots of scalar-tensor theory: an approximate history, 2005, arXiv:gr-qc/0506063.

[26] C.H. Brans. Mach's Principle and a Relativistic Theory of Gravitation. II. *Phys.Rev.*, 125:2194–2201, 1962.

[27] Bernd Bruegmann, Jose A. Gonzalez, Mark Hannam, Sascha Husa, and Ulrich Sperhake. Exploring black hole superkicks. *Phys.Rev.*, D77:124047, 2008, 0707.0135.

[28] H. A. Buchdahl. General relativistic fluid spheres. *Phys. Rev.*, 116(4):1027–1034, 1959.

[29] C.P. Burgess. Quantum gravity in everyday life: General relativity as an effective field theory. *Living Rev.Rel.*, 7:5, 2004, gr-qc/0311082.

[30] C.P. Burgess. Introduction to Effective Field Theory. *Ann.Rev.Nucl.Part.Sci.*, 57:329–362, 2007, hep-th/0701053.

[31] C.P. Burgess and J. Cloutier. Astrophysical Evidence for a Weak New Force? *Phys.Rev.*, D38:2944, 1988.

[32] Manuela Campanelli, Carlos O. Lousto, Yosef Zlochower, and David Merritt. Large merger recoils and spin flips from generic black-hole binaries. *Astrophys.J.*, 659:L5–L8, 2007, gr-qc/0701164.

[33] Manuela Campanelli, Carlos O. Lousto, Yosef Zlochower, and David Merritt. Maximum gravitational recoil. *Phys.Rev.Lett.*, 98:231102, 2007, gr-qc/0702133.

[34] W.B. Campbell, J. Macek, and T.A. Morgan. Relativistic Time Dependent Multipole Analysis for Scalar, Electromagnetic, and Gravitational Fields. *Phys.Rev.*, D15:2156–2164, 1977.

[35] S. Capozziello, P. Martin-Moruno, and C. Rubano. Physical non-equivalence of the Jordan and Einstein frames. *Phys.Lett.*, B689:117–121, 2010, 1003.5394.

[36] Vitor Cardoso, Sayan Chakrabarti, Paolo Pani, Emanuele Berti, and Leonardo Gualtieri. Floating and sinking: The Imprint of massive scalars around rotating black holes. *Phys.Rev.Lett.*, 107:241101, 2011, 1109.6021.

[37] Sean Carroll. *Spacetime and Geometry: An Introduction to General Relativity.* Pearson Education, Addison Wesley, 2004.

[38] Riccardo Catena, Massimo Pietroni, and Luca Scarabello. Einstein and Jordan reconciled: a frame-invariant approach to scalar-tensor cosmology. *Phys.Rev.*, D76:084039, 2007, astro-ph/0604492.

[39] Katerina Chatziioannou, Nicolas Yunes, and Neil Cornish. Model-Independent Test of General Relativity: An Extended post-Einsteinian Framework with Complete Polarization Content. 2012, 1204.2585.

[40] Takeshi Chiba, Tomohiro Harada, and Ken-ichi Nakao. Gravitational physics in scalar tensor theories: Tests of strong field gravity. *Prog.Theor.Phys.Suppl.*, 128:335–372, 1997.

[41] Neil Cornish, Laura Sampson, Nicolas Yunes, and Frans Pretorius. Gravitational Wave Tests of General Relativity with the Parameterized Post-Einsteinian Framework. *Phys.Rev.*, D84:062003, 2011, 1105.2088.

[42] T. Damour. Gravitational radiation and the motion of compact bodies. In N. Deruelle and T. Piran, editors, *Gravitational Radiation*, pages 59–144. North-Holland Publishing Company, 1983.

[43] T. Damour. Un nouveau test de la relativité générale. In S. Benenti, M. Ferraris, and M. Francaviglia, editors, *Proceedings of 'Journées relativistes 1983', Torino, May 5-8, 1983*, pages 89–110. Pitagora Editrice, 1985.

[44] T. Damour. The problem of motion in newtonian and einsteinian gravity. In S.W. Hawking and W. Israel, editors, *300 Years of Gravitation*, pages 128–198. Cambridge University Press, 1987.

[45] T. Damour and N. Deruelle. General relativistic celestial mechanics of binary systems. i. the post-newtonian motion. *Annales de l'Institut Henri Poincaré*, 43:107–132, 1985.

[46] T. Damour and N. Deruelle. General relativistic celestial mechanics of binary systems. ii. the post-newtonian timing formula. *Annales de l'Institut Henri Poincaré*, 44:263–292, 1986.

