ORDERED HJELMSLEV PLANES

By

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ABSTRACT

There are two equivalent ways to define order on ordinary affine planes; however generalizations of these definitions to A.H. planes yield two distinct definitions. We investigate the relationship between ordered A.H. planes and their ordered coordinate biternary rings. We introduce two new order relations: projective orderings of A.H. planes which are shown to be equivalent to strong orderings of the coordinate biternary rings of these planes and almost-strong orderings of biternary rings which are equivalent to strong orderings of the corresponding A.H. planes. In addition, we extend the axioms of order for projective planes to P.H. planes and discuss the properties of these order relations.

We now show that an A.H. plane embedded in an ordered P.H. plane is itself ordered.

We consider the projective completions constructed by Artmann, coordinatize them by means of biternary rings with additional ternary operations and prove various properties of the new ternary operators. We then show that although there exist strongly ordered projectively uniform A.H. planes which do not have ordered projective completions, we can always construct ordered projective completions of projectively ordered projectively uniform A.H. planes.

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CHAPTER 1

Introduction

Projective or affine Hjelmslev planes may be described as geometries in which more than one line may pass through two distinct points. They are defined by means of axiom systems and neighbour relations. Bacon [2] and Lorimer [13] have shown that affine Hjelmslev planes (henceforth called A.H. planes) may be coordinatized by algebraic structures called biternary rings.

Heyting [8] discusses ordered projective and ordered affine Hjelmslev planes. Thomas in [16] investigates ordered Desarguesian A.H. planes and shows that an ordering of such a plane is equivalent to the ordering of its coordinate A.H. ring. Laxton et al [10] extend this investigation, showing that the ordering of any A.H. plane is equivalent to the ordering of its coordinate biternary ring. In this paper they also introduce the notion of a strong ordering of an A.H. plane and a strong ordering of a biternary ring. Although the strong ordering of a biternary ring induces a strong ordering of its associated A.H. plane, they only prove the converse holds in the case of a translation plane.

In this thesis we continue the investigation of ordered Hjelmslev planes. In section 3, we define almost-strong orderings of biternary rings and projective orderings of A.H. planes. We then show that a strong ordering of an A.H. plane is equivalent to an almost-strong ordering of its associated biternary rings and a projective ordering of an A.H. plane is equivalent to a strong ordering of its associated biternary rings. In the Desarguesian case, these notions are all equivalent.

In section 4, we extend the axioms of order for an ordinary projective plane given by Heyting in [8] to projective Hjelmslev planes (henceforth called P.H. planes) and prove some properties of ordered P.H. planes. We also show that an A.H. plane embedded in an ordered P.H. plane is itself ordered.

In [8], Heyting also shows that the projective plane constructed by adding a line at infinity and all the points incident with it to an ordered affine plane is ordered. In section 6, we use the methods of Artmann [1] to construct an ordered projective completion for a projectively ordered, projectively uniform A.H. plane.

CHAPTER 2

Basic definitions and prerequisites.

2.1. Affine planes.

An incidence structure with parallelism is a structure $<\mathbb{P}, \mathbb{L}, I, ||>$ where $\mathbb{P}$ and $\mathbb{L}$ are disjoint sets whose elements are called points and lines respectively, $I \subset \mathbb{P} \times \mathbb{L}$ and $|| \subset \mathbb{L} \times \mathbb{L}$ is an arbitrary equivalence relation called parallelism. We write $P\, I\, \ell$ (P is incident with ℓ) if $(P, \ell) \in I$ and $\ell || m$ (ℓ is parallel to m) if $(\ell, m) \in ||$.

An (ordinary) affine plane is an incidence structure with parallelism $\mathcal{A} = <\mathbb{P}, \mathbb{L}, I\, ||>$ in which:

1) Any two points are incident with exactly one line.

Then two distinct lines either meet in one point or not at all. Two lines are parallel if they do not meet or are identical.

2) For each pair $(P, \ell) \in \mathbb{P} \times \mathbb{L}$, there exists a unique line $L(P,\ell)$ such that $P\, I\, L(P,\ell)$ and $\ell || L(P,\ell)$.

3) There exist three distinct non-collinear points.

In an incidence structure, points P and Q are called neighbours ($P \sim Q$) if there exist two distinct lines ℓ and m such that $P, Q\, I\, \ell, m$. Lines ℓ and m are called neighbours ($\ell \sim m$) if for each P on either of these lines there exists a point Q on the other such that $P \sim Q$. We define $\ell \wedge m = \{P \in \mathbb{P} \mid P\ I\ \ell, m\}$ and $|X|$ to be the cardinality of the set X.

An incidence structure with parallelism $\mathcal{H} = \langle \mathbf{P}, \mathbf{L}, I, \| \rangle$ is called an affine Hjelmselv plane (A.H. plane) if it satisfies the following axioms (cf. [1], 1).

(A1) Any two points are incident with at least one line.

(A2) If $\ell \wedge m \neq \emptyset$ $(\ell, m \in \mathbf{L})$, then $|\ell \wedge m| = 1$ if and only if $\ell \not\sim m$.

(A3) For each pair $(P, \ell) \in \mathbf{P} \times \mathbf{L}$, there exists a unique line $L(P, \ell)$ such that $P \, I \, L(P,\ell)$ and $\ell \| L(P,\ell)$.

(A4) There exist an affine plane $\mathcal{A}$ and an epimorphism (i.e., a surjective map which takes points to points and lines to lines and which preserves incidence and parallelism),

$\chi : \mathcal{H} \longrightarrow \mathcal{A}$ with the properties:

i) $P \sim Q$ if and only if $\chi(P) = \chi(Q)$;

ii) $\ell \sim m$ if and only if $\chi(\ell) = \chi(m)$;

iii) if $\ell \wedge m = \emptyset$, then $\chi(\ell) \| \chi(m)$.

Equivalently, an A.H. plane may be defined as an incidence structure with parallelism $\mathcal{H} = \langle \mathbf{P}, \mathbf{L}, I, \| \rangle$ satisfying the following set of axioms (cf. [14], 2.6).

(A1)' Any two points P and Q are incident with at least one line ℓ. If $P \not\sim Q$, we write $PQ = \ell$.

(A2)' There exist $P_1, P_2, P_3 \in \mathbf{P}$ such that $P_i P_j \not\sim P_i P_k$ where (i, j, k) is any permutation of $(1, 2, 3)$. $P_1P_2P_3$ is called a triangle.

(A3)' $\sim$ is transitive on $\mathbf{P}$.

(A4)' If $\ell \wedge m \neq \emptyset$ $(\ell, m \in \mathbf{L})$, then $|\ell \wedge m| = 1$ if and only if $\ell \not\sim m$.

(A5)' If $\ell \not\sim m$; $P, R \, I \, \ell$; $Q, R \, I \, m$ and $P \sim Q$, then $R \sim P, Q$.

(A6)' If $\ell \sim m$; $n \not\sim \ell$; $P \, I \, \ell, n$ and $Q \, I \, m, n$, then $P \sim Q$.

(A7)' If $\ell \| m$; $P \, I \, \ell, n$ and $\ell \not\sim n$, then $m \not\sim n$ and there exists Q such that $Q \, I \, m, n$.

(A8)' For each pair $(P, \ell) \in \mathbb{P} \times \mathbb{L}$, there exists a unique line $L(P,\ell)$ such that $P \, I \, L(P,\ell)$ and $\ell \| L(P,\ell)$.

The neighbour relations on $\mathbb{P}$ and $\mathbb{L}$ are equivalence relations and hence may be used to partition $\mathbb{P}$ and $\mathbb{L}$ into equivalence classes. Define $\overline{P} = \{Q \in \mathbb{P} \mid Q \sim P\}$ and $\overline{\ell} = \{m \in \mathbb{L} \mid m \sim \ell\}$. Let $\overline{\mathbb{P}} = \mathbb{P}/\sim$ and $\overline{\mathbb{L}} = \mathbb{L}/\sim$. Then $\overline{\mathcal{H}} = \langle \overline{\mathbb{P}}, \overline{\mathbb{L}}, \overline{I}, \overline{\|} \rangle$ (where $\overline{I}$ and $\overline{\|}$ are suitable restrictions of I and $\|$) is isomorphic to the underlying ordinary affine plane $\mathcal{A}$ (defined in (A4)) of $\mathcal{H}$; thus we may take $\mathcal{A} = \overline{\mathcal{H}}$.

Finally, if ℓ is a line of $\mathcal{H}$, let $\pi_\ell = \{m \in \mathbb{L} \mid m \| \ell\}$. For $\ell, m \in \mathbb{L}$, we write $\pi_\ell \sim \pi_m$ if there exist lines $g \in \pi_\ell$ and $h \in \pi_m$ such that $g \sim h$. Clearly, $\sim$ is an equivalence relation on $\{\pi_\ell \mid \ell \in \mathbb{L}\}$. If $\pi_\ell \sim \pi_m$, then for any $(g,h) \in \pi_\ell \times \pi_m$, either $g \wedge h = \emptyset$ or $g \sim h$. Let $L(P, \pi_\ell)$ be the unique line of π_ℓ through P.

An ordinary affine plane is an A.H. plane in which two points are neighbours if and only if they are identical.

Let $\mathcal{H}$ be an A.H. plane. We may coordinatize $\mathcal{H}$ in the following way. Select a triangle (cf. (A2)') O, X, Y of $\mathcal{H}$. Let $E = L(X, OY) \wedge L(Y, OX)$ and $k = OE$. Let $H = \{P \in \mathbb{P} \mid P \, I \, k\}$ and $\eta = H \cap \overline{O}$. The points of H are denoted by $a, b, c, \ldots$ except for O and E which are denoted 0 and and 1 respectively. To any point P, we assign the coordinates (x,y), where $x = L(P, OY) \wedge k$ and $y = L(P, OX) \wedge k$. Call $\mathbb{L}_1 = \{\ell \in \mathbb{L} \mid \pi_\ell \sim \pi_{OY}\}$ and $\mathbb{L}_2 = \{\ell \in \mathbb{L} \mid \pi_\ell \not\sim \pi_{OY}\}$, the sets of lines of the first and second kind respectively. If $\ell \in \mathbb{L}_1$ then we assign ℓ the coordinates $[u,v]_1$, where $L(O,\ell) \wedge YE = (u,1)$ and $\ell \wedge OX = (v,0)$. If $\ell \in \mathbb{L}_2$, we assign ℓ the coordinates $[m,n]_2$, where $L(O,\ell) \wedge XE = (1,m)$ and $\ell \wedge OE = (0,n)$.

We now define two ternary maps in the following way.

$$T_1 : H \times H \times H \longrightarrow H$$
$$(x,m,n) \rightsquigarrow T_1(x,m,n) = k \wedge L(L((0,n), O(1,m)) \wedge L(x,OY), OX)$$

$$T_0 : H \times \eta \times H \longrightarrow H$$
$$(y,u,v) \rightsquigarrow T_0(y,u,v) = k \wedge L(L((v,0), O(u,1)) \wedge L(y,OX), OY).$$

From the definitions, we may conclude $y = T_1(x,m,n)$ if and only if $(x,y)\ I\ [m,n]_2$; $x = T_0(y,u,v)$ if and only if $(x,y)\ I\ [u,v]_1$. (cf. [11]).

2.2. Biternary rings.

An algebraic system $\mathcal{R} = \langle R, T, 0, 1\rangle$, where R is a set, T is a ternary operator and 0,1 are distinct elements of R is a ternary ring if the following axioms hold (cf. [11], 2.7(a)).

($\mathcal{T}_0$) $T(m,0,n) = n = T(0,m,n)$, for all $m,n \in R$;

($\mathcal{T}_1$) $T(1,m,0) = m = T(m,1,0)$, for all $m \in R$;

($\mathcal{T}_2$) $T(a,m,x) = b$ is uniquely solvable for x, for all $a,m,b \in R$.

If $\mathcal{R}$ is a ternary ring then $a \neq 0$ is a left [right] divisor of zero if there exists $b \neq 0$ such that $T(a,b,0) = 0$ [$T(b,a,0) = 0$].

Let D_+ [D_-] be the set of 0 and the left [right] divisors of zero (cf. [11], 2.7(c)).

An algebraic system $\mathcal{B} = \langle R, T, T_0, 0, 1\rangle$ is a biternary ring if the following axioms hold (cf. [11], 2.7(e)).

(β1) $\mathcal{R} = \langle R, T, 0, 1\rangle$ is a ternary ring.

(β2) $\sim_R$ is an equivalence relation on R, where for $a, b \in R$ $a \sim_R b$ if and only if every x which satisfies the equation $a = T(x, 1, b)$ is an element of D_+.

(β3). $T_0 : R \times D_+ \times R \longrightarrow R$ with the properties

i) $T_0(m, 0, n) = n = T_0(0, p, n)$ for $p \in D_+$; $m, n \in R$;

ii) $T_0(1, u, 0) = u$, for $u \in D_+$;

iii) $T_0(a, m, x) = b$ is uniquely solvable for x, for all $a, b \in R$, $m \in D_+$.

(β4) $T(x, m_1, n_1) = T(x, m_2, n_2)$ is uniquely solvable for x if and only if $m_1 \not\sim_R m_2$.

(β5) The system $T(a_i, x, y) = b_i$; $i = 1,2$; uniquely determines the pair x, y if $a_1 \not\sim_R a_2$. If $a_1 \sim_R a_2$ and $b_1 \not\sim_R b_2$, then the system cannot be solved. If $a_1 \not\sim_R a_2$ and $b_1 \sim_R b_2$, then $x \in D_+$.

(β6) The system $y = T(x,m,n)$ and $x = T_0(y,u,v)$ where $u \in D_+$ determines uniquely the pair x, y.

(β7) If $a_1 \sim_R a_2$ and $b_1 \sim_R b_2$ and $(a_1,b_1) \neq (a_2,b_2)$, then one and only one of the systems $T(a_i,x,y) = b_i$, $T_0(b_i,u,v) = a_i$; $i = 1,2$; is solvable with respect to x, y or u, v. The solvable system has at least two solutions and

$x_1 \sim_R x_2$, $y_1 \sim_R y_2$ or $u_1 \sim_R u_2$, $v_1 \sim_R v_2$ according as the former or latter system is solvable.

(β8) The system $T_0(b_i, x, y) = a_i$; $i = 1,2$; determines uniquely $x \in D_+$ and $y \in R$ if $b_1 \not\sim_R b_2$ and $a_1 \sim_R a_2$; has no solutions for x and y if $a_1 \not\sim_R a_2$.

(β9) If $T_0(b, u_i, v_i) = a$; $i = 1,2$, then $v_1 \sim_R v_2$ and there exists at least one other pair a_1, b_1 such that $a_1 = T_0(b_i, u_i, v_i)$, $i = 1,2$.

(β10) The function T induces a function $\overline{T}$ in $R/\sim_R$ and $< R/\sim_R, \overline{T}, \overline{0}, \overline{1} >$ is a ternary field with $\overline{0} = \{z \mid z \sim_R 0\}$ and $\overline{1} = \{z \mid z \sim_R 1\}$ in the sense of Hall; cf. [7].

<u>2.2.1.</u> <u>Lemma</u>. The following properties hold in $\mathcal{B}$.

(β11) $T_1(x, m, n) = b$ is uniquely solvable for x if $m \not\sim_R 0$; for all $m, n, b \in R$.

(β12) $T_1(a, x, n) = b$ is uniquely solvable for x if $a \not\sim_R 0$; for all $a, n, b \in R$.

<u>Proof</u>. These properties are direct consequence of $(\mathcal{T}_1)$, $(\mathcal{T}_2)$ and (β5).

<u>2.2.2.</u> <u>Theorem</u>. (cf. [11], 2.8). $< H, T_1, T_0, 0, 1 >$ is a biternary ring where $D_+ = \eta$ and $\sim_R = \sim$.

<u>2.2.3</u> <u>Theorem</u>. (cf. [11], 2.9). Let $\mathcal{B} = < R, T_1, T_0, 0, 1 >$ be a biternary ring and let $\mathcal{H}(\mathcal{B}) = < \mathbb{P}, \mathbb{L}, \|, I >$ be the incidence structure defined by:

$\mathbb{P}' = R \times R$;

$\mathbb{L} = \mathbb{L}_1 \cup \mathbb{L}_2$. where

$\mathbb{L}_1 = \{[u,v]_1 \mid [u,v]_1 = \{(T_0(y,u,v),y) \mid y \in R\};\ (u,v) \in D_+ \times R\}$,

$\mathbb{L}_2 = \{[m,n]_2 \mid [m,n]_2 = \{(x, T_1(x,m,n)) \mid x \in R\};\ (m,n) \in R \times R\}$;

$[m,n]_i \parallel [u,v]_j$ if and only if $i = j$ and $m = u$;

I is set inclusion.

Then $\mathcal{H}(\mathcal{B})$ is an A.H. plane.

2.3. Projective planes.

An incidence structure $\mathcal{P} = \langle \mathbb{P}, \mathbb{L}, I \rangle$ is called an ordinary projective plane if it satisfies the following axioms.

1) Any two points are incident with a unique line.

2) Any two lines intersect exactly once.

3) There exist four points such that no three of them are collinear.

An incidence structure $\mathcal{P} = \langle \mathbb{P}, \mathbb{L}, I \rangle$ is called a projective Hjelmslev plane (P.H. plane) if it satisfies the following axioms (cf. [1], 1).

(H1) Any two points are incident with at least one line.

(H2) Any two lines intersect in at least one point.

Two points P,Q are neighbours $(P \sim Q)$ if they are joined by more than one line; two lines l,m are neighbours $(l \sim m)$ if they intersect in more than one point.

(H3) There exist an ordinary projective plane $\mathcal{P}'$ and an epimorphism (ie., a surjective map which takes points to points and lines to lines and preserves incidence) $\chi : \mathcal{P} \longrightarrow \mathcal{P}'$ with the properties

i) $P \sim Q$ if and only if $\chi(P) = \chi(Q)$;

ii) $\ell \sim m$ if and only if $\chi(\ell) = \chi(m)$.

Equivalently, a P.H. plane may be defined as an incidence structure $\mathcal{P} = \langle \mathbb{P}, \mathbb{L}, I \rangle$ with neighbour relations on $\mathbb{P}$ and $\mathbb{L}$ defined as above satisfying the following axioms. [cf. 9,1.1].

(H1)' Any two points P and Q are joined by at least one line ℓ. If $P \not\sim Q$, we write $PQ = \ell$.

(H2)' Any two lines ℓ and m intersect in at least one point P. If $\ell \not\sim m$, we write $\ell \wedge m = P$.

(H3)' There exist four pairwise non-neighbouring points P_1, P_2, P_3, P_4 such that $P_iP_j \not\sim P_iP_k$ for $i \neq j \neq k \neq i$; $i, j, k \in \{1, 2, 3, 4\}$.

(H4)' If $P\ I\ \ell, m, n$; $\ell \not\sim m$; $m \sim n$, then $\ell \not\sim n$.

(H5)' If $\ell \sim m$ and $n \not\sim \ell$, then $\ell \wedge n \sim m \wedge n$.

(H6)' If $P \sim Q$ and $Q \not\sim R$, then $PR \sim QR$.

As in 2.1, we define $\overline{P} = \{Q \in \mathbb{P} \mid Q \sim P\}$ and $\overline{\ell} = \{m \in \mathbb{L} \mid m \sim \ell\}$. Also as in 2.1, we may identify $\overline{\mathcal{P}} = \langle \mathbb{P}/\sim, \mathbb{L}/\sim, I/(\sim \times \sim) \rangle$ and the underlying ordinary projective plane $\mathcal{P}'$ of $\mathcal{P}$.

2.4. Uniform planes.

2.4.1. An A.H. (or P.H.) plane $\mathcal{H} = \langle \mathbb{P}, \mathbb{L}, I \rangle$ is uniform if and only if for any $P \in \mathbb{P}$, the incidence structure $\mathcal{A}(\overline{P}) = \langle \mathbb{P}_{\overline{P}}, \mathbb{L}_{\overline{P}}, I_{\overline{P}}, \|_{\overline{P}} \rangle$ where $\mathbb{P}_{\overline{P}} = \overline{P}$, $\mathbb{L}_{\overline{P}} = \{ \ell \in \mathbb{L} \mid \exists Q \in \overline{P};\ Q\, I\, \ell \}$, $I_{\overline{P}} = I \cap (\mathbb{P}_{\overline{P}} \times \mathbb{L}_{\overline{P}})$, is an ordinary affine plane. The pencil of lines in $\mathcal{A}(\overline{P})$ parallel to a given line g shall be denoted by $\pi_{P,g}$.

2.4.2. Lemma. An A.H. (or P.H.) plane $\mathcal{H} = \langle \mathbb{P}, \mathbb{L}, I \rangle$ with non-trivial neighbour relation is uniform if and only if for any $g, h \in \mathbb{L}$ and $P, Q \in \mathbb{P}$ with $g \sim h$; $P, Q\, I\, g$; $P\, I\, h$ and $P \sim Q$, then $Q\, I\, h$.

The affine case is proved in [14], Satz 2.12. A similar proof is valid in the projective case.

2.5. Projectively uniform A.H. planes.

An A.H. plane is projectively uniform if and only if it is uniform and whenever two neighbour lines fail to meet they are parallel [3].

2.5.1. Lemma. Any finite uniform A.H. plane is projectively uniform; cf. [3], Proposition 2.1.

CHAPTER 3

Ordered affine Hjelmslev planes

3.1. Ordered A.H. planes and biternary rings.

An ordering of an A.H. plane $\langle \mathbb{P}, \mathbb{L}, I, \| \rangle$ is a non-empty ternary relation ρ on $\mathbb{P}$ satisfying the following condition (cf. [10], 3.1).

(O1) $(P,Q,R) \in \rho$ implies P,Q,R are mutually distinct and collinear.

(O2) $(P,Q,R) \in \rho$ implies $(R,Q,P) \in \rho$.

(O3) $(P,Q,R) \in \rho$ implies $(P,R,Q) \notin \rho$.

(O4) If P,Q,R are mutually distinct and collinear, at least one of (P,Q,R), (P,R,Q), (R,P,Q) is in ρ.

(O5) If P,Q,R,S are mutually distinct and collinear and $(P,Q,R) \in \rho$, then either $(P,Q,S) \in \rho$ or $(S,Q,R) \in \rho$.

(O6) Non-degenerate parallel projections preserve order.

A parallel projection of an A.H. plane is a mapping from the points of a line ℓ to the points of a line m in the direction of a parallel pencil $\pi (\pi \not\sim \pi_m)$ given by $X \longrightarrow L(X,\pi) \wedge m$. It is called non-degenerate if it is bijective. It is bijective if and only if $\pi \not\sim \pi_\ell$. We shall denote a parallel projection from the points of ℓ onto the points of m in the direction of π_g by $\ell \xrightarrow[g]{} m$.

An ordered A.H. plane (O.A.H. plane) is a quintuplet $\mathcal{H} = \langle \mathbb{P}, \mathbb{L}, I, \|, \rho \rangle$ where ρ is an ordering of the A.H. plane

$\langle \mathbb{P}, \mathbb{L}, I, \| \rangle$ and some (hence each) line contains three mutually non-neighbouring points. We note finite A.H. planes cannot be ordered.

3.1.1. Lemma. An ordered A.H. plane has the following properties (cf. [10], 3.2).

(1) The words "at least" in (04) may be replaced by "exactly".

(2) $(P,Q,R), (Q,R,S) \in \rho$ imply $(P,Q,S), (P,R,S) \in \rho$.

(3) $(P,Q,R), (P,R,S) \in \rho$ imply $(P,Q,S), (Q,R,S) \in \rho$.

(4) Two of $(Q,P,R) \in \rho$, $(R,P,S) \in \rho$, $(S,P,Q) \in \rho$ exclude the third.

(5) $(P,Q,S), (P,R,S) \in \rho$ imply either $(Q,R,S) \in \rho$ or $(R,Q,S) \in \rho$.

3.1.2. Lemma. (cf. [10], 3.3). Let $\mathcal{H}$ be an ordered A.H. plane. Let $g_1, g_2, g_3, h, h' \in \mathbb{L}$; $h \| h'$; $g_i \not\approx g_j$ $(i \neq j;\ i,j \in \{1,2,3\})$. Let $O I g_i$; $P_i = g_i \wedge h$ $(i = 1,2,3)$. If either

(1) $P_i' = g_i \wedge h'$ $(i = 1,2,3)$ and $P_1' \neq 0$; or

(2) $0, 0' I h'$; $g_i' = L(0', g_i)$, $P_i' = g_i' \wedge h$ $(i = 1,2,3)$

then $(P_1, P_2, P_3) \in \rho$ implies $(P_1', P_2', P_3') \in \rho$.

3.1.3. An ordering of a biternary ring $\langle R, T_1, T_0, 0, 1\rangle$ is a transitive relation $<$ on R satisfying the following conditions (cf. [10], 4.1).

(OM1) For all $a, b \in R$, exactly one of $a < b$, $b < a$, $a = b$ holds.

(OM2) If $c < d$, then $T_1(a,m,c) < T_1(a,m,d)$, for any $a, m \in R$.

(OM3) If $m \not\sim_R n$, $m < n$ and $T_1(s,m,c) = T_1(s,n,d)$, then $s < t$ $[s > t]$ implies $T_1(t,m,c) < T_1(t,n,d)$ $[T_1(t,m,c) > T_1(t,n,d)]$ for all $c, d \in R$.

(OM4) If $T_1(a_i,m,d_i) = b_i$, $T_0(b_i,u,v) = a_i$ $(i=1,2)$, $m, u \in D_+$ and $d_1 < d_2$, then $b_1 < b_2$.

(OM5) Conditions (OM2) and (OM4) with T_1 and T_0 interchanged.

An <u>ordered biternary ring</u> is an algebraic structure $< R, T_1, T_0, 0, 1, < >$ where $< R, T_1, T_0, 0, 1 >$ is a biternary ring and $<$ is an ordering of this biternary ring.

<u>3.1.4</u>. <u>Lemma</u>. (cf. [10], 4.2). An ordered biternary ring $< R, T_1, T_0, 0, 1, < >$ has the following properties.

(1) If $m \not\sim_R n$, $T_1(s,m,c) = T_1(s,n,d)$ and $T_1(t,m,c) < T_1(t,n,d)$, then either $s < t$ and $m < n$ or $s > t$ and $m > n$.

(2) If $m \not\sim_R n$, $m < n$ and $0 < a$ $[0 > a]$, then $T_1(a,m,d) < T_1(a,n,d)$ $[T_1(a,m,d) > T_1(a,n,d)]$ for any $d \in R$.

(3) If $a < b$, $m \notin D_+$ and $0 < m$ $[0 > m]$, then $T_1(a,m,d) < T_1(b,m,d)$ $[T_1(a,m,d) > T_1(b,m,d)]$ for any $d \in R$.

(4) If $T_1(a_i,m,d_i) = b_i$, $T_0(b_i,u,v) = a_i$ $(i=1,2)$, $m, u \in D_+$ and $b_1 < b_2$, then $d_1 < d_2$.

(5) Property (4) with T_0 and T_1 interchanged.

(6) D_+ is convex (ie., if $a < b < c$ and $a, c \in D_+$, then $b \in D_+$).

(7) D_+ consists of infinitesimals (ie., if $a \notin D_+$, then $-a < x < a$ for all $x \in D_+$).

3.1.5. Theorem. (cf. [10], §6). An ordering of an A.H. plane $\mathcal{K}$ induces an ordering of each of its biternary rings.

The ordering induced on $\langle R, T_1, T_0, 0, 1\rangle$ by ρ is defined in the following way. Define $0 < 1$. For any $x \in R$, $x \neq 0,1$, define $0 < x$ if either $(0,x,1) \in \rho$ or $(0,1,x) \in \rho$ and $x > 0$ if $(x,0,1) \in \rho$. For any $x,y \in R$, $x,y \neq 0$ define $x < y$ if either: 1) $x < 0$ and $(x,0,y) \in \rho$ or $(x,y,0) \in \rho$; or 2) $x > 0$ and $(0,x,y) \in \rho$.

3.1.6. Theorem. (cf. [10], §7). An ordering of the biternary ring of an A.H. plane induces an ordering of the plane.

The ordering ρ is defined in the following way. For points $A = (a_1, a_2, 1)$, $B = (b_1, b_2, 1)$ on a line of the first kind define $A < B$ if and only if $a_2 < b_2$; if A,B are on a line of the second kind then define $A < B$ if and only if $a_1 < b_1$. Define $(A,B,C) \in \rho$ if and only if A,B,C are mutually distinct collinear points and $A < B < C$ or $C < B < A$.

3.2. Strongly ordered A.H. planes.

The definition of an ordered A.H. plane given in 3.1 is a generalization of Pickert's ordering for an ordinary affine plane (cf. [15], p.227); however another way to define an ordered (ordinary) affine plane is to require that: i) every line be totally ordered; ii) every parallel projection $\theta : \ell \longrightarrow m$ be either an order isomorphism (ie., $a < b$ if and only if $\theta(a) < \theta(b)$ for all $a,b \, I \, \ell$) or an order anti-isomorphism (ie., $a < b$ if and only if $\theta(b) < \theta(a)$, for all $a,b \, I \, \ell$). Thus the natural generalization of this definition to A.H. planes would be the following (cf. [10], 9.2).

<u>3.2.1</u>. An A.H. plane $\mathcal{H} = < \mathbb{P}, \mathbb{L}, I, || >$ is <u>strongly ordered</u> ($\mathcal{H}$ is a S.O.A.H. plane) if and only if

i) every line is totally ordered;

ii) every parallel projection $\theta : \ell \longrightarrow m$ is either an order homomorphism or an order antihomomorphism; ie., θ is a function (not necessarily a bijection) such that either $a < b$ implies $\theta(a) \leq \theta(b)$ for all $a, b\, I\, \ell$ or $a < b$ implies $\theta(b) \leq \theta(a)$, for all $a, b\, I\, \ell$;

iii) there is a line in $\mathbb{L}$ incident with at least three mutually non-neighbouring points.

<u>3.2.2</u>. <u>Lemma</u>. An A.H. plane is strongly ordered if and only if it is ordered and satisfies the following axiom.

(06)* If $\theta : \ell \longrightarrow m$ is any (possibly degenerate) parallel projection then for any $A, B, C\, I\, \ell$ with $(A,B,C) \in \rho$ either $(\theta(A), \theta(B), \theta(C)) \in \rho$ or two or more of the points $\theta(A)$, $\theta(B)$, $\theta(C)$ are equal. (We denote this by $(\theta(A), \theta(B), \theta(C)) \in \rho^*$ and say that ρ preserves order.)

<u>3.2.3</u>. <u>Lemma</u>. Let $\mathcal{H}$ be a strongly ordered A.H. plane. If $\theta : \ell \longrightarrow m$ is a parallel projection, $A, B, C\, I\, \ell$; $(A,B,C) \in \rho$ and $\theta(A) = \theta(C)$, then $\theta(B) = \theta(A)$.

<u>Proof</u>. Assume $\theta(B) \neq \theta(A)$. There exists $D\, I\, \ell$ such that $D \not\sim A$ and $(B,C,D) \in \rho$ or $(D,A,B) \in \rho$ without loss of generality, assume $(B,C,D) \in \rho$. Then $(B,C,D) \in \rho$ implies $(\theta(B), \theta(C), \theta(D)) \in \rho$ (as no two of these points are equal) and $(A,B,C) \in \rho$, $(B,C,D) \in \rho$

imply $(A,B,D) \in \rho$ which in turn implies $(\theta(A), \theta(B), \theta(D)) \in \rho$ (as no two of these points are equal). However $\theta(A) = \theta(C)$; thus $(\theta(B), \theta(A), \theta(D)) \in \rho$ and $(\theta(A), \theta(B), \theta(D)) \in \rho$; a contradiction.

3.3. Strongly and almost-strongly ordered biternary rings.

3.3.1. An ordered biternary ring $< R, T_1, T_0, 0, 1 >$ is strongly ordered if it also satisfies the following conditions (OM3)* and (OM3)*' (cf. [10], 9.4).

(OM3)* If $m < n$ and $T_1(s,m,c) = T_1(s,n,d)$, then $s < t$ $[s > t]$ implies $T_1(t,m,c) \leq T_1(t,n,d)$ $[T_1(t,m,c) \geq T_1(t,n,d)]$ for all $c, d \in R$.

(OM3)*' is just (OM3)* with T_1 replaced by T_0.

3.3.2. Lemma. A strongly ordered biternary ring has the following properties.

(1) If $T_i(s,m,c) = T_i(s,n,d)$ and $T_i(t,m,c) < T_i(t,n,d)$ $(i=0,1)$, then either $s < t$ and $m < n$ or $s > t$ and $m > n$.

(2) If $m < n$ and $0 < a$ $[0 > a]$, then $T_i(a,m,d) \leq T_i(a,n,d)$ $[T_i(a,m,d) \geq T_i(a,n,d)]$; $i=0,1$. If, in addition, $a \notin n$, then these are strict inequalities.

(3) If $a < b$ and $m > 0$ $[m < 0]$, then $T_i(a,m,d) \leq T_i(b,m,d)$ $[T_i(a,m,d) \geq T_i(b,m,d)]$ for any $d \in R$; $i=0,1$.

(4) If $m < n$ and $T_1(s,m,c) = T_1(s,n,d)$, then $s \not> t$, $s < t$ $[s > t]$ implies $T_1(t,m,c) < T_1(t,n,d)$ $[T_1(t,m,c) > T_1(t,n,d)]$ for $c, d \in R$.

Proof. These properties follow directly from the definition.

3.3.3. Remark. Although it is possible to show that the strong ordering of a biternary ring induces a strong ordering of its A.H. plane (cf. [10], 9.5), the converse is not true in general (cf. 3.3.8). If we insist that the A.H. plane be a translation plane, then a strong ordering of the A.H. plane does induce a strong ordering of its biternary rings (cf. [10], 9.6). We introduce two axioms that are slightly weaker than (OM3)* and (OM3)*'.

3.3.4. An ordered biternary ring is almost-strongly ordered if it satisfies axioms (OM3)** and (OM3)**'.

(OM3)** If $m \sim n$ and $T_1(s,m,c) = T_1(s,n,d)$ for any $c, d \in R$, then either

$$T_1(a,m,c) \leq T_1(a,n,d) \quad \text{and} \quad T_1(b,m,c) \geq T_1(b,n,d) \quad \text{for all} \quad a < s < b;$$

or

$$T_1(a,m,c) \geq T_1(a,n,d) \quad \text{and} \quad T_1(b,m,c) \leq T_1(b,n,d) \quad \text{for all} \quad a < s < b.$$

(OM3)**' is just (OM3)** with T_1 replaced by T_0.

3.3.5. Theorem. An almost-strong ordering of the biternary ring $\mathcal{B} = \langle R, T_1, T_0, 0, 1 \rangle$ induces a strong ordering of the A.H. plane $\mathcal{H}$ of $\mathcal{B}$.

Proof. Since $\mathcal{B}$ is almost-strongly ordered, it is ordered and hence $\mathcal{H}$ is ordered. Thus it is sufficient to verify (O6)*.

We shall show that the parallel projections

$$[m,n]_2 \xrightarrow{[p,0]_2} [0,0]_1 \quad \text{where } m \sim p \quad \text{and} \quad [m,n]_1 \xrightarrow{[p,0]_1} [0,0]_2$$

where $m \sim p \sim 0$ preserve order.

Take $a < b < c$. Then on $[m,n]_2$, we have

$((a,T_1(a,m,n)),\ (b,T_1(b,m,n)),\ (c,T_1(c,m,n))) \in \rho$. Let $L((b,T_1(b,m,n)),\ [p,0]_2) = [p,q]_2$. By (OM3)**, since $T_1(b,m,n) = T_1(b,p,q)$, either $T_1(a,m,n) \le T_1(a,p,q)$ and $T_1(c,m,n) \ge T_1(c,p,q)$ or $T_1(a,m,n) \ge T_1(a,p,q)$ and $T_1(c,m,n) \le T_1(c,p,q)$. Therefore, if $L((a,T_1(a,m,n)),\ [p,0]_2) = [p,r]_2$ and $L((c,T_1(c,m,n)),\ [p,0]_2) = [p,s]_2$, then either $r \le q \le s$ or $r \ge q \ge s$. In either case, the parallel projection preserves order.

Similarly, we can prove $[m,n]_1 \xrightarrow[{[p,q]_1}]{} [0,0]_2$ preserves order.

3.3.6. Lemma. Let $\mathcal{K}$ be a strongly ordered A.H. plane. Let $g_1, g_2, g_3, h, h' \in \mathbb{L}$; $h \| h'$; $O\ I\ g_i$; $P_i = g_i \wedge h$ $(i=1,2,3)$; thus $g_1 \not\sim h$. If $P_i' = g_i \wedge h'$ $(i=1,2,3)$ and $P_1' \ne O$, then $(P_1,P_2,P_3) \in \rho$ implies $(P_1', P_2', P_3') \in \rho^*$.

Proof. If $g_i \not\sim g_j$ for $i \ne j$; $i,j \in \{1,2,3\}$, then the result is given in 3.1.2. If any two of P_1', P_2', P_3' are equal, the result is given by definition. We may, therefore, assume $P_1' \ne P_2' \ne P_3' \ne P_1'$ and at least one pair g_i, g_j (and hence P_i', P_j') $(i,j \in \{1,2,3\},\ i \ne j)$ are neighbours.

Case 1: $(O,P_1,P_1') \in \rho$. Since $(P_1,P_2,P_3) \in \rho$, the parallel projection $h \xrightarrow[g_2]{} h'$ yields $(X,P_2',Y) \in \rho$ where $X = L(P_1,g_2) \wedge h'$ and $Y = L(P_3,g_2) \wedge h'$. However the (possibly degenerate) projection $g_1 \xrightarrow[g_2]{} h'$ yields $(P_2',X,P_1') \in \rho^*$ and the series of projections $g_1 \xrightarrow[h]{} g_3 \xrightarrow[g_2]{} h'$, $(P_2',Y,P_3') \in \rho^*$. Therefore as $P_2' \ne X, Y$, $(P_1',P_2',P_3') \in \rho$.

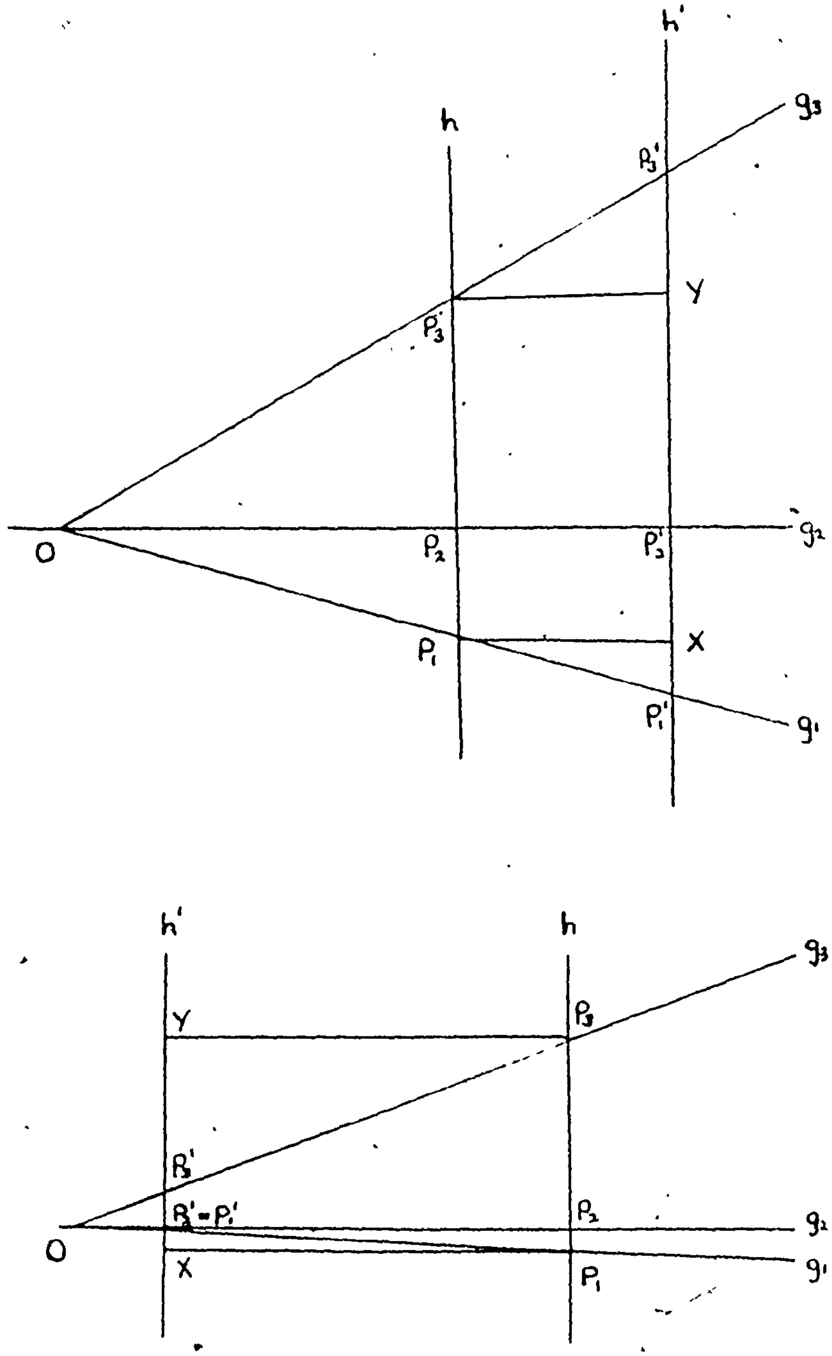


Figure 3.1
(Lemma 3.3.6, Case 1)

<u>Case 2</u>: $(O,P_1',P_1) \in \rho$. Using the same arguments as in Case 1, we obtain $(X,P_2',Y) \in \rho$, $(P_2,P_1',X) \in \rho^*$, $(P_2',P_3',Y) \in \rho^*$ with $P_1' \neq P_2' \neq P_3'$; hence $(P_1',P_2',P_3') \in \rho$.

<u>Case 3</u>: $(P_1,O,P_1') \in \rho$. Once again using the projections in Case 1, we obtain $(X,P_2',Y) \in \rho$, $(X,P_2',P_1') \in \rho^*$, $(P_3',P_2',Y) \in \rho^*$ with $P_1' \neq P_2' \neq P_3'$; hence $(P_1',P_2',P_3') \in \rho$.

<u>3.3.7</u>. <u>Theorem</u>. A strong ordering of an A.H. plane $\mathcal{H}$ induces an almost-strong ordering on each coordinate biternary ring $\mathcal{B}$ of $\mathcal{H}$.

<u>Proof</u>. Since a strong ordering of $\mathcal{H}$ is an ordering, it induces an ordering on each $\mathcal{B}$. We need only verify (OM3)** and (OM3)**'.

<u>Case 1</u>: Take $m \sim n \not\sim 0$ where $T_1(s,m,c) = T_1(s,n,d)$. Select some $t \not\sim s$, $t < s$. If $T_1(t,m,c) = T_1(s,m,c)$, then $(s,T_1(s,m,c))$, $(t,T_1(t,m,c))$ I $[m,c]_2$, $[0,T_1(s,m,c)]_2$; hence $[m,c]_2 \sim [0,T_1(s,m,c)]_2$; a contradiction. Similarly $T_1(t,n,d) \neq T_1(s,m,c)$. If $T_1(t,m,c) = T_1(t,n,d)$, then $(t,T_1(t,m,c))$, $(s,T_1(s,m,c))$ I $[m,c]_2$, $[n,d]_2$ which would imply $t \sim s$; a contradiction. Therefore $T_1(t,m,c)$, $T_1(t,n,d)$ and $T_1(s,m,c)$ are mutually distinct. Since $T_1(t,m,c) \sim T_1(t,n,d) \not\sim T_1(s,m,c)$, we have either $(T_1(t,m,c), T_1(t,n,d), T_1(s,m,c)) \in \rho$ or $(T_1(t,n,d), T_1(t,m,c), T_1(s,m,c)) \in \rho$ by 3.1.4(7) and 3.3.6.

First suppose that $T_1(t,m,c) < T_1(t,n,d) < T_1(s,m,c)$. Then by 3.3.6 for any $x \in H$, $x \neq s$, $(T_1(x,m,c), T_1(x,n,d), T_1(s,m,c)) \in \rho^*$ (the lines g_1, g_2, g_3 of 3.3.6 are $[m,c]_2$, $[n,d]_2$, $[0,T_1(s,m,c)]_2$ respectively). If $(x,s,t) \in \rho$, then $(T_1(x,m,c), T_1(s,m,c), T_1(t,m,c)) \in \rho^*$; if $(s,x,t) \in \rho$, then $(T_1(s,m,c), T_1(x,m,c), T_1(t,m,c)) \in \rho^*$; if

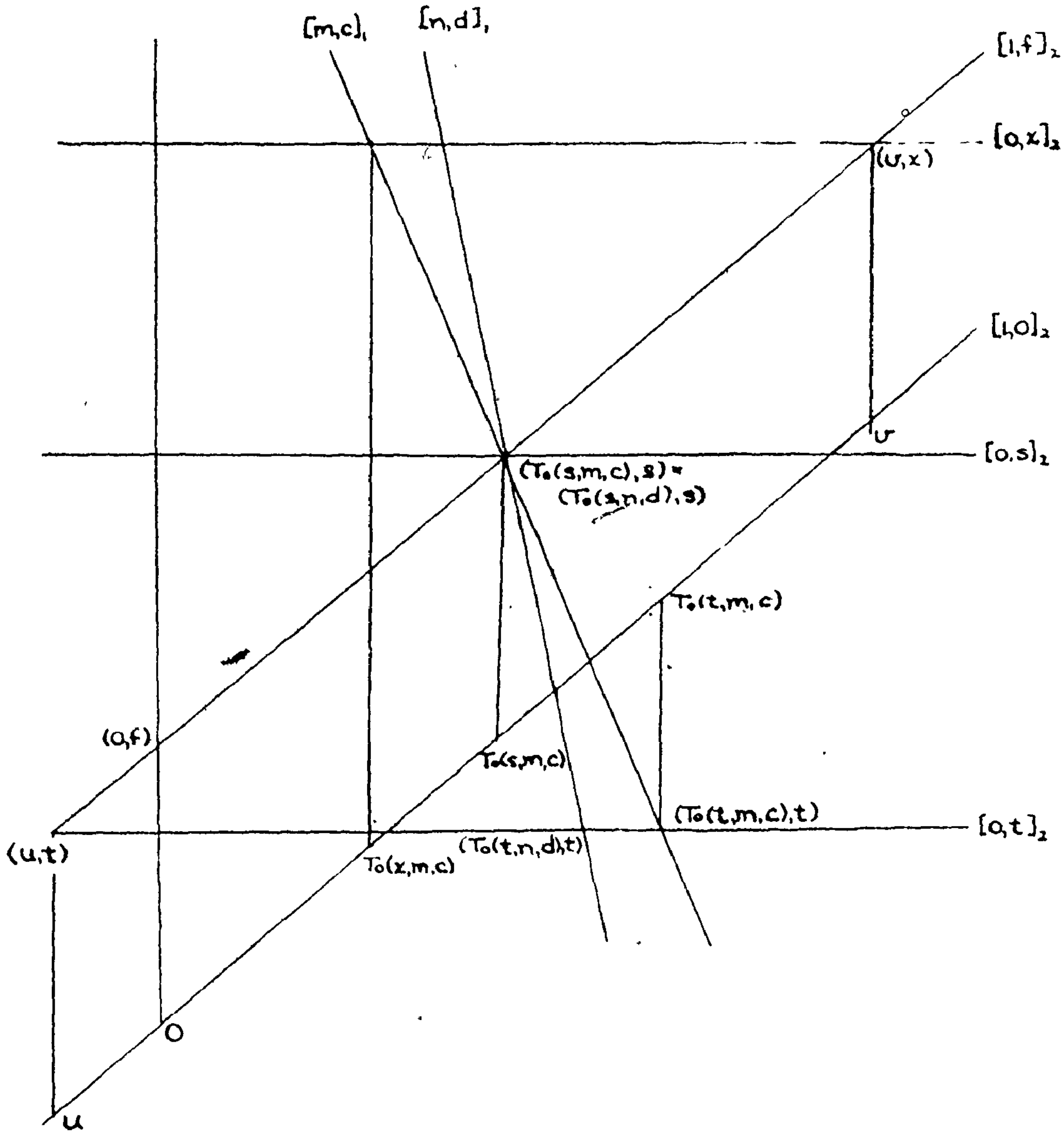


Figure 3.2
(Lemma 3.3.7, (OM3)**')

$(s,t,x) \in \rho$, then $(T_1(s,m,c), T_1(t,m,c), T_1(x,m,c)) \in \rho^*$. Therefore if $x < s$, $T_1(x,m,c) \leq T_1(x,n,d) \leq T_1(s,m,c)$ and if $x > s$, $T_1(x,m,c) \geq T_1(x,n,d) \geq T_1(s,m,c)$. Thus for a, b such that $a < s < b$, $T_1(a,m,c) \leq T_1(a,n,d)$ and $T_1(b,m,c) \geq T_1(b,n,d)$.

Similarly if $T_1(t,m,c) > T_1(t,n,d) > T_1(s,m,c)$, then $T_1(a,m,c) \geq T_1(a,n,d)$ and $T_1(b,m,c) \leq T_1(b,n,d)$ for all $a < s < b$; if $T_1(t,n,d) < T_1(t,m,c) < T_1(s,m,c)$, then $T_1(a,n,d) \leq T_1(a,m,c)$ and $T_1(b,n,d) \geq T_1(b,m,c)$ for all $a < s < b$; if $T_1(t,n,d) > T_1(t,m,c) > T_1(s,m,c)$, then $T_1(a,n,d) \geq T_1(a,m,c)$ and $T_1(b,n,d) \leq T_1(b,m,c)$ for all $a < s < b$.

Case 2: Take $m \sim n \sim 0$ where $T_1(s,m,c) = T_1(s,n,d)$. Then there exists a unique f such that $T_1(s,1,f) = T_1(s,m,c)$. Select some $t \not\sim s$, $t > s$. Then by (OM3), $T_1(t,m,c) < T_1(t,1,f)$. As $T_1(t,n,d) \sim T_1(t,m,c) \not\sim T_1(t,1,f)$, we have either $T_1(t,n,d) < T_1(t,m,c) < T_1(t,1,f)$ or $T_1(t,m,c) < T_1(t,n,d) < T_1(t,1,f)$.

Suppose $T_1(t,n,d) < T_1(t,m,c) < T_1(t,1,f)$. Then by 3.3.6, for any $x \in H$, $x \neq s$, $(T_1(x,n,d), T_1(x,m,c), T_1(x,1,f)) \in \rho^*$. However if $x < s$, (OM3) implies $T_1(x,1,f) < T_1(x,m,c) \leq T_1(x,n,d)$ and if $x > s$, $T_1(x,1,f) > T_1(x,m,c) \geq T_1(x,n,d)$. Therefore $T_1(a,m,c) \leq T_1(a,n,d)$ and $T_1(b,m,c) \geq T_1(b,n,d)$ for all $a < s < b$.

Similarly, if $T_1(t,m,c) < T_1(t,n,d) < T_1(t,1,f)$, then $T_1(a,m,c) \geq T_1(a,n,d)$ and $T_1(b,m,c) \leq T_1(b,n,d)$ for all $a < s < b$.

We now verify (OM3)**'.

Suppose $T_0(s,m,c) = T_0(s,n,d)$ where $m \neq n$. We may also assume $m \neq 0$. As $m, n \in \eta$, $c \sim T_0(s,m,c) = T_0(s,n,d) \sim d$. There exists a unique f such that $T_1(T_0(s,m,c),1,f) = s$. For any $t \not\sim s$, $t < s$, there exists a unique u such that $[0,t]_2 \wedge [1,f]_2 = (u,t)$; ie., $T_1(u,1,f) = t$. By 3.1.4(3), $u < T_0(s,m,c)$; otherwise $t = T_1(u,1,f) \geq T_1(T_0(s,m,c),1,f) = s$. Since $(u,t) \not\sim (T_0(s,m,c),s)$ and $[1,f]_2 \not\sim [0,t]_2$, we obtain $(u,t) \not\sim (T_0(s,m,c),t)$ and hence $u \not\sim T_0(s,m,c)$. As above, $u \not\sim T_0(s,m,c) \sim T_0(t,m,c)$ implies either $u < T_0(s,m,c) < T_0(t,m,c)$ or $u < T_0(t,m,c) < T_0(s,m,c)$.

