INFERENCE FOR A GAMMA STEP-STRESS MODEL UNDER CENSORING

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BY

LAILA ALKHALFAN, M.Sc.

McMaster University

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AUTHOR: Laila Alkhalfan
B.Sc., M.Sc.
Kuwait University

SUPERVISOR: Professor N. Balakrishnan

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Abstract

In reliability and life-testing experiments, one of the popular and commonly used strategies, that allows manufacturers and designers to identify, improve and control critical components, is called the Accelerated Life Test (ALT). The main idea of these tests is to investigate the product's reliability at higher than usual stress levels on test units to ensure earlier failure than what could result under the normal operating conditions. Stress can be induced by such factors as voltage, pressure, temperature, load or cycling rate.

ALT are applied using different types of accelerations such as high usage rate in which the compressed time testing is done through speed or by reducing off times. Another type of acceleration is the product design where the life of a unit can be accelerated through its size or its geometry. Stress loading is another type of acceleration that is applied using constant stress, step-stress, progressive stress, cyclic stress or random stress. Here, we discuss the step-stress model, which applies stress to each unit and increases the stress at pre-specified times during the experiment allowing us to obtain information about the parameters of the life distribution more quickly than under normal operating conditions.

In this thesis, we present the simple step-stress model (the situation in which there are only two stress levels) when the lifetimes at different stress levels follow the gamma distribution when the data are (Chapter 2) Type-II censored, (Chapter 3) Type-I censored, (Chapter 4) Progressively Type-II censored, and (Chapter 5) Progressively Type-I censored, as well as a multiple step-stress model under Type-I and Type-II censoring. The likelihood function is derived assuming a cumulative exposure model with gamma distributed lifetimes. The resulting likelihood equations do not have closed-form solutions, and so they need to be solved numerically. We then derive confidence intervals for the parameters using asymptotic normality of the maximum likelihood estimates and the parametric bootstrap method. In each case, the performance of the methods of inference developed here are examined by means of Monte Carlo simulation study and are also illustrated with some numerical examples.

Keywords and Phrases: Accelerated Testing; Step-Stress Models; Cumulative Exposure Model; Maximum Likelihood Estimation; Confidence Intervals; Bootstrap Method; Coverage Probability; Order Statistics; Type-I Censoring; Type-II Censoring; Progressive Type-I censoring; Progressive Type-II Censoring; Monte Carlo Simulation; Gamma Distribution; Fisher Information; Asymptotic Normality.

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Chapter 1

Introduction

1.1 Background

Observing the time to failure of a product under normal operating conditions is a relatively expensive and time-consuming process. This problem makes reliability engineers interested in a strategy that could test the lifetime of each product in a faster and economical way. One such important strategy for reliability testing is the accelerated life test, which tests each unit at higher stress levels than normal operating conditions to assure quicker failures. A model is then fitted to the accelerated failure times and then extrapolated to estimate the life distribution under normal operating conditions. Some of the important references in this area are Nelson (1980,1990), Meeker and Escobar (1998), and Bagdonavicius and Nikulin (2002).

Accelerated life test is usually done using constant stress, step stress, or linearly increasing stress levels. Unlike the constant stress experiment which could last for a long time, the step-stress experiment reduces test time and enables quicker failures.

It starts by applying low stress on each unit and after a pre-specified time, the stress is increased. So, the stress is sequentially increased to higher stress levels so that more and more units can fail at these elevated levels of stress. In this test scheme, there could be more than one change of stress level. For example, if we have n identical units placed on a life-test starting at an initial stress level x_1, then the stress level is increased to $x_2, x_3, \ldots, x_m$ at fixed times $\tau_1, \tau_2, \ldots, \tau_m$, respectively. The resulting failure times from such a test are observed in a naturally ordered manner and are called order statistics. These ordered failure times are then used to estimate the parameters of the distribution of failure times under normal operating conditions. Relating the level of stress with the parameters of the failure distribution at that stress level is then required, and one of the commonly used models for this purpose is called the cumulative exposure model.

In the literature, DeGroot and Goel (1979) introduced the tampered random variable model and discussed optimal tests under a Bayesian framework. Sedyakin (1966) proposed the cumulative exposure model in the simple step-stress model which was generalized by Bagdonavicius (1978) and Nelson (1980). Miller and Nelson (1983) obtained the optimal time for changing the stress level from x_1 to x_2, assuming exponentially distributed life times. Bai, Kim, and Lee (1989) extended the results of Miller and Nelson to the case of censoring. Bhattacharyya and Zanzawi (1989) proposed the tampered failure rate model which assumes that the effect of changing stress level is to multiply the initial failure rate function by a factor subsequent to the changed time. This tampered failure rate model was generalized by Madi (1993) from the simple step-stress model to the multiple step-stress case. Khamis and Higgins (1998) and Kateri and Balakrishnan (2008) discussed inferential methods for cumulative exposure model under Weibull distributed lifetimes. Alhadeed and Yang (2002, 2005) obtained the optimal design for the Khamis-Higgins model, and for the lognormal simple step-stress models. Xiong (1998) and Xiong and Milliken (1999) discussed inference of the exponential lifetimes assuming that the mean life is a log-linear function of the stress level. Even though the log-linear link function provides a simpler model, Watkins (2001) argued that it is preferable to work with the original exponential parameters. Gouno and Balakrishnan (2001) reviewed the development on step-stress accelerated life-tests. Gouno, Sen and Balakrishnan (2003) presented inference for step-stress models under the exponential distribution in the case of a progressively Type-I censored data. Xiong and Ji (2004) proposed an analysis of step-stress life tests based on grouped and censored data. Balakrishnan (2009) discussed exact inferential results for exponential step-stress models and some associated optimal accelerated life-tests. Tang (2003) reviewed multiple steps step-stress accelerated tests. Balakrishnan, Kundu, Ng and Kannan (2007) discussed the simple step-stress model under Type-II censoring under the exponential distribution. Balakrishnan, Xie and Kundu (2009) presented exact inference for the simple step-stress model from the exponential distribution when there is time constraint on the duration of the experiment. Balakrishnan and Xie (2007) discussed exact inference for a simple step-stress model with Type-II hybird censored data from the exponential distribution.

1.2 Order Statistics

It is clear that the failure data collected from life tests result in a naturally increasing order. So, the analysis of these types of lifetime data is done by using the theory of order statistics. The order statistics and their diverse applications have been revised extensively by Arnold, Balakrishnan and Nagraga (1992), Balakrishnan and Rao (1998a, b) and David and Nagraja (2003).

Let $X_1, X_2, \ldots, X_n$ denote a random sample from a continuous distribution having a probability density function (pdf) $f(x)$ and a distribution function $F(x)$. Let $Y_1 < Y_2 < \cdots < Y_n$ be the X_i's arranged in ascending order, so that Y_k is the k-th smallest. In particular, Y_1= minX_i and Y_n=max X_i, for $i = 1, 2, \ldots, n$. This way, $Y_1, Y_2, \ldots, Y_n$ are the order statistics from the X_i's. The distribution of Y_1 and Y_n can be obtained easily. The probability that $Y_n \leq x$ is the probability that $X_i \leq x$ for all i, which is $\prod_{i=1}^{n} P(X_i \leq x)$. But, $P(X_i \leq x)$ is $F(x)$ for all i, hence

$$F_{Y_n}(x) = [F(x)]^n \text{ and } f_{Y_n}(x) = n[F(x)]^{(n-1)}f(x).$$

Similarly,

$$P\{Y_1 > x\} = \prod_{i=1}^{n} P\{X_i > x\} = [1 - F(x)]^n.$$

Therefore,

$$F_{Y_1}(x) = 1 - [1 - F(x)]^n \text{ and } f_{Y_1}(x) = n[1 - F(x)]^{(n-1)}f(x).$$

To derive the density function of Y_k for $k = 2, 3, \ldots, n-1$, we consider the event

$x < Y_k \leq x + \delta x$. Then, there must be $k-1$ random variables less than x, one random variable between x and δx, and $n-k$ random variables greater than $x + \delta x$, where δx is relatively small. All other permutations will end up being of probability zero. The density function then can be found to be

$$f_{Y_k}(x)\delta x = \frac{n!}{(k-1)!(n-k)!}[F(x)]^{k-1} f(x)\delta x[1-F(x)]^{n-k},$$

so

$$f_{Y_k}(x) = \frac{n!}{(k-1)!(n-k)!}[F(x)]^{k-1}[1-F(x)]^{n-k} f(x).$$

Similar reasoning allows us to write down the joint density $f_{Y_k,Y_j}(x,y)$ of Y_k and Y_j, for $k < j$, as

$$\frac{n!}{(k-1)!(j-k-1)!(n-j)!}[F(x)]^{k-1}[F(y)-F(x)]^{j-k-1}[1-F(y)]^{n-j} f(x)f(y),$$

for $x < y$, and 0 elsewhere. The joint pdf of $Y_1, Y_2, \ldots, Y_n$ is similarly obtained as

$$f_{Y_1,Y_2,\ldots,Y_n}(x_1, x_2, \ldots, x_n) = n! \prod_{i=1}^{n} f(x_i), x_1 < x_2 < \cdots < x_n.$$

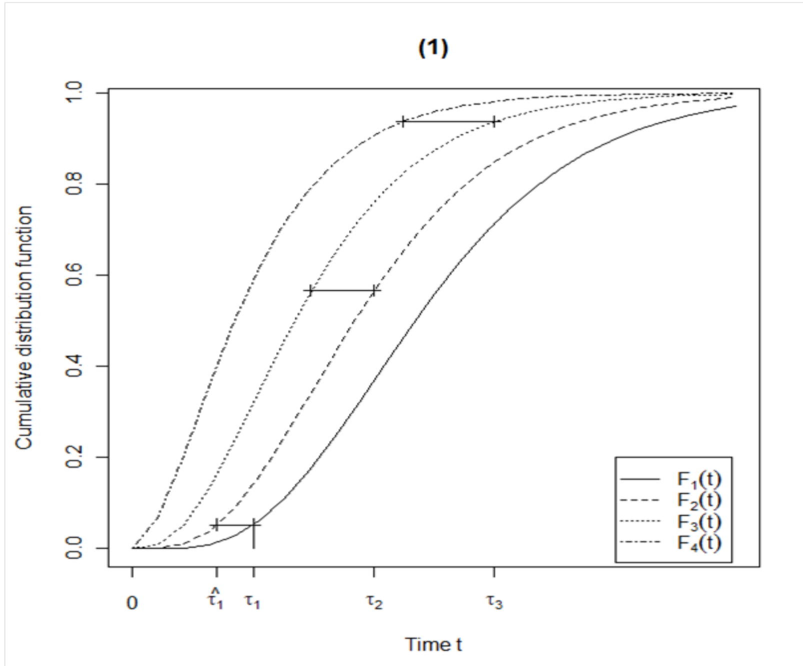
1.3 Cumulative Exposure Model

The cumulative exposure model, as mentioned before, is one of the most useful models in the analysis of step-stress experiments. This model relates the life distribution of the test units at one stress level to the distributions at preceding stress levels by assuming that the residual lives of the experimental units depend only on the cumulative exposure that the units have experienced, with no memory of how the stress was accumulated. Moreover, the surviving units will fail according to the cumulative distribution at the same stress level that is currently being tested at, but starting at the previous accumulated stress level.

Figure 1.1 explains the basic cumulative exposure model for a failure mode assuming that there are 4 stress levels (x_1, x_2, x_3, x_4) that are changed at fixed times (τ_1, τ_2, τ_3). We also assume that the lifetime distribution functions at stress levels x_1, x_2, x_3 and x_4 are F_1, F_2, F_3 and F_4, respectively, and that they all belong to the same family of distributions. Part (1) of the figure presents the four Cumulative Distribution Functions (CDFs) at the four stress levels. The experiment starts with n identical units, and each unit is subjected to an initial stress x_1 with lifetimes following the CDF $F_1(t)$. The time at which a unit failed will be collected and the un-failed units will continue until time τ_1 at which the stress is increased to x_2 and the units will follow the CDF $F_2(t)$, but it will start at the previously accumulated fraction failed. Similarly, when the stress is increased from x_2 to x_3 and from x_3 to x_4, and so on, the distributions of lifetimes would change to $F_3(t), F_4(t)$, and so on, with them starting suitably at the previously accumulated fraction failed. For example, the change of stress level from x_1 to x_2 is going to change the lifetime distribution at stress level x_2 from $F_2(t)$ to $F_2(t - \tau_1 + \hat{\tau}_1)$, where

$$F_1(\tau_1) = F_2(\hat{\tau}_1). \tag{1.3.1}$$

Assuming that $\theta_1, \theta_2, \theta_3$ and θ_4 are the scale parameters associated with F_1, F_2, F_3 and F_4, respectively, with the assumption that F_1, F_2, F_3 and F_4 all belong to the same family of distributions, and assuming absolute continuity of the cumulative



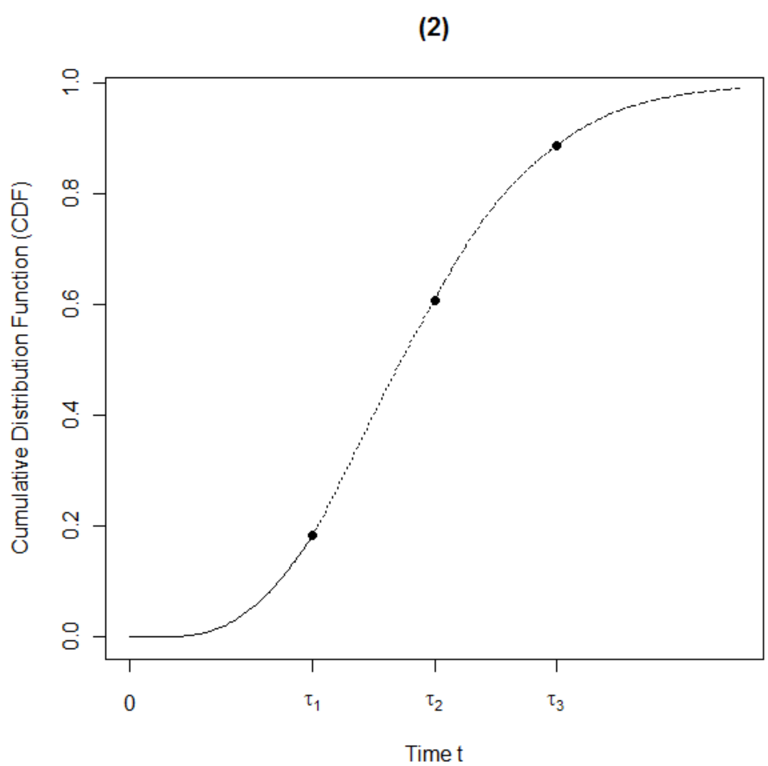


Figure 1.1: Basic Cumulative Exposure Model

distribution function of the lifetime, we find

$$\begin{aligned}
\hat{\tau}_1 &= \frac{\theta_2}{\theta_1}\tau_1, \\
\hat{\tau}_2 &= \frac{\theta_3}{\theta_2}\left[\tau_2 - \tau_1 + \frac{\theta_2}{\theta_1}\tau_1\right], \\
\hat{\tau}_3 &= \frac{\theta_4}{\theta_3}\left[\tau_3 - \tau_2 + \frac{\theta_3}{\theta_2}\left(\tau_2 - \tau_1 + \frac{\theta_2}{\theta_1}\tau_1\right)\right].
\end{aligned}$$

Thus, in general, for $k = 1, 2, 3, \ldots, m$, we have

$$\hat{\tau}_k = \frac{\theta_{k+1}}{\theta_k}\left(\tau_k - \tau_{k-1} + \hat{\tau}_{k-1}\right). \tag{1.3.2}$$

The resulting cumulative distribution function of the lifetime under the cumulative exposure model (shown in Part (2) of Figure 1.1), which consists of the segments of the CDFs at different stress levels, is given by

$$G(t) = \begin{cases}
G_1(t) = & F_1(t) \quad \text{for} \quad 0 < t < \tau_1 \\
G_2(t) = & F_2(t - \tau_1 + \hat{\tau}_1) \quad \text{for} \quad \tau_1 \leq t < \tau_2 \\
G_3(t) = & F_3(t - \tau_2 + \hat{\tau}_2) \quad \text{for} \quad \tau_2 \leq t < \tau_3 \\
G_4(t) = & F_4(t - \tau_3 + \hat{\tau}_3) \quad \text{for} \quad \tau_3 \leq t < \tau_4 \\
 & \vdots \qquad\qquad\qquad \vdots \\
G_k(t) = & F_k(t - \tau_{k-1} + \hat{\tau}_{k-1}) \quad \text{for} \quad \tau_{k-1} \leq t < \infty,
\end{cases}, \tag{1.3.3}$$

and the corresponding probability density function is given by

$$g(t) = \begin{cases} g_1(t) = & f_1(t) & \text{for} & 0 < t < \tau_1 \\ g_2(t) = & f_2(t-\tau_1+\hat{\tau}_1) & \text{for} & \tau_1 \le t < \tau_2 \\ g_3(t) = & f_3(t-\tau_2+\hat{\tau}_2) & \text{for} & \tau_2 \le t < \tau_3 \\ g_4(t) = & f_4(t-\tau_3+\hat{\tau}_3) & \text{for} & \tau_3 \le t < \tau_4 \\ & \vdots & & \vdots \\ g_k(t) = & f_k(t-\tau_{k-1}+\hat{\tau}_{k-1}) & \text{for} & \tau_{k-1} \le t < \infty. \end{cases} \tag{1.3.4}$$

In this thesis, we assume a gamma lifetime distribution at all stress levels, with common shape parameter α and scale parameter θ_i for distribution F_i. Then, the cumulative distribution function of the simple step-stress model, in which there are two stress levels x_1 and x_2, will become

$$G(t) = \begin{cases} G_1(t) = & F_1(t) & \text{for} & 0 < t < \tau_1 \\ G_2(t) = & F_2(t-\tau_1+\hat{\tau}_1) & \text{for} & \tau_1 \le t < \tau_2 \end{cases}, \tag{1.3.5}$$

where

$$F_1(t) = IG_{\frac{t}{\theta_1}}(\alpha) \text{ and } F_2(t) = IG_{\frac{t}{\theta_2}}(\alpha),$$

and $IG_t(\alpha)$ is the incomplete gamma ratio

$$IG_t(\alpha) = \int_0^t \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}\, dx,\ t > 0,\ \alpha > 0,$$

The corresponding probability density function (PDF) in this case will be

$$g(t) = \begin{cases} g_1(t) = & \frac{1}{\Gamma(\alpha)\ \theta_1^{\alpha}} t^{\alpha-1} e^{-\frac{t}{\theta_1}} & \text{for} \quad 0 < t < \tau_1 \\ g_2(t) = & \frac{1}{\Gamma(\alpha)\ \theta_2^{\alpha}} \left(t - \tau_1 + \frac{\theta_2}{\theta_1}\tau_1\right)^{\alpha-1} e^{-\frac{1}{\theta_2}\left(t-\tau_1+\frac{\theta_2}{\theta_1}\tau_1\right)} & \text{for} \quad \tau_1 \leq t < \tau_2 \end{cases} \tag{1.3.6}$$

Similarly, we can present forms of the cumulative distribution function and the probability density function of a gamma multiple step-stress model.

In this thesis, we consider the gamma distribution as a model of failure times because of its appealing features. This distribution is a very flexible function for the description of various lifetime events, given the variety of shapes it accommodates for different values of parameters. It has the structure of the exponential family, which means it has many of its important properties that are associated with sampling distributions. It is used to make realistic adjustments to the exponential distribution in representing life times. Another interesting property is the different patterns of its Hazard function. This function is used to calculate the failure rate for intervals of time. It increases monotonically for $\alpha > 1$, it is constant for $\alpha = 1$ and it decreases monotonically for $\alpha < 1$. For more details on the gamma distribution and its properties, one may refer to Johnson, Kotz and Balakrishnan (1994).

1.4 Types of Data Considered

Analysis of the successive failure times in life-testing experiments depend on the type of failure data that are collected. As mentioned before, there are different types of censoring schemes, and in this section we will briefly describe the ones that will be discussed in the subsequent chapters of this thesis.

1.4.1 Type-II Censored Samples

The Type-II censored sample is formed by terminating the life-testing experiment when a specified number of failures r are observed and the remaining $n - r$ units are censored. Fixing the number of failures would make the time to failure of a test unit random, and so the termination time would also be unknown prior to the experiment, which is a disadvantage of the Type-II censoring but it has the advantage of yielding the required number of failures from the life test.

1.4.2 Type-I Censored Samples

If the termination of a life-testing experiment is done at a prefixed time τ and lifetimes of units that are larger than τ are censored, then the observed failure data obtained from such an experiment form a Type-I censored sample. Controlling the duration of the experiment is actually an advantage, but fixing it may result in very few or even no failures before time τ, which is a disadvantage. Another disadvantage of this type of censoring is the randomness in the number of failures, which results in a complicated Maximum Likelihood Estimation (MLE) of parameters.

For more details on Type-I and Type-II censoring and associated inferential issues, one may refer to Lawless (2003), Nelson (1982), Cohen and Whitten (1988), Cohen (1991), and Balakrishnan and Cohen (1991).

1.4.3 Progressively Type-II Censored Samples

In progressive Type-II censoring, the experiment is terminated after reaching a fixed number of failures r, which is similar to the Type-II censoring. The difference here is that at the time of the first failure, R_1 of the $n - 1$ surviving units are randomly withdrawn or censored from the life-testing experiment. At the time of the next failure, R_2 of the $n-2-R_1$ surviving units are censored, and so on. Finally, at the time of the r-th failure, all the remaining $R_r = n - r - R_1 - \cdots - R_{r-1}$ surviving units are censored. Thus, the censoring takes place here progressively in r stages. It can be seen that this censoring scheme includes the complete sample situation when $R_1 = R_2 = \cdots = R_r = 0$ and $n = r$, and the conventional Type-II censoring scheme when $R_1 = R_2 = \cdots = R_{r-1} = 0$ and $R_r = n - r$. There have been many studies and discussions on procedures based on progressive Type-II censored sampling. One may refer to the works of Herd (1956). Cohen (1963, 1966, 1991), Nelson (1982), Cohen and Whitten (1988), Balakrishnan and Cohen (1991), Balakrishnan and Aggarwala (2000), Balasooriya, Saw and Gadag (2000), and Ng, Chan and Balakrishnan (2002, 2004), Balakrishnan (2007), all have studied some inferential procedures based on the progressively censored sample.

1.4.4 Progressively Type-I Censored Samples

In multiple step-stress model under the progressive Type-I censoring scheme, the experiment is terminated at a pre-specified time τ_m. We start with n identical units subjected to initial stress level x_1. Then, at a specified time τ_1, the stress is increased to x_2, at which R_1 surviving units are withdrawn from the experiment provided R_1 units are still surviving. After that at time τ_2, the stress is increased to x_3 and R_2 surviving units are removed from the test provided there are sufficient number of surviving units still available to be removed, and so on. In this test scheme, we fix the values $\tau_1, \tau_2, \ldots, \tau_m$ and $R_1, R_2, \ldots, R_{m-1}$ but R_m would be a random variable corresponding to whatever number of units remain at time τ_m. In the literature, there has been some discussions on such a progressive Type-I censoring; see, for example, Cohen (1963, 1991), Cohen and Whitten (1988), Balakrishnan and Cohen (1991), and Balakrishanan and Han (2008).

1.5 Scope of the Thesis

This thesis consists of eight chapters in addition to this Introduction chapter, out of which six chapters deal with various inferential issues concerning a simple step-stress model as well as m-step-stress model based on gamma lifetimes under different forms of censoring. Chapter eight presents the computational methods and the optimization algorithms that are used in this thesis. The final chapter presents some conclusions as well as further problems of interest in this direction.

In Chapter 2, we consider a simple step-stress model under Type-II censoring based on gamma lifetimes. In Section 2.2, the considered model is described. The maximum likelihood estimates (MLEs) are obtained using numerical methods in Section 2.3. The derivation of confidence intervals for the unknown parameters using both the approximate method, which uses the Fisher information matrix, and the parametric bootstrap method, are discussed in Section 2.4. In Section 2.5, a simulation study is presented to illustrate the performance of the maximum likelihood estimates and the confidence intervals which are obtained using the approximate and the bootstrap methods. Some illustrative examples are also presented.

In Chapter 3, we consider a simple step-stress model under Type-I censoring based on gamma lifetimes. In Section 3.2, the considered model is described. The maximum likelihood estimates (MLEs) are obtained using numerical methods in Section 3.3. The derivation of confidence intervals for the unknown parameters using the

Fisher information matrix with the asymptotic properties of MLEs and the parametric bootstrap method are discussed in Section 3.4. In Section 3.5, a simulation study is presented to illustrate the performance of the maximum likelihood estimates and the confidence intervals obtained by the two methods. An illustrative example is also presented.

In Chapter 4, we consider a simple step-stress model under progressive Type-II censoring based on gamma lifetimes. In Section 4.2, the considered model is discussed. The maximum likelihood function is derived and the MLEs are obtained in Section 4.3. After that, the derivation of the confidence intervals for the unknown parameters using both the approximate method, which uses the Fisher information matrix, and the parametric bootstrap method, are discussed in Section 4.4. In Section 4.5, we present a simulation study to illustrate the performance of the maximum likelihood estimates and the confidence intervals which are obtained using the approximate and the bootstrap methods and we also presented an illustrative example.

In Chapter 5, we consider a simple step-stress progressive Type-I censoring model. We assume that the lifetime data follows a gamma distribution. The considered model is discussed in Section 5.2. In Section 5.3, the maximum likelihood function is obtained with the MLEs. After that, the derivation of the confidence intervals for the unknown parameters using the Fisher information matrix with the asymptotic properties of MLEs and the parametric bootstrap method are discussed in Section 5.4. In Section 5.5, we present a simulation study to illustrate the performance of the maximum likelihood estimates and the confidence intervals obtained by the approximate and the bootstrap methods. An illustrative example is also presented.

In Chapter 6, we develop inference for the m-step-stress model under Type-II censoring with gamma distributed lifetimes. In Section 6.2, the considered model is described. The MLEs are obtained in Section 6.3. After numerically evaluating the MLEs, we construct confidence intervals for the unknown parameters by using two methods-the asymptotic method and the parametric bootstrap method-in Section 6.4. In Section 6.5, some simulation results and conclusions are presented. We also present the reduced parameter multiple step-stress model under Type-II censoring in Section 6.6. The MLEs for that model are derived and the confidence intervals are also constructed in Sections 6.6.1 and 6.6.2, respectively. In Section 6.6.3, a simulation study is carried out on the reduced parameter model with an illustrative example. In Section 6.7, we describe some life-stress relationships that can be used in this reduced parameter model.

In Chapter 7,we develop inference for the m-step-stress model under Type-I censoring with gamma distributed lifetimes. In Section 7.2, the considered model is described. The MLEs are obtained in Section 7.3. After numerically evaluating the MLEs, we construct confidence intervals for the unknown parameters by using two methods-the asymptotic method and the parametric bootstrap method-in Section 7.4. In Section 7.5, some simulation results and an illustrative example are presented. In Section 7.6, we present the reduced parameter multiple step-stress model under Type-I censoring. The MLEs for that model are derived and the confidence intervals are also constructed in Sections 7.6.1 and 7.6.2, respectively. In Section 7.6.3, a simulation study is carried out on the reduced parameter model and an illustrative example is presented.

Chapter 8 includes some information about the Sharcnet, which is used to accelerate the computation of the coverage probabilities. We use two algorithms, series and parallel. In Section 8.2, the structure and the facilities of the sharcnet are mentioned, and the performance of the parallel algorithm is tested. The *optim* function is used in R, which has the option of using different methods of optimization such as: Nelder and Mead, BFGS, CG, L-BFGS-B, SANN. Each method is explained explicitly in Section 8.3. After that, in Section 8.4, a comparison of these methods is made to optimize the likelihood function of the step stress model of Type-I censoring.

Finally, in Chapter 9, some conclusions are presented based on the work in this thesis. Some problems that are worth considering for further study are also presented.

Chapter 2

Simple Step-Stress Model Under Type-II Censoring

2.1 Introduction

In this chapter, we consider a simple step-stress model under Type-II censoring based on gamma lifetimes. In Section 2.2, the considered model is described. The maximum likelihood estimates (MLEs) are obtained using numerical methods in Section 2.3. The derivation of confidence intervals for the unknown parameters using both the approximate method, which uses the Fisher information matrix, and the parametric bootstrap method, are discussed in Section 2.4. In Section 2.5, a simulation study is presented to illustrate the performance of the maximum likelihood estimates and the confidence intervals which are obtained using the approximate and the bootstrap methods. Some illustrative examples are also presented.

2.2 Model Description

In simple step-stress model under Type-II censoring, we suppose that the failure time data come from a cumulative exposure model, and we consider a simple step stress model with stress levels x_1 and x_2 with Type-II censoring. The lifetime distributions at stress levels x_1 and x_2 are assumed to follow a gamma distribution with common shape parameter α and scale parameters θ_1 and θ_2, respectively. We have derived the corresponding Cumulative exposure distribution and PDF from Eqs. (1.3.5) and (1.3.6), respectively. In the simple step-stress model with Type-II censoring, we start with n identical units placed simultaneously on a life-test. Each unit will be subjected to an initial stress level x_1. After that, the experiment is run until a fixed time denoted by τ, at which time the stress level is changed to x_2. The experiment is continued until a specified number of failures r are observed. Let n_1 be the number of units that fail before τ and n_2 be the number of units that fail after τ, and so $r = n_1 + n_2$. If r failures occur before τ, then the test is terminated, and otherwise, the experiment continues after time τ until the required r failures are observed. The ordered failure times that are observed will be denoted by

$$\{t_{1:n} < \cdots < t_{n_1:n} < \tau \leq t_{n_1+1:n} < \cdots < t_{r:n}\}. \quad (2.2.1)$$

2.3 Maximum Likelihood Estimation

Considering the observed Type-II censored data given in (2.2.1), we can obtain the likelihood function, and then the maximum likelihood estimates (MLEs) of the unknown parameters α, θ_1 and θ_2 from it. The likelihood function based on the censored data in (2.2.1) [see Arnold, Balakrishnan and Nagraja (1992, p.161)] can be written as

$$L(\alpha, \theta_1, \theta_2|\mathbf{t}) = \frac{n!}{r!}\left\{\prod_{i=1}^{r} g\left(t_{i:n}\right)\left\{1 - G\left(t_{r:n}\right)\right\}^{n-r}\right\}, \quad 0 < t_{1:n} < \cdots < t_{r:n}, \quad (2.3.1)$$

where $r = n_1 + n_2$ and $\mathbf{t}$ is the vector of observed Type-II censored data. Using the cumulative exposure model in Eq. (1.3.5) and the corresponding PDF in Eq. (1.3.6), we obtain the likelihood function of α, θ_1 and θ_2 based on the observed Type-II censored sample in (2.2.1) as follows:

1. If $n_1 = r$ and $n_2 = 0$, the likelihood function in (2.3.1) becomes

$$\begin{aligned} L(\alpha, \theta_1, \theta_2|\mathbf{t}) &= \frac{n!}{r!}\left\{\prod_{i=1}^{r} g_1\left(t_{i:n}\right)\right\}\left\{1 - G_1(t_{r:n})\right\}^{n-r} \\ &= \frac{n!\left(\prod_{i=1}^{n_1} t_i\right)^{\alpha-1}}{r!\theta_1^{\alpha n_1}\left(\Gamma(\alpha)\right)_1^n} e^{-\frac{1}{\theta_1}\sum_{i=1}^{n_1} t_i}\left[1 - IG_{\frac{t_{n_1}}{\theta_1}}(\alpha)\right]^{n-r}, \\ &0 < t_{1:n} < \cdots < t_{r:n} < \infty; \end{aligned} \quad (2.3.2)$$

2. If $n_1 = 0$ and $n_2 = r$, the likelihood function in (2.3.1) becomes

$$
\begin{aligned}
L(\alpha, \theta_1, \theta_2|\mathbf{t}) &= \frac{n!}{r!}\left\{\prod_{i=1}^{r} g_2\left(t_{i:n}\right)\right\}\left\{1-G_2\left(t_{r:n}\right)\right\}^{n-r}, \\
&= \frac{n!}{r!}\frac{1}{\theta_2^{\alpha r}(\Gamma(\alpha))^r}\left(\prod_{i=n_1+1}^{r} y_i\right)^{\alpha-1} \\
&\times e^{-\frac{1}{\theta_2}\sum_{i=n_1+1}^{r} y_i}\left\{1-IG_{\frac{y_r}{\theta_2}}(\alpha)\right\}^{n-r}, \\
&\tau \le t_{1:n} < \cdots < t_{r:n} < \infty; \qquad (2.3.3)
\end{aligned}
$$

3. If $1 \le n_1 \le r-1$, the likelihood function in (2.3.1) becomes

$$
\begin{aligned}
L(\alpha, \theta_1, \theta_2|\mathbf{t}) &= \frac{n!}{n_1! n_2!}\left\{\prod_{i=1}^{n_1} g_1\left(t_{i:n}\right)\right\}\left\{\prod_{i=n_1+1}^{r} g_2\left(t_{i:n}\right)\right\} \\
&\times \left\{1-G_2\left(t_{r:n}\right)\right\}^{n-r} \\
&= \frac{n!}{n_1! n_2!}\frac{\left(\prod_{i=1}^{n_1} t_i\right)^{\alpha-1}\left(\prod_{n_1+1}^{r} y_i\right)^{\alpha-1}}{(\Gamma(\alpha))^r \theta_1^{\alpha n_1}\theta_2^{\alpha n_2}} \\
&\times e^{-\frac{1}{\theta_1}\sum_{i=1}^{n_1} t_i - \frac{1}{\theta_2}\sum_{i=n_1+1}^{r} y_i}\left(1-IG_{\frac{y_r}{\theta_2}}(\alpha)\right)^{n-r}, \\
0 < t_{1:n} &< \cdots < t_{n_1:n} < \tau \le t_{n_1+1:n} < \cdots < t_{r:n} < \infty, \qquad (2.3.4)
\end{aligned}
$$

where $r = n_1 + n_2, y_i = \ t_i - \tau + \frac{\theta_2}{\theta_1}\tau$.

From the likelihood functions in (2.3.2), (2.3.3) and (2.3.4), it is evident that the MLE of θ_1 does not exist if $n_1 = 0$, and the MLE of θ_2 does not exist if $n_1 = r$. The MLEs of θ_1 and θ_2 exist only when $1 \le n_1 \le r-1$, and may be obtained by maximizing the corresponding likelihood function in (2.3.4). In addition, since we are estimating the common shape parameter α, we need to have r to be at least 3. Maximizing the likelihood function for the parameters cannot be achieved analytically because there is no closed-form for the incomplete gamma function (IG). Numerically maximizing the likelihood function for the vector of parameters $(\alpha, \theta_1, \theta_2)$ seems to be the only choice. For this purpose, it is convenient to work with the log-likelihood function rather than the likelihood function in (2.3.4), which is given by

$$\begin{aligned} l(\alpha, \theta_1, \theta_2|\mathbf{t}) &= \ln L(\alpha, \theta_1, \theta_2|\mathbf{t}) \\ &= \ln(c) - r\ln\Gamma(\alpha) - \alpha n_1 \ln\theta_1 - \alpha n_2 \ln\theta_2 - \sum_{i=1}^{n_1} \frac{t_i}{\theta_1} \\ &- \sum_{i=n_1+1}^{r} \frac{y_i}{\theta_2} + (\alpha-1)\sum_{i=1}^{n_1} \ln t_i + (\alpha-1)\sum_{i=n_1+1}^{r} \ln y_i \\ &+ (n-r)\ln(1 - IG_{\frac{y_r}{\theta_2}}(\alpha)) \\ &0 < t_{1:n} < \cdots < t_{n_1:n} < \tau \leq t_{n_1+1:n} < \cdots < t_{r:n} < \infty, \end{aligned} \tag{2.3.5}$$

where $c = \frac{n!}{n_1! n_2!}$ and $y_i = \ t_i - \tau + \frac{\theta_2}{\theta_1}\tau$.

The maximum likelihood estimates must be derived numerically because there is no obvious simplification of the non-linear likelihood equations. Here, numerical likelihood maximization was carried out on the log-likelihood using R software. First, we used the log-likelihood function and started with initial values. Then, the function *optim* in R is used to maximize this log-likelihood function. After that, the estimates are found and their confidence intervals are constructed, using the Hessian matrix. We used the following algorithm to find the MLEs:

1. Simulate n order statistics from the uniform (0,1) distribution, $U_1, U_2, \ldots, U_n$.

2. Find n_1 such that $U_{n_1} \leq G_1(\tau) \leq U_{n_1+1}$.

3. For $i \leq n_1$, $T_i = \theta_1 G^{-1}(U_i)$, where $G(t) = \int_0^t \frac{1}{\Gamma(\alpha)} e^{-x} x^{\alpha-1} dx$.

4. For $n_1 + 1 \leq i \leq r$, $\quad T_i = \theta_2 \; G^{-1}(U_i) + \tau - \frac{\theta_2}{\theta_1}\tau$.

5. Compute the MLEs of $(\alpha,\ \theta_1,\ \theta_2)$ based on $T_1, T_2, \ldots, T_{n_1}, T_{n_1+1}, \ldots, T_r$, say $\widehat{\alpha}$, $\widehat{\theta}_1$ and $\widehat{\theta}_2$.

Differentiating the log-likelihood function in (2.3.5) with respect to α, θ_1 and θ_2, we obtain the following likelihood equations which need to be solved for finding the MLEs of α, θ_1 and θ_2:

$$\begin{aligned}\frac{\partial l(\alpha, \theta_1, \theta_2|\mathbf{t})}{\partial \alpha} &= -r\Psi(\alpha) - n_1 \ln\theta_1 - n_2 \ln\theta_2 + \sum_{i=1}^{n_1} \ln t_i + \sum_{n_1+1}^{r} \ln y_i \\ &+ \frac{(n-r)}{(1 - IG_s(\alpha))}\left[\Psi(\alpha) IG_s(\alpha) - \frac{1}{\Gamma(\alpha)}\int_0^s u^{\alpha-1}\ln(u) e^{-u} du\right] = 0,\end{aligned} \tag{2.3.6}$$

$$\begin{aligned}\frac{\partial l(\alpha, \theta_1, \theta_2|\mathbf{t})}{\partial \theta_1} &= -\frac{\alpha n_1}{\theta_1} + \frac{n_2 \tau}{\theta_1^2} - \frac{(\alpha-1)\theta_2\tau}{\theta_1^2}\sum_{i=n_1+1}^{r}\frac{1}{y_i} \\ &+ \frac{1}{\theta_1^2}\sum_{i=1}^{n_1} t_i + \frac{(n-r)\tau s^{\alpha-1}e^{-s}}{(1 - IG_s(\alpha))\Gamma(\alpha)\theta_1^2} = 0,\end{aligned} \tag{2.3.7}$$

$$\begin{aligned}\frac{\partial l(\alpha, \theta_1, \theta_2|\mathbf{t})}{\partial \theta_2} &= -\frac{\alpha n_2}{\theta_2} + \frac{1}{\theta_2^2}\sum_{i=n_1+1}^{r}(t_i - \tau) + \frac{(\alpha-1)\tau}{\theta_1}\sum_{i=n_1+1}^{r}\frac{1}{y_i} \\ &+ \frac{(n-r)(t_r - \tau)s^{\alpha-1}e^{-s}}{(1 - IG_s(\alpha))\Gamma(\alpha)\theta_2^2} = 0,\end{aligned} \tag{2.3.8}$$

where $y_i = t_i - \tau + \frac{\theta_2}{\theta_1}\tau, \quad s = \frac{t_r - \tau}{\theta_2} + \frac{\tau}{\theta_1}$ and $\Psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$.

2.4 Confidence Intervals

In this section, we present two different methods for constructing confidence intervals (CI) for the unknown parameters α, θ_1 and θ_2. The first method uses the asymptotic distributions of the MLEs to obtain approximate CIs for α, θ_1 and θ_2. The second method is based on a parametric bootstrap method.

2.4.1 Approximate Confidence Intervals

In this subsection, we present an approximate method which provides good probability coverages for large sample sizes and is easy to compute. Elements of Fisher information matrix of α, θ_1 and θ_2 were found numerically. Then, the asymptotic normality of MLEs is used to construct approximate confidence intervals for α, θ_1 and θ_2. Let I $(\alpha,\theta_1,\theta_2) = [\mathrm{I}_{ij}(\alpha,\theta_1,\theta_2)]$, for $i,\ j = 1,\ 2, 3$, denote the observed Fisher information matrix of α, θ_1 and θ_2, where

$$I_{ij}(\alpha,\theta_1,\theta_2) = -(\nabla^2 l(\alpha,\theta_1,\theta_2)). \tag{2.4.1}$$

Then, the observed Fisher information matrix (I) is given by

$$I = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}, \tag{2.4.2}$$

where

$$
\begin{aligned}
I_{11} &= \frac{\partial^2 l(\alpha,\theta_1,\theta_2)}{\partial\alpha^2} \\
&= -r\Psi^{'}(\alpha) + \frac{(n-r)}{[1-IG_s(\alpha)]^2}[(1-IG_s(\alpha))[2\Psi(\alpha)B_1(s) \\
&+ (\Psi^{'}(\alpha)-\Psi^2(\alpha))IG_s(\alpha) - B_2(s)] - [\Psi(\alpha)IG_s(\alpha) - B_1(s)]^2], \qquad (2.4.3)
\end{aligned}
$$

$$
\begin{aligned}
I_{12} &= \frac{\partial^2 l(\alpha,\theta_1,\theta_2)}{\partial\alpha\partial\theta_1} \\
&= -\frac{n_1}{\theta_1} - \frac{\theta_2\tau}{\theta_1^2}\sum_{i=n_1+1}^{r}\frac{1}{y_i} + \frac{(n-r)\tau s^{\alpha-1}e^{-s}}{[1-IG_s(\alpha)]^2\theta_1^2\Gamma(\alpha)} \\
&\times [(1-IG_s(\alpha))(\ln(s)) - \Psi(\alpha) + B_1(s)], \qquad (2.4.4)
\end{aligned}
$$

$$
\begin{aligned}
I_{13} &= \frac{\partial^2 l(\alpha,\theta_1,\theta_2)}{\partial\alpha\partial\theta_2} \\
&= -\frac{n_2}{\theta_2} + \frac{\tau}{\theta_1}\sum_{i=n_1+1}^{r}\frac{1}{y_i} + \frac{(n-r)(t_r-\tau)s^{\alpha-1}e^{-s}}{[1-IG_s(\alpha)]^2\theta_2^2\Gamma(\alpha)} \\
&\times [(1-IG_s(\alpha))\ln(s) - \Psi(\alpha) + B_1(s)], \qquad (2.4.5)
\end{aligned}
$$

$$
\begin{aligned}
I_{22} &= \frac{\partial^2 l(\alpha,\theta_1,\theta_2)}{\partial\theta_1^2} \\
&= \frac{\alpha n_1}{\theta_1^2} - \frac{2}{\theta_1^3}\sum_{i=1}^{n_1} t_i - \frac{2n_2\tau}{\theta_1^3} \\
&+ \frac{2(\alpha-1)\theta_2\tau}{\theta_1^3}\sum_{i=n_1+1}^{r}\frac{t_i-\tau+\frac{\theta_2}{2\theta_1}\tau}{y_i^2} \\
&+ \frac{(n-r)\tau s^{\alpha-1}e^{-s}}{[1-IG_s(\alpha)]^2\theta_1^4\Gamma(\alpha)} \\
&\times [(1-IG_s(\alpha))[(1-(\alpha-1)s^{-1})\tau-2\theta_1] - \frac{\tau s^{\alpha-1}e^{-s}}{\Gamma(\alpha)}],
\end{aligned} \tag{2.4.6}
$$

$$
\begin{aligned}
I_{23} &= \frac{\partial^2 l(\alpha,\theta_1,\theta_2)}{\partial\theta_1\partial\theta_2} \\
&= -\frac{(\alpha-1)\tau}{\theta_1^2}\sum_{i=n_1+1}^{r}\frac{t_i-\tau}{y_i^2} + \frac{(n-r)\tau(t_r-\tau)s^{\alpha-1}e^{-s}}{[1-IG_s(\alpha)]^2\theta_1^2\theta_2^2\Gamma(\alpha)} \\
&\times \left[(1-IG_s(\alpha))(1-(\alpha-1)s^{-1}) - \frac{s^{\alpha-1}e^{-s}}{\Gamma(\alpha)}\right],
\end{aligned} \tag{2.4.7}
$$

and

$$
\begin{aligned}
I_{33} &= \frac{\partial^2 l(\alpha,\theta_1,\theta_2)}{\partial\theta_2^2} \\
&= \frac{\alpha n_2}{\theta_2^2} - \frac{2}{\theta_2^3}\sum_{i=n_1+1}^{r}(t_i-\tau) - \frac{(\alpha-1)\tau^2}{\theta_1^2}\sum_{i=n_1+1}^{r}\frac{1}{y_i^2} \\
&+ \frac{(n-r)(t_r-\tau)s^{\alpha-1}e^{-s}}{[1-IG_s(\alpha)]^2\theta_2^4\Gamma(\alpha)} \\
&\times [(1-IG_s(\alpha))[(t_r-\tau)(1-(\alpha-1)s^{-1})-2\theta_2] - \frac{(t_r-\tau)s^{\alpha-1}e^{-s}}{\Gamma(\alpha)}],
\end{aligned} \tag{2.4.8}
$$

where

$$B_1(t) = \frac{1}{\Gamma(\alpha)} \int_0^t u^{\alpha-1} (\ln u) e^{-u} du,$$
$$B_2(t) = \frac{1}{\Gamma(\alpha)} \int_0^t u^{\alpha-1} \; (\ln u)^2 e^{-u} du.$$

It is known that $\mathrm{I}_{21} = \mathrm{I}_{12}$, $\mathrm{I}_{31} = \mathrm{I}_{13}$ and $\mathrm{I}_{32} = \mathrm{I}_{23}$. Now, the variances and coveriances of $\widehat{\alpha}$, $\widehat{\theta}_1$ and $\widehat{\theta}_2$ can be obtained through the observed Fisher information matrix as

$$Var \begin{bmatrix} \hat{\alpha} \\ \hat{\theta}_1 \\ \hat{\theta}_2 \end{bmatrix} = (I)^{-1} = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix}. \tag{2.4.9}$$

The asymptotic distribution of the maximum likelihood estimates are then given by $\frac{\widehat{\alpha}-\alpha}{\sqrt{V_{11}}} \sim N(0,1)$, $\frac{\widehat{\theta}_1-\theta_1}{\sqrt{V_{22}}} \sim N(0,1)$ and $\frac{\widehat{\theta}_2-\theta_2}{\sqrt{V_{33}}} \sim N(0,1)$, which can be used to construct $100\,(1-\alpha)\,\%$ confidence interval for α, θ_1 and θ_2, respectively. These confidence intervals are given by

$$\hat{\alpha} \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{11}}, \tag{2.4.10}$$

$$\hat{\theta}_1 \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{22}} \tag{2.4.11}$$

and

$$\hat{\theta}_2 \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{33}}, \tag{2.4.12}$$

where z_q is the q-th upper percentile of the standard normal distribution.

2.4.2 Bootstrap Confidence Intervals

Confidence intervals based on the parametric bootstrap sampling can be constructed. The following are the steps to generate the bootstrap confidence intervals:

1. Using the algorithm in Section 2.3, compute the MLEs of α, θ_1 and θ_2 based on $T_1, T_2, \ldots, T_{n_1}, T_{n_1+1}, \ldots, T_r$, say $\widehat{\alpha}$, $\widehat{\theta}_1$ and $\widehat{\theta}_2$.

2. The first r order statistics $U_1, U_2, \ldots, U_r$ from a sample from uniform (0,1) distribution are simulated next.

3. Find n_1 such that $U_{n_1} \leq F_1^*(\tau) \leq U_{n_1+1}$, where $F_1^*(\tau) = \int_0^{\frac{\tau}{\widehat{\theta}_1}} \frac{1}{\Gamma(\widehat{\alpha})} x^{\widehat{\alpha}-1} e^{-x} dx.$

4. For $1 \leq i \leq n_1$, $T_i = \widehat{\theta}_1 F^{*-1}(U_i)$, and for $n_1 + 1 \leq i \leq r$, $T_i = \widehat{\theta}_2 F^{*-1}(U_i) + \tau - \frac{\widehat{\theta}_2}{\widehat{\theta}_1}\tau$, where $F^*(t) = \int_0^t \frac{1}{\Gamma(\widehat{\alpha})} x^{\widehat{\alpha}-1} e^{-x} dx.$

5. Compute the MLEs of α, θ_1 and θ_2 based on $T_1, T_2, \ldots, T_{n_1}, T_{n_1+1}, \ldots, T_r$, say $\widehat{\alpha}^1, \widehat{\theta_1}^1$ and $\widehat{\theta_2}^1$.

6. Repeat steps 2-5 B times to obtain B sets of MLEs of α, θ_1 and θ_2 .

A two-sided $100(1-\alpha)\%$ bootstrap confidence interval of α, θ_1 and θ_2 are then given by

$$CI_\alpha = [\hat{\alpha} - z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\alpha}}}, \hat{\alpha} + z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\alpha}}}], \tag{2.4.13}$$

$$CI_{\theta_1} = [\hat{\theta}_1 - z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta_1}}}, \hat{\theta}_1 + z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta_1}}}] \tag{2.4.14}$$

and

$$CI_{\theta_2} = [\hat{\theta}_2 - z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta_2}}}, \hat{\theta}_2 + z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta_2}}}], \tag{2.4.15}$$

where $MSE_a = \text{var}(a) + (\text{bias}(a))^2$, and $\text{bias}(a) = \overline{a} - a$. The performance of the approximate confidence intervals and the bootstrap confidence intervals are evaluated using a simulation study in the next section followed by illustrative examples.

