Power Optimization for Amplify-and-Forward Half-Duplex Two-Way Relay System

POWER OPTIMIZATION FOR AMPLIFY-AND-FORWARD HALF-DUPLEX TWO-WAY RELAY SYSTEM

BY

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To Whom I Love And Who Love Me

Abstract

In this thesis, we consider an amplify and forward half-duplex two-way-relay wireless communication system. For such a system, we estimate the channel information using a particular test signal set. The actual transmitted signal from a normalized square QAM constellation is then detected by choosing the symbol in the QAM constellation closest in distance to it. We derive an error probability which is found to be signal dependent for this system. An optimum design problem under a transmission power constraint based on this signal dependent asymptotic formula is then formulated leading to the optimum transmission power condition. Simulations show that under this optimum power transmission condition the system indeed yields optimum performance.

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Acronyms

AF	Amplify-and-Forward
DF	Decode-and-Forward
QAM	Quadrature Amplitude Modulation
SER	Symbol Error Rate
IID	Independent and Identically Distributed
MIMO	Multiple Input Multiple Output
MISO	Multiple Input Single Output
SNR	Signal to Noise Ratio
LS	Least Square

Notation and abbreviations

a	Scalar a, lowercase letter denotes scalar
$\mathbf{a}$	Vector $\mathbf{a}$, boldface lowercase letter denotes column vector
$\mathbf{A}$	Matrix $\mathbf{A}$, boldface uppercase letter denotes matrix
$\mathbf{A}^T$	Transpose of Matrix $\mathbf{A}$
$\mathbf{A}^H$	Hermitian of Matrix $\mathbf{A}$
$\mathbb{E}[\cdot]$	Expectation operator
$\mathbf{I_T}$	T by T identity matirx
ln	Natural Logarithm
$[s(1), s(2), ..., s(p)]^T$	A length P vector
$\mathbf{x} \sim \mathcal{CN}(\mathbf{0}, \Sigma)$	Complex vector $\mathbf{x}$ is Gaussian distributed with zero mean and covariance matrix Σ
x^*	Conjugate of complex number x
j	$\sqrt{-1}$

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Chapter 1

Introduction

1.1 Wireless Communications

Wireless communications involve the transference of information between two or more points that are not physically connected. Distance between these points may be short, such as few meters, or may be as long as thousands or even millions of kilometers for which connection by wires is impractical. In wireless communication systems, signals are transmitted by electromagnetic wave propagations through the atmosphere. The presence of reflectors in the surrounding of the transmitter and the receiver may create multiple paths through which a transmitted signal may travel. The signals transmitted through these multiple paths often interfere with each other. As a result, the receiver sees the superposition of multiple copies of the transmitted signal, each going through a different path. Each signal copy will experience different attenuation, delay and phase shift while traveling from the source to the destination. This can result in either constructive or destructive interference. Strong destructive interference is frequently referred to as a deep fade. This is illustrated in Fig.1.1. Another kind of fading due to shadowing from obstacles may affect the wave propagation in the transmission path and is referred to as shadow fading.

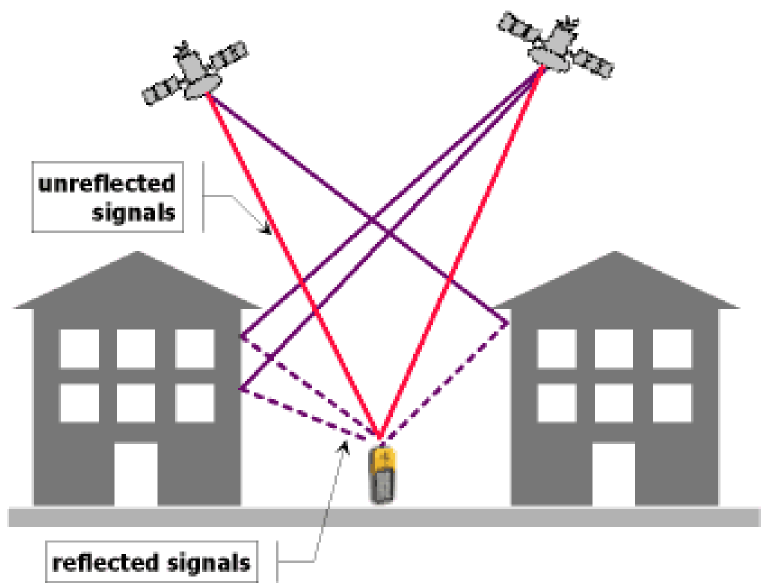


Figure 1.1: Multipath propagation

To mitigate channel fading, diversity techniques [15; 16; 17] which transmit and process multiple copies of the same signal through different fading channels is often employed. At the receiver, these different copies of the signal are then *combined* to reconstruct the complete transmitted signal. The probability of experiencing fading in this composite channel is then the probability when all the component channels simultaneously experiencing a fade, which is an unlikely event. Well-known forms of diversity include space diversity, time diversity, frequency diversity and polarization diversity [18; 19; 20; 21].

Time diversity implies that the same data is transmitted over different periods, so that different copies of the symbol undergo uncorrelated fading. Thus, there is a low probability of experiencing deep fades for the same data. Another strategy to achieve time diversity is to utilize bit-interleaving which adds a redundant error correction code such that the message is spread in time. However, since the same data have to be transmitted several times, the disadvantage of this technique is the decreasing of the data rate.

Frequency diversity is achieved by modulating the transmission signal by different carriers, thereby multiple copies of the signal appear in different frequency bands. Thus, it is very unlikely that the signals will suffer the same level of channel fading. Often, the receiver will rely on the strongest received signals as the reference of transmission.

Polarization is a property describing the orientation of oscillations of propagating waves. Polarization diversity combines pairs of antennas generating orthogonal polarizations, e.g. vertically and horizontally polarized waves. Different polarized signals undergo different polarization changes depending on the transmission media. If the channel fading over one polarized signal is severe, the signal may still be successfully transmitted over other signal copies.

Space diversity is obtained by employing multiple antennas that are physically separated from one another. Generally, a space in the order of several wavelengths is sufficient to achieve uncorrelated signals at the transmitter and/or receiver antennas. Therefore, the probability for experiencing deep fading for all uncorrelated signals is low.

The above diversity techniques are all employed in wireless communication systems. The purpose of these techniques is to achieve a more reliable transmission. The gain obtained by employing these diversity techniques, which is called diversity gain, is often measured by the slope at which the logarithmic error probability decreases with increasing of logarithm of the signal to noise ratio at the receiver. The larger the diversity gain, the steeper the slope, thus the more beneficial the diversity technique to the communication system.

1.2 Cooperative System and Cooperative Protocols

1.2.1 Cooperative System and Cooperative Diversity

In the previous section, various forms of diversities have been introduced. Among these diversity techniques, spatial diversity technique is particularly attractive, since it can be readily combined with other forms of diversity. The maximum gain obtained by exploiting the spatial diversity is measured by the product of the number of transmitter and receiver antennas. A system equipped with multiple transmitter and receiver antennas is referred to as a Multiple Input and Multiple Output (MIMO) system. Fig 1.2 illustrates a 2 x 2 MIMO system. A MIMO system offers significant increase in data throughput and link range without additional bandwidth or increased transmission power. It achieves this goal by spreading the same total transmission power over the antennas to achieve diversity gain that improves transmission reliability. Although MIMO system has such advantages, the requirement of multiple antennas in both transmitter and receiver could be quite an inconvenience for the mobile units in wireless communications, since installing multiple antennas would increase the size, complexity and cost. To overcome this limitation, another form of spatial diversity called cooperative diversity has recently been proposed for wireless communications. Cooperative diversity is a virtual multiple antenna technique which exploits multiple antenna signal processing advantages using the virtually aggregated multiple antennas each mounted on different user terminals. The basic idea of cooperative communications is that all mobile users in a wireless network can help each other to send signals to the destination cooperatively, which means each user's data information is sent out not only by the user, but also by other users. Thus, it is inherently more reliable for the destination to receive the transmitted information, since from a statistical point of view, the chance of having all transmission links to the destination failing is seldom. The discussion of cooperative communications can be traced back to the relay systems in the work of E. van der Meulen, T.M. Cover and A. El Gamal in 1970's [22; 23; 24]. However, after these works, relay systems received little attention for nearly two decades. A renewed interest has developed in the context of wireless communication systems in recent years [25; 26].

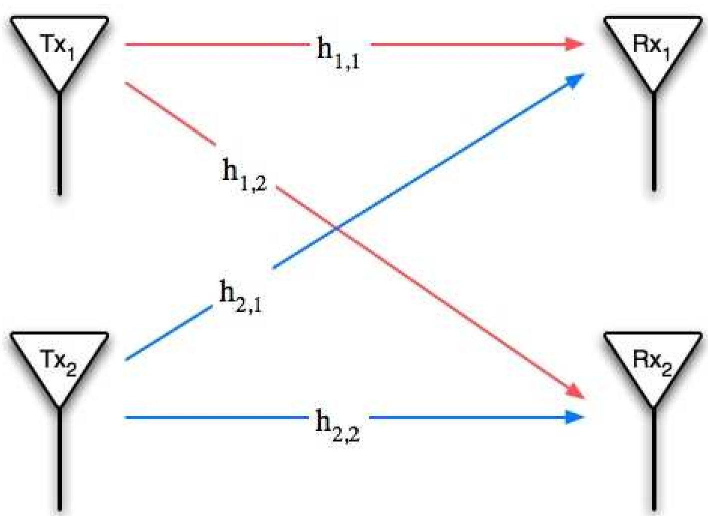


Figure 1.2: MIMO system

1.2.2 Cooperative Strategies

As pointed out in the previous section, in recent years cooperative systems have attracted much research interest due to their potential applications. Techniques, such as the designs of cooperative protocols or strategies for the cooperative systems have been considered in [27; 28; 29]. Recently, there are mainly three types of cooperative protocols: (i) fixed relaying schemes such as amplify-and-forward (AF) [7; 8; 9] and decode-and-forward (DF) [10; 11] protocols, (ii) selection relaying schemes which adaptively select the relays for transmission of the data based upon channel measurements between the cooperating terminals, and (iii) incremental relaying schemes that adaptively decide whether to transmit the data or not based upon the limited feedback from the destination terminal. These protocols employ different types of processing at the relay terminals, as well as different types of combining at the destination terminals.

In a relay system, the source node first transmits the information signal to the relay nodes and/or the destination node, then the relay nodes process and forward the received signal to the destination. For an AF protocol, the relay nodes re-transmit a scaled version of the signal that is received from the source node to the destination node. To stay within the power budget, the amplifier gain has to remain under a certain limit. This very much depends upon the fading environment. For a DF protocol, the relay nodes decode the message first, then check if errors have occurred or not. If the message is successfully decoded, it re-encodes the data symbol using a different code book and transmit it to the destination. Both full decoding and symbol-by-symbol decoding can be employed at the relay node. These options allow for trading off performance and complexity at the relay terminal. Comparing these two protocols, because AF protocol does not have the encoding and decoding functions, it has lower complexity than a DF protocol. Therefore, in most networks for which high complexity is not acceptable, AF protocol is adopted. However, AF protocol has a disadvantage of high noise especially when the channel between the source node and relay node is in severe fading. The following strategy is to combat this limitation.

The conventional assumptions in a multiple relay system are that all relay nodes participate in the transmission and that the available channel and power resources are equally distributed over all nodes. This approach which neglects the difference of performance of the relay, is clearly sub-optimal. On the other hand, selection relaying scheme in which the best end-to-end path between source and destination among M possible relays is first located before the transmission relays are selected has been proposed. Fig.1.3 illustrates the operation of this scheme. In [30] a simple distributed method for the selection relaying scheme is developed. This method requires no explicit communication among the relays, assumes no prior knowledge of network geometry and is based on instantaneous wireless channel measurements. The success (or failure) to select the best available path depends only on the statistics of the wireless channel. Utilizing this selection scheme, only the relay node which has the 'best' channel gain will forward the scaled version of the signal to the destination.

So far we have seen that in both fixed and selection relaying schemes, the relays repeat all the time. It does not make efficient use of the degrees of the freedom of the channel. To overcome this, incremental relaying protocol has been proposed. The main idea is that instead of repeating all the time, the transmission at the relay node depends on the feedback from the destination terminal. For example, a single bit may be used to indicate the success or failure of the direct transmission.

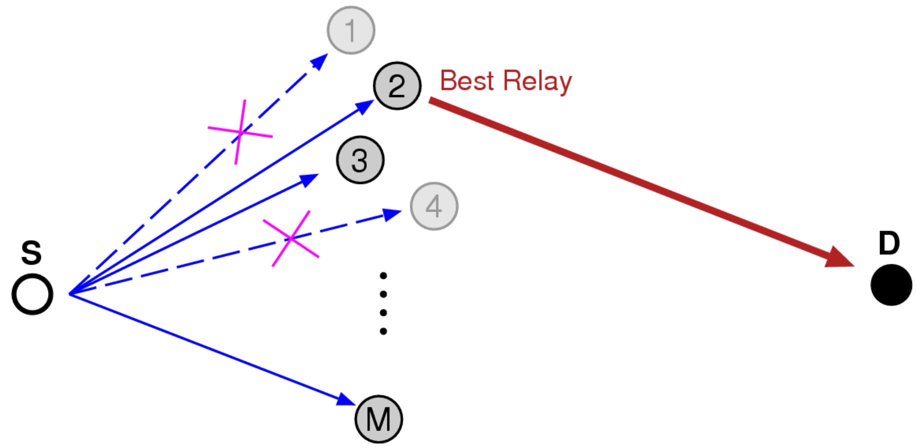


Figure 1.3: Demonstrate the selection scheme

Consider the following scheme utilizing feedback and AF protocol: First, the source node transmits the information signal to both relay and destination terminals. Then, the destination terminal sends out a single bit of feedback to the source and relays to indicate whether the direct transmission succeeds or fails. If the reception of the signal is successful, then a bit is sent from the destination to the source and relays indicating the success of the direct transmission, and the relays then do nothing. Otherwise, the feedback of a different bit from the destination would indicate a failure and the relays then forward the signal received from the source to the destination. Since the relays perform a transmission only when the failure feedback is received, this scheme make more efficient use of the degrees of freedom of the channel.

1.3 Motivation and Contribution of the Thesis

In the previous sections, we have introduced the background of wireless communications, especially the cooperative systems and cooperative protocols. In cooperative systems, the in-cell mobile users share the use of their antennas to create a virtual array through distributed transmission and signal processing. When channel state information (CSI) is available at the receiver, the performance for the coherent cooperative relay system with flat fading channels is characterized by the diversity gain function. Full diversity can be achieved by utilizing well designed pre-coders or space-time block codes.

However, full knowledge of channel state information at the relay nodes and the destination nodes, in practice, may be difficult to attain. The fading coefficients in mobile wireless communications may vary rapidly, thereby, full knowledge of CSI necessitates pilot test signals to be designed in the transmission scheme. In this thesis, we consider a two-way relay system in which two source terminals communicate with each other through a relay node using an AF protocol. The channel coefficients are assumed to be fast changing. The end terminals estimate the channel information, and then estimate the transmitted signals using the rest of the time slots. For this two-way relay system, we estimate the CSI using a particular test signal set. The main advantage of this system is that both end terminals can cancel the interference generated by its own transmission. The information signal is selected from a normalized square QAM constellation. This information signal is detected by choosing the symbol in the QAM constellation closest in distance to it. In the thesis, we examine the performance of such a system and derive a signal-dependent asymptotic formula for the error probability. This asymptotic formula verifies that full diversity gain is achieved. Finally, under a power constraint for the pilot signals, we formulate an optimum design problem, for which a solution of the optimum transmission power for the information signals can be obtained.

1.4 Structure of this Thesis

The thesis is structured as follow:

• In Chapter 1, we introduce the background and previous works for the wireless communication system, especially for the cooperative systems. We also present the motivation and primary contributions of this thesis.

• In Chapter 2, we first introduce system models for the amplify-and-forward half-duplex one-way relay systems which equipped with single relay. Then, we present the system model for the half duplex two-way relay system. Finally, a comparison between these systems and conventional MIMO system is discussed.

• Chapter 3, utilizing certain transmission scheme, a signal dependent asymptotic formula for the error probability is derived. Then, under a power constraint for the pilot signals, optimum design problems are formulated based on two different power allocation considerations. Finally, solutions of the optimum transmission power for both cases are calculated.

• In Chapter 4, simulation models and simulation results for both average-power-loading case and worst-case power loading case are presented. Furthermore, we compare these simulation results and the numerical results in previous sections.

• In Chapter 5, base on previous simulations and numerical results, we discuss the conclusion for this thesis and some suggestions are made for future work.

• Appendix, the proof for the lemmas and theorems and also some detailed derivations have been put into this section.

Chapter 2

System Model

In the previous chapter, we have given a general review of wireless communication systems, especially for relay systems. The earliest form of relay system is the one-way relay system which is considered by E. van der Meulen, T.M. Cover and A. El Gamal in 1970's. However, this transmission system can only apply to one directional transmission, in case users want to exchange information, then two-way transmission systems are required. Due to these potential applications, research in one-way relay systems have been extended to two-way relay systems in recent years. In this thesis, we also examine the half-duplex two-way relay system and base our design on it. Since the idea of two-way relay system is developed based on the one-way relay channel, for the sake of clarity, we first review the property and possible transmission schemes for half-duplex one-way relay systems before turning to the discussion of two-way relay systems. Finally, comparisons between these two systems with conventional MIMO systems are given.

2.1 One-Way Amplify-and-Forward Half-Duplex Relay System

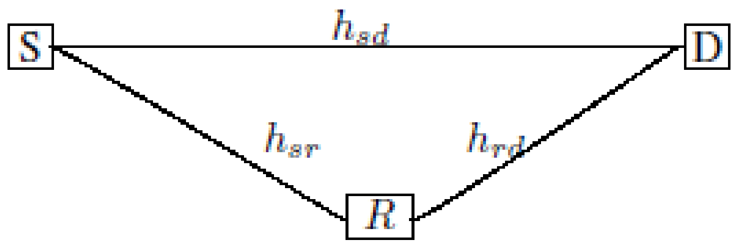


Figure 2.1: A single relay system

Fig. 2.1 shows a single relay system which is composed of a source node S, a destination node D and a relay node R. The relay node assists the transmission from the source node to the destination node in a half-duplex mode. Half-duplex means communications are supported in both directions, but only one direction at a time. Typically, once a party begins receiving a signal, it must wait for the transmitter to stop transmitting before replying, this requires the device to be a *transceiver*, so as to perform both transmission and reception. An example of the half-duplex system is a walkie-talkie. There are several benefits of using half-duplex over full-duplex, the most important one is its lower implementation complexity, since for a full-duplex system, simultaneous transmission and reception of signals requires precise design for the component. This advantage is also the main reason for our focus on the half-duplex mode in this thesis. In the last chapter, we have introduced cooperative protocols such as, the amplify-and forward (AF) and the decode-and-forward (DF) protocols. Since the AF protocol does not have the encoding and decoding functions, it has lower complexity than a DF protocol. Again, our focus in this thesis will be on the simpler AF protocol.

As shown in Fig.2.1, in this system, all nodes are equipped with single antenna. The channel gain from the source node to the destination node is denoted by h_{sd} , and those from the source node to the relay node and from the relay node to the destination node are denoted by h_{sr} and h_{rd} respectively. All channel gains are assumed to be independent, zero-mean, circularly symmetric, complex, Gaussian (CSCG) random variables having unit variance. In addition, all channel coefficients are assumed to be fixed during a period of observation. It is also assumed that terminals only have knowledge of the first and second order statistics of the *channel state information* (CSI), a term referred to the known properties of a communication link. This information describes the signal propagation conditions and represents the combined effect of the channel from the transmitter to the receiver. There are basically two levels of CSI, namely instantaneous CSI and statistical CSI.

Instantaneous CSI is also referred to as the *current channel condition*, i.e., knowing the impulse response of a channel. This instantaneous CSI gives an opportunity to adapt the transmitted signal to the impulse response and thereby optimize the received signal for spatial multiplexing or to achieve low bit error rates.