Suppose first that $u < T_0(s,m,c) < T_0(t,m,c)$. For any $x \in H$, $x \neq s, t$, there exists a unique v such that $T_1(v,1,f) = x$ and by 3.3.6, we have $(v, T_0(s,m,c), T_0(x,m,c)) \in \rho^*$. If $x < s$, then $v < T_0(s,m,c) \leq T_0(x,m,c)$ and if $x > s$; $v > T_0(s,m,c) \geq T_0(x,m,c)$ by 3.1.4(3). If $T_0(t,n,d) = T_0(s,m,c)$, we have the desired result. Otherwise, by 3.3.6, if $(T_0(t,n,d), T_0(s,m,c), T_0(t,m,c)) \in \rho$, then $(T_0(x,n,d), T_0(s,m,c), T_0(x,m,c)) \in \rho^*$ and $T_0(x,n,d) \leq T_0(x,m,c)$ when $x < s$, $T_0(x,n,d) \geq T_0(x,m,c)$ when $x > s$; if $(T_0(t,n,d), T_0(t,m,c), T_0(s,m,c)) \in \rho$, then $(T_0(x,n,d), T_0(x,m,c), T_0(s,m,c)) \in \rho^*$ and $T_0(x,m,c) \leq T_0(x,n,d)$ when $x < s$, $T_0(x,m,c) \geq T_0(x,n,d)$ when $x > s$; if $(T_0(t,m,c), T_0(t,n,d), T_0(s,m,c)) \in \rho$, then $(T_0(x,m,c), T_0(x,n,d), T_0(s,m,c)) \in \rho^*$ and $T_0(x,m,c) \geq T_0(x,n,d)$ when $x < s$, $T_0(x,m,c) \leq T_0(x,n,d)$ when $x > s$.

If $u < T_0(t,m,c) < T_0(s,m,c)$, then the result may be proved in a similar way.

3.3.8. An example. The following is an example of a strongly ordered A.H. plane with an almost-strongly ordered (but not strongly ordered) coordinate biternary ring.

Let $H = \mathbb{R} \times \mathbb{R}$ with coordinatewise addition and multiplication defined by

$$(A_1,A_2)\cdot(B_1,B_2) = (A_1B_1, A_1B_2 + A_2B_1) \quad \text{for any } A_1,A_2,B_1,B_2 \in \mathbb{R}.$$

Then H is an A.H. ring with unique maximal ideal $\eta = \{(0,A) \mid A \in \mathbb{R}\}$, unit $(1,0)$ (which may be written $\mathbf{1}$) and zero $(0,0)$ (written $\mathbf{0}$). Two elements of H, (A_1,A_2) and (B_1,B_2) are neighbours if and only if $(A_1,A_2) - (B_1,B_2) \in \eta$ (ie., $A_1 = B_1$) and we put $\overline{(A_1,A_2)} = \{(A_1,B_2) \mid B_2 \in \mathbb{R}\}$. In particular, $\bar{\mathbf{1}} = \{(1,A) \mid A \in \mathbb{R}\}$ and $\bar{\mathbf{0}} = \eta$. We define an order relation on H in the following way $(A_1,A_2) < (B_1,B_2)$ if and only if $A_1 < B_1$ or $A_1 = B_1$ and $A_2 < B_2$. We put $a = (A_1,A_2)$, $b = (B_1,B_2)$, etc.

Consider an incidence structure with parallelism $\mathcal{H} = \langle \mathbb{P}, \mathbb{L}, I, \| \rangle$ where

$\mathbb{P} = H \times H$;

$\mathbb{L} = \mathbb{L}_1 \cup \mathbb{L}_2 \cup \mathbb{L}_3$

$= \{L[m,n]_1 \mid m \in \eta,\ n \in H\} \cup \{L[m,n]_2 \mid (m,n) \in (H \times H) \setminus (\eta \times \bar{\mathbf{1}})\}$
$\cup \{L[m,n]_3 \mid (m,n) \in \eta \times \bar{\mathbf{1}}\}$

where $L[m,n]_1 = \{(ym + n, y) \mid y \in H\}$

$L[m,n]_2 = \{(x, xm + n) \mid x \in H\}$

$L[m,n]_3 = \{(x, -xm + n) \mid x \in H\}$

I is set inclusion;

$\| \subset \mathbb{L} \times \mathbb{L}$ such that

$L[m,n]_1 \| L[p,q]_1$ if and only if $m = p$,

$L[m,n]_1 \nparallel L[p,q]_i$ if $i = 1$,

$L[m,n]_i \| L[p,q]_j$ for $i,j, \in \{2,3\}$ if and only if $m = p$.

Then $\mathcal{H}$ is an A.H. plane.

We define a total ordering $<$ on each line of $\mathcal{H}$ by ordering the points on lines of the first kind by their second coordinates and the points on lines of the second or third kinds by their first coordinates. We define an order relation ρ on $\mathbb{P}$ by: for three mutually distinct collinear points (a_1,a_2), (b_1,b_2), (c_1,c_2), $((a_1,a_2), (b_1,b_2), (c_1,c_2)) \in \rho$ if and only if either $(a_1,a_2) < (b_1,b_2) < (c_1,c_2)$ or $(a_1,a_2) > (b_1,b_2) > (c_1,c_2)$. Then $\mathcal{H}$ is a strongly ordered A.H. plane.

We coordinatize $\mathcal{H}$ in the usual way (cf. 2.1 and 2.2) using the triangle $((0,0), (0,0))$, $((1,0), (0,0))$, $((0,0), (1,0))$ and the elements of $L[(1,0), (0,0)]_2 = \{((A_1,A_2), (A_1,A_2)) \mid A_1, A_2 \in \mathbb{R}\}$ (abbreviated as (A_1,A_2)) and hence the elements of H as coordinates. We observe that the point $((A_1,A_2), (B_1,B_2))$ has coordinates $((A_1,A_2), (B_1,B_2))$; the line $L[(0,A_2), (B_1,B_2)]_1$ has coordinates $[(0,A_2), (B_1,B_2)]_1$ and the line $L[(A_1,A_2), (B_1,B_2)]_i$ $(i = 2,3)$ has coordinates $[(A_1,A_2), B_1,B_2)]_2$. Therefore as $T_1 : H^3 \longrightarrow H$ is such that $y = T_1(a,b,c)$ if and only if $(a,y)\ I\ [b,c]_2$ we have

$$T_1(a,b,c) = \begin{cases} ab + c & \text{if } (b,c) \notin \eta \times \overline{I} \\ -ab + c & \text{if } (b,c) \in \eta \times \overline{I} \end{cases}$$

Clearly $D_+ = \eta$ and as $T_0 : H \times D_+ \times H \longrightarrow H$ is defined such that

$x = T_0(a,b,c)$ if and only if $(x,a)\ I\ [b,c]_1$ (cf. 2.1), we have $T_0(a,b,c) = ab + c$. Using the (lexiographic) ordering already defined on H, we have $< H, T_1, T_0, 0, 1, < >$ is an ordered biternary ring satisfying (OM3)*'. In addition, (OM3)* is satisfied for all choices of m, n, c, d except $(m,c), (n,d) \in \eta \times \bar{1}$.

Let $m = (0,M)$, $n = (0,N)$, $c = (1,C)$, $d = (1,D)$ with $m < n$ which implies $M < N$. Suppose $T_1(s,m,c) = T_1(s,n,d)$ for some $s \in H$; ie., $(1,-S_1M + C) = (1,-S_1N + D)$. Then for any $p > s$, $p \not\sim s$, we have $P_1 > S_1$. Hence

$$\begin{aligned} T_1(p,m,c) &= \bar{1}pm + c \\ &= (1,-P_1M + C) \\ &= (1,-S_1M + (-P_1 + S_1)M + C) \\ &= (1,-S_1N + D + (-P_1 + S_1)M) \\ &> (1,-S_1N + D + (-P_1 + S_1)N) \\ &= (1,-P_1N + D) \\ &= T_1(p,n,d). \end{aligned}$$

For any $p \sim s$, $T_1(p,m,c) = T_1(p,n,d)$ and for any $p < s$, $p \not\sim s$, $T_1(p,m,c) < T_1(p,n,d)$. Thus $< H, T_1, T_0, 0, 1, < >$ is an almost-strongly ordered biternary ring which is not strongly ordered.

3.3.9. Remark. An almost-strongly ordered A.H. ring (ie., the coordinate biternary ring of a translation plane) is strongly ordered.

3.3.10. Remark. An ordering ρ on an A.H. plane $\mathcal{H}$ induces an ordering $\bar{\rho}$ on the associated ordinary affine plane $\bar{\mathcal{H}}$. The ordered affine plane $\bar{\mathcal{H}}$ may be coordinatized by the ternary ring

$\bar{\mathcal{H}} = < \bar{H}, \bar{T}_1, \bar{0}, \bar{1} >$ (cf. [10], 5).

3.4. Projectively ordered A.H. planes.

3.4.1. An A.H. plane is said to be projectively ordered if it is strongly ordered and satisfies the following additional axiom.

(07) Let $A, B, C \; I \; m$; $D \not\sim X$ for any $X \; I \; m$. If $(A,B,C) \in \rho$, then $(L(E,AD) \wedge m, L(E,BD) \wedge m, L(E,CD) \wedge m) \in \rho$ for any $E \; I \; L(D,m)$.

3.4.2. Theorem. A projective ordering on an A.H. plane $\mathcal{H}$ induces a strong ordering on each of its coordinate biternary rings.

Proof. As a projective ordering on $\mathcal{H}$ is a strong ordering, it induces an almost-strong ordering on each coordinate biternary ring by 3.3.7. We now verify (OM3)* and (OM3)*'.

Consider $m < n$, where $T_1(s,m,c) = T_1(s,n,d)$ for some $s, c, d \in H$. If $m \not\sim n$, then (OM3) implies $T_1(a,m,c) > T_1(a,n,d)$ and $T_1(b,m,c) < T_1(b,n,d)$ for all $a < s < b$. Therefore let $m < n$ and $m \sim n$. Take any $t \in H$; $t \not\sim 0, s$; $t > s$. By (OM3)**, it will be sufficient to show $T_1(t,m,c) < T_1(t,n,d)$.

Case 1: Suppose $m, n \not\sim 0$. Then either $0 < m < n$ or $m < n < 0$. First we assume $0 < m < n$. Since $T_1(1,0,0) < T_1(1,m,0) < T_1(1,n,0)$, (OM3) and (OM3)** imply $(T_1(t,0,0),T_1(t,m,0),T_1(t,n,0)) \in \rho^*$.

If $T_1(s,m,0) = T_1(s,n,0)$ (which implies $s \sim 0$), then $T_1(s,m,0) \sim 0 \not\sim T_1(t,m,0) \sim T_1(t,n,0)$. By the convexity of η and the previous paragraph, $(T_1(t,0,T_1(s,m,0)), T_1(t,m,0), T_1(t,n,0)) \in \rho^*$.

If we assume $m < n < 0$, then we may use an argument similar to the one employed above.

<u>Case 2</u>: Suppose $m, n \sim 0$. Then $m < n < 1$.

If $T_1(s,m,0) = T_1(s,n,0)$ (which implies $s \sim 0$), then $T_1(1,m,0) < T_1(1,n,0)$ and (OM3)** imply $T_1(t,m,0) \leq T_1(t,n,0)$. However $T_1(t,m,0) \neq T_1(t,n,0)$ as $(s,T_1(s,m,0)) \in [m,0]_2 \wedge [n,0]_2$ and $s \not\sim t$. Let $w \in H$ such that $T_1(s,1,w) = T_1(s,m,0)$. Then by (OM3), $T_1(t,1,w) > T_1(t,n,0)$. By (O7) with $D = (s,T_1(s,m,0))$ and $E = (s,T_1(s,m,c))$, we obtain $(T_1(t,m,c), T_1(t,n,d), T_1(t,1,v)) \in \rho$ where $T_1(s,1,v) = T_1(s,m,c)$. By (OM3), $T_1(t,m,c) < T_1(t,1,v)$; hence $T_1(t,m,c) < T_1(t,n,d)$.

If $T_1(s,m,c) \neq T_1(s,n,0)$, then we use the method of Case 1 with $[0,T_1(s,m,u)]_2$ and $[0,T_1(s,m,c)]_2$ replaced by $[1,w]_2$ and $[1,v]_2$ respectively where $T_1(s,1,w) = T_1(s,m,u)$ and $T_1(s,1,v) = T_1(s,m,c)$.

Thus (OM3)* holds.

Now consider $m < n$ where $T_0(s,m,c) = T_0(s,n,d)$ for some $s, c, d \in H$. By definition $m, n \in \eta$. Then $m < n < 1$. Take any $t \in H$; $t \not\sim 0, s$ and $t > s$. By (OM3)**', it will be sufficient to show $T_0(t,m,c) < T_0(t,n,d)$.

If $T_0(s,m,0) = T_0(s,n,0)$ (which implies $s \sim 0$), then $(T_0(t,m,0), T_0(t,n,0), x) \in \rho$ where $(x,t) = [0,t]_2 \wedge L((T_0(s,m,0),s),[1,0]_2)$. By (O7) with $D = (T_0(s,m,0),s)$ and $E = (T_0(s,m,c),s)$, $(T_0(t,m,c), T_0(t,n,d), y) \in \rho$ where $(y,t) = [0,t]_2 \wedge L((T_0(s,m,c),s),[1,0]_2)$. However

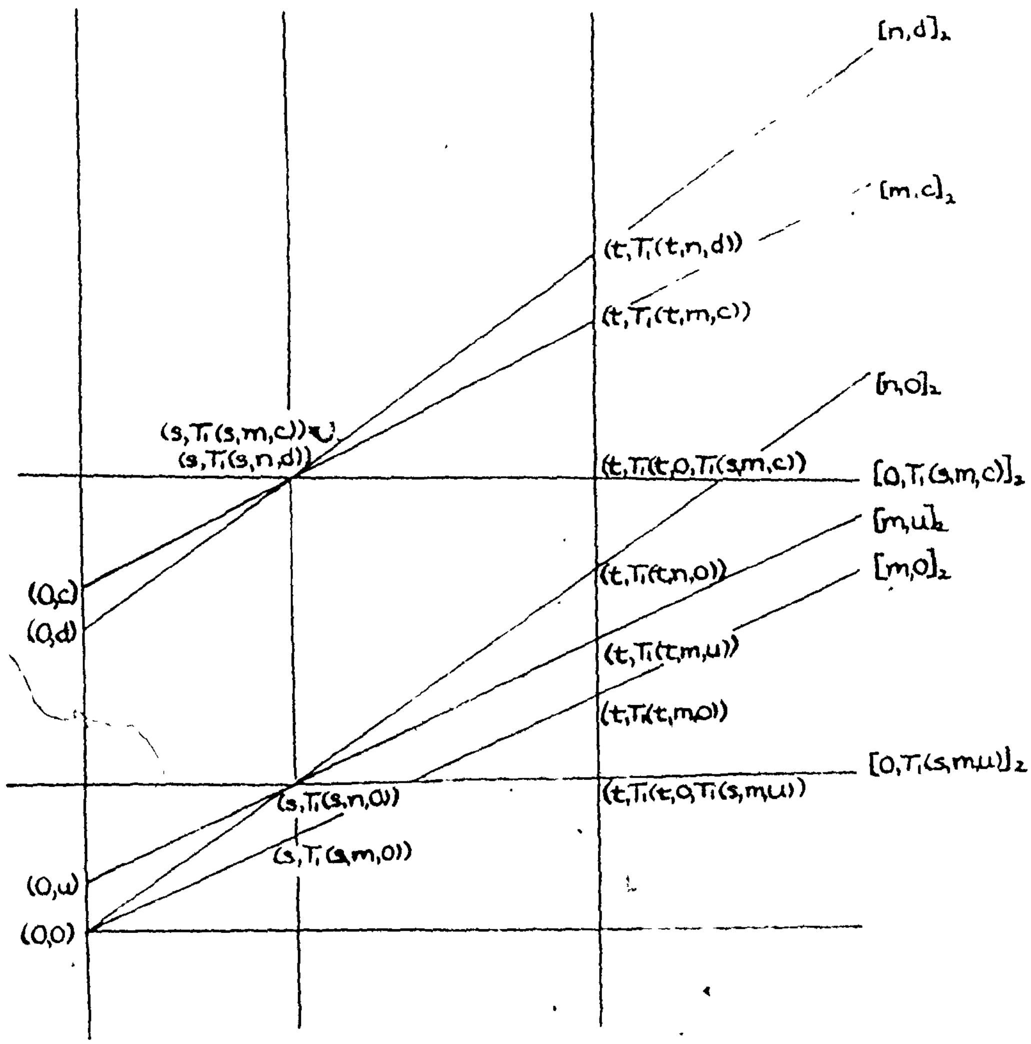


Figure 3.3
(Lemma 3.4.2,
Case 1, $T_1(s,m,0) < T_1(s,n,0)$)

However, as $(s,T_1(s,m,0)) \in [m,0]_2 \wedge [n,0]_2$ and $s \not\sim t$, $T_1(t,m,0) \neq T_1(t,n,0)$; hence $(T_1(t,0,T_1(s,m,0)), T_1(t,m,0), T_1(t,n,0)) \in \rho$. Using (07) with $D = (s,T_1(s,m,0))$ and $E = (s,T_1(s,m,c))$, we obtain $((t,T_1(t,0,T_1(s,m,0))), (t,T_1(t,m,c)), (t,T_1(t,n,d))) \in \rho$; ie., $(T_1(s,m,c), T_1(t,m,c), T_1(t,n,d)) \in \rho$. By 3.1.4(3), $T_1(s,m,c) < T_1(t,m,c)$; thus $T_1(t,m,c) < T_1(t,n,d)$.

Now assume $T_1(s,m,0) < T_1(s,n,0)$. Then $T_1(1,m,0) = m < n = T_1(1,n,0)$ and (OM3)** imply $s > 0$. Therefore $0 < s < t$. There exists $u \in H$ such that $T_1(s,m,u) = T_1(s,n,0)$. Since $T_1(s,m,u) = T_1(s,n,0) > T_1(s,m,0)$, (OM2) implies $u > 0$. As $((0,0), (s,T_1(s,n,0)), (t,T_1(t,n,0))) \in \rho$, the parallel projections $[n,0]_2 \xrightarrow{[m,0]_2} [0,t]_1 \xrightarrow{[0,0]_2} [1,0]_2$, yield $(T_1(t,m,0), T_1(t,m,u), T_1(t,n,0)) \in \rho^*$. Since $(s,T_1(s,m,u)) \in [m,u]_2 \wedge [n,0]_2$ and $s \not\sim t$ imply $T_1(t,m,u) \neq T_1(t,n,0)$ and (OM2) implies $T_1(t,m,0) < T_1(t,m,u)$, we have $T_1(t,m,0) < T_1(t,m,u) < T_1(t,n,0)$. By 3.1.4(3), $T_1(s,m,u) < T_1(t,m,u)$; hence $T_1(t,0,T_1(s,m,u)) < T_1(t,m,u) < T_1(t,n,0)$. By (07) with $D = (s,T_1(s,m,u))$ and $E = (s,T_1(s,m,c))$, $((t,T_1(t,0,T_1(s,m,c))), (t,T_1(t,m,c)), (t,T_1(t,n,d))) \in \rho$; ie., $(T_1(s,m,c), T_1(t,m,c), T_1(t,n,d)) \in \rho$. As $T_1(s,m,c) < T_1(t,m,c)$ by 3.1.4(3), we obtain $T_1(t,m,c) < T_1(t,n,d)$.

If $T_1(s,m,0) > T_1(s,n,0)$, then (OM3)** implies $s < 0$. Therefore either $s < 0 < t$ or $s < t < 0$. In either case, a discussion similar to the one in the previous paragraph yields the same result.

$T_0(t,m,c) \sim T_0(s,m,c) < y$; hence $T_0(t,m,c) < T_0(t,n,d)$.

If $T_0(s,m,0) < T_0(s,n,0)$, then (OM3)**' implies $0 < s < t$. Let $u \in H$ such that $T_0(s,m,u) = T_0(s,n,0)$; hence by (OM2)', $u > 0$. The parallel projections $[n,0]_1 \xrightarrow[{[m,0]_1}]{} [0,t]_2 \xrightarrow[{[0,0]_1}]{} [1,0]_2$ yield $(T_0(t,m,0), T_0(t,m,u), T_0(t,n,0)) \in \rho^*$; however (OM2)' and $(T_0(s,m,u),s) \in [m,u]_1 \wedge [n,0]_1$, $s \not\sim t$ imply $T_0(t,m,0) < T_0(t,m,u) < T_0(t,n,0)$. Let $(x,t) = [0,t]_2 \wedge L((T_0(s,m,u),s), [1,0]_2)$. Then $T_0(t,m,u) < T_0(t,n,0) < x$. Using (O7) with $D = (T_0(s,m,u),s)$ and $E = (T_0(s,m,c),s)$, we obtain $(T_0(t,m,c), T_0(t,n,d), y) \in \rho$ where $(y,t) = [0,t]_2 \wedge L((T_0(s,m,c),s),[1,0]_2)$. As $T_0(t,n,d) \sim T_0(s,m,c) < y$, $T_0(t,m,c) < T_0(t,n,d)$.

If $T_0(s,m,0) > T_0(s,n,0)$, then (OM3)** implies either $s < 0 < t$ or $s < t < 0$. In either case, a discussion similar to the one above yields the desired result.

Thus (OM3)*' holds.

<u>3.4.3</u>. <u>Theorem</u>. A strong ordering on the biternary ring $\mathcal{B}$ of an A.H. plane $\mathcal{H}$ induces a projective ordering on $\mathcal{H}$.

<u>Proof</u>. By [10], 9.5, a strong ordering on $\mathcal{B}$ induces a strong ordering on $\mathcal{H}$. We now verify (O7).

Take three points A, B, C I m such that $(A,B,C) \in \rho$ and points D and E such that $D \not\sim X$ for any X I m, E I $L(D,m)$. Clearly $E \not\sim X$ for any X I m also.

<u>Case 1</u>: If $A \not\sim B \not\sim C \not\sim A$, then $(L(E,AD) \wedge m, L(E,BD) \wedge m, L(E,CD) \wedge m) \in \rho$ by 3.1.2(2).

Case 2: At least two of A, B, C are neighbours and $\mathcal{m} = [p,q]_1$ for some $p \in \eta$, $q \in H$. Then $A = (T_0(a,p,q),a)$, $B = (T_0(b,p,q),b)$, $C = (T_0(c,p,q),c)$ for some $a, b, c \in H$ and AD, BD, CD are $[r_i,s_i]_2$ $(i=1,2,3)$ respectively for some r_i, $s_i \in H$. Since $(A,B,C) \in \rho$, either $a < b < c$ or $a > b > c$. Without loss of generality, we shall assume $a < b < c$. Since $D \; I \; [r_i,s_i]_2$, $D = (d,T_1(d,r_i,s_i))$ $(i=1,2,3)$ for some $d \in H$. As $D \not\sim X$ for any $X \; I \; \mathcal{m}$, $d \not\sim q \sim T_0(a,p,q) \sim T_0(b,p,q) \sim T_0(c,p,q)$.

First we assume $T_0(a,p,q) = T_0(b,p,q) = T_0(c,p,q)$. Then (OM3)* implies $r_1 < r_2 < r_3$ if $d < q$ and $r_1 > r_2 > r_3$ if $d > q$.

Now suppose at least two of $T_0(a,p,q)$, $T_0(b,p,q)$, $T_0(c,p,q)$ are distinct. Since $a < b < c$, the parallel projections

$$\mathcal{k} \xrightarrow[{[0,0]_2}]{} [p,q]_1 \xrightarrow[{[0,0]_1}]{} \mathcal{k}$$ yield

$(T_0(a,p,q),\ T_0(b,p,q),\ T_0(c,p,q)) \in \rho^*$. Since at least two of these are distinct, 3.2.3 implies $T_0(a,p,q) \neq T_0(c,p,q)$. As

$[p,q]_1 \sim [0,T_0(a,p,q)]_1 \not\sim [r_3,s_3]_2$,

$C = (T_0(c,p,q),c) \sim (T_0(a,p,q),T_1(T_0(a,p,q),r_3,s_3) \not\sim D$. Therefore, on the line $[r_3,s_3]_2$, either $(C,\ (T_0(a,p,q),T_1(T_0(a,p,q),r_3,s_3)),\ D) \in \rho$ or $((T_0(a,p,q),T_1(T_0(a,p,q),r_3,s_3)),\ C,\ D) \in \rho$. If the former [latter] holds, then the parallel projection

$$[r_3,s_3]_2 \xrightarrow[{[r_2,s_2]_2}]{} [0,T_0(a,p,q)]_1$$ yields

$(G,(T_0(a,p,q),T_1(T_0(a,p,q),r_3,s_3)),(T_0(a,p,q),T_1(T_0(a,p,q),r_2,s_2))) \in \rho^*$
$[((T_0(a,p,q),T_1(T_0(a,p,q),r_3,s_3)),G,(T_0(a,p,q),T_1(T_0(a,p,q),r_2,s_2))) \in \rho^*]$
where $G = L(C,[r_2,s_2]_2) \wedge [0,T_0(a,p,q)]_1$. However $D \not\sim C$ which implies $G \neq (T_0(a,p,q),T_1(T_0(a,p,q),r_2,s_2))$. Since $(A,B,C) \in \rho$, the

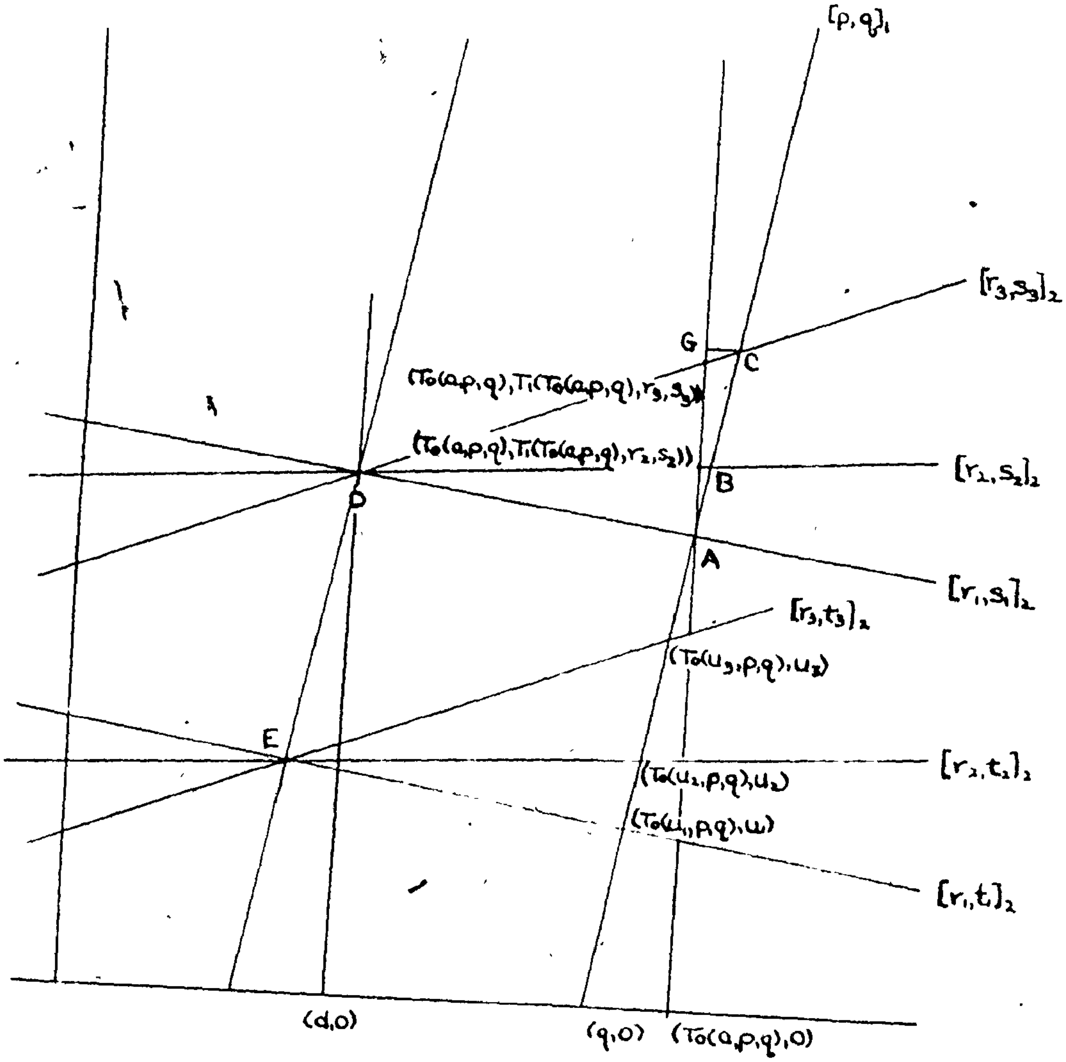


Figure 3.4
(Lemma 3.4.3, Case 2)

parallel projection $[p,q]_1 \xrightarrow[{[r_2,s_2]_2}]{} [0,T_0(a,p,q)]_1$ yields $(A,(T_0(a,p,q),T_1(T_0(a,p,q),r_2,s_2)),G) \in \rho$. Therefore $(A,(T_0(a,p,q),T_1(T_0(a,p,q),r_2,s_2)),(T_0(a,p,q),T_1(T_0(a,p,q),r_3,s_3))) \in \rho$. As above, (OM3)* implies $r_1 < r_2 < r_3$ if $d < q$ and $r_1 > r_2 > r_3$ if $d > q$.

Let $L(E,AD)$, $L(E,BD)$, $L(E,CD)$ be $[r_i,t_i]_2$ $(i=1,2,3)$ respectively and $E = (e,T_1(e,r_i,t_i))$ $(i=1,2,3)$. Let $[p,q]_1 \wedge [r_i,t_i]_2 = (T_0(u_i,p,q),u_i)$ $(i=1,2,3)$. Then by (OM3)*,

$$((T_0(u_1,p,q),T_1(T_0(u_1,p,q),r_1,t_1)),\ (T_0(u_1,p,q),T_1(T_0(u_1,p,q),r_2,t_2)),$$
$$(T_0(u_1,p,q),T_1(T_0(u_1,p,q),r_3,t_3))) \in \rho.$$

If $T_0(u_1,p,q) = T_0(u_2,p,q) = T_0(u_3,p,q)$, we are finished. Otherwise, $T_0(u_1,p,q) \neq T_0(u_3,p,q)$ as above and a discussion similar to the one used in the last paragraph implies

$(L(E,AD) \wedge m,\ L(E,BD) \wedge m,\ L(E,CD) \wedge m) \in \rho$.

<u>Case 3</u>: At least two of A, B, C are neighbours and $m = [p,q]_2$. Then $A = (a,T_1(a,p,q))$, $B = (b,T_1(b,p,q))$, $C = (c,T_1(c,p,q))$ for some $a, b, c \in H$. Since at least two of A, B, C are neighbours, we may assume $A \sim B$. (If $A \sim C$, then $(A,B,C) \in \rho$ implies $A \sim B \sim C$.) Then $AD \sim BD$ and both lines are of the form $[r_i,s_i]_2$ $(i=1,2)$ for some $r_1 \sim r_2 \not\sim p$, $s_i \in H$ or of the form $[r_i,s_i]_1$ $(i=1,2)$ for some $r_i \in n$, $s_i \in H$.

First, we assume AD and BD are $[r_i,s_i]_2$ $(i=1,2)$ respectively. Suppose also that CD is $[r_3,s_3]_2$ for some $r_3 \not\sim p$. Then $D = (d,T_1(d,r_i,s_i))$ $(i=1,2,3)$. If $(a,c,d) \in \rho$ or $(c,a,d) \in \rho$, we may use the methods of Case 2 to obtain the desired result. Since

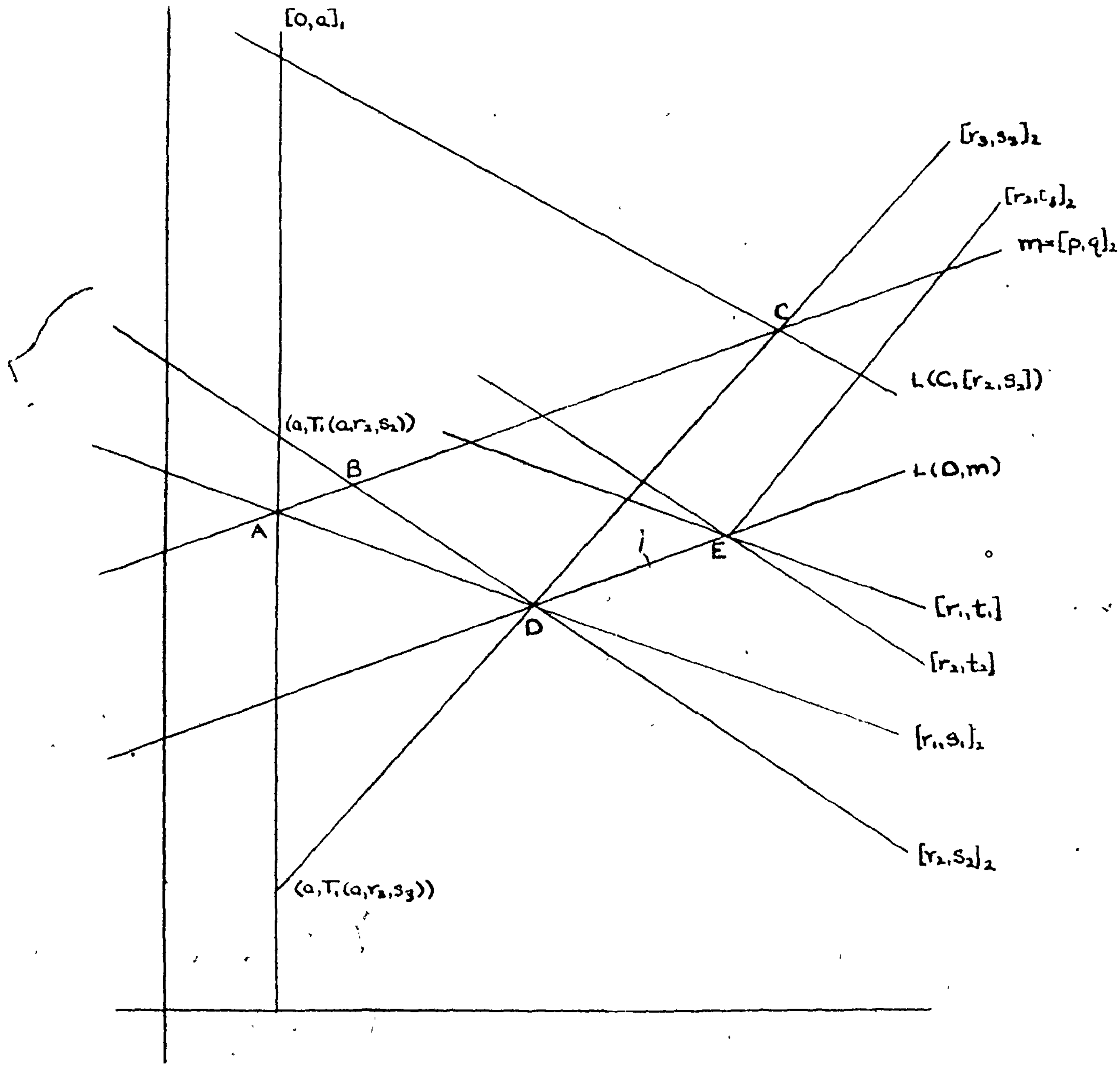


Figure 3.5
(Lemma 3.4.3, Case 3)

$A \sim C \not\sim D$ would imply $(a,c,d) \in \rho$ or $(c,a,d) \in \rho$ by 3.1.4(6), we need only consider the case where $A \not\sim C$ (ie., $[r_3,s_3]_2 \not\sim [r_1,s_1]_2$ also) and $(a,d,c) \in \rho$. As $(a,d,c) \in \rho$ implies

$((a,T_1(a,r_3,s_3)),\ (d,T_1(d,r_3,s_3)),\ C) \in \rho$, the parallel projection

$[r_3,s_3]_2 \xrightarrow[{[r_2,s_2]_2}]{} [0,a]_1$ yields

$(L(C,[r_2,s_2]_2) \wedge [0,a]_1,\ (a,T_1(a,r_2,s_2)),\ (a,T_1(a,r_3,s_3))) \in \rho$. The parallel projection $m \xrightarrow[{[r_2,s_2]_2}]{} [0,a]_1$ yields

$(L(C,[r_2,s_2]_2 \wedge [0,a]_1,\ (a,T_1(a,r_2,s_2)),\ A) \in \rho$. By (A5)',

$(a,T_1(a,r_3,s_3)) \not\sim A \sim (a,T_1(a,r_2,s_2))$; hence

$((a,T_1(a,r_3,s_3)),\ A,\ (a,T_1(a,r_2,s_2))) \in \rho$. By (OM3)*, either

$r_3 < r_1 < r_2$ or $r_3 > r_1 > r_2$.

Suppose $r_3 < r_1 < r_2$ and $a > d$ $[a < d]$. Then $(a,d,c) \in \rho$ implies $a > d > c$ $[a < d < c]$. We have $T_1(c,p,q) = T_1(c,r_3,s_3)$ and by (OM3), $T_1(a,r_3,s_3) < T_1(a,r_1,s_1) = T_1(a,p,q)$ and $a > c$ $[T_1(a,r_3,s_3) > T_1(a,r_1,s_1) = T_1(a,p,q)$ and $a < c]$; hence by (OM3)*, $r_3 < p$. Similarly, we have $T_1(a,p,q) = T_1(a,r_1,s_1)$ and $T_1(c,r_1,s_1) < T_1(c,r_3,s_3) = T_1(c,p,q)$ where $c < a$ $[T (c,r ,s) > T (c,r ,s) = T (c,p,q)$ where $c > a]$; hence by (OM3)* $r_1 > p$. Thus $r_3 < p < r_1 < r_2$.

Similarly, $r_3 > r_1 > r_2$ implies $r_3 > p > r_1 > r_2$.

Suppose $r_3 < p < r_1 < r_2$ $[r_3 > p > r_1 > r_2]$. Let $L(E,AD) \wedge m = (a',T_1(a',p,q))$, $L(E,BD) \wedge m = (b',T_1(b',p,q))$, $L(E,CD) \wedge m = (c',T_1(c',p,q))$ and $L(E,AD)$, $L(E,BD)$, $L(E,CD)$ be $[r_i,t_i]_2$ $(i=1,2,3)$ respectively. If $e < a'$, then

$T_1(e,p,q) > T_1(e,r_1,t_1) = T_1(e,r_3,t_3)$

$[T_1(e,p,q) < T_1(e,r_1,t_1) = T_1(e,r_3,t_3)]$ which by (OM3)* implies $c' < e$. However, $e < a'$ also implies

$T_1(e,r_2,t_2) = T_1(e,r_1,t_1) < T_1(e,p,q)$

$[T_1(e,r_2,t_2) = T_1(e,r_1,t_1) > T_1(e,p,q)]$ and

$T_1(a',r_2,t_2) > T_1(a',r_1,t_1) = T_1(a',p,q)$

$[T_1(a',r_2,t_2) < T_1(a',r_1,t_1) = T_1(a',p,q)]$; hence by (OM3)* $e < b'$ and $b' < a'$. Thus $c' < b' < a'$. Similarly, if $e > a$, we obtain $c' > b' > a'$. Hence $(L(E,AD) \wedge m, L(E,BD) \wedge m, L(E,CD) \wedge m) \in \rho$.

We still assume that AD and BD are $[r_i,s_i]_2$ $(i=1,2)$ respectively where $r_i \not\sim p$, $s_i \in H$, but suppose that $CD = [u,v]_1$ for some $u \in \eta$, $v \in H$. Then $C = (T_0(c,u,v),c)$ for some $c \in H$. Since $(A,B,C) \in \rho$ and $a, b \not\sim d \neq T_0(T_1(d,r_1,s_1),u,v) \sim T_0(c,u,v)$, we have $(a,b,d) \in \rho$.

Suppose $r_1 < r_2$ and $a < b < d$ $[a > b > d]$. Then

$T_1(b,p,q) = T_1(b,r_2,s_2) < T_1(b,r_1,s_1)$

$[T_1(b,p,q) = T_1(b,r_2,s_2) > T_1(b,r_1,s_1)]$ which implies $p < r_1$. Thus $p < r_1 < r_2$.

Suppose $r_2 < r_1$ and $a < b < d$ $[a > b > d]$. Then as above $p > r_1 > r_2$.

If $p < r_1 < r_2$ $[p > r_1 > r_2]$ and $e < a'$, then

$T_1(a',p,q) = T_1(a',r_1,t_1) < T_1(a',r_2,t_2)$

$[T_1(a',p,q) = T_1(a',r_1,t_1) > T_1(a',r_2,t_2)]$ which implies $b' < a'$. Since $b' < a'$, $T_1(b',r_1,t_1) < T_1(b',p,q) = T_1(b',r_2,t_2)$

$[T_1(b',r_1,t_1) > T_1(b',p,q) = T_1(b',r_2,t_2)]$; hence $b' > e$. Thus $e < b' < a'$. Similarly, if $e > a'$, we obtain $e > b' > a'$. However

$(e,b',a') \in \rho$ and $T_0(c',u,w) \sim w \sim T_0(T_1(e,r_1,s_1),u,w) = e \not\sim b', a'$, where $L(E,CD) = [u,w]_1$, imply $(T_0(c',u,w),b',a') \in \rho$ and $(L(E,AD) \wedge m, L(E,BD) \wedge m, L(E,CD)_a \wedge m) \in \rho$.

If AD and BD are of the form $[r_i,s_i]_1$ $(i=1,2)$, the methods used above along with (OM3)*' give the desired result.

3.4.4. Lemma. Let $\mathcal{H}$ be a projectively ordered A.H. plane. Let A, B, C I m such that $(A,B,C) \in \rho$ and D, E $\not\sim$ X for any X I m. Then $(L(E,AD) \wedge m, L(E,BD) \wedge m, L(E,CD) \wedge m) \in \rho$.

Proof. If E I L(D,m), then this is just (O7), so suppose E $\not{I}$ L(D,m).

Let $F = L(E,BD) \wedge L(D,m)$. Then by (O7), $(L(F,AD) \wedge m, L(F,BD) \wedge m, L(F,CD) \wedge m) \in \rho$. On the line L(E,BD) the points are ordered in one of three ways: 1) $(F, E, L(E,BD) \wedge m) \in \rho$; 2) $(E, F, L(E,BD) \wedge m) \in \rho$; 3) $(E, L(E,BD) \wedge m, F) \in \rho$.

Suppose first that $(F, E, L(E,BD) \wedge m) \in \rho$. The parallel projections $L(E,BD) \xrightarrow[AD]{} m$ and $L(E,BD) \xrightarrow[CD]{} m$ yield $(L(F,AD) \wedge m, L(E,AD) \wedge m, L(E,BD) \wedge m) \in \rho^*$ and $(L(F,CD) \wedge m, L(E,CD) \wedge m, L(E,BD) \wedge m) \in \rho^*$ respectively. Since $E \not\sim L(E,BD) \wedge m$, $L(E,AD) \wedge m \neq L(E,BD) \wedge m \neq L(E,CD) \wedge m$. Hence either $(L(F,AD) \wedge m, L(E,AD) \wedge m, L(E,BD) \wedge m) \in \rho$ or $L(F,AD) \wedge m = L(E,AD) \wedge m$ and either $(L(F,CD) \wedge m, L(E,CD) \wedge m, L(E,BD) \wedge m) \in \rho$ or $L(F,CD) \wedge m = L(E,CD) \wedge m$. Combining these results with $(L(F,AD) \wedge m, L(F,BD) \wedge m, L(F,CD) \wedge m) \in \rho$ we obtain $(L(E,AD) \wedge m, L(E,BD) \wedge m, L(E,CD) \wedge m) \in \rho$.

A similar discussion yields the same result in the other cases.

CHAPTER 4

Ordered P.H. planes and their embedded A.H. planes.

4.1. Ordered P.H. planes.

4.1.1. An ordering on a P.H. plane $\mathcal{P} = < \mathbb{P}, \mathbb{L}, I >$ is a quaternary relation on $\mathbb{P}$ satisfying the following axioms. If (A,B,C,D) lies in this relation, we write $AB \,\sigma\, CD$ (cf. [8], 7.2).

(OP1) If $AB \,\sigma\, CD$, then A, B, C, D are mutually distinct and collinear.

(OP2) $AB \,\sigma\, CD$ is equivalent to $CD \,\sigma\, AB$ and to $BA \,\sigma\, CD$.

(OP3) If A, B, C, D are mutually distinct and collinear, then one and only one of the relations $AB \,\sigma\, CD$, $AC \,\sigma\, BD$ or $AD \,\sigma\, BC$ holds.

(OP4) If $AC \,\sigma\, BD$ and $AD \,\sigma\, CE$, then $AD \,\sigma\, BE$, for any A, B, C, D, E mutually distinct and collinear points.

(OP5) At least one line is incident with at least four non-neighbouring points.

(OP6) If A, B, C, D are mutually distinct points of a line ℓ and A', B', C', D' are their images under a projection from centre P, then $AC \,\sigma\, BD$ implies $A'C' \,\sigma\, B'D'$.

A projection of a P.H. plane is a mapping from the points of a line ℓ to the points of a line m from a point P, such that $P \not\sim X$ for all $X \; I \; \ell, m$, given by $X \rightsquigarrow PX \wedge m$. We shall denote such a projection by $\ell \xrightarrow{P} m$ and call P the centre of the projection.

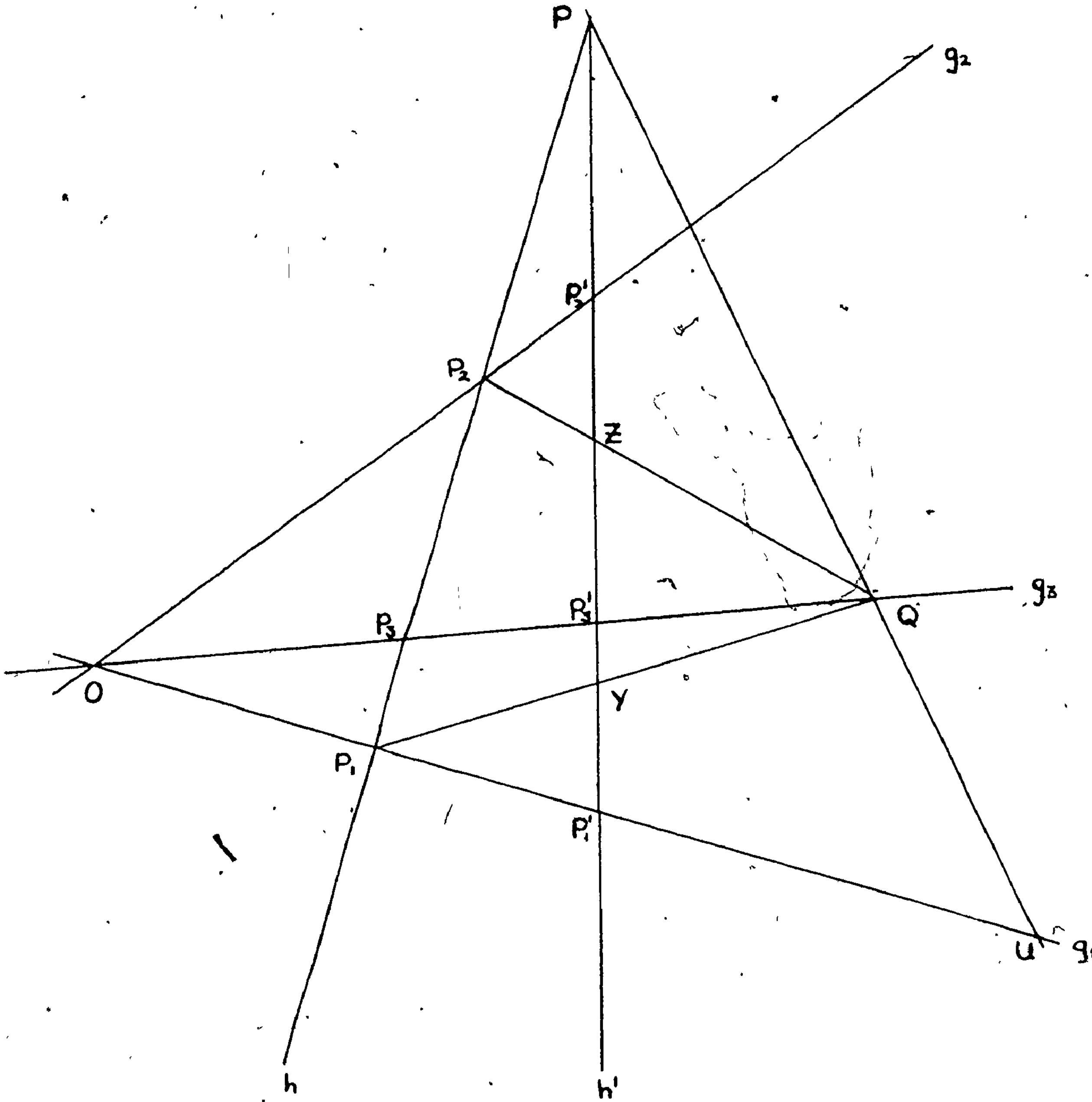


Figure 4.1
(Lemma 4.1.2)

4.1.2. Lemma. Let $g_1, g_2, g_3, h, h' \in \mathbb{L}$; $h \not\sim h'$; $P = h \wedge h'$; $g_i \not\sim g_j$ $(i \neq j;\ i,j \in \{1,2,3\})$; for $i = 1,2,3$, $g_i \not\sim h, h'$, $P_i = g_i \wedge h$, $P_i' = g_i \wedge h'$, $O\ I\ g_i$, $O \neq P_1'$, $P \not\sim P_i$. If $P_1P_2\ \sigma\ P_3P$, then $P_1'P_2'\ \sigma\ P_3'P$.

Proof. If $O \not\sim P_1, P_1'$, then the result is given by (OP6). We may therefore assume that O is a neighbour of P_1 or P_1'.

Take any $Q\ I\ g_3$ such that $Q \not\sim O, P_3, P_3'$. By (H6)', $Q \not\sim X$, for any $X\ I\ h, h'$. Since $P_1P_2\ \sigma\ P_3P$, the projection $h \xrightarrow{Q} h'$ yields $YZ\ \sigma\ P_3'P$ where $Y = P_1Q \wedge h'$ and $Z = P_2Q \wedge h'$. By (H6)', $P \not\sim X$, for any $X\ I\ g_1, g_2, g_3$. By (H5)', $PQ \not\sim g_1, g_2$. Let $U = PQ \wedge g_1$ and $V = PQ \wedge g_2$. We have three possibilities: 1) $OP_3'\ \sigma\ P_3Q$; 2) $OP_3\ \sigma\ P_3'Q$; 3) $OQ\ \sigma\ P_3P_3'$.

Suppose $OP_3'\ \sigma\ P_3Q$. Then the projections $g_3 \xrightarrow{P} g_1$ and $g_3 \xrightarrow{P} g_2$ yield $OP_1'\ \sigma\ P_1U$ and $OP_2'\ \sigma\ P_2V$ respectively. Since $OP_1'\sigma\ P_1U$, the projection $g_1 \xrightarrow{Q} h'$ yields $P_3'P_1'\ \sigma\ YP$. Since $OP_2'\ \sigma\ P_2V$, the projection $g_2 \xrightarrow{Q} h'$ yields $P_3'P_2'\ \sigma\ ZP$. Combining $YZ\ \sigma\ P_3'P$, $P_3'P_1'\ \sigma\ YP$ and $P_3'P_2'\ \sigma\ ZP$, we obtain $P_1'P_2'\ \sigma\ P_3'P$.

If $OP_3\ \sigma\ P_3'Q$, then the above discussion yields $P_3'Y\ \sigma\ P_1'P$ and $P_3'Z\ \sigma\ P_2'P$. Combining these with $YZ\ \sigma\ P_3'P$, we obtain $P_1'P_2'\ \sigma\ P_3'P$.

Finally, if $OQ\ \sigma\ P_3P_3'$, we obtain $P_3'P\ \sigma\ YP_1'$ and $P_3'P\ \sigma\ ZP_2'$; hence $P_1'P_2'\ \sigma\ P_3'P$.

4.1.3. Lemma. Let A, B, C, D be four mutually distinct, collinear points such that $A \sim B \not\sim C \not\sim D \not\sim A$. Then $AB \not\sigma CD$.