2.5 Simulation Study

A simulation study was carried out for different values of n, r and τ. The results are presented in Tables 2.1 to 2.4 and they are based on an average over 1000 replications.

In Tables 2.1 and 2.3, we can see that as τ increases the conditional failure probabilities occurring on the first level of stress in the interval $[0, \tau]$ increase as well. But those occurring on the other level of stress in the interval $[\tau, \infty]$ decrease as τ increases. This means that as τ increases, there will be more failures occurring before τ which means more information about θ_1 resulting in better inference about θ_1.

In Tables 2.2 and 2.4, we can see that as n and r increase the bias and MSE of each estimate decrease. We can also see that as τ increases the bias and MSE of both $\widehat{\alpha}$ and $\widehat{\theta_1}$ decrease, which agrees with the comment made above. It is also seen that the bias and MSE of $\widehat{\theta_2}$ are much smaller than those of $\widehat{\theta_1}$.

We also observe from Tables 2.2 and 2.4 that the estimated coverage probabilities of the confidence intervals obtained using the parametric bootstrap method are more closer to the nominal levels than those obtained by using the approximate method. The reason for this might be due to the fact that the asymptotic normality required for applying the approximate method may require much larger values of n and r. In Table 2.2, when n and r are small, the coverage probabilities using the approximate method for both α and θ_2 are above the nominal level while those for θ_1 are below the nominal level. When n and r get larger (see Table 2.4), the coverage probabilities

Table 2.1: Conditional failure probabilities for the step-stress test under Type-II censoring when $\alpha = 2$, $\theta_1 = e^1$, $\theta_2 = e^{.5}$, $n = 40$ and $r = 30$.

	Conditional Failure probabilities (in %)	
τ	$0 < t < \tau$	$\tau < t < \infty$
2	22.13	77.87
3	40.05	59.95
4	58.32	41.68

Table 2.2: The bias and MSE of the MLEs $\widehat{\alpha}, \widehat{\theta_1}$ and $\widehat{\theta_2}$ with estimated coverage probabilities (in %) of confidence intervals for a step-stress model under Type-II censoring for α, θ_1 and θ_2 based on 1000 simulations when $\alpha = 2, \theta_1 = e^1, \theta_2 = e^{0.5}, n = 40$ and $r = 30$.

				90% C.I.		95% C.I.		99% C.I.	
	τ	bias	MSE	Approx.	Boot	Approx.	Boot	Approx.	Boot
α	2	0.7201	4.6054	91.0	89.0	94.6	94.0	97.4	98.0
	3	0.3340	1.4608	96.5	92.0	97.9	95.0	99.6	99.2
	4	0.2541	0.6796	98.8	91.3	99.6	95.3	100	98.7
θ_1	2	1.1813	34.7807	75.9	88.4	82.1	92.4	90.8	97.0
	3	0.4485	7.1688	85.2	90.0	90.5	94.6	95.0	98.0
	4	0.1204	2.3709	88.8	89.7	92.9	94.5	97.6	97.9
θ_2	2	-0.0569	0.3582	97.0	89.0	98.4	93.8	99.6	97.9
	3	-0.0042	0.2601	99.1	89.5	99.5	94.2	100	98.7
	4	-0.0026	0.2896	99.6	92.2	99.8	95.5	99.9	98.6

become better but not for all values of τ. Thus, the bootstrap method, giving good coverage probabilities, is recommended for the purpose of constructing confidence intervals in this set-up.

Table 2.3: Conditional failure probabilities for the step-stress test under Type-II censoring when $\alpha = 2, \theta_1 = e^1, \theta_2 = e^{.5}, n = 60$ and $r = 50$.

	Conditional Failure probabilities (in %)	
τ	$0 < t < \tau$	$\tau < t < \infty$
2	20.23	79.77
3	36.02	63.98
4	51.93	48.07
5	65.83	34.17

2.5.1 Illustrative Examples

In this subsection, we consider two examples. The first example is based on a simulated data set. The second example involves a data set analysed earlier by Blakrishnan and Xie (2007). Both those examples are used to illustrate all the methods of inferences developed in this chapter.

Example 1

In this example, we consider the following data generated with $n = 40,\ \alpha = 2,\ \theta_1 = e^1 =\ 2.718282,\ \theta_2 = e^{.5} = 1.648721,\ \tau = 4$, and $r = 38$. The data are given in Table 2.5.

We consider here three different numbers of failures $r = 30,\ 35,\ 38$. The respective MLEs of $\alpha,\ \theta_1$ and θ_2 and their corresponding standard errors are calculated and are given in Table 2.6. It can be seen from Table 2.6 that the larger the r, the smaller the standard errors of $\widehat{\alpha}, \widehat{\theta}_1$ and $\widehat{\theta}_2$ as the estimation is then based on more failure times.

The confidence intervals for $\alpha,\ \theta_1$ and θ_2 obtained by the approximate method and the bootstrap method for different values of r are given in Table 2.7. From the presented results, it is seen that for θ_2 the bootstrap confidence intervals are narrower

Table 2.4: Estimated coverage probabilities (in %) of confidence intervals for a step-stress model under Type-II censoring for α, θ_1 and θ_2 based on 1000 simulations when $\alpha = 2, \theta_1 = e^1, \theta_2 = e^{0.5}, n = 60$ and $r = 50$.

				90% C.I.		95% C.I.		99% C.I.	
	τ	bias	MSE	Approx.	Boot	Approx.	Boot	Approx.	Boot
α	2	0.3634	1.3369	81.6	91.2	87.9	95.3	95.1	98.2
	3	0.2113	0.5950	92.7	91.3	96.4	95.8	99.0	98.8
	4	0.1552	0.3515	97.3	89.3	99	95.1	99.7	99.1
	5	0.1572	0.2932	97.8	90.0	99.3	95.6	100	98.9
θ_1	2	0.6361	9.0062	74.2	90.5	81.6	93.9	91.5	97.3
	3	0.2517	2.7744	83.8	91.0	89.7	95.0	95.8	98.7
	4	0.0969	1.4687	91.1	91.3	95.3	95.2	98.6	98.8
	5	0.0120	0.8455	90.6	90.6	94.6	94.8	98.7	99.0
θ_2	2	-0.0160	0.2231	98.1	90.7	99.3	94.6	99.9	98.2
	3	-0.0161	0.1463	89.4	89.5	92.2	94.8	96.1	98.5
	4	-0.0153	0.1422	99.6	89.4	99.9	94.2	100	98.5
	5	-0.0443	0.1686	89.9	90.4	94.2	94.7	97.7	98.9

Table 2.5: Simulated data for Example 1

Stress level	Failure times									
$\theta_1 = e^1$	0.360	0.963	1.093	1.579	1.583	1.912	2.055	2.204	2.588	2.763
	2.783	2.850	2.940	2.968	3.000	3.195	3.418	3.724	3.953	
$\theta_2 = e^{.5}$	4.040	4.191	4.322	4.443	4.481	4.808	4.920	5.129	5.248	5.537
	5.663	6.004	6.053	6.194	6.316	6.392	7.600	8.103	9.597	

than the approximate confidence interval, but that is not the case for θ_1. For α, some of the approximate confidence intervals did not include the true value.

Example 2

In this example, we consider the data analysed by Balakrishnan and Xie (2007). They used these data to illustrate the methods of inference for a simple step-stress model with Type-II hybrid censored data from the exponential distribution. The data are presented in Table 2.8. Here, we have $n = 35$, $\theta_1 = e^{3.25} = 25.79034$, $\theta_2 = e^{2.5} =$

Table 2.6: The MLEs of α, θ_1 and θ_2 and their standard errors

r	N_3	$\widehat{\alpha}$	se($\widehat{\alpha}$)	$\widehat{\theta_1}$	se($\widehat{\theta_1}$)	$\widehat{\theta_2}$	se($\widehat{\theta_2}$)
30	11	2.3753	0.3480	2.0460	0.2150	1.4140	0.9489
35	16	2.4189	0.1476	2.0100	0.2028	1.2390	0.5805
38	19	2.3293	0.1331	2.0878	0.1976	1.5514	0.4943

12.18249 and $\tau = 15$. The MLEs of α, θ_1 and θ_2 are obtained along with their interval estimates and they are given in Table 2.9 for different values of r. It can be seen from these results that the approximate confidence intervals are wider than the bootstrap confidence intervals for all the parameters α, θ_1 and θ_2. This once again shows that the bootstrap method is a better method for the interval estimation of the model parameters.

Table 2.7: Interval estimation for the simulated data in Example 1

		C.I. for α		
r	Method	90%	95%	99%
30	Approx C.I.	(1.8030, 2.9475)	(1.6934, 3.0572)	(1.4792, 3.2714)
	Bootstrap C.I.	(1.8997, 2.8509)	(1.8086, 2.9420)	(1.6305, 3.1201)
35	Approx C.I.	(2.1762, 2.6617)	(0.1013, 2.3766)	(2.0388, 2.7991)
	Bootstrap C.I.	(1.9065, 2.9314)	(1.8083, 3.0295)	(1.6165, 3.2214)
38	Approx C.I.	(2.1103, 2.5484)	(2.0683, 2.5904)	(1.9863, 2.6724)
	Bootstrap C.I.	(1.8332, 2.8255)	(1.7381, 2.9206)	(1.5523, 3.1063)
		C.I. for θ_1		
30	Approx C.I.	(1.6924, 2.3996)	(1.6246, 2.4673)	(1.4922, 2.5997)
	Bootstrap C.I.	(1.4043, 2.6876)	(1.2814, 2.8106)	(1.0411, 3.0508)
35	Approx C.I.	(1.6765, 2.3435)	(1.6126, 2.4074)	(1.4878, 2.5322)
	Bootstrap C.I.	(1.3136, 2.7064)	(1.1802, 2.8398)	(0.9194, 3.1006)
38	Approx C.I.	(1.7627, 2.4130)	(1.7004, 2.4753)	(1.5786, 2.5970)
	Bootstrap C.I.	(1.4014, 2.7742)	(1.2699, 2.9057)	(1.0129, 3.1627)
		C.I. for θ_2		
30	Approx C.I.	(0.0000, 2.9748)	(0.0000, 3.2738)	(0.0000, 3.8583)
	Bootstrap C.I.	(0.8546, 1.9734)	(0.7475, 2.0805)	(0.5380, 2.2900)
35	Approx C.I.	(0.2842, 2.1937)	(2.1297, 2.7082)	(0.0000, 2.7341)
	Bootstrap C.I.	(0.7694, 1.7085)	(0.6794, 1.7985)	(0.5036, 1.9743)
38	Approx C.I.	(0.7384, 2.3643)	(0.5826, 2.5201)	(0.2782, 2.8245)
	Bootstrap C.I.	(1.1245, 1.9782)	(1.0427, 2.0600)	(0.8829, 2.2198)

Table 2.8: The sample data for Example 2

$\alpha = 1$	Times-to-failure									
$\theta_1 = e^{3.25}$	0.22	0.35	1.27	1.67	2.22	3.79	5.78	8.43	9.27	10.34
	11.85	12.63	12.68	12.85	12.88	13.14				
$\theta_2 = e^{2.5}$	15.28	16.23	17.21	18.52	19.12	19.39	19.81	22.06	23.85	28.46
	28.65	28.97	30.02	31.42	35.45	36.25	57.40	58.46	115.14	

Table 2.9: Point and interval estimates of α, θ_1 and θ_2 based on the data in Table 2.8 with $n = 35$ and $\tau_1 = 15$ fixed and different choices of r

	r	Point est.		90% C.I.	95% C.I.	99% C.I.
α	20	0.8879	Approx.	(0.0000, 3.7781)	(0.0000, 4.3318)	(0.0000, 5.4140)
			Boot	(0.5787, 1.1970)	(0.5195, 1.2562)	(0.4038, 1.3720)
	25	0.9029	Approx.	(0.0000, 3.6559)	(0.0000, 4.1833)	(0.0000, 5.2141)
			Boot	(0.5937, 1.2120)	(0.5345, 1.2712)	(0.4187, 1.3870)
	32	0.9216	Approx.	(0.0000, 3.5141)	(0.0000, 4.0108)	(0.0000, 4.9815)
			Boot	(0.5894, 1.2539)	(0.5257, 1.3175)	(0.4013, 1.4420)
θ_1	20	30.6218	Approx.	(18.3192, 42.9244)	(15.9624, 45.2812)	(11.3561, 49.8876)
			Boot	(25.8225, 35.4211)	(24.9031, 36.3405)	(23.1061, 38.1375)
	25	29.5413	Approx.	(17.6349, 41.4477)	(15.3540, 43.7286)	(10.8960, 48.1866)
			Boot	(24.7420, 34.3406)	(23.8226, 35.2600)	(22.0256, 37.0570)
	32	28.6281	Approx.	(16.8428, 40.4134)	(14.5851, 42.6711)	(10.1724, 47.0838)
			Boot	(24.6586, 32.5976)	(23.8981, 33.3581)	(22.4119, 34.8443)
θ_2	20	16.4640	Approx.	(6.2449, 26.6830)	(4.2873, 28.6407)	(0.4611, 32.4669)
			Boot	(11.6017, 21.3262)	(10.6702, 22.2577)	(8.8497, 24.0783)
	25	14.9200	Approx.	(7.3494, 22.4904)	(5.8991, 23.9408)	(3.0645, 26.7754)
			Boot	(10.0577, 19.7822)	(9.1262, 20.7137)	(7.3057, 22.5342)
	32	14.0613	Approx.	(6.2646, 21.8579)	(4.7710, 23.3516)	(1.8518, 26.2708)
			Boot	(9.1364, 18.9861)	(8.1930, 19.9296)	(6.3490, 21.7736)

Chapter 3

Simple Step-Stress Model Under Type-I Censoring

3.1 Introduction

In this chapter, we consider a simple step-stress model under Type-I censoring based on gamma lifetimes. In Section 3.2, the considered model is described. The maximum likelihood estimates (MLEs) are obtained using numerical methods in Section 3.3. The derivation of confidence intervals for the unknown parameters using the Fisher information matrix with the asymptotic properties of MLEs and the parametric bootstrap method are discussed in Section 3.4. In Section 3.5, a simulation study is presented to illustrate the performance of the maximum likelihood estimates and the confidence intervals obtained by the two methods. An illustrative example is also presented.

3.2 Model Description

In the simple step-stress model under Type-I censoring, we suppose that the time to failure data come from a cumulative exposure model, and we consider a simple step stress model with stress levels x_1 and x_2. We start with n identical units placed simultaneously on a life-test. Each unit will be subjected to an initial stress level x_1. After that, the experiment is run until a fixed time denoted by τ_1, at which time the stress level is changed to x_2. The experiment is then terminated at a pre-fixed time τ_2. The lifetimes of units larger than τ_2 are censored. Let N_1 be the random number of units that fail before τ_1 and N_2 be the random number of units that fail between τ_1 and τ_2. If $N_1 = n$, the experiment is terminated, and otherwise it is continued until the pre-fixed time τ_2. The data observed are of the form

$$\{t_{1:n} < \cdots < t_{N_1:n} < \tau_1 \leq t_{N_1+1:n} < \cdots < t_{N_1+N_2:n} \leq \tau_2\}, \quad (3.2.1)$$

and we shall use $\mathbf{t}$ to denote the vector of ordered failure times.

3.3 Maximum Likelihood Estimation

Considering the observed Type-I censored data given in (3.2.1), we can obtain the likelihood function, and then the maximum likelihood estimates (MLEs) of the unknown parameters α, θ_1 and θ_2 from it. The likelihood function based on the censored data in (3.2.1) [see Arnold, Balakrishnan and Nagraja (1992, p.160)] can be written as

$$L(\alpha, \theta_1, \theta_2|\mathbf{t}) = \frac{n!}{(n-N)!} \left\{ \prod_{i=1}^{N} g\left(t_{i:n}\right) \left\{1 - G\left(\tau_2\right)\right\}^{n-N} \right\},$$

$$0 < t_{1:n} < \cdots < t_{N:n} < \tau_2, \quad (3.3.1)$$

where $N = N_1 + N_2$ and $\mathbf{t}$ is the vector of observed Type-I censored data. Using the cumulative exposure model in Eq. (1.3.5) and the corresponding PDF in Eq. (1.3.6), we obtain the likelihood function of α, θ_1 and θ_2 based on the observed Type-I censored sample in (3.2.1) as follows:

1. If $N_1 = n$ and $N_2 = 0$, the likelihood function in (3.3.1) becomes

$$\begin{aligned} L(\alpha, \theta_1, \theta_2|\mathbf{t}) &= n! \left\{ \prod_{i=1}^{N_1} g_1\left(t_{i:n}\right) \right\} \\ &= \frac{n! \left(\prod_{i=1}^{N_1} t_i \right)^{\alpha-1}}{\theta_1^{\alpha N_1} \Gamma(\alpha)^{N_1}} e^{-\frac{1}{\theta_1} \sum_{i=1}^{N_1} t_i}, \\ 0 &< t_{1:n} < \cdots < t_{N_1:n} < \tau_1; \end{aligned} \quad (3.3.2)$$

2. If $N_1 = 0$ and $N_2 = n$, the likelihood function in (3.3.1) becomes

$$L(\alpha,\theta_1,\theta_2|\mathbf{t}) = \frac{n!}{(n-N_2)!}\left\{\prod_{i=N_1+1}^{N} g_2\left(t_{i:n}\right)\right\}\left\{1-G_2\left(\tau_2\right)\right\}^{n-N_2},$$

$$= \frac{n!}{(n-N_2)!}\frac{1}{\theta_2^{\alpha N_2}(\Gamma(\alpha))^{N_2}}\left(\prod_{i=N_1+1}^{N} y_i\right)^{\alpha-1}$$

$$\times\, e^{-\frac{1}{\theta_2}\sum_{i=N_1+1}^{N_2} y_i}\left\{1-IG_x(\alpha)\right\}^{n-N_2},$$

$$\tau_1 \le t_{1:n} < \cdots < t_{N_2:n} < \tau_2; \tag{3.3.3}$$

3. If $1 \le N_1 \le N-1$, the likelihood function in (3.3.1) becomes

$$L(\alpha,\theta_1,\theta_2|\mathbf{t}) = \frac{n!}{(n-N)!}\left\{\prod_{i=1}^{N_1} g_1\left(t_{i:n}\right)\right\}\left\{\prod_{i=N_1+1}^{N} g_2\left(t_{i:n}\right)\right\}$$

$$\times\left\{1-G_2\left(\tau_2\right)\right\}^{n-N}$$

$$= \frac{n!}{(n-N)!}\frac{\left(\prod_{i=1}^{N_1} t_i\right)^{\alpha-1}\left(\prod_{N_1+1}^{N} y_i\right)^{\alpha-1}}{(\Gamma(\alpha))^{N}\theta_1^{\alpha N_1}\theta_2^{\alpha N_2}}$$

$$\times\, e^{-\frac{1}{\theta_1}\sum_{i=1}^{N_1} t_i-\frac{1}{\theta_2}\sum_{i=N_1+1}^{N} y_i}\left(1-IG_x(\alpha)\right)^{n-N},$$

$$0 < t_{1:n} < \cdots < t_{N_1:n} < \tau_1 \le t_{N_1+1:n} < \cdots < t_{N:n} < \tau_2, \tag{3.3.4}$$

where N$= N_1+N_2, y_i = \ t_i-\tau_1+\frac{\theta_2}{\theta_1}\tau_1$ and $x=\frac{1}{\theta_2}(\tau_2-\tau_1+\frac{\theta_2}{\theta_1}\tau_1)$.

From the likelihood functions in (3.3.2), (3.3.3) and (3.3.4), it is evident that the MLE of θ_1 does not exist if $N_1=0$, and the MLE of θ_2 does not exist if $N_1=n$. The MLEs of θ_1 and θ_2 exist only when $N_1 \ge 1$ and $N_2 \ge 1$, and may be obtained by maximizing the corresponding likelihood function in (3.3.4). Maximization of the likelihood function for the parameters cannot be done analytically because there is no closed-form for the incomplete gamma function (IG). Numerically maximizing the likelihood function for the vector of parameter $(\alpha, \theta_1, \theta_2)$ seems to be the only choice. For this purpose, it is convenient to work with the log-likelihood function rather than the likelihood function in (3.3.4), which is given by

$$\begin{aligned} l(\alpha, \theta_1, \theta_2|\mathbf{t}) &= \ln L(\alpha, \theta_1, \theta_2|\mathbf{t}) \\ &= \ln(c) - N \ln \Gamma(\alpha) - \alpha N_1 \ln \theta_1 - \alpha N_2 \ln \theta_2 - \sum_{i=1}^{N_1} \frac{t_i}{\theta_1} \\ &- \sum_{i=N_1+1}^{N} \frac{y_i}{\theta_2} + (\alpha - 1) \sum_{i=1}^{N_1} \ln t_i + (\alpha - 1) \sum_{i=N_1+1}^{N} \ln y_i \\ &+ (n - N) \ln(1 - IG_x(\alpha)) \\ & 0 < t_{1:n} < \cdots < t_{N_1:n} < \tau_1 \leq t_{N_1+1:n} < \cdots < t_{N:n} < \tau_2, \end{aligned} \tag{3.3.5}$$

where $c = \frac{n!}{(n-N)!}$.

The maximum likelihood estimates must be derived numerically because there is no obvious simplification of the non-linear likelihood equations. Here, numerical likelihood maximization was carried out on the log-likelihood using R software. First, we used the log-likelihood function and started with initial values. Then, the function *optim* in R is used to maximize this log-likelihood function. After that, the estimates are found and their confidence intervals are constructed, using the Hessian matrix. We used the following algorithm to find the MLEs:

1. Simulate n order statistics from the uniform (0,1) distribution, $U_1, U_2, \ldots, U_n$.

2. Find N_1 such that $U_{N_1} \leq G_1(\tau_1) \leq U_{N_1+1}$.

3. For $i \leq N_1$, $T_i = \theta_1 G^{-1}(U_i)$, where $G(t) = \int_0^t \frac{1}{\Gamma(\alpha)} e^{-x} x^{\alpha-1} dx$.

4. For $N_1 + 1 \leq i \leq N$, $\quad T_i = \theta_2\, G^{-1}(U_i) + \tau_1 - \frac{\theta_2}{\theta_1}\tau_1$.

5. Compute the MLEs of $(\alpha,\ \theta_1,\ \theta_2)$ based on $T_1, T_2, \ldots, T_{N_1}, T_{N_1+1}, \ldots, T_N$, say $\widehat{\alpha},\ \widehat{\theta}_1$ and $\widehat{\theta}_2$.

Differentiating the log-likelihood function in (3.3.5) with respect to $\alpha,\ \theta_1$ and θ_2, we attain the following likelihood equations which need to be solved for finding the MLEs of $\alpha,\ \theta_1$ and θ_2:

$$\begin{aligned}\frac{\partial l(\alpha,\theta_1,\theta_2|\mathbf{t})}{\partial\alpha} &= -N\Psi(\alpha) - N_1\ln\theta_1 - N_2\ln\theta_2 + \sum_{i=1}^{N_1}\ln t_i + \sum_{i=N_1+1}^{N}\ln y_i \\ &\quad + \frac{(n-N)}{(1-IG_s(\alpha))}\left[\Psi(\alpha)IG_s(\alpha) - \frac{1}{\Gamma(\alpha)}\int_0^s u^{\alpha-1}\ln(u)e^{-u}du\right] = 0,\end{aligned} \tag{3.3.6}$$

$$\begin{aligned}\frac{\partial l(\alpha,\theta_1,\theta_2|\mathbf{t})}{\partial\theta_1} &= -\frac{\alpha N_1}{\theta_1} + \frac{N_2\tau_1}{\theta_1^2} - \frac{(\alpha-1)\theta_2\tau_1}{\theta_1^2}\sum_{i=N_1+1}^{N}\frac{1}{y_i} + \frac{1}{\theta_1^2}\sum_{i=1}^{N_1} t_i \\ &\quad + \frac{(n-N)\tau_1 s^{\alpha-1}e^{-s}}{(1-IG_s(\alpha))\Gamma(\alpha)\theta_1^2} = 0,\end{aligned} \tag{3.3.7}$$

$$\frac{\partial l(\alpha,\theta_1,\theta_2|\mathbf{t})}{\partial\theta_2} = -\frac{\alpha N_2}{\theta_2} + \frac{1}{\theta_2^2}\sum_{i=N_1+1}^{N}(t_i-\tau_1) + \frac{(\alpha-1)\tau_1}{\theta_1}\sum_{i=N_1+1}^{N}\frac{1}{y_i} + \frac{(n-N)(\tau_2-\tau_1)s^{\alpha-1}e^{-s}}{(1-IG_s(\alpha))\Gamma(\alpha)\theta_2^2} = 0, \tag{3.3.8}$$

where $y_i = t_i - \tau_1 + \frac{\theta_2}{\theta_1}\tau_1$, $s = \frac{\tau_2-\tau_1}{\theta_2} + \frac{\tau_1}{\theta_1}$ and $\Psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$.

3.4 Confidence Intervals

In this section, we present two different methods for constructing confidence intervals (CI) for the unknown parameters α, θ_1 and θ_2. The first method uses the asymptotic distributions of the MLEs to obtain approximate CIs for α, θ_1 and θ_2. The second method is based on a parametric bootstrap method.

3.4.1 Approximate Confidence Intervals

In this subsection, we present an approximate method which provides good probability coverages for large sample sizes and provides easy computation. Elements of Fisher information matrix of α, θ_1 and θ_2 were found numerically. Then, the asymptotic normality of MLEs is used to construct approximate confidence intervals for α, θ_1 and θ_2.

Let I $(\alpha,\theta_1,\theta_2) = [\mathrm{I}_{ij}(\alpha,\theta_1,\theta_2)]$, for $i,\ j = 1,\ 2, 3$, denote the observed Fisher information matrix of α, θ_1 and θ_2, where

$$I_{ij}(\alpha,\theta_1,\theta_2) = -(\nabla^2 l(\alpha,\theta_1,\theta_2)). \tag{3.4.1}$$

Thus, then the observed Fisher information matrix (I) is given by

$$I = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}, \tag{3.4.2}$$

where

$$\begin{aligned} I_{11} &= \frac{\partial^2 l(\alpha, \theta_1, \theta_2)}{\partial \alpha^2} \\ &= -N\Psi^{'}(\alpha) + \frac{(n-N)}{[1-IG_s(\alpha)]^2}[(1-IG_s(\alpha))[2\Psi(\alpha)B_1(s) \\ &+ (\Psi^{'}(\alpha) - \Psi^2(\alpha))IG_s(\alpha) - B_2(s)] - [\Psi(\alpha)IG_s(\alpha) - B_1(s)]^2], \end{aligned} \tag{3.4.3}$$

$$\begin{aligned} I_{12} &= \frac{\partial^2 l(\alpha, \theta_1, \theta_2)}{\partial \alpha \partial \theta_1} \\ &= -\frac{N_1}{\theta_1} - \frac{\theta_2 \tau_1}{\theta_1^2} \sum_{i=N_1+1}^{N} \frac{1}{y_i} + \frac{(n-N)\tau_1 s^{\alpha-1} e^{-s}}{[1-IG_s(\alpha)]^2 \theta_1^2 \Gamma(\alpha)} \\ &\times [(1-IG_s(\alpha))(\ln(s)) - \Psi(\alpha) + B_1(s)], \end{aligned} \tag{3.4.4}$$

$$\begin{aligned} I_{13} &= \frac{\partial^2 l(\alpha, \theta_1, \theta_2)}{\partial \alpha \partial \theta_2} \\ &= -\frac{N_2}{\theta_2} + \frac{\tau_1}{\theta_1} \sum_{i=N_1+1}^{N} \frac{1}{y_i} + \frac{(n-N)(\tau_2 - \tau_1) s^{\alpha-1} e^{-s}}{[1-IG_s(\alpha)]^2 \theta_2^2 \Gamma(\alpha)} \\ &\times [(1-IG_s(\alpha)) \ln(s) - \Psi(\alpha) + B_1(s)], \end{aligned} \tag{3.4.5}$$

$$\begin{aligned}
I_{22} &= \frac{\partial^2 l(\alpha, \theta_1, \theta_2)}{\partial \theta_1^2} \\
&= \frac{\alpha N_1}{\theta_1^2} - \frac{2}{\theta_1^3} \sum_{i=1}^{N_1} t_i - \frac{2 N_2 \tau_1}{\theta_1^3} \\
&+ \frac{2(\alpha-1)\theta_2 \tau_1}{\theta_1^3} \sum_{i=N_1+1}^{N} \frac{t_i - \tau_1 + \frac{\theta_2}{2\theta_1}\tau_1}{y_i^2} \\
&+ \frac{(n-N)\tau_1 s^{\alpha-1} e^{-s}}{[1 - IG_s(\alpha)]^2 \theta_1^4 \Gamma(\alpha)} \\
&\times [(1 - IG_s(\alpha))[(1 - (\alpha-1)s^{-1})\tau - 2\theta_1] - \frac{\tau_1 s^{\alpha-1} e^{-s}}{\Gamma(\alpha)}], \qquad (3.4.5)
\end{aligned}$$

$$\begin{aligned}
I_{23} &= \frac{\partial^2 l(\alpha, \theta_1, \theta_2)}{\partial \theta_1 \partial \theta_2} \\
&= -\frac{(\alpha-1)\tau_1}{\theta_1^2} \sum_{i=N_1+1}^{N} \frac{t_i - \tau_1}{y_i^2} + \frac{(n-N)\tau_1(\tau_2 - \tau_1) s^{\alpha-1} e^{-s}}{[1 - IG_s(\alpha)]^2 \theta_1^2 \theta_2^2 \Gamma(\alpha)} \\
&\times \left[(1 - IG_s(\alpha))(1 - (\alpha-1)s^{-1}) - \frac{s^{\alpha-1} e^{-s}}{\Gamma(\alpha)} \right], \qquad (3.4.6)
\end{aligned}$$

and

$$\begin{aligned}
I_{33} &= \frac{\partial^2 l(\alpha, \theta_1, \theta_2)}{\partial \theta_2^2} \\
&= \frac{\alpha N_2}{\theta_2^2} - \frac{2}{\theta_2^3} \sum_{i=N_1+1}^{N} (t_i - \tau_1) - \frac{(\alpha-1)\tau_1^2}{\theta_1^2} \sum_{i=N_1+1}^{N} \frac{1}{y_i^2} \\
&+ \frac{(n-N)(\tau_2 - \tau_1) s^{\alpha-1} e^{-s}}{[1 - IG_s(\alpha)]^2 \theta_2^4 \Gamma(\alpha)} \\
&\times [(1 - IG_s(\alpha))[(\tau_2 - \tau_1)(1 - (\alpha-1)s^{-1}) - 2\theta_2] - \frac{(\tau_2 - \tau_1) s^{\alpha-1} e^{-s}}{\Gamma(\alpha)}], \qquad (3.4.7)
\end{aligned}$$

where

$$B_1(t) = \frac{1}{\Gamma(\alpha)} \int_0^t u^{\alpha-1} (\ln u) e^{-u} du,$$
$$B_2(t) = \frac{1}{\Gamma(\alpha)} \int_0^t u^{\alpha-1} \; (\ln u)^2 e^{-u} du.$$

It is known that $\mathrm{I}_{21} = \mathrm{I}_{12}$, $\mathrm{I}_{31} = \mathrm{I}_{13}$ and $\mathrm{I}_{32} = \mathrm{I}_{23}$. Now, the variances and covariances of $\widehat{\alpha}$, $\widehat{\theta}_1$ and $\widehat{\theta}_2$ can be obtained through the observed Fisher information matrix as

$$Var \begin{bmatrix} \hat{\alpha} \\ \hat{\theta}_1 \\ \hat{\theta}_2 \end{bmatrix} = (I)^{-1} = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix}. \tag{3.4.8}$$

The asymptotic distribution of the maximum likelihood estimates are then given by $\frac{\widehat{\alpha}-\alpha}{\sqrt{V_{11}}} \sim N(0,1)$, $\frac{\widehat{\theta}_1-\theta_1}{\sqrt{V_{22}}} \sim N(0,1)$ and $\frac{\widehat{\theta}_2-\theta_2}{\sqrt{V_{33}}} \sim N(0,1)$, which can be used to construct $100\,(1-\alpha)\,\%$ confidence interval for α, θ_1 and θ_2, respectively. These confidence intervals are given by

$$\hat{\alpha} \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{11}}, \tag{3.4.9}$$

$$\hat{\theta}_1 \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{22}} \tag{3.4.10}$$

and

$$\hat{\theta}_2 \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{33}}, \tag{3.4.11}$$

where z_q is the q-th upper percentile of the standard normal distribution.

3.4.2 Bootstrap Confidence Intervals

Confidence intervals based on the parametric bootstrap sampling can be constructed. The following are the steps to generate the bootstrap confidence intervals:

1. Using the algorithm in Section 3.3, compute the MLEs of α, θ_1 and θ_2 based on $T_1, T_2, \ldots, T_{N_1}, T_{N_1+1}, \ldots, T_N$, say $\widehat{\alpha}$, $\widehat{\theta}_1$ and $\widehat{\theta}_2$.

2. The n order statistics $U_1, U_2, \ldots, U_n$ from a sample from uniform (0,1) distribution are simulated next.

3. Find N_1 such that $U_{N_1} \leq F_1^*(\tau_1) \leq U_{N_1+1}$, where $F_1^*(\tau_1) = \int_0^{\frac{\tau_1}{\widehat{\theta}_1}} \frac{1}{\Gamma(\widehat{\alpha})} x^{\widehat{\alpha}-1} e^{-x} dx$.

4. For $1 \leq i \leq N_1$, $T_i = \widehat{\theta}_1 F^{*-1}(U_i)$, and for $N_1+1 \leq i \leq N$, $T_i = \widehat{\theta}_2 F^{*-1}(U_i) + \tau_1 - \frac{\widehat{\theta}_2}{\widehat{\theta}_1}\tau_1$, where $F^*(t) = \int_0^t \frac{1}{\Gamma(\widehat{\alpha})} x^{\widehat{\alpha}-1} e^{-x} dx$.

5. Compute the MLEs of α, θ_1 and θ_2 based on $T_1, T_2, \ldots, T_{N_1}, T_{N_1+1}, \ldots, T_N$, say $\widehat{\alpha}^1, \widehat{\theta_1}^1$ and $\widehat{\theta_2}^1$.

6. Repeat steps 2-5 B times to obtain B sets of MLEs of α, θ_1 and θ_2 .

A two-sided $100(1-\alpha)\%$ bootstrap confidence interval of α, θ_1 and θ_2 are then given by

$$CI_\alpha = [\hat{\alpha} - z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\alpha}}}, \hat{\alpha} + z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\alpha}}}], \tag{3.4.12}$$

$$CI_{\theta_1} = [\hat{\theta}_1 - z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta_1}}}, \hat{\theta}_1 + z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta_1}}}] \tag{3.4.13}$$

and

$$CI_{\theta_2} = [\hat{\theta}_2 - z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta_2}}}, \hat{\theta}_2 + z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta_2}}}], \tag{3.4.14}$$

where $MSE_a = \text{var}(a) + (\text{bias}(a))^2$, and $\text{bias}(a) = \bar{a} - a$. The performance of the approximate confidence intervals and the bootstrap confidence intervals are evaluated using a simulation study in the next section followed by an illustrative example.

3.5 Simulation Study

A simulation study was carried out for different values of τ_1 and τ_2. The results are presented in Tables 3.1 to 3.6 and they are based on an average over 1000 replications.

In Tables 3.1 and 3.3, we can see what is going on in our model when we take different values of τ_1 and τ_2. For example, when $\tau_1 = 3$ and $\tau_2 = 4$, this means that the gap between these times is small and so there will be fewer failures occurring in this interval compared to the number of failures occurring in the first interval, and that is exactly what we observe. But, when we increase the value of τ_2 to 6, which means that the gap between these times becomes larger, there will be more failures occurring in this interval. This means that there will be more information about θ_2, which will lead to better inference about θ_2. We also want to see what are possible values of τ_1 and τ_2 that will guarantee adequate numbers of failures in both intervals. We also can see that the failure probabilities at the first and second intervals add up to 100%. The reason for that is because as mentioned earlier, we only consider the case when $1 \leq N_1 \leq N - 1$, which means that these probabilities are conditional. They were calculated by dividing the number of failures at an interval by the total number of failures at both intervals.

In Tables 3.2 and 3.4, we see clearly that the MSEs of both $\widehat{\alpha}$ and $\widehat{\theta}_2$ are less than those of $\widehat{\theta}_1$. The MSEs of the three estimates $\widehat{\alpha}, \widehat{\theta}_1$ and $\widehat{\theta}_2$ are smaller when n is larger (see Table 3.4). If we look at the MSEs of $\widehat{\theta}_2$, we see that the wider the gap

Table 3.1: Conditional failure probabilities for the step-stress test under Type-I censoring when $\alpha = 2$, $\theta_1 = e^1$, $\theta_2 = e^{.5}$ and $n = 60$.

		Conditional failure probabilities (in %)	
τ_1	τ_2	$0 < t < \tau_1$	$\tau_1 < t < \tau_2$
3	4	59.83	40.17
	6	37.94	62.06
4	6	57.52	42.48
5	6	78.06	21.94
	7	67.85	32.15
6	8	75.82	24.18

between τ_1 and τ_2 the smaller the MSE of $\widehat{\theta}_2$. This is expected as explained above. Looking at the MSEs of $\widehat{\theta}_1$, we see that as τ_1 increases, the MSEs of $\widehat{\theta}_1$ decrease. This is understandable, since the larger the value of τ_1, the more information there will be about the parameter θ_1 and hence better inference. We also observe that taking different values of τ_2 does not effect the MSEs of $\widehat{\theta}_1$. This means that no matter when we stop the test, the information about θ_1 will depend completely on the value of τ_1.

Looking at the estimated coverage probabilities in Tables 3.2 and 3.4 obtained using the bootstrap, we can see that they are closer to the nominal levels than those obtained using the approximate method. The coverage probabilities obtained using the approximate method are unsatisfactory and this method would be suitable only when the sample size is large. We therefore recommend the bootstrap method for the construction of confidence intervals for the parameters of the model considered.

3.5.1 Illustrative Example

In this subsection, we consider the data generated with $n = 40$, $\alpha = 2$, $\theta_1 = e^1 = 2.718282$, $\theta_2 = e^{0.5}$, $\tau_1 = 3$ and $\tau_2 = 9$. The simulated data are given in Table 3.4.

Table 3.2: The bias and MSE of the estimates $\widehat{\alpha}, \widehat{\theta_1}$ and $\widehat{\theta_2}$ with estimated coverage probabilities (in %) of confidence intervals for a step-stress model under Type-I censoring for α, θ_1 and θ_2 based on 1000 simulations when $\alpha = 2, \theta_1 = e^1, \theta_2 = e^{0.5}$ and $n = 60$.

					90% C.I.		95% C.I.		99% C.I.	
	τ_1	τ_2	Bias	MSE	Approx.	Boot	Approx.	Boot	Approx.	Boot
α	3	4	0.2739	0.7969	98.6	91.3	99.3	95.4	99.7	99.2
		6	0.2124	0.5908	94.3	90.1	97.2	94.8	99.5	99.2
	4	6	0.1561	0.3501	98.5	90.9	99.0	95.6	99.7	98.3
	5	6	0.1017	0.2777	99.7	89.9	100	94.8	100	98.9
		7	0.1315	0.2709	98.6	89.8	99.6	94.9	100	99.0
	6	8	0.1017	0.2208	98.7	89.6	99.4	94.5	100	98.3
θ_1	3	4	0.1510	2.7861	89.1	90.5	93.2	94.9	97.3	98.6
		6	0.2037	2.3640	79.6	90.4	86.7	94.5	94.6	98.0
	4	6	0.0909	1.1771	92.1	90.4	94.9	95.1	98.6	98.3
	5	6	0.1028	0.8856	95.3	89.7	97.5	94.2	99.2	98.4
		7	0.0501	0.8607	93.7	89.7	96.7	94.2	98.9	98.4
	6	8	0.0197	0.5821	95.3	88.5	98.5	93.6	99.7	98.2
θ_2	3	4	0.1097	0.4743	94.8	91.2	97.1	96.1	99	98.6
		6	-0.0088	0.1601	96.1	90.7	97.9	94.9	99.3	98.7
	4	6	0.0372	0.2045	97.1	91.1	98.3	95.4	99.8	98.5
	5	6	0.1689	0.6323	96.2	92.8	97.5	96.1	99.1	98.4
		7	0.0604	0.2506	99.1	91.7	99.5	95.7	100	98.7
	6	8	0.0716	0.3585	96.8	91.7	98.3	95.0	99.4	98.8

We consider three different times $\tau_2 = 4,\ 6,\ 9$. The respective MLEs of $\alpha,\ \theta_1$ and θ_2 and their corresponding standard errors are calculated and are given in Table 3.6. It can be seen from Table 3.6 that the larger the τ_2, the smaller the standard errors of $\widehat{\alpha}, \widehat{\theta_1}$ and $\widehat{\theta_2}$.

The confidence intervals for $\alpha,\ \theta_1$ and θ_2 obtained by the approximate method and the bootstrap method for different values of τ_2 are given in Table 3.7. In this table, we can see that for α the bootstrap confidence intervals are narrower than the approximate confidence interval for all values of τ_2. We can also see that the

Table 3.3: Conditional failure probabilities for the step-stress test under Type-I censoring when $\alpha = 2$, $\theta_1 = e^{1}$, $\theta_2 = e^{.5}$ and $n = 100$.

		Conditional failure probabilities (in %)	
τ_1	τ_2	$0 < t < \tau_1$	$\tau_1 < t < \tau_2$
3	4	59.07	40.93
	6	38.38	61.62
4	6	57.66	42.34
5	6	78.26	21.74
	7	67.76	32.24
6	8	75.61	24.39

approximate confidence intervals for θ_1 are either too wide or do not include the true value, while the bootstrap confidence intervals are better in terms of coverage. For θ_2, the bootstrap confidence interval are again better except when $\tau_2 = 6$, in which case the approximate confidence interval is slightly narrower than the bootstrap confidence interval.

Table 3.4: The bias and MSE of the estimates $\widehat{\alpha}, \widehat{\theta_1}$ and $\widehat{\theta_2}$ with estimated coverage probabilities (in %) of confidence intervals for a step-stress model under Type-I censoring for α, θ_1 and θ_2 based on 1000 simulations when $\alpha = 2, \theta_1 = e^1, \theta_2 = e^{0.5}$ and $n = 100$.

					90% C.I.		95% C.I.		99% C.I.	
	τ_1	τ_2	Bias	MSE	Approx.	Boot	Approx.	Boot	Approx.	Boot
α	3	4	0.1248	0.3174	99.7	90.6	99.7	95.3	100	99.3
		6	0.0921	0.2406	96.9	90.5	98.5	95.1	99.8	99.4
	4	6	0.0873	0.2058	98.5	90.6	99.3	95.1	99.9	99.4
	5	6	0.0688	0.1583	99.7	89.7	100	94.9	100	99.3
		7	0.0693	0.1463	99.4	89.6	99.7	95.1	99.9	99.3
	6	8	0.0607	0.1260	98.9	89.5	99.8	94.5	99.9	98.9
θ_1	3	4	0.1282	1.3130	91.3	90.5	94.7	95.4	98.2	98.6
		6	0.1478	1.2148	80.6	90.5	88.0	95.5	95.6	98.8
	4	6	0.0985	0.8693	89.5	90.6	94.6	95.1	97.9	99.0
	5	6	0.0483	0.4719	94.3	90.2	96.6	94.7	99.7	99.0
		7	0.0335	0.4321	95.9	90.2	97.7	94.7	99.5	98.8
	6	8	0.0266	0.3812	95.1	90.2	98.2	94.7	99.3	98.4
θ_2	3	4	0.0481	0.2182	96.4	92.1	98.1	96.3	99.7	99.0
		6	0.0258	0.1087	95.6	91.0	97.4	95.5	99.3	99.4
	4	6	0.0357	0.1281	99.2	91.7	99.5	96.4	99.9	99.6
	5	6	0.1071	0.3182	98.8	92.2	99.2	96.3	99.8	99.4
		7	0.4321	0.1282	98.1	90.4	99.1	96.6	99.5	99.8
	6	8	0.0434	0.1600	97.9	91.5	99.1	96.1	99.8	99.4

Table 3.5: Simulated data for the illustrative example.

Stress level	Failure times									
$\theta_1 = e^1$	0.287	0.863	0.864	0.978	1.087	1.119	1.271	1.789	1.828	2.146
	2.164	2.238	2.331	2.528	2.839	2.916				
$\theta_2 = e^{.5}$	3.482	3.521	3.676	3.728	3.772	3.782	4.034	4.332	4.361	4.382
	4.403	4.403	4.546	4.909	4.945	5.656	5.776	6.250	6.446	6.568
	6.739	6.967								

Table 3.6: The MLEs of α, θ_1 and θ_2 and their standard errors.

τ_2	N_2	$\widehat{\alpha}$	se($\widehat{\alpha}$)	$\widehat{\theta_1}$	se($\widehat{\theta_1}$)	$\widehat{\theta_2}$	se($\widehat{\theta_2}$)
4	6	1.9238	1.4698	2.2923	1.8839	2.2645	0.6424
6	17	1.9532	0.4706	2.2577	0.3251	1.7112	0.1332
9	22	1.9590	0.2527	2.2522	0.1040	1.6473	0.2420

Table 3.7: Interval estimation for the simulated data presented in Table 3.5

		C.I. for α		
τ_2	Method	90%	95%	99%
4	Approx C.I.	(0.0000, 4.3414)	(0.0000, 6.1578)	(0.0000, 7.4882)
	Bootstrap C.I.	(1.4930, 2.3546)	(1.4105, 2.4371)	(1.2492, 2.5984)
6	Approx C.I.	(1.1791, 2.7273)	(1.5191, 2.3873)	(1.3827, 2.5237)
	Bootstrap C.I.	(1.5576, 2.3488)	(1.4818, 2.4246)	(1.3337, 2.5727)
9	Approx C.I.	(1.5434, 2.3746)	(1.8339, 2.0842)	(1.7946, 2.1235)
	Bootstrap C.I.	(1.5326, 2.3855)	(1.4509, 2.4672)	(1.2912, 2.6268)
		C.I. for θ_1		
4	Approx C.I.	(0.0000, 5.3910)	(0.0000, 5.9846)	(0.0000, 7.1448)
	Bootstrap C.I.	(1.5708, 3.0138)	(1.4326, 3.1520)	(1.1624, 3.4222)
6	Approx C.I.	(1.7230, 2.7924)	(1.6205, 2.8949)	(1.4203, 3.0951)
	Bootstrap C.I.	(1.6469, 2.8685)	(1.5299, 2.9855)	(1.3012, 3.2142)
9	Approx C.I.	(2.0811, 2.4233)	(2.0483, 2.4561)	(1.9843, 2.5201)
	Bootstrap C.I.	(1.6489, 2.8555)	(1.5333, 2.9711)	(1.3074, 3.1970)
		C.I. for θ_2		
4	Approx C.I.	(1.2080, 3.3211)	(1.0056, 3.5235)	(0.6100, 3.9191)
	Bootstrap C.I.	(1.5527, 2.9763)	(1.4164, 3.1127)	(1.1499, 3.3792)
6	Approx C.I.	(1.4920, 1.9303)	(1.4500, 1.9723)	(1.3679, 2.0544)
	Bootstrap C.I.	(1.3855, 2.0368)	(1.3231, 2.0992)	(1.2012, 2.2211)
9	Approx C.I.	(1.2494, 2.0453)	(1.1731, 2.1216)	(1.0241, 2.2706)
	Bootstrap C.I.	(1.3850, 1.9097)	(1.3347, 1.9600)	(1.2365, 2.0582)

Chapter 4

Simple Step-Stress Model under Progressive Type-II Censoring

4.1 Introduction

In this chapter, we consider a simple step-stress model under progressive Type-II censoring based on gamma lifetimes. In Section 4.2, the considered model is discussed. The maximum likelihood function is derived and the MLEs are obtained in Section 4.3. After that, the derivation of the confidence intervals for the unknown parameters using both the approximate method, which uses the Fisher information matrix, and the parametric bootstrap method, are discussed in Section 4.4. In Section 4.5, we present a simulation study to illustrate the performance of the maximum likelihood estimates and the confidence intervals which are obtained using the approximate and the bootstrap methods and we also present an illustrative example.

4.2 Model Description

Assume that the failure time data come from a cumulative exposure model, and we consider a simple step-stress model with stress levels x_1 and x_2 with progressive Type-II censoring. We also assume that the lifetime distribution at stress levels x_1 and x_2 follow a gamma distribution with common shape parameter α and scale parameters θ_1 and θ_2, respectively. The corresponding probability density function (PDF) and cumulative distribution function (CDF) are as given in Eqs. (1.3.6) and (1.3.5), respectively. The simple step-stress model with progressive Type-II censoring starts with n identical units placed at an initial stress level x_1 and at a pre-specified time τ the stress level is increased to x_2. The Progressive Type-II censoring is applied by fixing the total number of failures in the test, which is denoted by r, and by fixing number of un-failed units that are randomly removed from the test at each failure time, which is denoted by R_k, where $k = 1,\ 2,\ \cdots,\ r-1$, stands for the k^{th} failure. At the time of the first failure, R_1 of the $n-1$ surviving units are randomly removed from the test; at the time of the second failure, R_2 of the $n-2-R_1$ surviving units are randomly removed from the test, and so on; the test is continued until the r^{th} failure at which time all the remaining $R_r = n-r-R_1-\cdots-R_{r-1}$ surviving units are removed. If $R_1 = R_2 = \cdots = R_r = 0$, then $n = r$, which is the complete sample situation. If $R_1 = R_2 = \cdots = R_{r-1} = 0$, then $R_r = n-r$, which corresponds to the conventional Type-II censoring scheme discussed earlier in Chapter 2. Such a simple step-stress model under progressive Type-II censoring is presented in Figure 4.1.