[47] T. Damour and Alexander M. Polyakov. String theory and gravity. *Gen.Rel.Grav.*, 26:1171–1176, 1994, gr-qc/9411069.

[48] T. Damour and Alexander M. Polyakov. The String dilaton and a least coupling principle. *Nucl.Phys.*, B423:532–558, 1994, hep-th/9401069.

[49] Thibault Damour. Binary Systems as Test-beds of Gravity Theories. 2007, 0704.0749.

[50] Thibault Damour and Gilles Esposito-Farèse. Tensor-multi-scalar theories of gravitation. *Class. Quantum Grav.*, 9(9):2093, 1992.

[51] Thibault Damour and Gilles Esposito-Farese. Nonperturbative strong field effects in tensor - scalar theories of gravitation. *Phys.Rev.Lett.*, 70:2220–2223, 1993.

[52] Thibault Damour and Gilles Esposito-Farese. Tensor - scalar gravity and binary pulsar experiments. *Phys.Rev.*, D54:1474–1491, 1996, gr-qc/9602056.

[53] Thibault Damour and Gilles Esposito-Farese. Gravitational wave versus binary - pulsar tests of strong field gravity. *Phys.Rev.*, D58:042001, 1998, gr-qc/9803031.

[54] Thibault Damour and Joseph H. Taylor. Strong field tests of relativistic gravity and binary pulsars. *Phys.Rev.*, D45:1840–1868, 1992.

[55] Simon DeDeo and Dimitrios Psaltis. Towards New Tests of Strong-field Gravity with Measurements of Surface Atomic Line Redshifts from Neutron Stars. *Phys.Rev.Lett.*, 90:141101, 2003, astro-ph/0302095.

[56] Simon DeDeo and Dimitrios Psaltis. Testing strong-field gravity with quasiperiodic oscillations. *Bull. Am. Astron. Soc.*, 36, 2004, astro-ph/0405067.

[57] R. H. Dicke. Mach's principle and invariance under transformation of units. *Phys. Rev.*, 125(6):2163–2167, 1962.

[58] J.O. Dickey, P.L. Bender, J.E. Faller, X.X. Newhall, R.L. Ricklefs, et al. Lunar Laser Ranging: A Continuing Legacy of the Apollo Program. *Science*, 265:482–490, 1994.

[59] John F. Donoghue. General relativity as an effective field theory: The leading quantum corrections. *Phys.Rev.*, D50:3874–3888, 1994, gr-qc/9405057.

[60] John F. Donoghue. Introduction to the effective field theory description of gravity. 1995, gr-qc/9512024.

[61] Sergei Dubovsky and Victor Gorbenko. Black Hole Portal into Hidden Valleys. *Phys.Rev.*, D83:106002, 2011, 1012.2893.

[62] D.M. Eardley. Observable Effects of a Scalar Gravitational Field in a Binary Pulsar. *Astrophys.J.*, 196:L59–L62, 1975.

[63] Gilles Esposito-Farese. Binary pulsar tests of strong field gravity. 1996, gr-qc/9612039.

[64] Gilles Esposito-Farese. Comparing solar system, binary pulsar, and gravitational wave tests of gravity. 1999, gr-qc/9903058.

[65] Gilles Esposito-Farese. Tests of strong field gravity and gravitational radiation damping in binary pulsar systems. 2000, gr-qc/0011114.

[66] Gilles Esposito-Farese. Tests of scalar-tensor gravity. *AIP Conf.Proc.*, 736:35–52, 2004, gr-qc/0409081.

[67] Gilles Esposito-Farese. Binary pulsar tests of strong field gravity and gravitational radiation damping. *Proceedings World Scientific*, pages 647–666, 2005, gr-qc/0402007.

[68] Gilles Esposito-Farese. Motion in alternative theories of gravity. *Fundam.Theor.Phys.*, 162:461–489, 2011, 0905.2575.

[69] V. Faraoni and E. Gunzig. Einstein frame or Jordan frame? *Int.J.Theor.Phys.*, 38:217–225, 1999, astro-ph/9910176.

[70] Valerio Faraoni. *Cosmology in Scalar-Tensor Gravity*. Springer, 2004.

[71] M. Fierz. Über die physikalische Deutung der erweiterten Gravitationstheorie P. Jordans. *Helvetica Physica Acta*, 29:128–134, 1956.

[72] Eanna E. Flanagan. The Conformal frame freedom in theories of gravitation. *Class.Quant.Grav.*, 21:3817, 2004, gr-qc/0403063.

[73] E. Fomalont, S. Kopeikin, G. Lanyi, and J. Benson. Progress in Measurements of the Gravitational Bending of Radio Waves Using the VLBA. *Astrophys.J.*, 699:1395–1402, 2009, 0904.3992.