Proof. Assume $AB\ \sigma\ CD$. Let the line incident with A, B, C, D be ℓ. There exist lines m and n such that $A\ I\ m, n$ and $\ell \not\sim m \not\sim n \not\sim \ell$

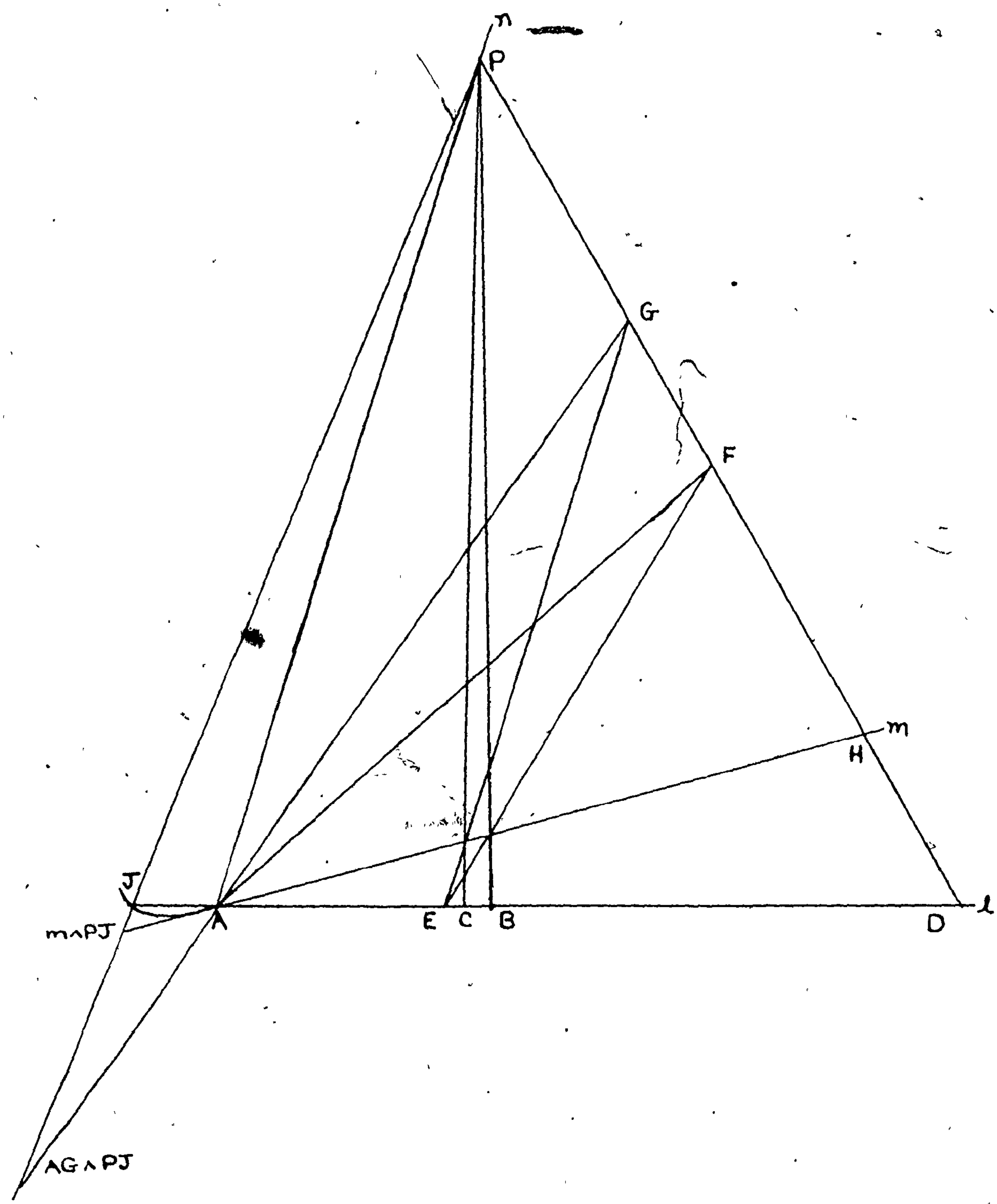


Figure 4.2
(Lemma 4.1.3)

and a point $P\ I\ n$ with $P \nsim A$. Then by (H6)', $P \nsim X$ for any $X\ I\ m, \ell$. Since $AB\ \sigma\ CD$, the projection $\ell \xrightarrow{P} m$ yields $A(PB \wedge m)\ \sigma\ (PC \wedge m)(PD \wedge m)$. As $A \sim B$, $PA \sim PB$ and $A \sim PB \wedge m$. By (OP5), there exists $E\ I\ \ell$ such that $E \nsim A, C, D$. By (H6)', $E \nsim X$ for any $X\ I\ m$, PD. Using the projection $m \xrightarrow{E} PD$, we obtain $DF\ \sigma\ GH$ where $F = E(PB \wedge m) \wedge PD$, $G = E(PC \wedge m) \wedge PD$ and $H = PD \wedge m$. By (H5)' and (H6)', $D \sim F \nsim G \nsim H \nsim D$. As $P \nsim D$, $P \nsim F$; as $H\ I\ m$, $P \nsim H$; as $C \nsim E$, $PC \nsim E(PC \wedge m)$, $P \nsim G$. As $A \nsim F$, there exists a unique line AF; however $A \nsim F \sim D$ implies $AF \sim \ell$ which implies there exists $J\ I\ \ell$, AF such that $J \neq A$. Clearly, $J \nsim C, D, E$. We have four possibilities: 1) $DG\ \sigma\ PH$ and $GH\ \sigma\ PF$; 2) $GH\ \sigma\ DP$ and $GF\ \sigma\ PH$; 3) $PD\ \sigma\ GH$ and $GP\ \sigma\ FH$; 4) $DH\ \sigma\ GP$ and $GH\ \sigma\ FP$.

Suppose $DG\ \sigma\ PH$ and $GH\ \sigma\ PF$. By 4.1.2 (with $h = PD$, $h' = PJ$, $g_1 = AG$, $g_2 = m$, $g_3 = \ell$), we obtain $J(AG \wedge PJ)\ \sigma\ P(m \wedge PJ)$. By 4.1.2 (with $h = PD$, $h' = PJ$, $g_1 = AG$, $g_2 = m$, $g_3 = AF$), we obtain $(AG \wedge PJ)(m \wedge PJ)\ \sigma\ PJ$; a contradiction.

The other three cases lead to a similar contradiction. Thus $AB \not\sigma CD$.

4.1.4. Lemma. Let A, B, C, D be four mutually distinct, collinear points such that $A \sim B \nsim C \sim D$. Then $AB \not\sigma CD$.

Proof. Suppose $AB\ \sigma\ CD$. By (OP5), there exists $X\ I\ AC$ such that $X \nsim A, C$. By (OP3) and (OP4), one of the following must hold: $AB\ \sigma\ XD$, $CD\ \sigma\ XA$ or $CD\ \sigma\ XB$; a contradiction to 4.1.3.

4.1.5. Lemma. Let $\mathcal{P}$ be an ordered P.H. plane. For any three

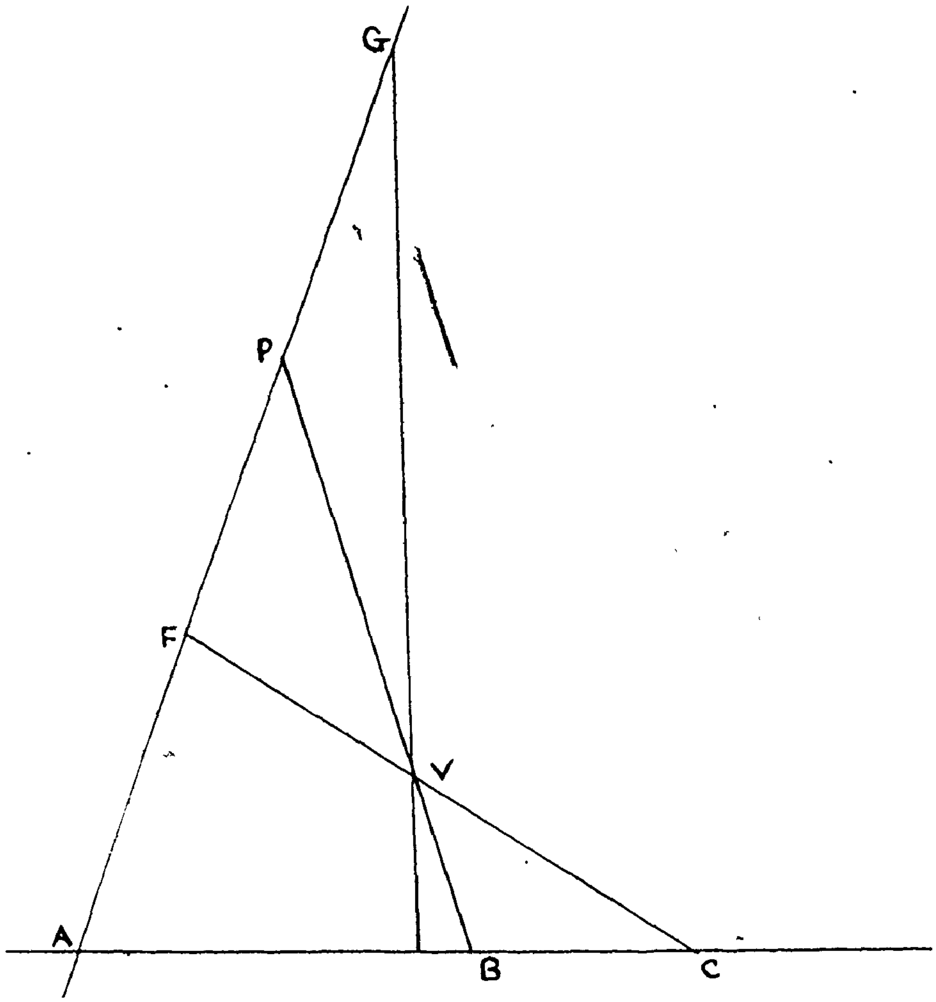


Figure 4.3
(Lemma 4.1.5,
$A \nless B \nless C \nless A$, Case 1)

distinct points A, B, C incident with some line ℓ, there exists a point $D \ I \ \ell$ such that $AB \ \sigma \ CD$.

Proof. As there exists a line incident with at least four non-neighbouring points, every line is incident with at least four non-neighbouring points.

First suppose A, B, C are pairwise not neighbours. There exists a point P such that $P \not\sim X$, for any $X \ I \ \ell$. Clearly, PA, PB, PC and ℓ are pairwise not neighbours. There also exist points $F, G \ I \ PA$ such that F, G, P, A are pairwise not neighbours. There are three possibilities: 1) $AP \ \sigma \ FG$; 2) $AF \ \sigma \ PG$; 3) $AG \ \sigma \ PF$.

Case 1: Suppose $AP \ \sigma \ FG$. If $F \sim C$, then $F \not\sim A$ implies $PA \sim \ell$ by (H6)'; a contradiction. If $PB \sim FC$, then $PB \not\sim PA$ implies $F \sim P$; a contradiction. Therefore there exists a unique point $V = PB \wedge FC$. If $F \sim V$, then $F \not\sim P$ implies $PA \sim PB$; another contradiction. If $FC \sim PA$, then $PA \not\sim PC$ implies $P \sim C$; again a contradiction. If $V \sim X$ for some $X \ I \ PA$, then $V \not\sim F$ implies $FC \sim PA$; a contradiction. If $V \sim C$, then $C \not\sim B$ implies $\ell \sim PB$; a contradiction. Finally, if $V \sim X$ for any $X \ I \ \ell$, then $V \not\sim C$ implies $FC \sim \ell$ which implies $F \sim A$ as $\ell \not\sim PA$; another contradiction. Therefore the map $PA \xrightarrow{V} \ell$ is a projection and by (OP6), $AP \ \sigma \ FG$ implies $AB \ \sigma \ C(GV \wedge \ell)$.

Case 2: Suppose $AF \ \sigma \ PG$. Using similar arguments to those employed in Case 1, we construct a unique point $V = FB \wedge PC$, such that $V \not\sim X$ for any $X \ I \ PA, \ell$. Under the projection $PA \xrightarrow{V} \ell$, $AF \ \sigma \ PG$ yields $AB \ \sigma \ C(GV \wedge \ell)$.

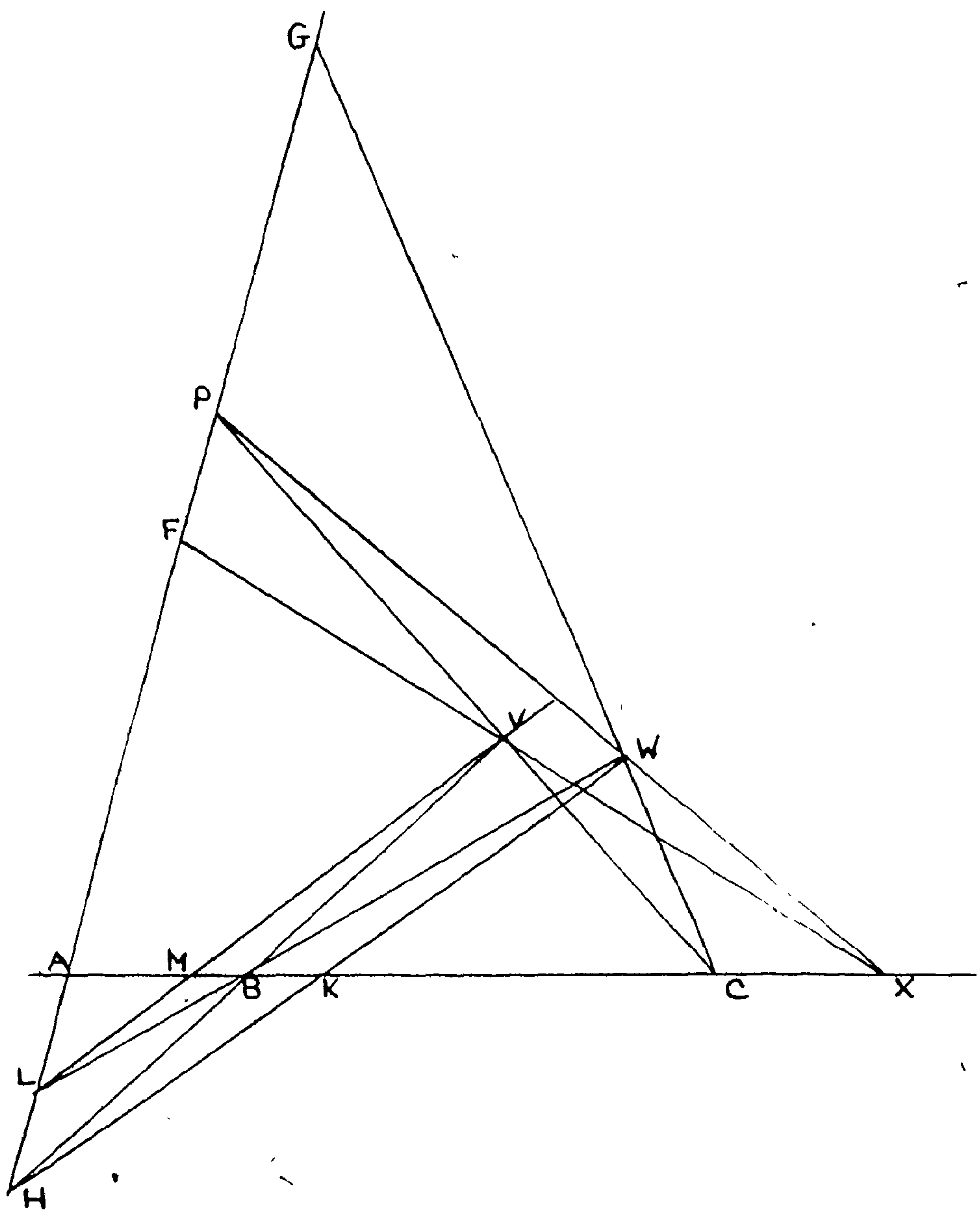


Figure 4.4
(Lemma 4.1.5,
A~B+C, Case 1)

Case 3: Suppose AG σ PF. As above, there exists a unique $V = PC \wedge BG$ such that $V \nmid X$ for any $X \, I \, PA, \ell$. Under the projection $PA \xrightarrow{V} \ell$, AG σ PF yields AB σ C(FV $\wedge \ell$).

Now suppose $A \sim B$ and $B \nmid C$. Then there exists $X \, I \, \ell$ such that $X \nmid A, C$. Then by 4.1.3, we have two possibilities: 1) AC σ BX or 2) AX σ BC. There exists P such that $P \nmid Q$ for any $Q \, I \, \ell$. In addition, there exist F, G I PA such that A, P, F, G are pairwise not neighbours. We have three possibilities: 1) AP σ FG; 2) AF σ PG; 3) AG σ PF.

First we assume AC σ BX.

Case 1: Suppose AP σ FG. As above, there exists a unique point $V = PC \wedge FX$ where $V \nmid Q$ for all $Q \, I \, \ell$, PA. Using the projection $\ell \xrightarrow{V} PA$, AC σ BX yields AP σ HF where $H = VB \wedge PA$ $(H \sim A)$. Therefore by (OP4) and 4.1.3, AP σ FG and AP σ HF imply AG σ HP. There exists $W = GC \wedge PX$ such that $W \nmid Q$ for any $Q \, I \, \ell$, PA. Then the projection $PA \xrightarrow{W} \ell$ yields AC σ KX where $K = HW \wedge \ell$ $(K \sim A)$. Hence either AB σ CK and we are finished or AK σ BC. Assume the latter holds. Using the projection $\ell \xrightarrow{W} PA$, we obtain AH σ LG where $L = WB \wedge PA$ $(L \sim A)$. In addition, AH σ LG and AG σ PH imply AH σ PL by (OP4). The projection $PA \xrightarrow{V} \ell$ then yields AB σ CM where $M = VL \wedge \ell$ $(M \sim A)$. Thus M is the required point.

Case 2: Suppose AF σ PG. There exists a unique point $V = FC \wedge GX$ such that $V \nmid Q$ for all $Q \, I \, \ell$, PA. Since AC σ BX, the projection $\ell \xrightarrow{V} PA$ yields AF σ HG where $H = BV \wedge PA$ $(H \sim A)$. From AF σ PG and AF σ HG, we obtain AP σ HF by (OP4) and 4.1.3. There exists a unique point $W = PC \wedge FX$ such that $W \nmid Q$ for any

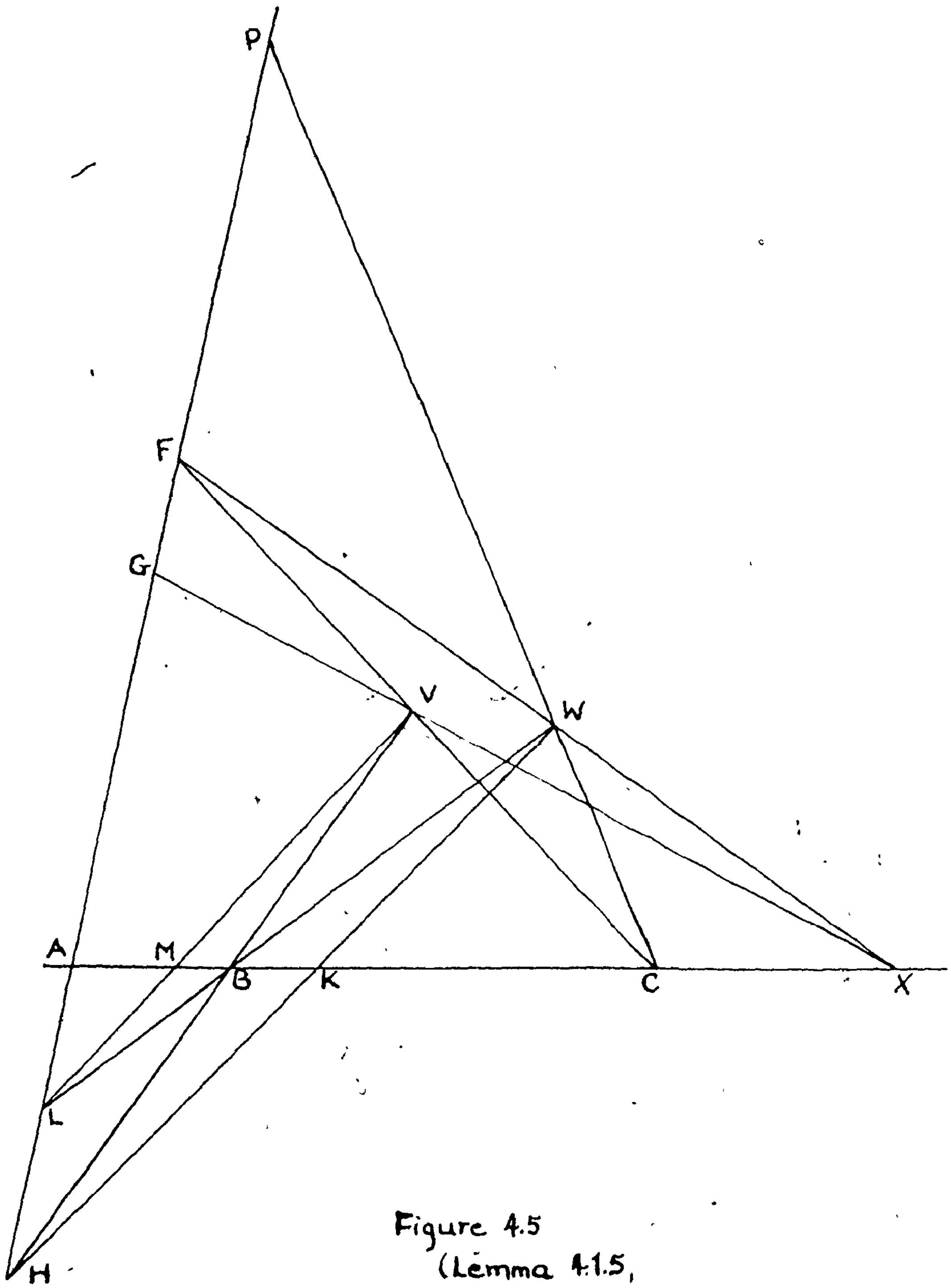


Figure 4.5
(Lemma 4.1.5,
A~B+C, Case 2)

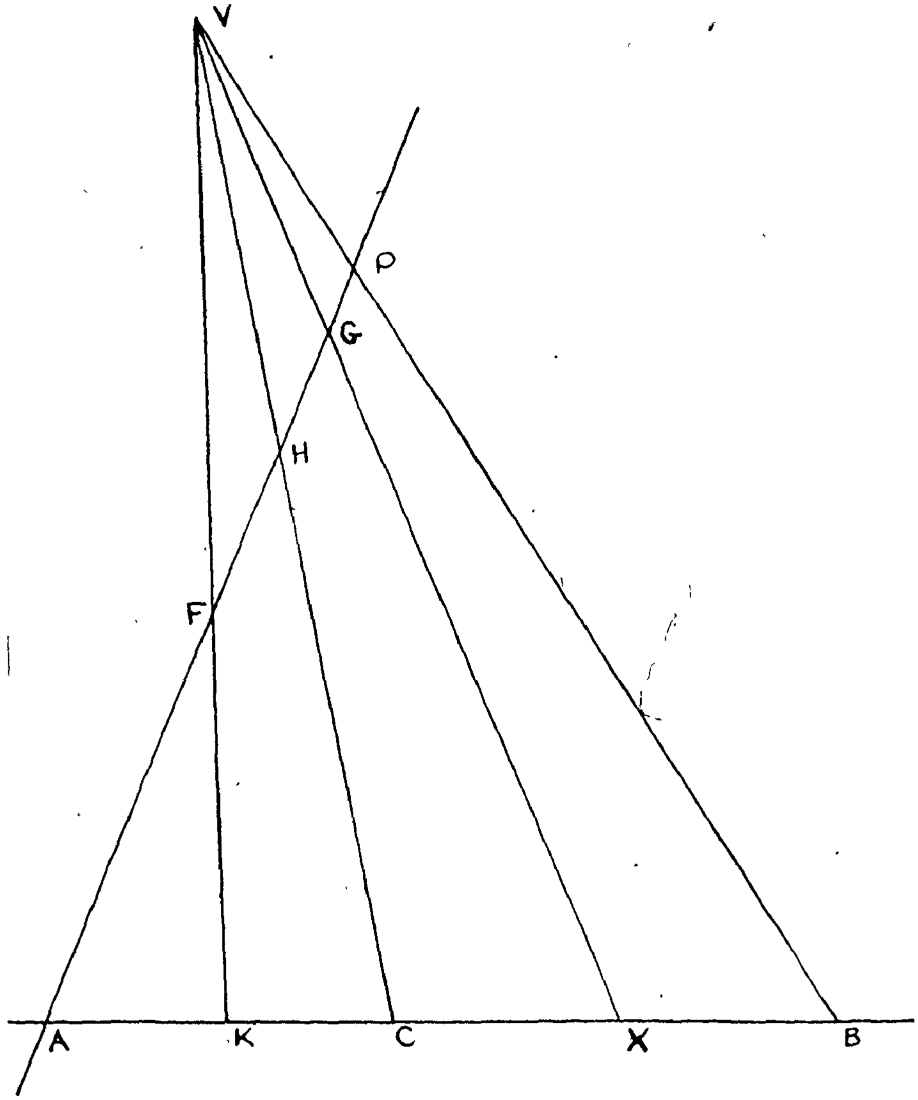


Figure 4.6
(Lemma 4.1.5,
A≁B, A~C, Case 1)

$Q \; I \; \ell$, PA. Then the projection $PA \xrightarrow{W} \ell$ yields $AC \; \sigma \; KX$ where $K = HW \wedge \ell \;\; (K \sim A)$. From $AC \; \sigma \; KX$ and $AC \; \sigma \; BX$, we may conclude by (OP4) that either $AB \; \sigma \; CK$ which implies K is the required point or $AK \; \sigma \; BC$. Suppose $AK \; \sigma \; BC$. Then the projection $\ell \xrightarrow{W} PA$ yields $AH \; \sigma \; LP$ where $L = BW \wedge PA \;\; (L \sim A)$. From $AH \; \sigma \; LP$ and $AP \; \sigma \; HF$, we obtain $AH \; \sigma \; FL$. Finally, using the projection $PA \xrightarrow{V} \ell$, this yields $AB \; \sigma \; CM$ where $M = LV \wedge \ell \;\; (M \sim A)$ and M is the required point.

<u>Case 3</u>: Suppose $AG \; \sigma \; PF$. Then we define $V = GC \wedge FX$ and $W = PC \wedge GX$. We may proceed as in Case 2.

Next suppose $A \not\sim B$, $A \sim C$. By (OP5), there exists $X \; I \; \ell$ such that $X \not\sim A, B$. By (OP3) and 4.1.3, either $AB \; \sigma \; CX$ and X is the required point or $AX \; \sigma \; BC$. Suppose $AX \; \sigma \; BC$. As above, there exist points P, F, G.

<u>Case 1</u>: Suppose $AP \; \sigma \; FG$. There exists $V = PB \wedge GX$ and $V \not\sim Q$ for any $Q \; I \; \ell$, PA. Since $AX \; \sigma \; BC$, the projection $\ell \xrightarrow{V} PA$ yields $AG \; \sigma \; PH$ where $H = CV \wedge PA \;\; (H \sim A)$. By (OP4), $AG \; \sigma \; PH$ and $AP \; \sigma \; FG$ imply $AP \; \sigma \; FH$; hence the projection $PA \xrightarrow{V} \ell$ yields $AB \; \sigma \; KC$ where $K = FV \wedge \ell$. Thus K is the required point.

<u>Case 2</u>: Suppose $AF \; \sigma \; PG$. Let $V = PX \wedge FB$. Then the argument used in Case 1 implies $K = VG \wedge \ell$ is the required point.

<u>Case 3</u>: Suppose $AG \; \sigma \; PF$. Let $V = BG \wedge FX$. Then $K = PV \wedge \ell$ is the required point.

Finally, suppose A, B, C are all neighbours. Then by (OP5), there exists $X \; I \; \ell$ such that $X \not\sim A$. We have three possibilities: 1) $AB \; \sigma \; CX$ which implies X is the required point; 2) $AX \; \sigma \; BC$;

3) $AX \sigma BC$. Suppose $AC \sigma BX$. By the above discussion, there exists $Y \, I \, \ell$ such that $AB \sigma XY$. Combining this with $AC \sigma BX$, we obtain $AB \sigma CY$ and Y is the required point. Similarly, if $AX \sigma BC$, there exists $Y \, I \, \ell$ such that $AB \sigma XY$, which implies $AB \sigma CY$ and Y is the required point.

4.2. The A.H. plane embedded in an ordered P.H. plane.

4.2.1. Let $\mathcal{P} = \langle \mathbb{P}, \mathbb{L}, I \rangle$ be a P.H. plane. For any $\ell \in \mathbb{L}$, define $\Sigma(\ell) = \{P \in \mathbb{P} \mid \text{there exists } m \in \bar{\ell} \text{ with } P \, I \, m\}$ and

$$\mathcal{H}(\ell) = \langle \mathbb{P}(\ell), \mathbb{L}(\ell), \hat{I}, \| \rangle$$

where

$\mathbb{P}(\ell) = \mathbb{P} \setminus \Sigma(\ell)$;

$\mathbb{L}(\ell) = \mathbb{L} \setminus \bar{\ell}$ (we write $m = \hat{m}$ if $m \in \mathbb{L}(\ell)$);

$\hat{m} \| \hat{n}$ if and only if there exists $P \in \mathbb{P}$ such that $P \, I \, \ell, m, n$;

$\hat{I} = I \cap (\mathbb{P}(\ell) \times \mathbb{L}(\ell))$.

4.2.2. Theorem. Let $\mathcal{P} = \langle \mathbb{P}, \mathbb{L}, I \rangle$ be an ordered P.H. plane. Then $\mathcal{H}(\ell) = \langle \mathbb{P}(\ell), \mathbb{L}(\ell), \hat{I}, \| \rangle$ is an ordered A.H. plane.

Proof. $\mathcal{H}(\ell)$ is an A.H. plane by [13].

For three mutually distinct, collinear points of $\mathcal{H}(\ell)$, we define $(A,B,C) \in \rho$ if and only if there exists a point $D \in \Sigma(\ell)$ such that $AC \sigma BD$. We now verify that ρ is an ordering of $\mathcal{H}(\ell)$.

(O1) and (O2) are clearly satisfied.

Before proving (O3), we observe that:

1) for any line $m \notin \bar{\ell}$, if $P, Q \, I \, m$ and $P, Q \in \Sigma(\ell)$, then $P \sim Q$;

2) if $P, Q \, I \, m \notin \bar{\ell}$, $P \sim Q$ and $P \in \Sigma(\ell)$, then $Q \in \Sigma(\ell)$.

To prove (O3) suppose $(A,B,C) \in \rho$. Then there exists $D \in \Sigma(\ell)$ such that $AC\ \sigma\ BD$. If, in addition, $(A,C,B) \in \rho$, then there would exist $E \sim D$ such that $AB\ \sigma\ CE$. Combining $AC\ \sigma\ BD$ and $AB\ \sigma\ CE$, we obtain $AB\ \sigma\ DE$; a contradiction to 4.1.3 or 4.1.4. Thus $(A,B,C) \in \rho$ implies $(A,C,B) \notin \rho$.

To prove (O4), consider three mutually distinct points $A, B, C \in \mathbb{P}(\ell)$ incident with a line $\hat{m}$. Then $m \notin \bar{\ell}$ and there exists a unique point $D = m \wedge \ell$ in $\mathcal{P}$. By (OP3), one of $AB\ \sigma\ CD$, $AC\ \sigma\ BD$ or $AD\ \sigma\ BC$ holds. These imply $(A,C,B) \in \rho$, $(A,B,C) \in \rho$ or $(B,A,C) \in \rho$ respectively.

To prove (O5), consider four mutually distinct, collinear points A, B, C, D of $\mathcal{K}(\ell)$ such that $(A,B,C) \in \rho$. Then there exists $E \in \Sigma(\ell)$ such that $AC\ \sigma\ BE$. By (OP3) and (OP4), either $DC\ \sigma\ BE$ which implies $(D,B,C) \in \rho$ or $AD\ \sigma\ BE$ which implies $(A,B,D) \in \rho$.

Finally, to prove (O6), we consider any non-degenerate parallel projection $\hat{m} \xrightarrow[\hat{g}]{} \hat{n}$. Consider any three mutually distinct points $A,B,C\ I\ \hat{m}$ such that $(A,B,C) \in \rho$. Let $m \wedge \ell = D$, $n \wedge \ell = E$ and $g \wedge \ell = P$. Then $AC\ \sigma\ BD$. The projection $m \xrightarrow{P} n$ yields $A'C'\ \sigma\ B'E$ where $A' = L(A,g) \wedge n$, etc. Therefore, $(A',B',C') \in \rho$.

CHAPTER 5

A projective completion of a projectively uniform A.H. plane.

5.1. Theorem. A projectively uniform A.H. plane may be extended to a uniform P.H. plane with a base line (cf. [1], [3]).

We shall not present a proof of this theorem, but rather shall restrict ourselves to a discussion of the resulting uniform P.H. plane with a base line which we call a projective completion $\mathcal{H}^*$ of the projectively uniform A.H. plane $\mathcal{H} = \langle \mathbb{P}, \mathbb{L}, I, || \rangle$. The construction itself is due to Artmann [1], but the fact that he used the additional condition that $\mathcal{H}$ be projectively uniform was observed by Bacon [3].

5.2. Preliminary results (cf. [1]).

5.2.1. Lemma. Let $\bar{P}$ be a point of $\bar{\mathcal{H}}$ (thus $\bar{P}$ is also a neighbour class of points of $\mathcal{H}$). Then

$$|\{\Pi_{\bar{P},g} \mid g \in \mathbb{L}_{\bar{P}}\}|$$
$$= |\{\ell \in \mathbb{L}_{\bar{P}} \mid Q\ I_{\bar{P}}\ \ell, \text{ for a fixed } Q \in \mathbb{P}_{\bar{P}}\}|$$
$$= |\{\bar{\ell} \in \bar{\mathbb{L}} \mid Q\ I\ \ell, \text{ for a fixed } Q \in \bar{P}\}|$$
$$= |\{\bar{\ell} \in \bar{\mathbb{L}} \mid \bar{P}\ \bar{I}\ \bar{\ell}\}|$$

Proof. For any $\Pi_{\bar{P},g}$ and any $Q \in \mathbb{P}_{\bar{P}}$, there exists a unique line $L(Q,\Pi_{\bar{P},g})$ of $\mathcal{A}(\bar{P})$ and $L(Q,\Pi_{\bar{P},g}) \neq L(Q,\Pi_{\bar{P},h})$ if $\Pi_{\bar{P},g} \neq \Pi_{\bar{P},h}$; hence $|\{\Pi_{\bar{P},g} \mid g \in \mathbb{L}_{\bar{P}}\}| = |\{\ell \in \mathbb{L}_{\bar{P}} \mid Q\ I_{\bar{P}}\ \ell, \text{ for a fixed } Q \in \mathbb{P}_{\bar{P}}\}|$.

Consider $\ell, m \in \mathbb{L}$ such that $Q\ I\ \ell, m$, some fixed $Q \in \bar{P}$, and $\ell \sim m$. Then as $\mathcal{H}$ is uniform, any point of $\mathbb{P}_{\bar{P}}$ incident with either ℓ or m is also incident with the other; hence the restrictions of ℓ

and m to $\mathcal{A}(\bar{P})$ are equal. Thus

$$|\{\ell \in \mathbb{L}_{\bar{P}} \mid Q\ I_{\bar{P}}\ \ell \text{ for a fixed } Q \in \mathbb{P}_{\bar{P}}\}|$$

$$= |\{\bar{\ell} \in \bar{\mathbb{L}} \mid Q\ I\ \ell \text{ for a fixed } Q \in \bar{P}\}|.$$

Clearly, $\{\bar{\ell} \in \bar{\mathbb{L}} \mid Q\ I\ \ell$ for a fixed $Q \in \bar{P}\} \subseteq \{\bar{\ell} \in \bar{\mathbb{L}} \mid \bar{P}\ \bar{I}\ \bar{\ell}\}$. If $\bar{P}\ \bar{I}\ \bar{\ell}$ and $Q \not I\ \ell$, then there exists $R \in \bar{P}$ with $R\ I\ \ell$. Since $Q \not I\ \ell$, any line through Q and R is not a neighbour of ℓ; hence by (A7)' $\bar{\ell} = \overline{L(Q,\ell)}$ and $Q\ I\ L(Q,\ell)$. Thus $|\{\bar{\ell} \in \bar{\mathbb{L}} \mid Q\ I\ \ell$ for a fixed $Q \in \bar{P}\}| = |\{\bar{\ell} \in \bar{\mathbb{L}} \mid \bar{P}\ \bar{I}\ \bar{\ell}\}|$.

<u>5.2.2</u>. <u>Corollary</u>. Let $\bar{P} \in \bar{\mathbb{P}}$. Then the affine planes $\bar{\mathcal{H}}$ and $\mathcal{A}(\bar{P})$ have the same order (where the <u>order of an affine plane</u> (not necessarily finite) is defined to be the cardinality of a line considered as a set of points); cf. [1], Lemma 2.

<u>5.2.3</u>. <u>Corollary</u>. The affine planes $\mathcal{A}(\bar{P})$, $\bar{P} \in \bar{\mathbb{P}}$, all have the same order.

<u>5.2.4</u>. <u>Corollary</u>. Let $\bar{P} \in \bar{P}$ and $h \in \mathbb{L}_{\bar{P}}$. Then
$|\{\bar{m} \in \bar{\mathbb{L}} \mid \bar{m} \| \bar{\ell}$ for a given $\bar{\ell} \in \bar{\mathbb{L}}\}| = |\{\Pi_{\bar{P},g} \mid g \in \mathbb{L}_{\bar{P}},\ \Pi_{\bar{P},g} \neq \Pi_{\bar{P},h}\}|$.

<u>5.2.5</u>. <u>Lemma</u>. Let ℓ be a line of $\mathcal{H}$. Then
$|\{\Pi_h \mid \Pi_h \sim \Pi_\ell\}| = |\{\Pi_h \mid h \in \bar{\ell}\}|$ is equal to the order of $\bar{\mathcal{H}}$; cf. [1], Lemma 3.

<u>Proof</u>. Select a point $P\ I\ \ell$. Then
$|\{\Pi_h \mid \Pi_h \sim \Pi_\ell\}| = |\{h \in \mathbb{L} \mid P\ I\ h,\ h \sim \ell\}| = |\{\Pi_h \mid h \in \bar{\ell}\}|$. Select a line m of $\mathcal{H}$ such that $P\ I\ m$ and $m \not\sim \ell$ and a point Q with $Q\ I\ \ell$ and $Q \not\sim P$. Take any $h \in \bar{\ell}$, $P\ I\ h$. Then h also meets $m' = L(Q,m)$ at a point $Q' \not\sim Q$ and different lines h give different points of m'.

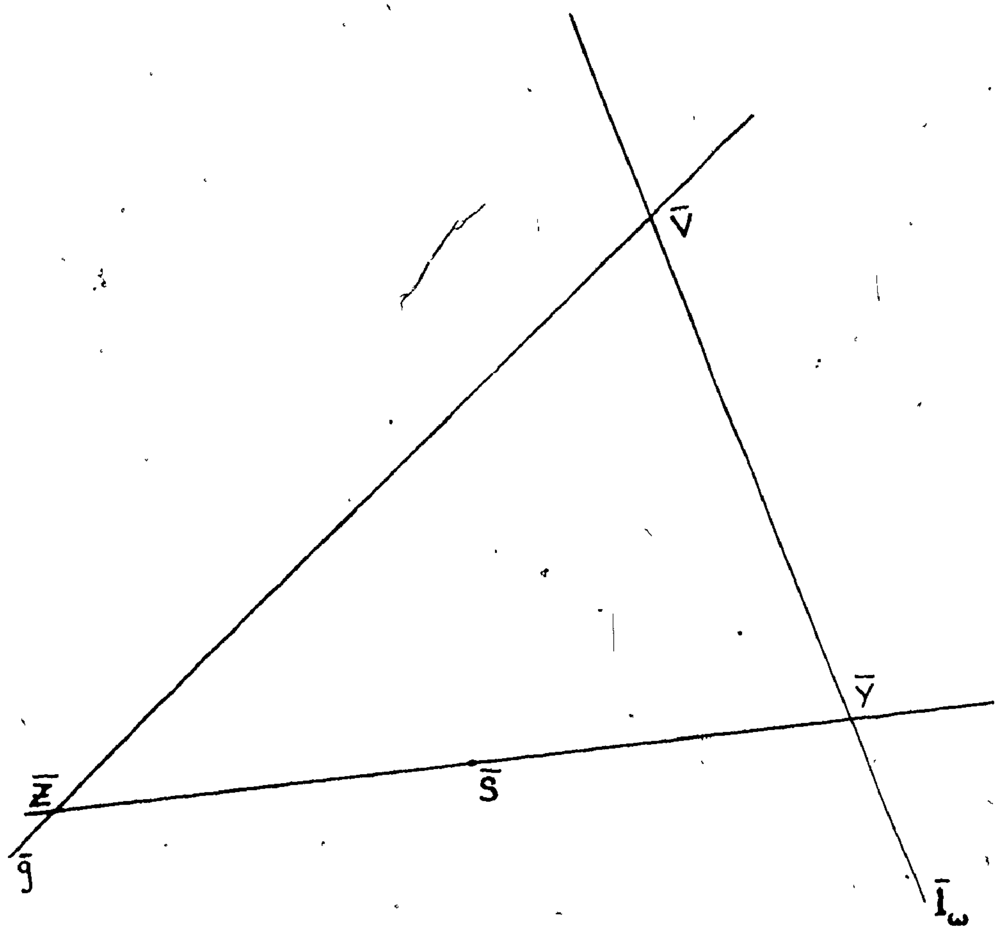


Figure 5.1
(Section 5.3.1)

Hence $|\{h \in \mathbb{L} \mid P\ I\ h,\ h \not\sim \ell\}| = |\{X \in \mathbb{P}_{\bar{Q}} \mid X\ I_{\bar{Q}}\ m'\}|$ which is equal to the order of the plane $\mathcal{A}(\bar{Q})$ and hence to the order of $\overline{\mathcal{H}}$.

5.3. The construction of a projective completion of $\mathcal{H}$.

Consider any projectively uniform A.H. plane $\mathcal{H} = \langle \mathbb{P}, \mathbb{L}, I, || \rangle$ with associated ordinary affine plane $\overline{\mathcal{H}} = \langle \bar{\mathbb{P}}, \bar{\mathbb{L}}, \bar{I}, \overline{||} \rangle$. Then $\overline{\mathcal{H}}$ can be extended to a projective plane $\overline{\mathcal{H}}^* = \langle \bar{\mathbb{P}}^*, \bar{\mathbb{L}}^*, \bar{I}^* \rangle$ by the addition of a single line $\bar{\ell}_\omega$ (the improper line of $\overline{\mathcal{H}}^*$) and the set of (improper) points incident with this line [15], 1.2. We shall use $\overline{\mathcal{H}}^*$ to construct a P.H. plane $\mathcal{H}^* = \langle \mathbb{P}^*, \mathbb{L}^*, I^* \rangle$ such that the neighbour relation of $\mathcal{H}^*$ is an extension of the neighbour relation of $\mathcal{H}$.

5.3.1. The points of a projective completion of $\mathcal{H}$. In the projective plane $\overline{\mathcal{H}}^*$, let $\bar{g}$ be an arbitrary but fixed line distinct from $\bar{\ell}_\omega$ and $\bar{S}$ an arbitrary but fixed point not incident with $\bar{\ell}_\omega$ or $\bar{g}$. (In the A.H. plane $\mathcal{H}$, for any Q, X such that $Q \in \bar{S}$ and X lies on a line of $\bar{g}$, we observe that $Q \not\sim X$.) Set $\bar{V} = \bar{g} \wedge \bar{\ell}_\omega$. For any point $\bar{Y}\ \bar{I}^*\ \bar{\ell}_\omega$, $\bar{Y} \neq \bar{V}$, there exists a unique $\bar{Z} = \bar{Y}\bar{S} \wedge \bar{g}$ and since $\mathcal{H}$ is uniform, $\mathcal{A}(\bar{Z})$ is an ordinary affine plane. We shall now define $\mathcal{A}(\bar{Y}) = \langle \mathbb{P}_{\bar{Y}}, \mathbb{L}_{\bar{Y}}, I_{\bar{Y}} \rangle$ to be the isomorphic copy of $\mathcal{A}(\bar{Z})$ under any arbitrary isomorphism $\psi_{\bar{Y}} : \mathcal{A}(\bar{Z}) \longrightarrow \mathcal{A}(\bar{Y})$. Finally, we let $\mathcal{A}(\bar{V}) = \langle \mathbb{P}_{\bar{V}}, \mathbb{L}_{\bar{V}}, I_{\bar{V}} \rangle$ be any ordinary affine plane which has the same order as $\overline{\mathcal{H}}$. We define $\mathbb{P}_\omega = \bigcup_{\bar{Y}\bar{I}\bar{\ell}_\omega} \mathbb{P}_{\bar{Y}}$, the set of improper points of the projective completion $\mathcal{H}^*$ and $\mathbb{P}^* = \mathbb{P} \cup \mathbb{P}_\omega$, the set of points of $\mathcal{H}^*$. (Here $\cup$ is disjoint union.)

5.3.2. The improper lines of the projective completion of $\mathcal{K}$. First we select a parallel pencil $\Pi_{\bar{V}}$ of $\mathcal{A}(\bar{V})$. By 5.2.5, there exists a bijection

$$\gamma : \{\Pi_{h} \mid h \in \bar{g}\} \longrightarrow \{\ell \in \mathbb{L}_{\bar{V}} \mid \ell \in \Pi_{\bar{V}}\}.$$

For any line $h \in \bar{g}$, as $h \in \mathbb{L}_{\bar{Z}}$ for every $\bar{Z}\ \bar{I}\ \bar{g}$, we define

$$h^* = \bigcup_{\bar{Y}\bar{I}\bar{\ell}_{\omega},\bar{Y}\neq\bar{V}} \{P \in \mathbb{P}_{\bar{Y}} \mid P\ I\ \psi_{\bar{Y}}(h)\} \cup \{P \in \mathbb{P}_{\bar{V}} \mid P\ I\ \gamma(\Pi_{h})\}.$$

(Thus each $P \in \mathbb{P}_{\bar{Y}}$ is equal to $\psi_{\bar{Y}}(Q)$ for some $Q\ I\ g$, $Q \in \bar{Z}$ and $\psi_{\bar{Y}}(\mathcal{A}(\bar{Z})) = \mathcal{A}(\bar{Y})$.) We select one line ℓ of $\bar{g}$ and call ℓ^* the central improper line of $\mathcal{K}^*$. We define $\mathbb{L}_{\omega} = \{h^* \mid h \in \bar{g}\}$, the set of improper lines of $\mathcal{K}^*$.

5.3.3. The proper lines of the projective completion of $\mathcal{K}$. We complete each line m of $\mathbb{L}$ by adjoining to m a set of improper points.

For any $\bar{X}$ in $\bar{\mathcal{K}}^*$ such that $\bar{X}\ \bar{I}^*\ \bar{\ell}_{\omega}$, we define $\lambda(\bar{X}) = \ell^* \cap \mathbb{P}_{\bar{X}}$; thus $\lambda(X) = \{P \in \mathbb{P}_{\bar{X}} \mid P\ I\ \psi_{\bar{X}}(\ell)\}$ if $\bar{X} \neq \bar{V}$ and $\lambda(\bar{V}) = \{P \in \mathbb{P}_{\bar{V}} \mid P\ I\ \lambda(\Pi_{\ell})\}$. The points of $\lambda(\bar{X})$ lie on one line $\lambda_{\bar{X}}$ of $\mathcal{A}(\bar{X})$ and $\ell^* = \bigcup_{\bar{X}\bar{I}\bar{\ell}_{\omega}} \lambda(\bar{X})$. From Lemma 5.2.4, there exists a bijection

$$\alpha_{\bar{X}} : \{\bar{h} \in \bar{\mathbb{L}}^* \setminus \{\bar{\ell}_{\omega}\} \mid \bar{X}\ \bar{I}^*\ \bar{h}\} \longrightarrow \{\Pi_{\bar{X},k} \mid \lambda_{\bar{X}} \notin \Pi_{\bar{X},k}\}.$$

For each $\bar{h} \in \bar{\mathbb{L}}^* \setminus \{\bar{\ell}_{\omega}\}$ and $\bar{X} = \bar{h} \wedge \bar{\ell}$, select a bijection (cf. 5.2.5 and 5.2.2).

$$\phi_{\bar{h},\bar{X}} : \{\Pi_{k} \mid k \in \bar{h}\} \longrightarrow \lambda(\bar{X})$$

with the property that for $h, m \in \mathbb{L}$ such that $h \| m$, $\bar{h} \neq \bar{m}$ and $\bar{X}\ \bar{I}\ \bar{h}, \bar{m}$ in $\bar{\mathcal{K}}^*$ (thus $\bar{h} \| \bar{m}$ in $\bar{\mathcal{K}}$) we have $\phi_{\bar{h},\bar{X}}(\Pi_{h}) = \phi_{\bar{m},\bar{X}}(\Pi_{m})$.

We may, therefore, write $\phi_{\bar{X}}$ instead of $\phi_{\bar{h},\bar{X}}$.

Let $h \in \mathbb{L}$ and $\bar{X} = \bar{h} \wedge \bar{l}_\omega$. The projective completion of h is

$h' = \{P \in \mathbb{P} \mid P \, I \, h\} \cup \{P \in \mathbb{P}_{\bar{X}} \mid P \, I_{\bar{X}} \, L(\phi_{\bar{X}}(\Pi_h), \alpha_{\bar{X}}(\bar{h}))\}$.

The set $\{h' \mid h \in \mathbb{L}\}$ is the <u>set of proper lines</u> of the projective completion.

<u>5.3.4</u>. <u>The lines of the projective completion</u>. We define $\mathbb{L}^* = \{h' \mid h \in \mathbb{L}\} \cup \mathbb{L}_\omega$, the <u>set of lines</u> of the projective completion $\mathcal{H}^*$.

<u>5.4</u>. <u>The neighbour relation in the projective completion</u>.

In a P.H. plane two points are defined to be neighbours if and only if they are joined by more than one line; two lines are defined to be neighbours if they intersect in more than one point. We shall now examine the relationship between the neighbour relations in $\mathcal{H}$ and $\mathcal{H}^*$.

<u>5.4.1</u>. <u>Lemma</u>. If $P, Q \in \mathbb{P}^* \setminus \mathbb{P}_\omega$, then $P \sim Q$ in $\mathcal{H}^*$ if and only if $P \sim Q$ in $\mathcal{H}$.

<u>5.4.2</u>. <u>Lemma</u>. If $P \in \mathbb{P}^* \setminus \mathbb{P}_\omega$ and $Q \in \mathbb{P}_\omega$, then $P \not\sim Q$.

<u>Proof</u>. Assume $P \sim Q$; ie., there exist two distinct lines of $\mathbb{L}^*$ through P and Q. As $P \in \mathbb{P}^* \setminus \mathbb{P}_\omega$, these two lines must be proper lines of $\mathbb{L}^*$, say h', k' $(h' \neq k')$. Since $P \, I \, h', k'$, we have $P \, I \, h, k$ and $h \not\parallel k$ in $\mathcal{H}$ and in $\bar{\mathcal{H}}^*$, $\bar{P} \, \bar{I}^* \, \bar{h}, \bar{k}$. If $\bar{h} \neq \bar{k}$, then $\bar{X} = \bar{h} \wedge \bar{l}_\omega \neq \bar{k} \wedge \bar{l}_\omega = \bar{Y}$ in $\bar{\mathcal{H}}^*$ which implies $h' \cap \mathbb{P}_\omega \subseteq \mathbb{P}_{\bar{X}}$, $k' \cap \mathbb{P}_\omega \subseteq \mathbb{P}_{\bar{Y}}$ and $Q \in \mathbb{P}_{\bar{X}} \cap \mathbb{P}_{\bar{Y}} = \emptyset$ (a contradiction). On the other hand, if $\bar{h} = \bar{k}$, then for $\bar{X} = \bar{h} \wedge \bar{l}_\omega$ in $\bar{\mathcal{H}}^*$, $h' \cap \mathbb{P}_{\bar{X}} = L(\phi_{\bar{X}}(\Pi_h), \alpha_{\bar{X}}(\bar{h}))$ and

$k' \cap \mathbb{P}_{\bar{X}} = L(\phi_{\bar{X}}(\Pi_k),\alpha_{\bar{X}}(\bar{k}))$; however these two lines are distinct and **parallel in** $\mathcal{A}(\bar{X})$ so $Q \in \emptyset$ (another contradiction).

5.4.3. Corollary. Two distinct lines $h', k' \in \mathbb{L}^* \setminus \mathbb{L}_\omega$ cannot meet in both proper and improper points.

5.4.4. Lemma. Let $h, k \in \mathbb{L}$. Then $h', k' \in \mathbb{L}^*$ and $h' \sim k'$ if and only if $h \sim k$.

Proof. First, assume $h \sim k$. By (A2) either $|h \wedge k| > 1$, which implies $h' \sim k'$, or $|h \wedge k| = 0$. If $h \sim k$ and $|h \wedge k| = 0$, then the projective uniformity of $\mathcal{H}$ implies $h||k$; hence if $\bar{X} = \bar{h} \wedge \bar{\ell}_\omega$, $\{P \in \mathbb{P}_{\bar{X}} \mid P\ I_{\bar{X}}\ L(\phi_{\bar{X}}(\Pi_h),\alpha_{\bar{X}}(h))\}\ I^*\ h', k'$ and $h' \sim k'$.