The corresponding cumulative exposure distribution and PDF are given in Eqs. (1.3.5) and (1.3.6) respectively. Let n_1 be the number of failures before time τ at stress level x_1, and n_2 be the number of failures after time τ at stress level x_2. With these notations, the following will be the observed progressively Type-II censored data:

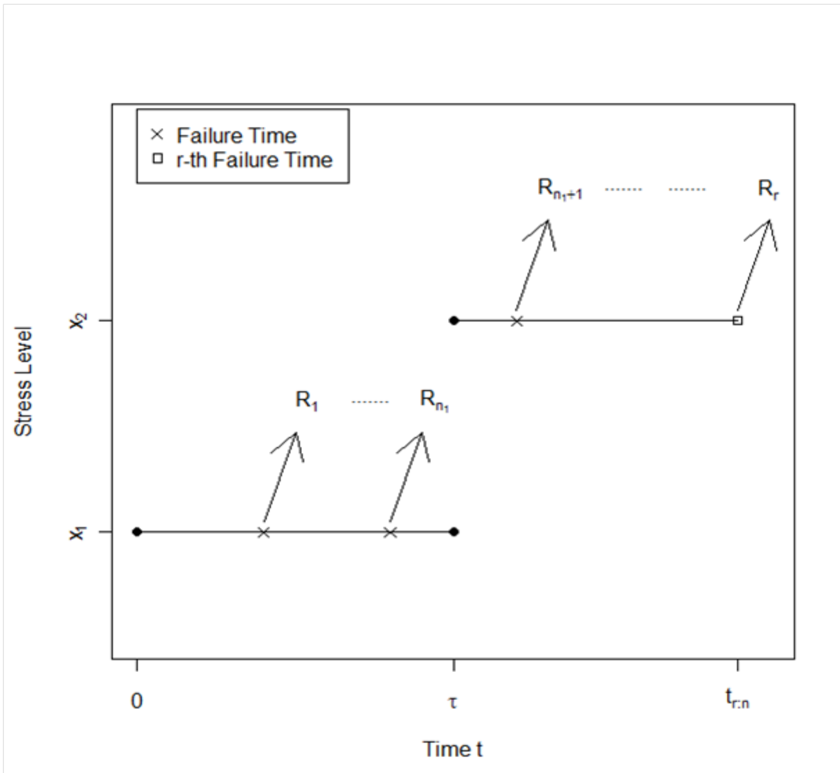


Figure 4.1: m-step-stress model under progressive Type-II censoring

$$\{t_{1:n} < \cdots < t_{n_1:n} \leq \tau < t_{n_1+1:n} < \cdots < t_{r:n}\}, \quad (4.2.1)$$

with the progressive censoring scheme $R = (R_1, \cdots, R_r)$, where $\sum_{j=1}^{r} R_j = n - r$.

4.3 Maximum Likelihood Estimation

In this section, the likelihood function is obtained based on the observed progressively Type-II censored data given in (4.2.1), and then the MLEs of the unknown parameters α, θ_1 and θ_2 are obtained. The Likelihood function of this censored sample [see Balakrishnan and Aggarwala (2000)] can be written as

$$L(\alpha,\theta_1,\theta_2|\mathbf{t}) = c_p \left\{ \prod_{i=1}^{r} g\left(t_{i:n}\right)\left\{1-G\left(t_{i:n}\right)\right\}^{R_i} \right\},$$

$$0 < t_{1:n} < \cdots < t_{r:n}, \quad (4.3.1)$$

where $r = n_1 + n_2$, $\mathbf{t}$ is the vector of observed progressively Type-II censored data, and

$$\begin{aligned} c_p &= n(n-1-R_1)(n-2-R_1-R_2)\cdots\left(n-r+1-\sum_{i=1}^{r-1} R_i\right) \\ &= \prod_{j=1}^{r}\left\{\sum_{r}^{i=j}(R_i+1)\right\} \\ &= \prod_{j=1}^{r} R_j^*. \end{aligned} \quad (4.3.2)$$

Using the cumulative exposure model in Eq. (1.3.5) and the corresponding PDF in Eq. (1.3.6), we obtain the likelihood function of α, θ_1 and θ_2 based on the observed progressively Type-II censored sample in (4.3.1) as follows:

1. If $n_1 = r$ and $n_2 = 0$, the likelihood function in (4.3.1) becomes

$$
\begin{aligned}
L(\alpha,\theta_1,\theta_2|\mathbf{t}) &= c_p\left\{\prod_{i=1}^{r} g_1\left(t_{i:n}\right)\left\{1-G_1\left(t_{i:n}\right)\right\}^{R_i}\right\} \\
&= \frac{c_p\left(\prod_{i=1}^{r} t_i\right)^{\alpha-1}}{\theta_1^{\alpha r}\left(\Gamma(\alpha)\right)^{r}} e^{-\frac{1}{\theta_1}\sum_{i=1}^{r} t_i} \prod_{i=1}^{r}\left\{1-IG_{\frac{t_i}{\theta_1}}(\alpha)\right\}^{R_i}, \\
& 0 < t_{1:n} < \cdots < t_{r:n} < \tau; \qquad (4.3.3)
\end{aligned}
$$

2. If $n_1 = 0$ and $n_2 = r$, the likelihood function in (4.3.1) becomes

$$
\begin{aligned}
L(\alpha,\theta_1,\theta_2|\mathbf{t}) &= c_p\left\{\prod_{i=1}^{r} g_2\left(t_{i:n}\right)\left\{1-G_2\left(t_{i:n}\right)\right\}^{R_i}\right\}, \\
&= \frac{c_p\left(\prod_{i=1}^{r} y_i\right)^{\alpha-1}}{\theta_2^{\alpha r}\left(\Gamma(\alpha)\right)^{r}} e^{-\frac{1}{\theta_2}\sum_{i=1}^{r} y_i} \prod_{i=1}^{r}\left\{1-IG_{\frac{y_i}{\theta_2}}(\alpha)\right\}^{R_i}, \\
& \tau < t_{1:n} < \cdots < t_{r:n} < \infty; \qquad (4.3.4)
\end{aligned}
$$

3. If $1 \leq n_1 \leq r-1$, the likelihood function in (4.3.1) becomes

$$
\begin{aligned}
L(\alpha,\theta_1,\theta_2|\mathbf{t}) &= c_p \left\{ \prod_{i=1}^{n_1} g_1\left(t_{i:n}\right)\left\{1-G_1\left(t_{i:n}\right)\right\}^{R_i} \right\} \\
&\times \left\{ \prod_{i=n_1+1}^{r} g_2\left(t_{i:n}\right)\left\{1-G_2\left(t_{i:n}\right)\right\}^{R_i} \right\} \\
&= \frac{c_p\left(\prod_{i=1}^{n_1} t_i\right)^{\alpha-1}\left(\prod_{i=n_1+1}^{r} y_i\right)^{\alpha-1}}{\theta_1^{\alpha n_1}\theta_2^{\alpha n_2}\left(\Gamma(\alpha)\right)^r} e^{-\frac{1}{\theta_1}\sum_{i=1}^{n_1} t_i} \\
&\times e^{-\frac{1}{\theta_2}\sum_{i=n_1+1}^{r} y_i} \prod_{i=1}^{n_1} \left\{1-IG_{\frac{t_i}{\theta_1}}(\alpha)\right\}^{R_i} \\
&\times \prod_{i=n_1+1}^{r} \left\{1-IG_{\frac{y_i}{\theta_2}}(\alpha)\right\}^{R_i}, \\
& 0<t_{1:n}<\cdots<t_{n_1:n}<\tau\leq t_{n_1+1:n}<\cdots<t_{r:n}<\infty, \qquad (4.3.5)
\end{aligned}
$$

where $r = n_1 + n_2$ and $y_i = t_i - \tau + \frac{\theta_2}{\theta_1}\tau$.

From the likelihood functions in (4.3.3), (4.3.4) and (4.3.5), it is evident that the MLE of θ_1 does not exist if $n_1 = 0$, and the MLE of θ_2 does not exist if $n_1 = r$. The MLEs of θ_1 and θ_2 exist only when $1 \leq n_1 \leq r-1$, and are obtained by maximizing the corresponding likelihood function in (4.3.5). In addition, since we are estimating the common shape parameter α, we need to have r to be at least 3. Maximizing the likelihood function for the parameters cannot be achieved analytically because there is no closed-form for the incomplete gamma function (IG). Numerically maximizing the likelihood function for the vector of parameter $(\alpha, \theta_1, \theta_2)$ seems to be the only choice. For this purpose, it is convenient to work with the log-likelihood function rather than the likelihood function in (4.3.5), which is given by

$$
\begin{aligned}
l(\alpha,\theta_1,\theta_2|\mathbf{t}) &= \ln L(\alpha,\theta_1,\theta_2|\mathbf{t}) \\
&= \ln(c_p) - r\ln\Gamma(\alpha) - \alpha n_1 \ln\theta_1 - \alpha n_2 \ln\theta_2 - \sum_{i=1}^{n_1}\frac{t_i}{\theta_1} \\
&- \sum_{i=n_1+1}^{r}\frac{y_i}{\theta_2} + (\alpha-1)\sum_{1}^{n_1}\ln t_i + (\alpha-1)\sum_{i=n_1+1}^{r}\ln y_i \\
&+ \sum_{i=1}^{n_1} R_i \ln(1 - IG_{\frac{t_i}{\theta_1}}(\alpha)) + \sum_{i=n_1+1}^{r} R_i \ln(1 - IG_{\frac{y_i}{\theta_2}}(\alpha)) \\
&0 < t_{1:n} < \cdots < t_{n_1:n} < \tau \le t_{n_1+1:n} < \cdots < t_{r:n} < \infty,
\end{aligned}
\tag{4.3.6}
$$

where $y_i = \ t_i - \tau + \frac{\theta_2}{\theta_1}\tau$.

The maximum likelihood estimates must be derived numerically because there is no obvious simplification of the non-linear likelihood equations. Here, numerical likelihood maximization was carried out on the log-likelihood using R software. First, we used the log-likelihood function and started with initial values. Then, the function *optim* in R is used to maximize this log-likelihood function. After that, the estimates are found and their confidence intervals are constructed, using the Hessian matrix. We used the following algorithm to find the MLEs:

(a) Simulate n order statistics from the uniform (0,1) distribution, $U_1, U_2, \ldots$,U_n.

(b) Find n_1 such that $U_{n_1} \le G_1(\tau) \le U_{n_1+1}$.

(c) For $i \le n_1$, $T_i = \theta_1 G^{-1}(U_i)$, where $G(t) = \int_0^t \frac{1}{\Gamma(\alpha)} e^{-x} x^{\alpha-1} dx$.

(d) For $n_1 + 1 \le i \le r$, $T_i = \theta_2\ G^{-1}(U_i) + \tau - \frac{\theta_2}{\theta_1}\tau$.

(e) Compute the MLEs of $(\alpha,\ \theta_1,\ \theta_2)$ based on $T_1, T_2, \ldots, T_{n_1}, T_{n_1+1}, \ldots, T_r$, and the censoring scheme $R = (R_1,\ R_2,\ \cdots,\ R_r)$, to obtain $\widehat{\alpha},\ \widehat{\theta}_1$and $\widehat{\theta}_2$.

Differentiating the log-likelihood function in (4.3.6) with respect to $\alpha,\ \theta_1,$ and θ_2, we obtain the following likelihood equations which need to be solved for finding the MLEs of $\alpha,\ \theta_1,$ and θ_2:

$$\begin{aligned}\frac{\partial l(\alpha,\theta_1,\theta_2|\mathbf{t})}{\partial\alpha} &= -r\Psi(\alpha) - n_1\ln\theta_1 - n_2\ln\theta_2 + \sum_{i=1}^{n_1}\ln t_i + \sum_{n_1+1}^{r}\ln y_i \\ &+ \sum_{i=1}^{n_1}\frac{R_i\left[\Psi(\alpha)IG_{\frac{t_i}{\theta_1}}(\alpha) - \frac{1}{\Gamma(\alpha)}\int_0^{\frac{t_i}{\theta_1}} u^{\alpha-1}\ln(u)e^{-u}du\right]}{(1 - IG_{\frac{t_i}{\theta_1}}(\alpha))} \\ &+ \sum_{i=n_1+1}^{r}\frac{R_i\left[\Psi(\alpha)IG_{\frac{y_i}{\theta_2}}(\alpha) - \frac{1}{\Gamma(\alpha)}\int_0^{\frac{y_i}{\theta_2}} u^{\alpha-1}\ln(u)e^{-u}du\right]}{(1 - IG_{\frac{y_i}{\theta_2}}(\alpha))} = 0,\end{aligned} \tag{4.3.7}$$

$$\begin{aligned}\frac{\partial l(\alpha,\theta_1,\theta_2|\mathbf{t})}{\partial\theta_1} &= -\frac{\alpha n_1}{\theta_1} + \frac{(r-n_1)\tau}{\theta_1^2} - \frac{(\alpha-1)\theta_2\tau}{\theta_1^2}\sum_{n_1+1}^{r}\frac{1}{y_i} + \sum_{i=1}^{n_1}\frac{t_i}{\theta_1^2} \\ &+ \sum_{i=1}^{n_1}\frac{R_i(t_i/\theta_1)^{\alpha-1}e^{-t_i/\theta_1}t_i}{(1 - IG_{\frac{t_i}{\theta_1}}(\alpha))\theta_1^2\Gamma(\alpha)} \\ &+ \sum_{i=n_1+1}^{r}\frac{R_i\tau(y_i/\theta_2)^{\alpha-1}e^{-y_i/\theta_2}}{(1 - IG_{\frac{y_i}{\theta_2}}(\alpha))\theta_1^2\Gamma(\alpha)} = 0,\end{aligned} \tag{4.3.8}$$

$$\frac{\partial l(\alpha,\theta_1,\theta_2|\mathbf{t})}{\partial \theta_2} = -\frac{\alpha n_2}{\theta_2} + \frac{1}{\theta_2^2}\sum_{i=n_1+1}^{r}(t_i-\tau) + \frac{(\alpha-1)\tau}{\theta_1}\sum_{i=n_1+1}^{r}\frac{1}{y_i}$$
$$+ \sum_{i=n_1+1}^{r}\frac{R_i(t_i-\tau)(y_i/\theta_2)^{\alpha-1}e^{-y_i/\theta_2}}{\left[1-IG_{y_i/\theta_2}(\alpha)\right]\Gamma(\alpha)\theta_2^2} = 0, \tag{4.3.9}$$

where $y_i = \ t_i - \tau + \frac{\theta_2}{\theta_1}\tau$ and $\Psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$.

4.4 Confidence Intervals

In this section, we present two different methods for constructing confidence intervals (CI) for the unknown parameters α, θ_1 and θ_2. The first method uses the asymptotic distributions of the MLEs to obtain approximate CIs for α, θ_1 and θ_2. The second method is based on a parametric bootstrap method.

4.4.1 Approximate Confidence Intervals

In this subsection, we present an approximate method which is easy to compute and provides good coverage probabilities for large sample sizes. Elements of Fisher information matrix of α, θ_1 and θ_2 were found numerically. Then, the asymptotic normality of MLEs is used to construct approximate confidence intervals for α, θ_1 and θ_2. Let I $(\alpha,\theta_1,\theta_2) = [\mathrm{I}_{ij}(\alpha,\theta_1,\theta_2)]$, for $i,\ j = 1,\ 2,\ 3$, denote the observed Fisher information matrix of α, θ_1 and θ_2, where

$$I_{ij}(\alpha,\theta_1,\theta_2) = -(\nabla^2 l(\alpha,\theta_1,\theta_2)). \tag{4.4.1}$$

Thus, then the observed Fisher information matrix (I) is given by

$$I = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}, \tag{4.4.2}$$

where

$$\begin{aligned} I_{11} &= \frac{\partial^2 l(\alpha, \theta_1, \theta_2)}{\partial \alpha^2} \\ &= -r\Psi^{'}(\alpha) + \sum_{i=1}^{n_1} \frac{R_i}{D_1^2}[D_1\{2\Psi(\alpha)\ B_1(t_i/\theta_1) - B_2(t_i/\theta_1) + (\Psi^{'}(\alpha) \\ &- \Psi^2(\alpha))IG_{t_i/\theta_1}(\alpha)\} - \{\Psi(\alpha)IG_{t_i/\theta_1}(\alpha) - B_1(t_i/\theta_1)\}^2] + \sum_{i=n_1+1}^{r} \frac{R_i}{D_2^2} \\ &\times [D_2\{2\Psi(\alpha)B_1(y_i/\theta_2) - B_2(y_i/\theta_2) + (\Psi^{'}(\alpha) - \Psi^2(\alpha))IG_{y_i/\theta_2}(\alpha)\} \\ &- \{\Psi(\alpha)IG_{y_i/\theta_2}(\alpha) - B_1(y_i/\theta_2)\}^2], \end{aligned} \tag{4.4.3}$$

$$\begin{aligned} I_{12} &= \frac{\partial^2 l(\alpha, \theta_1, \theta_2)}{\partial \alpha \partial \theta_1} \\ &= \frac{-n_1}{\theta_1} - \frac{\theta_2 \tau}{\theta_1^2} \sum_{i=n_1+1}^{r} \frac{1}{y_i} + \sum_{i=1}^{n_1} \frac{R_i (t_i/\theta_1)^{\alpha-1} e^{-t_i/\theta_1} t_i}{\theta_1^2 D_1^2 \Gamma(\alpha)} [D_1 \ln(t_i/\theta_1) \\ &- \Psi(\alpha) + B_1(t_i/\theta_1)] + \sum_{i=n_1+1}^{r} \frac{R_i \tau (y_i/\theta_2)^{\alpha-1} e^{-y_i/\theta_2}}{\theta_1^2 D_2^2 \Gamma(\alpha)} [D_2 \ln(y_i/\theta_2) \\ &- \Psi(\alpha) + B_1(y_i/\theta_2)], \end{aligned} \tag{4.4.4}$$

$$
\begin{aligned}
I_{13} &= \frac{\partial^2 l(\alpha,\theta_1,\theta_2)}{\partial\alpha\partial\theta_2} \\
&= -\frac{n_2}{\theta_2} + \frac{\tau}{\theta_1}\sum_{i=n_1+1}^{r}\frac{1}{y_i} + \sum_{i=n_1+1}^{r}\frac{R_i(t_i-\tau)(y_i/\theta_2)^{\alpha-1}e^{-y_i/\theta_2}}{\theta_2 D_2^2\Gamma(\alpha)} \\
&\times [D_2\ln(y_i/\theta_2) - \Psi(\alpha) + B_1(y_i/\theta_2)], \qquad (4.4.5)
\end{aligned}
$$

$$
\begin{aligned}
I_{22} &= \frac{\partial^2 l(\alpha,\theta_1,\theta_2)}{\partial\theta_1^2} \\
&= \frac{\alpha n_1}{\theta_1^2} - \frac{2}{\theta_1^3}\sum_{i=1}^{n_1} t_i - \frac{2(r-n_1)\tau}{\theta_1^3} + 2(\alpha-1)\frac{\theta_2\tau}{\theta_1^3}\sum_{i=n_1+1}^{r}\frac{1}{y_i^2}(t_i-\tau+\frac{\theta_2}{2\theta_1}\tau) \\
&+ \sum_{i=1}^{n_1}\frac{R_i(t_i/\theta_1)^{\alpha-1}e^{-t_i/\theta_1}t_i}{\theta_1^4\Gamma(\alpha)D_1^2}[D_1\{(1-(t_i/\theta_1)^{-1}(\alpha-1))t_i - 2\theta_1\} - \frac{1}{\Gamma(\alpha)} \\
&\times t_i(t_i/\theta_1)^{\alpha-1}e^{-t_i/\theta_1}] + \sum_{i=n_1+1}^{r}\frac{R_i\tau(y_i/\theta_2)^{\alpha-1}e^{-y_i/\theta_2}}{\theta_1^4\Gamma(\alpha)D_2^2}[D_2\{(1-(y_i/\theta_2)^{-1} \\
&\times (\alpha-1))\tau - 2\theta_1\} - \frac{\tau(y_i/\theta_2)^{\alpha-1}e^{-y_i/\theta_2}}{\Gamma(\alpha)}], \qquad (4.4.6)
\end{aligned}
$$

$$
\begin{aligned}
I_{23} &= \frac{\partial^2 l(\alpha,\theta_1,\theta_2)}{\partial\theta_1\partial\theta_2} \\
&= -\frac{(\alpha-1)\tau}{\theta_1^2}\sum_{i=n_1+1}^{r}\frac{t_i-\tau}{y_i^2} + \sum_{i=n_1+1}^{r}\frac{R_i\tau(t_i-\tau)(y_i/\theta_2)^{\alpha-1}e^{-y_i/\theta_2}}{\theta_1^2\theta_2^2 D_2^2\Gamma(\alpha)} \\
&\times \left[D_2(1-(y_i/\theta_2)^{-1}(\alpha-1)) - \frac{(y_i/\theta_2)^{\alpha-1}e^{-y_i/\theta_2}}{\Gamma(\alpha)}\right], \qquad (4.4.7)
\end{aligned}
$$

and

$$
\begin{aligned}
I_{33} &= \frac{\partial^2 l(\alpha, \theta_1, \theta_2)}{\partial \theta_2^2} \\
&= \frac{\alpha n_2}{\theta_2^2} - \frac{2}{\theta_2^3} \sum_{i=n_1+1}^{r} (t_i - \tau) - (\alpha - 1) \frac{\tau^2}{\theta_1^2} \sum_{i=n_1+1}^{r} \frac{1}{y_i^2} + \sum_{i=n_1+1}^{r} \frac{1}{\theta_2^4 D_2^2 \Gamma(\alpha)} \\
&\times R_i (t_i - \tau)(y_i/\theta_2)^{\alpha-1} e^{-y_i/\theta_2} [D_2 \{(1 - (y_i/\theta_2)^{-1}(\alpha - 1))(t_i - \tau) \\
&- 2\theta_2\} - \frac{1}{\Gamma(\alpha)} (t_i - \tau)(y_i/\theta_2)^{\alpha-1} e^{-y_i/\theta_2}],
\end{aligned} \tag{4.4.8}
$$

where

$$
\begin{aligned}
D_1 &= 1 - IG_{\frac{t_i}{\theta_1}}(\alpha), \\
D_2 &= 1 - IG_{\frac{y_i}{\theta_2}}(\alpha), \\
B_1(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t u^{\alpha-1} (\ln u) e^{-u} du, \\
B_2(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t u^{\alpha-1} \ (\ln u)^2 e^{-u} du.
\end{aligned}
$$

It is known that $I_{21} = I_{12}$, $I_{31} = I_{13}$ and $I_{32} = I_{23}$. Now, the variances and covariances of $\widehat{\alpha}$, $\widehat{\theta}_1$ and $\widehat{\theta}_2$ can be obtained through the observed Fisher information matrix as

$$
Var \begin{bmatrix} \hat{\alpha} \\ \hat{\theta}_1 \\ \hat{\theta}_2 \end{bmatrix} = (I)^{-1} = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix}. \tag{4.4.9}
$$

The asymptotic distribution of the maximum likelihood estimates are then given by $\frac{\widehat{\alpha}-\alpha}{\sqrt{V_{11}}} \sim N(0,1)$, $\frac{\widehat{\theta}_1-\theta_1}{\sqrt{V_{22}}} \sim N\,(0,1)$ and $\frac{\widehat{\theta}_2-\theta_2}{\sqrt{V_{33}}} \sim N\,(0,1)$, which can be used to construct $100\,(1-\alpha)\,\%$ confidence interval for α, θ_1 and θ_2, respectively. These confidence intervals are given by

$$\hat{\alpha} \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{11}}, \tag{4.4.10}$$

$$\hat{\theta}_1 \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{22}} \tag{4.4.11}$$

and

$$\hat{\theta}_2 \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{33}}, \tag{4.4.12}$$

where z_q is the q-th upper percentile of the standard normal distribution.

4.4.2 Bootstrap Confidence Intervals

Confidence intervals based on the parametric bootstrap sampling can be constructed. The following are the steps to generate the bootstrap confidence intervals:

(a) Using the algorithm in Section 4.3 and based on $T_1, T_2, \ldots, T_{n_1}, T_{n_1+1}, \ldots, T_r$, and the censoring scheme $R_1, R_2, \ldots, R_r$, we obtain the MLEs $\widehat{\alpha}$, $\widehat{\theta}_1$ and $\widehat{\theta}_2$.

(b) The first r order statistics $U_1, U_2, \ldots, U_r$ from a sample of size n from uniform (0,1) distribution are simulated.

(c) Find n_1 such that $U_{n_1} \leq F_1^*(\tau) \leq U_{n_1+1}$, where $F_1^*\,(\tau) = \int_0^{\frac{\tau}{\widehat{\theta}_1}} \frac{1}{\Gamma(\widehat{\alpha})} x^{\widehat{\alpha}-1} e^{-x} dx.$

(d) For $1 \leq i \leq n_1$, $T_i = \widehat{\theta}_1 F^{*-1}(U_i)$, and for $n_1 + 1 \leq i \leq r$,
$T_i = \widehat{\theta}_2 F^{*-1}(U_i) + \tau - \frac{\widehat{\theta}_2}{\widehat{\theta}_1}\tau$, where $F^*(t) = \int_0^t \frac{1}{\Gamma(\widehat{\alpha})} x^{\widehat{\alpha}-1} e^{-x} dx$.

(e) Compute the MLEs of α, θ_1 and θ_2 based on $T_1, T_2, \ldots, T_{n_1}, T_{n_1+1}, \ldots, T_r$, say $\widehat{\alpha}^1$, $\widehat{\theta}_1^1$ and $\widehat{\theta}_2^2$.

(f) Repeat steps (b)-(e) B times to obtain B sets of MLEs of α, θ_1 and θ_2 .

A two-sided $100(1-\alpha)\%$ bootstrap confidence interval of α, θ_1 and θ_2 are then given by

$$CI_\alpha = [\hat{\alpha} - z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\alpha}}}, \hat{\alpha} + z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\alpha}}}], \tag{4.4.13}$$

$$CI_{\theta_1} = [\hat{\theta}_1 - z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta}_1}}, \hat{\theta}_1 + z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta}_1}}] \tag{4.4.14}$$

and

$$CI_{\theta_2} = [\hat{\theta}_2 - z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta}_2}}, \hat{\theta}_2 + z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta}_2}}], \tag{4.4.15}$$

where $MSE_a = \text{var}(a) + (\text{bias}(a))^2$, and $\text{bias}(a) = \overline{a} - a$. The performance of the approximate confidence intervals and the bootstrap confidence intervals are evaluated using a simulation study in the next section followed by an illustrative example.

4.5 Simulation Study

A simulation study was carried out for different values of τ and different censoring schemes. The results are presented in Tables 4.1 to 4.6 and they are based on an average over 1000 replications.

Here, we consider different values of τ for different censoring schemes. It has been proven that the optimal censoring schemes for specific family of distribution including the exponential family is when $(R_1, R_2, \ldots, R_r) = (n - r, 0, \ldots, 0)$ or $(R_1, R_2, \ldots, R_r) = (0, 0, \ldots, n - r)$, (see Burkschat (2006)). Taking $(R_1, R_2, \ldots, R_r) = (0, 0, \ldots, n-r)$ means that all the un-failed units will be censored when reaching r failures in the test, which is the step-stress Type-II censoring model. We consider these two censoring schemes with some other cases. In Tables 4.1 and 4.3, we see that for different censoring schemes we got different failure probabilities. Looking at the same τ value at each censoring scheme shows how the failures are distributed in the test. We also see that as τ increases the failure probabilities occurring in the first stress level increase as well, which means the more information there will be about θ_1 resulting in better inference about it.

In Tables 4.2 and 4.4, we can see that the MSEs of both $\widehat{\alpha}$ and $\widehat{\theta_2}$ are less than those of $\widehat{\theta_1}$. The MSEs of the three estimates $\widehat{\alpha}, \widehat{\theta_1}$ and $\widehat{\theta_2}$ are smaller when n is larger (see Table 4.4). We observe that as τ increases the MSEs for both $\widehat{\alpha}$ and $\widehat{\theta_1}$ decrease. The MSEs for $\widehat{\theta_2}$, on the other hand, are larger for large values of τ, that is because for larger values of τ there will be less information about θ_2.

It is hard to decide on which censoring scheme is the best, so we fix the failure probabilities in which there are the same number of failures occurring on each interval. This results in different values of τ that are used to compare between each censoring scheme. In Table 4.2, at the censoring scheme $(24 * 0, 15)$, when $\tau = 3$, the failure probabilities are 48.69% and 51.31% in the intervals $[0, 3]$ and $[3, \infty]$ respectively. But for the censoring scheme $(15, 24 * 0)$, the failure probabilities are approximately the same as those for the previously mentioned censoring scheme when $\tau = 4$. By comparing the values of the MSEs of the estimates $\widehat{\alpha}, \widehat{\theta_1}$ and $\widehat{\theta_2}$ for the previously mentioned τ values for each censoring scheme, we conclude that when $n = 40$ (see Table 4.2) the best censoring scheme for both α and θ_1 is given by : $(8, 7, 23*0)$. But for θ_2 it is given by: $(15, 24*0)$. In Table 4.4, when $n = 100$ the best censoring scheme for both α and θ_2 is given by:$(30, 69 * 0)$. But for θ_1 it is given by:$(15, 15, 68 * 0)$.

In Tables 4.2 and 4.4, we see that for α the estimated coverage probabilities obtained using the bootstrap and the approximate method depend on the censoring scheme. For example, in Table 4.2 when the censoring scheme is $(15, 24*0)$ the coverage probabilities obtained using both methods are close to the nominal levels. On the other hand, when the censoring scheme is $(11 * 0, 7, 1, 7, 11 * 0)$ the coverage probabilities obtained using the bootstrap method are closer to the nominal levels. For θ_1 and θ_2 we can see that the coverage probabilities using the bootstrap method are more closer to the nominal levels than those obtained using the approximate method for each censoring scheme. We therefore recommend the bootstrap method for the construction of confidence intervals for the parameters of the model considered.

4.5.1 Illustrative Example

In this subsection, we consider the data generated with $n = 40,\ r = 25,\ \alpha = 2,\ \theta_1 = e^1 = 2.718282,\ \theta_2 = e^{0.5} = 1.648721$ and $\tau = 4$. The simulated data are given in Table 4.5.

We consider here the same data set with two different censoring schemes $R_1 =$

Table 4.1: Conditional failure probabilities for the step-stress progressive test under Type-II censoring when $\alpha = 2,\ \theta_1 = e^1,\ \theta_2 = e^{.5},\ n = 40$ and $r = 25$.

		Conditional failure probabilities (in %)	
Cens. Sche.	τ	$0 < t < \tau$	$\tau < t < \infty$
(24*0,15)	2	26.65	73.44
	3	48.69	51.31
	4	68.99	31.01
(15,24*0)	2	18.15	81.85
	3	31.41	68.59
	4	44.30	55.70
	5	55.18	44.82
(23*0,7,8)	2	26.83	73.17
	3	48.41	51.59
	4	68.72	31.28
(8,7,23*0)	2	27.13	72.87
	3	31.98	68.02
	4	45.14	54.86
	5	56.56	43.44
(11*0,7,1,7,11*0)	3	46.87	53.13
	4	60.03	39.97
	5	68.72	31.28
(9*0,3,2,2,1,2,2,3,9*0)	2	26.56	73.44
	3	45.88	54.12
	4	60.12	39.88
	5	69.80	30.20

$(8, 7, 0 * 23)$ and $R_2 = (11 * 0, 7, 1, 7, 0 * 11)$. The respective MLEs of $\alpha,\ \theta_1$ and θ_2 and their corresponding standard errors are calculated and are given in Tables 4.6 and 4.8. From these tables, we can see that the standard error of $\widehat{\alpha}$ decreased when we change the censoring scheme from R_1 to R_2, while the standard errors for both $\widehat{\theta_1}$ and $\widehat{\theta_2}$ increase.

The confidence intervals for $\alpha,\ \theta_1$ and θ_2 obtained by the approximate method and the bootstrap method are given in Tables 4.7 and 4.9. In Table 4.7, we see that the approximate confidence interval do not cover the true value of α and θ_2. We can also see that the bootstrap confidence interval for θ_1 is narrower than the approximate confidence interval. In Table 4.9, we can see that the approximate confidence intervals for α do not include the true value, while the bootstrap confidence intervals are better in terms of coverage. For θ_1, the approximate confidence interval is narrower than the bootstrap confidence interval. For θ_2, the bootstrap confidence interval is slightly narrower than the approximate confidence interval.

Table 4.2: Estimated coverage probabilities (in %) of confidence intervals for a step-stress model under progressive Type-II censoring for α, θ_1 and θ_2 based on 1000 simulations when $\alpha = 2,\ \theta_1 = e^1,\ \theta_2 = e^{0.5},\ n = 40$ and $r = 25$.

					90% C.I.		95% C.I.		99% C.I.	
	Censoring scheme	τ	Bias	MSE	Approx.	Boot	Approx.	Boot	Approx.	Boot
α	(24*0,15)	2	0.7333	5.2735	94.4	96.7	97.0	99.1	98.1	100
		3	0.3652	1.2232	97.6	92.9	98.9	98.6	99.6	99.8
		4	0.2298	0.6543	99.0	93.7	99.7	97.5	100	99.6
	(15,24*0)	2	0.8053	5.9580	81.2	97.9	86.8	99.3	93.2	100
		3	0.3928	1.6747	89.1	93.3	92.7	97.2	96.7	99.5
		4	0.3299	0.9228	89.0	90.6	93.9	94.7	98.2	98.7
		5	0.2337	0.6304	92.7	92.3	96.7	96.2	99.2	99.5
	(23*0,7,8)	2	0.8452	7.7846	94.0	96.3	96.5	99.0	98.4	100
		3	0.3531	1.2052	97.4	94.2	99.1	98.2	99.8	99.9
		4	0.2524	0.6743	99.3	94.0	99.9	97.5	100	99.6
	(8,7,23*0)	2	0.7486	5.4077	94.6	97.0	96.8	99.2	99.0	100
		3	0.3961	1.9437	87.2	92.5	91.8	97.9	97.1	99.7
		4	0.2802	0.8816	90.0	91.7	94.6	96.5	98.0	99.5
		5	0.2366	0.6618	92.4	92.4	96.3	96.7	99.6	99.3
	(11*0,7,1,7,11*0)	3	0.4395	1.4480	80.6	93.6	84.7	97.1	92.2	99.6
		4	0.3534	0.9418	91.1	90.0	93.5	95.0	97.5	98.7
		5	0.3064	0.7807	93.6	91.9	96.3	95.4	98.9	99.5
	(9*0,3,2,2,1,2,2,3,9*0)	2	0.5992	2.9624	72.1	96.5	78.0	98.5	87.3	100
		3	0.4020	1.3429	80.6	93.4	85.8	97.5	92.1	99.7
		4	0.2715	0.7172	89.1	90.8	93.4	95.8	98.3	98.8
		5	0.2970	0.6564	94.0	92.9	97.8	96.0	99.5	99.6
θ_1	(24*0,15)	2	1.3776	41.2470	72.2	92.3	79.2	96.0	88.0	99.3
		3	0.3105	6.2335	81.5	94.3	88.5	97.9	94.7	99.5
		4	0.1681	2.2429	87.3	96.5	92.2	99.4	97.1	99.7
	(15,24*0)	2	1.7548	58.4256	75.5	93.3	82.5	97.0	89.4	99.2
		3	0.5744	12.5923	85.6	91.1	90.0	94.7	96.2	98.7
		4	0.2205	4.7673	86.9	91.5	92.2	94.5	97.6	98.6
		5	0.1606	2.1183	90.8	91.9	94.7	95.7	98.5	99.3
	(23*0,7,8)	2	1.3392	40.2752	73.4	93.9	79.8	96.8	88.1	99.5
		3	0.4003	6.1278	81.2	93.7	86.7	97.5	93.4	99.6
		4	0.1338	1.9516	86.9	96..8	93.0	99.3	97.2	99.8
	(8,7,23*0)	2	0.9193	29.6945	76.0	92.0	80.5	96.1	89.1	99.5
		3	0.6157	12.8774	83.7	90.4	89.3	94.7	95.8	98.7
		4	0.3349	4.6906	86.6	91	92.2	95.8	96.9	99
		5	0.1186	2.0338	91.2	92.8	94.5	96.4	98.8	99.4
	(11*0,7,1,7,11*0)	3	0.2380	6.0303	80.6	90.5	87.3	94.6	94.4	98.1
		4	0.0271	2.7817	86	89.1	91.3	93.7	96.5	98.2
		5	-0.0149	1.8332	87.8	92.8	92.7	96.4	97.9	99.1
	(9*0,3,2,2,1,2,2,3,9*0)	2	1.1523	31.8881	75.4	92.8	81.1	96.7	90.4	99.1
		3	0.3201	5.5979	81.2	92.8	86.6	96.2	94.9	99.2
		4	0.1539	2.4235	85.4	90.3	91.3	95.3	97.6	98.4
		5	-0.0333	1.5615	87.5	93..9	93.7	96.8	98.0	99.6
θ_2	(24*0,15)	2	-0.0344	0.4499	94.0	95.6	96.7	98.2	99.2	99.8
		3	-0.0238	0.3648	98.1	90.7	99.1	96.3	99.8	99.0
		4	-0.0347	0.4726	99.1	90.5	99.3	94.3	100	98.9
	(15,24*0)	2	-0.0548	0.3301	96.8	93.4	98.3	97.3	99.6	99.8
		3	-0.0307	0.2372	98.4	89.0	99.5	94.5	99.9	99.1
		4	-0.0571	0.2265	99.4	91.2	99.8	96.1	100	99.1
		5	-0.0374	0.2479	99.3	89.6	99.5	94.3	100	98.3
	(23*0,7,8)	2	-0.0402	0.4807	93.8	94.1	95.7	97.9	98.8	99.7
		3	-0.0616	0.3261	98.6	92.1	99.2	95.6	99.8	99.5
		4	-0.0255	0.5766	96.4	90.3	98.2	94.8	99.8	98.8
	(8,7,23*0)	2	-0.0565	0.4091	95.7	93.3	98.3	97.6	99.2	99.2
		3	-0.0495	0.2250	99.2	91.4	99.7	95.6	99.8	98.5
		4	-0.0342	0.2610	99.2	88.9	99.6	94.9	99.9	98.4
		5	-0.0282	0.2735	99.2	90.7	99.9	95.3	100	99.0
	(11*0,7,1,7,11*0)	3	-0.0632	0.2652	99.0	89.0	99.5	94.3	99.9	97.7
		4	-0.0414	0.3023	99.0	88.6	99.3	95.0	99.9	97.9
		5	-0.0468	0.3402	99.0	91.0	99.8	95.1	100	98.6
	(9*0,3,2,2,1,2,2,3,9*0)	2	-0.0502	0.3499	97.4	92.8	98.2	97.0	99.4	99.6
		3	-0.0828	0.2331	99.2	91.4	99.6	95.0	100	98.5
		4	-0.0271	0.3082	99.4	90.4	99.7	94.8	100	98.9
		5	-0.0501	0.3447	99.3	91.0	99.7	95.1	99.9	98.7

Table 4.3: Conditional failure probabilities for the step-stress progressive test under Type-II censoring when $\alpha = 2,\ \theta_1 = e^1,\ \theta_2 = e^{.5},\ n = 100$ and $r = 70$.

		Conditional failure probabilities (in %)	
Cens. Sche.	τ	$0 < t < \tau$	$\tau < t < \infty$
(69*0,30)	2	23.86	76.14
	3	43.37	56.63
	4	61.67	38.33
(30,69*0)	2	17.21	82.79
	3	30.26	69.74
	4	43.28	56.72
	5	54.82	45.18
(68*0,15,15)	2	24.06	75.94
	3	43.07	56.93
	4	62.14	37.86
(15,15,68*0)	2	17.40	82.60
	3	30.63	69.37
	4	43.57	56.43
	5	55.10	44.90
(34*0,15,15,34*0)	3	42.83	57.17
	4	56.78	43.22
	5	65.61	34.39
(33*0,8,7,7,8,33*0)	2	23.56	76.44
	3	43.11	56.89
	4	56.75	43.25
	5	65.98	34.02

Table 4.4: Estimated coverage probabilities (in %) of confidence intervals for a step-stress model under progressive Type-II censoring for $\alpha,\ \theta_1$ and θ_2 based on 1000 simulations when $\alpha = 2,\ \theta_1 = e^1,\ \theta_2 = e^{0.5},\ n = 100$ and $r = 70$.

					90% C.I.		95% C.I.		99% C.I.	
	Censoring scheme	τ	Bias	MSE	Approx.	Boot	Approx.	Boot	Approx.	Boot
α	(69*0,30)	2	0.2180	0.5598	96.3	99.0	98.7	99..6	99.6	100
		3	0.1406	0.2982	98.5	98.6	99.5	99.6	100	100
		4	0.1044	0.2046	99.6	99.5	99.9	99.8	100	100
	(30,69*0)	2	0.1987	0.5750	82.2	88.9	88.7	92.5	95.3	97.5
		3	0.1380	0.3617	86.1	90.3	91.3	94.1	97.0	97.5
		4	0.0737	0.2001	90.4	91.3	95.2	94.9	99.5	98.2
		5	0.0976	0.2074	90.2	91.4	94.5	95.9	98.9	99.2
	(68*0,15,15)	2	0.2104	0.6288	95.1	98.3	97.9	99.3	99.5	100
		3	0.1209	0.2836	98.0	99.1	99.1	99.8	99.8	100
		4	0.0686	0.1641	99.8	99.1	99.9	99.9	100	100
	(15,15,68*0)	2	0.2398	0.7295	79.1	90.8	86.2	93.6	93.8	98.0
		3	0.1450	0.3842	84.8	89.8	90.6	94.5	96.4	97.7
		4	0.0874	0.2240	91.0	89.7	95.2	93.9	98.3	98.6
		5	0.0943	0.1837	90.5	92.9	95.2	97.0	99.8	99.4
	(34*0,15,15,34*0)	3	0.1765	0.3397	74.9	82.0	83.4	88.6	91.9	96.2
		4	0.1383	0.2221	91.8	92.6	95.6	96.4	98.9	99.1
		5	0.1153	0.1874	93.8	95.7	96.6	97.4	99.3	99.2
	(33*0,8,7,7,8,33*0)	2	0.2401	0.5946	70.1	85.9	77.7	91.1	87.9	94.8
		3	0.1230	0.2589	77.7	80.4	85.7	87.7	93.4	95.4
		4	0.1250	0.2118	91.9	93.2	95.9	96.6	98.8	98.9
		5	0.1065	0.1853	93.4	94.9	97.8	97.9	99.4	99.9
θ_1	(69*0,30)	2	0.3082	4.3639	75.0	91.6	83.3	95.1	92.4	98.5
		3	0.0940	1.3626	80.4	87.5	88.3	92.9	95.5	98.7
		4	0.0431	0.6835	86.1	88.9	92.6	93.3	97.7	99.2
	(30,69*0)	2	0.5028	6.7184	79.5	89.5	85.7	94.2	92.9	98.1
		3	0.1977	1.7785	82.8	87.9	90.0	94.2	96.0	98.2
		4	0.1481	1.0298	88.1	90.5	93.1	93.9	97.8	98.6
		5	0.0581	0.6842	90.4	91.3	94.7	95.9	98.6	99.3
	(68*0,15,15)	2	0.3944	5.3063	74.1	89.4	82.3	94.0	90.6	97.7
		3	0.1258	1.2835	81.3	89.2	88.3	94.5	96.4	98.7
		4	0.0598	0.6287	88.7	88.1	94.0	94.3	98.4	98.2
	(15,15,68*0)	2	0.4563	5.4316	75.9	91.1	83.3	94.5	90.6	98.9
		3	0.2206	2.1651	81.2	87.7	87.9	93.6	94.8	98.5
		4	0.1202	1.0267	87.5	87.7	93.4	93.9	97.9	98.8
		5	0.0363	0.6236	90.9	92.5	95.4	96.2	98.8	99.3
	(34*0,15,15,34*0)	3	0.0500	1.4805	81.0	88.0	88.0	92.1	95.9	98.2
		4	-0.0394	0.6476	94.5	88.9	97.2	95.7	99.3	98.3
		5	-0.0117	0.5920	89.7	91.1	94.3	94.2	98.1	98.6
	(33*0,8,7,7,8,33*0)	2	0.2932	4.0934	75.7	91.8	83.2	94.1	91.9	97.3
		3	0.1138	1.7159	84.2	86.7	90.2	92.8	96.7	98.3
		4	-0.0028	0.7231	86.1	89.1	91.8	95.1	97.7	98.9
		5	-0.0086	0.5640	87.3	89.4	92.9	94.9	98.6	99.4
θ_2	(69*0,30)	2	-0.0264	0.1430	97.2	91.5	98.7	95.5	99.4	99.4
		3	-0.0212	0.1129	94.2	96.7	96.6	98.6	99.3	99.8
		4	-0.0125	0.1308	99.4	97.6	99.6	99.1	100	99.7
	(30,69*0)	2	-0.0089	0.1207	84.6	89.1	91.4	94.5	97.6	98.8
		3	-0.0230	0.0865	99.3	95..3	99.6	98.3	99.9	99.4
		4	-0.0070	0.0798	97.4	97.0	98.8	98.6	100	99.8
		5	-0.0159	0.1061	99.7	97.3	99.9	98.9	100	99.5
	(68*0,15,15)	2	-0.0123	0.1584	96.0	90.0	98.1	95.5	99.4	98.6
		3	-0.0194	0.1056	95.5	97.4	98.7	98.6	100	99.9
		4	-0.0170	0.1137	97.7	98.2	99.5	99.4	99.9	99.8
	(15,15,68*0)	2	-0.0174	0.1184	97.8	92.1	99.0	96.5	99.8	99.2
		3	-0.0038	0.0924	98.6	96.3	99.6	98.3	99.9	99.8
		4	-0.0073	0.0868	99.3	97.7	99.6	99.1	99.9	99.8
		5	0.0048	0.0958	99.5	98.4	99.9	99.6	100	100
	(34*0,15,15,34*0)	3	-0.0411	0.1024	98.7	96.2	99.6	98.2	100	99.9
		4	-0.0342	0.0977	99.7	97.3	99.8	99.1	100	99.9
		5	-0.0209	0.1033	99.6	98.1	99.9	98.0	100	99.6
	(33*0,8,7,7,8,33*0)	2	-0.0340	0.1225	97.8	92.4	98.8	96.6	99.8	99.1
		3	-0.0154	0.0929	99.2	97.3	99.9	99.4	100	99.8
		4	-0.0210	0.0935	97.7	95.6	99.4	96.6	100	99.9
		5	-0.0016	0.1236	99.5	98.5	99.7	99.0	100	99.7

Table 4.5: Simulated data for the illustrative example.

Stress level	Failure times								
$\theta_1 = e^1$	0.805	0.918	1.574	2.428	2.714	2.889	3.006	3.328	3.783
$\theta_2 = e^{.5}$	4.443	4.455	4.501	4.601	4.787	4.884	5.266	5.566	5.803
	6.106	7.700	8.208	8.836	10.067	10.859	16.581		

Table 4.6: The MLEs of α, θ_1 and θ_2 and their standard errors with censoring scheme R_1.

r	n_2	$\widehat{\alpha}$	Se($\widehat{\alpha}$)	$\widehat{\theta_1}$	Se($\widehat{\theta_1}$)	$\widehat{\theta_2}$	Se($\widehat{\theta_2}$)
25	16	2.2201	0.1072	2.7619	0.2062	2.0439	0.0826

Table 4.7: Interval estimation for the simulated data presented in Table 4.5 when the censoring scheme is R_1.

	C.I. for α		
Method	90%	95%	99%
Approx C.I.	(2.0437, 2.3965)	(2.0099, 2.4303)	(1.9439, 2.4964)
Bootstrap C.I.	(1.5318, 2.9085)	(1.3999, 3.0404)	(1.1422, 3.2981)
	C.I. for θ_1		
Approx C.I.	(2.4227, 3.1010)	(2.3578, 3.1660)	(2.2308, 3.2930)
Bootstrap C.I.	(1.6802, 3.8436)	(1.4730, 4.0508)	(1.0680, 4.4558)
	C.I. for θ_2		
Approx C.I.	(1.9080, 2.1799)	(1.8819, 2.2059)	(1.8310, 2.2568)
Bootstrap C.I.	(1.5497, 2.5382)	(1.4550, 2.6329)	(1.2700, 2.8179)

Table 4.8: The MLEs of α, θ_1 and θ_2 and their standard errors with censoring scheme R_2.

r	n_2	$\widehat{\alpha}$	Se($\widehat{\alpha}$)	$\widehat{\theta_1}$	Se($\widehat{\theta_1}$)	$\widehat{\theta_2}$	Se($\widehat{\theta_2}$)
25	16	2.2559	0.0773	2.4447	0.2184	1.1252	0.4137

Table 4.9: Interval estimation for the simulated data presented in Table 4.5 when the censoring scheme is R_2.

	C.I. for α		
Method	90%	95%	99%
Approx C.I.	(2.1288, 2.3829)	(2.1045, 2.4073)	(2.0569, 2.4549)
Bootstrap C.I.	(1.5553, 2.9564)	(1.4211, 3.0906)	(1.1588, 3.3529)
	C.I. for θ_1		
Approx C.I.	(2.0854, 2.8041)	(2.0165, 2.8729)	(1.8820, 3.0075)
Bootstrap C.I.	(1.2953, 3.5942)	(1.0750, 3.8144)	(0.6446, 4.2448)
	C.I. for θ_2		
Approx C.I.	(0.5541, 1.6962)	(0.4447, 1.8056)	(0.2309, 2.0194)
Bootstrap C.I.	(0.5806, 1.6697)	(0.4763, 1.7740)	(0.2724, 1.9779)

Chapter 5

Simple Step-Stress Model under Progressive Type-I Censoring

5.1 Introduction

In this chapter, we consider a simple step-stress progressive Type-I censoring model, which was mentioned earlier in section 1.4.4. We assume that the lifetime data follow a gamma distribution. The considered model is discussed in Section 5.2. In Section 5.3, the maximum likelihood function is obtained with the MLEs. After that, the derivation of the confidence intervals for the unknown parameters using the Fisher information matrix with the asymptotic properties of MLEs and the parametric bootstrap method are discussed in Section 5.4. In Section 5.5, we present a simulation study to illustrate the performance of the maximum likelihood estimates and the confidence intervals obtained by the approximate and the bootstrap methods. An illustrative example is also presented.