[74] Paulo C.C. Freire, Norbert Wex, Gilles Esposito-Farese, Joris P.W. Verbiest, Matthew Bailes, et al. The relativistic pulsar-white dwarf binary PSR J1738+0333 II. The most stringent test of scalar-tensor gravity. 2012, 1205.1450.

[75] Yasunori Fujii and Kei-ichi Maeda. *The Scalar-Tensor Theory of Gravitation.* Cambridge University Press, 2007.

[76] Walter D. Goldberger. Les Houches lectures on effective field theories and gravitational radiation. 2007, hep-ph/0701129.

[77] Walter D. Goldberger and Andreas Ross. Gravitational radiative corrections from effective field theory. *Phys.Rev.*, D81:124015, 2010, 0912.4254.

[78] Walter D. Goldberger and Ira Z. Rothstein. An Effective field theory of gravity for extended objects. *Phys.Rev.*, D73:104029, 2006, hep-th/0409156.

[79] Walter D. Goldberger and Ira Z. Rothstein. Dissipative effects in the worldline approach to black hole dynamics. *Phys.Rev.*, D73:104030, 2006, hep-th/0511133.

[80] Walter D. Goldberger and Ira Z. Rothstein. Towers of Gravitational Theories. *Gen.Rel.Grav.*, 38:1537–1546, 2006, hep-th/0605238.

[81] J.A. Gonzalez, M.D. Hannam, U. Sperhake, Bernd Bruegmann, and S. Husa. Supermassive recoil velocities for binary black-hole mergers with antialigned spins. *Phys.Rev.Lett.*, 98:231101, 2007, gr-qc/0702052.

[82] Samuel E. Gralla, Abraham I. Harte, and Robert M. Wald. Bobbing and Kicks in Electromagnetism and Gravity. *Phys.Rev.*, D81:104012, 2010, 1004.0679.

[83] Tomohiro Harada. Stability analysis of spherically symmetric star in scalar - tensor theories of gravity. *Prog.Theor.Phys.*, 98:359–379, 1997, gr-qc/9706014.

[84] Tomohiro Harada. Neutron stars in scalar tensor theories of gravity and catastrophe theory. *Phys.Rev.*, D57:4802–4811, 1998, gr-qc/9801049.

[85] Gregory M. Harry. Advanced LIGO: The next generation of gravitational wave detectors. *Class.Quant.Grav.*, 27:084006, 2010.

[86] S.W. Hawking. Black holes in the Brans-Dicke theory of gravitation. *Commun.Math.Phys.*, 25:167–171, 1972.

[87] James Healy, Tanja Bode, Roland Haas, Enrique Pazos, Pablo Laguna, et al. Late Inspiral and Merger of Binary Black Holes in Scalar-Tensor Theories of Gravity. 2011, 1112.3928.

[88] John L. Safko Herbert Goldstein, Charles P. Poole. *Classical Mechanics*. Addison Wesley, 2001.

[89] M.W. Horbatsch and C.P. Burgess. Model-Independent Comparisons of Pulsar Timings to Scalar-Tensor Gravity. 2011, 1107.3585.

[90] M.W. Horbatsch and C.P. Burgess. Semi-Analytic Stellar Structure in Scalar-Tensor Gravity. *JCAP*, 1108:027, 2011, 1006.4411.

[91] M.W. Horbatsch and C.P. Burgess. Cosmic Black-Hole Hair Growth and Quasar OJ287. *JCAP*, 1205:010, 2012, 1111.4009.

[92] R.A. Hulse and J.H. Taylor. Discovery of a pulsar in a binary system. *Astrophys.J.*, 195:L51–L53, 1975.

[93] Russell A. Hulse. The discovery of the binary pulsar. *Rev.Mod.Phys.*, 66:699–710, 1994.

[94] R. Jackiw and S.Y. Pi. Chern-Simons modification of general relativity. *Phys.Rev.*, D68:104012, 2003, gr-qc/0308071.

[95] John David Jackson. *Classical Electrodynamics*. Wiley, 1998.

[96] Ted Jacobson. Primordial black hole evolution in tensor scalar cosmology. *Phys.Rev.Lett.*, 83:2699–2702, 1999, astro-ph/9905303.

[97] P. Jordan. Zum gegenwärtigen Stand der Diracschen kosmologischen Hypothesen. *Zeitschrift für Physik*, 157:112–121, 1959.

[98] Pascual Jordan. Formation of the stars and development of the universe. *Nature*, 164(4172):637–640, 1949.