Now assume $h' \sim k'$. By definition, there exist distinct points $P, Q \in \mathbb{P}^*$ with $P, Q\ I^*\ h', k'$. By 5.4.3, either $P, Q \in \mathbb{P}^* \setminus \mathbb{P}_\omega$ which implies $P, Q\ I\ h, k$ and $h \sim k$ or $P, Q \in \mathbb{P}_\omega$. Suppose $P, Q \in \mathbb{P}_\omega$. This implies $|h \wedge k| = 0$ and by (A4), iii), $\bar{h} \wedge \bar{\ell}_\omega = \bar{X} = \bar{k} \wedge \bar{\ell}_\omega$. Since $P, Q\ I_{\bar{X}}\ h' \cap \mathbb{P}_\omega, k' \cap \mathbb{P}_\omega$, we have
$L(\phi_{\bar{X}}(\Pi_h),\alpha_{\bar{X}}(\bar{h})) = h' \cap \mathbb{P}_\omega = k' \cap \mathbb{P}_\omega = L(\phi_{\bar{X}}(\Pi_k),\alpha_{\bar{X}}(\bar{k}))$. Hence $\alpha_{\bar{X}}(\bar{h}) = \alpha_{\bar{X}}(\bar{k})$ and $h \sim k$.

5.4.5. Lemma. Let $h, k \in \mathbb{L}$; $h, k \in \bar{g}$. Then $h^* \sim k^*$.

Proof. Since $h, k \in \bar{g}$, $h \sim k$ which implies by the projective uniformity of $\mathcal{H}$, that either $h||k$ or h and k intersect in $\mathcal{H}$.

If $h||k$, then $\{P \in \mathbb{P}_{\bar{V}} \mid P\ I\ \gamma(\Pi_h)\}\ I\ h^*, k^*$.

If h and k meet in some $Z \in \mathbb{P}$, then by the uniformity of $\mathcal{H}$, $\{P \in \mathbb{P}_{\bar{Z}} \mid P\ I_{\bar{Z}}\ h\} = \{P \in \mathbb{P}_{\bar{Z}} \mid P\ I_{\bar{Z}}\ k\}$. In $\bar{\mathcal{H}}^*$, there exists $\bar{X} \in \bar{\mathbb{P}}^*$ with $\bar{X} = \bar{Z}\bar{S} \wedge \bar{\ell}_\omega$. Therefore $\{P \in \mathbb{P}_{\bar{X}} \mid P\ I_{\bar{X}}\ \psi_{\bar{X}}(h)\}\ I\ h^*, k^*$.

5.4.6. Lemma. If $h' \in \mathbb{L}^* \setminus \mathbb{L}_\omega$ and $k^* \in \mathbb{L}_\omega$, then $h' \nmid k^*$.

Proof. First consider k and l (cf. 5.3.2). By the projective uniformity of $\mathcal{K}$, for any $\bar{X}\ \bar{I}^*\ \bar{q}$, either

$\{P \in \mathbb{P}_{\bar{X}} \mid P\ I_{\bar{X}}\ k\} \cap \{P \in \mathbb{P}_{\bar{X}} \mid P\ I_{\bar{X}}\ l\} = \emptyset$ or

$\{P \in \mathbb{P}_{\bar{X}} \mid P\ I_{\bar{X}}\ k\} = \{P \in \mathbb{P}_{\bar{X}} \mid P\ I_{\bar{X}}\ l\}$; i.e., in each $\mathcal{A}(\bar{X})$, $k \| _{\bar{X}} l$.

Therefore for any $\bar{Y}\ \bar{I}^*\ \bar{l}_\omega$, $\bar{Y} \neq \bar{V}$, $\psi_{\bar{Y}}(k) \| _{\bar{Y}} \psi_{\bar{Y}}(l) = \lambda_{\bar{Y}} \notin \alpha_{\bar{Y}}(m)$, for any $\bar{m} \in \mathbb{L}^* \setminus \{\bar{l}_\omega\}$, and $\gamma(\Pi_k) \| _{\bar{V}} \gamma(\Pi_l) = \lambda_{\bar{V}} \notin \alpha_{\bar{V}}(m)$, for any $m \in \mathbb{L}^* \setminus \{l_\omega\}$.

Now consider the lines h' and k^*. As all points incident with k^* are in $\mathbb{P}_\omega$, all points of $h' \wedge k^*$ must be in $\mathbb{P}_{\bar{Y}}$, where $\bar{Y} = \bar{h} \wedge \bar{l}_\omega$ in $\mathcal{K}^*$. Therefore if $\bar{Y} \neq \bar{V}$, $h' \wedge k^* = L(\phi_{\bar{Y}}(\Pi_h), \alpha_{\bar{Y}}(\bar{h})) \wedge \psi_{\bar{Y}}(k)$ which by the above discussion is a single point; if $\bar{Y} = \bar{V}$, $h' \wedge k^* = L(\phi_{\bar{V}}(\Pi_h), \alpha_{\bar{V}}(\bar{h})) \wedge \gamma(\Pi_k)$ which is also a single point.

5.4.7. Lemma. Let $P, Q \in \mathbb{P}_\omega$. Then $P \sim Q$ if and only if there exists $\bar{X} \in \bar{\mathbb{P}}^*$, $\bar{X}\ \bar{I}^*\ \bar{l}_\omega$ such that $P, Q \in \mathbb{P}_{\bar{X}}$

Proof. If $P \sim Q$, then the lines incident with both points are either all proper lines or improper lines.

Suppose $P, Q\ I\ h', k'$ $(h' \neq k')$. Then by 5.4.4, $h \sim k$. Hence $P, Q \in \mathbb{P}_{\bar{X}}$ where $\bar{X} = \bar{h} \wedge \bar{l}_\omega = \bar{k} \wedge \bar{l}_\omega$.

Now suppose $P, Q\ I\ h^*, k^*$. Then $h, k \in \bar{q}$ and there exists a unique $\bar{Y}\ \bar{I}^*\ \bar{q}$ such that $\{P \in \mathbb{P}_{\bar{Y}} \mid P\ I_{\bar{Y}}\ h\} = \{P \in \mathbb{P}_{\bar{Y}} \mid P\ I_{\bar{Y}}\ k\}$. We have $P, Q \in \mathbb{P}_{\bar{X}}$ where $\bar{X} = \bar{Y}\bar{S} \wedge \bar{l}_\omega$.

Conversely, suppose there exists $\bar{X} \in \bar{\mathbb{P}}^*$, $\bar{X}\ \bar{I}^*\ \bar{l}_\omega$ such that $P, Q \in \mathbb{P}_{\bar{X}}$. Let $\bar{Z} = \bar{X}\bar{S} \wedge \bar{q}$. Then $\psi_{\bar{X}}^{-1}(P), \psi_{\bar{X}}^{-1}(Q) \in \mathbb{P}_{\bar{Z}}$; hence there exist distinct lines $m, n \in \mathbb{L}$ such that $\psi_{\bar{X}}^{-1}(P), \psi_{\bar{X}}^{-1}(Q)\ I\ m, n$ (and $\{P \in \mathbb{P}_{\bar{Z}} \mid P\ I_{\bar{Z}}\ m\} = \{P \in \mathbb{P}_{\bar{Z}} \mid P\ I_{\bar{Z}}\ n\} \neq \emptyset$). Therefore

$\psi_{\bar{X}}(m)$, $\psi_{\bar{X}}(n) \in \mathbb{L}_{\bar{X}}$ and $P, Q\ I_{\bar{X}}\ \psi_{\bar{X}}(m) = \psi_{\bar{X}}(n)$ in $\mathcal{A}(\bar{X})$. If $\psi_{\bar{X}}(m) \| _{\bar{X}} \lambda_{\bar{X}}$, then $P, Q\ I^*\ m^*, n^*$ in $\mathcal{H}^*$ and if $\psi_{\bar{X}}(m) \not\|_{\bar{X}} \lambda_{\bar{X}}$, then there exists $h, k \in \mathbb{L}$ $(h \neq k)$ such that $h \sim k$, $h \| k$, $\bar{X} = \bar{h} \wedge \bar{\ell}_{\omega}$ and $h' = \{P \in \mathbb{P} \mid P\ I\ h\} \cup \{P \in \mathbb{P}_{\bar{X}} \mid P\ I_{\bar{X}}\ \psi_{\bar{X}}(m)\}$. Then $k' = \{P \in \mathbb{P} \mid P\ I\ k\} \cup \{P \in \mathbb{P}_{\bar{X}} \mid P\ I_{\bar{X}}\ \psi_{\bar{X}}(m)\}$ and $P, Q\ I^*\ h', k'$.

5.4.8. Corollary. Let $P, Q \in \mathbb{P}_{\omega}$ such that $P \not\sim Q$. Then $PQ \in \mathbb{L}_{\omega}$.

5.5. Remark.

From our discussion, it is clear that $\overline{\mathcal{H}^*}$ is isomorphic to the underlying projective plane $\overline{\mathcal{H}^*}$ of $\mathcal{H}^*$ (cf. 2.3). We may, therefore, use $\overline{\ell^*}$ to represent both the class of improper lines of $\mathcal{H}^*$ and the improper line, $\bar{\ell}_{\omega}$, of $\overline{\mathcal{H}^*}$. In addition, we may use $\bar{Q}$ to denote both a point of the improper line of $\overline{\mathcal{H}^*}$ $(\overline{\ell^*} = \bar{\ell}_{\omega})$ and to denote a neighbour class of improper points of $\mathcal{H}^*$ (and $\mathcal{A}(\bar{Q})$ to represent the affine plane defined over the points of $\bar{Q}$ in $\mathcal{H}^*$).

As there is a one-to-one correspondence between the lines h of $\mathcal{H}$ and the proper lines h' of $\mathcal{H}^*$, we may use h to represent either line. Generally, it will be clear from the context whether a line of $\mathcal{H}$ or a line of $\mathcal{H}^*$ is intended. In fact, we could have defined $\mathcal{H}^*$ in the following way. Construct $\mathbb{P}_{\omega}$ as in 5.3.1; define the bijections γ, $\alpha_{\bar{X}}$, $\phi_{\bar{X}}$ as in 5.3.2 and 5.3.3; and take

$\mathbb{L}_{\omega} = \{h^* = \bigcup_{\bar{X} \bar{I}^* \bar{\ell}_{\omega}, \bar{X} \neq \bar{V}} \{\psi_{\bar{X}}(h)\} \cup \{\gamma(\Pi_h)\} \mid h \in \bar{g}\}$. Then let $\mathbb{P}^* = \mathbb{P} \cup \mathbb{P}_{\omega}$, $\mathbb{L}^* = \mathbb{L} \cup \mathbb{L}_{\omega}$ and extend the incidence relation I to $I^* \subseteq \mathbb{P}^* \times \mathbb{L}^*$ where: 1) $I = I^*|_{\mathbb{P} \times \mathbb{L}}$; 2) for any $P \in \mathbb{P}_{\omega}$, $h \in \mathbb{L}$, $P\ I^*\ h$ if and only if

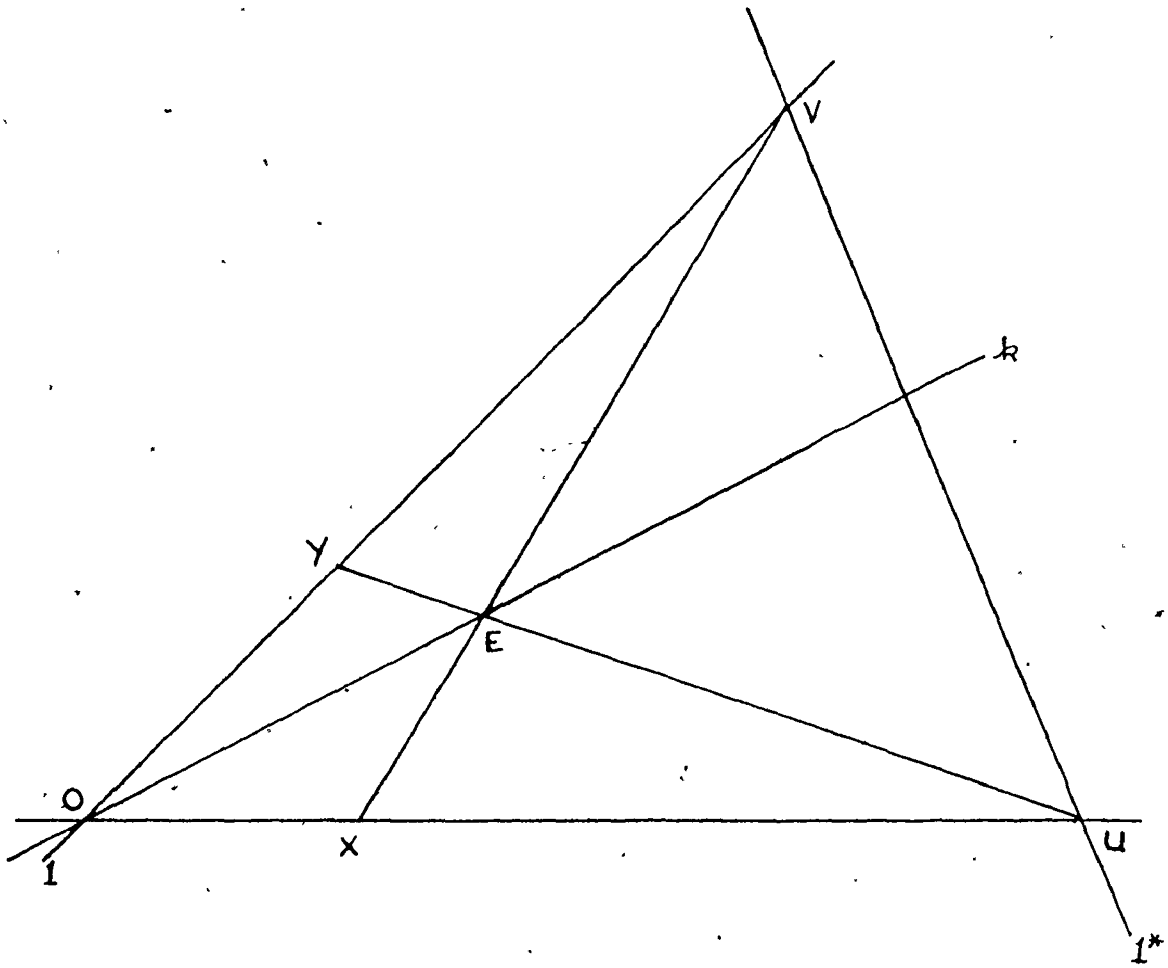


Figure 5.2
(Section 5.6)

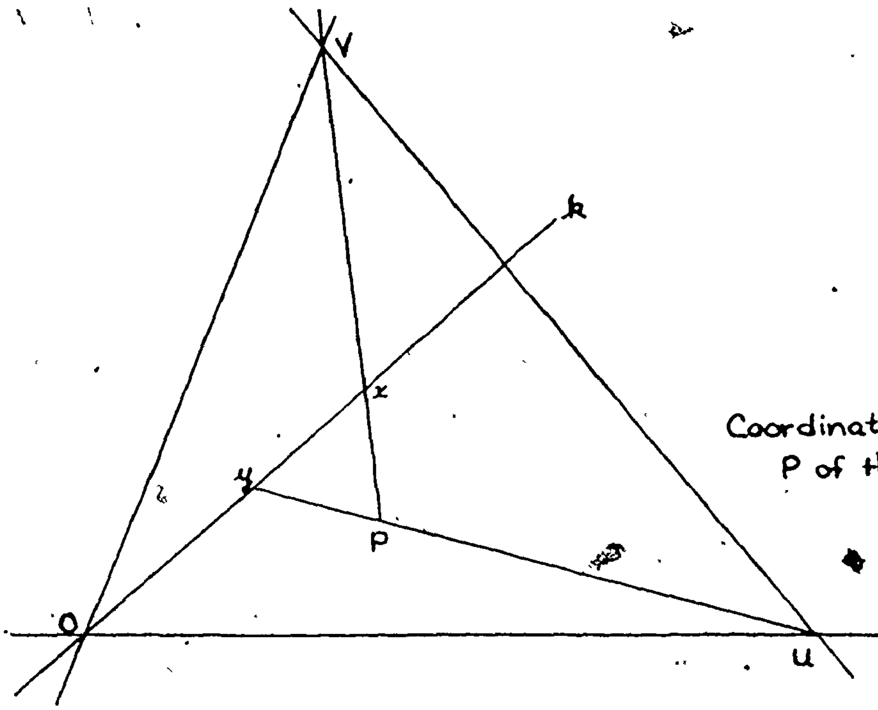
$P\ I_{\bar{X}}\ L(\phi_{\bar{X}}(\Pi_{h}), \alpha_{\bar{X}}(\bar{h}))$ where $X = h \wedge \ell_{\omega}$ and $P \in \mathbb{P}_{\bar{X}}$; 3) for any $P \in \mathbb{P}_{\omega}$, $h^* \in \mathbb{L}_{\omega}$, $P\ I^*\ h^*$ if and only if $P\ I_{\bar{X}}\ \psi_{\bar{X}}(h)$, for some $\bar{X}\ \bar{I}^*\ \bar{\ell}_{\omega}$, $\bar{X} \neq \bar{V}$ or $P\ I_{\bar{V}}\ \gamma(\Pi_{h})$; 4) for any $P \in \mathbb{P}$, $h^* \in \mathbb{L}_{\omega}$, $P\ \not{I}^*\ h^*$. Such a plane is isomorphic to the projective completion defined in 5.3.

Henceforth, the incidence relations of both $\mathcal{H}$ and $\mathcal{H}^*$ shall be denoted by I and the incidence relations of both $\bar{\mathcal{H}}$ and $\bar{\mathcal{H}}^*$ by $\bar{I}$. All lines of $\mathbb{L}^*$ shall be denoted by lower case script letters h, j, k,... . (The context will indicate whether we are dealing with proper or improper lines.)

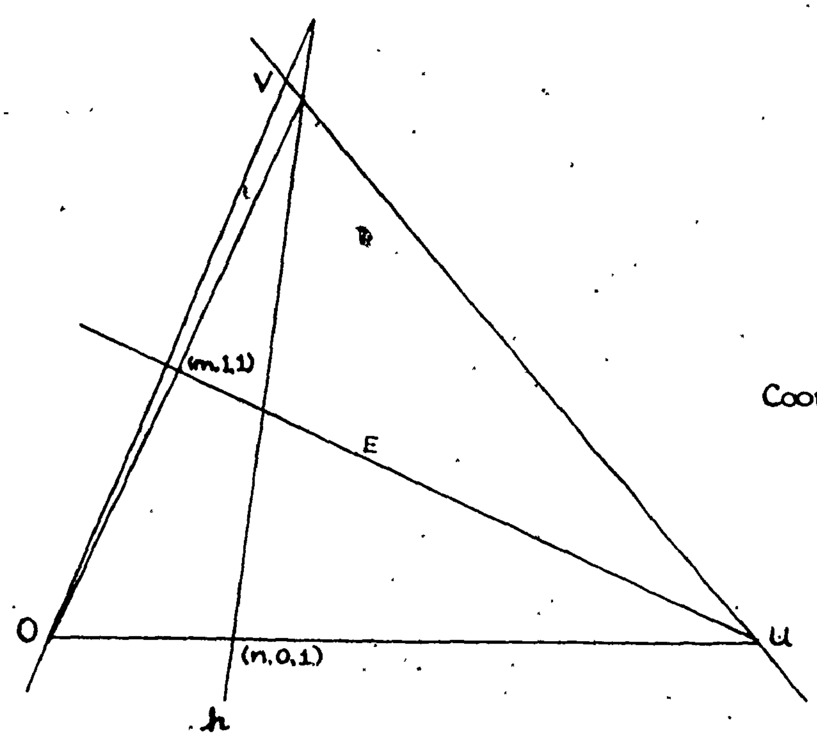
Finally, in $\mathcal{H}$ we use the notation $L(P,h)$ to denote the line through P parallel to h, Π_{h} to denote the pencil of lines parallel to h and $L(P,\Pi_{h})$ to denote the line of the parallel pencil Π_{h} which passes through the point P. These lines or classes of lines give rise to lines or classes of lines in $\mathcal{H}^*$; therefore in $\mathcal{H}^*$ we shall use $L(P,h)$, $h \in \mathbb{L}$, to denote the line of $\mathcal{H}^*$ which passes through P and is parallel to h in $\mathcal{H}$. We define Π_{h} and $L(P,\Pi_{h})$ similarly in $\mathcal{H}^*$.

5.6. A Coordinatization of $\mathcal{H}^*$.

We shall select a triangle OXY in $\mathcal{H}^*$ as follows. Choose O, Y I ℓ such that $O \nparallel Y$ and $\bar{O}, \bar{Y}\ \not{I}\ \overline{\ell^*}$. Let $\bar{U} = \bar{O}\bar{S} \wedge \bar{\ell}_{\omega}$ and put $U = \psi_{\bar{U}}(O)$. Now take any point X I OU such that $X \nparallel O, U$. Clearly, $X \nparallel P$ for any P I ℓ (by (H6)') or any P I ℓ^* and $Y \nparallel P$ for any P I OU, ℓ^*. There exists a unique point $E = L(X,\ell) \wedge L(Y,OX) \in \mathbb{P} \subset \mathbb{P}^*$ and a unique line k = OE with $k \nparallel \ell$, OU, ℓ^*. Let $H = \{P \in \mathbb{P}^* \setminus \mathbb{P}_{\omega} \mid P\ I\ k\}$. We use lower case Latin letters to denote the elements of H, with the exception of O and E which are denoted by

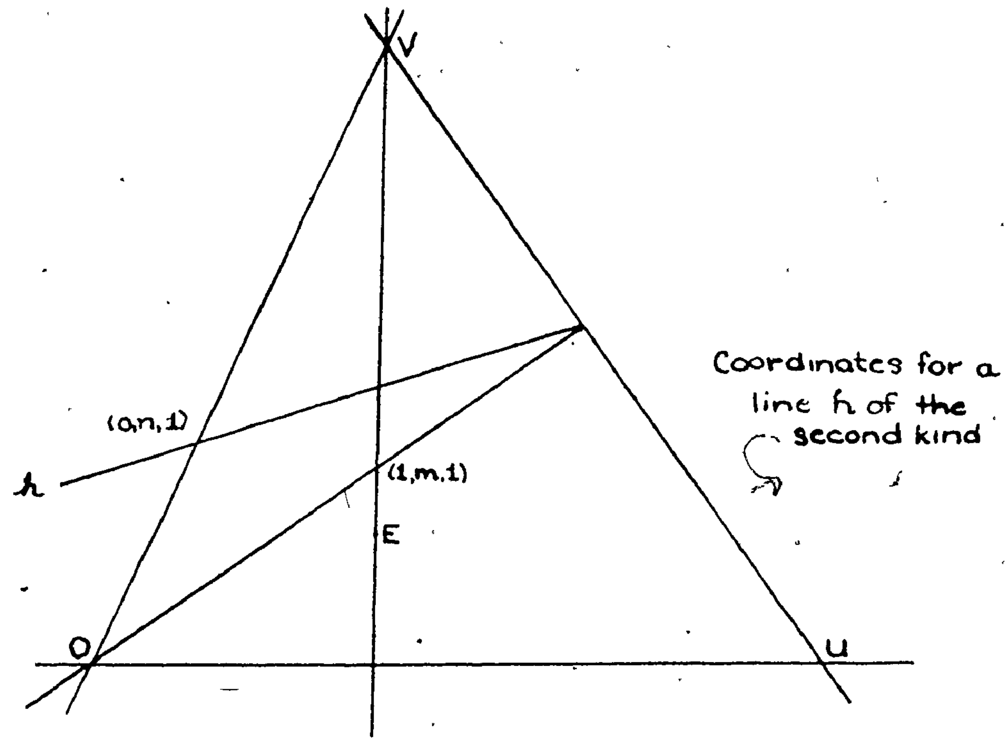


Coordinates for a point
P of the first kind



Coordinates for a line
h of the first kind

Figure 5.3
(Section 5.6)



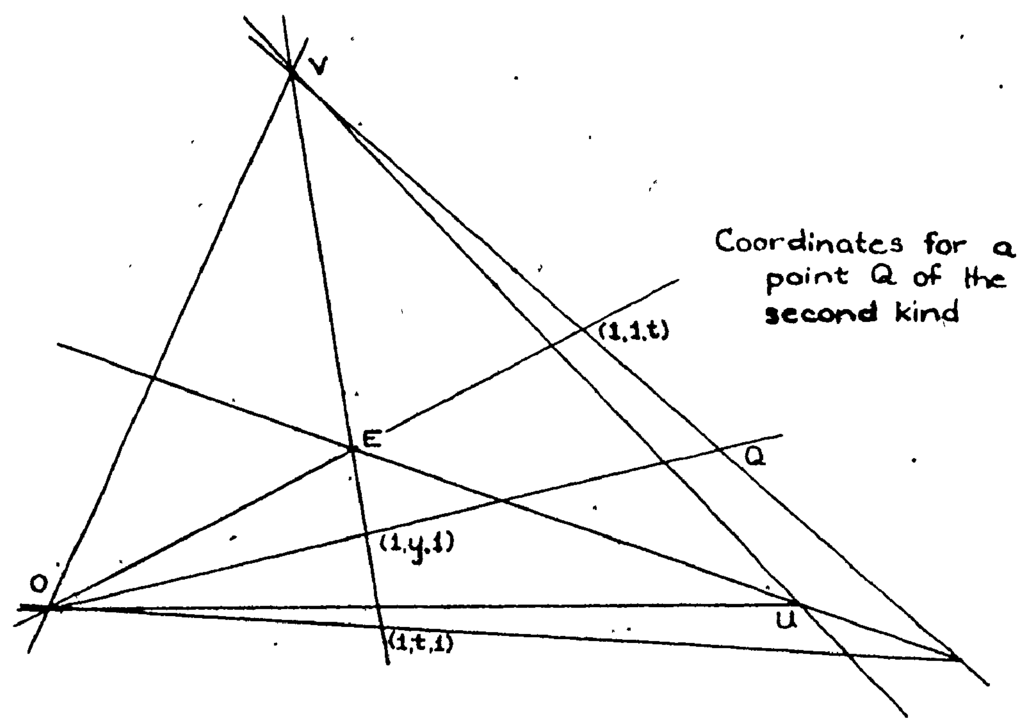


Figura 5.4
(Section 5.6)

0 and 1 respectively. The elements of H will be used to coordinatize $\mathcal{H}^*$. Let $\eta = \bar{0} \cap H$.

As $\ell \wedge \ell^* \in \bar{V}$, we may let $V = \ell \wedge \ell^*$. Then for any line $m || \ell$ in $\mathcal{H}$, in $\mathcal{H}^*$ we have

$\ell \wedge \ell^* = L(\phi_{\bar{V}}(\Pi_\ell), \alpha_{\bar{V}}(\ell)) \wedge \gamma(\Pi_\ell) = \phi_{\bar{V}}(\Pi_\ell) \in \ell \wedge m$; ie., if $m || \ell$ in $\mathcal{H}$, then $V \, I \, m$ in $\mathcal{H}^*$. Similarly, for any $m || OX$ in $\mathcal{H}$,

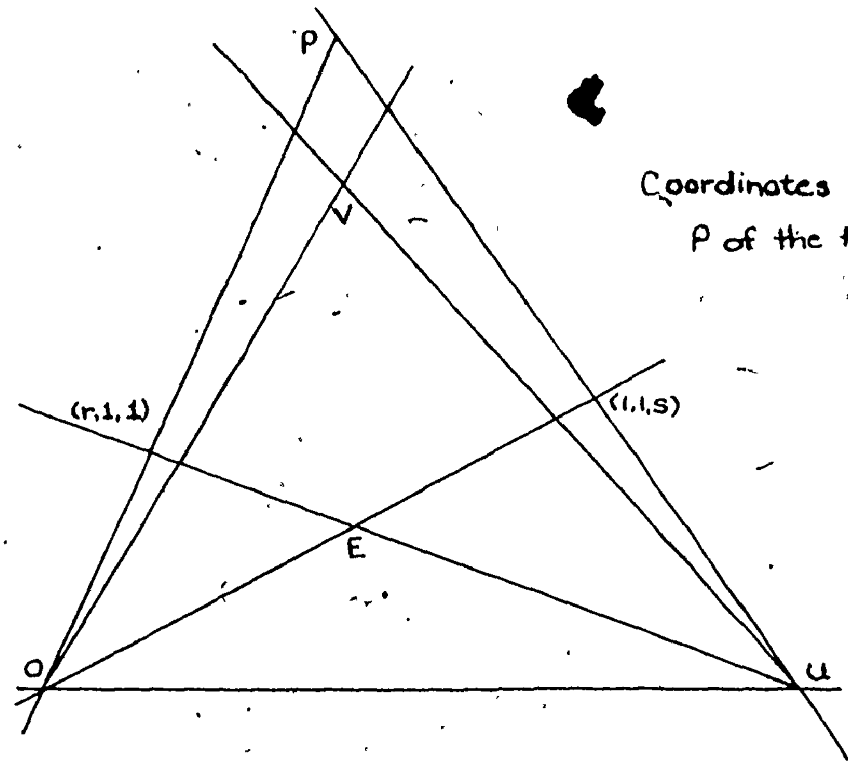
$U = OX \wedge \ell^* = L(\phi_{\bar{U}}(\Pi_{OX}), \alpha_{\bar{U}}(OX)) \wedge \psi_{\bar{U}}(\ell) = \phi_{\bar{U}}(\Pi_{OX}) \in OX \wedge m$ in $\mathcal{H}^*$.

To any point $P \in \mathbb{P}^* \setminus \mathbb{P}_\omega$, we assign the coordinates $(x,y,1)$ where $x = k \wedge PV$ and $y = k \wedge PU$. Thus $x, y \in H$. We also call such a proper point, a <u>point of the first kind</u>.

Now consider any line $h \in \mathbb{L}^* \setminus \mathbb{L}_\omega$. If $\bar{h} \wedge \bar{\ell}^* = \bar{V}$ (ie., if $\Pi_h \sim \Pi_\ell$ in $\mathcal{H}$), we call h a <u>line of the first kind</u> and assign it the coordinates $[1,m,n]$ where $(h \wedge \ell^*)O \wedge EU = (m,1,1)$ and $h \wedge OU = (n,0,1)$. It is clear that $[1,m,n]$ can represent a line of the first kind if and only if $m \in \eta$. If $\bar{h} \wedge \bar{\ell}^* \neq \bar{V}$, we call h a <u>line of the second kind</u> and assign it the coordinates $[m,1,n]$ where $(h \wedge \ell^*)O \wedge EV = (1,m,1)$ and $h \wedge \ell = (0,n,1)$.

If $P \, I \, OE$ and $P \in \mathbb{P}_\omega$, then $(PV \wedge UE)O \wedge EV = (1,s,1)$ for some $s \in \eta$. We assign P the coordinates $(1,1,s)$. To any $Q \in \mathbb{P}_\omega$, $\bar{Q} \neq \bar{V}$, we assign the coordinates $(1,y,t)$ $(y \in H,\ t \in \eta)$, where $QV \wedge OE = (1,1,t)$ and $QO \wedge EV = (1,y,1)$. We call such P and Q <u>points of the second kind</u>.

If $P \in \bar{V}$, we assign P the coordinates $(r,1,s)$ $(r, s \in \eta)$ where $PU \wedge OE = (1,1,s)$ and $PO \wedge UE = (r,1,1)$. Such a point P is called a <u>point of the third kind</u>.



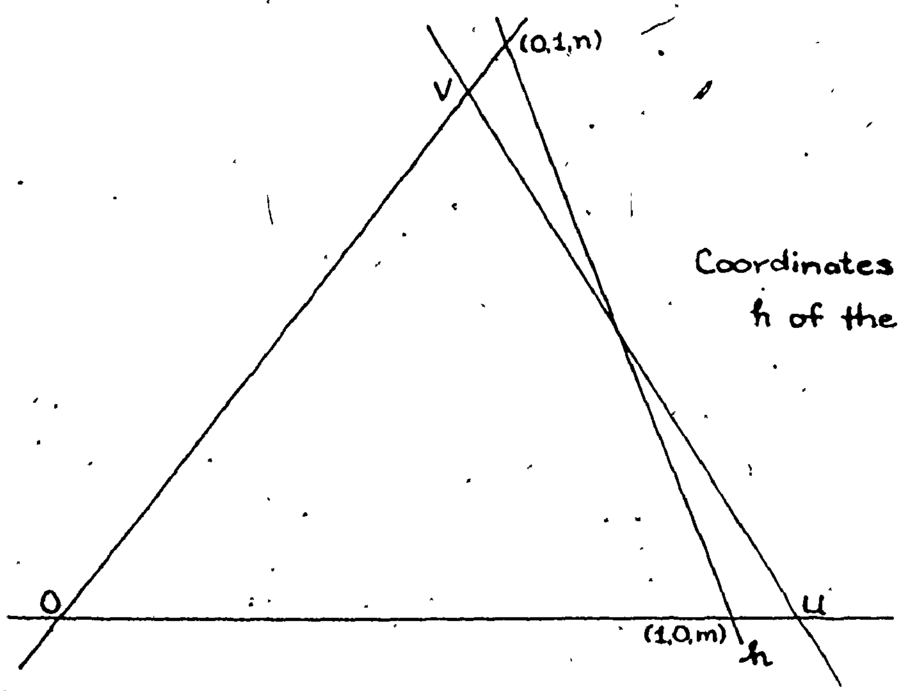


Figure 5.5
(Section 5.6)

Finally, we consider any line $h \in \mathbb{L}_\omega$. We call h a <u>line of the third kind</u> and assign it the coordinates $[m,n,1]$ $(m, n \in \eta)$ where $h \wedge OU = (1,0,m)$ and $h \wedge OV = (0,1,n)$.

<u>5.6.1</u>. <u>Remark</u>. This coordinatization is essentially the same as the one employed in [6] with two notable exceptions: the lines of the first and second kinds are interchanged and in each of these cases the coordinate m is defined in a slightly different manner. These changes enable us to obtain a marked similarity between the coordinates defined here for the proper points and lines and the coordinates defined for the corresponding points and lines of $\mathcal{H}$ in 2.1. The point $(x,y,1)$ corresponds to (x,y) in 2.1; the lines $[1,m,n]$ and $[m,1,n]$ correspond to $[m,n]_1$ and $[m,n]_2$, respectively.

<u>5.6.2</u>. <u>Lemma</u>. In $\mathcal{H}^*$,

$$\ell = OV = [1,0,0] = \{(0,y,1) \mid y \in H\} \cup \{(0,1,s) \mid s \in \eta\};$$

$$OU = [0,1,0] = \{(x,0,1) \mid x \in H\} \cup \{(1,0,s) \mid s \in \eta\};$$

$$OE = [1,1,0] = \{(x,x,1) \mid x \in H\} \cup \{(1,1,s) \mid s \in \eta\};$$

$$\ell^* = UV = [0,0,1] = \{(1,y,0) \mid y \in H\} \cup \{(t,1,0) \mid t \in \eta\};$$

$O = (0,0,1)$; $E = (1,1,1)$; $U = (1,0,0)$; $V = (0,1,0)$.

<u>Proof</u>. These follow directly from the definition of the coordinates.

<u>5.6.3</u>. <u>Lemma</u>. Two points (a_1,a_2,a_3) and (b_1,b_2,b_3) [Two lines $[a_1,a_2,a_3]$ and $[b_1,b_2,b_3]$] are neighbours if and only if $a_i \sim b_i$, for $i=1,2,3$.

<u>Proof</u>. First, we note that two points [lines] of different kinds cannot be neighbours by definition; cf. 5.4. For two points (a_1,a_2,a_3),

(b_1, b_2, b_3) [two lines $[a_1, a_2, a_3]$, $[b_1, b_2, b_3]$] of different kinds, there exists a pair x_i, y_i ($i \in \{1,2,3\}$) such that one of x_i, y_i is equal to 1 while the other lies in η.

For the points of the first kind and lines of the first and second kinds, the result follows from 5.6.1 and [11], 2.3, 2.4. By 5.4.5, all lines of the third kind (ie., $\{[m,n,1] \mid m, n \in \eta\}$) are neighbours and by definition their corresponding coordinates are neighbours. Similarly, all points of the third kind (ie., $\{(r,1,s) \mid r, s \in \eta\}$) belong to $\mathbb{P}_{\bar{V}}$, are neighbours and have neighbouring corresponding coordinates.

Finally, consider two points of the second kind $(1,a_2,a_3)$, $(1,b_2,b_3)$; a_2, $b_2 \in H$, a_3, $b_3 \in \eta$. Then by (H5)' and (H6)' $(1,a_2,a_3) \sim (1,b_2,b_3)$ if and only if $[a_2,1,0] = (0,0,1)(1,a_2,a_3) \sim (0,0,1)(1,b_2,b_3) = [b_2,1,0]$; hence if and only if $a_2 \sim b_2$.

<u>5.6.4</u>. <u>Lemma</u>. For any $m \in H$ [$m \in \eta$], $(1,m,0)$ I $[m,1,n]$ [$(m,1,0)$ I $[1,m,n]$] for all $n \in H$.

<u>Proof</u>. Consider any $[m,1,n]$, m, $n \in H$. By definition, $([m,1,n] \wedge \ell^*)O \wedge EV = (1,m,1)$ which implies $([m,1,n] \wedge \ell^*)$ I $O(1,m,1) = [m,1,0]$ and $[m,1,n] \wedge \ell^* = [m,1,0] \wedge \ell^* = [m,1,0] \wedge [0,0,1] = (1,m,0)$.

Now consider any $[1,m,n]$, $m \in \eta$, $n \in H$. By definition, $([1,m,n] \wedge \ell^*)O \wedge EU = (m,1,1)$ which implies $([1,m,n] \wedge \ell^*)$ I $[1,m,0]$ and $[1,m,n] \wedge \ell^* = [1,m,0] \wedge \ell^* = [1,m,0] \wedge [0,0,1] = (m,1,0)$.

5.7. The ternary operators defined on H.

5.7.1. We now use the incidence structure of $\mathcal{H}^*$ to define six ternary operators on H (cf. [6], 1.1).

By [11], 2.8.2, there exists a ternary operator

$$T_0 : H \times \eta \times H \longrightarrow H$$
$$(y,m,n) \rightsquigarrow T_0(y,m,n)$$

where $(T_0(y,m,n),y,1) = [0,1,y] \wedge [1,m,n]$ (cf. 2.1).

5.7.2. Lemma. For any $s \in \eta$ and any $[1,m,n]$ $(m \in \eta,\ n \in H)$, there exists a unique $r \in \eta$ such that $(r,1,s)\ I\ [1,m,n]$.

Proof. All points incident with $(1,1,s)U$ have last coordinate s. Since $[1,m,n] \cap \mathbb{P}_\omega \subseteq \bar{V}$, the unique (cf. 5.4.6) point $[1,m,n] \wedge (1,1,s)U$ is of the third kind; hence there exists a unique $r \in \eta$ with $(r,1,s)\ I\ [1,m,n]$.

5.7.3. We may therefore define a ternary operator

$$T_2 : \eta \times H \times \eta \longrightarrow \eta$$
$$(s,n,m) \rightsquigarrow T_2(s,n,m)$$

where $(T_2(s,n,m),1,s) = (1,1,s)U \wedge [1,m,n]$. Hence we have

$[1,m,n] = \{(T_0(y,m,n),y,1) \mid y \in H\} \cup \{(T_2(s,n,m),1,s) \mid s \in \eta\}$.

5.7.4. Similarly, there exist ternary operators

$$T_1 : H \times H \times H \longrightarrow H$$
$$(x,m,n) \rightsquigarrow T_1(x,m,n)$$

where $(x,T_1(x,m,n),1) = [1,0,x] \wedge [m,1,n]$ (cf. 2.1, [11], 2.8.1) and

$$T_3 : \eta \times H \times H \longrightarrow H$$
$$(t,n,m) \rightsquigarrow T_3(t,n,m)$$

where $(1,T_3(t,n,m),t) = (1,1,t)V \wedge [m,1,n]$. Hence

$[m,1,n] = \{(x,T_1(x,m,n),1) \mid x \in H\} \cup \{(1,T_3(t,n,m),t) \mid t \in \eta\}$.

5.7.5. Finally, we use the lines of the third kind to define two more ternary operators

$$T_4 : H \times \eta \times \eta \longrightarrow \eta$$

$$(y,n,m) \rightsquigarrow T_4(y,n,m)$$

where $(1,y,T_4(y,n,m)) = (1,y,1)0 \wedge [m,n,1]$ and

$$T_5 : \eta \times \eta \times \eta \longrightarrow \eta$$

$$(x,m,n) \rightsquigarrow T_5(x,m,n)$$

where $(x,1,T_5(x,m,n)) = (x,1,1)0 \wedge [m,n,1]$. Thus

$[m,n,1] = \{(1,y,T_4(y,n,m)) \mid y \in H\} \cup \{(x,1,T_5(x,m,n)) \mid x \in \eta\}$.

5.8. Properties of the ternary operators.

5.8.1. As in 2.2, $< H, T_1, T_0, 0, 1 >$ is a biternary ring.

5.8.2. Lemma. The following properties hold for $< H, T_0, T_1, T_2, T_3, T_4, T_5, 0, 1 >$.

i) $T_i(0,a,b) = b$; $i=0,1,2,3,4,5$; for all choices of a, b such that $T_i(0,a,b)$ is well-defined.

ii) $T_i(b,0,a) = a$; $i=0,1,2,3,4,5$; for all choices of a, b such that $T_i(b,0,a)$ is well-defined.

iii) $T_i(1,a,0) = a$; $i=0,1,4$; for all choices of a such that $T_i(1,a,0)$ is well-defined.

iv) $T_i(a,1,0) = a$; $i=1,3$; for all choices of a such that $T_i(a,1,0)$ is well-defined.

Proof. If $i = 0$ or 1, the results all follow from 5.8.1.

i) If $i = 2$, $a \in H$ and $b \in \eta$, then by definition $(T_2(0,a,b),1,0) = (1,1,0)U \wedge [1,b,a] = (b,1,0)$. If $i = 3$ and $a, b \in H$, then $(1,T_3(0,a,b),0) = (1,1,0)V \wedge [b,1,a] = (1,b,0)$. If $i = 4$ and $a, b \in \eta$, then $(1,0,T_4(0,a,b)) = (1,0,1)O \wedge [b,a,1] = (1,0,b)$. Finally, if $i = 5$ and $a, b \in \eta$, then $(0,1,T_5(0,a,b)) = (0,1,1)O \wedge [a,b,1] = [1,0,0] \wedge [a,b,1] = (0,1,b)$.

ii) If $i = 2$ and $a, b \in \eta$, then $(T_2(b,0,a),1,b) = (1,1,b)U \wedge [1,a,0]$ which implies that $((T_2(b,0,a),1,b)O = [1,a,0]$. Hence $(a,1,1) = [1,a,0] \wedge UE = (T_2(b,0,a),1,b)O \wedge UE = (T_2(b,0,a),1,1)$.

If $i = 3$, $a \in H$ and $b \in \eta$, then $(1,T_3(b,0,a),b) = (1,1,b)V \wedge [a,1,0]$ which implies $(1,T_3(b,0,a),b)O = [a,1,0]$. Hence $(1,a,1) = [a,1,0] \wedge EV = (1,T_3(b,0,a),b)O \wedge EV = (1,T_3(b,0,a),1)$.

If $i = 4$, $a \in \eta$ and $b \in H$, then $(1,b,T_4(b,0,a)) = (1,b,1)O \wedge [a,0,1] = (1,b,1)O \wedge (1,1,a)V$; however, since $(1,b,a)O \wedge EV = (1,b,1)$ and $(1,b,a)V \wedge OE = (1,1,a)$, $(1,b,a)\ I\ (1,b,1)O, (1,1,a)V$. Thus $(1,b,T_4(b,0,a)) = (1,b,a)$.

If $i = 5$ and $a, b \in \eta$, then $(b,1,T_5(b,0,a)) = (b,1,1)O \wedge [0,a,1]$. However $(b,1,a) = (b,1,1)O \wedge (b,1,a)U = (b,1,1)O \wedge (1,1,a)U = (b,1,T_5(b,0,a))$.

iii) Let $i = 4$ and $a \in \eta$. By definition,

$$(1,1,a)$$
$$= (0,1,a)U \wedge OE$$
$$= (0,1,a)(1,0,0) \wedge OE$$

$= [0,a,1] \wedge 0(1,1,1)$

$= (1,1,T_4(1,a,0))$.

iv) If $i = 3$, then $(1,1,a)V = [a,0,1]$. Hence

$(1,a,1)$

$= ((1,1,a)V \wedge UE)0 \wedge EV$

$= ([a,0,1] \wedge [0,1,1])0 \wedge EV$

$= (1,T_3(a,1,0),a)0 \wedge EV$.

Therefore,

$(1,T_3(a,1,0),a)$

$= (1,a,1)0 \wedge [a,0,1]$

$= [a,1,0] \wedge [a,0,1]$

$= (1,a,a)$.

Thus $T_3(a,1,0) = a$.

5.8.3. Lemma. $T_i(x,a,b) \sim b$; $i=2,3,4$; for all choices of x, a, b such that $T_i(s,a,b)$ is well-defined.

Proof. Consider $i = 2$, x, $b \in \eta$ and $a \in H$. Then $\ell^* \sim (1,1,x)U \nmid [1,b,a]$; hence

$(b,1,0) = \ell^* \wedge [1,b,a] \sim (1,1,x)U \wedge [1,b,a] = (T_2(x,a,b),1,x)$ by (H5)'.

Similarly, if $i = 3$, $x \in \eta$ and a, $b \in H$, then

$\ell^* \sim (1,1,x)V \nmid [b,1,a]$; hence

$(1,b,0) = \ell^* \wedge [b,1,a] \sim (1,1,x)V \wedge [b,1,a] = (1,T_3(x,a,b),x)$.

Finally, if $i = 4$, $x \in H$ and a, $b \in \eta$, then

$(1,x,0) = (1,x,1)0 \wedge \ell^* \sim (1,x,1)0 \wedge [b,a,1] = (1,x,T_4(x,a,b))$; hence

$T_4(x,a,b) \sim 0 \sim b$.

5.8.4. Lemma. $T_i(x,a,b) = b$; $i=0,1,2,3,4,5$; for all $x, a \in \eta$ and any b for which $T_i(x,a,b)$ is well-defined.

Proof. For $i = 0$, since $[1,a,b] \sim [1,0,b]$; $(b,0,1)\ I\ [1,a,b], [1,0,b]$; $(b,0,1) \sim (T_0(x,a,b),x,1)$ for any $x \in \eta$ (by the definition of T_0 and (H5)'), we have $(T_0(x,a,b),x,1)\ I\ [1,0,b]$ by the uniformity of $\mathcal{H}^*$; hence $(T_0(x,a,b),x,1) = (b,x,1)$.

Similarly, for $i = 1$, $[a,1,b] \sim [0,1,b]$; $(0,b,1)\ I\ [a,1,b], [0,1,b]$ and $(0,b,1) \sim (x,T_1(x,a,b),1)$ for any $x \in \eta$, imply $T_1(x,a,b) = b$. For $i = 2$, $[1,b,a] \sim [1,b,0]$; $(b,1,0)\ I\ [1,b,a], [1,b,0]$ (by 5.6.4) and $(b,1,0) \sim (T_2(x,a,b),1,x)$ (by 5.8.3) imply $T_2(x,a,b) = b$, by the uniformity of $\mathcal{H}^*$. For $i = 3$, $[b,1,a] \sim [b,1,0]$; $(1,b,0)\ I\ [b,1,a], [b,1,0]$ and $(1,T_3(x,a,b),x) \sim (1,b,0)$ imply $T_3(x,a,b) = b$. For $i = 4$, $[b,a,1] \sim [b,0,1]$; $(1,0,b)\ I\ [b,a,1], [b,0,1]$ and $(1,x,T_4(x,a,b)) \sim (1,0,b)$ for any $x \in \eta$ imply $T_4(x,a,b) = b$. Finally, if $i = 5$, $[a,b,1] \sim [0,b,1]$; $(0,1,b)\ I\ [a,b,1], [0,b,1]$ and $(0,1,b) = [a,b,1] \wedge [1,0,0] \sim [a,b,1] \wedge (x,1,1)0 = (x,1,T_5(x,a,b))$ for any $x \in \eta$, which imply $T_5(x,a,b) = b$.

5.8.5. Remark. Lemma 5.8.4 implies that we may ignore the ternary operator T_5 as it is just the identity map on the third variable. Therefore the set of points of $[m,n,1]$ given in 5.7.5 is $[m,n,1] = \{(1,y,T_4(y,m,n)) \mid y \in H\} \cup \{(x,1,n) \mid x \in \eta\}$.

5.8.6. Lemma. The equation $T_i(a,m,x) = b$; $i=0,1,2,3,4$; is uniquely sovable for x, for all a, m, b for which the equation is well-defined.

Proof. If $i = 0$ or 1, the result follows from 5.8.1.

Let $i = 2$. Then $a, b \in \eta$ and $m \in H$. There exists a unique line joining the points $(m,0,1)$ and $(b,1,a)$. This line must be of the first kind (as it contains points of the first and third kinds); hence there exists $x \in \eta$ such that $(b,1,a)$ I $[1,x,m]$ and $b = T_2(a,m,x)$ by 5.7.3.

Let $i = 3$. Then $m, b \in H$ and $a \in \eta$. There exists a unique line of the second kind joining the points $(1,b,a)$ and $(0,m,1)$; hence there exists a unique $x \sim b$ (by 5.8.3) such that $(1,b,a)$ I $[x,1,m]$ and $b = T_3(a,m,x)$ by 5.7.4.

Let $i = 4$. Then $a \in H$ and $b, m \in \eta$. There exists a unique line of the third kind joining the points $(1,a,b)$ and $(0,1,m)$; hence there exists a unique $x \in \eta$ such that $(1,a,b)$ I $[x,m,1]$ and $b = T_4(a,m,x)$ by 5.7.5.

5.8.7. Lemma. $T_i(x,m_1,n_1) = T_i(x,m_2,n_2)$; $i=2,3$; is uniquely solvable for x if and only if $n_1 \sim n_2$ and $m_1 \not\sim m_2$, for all m_1, m_2, n_1, n_2 for which $T_i(x,m_j,n_j)$ is well-defined $(i=2,3;\ j=1,2)$.

Proof. Let $i = 2$. Then $m_1, m_2 \in H$ and $n_1, n_2 \in \eta$; hence $n_1 \sim n_2$. If $m_1 \not\sim m_2$, then 5.6.3 implies $[1,n_1,m_1] \not\sim [1,n_2,m_2]$ and these lines have a unique intersection point. Moreover, (A7)' implies such an intersection point must be improper. By 5.7.3, this point is of the form $(T_2(x,m_1,n_1),1,x) = (T_2(x,m_2,n_2),1,x)$ for some unique $x \in \eta$. If $m_1 \sim m_2$, then $[1,n_1,m_1] \sim [1,n_2,m_2]$ and they cannot have a unique intersection point, so a unique solution x does not exist. (Actually in this case, if the two lines are distinct but not parallel in $\mathcal{H}$ (ie., $n_1 \neq n_2$), they must meet in a proper point by the projective uniformity of $\mathcal{H}$; hence by 5.4.2 no such x exists. If the two lines are parallel in $\mathcal{H}$, then they coincide in $\bar{V}$; hence any $x \in \eta$ is a solution.)

Let $i = 3$. Then $m_1, m_2, n_1, n_2 \in H$. If $n_1 \sim n_2$ and $m_1 \nsim m_2$, then the lines $[n_1,1,m_1]$ and $[n_2,1,m_2]$ have a unique intersection point and (A7)' implies that such an intersection point must be improper. By 5.7.4, this point is of the form $(1,T_3(x,m_1,n_1),x) = (1,T_3(x,m_2,n_2),x)$ for a unique $x \in \eta$. If $n_1 \nsim n_2$, then the lines $[n_1,1,m_1]$ and $[n_2,1,m_2]$ must meet in a single point of $\mathcal{H}$ by (A7)'. Hence by 5.4.2, there is no solution to the equation $T_3(x,m_1,n_1) = T_3(x,m_2,n_2)$. If $n_1 \sim n_2$ and $m_1 \sim m_2$, then the lines $[n_1,1,m_1]$ and $[n_2,1,m_2]$ cannot have a unique intersection point. If $n_1 \neq n_2$, then by the projective uniformity of $\mathcal{H}$, the lines $[n_1,1,m_1]$ and $[n_2,1,m_2]$ must meet in $\mathcal{H}$ and the equation has no solution; if $n_1 = n_2$, the lines must coincide in $\overline{(1,n_1,0)}$ and the equation has many solutions.