5.2 Model Description

The step-stress model with progressive Type-I censoring starts with n identical units placed at an initial stress level x_1. Then, at prefixed time τ_1, the number of failed units n_1 are counted and R_1 surviving units are removed from the test; starting from time τ_1, then $n - n_1 - R_1$ non-removed surviving units are put to a higher stress level x_2 and run until time τ_2, at which time the number of failures n_2 are counted and R_2 surviving units are removed from the test, and so on. At time τ_m, the number of failed units n_m are counted and the remaining $R_m = n - \sum_{i=1}^{m-1} n_i - \sum_{i=1}^{m-1} R_i$ surviving units are all removed. If at some point in the life-test not enough units are left for the planned censoring, then we remove all the surviving units at that point and terminate the experiment. Figure 5. 1 depict such an m-step-stress model

In this Chapter, for simplicity we only consider the simple step-stress model under the progressive Type-I censoring. we suppose that the failure time data comes from a cumulative exposure model and that the lifetime distribution in the simple step-stress model at stress levels x_1 and x_2 follow a gamma distribution with common shape parameter α and scale parameters θ_1 and θ_2, respectively. The corresponding cumulative exposure distribution and PDF are given in Eqs. (1.3.5) and (1.3.6), respectively. The observed progressively Type-I censored data is denoted by

$$\{t_{1:n} < \cdots < t_{n_1:n} < \tau_1 \leq t_{n_1+1:n} < \cdots < t_{n_1+n_2:n} < \tau_2\}, \quad (5.2.1)$$

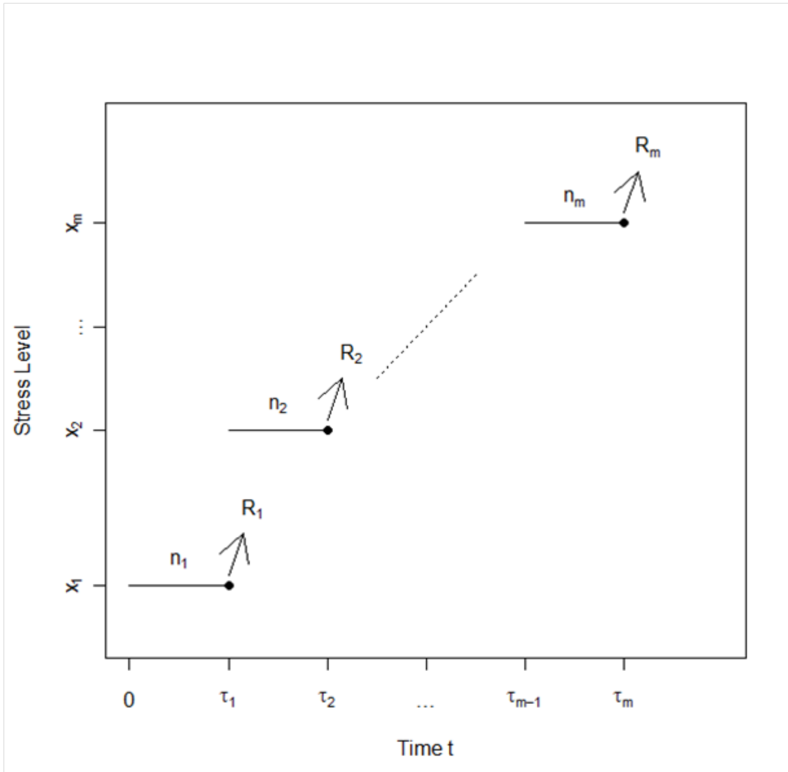


Figure 5.1: m step-stress model under progressive Type-I censoring

5.3 Maximum Likelihood Estimation

The likelihood function is obtained based on the observed progressively Type-I censored data given in (5.2.1), and the MLEs of the unknown parameters α, θ_1 and θ_2 are also obtained. The likelihood function based on the censored data in (5.2.1) [see Balakrishnan and Aggarwala (2000), page 119] can be written as

$$L(\alpha, \theta_1, \theta_2|\mathbf{t}) = C\left\{\prod_{i=1}^{N} g\left(t_{i:n}\right)\right\}\prod_{i=1}^{m}\{1-G\left(\tau_i\right)\}^{R_i},$$

$$0 < t_{1:n} < \cdots < t_{N:n} < \tau_2, \quad (5.3.1)$$

where C is the normalizing constant independent of α, θ_1 and θ_2, $N = n_1 + n_2$, $m = 2$ and $\mathbf{t}$ is the vector of observed progressive Type-I censored data. Using the cumulative exposure model in Eq. (1.3.5) and the corresponding PDF in Eq. (1.3.6), we obtain the likelihood function of α, θ_1 and θ_2 based on the observed progressive Type-I censored sample in (5.2.1) as follows:

(a) If $n_1 = n$ and $n_2 = 0$, the likelihood function in (5.3.1) becomes

$$\begin{aligned} L(\alpha, \theta_1, \theta_2 | t) &= C \left\{ \prod_{i=1}^{n_1} g_1\left(t_{i:n}\right) \right\} \left\{1 - G_1\left(\tau_1\right)\right\}^{R_1} \\ &= C \frac{\left(\prod_{i=1}^{n_1} t_i\right)^{\alpha-1}}{\theta_1^{\alpha n_1} \Gamma(\alpha)^{n_1}} e^{-\frac{1}{\theta_1} \sum_{i=1}^{n_1} t_i} \left[1 - IG_{\frac{\tau_1}{\theta 1}}(\alpha)\right]^{R_1} \\ & 0 < t_{1:n} < \cdots < t_{n_1:n} < \tau_1; \end{aligned} \tag{5.3.2}$$

(b) If $n_1 = 0$ and $n_2 = n$, the likelihood function in (5.3.1) becomes

$$\begin{aligned} L(\alpha, \theta_1, \theta_2 | t) &= C \left\{ \prod_{i=n_1+1}^{N} g_2\left(t_{i:n}\right) \right\} \left\{1 - G_2\left(\tau_2\right)\right\}^{R_2} \\ &= C \frac{\left(\prod_{i=n_1+1}^{N} y_i\right)^{\alpha-1}}{\theta_2^{\alpha n_2} \Gamma(\alpha)^{n_2}} e^{-\frac{1}{\theta_2} \sum_{i=n_1+1}^{n_2} y_i} \left[1 - IG_s(\alpha)\right]^{R_2} \\ & \tau_1 \leq t_{1:n} < \cdots < t_{n_2:n} < \tau_2; \end{aligned} \tag{5.3.3}$$

(c) If $1 \leq n_1 \leq N - 1$, the likelihood function in (5.3.1) becomes

$$
\begin{aligned}
L(\alpha, \theta_1, \theta_2|t) &= C \left\{ \prod_{i=1}^{n_1} g_1\left(t_{i:n}\right) \right\} \left\{ \prod_{i=n_1+1}^{N} g_2\left(t_{i:n}\right) \right\} \left\{1 - G_1\left(\tau_1\right)\right\}^{R_1} \\
&\times \left\{1 - G_2\left(\tau_2\right)\right\}^{R_2} \\
&= C \frac{\left(\prod_{i=1}^{n_1} t_i\right)^{\alpha-1} \left(\prod_{i=n_1+1}^{N} y_i\right)^{\alpha-1}}{\theta_1^{\alpha n_1} \Gamma(\alpha)^N \theta_2^{\alpha n_2}} e^{-\sum_{i=1}^{n_1} \frac{t_i}{\theta_1} - \sum_{i=n_1+1}^{N} \frac{y_i}{\theta_2}} \\
&\times \left[1 - IG_{\frac{\tau_1}{\theta 1}}(\alpha)\right]^{R_1} \left[1 - IG_s(\alpha)\right]^{R_2}, \\
&0 < t_{1:n} < \cdots < t_{n_1:n} < \tau_1 \leq t_{n_1+1:n} < \cdots < t_{N:n} < \tau_2,
\end{aligned} \tag{5.3.4}
$$

where $y_i = \; t_i - \tau_1 + \frac{\theta_2}{\theta_1}\tau_1$ and $s = \frac{1}{\theta_2}[\tau_2 - \tau_1 + \frac{\theta_2}{\theta_1}\tau_1]$.

It can be clearly seen from the likelihood functions in (5.3.2), (5.3.3) and (5.3.4) that the MLE of θ_1 does not exist if $n_1 = 0$, and the MLE of θ_2 does not exist if $n_1 = n$. The MLEs of θ_1 and θ_2 exist only when there are more than one failure occurring at each stress level, i.e. when $n_1 \geq 1$ and $n_2 \geq 1$, and may be obtained by maximizing the corresponding likelihood function in (5.3.4). Maximization of the likelihood function for the parameters cannot be done analytically because there is no closed-form for the incomplete gamma function (IG). Numerically maximizing the likelihood function for the vector of parameter $(\alpha, \theta_1, \theta_2)$ seems to be the only choice. For this purpose, it is convenient to work with the log-likelihood function rather than the likelihood function in (5.3.4), which is given by

$$
\begin{aligned}
l(\alpha,\theta_1,\theta_2|\mathbf{t}) &= \ln L(\alpha,\theta_1,\theta_2|\mathbf{t}) \\
&= \ln(C) - N\ln\Gamma(\alpha) - \alpha n_1 \ln\theta_1 - \alpha n_2 \ln\theta_2 - \sum_{1}^{n_1}\frac{t_i}{\theta_1} \\
&- \sum_{n_1+1}^{N}\frac{y_i}{\theta_2} + (\alpha-1)\sum_{1}^{n_1}\ln t_i + (\alpha-1)\sum_{n_1+1}^{N}\ln y_i \\
&+ R_1\ln(1 - IG_{\frac{\tau_1}{\theta_1}}(\alpha)) + R_2\ln(1 - IG_s(\alpha)) \\
& 0 < t_{1:n} < \cdots < t_{n_1:n} < \tau_1 \le t_{n_1+1:n} < \cdots < t_{N:n} < \tau_2, \qquad (5.3.5)
\end{aligned}
$$

The maximum likelihood estimates must be derived numerically because there is no obvious simplification of the non-linear likelihood equations. Here, numerical likelihood maximization was carried out on the log-likelihood using R software. First, we used the log-likelihood function and started with initial values. Then, the function *optim* in R is used to maximize this log-likelihood function. After that, the estimates are found and their confidence intervals are constructed, using the Hessian matrix. We used the following algorithm to find the MLEs:

(a) Simulate n order statistics from the uniform (0,1) distribution, $U_1, U_2, \ldots, U_n$.

(b) Find n_1 such that $U_{n_1} \le G_1(\tau_1) \le U_{n_1+1}$.

(c) For $i \le n_1$, $T_i = \theta_1 G^{-1}(U_i)$, where $G(t) = \int_0^t \frac{1}{\Gamma(\alpha)} e^{-x} x^{\alpha-1} dx$.

(d) For $n_1 + 1 \le i \le N$, $\quad T_i = \theta_2\ G^{-1}(U_i) + \tau_1 - \frac{\theta_2}{\theta_1}\tau_1$.

(e) Compute the MLEs of $(\alpha,\ \theta_1,\ \theta_2)$ based on $T_1, T_2, \ldots, T_{n_1}, T_{n_1+1}, \ldots, T_N$, and based on removing R_1 and R_2 surviving units from the test at τ_1 and

τ_2, respectively, to obtain $\widehat{\alpha}$, $\widehat{\theta}_1$and $\widehat{\theta}_2$.

Differentiating the log-likelihood function in (5.3.5) with respect to α, θ_1 and θ_2, we obtain the following likelihood equations which need to be solved for finding the MLEs of α, θ_1 and θ_2:

$$\begin{aligned}\frac{\partial l(\alpha,\theta_1,\theta_2|\mathbf{t})}{\partial\alpha} &= -N\Psi(\alpha) - n_1\ln\theta_1 - n_2\ln\theta_2 + \sum_{i=1}^{n_1}\ln t_i + \sum_{i=n_1+1}^{N}\ln y_i \\ &+ \frac{R_1}{(1-IG_{\frac{\tau_1}{\theta_1}}(\alpha))}\left[\Psi(\alpha)IG_{\frac{\tau_1}{\theta_1}}(\alpha) - \frac{1}{\Gamma(\alpha)}\int_0^{\frac{\tau_1}{\theta_1}} u^{\alpha-1}\ln(u)e^{-u}du\right] \\ &+ \frac{R_2}{(1-IG_s(\alpha))}\left[\Psi(\alpha)IG_s(\alpha) - \frac{1}{\Gamma(\alpha)}\int_0^{s} u^{\alpha-1}\ln(u)e^{-u}du\right] = 0,\end{aligned} \tag{5.3.6}$$

$$\begin{aligned}\frac{\partial l(\alpha,\theta_1,\theta_2|\mathbf{t})}{\partial\theta_1} &= -\frac{\alpha n_1}{\theta_1} + \frac{n_2\tau_1}{\theta_1^2} - \frac{(\alpha-1)\theta_2\tau_1}{\theta_1^2}\sum_{i=n_1+1}^{N}\frac{1}{y_i} + \sum_{i=1}^{n_1}\frac{t_i}{\theta_1^2} \\ &+ \frac{R_1(\tau_1/\theta_1)^\alpha e^{-\tau_1/\theta_1}}{(1-IG_{\frac{\tau_1}{\theta_1}}(\alpha))\theta_1\Gamma(\alpha)} + \frac{R_2\tau_1 s^{\alpha-1}e^{-s}}{(1-IG_s(\alpha))\theta_1^2\Gamma(\alpha)} = 0,\end{aligned} \tag{5.3.7}$$

$$\begin{aligned}\frac{\partial l(\alpha,\theta_1,\theta_2|\mathbf{t})}{\partial\theta_2} &= -\frac{\alpha n_2}{\theta_2} + \frac{1}{\theta_2^2}\sum_{i=n_1+1}^{N}(t_i-\tau_1) + \frac{(\alpha-1)\tau_1}{\theta_1}\sum_{i=n_1+1}^{N}\frac{1}{y_i} \\ &+ \frac{R_2(\tau_2-\tau_1)s^{\alpha-1}e^{-s}}{(1-IG_s(\alpha))\Gamma(\alpha)\theta_2^2} = 0,\end{aligned} \tag{5.3.8}$$

where $\Psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$.

5.4 Confidence Intervals

In this section, we present two different methods for constructing confidence intervals (CI) for the unknown parameters α, θ_1 and θ_2. The first method uses the asymptotic distributions of the MLEs to obtain approximate CIs for α, θ_1 and θ_2. The second method is based on a parametric bootstrap method.

5.4.1 Approximate Confidence Intervals

In this subsection, we present an approximate method which provides good probability coverages for large sample sizes and provides easy computation. Elements of Fisher information matrix of α, θ_1 and θ_2 were found numerically. Then, the asymptotic normality of MLEs is used to construct approximate confidence intervals for α, θ_1 and θ_2.

Let I $(\alpha,\theta_1,\theta_2) = [\mathrm{I}_{ij}(\alpha,\theta_1,\theta_2)]$, for $i,\ j = 1,\ 2,\ 3$, denote the observed Fisher information matrix of α, θ_1 and θ_2, where

$$I_{ij}(\alpha,\theta_1,\theta_2) = -(\nabla^2 l(\alpha,\theta_1,\theta_2)).$$

Thus, the observed Fisher information matrix (I) is given by

$$I = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}, \tag{5.4.1}$$

where

$$
\begin{aligned}
I_{11} &= \frac{\partial^2 l(\alpha,\theta_1,\theta_2)}{\partial\alpha^2} \\
&= -N\Psi^{'}(\alpha) + \frac{R_1}{D_1^2}[D_1\{2\Psi(\alpha)B_1(\frac{\tau_1}{\theta_1}) - B_2(\frac{\tau_1}{\theta_1}) + (\Psi^{'}(\alpha) - \Psi^2(\alpha)) \\
&\times IG_{\frac{\tau_1}{\theta_1}}(\alpha)\} - \{\Psi(\alpha)IG_{\frac{\tau_1}{\theta_1}}(\alpha) - B_1(\frac{\tau_1}{\theta_1})\}^2] + \frac{R_2}{D_2^2}[D_2\{2\Psi(\alpha)B_1(s) \\
&- B_2(s) + (\Psi^{'}(\alpha) - \Psi^2(\alpha))IG_s(\alpha)\} - \{\Psi(\alpha)IG_s(\alpha) - B_1(s)\}^2], \qquad (5.4.2)
\end{aligned}
$$

$$
\begin{aligned}
I_{12} &= \frac{\partial^2 l(\alpha,\theta_1,\theta_2)}{\partial\alpha\partial\theta_1} \\
&= \frac{-n_1}{\theta_1} - \frac{\theta_2\tau_1}{\theta_1^2}\sum_{i=n_1+1}^{N}\frac{1}{y_i} + \frac{R_1(\tau_1/\theta_1)^\alpha e^{-\tau_1/\theta_1}}{\theta_1 D_1^2\Gamma(\alpha)}[D_1\ln(\tau_1/\theta_1) \\
&- \Psi(\alpha) + B_1(\tau_1/\theta_1)] + \frac{R_2\tau_1 s^{\alpha-1}e^s}{\theta_1^2 D_2^2\Gamma(\alpha)}[D_2\ln(s) - \Psi(\alpha) + B_1(s)], \qquad (5.4.3)
\end{aligned}
$$

$$
\begin{aligned}
I_{13} &= \frac{\partial^2 l(\alpha,\theta_1,\theta_2)}{\partial\alpha\partial\theta_2} \\
&= -\frac{n_2}{\theta_2} + \frac{\tau_1}{\theta_1}\sum_{i=n_1+1}^{N}\frac{1}{y_i} + \frac{R_2(\tau_2-\tau_1)s^{\alpha-1}e^{-s}}{\theta_2^2 D_2^2\Gamma(\alpha)}[D_2\ln(s) \\
&- \Psi(\alpha) + B_1(s)], \qquad (5.4.4)
\end{aligned}
$$

$$\begin{aligned}
I_{22} &= \frac{\partial^2 l(\alpha, \theta_1, \theta_2)}{\partial \theta_1^2} \\
&= \frac{\alpha n_1}{\theta_1^2} - \frac{2}{\theta_1^3} \sum_{i=1}^{n_1} t_i - \frac{2 n_2 \tau_1}{\theta_1^3} + 2(\alpha - 1) \frac{\theta_2 \tau_1}{\theta_1^3} \sum_{i=n_1+1}^{N} \frac{1}{y_i^2} (t_i - \tau_1 + \frac{\theta_2}{2\theta_1} \tau_1) \\
&+ \frac{R_1 (\tau_1/\theta_1)^{\alpha} e^{-\tau_1/\theta_1}}{\theta_1^3 \Gamma(\alpha) D_1^2} \left[D_1 \{ (1 - (\tau_1/\theta_1)^{-1} (\alpha - 1)) \tau_1 - 2\theta_1 \} \right. \\
&- \left. \frac{\tau_1 (\tau_1/\theta_1)^{\alpha-1} e^{-\tau_1/\theta_1}}{\Gamma(\alpha)} \right] + \frac{R_2 \tau_1 s^{\alpha-1} e^{-s}}{\theta_1^4 \Gamma(\alpha) D_2^2} \left[D_2 \{ (1 - s^{-1}(\alpha - 1)) \tau_1 - 2\theta_1 \} \right. \\
&- \left. \frac{\tau_1 s^{\alpha-1} e^{-s}}{\Gamma(\alpha)} \right], \qquad (5.4.5)
\end{aligned}$$

$$\begin{aligned}
I_{23} &= \frac{\partial^2 l(\alpha, \theta_1, \theta_2)}{\partial \theta_1 \partial \theta_2} \\
&= -\frac{(\alpha - 1)\tau_1}{\theta_1^2} \sum_{i=n_1+1}^{N} \frac{t_i - \tau_1}{y_i^2} + \frac{R_2 \tau_1 (\tau_2 - \tau_1) s^{\alpha-1} e^{-s}}{\theta_1^2 \theta_2^2 D_2^2 \Gamma(\alpha)} \\
&\times \left[D_2 (1 - s^{-1}(\alpha - 1)) - \frac{s^{\alpha-1} e^{-s}}{\Gamma(\alpha)} \right], \qquad (5.4.6)
\end{aligned}$$

and

$$\begin{aligned}
I_{33} &= \frac{\partial^2 l(\alpha, \theta_1, \theta_2)}{\partial \theta_2^2} \\
&= \frac{\alpha n_2}{\theta_2^2} - \frac{2}{\theta_2^3} \sum_{i=n_1+1}^{N} (t_i - \tau_1) - (\alpha - 1) \frac{\tau_1^2}{\theta_1^2} \sum_{i=n_1+1}^{N} \frac{1}{y_i^2} + \frac{R_2 (\tau_2 - \tau_1) s^{\alpha-1} e^{-s}}{\theta_2^4 D_2^2 \Gamma(\alpha)} \\
&\times \left[D_2 \{ (1 - s^{-1}(\alpha - 1))(\tau_2 - \tau_1) - 2\theta_2 \} - \frac{(\tau_2 - \tau_1) s^{\alpha-1} e^{-s}}{\Gamma(\alpha)} \right], \qquad (5.4.7)
\end{aligned}$$

where

$$D_1 = 1 - IG_{\frac{\tau_1}{\theta_1}}(\alpha),$$
$$D_2 = 1 - IG_s(\alpha),$$
$$B_1(t) = \frac{1}{\Gamma(\alpha)} \int_0^t u^{\alpha-1} \, \ln(u) e^{-u} du,$$
$$B_2(t) = \frac{1}{\Gamma(\alpha)} \int_0^t u^{\alpha-1} \, (\ln u)^2 \, e^{-u} du.$$

It is known that $\mathrm{I}_{21} = \mathrm{I}_{12}$, $\mathrm{I}_{31} = \mathrm{I}_{13}$ and $\mathrm{I}_{32} = \mathrm{I}_{23}$. Now, the variances and covariances of $\widehat{\alpha}$, $\widehat{\theta}_1$ and $\widehat{\theta}_2$ can be obtained through the observed Fisher information matrix as

$$Var \begin{bmatrix} \hat{\alpha} \\ \hat{\theta}_1 \\ \hat{\theta}_2 \end{bmatrix} = (I)^{-1} = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix}. \tag{5.4.8}$$

The asymptotic distribution of the maximum likelihood estimates are then given by $\frac{\widehat{\alpha}-\alpha}{\sqrt{V_{11}}} \sim N(0,1)$, $\frac{\widehat{\theta}_1-\theta_1}{\sqrt{V_{22}}} \sim N(0,1)$ and $\frac{\widehat{\theta}_2-\theta_2}{\sqrt{V_{33}}} \sim N(0,1)$, which can be used to construct $100\,(1-\alpha)\,\%$ confidence interval for α, θ_1 and θ_2, respectively. These confidence intervals are given by

$$\hat{\alpha} \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{11}}, \tag{5.4.9}$$

$$\hat{\theta}_1 \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{22}} \tag{5.4.10}$$

and

$$\hat{\theta}_2 \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{33}}, \tag{5.4.11}$$

where z_q is the q-th upper percentile of the standard normal distribution.

5.4.2 Bootstrap Confidence Intervals

Confidence intervals based on the parametric bootstrap sampling can be constructed. The following are the steps to generate the bootstrap confidence intervals:

(a) First, we compute the MLEs of $\alpha,\ \theta_1$ and θ_2 based on $T_1, T_2, \ldots, T_{n_1}$, $T_{n_1+1}, \ldots, T_N$, say $\widehat{\alpha},\ \widehat{\theta}_1$ and $\widehat{\theta}_2$.

(b) The n order statistics $U_1, U_2, \ldots, U_n$ from a sample from uniform (0,1) distribution are simulated.

(c) Find n_1 such that $U_{n_1} \leq F_1^*(\tau_1) \leq U_{n_1+1}$, where $F_1^*(\tau_1) = \int_0^{\frac{\tau_1}{\widehat{\theta}_1}} \frac{1}{\Gamma(\widehat{\alpha})} x^{\widehat{\alpha}-1} e^{-x} dx$.

(d) For $1 \leq i \leq n_1$, $T_i = \widehat{\theta}_1 F^{*-1}(U_i)$, and for $n_1 + 1 = i = N$, $T_i = \widehat{\theta}_2 F^{*-1}(U_i) + \tau_1 - \frac{\widehat{\theta}_2}{\widehat{\theta}_1}\tau_1$, where $F^*(t) = \int_0^t \frac{1}{\Gamma(\widehat{\alpha})} x^{\widehat{\alpha}-1} e^{-x} dx$.

(e) Compute the MLEs of $\alpha,\ \theta_1$ and θ_2 based on $T_1, T_2, \ldots, T_{n_1}, T_{n_1+1}$, $\ldots, T_N$, say $\widehat{\alpha}^{(1)},\ \widehat{\theta}_1^{(1)}$ and $\widehat{\theta}_2^{(1)}$.

(f) Repeat steps (b)-(e) B times to obtain B sets of MLEs of $\alpha,\ \theta_1$ and θ_2 .

A two-sided $100(1-\alpha)\%$ bootstrap confidence interval of $\alpha,\ \theta_1$ and θ_2 are then given by

$$CI_{\alpha} = [\hat{\alpha} - z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\alpha}}}, \hat{\alpha} + z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\alpha}}}], \tag{5.4.12}$$

$$CI_{\theta_1} = [\hat{\theta}_1 - z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta_1}}}, \hat{\theta}_1 + z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta_1}}}] \tag{5.4.13}$$

and

$$CI_{\theta_2} = [\hat{\theta}_2 - z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta_2}}}, \hat{\theta}_2 + z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta_2}}}], \tag{5.4.14}$$

where $MSE_a = \text{var}\,(a) + (\text{bias}(a))^2$, and $\text{bias}\,(a) = \overline{a} - a$. The performance of the approximate confidence intervals and the bootstrap confidence intervals are evaluated using a simulation study in the next section followed by an illustrative example.

5.5 Simulation Study

A simulation study was carried out for different values of τ_1 and τ_2. We considered a modified progressive censoring scheme that was presented by Balakrishnan and Han (2008). The results are presented in Tables 5.1 to 5.6 and they are based on an average over 1000 replications.

In the progressive Type-I model a fixed proportion of surviving units are censored at the end of each stress level. we start with a vector of proportions $p^* = (p_1, p_2)$ where $0 < p_i < 1$ for $i = 1, 2$ and $p_1 + p_2 = 1$. Each p_i defines the proportion of surviving units to be censored at the end of each stress level. The number of censored items is given by:

$$c_i = \text{round}\ [(\text{n} - (\text{n}_1 + \text{n}_2))\ \text{p}_\text{i}],\ \text{for i} = 1, 2.$$

Here, n_1 and n_2 are both random while n and p_i are fixed and in this study we have consider $p_1 = 0.2$ and $p_2 = 0.8$.

In Tables 5.1 and 5.3, we see how the failure units are distributed in our model when different values of τ_1 and τ_2 are taken. We observe that the wider the gap between τ_1 and τ_2 the more failures occurring in this interval. we can also see that as τ_1 increases the failure probabilities in the first interval $[0, \tau_1]$ increase as well. This means that there will be more information about θ_1, which will lead to better inference about θ_1. We also can see that the failure probabilities at the first and second intervals add up to 100%. The reason for that is because as mentioned earlier, we only consider the case when $1 \leq n_1 \leq N-1$, which means that these probabilities are conditional. They were calculated by dividing the number of failures at an interval by the total number of failures at both intervals.

In Tables 5.2 and 5.4, we can see that the MSEs of both $\widehat{\alpha}$ and $\widehat{\theta_2}$ are less than those of $\widehat{\theta_1}$. These MSEs of the three estimates $\widehat{\alpha}, \widehat{\theta_1}$ and $\widehat{\theta_2}$ are smaller when n is larger (see Table 5.4). If we look at the MSEs of $\widehat{\theta_2}$, we see that the wider the gap between τ_1 and τ_2 the smaller the MSE of $\widehat{\theta_2}$. This is expected as explained above. Looking at the MSEs of $\widehat{\theta_1}$, we can see that as τ_1 increases, the MSEs of $\widehat{\theta_1}$ decrease. That is expected, since the larger the value of τ_1, the more information there will be about the parameter θ_1 and hence better inference. We also observe that taking different values of τ_2 do not effect the MSEs of $\widehat{\theta_1}$. This means that no matter when we stop the test the information about θ_1 will depend completely on the value of τ_1.

In Table 5.2, when $n = 40$ the estimated coverage probabilities obtained using the bootstrap method are closer to the nominal levels than those obtained using the approximate method for α, θ_1 and θ_2. The coverage probabilities obtained using the approximate method are unsatisfactory and way above the nominal levels for α and θ_2 but not for θ_1. We can see that for larger values of τ_1, the approximate method gave good coverage probabilities that are close to the nominal levels. In Table 5.4, when n is increased to 100, we can see that the coverage probabilities using the bootstrap method are closer to the nominal levels than those obtained using the approximate method for both α and θ_2. So, it is recommended to use the bootstrap method for the construction of confidence intervals for the parameters of the model considered.

Table 5.1: Conditional failure probabilities for the progressive step-stress test under Type-I censoring when $\alpha = 2,\ \theta_1 = e^1,\ \theta_2 = e^{.5},\ n = 40$ and $p_1 = 0.2$.

		Conditional Failure probabilities (in %)	
τ_1	τ_2	$0 < t < \tau_1$	$\tau_1 < t < \tau_2$
2	7	18.97	81.03
3	4	59.17	40.83
	6	38.36	61.64
	9	31.91	68.09
4	6	58.06	41.94
	9	45.92	54.08
5	6	78.17	21.83
	7	68.13	31.87
6	8	75.74	24.26

5.5.1 Illustrative Example

In this subsection, we consider the data generated with $n = 40,\ \alpha = 2,\ \theta_1 = e^1 = 2.718282,\ \theta_2 = e^{0.5},\ \tau_1 = 5,\ \tau_2 = 8$ and $p_1 = 0.2$. The simulated data are given in Table 5.5.

We consider three different times $\tau_2 = 6,\ 7,\ 8$. The respective MLEs of $\alpha,\ \theta_1$ and θ_2 and their corresponding standard errors are calculated and are given in Table 5.6. It can be seen from this Table that the larger the τ_2, the smaller the standard errors of $\widehat{\alpha}, \widehat{\theta_1}$ and $\widehat{\theta_2}$.

The confidence intervals for $\alpha,\ \theta_1$ and θ_2 obtained by the approximate method and the bootstrap method for different values of τ_2 are given in Table 5.7. In this table, we can see that for α the approximate confidence intervals are slightly narrower than the bootstrap confidence interval for all values of τ_2. We can also see that the approximate confidence intervals for θ_1 do not include the true value, while the bootstrap confidence intervals are better in terms of coverage. For θ_2, the bootstrap confidence interval are better except when $\tau_2 = 8$, in which case the approximate confidence interval is slightly narrower than the bootstrap confidence interval.

Table 5.2: Estimated coverage probabilities (in %) of confidence intervals for a step-stress progressive model under Type-I censoring for α, θ_1 and θ_2 based on 1000 simulations when $\alpha = 2,\ \theta_1 = e^1 = 2.72,\ \theta_2 = e^{.5} = 1.65,\ n = 40$ and $p_1 = 0.2$.

					90% C.I.		95% C.I.		99% C.I.	
	τ_1	τ_2	Bias	MSE	Approx.	Boot	Approx.	Boot	Approx.	Boot
α	2	7	0.7023	4.1958	73.3	90	81.0	95.1	90.4	98.3
	3	4	0.3412	1.2987	97.0	91.7	98.4	95.7	99.7	98.7
		6	0.3503	1.1067	92.0	90	95.8	94.7	98.7	98.2
		9	0.3805	1.0826	66.2	91	74.1	96.0	84.6	99
	4	6	0.2644	0.6979	97.3	90.9	98.9	94.6	99.6	98.4
		9	0.2261	0.6172	79.8	89.8	86.0	95.2	92.2	98.9
	5	6	0.2064	0.5155	98.9	90.7	99.5	95.5	99.9	99.1
		7	0.1986	0.5087	97.6	88.1	99.1	93.9	99.6	98.7
	6	8	0.1951	0.4345	97.9	88	99.2	93.4	99.9	98.5
θ_1	2	7	0.7364	17.6207	74.5	86.9	82.7	92.1	91.5	97.6
	3	4	0.3667	5.2797	83.0	90.1	88.7	94.9	95.0	97.7
		6	0.3341	6.8902	82.8	89.9	87.8	93.8	95.2	98.2
		9	0.1711	4.0882	82.8	90.9	88.5	94.8	95.2	98
	4	6	0.1308	2.2884	85.2	89.6	91.5	94.5	96.9	98
		9	0.1806	2.8785	87.8	89	92.9	95.2	96.9	98.6
	5	6	0.0708	1.3650	88.9	90.2	94.0	95.2	98.1	98.6
		7	0.0598	1.3414	90.3	88.9	93.8	94.1	98.4	98.7
	6	8	0.0216	1.0000	91.0	89.1	95.0	93.2	99.0	98.3
θ_2	2	7	-0.1518	0.2581	97.0	87.6	99.0	92.4	100	97.7
	3	4	-0.1165	0.6396	98.2	84.5	98.8	90.6	99.4	96.6
		6	-0.1481	0.2236	99.2	85.6	99.7	91.5	100	97.6
		9	-0.1011	0.1505	94.6	91.2	97.5	95.4	99.7	98.7
	4	6	-0.1387	0.2812	98.3	83.8	98.8	89.3	99.7	95.7
		9	-0.0878	0.1464	98.1	90	98.8	94.7	99.8	98.6
	5	6	-0.0255	0.9385	97.5	84.2	98.5	89.5	99.5	95.2
		7	-0.1295	0.3021	98.5	84.7	99.4	90.6	99.8	97.1
	6	8	-0.1042	0.5095	98.6	83.5	99.2	90.0	99.9	96.1

Table 5.3: Conditional Failure probabilities for the progressive step-stress test under Type-I censoring when $\alpha = 2,\ \theta_1 = e^1,\ \theta_2 = e^{.5},\ n = 100$ and $p_1 = 0.2$.

		Conditional failure probabilities (in %)	
τ_1	τ_2	$0 < t < \tau_1$	$\tau_1 < t < \tau_2$
2	7	18.82	81.18
3	4	59.32	40.68
	6	38.58	61.42
	9	32.07	67.93
4	6	57.84	42.16
	9	46.28	53.72
5	6	78.26	21.74
	7	67.69	32.31
6	8	75.48	24.52

Table 5.4: Estimated coverage probabilities (in %) of confidence intervals for a step-stress progressive model under Type-I censoring for α, θ_1 and θ_2 based on 1000 simulations when $\alpha = 2,\ \theta_1 = e^1 = 2.72,\ \theta_2 = e^{.5} = 1.65,\ n = 100$ and $p_1 = 0.2$.

					90% C.I.		95% C.I.		99% C.I.	
	τ_1	τ_2	bias	MSE	Approx.	Boot	Approx.	Boot	Approx.	Boot
α	2	7	0.2617	0.5377	76.0	94.9	83.2	96.4	92.5	98
	3	4	0.1122	0.2632	98.7	98	99.6	98.8	99.9	99.4
		6	0.1671	0.3401	92.8	98.1	96.6	98.8	98.4	98.9
		9	0.1407	0.2622	65.3	93.1	72.8	95.9	85.3	97.7
	4	6	0.0746	0.1757	98.2	92.1	99.7	94.5	99.9	97.3
		9	0.0917	0.1871	76.1	90	82.9	92.8	92.2	96.4
	5	6	0.0691	0.1425	99.4	97.3	100	98.1	100	99.1
		7	0.0649	0.1305	98.8	92.3	99.5	95.4	99.9	98.1
	6	8	0.0616	0.1311	97.9	97.5	99.1	98.2	99.9	99
θ_1	2	7	0.1038	2.6138	76.6	70.1	85.3	73.3	93.8	78.7
	3	4	0.1179	1.3645	85.2	84	90.1	89	96.4	94.3
		6	0.0517	1.2989	80.7	73.6	87.5	80.9	95.2	87.4
		9	0.0362	1.0793	84.3	71.3	91.3	76.9	97.0	84.4
	4	6	0.0755	0.7123	86.8	73.4	92.9	80.2	97.9	87.2
		9	0.0511	0.7323	85.7	77.9	92.5	84.4	97.8	91.4
	5	6	0.0247	0.4505	90.3	88.3	95.8	91.6	98.8	95.8
		7	0.0372	0.4340	91.1	79.8	95.2	86.2	98.9	94.2
	6	8	0.0377	0.4014	90.4	76.5	95.2	81	98.5	87.9
θ_2	2	7	-0.1423	0.0984	98.3	96.5	99.5	97.3	100	99.2
	3	4	-0.2053	0.1973	99.1	83.9	99.6	87.4	100	91.5
		6	-0.1624	0.0975	94.8	91.8	97.8	94.8	99.7	98.3
		9	-0.0764	0.0658	96.6	95.7	98.5	97.5	100	99.2
	4	6	-0.1753	0.1119	99.5	97.3	99.9	98.3	100	99.5
		9	-0.0768	0.0634	98.8	99.1	99.4	100	100	100
	5	6	-0.1930	0.2052	99.6	80.9	99.7	83.9	100	87.5
		7	-0.1844	0.1199	99.6	97.3	99.8	99.4	100	99.7
	6	8	-0.1674	0.1374	99.4	88.8	99.8	92.3	100	95.3

Table 5.5: Simulated data for the illustrative example.

Stress level	Failure times									
$\theta_1 = e^1$	0.179	0.955	1.521	1.738	1.753	1.960	2.284	2.293	2.628	2.637
	2.771	2.937	3.037	3.284	4.154	4.217	4.316	4.341	4.477	4.502
$\theta_2 = e^{.5}$	5.105	5.243	5.308	5.394	5.508	5.553	5.606	5.950	6.011	6.122
	6.797	6.889	7.649	7.761						

Table 5.6: The MLEs of α, θ_1 and θ_2 and their standard errors.

τ_2	n_2	$n_1 + n_2$	$\widehat{\alpha}$	Se($\widehat{\alpha}$)	$\widehat{\theta_1}$	Se($\widehat{\theta_1}$)	$\widehat{\theta_2}$	Se($\widehat{\theta_2}$)
6	8	28	1.8484	0.2741	3.2929	0.1350	1.1934	0.7789
7	12	32	1.8356	0.1755	3.3163	0.1319	1.2993	0.3884
8	14	34	1.8151	0.1162	3.3549	0.1291	1.5826	0.1989

Table 5.7: Interval estimation for the simulated data presented in Table 5.5.

		C.I. for α		
τ_2	Method	90%	95%	99%
6	Approx C.I.	(1.3976, 2.2993)	(1.3112, 2.3856)	(1.1424, 2.5545)
	Bootstrap C.I.	(1.3655, 2.3313)	(1.2730, 2.4239)	(1.0921, 2.6047)
7	Approx C.I.	(1.5470, 2.1243)	(1.4917, 2.1795)	(1.3836, 2.2876)
	Bootstrap C.I.	(1.3484, 2.3228)	(1.2550, 2.4162)	(1.0726, 2.5986)
8	Approx C.I.	(1.6240, 2.0061)	(1.5874, 2.0427)	(1.5159, 2.1143)
	Bootstrap C.I.	(1.3568, 2.2734)	(1.2690, 2.3612)	(1.0974, 2.5328)
		C.I. for θ_1		
6	Approx C.I.	(3.0709, 3.5150)	(3.0283, 3.5575)	(2.9452, 3.6407)
	Bootstrap C.I.	(2.6007, 3.9852)	(2.4681, 4.1178)	(2.2089, 4.3770)
7	Approx C.I.	(3.0994, 3.5331)	(3.0579, 3.5746)	(2.9767, 3.6558)
	Bootstrap C.I.	(2.6202, 4.0123)	(2.4869, 4.1456)	(2.2263, 4.4063)
8	Approx C.I.	(3.1426, 3.5673)	(3.1020, 3.6080)	(3.0225, 3.6874)
	Bootstrap C.I.	(2.6944, 4.0155)	(2.5678, 4.1421)	(2.3205, 4.3894)
		C.I. for θ_2		
6	Approx C.I.	(0.0000, 2.4745)	(0.0000, 2.7199)	(0.0000, 3.1996)
	Bootstrap C.I.	(0.5600, 1.8268)	(0.4386, 1.9482)	(0.2014, 2.1854)
7	Approx C.I.	(0.6605, 1.9381)	(0.5381, 2.0604)	(0.2990, 2.2996)
	Bootstrap C.I.	(0.8205, 1.7781)	(0.7287, 1.8698)	(0.5495, 2.0491)
8	Approx C.I.	(1.2555, 1.9096)	(1.1929, 1.9722)	(1.0704, 2.0947)
	Bootstrap C.I.	(1.1198, 2.0453)	(1.0311, 2.1340)	(0.8579, 2.3073)

Chapter 6

Multiple Step-Stress Model under Type-II Censoring

6.1 Introduction

In this chapter, we develop inference for the m-step-stress model under Type-II censoring with gamma distributed lifetimes. In Section 6.2, the considered model is described. The MLEs are obtained in Section 6.3. After numerically evaluating the MLEs, we construct confidence intervals for the unknown parameters by using two methods-the asymptotic method and the parametric bootstrap method-in Section 6.4. In Section 6.5, some simulation results and conclusions are presented. We also present the reduced parameter multiple step-stress model under Type-II censoring in Section 6.6. The MLEs for that model are derived and the confidence intervals are also constructed in Sections 6.6.1 and 6.6.2, respectively. In Section 6.6.3, a simulation study is carried out on the reduced parameter model with an illustrative example. In Section 6.7, we describe some life-stress relationships that can be used in this reduced parameter model.

6.2 Model Description

In this model, we assume that the failure time data come from a cumulative exposure model, and we consider the m-step-stress model with stress levels $x_1, x_2, \cdots, x_m$ under Type-II censoring with gamma distributed lifetimes. The lifetime distribution at stress level x_i, $i=1, 2, \cdots, m$, is gamma distribution with common shape parameter α and scale parameter θ_i.

The multiple step-stress Type-II censoring scheme is a generalization of the Type-II censoring discussed in Chapter 2. We start with n identical units at an initial stress level x_1, and at a fixed time τ the stress level is increased to x_2 and the successive failure times are recorded. Then, at the fixed time 2τ, the stress is increased to x_3, and so on. So, the stress level starts with x_1 and changed to $x_2, x_3, \cdots, x_m$ at fixed times $\tau, 2\tau, \ldots, (m-1)\tau$, respectively. The experiment is terminated when a fixed number of failures r are observed. Let N_k be the number of units that fail between $(k-1)\,\tau$ and $k\tau$ at stress level x_k for $k=1, 2, \cdots, m$. The observed censored sample is given by

$$t_1 < \cdots < t_{N_1} < \tau \leq t_{N_1+1} < \cdots < t_{N_1+N_2} < 2\tau \leq \cdots < \\ (m-1)\tau \leq t_{N_1+\cdots+N_{m-1}+1} < \cdots < t_r. \quad (6.2.1)$$

6.3 Maximum Likelihood Estimation

The likelihood function based on the observed Type-II censored data given in (6.2.1) is presented from which the MLEs of the unknown parameters $\alpha,\ \theta_1,\ \theta_2,$ $\cdots,\ \theta_m$ need to be obtained numerically. The likelihood function of this sample is given by

$$L(\theta|\mathbf{t}) = \frac{n!}{(n-r_m)!} \prod_{k=1}^{m} \left\{ \prod_{i_k=r_{k-1}+1}^{r_k} g_k(t_{i_k}) \right\} \{1 - G_m(t_{r_m})\}^{n-r_m},$$

$$\text{for } t_1 < \cdots < t_{N_1} < \tau \leq t_{N_1+1} < \cdots < t_{N_1+N_2} < 2\tau \leq \cdots < \\ (m-1)\tau \leq t_{N_1+\cdots+N_{m-1}+1} < \cdots < t_r, \quad (6.3.1)$$

where

$$r_0 = 0,$$
$$r_k = \sum_{i=1}^{k} N_i,\ k = 1, 2, \cdots, m-1,$$
$$N_m = r_m - r_{m-1},$$

and $\mathbf{t}$ is the vector of observed failure time data. The MLEs of $\alpha,\ \theta_1,\ \theta_2,\ \cdots,\ \theta_m$ exist only when all N_k'$s > 0$. Since we are considering a gamma lifetime distribution at all stress levels, with common shape parameter α and scale parameters θ_i for distribution F_i, the likelihood distribution function of the m-step-stress model is given by

$$\begin{aligned} L(\alpha, \theta_1, \theta_2, \cdots, \theta_m|\mathbf{t}) = &\frac{n!}{(n-r_m)!}\frac{1}{[\Gamma(\alpha)]^{r_m}}\prod_{k=1}^{m}(\theta_k)^{-N_k} \\ &\times \prod_{k=1}^{m}\left\{\prod_{i_k=r_{k-1}+1}^{r_k}(\zeta_{i_k})^{\alpha-1}\right\} \\ &\times exp\left\{-\sum_{k=1}^{m}\left[\sum_{i_k=r_{k-1}+1}^{r_k}\zeta_{i_k}\right]\right\} \\ &\times \{1 - IG_{\zeta_{r_m}}(\alpha)\}^{n-r_m}, \\ \text{for } t_1 < \cdots < t_{N_1} < \tau \le t_{N_1+1} < \cdots < t_{N_1+N_2} < 2\tau \le \cdots < \\ &(m-1)\tau \le t_{N_1+\cdots+N_{m-1}+1} < \cdots < t_r, \end{aligned} \quad (6.3.2)$$

where

$$\zeta_{i_k} = \frac{t_{i_k} - (k-1)\,\tau}{\theta_k} + \tau \sum_{j=1}^{k-1}\frac{1}{\theta_j}, \quad k = 1,\ 2,\ \cdots,\ m. \quad (6.3.3)$$

As before, it is convenient to work with the log-likelihood function rather than the likelihood function in (6.3.2), which is given by

$$\begin{aligned}
&l(\alpha, \theta_1, \theta_2, \cdots, \theta_m|\mathbf{t}) = \ln\left[\mathrm{L}\left(\alpha, \theta_1, \theta_2, \cdots, \theta_\mathrm{m}|\mathbf{t}\right)\right] \\
&= \ln\left(\frac{n!}{(n-r_m)!}\right) - r_m \ln\left(\Gamma(\alpha)\right) - \sum_{k=1}^{m} N_k \ln\left(\theta_k\right) + (\alpha-1)\sum_{k=1}^{m}\left\{\sum_{i_k=r_{k-1}+1}^{r_k} \ln\left(\zeta_{i_k}\right)\right\} \\
&\qquad - \sum_{k=1}^{m}\left(\sum_{i_k=r_{k-1}+1}^{r_k} \zeta_{i_k}\right) + (n-r_m)\ln\left(1 - IG_{\zeta_{r_m}}(\alpha)\right), \\
&\qquad \text{for } t_1 < \cdots < t_{N_1} < \tau \le t_{N_1+1} < \cdots < t_{N_1+N_2} < 2\tau \le \cdots < \\
&\qquad\qquad (m-1)\tau \le t_{N_1+\cdots+N_{m-1}+1} < \cdots < t_r. \quad (6.3.4)
\end{aligned}$$

Differentiating the log-likelihood function in (6.3.4) with respect to α and θ_k gives likelihood equations for finding the MLEs $\hat{\alpha}$ and $\hat{\theta}_k$. To find these MLEs, we will need the first and second partial derivatives of (6.3.4). The first partial derivatives are given by the following equations:

$$\begin{aligned}
\frac{\partial l(\alpha, \theta_1, \theta_2, \cdots, \theta_m|\mathbf{t})}{\partial \alpha} &= -r_m \psi(\alpha) + \sum_{k=1}^{m}\left\{\sum_{i_k=r_{k-1}+1}^{r_k} \ln\left(\zeta_{\mathrm{i_k}}\right)\right\} \\
&\quad + \frac{(n-r_m)}{\left[1 - IG_{\zeta_{r_m}}(\alpha)\right]}\left[\Psi(\alpha) IG_{\zeta_{r_m}}(\alpha) - B_1\left(\zeta_{r_m}\right)\right], \quad (6.3.5)
\end{aligned}$$

$$\frac{\partial l(\alpha,\theta_1,\theta_2,\cdots,\theta_m|\mathbf{t})}{\partial\theta_j} = -\frac{N_j}{\theta_j} + (\alpha-1)\sum_{k=1}^{m}\left\{\sum_{i_k=r_{k-1}+1}^{r_k}\frac{1}{\zeta_{i_k}}\frac{\partial}{\partial\theta_j}\left(\zeta_{i_k}\right)\right\}$$
$$-\sum_{k=1}^{m}\left(\sum_{i_k=r_{k-1}+1}^{r_k}\frac{\partial}{\partial\theta_j}\zeta_{i_k}\right) + \frac{(n-r_m)\left(\zeta_{r_m}\right)^{\alpha-1}e^{-\zeta_{r_m}}\tau}{\Gamma(\alpha)\theta_j^2\left[1-IG_{\zeta_{r_m}}(\alpha)\right]},$$
$$\text{for } j=1,2,\cdots,\ m-1, \quad (6.3.6)$$

$$\frac{\partial l(\alpha,\theta_1,\theta_2,\cdots,\theta_m|\mathbf{t})}{\partial\theta_m} = -\frac{N_m}{\theta_m} + (\alpha-1)\sum_{k=1}^{m}\left\{\sum_{i_k=r_{k-1}+1}^{r_k}\frac{1}{\zeta_{i_k}}\frac{\partial}{\partial\theta_m}\left(\zeta_{i_k}\right)\right\}$$
$$-\sum_{k=1}^{m}\left(\sum_{i_k=r_{k-1}+1}^{r_k}\frac{\partial}{\partial\theta_m}\zeta_{i_k}\right) + \frac{(n-r_m)\left(\zeta_{r_m}\right)^{\alpha-1}e^{-\zeta_{r_m}}(t_{r_m}-(m-1)\tau)}{\Gamma(\alpha)\theta_m^2\left[1-IG_{\zeta_{r_m}}(\alpha)\right]}, \quad (6.3.7)$$

where $\Psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$ and $B_1(v) = \int_0^v \frac{1}{\Gamma(\alpha)}\ln(u)\ u^{\alpha-1}e^{-u}du$.