[99] Kimmo Kainulainen and Daniel Sunhede. Dark energy from large extra dimensions. *Phys.Rev.*, D73:083510, 2006, astro-ph/0412609.

[100] Th. Kaluza. Zum Unitätsproblem der Physik. *Sitzungsberichte der Preußischen Akademie der Wissenschaften*, pages 966–972, 1921.

[101] Drew Keppel, David A. Nichols, Yanbei Chen, and Kip S. Thorne. Momentum Flow in Black Hole Binaries. I. Post-Newtonian Analysis of the Inspiral and Spin-Induced Bobbing. *Phys.Rev.*, D80:124015, 2009, 0902.4077.

[102] Justin Khoury and Amanda Weltman. Chameleon fields: Awaiting surprises for tests of gravity in space. *Phys.Rev.Lett.*, 93:171104, 2004, astro-ph/0309300.

[103] Oskar Klein. Quantentheorie und fünfdimensionale Relativitätstheorie. *Zeitschrift für Physik*, 37(12):895–906, 1926.

[104] Hideo Kodama and Hirotaka Yoshino. Axiverse and Black Hole. 2011, 1108.1365.

[105] M. Kramer, Ingrid H. Stairs, R.N. Manchester, M.A. McLaughlin, A.G. Lyne, et al. Tests of general relativity from timing the double pulsar. *Science*, 314:97–102, 2006, astro-ph/0609417.

[106] K. Lazaridis, N. Wex, A. Jessner, M. Kramer, B.W. Stappers, et al. Generic tests of the existence of the gravitational dipole radiation and the variation of the gravitational constant. 2009, 0908.0285.

[107] H. J. Lehto and M. J. Valtonen. OJ 287 Outburst Structure and a Binary Black Hole Model. *Astrophys.J.*, 460:207, March 1996.

[108] D.R. Lorimer. Binary and Millisecond Pulsars. *Living Rev.Rel.*, 11:8, 2008, 0811.0762.

[109] Geoffrey Lovelace, Yanbei Chen, Michael Cohen, Jeffrey D. Kaplan, Drew Keppel, et al. Momentum flow in black-hole binaries. II. Numer-

ical simulations of equal-mass, head-on mergers with antiparallel spins. *Phys.Rev.*, D82:064031, 2010, 0907.0869.

[110] A.G. Lyne, Marta Burgay, M. Kramer, A. Possenti, R.N. Manchester, et al. A Double - pulsar system - A Rare laboratory for relativistic gravity and plasma physics. *Science*, 303:1153–1157, 2004, astro-ph/0401086.

[111] Cristian Martinez and Ricardo Troncoso. Electrically charged black hole with scalar hair. *Phys.Rev.*, D74:064007, 2006, hep-th/0606130.

[112] Daniel V. Schroeder Michael E. Peskin. *An Introduction to Quantum Field Theory*. Basic Books, 1995.

[113] Birger Bergersen Michael Plischke. *Equilibrium Statistical Physics*. World Scientific, 1994.

[114] Charles W. Misner, Kip S. Thorne, and John A. Wheeler. *Gravitation*. W. H. Freeman Company, 1973.

[115] David F. Mota and John D. Barrow. Varying alpha in a more realistic Universe. *Phys.Lett.*, B581:141–146, 2004, astro-ph/0306047.

[116] Filipe Moura and Ricardo Schiappa. Higher-derivative corrected black holes: Perturbative stability and absorption cross-section in heterotic string theory. *Class.Quant.Grav.*, 24:361–386, 2007, hep-th/0605001.

[117] Alberto Nicolis, Riccardo Rattazzi, and Enrico Trincherini. The Galileon as a local modification of gravity. *Phys.Rev.*, D79:064036, 2009, 0811.2197.

[118] K. Nordtvedt. Testing relativity with laser ranging to the moon. *Phys.Rev.*, 170:1186–1187, 1968.

[119] Kenneth Nordtvedt. Equivalence Principle for Massive Bodies. 1. Phenomenology. *Phys.Rev.*, 169:1014–1016, 1968.

[120] Kenneth Nordtvedt. Equivalence Principle for Massive Bodies. 2. Theory. *Phys.Rev.*, 169:1017–1025, 1968.

[121] Kenneth Nordtvedt. Post-Newtonian metric for a general class of scalar-tensor gravitational theories and observational consequences. *The Astrophysical Journal*, 161:1059–1067, 1970.

[122] Jerome Novak. Neutron star transition to strong scalar field state in tensor scalar gravity. *Phys.Rev.*, D58:064019, 1998, gr-qc/9806022.

[123] Jerome Novak. Spherical neutron star collapse in tensor - scalar theory of gravity. *Phys.Rev.*, D57:4789–4801, 1998, gr-qc/9707041.