5.8.8. Remark. As all lines of the third kind are neighbours, the equation $T_4(x,m_1,n_1) = T_4(x,m_2,n_2)$ where $m_1, m_2, n_1, n_2 \in \eta$ never has a unique solution for x (cf. 5.7.5). The equation is solvable if $m_1 \neq m_2$. If $m_1 = m_2$ and $n_1 \neq n_2$, then the lines $[n_1,m_1,1]$ and $[n_2,m_2,1]$ coincide in $\bar{V}$; hence by 5.4.7, these lines do not intersect outside of $\bar{V}$ which implies that there is no solution to the equation $T_4(x,m_1,n_1) = T_4(x,m_2,n_2)$ by 5.7.5.

5.8.9. Lemma. The system $T_4(a_i,x,y) = b_i$ is uniquely solvable for the pair x, y if and only if $a_1 \not\sim a_2$.

Proof. By 5.6.3 and 5.4.8, there exists a unique line of the third kind through the pair of points $(1,a_1,b_1)$, $(1,a_2,b_2)$ if $a_1 \not\sim a_2$ and hence there is a unique solution to the system of equations $T_4(a_i,x,y) = b_i$. If $a_1 \sim a_2$, then $(1,a_1,b_1) \sim (1,a_2,b_2)$ and there are at least two lines through them. If these lines are of the third kind, the system $T_4(a_i,x,y) = b_i$ has at least two solutions; otherwise there are no solutions.

5.8.10. Remark. Neither the system $T_2(a_i,x,y) = b_i$ nor the system $T_3(a_i,x,y) = b_i$ is uniquely solvable as all the improper points incident with a given line of the first or second kind are neighbours.

5.8.11. Lemma. The system $y = T_3(x,m,n)$ and $x = T_4(y,u,v)$, where $m, n \in H$; $u, v \in \eta$ uniquely determines the pair x, y.

Proof. The lines $[n,1,m]$ and $[v,u,1]$ are not neighbours and hence have a unique intersection point, which by 5.7.4 and 5.7.5 must be of the second kind, say $(1,y,x)$. In addition, by 5.8.3, $y \sim n$ and $x \in \eta$ by definition.

5.8.12. Lemma. If $a_1 \sim a_2$; $b_1, b_2 \in \eta$ and $(1,a_1,b_1) \neq (1,a_2,b_2)$, then the system $b_i = T_4(a_i,u,v)$ $(i=1,2)$ is solvable with respect to u, v if and only if $b_1 = b_2$. The system $a_i = T_3(b_i,x,y)$ is solvable if and only if $b_1 \neq b_2$. The solvable system has at least two solutions with $x_1 \sim x_2$, $y_1 \sim y_2$ or $u_1 \sim u_2$, $v_1 \sim v_2$.

Proof. Consider the pair of points $(1,a_1,b_1)$, $(1,a_2,b_2)$ with

$a_1 \sim a_2$; $b_1, b_2 \in \eta$. If $b_1 = b_2$, then $(1,a_1,b_1)$, $(1,a_2,b_2)$ I $[b_1,0,1]$ and since $(1,a_1,b_1) \sim (1,a_2,b_2)$, there exists at least one other line of the third kind through them. Therefore, $b_i = T_4(a_i,u,v)$ has more than one solution. As there can be no line of the second kind through both points, $a_i = T_3(b_i,x,y)$ has no solution. If $b_1 \neq b_2$ and $(1,a_1,b_1)$, $(1,a_2,b_2)$ I $[u,v,1]$, for some $u, v \in \eta$, then by the uniformity of $\mathcal{K}^*$; $(1,a_1,b_1)$ I $[b_2,0,1]$; hence $b_1 = b_2$ (a contradiction). Therefore, $(1,a_1,b_1)$, $(1,a_2,b_2)$ must be incident with at least two lines of the second kind and $a_i = T_3(b_i,x,y)$ has at least two solutions.

5.8.13. Lemma. If $a_1, a_2, b_1, b_2 \in \eta$ and $b_1 \neq b_2$, then the system $T_2(b_i,u,v) = a_i$ has at least two solutions for u, v and $u_1 \sim u_2$, $v_1 \sim v_2$.

Proof. Consider the pair of points $(a_1,1,b_1)$, $(a_2,1,b_2)$. There exist at least two lines through these points and all lines through both points must be neighbours. By definition, these lines must be either of the first or third kind; however, if these points are incident with a line of the third kind, say $[u,v,1]$, then
$b_1 = T_4(a_1,v,u) = u = T_4(a_2,v,u) = b_2$; a contradiction. Therefore, these points are incident with lines of the form $[1,v,u]$ and the system $T_2(b_i,u,v) = a_i$ has at least two solutions.

5.8.14. Remark. As in the case of an A.H. plane and an associated biternary ring, it is possible to define an algebraic structure consisting of a set and five ternary operators which satisfy some of the properties in 5.8 and use it to construct a uniform P.H. plane (cf. Appendix I).

CHAPTER 6

The projective completion of a projectively ordered, projectively uniform A.H. plane

6.1. Consider any projectively ordered, projectively uniform A.H. plane $\mathcal{H} = \langle \mathbb{P}, \mathbb{L}, I, || \rangle$. If the neighbour relation on $\mathcal{H}$ is the trivial one, then the projective completion of $\mathcal{H}$ is just the ordered projective plane constructed by Heyting [8]. Henceforth, we may assume that $\mathcal{H}$ has a non-trivial neighbour relation. We may construct an arbitrary projective completion of $\mathcal{H}$ in the manner described in Chapter 5. However, in general, such a projective completion is not an ordered P.H. plane (cf. 7). By replacing the arbitrary bijections γ, $\phi_{\bar{X}}$ and $\alpha_{\bar{X}}$ used in the construction with special bijections and by restricting the choice of the arbitrary affine plane $\mathcal{A}(\bar{V})$, we are able to construct an ordered projective completion $\mathcal{H}^*$ for $\mathcal{H}$.

6.2. The construction of the ordered projective completion (cf. 5.3 and 4.1).

The underlying affine plane $\bar{\mathcal{H}}$ of $\mathcal{H}$ is ordered (cf. [10], 7) and may be extended to an ordered ordinary projective plane $\bar{\mathcal{H}}^*$ by the addition of a single line $\bar{l}_\omega$ (cf. [15] and [8]).

In $\bar{\mathcal{H}}^*$, let $\bar{g}$ be an arbitrary but fixed line distinct from $\bar{l}_\omega$ and $\bar{S}$ an arbitrary but fixed point not incident with $\bar{l}_\omega$, $\bar{g}$. Set $\bar{V} = \bar{g} \wedge \bar{l}_\omega$. For any $\bar{Y} \, \bar{I} \, \bar{l}_\omega$, $\bar{Y} \neq \bar{V}$, there exists a unique $\bar{Z} = \bar{Y}\bar{S} \wedge \bar{g}$. Let $\mathcal{A}(\bar{Y}) = \langle \mathbb{P}_{\bar{Y}}, \mathbb{L}_{\bar{Y}}, I_{\bar{Y}} \rangle$ be the isomorphic copy of $\mathcal{A}(\bar{Z})$ under any

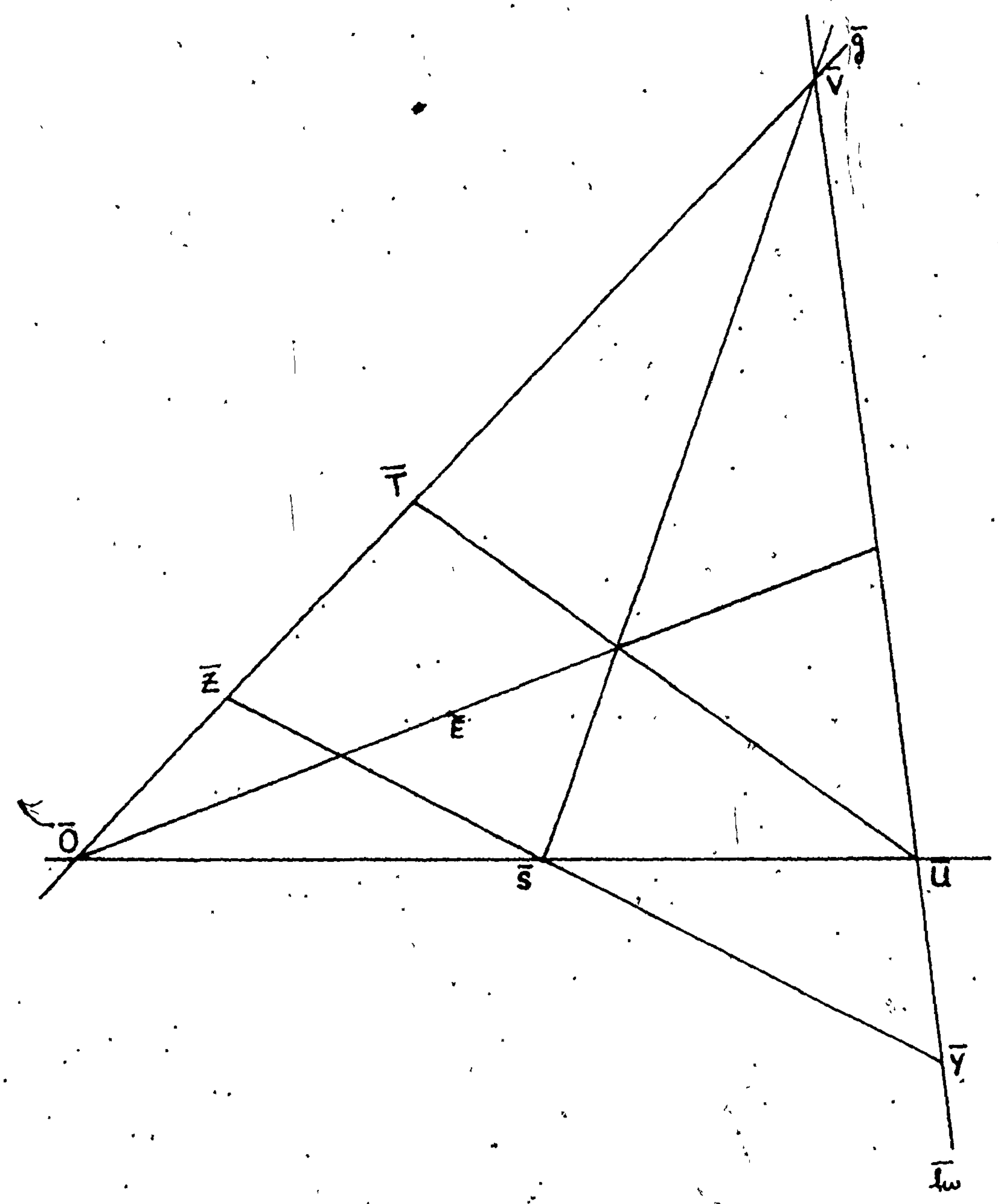


Figure 6.1
(Section 6.2)

isomorphism $\psi_{\bar{Y}} : \mathcal{A}(\bar{Z}) \longrightarrow \mathcal{A}(\bar{Y})$. Let $\bar{O}$ be an arbitrary, but fixed, point such that $\bar{O}\ \bar{I}\ \bar{g}$, $\bar{O} \neq \bar{V}$. Then there exists $\bar{U} = \bar{O}\bar{S} \wedge \bar{\ell}_\omega$. Let $\mathcal{A}(\bar{V}) = \langle \mathbb{P}_{\bar{V}}, \mathbb{L}_{\bar{V}}, I_{\bar{V}} \rangle$ be the isomorphic copy of $\mathcal{A}(\bar{O})$ under any isomorphism $\psi_{\bar{V}} : \mathcal{A}(\bar{O}) \longrightarrow \mathcal{A}(\bar{V})$. As in 5.3.1, we define $\mathbb{P}_\omega = \bigcup_{\bar{Y}\bar{I}\bar{\ell}_\omega} \mathbb{P}_{\bar{Y}}$ and $\mathbb{P}^* = \mathbb{P} \cup \mathbb{P}_\omega$.

Let $\bar{E}$ be an arbitrary, but fixed, point such that $\bar{E}\ \not{\bar{I}}\ \bar{\ell}_\omega, \bar{g}, \bar{O}\bar{U}$; ℓ an arbitrary, but fixed, line of $\mathcal{H}$ in $\bar{g}$; O an arbitrary, but fixed, element of $\bar{O}$, O I ℓ. Let $U = \psi_{\bar{U}}(O)$. Then take S to be an arbitrary, but fixed element of $\bar{S}$, S I OU.

We now construct the improper lines of $\mathcal{H}^*$. There exists $\bar{T} = (\bar{S}\bar{V} \wedge \bar{O}\bar{E})\bar{U} \wedge \bar{\ell}$ and $\bar{T} \neq \bar{O}, \bar{V}$. Select any point T in $\bar{T}$ such that T I ℓ. Choose a parallel pencil $\Pi_{\bar{V}}$ of $\mathcal{A}(\bar{V})$ such that $\psi_{\bar{V}}(OU) \notin \Pi_{\bar{V}}$. For any $h \in \mathbb{L}$, $h \sim \ell$, there exists $L(T,h) \wedge OU \in \bar{O}$. We define a bijection

$$\gamma : \{\Pi_h \mid h \in \bar{\ell}\} \longrightarrow \{m \in \mathbb{L}_{\bar{V}} \mid m \in \Pi_{\bar{V}}\}$$

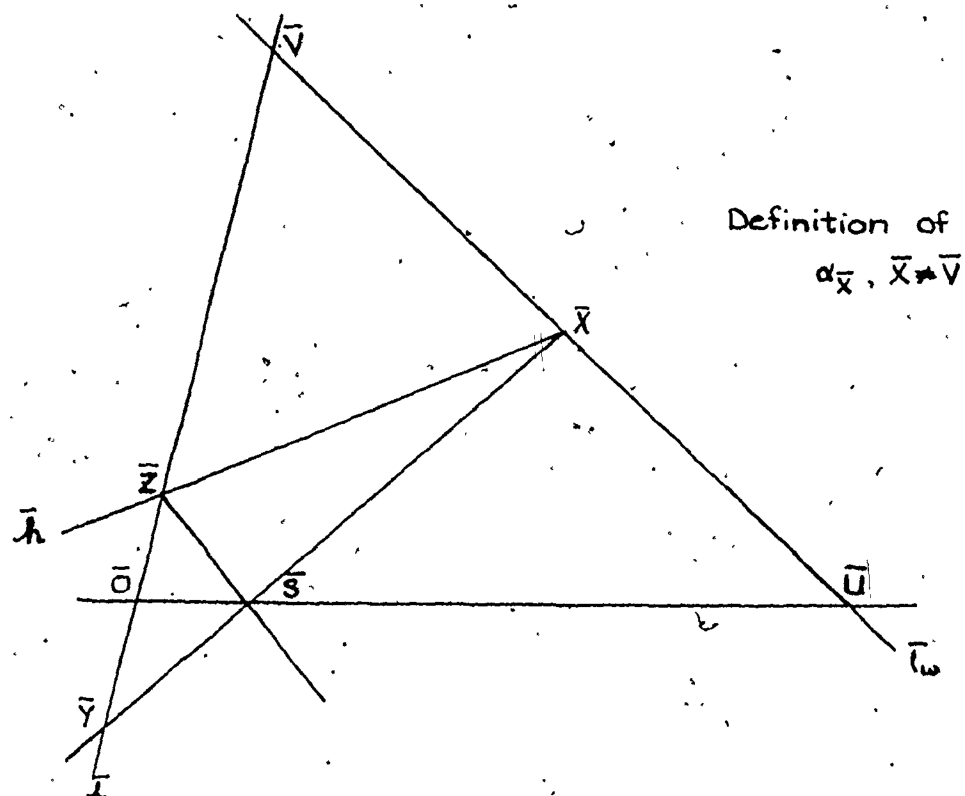
$$\Pi_h \rightsquigarrow \gamma(\Pi_h) = L(\psi_{\bar{V}}(L(T,h) \wedge OU), \Pi_{\bar{V}}).$$

Let $h^* = \bigcup_{\bar{Y}\bar{I}\bar{\ell}_\omega, \bar{Y} \neq \bar{V}} \{P \in \mathbb{P}_{\bar{Y}} \mid P\ I_{\bar{Y}}\ \psi_{\bar{Y}}(h)\} \cup \{P \in \mathbb{P}_{\bar{V}} \mid P\ I_{\bar{V}}\ \gamma(\Pi_h)\}$, for any $h \in \bar{\ell}$ and let $\mathbb{L}_\omega = \{h^* \mid h \in \bar{\ell}\}$.

Finally, we complete the lines of $\mathcal{H}$ by adding improper points to each proper line.

For any $\bar{X}$ in $\mathcal{H}^*$ such that $\bar{X}\ \bar{I}\ \bar{\ell}_\omega$, we define $\lambda(\bar{X}) = \ell^* \cap \mathbb{P}_{\bar{X}}$; thus $\lambda(\bar{X}) = \{P \in \mathbb{P}_{\bar{X}} \mid P\ I\ \psi_{\bar{X}}(\ell)\}$ if $\bar{X} \neq \bar{V}$ and $\lambda(\bar{V}) = \{P \in \mathbb{P}_{\bar{V}} \mid P\ I\ \gamma(\Pi_\ell)\}$. Let

$$\lambda_{\bar{X}} = \begin{cases} \psi_{\bar{X}}(\ell) & \text{if } \bar{X} \neq \bar{V} \\ \gamma(\Pi_\ell) & \text{otherwise.} \end{cases}$$



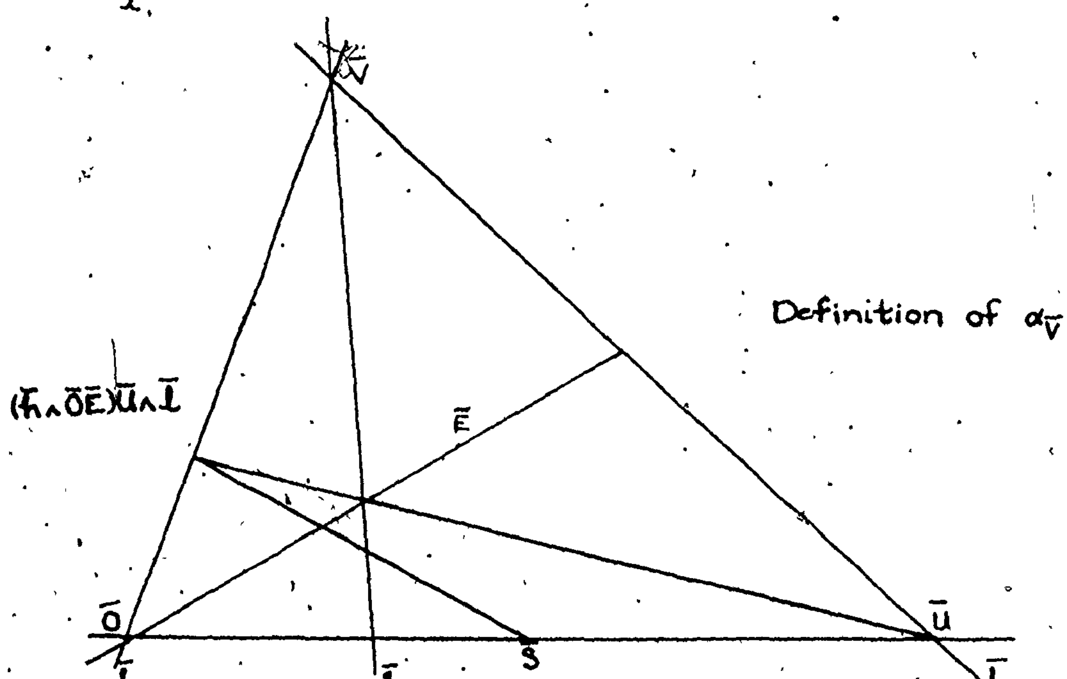


Figure 6.2
(Section 6.2)

We define a set of bijections $\alpha_{\bar{X}}$, for $\bar{X} \neq \bar{V}$, in the following way. For any $\bar{h} \in \bar{\mathbb{L}}^* \setminus \{\bar{\ell}_\omega\}$, such that $\bar{X} = \bar{h} \wedge \bar{\ell}_\omega$, there exists a point $\bar{Z} = \bar{h} \wedge \bar{\ell}$ and a line $\bar{Z}\bar{S}$. Since $\bar{Z}\bar{S}$ is a line of $\bar{\mathcal{H}}$, it is also a neighbour class of lines of $\mathcal{H}$. As $\bar{Z}\ \bar{I}\ \bar{Z}\bar{S}$, each line $m \in \bar{Z}\bar{S}$ of $\mathcal{H}$ is also a line of $\mathcal{A}(\bar{Z})$. For any two lines $m_1, m_2 \in \bar{Z}\bar{S}$, we have $m_1 || m_2$ by the uniformity of $\mathcal{H}$. Therefore $\bar{Z}\bar{S}$ is a parallel pencil of $\mathcal{A}(\bar{Z})$. There exists $\bar{Y} = \bar{X}\bar{S} \wedge \bar{\ell}$. Take any $m \in \bar{Z}\bar{S}$ and any $Y \in \bar{Y}$. Then $L(Y,m)$ (where this is the line through Y parallel to m in $\mathcal{H}$) is a line of $\mathcal{A}(\bar{Y})$ and $\Pi_{\bar{X},\psi_{\bar{X}}(L(Y,m))}$ is a parallel pencil of $\mathcal{A}(\bar{X})$. Since $\bar{S}\ \bar{I}\ \bar{\ell}$, $\Pi_{\bar{X},\psi_{\bar{X}}(L(Y,m))} \neq \Pi_{\psi_{\bar{X}}(\ell)}$. We define $\alpha_{\bar{X}}(\bar{h}) = \Pi_{\bar{X},\psi_{\bar{X}}(L(Y,m))}$.

<u>6.2.1</u>. <u>Lemma</u>. The map (where $\bar{X}\ \bar{I}\ \bar{\ell}_\omega$, $\bar{X} \neq \bar{V}$)

$$\alpha_{\bar{X}} : \{\bar{h} \in \bar{\mathbb{L}}^* \setminus \{\ell_\omega\} \mid \bar{X}\ \bar{I}\ \bar{h}\} \longrightarrow \{\Pi_{\bar{X},k} \mid \lambda_{\bar{X}} \notin \Pi_{\bar{X},k}\}$$

$$h \rightsquigarrow \alpha_{\bar{X}}(\bar{h}) = \Pi_{\bar{X},\psi_{\bar{X}}(L(Y,m))},$$

where $\Pi_{\bar{X},\psi_{\bar{X}}(L(Y,m))}$ is defined as above, is a bijection.

<u>Proof</u>. It is sufficient to show that this map is an injective function.

First we prove if $Y_1, Y_2 \in \bar{Y}$ and $m_1, m_2 \in \bar{Z}\bar{S}$, then $\Pi_{\bar{X},\psi_{\bar{X}}(L(Y_1,m_1))} = \Pi_{\bar{X},\psi_{\bar{X}}(L(Y_2,m_2))}$.

Since $m_1 ||_{\bar{Z}} m_2$, by the projective uniformity of $\mathcal{H}$, either $m_1 || m_2$ in $\mathcal{H}$ or there exists $P\ I\ m_1, m_2$, $P \notin \bar{Z}$ in $\mathcal{H}$. If $m_1 || m_2$, then $L(Y_1,m_1) || L(Y_2,m_2)$; hence $L(Y_1,m_1) ||_{\bar{Y}} L(Y_2,m_2)$. If there exists $P\ I\ m_1, m_2$, $P \in \bar{Z}$, then $L(Y_1,m_1) \sim L(Y_1,m_2)$ by (A7)'. By the uniformity of $\mathcal{H}$, these two lines coincide in $\bar{Y}$; hence $L(Y_1,m_1) ||_{\bar{Z}} L(Y_2,m_2)$. Since $\psi_{\bar{X}}$ is an isomorphism, the result follows.

Now we show that $\alpha_{\bar{X}}$ is injective.

Take two distinct lines $\bar{h}_1, \bar{h}_2 \in \bar{\mathbb{L}}^* \setminus \{\bar{\ell}_\omega\}$ such that $\bar{X}\ \bar{I}\ \bar{h}_1, \bar{h}_2$. Let $\bar{Z}_i = \bar{h}_i \wedge \bar{\ell}$ $(i = 1, 2)$. Then $\bar{Z}_1 \neq \bar{Z}_2$ and $\bar{Z}_1\bar{S} \neq \bar{Z}_2\bar{S}$. For any $m_i \in \bar{Z}_i\bar{S}$ $(i = 1, 2)$, $m_1 \nparallel m_2$. Therefore for any $Y \in \bar{Y}$, $L(Y,m_1) \nparallel L(Y,m_2)$ and Y is the unique point of intersection of these lines. Hence $L(Y,m_1) \not\parallel_{\bar{Y}} L(Y,m_2)$ and $\psi_{\bar{X}}(L(Y,m_1)) \not\parallel_{\bar{X}} \psi_{\bar{X}}(L(Y,m_2))$.

In addition, we define $\alpha_{\bar{V}}$ as follows. For any $\bar{h} \in \bar{\mathbb{L}}^* \setminus \{\bar{\ell}_\omega\}$ such that $\bar{V}\ \bar{I}\ \bar{h}$, there exists a neighbour class of lines $\bar{m}_o = ((\bar{h} \wedge \bar{O}\bar{E})\bar{U} \wedge \bar{\ell})\bar{S}$ of $\mathcal{H}$. Take any line m in this class. There exists $L(O,m)$ which is also a line of $\mathcal{A}(\bar{O})$ and the set of points of this line in $\bar{O}$ is independent of the choice of $m \in \bar{m}_o$ by the uniformity of $\mathcal{H}$. We define

$$\alpha_{\bar{V}} : \{\bar{h} \in \bar{\mathbb{L}}^* \setminus \{\bar{\ell}_\omega\} \mid \bar{V}\ \bar{I}\ \bar{h}\} \longrightarrow \{\Pi_{\bar{V},k} \mid \lambda_{\bar{V}} \not\subseteq \Pi_{\bar{V},k}\}$$

$$\bar{h} \rightsquigarrow \alpha_{\bar{V}}(\bar{h}) = \Pi_{\bar{V},\psi_{\bar{V}}(L(O,m))}.$$

Clearly, $\alpha_{\bar{V}}$ is a function. Since $\bar{h}_1, \bar{h}_2 \in \bar{\mathbb{L}}^* \setminus \{\bar{\ell}_\omega\}$, $\bar{V}\ \bar{I}\ \bar{h}_1, \bar{h}_2$ and $\bar{h}_1 \neq \bar{h}_2$ imply $((\bar{h}_1 \wedge \bar{O}\bar{E})\bar{U} \wedge \bar{\ell})\bar{S} \neq ((\bar{h}_2 \wedge \bar{O}\bar{E})\bar{U} \wedge \bar{\ell})\bar{S}$, $\alpha_{\bar{V}}$ is injective and hence bijective (cf. 5.2.4).

We define bijections $\phi_{\bar{X}}$ for $\bar{X}\ \bar{I}\ \bar{\ell}_\omega$ by

$$\phi_{\bar{X}} : \{\Pi_h \mid \bar{h}\ \bar{I}\ \bar{X}\} \longrightarrow \lambda(\bar{X})$$

$$\Pi_h \rightsquigarrow \phi_{\bar{X}}(\Pi_h) = \begin{cases} \psi_{\bar{X}}(L(S,h) \wedge \ell) & \text{if } \bar{X} \neq \bar{V}. \\ \psi_{\bar{V}}((L(T,h) \wedge OE)U \wedge \ell) & \end{cases}$$

Then the projective completion of the line $h \in \mathbb{L}$, where $\bar{h} \wedge \bar{\ell}_\omega = \bar{X}$, is

$$h' = \{P \in \mathbb{P} \mid P\ I\ h\} \cup \{P \in \mathbb{P}_{\bar{X}} \mid P\ I_{\bar{X}}\ L(\phi_{\bar{X}}(\Pi_h), \alpha_{\bar{X}}(\bar{h}))\}.$$

Let $\mathbb{L}^* = \{h' \mid h \in \mathbb{L}\} \cup \mathbb{L}_\omega$.

By 5.3, $\mathcal{H}^* = \langle \mathbb{P}^*, \mathbb{L}^*, I \rangle$ is a uniform P.H. plane. We may coordinatize $\mathcal{H}^*$ as in 5.6, using the triangle $O, X = VE \wedge OU, Y = UE \wedge \ell$.

Then $H = \{P \in \mathbb{P}^* \setminus \mathbb{P}_\omega \mid P \text{ I } OE\}$ with the ternary operators T_0, T_1 defined as in section 5.7, is a strongly ordered biternary ring (cf. 3.4.2). Additional ternary operators T_2, T_3, T_4 may also be defined on H as in 5.7 and they satisfy all the properties given in section 5.8. We shall prove they also satisfy certain order properties in 6.4, but first we investigate the orderings $\rho|_{\bar{X}}$ induced on the affine planes $\mathcal{A}(\bar{X})$ by the ordering ρ of $\mathcal{H}$.

6.3. The induced order relation on each $\mathcal{A}(\bar{X})$

6.3.1. Let $P \in \mathbb{P}^* \setminus \mathbb{P}_\omega$. We may define an order relation $\rho|_{\bar{P}}$ in the following way. For any $A, B, C \in \mathbb{P}_{\bar{P}}$, $(A,B,C) \in \rho|_{\bar{P}}$ if and only if $(A,B,C) \in \rho$.

6.3.2. Lemma. $\mathcal{A}(\bar{P}) = \langle \mathbb{P}_{\bar{P}}, \mathbb{L}_{\bar{P}}, I_{\bar{P}}, ||_{\bar{P}}, \rho|_{\bar{P}} \rangle$ is an ordered affine plane.

Proof. By 2.4.1, $\mathcal{A}(\bar{P})$ is an ordinary affine plane, so it will be sufficient to show $\rho|_{\bar{P}}$ is an ordering of $\mathcal{A}(\bar{P})$ and each line contains three distinct points (cf. 3.1.).

As ρ is an ordering of $\mathcal{H}$, axioms (O1) - (O5) are clearly satisfied. To prove (O6) holds, we need merely show that a parallel projection in $\mathcal{A}(\bar{P})$ is a parallel projection of $\mathcal{H}$. Consider any two lines $m_1, m_2 \in \mathbb{L}_{\bar{P}}$ such that $m_1 ||_{\bar{P}} m_2$. If $m_1 || m_2$ in $\mathcal{H}$, the result is clear, so suppose $m_1 \nparallel m_2$. Then for any $Q \in \bar{P}$, $Q \text{ I } m_1$, there exists $L(Q,m_2)$. If $L(Q,m_2) \nsim m_1$, then by (A7)' and (A5)', $m_1 \nsim m_2$ and there exists $R = m_1 \wedge m_2 \in \bar{\mathbb{P}}$, a contradiction. Hence $L(Q,m_2) \sim m_1$ and by the uniformity of $\mathcal{H}$, these two lines must coincide in $\mathcal{A}(\bar{P})$. Hence we may replace m_1 by $L(Q,m_2)$. Thus all parallel projections of $\mathcal{A}(\bar{P})$

are also parallel projections of $\mathcal{H}$.

We now show that there exist at least three points on one line. Consider the ordinary affine plane $\mathcal{A}(\bar{O})$ and the line $[0,1,0]$ of $\mathcal{A}(\bar{O})$. Since the neighbour relation on $\mathcal{H}$ is non-trivial, $\eta \neq \{0\}$ and there exists $a \in \eta$, $a \neq 0$. Then the points $(0,0,1)$, $(a,0,1)\ I_{\bar{O}}\ [0,1,0]$. By ($\beta$1), there exists a unique $b \in H$ such that $T_1(a,1,b) = 0$ and by (β2) and (β1), $b = T_1(0,1,b) \sim T_1(a,1,b) = 0$; ie., $b \in \eta$. If $0 < a$ $[0 > a]$, then 3.1.4(3) implies $b = T_1(0,1,b) < T_1(a,1,b) = 0$ $[b = T_1(0,1,b) > T_1(a,1,b) = 0]$ and $b \neq 0,a$; hence in $\mathcal{A}(\bar{O})$ the three mutually distinct points $(b,0,1)$, $(0,0,1)$, $(a,0,1)\ I_{\bar{O}}\ [0,1,0]$. Then all lines of $\mathcal{A}(\bar{O})$ are incident with at least three points and by 5.2.2, all lines of every $\mathcal{A}(\bar{P})$, $P \in \mathbb{P}^* \setminus \mathbb{P}_\omega$ are incident with at least three points.

6.3.3. We now use the order relations $\rho|_{\bar{P}}$ on $\mathcal{A}(\bar{P})$, $P \in \mathbb{P}^* \setminus \mathbb{P}_\omega$ to induce order relations on each $\mathcal{A}(\bar{X})$, $X \in \mathbb{P}_\omega$ in the following way. For three points $A, B, C \in \mathbb{P}_{\bar{X}}$, $(A,B,C) \in \rho|_{\bar{X}}$ if and only if $(\psi_{\bar{X}}^{-1}(A), \psi_{\bar{X}}^{-1}(B), \psi_{\bar{X}}^{-1}(C)) \in \rho|_{\overline{(\psi_{\bar{X}}^{-1}(X))}}$. It is clear that each $\mathcal{A}(\bar{X}) = \langle \mathbb{P}_{\bar{X}}, \mathbb{L}_{\bar{X}}, I_{\bar{X}}, ||_{\bar{X}}, \rho|_{\bar{X}} \rangle$ is an ordered affine plane.

<u>6.3.4</u>. Let $P \in \mathbb{P}_\omega$, $\bar{P} \neq \bar{V}$. We call $\psi_{\bar{P}}^{-1}(P)$ which is a point of $\mathcal{A}(\overline{\psi_{\bar{P}}^{-1}(P)})$, the <u>preimage</u> of P. Similarly, for any $m \in \mathbb{L}_{\bar{P}}$, we call $\psi_{\bar{P}}^{-1}(m)$ (which is a line of $\mathcal{A}(\overline{\psi_{\bar{P}}^{-1}(P)})$) the <u>preimage</u> of m. If $h^* \in \mathbb{L}_\omega$, then

$$h^* = \bigcup_{\bar{Y} I \ell_\omega, \bar{Y} \neq \bar{V}} \{Q \in \mathbb{P}_{\bar{Y}} \mid Q\ I_{\bar{Y}}\ \psi_{\bar{Y}}(h)\} \cup \{Q \in \mathbb{P}_{\bar{V}} \mid Q\ I_{\bar{V}}\ \gamma(\Pi_h)\}$$

(cf. 6.2). So each $\psi_{\bar{Y}}(h)$ has h (considered as a line of $\mathcal{A}(\overline{\psi_{\bar{Y}}^{-1}(Y)})$)

as its preimage. We therefore call the line h of $\mathcal{H}$ the <u>preimage</u> of h^*. For any line $m \in \mathcal{L}^*$, let $m|_{\bar{Q}} = \{R \in \bar{Q} \mid R\ I\ m\}$. Hence $m|_{\bar{Q}}$ is actually a line of $\mathcal{A}(\bar{Q})$ considered as a point set and we may also use $m|_{\bar{Q}}$ to represent the line of $\mathcal{A}(\bar{Q})$ through these points.

<u>6.3.5</u>. The point $\bar{S}$ in $\overline{\mathcal{H}}$ played an important role in the construction of $\mathcal{H}^*$ from $\mathcal{H}$ (cf. 5.3.1 and 6.2); therefore the location of $\bar{S}$ in $\overline{\mathcal{H}}$ (expressed in terms of the relation $\bar{\rho}$ with respect to other known points of $\overline{\mathcal{H}}$) also plays an important role in the relationship between the improper points and their preimages in $\mathcal{H}$. By the definition, $\psi_{\bar{U}}(0) = U$, $\bar{S}\ \bar{I}\ \bar{O}\bar{U}$, $\bar{S} \neq \bar{O},\bar{U}$; hence $\bar{S}$ must be of the form $\overline{(s,0,1)}$ where $s \in H \setminus \eta$. Without loss of generality, we may let $S = (s,0,1)$. There are two possibilities: either $s > 0$ or $s < 0$.

<u>6.3.6</u>. <u>Lemma</u>. Let $m < p$ and $(0,n,1)$ and $(0,q,1)$ be the preimages of $(1,m,0)$ $(1,p,0)$ respectively. Then $s > 0$ [$s < 0$] implies $n > q$ [$n < q$]. In addition, $m \not\sim p$ if and only if $n \not\sim q$.

<u>Proof</u>. First suppose $m \not\sim p$ and $m < p$.

By definition, the preimage of $(1,m,0)$ is in the neighbour class $\overline{(1,m,0)}\bar{S} \wedge \bar{\mathcal{L}} = \overline{(0,n_1,1)}$ for some $n_1 \in H$; in fact, as $(1,m,0)\ I\ \mathcal{L}^*$, the preimage of $(1,m,0)$ is $(0,n,1)$ for some $n \in \bar{n}_1$. Similarly, the preimage of $(1,p,0)$ is $(0,q,1)$ for some $q \in H$. As $m \not\sim p$, $\overline{(1,m,0)}\bar{S} \wedge \bar{\mathcal{L}} \neq \overline{(1,p,0)}\bar{S} \wedge \bar{\mathcal{L}}$; hence $n \not\sim q$. The lines $[m,1,n]$ and $[p,1,q]$ both pass through the neighbour class $\bar{S}$ (since by definition, $\overline{(1,m,0)}\bar{S} = \overline{(1,m,0)}\,\overline{(0,n,1)}$ and $\overline{(1,p,0)}\bar{S} = \overline{(1,p,0)}\,\overline{(0,q,1)}$). Since $m \not\sim p$, $[m,1,0] \not\sim [p,1,0]$; hence by (A7)', $[m,1,n] \not\sim [p,1,q]$ and there exists $a \in H$ such that $(a,T_1(a,m,n),1) = (a,T_1(a,p,q),1) = [m,1,n] \wedge [p,1,q]$.

By (A5)', $a \sim s \nmid 0$. Therefore $a > 0$ [$a < 0$] and $n = T_1(0,m,n) > T_1(0,p,q) = q$ [$n = T_1(0,m,n) < T_1(0,p,q) = q$] by (OM3).

Now suppose $m \sim p$ and $m < p$.

Then the preimages of $(1,m,0)$ and $(1,p,0)$ are in the neighbour class $\overline{(1,m,0)}\bar{S} \wedge \bar{l} = \overline{(1,p,0)}\bar{S} \wedge \bar{L}$. As $(1,m,0)$, $(1,p,0)$ I l^*, their preimages are of the form $(0,n,1)$, $(0,q,1)$ respectively with $n \sim q$. By 2.2, $(1,m,0)$ I $[m,1,0]$ as well as l^*. However by definition, $\phi_{\overline{(1,m,0)}}(\Pi_{[m,1,0]})$ I $[m,1,0], l^*$ also. Therefore

$$\begin{aligned}\psi_{\overline{(1,m,0)}}((0,n,1)) &= (1,m,0)\\ &= \phi_{\overline{(1,m,0)}}(\Pi_{[m,1,0]})\\ &= \psi_{\overline{(1,m,0)}}(L(S,[m,1,0]) \wedge l),\end{aligned}$$

which implies n is the unique value for which $T_1(s,m,n) = 0$. Similarly, q is the unique value for which $T_1(s,p,q) = 0$. If $s > 0$ [$s < 0$], then 3.3.2(4) implies $n = T_1(0,m,n) > T_1(0,p,q) = q$ [$n = T_1(0,m,n) < T_1(0,p,q) = q$].

<u>6.3.7</u>. <u>Lemma</u>. Let $m, p \in \eta$ such that $m < p$ and let $\psi_{\bar{V}}^{-1}((m,1,0)) = (0,n,1)$, $\psi_{\bar{V}}^{-1}((p,1,0)) = (0,q,1)$. Then $s > 0$ [$s < 0$] implies $n > q$ [$n < q$].

<u>Proof</u>. By definition, $\bar{T} = (\bar{S}\bar{V} \wedge \bar{O}\bar{E})\bar{U} \wedge \bar{l}$ and T is any point of $\bar{T}$ incident with l. By the uniformity of $\mathcal{H}$, for any line h such that $\bar{h}$ I $\bar{V}$, $L(T,h) = L((0,s,1),h)$. Consider the line $[1,m,0]$. Let $t \in \eta$ such that $T_0(s,m,t) = 0$. Then $[1,m,t] \wedge [1,1,0] = (t,t,1)$ by 3.8.8. By definition,

$$\psi_{\bar{V}}((0,t,1)) = \psi_{\bar{V}}((L((0,s,1),[1,m,0]) \wedge OE)U \wedge l)$$

$$= \phi_{\bar{V}}(\Pi_{[1,m,0]})$$
$$= [1,m,0] \wedge \ell^*$$
$$= (1,m,0)$$
$$= \psi_{\bar{V}}((0,n,1));$$

ie., $t = n$ and $T_0(s,m,n) = 0$. Similarly, $(p,1,0) = \psi_{\bar{V}}((0,q,1))$ where $T_0(s,p,q) = 0$.

Suppose $s > 0$ [$s < 0$]. Since $m < p$, 3.3.2 implies $n = T_0(0,m,n) > T_0(0,p,q) = q$ [$n = T_0(0,m,n) < T_0(0,p,q) = q$].

6.4. Order properties for ternary operators.

6.4.1. In the following section, we shall make use of the preimages of some improper points and lines to prove various order properties for the ternary operators T_2, T_3, T_4. We shall use the following pairs of improper points/lines and preimages with the possible addition of subscripts when the need arises.

As in Lemma 6.3.6, a point $(1,m,0) = \psi_{\overline{(1,m,0)}}(L(S,[m,1,0]) \wedge \ell)$ and hence has as preimage the point $(0,n,1)$ where $T_1(s,m,n) = 0$. Let $(1,1,0)$ and $(1,r,0)$ have preimages $(0,p,1)$ and $(0,u,1)$ respectively, where $T_1(s,1,p) = 0$ and $T_1(s,r,u) = 0$.

Now consider a line $[m,1,0]$ of $\mathcal{H}^*$. By definition, $[m,1,0]|_{\overline{(1,m,0)}} = L(\phi_{\overline{(1,m,0)}}(\Pi_{[m,1,0]}), \alpha_{\overline{(1,m,0)}}([m,1,0]))$. As $(0,n,1)$ is the preimage of $(1,m,0)$, $\vec{I} \wedge [\overline{m,1,0}] = (\overline{0,0,1})$ and $(\overline{0,0,1})\bar{S} = \overline{OU} = [\overline{0,1,0}]$, we obtain

$$\alpha_{\overline{(1,m,0)}}([\overline{m,1,0}]) = \Pi_{\overline{(1,m,0)}}, \psi_{\overline{(1,m,0)}}([0,1,n]).$$

Hence

$$[m,1,0]|_{\overline{(1,m,0)}} = L((1,m,0), \Pi_{\overline{(1,m,0)}}, \psi_{\overline{(1,m,0)}}([0,1,n])$$

$$= \psi_{\overline{(1,m,0)}}(L((0,1,n),\Pi_{(0,n,1)},[0,1,n]))$$
$$= \psi_{\overline{(1,m,0)}}([0,1,n]|_{\overline{(0,1,n)}}).$$

Thus the preimage of $[m,1,0]|_{\overline{(1,m,0)}}$ is $[0,1,n]|_{\overline{(0,n,1)}}$.

Similarly, the lines $[1,1,0]|_{\overline{(1,1,0)}}$ and $[0,1,0]|_{\overline{(1,0,0)}}$ have the preimages $[0,1,p]|_{\overline{(0,p,1)}}$ and $[0,1,0]|_{\overline{(0,0,1)}}$ respectively.

Now consider a line $[\alpha,\beta,1]$ of $\mathcal{H}^*$. Since it is an improper line, it has as preimage a line of the form $[1,b,a]$ where $a,b \in \eta$. Now consider the line $[0,\beta,1]$. Since $[\alpha,\beta,1]$ and $[0,\beta,1]$ coincide in $\bar{V}$ (cf. 5.8.5) and do not meet elsewhere, their preimages in $\mathcal{H}$ cannot meet. Since both preimages belong to $\bar{l}$, they must be parallel by the projective uniformity of $\mathcal{H}$. However, $(1,0,0)$ I $[0,\beta,1]$ which implies that the preimage of $[0,\beta,1]$ must pass through $(0,0,1)$. Thus the preimage of $[0,\beta,1]$ must be $[1,b,0]$. Next consider the line $[\alpha,0,1]$. The lines l^* and $[\alpha,0,1]$ coincide in $\bar{V}$ and fail to meet elsewhere; hence the preimage of $[\alpha,0,1]$ cannot meet l and is therefore, by the projective uniformity of $\mathcal{H}$, parallel to l. In addition, as $[0,1,0] \wedge [\alpha,\beta,1] = (1,0,\alpha)$ I $[\alpha,0,1]$, $[0,1,0]|_{\overline{(0,0,1)}} \wedge [1,b,a] = (a,0,1)$ (ie., $(a,0,1)$ is the primage of $(1,0,\alpha)$) must be incident with the preimage of $[\alpha,0,1]$. Thus $[1,0,a]$ is the preimage of $[\alpha,0,1]$.

Since $(1,1,\alpha) = [\alpha,0,1] \wedge [1,1,0]$, the preimage of $(1,1,\alpha)$ is $[1,0,a] \wedge [0,1,p] = (a,p,1)$ and since $(1,1,\beta) = [0,\beta,1] \wedge [1,1,0]$, the preimage of $(1,1,\beta)$ is $[1,b,0] \wedge [0,1,p] = (T_0(p,b,0),p,1)$.

For reference we tabulate these results.

	PREIMAGE	IMAGE
POINTS	$(0,n,1)$	$(1,m,0)$
	$(0,p,1)$	$(1,1,0)$
	$(0,u,1)$	$(1,r,0)$
	$(a,p,1)$	$(1,1,\alpha)$
	$(T_0(p,b,0),p,1)$	$(1,1,\beta)$
	$(a,0,1)$	$(1,0,\alpha)$
PROPER LINES	$[0,1,n]\|\overline{(0,n,1)}$	$[m,1,0]\|\overline{(1,m,0)}$
	$[0,1,p]\|\overline{(0,p,1)}$	$[1,1,0]\|\overline{(1,1,0)}$
	$[0,1,0]\|\overline{(0,0,1)}$	$[0,1,0]\|\overline{(1,0,0)}$
IMPROPER LINES	$[1,b,a]$	$[\alpha,\beta,1]$
	$[1,b,0]$	$[0,\beta,1]$
	$[1,0,a]$	$[\alpha,0,1]$

Table 6.4.1.

6.4.2. Lemma. If $\alpha_1, \alpha_2, \alpha_3 \in \eta$ such that $(\alpha_1,\alpha_2,\alpha_3) \in \rho$, then $((1,T_3(\alpha_1,q,m),\alpha_1),(1,T_3(\alpha_2,q,m),\alpha_2),(1,T_3(\alpha_3,q,m),\alpha_3)) \in \rho|\overline{(1,m,0)}$.

Proof. Since $(\alpha_1,\alpha_2,\alpha_3) \in \rho$, we have $((1,\alpha_1,0),(1,\alpha_2,0),(1,\alpha_3,0)) \in \rho|_{\bar{U}}$. The three lines $[\alpha_i,1,0]$ $(i=1,2,3)$ meet at $(0,0,1)$; hence they cannot meet in $\mathcal{A}(\bar{U})$. Thus $[\alpha_i,1,0]|_{\bar{U}} \,||_{\bar{U}}\, [\alpha_j,1,0]|_{\bar{U}}$, for all $i,j \in \{1,2,3\}$. The parallel projections $\ell^*|_{\bar{U}} \xrightarrow{[\alpha_i,1,0]|_{\bar{U}}} [0,1,1]|_{\bar{U}} \xrightarrow{\ell^*|_{\bar{U}}} [0,1,0]|_{\bar{U}}$ in $\mathcal{A}(\bar{U})$ yield $((1,0,\alpha_1),(1,0,\alpha_2),(1,0,\alpha_3)) \in \rho|_{\bar{U}}$. By definition $(1,0,\alpha_i)V = [\alpha_i,0,1]$ $(i=1,2,3)$.

Consider the preimages of the lines $[\alpha_i,0,1]$ $(i=1,2,3)$ and $[m,1,0]$ (cf. 6.4.1) and their intersections. Clearly, $(1,m,\alpha_i) = [\alpha_i,0,1] \wedge [m,1,0]$ has preimage $[1,0,a_i] \wedge [0,1,n]|_{\overline{(0,n,1)}} = (a_i,n,1)$. Since $((1,0,\alpha_1),(1,0,\alpha_2),(1,0,\alpha_3)) \in \rho|_{\bar{U}}$, $((a_1,0,1),(a_2,0,1),(a_3,0,1)) \in \rho|_{\bar{0}}$ by 6.3.7; ie., $(a_1,a_2,a_3) \in \rho$. Therefore $((a_1,n,1),(a_2,n,1),(a_3,n,1)) \in \rho|_{\overline{(0,n,1)}}$ and by 6.3.4 $((1,m,\alpha_1),(1,m,\alpha_2),(1,m,\alpha_3)) \in \rho|_{\overline{(1,m,0)}}$. As the lines $[\alpha_i,0,1]$ meet in $\bar{V}$, they cannot meet in $\mathcal{A}(\overline{(1,m,0)})$ and hence are parallel there. Using the parallel projection $[m,1,0]|_{\overline{(1,m,0)}} \xrightarrow{[\alpha_i,0,1]} [m,1,q]|_{\overline{(1,m,0)}}$, we obtain

$$((1,T_3(\alpha_1,q,m),\alpha_1),(1,T_3(\alpha_2,q,m),\alpha_2),(1,T_3(\alpha_3,q,m),\alpha_3)) \in \rho|_{\overline{(1,m,0)}}.$$

<u>6.4.3</u>. <u>Lemma</u>. Let $\alpha, \beta \in \eta$ and $m_1, m_2, m_3 \in H$ such that $(m_1,m_2,m_3) \in \rho$ and $m_1 \not\sim m_2 \not\sim m_3 \not\sim m_1$. Then $(T_4(m_1,\beta,\alpha),T_4(m_2,\beta,\alpha),T_4(m_3,\beta,\alpha)) \in \rho$.

<u>Proof</u>. Since $(1,m_i,T_4(m_i,\beta,\alpha)) = [m_i,1,0] \wedge [\alpha,\beta,1]$, it has as preimage the point $[0,1,n_i] \wedge [1,b,a] = (T_0(n_i,b,a),n_i,1)$ $(i=1,2,3)$. In addition, $[T_4(m_i,\beta,\alpha),0,1] = (1,m_i,T_4(m_i,\beta,\alpha))V$, so its preimage is $L((T_0(n_i,b,a),n_i,1),\iota) = [1,0,T_0(n_i,b,a)]$ $(i=1,2,3)$. Therefore, the point $(1,0,T_4(m_i,\beta,\alpha)) = [0,1,0] \wedge [T_4(m_i,\beta,\alpha),0,1]$ has as preimage the point $[0,1,0] \wedge [1,0,T_0(n_i,b,a)] = (T_0(n_i,b,a),0,1)$.

Since $(m_1,m_2,m_3) \in \rho$ and $m_1 \not\sim m_2 \not\sim m_3 \not\sim m_1$, 6.3.6 implies $(n_1,n_2,n_3) \in \rho$ and $n_1 \not\sim n_2 \not\sim n_3 \not\sim n_1$. By 3.3.3(3), $(T_0(n_1,b,a),T_0(n_2,b,a),T_0(n_3,b,a)) \in \rho$; hence $((T_0(n_1,b,a),0,1),(T_0(n_2,b,a),0,1),(T_0(n_3,b,a),0,1)) \in \rho|_{\bar{0}}$ and by 6.3.3

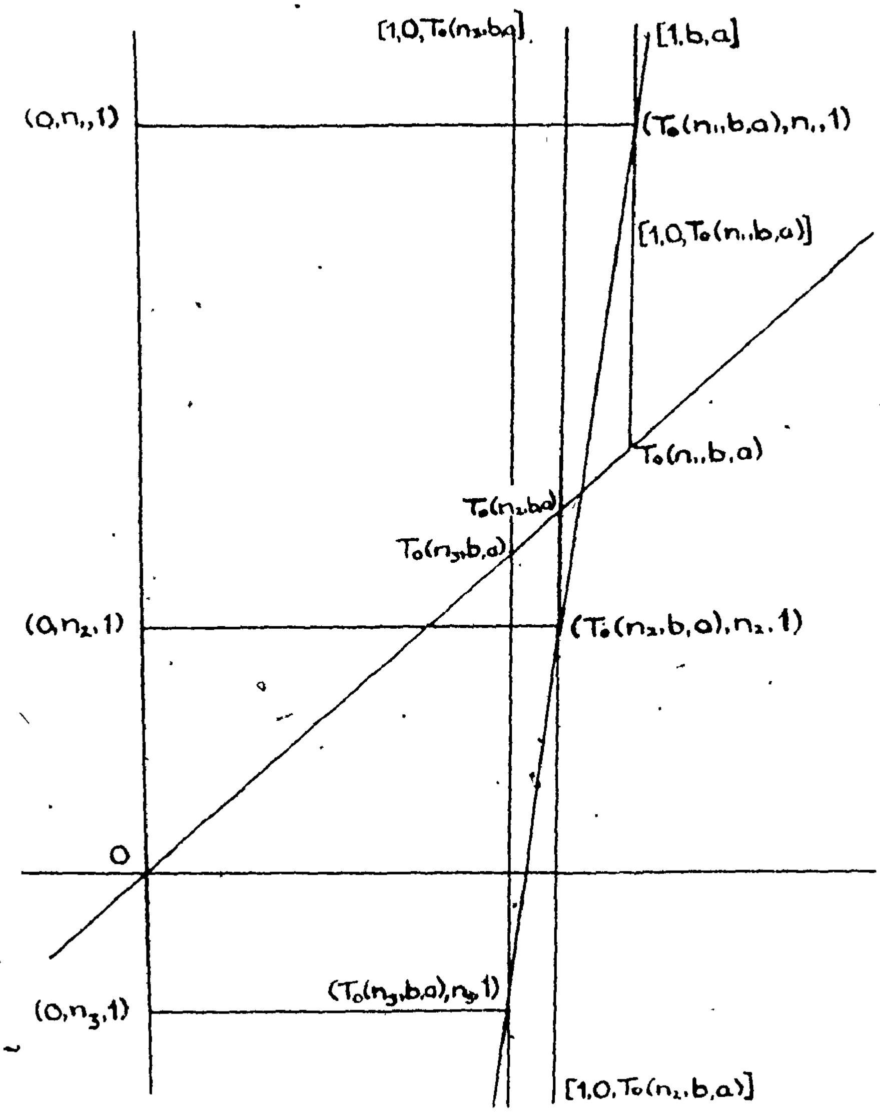


Figure 6.3
(Lemma 6.4.3)

$((1,0,T_4(m_1,\beta,\alpha)),(1,0,T_4(m_2,\beta,\alpha)),(1,0,T_4(m_3,\beta,\alpha))) \in \rho|_{\bar{U}}$. Therefore $(T_4(m_1,\beta,\alpha),T_4(m_2,\beta,\alpha),T_4(m_3,\beta,\alpha)) \in \rho$.