Statistical CSI means that a statistical characterization of the channel is known. This description may include, for example, the type of fading distribution, the average channel gain, the line-of-sight component, and statistics of the spatial correlation. As with instantaneous CSI, this information can be used for partial transmission optimization.

CSI acquisition is practically limited by how fast the channel conditions are changing. In fast fading systems where channel conditions vary rapidly under the transmission of information symbols, estimation of instantaneous CSI needs to be performed on a short-term basis. A popular approach is so-called training sequence (or pilot sequence), where a known signal is transmitted and the channel matrix is estimated using the combined knowledge of the transmitted and received signal.Let the training sequence be denoted by $\mathbf{p}_1, ..., \mathbf{p}_n$, and combining the received training signals $\mathbf{y}_i$ for $i = 1, ..., n$, we obtain

$$\mathbf{Y} = \mathbf{HP} + \mathbf{W}$$

with the training matrix $\mathbf{P} = [\mathbf{p}_1, ..., \mathbf{p}_n]$ and the noise matrix $\mathbf{W} = [\mathbf{w}_1, ..., \mathbf{w}_n]$. Utilizing the least-square estimator [37], we have

$$\hat{\mathbf{H}} = \mathbf{YP}^H(\mathbf{PP}^H)^{-1}$$

where $(\cdot)^H$ denotes the conjugate transpose, and $\hat{\mathbf{H}}$ denotes the estimated value for $\mathbf{H}$. This technique will be utilized in later chapter. Since we have mentioned before that the CSI acquisition is practically limited by how fast the fading system is, let us introduce the fading property and examine how it affects the transmission. In wireless communications, there are two kinds of fading channels, flat fading and frequency-selective fading. Flat fading transmission system means the coherence bandwidth of the channel is larger than the bandwidth of the signal. Therefore, all frequency components of the signal will experience the same magnitude of fading. In contrast, for frequency-selective fading the coherence bandwidth of the channel is smaller than the bandwidth of the signal. Therefore, different frequency components of the signal experience uncorrelated fading. Since different frequency components of the signal are affected independently, it is highly unlikely that all parts of the signal will be simultaneously affected by a deep fade. Certain modulation schemes such as OFDM and CDMA are developed to employing frequency diversity to provide robustness to fading. In this thesis we focus on flat fading transmission.

Based on all previous assumptions, we have following considerations of cooperative strategies for the single relay AF system. There are mainly three different strategies being proposed. The operations of these protocols are shown in Table.2.1. Protocol 1 is referred to as a non-orthogonal AF protocol. It has been shown [33] that this protocol can achieve the optimal diversity-multiplexing tradeoff for relay systems with single antenna. Protocol 2 is referred to as an orthogonal AF protocol [29], and protocol 3 is proposed in [35]. For these three strategies, the signal transmission is carried out in a block-based fashion. Now, if we denote the transmitting data block by

$$\mathbf{x_1} = [\mathbf{x}_I^T \quad \mathbf{x}_{II}^T]^T \tag{2.1}$$

with

$$\begin{aligned} \mathbf{x}_I &= [x(1), ..., x(T)] \\ \mathbf{x}_{II} &= [x(T+1), ..., x(T')] \end{aligned}$$

The input-output relation of all three protocols can be expressed in the form of

$$\mathbf{z_1} = \sqrt{E_r}\begin{pmatrix} h_{sd}\mathbf{A_1} & \mathbf{0} \\ a_1 h_{sr} h_{rd}\mathbf{A_2} & h_{sd}\mathbf{A_3} \end{pmatrix}\begin{pmatrix} \mathbf{x_I} \\ \mathbf{x_{II}} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ a_1 h_{rd}\mathbf{B} & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{n_I} \\ \mathbf{n_{II}} \end{pmatrix} + \begin{pmatrix} \eta_{\mathbf{I}} \\ \eta_{\mathbf{II}} \end{pmatrix} \tag{2.2}$$

where E_r denotes the average power per transmitting symbol, a_1 denotes the amplification coefficient at the relay node, $\mathbf{z_1} = [z(1), ..., z(T')]^T$ represents the receiving vector at the destination. The transmission signal vector $\mathbf{x_1}$ is given in Eq.(2.1). Noise vectors $\mathbf{n_I}$, $\mathbf{n_{II}}$ and $\eta_{\mathbf{I}}$, $\eta_{\mathbf{II}}$ in Eq.(2.2) represent the i.i.d. Gaussian noise vectors in the relay and direct paths during the 1st T and rest $(T'-T)$ slots. $\mathbf{A_1}$ and $\mathbf{A_2}$ is a $T \times T$ identity matrix, and $\mathbf{A_3}$ is $(T'-T)\times(T'-T)$ identity matrix, with T' represent the transmission symbol length in two consecutive data block, and T represent the transmission symbol length in the first data block. $\mathbf{B}$ stands for a $(T'-T)\times T$ identity matrix. [33] shows that the only requirement on $\mathbf{B}$ such that the protocols described by Eq. (2.2) could potentially achieve the optimal diversity-multiplexing tradeoff are for $\mathbf{B}$ to be square (T=T'/2). Thus, it is means that there is a data receiving period for the relay node before it forwards the received data to the destination. When this period is half of the length of whole data block, the transmission achieves the optimum diversity-multiplexing tradeoff.

Table 2.1: Three protocols for AF relay system

Time slots / protocols	1	2	3
1st	$S \to R$ $S \to D$	$S \to R$ $S \to D$	$S \to R$
2nd	$S \to D$ $R \to D$	$R \to D$	$S \to D$ $R \to D$

For Protocol 1 (non-orthogonal AF protocol), in the first T time slots (time slot is defined as the time required for the transmission of one symbol), the source node transmits T data symbols to both the destination and the relay node (each data symbol takes 1 time slot). And in the second T time slots, the source transmits another different T data symbols to the destination, and the relay node simply amplifies and forwards what it received from the first T time slots to the destination. Therefore, in the second T time slots, the destination node combines whatever it received from both source and relay node. The transmission rate which is defined as the number of symbols transmitted per channel use is 1 for this transmission scheme.

For Protocol 2 (Orthogonal AF protocol), the operation is quite similar to Protocol 1. In the first T time slots, the source node transmits T data symbols to both the destination and the relay node. In the second T time slots, the relay node amplifies and forwards the signal to the destination. However, there is no transmission from the source node to the destination, which means in $2T$ time slots there are only T symbols being transmitted. Therefore, the transmission rate for this scheme is 1/2.

For Protocol 3, in the first T time slots, symbols are transmitted from source node to the relay only. And in the second T time slots, the source node transmit the same T symbols to destination, and the relay node amplifies and forwards what it received from the first time slot to the destination. Therefore, the transmission rate for this scheme is 1/2.

By comparing these three protocols, the non-orthogonal protocol provides high symbol rate and it has the most general transmission pattern since the other two protocols are just special cases of it. A detail analysis for the relay system having this non-orthogonal protocol in terms of channel capacity and error performance can be found in [7; 8; 9] .

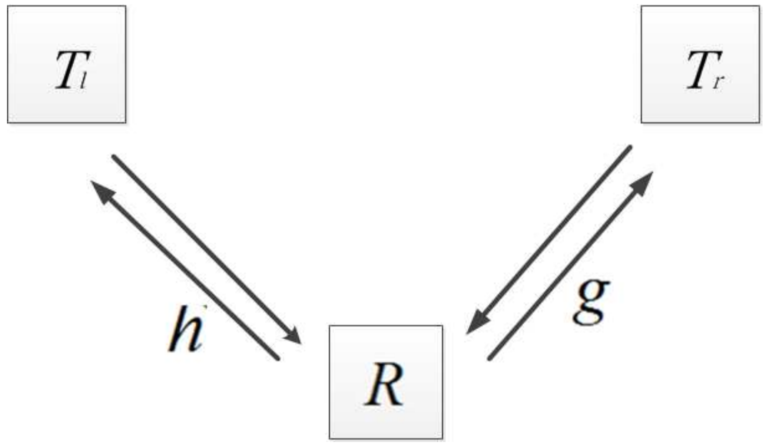


Figure 2.2: A two-way relay system

2.2 Two-Way Amplify-and-Forward Half-Duplex Relay System

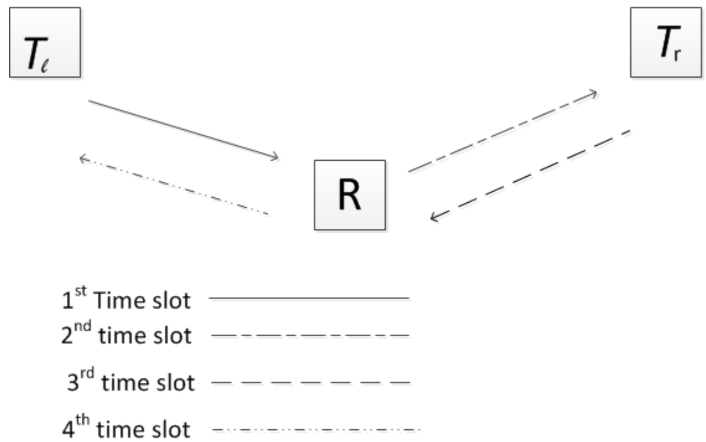
Fig. 2.2 illustrates the two-way relay channel which is composed of two source nodes T_ℓ, T_r and a relay node R. Each node is equipped with a single antenna. h and g denote channel gains from T_ℓ to R and from T_r to R respectively. Both h and g are assumed to be independent, zero-mean, circularly symmetric, complex, Gaussian (CSCG) random variables having unit variance. In addition, all channel coefficients are assumed to be fixed during a period of observation. It is also assumed that only the first and second order statistics of the channel gains are known at the terminals. For this two-way amplify-and-forward half-duplex relay system, four protocols have been proposed so far [4; 34]. Again, we consider the block-based communication in which a time slot is defined as the time required for the transmission of 1 symbol.

Protocol I is also referred to as a traditional transmission scheduling scheme. For this transmission scheme, interference is usually avoided by prohibiting the overlapping of signals in the same time slot. A possible transmission schedule is given in

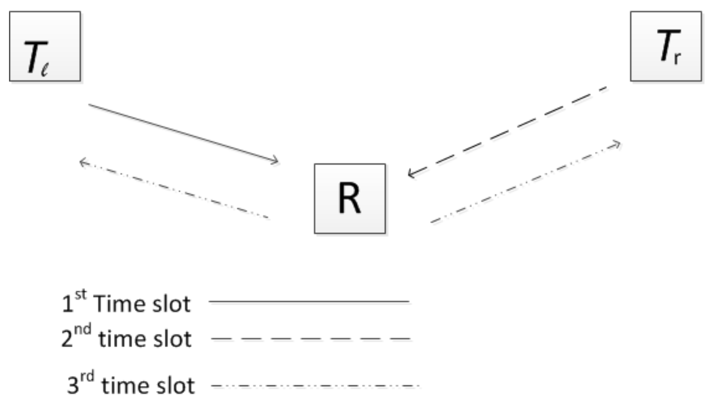
Fig. 2.3(a). In the first T time slots, the node T_ℓ transmits T data symbols to the relay node R. In the second T time slots the relay node simply amplifies and forwards what it received from the first T time slots to T_r. This process finishes the signal transmission from node T_l to node T_r. In the third and fourth T time slots, transmission is just in the reverse direction to complete the signal transmission from T_r to T_ℓ. A total of four T time slots are needed for exchanging of $2T$ data symbols in opposite directions.

Protocol II (straightforward network coding scheme), Fig.2.3(b) illustrates the idea for this protocol. In the first T time slots, T_ℓ transmits T data symbols to the relay node R. In the second T time slots, node T_r transmits another T data symbols to the relay node. Finally, in the third T time slots, the relay node first combines what it received from those two terminals, and then forwards a scaled version of this symbol to both T_ℓ and T_r. A total of three T time slots are needed, for a throughput improvement of 33 percents over the traditional transmission scheduling scheme.

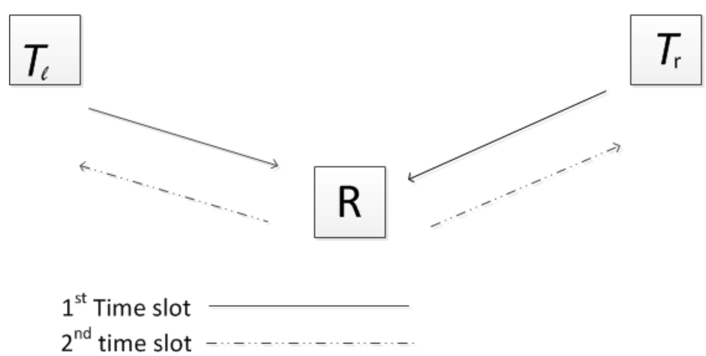
Protocol III (physical-layer network coding (PNC)) is the most efficient transmission scheme among these three protocols. As shown in Fig.2.3(c), in this protocol only $2T$ time slots are need to complete the information exchanging process. In the first T time slot (the multiple access phase (MAC)), both T_ℓ and T_r transmit data symbols to the relay node R. Then, in the second T time slots (the Broadcasting phase), the relay node forwards the received signals to both end terminals to complete this information exchanging process.



(a) traditional transmission scheduling scheme



(b) straightforward network coding scheme



(c) physical-layer network coding (PNC) scheme

Figure 2.3: Three protocols for two-way relay system

Comparing these three protocols, since the system models are the same for every protocol (two-way relay system without direct channel), the maximum diversity of all three protocols should be the same. Therefore, one may conclude that Protocol III always outperforms Protocol I and Protocol II in terms of diversity-multiplexing trade-off. Besides these three protocols (for the two-way relay system without direct channel between two end terminals), protocols for two-way relay systems with the direct channels are also proposed [34]. Fig.2.4 illustrates the two-way relay system which contains the direct link between the two end terminals. For this system, under the same assumptions as the systems without direct link, the operation of the protocol is shown as following. In the first time slot, node T_ℓ transmits T data symbols to both the relay node R and node T_r. In the second time slot, node T_r transmits another T data symbols to the relay node R and node T_ℓ. Finally, in the third time slot, relay node R combines what it received from the first two time slots and forwards it to both end terminals T_ℓ and T_r. Comparing this protocol with protocol III, with the benefit of the direct link, this protocol may achieve higher diversity than Protocol III since signals might be transmitted through multiple path to achieve spatial diversity. However, our consideration in this thesis is for the case when the two end terminals want to exchange information, but they are out of range of each other. Thus, no direct link is available. Therefore, Protocol III is the protocol on which our attention

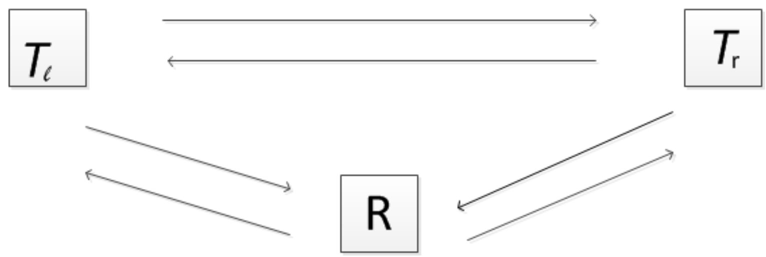


Figure 2.4: A two-way relay system with direct channel

is focused. For this protocol, if we denote respectively the transmitting data blocks for T_ℓ and T_r by

$$\mathbf{x} = [x(1), ..., x(T)]$$
$$\mathbf{y} = [y(1), ..., y(T)]$$

The input-output relation can be expressed as

$$z_\ell(T+p) = \sqrt{E_t} a_2 h[hx(p) + gy(p) + n(p)] + \eta_\ell(T+p) \tag{2.3}$$
$$z_r(T+p) = \sqrt{E_t} a_2 g[hx(p) + gy(p) + n(p)] + \eta_r(T+p) \tag{2.4}$$

where $p = 1, ..., T$, $z_\ell(\cdot)$ and $z_r(\cdot)$ denote the received signals for terminal T_ℓ and T_r respectively. n is the noise received at the relay node, and $\eta_\ell(\cdot)$ and $\eta_r(\cdot)$ respectively denote the noise received at left and right terminals. All noise are independent and identically distributed zero-mean circularly symmetric Gaussian with variance σ^2. $\sqrt{E_t}$ represents the average power per transmitted symbol. a_2 denotes the amplification coefficient at the relay node. This amplification coefficient controls the strength of relayed signal and is constrained such that the average power for the relayed signal does not exceed the power budget available at the relay node. Writing Eq.(2.3) and Eq.(2.4) in a matrix form, we have

$$\mathbf{z} = \begin{pmatrix} \mathbf{z}_\ell \\ \mathbf{z_r} \end{pmatrix} = \sqrt{E_t} a_2 \begin{pmatrix} h^2\mathbf{I_T} & hg\mathbf{I_T} \\ hg\mathbf{I_T} & g^2\mathbf{I_T} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} + a_2 \begin{pmatrix} h\mathbf{I_T} & \mathbf{0} \\ g\mathbf{I_T} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{n} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \boldsymbol{\eta}_\ell \\ \boldsymbol{\eta}_r \end{pmatrix} \tag{2.5}$$
$$= \sqrt{E_t} a_2 \mathbf{Ht} + \mathbf{v} \tag{2.6}$$

where the receiving vector $\mathbf{z}_\ell = [z_\ell(T), ..., z_\ell(2T)]^T$ and $\mathbf{z_r} = [z_r(T), ..., z_r(2T)]^T$. The transmitting data vector $\mathbf{t}$ is given by $[\mathbf{x}, \mathbf{y}]^T$, and the noise vector $\mathbf{n}$, $\boldsymbol{\eta}_\ell$ and $\boldsymbol{\eta}_r$ are respectively the i.i.d. Gaussian noise vectors in MAC phase and broadcast phase. This renders the equivalent noise at terminal T_ℓ and T_r being the sum of the two components such that $\mathbf{v} = [\mathbf{v}_\ell, \mathbf{v_r}]^T$, with $\mathbf{v}_\ell = a_2 h\mathbf{n} + \boldsymbol{\eta}_\ell$, $\mathbf{v_r} = a_2 g\mathbf{n} + \boldsymbol{\eta}_r$ and $\mathbf{v}_\ell, \mathbf{v_r} \sim \mathcal{CN}(\mathbf{0}, \sigma^2\boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma}$ is the covariance matrix. The channel matrix is given by

$$\mathbf{H} = \begin{pmatrix} h^2\mathbf{I_T} & hg\mathbf{I_T} \\ hg\mathbf{I_T} & g^2\mathbf{I_T} \end{pmatrix} \tag{2.7}$$

with the expression of this channel matrix $\mathbf{H}$, if the transmitting data block $\mathbf{t}$ is a well designed training sequence, then utilizing the estimation method given in Section 2.1 , estimated value of the channel coefficients could be obtained.

2.3 Comparison with Conventional MIMO Systems

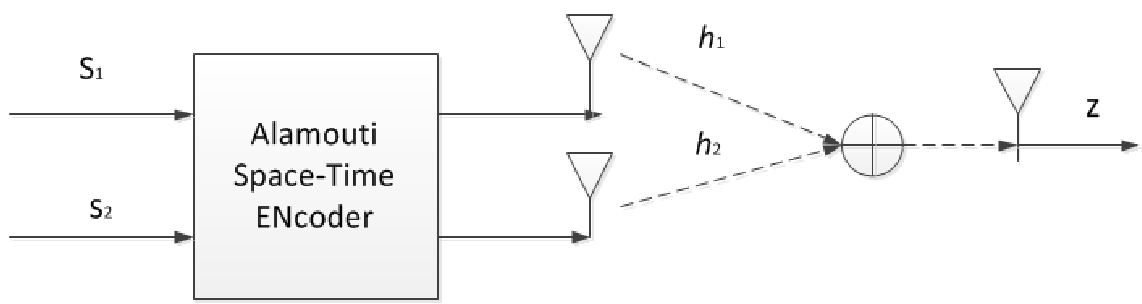


Figure 2.5: A two-way relay system

As pointed out in the previous chapter, a relay system is not a precise equivalence to a conventional MIMO system, although it can be regarded as a virtual MIMO system in the sense that the collaborative users create a virtual array by distributed signal processing and transmission. In this section, we present a comparison of the models of relay systems to the conventional MISO system (2 transmitter antennas and one receiver antenna). For the sake of clarity, the comparison is carried out using the Alamouti STBC channel model for the MISO system. We assume that the MISO system has the same symbol rate as the relay systems, i.e., 1 symbol per channel use. The transmission is carried out by blocks of length 2 ($T = 1$). Fig. 2.5 illustrates this MISO system. h_1 and h_2 are respective channel coefficients from Transmitters 1 and 2 to the receiver. Both h_1 and h_2 remain constant within a period of observation. Then, the input-output relation for the this MISO system can be written as

$$\mathbf{z_m} = \mathbf{H_m}\mathbf{x_m} + \mathbf{n_m} \tag{2.8}$$

where the subscript $(\cdot)_m$ indicates that the quantities are related to the MISO system. $\mathbf{z_m}$ is the 2 x 1 received data vector, $\mathbf{x_m}$ is the 2 x 1 transmitted data vector, the noise vector $\mathbf{n_m} = [n_1 \quad n_2]^T$ and the channel matrix $\mathbf{H_m}$ is given by

$$\mathbf{H_m} = \begin{pmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{pmatrix} \tag{2.9}$$

The differences between models of relay systems and this MISO system are noted as follows:

- Channel matrix: The channel matrix of both one-way relay and two-way relay systems involve the products of Gaussian random variables. Therefore, the entries in channel matrix for the relay systems are no longer Gaussian. In contrast, the channel matrix of the conventional MIMO system can still be assumed Gaussian.