[124] Jerome Novak and Jose M. Ibanez. Gravitational waves from the collapse and bounce of a stellar core in tensor scalar gravity. *Astrophys.J.*, 533:392–405, 2000, astro-ph/9911298.

[125] Paolo Pani, Emanuele Berti, Vitor Cardoso, and Jocelyn Read. Compact stars in alternative theories of gravity. Einstein-Dilaton-Gauss-Bonnet gravity. *Phys.Rev.*, D84:104035, 2011, 1109.0928.

[126] Paolo Pani and Vitor Cardoso. Are black holes in alternative theories serious astrophysical candidates? The Case for Einstein-Dilaton-Gauss-Bonnet black holes. *Phys.Rev.*, D79:084031, 2009, 0902.1569.

[127] Paolo Pani, Vitor Cardoso, and Leonardo Gualtieri. Gravitational waves from extreme mass-ratio inspirals in Dynamical Chern-Simons gravity. *Phys.Rev.*, D83:104048, 2011, 1104.1183.

[128] Matthew Pitkin, Stuart Reid, Sheila Rowan, and Jim Hough. Gravitational Wave Detection by Interferometry (Ground and Space). *Living Rev.Rel.*, 4:5, 2011, 1102.3355.

[129] Rafael A. Porto. Post-Newtonian corrections to the motion of spinning bodies in NRGR. *Phys.Rev.*, D73:104031, 2006, gr-qc/0511061.

[130] Rafael A. Porto. New results at 3PN via an effective field theory of gravity. pages 2493–2496, 2007, gr-qc/0701106.

[131] Rafael A. Porto. Absorption effects due to spin in the worldline approach to black hole dynamics. *Phys.Rev.*, D77:064026, 2008, 0710.5150.

[132] Rafael A. Porto. Next to leading order spin-orbit effects in the motion of inspiralling compact binaries. *Class.Quant.Grav.*, 27:205001, 2010, 1005.5730.

[133] Rafael A. Porto, Andreas Ross, and Ira Z. Rothstein. Spin induced multipole moments for the gravitational wave flux from binary inspirals to third Post-Newtonian order. *JCAP*, 1103:009, 2011, 1007.1312.

[134] Rafael A. Porto, Andreas Ross, and Ira Z. Rothstein. Spin induced multipole moments for the gravitational wave amplitude from binary inspirals to 2.5 Post-Newtonian order. 2012, 1203.2962.

[135] Rafael A. Porto and Ira Z. Rothstein. The Hyperfine Einstein-Infeld-Hoffmann potential. *Phys.Rev.Lett.*, 97:021101, 2006, gr-qc/0604099.

[136] Rafael A. Porto and Ira Z. Rothstein. Comment on 'On the next-to-leading order gravitational spin(1) - spin(2) dynamics' by J. Steinhoff et al. 2007, 0712.2032.

[137] Rafael A Porto and Ira Z. Rothstein. Next to Leading Order Spin(1)Spin(1) Effects in the Motion of Inspiralling Compact Binaries. *Phys.Rev.*, D78:044013, 2008, 0804.0260.

[138] Rafael A. Porto and Ira Z. Rothstein. Spin(1)Spin(2) Effects in the Motion of Inspiralling Compact Binaries at Third Order in the Post-Newtonian Expansion. *Phys.Rev.*, D78:044012, 2008, 0802.0720.

[139] Frans Pretorius. Binary Black Hole Coalescence. 2007, 0710.1338.

[140] Dimitrios Psaltis. Probes and Tests of Strong-Field Gravity with Observations in the Electromagnetic Spectrum. *Living Rev.Rel.*, 11:9, 2008, 0806.1531.

[141] Dimitrios Psaltis. Testing General Metric Theories of Gravity with Bursting Neutron Stars. *Phys.Rev.*, D77:064006, 2008, 0704.2426.

[142] M. Punturo, M. Abernathy, F. Acernese, B. Allen, N. Andersson, et al. The Einstein Telescope: A third-generation gravitational wave observatory. *Class.Quant.Grav.*, 27:194002, 2010.

[143] T. Pursimo, L. O. Takalo, A. Sillanpää, M. Kidger, H. J. Lehto, J. Heidt, P. A. Charles, et al. Intensive monitoring of OJ 287. *Astron. Astrophys. Suppl.*, 146:141–155, October 2000.

[144] A. Ronveaux. *Heun's Differential Equations*. Oxford University Press, 1995.

[145] Andreas Ross. Multipole expansion at the level of the action. 2012, 1202.4750. 39 pages.