The maximum likelihood estimates must be obtained numerically because there is no obvious simplification of the above non-linear likelihood equations. Here, numerical maximization is carried out on the log-likelihood using R software. First, we use the log-likelihood function and start with initial values. Then, the function *optim* in R is used to maximize this log-likelihood function. After that, the estimates are found and their confidence intervals are constructed, using the Hessian matrix. The following is the algorithm used in order to find the MLEs in the case when $m = 3$:

(a) Simulate n order statistics from the uniform (0,1) distribution, $U_1, U_2, \ldots, U_n$.

(b) Find N_1 such that $U_{N_1} \le G_1(\tau_1) \le U_{N_1+1}$ where $G_1(\tau_1) = \int_0^{\frac{\tau_1}{\theta_1}} \frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x}dx$.

(c) For $i \leq N_1$, $T_i = \theta_1 F^{-1}(U_i)$, where $F(t) = \int_0^t \frac{1}{\Gamma(\alpha)} e^{-x} x^{\alpha-1} dx$.

(d) Find N_2 such that $U_{N_1+N_2} \leq G_2(\tau_2) \leq U_{N_1+N_2+1}$ where $G_2(\tau_2) = \int_0^{\frac{\tau_2 - \tau_1 + \frac{\theta_2}{\theta_1}\tau_1}{\theta_2}} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} dx$.

(e) For $N_1 + 1 \leq i \leq N_1 + N_2, \quad T_i = \theta_2 \ F^{-1}(U_i) + \tau_1 - \frac{\theta_2}{\theta_1}\tau_1$.

(f) For $N_1 + N_2 + 1 \leq i \leq r, \quad T_i = \theta_3 \ F^{-1}(U_i) + \tau_2 - \frac{\theta_3}{\theta_2}(\tau_2 - \tau_1 + \frac{\theta_2}{\theta_1}\tau_1)$.

(g) Compute the MLEs of $(\alpha, \ \theta_1, \ \theta_2, \ \theta_3)$ based on $T_1, T_2, \ldots, T_{N_1}, T_{N_1+1}, \ldots, T_{N_1+N_2}, T_{N_1+N_2+1}, \cdots T_r$, say $\widehat{\alpha}, \ \widehat{\theta}_1, \ \widehat{\theta}_2$ and $\widehat{\theta}_3$.

6.4 Confidence Intervals

This section contains two different methods for constructing confidence intervals (CI) for the unknown parameters $\alpha, \ \theta_1, \ \theta_2, \ \cdots, \ \theta_m$. We use the asymptotic distribution of the MLEs to obtain the approximate CIs for $\alpha, \ \theta_1, \ \theta_2, \ \cdots, \ \theta_m$. The second method is based on a parametric bootstrap method.

6.4.1 Approximate Confidence Intervals

We present an approximate method which provides good coverage probabilities for large sample sizes and facilitates easy computation. Elements of Fisher information matrix are found numerically for different values of m, and then the asymptotic normality of the MLEs is used to construct the approximate confidence intervals for $\alpha, \ \theta_1, \ \theta_2, \ \cdots, \ \theta_m$.

Let $I(\alpha, \theta_1, \theta_2, \cdots, \theta_m) = [I_{ij}(\alpha, \theta_1, \theta_2, \cdots, \theta_m)]$, for i, j $= 1, 2, \cdots,$ m, denote the observed Fisher information matrix of $\alpha, \theta_1, \theta_2, \cdots, \theta_m$, where

$$I_{ij}(\alpha, \theta_1, \theta_2, \cdots, \theta_m) = -\left(\nabla^2 l(\alpha, \theta_1, \theta_2, \cdots, \theta_m)\right). \tag{6.4.1}$$

For simplicity, we take $m = 3$, then the observed Fisher information matrix (I) is given by

$$I = \begin{bmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{bmatrix}, \tag{6.4.2}$$

where

$$I_{11} = \frac{\partial^2 l(\alpha, \theta_1, \theta_2, \theta_3)}{\partial \alpha^2} = -r_3 \psi^{'}(\alpha) + \frac{(n - r_3)}{\left(1 - IG_{\zeta_{r_3}}(\alpha)\right)^2}$$
$$\times \left[(1 - IG_{\zeta_{r_3}}(\alpha)) \left[2\psi(\alpha) B_1(\zeta_{r_3}) + IG_{\zeta_{r_3}}(\psi^{'}(\alpha) - \psi^2(\alpha)) - B_2(\zeta_{r_3})\right]\right.$$
$$\left. - \left[\psi(\alpha) IG_{\zeta_{r_3}}(\alpha) - B_1(\zeta_{r_3})\right]^2\right], \tag{6.4.3}$$

$$I_{12} = \frac{\partial^2 l(\alpha, \theta_1, \theta_2, \theta_3)}{\partial \alpha \partial \theta_1} = -\frac{N_1}{\theta_1} - \frac{\tau}{\theta_1^2} \sum_{i=r_1+1}^{r_2} \frac{1}{\zeta_{i_2}} - \frac{\tau}{\theta_1^2} \sum_{i=r_2+1}^{r_3} \frac{1}{\zeta_{i_3}}$$
$$+ \frac{(n - r_3)(\zeta_{r_3})^{\alpha-1} e^{-\zeta_{r_3}} \tau}{\Gamma(\alpha) \theta_1^2 \left(1 - IG_{\zeta_{r_3}}(\alpha)\right)^2} \left[\left(1 - IG_{\zeta_{r_3}}(\alpha)\right) \ln(\zeta_{r_3}) - \psi(\alpha) + \mathrm{B}_1(\zeta_{r_3})\right], \tag{6.4.4}$$

$$
\begin{aligned}
I_{13} = \frac{\partial^2 l(\alpha, \theta_1, \theta_2, \theta_3)}{\partial\alpha \ \partial\theta_2} &= -\frac{1}{\theta_2^2} \sum_{i=r_1+1}^{r_2} \frac{(t_i - \tau)}{\zeta_{i_2}} - \frac{\tau}{\theta_2^2} \sum_{i=r_2+1}^{r_3} \frac{1}{\zeta_{i_3}} \\
&+ \frac{(n-r_3)(\zeta_{r_3})^{\alpha-1} e^{-\zeta_{r_3}} \tau}{\Gamma(\alpha)\theta_2^2 \left(1 - IG_{\zeta_{r_3}}(\alpha)\right)^2} \\
&\left[\left(1 - IG_{\zeta_{r_3}}(\alpha)\right) \ln(\zeta_{\mathrm{r}_3}) - \psi(\alpha) + \mathrm{B}_1(\zeta_{\mathrm{r}_3})\right], \quad (6.4.5)
\end{aligned}
$$

$$
\begin{aligned}
I_{14} = \frac{\partial^2 l(\alpha, \theta_1, \theta_2, \theta_3)}{\partial\alpha \ \partial\theta_3} &= -\frac{1}{\theta_3^2} \sum_{i=r_2+1}^{r_3} \frac{(t_i - 2\tau)}{\zeta_{i_3}} \\
+ \frac{(n-r_3)(\zeta_{r_3})^{\alpha-1} e^{-\zeta_{r_3}} (t_{r_3} - 2\tau)}{\Gamma(\alpha)\theta_3^2 \left(1 - IG_{\zeta_{r_3}}(\alpha)\right)^2} &\left[\left(1 - IG_{\zeta_{r_3}}(\alpha)\right) \ln(\zeta_{\mathrm{r}_3}) - \psi(\alpha) + \mathrm{B}_1(\zeta_{\mathrm{r}_3})\right], \quad (6.4.6)
\end{aligned}
$$

$$
\begin{aligned}
I_{22} = \frac{\partial^2 l(\alpha, \theta_1, \theta_2, \theta_3)}{\partial\theta_1^2} &= \frac{\alpha N_1}{\theta_1^2} + \frac{(\alpha-1)\tau}{\theta_1^4 \ \theta_2} \sum_{i=r_1+1}^{r_2} \frac{2 \ \theta_1 t_i + (\theta_2 - 2\theta_1)\tau}{\zeta_{i_2}^2} \\
&+ \frac{(\alpha-1)\tau}{\theta_1^4 \ \theta_2 \ \theta_3} \sum_{i=r_2+1}^{r_3} \frac{(2\theta_1\theta_2)(t_i - 2\tau) + (\theta_2 + 2\theta_1)\theta_3\tau}{\zeta_{i_3}^2} \\
&- \frac{2}{\theta_1^3} \left[(N_2 + N_3)\tau + \sum_{i=1}^{r_1} t_i\right] + \frac{(n-r_3)(\zeta_{r_3})^{\alpha-1} e^{-\zeta_{r_3}} \tau}{\Gamma(\alpha)\theta_1^4 \left(1 - IG_{\zeta_{r_3}}(\alpha)\right)^2} \\
&[\left(1 - IG_{\zeta_{r_3}}(\alpha)\right) [\left(1 - (\alpha-1)(\zeta_{r_3})^{-1}\right) \tau - 2\theta_1] \\
&- \frac{\tau}{\Gamma(\alpha)} (\zeta_{r_3})^{\alpha-1} e^{-\zeta_{r_3}}], \quad (6.4.7)
\end{aligned}
$$

$$
\begin{aligned}
I_{23} = \frac{\partial^2 l(\alpha,\theta_1,\theta_2,\theta_3)}{\partial\theta_1\partial\theta_2} &= -\frac{(\alpha-1)\tau}{\theta_1^2\theta_2^2}\sum_{i=r_1+1}^{r_2}\frac{t_i-\tau}{(\zeta_{i_2})^2} - \frac{(\alpha-1)\tau^2}{\theta_1^2\theta_2^2}\sum_{i=r_2+1}^{r_3}\frac{1}{\zeta_{i_3}^2} \\
&+ \frac{(n-r_3)(\zeta_{r_3})^{\alpha-1}e^{-\zeta_{r_3}}\tau^2}{\Gamma(\alpha)\theta_1^2\theta_2^2\left(1-IG_{\zeta_{r_3}}(\alpha)\right)^2} \\
&\times\left[\left(1-IG_{\zeta_{r_3}}(\alpha)\right)\left(1-(\alpha-1)(\zeta_{r_3})^{-1}\right) - \frac{1}{\Gamma(\alpha)}(\zeta_{r_3})^{\alpha-1}e^{-\zeta_{r_3}}\right], \quad (6.4.8)
\end{aligned}
$$

$$
\begin{aligned}
I_{24} = \frac{\partial^2 l(\alpha,\theta_1,\theta_2,\theta_3)}{\partial\theta_1\,\partial\theta_3} &= -\frac{(\alpha-1)\tau}{\theta_1^2\theta_3^2}\sum_{i=r_2+1}^{r_3}\frac{t_i-2\tau}{\zeta_{i_3}^2} \\
&+ \frac{(n-r_3)(\zeta_{r_3})^{\alpha-1}e^{-\zeta_{r_3}}(t_r-2\tau)\tau}{\Gamma(\alpha)\theta_1^2\theta_3^2\left(1-IG_{\zeta_{r_3}}(\alpha)\right)^2} \\
&\times\left[\left(1-IG_{\zeta_{r_3}}(\alpha)\right)\left(1-(\alpha-1)(\zeta_{r_3})^{-1}\right) - \frac{1}{\Gamma(\alpha)}(\zeta_{r_3})^{\alpha-1}e^{-\zeta_{r_3}}\right], \quad (6.4.9)
\end{aligned}
$$

$$
\begin{aligned}
I_{33} = \frac{\partial^2 l(\alpha,\theta_1,\theta_2,\theta_3)}{\partial\theta_2^2} &= \frac{N_2}{\theta_2^2} + \frac{2(\alpha-1)}{\theta_3^2}\sum_{i=r_1+1}^{r_2}\frac{1}{(\zeta_{i2})^2} \\
&\left[(t_i-\tau)\left(\frac{(t_i-\tau)}{2\theta_2}+\frac{\tau}{\theta_1}\right)\right] + \frac{2(\alpha-1)\tau}{\theta_2^3}\sum_{i=r_2+1}^{r_3}\frac{1}{(\zeta_{i_3})^2}\left[\frac{(t_i-2\tau)}{\theta_3}+\frac{\tau}{\theta_1}+\frac{\tau}{2\theta_2}\right] \\
&- \frac{2}{\theta_2^3}\left[N_3\tau+\sum_{i=r_1+1}^{r_2}(t_i-\tau)\right] + \frac{(n-r_3)(\zeta_{r_3})^{\alpha-1}e^{-\zeta_{r_3}}\tau}{\Gamma(\alpha)\theta_2^4\left(1-IG_{\zeta_{r_3}}(\alpha)\right)^2} \\
&\times\left[(1-IG_{\zeta_{r_3}}(\alpha))\left\{(1-(\alpha-1)(\zeta_{r_3})^{-1})\tau-2\theta_2\right\} - \frac{\tau}{\Gamma(\alpha)}(\zeta_{r_3})^{\alpha-1}e^{-\zeta_{r_3}}\right],
\end{aligned}
$$

(6.4.10)

$$I_{34} = \frac{\partial^2 l(\alpha,\theta_1,\theta_2,\theta_3)}{\partial\theta_2\,\partial\theta_3} = -\frac{(\alpha-1)\tau}{\theta_2^2\theta_3^2}\sum_{i=r_2+1}^{r_3}\frac{t_i-2\tau}{\zeta_{i_3}^2} + \frac{(n-r_3)(\zeta_{r_3})^{\alpha-1}e^{-\zeta_{r_3}}\tau(t_r-2\tau)}{\Gamma(\alpha)\theta_2^2\theta_3^2\left(1-IG_{\zeta_{r_3}}(\alpha)\right)^2} \times\left[\left(1-IG_{\zeta_{r_3}}(\alpha)\right)\left(1-(\alpha-1)(\zeta_{r_3})^{-1}\right)-\frac{1}{\Gamma(\alpha)}(\zeta_{r_3})^{\alpha-1}e^{-\zeta_{r_3}}\right], \quad (6.4.11)$$

and

$$I_{44} = \frac{\partial^2 l(\alpha,\theta_1,\theta_2,\theta_3)}{\partial\theta_3^2} = \frac{N_3}{\theta_3^2} + \frac{(\alpha-1)}{\theta_3^2}\sum_{i=r_2+1}^{r_3}\frac{1}{(\zeta_{i3})^2} \left[(t_i-2\tau)\left(\frac{(t_i-2\tau)}{2\theta_3}+\frac{\tau}{\theta_1}+\frac{\tau}{\theta_2}\right)\right] - \frac{2}{\theta_3^3}\sum_{i=r_2+1}^{r_3}(t_i-2\tau) + \frac{(n-r_3)(\zeta_{r_3})^{\alpha-1}e^{-\zeta_{r_3}}(t_r-2\tau)}{\Gamma(\alpha)\theta_3^4\left(1-IG_{\zeta_{r_3}}(\alpha)\right)^2}[(1-IG_{\zeta_{r_3}}(\alpha)) \times\left\{(1-(\alpha-1)(\zeta_{r_3})^{-1})(t_r-2\tau)-2\theta_3\right\} - \frac{(t_r-2\tau)}{\Gamma(\alpha)}(\zeta_{r_3})^{\alpha-1}e^{-\zeta_{r_3}}]. \quad (6.4.12)$$

where

$$B_1(t) = \frac{1}{\Gamma(\alpha)}\int_0^t u^{\alpha-1}(\ln\ u)e^{-u}du,$$
$$B_2(t) = \frac{1}{\Gamma(\alpha)}\int_0^t u^{\alpha-1}\ (\ln\ u)^2e^{-u}du.$$

It is known that $I_{21}=I_{12}, I_{31}=I_{13}, I_{32}=I_{23}, I_{41}=I_{14}, I_{42}=I_{24}$ and $I_{43}=I_{34}$. Now, the variances and covariances of $\widehat{\alpha}$, $\widehat{\theta}_1$, $\widehat{\theta}_2$ and $\widehat{\theta}_3$ can be obtained through the observed Fisher information matrix as

$$Var\begin{bmatrix}\hat{\alpha}\\ \hat{\theta}_1\\ \hat{\theta}_2\\ \hat{\theta}_3\end{bmatrix}=(I)^{-1}=\begin{bmatrix}V_{11} & V_{12} & V_{13} & V_{14}\\ V_{21} & V_{22} & V_{23} & V_{24}\\ V_{31} & V_{32} & V_{33} & V_{34}\\ V_{41} & V_{42} & V_{43} & V_{44}\end{bmatrix}. \tag{6.4.13}$$

The asymptotic distribution of the maximum likelihood estimates are then given by $\frac{\widehat{\alpha}-\alpha}{\sqrt{V_{11}}} \sim N(0,1)$, $\frac{\widehat{\theta}_1-\theta_1}{\sqrt{V_{22}}} \sim N\left(0,1\right)$, $\frac{\widehat{\theta}_2-\theta_2}{\sqrt{V_{33}}} \sim N\left(0,1\right)$, and $\frac{\widehat{\theta}_3-\theta_3}{\sqrt{V_{44}}} \sim N\left(0,1\right)$, which can be used to construct $100\left(1-\alpha\right)\%$ confidence intervals for the parameters $\alpha,\ \theta_1,\ \theta_2$ and θ_3, respectively. These confidence intervals are given by

$$\hat{\alpha} \pm z_{1-\frac{\alpha}{2}}\sqrt{V_{11}}, \tag{6.4.14}$$

$$\hat{\theta}_1 \pm z_{1-\frac{\alpha}{2}}\sqrt{V_{22}}, \tag{6.4.15}$$

$$\hat{\theta}_2 \pm z_{1-\frac{\alpha}{2}}\sqrt{V_{33}} \tag{6.4.16}$$

and

$$\hat{\theta}_3 \pm z_{1-\frac{\alpha}{2}}\sqrt{V_{44}}, \tag{6.4.17}$$

where z_q is the q-th upper percentile of the standard normal distribution.

6.4.2 Bootstrap Confidence Intervals

Confidence intervals based on the parametric bootstrap sampling can be constructed. The following are the steps to generate the bootstrap confidence intervals in the case when $m = 3$:

(a) Compute the MLEs of $\alpha,\ \theta_1,\ \theta_2$ and θ_3, by using the method described in

Section 6.3, based on $T_1, T_2, \ldots, T_{N_1}, T_{N_1+1}, \ldots, T_{N_1+N_2}, T_{N_1+N_2+1}, \cdots, T_r$, denoted by $\widehat{\alpha},\ \widehat{\theta}_1,\ \widehat{\theta}_2$ and $\widehat{\theta}_3$.

(b) Simulate n order statistics from the uniform (0,1) distribution.

(c) Find N_1 such that $U_{N_1} \leq G_1^*(\tau_1) \leq U_{N_1+1}$, where $G_1^*(\tau_1) = F_1^*(\tau_1) = \int_0^{\frac{\tau_1}{\widehat{\theta}_1}} \frac{1}{\Gamma(\widehat{\alpha})} x^{\widehat{\alpha}-1} e^{-x} dx$.

(d) For $i \leq N_1$, $T_i = \widehat{\theta}_1 F^{*-1}(U_i)$, where $F^*(t) = \int_0^t \frac{1}{\Gamma(\widehat{\alpha})} x^{\widehat{\alpha}-1} e^{-x} dx$.

(e) Find N_2 such that $U_{N_1+N_2} \leq G_2^*(\tau_2) \leq U_{N_1+N_2+1}$, where $G_2^*(\tau_2) = F_2^*(\tau_2) = \int_0^{\frac{\tau_2-\tau_1+\frac{\widehat{\theta}_2}{\widehat{\theta}_1}\tau_1}{\widehat{\theta}_1}} \frac{1}{\Gamma(\widehat{\alpha})} x^{\widehat{\alpha}-1} e^{-x} dx$.

(f) For $N_1 + 1 \leq i \leq N_1 + N_2$, $\quad T_i = \widehat{\theta}_2\ F^{*-1}(U_i) + \tau_1 - \frac{\widehat{\theta}_2}{\widehat{\theta}_1}\tau_1$.

(g) For $N_1 + N_2 + 1 \leq i \leq r$, $\quad T_i = \widehat{\theta}_3\ F^{-1}(U_i) + \tau_2 - \frac{\widehat{\theta}_3}{\widehat{\theta}_2}(\tau_2 - \tau_1 + \frac{\widehat{\theta}_2}{\widehat{\theta}_1}\tau_1)$.

(h) Compute the MLEs of $(\alpha,\ \theta_1,\ \theta_2,\ \theta_3)$ based on $T_1, T_2, \ldots, T_{N_1}, T_{N_1+1}, \cdots,$ $T_{N_1+N_2}, T_{N_1+N_2+1}, \cdots, T_r$, say $\widehat{\alpha}(1),\ \widehat{\theta}_1^{(1)},\ \widehat{\theta}_2^{(1)}$ and $\widehat{\theta}_3^{(1)}$.

(i) Repeat steps (b)-(h) B times to obtain B sets of MLEs of $\alpha,\ \theta_1,\ \theta_2$ and θ_3 .

A two-sided $100(1-\alpha)\%$ bootstrap confidence interval of $\alpha,\ \theta_1,\ \theta_2$ and θ_3 are then given by

$$CI_\alpha = [\hat{\alpha} - z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\alpha}}}, \hat{\alpha} + z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\alpha}}}], \tag{6.4.18}$$

$$CI_{\theta_1} = [\hat{\theta}_1 - z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta}_1}}, \hat{\theta}_1 + z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta}_1}}], \tag{6.4.19}$$

$$CI_{\theta_2} = [\hat{\theta}_2 - z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta}_2}}, \hat{\theta}_2 + z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta}_2}}] \tag{6.4.20}$$

and

$$CI_{\theta_3} = [\hat{\theta}_3 - z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta}_3}}, \hat{\theta}_1 + z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta}_3}}], \tag{6.4.21}$$

where the $MSE_a = \text{var}(a) + (\text{bias}(a))^2$, and $\text{bias}(a) = \overline{a} - a$. The performance of the approximate confidence intervals and the bootstrap confidence intervals are evaluated by using a simulation study in the next section followed by an illustrative example.

6.5 Simulation Study

A simulation study is carried out for different values of n, r, τ_1 and τ_2. The results are presented in Tables 6.1 to 6.4, and they are based on an average over 1000 replications.

In Tables 6.1 and 6.3, we can see that as τ_1 increases, the conditional failure probabilities in the interval $[0, \tau_1]$ increase as well. It can also be seen that the larger the gap between τ_1 and τ_2 the larger the failure probabilities in the interval $[\tau_1, \tau_2]$, and the smaller the failure probabilities in the the interval $[\tau_2, \infty]$.

In Tables 6.2 and 6.4, we can see that as n and r increase the bias and MSE of all estimates decrease. We can also see that as τ_1 increases the bias and MSE of both $\widehat{\alpha}$ and $\widehat{\theta_1}$ decrease. It is also observed that the bias and MSE of $\widehat{\theta_1}$ are much larger than those of $\widehat{\alpha}, \widehat{\theta_2}$ and $\widehat{\theta_3}$. We can also see that as the gap between τ_1 and τ_2 increases, the MSE of $\widehat{\theta_2}$ decrease and those of $\widehat{\theta_3}$ increase, and this is as expected, of course.

We also observe from Tables 6.2 and 6.4 that the estimated coverage probabilities of the confidence intervals obtained by using the parametric bootstrap method are much closer to the nominal levels than those obtained by using the approximate method for all the parameters. The reason for this might be due

Table 6.1: Conditional failure probabilities for the multiple step-stress test under Type-II censoring when $\alpha = 2,\ \theta_1 = e^{1.5},\ \theta_2 = e^{1},\ \theta_3 = e^{.5}\ n = 150$ and $r = 100$.

		Conditional failure probabilities (in %)		
τ_1	τ_2	$0 < t < \tau_1$	$\tau_1 < t < \tau_2$	$\tau_2 < t < \infty$
2	5	11.27	57.36	31.37
3	5	21.67	39.72	39.59
	6	21.89	57.47	20.64
4	6	33.59	38.77	27.64
	7	33.51	55.05	11.44
5	6	46.16	19.30	34.54
	7	45.97	36.77	17.25
6	7	57.97	18.07	23.95
7	8	69.30	16.66	14.04

to the fact that the asymptotic normality required for the approximate method would require much larger values of n and r. It is also seen that, for the same values of both τ_1 and τ_2, in Tables 6.2 and 6.4, the coverage probabilities for the bootstrap method are much closer to the nominal level. From these findings, we would recommend the use of the bootstrap method for the construction of confidence intervals for the model parameters.

6.5.1 Illustrative Example

In this subsection, we consider the data generated with $n = 40,\ r = 38,\ \alpha = 2,\ \theta_1 = e^{1.5} = 4.481689,\ \theta_2 = e^{1} = 2.718282,\ \theta_3 = e^{0.5} = 1.648721,\ \tau_1 = 5$ and $\tau_2 = 7$. The simulated data are presented in Table 6.5.

We consider three different numbers of failures $r = 30,\ 35,\ 38$. The respective MLEs of $\alpha,\ \theta_1,\ \theta_2$ and θ_3 and their corresponding standard errors are calculated and are given in Table 6.6. It can be seen from Table 6.6 that the larger the r,

Table 6.2: The bias and MSEs of the MLEs $\widehat{\alpha},$, $\widehat{\theta_1}$, $\widehat{\theta_2}$ and $\widehat{\theta_3}$ with the estimated coverage probabilities (in %) of confidence intervals for a multiple step-stress model under Type-II censoring for $\alpha,\ \theta_1,\ \theta_2$ and θ_3 based on 1000 simulations when $\alpha = 2,\ \theta_1 = e^{1.5},\ \theta_2 = e^1,\ \theta_3 = e^{.5},\ n = 150$ and $r = 100$.

					90% C.I.		95% C.I.		99% C.I.	
	τ_1	τ_2	bias	MSE	Approx.	Boot	Approx.	Boot	Approx.	Boot
α	2	5	0.2273	0.9652	96.4	90	98.2	94.9	99.3	98.4
	3	5	0.1789	0.3978	98.2	91.6	99.3	95.9	100	98.6
		6	0.1690	0.4329	97.9	89.2	99.2	94	99.9	98.3
	4	6	0.1062	0.2384	99.2	89.8	99.8	94.1	100	98.8
		7	0.1138	0.2362	99.6	89.1	99.9	93.8	100	99.2
	5	6	0.0850	0.1574	99.6	91.6	99.8	96.1	100	99.2
		7	0.0837	0.1777	99.6	90.8	100	95.3	99.9	98.8
	6	7	0.0589	0.1354	99.8	90.3	99.9	95.4	100	99.2
	7	8	0.0565	0.1073	99.8	89.9	100	95.3	100	99.3
θ_1	2	5	1.5995	42.5039	87.5	88.5	92	93.4	96.7	97.4
	3	5	0.3119	6.7702	93.2	90.9	96.8	95	99.2	98.4
		6	0.4099	8.8985	90.9	89	95.3	92.9	98.8	97.7
	4	6	0.2044	3.4123	95.6	89	97.8	94.6	99.5	98.2
		7	0.1933	3.4468	95.8	89.3	98.3	93.6	99.7	98.5
	5	6	0.0602	1.8295	98.5	90.7	99.4	96.1	99.9	98.9
		7	0.1228	2.1528	96.4	91.1	98.6	95.4	99.8	99
	6	7	0.1040	1.4797	98.5	90.1	99.7	95.6	100	99.4
	7	8	0.0393	0.9971	98.4	90.3	99.5	94.3	100	98.9
θ_2	2	5	0.1183	0.8601	92	90.6	95.7	94.5	98.4	98.5
	3	5	0.0215	0.5269	99	91.2	99.7	94.9	100	98.8
		6	0.0148	0.3896	99.4	88.4	99.7	93.6	100	98.5
	4	6	0.0559	0.3997	99.2	88.1	99.8	94.1	99.9	98.7
		7	0.0212	0.2628	99.4	90.3	99.8	94.5	100	99
	5	6	0.1365	0.5945	99.6	92	99.8	96.2	100	99.5
		7	0.0491	0.3187	99.8	89.5	99.9	94.1	100	98.7
	6	7	0.1913	0.7689	99.1	92.1	99.9	95.6	99.9	99.4
	7	8	0.1877	0.7402	99.6	92	99.9	95.7	100	99.5
θ_3	2	5	0.0068	0.1820	98.2	96.7	99.1	98.2	100	99.8
	3	5	-0.0188	0.1211	98.5	98.2	99.4	99.2	100	100
		6	-0.0102	0.1922	97.6	89.6	98.1	94.8	99.5	98.6
	4	6	-0.0273	0.1360	98.9	97.6	99.6	98.7	100	100
		7	-0.0347	0.4106	99	94	99.5	97.6	99.8	99.5
	5	6	0.0004	0.1053	99.4	97.4	99.8	98.7	100	99.9
		7	-0.0169	0.1901	99.8	92.5	100	96.3	100	99.4
	6	7	-0.0095	0.1387	99.6	97.8	99.7	99.5	100	100
	7	8	-0.0257	0.2519	99.7	93.7	100	97.6	100	99.7

Table 6.3: Conditional failure probabilities for the multiple step-stress test under Type-II censoring when $\alpha = 2,\ \theta_1 = e^{1.5},\ \theta_2 = e^{1},\ \theta_3 = e^{.5}\ n = 250$ and $r = 170$.

		Conditional failure probabilities (in %)		
τ_1	τ_2	$0 < t < \tau_1$	$\tau_1 < t < \tau_2$	$\tau_2 < t < \infty$
2	5	10.88	56.42	32.70
3	5	21.34	38.92	39.74
	6	21.34	56.15	22.51
4	6	32.99	38.24	28.76
	7	33.06	53.99	12.95
5	6	44.79	19.29	35.92
	7	45.08	36.15	18.77
6	7	56.79	17.77	25.43
7	8	68.09	16.55	15.36

the smaller the standard errors of $\widehat{\alpha}, \widehat{\theta}_1,\ \widehat{\theta}_2$ and $\widehat{\theta}_3$.

The confidence intervals for $\alpha,\ \theta_1,\ \theta_2$ and θ_3 obtained by the approximate method and the bootstrap method for different values of r are given in Table 6.7. From this table, we can see that for α the approximate confidence intervals are unsatisfactory and are wider than the bootstrap confidence intervals. There is only one case when $r = 35$ in which the approximate confidence interval is narrower than the bootstrap confidence interval. In almost all cases and for all the parameters, the bootstrap confidence intervals are narrower than the approximate confidence intervals for every value of r.

Table 6.4: The bias and MSEs of the MLEs $\widehat{\alpha},$, $\widehat{\theta_1}$, $\widehat{\theta_2}$ and $\widehat{\theta_3}$ with the estimated coverage probabilities (in %) of confidence intervals for a multiple step-stress model under Type-II censoring for α, θ_1 and θ_2 based on 1000 simulations when $\alpha = 2,\ \theta_1 = e^{1.5},\ \theta_2 = e^{1},\ \theta_3 = e^{.5},\ n = 250$ and $r = 170$.

					90% C.I.		95% C.I.		99% C.I.	
	τ_1	τ_2	bias	MSE	Approx.	Boot	Approx.	Boot	Approx.	Boot
α	2	5	0.1598	0.3859	96.7	90	98.5	94.9	99.8	98.4
	3	5	0.0984	0.2090	98.4	91.6	99.2	95.9	100	98.6
		6	0.0832	0.1958	98.7	89.2	99.7	94	100	98.3
	4	6	0.0694	0.1346	98.8	89.8	99.7	94.1	100	98.8
		7	0.0687	0.1287	99.6	89.1	99.9	93.8	100	99.2
	5	6	0.0549	0.0990	99.3	88.8	99.9	94.8	100	99
		7	0.0468	0.0901	99.7	90.8	99.9	95.3	100	98.8
	6	7	0.0443	0.0707	99.9	90.3	100	95.4	100	99.2
	7	8	0.0320	0.0577	100	89.9	100	95.3	100	99.3
θ_1	2	5	0.6983	15.7358	88	88.5	92.9	93.4	97.5	97.4
	3	5	0.2138	3.6947	92.5	90.9	96.2	95	99	98.4
		6	0.2690	3.7113	92.9	89	96.7	92.9	99.4	97.7
	4	6	0.1103	2.1135	95	89	97.9	94.6	99.5	98.2
		7	0.0866	1.7608	95.8	89.3	98.7	93.6	99.8	98.5
	5	6	0.0807	1.2279	97.1	90.8	98.5	94.7	99.8	98.8
		7	0.0673	1.1381	97.8	91.1	99.3	95.4	99.9	99
	6	7	0.0255	0.7733	98.6	90.1	99.8	95.6	100	99.4
	7	8	0.0203	0.5751	99.1	90.3	99.8	94.3	100	98.9
θ_2	2	5	0.0201	0.4654	93.4	90.6	96.7	94.5	99.1	98.5
	3	5	0.0251	0.3240	97.5	91.2	99.1	94.9	99.8	98.8
		6	0.0382	0.2626	96.9	88.4	98.4	93.6	99.9	98.5
	4	6	0.0159	0.2286	99.3	88.1	99.9	94.1	100	98.7
		7	0.0143	0.1609	99.3	90.3	99.8	94.5	100	99
	5	6	0.0524	0.3621	99.8	90.8	100	95.1	100	99.5
		7	0.0271	0.1897	100	89.5	100	94.1	100	98.7
	6	7	0.0944	0.3201	99.9	92.1	100	95.6	100	99.4
	7	8	0.0691	0.3600	99.8	92	99.8	95.7	99.9	99.5
θ_3	2	5	-0.0278	0.1051	98.7	96.7	99.4	98.2	100	99.8
	3	5	-0.0120	0.0709	99	98.2	99.8	99.2	99.9	100
		6	-0.0066	0.1096	99	89.6	99.5	94.8	99.9	98.6
	4	6	-0.0023	0.0780	99.4	97.6	99.8	98.7	100	100
		7	-0.0097	0.1627	99.3	94	99.8	97.6	99.9	99.5
	5	6	-0.0057	0.0566	99.7	96.7	100	99	100	100
		7	-0.0228	0.0956	99.9	92.5	99.9	96.3	100	99.4
	6	7	-0.0161	0.0730	99.6	97.8	99.7	99.5	99.8	100
	7	8	-0.0203	0.1257	98	93.7	99	97.6	99.9	99.7

Table 6.5: Simulated data for the illustrative example.

Stress level	Failure times								
$\theta_1 = e^{1.5}$	0.374	1.022	1.545	1.983	2.099	2.100	2.111	2.999	3.087
	3.165	3.370	3.371	3.829	4.295	4.450	4.644		
$\theta_2 = e^1$	5.093	5.216	5.459	5.629	5.714	5.831	6.330	6.778	6.890
$\theta_3 = e^{.5}$	7.400	7.711	7.804	8.059	8.280	8.536	8.925	9.537	9.664
	9.734	9.912	11.478	12.332					

Table 6.6: The MLEs of α, θ_1, θ_2 and θ_3 and their standard errors.

r	N_3	$\widehat{\alpha}$	Se($\widehat{\alpha}$)	$\widehat{\theta_1}$	Se($\widehat{\theta_1}$)	$\widehat{\theta_2}$	Se($\widehat{\theta_2}$)	$\widehat{\theta_3}$	Se($\widehat{\theta_3}$)
30	5	1.8187	0.5079	4.0855	0.8472	2.8426	0.7852	2.4925	0.2723
35	10	1.8417	0.2154	4.0208	0.8407	2.8335	0.7753	2.1881	0.1565
38	13	1.8352	0.0370	4.0385	0.8345	2.8356	0.7658	2.2964	0.1081

6.6 The Reduced-Parameter Model

In this section, we consider a re-parametrization of the m-step-stress model, in which θ_i is assumed to satisfy a log-linear link function of the form

$$\ln\theta_i = a - bx_i, i = 1, 2, \cdots, m, \tag{6.5.1}$$

where a and b are unknown parameters and we need to develop inference only for these two parameters instead of for the original m parameters $\theta_1, \theta_2, \ldots, \theta_m$.

6.6.1 Maximum Likelihood Estimation

The likelihood function is obtained based on the observed Type-II censored data in (6.2.1), and from it the MLEs of the three unknown parameters α, a and b are then obtained numerically. The log-likelihood function of this sample can

Table 6.7: Interval estimation for the simulated data presented in Table 6.5.

		C.I. for α		
r	Method	90%	95%	99%
30	Approx C.I.	(0.9833, 2.6541)	(0.8233, 2.8141)	(0.5105, 3.1269)
	Bootstrap C.I.	(1.2581, 2.3792)	(1.1508, 2.4866)	(0.9409, 2.6965)
35	Approx C.I.	(1.4874, 2.1960)	(1.4196, 2.2638)	(1.2869, 2.3965)
	Bootstrap C.I.	(1.3064, 2.3770)	(1.2038, 2.4796)	(1.0034, 2.6800)
38	Approx C.I.	(1.7744, 1.8961)	(1.7627, 1.9077)	(1.7400, 1.9305)
	Bootstrap C.I.	(1.3141, 2.3563)	(1.2143, 2.4561)	(1.0192, 2.6512)
		C.I. for θ_1		
30	Approx C.I.	(2.6920, 5.4791)	(2.4250, 5.7460)	(1.9032, 6.2678)
	Bootstrap C.I.	(3.2526, 4.9184)	(3.0931, 5.0780)	(2.7812, 5.3898)
35	Approx C.I.	(2.6380, 5.4037)	(2.3730, 5.6686)	(1.8553, 6.1863)
	Bootstrap C.I.	(3.2419, 4.7997)	(3.0927, 4.9489)	(2.8011, 5.2405)
38	Approx C.I.	(2.6658, 5.4112)	(2.4028, 5.6742)	(1.8888, 6.1882)
	Bootstrap C.I.	(3.2660, 4.8110)	(3.1180, 4.9590)	(2.8288, 5.2482)
		C.I. for θ_2		
30	Approx C.I.	(1.5511, 4.1341)	(1.3037, 4.3816)	(0.8201, 4.8651)
	Bootstrap C.I.	(2.2144, 3.4709)	(2.0940, 3.5913)	(1.8588, 3.8265)
35	Approx C.I.	(1.5583, 4.1086)	(1.3140, 4.3529)	(0.8366, 4.8303)
	Bootstrap C.I.	(2.2149, 3.4520)	(2.0964, 3.5705)	(1.8649, 3.8021)
38	Approx C.I.	(1.5759, 4.0954)	(1.3346, 4.3367)	(0.8629, 4.8083)
	Bootstrap C.I.	(2.2073, 3.4640)	(2.0869, 3.5844)	(1.8517, 3.8196)
		C.I. for θ_3		
30	Approx C.I.	(2.0446, 2.9404)	(1.9588, 3.0262)	(1.7911, 3.1939)
	Bootstrap C.I.	(1.3668, 3.6183)	(1.1511, 3.8339)	(0.7296, 4.2554)
35	Approx C.I.	(1.9306, 2.4457)	(1.8812, 2.4950)	(1.7848, 2.5914)
	Bootstrap C.I.	(1.5771, 2.7991)	(1.4601, 2.9162)	(1.2313, 3.1449)
38	Approx C.I.	(2.1186, 2.4742)	(2.0845, 2.5083)	(2.0179, 2.5749)
	Bootstrap C.I.	(1.7716, 2.8212)	(1.6711, 2.9217)	(1.4746, 3.1182)

be written as

$$l(\alpha, a, b|\mathbf{t}) = \ln\left[\mathrm{L}\left(\alpha, \mathrm{a}, \mathrm{b}|\mathbf{t}\right)\right]$$
$$= \ln\left(\frac{n!}{(n-r_m)!}\right) - r_m \ln\left(\Gamma(\alpha)\right) - \sum_{k=1}^{m} N_k(a - bx_k) + (\alpha - 1)\sum_{k=1}^{m}\left\{\sum_{i_k=r_{k-1}+1}^{r_k} \ln\left(\zeta_{i_k}^*\right)\right\}$$
$$- \sum_{k=1}^{m}\left(\sum_{i_k=r_{k-1}+1}^{r_k} \zeta_{i_k}^*\right) + (n - r_m)\ln\left(1 - IG_{\zeta_{r_m}^*}(\alpha)\right),$$
$$\text{for } t_1 < \cdots < t_{N_1} < \tau \le t_{N_1+1} < \cdots < t_{N_1+N_2} < 2\tau \le \cdots <$$
$$(m-1)\tau \le t_{N_1+\cdots+N_{m-1}+1} < \cdots < t_r, \quad (6.5.2)$$

where

$$\zeta_{i_k}^* = (t_{i_k} - (k-1)\,\tau)e^{-a+bx_k} + \tau e^{-a}\sum_{j=1}^{k-1} e^{b\ x_j}, \quad k = 1,\ 2,\ \cdots,\ m. \quad (6.5.3)$$

Now, instead of differentiating the log-likelihood function with respect to α and θ_i for $i = 1, 2, \cdots, m$, we differentiate (6.5.2) with respect to α, a and b. As before, we will need the first and second partial derivatives of (6.5.2), but here with respect to the parameters α, a and b. The first partial derivatives are given by the following equations:

$$\frac{\partial l(\alpha, a, b|\mathbf{t})}{\partial \alpha} = -r_m\psi(\alpha) + \sum_{k=1}^{m}\left\{\sum_{i_k=r_{k-1}+1}^{r_k} \ln\left(\zeta_{i_k}^*\right)\right\}$$
$$+ \frac{(n-r_m)}{1 - IG_{\zeta_{r_m}^*}(\alpha)}\left[\Psi(\alpha) IG_{\zeta_{r_m}^*}(\alpha) - B_1\left(\zeta_{r_m}^*\right)\right], \quad (6.5.4)$$

$$\frac{\partial l(\alpha,a,b|\mathbf{t})}{\partial a} = -\sum_{k=1}^{m} N_k + (\alpha-1)\sum_{k=1}^{m}\left\{\sum_{i_k=r_{k-1}+1}^{r_k}(-1)\right\}$$
$$-\sum_{k=1}^{m}\left(\sum_{i_k=r_{k-1}+1}^{r_k} -\zeta^*_{i_k}\right) + \frac{(n-r_m)\left(\zeta^*_{r_m}\right)^{\alpha} e^{-\zeta^*_{r_m}}}{\Gamma(\alpha)\left[1-IG_{\zeta^*_{r_m}}(\alpha)\right]}, \tag{6.5.5}$$

and

$$\frac{\partial l(\alpha,a,b|\mathbf{t})}{\partial b} = \sum_{k=1}^{m} N_k x_k + (\alpha-1)\sum_{k=1}^{m}\left\{\sum_{i_k=r_{k-1}+1}^{r_k}\frac{1}{\zeta^*_{i_k}}A_1(t_{i_k})\right\}$$
$$-\sum_{k=1}^{m}\left\{\sum_{i_k=r_{k-1}+1}^{r_k} A_1(t_{i_k})\right\} - \frac{(n-r_m)\zeta^{*\alpha-1}_{r_m} e^{-\zeta^*_{r_m}}}{\Gamma(\alpha)\left[1-IG_{\zeta^*_{r_m}}(\alpha)\right]}A_1(t_{r_m}), \tag{6.5.6}$$

where

$$\Psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)},$$
$$B_1(v) = \int_0^v \frac{1}{\Gamma(\alpha)}\ln(u)\ \ u^{\alpha-1}e^{-u}du,$$
$$A_1(t_{i_k}) = (t_{i_k} - (k-1)\tau)x_k e^{-a+bx_k} + \tau e^{a}\sum_{j=1}^{k-1} x_j e^{bx_j}.$$

Since there is no obvious simplification of the above non-linear likelihood equations, the maximum likelihood estimates of the parameters need to be obtained numerically as mentioned earlier. So, the required numerical maximization was carried out on the log-likelihood using the R software. First, we use the log-likelihood function and start with initial values. Then, the function *optim* in R is used to maximize this log-likelihood function. After that, the estimates are found and their confidence intervals are constructed, using the Hessian matrix. Since for the complete parametrization we used $m = 3$, so we will consider the same setting here. The following is the algorithm used to find the MLEs:

(a) Simulate n order statistics from the uniform (0,1) distribution, $U_1, U_2, \ldots, U_n$.

(b) Find N_1 such that $U_{N_1} \leq G_1(\tau_1) \leq U_{N_1+1}$ where
$G_1(\tau_1) = \int_0^{\tau_1 e^{-a+bx_1}} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} dx.$

(c) For $i \leq N_1$, $T_i = e^{a-bx_1} F^{-1}(U_i)$, where $F(t) = \int_0^t \frac{1}{\Gamma(\alpha)} e^{-x} x^{\alpha-1} dx.$

(d) Find N_2 such that $U_{N_1+1} \leq G_2(\tau_2) \leq U_{N_1+N_2}$ where
$G_2(\tau_2) = \int_0^{\frac{\tau_2-\tau_1+e^{-b(x_2-x_1)}}{e^{a-bx_2}}} \frac{1}{\Gamma} x^{\alpha-1} e^{-x} dx.$

(e) For $N_1 + 1 \leq i \leq N_1 + N_2$, $\quad T_i = e^{a-bx_2}\ F^{-1}(U_i) + \tau_1(1 - e^{-b(x_2-x_1)}).$

(f) For $N_1 + N_2 + 1 \leq i \leq r$, $\quad T_i = e^{a-bx_3}\ F^{-1}(U_i) + \tau_2 - e^{-b(x_3-x_2)}(\tau_2 - \tau_1(1 - e^{-b(x_2-x_1)})).$

(g) Compute the MLEs of (α, a, b) based on $T_1, T_2, \ldots, T_{N_1}, T_{N_1+1}, \ldots,$
$T_{N_1+N_2}, T_{N_1+N_2+1}, \cdots T_r$, say $(\widehat{\alpha},\ \hat{a},\ \hat{b})$.

6.6.2 Confidence Intervals

As in Section 6.4, we will use two different methods for constructing confidence intervals (CI) for the unknown parameters α, a and b. The asymptotic distributions of the MLEs is then used to obtain the approximate CIs for α, a and b.

Here again, we use the parametric bootstrap method as the second method for constructing confidence intervals for α, a and b.

Approximate Confidence Intervals

We present an approximate method which provides good coverage probabilities for large sample sizes and also facilitates easy computation. Elements of Fisher information matrix are found numerically for different values of m, and then the asymptotic normality of the MLEs is used to construct the approximate confidence intervals for α, a and b.