<u>6.4.4</u>. <u>Lemma</u>. Let $\alpha, \beta \in \eta \setminus \{0\}$; $m \in H \setminus \eta$ such that $(1,m,0)$ I $[\alpha,\beta,1]$. Then $(\alpha,0,\beta) \in \rho$ when $m > 0$ and α, β lie on the same side of 0 when $m < 0$.

<u>Proof</u>. If $m \in \eta$, then $[\alpha,\beta,1]$ and $[0,0,1]$ would coincide in $\overline{(1,m,0)} = \bar{U}$ and $(1,0,\alpha) = [\alpha,\beta,1] \wedge [0,1,0] = [0,0,1] \wedge [0,1,0] = (1,0,0)$; a contradiction. Hence $m \notin \eta$.

<u>Case 1</u>: Suppose $m > 0$. Then n and p (cf. 6.4.1) must lie on the same side of 0 by 6.3.6. If $a > 0$ $[a < 0]$, then by (OM2)', $T_0(n,b,0) < T_0(n,b,a)$ $[T_0(n,b,0) > T_0(n,b,a)]$. However, since $T_0(0,0,0) = T_0(0,b,0)$ and $T_0(n,0,0) = 0 = T_0(n,b,a) > T_0(n,b,0)$ $[T_0(n,0,0) < T_0(n,b,0)]$, (OM3)*' implies $T_0(p,0,0) > T_0(p,b,0)$ $[T_0(p,0,0) < T_0(p,b,0)]$ as n and p lie on the same side of 0. Therefore $((a,T_1(a,q,p),1),(0,p,1),(T_0(p,b,0),T_1(T_0(p,b,0),q,p),1)) \in \rho|_{\overline{(0,p,1)}}$ and $((1,1,\alpha),(1,1,0),(1,1,\beta)) \in \rho|_{\overline{(1,1,0)}}$; hence $(\alpha,0,\beta) \in \rho$.

<u>Case 2</u>: If $m < 0$, an analagous proof gives the desired result.

<u>6.4.5</u>. <u>Lemma</u>. Let $\alpha_1, \alpha_2, a_1, a_2 \in \eta$ such that $[1,0,a_i]$ is the preimage of $[\alpha_i,0,1]$ $(i=1,2)$. Then $\alpha_1 < \alpha_2$ implies $a_1 < a_2$.

<u>Proof</u>. Take the unique element v of H such that $T_1(s,v,1) = 0$. By (OM3)*, if $s > 0$ $[s < 0]$, then $v < 0$ $[v > 0]$. Since

$$\begin{aligned}[0,1,1]|_{\bar{U}} &= L(\phi_{\bar{U}}(\Pi_{[0,1,1]}),\alpha_{\bar{U}}(\overline{[0,1,1]})) \\ &= L((1,0,0),\Pi_{\bar{U}},\psi_{\bar{U}}(L(0,[v,1,1]))) \\ &= \psi_{\bar{U}}(L((0,0,1),[v,1,0]),\end{aligned}$$

the preimage of $[0,1,1]|_{\bar{U}}$ is $[v,1,0]|_{\bar{O}}$. Hence

$(1,\alpha_i,\alpha_i) = [0,1,1] \wedge [\alpha_i,0,1]$ has the preimage $[v,1,0] \wedge [1,0,a_i] = (a_i,T_1(a_i,v,0),1)$.

First we show if $\alpha_i > 0$, then $a_i > 0$.

The line $[\alpha_i,1,0]$ meets $[0,1,0]$ in $\bar{0}$ (hence $[\alpha_i,1,0] \| _{\bar{U}} [0,1,0]$) and is incident with $(1,\alpha_i,\alpha_i)$; thus the preimage of $[\alpha_i,1,0]$ is $L((a_i,T_1(a_i,v,0),1),[0,1,0]) = [0,1,T_1(a_i,v,0)]$. Since $(1,\alpha_i,0) = [\alpha_i,1,0] \wedge \ell^*$, its preimage is $[0,1,T_1(a_i,v,0)] \wedge \ell = (0,T_1(a_i,v,0),1)$. However by 6.3.6, if $s > 0$ $[s < 0]$, then $T_1(a_i,v,0) < 0$ $[T_1(a_i,v,0) > 0]$. As $v < 0$ $[v > 0]$, (OM3)* implies $a_i > 0$.

Similarly if $\alpha_i < 0$, then $a_i < 0$.

Finally if $\alpha_1 < \alpha_2$ and $\alpha_1 \neq 0 \neq \alpha_2$, then one of $0 < \alpha_1 < \alpha_2$, $\alpha_1 < 0 < \alpha_2$ or $\alpha_1 < \alpha_2 < 0$ holds. If $0 < \alpha_1 < \alpha_2$, then by 6.3.6 $((1,0,0),(1,\alpha_1,0),(1,\alpha_2,0)) \in \rho|_{\bar{U}}$. The parallel projection $\ell \xrightarrow[{[0,1,0]}]{} [0,1,1]$ in $\mathcal{A}(\bar{U})$ yields $((1,0,0),(1,\alpha_1,\alpha_1),(1,\alpha_2,\alpha_2)) \in \rho|_{\bar{U}}$; hence $((0,0,1),(a_1,T_1(a_1,v,0),1),(a_2,T_1(a_2,v,0),1)) \in \rho|_{\bar{0}}$ and $(0,a_1,a_2) \in \rho$. Similarly $\alpha_1 < 0 < \alpha_2$ or $\alpha_1 < \alpha_2 < 0$ yield $(a_1,0,a_2) \in \rho$ or $(a_1,a_2,0) \in \rho$ respectively. Therefore, by the above discussion, we obtain $a_1 < a_2$.

<u>6.4.6</u>. <u>Lemma</u>. Let α_1, α_2, β_1, $\beta_2 \in \eta$ and r_1, r_2, $m \in H$ such that $m \nmid r_1 \sim r_2$. If $(0,\beta_1,\beta_2) \in \rho$, then $(T_4(m,0,0),T_4(m,\beta_1,\alpha_1),T_4(m,\beta_2,\alpha_2)) \in \rho$.

<u>Proof</u>. We consider the preimages of these points and lines (cf. 6.4.1). Since $(1,m,T_4(m,\beta_i,\alpha_i)) = [m,1,0] \wedge [\alpha_i,\beta_i,1]$, its preimage is

$[0,1,n] \wedge [1,b_i,a_i] = (T_0(n,b_i,a_i),n,1)$ $(i=1,2)$. By 6.4.2, $(0,\beta_1,\beta_2) \in \rho$ implies $((1,1,0),(1,1,\beta_1),(1,1,\beta_2)) \in \rho|_{\overline{(1,1,0)}}$ which implies $((0,p,1),(T_0(p,b_1,0),p,1),(T_0(p,b_2,0),p,1)) \in \rho|_{\overline{(0,p,1)}}$. Hence $(0,T_0(p,b_1,0),T_0(p,b_2,0)) \in \rho$ which by (OM3)*' implies $(0,b_1,b_2) \in \rho$. Since $m \nmid r_1 \sim r_2$, we obtain $n \nmid u_1 \sim u_2$ and $T_0(u_1,b_1,a_1) = 0 = T_0(u_2,b_2,a_2) = T_0(u_1,b_2,a_2)$. By (OM3)*', $(T_0(n,0,0),T_0(n,b_1,a_1),T_0(n,b_2,a_2)) \in \rho$; therefore $((T_0(n,0,0),n,1),(T_0(n,b_1,a_1),n,1),(T_0(n,b_2,a_2),n,1)) \in \rho|_{\overline{(0,n,1)}}$ and $((1,m,T_4(m,0,0)),(1,m,T_4(m,\beta_1,\alpha_1)),(1,m,T_4(m,\beta_2,\alpha_2)) \in \rho|_{\overline{(1,m,0)}}$. Thus by 6.4.2, $(T_4(m,0,0),T_4(m,\beta_1,\alpha_1),T_4(m,\beta_2,\alpha_2)) \in \rho$.

We shall now prove two order properties for the ternary operator T_4. They correspond to (OM2) and (OM3)*.

<u>6.4.7</u>. <u>Lemma</u>. Let $r \in H$ and $\alpha_1, \alpha_2, \beta \in \eta$ such that $\alpha_1 < \alpha_2$. Then $T_4(r,\beta,\alpha_1) < T_4(r,\beta,\alpha_2)$.

<u>Proof</u>. If either $r \in \eta$ or $\beta = 0$, $T_4(r,\beta,\alpha_i) = \alpha_i$ $(i=1,2)$ and we have the desired result. Therefore we may assume that $r \notin \eta$ and $\beta \neq 0$.

Suppose $\alpha_1 < \alpha_2$ and $\beta > 0$ $[\beta < 0]$. Consider the preimages (cf. 6.4.1). As $[1,0,a_i]$ is the preimage of $[\alpha_i,0,1]$ $(i=1,2)$, 6.4.5 implies $a_1 < a_2$; hence by (OM2)', $T_0(u,b,a_1) < T_0(u,b,a_2)$. However as $[T_4(r,\beta,\alpha_i),0,1] = (1,r,T_4(r,\beta,\alpha_i))V = ([r,1,0] \wedge [\alpha_i,\beta,1])V$, it has the preimage $L(([0,1,u] \wedge [1,b,a_i]),\ell) = [1,0,T_0(u,b,a_i)]$ $(i=1,2)$. However by 6.4.5, $T_0(u,b,a_1) < T_0(u,b,a_2)$ implies $T_4(r,\beta,\alpha_1) < T_4(r,\beta,\alpha_2)$.

<u>6.4.8</u>. <u>Lemma</u>. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \eta$ and $m \in H$ such that

$T_4(m,\beta_1,\alpha_1) = T_4(m,\beta_2,\alpha_2)$. If $\beta_1 < \beta_2$, then

$$T_4(x,\beta_1,\alpha_1) \leq T_4(x,\beta_2,\alpha_2) \text{ for any } x > m \text{ and}$$

$$T_4(x,\beta_1,\alpha_1) \geq T_4(x,\beta_2,\alpha_2) \text{ for any } x < m.$$

Proof. First we take $\beta_1 = 0$. Since $\beta_2 \neq \beta_1 = 0$, there exists $(1,r,0)$ I $[\alpha_2,\beta_2,1]$. By 6.4.4, $0 < \beta_2$ implies $\alpha_2 < 0$ if $r > 0$ and $r \nsim 0$ and $\alpha_2 > 0$ if $r < 0$ and $r \nsim 0$. If $r \sim 0$, then $\alpha_2 = 0$.

Case 1: Let $r \nsim 0$.

(i) Suppose first $m \sim r$. Then $\alpha_1 = T_4(m,0,\alpha_1) = 0 = T_4(m,\beta_2,\alpha_2)$. By 6.4.4, $0 > \alpha_2 = T_4(0,\beta_2,\alpha_2)$ if $m > 0$ $[0 < \alpha_2 = T_4(0,\beta_2,\alpha_2)$ if $m < 0]$. Take any $x \in H$. If $x \sim m$, then $T_4(x,\beta_2,\alpha_2) = 0$. If $x \in \eta$, then 5.8.4 implies $T_4(x,\beta_2,\alpha_2) = \alpha_2 < 0$ when $m > x$ and $T_4(x,\beta_2,\alpha_2) = \alpha_2 > 0$ when $m < x$. Suppose $x \nsim m, 0$. If $(x,m,0) \in \rho$, then $(T_4(x,\beta_2,\alpha_2),T_4(x,\beta_2,\alpha_2),T_4(0,\beta_2,\alpha_2)) = (T_4(x,\beta_2,\alpha_2),0,\alpha_2) \in \rho$ by 6.4.3. Similarly, if $(m,x,0) \in \rho$ or $(x,0,m) \in \rho$, then $(0,T_4(x,\beta_2,\alpha_2),\alpha_2) \in \rho$ or $(T_4(x,\beta_2,\alpha_2),\alpha_2,0) \in \rho$ respectively. Therefore $T_4(x,\beta_2,\alpha_2) \geq 0$ when $x > m$ and $T_4(x,\beta_2,\alpha_2) \leq 0$ when $x < m$.

(ii) Suppose $m \sim 0$. Then $\alpha_1 = T_4(m,0,\alpha_1) = T_4(m,\beta_2,\alpha_2) = T_4(0,\beta_2,\alpha_2) = \alpha_2$. Take any $x \in H$. If $x \sim m$, then $T_4(x,0,\alpha_1) = T_4(m,0,\alpha_1) = T_4(m,\beta_2,\alpha_2) = T_4(x,\beta_2,\alpha_2)$. If $x \sim r$, then $T_4(x,0,\alpha_1) = \alpha_1 = \alpha_2 < 0 = T_4(x,\beta_2,\alpha_2)$ when $r > 0$ and $T_4(x,0,\alpha_1) = \alpha_1 = \alpha_2 > 0 = T_4(x,\beta_2,\alpha_2)$ when $r < 0$. If $x \nsim m, r$, then one of $(x,m,r) \in \rho$, $(m,x,r) \in \rho$ or $(x,r,m) \in \rho$ which implies, by 6.4.3, $(T_4(x,\beta_2,\alpha_2),T_4(m,\beta_2,\alpha_2),T_4(r,\beta_2,\alpha_2)) = (T_4(x,\beta_2,\alpha_2),\alpha_2,0) \in \rho$, $(\alpha_1,T_4(x,\beta_2,\alpha_2),0) \in \rho$ or $(T_4(x,\beta_2,\alpha_2),0,\alpha_1) \in \rho$ respectively. Therefore $T_4(x,\beta_2,\alpha_2) \geq T_4(x,0,\alpha_1)$ if $x > m$ and $T_4(x,\beta_2,\alpha_2) \leq T_4(x,0,\alpha_1)$ if $x < m$.

(iii) Now suppose $m \not\sim r \not\sim 0 \not\sim m$. If $(m,0,r) \in \rho$, then $(T_4(m,\beta_2,\alpha_2), T_4(0,\beta_2,\alpha_2), T_4(r,\beta_2,\alpha_2)) = (\alpha_1,\alpha_2,0) \in \rho$. If $(m,r,0) \in \rho$, then $(\alpha_1,0,\alpha_2) \in \rho$. If $(0,m,r) \in \rho$, then $(\alpha_2,\alpha_1,0) \in \rho$. Hence if $m > 0$, then $\alpha_1 > \alpha_2$ and if $m < 0$, then $\alpha_1 < \alpha_2$. Consider any $x \in H$. If $x \sim m$, then $T_4(x,\beta_2,\alpha_2) = \alpha_1$. If $x \sim 0$, then $T_4(x,\beta_2,\alpha_2) = \alpha_2$ which is greater than α_1 if $m < 0$ and is less than α_1 if $m > 0$. Suppose $x \not\sim 0$, m. Then $(x,m,0) \in \rho$ implies $(T_4(x,\beta_2,\alpha_2), T_4(m,\beta_2,\alpha_2), T_4(0,\beta_2,\alpha_2)) = (T_4(x,\beta_2,\alpha_2),\alpha_1,\alpha_2) \in \rho$ by 6.4.3. Similarly, $(x,0,m) \in \rho$ implies $(T_4(x,\beta_2,\alpha_2),\alpha_2,\alpha_1) \in \rho$ and $(0,x,m) \in \rho$ implies $(\alpha_2,T_4(x,\beta_2,\alpha_2),\alpha_1) \in \rho$. Combining these results we obtain $T_4(x,\beta_2,\alpha_2) \geq \alpha_1$ when $x > m$ and $T_4(x,\beta_2,\alpha_2) \leq \alpha_1$ when $x < m$.

<u>Case 2</u>: Suppose $r \sim 0$. Then $\alpha_2 = 0$.

(i) If $m \sim 1$, then $\alpha_1 = T_4(m,0,\alpha_1) = T_4(m,\beta_2,0) = T_4(1,\beta_2,0) = \beta_2$; hence $\alpha_1 > 0$. Consider any $x \in H$. If $x \sim 1$, then $T_4(x,\beta_2,0) = \alpha_1$. If $x \sim 0$, then $T_4(x,\beta_2,0) = 0 < \alpha_1$ and $x < m$. Suppose $x \not\sim 0, 1$. Then $(0,1,x) \in \rho$ implies $(T_4(0,\beta_2,0), T_4(1,\beta_2,0), T_4(x,\beta_2,0)) = (0,\alpha_1,T_4(x,\beta_2,0)) \in \rho$; $(0,x,1) \in \rho$ implies $(0,T_4(x,\beta_2,0),\alpha_1) \in \rho$ and $(x,0,1) \in \rho$ implies $(T_4(x,\beta_2,0),0,\alpha_1) \in \rho$. Hence $\alpha_1 \leq T_4(x,\beta_2,0)$ when $x > m$ and $\alpha_1 \geq T_4(x,\beta_2,0)$ if $x < m$.

(ii) If $m \sim 0$, then $\alpha_1 = 0$ also. Consider any $x \in H$. If $x \sim 0$, then $T_4(x,\beta_2,0) = 0$ and if $x \sim 1$, then $T_4(x,\beta_2,0) = \beta_2 > 0$. Suppose $x \not\sim 0, 1$. If $0 < 1 < x$, then $0 < \beta_2 < T_4(x,\beta_2,0)$. If $0 < x < 1$, then $0 < T_4(x,\beta_2,0) < \beta_2$ and if $x < 0 < 1$, then $T_4(x,\beta_2,0) < 0 < \beta_2$. Therefore $T_4(x,\beta_2,0) \geq 0$ when $x > m$ and

$T_4(x,\beta_2,0) \leq 0$ when $x < m$.

(iii) If $m \nsim 0, 1$, then one of the following holds: $0 < m < 1$ which implies $(T_4(0,\beta_2,0),T_4(m,\beta_2,0),T_4(1,\beta_2,0)) = (0,\alpha_1,\beta_2) \in \rho$ (hence $0 < \alpha_1 < \beta_2$); or $m < 0$ which implies $\alpha_1 < 0 < \beta_2$; or $1 < m$ which implies $0 < \beta_2 < \alpha_1$. Hence $\alpha_1 > 0$ when $m > 0$ and $\alpha_1 < 0$ when $m < 0$. Consider any $x \in H$. If $x \sim 0$, then $T_4(x,\beta_2,0) = 0$ which is greater than α_1 when $x > m$ and is less than α_1 when $x < m$. If $x \sim m$, then $T_4(x,\beta_2,0) = \alpha_1$. If $x \nsim 0, m$, then $(0,x,m) \in \rho$ and $(T_4(0,\beta_2,0),T_4(x,\beta_2,0),T_4(m,\beta_2,0)) = (0,T_4(x,\beta_2,0),\alpha_1) \in \rho$ or $(0,m,x) \in \rho$ and $(0,\alpha_1,T_4(x,\beta_2,0)) \in \rho$ or $(x,0,m) \in \rho$ and $(T_4(x,\beta_2,0),0,\alpha_1) \in \rho$. Therefore $T_4(x,\beta_2,0) \geq \alpha_1$ when $x > m$ and $T_4(x,\beta_2,0) \leq \alpha_1$ when $x < m$.

If $\beta_2 = 0 \neq \beta_1$, then a discussion similar to the one above yields the desired result.

Now suppose $\beta_1 \neq 0 \neq \beta_2$. Then there exist $r_1, r_2 \in H$ such that $(1,r_i,0) \; I \; [\alpha_i,\beta_i,1]$ $(i=1,2)$. Suppose $0 < \beta_1 < \beta_2$.

<u>Case 1</u>: Let $r_1 \sim r_2 \sim 0$. Then $m \in \eta$ also and $\alpha_1 = \alpha_2 = 0$. Consider any $x \in H$. If $x \sim 0$, then $T_4(x,\beta_2,0) = 0 = T_4(x,\beta_1,0)$. If $x \nsim 0$, then by 6.4.6 $(0,T_4(x,\beta_1,0),T_4(x,\beta_2,0)) \in \rho$. However if $x > 0$ (ie., x and 1 lie on the same side of 0), then $T_4(x,\beta_1,0)$, $T_4(1,\beta_1,0) = \beta_1 > 0$ by 6.4.3; hence $0 < T_4(x,\beta_1,0) < T_4(x,\beta_2,0)$. If $x < 0$, then $(T_4(x,\beta_1,0),0,\beta_1) \in \rho$; hence $T_4(x,\beta_2,0) < T_4(x,\beta_1,0) < 0$. Therefore $T_4(x,\beta_2,0) \geq T_4(x,\beta_1,0)$ when $x > m$ and $T_4(x,\beta_2,0) \leq T_4(x,\beta_1,0)$ when $x < m$.

<u>Case 2</u>: Suppose $r_1 \sim r_2 \nsim 0$. Then $m \sim r_1$, r_2. If $r_1 > 0$

$[r_1 < 0]$, then $\alpha_1, \alpha_2 < 0$ $[\alpha_1, \alpha_2 > 0]$ by 6.4.4. By 6.4.6 with $x = 0$, we have $(\alpha_2, \alpha_1, 0) \in \rho$. Hence $\alpha_2 < \alpha_1$ if $m > 0$ and $\alpha_2 > \alpha_1$ if $m < 0$. Consider any $x \in H$. If $x \sim r$, then $T_4(x,\beta_2,\alpha_2) = 0 = T_4(x,\beta_1,\alpha_1)$. If $x \sim 0$, then $T_4(x,\beta_2,\alpha_2) = \alpha_2$ and $T_4(x,\beta_1,\alpha_1) = \alpha_1$. If $x \nsim r, 0$, then by 6.4.6, $(T_0(x,0,0),T_0(x,\beta_1,\alpha_1),T_0(x,\beta_2,\alpha_2)) \in \rho$. However by 6.4.3, $(x,m,0) \in \rho$ implies $(T_4(x,\beta_2,\alpha_2),T_4(m,\beta_2,\alpha_2),T_4(0,\beta_2,\alpha_2)) = (T_4(x,\beta_2,\alpha_2),0,\alpha_2) \in \rho$; $(m,x,0) \in \rho$ implies $(0,T_4(x,\beta_2,\alpha_2),\alpha_2) \in \rho$ and $(m,0,x) \in \rho$ implies $(0,\alpha_2,T_4(x,\beta_2,\alpha_2)) \in \rho$. Combining these results, we obtain $T_4(x,\beta_2,\alpha_2) \geq T_4(x,\beta_1,\alpha_1)$ when $x > m$ and $T_4(x,\beta_2,\alpha_2) \leq T_4(x,\beta_1,\alpha_1)$ when $x < m$.

Case 3: Suppose $r_1 \nsim r_2$. Then $r_1, r_2 \nsim m$. We shall use pre-images (cf. 6.4.1) to show the desired result.

Since $0 < \beta_1 < \beta_2$, we have $((1,1,0),(1,1,\beta_1),(1,1,\beta_2)) \in \rho|_{\overline{(1,1,0)}}$ and $((0,p,1),(T_0(p,b_1,0),p,1),(T_0(p,b_2,0),p,1)) \in \rho|_{\overline{(0,p,1)}}$ by definition. Hence $(0,T_0(p,b_1,0),T_0(p,b_2,0)) \in \rho$. By (OM3)*', we obtain $(0,b_1,b_2) \in \rho$. There exists $c_1 \in \eta$ such that $(0,u_2,1)\ I\ [1,b_1,c_1]$.

Assume first that $0 < b_1 < b_2$. If $u_2 < u_1$ $[u_2 > u_1]$, then $T_0(u_1,b_1,c_1) > T_0(u_1,0,0) = 0 = T_0(u_1,b_1,c_1)$ $[T_0(u_1,b_1,c_1) < T_0(u_1,0,0) = 0 = T_0(u_1,b_1,c_1)]$; hence $c_1 > a_1$ $[c_1 < a_1]$. Therefore for any $y \geq u_2$ $[y \leq u_2]$, $T_0(y,b_2,a_2) \geq T_0(y,b_1,c_1) > T_0(y,b_1,a_1)$ $[T_0(y,b_2,a_2) \leq T_0(y,b_1,c_1) < T_0(y,b_1,a_1)]$ which implies $n < u_2$ $[n > u_2]$ since $T_0(n,b_2,a_2) = T_0(n,b_1,a_1)$. Thus $(n,u_2,u_1) \in \rho$.

If we assume $0 > b_1 > b_2$, then a discussion similar to the one

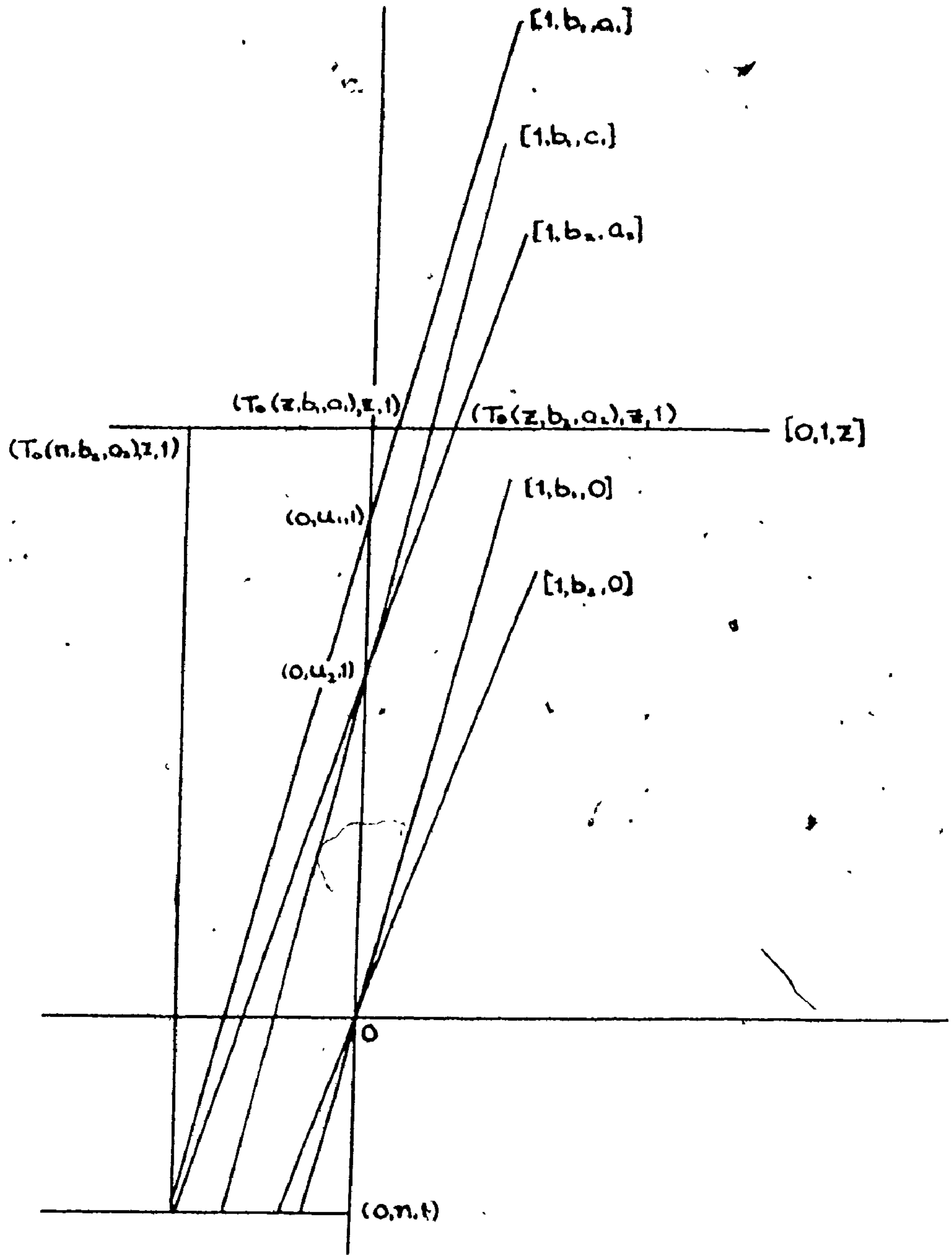


Figure 6.4
(Lemma 6.4.8, $\beta_1 \neq 0 \neq \beta_2$,
Case 3)

above also yields $(n,u_2,u_1) \in \rho$.

Thus $(0,\beta_1,\beta_2) \in \rho$ implies $(m,r_2,r_1) \in \rho$.

Consider any $x \in H$. Since $x \sim m$ implies $T_4(x,\beta_1,\alpha_1) = T_4(x,\beta_2,\alpha_2)$, we may assume $x \nsim m$. There exists $z \in H$, $z \nsim n$, such that $(0,z,1)$ is the preimage of $(1,x,0)$. Since $(0,b_1,b_2) \in \rho$, we have

$(T_0(z,0,T_0(n,b_2,a_2)),T_0(z,b_1,a_1),T_0(z,b_2,a_2)) \in \rho$ which implies $(T_4(x,0,T_4(m,\beta_2,\alpha_2)),T_4(x,\beta_1,\alpha_1),T_4(x,\beta_2,\alpha_2)) \in \rho$.

Now $(m,r_2,r_1) \in \rho$. Suppose first $m < r_2 < r_1$. By a previous case, $m < r_2$ implies $T_4(m,\beta_2,\alpha_2) < T_4(m,0,0) = 0$. If $x < m$, then by 6.4.3,

$(T_4(x,\beta_2,\alpha_2),T_4(m,\beta_2,\alpha_2),T_4(r_2,\beta_2,\alpha_2)) = (T_4(x,\beta_2,\alpha_2),T_4(m,\beta_2,\alpha_2),0) \in \rho$. Therefore $T_4(x,\beta_2,\alpha_2) \leq T_4(x,\beta_1,\alpha_1)$ for any $x < m$. Similarly, for any $x > m$, $T_4(x,\beta_2,\alpha_2)$ and $T_4(r_2,\beta_2,\alpha_2) = 0$ lie on the same side of $T_4(m,\beta_2,\alpha_2)$; hence $T_4(x,\beta_2,\alpha_2) \geq T_4(x,\beta_1,\alpha_1)$ for any $x > m$.

A similar discussion yields the desired result when $m > r_2 > r_1$.

The next two lemmas give order properties corresponding to (OM2) and (OM3)* for the ternary operator T_3.

6.4.9. Lemma. Let $\alpha \in \eta$ and m_1, m_2, $q \in H$ such that $m_1 < m_2$. Then $T_3(\alpha,q,m_1) < T_3(\alpha,q,m_2)$.

Proof. If $m_1 \nsim m_2$, then $T_3(\alpha,q,m_1) \sim m_1 \nsim m_2 \sim T(\alpha,q,m_2)$ and $T_3(\alpha,q,m_1) < T_3(\alpha,q,m_2)$.

Suppose $m_1 \sim m_2$. Consider the preimages of $[m_i,1,q]$, $i=1,2$. Take $v \in H$ such that $T_1(s,v,q) = 0$. Then

$$[m_i,1,q]|_{\overline{(1,m_i,0)}} = L(\phi_{\overline{(1,m_i,0)}}(\Pi_{[m_i,1,q]}),\alpha_{\overline{(1,m_i,0)}}([\overline{m_i,1,q}]))$$
$$= L((1,m_i,0),\Pi_{\overline{(1,m_i,0)}},\psi_{\overline{(1,m_i,0)}}(L((0,n,1),[v,1,q])))$$
$$= \psi_{\overline{(1,m_i,0)}}((0,n_i,1)[v,1,n])$$
$$= \psi_{\overline{(1,m_i,0)}}([v,1,n_i]).$$

The preimage of $(1,T_3(\alpha,q,m_i),\alpha) = [\alpha,0,1] \wedge [m_i,1,q]$ is $[1,0,a] \wedge [v,1,n_i] = (a,T_1(a,v,n_i),1)$. By (OM2), $n_1 < n_2$ if and only if $T_1(a,v,n_1) < T_1(a,v,n_2)$. Since

$$(1,T_3(\alpha,q,m_i),0) = \ell^* \wedge [T_3(\alpha,q,m_i)1,0]$$
$$= \ell^* \wedge L((1,T_3(\alpha,q,m_i),\alpha),\Pi_{\overline{(1,m_i,0)}},[m_i,1,0]),$$

its preimage is

$\ell \wedge L((a,T_1(a,v,n_i),1),\Pi_{\overline{(0,n_i,1)}},[0,1,n_i]) = (0,T_1(a,v,n_i),1)$. Hence by 6.3.6, $m_1 < m_2$ implies $T_3(\alpha,q,m_1) < T_3(\alpha,q,m_2)$.

<u>6.4.10</u>. <u>Lemma</u>. Let $\alpha \in \eta$ and $m_1, m_2, q_1, q_2 \in H$ such that $T_3(\alpha,q_1,m_1) = T_3(\alpha,q_2,m_2)$. If $q_1 < q_2$, then for any $x \in \eta$,

$$T_3(x,q_1,m_1) \leq T_3(x,q_2,m_2) \quad \text{when} \quad x > \alpha \quad \text{and}$$
$$T_3(x,q_1,m_1) \geq T_3(x,q_2,m_2) \quad \text{when} \quad x \lessgtr \alpha.$$

<u>Proof</u>. If $q_1 \sim q_2$, then by the uniformity of $\mathcal{H}^*$, $[m_1,1,q_1]$ and $[m_2,1,q_2]$ coincide in the neighbour class $\overline{(1,m_1,0)}$; hence for any $x \in \eta$, $T_3(x,q_1,m_1) = T_3(x,q_2,m_2)$. So we may assume $n_1 \nsim n_2$.

As in the previous lemma, the preimage of $[m_i,1,q_i]$ is $[v_i,1,n_i]$ where $T_1(s,v_i,q_i) = 0$. Since $q_1 < q_2$, $T_1(s,v_1,q_1) = T_1(s,v_2,q_2) > T_1(s,v_2,q_1)$ by (OM2); hence if $s > 0$ $[s < 0]$, $v_1 > v_2 [v_1 < v_2]$. The preimage of $(1,T_3(\alpha,q_i,m_i),\alpha) = [m_i,1,q_i] \wedge [\alpha,0,1]$ is

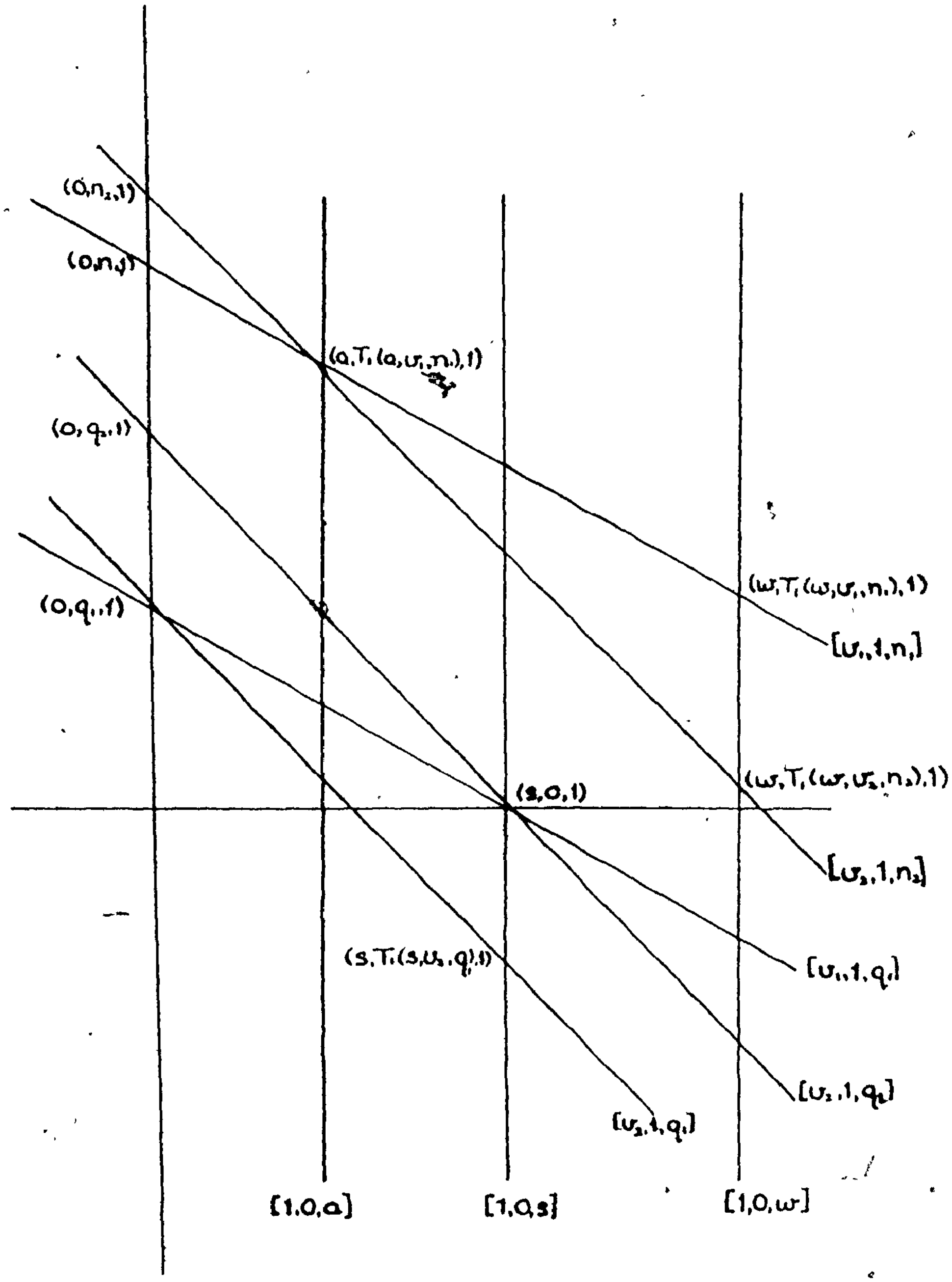


Figure 6.5
(Lemma 6.4.10)

$[v_i,1,n_i] \wedge [1,0,a] = (a,T_*(a,v_i,n_i),1)$ for $i=1,2$. Take any $x \in \eta$. The preimage of $[x,0,1]$ is $[1,0,w]$ for some $w \in \eta$. Suppose $x > \alpha$. Then $w > a$ by 6.4.5. Since $T_1(a,v_1,n_1) = T_1(a,v_2,n_2)$, (OM3)* implies $T_1(w,v_1,n_1) \geq T_1(w,v_2,n_2)$ $[T_1(w,v_1,n_1) \leq T_1(w,v_2,n_2)]$ if $s > 0$ $[s < 0]$. As $(0,T_1(w,v_i,n_i),1)$ is the preimage of $(1,T_3(x,q_i,m_i),0)$ $(i=1,2)$, we obtain $T_3(x,q_1,n_1) \leq T_3(x,q_2,n_2)$ by 6.3.6. Similarly, for any $x < \alpha$, $T_3(x,q_1,n_1) \geq T_3(x,q_2,n_2)$.

6.4.11. In 6.4.1, we discussed the preimages of various improper points which were not in $\bar{V}$, of various lines in $\mathcal{A}(\bar{X})$, $\bar{X} \neq \bar{V}$ and of the improper lines. We then used these preimages to prove some order properties for the ternary operators T_3 and T_4. We now examine the inverse images under $\psi_{\bar{V}}$ of the points and lines of $\mathcal{A}(\bar{V})$ and use their properties to prove order properties corresponding to (OM2) and (OM3)* for the ternary operator T_2.

As in 6.3.7, $\psi_{\bar{V}}^{-1}((w,1,0)) = (0,x,1)$ where $T_0(s,w,x) = 0$. Now consider the line $[1,w,0]$ of $\mathcal{H}^*$ (cf. 6.3.7). Then

$$\begin{aligned}
[1,w,0]|_{\bar{V}} &= L(\phi_{\bar{V}}(\Pi_{[1,w,0]}),\alpha_{\bar{V}}(\overline{[1,w,0]})) \\
&= L((w,1,0),\Pi_{\bar{V},\psi_{\bar{V}}(L(0,[0,1,0]))}) \\
&= \psi_{\bar{V}}(L((0,x,1),[0,1,0]|_{\bar{0}})) \\
&= \psi_{\bar{V}}([0,1,x]|_{\bar{0}}),
\end{aligned}$$

where $T_0(s,w,x) = 0$. In particular, $[1,0,0]|_{\bar{V}} = \psi_{\bar{V}}([0,1,0]|_{\bar{0}})$.

Since $\bar{T} = (\overline{SV} \wedge \overline{OE})\bar{U} \wedge \bar{I}$ and $T\ I\ \ell$ (cf. 6.2), $T = (0,t,1)$ for some $t \neq 0$ and $t > 0$ if $s > 0$, $t < 0$ if $s < 0$. Now consider a line $[0,\beta,1]$. By 6.4.1, the preimage of $[0,\beta,1]$ in $\mathcal{H}$ is $[1,b,0]$. Hence

$$
\begin{aligned}
&[0,\beta,1] \\
&= [1,b,0]^* \\
&= \bigcup_{\bar{Y} I \bar{1}^*, \bar{Y} \neq \bar{V}} \{P \in \mathbb{P}_{\bar{Y}} \mid P\ I_{\bar{Y}}\ \psi_{\bar{Y}}([1,b,0])\} \cup \{P \in \mathbb{P}_{\bar{V}} \mid P\ I_{\bar{V}}\ \gamma(\Pi_{[1,b,0]})\}.
\end{aligned}
$$

Let $c \in \eta$ such that $T_0(t,b,c) = 0$. Then

$$
\begin{aligned}
[0,\beta,1]|_{\bar{V}} &= \gamma(\Pi_{[1,b,0]}) \\
&= L(\psi_{\bar{V}}((c,0,1),\Pi_{\bar{V}}) \\
&= \psi_{\bar{V}}([1,0,c]).
\end{aligned}
$$

Since $(0,1,\beta) = [0,\beta,1] \wedge \ell$, we obtain

$\psi_{\bar{V}}^{-1}((0,1,\beta)) = [1,0,c] \wedge [0,1,0] = (c,0,1).$

Finally, consider the line $[1,w,v]$. As above, take x such that $T_0(s,w,x) = 0$ and take $z \in H$ such that $T_1(s,z,v) = 0$. Then

$$
\begin{aligned}
[1,w,v]|_{\bar{V}} &= L(\phi_{\bar{V}}(\Pi_{[1,w,v]}),\alpha_{\bar{V}}(\overline{[1,w,v]})) \\
&= L(\psi_{\bar{V}}((0,x,1)),\Pi_{\bar{V}},\psi_{\bar{V}}([z,1,0]|_{\bar{0}})) \\
&= \psi_{\bar{V}}([z,1,x]).
\end{aligned}
$$

Since $(T_2(\beta,v,w),1,\beta) = [0,\beta,1] \wedge [1,w,v]$, we obtain

$\psi_{\bar{V}}^{-1}((T_2(\beta,v,w),1,\beta)) = [1,0,c] \wedge [z,1,x] = (c,T_1(c,z,x),1)$ and since

$(T_2(\beta,v,w),1,0) = \ell^* \wedge L((T_2(\beta,v,w),1,\beta),\Pi_{\bar{V},[1,0,0]})$; we obtain

$$
\begin{aligned}
\psi_{\bar{V}}^{-1}((T_2(\beta,v,w),1,0)) &= \ell \wedge L((c,T_1(c,z,x),1),\Pi_{\bar{0},[0,1,0]}) \\
&= \ell \wedge [0,1,T_1(c,z,x)]|_{\bar{0}} \\
&= (0,T_1(c,z,x),1).
\end{aligned}
$$

<u>6.4.12</u>. <u>Lemma</u>. Let $w_1, w_2, \beta \in \eta$ and $v \in H$ such that $w_1 < w_2$. Then $T_2(\beta,v,w_1) < T_2(\beta,v,w_2)$.

<u>Proof</u>. We shall consider the preimages of appropriate points and lines of $\mathcal{A}(\bar{V})$ under the isomorphism $\psi_{\bar{V}}$ (cf. 6.4.11).

Suppose $s > 0$ $[s < 0]$. As $T_0(s,w_1,x_1) = T_0(s,w_2,x_2) = 0$ and

$w_1 < w_2$, 3.3.2(4) implies $x_1 = T_0(0,w_1,x_1) > T_0(0,w_2,x_2) = x_2$ $[x_1 = T_0(0,w_1,x_1) < T_0(0,w_2,x_2) = x_2]$. By the above discussion, $\psi_{\bar{V}}^{-1}((T_2(\beta,v,w_i),1,0)) = (0,T_1(c,z,x_i),1)$ and $\psi_{\bar{V}}^{-1}((w_i,1,0)) = (0,x_i,1)$; hence by (OM2), $T_1(c,z,x_1) > T_1(c,z,x_2)$ $[T_1(c,z,x_1) < T_1(c,z,x_2)]$ and by 6.3.7, $T_2(\beta,v,w_1) < T_2(\beta,v,w_2)$.

<u>6.4.13</u>. <u>Lemma</u>. Let β_1, β_2, c_1, $c_2 \in \eta$ such that $\psi_{\bar{V}}^{-1}((0,1,\beta_i)) = (c_i,0,1)$ $(i=1,2)$. If $\beta_1 < \beta_2$, then $c_1 < c_2$.

<u>Proof</u>. We consider the preimages of various points and lines (cf. 6.4.1 and 6.4.11). As $[1,b_i,0]$ and $[0,1,p]|_{\overline{(0,p,1)}}$ are the preimages of $[0,\beta_i,1]$ and $[1,1,0]|_{\overline{(1,1,0)}}$ respectively, $[\beta_i,0,1]$ has the preimage $[1,0,T_0(p,b_i,0)]$ in $\mathcal{H}$. By 6.4.5, $\beta_1 < \beta_2$ implies $T_0(p,b_1,0) < T_0(p,b_2,0)$. However if $s > 0$ $[s < 0]$, then $p < 0$ $[p > 0]$ by 6.3.6; hence $b_1 > b_2$ $[b_1 < b_2]$ by (OM3)*. In addition, $s > 0$ $[s < 0]$ implies $t > 0$ $[t < 0]$ by 6.4.11. Since $T_0(t,b_i,c_i) = 0$, (OM3)* implies $c_1 = T_0(0,b_1,c_1) < T_0(0,b_2,c_2) = c_2$.

<u>6.4.14</u>. <u>Lemma</u>. Let β_1, β_2, β_3, $w \in \eta$ and $v \in H$ such that $(\beta_1,\beta_2,\beta_3) \in \rho$. Then

$((T_2(\beta_1,v,w),1,\beta_1),(T_2(\beta_2,v,w),1,\beta_2),(T_2(\beta_3,v,w),1,\beta_3)) \in \rho|_{\bar{V}}$.

<u>Proof</u>. Since $(\beta_1,\beta_2,\beta_3) \in \rho$, $(b_1,b_2,b_3) \in \rho$ where $[1,b_i,0]$ is the preimage of $[0,\beta_i,1]$ in $\mathcal{H}$ $(i=1,2)$. Take c_1, c_2, $c_3 \in \eta$ such that $0 = T_0(t,b_i,c_i)$ $(i=1,2,3)$. Then by 3.3.2(4), $(c_1,c_2,c_3) \in \rho$; hence $((c_1,0,1),(c_2,0,1),(c_3,0,1)) \in \rho|_{\overline{(0,0,1)}}$ and $((0,1,\beta_1),(0,1,\beta_2),(0,1,\beta_3)) \in \rho|_{\bar{V}}$.

<u>6.4.15</u>. <u>Lemma</u>. Let w_1, w_2, $\beta \in \eta$ and v_1, $v_2 \in H$ such that

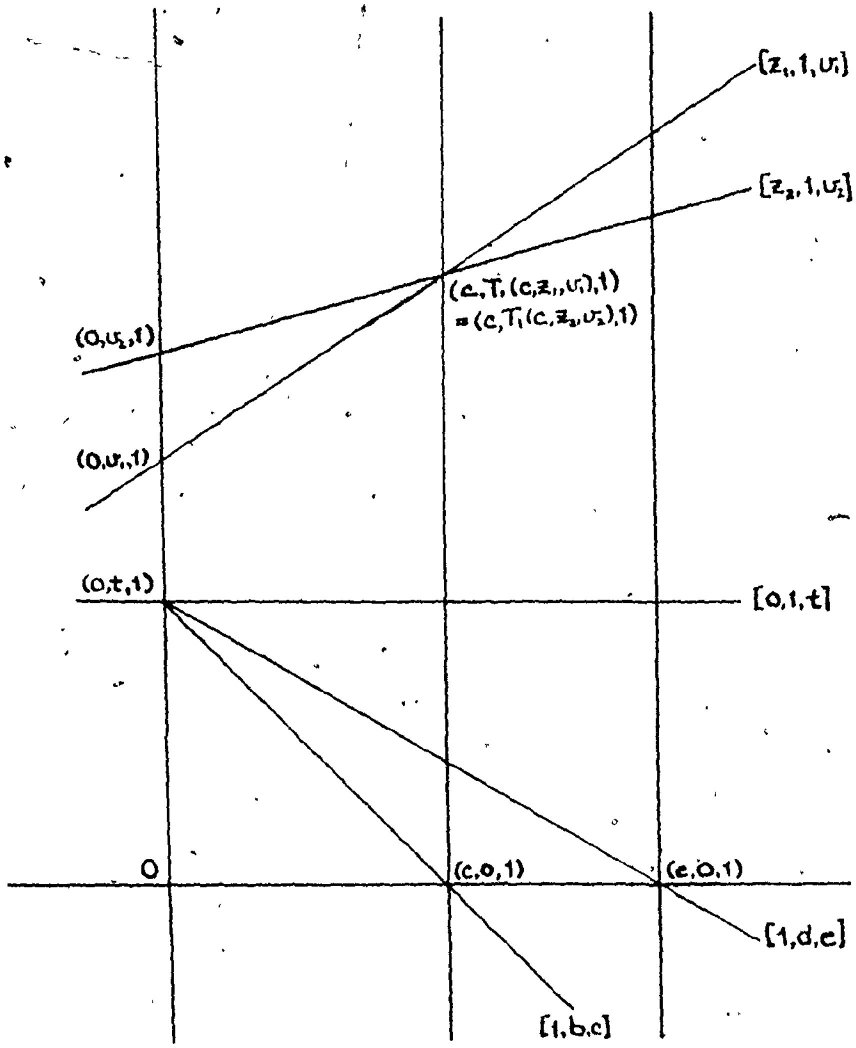


Figure 6.6
(Lemma 6.4.15)

$T_2(\beta,v_1,w_1) = T_2(\beta,v_2,w_2)$. If $v_1 < v_2$, then for any $y \in \eta$,

$$T_2(y,v_1,w_1) \leq T_2(y,v_2,w_2) \text{ when } y > \beta \text{ and}$$
$$T_2(y,v_1,w_1) \geq T_2(y,v_2,w_2) \text{ when } y < \beta.$$

Proof. If $v_1 \sim v_2$, then $[1,w_1,v_1]$ and $[1,w_2,v_2]$ would coincide in the neighbour class $\bar{V}$; hence for any $y \in \eta$, $T_2(y,v_1,w_1) = T_2(y,v_2,w_2)$. We may therefore assume $v_1 \not\sim v_2$.