[146] Marcelo Salgado. The Cauchy problem of scalar tensor theories of gravity. *Class.Quant.Grav.*, 23:4719–4742, 2006, gr-qc/0509001.

[147] Marcelo Salgado, David Martinez-del Rio, Miguel Alcubierre, and Dario Nunez. Hyperbolicity of scalar-tensor theories of gravity. *Phys.Rev.*, D77:104010, 2008, 0801.2372.

[148] Marcelo Salgado, Daniel Sudarsky, and Ulises Nucamendi. On spontaneous scalarization. *Phys.Rev.*, D58:124003, 1998, gr-qc/9806070.

[149] Marcelo Salgado, Daniel Sudarsky, and Ulises Nucamendi. The Violation of the weak energy condition, is it generic of spontaneous scalarization? *Phys.Rev.*, D70:084027, 2004, gr-qc/0402126.

[150] A. Salmona. Effect of Gravitational Scalar Field on High-Density Star Structure. *Phys.Rev.*, 154:1218–1223, 1967.

[151] B.S. Sathyaprakash and B.F. Schutz. Physics, Astrophysics and Cosmology with Gravitational Waves. *Living Rev.Rel.*, 12:2, 2009, 0903.0338.

[152] Paul D. Scharre and Clifford M. Will. Testing scalar tensor gravity using space gravitational wave interferometers. *Phys.Rev.*, D65:042002, 2002, gr-qc/0109044.

[153] Karl Schwarzschild. Über das Gravitationsfeld einer Kugel aus inkompressibler Flüssigkeit. *Sitzungsberichte Preuss. Akad. Wiss.*, pages 424–434, 1916.

[154] Irwin I. Shapiro. Fourth Test of General Relativity. *Phys.Rev.Lett.*, 13:789–791, 1964.

[155] S.S. Shapiro, J.L. Davis, D.E. Lebach, and J.S. Gregory. Measurement of the Solar Gravitational Deflection of Radio Waves using Geodetic Very-Long-Baseline Interferometry Data, 1979-1999. *Phys.Rev.Lett.*, 92:121101, 2004.

[156] A. Sheykhi, M. Allahverdizadeh, Y. Bahrampour, and M. Rahnama. Asymptotically flat charged rotating dilaton black holes in higher dimensions. *Phys.Lett.*, B666:82–85, 2008, 0805.4464.

[157] A. Sillanpaa, S. Haarala, M.J. Valtonen, B. Sundelius, and G.G. Byrd. OJ 287 - Binary pair of supermassive black holes. *Astrophys.J.*, 325:628–634, 1988.

[158] T. Singh and L. N. Rai. Scalar-tensor theories of gravitation: Foundations and prospects. *General Relativity and Gravitation*, 15(9):875–902, 1983.

[159] Carlos F. Sopuerta and Nicolas Yunes. Extreme and Intermediate-Mass Ratio Inspirals in Dynamical Chern-Simons Modified Gravity. *Phys.Rev.*, D80:064006, 2009, 0904.4501.

[160] Hajime Sotani and Kostas D. Kokkotas. Probing strong-field scalar-tensor gravity with gravitational wave asteroseismology. *Phys.Rev.*, D70:084026, 2004, gr-qc/0409066.

[161] Hajime Sotani and Kostas D. Kokkotas. Stellar oscillations in scalar-tensor theory of gravity. *Phys.Rev.*, D71:124038, 2005, gr-qc/0506060.

[162] Thomas P. Sotiriou and Valerio Faraoni. Black holes in scalar-tensor gravity. *Phys.Rev.Lett.*, 108:081103, 2012, 1109.6324. v1: 4 pages/ v2: typos corrected, published version.

[163] Ulrich Sperhake, Emanuele Berti, Vitor Cardoso, Frans Pretorius, and Nicolas Yunes. Superkicks in ultrarelativistic encounters of spinning black holes. *Phys.Rev.*, D83:024037, 2011, 1011.3281.

[164] Ingrid H. Stairs. Testing general relativity with pulsar timing. *Living Rev.Rel.*, 6:5, 2003, astro-ph/0307536.

[165] Ivan Zh. Stefanov, Stoytcho S. Yazadjiev, and Michail D. Todorov. Phases of 4D scalar-tensor black holes coupled to Born-Infeld nonlinear electrodynamics. *Mod.Phys.Lett.*, A23:2915–2931, 2008, 0708.4141.

[166] Leo C. Stein and Nicolas Yunes. Effective Gravitational Wave Stress-energy Tensor in Alternative Theories of Gravity. *Phys.Rev.*, D83:064038, 2011, 1012.3144.