- Noise: For both one-way and two-way relay systems, the equivalent noise at the receiver is also a function of the channel gain from relay node to destination. In addition, it may or may not be white any more. However, the noise in conventional MIMO systems can usually be assumed to be white Gaussian and it is usually not a function of the channel gains.

Chapter 3

Performance Analysis of the AF Two-way Relay System

In this chapter, the transmission scheme and probability of error for the AF two-way relay system are discussed and analysed. Our attention is focused on Protocol III which is the most efficient strategy for this system. Since in this thesis we assume that terminals only has knowledge of the first and second order statistics of the channel state information. We estimate the channel information using a particular test signal set first. The transmitted information signal is selected from a normalized square QAM constellation and is detected by choosing the symbol in the QAM constellation closest in distance to it. Finally, we derive a signal dependent asymptotic formula for the error probability.

3.1 Channel Estimation

A diagram of a two-way relay system is presented in Fig. 2.2. In this system, the relay node R assists information exchanging for the two end terminals T_ℓ and T_r. All nodes are equipped with a single antenna. h and g denote channel gains from T_ℓ to R and from T_r to R respectively. Both h and g are assumed to be independent, zero-mean, circularly symmetric, complex, Gaussian (CSCG) random variables having unit variance. As shown in Chapter 2, the transmission for this two-way relay system is carried out in a block-based fashion, each block is of length $2T$, $T \geq 1$. In the following discussion, we assume, for simplicity of illustration, that $T = 1$, i.e., one information exchanging process in the opposite direction is complete in two time slots. Also, we assume that the amplification coefficient a, is unity, i.e., $a = 1$. The information signals transmitted are assumed to be equally probable from a normalized square QAM constellation. The channel coefficients are fast changing, but are fixed for at least 6 time slots, after which they may change to new independent values which are then fixed for at least another 6 time slots, and so on. The end terminals estimate the channel information within the first 4 time slots, and then detect the transmitted signals using the rest of the time slots. Due to the symmetry of this two-way relay system, the following discussion will focus on the reception of the signal at the left terminal T_ℓ only, knowing that the signal on the right terminal will have a similar expression.

During the first time slot, both terminals transmit testing signals to the relay node. The received signal at the relay node can be written as

$$r_1 = hx_1 + gy_1 + n_1 \tag{3.1}$$

where x_1 and y_1 are testing signals transmitted from terminal T_ℓ and T_r respectively, and n_1 is the additive noise in the first time slot. In particular, we set $x_1 = \sqrt{p_1}$ and $y_1 = 0$, with p_1 being the transmission power of signal x_1.

During the second time slot, the relay node R amplifies and forwards its received signal r_1 of Eq. (3.1) to both terminals T_ℓ and T_r. Thus the received symbol at left terminal is

$$\begin{aligned} z_{\ell 1} &= h(hx_1 + gy_1 + n_1) + \eta_{\ell 1} \\ &= \sqrt{p_1}h^2 + hn_1 + \eta_{\ell 1} \end{aligned} \tag{3.2}$$

where $\eta_{\ell 1}$ is the noise in the transmission path from the relay node R to T_ℓ during the second time slot.

In the third and fourth time slots, we follow the same transmission pattern. However, we set the testing signal $x_2 = 0$ and $y_2 = \sqrt{q_1}$ where q_1 is the transmission power of signal y_2. Thus, the received signal at the relay and the subsequent received signal at the left transmitter are respectively

$$r_2 = hx_2 + gy_2 + n_2 \tag{3.3a}$$

$$\begin{aligned} z_{\ell 2} &= h(hx_2 + gy_2 + n_2) + \eta_{\ell 2} \\ &= \sqrt{q_1}hg + hn_2 + \eta_{\ell 2} \end{aligned} \tag{3.3b}$$

where n_2 is the total additive noise received by the relay node during the third time slot transmission from both end terminals, and $\eta_{\ell 2}$ is the noise in the transmission path from R to T_ℓ during the fourth time slot. We assume all additive noise in the transmission channels to be independent and identically distributed (IID) zero mean circular Gaussian with variance σ^2. The channel information h^2 and hg can now be estimated with these two received signals $z_{\ell 1}$ and $z_{\ell 2}$ using the least squares method [38] such that

$$\arg\min_{h^2} \ |z_{\ell 1} - \sqrt{p_1}h^2| \ \Rightarrow \ \hat{h}^2 = \frac{z_{\ell 1}}{\sqrt{p_1}} \tag{3.4a}$$

$$\arg\min_{hg} \ |z_{\ell 2} - \sqrt{q_1}hg| \ \Rightarrow \ \widehat{hg} = \frac{z_{\ell 2}}{\sqrt{q_1}} \tag{3.4b}$$

where $\hat{h}^2$ and $\widehat{hg}$ are estimated value of h^2 and hg respectively. After this channel estimation process, in the following two time slots we transmit the information signals which are selected with equal probability from a normalized square QAM constellation. Thus, following the same transmission manner, the received signal is

$$\begin{aligned} z_\ell &= \sqrt{p_2}h^2x + \sqrt{q_2}hgy + hn + \eta_\ell \\ &= \sqrt{p_2}\hat{h}^2x + \sqrt{q_2}\widehat{hg}y + \overline{\eta} \end{aligned} \tag{3.5}$$

where

$$\overline{\eta} = (h^2 - \hat{h}^2)x\sqrt{p_2} + (hg - \widehat{hg})y\sqrt{q_2} + hn + \eta_\ell \tag{3.6}$$

p_2 and q_2 are the transmission power for the signal x and y respectively such that the constraints on the transmission power are $p_2 = 1 - p_1$, $q_2 = 1 - q_1$, and n and η_ℓ are respectively the noise in the transmission path from the two end terminals to the relay node R and from R to T_ℓ. We can regard $\overline{\eta}$ in Eq (3.5) as noise. Combining

Eq.(3.2) and Eq.(3.4a)

$$\begin{cases} z_{\ell 1} = \sqrt{p_1}h^2 + hn_1 + \eta_{\ell 1} \\ \hat{h}^2 = \frac{z_{\ell 1}}{\sqrt{p_1}} \end{cases} \Rightarrow \quad \widehat{h}^2 - h^2 = \frac{h}{\sqrt{p_1}}n_1 + \frac{1}{\sqrt{p_1}}\eta_{\ell,1}$$

Combining Eq.(3.3b) and Eq.(3.4b), we obtain

$$\begin{cases} z_{\ell 2} = \sqrt{q_1}hg + hn_2 + \eta_{\ell 2} \\ \widehat{hg} = \frac{z_{\ell 2}}{\sqrt{q_1}} \end{cases} \Rightarrow \quad \widehat{hg} - hg = \frac{h}{\sqrt{q_1}}n_2 + \frac{1}{\sqrt{q_1}}\eta_{\ell,2}$$

Thus, under the assumption that all additive noise in the transmission channels to be independent and identically distributed (IID) zero mean circular Gaussian with variance σ^2. And from the expression of $\overline{\eta}$ in Eq. (3.6), we can obtain the variance of the noise as

$$\sigma_{\overline{\eta}}^2 = \left(\frac{p_2|x|^2}{p_1} + \frac{q_2|y|^2}{q_1} + 1\right)(|h|^2 + 1)\sigma^2 \tag{3.7}$$

3.2 Probability of Error

The probability of error for a standard QAM constellation can be found in [12]. Let $\xi = |h|^2$ and let $\mathcal{Q}$ being a normalized square QAM constellation. Then, for a given channel realization ξ and a given received testing signal $z_{\ell 2}$, the error probability which is defined as the probability of deciding in favor of $\hat{y} \neq y$, $\hat{y}$ and $y \in \mathcal{Q}$, should be in the form of $Q\left(\frac{d}{\sqrt{2}\sigma_{\overline{\eta}}}\right)$ ($Q(\cdot)$ stands for the Q-function, we will give detail expression later), where d is the minimum distance of this QAM constellation. $\sigma_{\overline{\eta}}$ is the variance of the receiving noise. However, in section 3.1, Eq.(3.7) shows that the variance of the noise for this two-way relay system depends on the transmission signal y, which means we have to consider different variances according to different transmission signals. Therefore, we first examine the energy property of a square QAM constellation first. Fig. 3.1 illustrates a 64-QAM constellation (in this particular case $M = 64$, however, we derive all formulas in general case). Let us first group these symbol points according to the different signal powers.

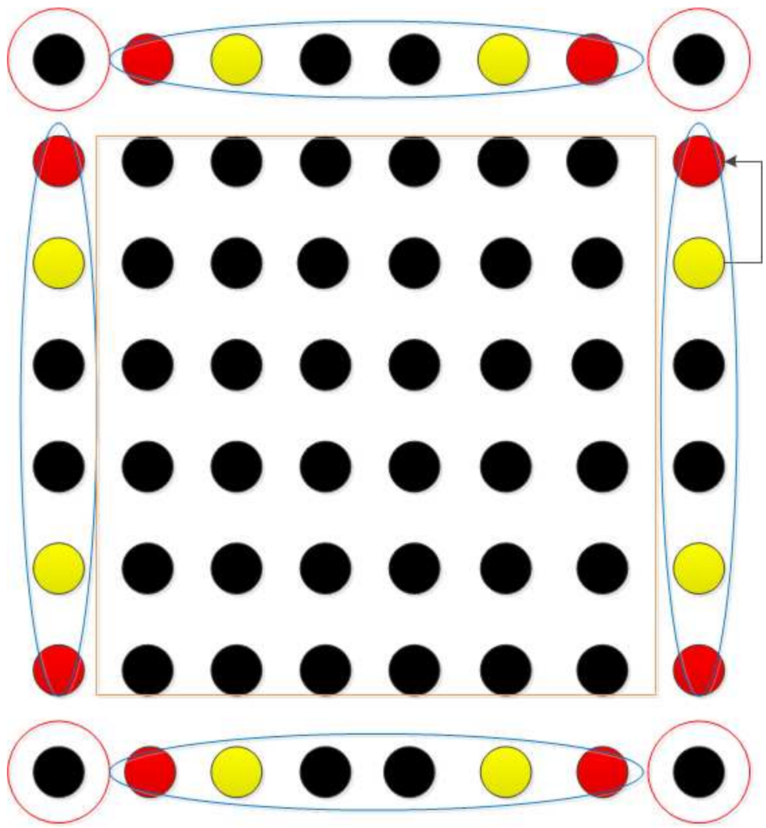


Figure 3.1: 64-QAM constellation ($M = 64$)

Corner points: In Fig. 3.1, those 4 points which are in red circles are corner points, we denote this group by $\mathcal{Q}_c$. These symbols at the corners all have the same transmission power $|y|^2 = \frac{2}{\bar{E}}(\sqrt{M} - 1)^2$. If any of these corner point symbol is transmitted, the received noise all have the same variance. Then, for a given channel realization and a given received testing signal $z_{\ell 2}$, the correct detection probability for points in this group is given by

$$P_{c,c}|\xi, z_{\ell 2} = \left(1 - Q\left(\frac{d}{\sqrt{2}\sigma_{\overline{\eta}}}\right)\right)^2 \tag{3.8}$$

where $P_{c,(\cdot)}$ indicates that this quantities represent the correct probabilities. d represents the minimal distance for the M-ary QAM constellation. The proof of Eq. (3.8) is given in Appendix A. Eq. (3.8) shows that this correct probability is depending on the noise variance $\sigma_{\overline{\eta}}$ which apparently depends on signal transmission energy $|y|^2$.

Edge Points: In Fig. 3.1, those points which are circled by blue ellipses are in the edge point group, we denote this group by $\mathcal{Q}_e$. Although all points are in the same group, in this case, not all edge points have the same power. Each corner point has 2 nearest points, totally we have 8 points, and they all have same power $|y|^2 = \frac{1}{\bar{E}}[(\sqrt{M}-1)^2 + (\sqrt{M}-3)^2]$. And for these 8 points, one nearest point for each one on this layer, and these 8 point have the same variance again. Following this manner, one could conclude that the edge points are divided into different groups, each group contains 8 points. Signal powers for edge points follow the following equation

$$|y|^2 = \frac{1}{\bar{E}}[(\sqrt{M}-1)^2 + (\sqrt{M}-m)^2], m = 3, 5, 7...(\sqrt{M}-1)$$

Although each 8 points have different variances, they all follow the same formula of correct detection probability which is given by

$$P_{c,e}|\xi, z_{\ell 2} = \left(1 - Q\left(\frac{d}{\sqrt{2}\sigma_{\overline{\eta}}}\right)\right) * \left(1 - 2Q\left(\frac{d}{\sqrt{2}\sigma_{\overline{\eta}}}\right)\right) \tag{3.9}$$

The proof of Eq. (3.9) is given in Appendix A.

Inner Points: In Fig. 3.1, all points except that in out layer are inner points, we denote the inner points group as $\mathcal{Q}_i$. Similar as the edge points group, for this group, different points contain different signal powers, we list the formula below

$$|y|^2 = \frac{1}{\bar{E}}[(\sqrt{M} - n)^2 + (\sqrt{M} - m)^2], n = 3, 5, 7...(\sqrt{M} - 1), m = n, n + 2...(\sqrt{M} - 1)$$

and the correct decision formula for points in this group is

$$P_{c,i}|\xi, z_{\ell 2} = \left(1 - 2Q\left(\frac{d}{\sqrt{2}\sigma_{\overline{\eta}}}\right)\right)^2 \tag{3.10}$$

The proof of Eq. (3.10) is given in Appendix A. Taking average of all possible transmission signals in an M-ary QAM constellation, the average correct probability is given by

$$\begin{aligned}
P_c|\xi, z_{\ell 2} &= \frac{1}{M}(\sum_{y \in Q_c} P_{c,c}|\xi, z_{\ell 2} + \sum_{y \in Q_e} P_{c,e}|\xi, z_{\ell 2} + \sum_{y \in Q_i} P_{c,i}|\xi, z_{\ell 2}) \\
&= \frac{1}{M}\left(\sum_{y \in Q_c}(1 - Q(\frac{d}{\sqrt{2}\sigma_{\overline{\eta}}}))^2 + \sum_{y \in Q_e}(1 - Q(\frac{d}{\sqrt{2}\sigma_{\overline{\eta}}})) * (1 - 2Q(\frac{d}{\sqrt{2}\sigma_{\overline{\eta}}})) + \sum_{y \in Q_i}(1 - 2Q(\frac{d}{\sqrt{2}\sigma_{\overline{\eta}}}))^2\right) \\
&= 1 - \frac{1}{M}\Big(\sum_{y \in \mathcal{Q}_c} \phi_1(y) + \sum_{y \in \mathcal{Q}_e} \phi_2(y) + \sum_{y \in \mathcal{Q}_i} \phi_3(y)\Big)
\end{aligned}$$

where

$$\begin{aligned}\phi_1(y) &= 2Q\left(\frac{d}{\sqrt{2}\sigma_{\overline{\eta}}}\right) - Q^2\left(\frac{d}{\sqrt{2}\sigma_{\overline{\eta}}}\right)\\ \phi_2(y) &= 3Q\left(\frac{d}{\sqrt{2}\sigma_{\overline{\eta}}}\right) - 2Q^2\left(\frac{d}{\sqrt{2}\sigma_{\overline{\eta}}}\right)\\ \phi_3(y) &= 4Q\left(\frac{d}{\sqrt{2}\sigma_{\overline{\eta}}}\right) - 4Q^2\left(\frac{d}{\sqrt{2}\sigma_{\overline{\eta}}}\right)\end{aligned}$$

Obviously, the error probability is given by $1 - P_c|\xi, z_{\ell 2}$, which is

$$P(e|\xi, z_{\ell 2}) = \frac{1}{M}\Big(\sum_{y\in\mathcal{Q}_c}\phi_1(y) + \sum_{y\in\mathcal{Q}_e}\phi_2(y) + \sum_{y\in\mathcal{Q}_i}\phi_3(y)\Big)$$

The average error probability is the expected value of $P(e|\xi, z_{\ell 2})$ taken with respect to ξ and $z_{\ell 2}$, i.e.

$$\begin{aligned}\bar{P}_e &= \mathbb{E}\Bigg(\frac{1}{M}\Big(\sum_{y\in\mathcal{Q}_c}[\phi_1(y)] + \sum_{y\in\mathcal{Q}_e}[\phi_2(y)] + \sum_{y\in\mathcal{Q}_i}[\phi_3(y)]\Big)\Bigg)\\ &= \frac{1}{M}\Big(\sum_{y\in\mathcal{Q}_c}\mathbb{E}[\phi_1(y)] + \sum_{y\in\mathcal{Q}_e}\mathbb{E}[\phi_2(y)] + \sum_{y\in\mathcal{Q}_i}\mathbb{E}[\phi_3(y)]\Big)\end{aligned} \tag{3.11}$$

To further simplify this expression, the first step is to take expected value of $Q(\frac{d}{\sqrt{2}\sigma_{\overline{\eta}}})$ and $Q^2(\frac{d}{\sqrt{2}\sigma_{\overline{\eta}}})$ with respect to ξ and $z_{\ell 2}$. By definition [39]

$$Q(t) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{-\frac{t^2}{2\sin^2\theta}}\,d\theta \tag{3.12}$$

$$Q^2(t) = \frac{1}{\pi}\int_0^{\frac{\pi}{4}} e^{-\frac{t^2}{2\sin^2\theta}}\,d\theta \tag{3.13}$$

where

$$t^2 = \frac{d^2}{2\sigma_{\overline{\eta}}^2} \tag{3.14}$$

From Eq.(3.5), $z_\ell - \sqrt{p_2}\hat{h}^2 x = \sqrt{q_2}\widehat{hg}y + \overline{\eta}$, comparing with the standard AWGN channel model $z_{stan} = y_{stan} + \eta_{stan}$. The transmission signal y in our case is scaled by factor $\sqrt{q_2}\widehat{hg}$. Therefore, the minimal distance d should be scaled by the same factor, we obtain

$$\begin{aligned} d &= \frac{2\sqrt{q_2}\widehat{hg}}{\sqrt{\bar{E}}} = \frac{2\sqrt{q_2}}{\sqrt{\bar{E}}}\frac{z_{\ell,2}}{\sqrt{q_1}} \\ d^2 &= \frac{4q_2}{\bar{E}q_1}|z_{\ell,2}|^2 \end{aligned} \tag{3.15}$$

Substitute Eq. (3.15) and Eq. (3.7) to Eq.(3.14), we have:

$$t^2 = \frac{d^2}{2\sigma_{\overline{\eta}}^2} = \frac{2\frac{q_2}{q_1}|z_{\ell,2}|^2}{\bar{E}(\frac{p_2|x|^2}{p_1} + \frac{q_2|y|^2}{q_1} + 1)(\xi+1)\sigma^2}$$