Let $I(\alpha,\ a,\ b) = [I_{ij}(\alpha,\ a,\ b)]$, for $i,\ j = 1,\ 2,\ 3$, denote the observed Fisher information matrix of $\alpha,\ a$ and b, where

$$I_{ij}(\alpha, a, b) = -\left(\nabla^2 l(\alpha, a, b)\right). \tag{6.5.7}$$

The observed Fisher information matrix (I) is given by

$$I = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}, \tag{6.5.8}$$

where

$$I_{11} = \frac{\partial^2 l(\alpha, a, b)}{\partial \alpha^2} = -r_3 \Psi^{'}(\alpha) + \frac{(n - r_3)}{\left(1 - IG_{\zeta^*_{r_3}}(\alpha)\right)^2}$$
$$\times [\left(1 - IG_{\zeta^*_{r_3}}(\alpha)\right) \left[2\Psi(\alpha) B_1(\zeta^*_{r_3}) + IG_{\zeta^*_{r_3}}(\Psi^{'}(\alpha) - \Psi^2(\alpha)) - B_2(\zeta^*_{r_3})\right]$$
$$- \left[\Psi(\alpha) IG_{\zeta^*_{r_3}}(\alpha) - B_1(\zeta^*_{r_3})\right]^2], \quad (6.5.9)$$

$$I_{12} = \frac{\partial^2 l(\alpha, a, b)}{\partial \alpha \partial a} = \sum_{k=1}^{3} \sum_{i=r_{k-1}+1}^{r_k} (-1) + \frac{(n - r_3)(\zeta^*_{r_3})^{\alpha} e^{-\zeta^*_{r_3}}}{\Gamma(\alpha) \left(1 - IG_{\zeta^*_{r_3}}(\alpha)\right)^2}$$
$$\times [\left(1 - IG_{\zeta^*_{r_3}}(\alpha)\right) \ln(\zeta^*_{r_3}) - \Psi(\alpha) + B_1(\zeta^*_{r_3})], \quad (6.5.10)$$

$$I_{13} = \frac{\partial^2 l(\alpha, a, b)}{\partial \alpha \, \partial b} = \sum_{k=1}^{3} \left\{ \sum_{i=r_{k-1}+1}^{r_k} \frac{1}{\zeta^*_{i_k}} A_1(t_{i_k}) \right\}$$
$$+ \frac{(n - r_3)(\zeta^*_{r_3})^{\alpha - 1} e^{-\zeta^*_{r_3}} A_1(t_{r_3})}{\Gamma(\alpha) \left(1 - IG_{\zeta^*_{r_3}}(\alpha)\right)^2} \left[\Psi(\alpha) - \left[1 - IG_{\zeta^*_{r_3}}(\alpha)\right] \ln \zeta^*_{r_3} - B_1(\zeta^*_{r_3})\right], \quad (6.5.11)$$

$$I_{22} = \frac{\partial^2 l(\alpha, a, b)}{\partial a^2} = -\sum_{k=1}^{3} \sum_{i=r_{k-1}+1}^{r_k} \zeta^*_{i_k} + \frac{(n - r_3)(\zeta^*_{r_3})^{\alpha} e^{-\zeta^*_{r_3}}}{\Gamma(\alpha) \left(1 - IG_{\zeta^*_{r_3}}(\alpha)\right)^2}$$
$$\times \left[\left(1 - IG_{\zeta^*_{r_3}}(\alpha)\right) \left(\zeta^*_{r_3} - \alpha\right) - \frac{1}{\Gamma(\alpha)} (\zeta^*_{r_3})^{\alpha} e^{-\zeta^*_{r_3}}\right], \quad (6.5.12)$$

$$I_{23} = \frac{\partial^2 l(\alpha, a, b)}{\partial a \partial b} = -\sum_{k=1}^{3} \sum_{i=r_{k-1}+1}^{r_k} (-A_1(t_{i_k})) + \frac{(n-r_3)(\zeta^*_{r_3})^{\alpha} e^{-\zeta^*_{r_3}} A_1(t_{r_m})}{\Gamma(\alpha)\left(1 - IG_{\zeta^*_{r_3}}(\alpha)\right)^2}$$
$$\times \left[\left(1 - IG_{\zeta^*_{r_3}}(\alpha)\right)\left(\alpha \zeta^{*-1}_{r_3} - 1\right) + \frac{1}{\Gamma(\alpha)}(\zeta^*_{r_3})^{\alpha-1} e^{-\zeta^*_{r_3}}\right], \quad (6.5.13)$$

and

$$I_{33} = \frac{\partial^2 l(\alpha, a, b)}{\partial b_2} = (\alpha - 1)\sum_{k=1}^{3} \sum_{i=r_{k-1}+1}^{r_k} \frac{1}{\zeta^{*2}_{i_k}} \left\{\zeta^*_{i_k} A_2(t_{i_k}) - (A_1(t_{i_k}))^2\right\}$$
$$-\sum_{k=1}^{m} \sum_{i_k=r_{k-1}+1}^{r_k} A_2(t_{i_k}) - \frac{(n-r_3)(\zeta^*_{r_3})^{\alpha-1} e^{-\zeta^*_{r_3}}}{\Gamma(\alpha)\left(1 - IG_{\zeta^*_{r_3}}(\alpha)\right)^2}[(1 - IG_{\zeta^*_{r_3}}(\alpha))$$
$$\left\{\left((\alpha-1)\zeta^{*-1}_{r_3} - 1\right)(A_1(t_{r_3}))^2 + A_2(t_{r_3})\right\} + \frac{1}{\Gamma(\alpha)}(\zeta^*_{r_3})^{\alpha-1} e^{-\zeta^*_{r_3}} (A_1(t_{r_3}))^2],$$
$$(6.5.14)$$

where

$$B_1(t) = \frac{1}{\Gamma(\alpha)} \int_0^t u^{\alpha-1} (\ln u) e^{-u} du,$$
$$B_2(t) = \frac{1}{\Gamma(\alpha)} \int_0^t u^{\alpha-1} (\ln u)^2 e^{-u} du,$$
$$A_1(t_{i_k}) = (t_{i_k} - (k-1)\tau) x_k e^{-a+bx_k} + \tau e^a \sum_{j=1}^{k-1} x_j e^{bx_j},$$
$$A_2(t_{i_k}) = (t_{i_k} - (k-1)\tau) x_k^2 e^{-a+bx_k} + \tau e^a \sum_{j=1}^{k-1} x_j^2 e^{bx_j}.$$

It is known that $I_{21} = I_{12}, I_{31} = I_{13}$ and $, I_{32} = I_{23}$. Now, the variances and covariances of $\widehat{\alpha}$, $\widehat{a}$ and $\widehat{b}$ can be obtained through the observed Fisher information matrix as

$$Var\begin{bmatrix}\hat{\alpha}\\ \hat{a}\\ \hat{b}\end{bmatrix}=(I)^{-1}=\begin{bmatrix}V_{11} & V_{12} & V_{13}\\ V_{21} & V_{22} & V_{23}\\ V_{31} & V_{32} & V_{33}\end{bmatrix}. \tag{6.5.15}$$

The asymptotic distribution of the maximum likelihood estimates are then given by $\frac{\widehat{\alpha}-\alpha}{\sqrt{V_{11}}} \sim N(0,1)$, $\frac{\widehat{a}-a}{\sqrt{V_{22}}} \sim N(0,1)$ and $\frac{\widehat{b}-b}{\sqrt{V_{33}}} \sim N(0,1)$, which can be used to construct $100\,(1-\alpha)\,\%$ confidence intervals for the parameters α, a and b, respectively. These confidence intervals are given by

$$\hat{\alpha} \pm z_{1-\frac{\alpha}{2}}\sqrt{V_{11}}, \tag{6.5.16}$$

$$\hat{a} \pm z_{1-\frac{\alpha}{2}}\sqrt{V_{22}} \tag{6.5.17}$$

and

$$\hat{b} \pm z_{1-\frac{\alpha}{2}}\sqrt{V_{33}}, \tag{6.5.18}$$

where z_q is the q-th upper percentile of the standard normal distribution.

Bootstrap Confidence Intervals

Confidence intervals based on the parametric bootstrap sampling can be constructed. The following are the steps to generate the bootstrap confidence intervals for the case when $m = 3$:

(a) Compute the MLEs of α, a and b, by using the method described in Section 6.5.1, based on $T_1, T_2, \ldots, T_{N_1}, T_{N_1+1}, \ldots, T_{N_1+N_2}, T_{N_1+N_2+1}, \cdots, T_r$, denoted by $\widehat{\alpha}$, $\widehat{a}$ and $\widehat{b}$.

(b) Simulate n order statistics from the uniform (0,1) distribution.

(c) Find N_1 such that $U_{N_1} \leq G_1^*(\tau_1) \leq U_{N_1+1}$, where $G_1^*(\tau_1) = F_1^*(\tau_1) = \int_0^{\frac{\tau_1}{e^{\widehat{a}-\widehat{b}x_1}}} \frac{1}{\Gamma(\widehat{\alpha})} x^{\widehat{\alpha}-1} e^{-x} dx$.

(d) For $i \leq N_1$, $T_i = e^{\widehat{a}-\widehat{b}x_1} F^{*-1}(U_i)$, where $F^*(t) = \int_0^t \frac{1}{\Gamma(\widehat{\alpha})} x^{\widehat{\alpha}-1} e^{-x} dx$.

(e) Find N_2 such that $U_{N_1+N_2} \leq G_2^*(\tau_2) \leq U_{N_1+N_2+1}$, where $G_2^*(\tau_2) = F_2^*(\tau_2) = \int_0^y \frac{1}{\Gamma(\widehat{\alpha})} x^{\widehat{\alpha}-1} e^{-x} dx$, where $y = (\tau_2 - \tau_1) e^{-\widehat{a}+\widehat{b}x_2} + \tau_1 e^{-\widehat{a}+\widehat{b}x_1}$.

(f) For $N_1 + 1 \leq i \leq N_1 + N_2$, $\quad T_i = e^{\widehat{a}-\widehat{b}x_2} F^{*-1}(U_i) + \tau_1 - \tau_1 e^{-\widehat{b}(x_2-x_1)}$.

(g) For $N_1 + N_2 + 1 \leq i \leq r$, $\quad T_i = e^{\widehat{a}-\widehat{b}x_3} F^{-1}(U_i) + \tau_2 - e^{-\widehat{b}(x_3-x_2)}(\tau_2 - \tau_1 + \tau_1 e^{-\widehat{b}(x_2-x_1)})$.

(h) Compute the MLEs of (α, a, b) based on $T_1, T_2, \ldots, T_{N_1}, T_{N_1+1}, \cdots, T_{N_1+N_2}$, $T_{N_1+N_2+1}, \cdots, T_r$, say $\widehat{\alpha}^1, \widehat{a}^{(1)}$ and $\widehat{b}^{(1)}$.

(i) Repeat steps (b)-(h) B times to obtain B sets of MLEs of α, a and b.

A two-sided $100(1-\alpha)\%$ bootstrap confidence interval of α, a and b are then given by

$$CI_\alpha = [\hat{\alpha} - z_{1-\frac{\alpha}{2}} \sqrt{MSE_{\hat{\alpha}}}, \hat{\alpha} + z_{1-\frac{\alpha}{2}} \sqrt{MSE_{\hat{\alpha}}}], \tag{6.5.19}$$

$$CI_a = [\hat{a} - z_{1-\frac{\alpha}{2}} \sqrt{MSE_{\hat{a}}}, \hat{a} + z_{1-\frac{\alpha}{2}} \sqrt{MSE_{\hat{a}}}] \tag{6.5.20}$$

and

$$CI_b = [\hat{b} - z_{1-\frac{\alpha}{2}} \sqrt{MSE_{\hat{b}}}, \hat{b} + z_{1-\frac{\alpha}{2}} \sqrt{MSE_{\hat{b}}}], \tag{6.5.21}$$

where the $MSE_s = \mathrm{var}(s) + (\mathrm{bias}(s))^2$, and $\mathrm{bias}(s) = \overline{s} - s$. The performance of the approximate confidence intervals and the bootstrap confidence intervals are evaluated by using a simulation study in the next subsection followed by an illustrative example.

6.6.3 Simulation Study

A simulation study is carried out for different values of τ_1 and τ_2. The results are presented in Tables 6.8 to 6.11, and they are based on an average over 1000 replications.

In Tables 6.8 and 6.10, we can see how the failures are distributed in our model when we take different values of τ_1 and τ_2. We see that as τ_1 increases, the failure probability in the first interval also increases. It is seen that as the gap between τ_1 and τ_2 increases the failure probabilities in the second interval $[\tau_1, \tau_2]$ increase, while those in the third interval $[\tau_2, \infty]$ decrease.

In Tables 6.9 and 6.11, we see clearly that the MSEs of $\widehat{b}$ are less than those of $\widehat{\alpha}$ and $\widehat{a}$. If we look at the MSEs of $\widehat{\alpha}$ and $\widehat{a}$, we observe that the wider the gap between τ_1 and τ_2, the smaller the MSE. On the other hand, the MSEs of $\widehat{b}$ slightly increase as the gap between τ_1 and τ_2 increases. By comparing the MSEs in Tables 6.9 and 6.11, we see that for larger n we have smaller MSE values. In Table 6.9, it is seen that the coverage probabilities obtained by using the bootstrap method are closer to the nominal levels than those obtained by using the approximate method. These coverage probabilities based on the bootstrap method are equal or below the nominal levels for α while they are above the nominal levels for a and b. As both n and r increase (see Table 6.11), the coverage probabilities based on the bootstrap method get closer to the nominal levels.

Table 6.8: Conditional failure probabilities for the reduced multiple step-stress test under Type-II censoring when $\alpha = 2,\ \ x_1 = 1,\ x_2 = 1.5,\ x_3 = 2.5,\ a = 4,\ b = 2,\ n = 150$ and $r = 100$.

		Conditional failure probabilities (in %)		
τ_1	τ_2	$0 < t < \tau_1$	$\tau_1 < t < \tau_2$	$\tau_2 < t < \infty$
4	6	15.22	39.31	45.48
	7	15.60	58.16	26.24
5	6	22.04	19.73	58.23
	7	22.30	39.80	37.90
	8	22.13	57.56	20.31
6	7	29.28	20.26	50.46
	8	29.19	39.39	31.43
7	8	36.70	19.93	43.37

Illustrative Example

In this subsection, we consider the data generated with $n = 40,\ \alpha = 2,\ a = 4,\ b = 2,\ x_1 = 1,\ x_2 = 1.5,\ x_3 = 2.5, \tau_1 = 5,\ \tau_2 = 7$ and $r = 38$. The simulated data are presented in Table 6.12.

We consider three different values for the total number of failures: $r = 30,\ 35,\ 38$. The respective MLEs of $\alpha,\ a$ and b and their corresponding standard errors are calculated and are given in Table 6.13. It is seen from this table that as r increases, the standard errors of $\widehat{a}$ and $\widehat{b}$ decrease. The confidence intervals for $\alpha,\ a$ and b obtained by the approximate method and the bootstrap method for different values of r are given in Table 6.14. From this table, we can see that the bootstrap confidence intervals are narrower than the approximate confidence intervals for all values of r.

Recalling the simulation study for the full multiple step-stress model under Type-II censoring, we faced some problems in the coverage probabilities for θ_3.

Table 6.9: Estimated coverage probabilities (in %) of confidence intervals for the reduced multiple step-stress model under Type-II censoring for $\alpha,\ a$ and b based on 1000 simulations when $\alpha = 2,\ x_1 = 1,\ x_2 = 1.5, x_3 = 2.5,\ a = 4,\ b = 2,\ n = 150$ and $r = 100$.

					90% C.I.		95% C.I.		99% C.I.	
	τ_1	τ_2	bias	MSE	Approx.	Boot	Approx.	Boot	Approx.	Boot
α	4	6	0.1556	0.2976	90.8	90.4	94.8	95.3	98.8	99
		7	0.0921	0.1894	94.4	89.7	97.2	95.9	99.5	99.1
	5	6	0.1109	0.2677	90.2	90.8	94.9	95.1	98.4	98.9
		7	0.0905	0.2432	93.2	89.9	96.1	95.2	99.2	99.4
		8	0.0686	0.1771	95	90.9	97.7	96.1	99.5	99.5
	6	7	0.0922	0.2069	92.8	89.8	96.5	95.1	99.2	99
		8	0.1141	0.2021	94.1	92.1	97.2	95.7	99.3	99.2
	7	8	0.1014	0.1916	94.1	90.3	97.4	95.9	99.4	99.5
a	4	6	-0.0607	0.3662	99.7	93.5	99.8	96.9	100	99.8
		7	-0.0270	0.3054	99.9	93.4	100	96.7	100	99.4
	5	6	-0.0123	0.3631	99.4	92.5	99.7	97.8	100	99.5
		7	-0.0243	0.2826	99.7	91.3	99.9	96.7	99.9	99.4
		8	0.0094	0.2752	99.9	93.6	100	96.5	100	99
	6	7	-0.0129	0.2733	99.9	92.9	100	96.9	100	99.5
		8	-0.0454	0.2475	99.9	92.3	100	97.2	100	99.8
	7	8	-0.0348	0.2057	100	92	100	95.8	100	99.4
b	4	6	-0.0095	0.0496	100	91.9	100	96.5	100	99.7
		7	0.0007	0.0568	100	91.9	100	96.1	100	99.3
	5	6	0.0031	0.0434	100	91	100	96.2	100	99.4
		7	-0.0033	0.0420	100	90.7	100	95.2	100	98.9
		8	0.0165	0.0593	100	92.5	100	96.9	100	98.8
	6	7	0.0052	0.0361	100	92.1	100	96.5	100	99.5
		8	-0.0086	0.0411	100	91.4	100	96.4	100	99.5
	7	8	-0.0046	0.0296	100	91.5	100	95.9	100	98.9

Table 6.10: Conditional failure probabilities for the reduced multiple step-stress test under Type-II censoring when
$\alpha = 2, \quad x_1 = 1, \; x_2 = 1.5, x_3 = 2.5, \; a = 4, b = 2, \quad n = 250$ and $r = 170$.

		Conditional failure probabilities (in %)		
τ_1	τ_2	$0 < t < \tau_1$	$\tau_1 < t < \tau_2$	$\tau_2 < t < \infty$
4	6	15.15	38.24	46.61
	7	15.27	56.82	27.91
5	6	21.68	19.58	58.74
	7	21.65	38.94	39.41
	8	21.75	56.39	21.87
6	7	28.71	19.81	51.49
	8	28.80	38.59	32.61
7	8	35.84	19.76	44.40

Moreover, the use of the full model becomes complicated if the number of stress levels increase. But we observe that in the reduced multiple step-stress model under Type-II censoring, although the approximate method is not satisfactory, the bootstrap method gives very good coverage probabilities for all parameters. Here, for simplicity, we have presented the results only for three steps in which we have 4 parameters for the full model and 3 parameters in the reduced model. It is evident that the reduced model is simpler and more convenient to use since we need to estimate only three parameters, α, a and b, no matter how many steps we consider in the step-stress test. These suggest that it is advisable to use the reduced model for a multiple step-stress model than the full model due to its simplicity and computation ease.

Table 6.11: Estimated coverage probabilities (in %) of confidence intervals for the reduced multiple step-stress model under Type-II censoring for α, a and b based on 1000 simulations when $\alpha = 2,\ x_1 = 1,\ x_2 = 1.5,\ x_3 = 2.5,\ a = 4,\ b = 2,\ n = 250$ and $r = 170$.

					90% C.I.		95% C.I.		99% C.I.	
	τ_1	τ_2	bias	MSE	Approx.	Boot	Approx.	Boot	Approx.	Boot
α	4	6	0.0434	0.1223	92.9	91	96.4	94.7	99.3	99.6
		7	0.0381	0.0943	95.6	88.5	97.9	94.7	99.5	99.3
	5	6	0.0862	0.1485	90.6	89.9	94.8	95.1	98.9	98.8
		7	0.0730	0.1189	93.2	88.9	96.7	95.1	99.5	99.1
		8	0.0401	0.0910	95.5	90.2	98	95.3	99.5	99
	6	7	0.0649	0.1125	93.1	89.8	97.1	94.9	99.5	98.9
		8	0.0510	0.1048	94.1	89	97.6	94.7	99.6	98.5
	7	8	0.0529	0.0997	94.9	89.4	97.8	94.4	99.9	98.9
a	4	6	0.0084	0.2139	99.9	92.9	100	96.9	100	99.6
		7	-0.0066	0.1642	99.9	91.8	100	96.2	100	99.6
	5	6	-0.0424	0.1985	100	91.9	100	95.9	100	99.7
		7	-0.0329	0.1702	99.8	91.2	100	95.6	100	99.3
		8	0.0048	0.1521	100	91.3	100	95.4	100	99.5
	6	7	-0.0207	0.1500	100	91.4	100	96.1	100	99.2
		8	-0.0117	0.1553	100	89.6	100	94.6	100	99.3
	7	8	-0.0104	0.1268	100	90.5	100	96.4	100	99.4
b	4	6	0.0050	0.0311	100	92.2	100	96.2	100	99.5
		7	-0.0002	0.0300	100	92.3	100	96.9	100	99.1
	5	6	-0.0110	0.0238	100	90.7	100	95.3	100	99.6
		7	-0.0077	0.0254	100	90.9	100	95.5	100	99
		8	0.0095	0.0314	100	90.2	100	96	100	99.3
	6	7	-0.0007	0.0199	100	92.1	100	96.4	100	99
		8	0.0002	0.0257	100	89.6	100	94.8	100	99.3
	7	8	0.0021	0.0181	100	91.4	100	95.5	100	99.1

Table 6.12: Simulated data for the illustrative example.

Stress level	Failure times									
$x_1 = 1$	0.853	1.754	2.352	2.567	2.665	3.802	4.711			
$x_2 = 1.5$	5.113	5.118	5.302	5.307	5.410	5.413	5.624	5.794	6.061	6.218
	6.384	6.457	6.835	6.900	6.996					
$x_3 = 2.5$	7.019	7.033	7.040	7.119	7.139	7.145	7.154	7.207	7.245	7.271
	7.295	7.357	7.425	7.740	7.822	7.898				

Table 6.13: The MLEs of α, a and b and their standard errors.

r	N_3	$\widehat{\alpha}$	Se($\widehat{\alpha}$)	$\widehat{a}$	Se($\widehat{a}$)	$\widehat{b}$	Se($\widehat{b}$)
30	8	1.8767	0.5447	4.0999	0.3953	2.2094	0.3124
35	13	1.8861	0.3826	4.0709	0.3825	2.1906	0.2937
38	16	1.9008	0.4523	3.8613	0.3702	2.0270	0.2760

6.7 Life-Stress Relationships

The representation of the life-stress relationships (through a log-linear link) mentioned earlier in Section 6.5 is a mathematical representation. In this section, we present the most widely used physical relationships such as the Arrehenius relationship for temperature-accelerated tests and the Inverse Power Law. These are all transformable to log-linear link forms. We also describe some other combination relationships where, instead of having only one type of stress, we have two types of stresses on each step.

6.7.1 Arrhenius Life-Tempreture Relationship

The Arrhenius life relationship is used to model product life as a function of temperature. Based on the Arrhenius Law for simple chemical-reaction rates, the relationship is used to describe many products that fail due to chemical reaction or metal diffusion. The relationship is adequate over some range of

Table 6.14: Interval estimation for the simulated data presented in Table 6.12.

		C.I. for α		
r	Method	90%	95%	99%
30	Approx C.I.	(0.9807, 2.7727)	(0.8091, 2.9443)	(0.4736, 3.2798)
	Bootstrap C.I.	(1.2739, 2.4795)	(1.1585, 2.5949)	(0.9328, 2.8206)
35	Approx C.I.	(1.0506, 2.7216)	(0.8906, 2.8816)	(0.5777, 3.1944)
	Bootstrap C.I.	(1.2568, 2.5153)	(1.1363, 2.6359)	(0.9007, 2.8715)
38	Approx C.I.	(1.1568, 2.6447)	(1.0143, 2.7872)	(0.7358, 3.0657)
	Bootstrap C.I.	(1.2766, 2.5250)	(1.1570, 2.6445)	(0.9233, 2.8783)
		C.I. for a		
30	Approx C.I.	(3.4497, 4.7501)	(3.3252, 4.8747)	(3.0817, 5.1181)
	Bootstrap C.I.	(3.7019, 4.4979)	(3.6257, 4.5742)	(3.4767, 4.7232)
35	Approx C.I.	(3.4418, 4.7001)	(3.3212, 4.8206)	(3.0856, 5.0562)
	Bootstrap C.I.	(3.6462, 4.4956)	(3.5649, 4.5770)	(3.4058, 4.7360)
38	Approx C.I.	(3.2523, 4.4703)	(3.1356, 4.5869)	(2.9076, 4.8149)
	Bootstrap C.I.	(3.4144, 4.3081)	(3.3288, 4.3937)	(3.1615, 4.5610)
		C.I. for b		
30	Approx C.I.	(1.6956, 2.7232)	(1.5971, 2.8217)	(1.4047, 3.0141)
	Bootstrap C.I.	(1.9125, 2.5063)	(1.8556, 2.5632)	(1.7444, 2.6744)
35	Approx C.I.	(1.7075, 2.6737)	(1.6150, 2.7662)	(1.4341, 2.9471)
	Bootstrap C.I.	(1.2568, 2.5153)	(1.8538, 2.5274)	(1.7479, 2.6333)
38	Approx C.I.	(1.5730, 2.4810)	(1.4860, 2.5680)	(1.3160, 2.7380)
	Bootstrap C.I.	(1.7359, 2.3182)	(1.6801, 2.3739)	(1.5711, 2.4829)

temperature.

According to the Arrhenius rate law, the rate of a simple (first-order) chemical reaction depends on temperature as follows:

$$t_p = A'\exp[-E/(kV)], \tag{6.6.1}$$

where

t_p is the nominal life which can represent any percentile that is chosen according to the assumed underline distribution,

E is the activation energy of the reaction, usually in electron-volts,

k is Boltzmann's constant, 8.6171 $\times 10^{-}5$ electron-volts per °C,

V is the absolute Kelvin temperature which equals the Centigrade temperature plus 273.16 degrees,

A' is a constant that is a characteristic to the product failure mechanism and test conditions.

The Arrhenius life-temperature relationship can be transformed to a log-linear link form in (6.5.1) by setting

$$x_i = 1/V_i. \tag{6.6.2}$$

6.7.2 Inverse Power Law Relationship

The relationship is sometimes called the inverse power law or simply the power law. This type of relationship models product life as a function of an accelerating stress. Assuming that the accelerating stress variable V is positive, then the inverse power relationship is given by

$$t_p = A/V^{\gamma_1}, \tag{6.6.3}$$

where A and γ_1 are parameters of some characteristics of the product, specimen geometry and fabrication, the test method, etc. Some other equivalent forms of this law are

$$t_p = (A'/V)^{\gamma_1} \text{ and } t_p = A''(V_0/V)^{\gamma_1},$$

where V_0 is a specified standard level of stress. The parameter γ_1 is called the power or exponent. We can transform the inverse power law to a log-linear link form in (6.5.1) by setting

$$x_i = \ln(V_i). \tag{6.6.4}$$

The inverse power law relationship is used for modelling different types of stresses, as described in the following models:

(a) Coffin-Manson relationship:

The inverse power law relationship is used to model fatigue failure of metals subjected to thermal cycling. The number N of cycles to failure as a function of the temperature range δV of the thermal cycle is given by

$$N = A/(\delta V)^B, \tag{6.6.5}$$

where A and B are constants representing characteristics of the metal and test method and cycle. This relationship has been used for mechanical and electronic components. For metals, the B is close to 2 while for plastic encapsulate for microelectronics, B is close to 5.

(b) Palmgren's equation:

Life tests of roller and ball bearings employ high mechanical load. Life in millions of revolutions as a function of load is represented by Palmgren's equation for the 10th percentile B_{10} of the life distribution, namely,

$$B_{10} = (C/P)^p, \tag{6.6.6}$$

where C is the bearing capacity, p is the power, B_{10} is the "B-ten" bearing life, and P is the load in pounds.

(c) Taylor's model:

In Taylor's model for the median life τ of cutting tools, we have

$$\tau = A/V^m, \tag{6.6.7}$$

where V is the cutting velocity (feet/sec), and A and m are constants depending on the tool material, geometry, etc. For high strength steels, it is known that $m \approx 8$, while $m \approx 4$ for carbides, and $m \approx 2$ for ceramics.

6.7.3 Some Other Combination Relationships

In all previous cases, we only consider the life-stress relationship for only one type of stress factor. Now, we introduce some relationships with two types of stresses such as Temperature-Non thermal model. Before considering these models, we first need to generalize the log-linear link function on (6.5.1) to the form

$$\ln\theta_i = a - b_1 x_i - b_2 y_i, i = 1, 2, \cdots, m. \tag{6.6.8}$$

Here, a, b_1 and b_2 are coefficients that we need to develop inference for, and x_i and y_i are accelerating stress levels of two different stress factors.

It is evident that the function in (6.6.8) is linear in both stress factors, and these type of relationships are mostly used since they are mathematically convenient and also physically adequate. We describe below some special cases of the log-linear relationship of this form:

(a) Electromigration:

High current densities in aluminium conductors, which fail from electromigration, promote movement of aluminium atoms which results in voids or extrusions. Accelerated tests of this phenomenon employ elevated temperature T and current density J. Black's formula for median life τ of such conductors is given by the Erying relationship

$$\tau = AJ^{-n} \exp[E/(kT)], \tag{6.6.9}$$

where we may set $x_i = -\ln J_i$ and $y_i = \frac{1}{T}$ to get the log-linear link form in (6.6.8).

(b) Temperature-Humidity test:

Many accelerated life-tests on epoxy packaging for electronics employ high temperature and humidity. Peck surveyed such testing and proposed an

Eyring relationship for life, called Peck's relationship, given by

$$\tau = A(RH)^{-n} \exp[E/(kT)], \tag{6.6.10}$$

where RH is the relative humidity. In this relationship, we may set $x_i = -\ln(RH)$ and $y_i = \frac{1}{T}$ to get the log-linear link form in (6.6.8).

Chapter 7

Multiple Step-Stress Model under Type-I Censoring

7.1 Introduction

In this chapter, we develop inference for the m-step-stress model under Type-I censoring with gamma distributed lifetimes. In Section 7.2, the considered model is described. The MLEs are obtained in Section 7.3. After numerically evaluating the MLEs, we construct confidence intervals for the unknown parameters by using two methods-the asymptotic method and the parametric bootstrap method-in Section 7.4. In Section 7.5, some simulation results and an illustrative example are presented. In Section 7.6, we present the reduced parameter multiple step-stress model under Type-I censoring. The MLEs for that model are derived and the confidence intervals are also constructed in Sections 7.6.1 and 7.6.2, respectively. In Section 7.6.3, a simulation study is carried out on the reduced parameter model and an illustrative example is presented.

7.2 Model Description

In this model, we assume that the failure time data come from a cumulative exposure model, and we consider the m-step-stress model with stress levels $x_1, x_2, \cdots, x_m$ under Type-I censoring with gamma distributed lifetimes. The lifetime distribution at stress level $x_i, i = 1, 2, \cdots, m$, is gamma distribution with common shape parameter α and scale parameter θ_i.

This type of censoring scheme is a generalization of the Type-I censoring discussed in Chapter 3. We start with n identical units at an initial stress level x_1, and at a fixed time τ the stress level is increased to x_2 and the successive failure times are recorded. Then, at the fixed time 2τ, the stress is increased to x_3, and so on. So, the stress level starts with x_1 and changed to $x_2, x_3, \cdots, x_m$ at fixed times $\tau, 2\tau, \cdots, (m-1)\tau$, respectively. The experiment is then terminated at time $m\tau$, where $\tau < 2\tau < \cdots < m\tau$ are fixed in advance. The lifetimes of units larger than $m\tau$ are censored. Let N_k be the random number of units that fail between $(k-1)\tau$ and $k\tau$ at stress level x_k for $k = 1, 2, \cdots, m$. The observed censored sample is given by

$$t_1 < \cdots < t_{N_1} < \tau \leq t_{N_1+1} < \cdots < t_{N_1+N_2} < 2\tau \leq \cdots < (m-1)\tau \leq t_{N_1+\cdots+N_{m-1}+1} < \cdots < m\tau. \quad (7.2.1)$$

7.3 Maximum Likelihood Estimation

The likelihood function based on the observed Type-I censored data in (7.2.1) is presented from which the MLEs of the unknown parameters $\alpha, \theta_1, \theta_2, \cdots, \theta_m$ need to be obtained numerically. The likelihood function of this sample is given by

$$L(\theta|\mathbf{t}) = \frac{n!}{(n-r_m)!} \prod_{k=1}^{m} \left\{ \prod_{i_k=r_{k-1}+1}^{r_k} g_k(t_{i_k}) \right\} \{1 - G_m(m\tau)\}^{n-r_m},$$

$$\text{for } 0 < t_1 < \cdots < t_{r_1} < \tau \leq t_{r_1+1} < \cdots < t_{r_2} < 2\tau \leq \cdots <$$

$$(m-1)\tau \leq t_{r_{m-1}+1} < \cdots < m\tau, \text{ and } 0 < r_m < n, \quad (7.3.1)$$

where

$$r_0 = 0,$$
$$r_k = \sum_{i=1}^{k} N_i, \; k = 1, 2, \cdots, m,$$

and $\mathbf{t}$ is the vector of observed failure time data. The MLEs of $\alpha, \theta_1, \theta_2, \cdots, \theta_m$ exist only when all N_k'$s > 0$. Since we are considering a gamma lifetime distribution at all stress levels, with common shape parameter α and scale parameters θ_i for distribution F_i, the likelihood function of the m-step-stress model is given by

$$\begin{aligned} L(\alpha, \theta_1, \theta_2, \cdots, \theta_m|\mathbf{t}) = &\frac{n!}{(n-r_m)!}\frac{1}{[\Gamma(\alpha)]^{r_m}}\prod_{k=1}^{m}(\theta_k)^{-N_k} \\ &\times \prod_{k=1}^{m}\left\{\prod_{i_k=r_{k-1}+1}^{r_k}(\zeta_{i_k})^{\alpha-1}\right\} \\ &\times \exp\left\{-\sum_{\mathrm{k}=1}^{\mathrm{m}}\left[\sum_{\mathrm{i_k}=\mathrm{r_{k-1}}+1}^{\mathrm{r_k}}\zeta_{\mathrm{i_k}}\right]\right\} \\ &\times \{1 - IG_{\gamma_m}(\alpha)\}^{n-r_m}, \end{aligned}$$

$$\text{for } 0 < t_1 < \cdots < t_{r_1} < \tau \le t_{r_1+1} < \cdots < t_{r_2} < 2\tau \le \cdots < (m-1)\tau \le t_{r_{m-1}+1} < \cdots < m\tau, \quad (7.3.2)$$

where

$$\zeta_{i_k} = \frac{t_{i_k} - (k-1)\,\tau}{\theta_k} + \tau\,\sum_{j=1}^{k-1}\frac{1}{\theta_j}, \quad k = 1, 2, \cdots, m, \quad (7.3.3)$$

and

$$\gamma_m = \tau\sum_{j=1}^{m}\frac{1}{\theta_j}. \quad (7.3.4)$$

As before, it is convenient to work with the log-likelihood function rather than the likelihood function in (7.3.2), which will be given by

$$
\begin{aligned}
&l(\alpha,\theta_1,\theta_2,\cdots,\theta_m|\mathbf{t}) = \ln\left[L\left(\alpha,\theta_1,\theta_2,\cdots,\theta_m|\mathbf{t}\right)\right] \\
&= \ln\left(\frac{n!}{(n-r_m)!}\right) - r_m \ln\left(\Gamma(\alpha)\right) - \sum_{k=1}^{m} N_k \ln\left(\theta_k\right) + (\alpha-1)\sum_{k=1}^{m}\left\{\sum_{i_k=r_{k-1}+1}^{r_k} \ln\left(\zeta_{i_k}\right)\right\} \\
&\qquad - \sum_{k=1}^{m}\left(\sum_{i_k=r_{k-1}+1}^{r_k} \zeta_{i_k}\right) + (n-r_m)\ln\left(1 - IG_{\gamma_m}(\alpha)\right), \\
&\qquad \text{for } 0 < t_1 < \cdots < t_{r_1} < \tau \le t_{r_1+1} < \cdots < t_{r_2} < 2\tau \le \cdots < \\
&\qquad\qquad (m-1)\tau \le t_{r_{m-1}+1} < \cdots < m\tau. \qquad (7.3.5)
\end{aligned}
$$

Differentiating the log-likelihood function in (7.3.5) with respect to α and θ_k gives likelihood equations for finding the MLEs $\widehat{\alpha}$ and $\widehat{\theta_k}$. To find these MLEs, we need the first and second partial derivatives of (7.3.5). The first partial derivatives are given by the following equations:

$$
\begin{aligned}
\frac{\partial l(\alpha,\theta_1,\theta_2,\cdots,\theta_m|\mathbf{t})}{\partial\alpha} &= -r_m\psi(\alpha) + \sum_{k=1}^{m}\left\{\sum_{i_k=r_{k-1}+1}^{r_k} \ln\left(\zeta_{i_k}\right)\right\} \\
&\quad + \frac{(n-r_m)}{1 - IG_{\gamma_m}(\alpha)}\left[\Psi(\alpha) IG_{\gamma_m}(\alpha) - B_1\left(\gamma_m\right)\right], \qquad (7.3.6)
\end{aligned}
$$

$$\frac{\partial l(\alpha,\theta_1,\theta_2,\cdots,\theta_m|\mathbf{t})}{\partial\theta_j} = -\frac{N_j}{\theta_j} + (\alpha-1)\sum_{k=1}^{m}\left\{\sum_{i_k=r_{k-1}+1}^{r_k}\frac{1}{\zeta_{i_k}}\frac{\partial}{\partial\theta_j}(\zeta_{i_k})\right\}$$
$$-\sum_{k=1}^{m}\left(\sum_{i_k=r_{k-1}+1}^{r_k}\frac{\partial}{\partial\theta_j}\zeta_{i_k}\right) + \frac{(n-r_m)(\gamma_m)^{\alpha-1}e^{-\gamma_m}\tau}{\Gamma(\alpha)\theta_j^2\left[1-IG_{\gamma_m}(\alpha)\right]},$$
$$\text{for } j=1,2,\cdots,\ m, \quad (7.3.7)$$

where $\Psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$ and $B_1(v) = \int_0^v \frac{1}{\Gamma(\alpha)}\ln(u)\ u^{\alpha-1}e^{-u}du$.

The maximum likelihood estimates must be obtained numerically because there is no obvious simplification of the above non-linear likelihood equations. Here, numerical maximization is carried out on the log-likelihood using R software. First, we use the log-likelihood function and start with initial values. Then, the function *optim* in R is used to maximize this log-likelihood function. After that, the estimates are found and their confidence intervals are constructed, using the Hessian matrix. The following is the algorithm used in order to find the MLEs in the case when $m = 3$:

(a) Simulate n order statistics from the uniform (0,1) distribution, $U_1, U_2, \ldots, U_n$.

(b) Find N_1 such that $U_{N_1} \le G_1(\tau_1) \le U_{N_1+1}$ where $G_1(\tau_1) = \int_0^{\frac{\tau_1}{\theta_1}} \frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x}dx$.

(c) For $i \le N_1$, $T_i = \theta_1 F^{-1}(U_i)$, where $F(t) = \int_0^t \frac{1}{\Gamma(\alpha)}e^{-x}x^{\alpha-1}dx$.

(d) Find N_2 such that $U_{N_1+N_2} \le G_2(\tau_2) \le U_{N_1+N_2+1}$ where $G_2(\tau_2) = \int_0^{\frac{\tau_2-\tau_1+\frac{\theta_2}{\theta_1}\tau_1}{\theta_2}} \frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x}dx$.

(e) For $N_1+1 \le i \le N_1+N_2$, $\quad T_i = \theta_2\ F^{-1}(U_i) + \tau_1 - \frac{\theta_2}{\theta_1}\tau_1$.

(f) Find N_3 such that $U_{N_1+N_2+N_3} \leq G_3(\tau_3) \leq U_{N_1+N_2+N_3+1}$ where $G_3(\tau_3) = \int_0^u \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} dx$ where $u = \tau_3 - \tau_2 + \frac{\theta_3}{\theta_2}(\tau_2 - \tau_1 + \frac{\theta_2}{\theta_1}\tau_1)$.

(g) For $N_1+N_2+1 \leq i \leq N_1+N_2+N_3$, $T_i = \theta_3 F^{-1}(U_i) + \tau_2 - \frac{\theta_3}{\theta_2}(\tau_2 - \tau_1 + \frac{\theta_2}{\theta_1}\tau_1)$.

(h) Compute the MLEs of $(\alpha,\ \theta_1,\ \theta_2,\ \theta_3)$ based on $T_1, T_2, \ldots, T_{N_1}, T_{N_1+1}, \ldots, T_{N_1+N_2}, T_{N_1+N_2+1}, \cdots, T_{N_1+N_2+N_3}$, say $\hat{\alpha},\ \hat{\theta}_1,\ \hat{\theta}_2$ and $\hat{\theta}_3$.

7.4 Confidence Intervals

In this section, we consider two different methods for constructing confidence intervals (CI) for the unknown parameters $\alpha,\ \theta_1, \theta_2, \cdots, \theta_m$. The first method uses the asymptotic distributions of the MLEs to obtain approximate CIs for $\alpha,\ \theta_1,\ \theta_2,\ \cdots,\ \theta_m$. The second method is based on a parametric bootstrap method.

7.4.1 Approximate Confidence Intervals

We present an approximate method which provides good coverage probabilities for large sample sizes and facilitates easy computation. Elements of Fisher information matrix are found numerically for different values of m, and then the asymptotic normality of the MLEs is used to construct the approximate confidence intervals for $\alpha, \theta_1, \theta_2, \cdots, \theta_m$.

Let $I(\alpha, \theta_1, \theta_2, \cdots, \theta_m) = [I_{ij}(\alpha, \theta_1, \theta_2, \cdots, \theta_m)]$, for $i, j = 1, 2, \cdots, m$, denote the observed Fisher information matrix of $\alpha, \theta_1,\ \theta_2, \cdots, \theta_m$, where

$$I_{ij}(\alpha, \theta_1, \theta_2, \cdots, \theta_m) = -\left(\nabla^2 l(\alpha, \theta_1, \theta_2, \cdots, \theta_m)\right). \tag{7.4.1}$$

For simplicity, we take $m = 3$, then the observed Fisher information matrix (I) is given by

$$I = \begin{bmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{bmatrix}, \tag{7.4.2}$$

where

$$\begin{aligned} I_{11} = \frac{\partial^2 l(\alpha,\theta_1,\theta_2,\theta_3)}{\partial\alpha^2} &= -r_3\psi^{'}(\alpha) + \frac{(n-r_3)}{(1-IG_{\gamma_3}(\alpha))^2} \\ &\times [(1-IG_{\gamma_3}(\alpha))\left[2\psi(\alpha)B_1(\gamma_3) + IG_{\gamma_3}(\psi^{'}(\alpha)-\psi^2(\alpha)) - B_2(\gamma_3)\right] \\ &- [\psi(\alpha)IG_{\gamma_3}(\alpha) - B_1(\gamma_3)]^2], \end{aligned} \tag{7.4.3}$$

$$\begin{aligned} I_{12} = \frac{\partial^2 l(\alpha,\theta_1,\theta_2,\theta_3)}{\partial\alpha\partial\theta_1} &= -\frac{N_1}{\theta_1} - \frac{\tau}{\theta_1^2}\sum_{i=r_1+1}^{r_2}\frac{1}{\zeta_{i_2}} - \frac{\tau}{\theta_1^2}\sum_{i=r_2+1}^{r_3}\frac{1}{\zeta_{i_3}} \\ &+ \frac{(n-r_3)(\gamma_3)^{\alpha-1}e^{\gamma_3}\tau}{\Gamma(\alpha)\theta_1^2\,(1-IG_{\gamma_3}(\alpha))^2} \\ &\times [(1-IG_{\gamma_3}(\alpha))\ln(\gamma_3) - \psi(\alpha) + \mathrm{B}_1(\gamma_3)], \end{aligned} \tag{7.4.4}$$

$$\begin{aligned} I_{13} = \frac{\partial^2 l(\alpha,\theta_1,\theta_2,\theta_3)}{\partial\alpha\;\partial\theta_2} &= -\frac{1}{\theta_2^2}\sum_{i=r_1+1}^{r_2}\frac{(t_{i2}-tau)}{\zeta_{i_2}} - \frac{\tau}{\theta_2^2}\sum_{i=r_2+1}^{r_3}\frac{1}{\zeta_{i_3}} \\ &+ \frac{(n-r_3)(\gamma_3)^{\alpha-1}e^{-\gamma_3}\tau}{\Gamma(\alpha)\theta_2^2\,(1-IG_{\gamma_3}(\alpha))^2} \\ &\times [(1-IG_{\gamma_3}(\alpha))\ln(\gamma_3) - \psi(\alpha) + \mathrm{B}_1(\gamma_3)], \end{aligned} \tag{7.4.5}$$

$$I_{14} = \frac{\partial^2 l(\alpha,\theta_1,\theta_2,\theta_3)}{\partial\alpha\ \partial\theta_3} = -\frac{1}{\theta_3^2}\sum_{i=r_2+1}^{r_3}\frac{(t_{i_3}-2\tau)}{\zeta_{i_3}}$$
$$+\frac{(n-r_3)(\gamma_3)^{\alpha-1}e^{-\gamma_3}\tau}{\Gamma(\alpha)\theta_3^2\left(1-IG_{\gamma_3}(\alpha)\right)^2}\left[(1-IG_{\gamma_3}(\alpha))\ln(\gamma_3)-\psi(\alpha)+\mathrm{B}_1(\gamma_3)\right], \quad (7.4.6)$$

$$I_{22} = \frac{\partial^2 l(\alpha,\theta_1,\theta_2,\theta_3)}{\partial\theta_1^2} = \frac{\alpha N_1}{\theta_1^2}+\frac{(\alpha-1)\tau}{\theta_1^4\ \theta_2}\sum_{i=r_1+1}^{r_2}\frac{2\ \theta_1 t_i+(\theta_2-2\theta_1)\tau}{\zeta_{i_2}^2}$$
$$+\frac{(\alpha-1)\tau}{\theta_1^4\ \theta_2\ \theta_3}\sum_{i=r_2+1}^{r_3}\frac{(2\theta_1\theta_2)(t_i-2\tau)+(\theta_2+2\theta_1)\theta_3\tau}{\zeta_{i_3}^2}$$
$$-\frac{2}{\theta_1^3}\left[(N_2+N_3)\tau+\sum_{i=1}^{r_1}t_i\right]+\frac{(n-r_3)(\gamma_3)^{\alpha-1}e^{-\gamma_3}\tau}{\Gamma(\alpha)\theta_1^4\left(1-IG_{\gamma_3}(\alpha)\right)^2}$$
$$\times\left[(1-IG_{\gamma_3}(\alpha))\left\{\left(1-(\alpha-1)(\gamma_3)^{-1}\right)\tau-2\theta_1\right\}\right.$$
$$\left.-\frac{\tau}{\Gamma(\alpha)}(\gamma_3)^{\alpha-1}e^{-\gamma_3}\right], \quad (7.4.7)$$

$$I_{23} = \frac{\partial^2 l(\alpha,\theta_1,\theta_2,\theta_3)}{\partial\theta_1\partial\theta_2} = -\frac{(\alpha-1)\tau}{\theta_1^2\theta_2^2}\sum_{i=r_1+1}^{r_2}\frac{t_i-\tau}{(\zeta_{i_2})^2}-\frac{(\alpha-1)\tau^2}{\theta_1^2\theta_2^2}\sum_{i=r_2+1}^{r_3}\frac{1}{\zeta_{i_3}^2}$$
$$+\frac{(n-r_3)(\gamma_3)^{\alpha-1}e^{-\gamma_3}\tau^2}{\Gamma(\alpha)\theta_1^2\theta_2^2\left(1-IG_{\gamma_3}(\alpha)\right)^2}$$
$$\times\left[(1-IG_{\gamma_3}(\alpha))\left(1-(\alpha-1)(\gamma_3)^{-1}\right)-\frac{1}{\Gamma(\alpha)}(\gamma_3)^{\alpha-1}e^{-\gamma_3}\right], \quad (7.4.8)$$

$$I_{24} = \frac{\partial^2 l(\alpha, \theta_1, \theta_2, \theta_3)}{\partial\theta_1\ \partial\theta_3} = -\frac{(\alpha-1)\tau}{\theta_1^2\theta_3^2}\sum_{i=r_2+1}^{r_3}\frac{t_i - 2\tau}{\zeta_{i_3}^2}$$
$$+\frac{(n-r_3)(\gamma_3)^{\alpha-1}e^{-\gamma_3}\tau^2}{\Gamma(\alpha)\theta_1^2\theta_3^2\left(1-IG_{\gamma_3}(\alpha)\right)^2}$$
$$\times\left[(1-IG_{\gamma_3}(\alpha))\left(1-(\alpha-1)(\gamma_3)^{-1}\right)-\frac{1}{\Gamma(\alpha)}(\gamma_3)^{\alpha-1}e^{-\gamma_3}\right], \quad (7.4.9)$$

$$I_{33} = \frac{\partial^2 l(\alpha, \theta_1, \theta_2, \theta_3)}{\partial\theta_2^2} = \frac{N_2}{\theta_2^2} - \frac{(\alpha-1)}{\theta_2^4}\sum_{i=r_1+1}^{r_2}\frac{1}{(\zeta_{i2})^2}[(t_{i_2}-\tau)$$
$$\times\left[(t_{i_2}-\tau)+\frac{2\theta_2\tau}{\theta_1}\right]] + \frac{(\alpha-1)\tau}{\theta_1\theta_2^4\theta_3}\sum_{i=r_2+1}^{r_3}\frac{(2\theta_1\theta_2)(t_i-2\tau)+(2\theta_2+\theta_1)\theta_3\tau}{(\zeta_{i_3})^2}$$
$$-\frac{2}{\theta_2^3}\left[N_3\tau+\sum_{i=r_1+1}^{r_2}(t_i-\tau)\right]+\frac{(n-r_3)(\gamma_3)^{\alpha-1}e^{-\gamma_3}\tau}{\Gamma(\alpha)\theta_2^4\left(1-IG_{\gamma_3}(\alpha)\right)^2}$$
$$\times\left[(1-IG_{\gamma_3}(\alpha))\left\{(1-(\alpha-1)(\gamma_3)^{-1})\tau-2\theta_2\right\}-\frac{\tau}{\Gamma(\alpha)}(\gamma_3)^{\alpha-1}e^{-\gamma_3}\right], \quad (7.4.10)$$

$$I_{34} = \frac{\partial^2 l(\alpha, \theta_1, \theta_2, \theta_3)}{\partial\theta_2\ \partial\theta_3} = -\frac{(\alpha-1)\tau}{\theta_2^2\theta_3^2}\sum_{i=r_2+1}^{r_3}\frac{t_i - 2\tau}{\zeta_{i_3}^2}$$
$$+\frac{(n-r_3)(\gamma_3)^{\alpha-1}e^{-\gamma_3}\tau^2}{\Gamma(\alpha)\theta_2^2\theta_3^2\left(1-IG_{\gamma_3}(\alpha)\right)^2}$$
$$\times\left[(1-IG_{\gamma_3}(\alpha))\left(1-(\alpha-1)(\gamma_3)^{-1}\right)-\frac{1}{\Gamma(\alpha)}(\gamma_3)^{\alpha-1}e^{-\gamma_3}\right], \quad (7.4.11)$$

and

$$I_{44} = \frac{\partial^2 l(\alpha,\theta_1,\theta_2,\theta_3)}{\partial\theta_3^2} = \frac{N_3}{\theta_3^2} + \frac{(\alpha-1)}{\theta_3^4}\sum_{i=r_2+1}^{r_3}\frac{(t_{i_3}-2\tau)}{(\zeta_{i3})^2}$$
$$\times\left[(t_{i-3}-2\tau)+\frac{2\theta_3}{\theta_2}\tau+\frac{2\theta_3}{\theta_1}\tau\right]-\frac{2}{\theta_3^3}\sum_{i=r_2+1}^{r_3}(t_i-2\tau)+\frac{(n-r_3)(\gamma_3)^{\alpha-1}e^{-\gamma_3}\tau}{\Gamma(\alpha)\theta_3^4\left(1-IG_{\gamma_3}(\alpha)\right)^2}$$
$$\times\left[(1-IG_{\gamma_3}(\alpha))\left\{(1-(\alpha-1)(\gamma_3)^{-1})\tau-2\theta_3\right\}\right.$$
$$\left.-\frac{\tau}{\Gamma(\alpha)}(\gamma_3)^{\alpha-1}e^{-\gamma_3}\right], \quad (7.4.12)$$

where $B_2(v)=\int_0^v \frac{1}{\Gamma(\alpha)}(\ln(u))^2\ u^{\alpha-1}e^{-u}du$ and $B_1(v)$ is as given before.