[167] B. Sundelius, M. Wahde, H. J. Lehto, and M. J. Valtonen. A Numerical Simulation of the Brightness Variations of OJ 287. *Astrophys.J.*, 484:180, July 1997.

[168] J.H. Taylor and J.M. Weisberg. A new test of general relativity: Gravitational radiation and the binary pulsar PS R 1913+16. *Astrophys.J.*, 253:908–920, 1982.

[169] Joseph H. Taylor. Binary pulsars and relativistic gravity. *Rev.Mod.Phys.*, 66:711–719, 1994.

[170] Joseph H. Taylor and J.M. Weisberg. Further experimental tests of relativistic gravity using the binary pulsar PSR 1913+16. *Astrophys.J.*, 345:434–450, 1989.

[171] K.S. Thorne. Multipole Expansions of Gravitational Radiation. *Rev.Mod.Phys.*, 52:299–339, 1980.

[172] Tooru Tsuchida, Go Kawamura, and Kazuya Watanabe. A maximum mass-to-size ratio in scalar-tensor theories of gravity. *Progress of Theoretical Physics*, 100(2):291–313, 1998.

[173] A.I. Vainshtein. To the problem of nonvanishing gravitation mass. *Phys.Lett.*, B39:393–394, 1972.

[174] M. Valtonen, M. Kidger, H. Lehto, and G. Poyner. The structure of the October/November 2005 outburst in OJ287 and the precessing binary black hole model. *Astron. Astrophys.*, 477:407–412, January 2008.

[175] M. Valtonen, S. Mikkola, H.J. Lehto, T. Hyvonen, K. Nilsson, et al. Measuring Black Hole Spin in OJ287. 2010, 1001.1284.

[176] M. J. Valtonen. The OJ 287 binary model and the expected outburst in November 1995. In L. O. Takalo, editor, *Workshop on Two Years of Intensive Monitoring of OJ 287 and 3C66A*, page 64, 1996.

[177] M. J. Valtonen. New Orbit Solutions for the Precessing Binary Black Hole Model of OJ 287. *Astrophys.J.*, 659:1074–1081, April 2007.

[178] M. J. Valtonen and H. J. Lehto. Outbursts in OJ 287: A New Test for the General Theory of Relativity. *Astrophys.J.*, 481:L5, May 1997.

[179] M. J. Valtonen, H. J. Lehto, A. Sillanpää, K. Nilsson, S. Mikkola, R. Hudec, M. Basta, H. Teräsranta, S. Haque, and H. Rampadarath. Predicting the Next Outbursts of OJ 287 in 2006-2010. *Astrophys.J.*, 646:36–48, July 2006.

[180] M. J. Valtonen, H. J. Lehto, L. O. Takalo, and A. Sillanpää. Testing the 1995 Binary Black Hole Model of OJ287. *Astrophys.J.*, 729:33, March 2011.

[181] M. J. Valtonen, S. Mikkola, H. J. Lehto, T. Hyvönen, K. Nilsson, D. Merritt, A. Gopakumar, H. Rampadarath, R. Hudec, M. Basta, and R. Saunders. Measuring black hole spin in OJ287. *Celestial Mechanics and Dynamical Astronomy*, 106:235–243, March 2010, 1001.1284.

[182] M. J. Valtonen, K. Nilsson, A. Sillanpää, L. O. Takalo, H. J. Lehto, W. C. Keel, S. Haque, D. Cornwall, and A. Mattingly. The 2005 November Outburst in OJ 287 and the Binary Black Hole Model. *Astrophys.J.*, 643:L9–L12, May 2006.

[183] M. J. Valtonen, K. Nilsson, C. Villforth, H. J. Lehto, L. O. Takalo, E. Lindfors, A. Sillanpää, et al. Tidally Induced Outbursts in OJ 287 during 2005-2008. *Astrophys.J.*, 698:781–785, June 2009.

[184] Mauri Valtonen and Stefano Ciprini. OJ 287 binary black hole system. 2011, 1112.1162.

[185] M.J. Valtonen, H.J. Lehto, K. Nilsson, J. Heidt, L.O. Takalo, et al. A massive binary black-hole system in OJ 287 and a test of general relativity. *Nature*, 452:851–853, 2008, 0809.1280.

[186] M.J. Valtonen, S. Mikkola, H.J. Lehto, A. Gopakumar, R. Hudec, et al. Testing black hole no-hair theorem with OJ287. *Astrophys.J.*, 742:22, 2011, 1108.5861.

[187] M.J. Valtonen, S. Mikkola, D. Merritt, A. Gopakumar, H.J. Lehto, et al. Measuring the spin of the primary black hole in OJ287. *Astrophys.J.*, 709:725–732, 2010, 0912.1209.