Substitute the above equation to the Eq. (3.12) and (3.13), we obtain

$$\begin{aligned} Q(\frac{d}{\sqrt{2}\sigma_{\overline{\eta}}}) &= \frac{1}{\pi}\int_0^{\pi/2} e^{\frac{-\frac{q_2}{q_1}|z_{\ell,2}|^2}{\bar{E}(\frac{p_2|x|^2}{p_1}+\frac{q_2|y|^2}{q_1}+1)(\xi+1)\sin^2\theta\sigma^2}} d\theta \\ Q^2(\frac{d}{\sqrt{2}\sigma_{\overline{\eta}}}) &= \frac{1}{\pi}\int_0^{\pi/4} e^{\frac{-\frac{q_2}{q_1}|z_{\ell,2}|^2}{\bar{E}(\frac{p_2|x|^2}{p_1}+\frac{q_2|y|^2}{q_1}+1)(\xi+1)\sin^2\theta\sigma^2}} d\theta \end{aligned}$$

From Eq. (3.3b), given ξ the probability density function of $z_{\ell 2}$ takes on the form

$$\begin{aligned}\vartheta(z_{\ell 2}|\xi) &= \frac{1}{\sqrt{2\pi(q_1\xi + (\xi+1)\sigma_{re}^2)}} e^{\frac{-z_{\ell 2re}^2}{2(q_1|h|^2+(|h|^2+1)\sigma_{re}^2)}} \frac{1}{\sqrt{2\pi(q_1\xi + (\xi+1)\sigma_{im}^2)}} e^{\frac{-z_{\ell 2im}^2}{2(q_1|h|^2+(|h|^2+1)\sigma_{im}^2)}} \\ &= \frac{1}{\pi(2q_1\xi + (\xi+1)\sigma^2)} e^{\frac{-|z_{\ell 2}|^2}{2q_1\xi+(\xi+1)\sigma^2}}\end{aligned}$$

Since h is a zero mean complex Gaussian random variable with unit variance, then the PDF of ξ is exponential distributed [13], i.e., $\vartheta(\xi) = e^{-\xi}$ for $0 \leq \xi \leq \infty$. The expected value of $Q(\frac{d}{\sqrt{2}\sigma_{\overline{\eta}}})$ and $Q^2(\frac{d}{\sqrt{2}\sigma_{\overline{\eta}}})$ with respect to ξ and $z_{\ell 2}$ is given by

$$\mathbb{E}[Q(\frac{d}{\sqrt{2}\sigma})] = \frac{1}{\pi^2}\int_0^{\pi/2}\int_0^{\infty}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sin\theta^2}}\vartheta(z_{\ell 2}|\xi)\vartheta(\xi)d_{z_{\ell 2}}d\xi d\theta \tag{3.16}$$

$$\mathbb{E}[Q^2(\frac{d}{\sqrt{2}\sigma})] = \frac{1}{\pi^2}\int_0^{\pi/4}\int_0^{\infty}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sin\theta^2}}\vartheta(z_{\ell 2}|\xi)\vartheta(\xi)d_{z_{\ell 2}}d\xi d\theta \tag{3.17}$$

Based on previous equations we immediately obtain the following two lemmas

Lemma 1. *The approximate average of the Q-function with respect to the channel estimation signal is given by:*

$$\mathbb{E}\left[Q\left(\frac{d}{\sqrt{2}\sigma_{\bar{\eta}}}\right)\right] = \alpha_0\rho^{-1}\ln\rho + \alpha_1\rho^{-1} + O(\rho^{-2}\ln\rho) \tag{3.18}$$

where ρ is the signal to noise ratio defined as $\frac{1}{\sigma^2}$, and,

$$\begin{aligned}\alpha_0 &= \frac{A}{8q_2}\\ \alpha_1 &= -\frac{1}{2}\left(\frac{(\gamma-1)A}{4q_2} + \frac{A}{4q_2}(\ln b - \ln 2q_2)\right.\\ &\left.+\frac{-A}{2q_2}\left(-\ln\frac{1+\sqrt{\frac{q_2}{q_1 b}}}{2} - \frac{1}{1+\sqrt{\frac{q_2}{q_1 b}}} + \frac{1}{2}\right)\right)\end{aligned}$$

with γ being the Euler constant and

$$\begin{aligned}A &= \bar{E}(\frac{p_2|x|^2}{p_1} + \frac{q_2|y|^2}{q_1} + 1)\\ b &= A + \frac{q_2}{q_1}\end{aligned}$$

Here, $\bar{E}$ denotes the average symbol energy for an M-ary QAM constellation.

Lemma 2. *The approximate average of the Q^2 function with respect to the channel estimation signal is given by:*

$$\mathbb{E}\left[Q^2\left(\frac{d}{\sqrt{2}\sigma_{\bar{\eta}}}\right)\right] = \beta_0\rho^{-1}\ln\rho + \beta_1\rho^{-1} + O(\rho^{-2}\ln\rho) \tag{3.19}$$

where,

$$\begin{aligned}\beta_0 &= \frac{A}{16q_2}\\ \beta_1 &= -\frac{1}{4}\left((\frac{\gamma}{2} - \frac{1}{2} - \frac{1}{\pi})\frac{A}{2q_2} + \frac{A}{4q_2}(\ln b - \ln 2q_2)\right.\\ &\left.+\frac{-A}{2q_2}\left(-\ln\frac{1+\sqrt{\frac{q_2}{q_1 b}}}{2} - \frac{1}{1+\sqrt{\frac{q_2}{q_1 b}}} + \frac{1}{2}\right)\right)\end{aligned}$$

The proofs of the above lemmas are in Appendix B and C. Applying Lemma 1 and Lemma 2 to Eq. (3.11), after some simplifications, we obtain the following theorem and corollary:

Theorem 1. *The error probability for the two-way-relay system be defined by Eq.* (3.11). *With the results in Lemmas 1 and 2, this average error probability for the left terminal receiver has the following asymptotic form:*

$$\bar{P}_{e\ell}(x) = C_{\ell 0}\rho^{-1}\ln\rho + C_{\ell 1}\rho^{-1} + O(\rho^{-2}\ln\rho) \tag{3.20}$$

where,

$$\begin{aligned}
C_{\ell 0} &= \frac{\bar{E}}{4q_2}\frac{M-1}{M}\left(\frac{p_2|x|^2}{p_1}+1\right) + \frac{\bar{E}}{4q_1}\left(1-\frac{E_c}{4M}\right)\\
C_{\ell 1} &= \frac{\bar{E}}{2q_2}(\frac{p_2|x|^2}{p_1}+1)\bigg((-\frac{M}{2}+\frac{1}{2})\ln 2q_2 + (C_{02}-\frac{1}{2})M\\
&\quad +(4C_{01}-2C_{02}-1)\sqrt{M} + 4C_{00} - 8C_{01} + 4C_{02} + 2\bigg)\\
&\quad + \frac{\bar{E}}{2q_1}\bigg((C_{00}-\frac{3}{8}\ln 2q_2)E_c + (C_{01}-\frac{1}{2}\ln 2q_2)E_e\\
&\quad +(C_{02}-\frac{1}{2}\ln 2q_2)E_i\bigg)\\
&\quad + \sum_{y\in\mathcal{Q}_c}\frac{A}{2q_2}\left(\frac{3}{8}\ln b - \frac{3}{4}\Big(-\ln\frac{1+\sqrt{\frac{q_2}{q_1 b}}}{2} - \frac{1}{1+\sqrt{\frac{q_2}{q_1 b}}}\Big)\right)\\
&\quad + \sum_{y\in\mathcal{Q}_e}\frac{A}{2q_2}\left(\frac{1}{2}\ln b - \Big(-\ln\frac{1+\sqrt{\frac{q_2}{q_1 b}}}{2} - \frac{1}{1+\sqrt{\frac{q_2}{q_1 b}}}\Big)\right)\\
&\quad + \sum_{y\in\mathcal{Q}_i}\frac{A}{2q_2}\left(\frac{1}{2}\ln b - \Big(-\ln\frac{1+\sqrt{\frac{q_2}{q_1 b}}}{2} - \frac{1}{1+\sqrt{\frac{q_2}{q_1 b}}}\Big)\right)
\end{aligned}$$

with $C_{00} = \frac{3\gamma}{8} - \frac{3}{4} + \frac{1}{4\pi}$, $C_{01} = \frac{\gamma}{2} - 1 + \frac{1}{2\pi}$ and $C_{02} = \frac{\gamma}{2} - 1 + \frac{1}{\pi}$. E_c, E_e and E_i are the corner energy, edge energy and inner energy respectively. This theorem can be verified by simple calculation and thus, its proof is omitted.

Corollary 1. *Parallel to theorem 1, we obtain asymptotic formula of the error probability for the right side of the two-way relay system as:*

$$\bar{P}_{er}(y) = C_{r0}\rho^{-1}\ln\rho + C_{r1}\rho^{-1} + O(\rho^{-2}\ln\rho) \tag{3.21}$$

where

$$\begin{aligned}
C_{r0} &= \frac{\bar{E}}{4p_2}\frac{M-1}{M}\left(\frac{q_2|y|^2}{q_1}+1\right) + \frac{\bar{E}}{4p_1}\left(1-\frac{E_c}{4M}\right)\\
C_{r1} &= \frac{\bar{E}}{2p_2}\left(\frac{q_2|y|^2}{q_1}+1\right)\Big((-\frac{M}{2}+\frac{1}{2})\ln 2p_2 + (C_{02}-\frac{1}{2})M\\
&\quad +(4C_{01}-2C_{02}-1)\sqrt{M} + 4C_{00} - 8C_{01} + 4C_{02} + 2\Big)\\
&\quad + \frac{\bar{E}}{2p_1}\Big((C_{00}-\frac{3}{8}\ln 2p_2)E_c + (C_{01}-\frac{1}{2}\ln 2p_2)E_e\\
&\quad +(C_{02}-\frac{1}{2}\ln 2p_2)E_i\Big)\\
&\quad + \sum_{x\in\mathcal{Q}_c}\frac{A}{2p_2}\left(\frac{3}{8}\ln\bar{b} - \frac{3}{4}\Big(-\ln\frac{1+\sqrt{\frac{p_2}{p_1\bar{b}}}}{2} - \frac{1}{1+\sqrt{\frac{p_2}{p_1\bar{b}}}}\Big)\right)\\
&\quad + \sum_{x\in\mathcal{Q}_e}\frac{A}{2p_2}\left(\frac{1}{2}\ln\bar{b} - \Big(-\ln\frac{1+\sqrt{\frac{p_2}{p_1\bar{b}}}}{2} - \frac{1}{1+\sqrt{\frac{p_2}{p_1\bar{b}}}}\Big)\right)\\
&\quad + \sum_{x\in\mathcal{Q}_i}\frac{A}{2p_2}\left(\frac{1}{2}\ln\bar{b} - \Big(-\ln\frac{1+\sqrt{\frac{p_2}{p_1\bar{b}}}}{2} - \frac{1}{1+\sqrt{\frac{p_2}{p_1\bar{b}}}}\Big)\right)
\end{aligned}$$

where $\bar{b} = A + \frac{p_2}{p_1}$

The above theorem and corollary show us the asymptotic formulas of average probability of error for a two-way relay system in which p_1 and q_1 are the power of the testing signals x_1 and y_2 and p_2 and q_2 are the power of the transmitting signals

x and y.

Comparing the result in Theorem 1 and Corollary 1 with the asymptotic formula of the error probability for a conventional MISO system, we can conclude that the error probability for a conventional MISO system is often characterized by the diversity gain which is defined as the slope at which the error probability decreases with the logarithm of SNR. However, Theorem 1 and Corollary 1, show that the asymptotic formula of the error probability for an AF two-way relay system is no longer simply a power function of SNR, it involves the term $\ln(SNR)$ which is a function of the logarithm of SNR. This is caused by the term which involves the product of two Gaussian random variables in the channel matrix. Because of this term the entries of the channel matrix of this two-way relay system are no longer Gaussian as in the case for conventional MISO system. This $\ln(SNR)$ factor in the diversity gain function has also been observed in [36]

Chapter 4

Optimum Transmission Power

In the last chapter we have derived the asymptotic formula of the error probability for both terminals. Our goal in this chapter is to find a combination of transmission power which yields minimum error probability subject to a total transmission power constraint. Since these error probabilities depend on the transmission signals, in the following, we divide the optimization problem into two separate cases: the *average power loading* optimization, and the *worst-case power loading* optimization.

4.1 Average Power Loading Optimization

In this case, since we are transmitting from an M-ary QAM constellation in which each signal may have different power, thus, we take the average of the error probability for all signals from both left and right terminals. Substituting equations from Theorem 1 and Corollary 1. This results in a signal independent error probability equation such that,

$$\begin{aligned}\bar{P}_{ea} &= \frac{1}{2}\left(\frac{1}{M}\sum_{x\in\mathcal{Q}}\bar{P}_{e\ell}(x) + \frac{1}{M}\sum_{y\in\mathcal{Q}}\bar{P}_{er}(y)\right)\\ &= \frac{1}{2}\left(\bar{P}_{e\ell} + \bar{P}_{er}\right)\\ &= F(p_1,p_2,q_1,q_2)\rho^{-1}\ln\rho\end{aligned}$$

where

$$\begin{aligned}\bar{P}_{e\ell} &= \frac{1}{M}\sum_{x\in\mathcal{Q}} C_{\ell 0}\rho^{-1}\ln\rho = \left[\frac{\bar{E}(M-1)}{4Mq_2}(1+\frac{p_2}{p_1}) + \frac{\bar{E}}{4q_1}(1-\frac{E_c}{4M})\right]\rho^{-1}\ln\rho\\ \bar{P}_{er} &= \frac{1}{M}\sum_{y\in\mathcal{Q}} C_{r0}\rho^{-1}\ln\rho = \left[\frac{\bar{E}(M-1)}{4Mp_2}(1+\frac{q_2}{q_1}) + \frac{\bar{E}}{4p_1}(1-\frac{E_c}{4M})\right]\rho^{-1}\ln\rho\\ F(p_1,p_2,q_1,q_2) &= \frac{1}{2}\left(A_1\left(\frac{p_1+p_2}{p_1q_2}+\frac{q_1+q_2}{q_1p_2}\right)+A_2\left(\frac{1}{q_1}+\frac{1}{p_1}\right)\right) \qquad (4.1)\end{aligned}$$

with $A_1 = \frac{\bar{E}}{4}\frac{M-1}{M}$ and $A_2 = \frac{\bar{E}}{4}\left(1-\frac{E_c}{4M}\right)$. Since we define $\rho = \frac{1}{\sigma^2}$ which is independent of p_1, p_2, q_1, q_2, optimization of $\bar{P}_{ea}$ is equivalent to optimization of $F(p_1,p_2,q_1,q_2)$. Therefore, imposing the power constraints $p_1 = 1-p_2$ and $q_1 = 1-q_2$ into Eq. (4.1), our design problem becomes

$$\min_{p_2,q_2} \quad F(p_2,q_2) \qquad (4.2)$$

Taking derivatives of Eq. (4.2) with respect to p_2 and q_2, and equating both results to zero, we have

$$\frac{\partial F(p_2,q_2)}{\partial p_2} = \frac{1}{2}\left(\frac{-A_1}{p_2^2}+\frac{A_1}{(1-p_2)^2q_2}-\frac{A_1q_2}{(1-q_2)p_2^2}+\frac{A_2}{(1-p_2)^2}\right) = 0 \qquad (4.3)$$

$$\frac{\partial F(p_2,q_2)}{\partial q_2} = \frac{1}{2}\left(\frac{-A_1}{q_2^2}+\frac{A_1}{(1-q_2)^2p_2}-\frac{A_1p_2}{(1-p_2)q_2^2}+\frac{A_2}{(1-q_2)^2}\right) = 0 \qquad (4.4)$$

Simplifying the above two equations, first, we subtract Eq. (4.4) from Eq. (4.3), we have

$$\frac{\partial F(p_2,q_2)}{\partial p_2}-\frac{\partial F(p_2,q_2)}{\partial q_2}=(p_2-q_2)[A_1(f_1(p_2,q_2)+f_2(p_2,q_2))+A_2f_3(p_2,q_2)]=0 \tag{4.5}$$

where:

$$\begin{aligned}
f_1(p_2,q_2)&=\frac{p_2+q_2}{p_2^2q_2^2}\\
f_2(p_2,q_2)&=\frac{1-p_2q_2}{(1-p_2)^2(1-q_2)^2p_2q_2}+\frac{p_2^2+p_2q_2+q_2^2-p_2q_2(p_2+q_2)}{(1-p_2)(1-q_2)p_2^2q_2^2}\\
f_3(p_2,q_2)&=\frac{2-p_2-q_2}{(1-p_2)^2(1-q_2)^2}
\end{aligned}$$

Obviously, one of the solution for Eq.(4.5) is $p_2=q_2$. Next, we have to check if this solution is unique. Let us examine the second bracket in Eq. (4.5): $A_1(f_1(p_2,q_2)+f_2(p_2,q_2))+A_2f_3(p_2,q_2)$. From the expressions of A_1, and A_2, it is easy to prove that, they are all positive. Also, because $0\le p_2,q_2\le 1$, obviously, both $f_1(p_2,q_2)$ and $f_3(p_2,q_2)$ are positive. Now, let us check $f_2(p_2,q_2)$

$$\begin{aligned}
f_2(p_2,q_2)&=\frac{1-p_2q_2}{(1-p_2)^2(1-q_2)^2p_2q_2}+\frac{p_2^2+p_2q_2+q_2^2-p_2q_2(p_2+q_2)}{(1-p_2)(1-q_2)p_2^2q_2^2}\\
&\ge\frac{1-p_2q_2}{(1-p_2)^2(1-q_2)^2p_2q_2}+\frac{3p_2q_2-p_2q_2(p_2+q_2)}{(1-p_2)(1-q_2)p_2^2q_2^2}\\
&\ge\frac{1-p_2q_2}{(1-p_2)^2(1-q_2)^2p_2q_2}+\frac{3p_2q_2-2p_2q_2}{(1-p_2)(1-q_2)p_2^2q_2^2}\\
&=\frac{1-p_2q_2}{(1-p_2)^2(1-q_2)^2p_2q_2}+\frac{p_2q_2}{(1-p_2)(1-q_2)p_2^2q_2^2}\\
&\ge 0
\end{aligned}$$

Since each term for the expression $A_1(f_1(p_2, q_2) + f_2(p_2, q_2)) + A_2 f_3(p_2, q_2)$ is positive, the whole equation is always positive. Therefore, we can conclude that $p_2 = q_2$ is the unique solution for Eq.(4.5). Under this condition, solving Eq. (4.4) or Eq. (4.3), the unique optimum solution is achieved when

$$p_2 = q_2 = \frac{-A_1 + \sqrt{A_1^2 + A_1 A_2}}{A_2} \tag{4.6}$$

Next, let's prove that the power constraint $0 < p_2, q_2 < 1$ is satisfied. It is easy to see from Eq. (4.6) that $p_2, q_2 \geq 0$.Now, for

$$\begin{aligned} p_2 = q_2 = \frac{-A_1 + \sqrt{A_1^2 + A_1 A_2}}{A_2} &< 1 \\ \Rightarrow \sqrt{A_1^2 + A_1 A_2} &< A_1 + A_2 \\ A_1^2 + A_1 A_2 &< A_1^2 + 2A_1 A_2 + A_2^2 \\ 0 &< A_1 A_2 + A_2^2 \end{aligned}$$

since $A_1, A_2 \geq 0$, the above equation is true, $p_2, q_2 \leq 1$ is proved. Combining the last two inequalities, we conclude that $0 \leq p_2, q_2 \leq 1$. Comparing Eq.(4.6) with the value of the end points, we can observe from the objective function, at each endpoints this function goes to infinite. Thus, we complete the proof that our optimum point is the minimum point.