Consider the line $[0,y,1]$. Using the methods of 6.4.1 and 6.4.11, we obtain $[0,y,1]|_{\bar{V}} = \psi_{\bar{V}}([1,0,e]|_{\bar{0}})$ where $[1,d,0]$ is the preimage of $[0,y,1]$ in $\mathcal{H}$ and $0 = T_0(t,d,e)$. As in 6.4.11, $[0,\beta,1]|_{\bar{V}} = \psi_{\bar{V}}([1,0,c]|_{\bar{0}})$ where $[1,b,0]$ is the preimage of $[0,\beta,1]$ in $\mathcal{H}$ and $0 = T_0(t,b,c)$. Also as in 6.4.11,

$\psi_{\bar{V}}^{-1}((T_2(\beta,v_i,w_i),1,\beta)) = (c,T_1(c,z_i,x_i),1)$ and

$\psi_{\bar{V}}^{-1}((T_2(y,v_i,w_i),1,y)) = (e,T_1(e,z_i,x_i),1)$ where $T_0(s,w_i,x_i) = 0$ and $T_1(s,z_i,v_i) = 0$ $(i=1,2)$.

Suppose $s > 0$ $[s < 0]$. Then $v_1 < v_2$, $T_1(0,z_1,v_1) = v_1 < v_2 = T_1(0,z_2,v_2)$ and 3.3.2(1) imply $z_1 > z_2$ $[z_1 < z_2]$. Since $T_2(\beta,v_1,w_1) = T_2(\beta,v_2,w_2)$, we have $T_1(c,z_1,x_1) = T_1(c,z_2,x_2)$. If $y > \beta$, then 6.4.13 implies $e > c$; hence (OM3)* implies $T_1(e,z_1,x_1) > T_1(e,z_2,x_2)$ $[T_1(e,z_1,x_1) < T_1(e,z_2,x_2)]$ and 6.3.7 implies $T_2(y,v_1,w_1) < T_2(y,v_2,w_2)$ as $(T_2(y,v_i,w_i),1,0) = \psi_{\bar{V}}((0,T_2(e,z_i,x_i),1))$.

6.4.16. Corollary. Let w, $\beta \in \eta$ and $v_1, v_2 \in H$ such that $v_1 < v_2$. If $\beta > 0$ $[\beta < 0]$, then $T_2(\beta,v_1,w) \leq T_2(\beta,v_2,w)$ $[T_2(\beta,v_1,w) \geq T_2(\beta,v_2,w)]$. If $v_1 \not\sim v_2$, then these inequalities are strict inequalities.

6.4.17. Corollary. Let β_1, β_2, $w \in \eta$ and $v \in H \setminus \eta$ such that $\beta_1 < \beta_2$. Then $T_2(\beta_1,v,w) < T_2(\beta_2,v,w)$ if $v > 0$ and $T_2(\beta_1,v,w) > T_2(\beta_2,v,w)$ if $v < 0$.

6.5. The behavior of the points of $\mathcal{H}^*$ under projections.

We now use the order properties of the ternary operators to investigate the behavior of the points of $\mathcal{H}^*$ under various kinds of projections.

6.5.1. Projections with improper centres. In this section, we investigate the behavior of the points of $\mathcal{H}^*$ under three types of projections:

$$[1,0,0] \xrightarrow{(1,u,t)} [1,m,n], \text{ where } m,t \in \eta;\ u,\ n \in H;$$

$$[1,0,0] \xrightarrow{(1,u,t)} [m,1,n], \text{ where } t \in \eta;\ m,\ n,\ u \in H;$$

$$[0,1,0] \xrightarrow{(u,1,t)} [m,1,n], \text{ where } u,\ t \in \eta;\ m,\ n \in H.$$

We then use the information we obtain to determine the behavior of the points of $\mathcal{H}^*$ under any projection with improper centre.

By 5.4.8, a pair of non-neighbouring improper points is joined by an improper line; hence the image of an improper point under a projection with an improper centre is another improper point. In addition, as a proper point and an improper point are joined by a proper line, the line between a proper point and the centre (which is improper) must be a proper line which passes through the neighbour class of the centre and hence does not pass through the neighbour class containing the intersection of the range line with ℓ^*. Thus proper points are mapped to proper points. We may therefore split each of these projections into two separate maps: one on proper points and one on improper points.

First, we define total orderings on the points of the lines in $\mathbb{L}$ (ie., on the proper points of the proper lines) and on the proper lines of each ordinary affine plane $\mathcal{A}(\bar{X})$ where $\bar{X}\ \bar{I}\ \bar{\ell}^*$ in the following way:

for $(a_1,b_1,1), (a_2,b_2,1)\ I\ [1,m,n]$,

$(a_1,b_1,1) < (a_2,b_2,1)$ if and only if $b_1 < b_2$;

for $(a_1,b_1,1), (a_2,b_2,1)\ I\ [m,1,n]$,

$(a_1,b_1,1) < (a_2,b_2,1)$ if and only if $a_1 < a_2$;

for $(a_1,1,b_1), (a_2,1,b_2)\ I_{\bar{V}}\ [1,m,n]$,

$(a_1,1,b_1) < (a_2,1,b_2)$ if and only if $b_1 < b_2$;

for $(1,a_1,b_1), (1,a_2,b_2)\ I_{\overline{(1,a_1,b_1)}}\ [m,1,n]$,

$(1,a_1,b_1) < (1,a_2,b_2)$ if and only if $b_1 < b_2$.

We say order is <u>preserved</u> under the projection θ (restricted to either the proper or improper points) if $P < Q$ implies $\theta(P) < \theta(Q)$ and order is <u>reversed</u> if $P < Q$ implies $\theta(P) > \theta(Q)$ where $<$ is the total ordering defined above.

6.5.2. Lemma. A projection $[1,0,0] \xrightarrow{(1,u,t)} [1,m,n]$ where $t, m \in \eta$ and $u, n \in H_o$ preserves the order of the improper points.

Proof. Take any $(0,1,\beta_1), (0,1,\beta_2)\ I\ [1,0,0]$ with $\beta_1 < \beta_2$. Then $(0,1,\beta_i)(1,u,t) = [\alpha_i,\beta_i,1]$ for some $\alpha_i \in \eta$ $(i=1,2)$ by 5.4.8. However $[\alpha_i,\beta_i,1] \wedge' [1,m,n] = (T_2(\beta_i,n,m),1,\beta_i)$ $(i=1,2)$ by 5.8.5 and 5.7.3. Thus $(T_2(\beta_1,n,m),1,\beta_1) < (T_2(\beta_2,n,m),1,\beta_2)$ and order is preserved.

From the above proof, we may also conclude:

6.5.3. Corollary. A projection $[1,m,n] \xrightarrow{(1,u,t)} [1,0,0]$

where $m, t \in \eta$ and $u, n \in H$ preserves the order of the improper points.

Combining 6.5.2 and 6.5.3, we obtain

6.5.4. Corollary. A projection $[1,m,n] \xrightarrow{(1,u,t)} [1,p,q]$ where $m, p, t \in \eta$ and $n, q, u \in H$ preserves the order of the improper points.

6.5.5. Lemma. A projection $[1,0,0] \xrightarrow{(1,u,t)} [1,m,n]$, where $m, t \in \eta$ and $n, u \in H$ preserves the order of the proper points.

Proof. Take any $(0,b_1,1)$, $(0,b_2,1)$ I $[1,0,0]$ with $b_1 < b_2$. Then $(0,b_i,1)(1,u,t) = [p_i,1,b_i]$ for some $p_i \sim u$ $(i=1,2)$.

First suppose $b_1 \sim b_2$. Then $[p_1,1,b_1] \sim [p_2,1,b_2]$ and the two lines must coincide in $(\overline{1,u,t})$; hence $p_1 = p_2$. Clearly,

$[p_1,1,b_i] \wedge [1,0,n] = (n,T_1(n,p_1,b_i),1)$ and

$[0,1,T_1(n,p_1,b_i)] \wedge [1,m,n] = (T_0(T_1(n,p_1,b_i),m,n),T_1(n,p_1,b_i),1)$.

As $[1,m,n]$ and $[1,0,T_0(T_1(n,p_1,b_1),m,n)]$ coincide in

$\overline{(T_0(T_1(n,p_1,b_1),m,n),T_1(n,p_1,b_1),1)} = \overline{(n,T_1(n,p_1,b_1),1)}$ and

$(T_0(T_1(n,p_1,b_2),m,n),T_1(n,p_1,b_2),1) \in \overline{(n,T_1(n,p_1,b_1),1)}$, we have

$T_0(T_1(n,p_1,b_1),m,n) = T_0(T_1(n,p_1,b_2),m,n)$. Hence

$T_1(T_0(T_1(n,p_1,b_1),m,n),p_1,b_1) < T_1(T_0(T_1(n,p_1,b_2),m,n),p_1,b_2)$ and order is preserved.

Now suppose $b_1 \not\sim b_2$. Then

$$\begin{aligned}[p_1,1,b_1] \wedge [1,0,n] &= (n,T_1(n,p_1,b_1) \\ &\sim (n,T_1(n,p_2,b_1),1) \\ &= [p_2,1,b_1] \wedge [1,0,n]\end{aligned}$$

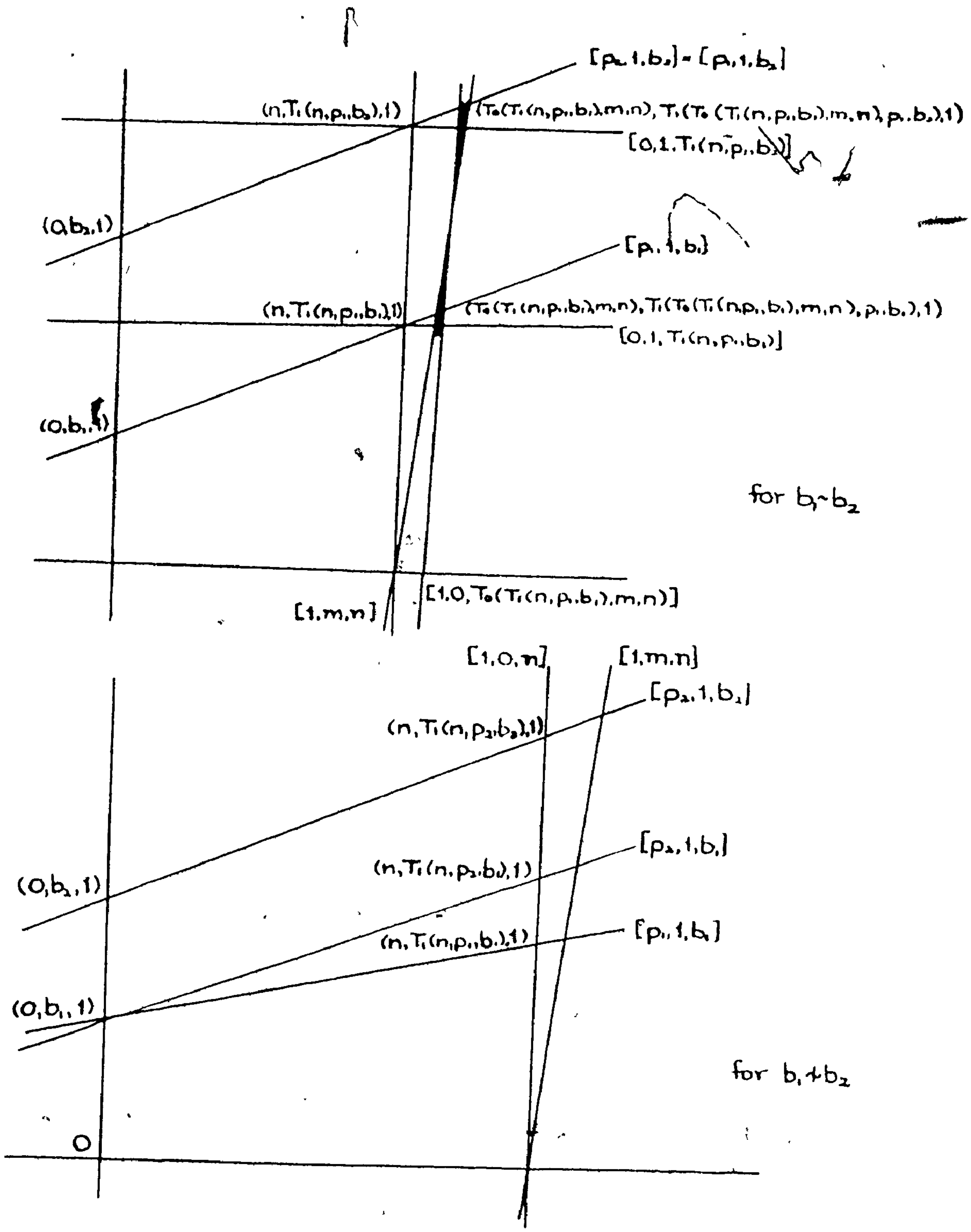


Figure 6.7
(Lemma 6.5.5)

$$\nparallel [p_2,1,b_2] \wedge [1,0,n]$$
$$= (n,T_1(n,p_2,b_2),1)$$

which implies $T_1(n,p_1,b_1) < T_1(n,p_2,b_2)$. As

$$[p_1,1,b_1] \wedge [1,m,n] \sim [p_2,1,b_1] \wedge [1,m,n]$$
$$\sim [p_2,1,b_1] \wedge [1,0,n]$$
$$\nparallel [p_2,1,b_2] \wedge [1,0,n]$$
$$\sim [p_2,1,b_2] \wedge [1,m,n],$$

it follows that $[p_1,1,b_1] \wedge [1,m,n] < [p_2,1,b_2] \wedge [1,m,n]$.

6.5.6. Corollary. A projection $[1,m,n] \xrightarrow{(1,u,t)} [1,0,0]$, where $m, t \in \eta$ and $u, n \in H$, preserves the order of the proper points.

6.5.7. Corollary. A projection $[1,m,n] \xrightarrow{(1,u,t)} [1,p,q]$, where $m, p, t \in \eta$ and $n, q, u \in H$, preserves the order of the proper points.

6.5.8. Lemma. A projection $[1,0,0] \xrightarrow{(1,u,t)} [m,1,n]$, where $t \in \eta$ and $m, n, u \in H$, $m \nparallel u$, preserves the order of the improper points if $m > u$ and reverses their order if $m < u$.

Proof. Consider two points $(0,1,\beta_1)$, $(0,1,\beta_2)$ I $[1,0,0]$ with $\beta_1 < \beta_2$. Then $(0,1,\beta_i)(1,u,t) = [\alpha_i,\beta_i,1]$ for some $\alpha_i \in \eta$ $(i=1,2)$. Since $T_4(u,\beta_1,\alpha_1) = t = T_4(u,\beta_2,\alpha_2)$, 6.4.8 implies that for any $m > u$ $[m < u]$, $m \nparallel u$, $T_4(m,\beta_1,\alpha_1) < T_4(m,\beta_2,\alpha_2)$ $[T_4(m,\beta_1,\alpha_1) > T_4(m,\beta_2,\alpha_2)]$. Now $[\alpha_i,\beta_i,1] \wedge [m,1,0] = (1,m,T_4(m,\beta_i,\alpha_i)) \sim [\alpha_i,\beta_i,1] \wedge [m,1,n]$ and $[\alpha_i,\beta_i,1] \sim [T_4(m,\beta_i,\alpha_i),0,1]$ imply

$$[\alpha_i,\beta_i,1] \wedge [m,1,n] = [T_4(m,\beta_i,\alpha_i),0,1] \wedge [m,1,n]$$
$$= (1,T_3(T_4(m,\beta_i,\alpha_i),n,m),T_4(m,\beta_i,\alpha_i))$$

by the uniformity of $\mathcal{H}$*.

6.5.9. Lemma. A projection $[1,0,0] \xrightarrow{(1,u,t)} [m,1,n]$, where $t \in \eta$ and $m, n, u \in H$, $m \nsim u$, preserves the order of the proper points if $m > u$ and reverses their order if $m < u$.

Proof. Take any $(0,b_1,1)$, $(0,b_2,1)$ I $[1,0,0]$ such that $b_1 < b_2$. The line $(0,b_i,1)(1,u,t) = [p_i,1,b_i]$ for some $p_i \sim u$ $(i=1,2)$. Take $x_i \in H$ such that $[p_i,1,b_i] \wedge [m,1,n] = (x_i,T_1(x_i,m,n),1)$.

Suppose first that $b_1 \sim b_2$. Then by the proof of 6.5.5, $p_1 = p_2$. If $m > u$ $[m < u]$, then $m > p$ $[m < p]$ also and for any $y \geq x_2$ $[y \leq x_2]$, $T_1(y,m,n) \geq T_1(y,p_1,b_2) > T_1(y,p_1,b_1)$. Hence $x_1 < x_2$ $[x_1 > x_2]$, and the order is preserved [reversed].

Now suppose $b_1 \nsim b_2$. Let $x_3 \in H$ such that $(x_3,T_1(x_3,m,n),1) = [m,1,n] \wedge [p_1,1,b_2]$. Then $x_3 \sim x_2 \nsim x_1$. As above if $m > u$ $[m < u]$, then for any $y \geq x_3$ $[y \leq x_3]$, $T_1(y,m,n) \geq T_1(y,p_1,b_2) > T_1(y,p_1,b_1)$ which implies $x_1 < x_3$ $[x_1 > x_3]$. However as $x_3 \sim x_2 \nsim x_1$, this gives $x_1 < x_2$ $[x_1 > x_2]$, and the order is preserved [reversed].

6.5.10. Corollary (to 6.5.8 and 6.5.9). A projection $[m,1,n] \xrightarrow{(1,u,t)} [1,0,0]$, where $t \in \eta$ and $m, n, u \in H$, $m \nsim u$, preserves the order of the improper [proper] points if $m > u$ and reverses their order if $m < u$.

6.5.11. Lemma. A projection $[m,1,n] \xrightarrow{(1,u,t)} [p,1,q]$, where $m, n, u, p, q \in H$, $t \in \eta$ and $m \nsim u \nsim p$ preserves the order of the improper [proper] points if p and m lie on the same side of u and reverses their order if $(p,u,m) \in \rho$.

Proof. If p and m lie on the same side of u, then the projections $[m,1,n] \xrightarrow{(1,u,t)} [1,0,0]$ and $[1,0,0] \xrightarrow{(1,u,t)} [p,1,q]$ either both preserve the order of the improper [proper] points or both reverse the order of the improper [proper] points; hence their composition must preserve order. If $(p,u,m) \in \rho$, then one of these projections preserves order and the other reverses; hence their composition reverses the order of the improper [proper] points.

6.5.12. Lemma. A projection $[0,1,0] \xrightarrow{(u,1,t)} [m,1,n]$, where $u, t \in \eta$ and $m, n \in H$, preserves the order of the improper points.

Proof. Consider two points $(1,0,\alpha_1)$, $(1,0,\alpha_2)$ I $[0,1,0]$ with $\alpha_1 < \alpha_2$. Then $(1,0,\alpha_i)(u,1,t) = [\alpha_i,t,1]$ by 5.8.5 $(i=1,2)$ and by the uniformity of $\mathcal{K}^*$ $[\alpha_i,t,1]$ and $[T_4(m,t,\alpha_i),0,1]$ coincide in $\overline{(1,m,0)}$. Hence $[\alpha_i,t,1] \wedge [m,1,n] = (1,T_3(T_4(m,t,\alpha_i),n,m),T_4(m,t,\alpha_i))$. By 6.4.7, $T_4(m,t,\alpha_1) < T_4(m,t,\alpha_2)$ and order is preserved.

6.5.13. Lemma. A projection $[0,1,0] \xrightarrow{(u,1,t)} [m,1,n]$, where $u, t \in \eta$ and $m, n \in H$, preserves the order of the proper points.

Proof. Take any $(a_1,0,1)$, $(a_2,0,1)$ I $[0,1,0]$ with $a_1 < a_2$. The line $(a_i,0,1)(u,1,t) = [1,p_i,a_i]$ for some $p_i \in \eta$ $(i=1,2)$.

First suppose $a_1 \sim a_2$. Then $[1,p_1,a_1] \sim [1,p_2,a_2]$; hence they coincide in $\bar{V}$ and $p_1 = p_2$. Clearly,

$[1,0,a_i] \wedge [m,1,n] = (a_i,T_1(a_i,m,n),1)$ and

$[0,1,T_1(a_i,m,n)] \wedge [1,p_i,a_i] = (T_0(T_1(a_i,m,n),p_i,a_i),T_1(a_i,m,n),1)$

$(i=1,2)$. As the lines $[1,0,T_0(T_1(a_i,m,n),p_i,a_i]$ and $[1,p_i,a_i]$ coincide in the neighbour class

$\overline{(T_0(T_1(a_1,m,n),p_1,a_i),T_1(a_1,m,n),1)} = \overline{(a_i,T_1(a_1,m,n),1)}$ and $[1,p_1,a_i] \wedge [m,1,n] \in \overline{(a_i,T_1(a_1,m,n),1)}$, we have

$$[1,p_1,a_i] \wedge [m,1,n]$$
$$= (T_0(T_1(a_1,m,n),p_1,a_i),T_1(T_0(T_1(a_1,m,n),p_1,a_i),m,n),1).$$

Since $T_0(T_1(a_1,m,n),p_1,a_1) < T_0(T_1(a_1,m,n),p_1,a_2)$ by (OM2)', the order is preserved.

Now suppose $a_1 \nmid a_2$. Clearly,

$[1,p_1,a_1] \wedge [m,1,n] \nmid [1,p_2,a_2] \wedge [m,1,n] \sim [1,p_1,a_2] \wedge [m,1,n]$, so it will be sufficient to show that

$[1,p_1,a_1] \wedge [m,1,n] < [1,p_1,a_2] \wedge [m,1,n]$. Using the same method as above, we obtain

$$[1,p_1,a_i] \wedge [m,1,n]$$
$$= (T_0(T_1(a_i,m,n),p_1,a_i),T_1(T_0(T_1(a_i,m,n),p_1,a_i),m,n),1).$$

However $T_0(T_1(a_1,m,n),p_1,a_1) \sim a_1 < a_2 \sim T_0(T_1(a_2,m,n),p_1,a_2)$.

6.5.14. Corollary (to 6.5.12 and 6.5.13). A projection $[m,1,n] \xrightarrow{(u,1,t)} [1,0,0]$, where $u, t \in \eta$ and $m, n \in H$, preserves the order of the improper [proper] points.

6.5.15. Corollary (to 6.5.12 and 6.5.13). A projection $[m,1,n] \xrightarrow{(u,1,t)} [p,1,q]$, where $u, t \in \eta$ and $m, n, p, q \in H$, preserves the order of the improper [proper] points.

6.5.16. Projections with proper centres. We now examine the projections with proper centres. We shall investigate the behavior of the points under the following types of projections:

$$[1,0,0] \xrightarrow{(a,b,1)} [1,m,n], \text{ where } a \notin \eta,\ a \nmid n;$$

$[1,0,1] \xrightarrow{(a,b,1)} [1,m,n]$, where $a \in \eta, n \notin \eta$;

$[1,0,0] \xrightarrow{(a,b,1)} [m,1,n]$, where $a \notin \eta$, $b \nmid T_1(a,m,n)$;

$[1,0,1] \xrightarrow{(a,b,1)} [m,1,n]$, where $a \in \eta$, $b \nmid n$;

$[1,0,0] \xrightarrow{(a,b,1)} [m,n,1]$, where $a \notin \eta$;

$[1,0,1] \xrightarrow{(a,b,1)} [m,n,1]$, where $a \in \eta$.

We shall use the information we obtain to determine the behavior of the points under any projection with proper centre.

We can use the total orderings defined in 6.5.1 to define total orderings on subsets of proper points incident with proper lines by using the restriction of the total ordering to that subset.

We also define a total ordering on the points of the second kind incident with an improper line (and subsequently on subsets of this) in the following way:

for $(1,a_1,b_1)$, $(1,a_2,b_2)$ I $[m,n,1]$,

$$(1,a_1,b_1) < (1,a_2,b_2) \text{ if and only if } a_1 < a_2.$$

Finally we define a total ordering on the lines of $\Pi_{\bar{V}}$ in $\mathcal{A}(\bar{V})$:

for $(a_1,1,b)$, $(a_2,1,b)$ I $[m,b,1]$,

$$(a_1,1,b) < (a_2,1,b) \text{ if and only if } a_1 < a_2.$$

<u>6.5.17</u>. Consider a projection $[1,0,0] \xrightarrow{(a,b,1)} [1,m,n]$, where $a \notin \eta$, $a \nmid n$. For any $(0,1,\beta)$ I $[1,0,0]$, $(0,1,\beta)(a,b,1) = [1,p,q]$ for some $p \in \eta$, $q \sim a$. Since $q \sim a \nmid n$, $[1,p,q] \wedge [1,m,n] = (T_2(t,n,m),1,t)$ for some $t_0 \in \eta$ by (β9). In addition, for any $(0,c,1)$ I $[1,0,0]$, there exists $p \in H$ such that $T_1(a,p,c) = b$ and $[1,m,n]$ meets $[p,1,c]$ in a proper point (cf. (β6)). Hence the projection maps proper points to proper points and improper points to improper points.

6.5.18. Lemma. A projection $[1,0,0] \xrightarrow{(a,b,1)} [1,m,n]$, where $a, b, n \in H$ and $m \in \eta$ such that $a \notin \eta$ and $a \nmid n$, preserves the order of the improper points when n and 0 lie on the same side of a and reverses their order when $(n,a,0) \in \rho$.

Proof. Take two points $(0,1,\beta_1)$, $(0,1,\beta_2)$ I $[1,0,0]$ such that $\beta_1 < \beta_2$. The line $(0,1,\beta_i)(a,b,1) = [1,p_i,q_i]$ for some $p_i \in \eta$, $q_i \sim a$ $(i=1,2)$. Since $q_1 \sim a \sim q_2$, the lines $[1,p_1,q_1]$ and $[1,p_1,q_2]$ coincide in $\bar{V}$. Let

$[1,p_i,q_i] \wedge [1,m,n] = [1,p_i,q_2] \wedge [1,m,n] = (T_2(t_i,n,m),1,t_i)$.

Case 1: Suppose $n, 0 < a$. Then $n, 0 < q_2$ also. By 6.4.17, $T_2(\beta_2,q_2,p_1) > T_2(\beta_1,q_2,p_1) = T_2(\beta_2,q_2,p_2)$; hence by 6.4.12, $p_1 > p_2$. Since $T_2(t_1,n_1,m) = T_2(t_1,q_2,p_1)$, 6.4.15 implies for any $x \leq t_1$ $T_2(x,n,m) \geq T_2(x,q_2,p_1) > T_2(x,q_2,p_2)$. However $T_2(t_2,n,m) = T_2(t_2,q_2,p_2)$, so $t_1 < t_2$.

Case 2: Suppose $n, 0 > a$. Then a discussion similar to the one used in Case 1, yields $t_1 < t_2$.

Case 3: Suppose $0 < a < n$. Then $0 < q_2 < n$ also. By 6.4.17, $T_2(\beta_2,q_2,p_1) > T_2(\beta_1,q_2,p_1) = T_2(\beta_2,q_2,p_2)$; hence by 6.4.12, $p_1 > p_2$. Since $T_2(t_1,n,m) = T_2(t_1,q_2,p_1)$, 6.4.15 implies for any $x \geq t_1$, $T_2(x,n,m) \geq T_2(x,q_2,p_1) > T_2(x,q_2,p_2)$. However $T_2(t_2,n,m) = T_2(t_2,q_2,p_2)$, so $t_2 < t_1$.

Case 4: Suppose $n < a < 0$. Then a discussion similar to the one used in Case 3, yields $t_2 < t_1$.

6.5.19. Lemma. A projection $[1,0,0] \xrightarrow{(a,b,1)} [1,m,n]$, where $a, b, n \in H$ and $m \in \eta$ such that $a \notin \eta$ and $a \nmid n$, preserves the order of the proper points when n and 0 lie on the same side of a and reverses their order iff $(n,a,0) \in \rho$.

Proof. Take two points $(0,c_1,1)$, $(0,c_2,1)$ I $[1,0,0]$ with $c_1 < c_2$. The line $(0,c_i,1)(a,b,1) = [p_i,1,c_i]$ for some $p_i \in H$ $(i=1,2)$. Using the methods of 6.5.12, we obtain

$$[p_i,1,c_i] \wedge [1,m,n]$$
$$= (T_0(T_1(n,p_i,c_i),m,n), T_1(T_0(T_1(n,p_i,c_i),m,n),p_i,c_i),1).$$

If $c_1 \sim c_2$, then (A5)' implies $p_1 \sim p_2$; hence $T_0(T_1(n,p_1,c_1),m,n) = T_0(T_1(n,p_2,c_2),m,n)$ by the uniformity of $\mathcal{H}^*$.

Case 1: Suppose $n, 0 < a$. Then

$T_1(a,p_2,c_2) = T_1(a,p_1,c_1) < T_1(a,p_1,c_2)$ implies $p_2 < p_1$ by (OM3)*.

If $c_1 \sim c_2$, then $T_0(T_1(n,p_1,c_1),m,n) \sim n < a$ and $p_2 < p_1$ imply

$$T_1(T_0(T_1(n,p_1,c_1),m,n),p_1,c_1) < T_1(T_0(T_1(n,p_1,c_1),m,n),p_2,c_2)$$
$$= T_1(T_0(T_1(n,p_2,c_2),m,n),p_2,c_2).$$

If $c_1 \not\sim c_2$, then $[p_1,1,c_1] \not\sim [p_2,1,c_2]$ and (A5)' implies $T_1(T_0(T_1(n,p_1,c_1),m,n),p_1,c_1) \not\sim T_1(T_0(T_1(n,p_2,c_2),m,n),p_2,c_2)$. Since $T_0(T_1(n,p_2,c_2),m,n) \sim n < a$, (OM3)* implies

$$T_1(T_0(T_1(n,p_2,c_2),m,n),p_2,c_2) > T_1(T_0(T_1(n,p_2,c_2),m,n),p_1,c_1)$$
$$\sim T_1(T_0(T_1(n,p_1,c_1),m,n),p_1,c_1).$$

Hence $T_1(T_0(T_1(n,p_2,c_2),m,n),p_2,c_2) > T_1(T_0(T_1(n,p_1,c_1),m,n),p_1,c_1)$.

Case 2: Suppose $n, 0 > a$. Then a discussion similar to the one used in Case 1 yields

$T_1(T_0(T_1(n,p_2,c_2),m,n),p_2,c_2) > T_1(T_0(T_1(n,p_1,c_1),m,n),p_1,c_1)$.

Case 3: Suppose $n < a < 0$. Then

$T_1(a,p_2,c_2) = T_1(a,p_1,c_1) < T_1(a,p_1,c_2)$ implies $p_2 > p_1$ by (OM3)*.

If $c_1 \sim c_2$, then $T_0(T_1(n,p_1,c_1),m,n) \sim n < a$ and $p_2 > p_1$ imply

$$T_1(T_0(T_1(n,p_1,c_1),m,n),p_1,c_1) > T_1(T_0(T_1(n,p_1,c_1),m,n),p_2,c_2)$$
$$= T_1(T_0(T_1(n,p_2,c_2),m,n),p_2,c_2).$$

If $c_1 \not\sim c_2$, then as in Case 1,

$$T_1(T_0(T_1(n,p_2,c_2),m,n),p_2,c_2) \not\sim T_1(T_0(T_1(n,p_1,c_1),m,n),p_1,c_1)$$
$$\sim T_1(T_0(T_1(n,p_2,c_2),m,n),p_1,c_1).$$

Since $T_0(T_1(n,p_2,c_2),m,n) \sim n < a$, (OM3)* implies $T_1(T_0(T_1(n,p_2,c_2),m,n),p_1,c_1) > T_1(T_0(T_1(n,p_2,c_2),m,n),p_2,c_2)$; hence $T_1(T_0(T_1(n,p_1,c_1),m,n),p_1,c_1) > T_1(T_0(T_1(n,p_2,c_2),m,n),p_2,c_2)$.

Case 4: Suppose $0 < a < n$. Then a discussion similar to the one used in Case 3 yields the same result.

6.5.20. Now consider a projection $[1,0,1] \xrightarrow{(a,b,1)} [1,m,n]$, where $a \in \eta$ and $n \notin \eta$. It also maps proper points to proper points and improper points to improper points.

6.5.21. Lemma. A projection $[1,0,1] \xrightarrow{(a,b,1)} [1,m,n]$, where $m, a \in \eta$, $b \in H$ and $n \in H \setminus \eta$, preserves the order of the improper points when $n > 0$ and reverses their order when $n < 0$.

Proof. Take any two points $(T_2(\beta_1,1,0),1,\beta_1)$, $(T_2(\beta_2,1,0),1,\beta_2)$ on the line $[1,0,1]$ such that $\beta_1 < \beta_2$. Then the line $(T_2(\beta_i,1,0),1,\beta_i)(a,b,1) = [1,p_i,q_i]$ for some p_i, $q_i \in \eta$ $(i=1,2)$. As $q_1 \sim 0 \sim q_2$, the lines $[1,p_i,q_i]$ and $[1,p_i,0]$ coincide in the neighbour class $\bar{V}$ $(i=1,2)$. Let

$[1,p_i,q_i] \wedge [1,m,n] = [1,p_i,0] \wedge [1,m,n] = (p_i,1,t_i)$ for some $t_i \in \eta$.

$(i=1,2)$.

Case 1: Suppose $n > 0$. Since $1 > 0$, 6.4.17 implies $T_2(\beta_1,1,0) < T_2(\beta_2,1,0)$. However as $(T_2(\beta_i,1,0),1,\beta_i)\ I\ [1,p_i,0]$, $p_i = T_2(\beta_i,1,0)$, so $p_1 < p_2$. Therefore $T_2(t_1,n,m) = p_1 < p_2 = T_2(t_2,n,m)$ which implies by 6.4.17 that $t_1 < t_2$.

Case 2: Suppose $n < 0$. Then a discussion similar to the one used in Case 1 yields $t_1 > t_2$.

6.5.22. Lemma. A projection $[1,0,1] \xrightarrow{(a,b,1)} [1,m,n]$, where $m, a \in \eta$, $b \in H$ and $n \in H \setminus \eta$, preserves the order of the proper points when $n > 0$ and reverses their order when $n < 0$.

Proof. Take any two points $(1,c_1,1), (1,c_2,1)\ I\ [1,0,1]$ such that $c_1 < c_2$. Since $T_1(1,p_1,q_1) = c_1 < c_2 = T_1(1,p_2,q_2)$ and $1 > a$, (OM3)* implies $p_1 < p_2$. Using the methods of 6.5.12, we obtain

$$[p_i,1,q_i] \wedge [1,m,n]$$
$$= ((T_0(T_1(n,p_i,q_i),m,n),T_1(T_0(T_1(n,p_i,q_i),m,n),p_i,q_i),1).$$

If $c_1 \sim c_2$, (A5)' implies $p_1 \sim p_2$; hence $T_0(T_1(n,p_1,q_1),m,n) = T_0(T_1(n,p_2,q_2),m,n)$. Therefore $T_1(T_0(T_1(n,p_1,q_1),m,n),p_1,q_1) < T_1(T_0(T_1(n,p_2,q_2),m,n),p_2,q_2)$ when $n > 0$ and $T_1(T_0(T_1(n,p_1,q_1),m,n),p_1,q_1) > T_1(T_0(T_1(n,p_2,q_2),m,n),p_2,q_2)$ when $n < 0$.

If $c_1 \nsim c_2$, then (A6)' implies $p_1 \nsim p_2$ and (A5)' gives

$$T_1(T_0(T_1(n,p_2,q_2),m,n),p_2,q_2) \nsim T_1(T_0(T_1(n,p_1,q_1),m,n),p_1,q_1)$$
$$\sim T_1(T_0(T_1(n,p_2,q_2),m,n),p_1,q_1).$$

Therefore by (OM3)*,

$T_1(T_0(T_1(n,p_2,q_2),m,n),p_2,q_2) > T_1(T_0(T_1(n,p_1,q_1),m,n),p_1,q_1)$ when $n > 0$ and

$T_1(T_0(T_1(n,p_2,q_2),m,n),p_2,q_2) < T_1(T_0(T_1(n,p_1,q_1),m,n),p_1,q_1)$ when $n < 0$.

We may obtain corollaries similar to 6.5.3 and 6.5.4 for Lemmas 6.5.18, 6.5.19, 6.5.21, 6.5.22. Combining all this information, we obtain the following lemma.

6.5.23. Lemma. Let $n, q, a, b \in H$ and $m, p \in \eta$ such that $a \nmid n, q$. Then the projection $[1,m,n] \xrightarrow{(a,b,1)} [1,p,q]$ preserves the order of the improper [proper] points when n and q lie on the same side of a and reverses their order when $(n,a,q) \in \rho$.

6.5.24. Next we consider a projection $[1,0,0] \xrightarrow{(a,b,1)} [m,1,n]$ where $a \notin \eta$ and $b \nmid T_1(a,m,n)$. There exists a unique $u \in H$ such that $b \sim T_1(a,m,u)$. Clearly $u \nmid n$. Then we may partition the domain line of the projection into four segments (see below) and each segment is mapped to a segment of the range line. We shall investigate each submap individually.

$$\{(0,c,1) \mid c \nmid u;\ (c,u,n) \notin \rho\} \longrightarrow \{(x,T_1(x,m,n),1) \mid x \nmid a;\ (x,a,0) \notin \rho\}$$

$$\{(0,c,1) \mid c \nmid u;\ (c,u,n) \in \rho\} \longrightarrow \{(x,T_1(x,m,n),1) \mid x \nmid a;\ (x,a,0) \in \rho\}$$

$$\{(0,c,1) \mid c \sim u\} \longrightarrow \{(1,T_3(t,n,m),t) \mid t \in \eta\}$$

$$\{(0,1,t) \mid t \in \eta\} \longrightarrow \{(x,T_1(x,m,n),1) \mid x \sim a\}.$$

6.5.25. Lemma. The map

$$\{(0,c,1) \mid c \nmid u;\ (c,u,n) \notin \rho\} \longrightarrow \{(x,T_1(x,m,n),1) \mid x \nmid a;\ (x,a,0) \notin \rho\}$$

preserves order if $a > 0$ and $u < n$ or if $a < 0$ and $u > n$; it reverses order if $a > 0$ and $u > n$ or if $a < 0$ and $u < n$.

Proof. Take any $(0,c_1,1)$, $(0,c_2,1)$ from the set $\{(0,c,1) \mid c \nmid u;\ (c,u,n) \notin \rho\}$ such that $c_1 < c_2$. Then $(0,c_i,1)(a,b,1) = [p_i,1,c_i]$ for some $p_i \nmid m$ $(i=1,2)$. Since $p_i \nmid m$, there exists $x_i \in H$ such that $[p_i,1,c_i] \wedge [m,1,n] = (x_i,T_1(x_i,m,n),1)$ $(i=1,2)$.

Case 1: Suppose $u < n$ and $a > 0$ $[a < 0]$. Then $u < c_1 < c_2$. As $T_1(0,p_i,c_i) = c_i > u = T_1(0,m,u)$, (OM3)* implies $p_i < m$ $[p_i > m]$. Therefore, for any $y \geq a$ $[y \leq a]$, (OM3)* and (OM2) imply $T_1(y,p_i,c_i) \leq T_1(y,m,u) < T_1(y,m,n)$. However $T_1(x_i,p_i,c_i) = T_1(x_i,m,n)$, so $x_i < a$ $[x_i > a]$. In addition, as $T_1(0,p_1,c_1) = c_1 < c_2 = T_1(0,p_2,c_2)$, (OM3)* implies $p_1 > p_2$ $[p_1 < p_2]$. Hence for any y such that $x_2 \leq y < a$ $[x_2 \geq y > a]$, $T_1(y,p_1,c_1) < T_1(y,p_2,c_2) \leq T_1(y,m,n)$ which implies $x_1 < x_2$ $[x_1 > x_2]$.

Case 2: Suppose $u > n$ and $a > 0$ $[a < 0]$. Then a discussion similar to the used above yields $x_1 > x_2$ $[x_1 < x_2]$.

6.5.26. Lemma. The map $\{(0,c,1) \mid c \nmid u;\ (c,u,n) \in \rho\} \longrightarrow \{(x,T_1(x,m,n),1) \mid x \nmid a;\ (x,a,0) \in \rho\}$ preserves order if $a > 0$ and $u < n$ or if $a < 0$ and $u > n$; it reverses order if $a > 0$ and $u > n$ or if $a < 0$ and $u < n$.

Proof. A similar argument to that used in the proof of 6.5.25 gives the desired result.

6.5.27. Lemma. The map

$$\{(0,c,1) \mid c \sim u\} \longrightarrow \{(1,T_3(t,n,m),t) \mid t \in \eta\}$$

preserves the order if $a > 0$ and $u > n$ or if $a < 0$ and $u < n$;

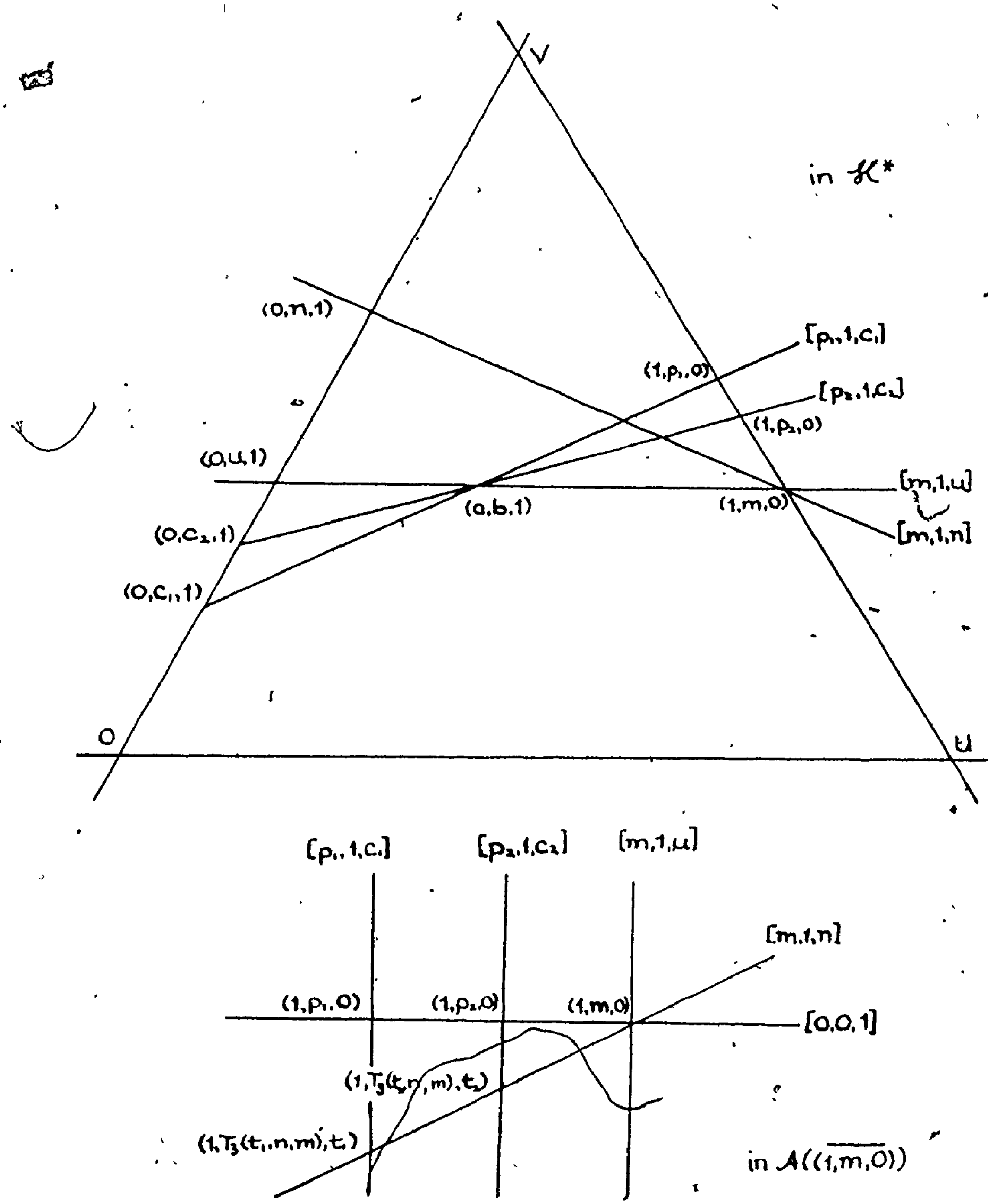


Figure 6.8
(Lemma 6.5.27)

it reverses the order if $a > 0$ and $u < n$ or if $a < 0$ and $u > n$.

Proof. Consider any $(0,c,1)$ such that $c \sim u$ and $c > u$. Then $(0,c,1)(a,b,1) = [p,1,c]$ where $p \sim m$. Since $c \sim u$, the lines $[p,1,c]$ and $[p,1,u]$ coincide in the neighbour class $(\overline{1,m,0})$. Let $[p,1,c] \wedge [m,1,n] = [p,1,u] \wedge [m,1,n] = (1,T_3(t,n,m),t)$, $t \in \eta$.

Case 1: Suppose $u > n$ and $a > 0$ $[a < 0]$. Then $T_1(a,p,c) = T_1(a,m,u)$ and $T_1(0,p,c) = c > u = T_1(0,m,u)$ imply $p < m$ $[p > m]$. Since $T_3(t,n,m) = T_3(t,u,p)$ and $T_3(0,n,m) = m > p = T_3(0,u,p)$ $[T_3(0,n,m) = m < p = T_3(0,u,p)]$, 6.4.10 implies $t > 0$ $[t < 0]$.

Case 2: Suppose $u < n$ and $a > 0$ $[a < 0]$. Then a similar discussion to the one above yields $t < 0$ $[t > 0]$.

Similarly, if $c < u$, then $t < 0$ when $u > n$ and $a > 0$ or $u < n$ and $a < 0$ and $t > 0$ when $u < n$ and $a > 0$ or $u > n$ and $a < 0$.

Now consider any two points $(0,c_1,1)$, $(0,c_2,1)$ such that $c_1, c_2 \sim u$, $c_1 \neq c_2 \neq u \neq c_1$. Then one of $(c_1,c_2,u) \in \rho$, $(c_2,c_1,u) \in \rho$ or $(c_1,u,c_2) \in \rho$ holds. Let $(0,c_i,1)(a,b,1) = [p_i,1,c_i]$ and $[p_i,1,c_i] \wedge [m,1,n] = [p_i,1,u] \wedge [m,1,n] = (1,T_3(t_i,n,m),t_i)$ as above. If $(c_1,c_2,u) \in \rho$, then $(p_1,p_2,m) \in \rho$ by (OM3)*; hence in $\mathcal{A}((\overline{1,m,0}))$ $((1,p_1,0),(1,p_2,0),(1,m,0)) \in \rho|_{(\overline{1,m,0})}$ and the parallel projection $[0,0,1]|_{(\overline{1,m,0})} \xrightarrow{[m,1,u]} [m,1,n]|_{(\overline{1,m,0})}$ in $\mathcal{A}((\overline{1,m,0}))$ yields $((1,T_3(t_1,n,m),t_1),(1,T_3(t_2,n,m),t_2),(1,m,0)) \in \rho|_{(\overline{1,m,0})}$. Therefore $(t_1,t_2,0) \in \rho$ by 6.4.2. Similarly, if $(c_2,c_1,u) \in \rho$ or $(c_1,u,c_2) \in \rho$, then $(t_2,t_1,0) \in \rho$ or $(t_1,0,t_2) \in \rho$ respectively.

Combining these results with Cases 1 and 2 above, we obtain the desired result.

6.5.28. Lemma. The map

$$\{(0,1,t) \mid t \in \eta\} \longrightarrow \{(x,T_1(x,m,n),1) \mid x \sim a\}$$

preserves order if $a > 0$ and $u > n$ or if $a < 0$ and $u < n$; it reverses order if $a > 0$ and $u < n$ or if $a < 0$ and $u > n$.

Proof. Consider two points $(0,1,t_1)$, $(0,1,t_2)$, where $t_1, t_2 \in \eta$ and $t_1 < t_2$. Then $(0,1,t_i)(a,b,1) = [1,p_i,q_i]$ for some $p_i \in \eta$, $q_i \sim a$ $(i=1,2)$. Since $q_1 \sim a \sim q_2$, the line $[1,p_i,q_i]$ coincides with the line $[1,p_i,a]$ in $\bar{V}$.

Case 1: Suppose $u > n$ and $a > 0$ $[a < 0]$. Then

$T_2(t_2,a,p_1) > T_2(t_1,a,p_1) = T_2(t_2,a,p_2)$

$[T_2(t_2,a,p_1) < T_2(t_1,a,p_1) = T_2(t_2,a,p_2)]$ by 6.4.17; hence $p_1 > p_2$ $[p_1 < p_2]$ by 6.4.12. As in 6.5.13,

$$[1,p_i,q_i] \wedge [m,1,n]$$
$$= (T_0(T_1(q_i,m,n),p_i,q_i),T_1(T_0(T_1(q_i,m,n),p_i,q_i),m,n),1).$$

However $T_1(q_1,m,n) \sim T_1(q_2,m,n)$; hence

$T_0(T_1(q_1,m,n),p_1,q_1) = T_0(T_1(q_2,m,n),p_1,q_1)$. Since $q_2 \sim a$,

$T_1(q_2,m,n) \sim T_1(a,m,n) < T_1(a,m,u) = b$. Therefore by (OM3)*',

$T_0(T_1(q_2,m,n),p_1,q_1) < T_0(T_1(q_2,m,n),p_2,q_2)$

$[T_0(T_1(q_2,m,n),p_1,q_1) > T_0(T_1(q_2,m,n),p_2,q_2)]$.

Case 2: Suppose $u < n$ and $a > 0$ $[a < 0]$. Then using the same methods as above, we obtain $T_0(T_1(q_2,m,n),p_1,q_1) > T_0(T_1(q_2,m,n),p_2,q_2)$ $[T_0(T_1(q_2,m,n),p_1,q_1) < T_0(T_1(q_2,m,n),p_2,q_2)]$.

6.5.29. Consider the projection $[1,0,1] \xrightarrow{(a,b,1)} [m,1,n]$, where $a \in \eta$, $b \nmid n$. There exists $u \in H$, $u \nmid n$ such that $b = T_1(a,m,u)$. As in 6.5.24, we may split this into four maps as follows:

$$\{(1,c,1) \mid c \nmid T_1(1,m,u);\ (c,T_1(1,m,u),T_1(1,m,n)) \notin \rho\} \longrightarrow \{(x,T_1(x,m,n),1) \mid x \nmid a;\ (x,a,0) \notin \rho\};$$

$$\{(1,c,1) \mid c \nmid T_1(1,m,u);\ (c,T_1(1,m,u),T_1(1,m,n)) \in \rho\} \longrightarrow \{(x,T_1(x,m,n),1) \mid x \nmid a;\ (x,a,0) \in \rho\};$$

$$\{(1,c,1) \mid c \sim T_1(1,m,u)\} \longrightarrow \{(1,T_3(t,n,m),t) \mid t \in \eta\}$$

$$\{(T_2(t,1,0),1,t) \mid t \in \eta\} \longrightarrow \{(x,T_1(x,m,n),1) \mid x \sim a\}.$$

We may use the methods of 6.5.25 - 6.5.28 to obtain the following:

The first two maps preserve order if $u > n$ and reverse order if $u < n$. The last two maps preserve order if $u < n$ and reverse order if $u > n$.

6.5.30. Now consider the projection $[1,0,0] \xrightarrow{(a,b,1)} [m,n,1]$, where $a \notin \eta$. Here the proper points of $[1,0,0]$ are mapped to the points of the second kind incident with $[m,n,1]$ and the improper points of $[1,0,0]$ are mapped to the points of $\bar{V}$ incident with $[m,n,1]$; ie.,

$$\{(0,c,1) \mid c \in H\} \longrightarrow \{(1,u,T_4(u,n,m)) \mid u \in H\} \text{ and}$$

$$\{(0,1,t) \mid t \in H\} \longrightarrow \{(v,1,n) \mid v \in \eta\}.$$

6.5.31. Lemma. The map

$$\{(0,c,1) \mid c \in H\} \xrightarrow{(a,b,1)} \{(1,u,T_4(u,n,m)) \mid u \in H\}$$

preserves order if $a < 0$ and reverses order if $a > 0$.

Proof. Take two points $(0,c_1,1)$, $(0,c_2,1)$ I $[1,0,0]$ such that $c_1 < c_2$. Then $(0,c_i,1)(a,b,1) = [p_i,1,c_i]$ for some $p_i \in H$ $(i=1,2)$.

Suppose $a > 0$ [$a < 0$]. Then $T_1(a,p_1,c_1) = T_1(a,p_2,c_2)$ and $T_1(0,p_1,c_1) = c_1 < c_2 = T_1(0,p_2,c_2)$ imply $p_1 > p_2$ [$p_1 < p_2$] by (OM3)*.