It is known that $I_{21}=I_{12}, I_{31}=I_{13}, I_{32}=I_{23}, I_{41}=I_{14}, I_{42}=I_{24}$ and $I_{43}=I_{34}$. Now, the variances and covariances of $\widehat{\alpha},\ \widehat{\theta}_1,\ \widehat{\theta}_2$ and $\widehat{\theta}_3$ can be obtained through the observed Fisher information matrix as

$$Var\begin{bmatrix}\hat{\alpha}\\ \hat{\theta}_1\\ \hat{\theta}_2\\ \hat{\theta}_3\end{bmatrix} = (I)^{-1} = \begin{bmatrix}V_{11} & V_{12} & V_{13} & V_{14}\\ V_{21} & V_{22} & V_{23} & V_{24}\\ V_{31} & V_{32} & V_{33} & V_{34}\\ V_{41} & V_{42} & V_{43} & V_{44}\end{bmatrix}. \quad (7.4.13)$$

The asymptotic distribution of the maximum likelihood estimates are then given by $\frac{\widehat{\alpha}-\alpha}{\sqrt{V_{11}}}\sim N(0,1),\ \frac{\widehat{\theta}_1-\theta_1}{\sqrt{V_{22}}}\sim N(0,1),\ \frac{\widehat{\theta}_2-\theta_2}{\sqrt{V_{33}}}\sim N(0,1)$, and $\frac{\widehat{\theta}_3-\theta_3}{\sqrt{V_{44}}}\sim N(0,1)$, which can be used to construct $100(1-\alpha)\%$ confidence intervals for the parameters $\alpha,\ \theta_1,\ \theta_2$ and θ_3, respectively. These confidence intervals are given by

$$\hat{\alpha}\pm z_{1-\frac{\alpha}{2}}\sqrt{V_{11}}, \quad (7.4.14)$$

$$\hat{\theta}_1 \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{22}}, \tag{7.4.15}$$

$$\hat{\theta}_2 \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{33}} \tag{7.4.16}$$

and

$$\hat{\theta}_3 \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{44}}, \tag{7.4.17}$$

where z_q is the q-th upper percentile of the standard normal distribution.

7.4.2 Bootstrap Confidence Intervals

Confidence intervals based on the parametric bootstrap sampling can be constructed. The following are the steps to generate the bootstrap confidence intervals in the case when $m = 3$:

(a) Compute the MLEs of $\alpha,\ \theta_1,\ \theta_2$ and θ_3, by using the method described in Section 7.3, based on $T_1, T_2, \ldots, T_{N_1}, T_{N_1+1}, \ldots, T_{N_1+N_2}, T_{N_1+N_2+1}, \cdots,$ $T_{N_1+N_2+N_3}$, denoted by $\widehat{\alpha},\ \widehat{\theta}_1,\ \widehat{\theta}_2$ and $\widehat{\theta}_3$.

(b) Simulate n order statistics from the uniform (0,1) distribution.

(c) Find N_1 such that $U_{N_1} \leq G_1^*(\tau_1) \leq U_{N_1+1}$, where $G_1^*(\tau_1) = F_1^*(\tau_1) = \int_0^{\frac{\tau_1}{\widehat{\theta}_1}} \frac{1}{\Gamma(\widehat{\alpha})} x^{\widehat{\alpha}-1} e^{-x} dx$.

(d) For $i \leq N_1$, $T_i = \hat{\theta}_1 F^{*-1}(U_i)$, where $F^*(t) = \int_0^t \frac{1}{\Gamma(\hat{\alpha})} x^{\hat{\alpha}-1} e^{-x} dx$.

(e) Find N_2 such that $U_{N_1+N_2} \leq G_2^*(\tau_2) \leq U_{N_1+N_2+1}$, where $G_2^*(\tau_2) = F_2^*(\tau_2) = \int_0^{\frac{\tau_2 - \tau_1 + \frac{\hat{\theta}_2}{\hat{\theta}_1}\tau_1}{\hat{\theta}_1}} \frac{1}{\Gamma(\hat{\alpha})} x^{\hat{\alpha}-1} e^{-x} dx$.

(f) For $N_1 + 1 \leq i \leq N_1 + N_2$, $T_i = \hat{\theta}_2\ F^{*-1}(U_i) + \tau_1 - \frac{\hat{\theta}_2}{\hat{\theta}_1}\tau_1$.

(g) Find N_3 such that $U_{N_1+N_2+N_3} \leq G_3^*(\tau_3) \leq U_{N_1+N_2+N_3+1}$, where $G_3^*(\tau_3) = F_3^*(\tau_3) = \int_0^y \frac{1}{\Gamma(\widehat{\alpha})} x^{\widehat{\alpha}-1} e^{-x} dx$ and $y = \frac{1}{\widehat{\theta_3}}(\tau_3 - \tau_2 + \frac{\widehat{\theta_3}}{\widehat{\theta_2}}(\tau_2 - \tau_1 + \frac{\widehat{\theta_2}}{\widehat{\theta_1}}\tau_1))$.

(h) For $N_1+N_2+1 \leq i \leq N_1+N_2+N_3$, $T_i = \hat{\theta}_3 F^{-1}(U_i) + \tau_2 - \frac{\hat{\theta}_3}{\hat{\theta}_2}(\tau_2 - \tau_1 + \frac{\hat{\theta}_2}{\hat{\theta}_1}\tau_1)$.

(i) Compute the MLEs of $(\alpha, \theta_1, \theta_2, \theta_3)$ based on $T_1, T_2, \ldots, T_{N_1}, T_{N_1+1}, \cdots, T_{N_1+N_2}, T_{N_1+N_2+1}, \cdots, T_{N_1+N_2+N_3}$, say $\widehat{\alpha}^{(1)}$, $\widehat{\theta}_1^{(1)}$, $\widehat{\theta}_2^{(1)}$ and $\widehat{\theta}_3^{(1)}$.

(j) Repeat steps (b)-(h) B times to obtain B sets of MLEs of α, θ_1, θ_2 and θ_3 .

A two-sided $100(1-\alpha)\%$ bootstrap confidence interval of α, θ_1, θ_2 and θ_3 are then given by

$$CI_\alpha = [\hat{\alpha} - z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\alpha}}}, \hat{\alpha} + z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\alpha}}}], \tag{7.4.18}$$

$$CI_{\theta_1} = [\hat{\theta}_1 - z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta}_1}}, \hat{\theta}_1 + z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta}_1}}], \tag{7.4.19}$$

$$CI_{\theta_2} = [\hat{\theta}_2 - z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta}_2}}, \hat{\theta}_2 + z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta}_2}}] \tag{7.4.20}$$

and

$$CI_{\theta_3} = [\hat{\theta}_3 - z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta}_3}}, \hat{\theta}_1 + z_{1-\frac{\alpha}{2}}\sqrt{MSE_{\hat{\theta}_3}}], \tag{7.4.21}$$

where the $MSE_a = \text{var}(a) + (\text{bias}(a))^2$, and $\text{bias}(a) = \overline{a} - a$. The performance of the approximate confidence intervals and the bootstrap confidence intervals are evaluated using a simulation study in the next section followed by an illustrative example.

7.5 Simulation Study

A simulation study is carried out for different values of τ_1, τ_2 and τ_3. The results are presented in Tables 7.1 to 7.4, and they are based on an average over 1000 replications.

In Tables 7.1 and 7.3, we see how the failures are distributed in our model when we take different values of τ_1, τ_2 and τ_3. We observe that as τ_1 increases, the failure probabilities in the first interval also increase. It can also be seen that the larger the gap between τ_1 and τ_2 the larger the failure probabilities in the second interval $[\tau_1, \tau_2]$, and the smaller the failure probabilities in the third interval $[\tau_2, \tau_3]$. We also observe that the larger the gap between τ_2 and τ_3 the smaller the failure probabilities in the first and the second intervals and the larger the failure probabilities in the third interval. We also can see that the failure probabilities at the first, second and third intervals add up to 100%. The reason for that is because as mentioned earlier, we only consider the case when all N_k'$s > 0$, for $k = 1, 2, 3$, which means that these probabilities are conditional. They were calculated by dividing the number of failures at an interval by the total number of failures at all three intervals.

In Tables 7.2 and 7.4, we see that the MSEs of $\widehat{\theta}_3$ are less than those of $\widehat{\alpha}$, $\widehat{\theta}_1$ and $\widehat{\theta}_2$. The MSEs of the estimates $\widehat{\alpha}$, $\widehat{\theta}_1$, θ_2 and $\widehat{\theta}_3$ are smaller when n is larger (see Table 7.4). We can see that as τ_1 increases the MSEs of $\widehat{\alpha}$ and $\widehat{\theta}_1$ decrease. We also observe that the wider the gap between τ_1 and τ_2 the smaller the MSEs of $\widehat{\alpha}$, $\widehat{\theta}_1$ and $\widehat{\theta}_2$, and the larger the MSE of $\widehat{\theta}_3$. If the third interval $[\tau_2, \tau_3]$ increases, then the MSEs of $\widehat{\alpha}$ and $\widehat{\theta}_3$ decrease, and the MSE of $\widehat{\theta}_2$ decrease.

Table 7.1: Conditional failure probabilities for the multiple step-stress test under Type-I censoring when $\alpha = 2,\ \theta_1 = e^{1.5},\ \theta_2 = e^{1},\ \theta_3 = e^{.5}$ and $n = 150$.

			Conditional failure probabilities (in %)		
τ_1	τ_2	τ_3	$0 < t < \tau_1$	$\tau_1 < t < \tau_2$	$\tau_2 < t < \tau_3$
2	4	5	13.95	48.07	37.98
3	4	5	29.62	27.21	43.17
		8	16.93	15.46	67.61
	7	9	17.05	57.37	25.58
4	5	9	25.41	15.19	59.40
	6	9	26.32	30.11	43.57
5	6	10	34.03	14.33	51.64
	7	10	34.98	27.88	37.14

In Tables 7.2 and 7.4, we observe that the coverage probabilities of the confidence intervals obtained by using the parametric bootstrap method are much closer to the nominal levels than those obtained by using the approximate method for almost all the parameters. From these findings, we would recommend the use of the bootstrap method for the construction of confidence intervals for the model parameters.

7.5.1 Illustrative Example

In this subsection, we consider the data generated with $n = 40$, $\alpha = 2$, $\theta_1 = e^{1.5} = 4.481689$, $\theta_2 = e^{1} = 2.718282$, $\theta_3 = e^{.5} = 1.648721$, $\tau_1 = 5$, $\tau_2 = 7$ and $\tau_3 = 12$. The simulated data are presented in Table 7.5.

We consider three different times $\tau_3 = 10,\ 11,\ 12$. The respective MLEs of α, θ_1, θ_2 and θ_3 and their corresponding standard errors are calculated and are given in Table 7.6. It can be seen from Table 7.6 that the standard errors of $\widehat{\alpha}$ are the smallest compared to the others, while the standard errors of $\widehat{\theta}_2$ are the

Table 7.2: Estimated coverage probabilities (in %) of confidence intervals for a multiple step-stress model under Type-I censoring for $\alpha, \theta_1, \theta_2$ and θ_3 based on 1000 simulations when $\alpha = 2,\ \theta_1 = e^{1.5},\ \theta_2 = e^{1},\ \theta_3 = e^{.5}$ and $n = 150$.

	τ_1	τ_2	τ_3	Bias	MSE	Approx.	Boot	Approx.	Boot	Approx.	Boot
α	2	4	5	0.3186	1.1271	97.1	91.1	98.6	94.7	99.4	97.9
	3	4	5	0.1739	0.4463	99.2	90.3	99.7	94.7	100	99
			8	0.1760	0.4138	86.9	91.5	91.4	96	96.4	99.2
		7	9	0.1324	0.3741	91.1	90.3	95.5	95.3	98.5	99
	4	5	9	0.1118	0.2530	85.3	90.8	90.7	94.4	96.9	98.3
		6	9	0.1012	0.2335	91.1	91.1	95.4	95.5	99	98.2
	5	6	10	0.0849	0.1656	84.9	88.3	90.6	92.6	97.7	97.9
		7	10	0.0888	0.1670	91.5	88.7	95.7	93.4	99.1	98
θ_1	2	4	5	1.2764	37.7601	87	89.8	92.3	93.1	97.1	97.3
	3	4	5	0.4395	8.2510	92	89.2	95.9	94.7	99.1	98.6
			8	0.4300	12.9653	92.9	90.6	96.4	95.3	99	99.1
		7	9	0.5047	7.9598	93.1	89.2	96.9	95.2	99	97.8
	4	5	9	0.2022	3.5580	94.5	90.4	97.5	94.6	99.7	98.4
		6	9	0.1766	3.0939	96.4	90.5	98.6	94.8	99.9	98.4
	5	6	10	0.0859	1.9731	98.2	88.3	99.5	93.1	99.9	97.6
		7	10	0.0511	1.8297	97.2	88.7	99.4	93.4	100	97.7
θ_2	2	4	5	0.0746	1.1082	94.9	89.6	97.3	94.2	99.2	98.4
	3	4	5	0.1204	0.9940	98.7	90	99.4	95.2	100	98.8
			8	0.1233	1.0467	98.9	92.5	99.4	95.9	99.7	98.9
		7	9	0.0187	0.3439	98.3	90.7	99.4	95	99.9	98.9
	4	5	9	0.1128	0.6695	99.6	92.5	99.8	96.3	100	99.2
		6	9	0.0544	0.3671	99.7	90	99.8	95.3	100	98.5
	5	6	10	0.1501	0.7300	99.2	90.9	99.7	96	99.9	98.9
		7	10	0.0279	0.2930	99.7	90.8	100	95.3	100	98.8
θ_3	2	4	5	0.0337	0.2792	97.5	95.1	98.3	97.8	99.4	99.6
	3	4	5	0.0317	0.1970	91.5	97	96	98.9	98.8	99.8
			8	-0.0061	0.0826	97.3	97.7	98.9	99.1	99.9	100
		7	9	0.0215	0.1266	95.8	96.6	98.3	98.4	99.9	99.8
	4	5	9	-0.0165	0.0637	99.2	95.9	99.8	98.4	100	99.8
		6	9	-0.0004	0.0741	99.5	96.1	99.8	98.2	100	99.7
	5	6	10	-0.0003	0.0544	99.5	97	99.9	98.4	100	99.9
		7	10	0.0019	0.0703	97.3	96.3	98.5	98.5	99.8	99.9

Table 7.3: Conditional failure probabilities for the multiple step-stress test under Type-I censoring when $\alpha = 2,\ \theta_1 = e^{1.5},\ \theta_2 = e^{1},\ \theta_3 = e^{.5}$ and $n = 250$.

			Conditional failure probabilities (in %)		
τ_1	τ_2	τ_3	$0 < t < \tau_1$	$\tau_1 < t < \tau_2$	$\tau_2 < t < \tau_3$
2	4	5	14.05	47.96	37.99
3	4	5	29.59	27.25	43.17
		8	16.91	15.38	67.71
	7	9	16.96	57.58	25.46
4	5	9	25.48	15.28	59.24
	6	8	29.04	33.53	37.43
		9	26.20	30.30	43.50
5	6	10	34.05	14.45	51.49
	7	8	43.38	35.08	21.54
		10	34.74	27.90	37.36

largest. We also can see that for each estimate the standard error is the largest when $\tau_3 = 12$, at which case the total number of failures is 40, and since $n = 40$ so this means that there are no censored data at this case.

The confidence intervals for $\alpha,\ \theta_1,\ \theta_2$ and θ_3 obtained by the approximate method and the bootstrap method for different values of τ_3 are given in Table 7.7. From this table, we can see that for θ_1 the bootstrap confidence intervals are narrower than the approximate confidence intervals. For both α and θ_3 the approximate confidence intervals are slightly narrower than the bootstrap confidence intervals.

Table 7.4: Estimated coverage probabilities (in %) of confidence intervals for a multiple step-stress model under Type-I censoring for $\alpha, \theta_1, \theta_2$ and θ_3 based on 1000 simulations when $\alpha = 2,\ \theta_1 = e^{1.5},\ \theta_2 = e^{1},\ \theta_3 = e^{.5}$ and $n = 250$.

						90% C.I.		95% C.I.		99% C.I.	
	τ_1	τ_2	τ_3	bias	MSE	Approx.	Boot	Approx.	Boot	Approx.	Boot
α	2	4	5	0.1760	0.5002	97.3	89.9	98.7	95.1	99.8	98.6
	3	4	5	0.0986	0.2029	99.5	89.9	99.9	94.4	100	98.9
			8	0.0786	0.1782	87.3	90	92.9	94.4	98.2	98.8
		7	9	0.0637	0.1793	92.1	91.7	95.5	95	98.7	98.8
	4	5	9	0.0533	0.1199	95.5	89.7	98.1	95.3	99.5	98.9
		6	8	0.0514	0.1211	97.6	90.8	99.6	95.1	100	98.2
			9	0.0635	0.1237	90.9	89.7	95.5	95.5	98.7	98.8
	5	6	10	0.0431	0.0853	86.4	89.1	92.4	94.5	98.7	98.9
		7	8	0.0450	0.0925	99.6	88.9	100	93.6	100	97.9
			10	0.0387	0.0923	91.4	89.8	95.7	95	99.1	99
θ_1	2	4	5	0.7329	16.0239	86.9	90	91.4	94.7	96.8	98.3
	3	4	5	0.1986	3.5961	93.8	89.4	97.4	94	99.5	98.3
			8	0.2390	3.4936	93.8	89.6	96.8	94.5	99.5	98.4
		7	9	0.3416	3.6475	93.4	90.5	96.3	94.7	99.1	98.5
	4	5	9	0.1422	1.8799	95.5	90.1	98.5	95.4	99.9	98.7
		6	8	0.1432	1.8789	94.3	90.3	98.3	94.4	99.8	98.5
			9	0.0832	1.7071	95.7	90	98.7	94.9	99.8	98.7
	5	6	10	0.0614	1.0964	97.9	89.4	99.4	94.7	99.9	98.9
		7	8	0.0822	1.1618	97.7	89.3	99.2	93.3	99.8	97.8
			10	0.1005	1.1796	97.5	90.1	98.7	94.5	100	98.8
θ_2	2	4	5	0.0562	0.6980	93.5	91.7	96.2	96.3	98.9	98.3
	3	4	5	0.0526	0.4910	99.4	90.4	99.8	95.7	99.8	99
			8	0.1014	0.5423	98.5	88.5	99.5	94.6	100	98.7
		7	9	0.0107	0.2044	98.3	90.1	99.3	94.8	99.9	98.9
	4	5	9	0.0694	0.4026	99.4	89.8	99.5	94.9	99.9	98.5
		6	8	0.0290	0.2052	99.9	89.9	100	95.1	100	99.2
			9	0.0093	0.2162	99.9	89.7	100	95.1	100	99.2
	5	6	10	0.0568	0.3071	99.9	90.2	100	94.6	100	98.9
		7	8	0.0082	0.1818	99.8	89.7	99.9	94.6	100	98.7
			10	0.0330	0.1848	99.8	89.7	99.9	94.6	100	98.7
θ_3	2	4	5	0.0101	0.1450	97.7	96.1	98.9	98.3	99.9	99.6
	3	4	5	0.0155	0.1064	92.1	96.9	96.7	98.9	99.2	100
			8	-0.0016	0.0526	96.8	96.5	98.3	98.3	99.8	100
		7	9	0.0248	0.0676	99.8	95.6	99.9	97.8	100	99.8
	4	5	9	0.0023	0.0372	93	96.6	96.3	98.3	99.1	99.7
		6	8	0.0232	0.0568	96	96.7	98.5	98.3	99.8	99.7
			9	-0.0047	0.0420	96.3	95.7	98.9	98.2	99.8	99.9
	5	6	10	-0.0052	0.0339	96.7	96.8	99.2	98.4	99.9	99.8
		7	8	0.0227	0.0896	98.6	96.5	99.2	98.4	99.8	100
			10	0.0049	0.0426	100	96.3	100	98.8	100	99.8

Table 7.5: Simulated data for the illustrative example.

Stress level	Failure times									
$\theta_1 = e^{1.5}$	0.821	0.909	1.718	2.208	2.322	2.330	2.431	2.832	2.897	3.462
	3.882	4.179	4.635							
$\theta_2 = e^{1}$	5.009	5.190	5.312	5.535	5.602	5.644	5.687	5.951	5.997	6.174
	6.463	6.547	6.785							
$\theta_3 = e^{.5}$	7.080	7.282	7.284	7.986	8.268	8.693	8.756	8.906	9.026	9.489
	9.537	10.338	11.144	11.593						

Table 7.6: The MLEs of α, θ_1, θ_2 and θ_3 and their standard errors.

τ_3	$\widehat{\alpha}$	Se($\widehat{\alpha}$)	$\widehat{\theta_1}$	Se($\widehat{\theta_1}$)	$\widehat{\theta_2}$	Se($\widehat{\theta_2}$)	$\widehat{\theta_3}$	Se($\widehat{\theta_3}$)
10	1.7740	0.1691	4.9798	0.7611	2.0140	1.5266	1.7228	0.2372
11	1.7699	0.1117	4.9967	0.7606	2.0160	1.5238	1.7449	0.2093
12	1.8172	0.1898	4.8097	0.7677	1.9929	1.5507	1.5294	0.2336

7.6 The Reduced-Parameter Model

In this section, we consider a re-parametrization of the m-step-stress model, in which θ_i is assumed to satisfy a log-linear link function of the form

$$\ln\theta_i = a - bx_i, i = 1, 2, \cdots, m, \quad (7.5.1)$$

where a and b are unknown parameters and we need to develop inference only for these two parameters instead of for the original m parameters $\theta_1, \theta_2, \ldots, \theta_m$.

7.6.1 Maximum Likelihood Estimation

The likelihood function is obtained based on the observed Type-I censored data in (7.2.1), and from it the MLEs of the three unknown parameters α, a and b are then obtained numerically. The log-likelihood function of this sample can

Table 7.7: Interval estimation for the simulated data presented in Table 7.5.

		C.I. for α		
τ_3	Method	90%	95%	99%
10	Approx C.I.	(1.4959, 2.0520)	(1.4427, 2.1130)	(1.3385, 2.2094)
	Bootstrap C.I.	(1.1728, 2.3752)	(1.0576, 2.4903)	(0.8325, 2.7155)
11	Approx C.I.	(1.5862, 1.9535)	(1.5510, 1.9887)	(1.4822, 2.0575)
	Bootstrap C.I.	(1.1613, 2.3784)	(1.0447, 2.4950)	(0.8169, 2.7229)
12	Approx C.I.	(1.5050, 2.1295)	(1.4451, 2.1893)	(1.3282, 2.3063)
	Bootstrap C.I.	(1.2507, 2.3838)	(1.1421, 2.4924)	(0.9300, 2.7044)
		C.I. for θ_1		
10	Approx C.I.	(3.7279, 6.2317)	(1.4427, 2.1053)	(3.0193, 6.9403)
	Bootstrap C.I.	(4.0490, 5.9106)	(3.8708, 6.0889)	(3.5223, 6.4374)
11	Approx C.I.	(3.7457, 6.2477)	(3.5060, 6.4874)	(3.0377, 6.9558)
	Bootstrap C.I.	(4.0206, 5.9728)	(3.8336, 6.1598)	(3.4681, 6.5253)
12	Approx C.I.	(3.5489, 6.0705)	(3.3074, 6.3120)	(2.8353, 6.7841)
	Bootstrap C.I.	(3.8811, 5.7383)	(3.7032, 5.9162)	(3.3555, 6.2639)
		C.I. for θ_2		
10	Approx C.I.	(0.0000, 4.5251)	(0.0000, 5.0063)	(0.0000, 5.9465)
	Bootstrap C.I.	(1.4193, 2.6086)	(1.3054, 2.7225)	(1.0828, 2.9452)
11	Approx C.I.	(0.0000, 4.5224)	(0.0000, 5.0026)	(0.0000, 5.9410)
	Bootstrap C.I.	(1.4299, 2.6021)	(1.3176, 2.7144)	(1.0981, 2.9338)
12	Approx C.I.	(0.0000, 4.5436)	(0.0000, 5.0323)	(0.0000, 5.9873)
	Bootstrap C.I.	(1.4387, 2.5471)	(1.3326, 2.6533)	(1.1251, 2.8608)
		C.I. for θ_3		
10	Approx C.I.	(1.3327, 2.1130)	(1.2579, 2.1878)	(1.1118, 2.3339)
	Bootstrap C.I.	(1.1217, 2.3240)	(1.0065, 2.4392)	(0.7814, 2.6643)
11	Approx C.I.	(1.4007, 2.0891)	(1.3348, 2.1551)	(1.2059, 2.2840)
	Bootstrap C.I.	(1.1636, 2.3263)	(1.0522, 2.4377)	(0.8345, 2.6553)
12	Approx C.I.	(1.1452, 1.9135)	(1.0716, 1.9871)	(0.9278, 2.1310)
	Bootstrap C.I.	(1.0113, 2.0474)	(0.9121, 2.1466)	(0.7181, 2.3406)

be written as

$$l(\alpha, a, b|\mathbf{t}) = \ln\left[L\left(\alpha, a, b|\mathbf{t}\right)\right]$$
$$= \ln\left(\frac{n!}{(n-r_m)!}\right) - r_m \ln\left(\Gamma(\alpha)\right) - \sum_{k=1}^{m} N_k(a - bx_k) + (\alpha - 1)\sum_{k=1}^{m}\left\{\sum_{i_k=r_{k-1}+1}^{r_k} \ln\left(\zeta_{i_k}^{*}\right)\right\}$$
$$-\sum_{k=1}^{m}\left(\sum_{i_k=r_{k-1}+1}^{r_k} \zeta_{i_k}^{*}\right) + (n - r_m)\ln\left(1 - IG_{\gamma_m^{*}}(\alpha)\right),$$
$$\text{for } t_1 < \cdots < t_{N_1} < \tau \le t_{N_1+1} < \cdots < t_{N_1+N_2} < 2\tau \le \cdots <$$
$$(m-1)\tau \le t_{N_1+\cdots+N_{m-1}+1} < \cdots < t_{N_1+N_2+\cdots+N_m} < m\tau, \quad (7.5.2)$$

where

$$\zeta_{i_k}^{*} = (t_{i_k} - (k-1)\,\tau)e^{-a+bx_k} + \tau e^{-a} \sum_{j=1}^{k-1} e^{b\ x_j}, \text{for } k = 1,\ 2,\ \cdots,\ m, \quad (7.5.3)$$

and

$$\gamma_m^{*} = \tau\ e^{-a} \sum_{j=1}^{m} e^{bx_j}. \quad (7.5.4)$$

Now, instead of differentiating the log-likelihood function with respect to α and θ_i for $i = 1, 2, \cdots, m$, we differentiate (7.5.2) with respect to α, a and b. As before, we will need the first and second partial derivatives of (7.5.2), but here with respect to the parameters α, a and b. The first partial derivatives are given by the following equations:

$$\frac{\partial l(\alpha, a, b|\mathbf{t})}{\partial \alpha} = -r_m \psi(\alpha) + \sum_{k=1}^{m} \left\{ \sum_{i_k=r_{k-1}+1}^{r_k} \ln\left(\zeta^*_{\mathrm{i_k}}\right) \right\} + \frac{(n-r_m)}{1 - IG_{\gamma^*_m}(\alpha)} \left[\Psi(\alpha) IG_{\zeta^*_m}(\alpha) - B_1\left(\gamma^*_m\right) \right], \quad (7.5.5)$$

$$\frac{\partial l(\alpha, a, b|\mathbf{t})}{\partial a} = -\sum_{k=1}^{m} N_k + (\alpha - 1) \sum_{k=1}^{m} \left\{ \sum_{i_k=r_{k-1}+1}^{r_k} (-1) \right\} - \sum_{k=1}^{m} \left(\sum_{i_k=r_{k-1}+1}^{r_k} -\zeta^*_{i_k} \right) + \frac{(n-r_m)\left(\gamma^*_m\right)^{\alpha} e^{-\gamma^*_m}}{\Gamma(\alpha)\left[1 - IG_{\gamma^*_m}(\alpha)\right]}, \quad (7.5.6)$$

and

$$\frac{\partial l(\alpha, a, b|\mathbf{t})}{\partial b} = \sum_{k=1}^{m} N_k x_k + (\alpha - 1) \sum_{k=1}^{m} \left\{ \sum_{i_k=r_{k-1}+1}^{r_k} \frac{1}{\zeta^*_{i_k}} A_1(t_{i_k}) \right\} - \sum_{k=1}^{m} \left\{ \sum_{i_k=r_{k-1}+1}^{r_k} A_1(t_{i_k}) \right\} - \frac{(n-r_m)\gamma^{*\alpha-1}_m e^{-\gamma^*_m}}{\Gamma(\alpha)\left[1 - IG_{\gamma^*_m}(\alpha)\right]} [\tau e^{-a} \sum_{j=1}^{m} x_j e^{b x_j}], \quad (7.5.7)$$

where

$$\Psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)},$$
$$B_1(v) = \int_0^v \frac{1}{\Gamma(\alpha)} \ln(u)\ u^{\alpha-1} e^{-u} du,$$
$$A_1(t_{i_k}) = (t_{i_k} - (k-1)\tau) x_k e^{-a+bx_k} + \tau e^a \sum_{j=1}^{k-1} x_j e^{bx_j}.$$

Since there is no obvious simplification of the above non-linear likelihood equations, the maximum likelihood estimates of the parameters need to be obtained numerically as mentioned earlier. So, the required numerical maximization was carried out by using the R software. First, we use the log-likelihood function and start with initial values. Then, the function *optim* in R is used to maximize this log-likelihood function. After that, the estimates are found and their confidence intervals are constructed, using the Hessian matrix. Since for the complete parametrization we used $m = 3$, so we will consider the same setting here. The following is the algorithm used to find the MLEs:

(a) Simulate n order statistics from the uniform (0,1) distribution, $U_1, U_2, \ldots, U_n$.

(b) Find N_1 such that $U_{N_1} \leq G_1(\tau_1) \leq U_{N_1+1}$, where $G_1(\tau_1) = \int_0^{\tau_1 e^{-a+bx_1}} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} dx$.

(c) For $i \leq N_1$, $T_i = e^{a-bx_1} F^{-1}(U_i)$, where $F(t) = \int_0^t \frac{1}{\Gamma(\alpha)} e^{-x} x^{\alpha-1} dx$.

(d) Find N_2 such that $U_{N_1+1} \leq G_2(\tau_2) \leq U_{N_1+N_2}$, where $G_2(\tau_2) = \int_0^{y_1} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} dx$ and $y_1 = \frac{\tau_2 + \tau_1 e^{-b(x_2-x_1)} \tau_1}{e^{a-bx_2}}$.

(e) For $N_1 + 1 \leq i \leq N_1 + N_2$, $\quad T_i = e^{a-bx_2}\ F^{-1}(U_i) + \tau_1(1 - e^{-b(x_2-x_3)})$.

(f) Find N_3 such that $U_{N_1+N_2+N_3} \leq G_3(\tau_3) \leq U_{N_1+N_2+N_3+1}$ where $G_3(\tau_3) = \int_0^{y_2} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} dx$ and $y_2 = \frac{\tau_3 - \tau_2 + e^{-b(x_3-x_2)}(\tau_2 + \tau_1 e^{-b(x_2-x_1)}\tau_1)}{e^{a-bx_3}}$.

(g) For $N_1 + N_2 + 1 \leq i \leq N_1 + N_2 + N_3$, $T_i = e^{a-bx_3} \; F^{-1}(U_i) + \tau_2 - e^{-b(x_3-x_2)}(\tau_2 - \tau_1(1 - e^{-b(x_2-x_1)}))$.

(h) Compute the MLEs of (α, a, b) based on $T_1, T_2, \ldots, T_{N_1}, T_{N_1+1}, \ldots, T_{N_1+N_2}, T_{N_1+N_2+1}, \cdots, T_{N_1+N_2+N_3}$, say $\widehat{\alpha}$, $\hat{a}$, $\hat{b}$.

7.6.2 Confidence Intervals

As in Section 7.4, we will use two different methods for constructing confidence intervals (CI) for the unknown parameters α, a and b. The asymptotic distributions of the MLEs is then used to obtain the approximate CIs for α, a and b. Here again, we use the parametric bootstrap method as the second method for constructing confidence intervals for α, a and b.

Approximate Confidence Intervals

We present an approximate method which provides good coverage probabilities for large sample sizes and also facilitates easy computation. Elements of Fisher information matrix are found numerically for different values of m, and then the asymptotic normality of the MLEs is used to construct the approximate confidence intervals for α, a and b.

Let$I(\alpha, a, b) = [I_{ij}(\alpha, a, b)]$, for $i,\ j = 1, 2, 3$, denote the observed Fisher information matrix of $\alpha,\ a$ and b, where

$$I_{ij}(\alpha, a, b) = -\left(\nabla^2 l(\alpha, a, b)\right). \tag{7.5.8}$$

The observed Fisher information matrix (I) is given by

$$I = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}, \tag{7.5.9}$$

where

$$I_{11} = \frac{\partial^2 l(\alpha, a, b)}{\partial \alpha^2} = -r_3 \Psi^{'}(\alpha) + \frac{(n - r_3)}{\left(1 - IG_{\gamma_3^*}(\alpha)\right)^2} \left[\left(1 - IG_{\gamma_3^*}(\alpha)\right)\right.$$
$$\left[2\Psi(\alpha)B_1(\gamma_3^*) + IG_{\gamma_3^*}(\Psi^{'}(\alpha) - \Psi^2(\alpha)) - B_2(\gamma_3^*)\right]$$
$$\left. - \left[\Psi(\alpha) IG_{\gamma_3^*}(\alpha) - B_1(\gamma_3^*)\right]^2 \right., \tag{7.5.10}$$

$$I_{12} = \frac{\partial^2 l(\alpha, a, b)}{\partial \alpha \partial a} = \sum_{k=1}^{3} \sum_{i=r_{k-1}+1}^{r_k} (-1) + \frac{(n - r_3)(\gamma_3^*)^{\alpha} e^{-\gamma_3^*}}{\Gamma(\alpha)\left(1 - IG_{\gamma_3^*}(\alpha)\right)^2}$$
$$\left[\left(1 - IG_{\gamma_3^*}(\alpha)\right)\ln(\gamma_3^*) - \Psi(\alpha) + \mathrm{B}_1(\gamma_3^*)\right], \tag{7.5.11}$$

$$I_{13} = \frac{\partial^2 l(\alpha, a, b)}{\partial\alpha\ \partial b} = \sum_{k=1}^{3} \left\{ \sum_{i=r_{k-1}+1}^{r_k} \frac{1}{\zeta^*_{i_k}} A_1(t_{i_k}) \right\}$$
$$+ \frac{(n-r_3)\gamma_3^{*\alpha-1} e^{-\gamma_3^*} \tau\ e^{-a} \sum\limits_{j=1}^{3} x_j\ e^{b\ x_j}}{\Gamma(\alpha)\left(1 - IG_{\gamma_3^*}(\alpha)\right)^2}$$
$$\times \left\{ \Psi(\alpha) - \left(1 - IG_{\gamma_3^*}(\alpha)\right) \ln\gamma_3^* - \mathrm{B}_1(\gamma_3^*) \right\}, \quad (7.5.12)$$

$$I_{22} = \frac{\partial^2 l(\alpha, a, b)}{\partial a^2} = -\sum_{k=1}^{3} \sum_{i=r_{k-1}+1}^{r_k} \zeta^*_{r_3} + \frac{(n-r_3)(\gamma_3^{*\alpha}) e^{-\gamma_3^*}}{\Gamma(\alpha)\left(1 - IG_{\gamma_3^*}(\alpha)\right)^2}$$
$$\times \left[\left(1 - IG_{\gamma_3^*}(\alpha)\right)(\gamma_3^* - \alpha) - \frac{1}{\Gamma(\alpha)}(\gamma_3^{*\alpha}) e^{-\gamma_3^*} \right], \quad (7.5.13)$$

$$I_{23} = \frac{\partial^2 l(\alpha, a, b)}{\partial a \partial b} = -\sum_{k=1}^{3} \sum_{i=r_{k-1}+1}^{r_k} (-A_1(t_{i_k})) + \frac{(n-r_3)(\gamma_3^{*\alpha}) e^{-\gamma_3^*} \tau e^{-a} \sum_{j=1}^{3} x_j e^{b x_j}}{\Gamma(\alpha)\left(1 - IG_{\gamma_3^*}(\alpha)\right)^2}$$
$$\times \left[\left(1 - IG_{\gamma_3^*}(\alpha)\right)\left((\alpha\gamma_3^{*-1} - 1\right) + \frac{1}{\Gamma(\alpha)}(\gamma_3^{*\alpha-1}) e^{-\gamma_3^*} \right], \quad (7.5.14)$$

and

$$I_{33} = \frac{\partial^2 l(\alpha, a, b)}{\partial b_2} = (\alpha - 1)\sum_{k=1}^{3}\sum_{i=r_{k-1}+1}^{r_k} \frac{1}{\zeta_{i_k}^{*2}} \left\{\zeta_{i_k}^{*} A_2(t_{i_k}) - (A_1(t_{i_k}))^2\right\}$$
$$-\sum_{k=1}^{3}\sum_{i=r_{k-1}+1}^{r_k} A_2(t_{i_k}) - \frac{(n-r_3)\tau e^{-a}\gamma_3^{*\alpha-1}e^{-\gamma_3^*}}{\Gamma(\alpha)\left(1 - IG_{\gamma_3^*}(\alpha)\right)^2}$$
$$\times\ [(1 - IG_{\gamma_3^*}(\alpha))\left\{\sum_{j=1}^{3} x_j^2 e^{bx_j} + \tau e^{-a}\left((\alpha-1)\gamma_3^{*-1} - 1\right)\left(\sum_{j=1}^{3} x_j e^{bx_j}\right)^2\right\}$$
$$+\frac{1}{\Gamma(\alpha)}\gamma_3^{*\alpha-1}e^{-\gamma_{r_3}^*}\tau e^{-a}\left(\sum_{j=1}^{3} x_j e^{bx_j}\right)^2], \quad (7.5.15)$$

where

$$B_1(v) = \int_0^v \frac{1}{\Gamma(\alpha)}(\ln\, u)\ u^{\alpha-1}e^{-u}du,$$
$$B_2(v) = \int_0^v \frac{1}{\Gamma(\alpha)}(\ln\, u)^2 u^{\alpha-1}e^{-u}du,$$
$$A_1(t_{i_k}) = (t_{i_k} - (k-1)\tau)x_k e^{-a+bx_k} + \tau e^a \sum_{j=1}^{k-1} x_j e^{bx_j},$$
$$A_2(t_{i_k}) = (t_{i_k} - (k-1)\tau)x_k^2 e^{-a+bx_k} + \tau e^a \sum_{j=1}^{k-1} x_j^2 e^{bx_j}.$$

It is known that $I_{21} = I_{12}, I_{31} = I_{13}$ and $, I_{32} = I_{23}$. Now, the variances and covariances of $\widehat{\alpha}$, $\widehat{a}$ and $\widehat{b}$ can be obtained through the observed Fisher information matrix as

$$Var \begin{bmatrix} \hat{\alpha} \\ \hat{a} \\ \hat{b} \end{bmatrix} = (I)^{-1} = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix}. \tag{7.5.16}$$

The asymptotic distribution of the maximum likelihood estimates are then given by $\frac{\widehat{\alpha}-\alpha}{\sqrt{V_{11}}} \sim N(0,1)$, $\frac{\widehat{a}-a}{\sqrt{V_{22}}} \sim N(0,1)$ and $\frac{\widehat{b}-b}{\sqrt{V_{33}}} \sim N(0,1)$, which can be used to construct $100\,(1-\alpha)\,\%$ confidence intervals for α, a and b, respectively. These confidence intervals are given by

$$\hat{\alpha} \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{11}}, \tag{7.5.17}$$

$$\hat{a} \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{22}} \tag{7.5.18}$$

and

$$\hat{b} \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{33}}, \tag{7.5.19}$$

where z_q is the q-th upper percentile of the standard normal distribution.

Bootstrap Confidence Intervals

Confidence intervals based on the parametric bootstrap sampling can be constructed. The following are the steps to generate the bootstrap confidence intervals for the case when $m = 3$:

(a) Compute the MLEs of α, a and b, using the method described in Section 7.3, based on $T_1, T_2, \ldots, T_{N_1}, T_{N_1+1}, \ldots, T_{N_1+N_2}, T_{N_1+N_2+1}, \cdots, T_{N_1+N_2+N_3}$, denoted by $\widehat{\alpha}, \widehat{a}$ and $\widehat{b}$.

(b) Simulate n order statistics from the uniform (0,1) distribution, denoted by $U_1, U_2, \ldots, U_n$.

(c) Find N_1 such that $U_{N_1} \leq G_1^*(\tau_1) \leq U_{N_1+1}$, where $G_1^*(\tau_1) = F_1^*(\tau_1) = \int_0^{y_1} \frac{1}{\Gamma(\widehat{\alpha})} x^{\widehat{\alpha}-1} e^{-x} dx$ where $y_1 = \frac{\tau_1}{e^{\widehat{a}-\widehat{b}x_1}}$.

(d) For $i \leq N_1$, $T_i = e^{\widehat{a}-\widehat{b}x_1} F^{*-1}(U_i)$, where $F^*(t) = \int_0^t \frac{1}{\Gamma(\hat{\alpha})} x^{\hat{\alpha}-1} e^{-x} dx$.

(e) Find N_2 such that $U_{N_1+N_2} \leq G_2^*(\tau_2) \leq U_{N_1+N_2+1}$, where $G_2^*(\tau_2) = F_2^*(\tau_2) = \int_0^{y_2} \frac{1}{\Gamma(\hat{\alpha})} x^{\hat{\alpha}-1} e^{-x} dx$, and $y_2 = (\tau_2 - \tau_1) e^{-\widehat{a}+\widehat{b}x_2} + \tau_1 e^{-\widehat{a}+\widehat{b}x_1}$.

(f) For $N_1 + 1 \leq i \leq N_1 + N_2$, $\quad T_i = e^{\widehat{a}-\widehat{b}x_2} F^{*-1}(U_i) + \tau_1 - e^{-\widehat{b}(x_2-x_1)} \tau_1$.

(g) Find N_3 such that $U_{N_1+N_2+N_3} \leq G_3^*(\tau_3) \leq U_{N_1+N_2+N_3+1}$, where $G_3^*(\tau_3) = F_3^*(\tau_3) = \int_0^{y_3} \frac{1}{\Gamma(\hat{\alpha})} x^{\hat{\alpha}-1} e^{-x} dx$, and $y_3 = e^{\widehat{a}-\widehat{b}x_3}(\tau_3 - \tau_2 + e^{-\widehat{b}(x_3-x_2)}(\tau_2 - \tau_1 + e^{-\widehat{b}(x_2-x_1)} \tau_1))$.

(h) For $N_1 + N_2 + 1 \leq i \leq N_1 + N_2 + N_3$, $\quad T_i = e^{\widehat{a}-\widehat{b}x_3} F^{-1}(U_i) + \tau_2 - e^{-\widehat{b}(x_3-x_2)}(\tau_2 - \tau_1 + e^{-\widehat{b}(x_2-x_1)} \tau_1)$.

(i) Compute the MLEs of (α, a, b) based on $T_1, T_2, \ldots, T_{N_1}, T_{N_1+1}, \cdots,$ $T_{N_1+N_2}, T_{N_1+N_2+1}, \cdots, T_{N_1+N_2+N_3}$, say $\widehat{\alpha}^(1), \widehat{a}^{(1)}$ and $\widehat{b}^{(1)}$.

(j) Repeat steps (b)-(h) B times to obtain B sets of MLEs of α, a and b.

A two-sided $100(1-\alpha)\%$ bootstrap confidence interval of α, a and b are then given by

$$CI_\alpha = [\hat{\alpha} - z_{1-\frac{\alpha}{2}} \sqrt{MSE_{\hat{\alpha}}}, \hat{\alpha} + z_{1-\frac{\alpha}{2}} \sqrt{MSE_{\hat{\alpha}}}], \tag{7.5.20}$$

$$CI_a = [\hat{a} - z_{1-\frac{\alpha}{2}} \sqrt{MSE_{\hat{a}}}, \hat{a} + z_{1-\frac{\alpha}{2}} \sqrt{MSE_{\hat{a}}}] \tag{7.5.21}$$

and

$$CI_b = [\hat{b} - z_{1-\frac{\alpha}{2}} \sqrt{MSE_{\hat{b}}}, \hat{b} + z_{1-\frac{\alpha}{2}} \sqrt{MSE_{\hat{b}}}], \tag{7.5.22}$$

where $MSE_s = \text{var}(s) + (\text{bias}(s))^2$, and $\text{bias}(s) = \overline{s} - s$. The performance of the approximate confidence intervals and the bootstrap confidence intervals are evaluated by using a simulation study in the next subsection followed by an illustrative example.

7.6.3 Simulation Study

A simulation study is carried out for different values of $\tau_1,\ \tau_2$ and τ_3. The results are presented in Tables 7.8 to 7.11, and they are based on an average over 1000 replications.

In Tables 7.8 and 7.10, we can see how the failures are distributed in our model when we take different values of $\tau_1,\ \tau_2$ and τ_3. We observe that as τ_1 increases, the failure probability in the first interval also increases. It is seen that as the gap between τ_1 and τ_2 increases the failure probabilities in the second interval $[\tau_1, \tau_2]$ increase, while those in the third interval $[\tau_2, \tau_3]$ decrease. And when the gap between τ_2 and τ_3 increases the failure probabilities in both the first and the second intervals decrease, while those in the third interval increase. Here again, we observe that the failure probabilities at the first, second and third intervals add up to 100%. The reason for that is because as mentioned earlier, we only consider the case when all N_k's > 0, for $k = 1, 2, 3$, which means that these probabilities are conditional. They were calculated by dividing the number of failures at an interval by the total number of failures at all three intervals.

In Tables 7.9 and 7.11, we see that the MSEs of $\widehat{b}$ are less than those of $\widehat{\alpha}$ and $\widehat{a}$. If we look at the MSEs of $\widehat{\alpha},\ \widehat{a}$ and $\widehat{b}$, we see that the wider the gap between τ_1 and τ_2 the smaller the MSEs of all the estimates. When only increasing the

Table 7.8: Conditional failure probabilities for the multiple step-stress test Type-I censoring when $\alpha = 2,\ a = 4,\ b = 2,\ x_1 = 1,\ x_2 = 1.5,\ x_3 = 2.5$ and $n = 150$.

			Conditional failure probabilities (in %)		
τ_1	τ_2	τ_3	$0 < t < \tau_1$	$\tau_1 < t < \tau_2$	$\tau_2 < t < \tau_3$
4	5	9	10.21	12.68	77.11
	6	8	10.48	26.51	63.01
		9	10.55	26.06	63.39
	8	10	10.40	49.50	40.10
5	6	10	14.76	13.16	71.98
	7	8	16.17	29.01	54.83
		10	14.71	26.60	58.69
6	7	8	21.70	14.89	63.41
8	9	10	31.94	14.39	53.67

value of τ_3 and keeping the values of τ_1 and τ_2 the same, we can see that the MSEs of all estimates are approximately the same. By comparing the MSEs in Tables 7.9 and 7.11, we see that for larger n we have smaller MSE values. In Table 7.9, it is seen that the coverage probabilities obtained by using the bootstrap method are closer to the nominal levels than those obtained by using the approximate method. These coverage probabilities based on the bootstrap method are equal or below the the nominal levels for α while they are equal or above the nominal levels for a and b. As both n and r increase (see Table 7.11), the coverage probabilities based on the bootstrap method get closer to the nominal levels.

Illustrative Example

In this subsection, we consider the data generated with $n = 40,\ \ \alpha = 2,\ \ a = 4,\ b = 2,\ x_1 = 1,\ \ x_2 = 1.5\ ,\ x_3 = 2.5\ ,\ \tau_1 = 5\ ,\ \tau_2 = 7$ and $\tau_3 = 9$. The

Table 7.9: Estimated coverage probabilities (in %) of confidence intervals for the reduced multiple step-stress model under Type-I censoring for α, a and b based on 1000 simulations when $\alpha = 2,\ a = 4,\ b = 2,\ x_1 = 1,\ x_2 = 1.5,\ x_3 = 2.5$ and $n = 150$.