4.2 Worst-Case Power Loading Optimization

In this case we try to minimize the worst case error probability caused by a given transmission signal. Referring to Eqs. (3.20) and (3.21), we see that the probabilities of error are dominated by the energy of the signals $|x|^2$ and $|y|^2$. Therefore, the worst error probability is caused by the signal containing the maximum energy in the square QAM, i.e., the signal at one of the corner points. Substituting the corresponding energy equation to Eq. (3.20) and Eq. (3.21), we obtain

$$\bar{P}_{e\ell\max} = \Phi_\ell(p_1, p_2, q_1, q_2)\rho^{-1}\ln\rho$$

$$\bar{P}_{er\max} = \Phi_r(p_1, p_2, q_1, q_2)\rho^{-1}\ln\rho$$

where

$$\Phi_\ell(p_1, p_2, q_1, q_2) = \frac{K_1 p_2}{p_1 q_2} + \frac{K_2}{q_2} + \frac{K_3}{q_1} \tag{4.7a}$$

$$\Phi_r(p_1, p_2, q_1, q_2) = \frac{K_1 q_2}{q_1 p_2} + \frac{K_2}{p_2} + \frac{K_3}{p_1} \tag{4.7b}$$

with $K_1 = \frac{\bar{E}E_c(M-1)}{16M}$, $K_2 = \frac{\bar{E}(M-1)}{4M}$ and $K_3 = \frac{\bar{E}}{4}(1 - \frac{E_c}{4M})$. Similar to the first case, since ρ is independent of p_1, p_2, q_1, q_2, optimization of $\bar{P}_{ea}$ is equivalent to optimization of $\Phi_\ell(p_1, p_2, q_1, q_2)$ or $\Phi_r(p_1, p_2, q_1, q_2)$. We impose the power constraints $p_1 = 1 - p_2$, and $q_1 = 1 - q_2$. This yields the objective functions for the worst-case power loading case $\Phi_\ell(p_2, q_2)$ and $\Phi_r(p_2, q_2)$ respectively for the left and right terminals. We now compare these two error probabilities so that we may obtain the conditions under which one is greater than the other. Thus, we write

$$\Phi_\ell(p_2, q_2) - \Phi_r(p_2, q_2) = K_c(p_2 - q_2)\Phi_c(p_2, q_2) \tag{4.8}$$

where

$$K_c = \frac{1}{(1-p_2)(1-q_2)p_2q_2}$$
$$\Phi_c(p_2,q_2) = K_1(p_2+q_2-p_2q_2) - K_3p_2q_2 + K_2(1-p_2-q_2+p_2q_2)$$

Back to Eq. (4.8), in order to have $\bar{P}_{e\ell max}$ greater than $\bar{P}_{ermax}$, we have to prove the following inequality is true

$$\Phi_\ell(p_2,q_2) - \Phi_r(p_2,q_2) = K_c(p_2-q_2)\Phi_c(p_2,q_2) \geq 0$$

Obviously, K_c is greater than zero. Therefore during the following derivation we just neglect it at this moment.

1. If $p_2 \geq q_2$,

$$K_1(p_2+q_2-p_2q_2) + K_2(1-p_2-q_2+p_2q_2) - K_3p_2q_2 \geq 0$$
$$\Rightarrow [(-K_1+K_2-K_3)q_2 + (K_1-K_2)]p_2 + (K_1-K_2)q_2 + K_2 \geq 0$$

   (a) If

$$(-K_1+K_2-K_3)q_2 + (K_1-K_2) > 0$$

$$\Rightarrow \begin{cases} q_2 < \frac{K_1-K_2}{K_1-K_2+K_3} \\ p_2 \geq \frac{-(K_1-K_2)q_2-K_2}{(-K_1+K_2-K_3)q_2+(K_1-K_2)} \end{cases} \tag{4.9}$$

since $K_1 \geq K_2 \geq K_3 > 0$, we obtain

$$\frac{-(K_1 - K_2)q_2 - K_2}{(-K_1 + K_2 - K_3)q_2 + (K_1 - K_2)} < 0$$

Combining Eq. (4.9) and the condition $p_2 \geq q_2 \geq 0$, we obtain the following range

$$\Rightarrow \begin{cases} q_2 < \frac{K_1 - K_2}{K_1 - K_2 + K_3} \\ 0 \leq q_2 \leq p_2 \leq 1 \end{cases} \tag{4.10}$$

(b) If

$$(-K_1 + K_2 - K_3)q_2 + (K_1 - K_2) < 0$$

$$\Rightarrow \begin{cases} q_2 > \frac{K_1 - K_2}{K_1 - K_2 + K_3} \\ p_2 \leq \frac{-(K_1 - K_2)q_2 - K_2}{(-K_1 + K_2 - K_3)q_2 + (K_1 - K_2)} \end{cases} \tag{4.11}$$

since we are under the condition $p_2 \geq q_2$, the expression of p_2 should be greater than q_2 as well.

$$\frac{-(K_1 - K_2)q_2 - K_2}{(-K_1 + K_2 - K_3)q_2 + (K_1 - K_2)} - q_2 \geq 0$$

Solving this inequality, we have the following range for q_2

$$\frac{K_1 - K_2 - \sqrt{K_1^2 - K_1K_2 + K_2K_3}}{K_1 - K_2 + K_3} \leq q_2 \leq \frac{K_1 - K_2 + \sqrt{K_1^2 - K_1K_2 + K_2K_3}}{K_1 - K_2 + K_3}$$

Combining Eq. (4.11) and the above inequality, we obtain the following constraint for q_2 and p_2

$$\frac{K_1-K_2}{K_1-K_2+K_3} \leq q_2 \leq \frac{K_1-K_2+\sqrt{K_1^2-K_1K_2+K_2K_3}}{K_1-K_2+K_3}$$
$$q_2 \leq p_2 \leq \frac{-(K_1-K_2)q_2-K_2}{(-K_1+K_2-K_3)q_2+(K_1-K_2)}$$

because

$$\frac{K_1-K_2+\sqrt{K_1^2-K_1K_2+K_2K_3}}{K_1-K_2+K_3} > 1$$
$$\Rightarrow \frac{K_1-K_2}{K_1-K_2+K_3} \leq q_2 \leq 1$$

and

$$\frac{-(K_1-K_2)q_2-K_2}{(-K_1+K_2-K_3)q_2+(K_1-K_2)} \geq 1$$
$$\Rightarrow q_2 \leq p_2 \leq 1$$

Finally, we have the range in this case:

$$\Rightarrow \begin{cases} \frac{K_1-K_2}{K_1-K_2+K_3} \leq q_2 \leq 1 \\ 0 \leq q_2 \leq p_2 \leq 1 \end{cases} \tag{4.12}$$

Combining Eq. (4.10) and Eq. (4.12), we conclude that the range for $\bar{P}_{e\ell max}$ greater than $\bar{P}_{ermax}$ under the condition $p_2 \geq q_2$ is

$$0 \leq q_2 \leq p_2 \leq 1 \tag{4.13}$$

2. If $p_2 \leq q_2$

$$K_1(p_2 + q_2 - p_2q_2) + K_2(1 - p_2 - q_2 + p_2q_2) - K_3p_2q_2 \leq 0$$

$$[(-K_1 + K_2 - K_3)q_2 + (K_1 - K_2)]p_2 + (K_1 - K_2)q_2 + K_2 \leq 0$$

   (a) If

$$(-K_1 + K_2 - K_3)q_2 + (K_1 - K_2) > 0$$

$$\Rightarrow \begin{cases} q_2 < \frac{K_1 - K_2}{K_1 - K_2 + K_3} \\ p_2 \leq \frac{-(K_1 - K_2)q_2 - K_2}{(-K_1 + K_2 - K_3)q_2 + (K_1 - K_2)} \end{cases} \tag{4.14}$$

   since Eq. (4.14) is negative for sure, obviously, p_2 is out of range ($0 \leq p_2 \leq 1$).

   (b) If

$$(-K_1 + K_2 - K_3)q_2 + (K_1 - K_2) < 0$$

$$\Rightarrow \begin{cases} q_2 > \frac{K_1 - K_2}{K_1 - K_2 + K_3} \\ p_2 \geq \frac{-(K_1 - K_2)q_2 - K_2}{(-K_1 + K_2 - K_3)q_2 + (K_1 - K_2)} \end{cases}$$

   We have checked in case 1(b),that

$$\frac{-(K_1 - K_2)q_2 - K_2}{(-K_1 + K_2 - K_3)q_2 + (K_1 - K_2)} \geq 1$$

Therefore, p_2 is out of range again.

According to above derivation, we can conclude that the condition for $\Phi_l(p_2, q_2) \geq \Phi_r(p_2, q_2)$ is expressed in Eq. (4.13). In this range, our design problem becomes

$$\min_{0 \leq q_2 \leq p_2 \leq 1} \quad \Phi_\ell(p_2, q_2)$$

Taking derivative with respect to p_2, we have

$$\frac{\partial \Phi_\ell(p_2, q_2)}{\partial p_2} = \frac{K_1}{(1-p_2)^2 q_2} \tag{4.15}$$

Since Eq. (4.15) is always positive, we can conclude that $\Phi_\ell(p_2, q_2)$ is an increasing function with respect to p_2. Thus, the minimum error probability occurs at minimum power p_2. Now, $q_2 \leq p_2$, hence minimum power occurs at $p_2 = q_2$. Substitute this condition into Eq. (4.7a), we obtain

$$\Phi_\ell(q_2) = \frac{K_1 + K_3}{(1-q_2)} + \frac{K_2}{q_2}$$

Equating the derivative of $\Phi_\ell(q_2)$ with respect to q_2 to zero, we have

$$\frac{\partial \Phi_\ell(q_2)}{\partial q_2} = \frac{-K_2}{q_2^2} + \frac{K_1 + K_3}{(1-q_2)^2} = 0$$

This equation leads to the condition of minimum error probability so that

$$p_2 = q_2 = \frac{-K_2 + \sqrt{K_1 K_2 + K_2 K_3}}{K_1 - K_2 + K_3} \tag{4.16}$$

In the following discussion, we analyze the condition for $\bar{P}_{ermax}$ greater than $\bar{P}_{e\ell max}$. Following the manner as the last case, refer back to Eq. (4.8). In order to have $\bar{P}_{ermax}$ greater than $\bar{P}_{e\ell max}$, the following inequality has to be true

$$(p_2 - q_2)[K_1(p_2 + q_2 - p_2q_2) + K_2(1 - p_2 - q_2 + p_2q_2) - K_3p_2q_2] \leq 0$$

1. If $p_2 \leq q_2$, then

$$\begin{aligned} &K_1(p_2 + q_2 - p_2q_2) + K_2(1 - p_2 - q_2 + p_2q_2) - K_3p_2q_2 \geq 0 \\ \Rightarrow&[(-K_1 + K_2 - K_3)q_2 + (K_1 - K_2)]p_2 + (K_1 - K_2)q_2 + K_2 \geq 0 \end{aligned}$$

   (a) If

$$(-K_1 + K_2 - K_3)q_2 + (K_1 - K_2) > 0$$

$$\Rightarrow \begin{cases} q_2 < \frac{K_1 - K_2}{K_1 - K_2 + K_3} \\ p_2 \geq \frac{-(K_1 - K_2)q_2 - K_2}{(-K_1 + K_2 - K_3)q_2 + (K_1 - K_2)} \end{cases}$$

   these conditions has been checked in Eq. (4.9) already, thus, we will omit the proof and get the following range:

$$\Rightarrow \begin{cases} q_2 < \frac{K_1-K_2}{K_1-K_2+K_3} \\ 0 \leq p_2 \leq q_2 \leq 1 \end{cases} \tag{4.17}$$

(b) If

$$(-K_1 + K_2 - K_3)q_2 + (K_1 - K_2) < 0$$

$$\Rightarrow \begin{cases} q_2 > \frac{K_1-K_2}{K_1-K_2+K_3} \\ p_2 \leq \frac{-(K_1-K_2)q_2-K_2}{(-K_1+K_2-K_3)q_2+(K_1-K_2)} \end{cases}$$

following the same argument as Eq. (4.11), and the range for this case will be:

$$\Rightarrow \begin{cases} \frac{K_1-K_2}{K_1-K_2+K_3} \leq q_2 \leq 1 \\ 0 \leq p_2 \leq q_2 \leq 1 \end{cases} \tag{4.18}$$

Combining Eq. (4.17) and Eq. (4.18) , we can conclude that the range for $\bar{P}_{ermax} > \bar{P}_{e\ell max}$ is

$$0 \leq p_2 \leq q_2 \leq 1 \tag{4.19}$$

2. If $q_2 \leq p_2$

We have proved in the previous case that this is exact the range for $\bar{P}_{e\ell max}$

greater than $\bar{P}_{ermax}$.

From Eq. (4.13) and Eq. (4.19), we further prove the symmetry of this system. Thus, due to the symmetry of this system, we can conclude that the range for $\Phi_r(p_2, q_2) \geq \Phi_\ell(p_2, q_2)$ is $0 \leq p_2 \leq q_2 \leq 1$ leading to the same optimum condition of $q_2 = p_2$ as in Eq. (4.16). Comparing Eqs. (4.6) and (4.16), we note that even the optimum conditions for the two cases are similar such that $p_2 = q_2$, the actual transmission power in the two cases are different due to different optimization criteria.

Chapter 5

Simulation Result

5.1 Simulation Model

In this chapter, we compare the error performance of this two-way relay system operating at optimum transmission power for both cases of average and worst-case power loadings with the error performance of same system operating at other transmission powers values. The comparison is evaluated under different SNR. The system is tested with transmitted signals being selected from 4-QAM and 16QAM constellations. The channel g and h are assumed to be CSCG random variables with unit variance, and the noise is IID with variance σ^2. The SNR is defined as $\rho = \frac{1}{\sigma^2}$. At the receiver, we employ a detection such that we choose the QAM symbol closest to the received signal as the detected symbol. The performance comparison of the different cases are shown in Figs.2 to 4, for which the optimum transmission power are $p_2 = q_2 = 0.414$, $p_2 = q_2 = 0.414$, $p_2 = q_2 = 0.417$, $p_2 = q_2 = 0.376$, respectively. It can be observed that in all the cases, the optimized system yields the best performance, showing that indeed, the transmission power obtained is the optimum value for the system.