[188] Robert V. Wagoner. Scalar-tensor theory and gravitational waves. *Phys. Rev. D*, 1(12):3209–3216, 1970.

[189] Robert M. Wald. *General Relativity*. The University of Chicago Press, 1984.

[190] G. N. Watson. *A Treatise on the Theory of Bessel Functions*. Merchant Books, 2008.

[191] Steven Weinberg. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*. Wiley, 1972.

[192] J.M. Weisberg, D.J. Nice, and J.H. Taylor. Timing Measurements of the Relativistic Binary Pulsar PSR B1913+16. *Astrophys.J.*, 722:1030–1034, 2010, 1011.0718.

[193] Joel M. Weisberg and Joseph H. Taylor. Relativistic binary pulsar B1913+16: Thirty years of observations and analysis. *ASP Conf.Ser.*, 328:25, 2005, astro-ph/0407149.

[194] A.W. Whinnett. Spontaneous scalarization and boson stars. *Phys.Rev.*, D61:124014, 2000, gr-qc/9911052.

[195] A.W. Whinnett and Diego F. Torres. Spontaneous violation of the energy conditions. *Astrophys.J.*, 603:L133–L136, 2004, astro-ph/0401521.

[196] Clifford M. Will. *Theory and Experiment in Gravitational Physics*. Cambridge University Press, 1993.

[197] Clifford M. Will. Testing scalar - tensor gravity with gravitational wave observations of inspiraling compact binaries. *Phys.Rev.*, D50:6058–6067, 1994, gr-qc/9406022.

[198] Clifford M. Will. The confrontation between general relativity and experiment. *Living Rev. Rel.*, 9:3, 2005, arXiv:gr-qc/0510072.

[199] Clifford M. Will. Resource Letter PTG-1: Precision Tests of Gravity. 2010, 1008.0296.

[200] Clifford M. Will and Douglas M. Eardley. Dipole gravitational radiation in Rosen's theory of gravity - Observable effects in the binary system PSR 1913+16. *Astrophys.J.*, 212:L91–L94, 1977.

[201] Clifford M. Will and Nicolas Yunes. Testing alternative theories of gravity using LISA. *Class.Quant.Grav.*, 21:4367, 2004, gr-qc/0403100.

[202] Clifford M. Will and Helmut W. Zaglauer. Gravitational Radiation, Close Binary Systems, and the Brans-Dicke Theory of Gravity. *Astrophys.J.*, 346:366, 1989.

[203] C.M. Will. Gravitational Radiation from Binary Systems in Alternative Metric Theories of Gravity: Dipole Radiation and the Binary Pulsar. *Astrophys.J.*, 214:826–839, 1977.

[204] James G. Williams, Slava G. Turyshev, and Dale H. Boggs. Progress in lunar laser ranging tests of relativistic gravity. *Phys.Rev.Lett.*, 93:261101, 2004, gr-qc/0411113.

[205] Kent Yagi, Leo C. Stein, Nicolas Yunes, and Takahiro Tanaka. Post-Newtonian, Quasi-Circular Binary Inspirals in Quadratic Modified Gravity. *Phys.Rev.*, D85:064022, 2012, 1110.5950.

[206] Kent Yagi and Takahiro Tanaka. Constraining alternative theories of gravity by gravitational waves from precessing eccentric compact binaries with LISA. *Phys.Rev.*, D81:064008, 2010, 0906.4269.

[207] Nicolas Yunes, Paolo Pani, and Vitor Cardoso. Gravitational Waves from Extreme Mass-Ratio Inspirals as Probes of Scalar-Tensor Theories. 2011, 1112.3351.

[208] Nicolas Yunes and Frans Pretorius. Dynamical Chern-Simons Modified Gravity. I. Spinning Black Holes in the Slow-Rotation Approximation. *Phys.Rev.*, D79:084043, 2009, 0902.4669.

[209] Nicolas Yunes and Frans Pretorius. Fundamental Theoretical Bias in Gravitational Wave Astrophysics and the Parameterized Post-Einsteinian Framework. *Phys.Rev.*, D80:122003, 2009, 0909.3328.

[210] Nicolas Yunes and Carlos F. Sopuerta. Perturbations of Schwarzschild Black Holes in Chern-Simons Modified Gravity. *Phys.Rev.*, D77:064007, 2008, 0712.1028.

[211] Nicolas Yunes and Leo C. Stein. Non-Spinning Black Holes in Alternative Theories of Gravity. *Phys.Rev.*, D83:104002, 2011, 1101.2921. 7 pages, submitted to PRD.