						90% C.I.		95% C.I.		99% C.I.	
	τ_1	τ_2	τ_3	Bias	MSE	Approx.	Boot	Approx.	Boot	Approx.	Boot
α	4	5	9	-0.1421	0.2265	87.9	84.4	94.2	90.6	98.7	97.1
		6	8	0.0842	0.2341	99.3	92.3	99.9	95.9	100	98.4
			9	0.0196	0.2115	97.7	86.4	99.4	92.5	99.8	97.9
		8	10	0.0666	0.1514	89.5	88.9	93.4	94.4	98.1	99.3
	5	6	10	-0.0814	0.2095	89.9	86.6	95.4	92.8	99.5	98
		7	8	0.0789	0.1928	95.3	89.6	97.5	95.1	98.9	99
			10	0.0332	0.1827	98.6	90.3	99.6	95	100	99.2
	6	7	8	0.0953	0.1939	95.2	89.7	97.3	94.8	99.7	99.1
	8	9	10	0.0783	0.1392	99.1	88.9	99.7	94.3	100	98.7
a	4	5	9	0.2921	0.6102	87.9	88.5	80	93.3	89.2	97.9
		6	8	-0.0260	0.3554	83.3	92.5	89.2	96.6	96.8	99.5
			9	0.0091	0.3581	82.7	89.3	89.4	94.4	96.3	98.9
		8	10	-0.0526	0.2317	90.9	91.8	94.7	96.2	98.7	99.7
	5	6	10	0.1556	0.4290	78.1	89.5	85.9	94.7	94.8	98.2
		7	8	-0.0271	0.2811	88.5	92.8	93.6	97.1	98	99.7
			10	-0.0092	0.2903	86	90.8	92.8	96.3	98.2	99.9
	6	7	8	-0.0299	0.2484	91.1	91.9	95.2	97	98.8	99.3
	8	9	10	-0.0308	0.1742	94.9	89.9	98	94.7	99.7	99.6
b	4	5	9	0.0729	0.0687	70.3	90.4	78.1	95.5	89.8	99
		6	8	-0.0119	0.0462	80.8	90.9	87	95.9	95.1	99.1
			9	-0.0211	0.0468	80.3	89.9	86.3	94	95	98.9
		8	10	-0.0341	0.0384	86.4	91.5	92.1	95.8	98.9	99.6
	5	6	10	0.0229	0.0505	77	91.1	84	94.9	93.8	98.7
		7	8	-0.0096	0.0381	87.1	90.7	93.1	97.1	97.1	99.6
			10	-0.0261	0.0393	83.5	89.9	90.8	95.4	97.4	99.5
	6	7	8	-0.0044	0.0309	89.5	90.7	94.2	96	98.8	99.4
	8	9	10	-0.0070	0.0251	92.6	90	96.6	94.2	99.7	99.5

Table 7.10: Conditional failure probabilities for the multiple step-stress test under Type-I censoring when $\alpha = 2,\ a = 4,\ b = 2,\ x_1 = 1,\ x_2 = 1.5,\ x_3 = 2.5$ and $n = 250$.

			Conditional failure probabilities (in %)		
τ_1	τ_2	τ_3	$0 < t < \tau_1$	$\tau_1 < t < \tau_2$	$\tau_2 < t < \tau_3$
4	5	9	10.30	12.70	77.00
	6	8	10.36	26.48	63.16
		9	10.28	26.27	63.45
	8	10	10.30	49.87	39.83
5	6	10	14.80	13.30	71.90
	7	8	16.07	29.02	54.91
		10	14.85	26.56	58.59
6	7	8	21.60	15.00	63.39
8	9	10	31.99	14.23	53.78

simulated data are presented in Table 7.12.

We consider two different times $\tau_3 = 5,\ 7$. The respective MLEs of $\alpha,\ a$ and b and their corresponding standard errors are calculated and are given in Table 7.13. It is seen from this table that as τ_3 increases, the standard errors of all estimates $\widehat{\alpha}, \widehat{a}$ and $\widehat{b}$ decrease. The confidence intervals for $\alpha,\ a$ and b obtained by the approximate method and the bootstrap method for different values of τ_3 are given in Table 7.14. Although there is a slight difference between the two intervals, we observe that the bootstrap confidence intervals are narrower than the approximate confidence intervals for each value of τ_3.

Recalling the simulation study for the full multiple step-stress model under Type-I censoring, we faced some problems in the coverage probabilities for θ_3. Moreover, the use of the full model becomes complicated if the number of stress levels increase. But we observe that in the reduced multiple step-stress model under Type-I censoring, although the approximate method is not satisfactory,

Table 7.11: Estimated coverage probabilities (in %) of confidence intervals for the reduces multiple step-stress model under Type-I censoring for α, a and b based on 1000 simulations when $\alpha = 2,\ a = 4,\ b = 2,\ x_1 = 1,\ x_2 = 1.5,\ x_3 = 2.5$ and $n = 250$.

	τ_1	τ_2	τ_3	Bias	MSE	Approx.	Boot	Approx.	Boot	Approx.	Boot
α	4	5	9	-0.0808	0.1394	94.9	86.4	98.2	92.7	99.7	97.3
		6	8	0.0545	0.1234	93.3	89.7	96.3	95.7	99	99.1
			9	0.0132	0.1258	84	89.4	90.2	93.3	95.9	98.4
		8	10	0.0554	0.0912	93.9	88.7	97	94	99.7	98.2
	5	6	10	-0.0590	0.1131	96.5	88.4	98.8	92.7	99.9	98.4
		7	8	0.0716	0.1257	96.5	89	98.7	94.2	99.5	99.2
			10	0.0258	0.1156	85.6	89.1	91.1	93.7	96.1	99
	6	7	8	0.0614	0.1137	98.1	88.5	99	94.6	99.6	98.7
	8	9	10	0.0316	0.0795	99.2	89.2	99.5	94.6	100	98.5
a	4	5	9	0.1593	0.3410	80.7	87.1	88.3	93.6	95.7	98
		6	8	-0.0137	0.2004	90.9	90.7	95.5	96	99.3	99.4
			9	0.0182	0.2287	88.6	90.3	94.5	94.9	98.5	98.7
		8	10	-0.0499	0.1505	95.5	90	98.7	94.8	99.9	98
	5	6	10	0.0980	0.2425	88.1	90	93.8	94.3	98.1	98.8
		7	8	-0.0408	0.1721	93.1	91.1	96.8	95.8	99.5	99.7
			10	-0.0136	0.1712	92.8	89.1	97.1	94.6	99.6	99.2
	6	7	8	-0.0231	0.1502	95.3	90.7	97.9	95.3	99.8	98.9
	8	9	10	-0.0022	0.0987	98.2	89.7	99.3	94.2	100	98.6
b	4	5	9	0.0362	0.0403	75.1	89.1	83.9	95.6	93.2	99.2
		6	8	-0.0035	0.0258	87.5	90.8	92.4	95.5	97.9	99
			9	-0.0046	0.0293	84.8	89.9	90.8	94	97.1	98.6
		8	10	-0.0252	0.0248	90.5	89.9	94.8	94.9	99	98.8
	5	6	10	0.0147	0.0297	84	90.7	89.4	95.1	96.4	99.1
		7	8	-0.0137	0.0229	89.3	90.8	94.4	95.7	99.5	99.4
			10	-0.0170	0.0230	88.7	89.7	94.8	94.3	99	99.2
	6	7	8	-0.0060	0.0181	93	90.5	96.5	95.3	99.5	98.7
	8	9	10	0.0008	0.0137	96.6	90.7	98.9	94.7	99.8	99

the bootstrap method gives very good coverage probabilities for all parameters. Here, for simplicity, we have presented the results only for three steps in which we have 4 parameters, for the full model and 3 parameters in the reduced model. it is evident that the reduced model is simpler and more convenient to use since we need to estimate only three parameters, α, a and b, no matter how many steps we consider in the step-stress test. These suggest that, it is advisable to use the reduced model for a multiple step-stress model than the full model due to its simplicity and computational ease.

Table 7.12: Simulated data for the illustrative example.

Stress level	Failure times									
$x_1 = 1$	1.998	2.526	3.261	3.369	3.910	3.958	4.465			
$x_2 = 1.5$	5.545	6.299	6.312	6.865	6.947	6.964	6.987			
$x_3 = 2.5$	7.002	7.005	7.020	7.030	7.036	7.089	7.093	7.098	7.143	7.151
	7.228	7.276	7.297	7.317	7.347	7.371	7.465	7.521	7.538	7.558
	7.718	7.755	7.8250	7.955	8.009					

Table 7.13: The MLEs of α, a and b and their standard errors.

τ_3	$\widehat{\alpha}$	Se($\widehat{\alpha}$)	$\widehat{a}$	Se($\widehat{a}$)	$\widehat{b}$	Se($\widehat{b}$)
8	1.9845	0.4274	4.3899	0.3473	2.2372	0.2430
9	1.8438	0.2109	4.5000	0.3439	2.2420	0.2184

Table 7.14: Interval estimation for the simulated data presented in Table 7.12

		C.I. for α		
τ_3	Method	90%	95%	99%
8	Approx C.I.	(1.2815, 2.6875)	(1.1468, 2.8222)	(0.8836, 3.0854)
	Bootstrap C.I.	(1.2588, 2.7101)	(1.1198, 2.8491)	(0.8481, 3.1208)
9	Approx C.I.	(1.4969, 2.1905)	(1.4304, 2.2571)	(1.3005, 2.3870)
	Bootstrap C.I.	(1.1228, 2.5647)	(0.9847, 2.7028)	(0.7148, 2.9727)
		C.I. for a		
8	Approx C.I.	(3.8186, 4.9612)	(3.7091, 5.0706)	(3.4952, 5.2845)
	Bootstrap C.I.	(3.8922, 4.8875)	(3.7968, 4.9829)	(3.6105, 5.1692)
9	Approx C.I.	(3.9343, 5.0656)	(3.8260, 5.1740)	(3.6142, 5.3858)
	Bootstrap C.I.	(3.9982, 5.0018)	(3.9020, 5.0979)	(3.7141, 5.2858)
		C.I. for b		
8	Approx C.I.	(1.8375, 2.6368)	(1.7609, 2.7134)	(1.6113, 2.8630)
	Bootstrap C.I.	(1.9238, 2.5505)	(1.8638, 2.6106)	(1.7464, 2.7279)
9	Approx C.I.	(1.8487, 2.6353)	(1.7734, 2.7107)	(1.6261, 2.8579)
	Bootstrap C.I.	(1.8827, 2.6014)	(1.8138, 2.6703)	(1.6793, 2.8048)

Chapter 8

Computational Methods

8.1 Introduction

This chapter includes some information about the Sharcnet, which is used to accelerate the computation of the coverage probabilities. We use two algorithms, series and parallel. In Section 8.2, the structure and the facilities of the sharcnet are mentioned, and the performance of the parallel algorithm is tested. The *optim* function is used in R, which has the option of using different methods of optimization such as: Nelder and Mead, BFGS, CG, L-BFGS-B, SANN. Each method is explained explicitly in Section 8.3. After that, in Section 8.4, a comparison of these methods is made to optimize the likelihood function of the step stress model of Type-I censoring.

8.2 High Performance Computing (HPC)

A large part of this thesis considers the computational aspect since there are no closed-form solution for the MLEs, which have to be determined numerically. Another obstacle is the intensive computations for finding the coverage probabilities of the bootstrap confidence intervals. The time it took to complete the bootstrap step is between 4 to 6 hours not forgetting the massive memory that it required. So, it is necessary to accelerate those intensive computations, which can not be done even on today's leading desktop systems. Luckily, we had the opportunity to use Sharcnet, which is a consortium of Canadian academic institutions who share a network of high performance computers. It exists to enable world-class computational research so as to accelerate the production of research results by providing unattainable computing resources, building common computing environments, and promoting remote collaboration and research.

HPC is concerned with varied issues such as hardware, algorithms and software. In hardware, communicating working processors is the most difficult idea since interconnecting hardware is a very complex task. Also, sharing memory is easy to say but hard to realize as systems scale. In software, although parallel algorithms are well understood, applying those algorithms in software is non-trivial. Building parallel machines is used to communicate multiple processors, which can be done in the following ways:

(a) Symmetric Multiprocessors (SMP) in which the memory is shared with uniform memory access (UMA). In this system, each processor executes different data with capability of sharing common resources connected by a system bus or a crossbar. Speeding up the shared memory data access and reducing the system bus traffic are both achieved because of the private high speed memory known as cache memory associated to each processor.

(b) Non-Uniform Memory Access (NUMA), which provides separate memory for each processor, avoiding the performance impact when several processors attempt to address the same memory. For problems involving spread data, NUMA can improve the performance over a single shared memory by a factor of roughly the number of processors. Of course, not all data end up being confined to a single task, which means that more than one processor may be required to perform the same data. To handle these cases, NUMA systems include additional hardware or software to move data between banks. This operation slows the processors attached to those banks, so the overall speed increase due to the heavily dependence of NUMA on the exact nature of tasks that are running.

(c) Clusters in which the components are connected through a fast local area networks where each node runs its own sets of operating system. Those clusters are the result of convergence of some of the computing trends such as the availability of low cost microprocessors, high speed networks, and software for high performance computing. There are different designs of the cluster depending on how tightly coupled the individual nodes may be. The Beowulf system is one of the designs in which the application programs never see the computational nodes, also known as slave computers, but only interact with the Master computer. This specific Master computer handles the scheduling and management of the slaves and it has two network interfaces. One of those networks communicates with the private Beowulf network for the slaves and the other for the general purpose network for the organization.

The Sharcnet systems are built in clusters and share memory and offers running programs in parallel across multiple machines or in series. If the program was written without parallelism in mind, then there is very little that can be done to run it automatically in parallel. Some compilers are able to translate some serial portion of a program, such as loops, into equivalent parallel code. Also, some libraries are able to use parallelism internally, without any change in the user's program. For this to work, the program needs to spend most of its time in the library which doesn't speed up the program itself. So, to gain the true parallelism and scalability, the code must be rewritten using the message passing interface (MPI) library (Rmpi library in R software). However, it is not always the case that running a single program will be faster using parallelism. Often, one might want to run many different configurations of a program, differing only in a set of input parameters which is exactly what we have. Since it is possible to implement this kind of loosely-coupled parallelism using MPI, it is often less efficient and more difficult, so it's usually best to start out doing this as a series of independent serial jobs.

To use the Rmpi library, we had different methods to transform the code into a parallel algorithm, and they are : the Brute Force method, the task push method, and the task pull method. The basic steps of all these three methods is to first divide the problem into multiple sub-problems so that a slave will be spawned to handle one of these problems. After that, in the Master part, we write a code for the common data and functions that are used for each problem. Then, again in the Master we send the common data, and the functions to the slaves. Finally, we call back the results from each slave and close those slaves.

Paralleling the algorithm, that is used to compute the coverage probabilities of the bootstrap confidence intervals, is challenging. The problem is the dependency of many parameters that are passing between processors. So, instead of paralleling the whole algorithm, we only translate some serial portion of it into equivalent parallel code. However, it is still expensive timewise. So, we run multiple independent serial jobs instead of running one parallel job, which worked very well and accelerated the computation. Instead of waiting 4-6 hours for only one result, the same amount of time resulted in 10 to 15 results.

8.2.1 Parallel Computing

In this subsection, the performance of the parallel algorithm is evaluated. The task is to generate 1000 MLEs of α, θ_1 and θ_2 for the simple step-stress model under Type-II censoring. The log-likelihood function is maximized starting at some initial values for the parameters. We first start by spawning a number of slave processes to perform the work. After that, we send the data and the functions needed to perform the task to all slave processors. Then we call back the results, which in our case the MLEs of α, θ_1 and θ_2 from each slave.

The performance measures that are used to study the scalability of the parallel algorithm is as follows:

Parallel Execution Time T_p: The time elapsed from the moment a parallel computation starts to the moment the last processor finishes execution.

Table 8.1: Performance measures for different number of processors in the parallel algorithm.

N	loop size	T_p	S	Sf	e
1	1000	8.107	1.0000	-	100%
2	500	7.231	1.1211	0.7840	56.06%
4	250	6.387	1.2693	0.6818	31.73%
5	200	5.370	1.5097	0.5780	30.19%
8	125	4.720	1.7176	0.5225	21.47%
10	100	4.942	1.6404	0.5662	16.40%
20	50	4.647	1.7446	0.5507	8.72%

Speedup S: The ratio of the serial execution time of the fastest known serial algorithm (T_1) to the parallel execution time of the chosen algorithm (T_p), which is given by

$$S = \frac{T_1}{T_p}. \tag{8.2.1}$$

Serial Fraction Sf: The ratio of the serial component of an algorithm, which is the part of the algorithm that cannot be paralleled and had to be executed on a single processor, to its execution time on one processor. It is given by

$$Sf = \frac{1/S - 1/N}{1 - 1/N}, \tag{8.2.2}$$

where N is the number of processors.

Efficiency e: The ratio of speedup to the number of processors and is used to normalize the speedup value to a certain percentage. It is given by

$$e = \frac{S}{N}. \tag{8.2.3}$$

The algorithm is implemented in two different versions, namely, series and parallel. The serial algorithm, mentioned in Chapter 3, is used as the base for the parallel version. In this algorithm, we use the Brute force method to communicate the master with each slave. The parallel algorithm proceeds in the following manner:

(a) Load the Rmpi package and spawn the slaves;

(b) Write the parameters and the likelihood function;

(c) Combine the series algorithm as one function, including the loop at which n order statistics are simulated and the MLEs are calculated;

(d) Send the parameters and the functions to each slave;

(e) Run the code on each slave and generate some equal number of MLEs;

(f) Send the set of MLEs from each slave back to the master;

(g) The master receives an equal number of MLEs from each slave;

(h) Close the slaves and quit.

The function that is used to transmit an R command from the master to all spawned slave processors is the `mpi.bcast.cmd`. This function is the MPI name of the one-to-all broadcast operation, which is used to send identical data to all other processors or to a subset of them. The function that is used to gather each slave's results as a list to the master, is the `mpi.gather.Robj`. This function is the MPI name of the gather operation, that is the dual of one-to-all personalized operation, and it collects a unique message from each processor.

In Table 8.1, it is seen that as the number of processors increases, the parallel execution time decreases, but after using 8 processors, it is almost equal. This can be seen clearly in Figure 8.1. We can also observe, from Table 8.1, that the speedup increases as the number of processors increase. However, it is not a linear speedup, as seen in Figure 8.2. It is also seen that when we use 10 processors the parallel execution time is larger than using 8 processors. The smallest parallel execution time occurs when using 20 processors. The efficiency decreases as the number of processors increases, which means that small gains in speedup come at the cost of inefficient machine use. It is noticeable that we are finding only 1000 estimates, as this number increases the efficiency will be more satisfactory.

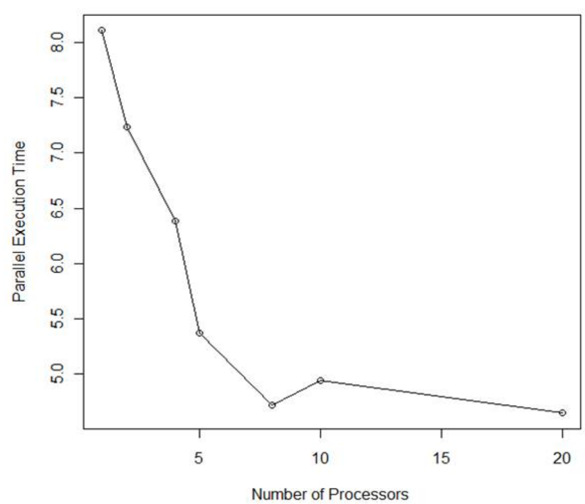


Figure 8.1: The parallel execution time (in seconds) vs. the number of processors.

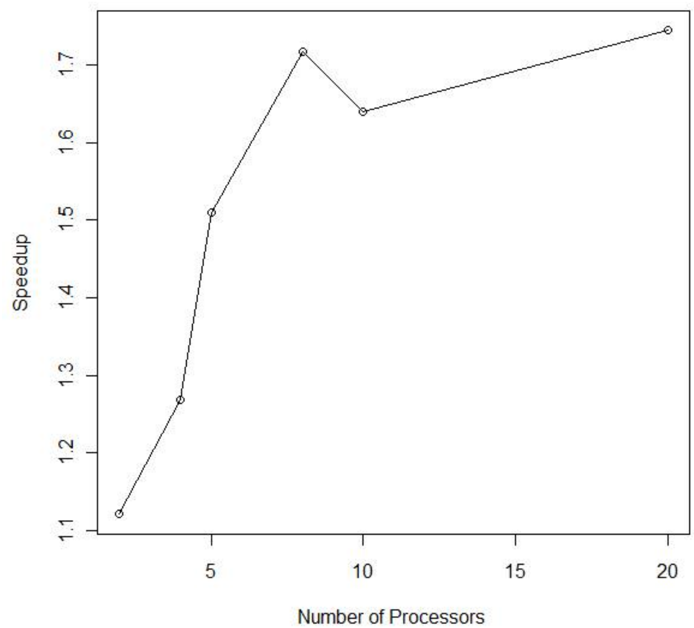


Figure 8.2: The speedup vs. number of processors.

8.3 Overview of Algorithms

All optimization algorithms require a starting point which is usually denoted by x_0. Then, each algorithm generates a sequence of iterates $\{x_k\}_{k=0}^{\infty}$ that are terminated when either no more progress can be made or when an optimum point has been reached and approximated with sufficient accuracy. Starting with the initial value x_0 and moving to the final result can be done in different strategies. There are two types of algorithms: a derivative-free optimization algorithm and a derivative-based optimization algorithm.

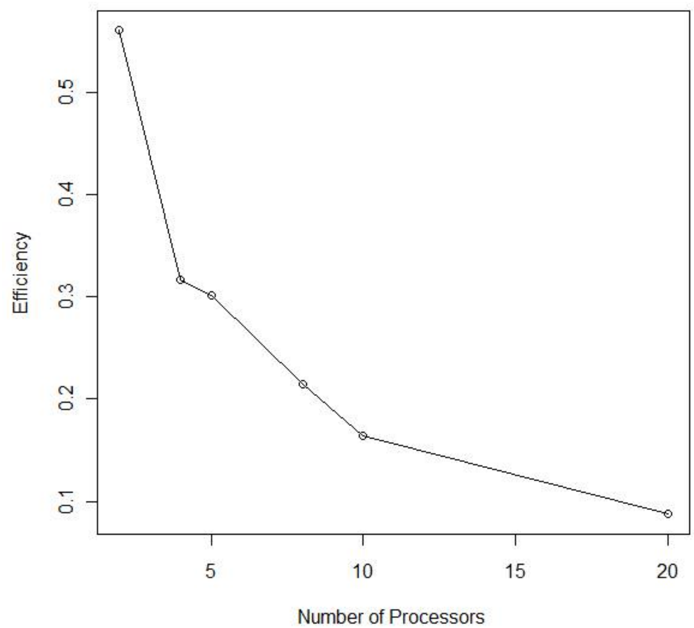


Figure 8.3: Efficiency vs. number of processors

8.3.1 Derivative Free Optimization (DFO) Algorithms

Those algorithms differ in the way they use the sampled function values to determine the new iterate. The most widely used DFO methods include the simplex-reflection method of Nelder and Mead, and simulated annealing. In practice, it is not always the case that the derivatives are available due to many different reasons. One of those reasons might be unknown analytical form of the objective function f, for example, the evaluation of $f(x)$ can be the result of an experimental measurement or a stochastic simulation. Another reason might be the difficulties of coding the function's derivatives which might be time consuming or impractical. One might think of using automatic differentiation tools to solve that problem but those tools may not be applicable in all cases. Having $f(x)$ being provided in the form of binary computer code is one of these cases. Even when the code is available but is written in a combination of languages, then the automatic tools cannot be applied.

Nelder-Mead Method

This simplex reflection method is the most popular DFO method that was proposed by John Nelder and Roger Mead (1965). Given a simplex S with vertices $\{z_1, z_2, ..., z_{n+1}\}$, the algorithm starts by seeking to remove the vertex with the worst function value and replace it with another point with a better value. This new point is obtained by reflecting, expanding or contracting the simplex along the line joining the worst vertex with the centroid of the remaining vertices. If no better point is found in that manner, then the vertex with the best function value is retained, and the simplex is shrunk by moving all other vertices toward this value. Although the performance of the Nelder-Mead algorithm is often reasonable, a stagnation could be encountered at non-optimal points. Restarting the algorithm could be used as a solution, but it could cost some time. It is also notable that the average function value given by

$$\frac{1}{n+1}\sum_{i=1}^{n+1} f(x_i) \tag{8.3.1}$$

will decrease at each step of the algorithm. Even if the function f is convex, the shrinkage step is guaranteed to decrease the average function value.

Simulated Annealing

This simulated annealing (SANN) method is a probabilistic heuristic method for the global optimization problems of locating a good approximation to the global optimum of a given function in a large search set of possible solutions. The basic idea is to generate a path from one solution to another nearby solution which will eventually lead to the optimum solution. Each step of the SANN algorithm replaces the current solution by a random solution. The new solution may then be accepted with a probability that depends on the difference between the corresponding function values. This probabilistic acceptance function is given by

$$P(i-j) = \begin{cases} 1 & \text{if} \quad f(j) \leq f(i) \\ \exp(\frac{\mathrm{f(i)-f(j)}}{\mathrm{c}}) & \text{if} \quad f(j) > f(i) \end{cases}, \qquad (8.3.2)$$

where c is a control parameter analogous to temperature in a physical system. So, steps are not strictly required to produce improved solution, but each step has a certain probability of leading to improvement. At the start, all steps are equally likely but as the algorithm progresses, the tolerance for the worst solutions decreases eventually to the point where only improvements are accepted. That is what makes it attain the optimum solution without trapping in local optima. It is noted that the SANN method depends critically on the settings of the control parameters so a well-formulated implementation will guarantee an answer, but it might take a long time to be attained. Although it is not a general purpose method, it is very useful in getting a good value on a very rough surface.

8.3.2 Gradient-Based Optimization Algorithms

The line search is one of those strategies that are used to move from the current point x_k to a new iterate x_{k+1}. If the aim is to find the minimum (maximum) of a function f, then the line search strategy will choose a direction p_k and search along this direction from the current iterate x_k for a new iterate with a lower (higher) function value. The distance to move along p_k can be found by approximately solving the following one-dimensional minimization problem to find a step length α:

$$min_{\alpha>0} f(x_k + \alpha p_k). \tag{8.3.3}$$

Since solving (8.3.3) exactly may be too expensive and unnecessary, the line search algorithm generates a limited number of trial step lengths until it finds one that approximates the minimum. Then a new search direction and step length are computed at the new point, and the process is repeated. This means that at each iteration of a line search method, a search direction p_k is computed and then the length that is needed to be moved in that direction is decided, and that iteration is given by

$$x_{k+1} = x_k + \alpha_k p_k. \tag{8.3.4}$$

Knowing the fact that a function f is guaranteed to be reduced along a descent direction p_k, one for which $p_k^T \nabla f_k < 0$, most of the line search algorithms require a descent direction which often has the following form:

$$p_k = -B_k^{-1} \nabla f_k, \tag{8.3.5}$$

where B_k is a symmetric and non-singular matrix.

The major issue in the line search method is to choose a direction p_k and the following are the different search directions for that method.

The steepest descent direction

This method is the most obvious choice for search direction for a line search method. It moves along $p_k = -\nabla f_k$ at every step. Although it is an advantage that it only requires calculation of the gradient ∇f_k and not the second derivatives, it can be extremely slow on difficult problems.

The Newton direction

This direction is derived from the second-order Tylor series approximation to $f(x_k + p)$, and it is given by

$$p_k = -(\nabla^2 f_k)^{-1} \nabla f_k. \tag{8.3.6}$$

In order for this direction method to be suitable, it must satisfy the descent property $p_k^T \nabla f_k < 0$, which means that the Hessian $\nabla^2 f_k$ must be positive definite, otherwise $(\nabla^2 f_k)^{-1}$ may not exist. Methods that use the Newton direction have a fast rate of local convergence, typically quadratic. Convergence to high accuracy often occurs in just a few iterations, after a neighbourhood of the solution is reached. Since explicit computation of the Hessian $\nabla^2 f_k$ can sometimes be a burdensome, error-prone and expensive process, the need of the Hessian would be in this case the main disadvantage.

The Quasi-Newton directions

This direction is the best solution of the Newton direction problem. It does not require computation of the Hessian and yet attains a super-linear rate of convergence. Instead of the true Hessian $\nabla^2 f_k$, an approximation B_k is obtained and is updated after each step. The new Hessian approximation B_{k+1} is chosen to satisfy the following condition, known as the secant equation:

$$B_{k+1} s_k = y_k, \tag{8.3.7}$$

where

$$s_k = x_{k+1} - x_k, \; y_k = \nabla f_{k+1} - \nabla f_k. \tag{8.3.8}$$

Two of the most popular formulas for updating the Hessian approximation B_k are the symmetric-rank-one formula and the BFGS formula. The quasi-Newton search direction is obtained by using B_k in place of the exact Hessian in (8.3.6), that is,

$$p_k = -B_k^{-1} \nabla f_k. \tag{8.3.9}$$

(a) The BFGS Method:

It is one of the most popular quasi-Newton algorithms which was published in 1970 by Broyden, Fletcher, Goldfarb and Shanno. Each step of the BFGS method has the form

$$x_{k+1} = x_k - \alpha_k H_k \nabla f_k, \tag{8.3.10}$$

where α_k is the step length and H_k is updated at every iteration by means of the formula

$$H_{k+1} = V_k^T H_k V_k + \rho_k s_k s_k^T, \tag{8.3.11}$$

where

$$\rho_k = \frac{1}{y_k^T s_k}, \; V_k = I - \rho_k y_k s_k^T. \tag{8.3.12}$$

where s_k and y_k are as given in (8.3.8). The cost of the performance of each iteration is of $O(n^2)$ arithmetic operations plus the cost of function and gradient evaluations, which is lower than the cost of performance in linear system solves or matrix matrix operations which is of $O(n^3)$. This BFGS algorithm is robust and its rate of convergence is super-linear, which is fast for most practical purposes. It can work with the Hessian approximation B_k instead of H_k, and the update formula for B_k is given by

$$B_{k+1} = B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{y_k y_k^T}{y_k^T s_k}.$$

One of the significant advantages of the BFGS formula is having a very effective self-correcting property. In the case of incorrectly estimating the curvature in the objective function and slowing down the iteration, the Hessian approximation will tend to correct itself within a few steps. Unfortunately, the inverse Hessian approximation H_k will generally be dense, so the cost of storing and manipulating is too expensive when the number of variables is large.

(b) Limited Memory BFGS method (L-BFGS):

This method is based on the BFGS formula and it is useful for solving large problems whose Hessian matrices cannot be computed at a reasonable cost. Unlike the original BFGS method which stores a dense $n \times n$ approximation, the L-BFGS method stores a few vectors of length n that represent the approximation implicitly.

Storing a modified version of H_k implicitly is done by storing a certain number, say m, of the vector pairs $\{s_i, y_i\}$ used in (8.3.6), (8.3.11) and (8.3.12). After the new iterate is computed, the oldest vector pair in the set of pairs $\{s_i, y_i\}$ is replaced by the new pair $\{s_k, y_k\}$ obtained from the current step. This way, the set of vector pairs includes curvature information from the m most recent iterations. It has been shown that modest values of m (say between 3 to 20) produced satisfactory results.

This method is the best choice for large problems in which the true Hessian is not sparse. It outperforms the Newton method and the Hessian-free Newton methods such as Newton-conjugate gradient approaches. The main drawback of this method is that it converges slowly on ill-conditioned problems, especially when the Hessian matrix contains a wide distribution of eigenvalues.

The nonlinear conjugate gradient direction

They have the form

$$p_k = -\nabla f_k + \beta_k p_{k-1}, \tag{8.3.13}$$

where β_k is a scalar that ensures that p_k and p_{k-1} are conjugate. Conjugate gradient methods were originally designed to solve systems of linear equations $Ax = b$, where the coefficient matrix A is symmetric and positive definite. Solving this linear system is equivalent to minimizing the convex quadratic function defined by

$$\Phi(x) = \frac{1}{2}x^T Ax - b^T x.$$

Hence, extensions of these algorithms to more general types of unconstrained minimization problems were investigated. One of the advantages of these nonlinear conjugate gradient directions is that they are more effective than the steepest decent direction and are almost as simple to compute. Another advantage is that they do not require storage of matrices. However, as a drawback, these directions do not attain the fast convergence rates of Newton or quasi-Newton methods.

The Conjugate Gradient Method It is a conjugate direction method with a special property. When generating the set of conjugate vectors and trying to compute a new vector p_k, it does not need to know all the previous elements $p_0, p_1, \ldots, p_{k-2}$ of the conjugate set. It only uses the immediate previous vector p_{k-1} since p_k will automatically be conjugate to all other vectors, which means this strong property would result in requiring the method to have fewer storage space and computations.

In the conjugate gradient method, each direction p_k is chosen to be a linear combination of the previous direction p_{k-1} and the negative residual $-r_k$ given by

$$r_k = Ax_k - b, \tag{8.3.14}$$

and the new direction p_k is given by

$$p_k = -r_k + \beta_k p_{k-1}, \tag{8.3.15}$$

where β_k is determined by satisfying the conjugate property of p_{k-1} and p_k with respect to A.

8.4 Illustrative example

As in Chapter 3, we consider the simple gamma step-stress model under Type-I censoring and the time to failure data coming from a cumulative exposure model with stress levels x_1 and x_2. We start with n identical units subjected to an initial stress level x_1. After a pre-specified time τ_1, the stress in increased to x_2. The experiment is terminated at a pre-fixed time τ_2. We denote N_1 for the random number of units that fail before τ_1, and N_2 for the random number of units failing between τ_1 and τ_2. The likelihood function that we are optimizing, with respect to α, θ_1 and θ_2, is given by

$$L(\alpha,\theta_1,\theta_2|\mathbf{t}) = \frac{n!}{(n-N)!}\frac{(\prod_{1}^{N_1} t_i)^{\alpha-1}(\prod_{N_1+1}^{N} y_i)^{\alpha-1}}{\Gamma(\alpha)^N \theta_1^{\alpha N_1}\theta_2^{\alpha N_2}} \times e^{-\frac{1}{\theta_1}\sum_{1}^{N_1} t_i - \frac{1}{\theta_2}\sum_{N_1+1}^{N} y_i}(1-IG_{t^*}(\alpha))^{n-N}$$

$$0 < t_{1:n} < ... < t_{N_1:n} < \tau_1 \le t_{N_{1+1:n}<...<t_{N:n}<\tau_2}, \quad (8.4.1)$$

where $N = N_1 + N_2, y_i = t_i - \tau_1 + \frac{\theta_2}{\theta_1}\tau_1$ and $t^* = \frac{\tau_2-\tau_1+\frac{\theta_2}{\theta_1}\tau_1}{\theta_2}$.

For simplicity, we consider the log-likelihood function which is given by

$$\begin{aligned} l(\alpha,\theta_1,\theta_2|t) &= \ln[L(\alpha,\theta_1,\theta_2|t)] \\ &= \ln(c) - N\ln\Gamma(\alpha) - \alpha N_1 \ln\theta_1 - \alpha N_2 \ln\theta_2 - \frac{1}{\theta_1}\sum_{1}^{N_1} t_i - \frac{1}{\theta_2}\sum_{N_1+1}^{N} y_i \\ &\quad + (\alpha-1)\sum_{1}^{N_1}\ln t_i + (\alpha-1)\sum_{N_1+1}^{N}\ln y_i + (n-N)\ln(1-IG_{t^*}(\alpha)). \quad (8.4.2) \end{aligned}$$

The aim here is to maximize this non-linear log-likelihood function with respect to the parameters α, θ_1 and θ_2. Since the first- and second-order partial derivatives, as shown in Chapter 3, have explicit forms, we can exclude both the Nelder-Mead and SANN methods. Those two methods are most useful in the case when there is no explicit form of the Hessian matrix. So, we now want to find the most suitable gradient-based method for optimizing our function.

The `optim` function is given by

Table 8.2: MLE's of α, θ_1 and θ_2 and their standard errors by using different optimization methods.

	$\hat{\alpha}$	$\widehat{\theta_1}$	$\widehat{\theta_2}$
Nelder-Mead	1.95294600	3.07289500	1.09714300
BFGS	1.95327973	3.07193982	1.09723571
CG	1.95387008	3.07033605	1.09702448
L-BFGS-B	1.95321910	3.07206719	1.09724170
SANN	1.95597641	3.03686709	1.08175779

```
optim(par, fn, gr = NULL, ...,method = c("Nelder-Mead", "BFGS",
"CG", "L-BFGS-B", "SANN"),lower = -Inf, upper = Inf,
control = list(), hessian = FALSE)
```

The *optim* function takes the objective function to be optimized and optimizes it by starting with the initial values provided for the parameters. This function performs minimization by default, but it will maximize if control is negative. Considering the log-likelihood function in (8.4.2) as the objective function and maximizing it with respect to α, θ_1 and θ_2, would result in the MLE's $\widehat{\alpha}, \widehat{\theta_1}$ and $\widehat{\theta_2}$, respectively.
In this example, we take $n = 40, \alpha = 2, \theta_1 = e^1 = 2.718282, \theta_2 = e^{0.5} = 1.648721, \tau_1 = 4$ and $\tau_2 = 8$. The initial values that are used in the `optim` function are $\alpha = 2.1, \theta_1 = 2.5$ and $\theta_2 = 1.5$. The MLE's are found by using the different optimization methods, including the Nelder-Mead and SANN methods, and the results are given in Table 8.2.

In Table 8.2, we see that all the methods converge, even the Nelder-Mead and the SANN method, and that the MLEs obtained by all of them are almost equal. In Table 8.3, we observe that the MSEs of $\widehat{\alpha}$ and $\widehat{\theta_1}$ are the smallest when using the

Table 8.3: MLE's of α, θ_1 and θ_2 and their standard errors by using different optimization methods.

	bias($\hat{\alpha}$)	MSE($\hat{\alpha}$)	bis($\widehat{\theta_1}$)	MSE($\widehat{\theta_1}$)	bis($\widehat{\theta_2}$)	MSE($\widehat{\theta_2}$)
Nelder-Mead	-0.0471	0.0112	0.3546	0.1841	-0.5516	0.3060
BFGS	-0.0467	0.0112	0.3537	0.1833	-0.5515	0.3059
CG	-0.0461	0.0111	0.3521	0.1819	-0.5517	0.3062
L-BFGS-B	-0.0468	0.0112	0.3538	0.1834	-0.5515	0.3059
SANN	-0.0440	0.0104	0.3186	0.1545	-0.5670	0.3231

SANN method, but the difference is very small. So, it is not clear which method gives the best estimates.

Table 8.4 shows the CPU time of each optimization method. In this table, we see that the fastest methods are BFGS and the L-BFGS-B, and the slowest method is the SANN method. The `Optim` function returns list of values of which one is 'counts', which gives the number of times `optim` called the function and its gradient while obtaining the optimum value. Table 8.5 shows these values. From this table, we observe that the counts for calling the function using the L-BFGS-B method is the smallest and that for calling the gradient is the smallest using the BFGS method. The SANN method, by default, searches for a finer value of the parameter 10000 times even if it reaches the optimum before. So, we can conclude that the BGFS and the L-BFGS-B methods are the best optimization methods to optimize the likelihood function in (8.4.2). This function is optimized with respect to three parameters, α, θ_1 and θ_2, and so it is more convenient to use the BFGS method. But, in the case of the multiple step-stress model with more than 3 steps, it might be more convenient to use the L-BFGS-B method.

Table 8.4: CPU time (in seconds) for finding MLE's of α, θ_1 and θ_2 by using different optimization methods.

	Used	System	Elapsed
Nelder-Mead	0.06	0.06	0.13
BFGS	0.04	0.01	0.06
CG	0.20	0.20	0.22
L-BFGS-B	0.06	0.01	0.06
SANN	2.33	0.02	2.36

Table 8.5: The number of times `optim` called the function and its gradient using different optimization methods.

	Function	gradient
Nelder-Mead	104	NA
BFGS	25	12
CG	270	101
L-BFGS-B	14	14
SANN	10000	NA

Chapter 9

Conclusions and Further Research

9.1 Concluding Remarks

In this thesis, we have considered the simple step-stress model with two stress levels, which applies stress to each unit and changes the stress at a pre-specified time during the life-testing experiment. This model is studied when the observed failure time data are (1) Type-II censored, (2) Type-I censored, (3) Progressively Type-II censored, and (4) Progressively Type-I censored. We have also discussed the multiple step-stress model under Type-II and Type-I censoring. The likelihood functions have been derived assuming a cumulative exposure model with gamma distributed lifetimes with common shape parameter α and scale parameters θ_1, θ_2, (and θ_3). The likelihood equations needed to be solved numerically since they do not allow closed-form solutions. An optimization program has been used for maximizing the log-likelihood function to obtain the MLEs $\widehat{\alpha}, \widehat{\theta}_1, \widehat{\theta}_2$ (and $\widehat{\theta}_3$). This optimization is achievable by different optimization methods such as Nelder-Mead, BFGS, CG, L-BFGS-B and SANN, which have all been examined in order to find the best method that is most convenient for optimizing the pertinent likelihood function. Confidence intervals for the parameters $\alpha, \theta_1, \theta_2$ (and θ_3) have been discussed by two methods: the approximate method by using the asymptotic normality of the maximum likelihood estimates, and the parametric bootstrap method. All the methods of inference developed here have been demonstrated by an empirical study as well as through many numerical examples.

It has been shown that the best optimization method for determining the maximum likelihood estimates of the model parameters is the BFGS method. From the simulation studies that have been carried out, we have observed that the approximate method of constructing confidence intervals is unsatisfactory in terms of coverage probabilities, especially when the sample size is not large. On the other hand, the parametric bootstrap method almost always maintains its coverage probabilities at the nominal level. Some examples have also been presented to illustrate these methods of inference and to support the conclusions made.

In the m-step-stress model under Type-II and Type-I censoring, we have considered a reduced model by assuming a log-linear link function, which links the scale parameter θ_i of the lifetime distribution to the stress level x_i. This mathematical representation is convenient for modeling as well as computation. Moreover, many known physical relationships can be transformed to this log-linear link form. The MLEs have been derived for the full-parameter model as well as the reduced-parameter model when the data are Type-II and Type-I censored. We have also constructed confidence intervals in this case by using the approximate method and the parametric bootstrap method. Here again, all these inferential results have been illustrated by empirical study and some numerical examples.

A large part of this thesis is regarding the computational aspect, since there are no closed-form solutions for the MLEs and that they need to be determined numerically. We faced some problems while computing the coverage probabilities of the parametric bootstrap confidence intervals. The main problem is the time it takes to complete the bootstrap step, and we made use of Sharcnet for completing this computational task. This network and a discussion on how it was utilized for the computational work in this thesis has been presented as well.

9.2 Further Research

Although we have covered here many important and interesting aspects of the gamma step-stress model, we still have many more theoretical as well as practical problems that are worth considering for further study. Below are some of the problems that deserve further attention:

1. We have considered the Type-II, Type-I, Progressive Type-II and Progressive Type-I censoring and developed corresponding inferential results. As an extension, other types of censoring could be considered such as the Hybrid Type-I censoring and the Hybrid Type-II censoring. In the Hybrid Type-I censoring, the experiment is terminated at a pre-fixed time (say τ_m) if a specific number of failures (say r) occur after the time τ_m, and otherwise the test is terminated as soon as the r-th failure occurs. Thus, the termination time of the life-test is $\tau_m^* = min\{\tau_m, T_{r,n}\}$. On the other hand, in the Hybrid Type-II censoring, the life-test is terminated at a pre-fixed time τ_m if the r-th failure occurs before time τ_m, and otherwise the test is terminated as soon as the r-th failure occurs.

Thus, the termination time of the life-test in this case is $\tau_m^* = max\{\tau_m, T_{r,n}\}$. It will then be of interest to develop the inferential procedures under these two hybrid censored samples.

2. Here, we have discussed the direct way of the maximum likelihood estimation for the model parameters. One can develop an EM algorithm for this purpose. This is an iterative method requiring two primary calculations on each iteration: Computation of a specific conditional expectation of the log-likelihood (E-step) and then maximization of this expectation over the relevant parameters (M-step). Another method is the Monte Carlo EM method, which is a modification of the EM algorithm wherein the required expectation in the E-step is computed numerically be means of Monte Carlo simulations. A comparison between these methods may then be done by means of computational efficiency, time and cost.

3. Use of informative prior(s) for computing the Bayesian estimates of the model parameters may be of interest as well. It is well known that in general if a proper prior information on model parameters is available, it is better to use the informative prior(s) than the non-informative prior(s).

4. There has been some work on predicting the times to failure of censored items for a simple step-stress model with progressive censoring when the failure times are exponentially distributed. This prediction result could be generalized to the case of the step-stress test under the gamma distribution that has been considered here.

5. Through out this thesis, we have relied on Nelson's cumulative exposure model for step-stress tests. Unfortunately, one of the consequences of this model is that

the corresponding hazard function has discontinuities at the points at which the stress levels are changed as the hazard rate jumps at those points. For this reason, Kannan, Kundu and Balakrishnan (2010) discussed a step-stress model with lagged effects which allows for a lag period before the effects of the change in stress are observed. Incidentally, this model also results in a continuous hazard function. The work of these authors for this model based on the exponential lifetime distribution can be generalized to the gamma case.

6. As mentioned in this thesis, there are different types of acceleration, and in this thesis, we have only considered the simple and multiple step-stress tests. Consideration of some other types of stress loading such as progressive stress loading, cyclic stress loading, or random stress loading would be of great interest since these types of stress loading are widely used in reliability analysis.

Appendix A

Series and Parallel Algorithms

Below are R series and parallel algorithms for computing 1000 MLEs of the parameters α, θ_1 and θ_2 of the simple step-stress model under Typ-II censoring.

The series algorithm:

```
rm(.Random.seed)

n<-40

r=30

shape<-2

theta1<-exp(1) # (=2.718282)

theta2<-exp(.5) #( =1.648721)


m0<-function(x)

{

  loglik<- function (x)

  {-(  -r*lgamma(x[1])-x[1]*N1*log(x[2])-x[1]*N2*log(x[3])
```

```
        +(x[1]-1)*sum(log(cc[1:N1]))+(x[1]-1)*sum(log(cc[w:r]-tau1
        +(x[3]/x[2])*tau1))-(1/x[2])*sum(cc[1:N1])-(1/x[3])*sum(cc[w:r]
        -tau1+(x[3]/x[2])*tau1)+(n-r)*log(1-pgamma((cc[r]-tau1+(x[3]/x[2])
        *tau1)/x[3],x[1],1)))
  }
  W=optim(c(2.1,3,2), loglik, method="BFGS", hessian= TRUE)
  W$par
}

estimates<-list()
g<-1000
for (k in 1:g){
          U<-runif(n)
          Us<-sort(U)
          tau1<-4
          F1tau<- pgamma((tau1/theta1),shape,1)
          N1<-sum(Us<F1tau)
          if (N1>=n | N1<=0|N1>=r)next
          X1<- Us[1:N1]
          Ti<- theta1*qgamma(X1,shape,1)
          w=N1+1
          V<- Us[w:r]
          N2<- length(V)
          V1<-theta2*qgamma(V,shape,1)
```

```
          Tj<-V1+tau1-(theta2/theta1)*tau1
          c1<-matrix(Ti,ncol=1)
          c2<-matrix(Tj,ncol=1)
          cc<-data.frame(rbind(c1,c2))[,1]
          estimates[[k]]<- as.data.frame(t(m0(cc)))
          }
          R<- do.call(rbind, estimates)
          R
```

The parallel algorithm:

```
library(Rmpi)
mpi.spawn.Rslaves(nslaves=5)
mpi.setup.rngstream(seed=1:6)
mpi.bcast.cmd( id <- mpi.comm.rank() )
mpi.bcast.cmd( np <- mpi.comm.size() )
mpi.bcast.cmd( host <- mpi.get.processor.name() )
result <- mpi.remote.exec(paste("I am", id, "of", np, "running on", host))
print(unlist(result))


n<-40
r=30
shape<-2
theta1<-exp(1) # (=2.718282)
theta2<-exp(.5) #( =1.648721)
```

```
m0<-function(x)
{
  loglik<- function (x)
     {
     -(
        -r*lgamma(x[1])-x[1]*N1*log(x[2])-x[1]*N2*log(x[3])
        +(x[1]-1)*sum(log(cc[1:N1]))+(x[1]-1)*sum(log(cc[w:r]-tau1
        +(x[3]/x[2])*tau1))-(1/x[2])*sum(cc[1:N1])-(1/x[3])*sum(cc[w:r]
        -tau1+(x[3]/x[2])*tau1)+(n-r)*log(1-pgamma((cc[r]-tau1
        +(x[3]/x[2])*tau1)/x[3],x[1],1)))
     }
     W=optim(c(2.1,3,2), loglik, method="BFGS", hessian= TRUE)
     W$par
}


loop1=function(n){
 m0<-function(x)
{
  loglik<- function (x)
     {
     -(
        -r*lgamma(x[1])-x[1]*N1*log(x[2])-x[1]*N2*log(x[3])
```

```
        +(x[1]-1)*sum(log(cc[1:N1]))+(x[1]-1)*sum(log(cc[w:r]-tau1
        +(x[3]/x[2])*tau1))-(1/x[2])*sum(cc[1:N1])-(1/x[3])*sum(cc[w:r]
        -tau1+(x[3]/x[2])*tau1)+(n-r)*log(1-pgamma((cc[r]-tau1
        +(x[3]/x[2])*tau1)/x[3],x[1],1)))
     }
     W=optim(c(2.1,3,2), loglik, method="BFGS", hessian= TRUE)
     W$par
}

estimates<-list()
g<-400
i1=0
repeat{   U<-runif(n)
          Us<-sort(U)
          tau1<-4
          F1tau<- pgamma((tau1/theta1),shape,1)
          N1<-sum(Us<F1tau)
          if (N1>=n | N1<=0|N1>=r)next
          i1=i1+1
          X1<- Us[1:N1]
          Ti<- theta1*qgamma(X1,shape,1)
          w=N1+1
          V<- Us[w:r]
          N2<- length(V)
```

```
          V1<-theta2*qgamma(V,shape,1)
          Tj<-V1+tau1-(theta2/theta1)*tau1
          c1<-matrix(Ti,ncol=1)
          c2<-matrix(Tj,ncol=1)
          cc<-data.frame(rbind(c1,c2))[,1]
          estimates[[i1]]<- as.data.frame(t(m0(cc)))
          if(i1 == g) break
          }
          R<- do.call(rbind, estimates)
         R2=data.frame(R)
         R2
          }
x <- loop1(40)
mpi.bcast.cmd(x <- loop1(40))
mpi.bcast.cmd(mpi.gather.Robj(x))
y <- mpi.gather.Robj(x)
y
mpi.bcast.cmd(n)
mpi.bcast.cmd(r)
mpi.bcast.cmd(shape)
mpi.bcast.cmd(theta1)
mpi.bcast.cmd(theta2)
mpi.bcast.cmd(m0)
mpi.bcast.cmd(loop1)
```

```
y[,1]

proc.time()

mpi.close.Rslaves()

mpi.quit(save="no")
```

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