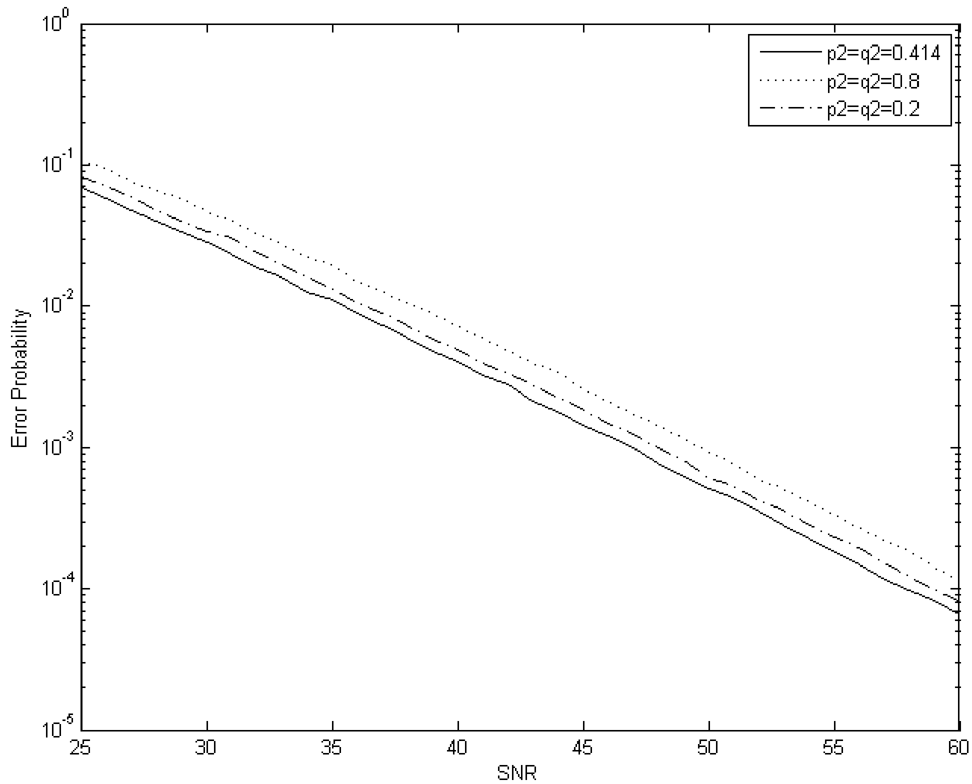


Figure 5.1: Performance comparison for 4-QAM in average power loading case

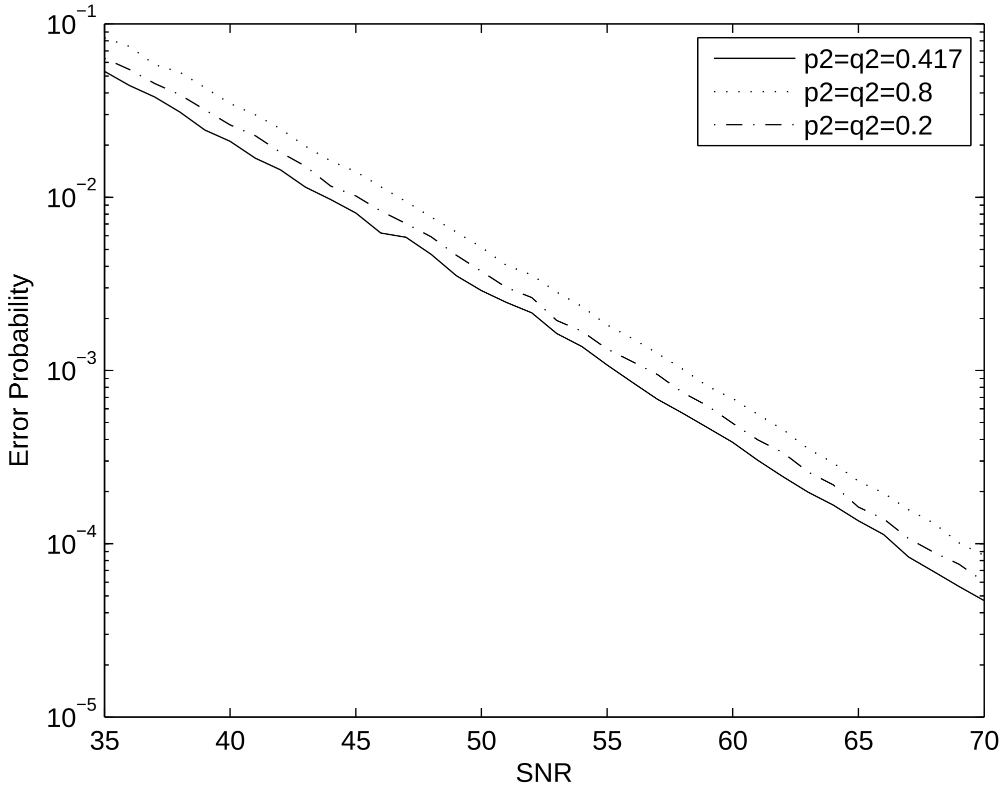


Figure 5.2: Performance comparison for 16-QAM in average power loading case

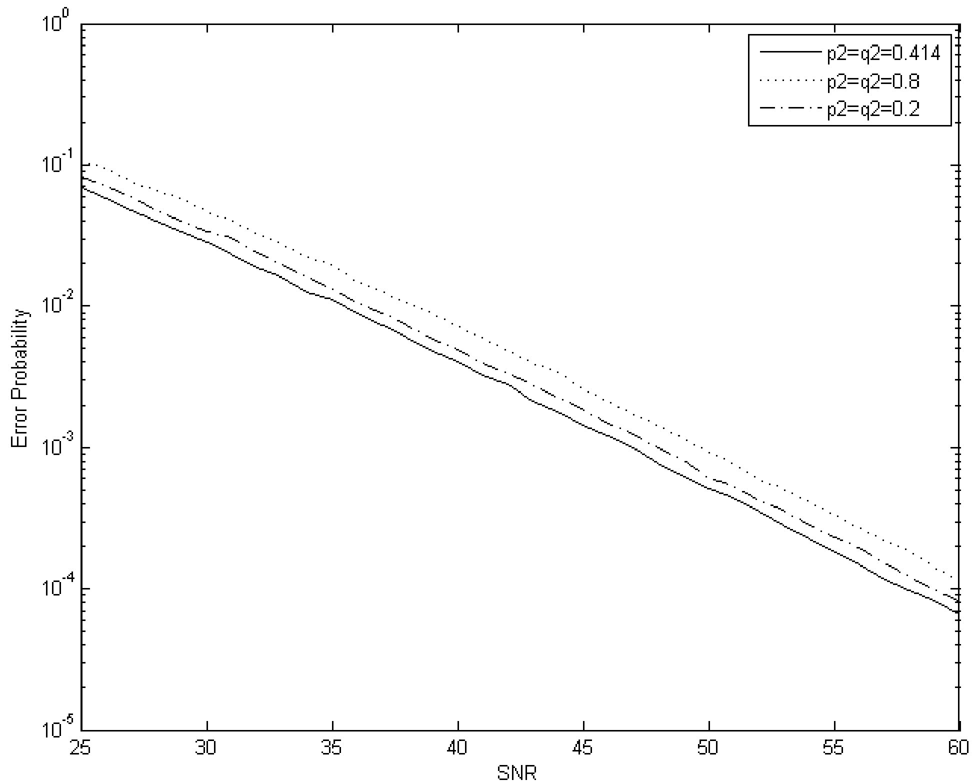


Figure 5.3: Performance comparison for 4-QAM: worst-case power loading

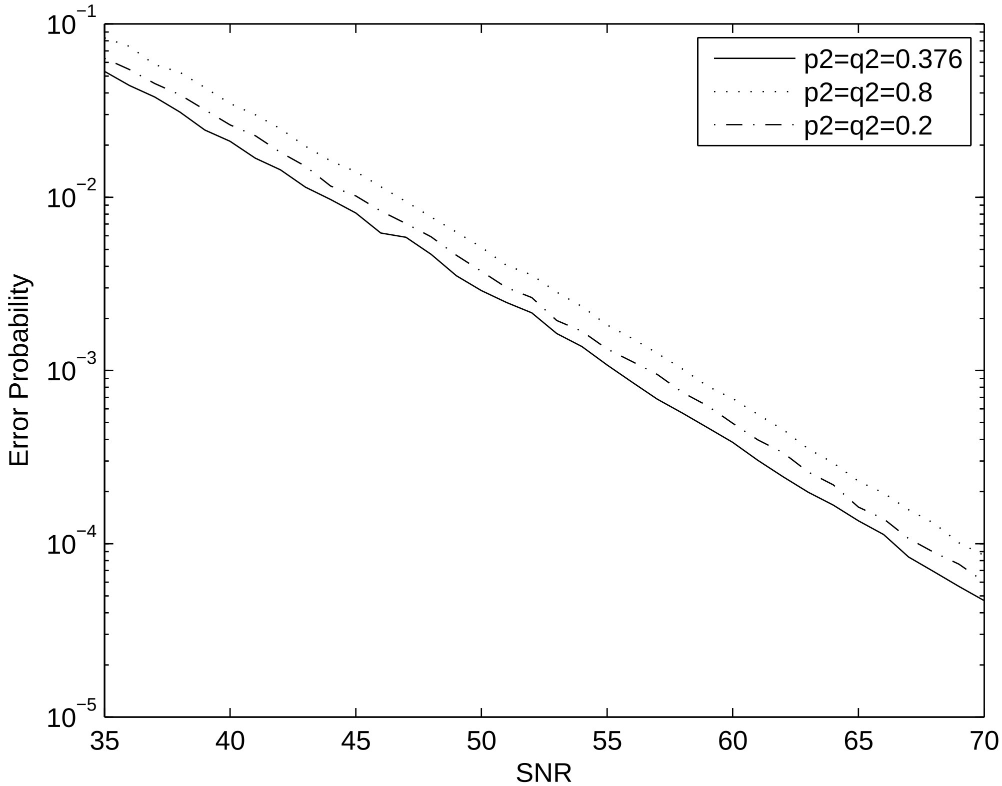


Figure 5.4: Performance comparison for 16-QAM: worst-case power loading

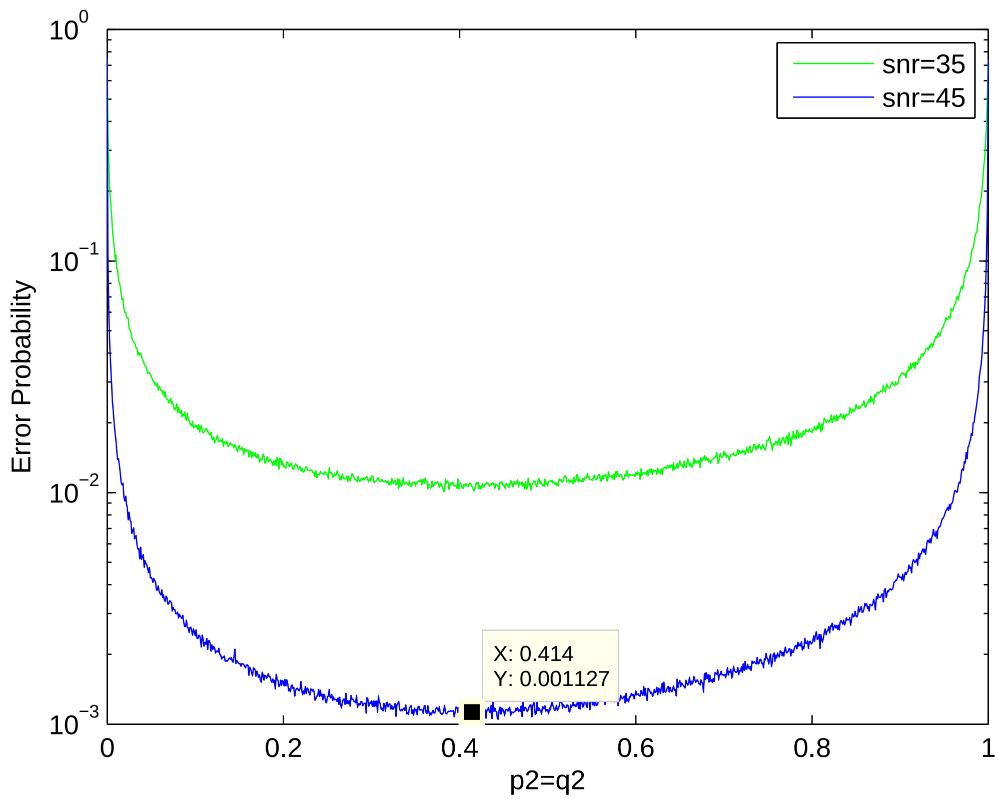


Figure 5.5: Performance comparison for 4-QAM in average power loading case

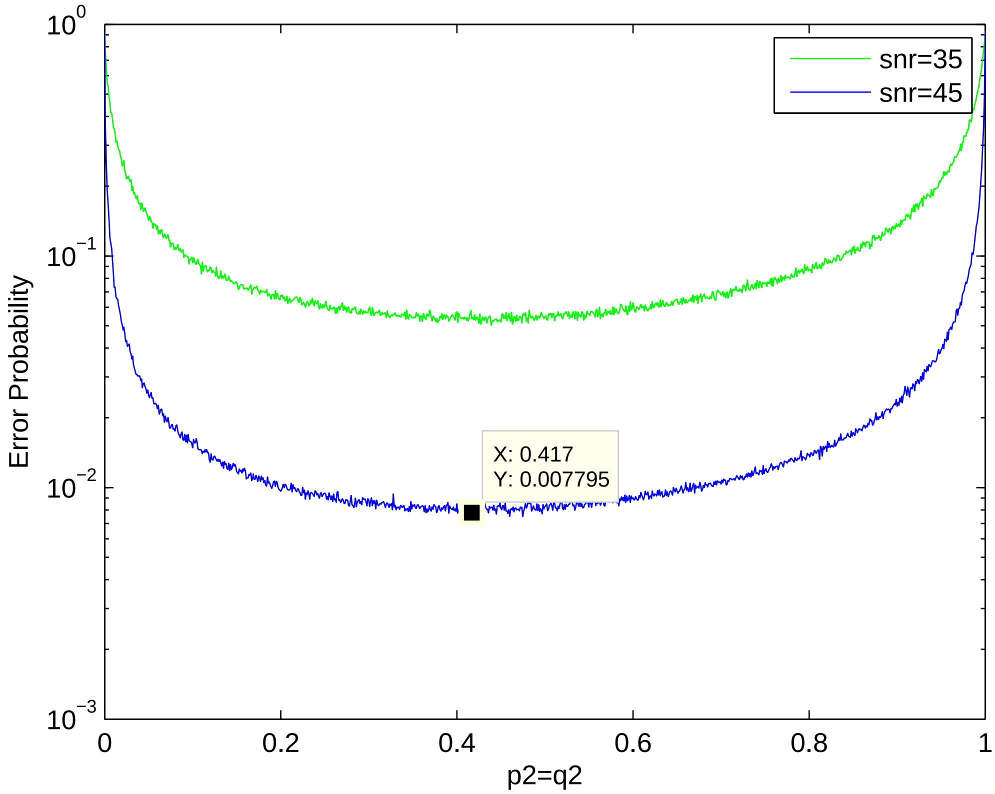


Figure 5.6: Performance comparison for 16-QAM in average power loading case

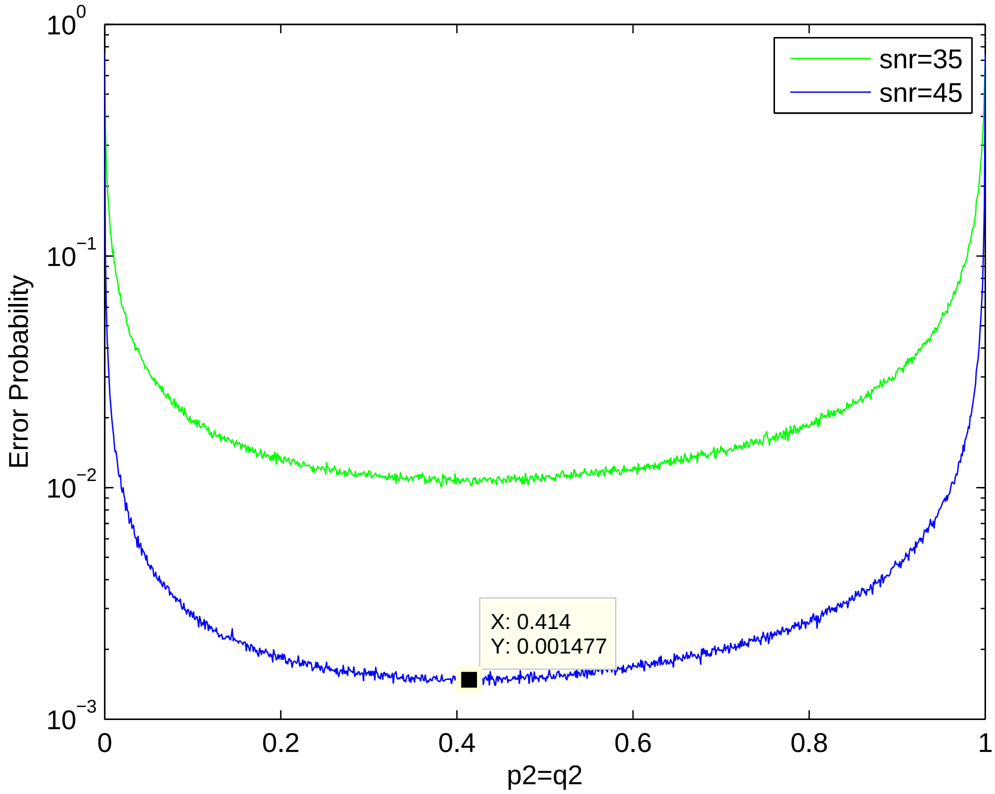


Figure 5.7: Performance comparison for 4-QAM: worst-case power loading

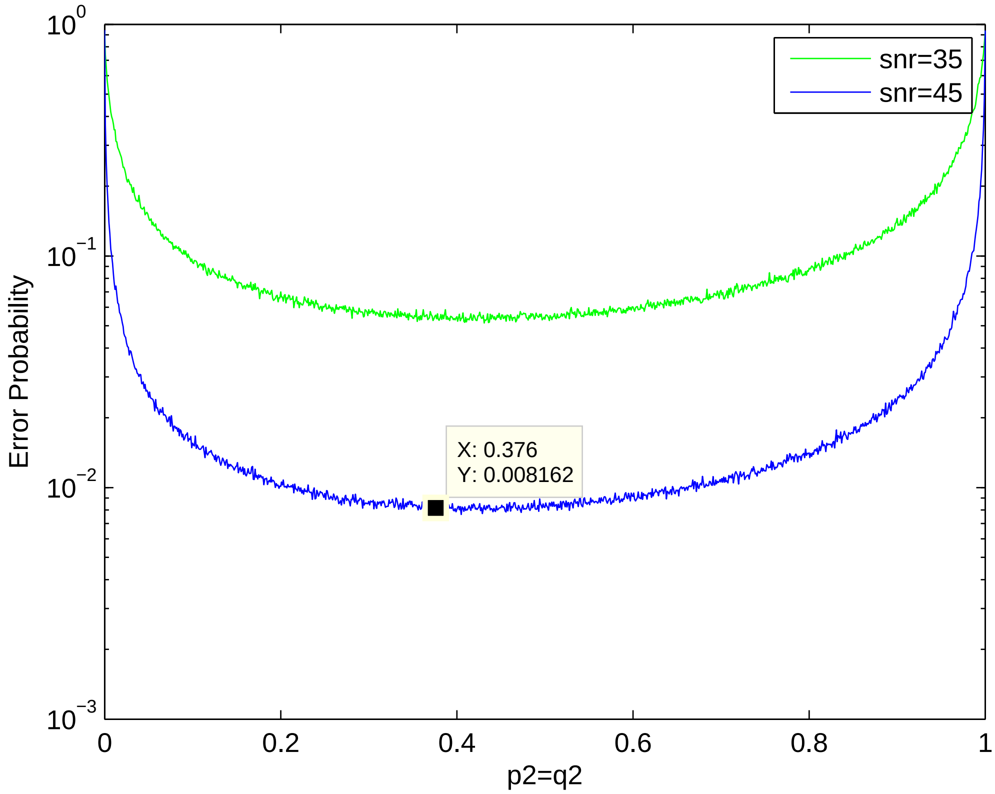


Figure 5.8: Performance comparison for 16-QAM: worst-case power loading

Chapter 6

Conclusion and Future Work

6.1 Conclusion

In this thesis, the amplify-and-forward half-duplex two-way relay system is studied. The system operates in a half-duplex mode. It is assumed that all channel gains are zero mean, circularly symmetric Gaussian, and they do not change during the period of observation. The short-term channel state information is not available at the end terminals, only the second order statistics is available. The amplification coefficient at the relay node is assumed to be unity. Noise is assumed to be zero mean, circular Gaussian and independent and identically distributed. We analyzed the performance of this AF half-duplex two-way relay system. In particular, we derive the probability of error which is signal dependent. Utilizing this error probability expression, we also examine the optimum transmission power design problem under a power constraint. Since the error probability is signal dependent, the optimum transmission power problem is studied in two cases: average-power-loading optimization and worst-case power loading optimization. The optimum solution for both cases are derived. Computer simulations verify that these conditions indeed yield optimum performance for the system.

6.2 Future Work

There are several possibilities for future work in this area.

1. In this thesis, we are considering the case that two end terminals want to exchange the information, but they are out of range of each other. Therefore, we consider the system in which a relay helps the information exchanging process between these two end terminals, and there is no direct link between the terminals. We may consider the system which contains a direct link between the terminals, and develop a transmission strategy for such a system.

2. In this thesis we consider a symmetric system. However, in the future we may consider the system that two end terminals are not exactly symmetric to each other, i.e., noise variances are not the same. This may cause the final optimization result becomes nonsymmetric as well.

3. Here, we consider the half-duplex two-way relay system with single relay. It is possible to extend it to multiple relay case.

Appendix A

Proof for Error Probability Formula

The received signal expression for the left terminal for our system is given by Eq. (3.5)

$$z_\ell - \sqrt{p_2}\hat{h}^2 x = \sqrt{q_2}\widehat{hg}y + \overline{\eta}$$

For the sake of simplicity, let

$$\begin{aligned} z_\ell - \sqrt{p_2}\hat{h}^2 x &= \overline{z}_\ell \\ \sqrt{q_2}\widehat{hg}y &= s \end{aligned}$$

we obtain

$$\overline{z}_\ell = s + \overline{\eta}$$

we regard $\overline{z}_\ell$ as the received signal, s as the transmitting signal and $\overline{\eta}$ as noise, it is proved that $\overline{\eta}$ has zero mean and variance $\sigma^2_{\overline{\eta}}$.

1. If s is independently and equally likely chosen from an M-ary PAM constellation $\mathcal{Q}_p$.

   Decision rule is given by:

   $$\hat{s} = arg_{s\in\mathcal{Q}_p}\min|\overline{z}_\ell - s|$$

   Notice that M-ary PAM constellation can be represented by $\mathcal{Q}_{\surd} = \{(k - \frac{M-1}{2})d\}_{k=0}^{M-1}$. Let $s = (k - \frac{M-1}{2})d$. Then, the optimal estimate is given by $\hat{s} = (\hat{k} - \frac{M-1}{2})d$.

   Decision regions: The correct decision regions for detecting $s = (k - \frac{M-1}{2})d$ are given as follows:

   $$\Gamma_0 = \{\overline{z}_\ell : \overline{z}_\ell \leq -\frac{(M-1)d}{2} + \frac{d}{2}\} \qquad \text{for the left edge point}$$
   $$\Gamma_{M-1} = \{\overline{z}_\ell : \overline{z}_\ell \geq \frac{(M-1)d}{2} - \frac{d}{2}\} \qquad \text{for the right edge point}$$
   $$\Gamma_k = \{\overline{z}_\ell : (k - \frac{M-1}{2})d - \frac{d}{2} < y < (k - \frac{M-1}{2})d + \frac{d}{2}\} \qquad \text{for the } k\text{th inner point}$$

   Symbol error probability: Note that the conditional probability density function of the received signal $\overline{z}_\ell$ given $s = s_k$ is given by

   $$\vartheta(\overline{z}_\ell|s_k) = \frac{1}{\sqrt{2\pi\sigma^2_{\overline{\eta}}}}e^{-\frac{(\overline{z}_\ell - s_k)^2}{2\sigma^2_{\overline{\eta}}}}$$

Hence, the conditional probability of making correct decision is determined by

$$P_{c|s_k} = \int_{\Gamma_k} \vartheta(\overline{z}_\ell | s_k) d\overline{z}_\ell$$

In order to evaluate this integral, we consider the following three cases:

(a) For the left edge point, the conditional probability of making correct decision on $s = s_0 = -\frac{(M-1)d}{2}$ is given by

$$\begin{aligned} P_{c|s_0} &= \int_{\Gamma_0} \vartheta(\overline{z}_\ell | s_0) d\overline{z}_\ell \\ &= \int_{-\infty}^{-\frac{(M-1)d}{2}+\frac{d}{2}} \frac{1}{\sqrt{2\pi\sigma_{\overline{\eta}}^2}} e - \frac{(\overline{z}_\ell + \frac{(M-1)d}{2})^2}{2\sigma_{\overline{\eta}}^2} d\overline{z}_\ell \\ &= 1 - Q(\frac{d}{2\sigma_{\overline{\eta}}}) \end{aligned}$$

(b) For the right edge point, the conditional probability of making correct decision on $s = s_{M-1} = \frac{(M-1)d}{2}$ is given by

$$\begin{aligned} P_{c|s_{M-1}} &= \int_{\frac{(M-1)d}{2}-\frac{d}{2}}^{\infty} \frac{1}{\sqrt{2\pi\sigma_{\overline{\eta}}^2}} e - \frac{(\overline{z}_\ell - \frac{(M-1)d}{2})^2}{2\sigma_{\overline{\eta}}^2} d\overline{z}_\ell \\ &= 1 - Q(\frac{d}{2\sigma_{\overline{\eta}}}) \end{aligned}$$

(c) For the kth inner point, the conditional probability of making correct decision on $s = s_k = k - \frac{(M-1)d}{2}$ is given by

$$P_{c|s_k} = \int_{k-\frac{(M-1)d}{2}-\frac{d}{2}}^{k-\frac{(M-1)d}{2}+\frac{d}{2}} \frac{1}{\sqrt{2\pi\sigma_{\overline{\eta}}^2}} e - \frac{(\overline{z}_\ell - \frac{(M-1)d}{2} - k)^2}{2\sigma_{\overline{\eta}}^2} d\overline{z}_\ell$$

$$= 1 - 2Q(\frac{d}{2\sigma_{\overline{\eta}}})$$

2. If s is independently and equally likely chosen from an M-ary QAM constellation $\mathcal{Q}_q$.

Decision rule is given by:

$$\hat{s} = arg_{s \in \mathcal{Q}_q} \min |\overline{z}_\ell - s|$$

let $\overline{z}_\ell = \overline{z}_{\ell re} + j\overline{z}_{\ell im}$ and $s = s_{re} + js_{im}$. Since $|\overline{z}_\ell - s| = \sqrt{(\overline{z}_{\ell re} - sre)^2 + (\overline{z}_{\ell im} - sim)^2}$ and s_{re} and s_{im} are independent, the optimization problem is equivalent to the following two optimization problems

$$\hat{s}_{re} = arg_{s_{re} \in \mathcal{Q}_p} \min |\overline{z}_{\ell re} - s_{re}|$$

$$\hat{s}_{im} = arg_{s_{im} \in \mathcal{Q}_p} \min |\overline{z}_{\ell im} - s_{im}|$$

where $\mathcal{Q}_p$ denotes the $\sqrt{M}$-ary PAM constellation. If we let $s_{re} = (m - \frac{\sqrt{M}-1}{2})d$ and $s_{im} = (n - \frac{\sqrt{M}-1}{2})d$, where $m, n = 0, 1, 2..., \sqrt{M} - 1$. Therefore, the optimal estimate of s is given by $\hat{s} = \hat{s}_{re} + j\hat{s}_{im}$, where

$$\hat{s}_{re} = (\hat{m} - \frac{\sqrt{m}-1}{2})d$$
$$\hat{s}_{im} = (\hat{n} - \frac{\sqrt{m}-1}{2})d$$

Decision regions: The correct decision regions for detecting $s_{m,n} = (m - \frac{M-1}{2})d + j(n - \frac{M-1}{2})d$ are given as follows:

(a) For four corner points

$$\Gamma_{0,0} = \Gamma_0 \times \Gamma_0$$
$$\Gamma_{\sqrt{M}-1,0} = \Gamma_{\sqrt{M}-1} \times \Gamma_0$$
$$\Gamma_{0,\sqrt{M}-1} = \Gamma_0 \times \Gamma_{\sqrt{M}-1}$$
$$\Gamma_{\sqrt{M}-1,\sqrt{M}-1} = \Gamma_{\sqrt{M}-1} \times \Gamma_{\sqrt{M}-1}$$

where notation Γ_k denotes the kth decision region for the kth PAM constellation point.

(b) For edge points

$$\Gamma_{0,k} = \Gamma_0 \times \Gamma_k$$
$$\Gamma_{k,0} = \Gamma_k \times \Gamma_0$$
$$\Gamma_{\sqrt{M}-1,k} = \Gamma_{\sqrt{M}-1} \times \Gamma_k$$
$$\Gamma_{k,\sqrt{M}-1} = \Gamma_k \times \Gamma_{\sqrt{M}-1}$$

for $1 \leq k \leq \sqrt{M} - 2$

(c) For inner points

$$\Gamma_{m,n} = \Gamma_m \times \Gamma_n$$

for $1 \leq m, n \leq \sqrt{M} - 2$

Symbol error probability: Note that the conditional probability density function of the received signal $\overline{z}_\ell$ given $s = s_{m,n}$ is the joint Gaussian distribution

$$\vartheta(\overline{z}_\ell | s_{m,n}) = \frac{1}{2\pi\sigma_{\overline{\eta}}^2} e^{-\frac{((\overline{z}_{\ell re} - s_m)^2) + ((\overline{z}_{\ell im} - s_n)^2)}{2\sigma_{\overline{\eta}}^2}}$$

where s_m and s_n denote the real and imaginary parts of $s_{M,n}$, respectively.

Hence, the conditional probability of making correct decision on the QAM symbol is determined by

$$
\begin{aligned}
P_{c|s_{m,n}} &= \int_{\Gamma_{m,n}} \vartheta(\overline{z}_\ell|s_{m,n}) d\overline{z}_{\ell re} d\overline{z}_{\ell im} \\
&= \int_{\Gamma_m} \vartheta(\overline{z}_{\ell re}|s_m) d\overline{z}_{\ell re} \int_{\Gamma_n} \vartheta(\overline{z}_{\ell im}|s_n) d\overline{z}_{\ell im} \\
&= P_{c|s_m} \times P_{c|s_n}
\end{aligned}
$$

Now, we can make use of the result for the PAM constellation by considering the following three cases

(a) For corner points

$$P_{c,c} = P_{c|s_m} \times P_{c|s_n} = (1 - Q(\frac{d}{2\sigma_{\overline{\eta}}}))^2$$

(b) For edge points

$$P_{c,e} = P_{c|s_m} \times P_{c|s_n} = (1 - Q(\frac{d}{2\sigma_{\overline{\eta}}})) \times (1 - 2Q(\frac{d}{2\sigma_{\overline{\eta}}}))$$

(c) For inner points

$$P_{c,i} = P_{c|s_m} \times P_{c|s_n} = (1 - 2Q(\frac{d}{2\sigma_{\overline{\eta}}}))^2$$

Appendix B

Proof of Lemma 1

In this chapter, an asymptotic expression for the expected value of the Q-function with respect to the channel estimation signal is derived. From Eq. 3.16, the expected value of Q-function for the two-way relay system for this thesis can be written as

$$\mathbb{E}\left[Q\left(\frac{d}{\sqrt{2}\sigma_{\bar{\eta}}}\right)\right] = \frac{1}{\pi}\int_0^\infty \int_0^{\pi/2} \int_{-\infty}^\infty e^{-\frac{t^2}{2\sin\theta^2}} \vartheta(z_{\ell 2}|\xi)\vartheta(\xi) d_{z_{\ell 2}} d\theta d\xi \tag{B.1}$$

where

$$t^2 = \frac{d^2}{2\sigma_{\bar{\eta}}^2} = \frac{2\frac{q_2}{q_1}|z_{\ell,2}|^2}{\bar{E}(\frac{p_2|x|^2}{p_1} + \frac{q_2|y|^2}{q_1} + 1)(\xi+1)\sigma^2}$$

$$\vartheta(z_{\ell 2}|\xi) = \frac{1}{\pi(2q_1\xi + (\xi+1)\sigma^2)} e^{\frac{-|z_{\ell 2}|^2}{2q_1\xi + (\xi+1)\sigma^2}}$$

$$\vartheta(\xi) = e^{-\xi}$$

Substituting the above expressions to Eq. 3.16, we obtain

$$\mathbb{E}\left[Q\left(\frac{d}{\sqrt{2}\sigma_{\bar{\eta}}}\right)\right] = \frac{1}{\pi^2}\int_0^{\infty}\int_0^{\pi/2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{2q_1\xi+(\xi+1)\sigma^2} e^{\frac{-\frac{q_2}{q_1}|z_{\ell 2}|^2}{\bar{E}(\frac{p_2|x|^2}{p_1}+\frac{q_2|y|^2}{q_1}+1)(\xi+1)\sin^2\theta\sigma^2}} e^{\frac{-|z_{\ell 2}|^2}{2q_1\xi+(\xi+1)\sigma^2}-\xi} d_{z_{\ell 2}}} d\theta d\xi$$

Integrating the expression in the bracket first, we have

$$\begin{aligned} M &= \int_{-\infty}^{\infty} \frac{1}{2q_1\xi+(\xi+1)\sigma^2} e^{\frac{-\frac{q_2}{q_1}|z_{\ell 2}|^2}{\bar{E}(\frac{p_2|x|^2}{p_1}+\frac{q_2|y|^2}{q_1}+1)(\xi+1)\sin^2\theta\sigma^2}} e^{\frac{-|z_{\ell 2}|^2}{2q_1\xi+(\xi+1)\sigma^2}-\xi} d_{z_{\ell 2}} \\ &= \frac{e^{-\xi}}{2q_1\xi+(\xi+1)\sigma^2}\int_{-\infty}^{\infty} e^{-\left(\frac{\frac{q_2}{q_1}}{\bar{E}(\frac{p_2|x|^2}{p_1}+\frac{q_2|y|^2}{q_1}+1)(\xi+1)\sin^2\theta\sigma^2}+\frac{1}{2q_1\xi+(\xi+1)\sigma^2}\right)|z_{\ell 2}|^2} d_{z_{\ell 2}} \\ &= \frac{e^{-\xi}}{2q_1\xi+(\xi+1)\sigma^2}\int_{-\infty}^{\infty} e^{-\left(\frac{\frac{q_2}{q_1}}{\bar{E}(\frac{p_2|x|^2}{p_1}+\frac{q_2|y|^2}{q_1}+1)(\xi+1)\sin^2\theta\sigma^2}+\frac{1}{2q_1\xi+(\xi+1)\sigma^2}\right)z_{re}^2} d_{zre} \\ &+ \frac{e^{-\xi}}{2q_1\xi+(\xi+1)\sigma^2}\int_{-\infty}^{\infty} e^{-\left(\frac{\frac{q_2}{q_1}}{\bar{E}(\frac{p_2|x|^2}{p_1}+\frac{q_2|y|^2}{q_1}+1)(\xi+1)\sin^2\theta\sigma^2}+\frac{1}{2q_1\xi+(\xi+1)\sigma^2}\right)z_{im}^2} d_{zim} \end{aligned}$$

Since

$$\int_{-\infty}^{\infty} e^{-Ax^2}dx = \sqrt{\frac{\pi}{A}}$$

$$M = \frac{\pi e^{-\xi}}{2q_1\xi+(\xi+1)\sigma^2}\left[\frac{\frac{q_2}{q_1}}{\bar{E}(\frac{p_2|x|^2}{p_1}+\frac{q_2|y|^2}{q_1}+1)(\xi+1)\sin^2\theta\sigma^2}+\frac{1}{2q_1\xi+(\xi+1)\sigma^2}\right]^{-1}$$

Substituting the above equation to the original expression, we obtain

$$\mathbb{E}\left[Q\left(\frac{d}{\sqrt{2}\sigma_{\bar{\eta}}}\right)\right] \tag{B.2}$$

$$= \frac{1}{\pi}\int_0^{\infty}\int_0^{\pi/2} \underbrace{\left[\frac{\frac{q_2}{q_1}}{\bar{E}(\frac{p_2|x|^2}{p_1}+\frac{q_2|y|^2}{q_1}+1)(\xi+1)\sin^2\theta\sigma^2}+\frac{1}{2q_1\xi+(\xi+1)\sigma^2}\right]^{-1}\frac{e^{-\xi}}{2q_1\xi+(\xi+1)\sigma^2} d\theta\, d\xi} \tag{B.3}$$

Integrating the expression in the bracket in Eq. (B.2), and letting

$$\Psi = \frac{\frac{q_2}{q_1}}{\bar{E}(\frac{p_2|x|^2}{p_1} + \frac{q_2|y|^2}{q_1} + 1)(\xi + 1)\sigma^2}$$
$$\Phi = \frac{1}{2q_1\xi + (\xi + 1)\sigma^2}$$

we have

$$\begin{aligned}
N &= \Phi e^{\xi} \int_0^{\frac{\pi}{2}} \left[\frac{\Psi}{\sin^2\theta} + \Phi\right]^{-1} d\theta \\
&= \Phi e^{\xi} \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{\Phi\sin^2\theta + \Psi} d\theta \\
&= e^{\xi} \int_0^{\frac{\pi}{2}} \left[1 - \frac{\frac{\Psi}{\Phi}}{\sin^2\theta + \frac{\Psi}{\Phi}}\right] d\theta \\
&= \frac{\pi}{2}\left(1 - \sqrt{\frac{1}{1 + \frac{\Phi}{\Psi}}}\right)
\end{aligned}$$

Substituting the above integration result to Eq. (B.2), we obtain

$$
\begin{aligned}
\mathbb{E}\left[Q\left(\frac{d}{\sqrt{2}\sigma_{\bar{\eta}}}\right)\right] &= \frac{1}{2}\int_0^\infty e^{-\xi}\left(1-\frac{1}{\sqrt{1+\frac{\bar{E}(\frac{p_2|x|^2}{p_1}+\frac{q_2|y|^2}{q_1}+1)(\xi+1)}{(\frac{2q_2}{\sigma^2}+\frac{q_2}{q_1})\xi+\frac{q_2}{q_1}}}}\right)d\xi \\
&= \frac{1}{2}-\frac{1}{2}[\int_0^\infty e^{-\xi}\frac{1}{\sqrt{1+\frac{A(\xi+1)}{(\frac{2q_2}{\sigma^2}+\frac{q_2}{q_1})\xi+\frac{q_2}{q_1}}}}] \\
&= \frac{1}{2}-\frac{1}{2}[\int_0^\infty e^{-\xi}\sqrt{\frac{c\xi+d}{a\xi+b}}d\xi] \\
&= \frac{1}{2}-\frac{1}{2}\sqrt{\frac{c}{a}}[\int_0^\infty e^{-\xi}\sqrt{1+\frac{ad-bc}{c(a\xi+b)}}d\xi] \\
&= \frac{1}{2}-\frac{1}{2}\sqrt{\frac{c}{a}}[\int_0^\infty e^{-\xi}\sqrt{1+\frac{-2Aq_2\rho}{ac\xi+bc}}d\xi]
\end{aligned}
$$

with

$$
\begin{aligned}
\rho &= \frac{1}{\sigma^2} \\
A &= \bar{E}(\frac{p_2|x|^2}{p_1}+\frac{q_2|y|^2}{q_1}+1) \\
c &= 2q_2\rho+\frac{q_2}{q_1} \\
d &= \frac{q_2}{q_1} \\
a &= A+2q_2\rho+\frac{q_2}{q_1} \\
b &= A+\frac{q_2}{q_1}
\end{aligned}
$$

We can employ binomial series to simplify this integration. Let's review the concept for binomial series first: Binomial series is the Taylor series at $x = 0$ of the function f given by $f(x) = (1 + x)^a$, where a is an arbitrary complex number. Explicitly,

$$(1+x)^a = \sum_{k=0}^{\infty} \begin{pmatrix} a \\ k \end{pmatrix} x^k, \qquad |x| \leq 1$$

Before applying the formula, let's check the condition for convergence: which is $|\frac{-2Aq_2\rho}{ac\xi+bc}| \leq 1$.

$$\begin{aligned} \frac{ad-bc}{c(a\xi+b)} &= \frac{-2Aq_2\rho}{ac\xi+bc} \\ &> \frac{-2Aq_2\rho}{cb} \\ &= \frac{-2Aq_2\rho}{(2q_2\rho+\frac{q_2}{q_1})(A+\frac{q_2}{q_1})} \\ &> -1 \end{aligned}$$

Because all values such as A, q_2, ρ, a, b, c, ξ are nonnegative, we get:

$$\frac{-2Aq_2\rho}{ac\xi+bc} < 0$$

Combining previous two equations we proved the condition for convergence. Employing the binomial series to our equation, we can express the original equation in the form $C_0\rho^{-1}\ln\rho + C_1\rho^{-1} + O(\rho^{-2}\ln\rho)$.

$$\begin{aligned}
&\mathbb{E}\left[Q\left(\frac{d}{\sqrt{2}\sigma_{\bar{\eta}}}\right)\right] \\
&= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{c}{a}}[\int_0^\infty e^{-\xi}\sum_{k=0}^{\infty}\left(\begin{array}{c}\frac{1}{2}\\ k\end{array}\right)(\frac{-2Aq_2\rho}{ac\xi+bc})^k d\xi] \\
&= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{c}{a}}(\frac{-2Aq_2\rho}{ac})^k[\int_0^\infty e^{-\xi}\sum_{k=0}^{\infty}\left(\begin{array}{c}\frac{1}{2}\\ k\end{array}\right)\left(\frac{1}{\xi+\frac{b}{a}}\right)^k d\xi] \\
&= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{c}{a}}\left(1 - \frac{Aq_2\rho}{ac}(e^{\frac{b}{a}}Ei(\frac{b}{a})) + \sum_{k=2}^{\infty}\left(\begin{array}{c}\frac{1}{2}\\ k\end{array}\right)(\frac{-2Aq_2\rho}{ac})^k\int_0^\infty e^{-\xi}\left(\frac{1}{\xi+\frac{b}{a}}\right)^k d\xi\right)
\end{aligned}$$

where $Ei(x) = \int_x^\infty \frac{e^{-u}}{u}du$ is the exponential integral and it can be evaluated as,

$$Ei(x) = -\left(\gamma + \ln x + \sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n!n}\right)$$

with γ being the Euler constant. Also by employing the following formula,

$$\int_0^\infty \frac{e^{-x}}{(x+\beta)^k}dx = \frac{1}{(k-1)!}\sum_{m=1}^{k-1}(m-1)!(-1)^{k-m-1}\beta^{-m} + \frac{(-1)^{k-1}}{(k-1)!}e^{\beta}Ei(\beta)$$

Then the original equation has the form

$$\mathbb{E}\left[Q\left(\frac{d}{\sqrt{2}\sigma_{\bar{\eta}}}\right)\right] = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{c}{a}}\left(1 + \frac{\gamma Aq_2\rho}{ac} + \frac{Aq_2\rho}{ac}\ln\frac{b}{a} + \sum_{k=2}^{\infty}\left(\begin{array}{c}\frac{1}{2}\\ k\end{array}\right)(\frac{-2Aq_2\rho}{c})^k\frac{1}{ab^{k-1}(k-1)}\right) + O(p^{-2}\ln p)$$

To simplify the above equation, we need to employ the follow formula

$$\frac{1}{1+x} = 1 + x + x^2 + x^3 + ... \qquad |x| < 1$$

With the help of binomial expansion and the above formula, we expand $\frac{1}{ac}$, $\frac{1}{c}$,$\frac{1}{a}$ and $\sqrt{\frac{a}{c}}$ in terms of ρ^{-n}

$$\begin{aligned}
\frac{1}{c} &= \frac{1}{2q_2\rho + \frac{q_2}{q_1}} = \frac{1}{2q_2\rho}\frac{1}{(1+\frac{1}{2q_1\rho})} = \frac{1}{2q_2\rho} + O(\rho^{-2}) \\
\frac{1}{a} &= \frac{1}{A + 2q_2\rho + \frac{q_2}{q_1}} = \frac{1}{2q_2\rho}\frac{1}{(1+\frac{\frac{q_2}{q_1}+A}{2q_2\rho})} = \frac{1}{2q_2\rho} + O(\rho^{-2}) \\
\frac{1}{ac} &= \frac{1}{(A + 2q_2\rho + \frac{q_2}{q_1})(2q_2\rho + \frac{q_2}{q_1})} = \frac{1}{4q_2^2\rho^2}\frac{1}{(1+\frac{W}{4q_2^2\rho^2})} = \frac{1}{4q_2^2\rho^2} + O(\rho^{-3})
\end{aligned}$$

where

$$\begin{aligned}
W &= (A + \frac{q_2}{q_1})2q_2\rho + \frac{q_2}{q_1}(A + 2q_2\rho + \frac{q_2}{q_1}) \\
\sqrt{\frac{c}{a}} &= \sqrt{\frac{2q_2\rho + \frac{q_2}{q_1}}{A + 2q_2\rho + \frac{q_2}{q_1}}} = \sqrt{1 + \frac{-A}{A + 2q_2\rho + \frac{q_2}{q_1}}}
\end{aligned}$$

utilizing binomial expansion, we obtain

$$\sqrt{\frac{c}{a}} = 1 - \frac{1}{2}(\frac{A}{A + 2q_2\rho + \frac{q_2}{q_1}}) + O\rho^2 = 1 - \frac{1}{2}\frac{A}{2q_2\rho + O\rho^2}$$

Substituting the above equations to the original equation, we can express the equation in the form $C_0\rho^{-1}\ln\rho + C_1\rho^{-1} + O(\rho^{-2}\ln\rho)$.

$$\begin{aligned}
&\mathbb{E}\left[Q\left(\frac{d}{\sqrt{2}\sigma_{\bar{\eta}}}\right)\right]\\
&= \frac{A}{8q_2}\rho^{-1}\ln\rho - \frac{1}{2}\left(\frac{(\gamma-1)A}{4q_2} + \frac{A}{4q_2}(\ln b - \ln 2q_2) + \sum_{k=2}^{\infty}\binom{\frac{1}{2}}{k}\frac{(-A)^k}{2b^{k-1}(k-1)q_2}\right)\rho^{-1} + O(\rho^{-2}\ln\rho)\\
&= \frac{A}{8q_2}\rho^{-1}\ln\rho - \frac{1}{2}\left(\frac{(\gamma-1)A}{4q_2} + \frac{A}{4q_2}(\ln b - \ln 2q_2) + \frac{-A}{2q_2}\left(-\ln\frac{1+\sqrt{\frac{q_2}{q_1 b}}}{2} - \frac{1}{1+\sqrt{\frac{q_2}{q_1 b}}} + \frac{1}{2}\right)\right)\rho^{-1} + O(\rho^{-2}\ln\rho)
\end{aligned}$$

Appendix C

Proof of Lemma 2

From Eq. 3.16, the expected value of the function Q^2 for the two-way relay system for this thesis can be written as

$$\mathbb{E}\left[Q^2\left(\frac{d}{\sqrt{2}\sigma_{\bar{\eta}}}\right)\right]$$

$$= \frac{1}{\pi}\int_0^{\infty}\int_0^{\pi/4}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sin\theta^2}}\vartheta(z_{\ell 2}|\xi)\vartheta(\xi)d_{z_{\ell 2}}d\xi d\theta$$

$$= \frac{1}{\pi^2}\int_0^{\infty}\int_0^{\pi/4}\int_{-\infty}^{\infty} \frac{1}{2q_1\xi+(\xi+1)\sigma^2} e^{\frac{-\frac{q_2}{q_1}|z_{\ell 2}|^2}{\bar{E}(\frac{p_2|x|^2}{p_1}+\frac{q_2|y|^2}{q_1}+1)(h^2+1)\sin^2\theta\sigma^2}} e^{\frac{-|z_{\ell 2}|^2}{2q_1\xi+(\xi+1)\sigma^2}-\xi} d_{z_{\ell 2}}d\xi d\theta$$

$$= \frac{1}{\pi}\int_0^{\infty}\int_0^{\pi/4}\left[\frac{\frac{q_2}{q_1}}{\bar{E}(\frac{p_2|x|^2}{p_1}+\frac{q_2|y|^2}{q_1}+1)(h^2+1)\sin^2\theta\sigma^2}+\frac{1}{2q_1\xi+(\xi+1)\sigma^2}\right]^{-1}\frac{e^{-\xi}}{2q_1\xi+(\xi+1)\sigma^2}d\theta d\xi$$

$$= \frac{1}{\pi}\int_0^{\infty} e^{-\xi}\left(\frac{\pi}{4}-\frac{\arctan\sqrt{1+\frac{\bar{E}(\frac{p_2|x|^2}{p_1}+\frac{q_2|y|^2}{q_1}+1)(\xi+1)}{(\frac{2q_2}{\sigma^2}+\frac{q_2}{q_1})\xi+\frac{q_2}{q_1}}}}{\sqrt{1+\frac{\bar{E}(\frac{p_2|x|^2}{p_1}+\frac{q_2|y|^2}{q_1}+1)(\xi+1)}{(\frac{2q_2}{\sigma^2}+\frac{q_2}{q_1})\xi+\frac{q_2}{q_1}}}}\right)d\xi$$

In order to simplify this expression, first, we should expand the *arctan* term which will employ the the following formula

$$\arctan x = \frac{\pi}{2} - \arctan\frac{1}{x}, \qquad \text{if} \quad x > 0$$

With the help of this formula and mathematical manipulation, we simplify the expression and extract the $O(\rho^{-2})$ term

$$\begin{aligned}
&\arctan\sqrt{1+\frac{\bar{E}(\frac{p_2|x|^2}{p_1}+\frac{q_2|y|^2}{q_1}+1)(\xi+1)}{(\frac{2q_2}{\sigma^2}+\frac{q_2}{q_1})\xi+\frac{q_2}{q_1}}} \\
&= \frac{\pi}{2} - \arctan\frac{1}{\sqrt{1+\frac{\bar{E}(\frac{p_2|x|^2}{p_1}+\frac{q_2|y|^2}{q_1}+1)(\xi+1)}{(\frac{2q_2}{\sigma^2}+\frac{q_2}{q_1})\xi+\frac{q_2}{q_1}}}} \\
&= \frac{\pi}{2} - \arctan\sqrt{\frac{c}{a}}\sqrt{1+\frac{-2Aq_2\rho}{ac\xi+bc}} \\
&= \frac{\pi}{2} - \arctan\sqrt{\frac{c}{a}}\left(1-\frac{Aq_2\rho}{ac\xi+bc}\right) - \left(\arctan\sqrt{\frac{c}{a}}\sqrt{1+\frac{-2Aq_2\rho}{ac\xi+bc}} - \arctan\sqrt{\frac{c}{a}}\left(1-\frac{Aq_2\rho}{ac\xi+bc}\right)\right)
\end{aligned}$$

Now, let's look at the term $\arctan\sqrt{\frac{c}{a}}\sqrt{1+\frac{-2Aq_2\rho}{ac\xi+bc}} - \arctan\sqrt{\frac{c}{a}}(1-\frac{Aq_2\rho}{ac\xi+bc})$. Since *arctan* is an odd function. By applying the arctangent addition formula which is list below

$$\arctan u + \arctan v = \arctan\frac{u+v}{1-uv}, \qquad uv \neq 1$$

we have the following result

$$
\begin{aligned}
&\arctan\sqrt{\frac{c}{a}}\sqrt{1+\frac{-2Aq_2\rho}{ac\xi+bc}}-\arctan\sqrt{\frac{c}{a}}(1-\frac{Aq_2\rho}{ac\xi+bc})\\
&=\arctan\frac{\sqrt{\frac{c}{a}}\sqrt{1+\frac{-2Aq_2\rho}{ac\xi+bc}}-\sqrt{\frac{c}{a}}(1-\frac{Aq_2\rho}{ac\xi+bc})}{1+\sqrt{\frac{c}{a}}\sqrt{1+\frac{-2Aq_2\rho}{ac\xi+bc}}\sqrt{\frac{c}{a}}(1-\frac{Aq_2\rho}{ac\xi+bc})}\\
&\leq\sqrt{\frac{c}{a}}\sqrt{1+\frac{-2Aq_2\rho}{ac\xi+bc}}-\sqrt{\frac{c}{a}}(1-\frac{Aq_2\rho}{ac\xi+bc})\\
&=\sqrt{\frac{c}{a}}\left(1+\frac{1}{2}(\frac{-2Aq_2\rho}{ac\xi+bc})+\sum_{k=2}^{\infty}\left(\begin{array}{c}\frac{1}{2}\\ k\end{array}\right)(\frac{-2Aq_2\rho}{ac\xi+bc})^k-(1+\frac{1}{2}(\frac{-2Aq_2\rho}{ac\xi+bc}))\right)\\
&=\sqrt{\frac{c}{a}}\sum_{k=2}^{\infty}\left(\begin{array}{c}\frac{1}{2}\\ k\end{array}\right)(\frac{-2Aq_2\rho}{ac\xi+bc})^k\\
&=O(\rho^{-2})
\end{aligned}
$$

which means

$$
\arctan\sqrt{1+\frac{\bar{E}(\frac{p_2|x|^2}{p_1}+\frac{q_2|y|^2}{q_1}+1)(\xi+1)}{(\frac{2q_2}{\sigma^2}+\frac{q_2}{q_1})\xi+\frac{q_2}{q_1}}}=\frac{\pi}{2}-\arctan\sqrt{\frac{c}{a}}(1-\frac{Aq_2\rho}{ac\xi+bc})+O(\rho^{-2})
$$

After substituting the above expression into the original $\mathbb{E}\left[Q^2\left(\frac{d}{\sqrt{2}\sigma_{\bar{\eta}}}\right)\right]$ equation we obtain

$$
\begin{aligned}
&\mathbb{E}\left[Q^2\left(\frac{d}{\sqrt{2}\sigma_{\bar{\eta}}}\right)\right] \\
&= \frac{1}{4} - \frac{1}{2}\int_0^\infty \frac{e^{-\xi}}{\sqrt{1+\frac{A(\xi+1)}{(\frac{2q_2}{\sigma^2}+\frac{q_2}{q_1})\xi+\frac{q_2}{q_1}}}}d\xi + \frac{1}{\pi}\int_0^\infty e^{-\xi}\frac{\arctan\sqrt{\frac{c}{a}}(1-\frac{Aq_2\rho}{ac\xi+bc})}{\sqrt{1+\frac{A(\xi+1)}{(\frac{2q_2}{\sigma^2}+\frac{q_2}{q_1})\xi+\frac{q_2}{q_1}}}}d\xi \\
&= \frac{1}{4} - \frac{1}{2}\int_0^\infty \frac{e^{-\xi}}{\sqrt{1+\frac{A(\xi+1)}{(\frac{2q_2}{\sigma^2}+\frac{q_2}{q_1})\xi+\frac{q_2}{q_1}}}}d\xi + \frac{1}{\pi}\arctan\sqrt{\frac{c}{a}}\int_0^\infty e^{-\xi}\frac{1}{\sqrt{1+\frac{A(\xi+1)}{(\frac{2q_2}{\sigma^2}+\frac{q_2}{q_1})\xi+\frac{q_2}{q_1}}}}d\xi \\
&+ \frac{1}{\pi}\int_0^\infty e^{-\xi}\frac{\arctan\sqrt{\frac{c}{a}}(1-\frac{Aq_2\rho}{ac\xi+bc}) - \arctan\sqrt{\frac{c}{a}}}{\sqrt{1+\frac{A(\xi+1)}{(\frac{2q_2}{\sigma^2}+\frac{q_2}{q_1})\xi+\frac{q_2}{q_1}}}}d\xi
\end{aligned}
$$

Following the same argument as the last expression we can have:

$$
\begin{aligned}
&\arctan\sqrt{\frac{c}{a}}(1-\frac{Aq_2\rho}{ac\xi+bc}) - \arctan\sqrt{\frac{c}{a}} \\
&\leq \sqrt{\frac{c}{a}}(\frac{Aq_2\rho}{ac\xi+bc}) \\
&= O(\rho^{-1})
\end{aligned}
$$

Also

$$
\begin{aligned}
\frac{1}{\sqrt{1+\frac{A(\xi+1)}{(\frac{2q_2}{\sigma^2}+\frac{q_2}{q_1})\xi+\frac{q_2}{q_1}}}} &= \sqrt{\frac{c\xi+d}{a\xi+b}} \\
&= \sqrt{\frac{c}{a}}\left(\sqrt{\frac{\xi+\frac{d}{c}}{\xi+\frac{b}{a}}}+1-1\right) \\
&= \sqrt{\frac{c}{a}}+\sqrt{\frac{c}{a}}\left(\sqrt{1+\frac{-2Aq_2\rho}{ac\xi+bc}}-1\right) \\
&= \sqrt{\frac{c}{a}}+\sqrt{\frac{c}{a}}\sum_{k=1}^{\infty}\left(\begin{array}{c}\frac{1}{2}\\ k\end{array}\right)(\frac{-2Aq_2\rho}{ac\xi+bc})^k \\
&= \sqrt{\frac{c}{a}}+O(\rho^{-1})
\end{aligned}
$$

The original $\mathbb{E}\left[Q^2\left(\frac{d}{\sqrt{2}\sigma_{\bar{\eta}}}\right)\right]$ can be expressed as the following expression:

$$
\begin{aligned}
\mathbb{E}\left[Q^2\left(\frac{d}{\sqrt{2}\sigma_{\bar{\eta}}}\right)\right] &= \frac{1}{4}-\frac{1}{2}\int_0^{\infty}\frac{e^{-\xi}}{\sqrt{1+\frac{A(\xi+1)}{(\frac{2q_2}{\sigma^2}+\frac{q_2}{q_1})\xi+\frac{q_2}{q_1}}}}d\xi+\frac{1}{\pi}\arctan\sqrt{\frac{c}{a}}\int_0^{\infty}e^{-\xi}\frac{1}{\sqrt{1+\frac{A(\xi+1)}{(\frac{2q_2}{\sigma^2}+\frac{q_2}{q_1})\xi+\frac{q_2}{q_1}}}} \\
&\quad+\frac{1}{\pi}\sqrt{\frac{c}{a}}\int_0^{\infty}e^{-\xi}(\arctan\sqrt{\frac{c}{a}}(1-\frac{Aq_2\rho}{ac\xi+bc})-\arctan\sqrt{\frac{c}{a}})d\xi \\
&= \frac{1}{4}-\frac{1}{2}\int_0^{\infty}\frac{e^{-\xi}}{\sqrt{1+\frac{A(\xi+1)}{(\frac{2q_2}{\sigma^2}+\frac{q_2}{q_1})\xi+\frac{q_2}{q_1}}}}d\xi+\frac{1}{\pi}\arctan\sqrt{\frac{c}{a}}\int_0^{\infty}e^{-\xi}\frac{1}{\sqrt{1+\frac{A(\xi+1)}{(\frac{2q_2}{\sigma^2}+\frac{q_2}{q_1})\xi+\frac{q_2}{q_1}}}} \\
&\quad-\frac{1}{\pi}\sqrt{\frac{c}{a}}\arctan\sqrt{\frac{c}{a}}\int_0^{\infty}e^{-\xi}d\xi+\frac{1}{\pi}\sqrt{\frac{c}{a}}\int_0^{\infty}e^{-\xi}\arctan\sqrt{\frac{c}{a}}(1-\frac{Aq_2\rho}{ac\xi+bc})d\xi
\end{aligned}
$$

In order to further simplify the equation, with the help of the following formula, we can express $\arctan\sqrt{\frac{c}{a}}(1-\frac{Aq_2\rho}{ac\xi+bc})$ in the form of $C_0+C_1\rho^{-1}+O(\rho^{-2})$,

$$\arctan\xi=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\xi^{2n+1}, \qquad |\xi|\leq 1 \quad \xi\neq i,-i.$$

By applying this formula to the *arctan* term, we have the following expression:

$$\begin{aligned}\arctan\sqrt{\frac{c}{a}}(1-\frac{Aq_2\rho}{ac\xi+bc}) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}(\sqrt{\frac{c}{a}})^{2n+1}\left(1-\frac{Aq_2\rho}{ac\xi+bc}\right)^{2n+1}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}(\sqrt{\frac{c}{a}})^{2n+1}\left(1-(2n+1)\frac{Aq_2\rho}{ac\xi+bc}\right)+O(\rho^{-2})\end{aligned}$$

Note:

$$\begin{aligned}&-1\leq\frac{-2Aq_2\rho}{ac\xi+bc}\leq 0\\ \Rightarrow&-\frac{1}{2}\leq\frac{-Aq_2\rho}{ac\xi+bc}\leq 0\\ \Rightarrow&\frac{1}{2}\leq 1-\frac{Aq_2\rho}{ac\xi+bc}\leq 1\\ \Rightarrow&\left|\sqrt{\frac{c}{a}}(1-\frac{Aq_2\rho}{ac\xi+bc})\right|\leq 1\end{aligned}$$

Substituting this expression in the last $\mathbb{E}\left[Q^2\left(\frac{d}{\sqrt{2}\sigma_{\bar{\eta}}}\right)\right]$ equation, we get:

$$
\begin{aligned}
\mathbb{E}\left[Q^2\left(\frac{d}{\sqrt{2}\sigma_{\bar{\eta}}}\right)\right] &= \frac{1}{4} - (\frac{1}{2} - \frac{1}{\pi}\arctan\sqrt{\frac{c}{a}})\int_0^\infty \frac{e^{-\xi}}{\sqrt{1+\frac{A(\xi+1)}{(\frac{2q_2}{\sigma^2}+\frac{q_2}{q_1})\xi+\frac{q_2}{q_1}}}}d\xi - \frac{1}{\pi}\sqrt{\frac{c}{a}}\arctan\sqrt{\frac{c}{a}} \\
&+ \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}(\sqrt{\frac{c}{a}})^{2n+2}\int_0^\infty e^{-\xi}\left(1-(2n+1)\frac{Aq_2\rho}{ac\xi+bc}\right)d\xi \\
&= \frac{1}{4} - (\frac{1}{2} - \frac{1}{\pi}\arctan\sqrt{\frac{c}{a}})\int_0^\infty \frac{e^{-\xi}}{\sqrt{1+\frac{A(\xi+1)}{(\frac{2q_2}{\sigma^2}+\frac{q_2}{q_1})\xi+\frac{q_2}{q_1}}}}d\xi - \frac{1}{\pi}\sqrt{\frac{c}{a}}\arctan\sqrt{\frac{c}{a}} \\
&+ \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}(\sqrt{\frac{c}{a}})^{2n+2} - \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n(\sqrt{\frac{c}{a}})^{2n+2}\frac{Aq_2\rho}{ac}\int_0^\infty \frac{e^{-\xi}}{\xi+\frac{b}{c}}d\xi \\
&= \frac{1}{4} - (\frac{1}{2} - \frac{1}{\pi}\arctan\sqrt{\frac{c}{a}})\int_0^\infty \frac{e^{-\xi}}{\sqrt{1+\frac{A(\xi+1)}{(\frac{2q_2}{\sigma^2}+\frac{q_2}{q_1})\xi+\frac{q_2}{q_1}}}}d\xi - \frac{1}{\pi}\sqrt{\frac{c}{a}}\arctan\sqrt{\frac{c}{a}} \\
&+ \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}(\sqrt{\frac{c}{a}})^{2n+2} - \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n(\sqrt{\frac{c}{a}})^{2n+2}\frac{Aq_2\rho}{ac}(e^{\frac{b}{c}}Ei(\frac{b}{c}))
\end{aligned}
$$

Since

$$
\begin{aligned}
&\frac{1}{\pi}\sqrt{\frac{c}{a}}\arctan\sqrt{\frac{c}{a}} = \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}(\sqrt{\frac{c}{a}})^{2n+2} \\
&\frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n(\sqrt{\frac{c}{a}})^{2n+2}\frac{Aq_2\rho}{ac}(e^{\frac{b}{c}}Ei(\frac{b}{c})) = O(\rho^{-2})
\end{aligned}
$$

We have the final expression for $\mathbb{E}\left[Q^2\left(\frac{d}{\sqrt{2}\sigma_{\bar{\eta}}}\right)\right]$ as

$$\mathbb{E}\left[Q^2\left(\frac{d}{\sqrt{2}\sigma_{\bar{\eta}}}\right)\right] =$$

$$=\frac{1}{4}-(\frac{1}{4}+\frac{A}{8\pi q_2\rho})\int_0^\infty \frac{e^{-\xi}}{\sqrt{1+\frac{A(\xi+1)}{(\frac{2q_2}{\sigma^2}+\frac{q_2}{q_1})\xi+\frac{q_2}{q_1}}}}d\xi$$

$$=\frac{A}{16q_2}\rho^{-1}\ln\rho-\frac{1}{4}\Bigg((\frac{\gamma}{2}-\frac{1}{2}-\frac{1}{\pi})\frac{A}{2q_2}+\frac{A}{4q_2}(\ln b-\ln 2q_2)+\sum_{k=2}^{\infty}\begin{pmatrix}\frac{1}{2}\\ k\end{pmatrix}\frac{(-A)^k}{2b^{k-1}(k-1)q_2}\Bigg)\rho^{-1}+O(\rho^{-2}\ln\rho)$$

$$=\frac{A}{16q_2}\rho^{-1}\ln\rho-\frac{1}{4}\Bigg((\frac{\gamma}{2}-\frac{1}{2}-\frac{1}{\pi})\frac{A}{2q_2}+\frac{A}{4q_2}(\ln b-\ln 2q_2)+\frac{-A}{2q_2}\Bigg(-\ln\frac{1+\sqrt{\frac{q_2}{q_1 b}}}{2}-\frac{1}{1+\sqrt{\frac{q_2}{q_1 b}}}+\frac{1}{2}\Bigg)\Bigg)\rho^{-1}$$

$$+O(\rho^{-2}\ln\rho)$$

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