If $c_1 \nsim c_2$, then $[p_1,1,c_1] \nsim [p_2,1,c_2]$ and $(a,b,1)\ I\ [p_1,1,c_1], [p_2,1,c_2]$; hence $p_1 \nsim p_2$ by (B4). Since $[p_i,1,0] \wedge [m,n,1] = (1,p_i,T_4(p_i,m,n))$ and $[m,n,1]$ coincides with $[T_4(p_i,n,m),0,1]$ in $\overline{(1,p_i,0)}$, we obtain

$[p_i,1,c_i] \wedge [m,n,1] = (1,T_3(T_4(p_i,n,m),c_i,p_i),T_4(p_i,n,m))$ $(i=1,2)$.

However $T_3(T_4(p_1,n,m),c_1,p_1) \sim p_1 > p_2 \sim T_3(T_4(p_2,n,m),c_2,p_2)$ [$T_3(T_4(p_1,n,m),c_1,p_1) \sim p_1 < p_2 \sim T_3(T_4(p_2,n,m),c_2,p_2)$] and order is reversed [preserved].

If $c_1 \sim c_2$, then (A5)' implies $[p_1,1,c_1] \sim [p_2,1,c_2]$; hence $p_1 \sim p_2$. Therefore $T_4(p_1,n,m) = T_4(p_1,n,m)$ by the uniformity of $\mathcal{K}$. Hence all three lines $[m,n,1]$, $[T_4(p_i,n,m),0,1]$ $(i=1,2)$ coincide in $\overline{(1,p_1,0)} = \overline{(1,p_2,0)}$. As $c_1 \sim c_2$, the lines $[p_1,1,c_1]$ and $[p_1,1,c_2]$ also coincide in $\overline{(1,p_1,0)}$. Therefore

$T_3(T_4(p_1,n,m),c_1,p_1) = T_3(T_4(p_2,n,m),c_2,p_2) > T_3(T_4(p_2,n,m),c_2,p_2)$ [$T_3(T_4(p_1,n,m)c_1,p_1) = T_3(T_4(p_2,n,m),c_2,p_1) < T_3(T_4(p_2,n,m),c_2,p_2)$] by 6.4.9 and order is reversed [preserved].

<u>6.5.32</u>. <u>Lemma</u>. The map

$$\{(0,1,t) \mid t \in \eta\} \xrightarrow{(a,b,1)} \{(v,1,n) \mid v \in n\}$$

preserves order if $a < 0$ and reverses order if $a > 0$.

<u>Proof</u>. Take two points $(0,1,t_1)$, $(0,1,t_2)\ I\ [1,0,0]$, where $t_1 < t_2$. Then $(0,1,t_i)(a,b,1) = [1,p_i,q_i]$ for some $p_i \in \eta$, $q_i \sim a$ $(i=1,2)$. Suppose $a > 0$ [$a < 0$]. Then the two lines $[1,p_i,q_i]$ and $[1,p_i,a]$ coincide in $\bar{V}$ $(i=1,2)$. Hence

$T_2(t_2,a,p_2) = T_2(t_1,a,p_1) < T_2(t_2,a,p_1)$

$[T_2(t_2,a,p_2) = T_2(t_1,a,p_1) > T_2(t_2,a,p_1)]$ by 6.4.17 and $p_2 < p_1$ $[p_2 > p_1]$, by 6.4.12. However by 5.8.5,

$[1,p_i,q_i] \wedge [m,n,1] = (T_2(n,q_i,p_i),1,n) = (T_2(n,a,p_i),1,n)$ $(i=1,2)$. As $T_2(n,a,p_1) > T_2(n,a,p_2)$ $[T_2(n,a,p_1) < T_2(n,a,p_2)]$ by 6.4.12, order is reversed [preserved].

6.5.33. Finally, we consider the projection $[1,0,1] \xrightarrow{(a,b,1)} [m,n,1]$, where $a \in \eta$. This projection also maps the proper points of the domain line to points of the second kind incident with the range line and the improper points of the domain line to points of $\bar{V}$ incident with the range line (cf. 6.5.30); ie.,

$$\{(1,c,1) \mid c \in H\} \longrightarrow \{(1,u,T_4(u,n,m)) \mid u \in H\} \text{ and}$$

$$\{(T_2(t,1,0),1,t) \mid t \in \eta\} \longrightarrow \{(v,1,n) \mid v \in \eta\}.$$

6.5.34. Lemma. The map

$$\{(1,c,1) \mid c \in H\} \xrightarrow{(a,b,1)} \{(1,u,T_4(u,n,m)) \mid u \in H\};\ a \in \eta;$$

preserves order.

Proof. Take any two points $(1,c_1,1)$, $(1,c_2,1)$ I $[1,0,1]$ such that $c_1 < c_2$. Then $(1,c_i,1)(a,b,1) = [p_i,1,q_i]$ for some $p_i \in H$, $q_i \sim b$. Since $T_1(1,p_1,q_1) = c_1 < c_2 = T_1(1,p_2,q_2)$ and $a \sim 0 < 1$, (OM3)* implies $p_1 < p_2$. As in 6.5.31,

$[p_i,1,q_i] \wedge [m,n,1] = (1,T_3(T_4(p_i,n,m),q_i,p_i),T_4(p_i,n,m))$ $(i=1,2)$.

If $c_1 \not\sim c_2$, then $p_1 \not\sim p_2$ and $T_3(T_4(p_1,n,m),q_1,p_1) \sim p_1 < p_2 \sim T_3(T_4(p_2,n,m),q_2,p_2)$. Hence order is preserved.

If $c_1 \sim c_2$, then $p_1 \sim p_2$ and $T_4(p_1,n,m) = T_4(p_2,n,m)$. In addition, as $q_1 \sim b \sim q_2$, the lines $[p_1,1,q_1]$ and $[p_1,1,q_2]$ coincide in $\overline{(1,p,0)}$. Therefore

$T_3(T_4(p_1,n,m),q_1,p_1) = T_3(T_4(p_2,n,m),q_2,p_1) < T_3(T_4(p_2,n,m),q_2,p_2)$ and order is preserved.

6.5.35. Lemma. The map

$$\{(T_2(t,1,0),1,t) \mid t \in \eta\} \xrightarrow{(a,b,1)} \{(v,1,n) \mid v \in \eta\};\ a \in \eta;$$

preserves order.

Proof. Take two points $(T_2(t_1,1,0),1,t_1)$, $(T_2(t_2,1,0),1,t_2)$ I $[1,0,1]$ such that $t_1 < t_2$. Then $(T_2(t_i,1,0),1,t_i)(a,b,1) = [1,p_i,q_i]$ for some p_i, $q_i \in \eta$. Since $q_i \in \eta$, the lines $[1,p_i,q_i]$ and $[1,p_i,0]$ coincide in $\bar{V}$ $(i=1,2)$; hence $T_2(t_i,1,0) = T_2(t_i,q_i,p_i) = T_2(t_i,0,p_i) = p_i$. By 6.4.15, $p_1 = T_2(t_1,1,0) < T_2(t_2,1,0) = p_2$. However $[1,p_i,0] \wedge [m,n,1] = (p_i,1,n)$, so order is preserved.

6.5.36. We note that any projection between improper lines has a proper centre and preserves the order of the second kind points and of the points in $\bar{V}$.

As in 6.5.3, the inverse of each of the projections discussed behaves in the same manner as the projection itself. We can obtain any projection with a proper centre by composing one of the projections discussed in this section with the inverse of another such projection. Hence we can determine the behavior of various sets of points under any projection with proper centre.

6.6. The ordering on $\mathcal{K}^*$.

We shall now use the information we have compiled to define a total ordering on the entire point set of each line of $\mathcal{K}^*$ and then use this total ordering to define a cyclic ordering on the P.H. plane $\mathcal{K}^*$. First, we define the notion of ordering by sets.

6.6.1. Let $A = \bigcup_{i=1}^{n} A_i$ be the union of disjoint totally ordered sets $(A_i, <_i)$; $i = 1,2,\ldots,n$. For $x, y \in A$, we define $x < y$ if either:

1) $x, y \in A_i$ and $x <_i y$; or

2) $x \in A_i$, $y \in A_j$ and $i < j$.

We say that A is ordered setwise.

6.6.2. Consider any line $[1,m,n]$. Then we order the points of $[1,m,n]$ setwise as follows:

$$[1,m,n] = \{(T_2(t,n,m),1,t) \mid t \in \eta\} \cup \{(T_0(x,m,n),x,1) \mid x \in H\}$$

with $(T_2(t_1,n,m),1,t_1) \prec (T_2(t_2,n,m),1,t_2)$ if and only if $t_1 > t_2$ and $(T_0(x_1,m,n),x_1,1) \prec (T_0(x_2,m,n),x_2,1)$ if and only if $x_1 < x_2$.

A line $[m,1,n]$ is ordered setwise by:

$$[m,1,n] = \{(1,T_3(t,n,m),t) \mid t \in \eta\} \cup \{(x,T_1(x,m,n),1) \mid x \in H\}$$

with $(1,T_3(t_1,n,m),t_1) \prec (1,T_3(t_2,n,m),t_2)$ if and only if $t_1 > t_2$ and $(x_1,T_1(x_1,m,n),1) \prec (x_2,T_1(x_2,m,n),1)$ if and only if $x_1 < x_2$.

Finally, a line $[m,n,1]$ is ordered setwise by:

$$[m,n,1] = \{(t,1,n) \mid t \in \eta\} \cup \{(1,x,T_4(x,n,m)) \mid x \in H\}$$

with $(t_1,1,n) \prec (t_2,1,n)$ if and only if $t_1 > t_2$ and $(1,x_1,T_4(x_1,n,m)) \prec (1,x_2,T_4(x_2,n,m))$ if and only if $x_1 < x_2$.

Let $\prec$ also denote the entire total ordering on each line.

6.6.3. The cyclic ordering on $\mathcal{H}^*$. We define a quaternary relation on $\mathbb{P}$ in the following way. Four points A, B, C, D are in this relation (written AB σ CD) if and only if they are mutually distinct, collinear and one of the following holds:

$$A \prec C \prec B \prec D$$

$$A \prec D \prec B \prec C$$

$$B \prec C \prec A \prec D$$

$$B \prec D \prec A \prec C$$

$$C \prec B \prec D \prec A$$

$$C \prec A \prec D \prec B$$

$$D \prec A \prec C \prec B$$

$$D \prec B \prec C \prec A.$$

We say A and B separate C and D.

6.6.4. Theorem. $\mathcal{H}^* = \langle \mathbb{P}^*, \mathbb{L}^*, I, \sigma \rangle$ is an ordered P.H. plane.

Proof. Axioms (OP1) through (OP4) follow immediately from the definition of σ.

Since $\mathcal{H}$ is an ordered A.H. plane, each proper line of $\mathcal{H}^*$ is incident with at least three non-neighbouring proper points (cf. 4.1). The fourth non-neighbouring point on a proper line of $\mathcal{H}^*$ would be its intersection with ℓ^*. Thus (OP5) holds.

Axiom (OP6) follows from 6.5.

CHAPTER 7

Examples and Counterexamples

In Chapter 6, we constructed an ordered projective completion for a projectively ordered, projectively uniform A.H. plane. In this chapter we provide an example of a strongly ordered, but not projectively ordered, projectively uniform A.H. plane which has no ordered projective completion. We also give an example to show that an arbitrary projective completion of a projectively ordered, projectively uniform A.H. plane is not necessarily ordered. Finally, we present an example of a projectively ordered, projectively uniform A.H. plane and an ordered projective completion.

7.1. A strongly ordered projectively uniform A.H. plane with no ordered projective completion.

In this section, we present an example to show that, in general, strongly ordered, projectively uniform A.H. planes do not have ordered projective completions.

Consider the example given in 3.3.8. Suppose that $\mathcal{H}$ has a projective completion $\mathcal{H}^*$ with order relation σ. Then $\bar{\mathcal{H}}^*$ is an ordered projective plane with improper line $\bar{\ell}_\omega$.

Take four points $(\mathbf{0},\mathbf{0},\mathbf{1})$, $(\mathbf{0},\mathbf{1},\mathbf{1})$, $(\mathbf{0},(1,1),\mathbf{1})$, $(\mathbf{0},\mathbf{1},\mathbf{0})$ incident with the line $[\mathbf{1},\mathbf{0},\mathbf{0}]$. Since

$(\mathbf{0},\mathbf{0},\mathbf{1}) \not\sim (\mathbf{0},\mathbf{1},\mathbf{1}) \sim (\mathbf{0},(1,1),\mathbf{1}) \not\sim (\mathbf{0},\mathbf{1},\mathbf{0}) \not\sim (\mathbf{0},\mathbf{0},\mathbf{1})$, 4.1.3 implies $(\mathbb{0},\mathbb{1},\mathbb{1})\,(\mathbb{0},(1,1),\mathbb{1}) \not\sigma (\mathbb{0},\mathbb{0},\mathbb{1})\,(\mathbb{0},\mathbb{1},\mathbb{0})$; hence either $(\mathbb{0},\mathbb{1},\mathbb{1})\,(\mathbb{0},\mathbb{1},\mathbb{0})\ \sigma\ (\mathbb{0},\mathbb{0},\mathbb{1})\,(\mathbb{0},(1,1),\mathbb{1})$ or $(\mathbb{0},\mathbb{1},\mathbb{1})\,(\mathbb{0},\mathbb{0},\mathbb{1})\ \sigma\ (\mathbb{0},\mathbb{1},\mathbb{0})\,(\mathbb{0},(1,1),\mathbb{1})$. Assume first that the former holds.

The projections

$$[\mathbb{1},\mathbb{0},\mathbb{0}] \xrightarrow{(\mathbb{1},\mathbb{0},\mathbb{1})} [\mathbb{1},\mathbb{1},\mathbb{0}] \xrightarrow{(\mathbb{0},\mathbb{1},\mathbb{1})} [\mathbb{0},\mathbb{1},\mathbb{0}]$$

$$\xrightarrow{(\mathbb{1},\mathbb{1},\mathbb{0})} [\mathbb{1},\mathbb{0},\mathbb{1}] \xrightarrow{(\mathbb{1},\mathbb{0},\mathbb{0})} [\mathbb{1},\mathbb{0},\mathbb{0}]$$

yield $(\mathbb{0},\mathbb{0},\mathbb{1})\,(\mathbb{0},\mathbb{1},\mathbb{0})\ \sigma\ (\mathbb{0},\mathbb{1},\mathbb{1})\,(\mathbb{0},(0,-1),\mathbb{1})$. Using the projections

$$[\mathbb{1},\mathbb{0},\mathbb{0}] \xrightarrow{(\mathbb{1},(-1,-1),\mathbb{0})} [\mathbb{1},\mathbb{0},\mathbb{1}] \xrightarrow{(\mathbb{1},\mathbb{0},\mathbb{0})} [\mathbb{1},\mathbb{0},\mathbb{0}],$$

$(\mathbb{0},\mathbb{1},\mathbb{1})\,(\mathbb{0},\mathbb{1},\mathbb{0})\ \sigma\ (\mathbb{0},\mathbb{0},\mathbb{1})\,(\mathbb{0},(1,1),\mathbb{1})$ gives $(\mathbb{0},(0,-1),\mathbb{1})\,(\mathbb{0},\mathbb{1},\mathbb{0})\ \sigma\ (\mathbb{0},(-1,-1),\mathbb{1})\,(\mathbb{0},\mathbb{0},\mathbb{1})$. If X and Y are any pair of distinct points selected from the set $\{(\mathbb{0},\mathbb{0},\mathbb{1}), (\mathbb{0},(0,-1),\mathbb{1}), (\mathbb{0},\mathbb{1},\mathbb{1}), (\mathbb{0},(1,1),\mathbb{1}), (\mathbb{0},\mathbb{1},\mathbb{0})\}$, then $(\mathbb{0},(-1,-1),\mathbb{1})\,(\mathbb{0},(-1,0),\mathbb{1}) \not\sigma XY$ by 4.1.3 and 4.1.4. Therefore $(\mathbb{0},(0,-1),\mathbb{1})\,(\mathbb{0},\mathbb{1},\mathbb{0})\ \sigma\ (\mathbb{0},\mathbb{0},\mathbb{1})\,(\mathbb{0},(-1,0),\mathbb{1})$.

Using the projection $[\mathbb{1},\mathbb{0},\mathbb{0}] \xrightarrow{(\mathbb{1},\mathbb{1},\mathbb{1})} [\mathbb{0},\mathbb{0},\mathbb{1}]$, $(\mathbb{0},\mathbb{1},\mathbb{1})\,(\mathbb{0},\mathbb{1},\mathbb{0})\ \sigma\ (\mathbb{0},\mathbb{0},\mathbb{1})\,(\mathbb{0},(1,1),\mathbb{1})$ gives $(\mathbb{1},\mathbb{0},\mathbb{0})\,(\mathbb{0},\mathbb{1},\mathbb{0})\ \sigma\ (\mathbb{1},\mathbb{1},\mathbb{0})\,(\mathbb{1},(0,1),\mathbb{1})$. However since $(\mathbb{0},(0,-1),\mathbb{1})\,(\mathbb{0},\mathbb{1},\mathbb{0})\ \sigma\ (\mathbb{0},\mathbb{0},\mathbb{1})\,(\mathbb{0},(-1,0),\mathbb{1})$, the projection $[\mathbb{1},\mathbb{0},\mathbb{0}] \xrightarrow{(\mathbb{1},\mathbb{0},\mathbb{1})} [\mathbb{0},\mathbb{0},\mathbb{1}]$ yields $(\mathbb{1},(0,1),\mathbb{1})\,(\mathbb{0},\mathbb{1},\mathbb{0})\ \sigma\ (\mathbb{1},\mathbb{0},\mathbb{0})\,(\mathbb{1},\mathbb{1},\mathbb{0})$; a contradiction.

A discussion similar to the one used above also results in a contradiction if we assume $(\mathbb{0},\mathbb{1},\mathbb{1})\,(\mathbb{0},\mathbb{0},\mathbb{1})\ \sigma\ (\mathbb{0},\mathbb{1},\mathbb{0})\,(\mathbb{0},(1,1),\mathbb{1})$.

Thus $\mathcal{H}^*$ is not ordered.

7.2. A projective completion of a projectively ordered, projectively uniform A.H. plane which is not ordered.

In Chapter 6, we used special bijections γ, $\phi_{\bar{X}}$, $\alpha_{\bar{X}}$ to construct an ordered projective completion of a projectively uniform, projectively ordered A.H. plane. In this section, we present an example to show that all projective completions of such a plane are not ordered.

Let $H = \mathbb{R} \times \mathbb{R}$ with coordinatewise addition and multiplication defined by $(A_1,A_2) \cdot (B_1,B_2) = (A_1B_1, A_1B_2 + A_2B_1)$. Then, as in 3.3.8, H is an A.H. ring with unique maximal ideal $\eta = \{(0,A) \mid A \in \mathbb{R}\}$, unit $\mathbb{1} = (1,0)$ and zero $\mathbb{0} = (0,0)$. We may order H lexiographically. The incidence structure $\mathcal{H}(H)$ constructed as in 2.2.3, is a projectively ordered, projectively uniform A.H. plane. Take $\mathcal{L} = [\mathbb{0},\mathbb{0}]_1$ and $S = (\mathbb{1},\mathbb{0})$. We may use $\mathcal{L}$ and S to construct the set of improper points for a projective completion $\mathcal{H}^*$ of $\mathcal{H}(H)$ as in 6.2. Since $\bar{T} = (\bar{S}\bar{V} \wedge \bar{O}\bar{E})\bar{U} \wedge \bar{\mathcal{L}}$ and T is any element of $\bar{T}$ incident with $\mathcal{L}$, let $T = (\mathbb{0},\mathbb{1})$. Define γ, $\phi_{\bar{X}}$ for $\bar{X}\ \bar{I}\ \bar{\mathcal{L}}_\omega$, $\bar{X} \neq \bar{V}$, and $\alpha_{\bar{Y}}$ for $\bar{Y}\ \bar{I}\ \bar{\mathcal{L}}_\omega$ as in 6.2. We shall define $\phi_{\bar{V}}$ in the following way.

$$\phi_{\bar{V}} : \{\Pi_{h} \mid \bar{V}\ \bar{I}\ \bar{h}\} \longrightarrow \lambda(\bar{V})$$

$$\Pi_{h} \rightsquigarrow \phi_{\bar{V}}(\Pi_{h}) = \psi_{\bar{V}}((L((\mathbb{0},-\mathbb{1}),h) \wedge OE)U \wedge \mathcal{L})$$

We may use these bijections to construct the improper lines and to complete the proper lines of $\mathcal{H}^*$. Using the triangle $(\mathbb{0},\mathbb{0})$, $(\mathbb{1},\mathbb{0})$, $(\mathbb{0},\mathbb{1})$, we may coordinatize $\mathcal{H}^*$ in the manner of

5.6. The points O, S, T, $(\mathbf{0},-\mathbf{1})$ have the coordinates $(\mathbf{0},\mathbf{0},\mathbf{1})$, $(\mathbf{1},\mathbf{0},\mathbf{1})$, $(\mathbf{0},\mathbf{1},\mathbf{1})$, $(\mathbf{0},-\mathbf{1},\mathbf{1})$ respectively. Using the definitions of γ, $\phi_{\bar{X}}$, $\alpha_{\bar{X}}$ $(\bar{X}\ \bar{I}\ \bar{L}_{\omega})$, we obtain the following results.

For $m, n \in H$,

$$(\mathbf{1},m,\mathbf{0})$$
$$= [m,\mathbf{1},\mathbf{0}] \wedge \ell^*$$
$$= \phi_{\overline{(\mathbf{1},m,\mathbf{0})}}(\Pi_{[m,\mathbf{1},\mathbf{0}]})$$
$$= \psi_{\overline{(\mathbf{1},m,\mathbf{0})}}(L((\mathbf{1},\mathbf{0},\mathbf{1}),[m,\mathbf{1},\mathbf{0}]) \wedge \ell)$$
$$= \psi_{\overline{(\mathbf{1},m,\mathbf{0})}}((\mathbf{0},-m,\mathbf{1}))$$

$$[m,\mathbf{1},\mathbf{0}]|_{\overline{(\mathbf{1},m,\mathbf{0})}}$$
$$= L(\phi_{\overline{(\mathbf{1},m,\mathbf{0})}}(\Pi_{[m,\mathbf{1},\mathbf{0}]}),\alpha_{\overline{(\mathbf{1},m,\mathbf{0})}}(\overline{[m,\mathbf{1},\mathbf{0}]}))$$
$$= L((\mathbf{1},m,\mathbf{0}),\Pi_{\overline{(\mathbf{1},m,\mathbf{0})}},\psi_{\overline{(\mathbf{1},m,\mathbf{0})}}(L((\mathbf{0},-m,\mathbf{1}),[\mathbf{0},\mathbf{1},\mathbf{0}])))$$
$$= \psi_{\overline{(\mathbf{1},m,\mathbf{0})}}([\mathbf{0},\mathbf{1},-m]|_{\overline{(\mathbf{0},-m,\mathbf{1})}})$$

$$[m,\mathbf{1},n]|_{\overline{(\mathbf{1},m,\mathbf{0})}}$$
$$= L(\phi_{\overline{(\mathbf{1},m,\mathbf{0})}}(\Pi_{[m,\mathbf{1},n]}),\alpha_{\overline{(\mathbf{1},m,\mathbf{0})}}(\overline{[m,\mathbf{1},n]}))$$
$$= \psi_{\overline{(\mathbf{1},m,\mathbf{0})}}([-n,\mathbf{1},-m]|_{\overline{(\mathbf{0},-m,\mathbf{1})}})$$

For $w \in \eta$, $v \in H$,

$$(w,\mathbf{1},\mathbf{0})$$
$$= [\mathbf{1},w,\mathbf{0}] \wedge \ell^*$$
$$= \phi_{\bar{V}}(\Pi_{[\mathbf{1},w,\mathbf{0}]})$$
$$= \psi_{\bar{V}}(L((\mathbf{0},-\mathbf{1},\mathbf{1}),[\mathbf{1},w,\mathbf{0}]) \wedge OE)U \wedge \ell)$$
$$= \psi_{\bar{V}}((\mathbf{0},w,\mathbf{1}))$$

$$[\mathbf{1},w,\mathbf{0}]|_{\bar{V}}$$
$$= L(\phi_{\bar{V}}(\Pi_{[\mathbf{1},w,\mathbf{0}]}),\alpha_{\bar{V}}(\overline{[\mathbf{1},w,\mathbf{0}]}))$$
$$= L((w,\mathbf{1},\mathbf{0}),\alpha_{\bar{V}}(\overline{[\mathbf{1},\mathbf{0},\mathbf{0}]}))$$
$$= L((w,\mathbf{1},\mathbf{0}),\Pi_{\bar{V}},\psi_{\bar{V}}(L(0,[\mathbf{0},\mathbf{1},\mathbf{0}]|_{\bar{0}})))$$
$$= \psi_{\bar{V}}([\mathbf{0},\mathbf{1},w]|_{\bar{0}})$$

$$[\mathbf{1},w,v]|_{\bar{V}}$$
$$= L((w,\mathbf{1},\mathbf{0}),\alpha_{\bar{V}}(\overline{[\mathbf{1},\mathbf{0},v]}))$$
$$= \psi_{\bar{V}}([-v,\mathbf{1},w]).$$

By the above discussion, $(\mathbf{1},t,\mathbf{0}) = \psi_{\bar{U}}((\mathbf{0},-t,\mathbf{1}))$ and

$$(\mathbf{1},t,t)$$
$$= \psi_{\bar{U}}([-\mathbf{1},\mathbf{1},\mathbf{0}]|_{\bar{0}} \wedge [\mathbf{0},\mathbf{1},-t]|_{\bar{0}})$$
$$= \psi_{\bar{U}}((t,-t,\mathbf{1}));$$

hence the preimage of $[t,\mathbf{0},\mathbf{1}] = (\mathbf{1},t,t)V$ in $\mathcal{K}$ is $L((t,-t,\mathbf{1}),[\mathbf{1},\mathbf{0},\mathbf{0}]) = [\mathbf{1},\mathbf{0},t]$ (ie., $[t,\mathbf{0},\mathbf{1}]|_{\bar{Y}} = \psi_{\bar{Y}}([\mathbf{1},\mathbf{0},t]|_{\psi_{\bar{Y}}^{-1}(\bar{Y})})$). Therefore

$$(\mathbf{1},\mathbf{1},t)$$
$$= [\mathbf{1},\mathbf{1},\mathbf{0}]|_{\overline{(\mathbf{1},\mathbf{1},\mathbf{0})}} \wedge [t,\mathbf{0},\mathbf{1}]|_{\overline{(\mathbf{1},\mathbf{1},\mathbf{0})}}$$
$$= \psi_{\overline{(\mathbf{1},\mathbf{1},\mathbf{0})}}([\mathbf{0},\mathbf{1},-\mathbf{1}]|_{\overline{(\mathbf{0},-\mathbf{1},\mathbf{1})}} \wedge [\mathbf{1},\mathbf{0},t]|_{\overline{(\mathbf{0},-\mathbf{1},\mathbf{1})}})$$
$$= \psi_{\overline{(\mathbf{1},\mathbf{1},\mathbf{0})}}((t,-\mathbf{1},\mathbf{1}))$$

and the preimage of $[\mathbf{0},t,\mathbf{1}] = (\mathbf{1},\mathbf{1},t)(\mathbf{1},\mathbf{0},\mathbf{0})$ in $\mathcal{K}$ is $(t,-\mathbf{1},\mathbf{1})(\mathbf{0},\mathbf{0},\mathbf{1}) = [\mathbf{1},-t,\mathbf{0}]$. Hence

$$[\mathbf{0},t,\mathbf{1}]$$
$$= \bigcup_{\bar{Y} I \bar{L}_\omega,\, \bar{Y}\neq\bar{V}} \{P \in \mathbb{P}_{\bar{Y}} \mid P\, I_{\bar{Y}}\, \psi_{\bar{Y}}([\mathbf{1},-t,\mathbf{0}])\}$$
$$\cup \{P \in \mathbb{P}_{\bar{V}} \mid P\, I_{\bar{V}}\, \gamma(\Pi_{[\mathbf{1},-t,\mathbf{0}]})\}$$

and

$$[\mathbf{0},t,\mathbf{1}\,\mathcal{N}_{\bar{V}}$$

$$= \gamma(\Pi_{[\mathbf{1},-t,\mathbf{0}]})$$

$$= L(\psi_{\bar{V}}(L((\mathbf{0},\mathbf{1},\mathbf{1}),[\mathbf{1},-t,\mathbf{0}]) \wedge OU),\Pi_{\bar{V}})$$

$$= L(\psi_{\bar{V}}((t,\mathbf{0},\mathbf{1})),\Pi_{\bar{V}})$$

$$= \psi_{\bar{V}}([\mathbf{1},\mathbf{0},t]|_{\bar{0}})$$

for any $t \in \eta$.

We may, therefore, conclude that

$$\psi_{\overline{(\mathbf{1},m,\mathbf{0})}}((\mathbf{0},-T_3(t,n,m),\mathbf{1}))$$

$$= (\mathbf{1},T_3(t,n,m),\mathbf{0})$$

$$= \ell^* \wedge L((\mathbf{1},T_3(t,n,m),t),\Pi_{\overline{(\mathbf{1},m,\mathbf{0})},[m,\mathbf{1},\mathbf{0}]})$$

$$= \ell^* \wedge L([t,\mathbf{0},\mathbf{1}]|_{\overline{(\mathbf{1},m,\mathbf{0})}} \wedge [m,\mathbf{1},n]|_{\overline{(\mathbf{1},m,\mathbf{0})}},$$
$$\Pi_{\overline{(\mathbf{1},m,\mathbf{0})},[m,\mathbf{1},\mathbf{0}]})$$

$$= \ell^* \wedge \psi_{\overline{(\mathbf{1},m,\mathbf{0})}}(L([\mathbf{1},\mathbf{0},t]|_{\overline{(\mathbf{0},-m,\mathbf{1})}} \wedge [-n,\mathbf{1},-m]|_{\overline{(\mathbf{0},-m,\mathbf{1})}},$$
$$= \qquad [\mathbf{0},\mathbf{1},-m]|_{\overline{(\mathbf{0},-m,\mathbf{1})}})$$

$$= \ell^* \wedge \psi_{\overline{(\mathbf{1},m,\mathbf{0})}}((t,-tn-m,\mathbf{1}),[\mathbf{0},\mathbf{1},-m]|_{\overline{(\mathbf{0},-m,\mathbf{1})}})$$

$$= \psi_{\overline{(\mathbf{1},m,\mathbf{0})}}(1 \wedge [\mathbf{0},\mathbf{1},-tn-m])$$

$$= \psi_{\overline{(\mathbf{1},m,\mathbf{0})}}((\mathbf{0},-tn-m,\mathbf{1}))$$

(ie., $T_3(t,n,m) = tn + m$) for any $t \in \eta$; $m, n \in H$ and

$$\psi_{\bar{V}}((\mathbf{0},T_2(t,v,w),\mathbf{1}))$$

$$= (T_2(t,v,w),\mathbf{1},\mathbf{0})$$

$$= \ell^* \wedge L((T_2(t,v,w),\mathbf{1},t),\Pi_{\bar{V},\ell|_{\bar{V}}})$$

$$= \ell^* \wedge L(([\mathbf{0},t,\mathbf{1}]|_{\bar{V}} \wedge [\mathbf{1},w,v]|_{\bar{V}}),\Pi_{\bar{V},\ell|_{\bar{V}}})$$

$$= \ell^* \wedge \psi_{\bar{V}}(L([\mathbf{1},\mathbf{0},t]|_{\bar{0}} \wedge [-v,\mathbf{1},w]|_{\bar{0}},[\mathbf{0},\mathbf{1},\mathbf{0}]|_{\bar{V}}))$$

$$= \psi_{\bar{V}}(\ell \wedge [\mathbf{0},\mathbf{1},-tv + w])$$

$= \psi_{\bar{V}}((\mathbf{0}, -tv + w, \mathbf{1}))$

(ie., $T_2(t,v,w) = -tv + w$) for any $t, w \in \eta$, $v \in H$. We note that this implies that the inequalities are reversed in 6.4.15.

Consider four points $(\mathbf{0},\mathbf{1},\mathbf{0})$, $(\mathbf{0},\mathbf{1},(0,1))$, $(\mathbf{0},\mathbf{1},\mathbf{1})$, $(\mathbf{0},\mathbf{0},\mathbf{1})$ incident with the line $[\mathbf{1},\mathbf{0},\mathbf{0}]$. By 4.1.3, $(\mathbf{0},\mathbf{1},\mathbf{0})\ (\mathbf{0},\mathbf{1},(0,1)) \not\sigma\ (\mathbf{0},\mathbf{1},\mathbf{1})\ (\mathbf{0},\mathbf{0},\mathbf{1})$, so either $(\mathbf{0},\mathbf{1},\mathbf{0})\ (\mathbf{0},\mathbf{1},\mathbf{1})\ \sigma\ (\mathbf{0},\mathbf{0},\mathbf{1})\ (\mathbf{0},\mathbf{1},(0,1))$ or $(\mathbf{0},\mathbf{1},\mathbf{0})\ (\mathbf{0},\mathbf{0},\mathbf{1})\ \sigma\ (\mathbf{0},\mathbf{1},\mathbf{1})\ (\mathbf{0},\mathbf{1},(0,1))$. Assume first that the former holds.

Using the projections

$$[\mathbf{1},\mathbf{0},\mathbf{0}] \xrightarrow{(\mathbf{1},\mathbf{0},\mathbf{1})} [\mathbf{0},\mathbf{0},\mathbf{1}] \xrightarrow{(\mathbf{0},\mathbf{0},\mathbf{1})} [\mathbf{0},\mathbf{1},\mathbf{1}] \xrightarrow{(\mathbf{0},\mathbf{1},\mathbf{0})} [\mathbf{0},\mathbf{1},\mathbf{0}] \xrightarrow{(\mathbf{1},-\mathbf{1},\mathbf{0})} [\mathbf{1},\mathbf{0},\mathbf{0}]$$

we obtain $(\mathbf{0},\mathbf{0},\mathbf{1})\ (\mathbf{0},-\mathbf{1},\mathbf{1})\ \sigma\ (\mathbf{0},\mathbf{1},\mathbf{0})\ (\mathbf{0},(0,1),\mathbf{1})$. In addition, the projections

$$[\mathbf{1},\mathbf{0},\mathbf{0}] \xrightarrow{(\mathbf{1},(0,-1),\mathbf{0})} [\mathbf{1},\mathbf{0},\mathbf{1}] \xrightarrow{(\mathbf{1},\mathbf{0},\mathbf{0})} [\mathbf{1},\mathbf{0},\mathbf{0}]$$

yield $(\mathbf{0},(0,1),\mathbf{1})\ (\mathbf{0},(-1,-1),\mathbf{1})\ \sigma\ (\mathbf{0},\mathbf{1},\mathbf{0})\ (\mathbf{0},\mathbf{0},\mathbf{1})$. However as $(\mathbf{0},-\mathbf{1},\mathbf{1})\ (\mathbf{0},(-1,-1),\mathbf{1}) \not\sigma\ 0\ V$ by 4.1.3, $(\mathbf{0},(0,-1),\mathbf{1})\ (\mathbf{0},-\mathbf{1},\mathbf{1})\ \sigma\ (\mathbf{0},\mathbf{1},\mathbf{0})\ (\mathbf{0},\mathbf{0},\mathbf{1})$. Using the projections

$$[\mathbf{1},\mathbf{0},\mathbf{0}] \xrightarrow{(\mathbf{1},\mathbf{0},\mathbf{0})} [\mathbf{1},\mathbf{0},\mathbf{1}] \xrightarrow{(\mathbf{0},\mathbf{0},\mathbf{1})} [\mathbf{0},\mathbf{0},\mathbf{1}] \xrightarrow{(\mathbf{0},\mathbf{1},\mathbf{1})} [\mathbf{0},\mathbf{1},\mathbf{0}] \xrightarrow{(\mathbf{0},\mathbf{1},\mathbf{0})} [\mathbf{1},\mathbf{1},\mathbf{0}] \xrightarrow{(\mathbf{1},\mathbf{0},\mathbf{0})} [\mathbf{1},\mathbf{0},\mathbf{0}],$$

we obtain $(\mathbf{0},\mathbf{1},(0,1))\ (\mathbf{0},\mathbf{1},\mathbf{1})\ \sigma\ (\mathbf{0},\mathbf{0},\mathbf{1})\ (\mathbf{0},\mathbf{1},\mathbf{0})$; a contradiction.

A similar contradiction results when we assume

$(\mathbb{O},\mathbb{1},\mathbb{O})$ $(\mathbb{O},\mathbb{O},\mathbb{1})$ σ $(\mathbb{O},\mathbb{1},\mathbb{1})$ $(\mathbb{O},\mathbb{1},(0,1))$.

7.3. An ordered projective completion of a projectively ordered, projectively uniform A.H. plane.

In 7.2, we constructed an unordered projective completion for $\mathcal{H}(H)$. If we replace the definition of $\phi_{\bar{V}}$ used there by the original definition (cf. 6.2),

$$\phi_{\bar{V}} : \{\Pi_{h} \mid \bar{V}\ \bar{I}\ \bar{h}\} \longrightarrow \lambda(\bar{V})$$

$$\Pi_{h} \rightsquigarrow \phi_{\bar{V}}(\Pi_{h}) = \psi_{\bar{V}}((L(T,h) \wedge OE)U \wedge \mathcal{L}),$$

we obtain for $w, t \in \eta$, $v \in H$,

$$(w,\mathbb{1},\mathbb{O}) = \psi_{\bar{V}}((\mathbb{O},w,\mathbb{1}))$$

$$[\mathbb{1},w,\mathbb{O}]|_{\bar{V}} = \psi_{\bar{V}}([\mathbb{O},\mathbb{1},-w]|_{\bar{0}})$$

$$[\mathbb{1},w,v]|_{\bar{V}} = \psi_{\bar{V}}([-v,\mathbb{1},-w]|_{\bar{0}})$$

and $T_2(t,v,w) = tv + w$.

The projective completion $\mathcal{H}^*$ constructed using these bijections is ordered by the ordering σ defined in 6.6.

We may use the Desarguesian plane $\mathcal{H}(H)$ to construct a non-Desarguesian A.H. plane $\mathcal{H}' = \langle \mathbb{P}, \mathbb{L}, I, || \rangle$ (cf. [4]) where

$$\mathbb{P} = H \times H;$$

$$\mathbb{L} = \{[m,n]_1 \mid m \in \eta,\ n \in H\} \cup \{[m,n]_2 \mid m \in \eta,\ n \in H\} \cup$$
$$\{[m,n]_2 \mid m \in H \setminus \eta,\ m \leq 0,\ n \in H\} \cup$$
$$\{[m,n]_3 \mid m \in H \setminus \eta,\ m > 0,\ n \in H\},$$

with

$$[m,n]_1 = \{(xm + n,x) \mid x \in H\},$$

$$[m,n]_2 = \{(x,xm + n) \mid x \in H\},$$

$$[m,n]_3 = \{(x, xm + n) \mid x + nm^{-1} \geq 0\} \cup \{(x, \mathbb{2}\,xm + \mathbb{2}\,n) \mid x + nm^{-1} < 0\};$$

I is set inclusion;

$[m,n]_i || [u,v]_j$ if and only if $i = j$ and $m = u$.

Then the projective completion $\mathcal{K}'^*$ of $\mathcal{K}'$ may be constructed using $S = (\mathbb{1}, \mathbb{0})$, $\ell = [\mathbb{0}, \mathbb{0}]_1$ and $T = (\mathbb{0}, \mathbb{1})$. This projective completion is also ordered by the order relation given in 6.6.

APPENDIX I

Uniform quinternary rings and uniform P.H. planes.

An algebraic structure $\mathcal{R} = \langle R, T_0, T_1, T_2, T_3, T_4, 0, 1 \rangle$ is a uniform quinternary ring if the following axioms hold.

(Q1) $\langle R, T_1, T_0, 0, 1 \rangle$ is a biternary ring. We define D_+ and $\sim_R$ as in 5.8.

We define T_2, T_3, T_4 to be ternary operators such that

$$T_2 : D_+ \times R \times D_+ \longrightarrow D_+$$

$$T_3 : D_+ \times R \times R \longrightarrow R$$

$$T_4 : R \times D_+ \times D_+ \longrightarrow D_+$$

(Q2) $T_i(a,m,x) = b$; $i=2,3,4$; is uniquely solvable for x, for all choices of a, m, b such that $T_i(a,m,x) = b$ is well-defined.

(Q3) The equation $T_i(x,n_1,m_1) = T_i(x,n_2,m_2)$; $i=2,3$; is uniquely solvable for x if and only if $n \not\sim_R q$ and $m \sim_R p$.

(Q4) The system $x = T_3(y,n,m)$, $y = T_4(x,q,p)$ determines the pair x, y uniquely.

(Q5) The system $T_4(a_i,x,y) = b_i$ is uniquely solvable for x, y if and only if $a_1 \not\sim_R a_2$.

(Q6) If $a_1 \sim_R a_2$, then for $(a_1,b_1) \neq (a_2,b_2)$, one and only one of the systems $a_i = T_3(b_i,x,y)$ or $b_i = T_4(a_i,u,v)$ is solvable for x, y or u, v. The solvable system has at least two solutions and $x_1 \sim_R x_2$, $y_1 \sim_R y_2$ or $u_1 \sim_R u_2$, $v_1 \sim_R v_2$ depending whether the first or second system is solvable.

(Q7) If $b_1 \neq b_2$, then the system $a_i = T_2(b_i,x,y)$ has at least two solutions for x, y and $x_1 \sim_R x_2$.

(Q8) $T_i(a,m,n) \sim_R n$ for $i=2,3,4$ and any a, m, n such that $T_i(a,m,n)$ is well-defined; if $a \in D_+$ and $m_1 \sim_R m_2$, then $T_i(a,m_1,n) = T_i(a,m_2,n)$ for $i=0,1,2,3,4$ (some of these conditions are automatically satisfied by the definition of T_i).

(Q9) $T_i(x,m_1,n_1) = T_i(x,m_2,n_2)$; $i=0,1,4$; is solvable for x if $m_1 \sim_R m_2$, $m_1 \neq m_2$ and $n_1 \sim_R n_2$.

We shall use a uniform quinternary ring to construct a uniform P. H. plane in the following way. Let

$$\mathbb{P} = \mathbb{P}_1 \cup \mathbb{P}_2 \cup \mathbb{P}_3$$
$$= \{(a,b,1) \mid a,\ b \in R\} \cup \{(1,a,b) \mid a \in R,\ b \in D_+\} \cup \{(a,1,b) \mid a,\ b \in D_+\};$$

$$\mathbb{L} = \mathbb{L}_1 \cup \mathbb{L}_2 \cup \mathbb{L}_3$$
$$= \{[1,m,n] \mid m \in D_+,\ n \in R\} \cup \{[m,1,n] \mid m,\ n \in R\} \cup \{[m,n,1] \mid m,\ n \in D_+\};$$

$I \subseteq \mathbb{P} \times \mathbb{L}$ where

$(a,b,1)\ I\ [1,m,n]$ if and only if $a = T_0(b,m,n)$,

$(a,b,1)\ I\ [m,1,n]$ if and only if $b = T_1(a,m,n)$,

$(a,b,1)\ \not I\ [m,n,1]$,

$(1,a,b)\ \not I\ [1,m,n]$,

$(1,a,b)\ I\ [m,1,n]$ if and only if $a = T_3(b,n,m)$,

$(1,a,b)\ I\ [m,n,1]$ if and only if $b = T_4(a,n,m)$,

$(a,1,b)\ I\ [1,m,n]$ if and only if $a = T_2(b,n,m)$,

$(a,1,b)\ \not I\ [m,1,n]$

$(a,1,b)\ I\ [m,n,1]$ if and only if $b = n$.

In the following sequence of lemmas, we show that $\mathcal{P} = \langle \mathbb{P}, \mathbb{L}, I \rangle$ is a uniform P.H. plane.

AI.1. Lemma. (H1) Two points are joined by at least one line.

Proof. Consider distinct points $(a_1,b_1,1)$ and $(a_2,b_2,1)$ where $a_1, a_2, b_1, b_2 \in R$.

If $a_1 \not\sim_R a_2$, then (β5) implies the existence of a unique pair x, y in R such that $T_I(a_i,x,y) = b_i$ for $i=1,2$. Hence $(a_i,b_i,1)\ I\ [x,1,y]$; $i=1,2$. By (β8), the system $T_0(b_i,x,y) = a_i$; $i=1,2$; has no solutions for x and y as $a_1 \not\sim_R a_2$; hence $(a_1,b_1,1)$ $(a_2,b_2,1)$ cannot be on a line of $\mathbb{L}_1$. By definition, these points cannot be incident with a line of $\mathbb{L}_3$.

If $a_1 \sim_R a_2$ and $b_1 \not\sim_R b_2$, then (β8) implies the existence of a unique pair $(x,y) \in D_+ \times R$ such that $T_0(b_i,x,y) = a_i$ for $i=1,2$; hence $(a_i,b_i,1)\ I\ [1,x,y]$ for $i=1,2$. By (β5), the system $T_1(a_i,x,y) = b_i$; $i=1,2$; cannot be solved for x and y as $a_1 \sim_R a_2$ and $b_1 \not\sim_R b_2$; hence both $(a_1,b_1,1)$ and $(a_2,b_2,1)$ cannot be incident with a line of $\mathbb{L}_2$. Points of $\mathbb{P}_1$ cannot be incident with lines of $\mathbb{L}_3$.

Finally, if $a_1 \sim_R a_2$ and $b_1 \sim_R b_2$, by (β7) one and only one of the systems $T_1(a_i,x,y) = b_i$ or $T_0(b_i,x,y) = a_i$ is solvable and the solvable system has at least two solutions. Therefore, $(a_1,b_1,1)$, $(a_2,b_2,1)$ are incident with either two lines of the form $[x,1,y]$ or two lines of the form $[1,x,y]$. Points of $\mathbb{P}_1$ cannot be incident with lines of $\mathbb{L}_3$.

We now consider distinct points $(a,b,1)$, $(1,c,d)$ where $a, b, c \in R$ and $d \in D_+$. By (τ_2), there exists a unique n such that $T_1(a,c,n) = b$ and by (Q2) there exists a unique m such that $T_3(d,n,m) = c$. By (Q8), $c = T_3(d,n,m) \sim_R m$. In addition, (τ_2) implies that there is a unique p such that $T_1(a,m,p) = b$. By (β4) the equation $T_1(x,c,n) = T_1(x,m,p)$ has a unique solution for x if and only if $c \not\sim_R m$; however as $c \sim_R m$ and a is a solution of the equation there must be another solution $a_1 \neq a_2$. Put $b_1 = T_1(a_1,c,n)$. By (β5), $a_1 \sim_R a$ and $b_1 \sim_R b$. Since c, n and m, p are both solutions of the system $b = T_1(a,x,y)$, $b_1 = T_1(a_1,x,y)$, (β7) implies $n \sim_R p$. Therefore by (Q8), $T_3(d,p,m) = T_3(d,n,m) = c$. Thus $(a,b,1)$, $(1,c,d)$ I $[m,1,p]$. We now prove there is only one line joining the points $(a,b,1)$ and $(1,c,d)$. By the definition of I, only lines of $\mathbb{L}_2$ may pass through both $(a,b,1)$ and $(1,c,d)$. Suppose there exists another line, say $[q,1,r]$ through these points. Then by (Q8), $q \sim_R T_3(d,r,q) = c = T_3(d,p,m) \sim_R m$. Using (β7) as we did in the above argument, we see that $r \sim_R p$ also. Therefore by (Q8) $T_3(d,r,m) = T_3(d,p,m) = c = T_3(d,r,q)$; hence (Q2) implies $m = q$. Since $(a,b,1)$ I $[m,1,p]$, $[q,1,r]$, we have $T_1(a,m,p) = b = T_1(a,q,r) = T_1(a,m,r)$ and by (τ_2) $r = p$.

Now we consider the points $(a,b,1)$ and $(c,1,d)$ where $a, b \in R$ and $c, d \in D_+$. By (Q2), there exists a unique $m \in D_+$ such that

$T_2(d,a,m) = c$ and by (β3) the equation $T_0(b,m,n) = a$ is uniquely solvable for n. Moreover, by (β3) and (β9), $n \sim_R a$; hence by (Q8) $T_2(d,n,m) = T_2(d,a,m) = c$. Thus $(a,b,1), (c,1,d)\ I\ [1,m,n]$. Suppose $(a,b,1), (c,1,d)$ are incident with some other line. By definition, this line must be of the form $[1,p,q]$. By (β9), $q \sim_R n$. Then (Q8) implies $T_2(d,q,m) = T_2(d,n,m) = c = T_2(d,q,p)$. By (Q2) $m = p$. Since $(a,b,1)\ I\ [1,m,n], [1,p,q]$, we have $T_0(b,m,n) = a = T_0(b,p,q) = T_0(b,m,q)$ and by (β3) $n = q$.

Next we consider the distinct points $(1,a_1,b_1)$ and $(1,a_2,b_2)$ for $a_1, a_2 \in R$ and $b_1, b_2 \in D_+$.

If $a_1 \not\sim_R a_2$, then the system $b_i = T_4(a_i,x,y)$; $i=1,2$; is uniquely solvable for x and y by (Q5); hence $(1,a_1,b_1), (1,a_2,b_2)\ I\ [y,x,1]$. By definition, points of $\mathbb{P}_2$ cannot be incident with lines of $\mathbb{L}_1$. The points $(1,a_1,b_1), (1,a_2,b_2)$ cannot be incident with a line $[m,1,n]$ of $\mathbb{L}_2$, since (Q3) implies the system $u = T_3(v,n,m)$, $v = T_4(u,x,y)$ is uniquely solvable for u and v.

If $a_1 \sim_R a_2$, then by (Q6) one and only one of the systems $a_i = T_3(b_i,x,y)$; $i=1,2$; or $b_i = T_4(a_i,x,y)$; $i=1,2$; is solvable for x and y. The solvable system has at least two solutions and $(1,a_1,b_1)$ $(1,a_2,b_2)$ are incident with $[y,1,x]$ or $[y,x,1]$ according as the first or second system is solvable. These are the only kinds of lines through $(1,a_1,b_1)$ and $(1,a_2,b_2)$ as points of $\mathbb{P}_2$ cannot be incident with lines of $\mathbb{L}_1$.

Now we consider the points $(1,a,b), (c,1,d)$, where $a \in R$ and $b, c, d \in D_+$. By (Q2), there exists a unique m such that $T_4(a,d,m) = b$.

Then as $(c,1,d)\ I\ [m,d,1]$ by definition, both points are incident with the unique line $[m,d,1]$.

Finally, consider the distinct points $(a_1,1,b_1)$ and $(a_2,1,b_2)$ where $a_1, a_2, b_1, b_2 \in D_+$. If $b_1 = b_2$, then $(a_1,1,b_1), (a_2,1,b_2)\ I\ [m,b_1,1]$ for all $m \in D_+$. In addition, $(a_1,1,b_1)$ and $(a_2,1,b_2)$ cannot be incident with a line $[1,p,q]$ of $\mathbb{L}_1$ as $T_2(b_2,q,m) = a_2 \neq a_1 = T_2(b_1,q,p) = T_2(b_2,q,p)$. If $b_1 \neq b_2$, then the system $a_i = T_2(b_i,x,y)$ has at least two solutions for x and y by (Q7) and $(a_1,1,b_1), (a_2,1,b_2)\ I\ [1,y,x]$ for any such x and y. These points cannot be incident with a line of $\mathbb{L}_3$ by definition. Points of $\mathbb{P}_3$ cannot be incident with lines of $\mathbb{L}_2$.

AI.2. Remark. We define two points (a_1,a_2,a_3), (b_1,b_2,b_3) of $\mathcal{P}$ to be neighbours (we write $(a_1,a_2,a_3) \sim (b_1,b_2,b_3)$) if there exist more than one line through them. From the proof of the previous lemma, it is clear that $(a_1,a_2,a_3) \sim (b_1,b_2,b_3)$ if and only if $a_i \sim_R b_i$ $(i=1,2,3)$. Since $\sim$ is an equivalence relation, we may write $\overline{(a_1,a_2,a_3)}$ for the equivalence class of $\mathbb{P}$ containing (a_1,a_2,a_3). It is clear that $\overline{(a_1,a_2,a_3)} = \overline{(b_1,b_2,b_3)}$ if and only if $\bar{a}_i = \bar{b}_i$ $(i=1,2,3)$.

AI.3. Lemma. (H2). Two lines intersect in at least one point.

Proof. Consider first the pair of distinct lines $[1,m_1,n_1]$, $[1,m_2,n_2]$ where $m_1, m_2 \in D_+$ and $n_1, n_2 \in R$.

If $n_1 \not\sim_R n_2$, then by (Q3), there exists a unique $a \in D_+$ such that $T_2(a,n_1,m_1) = T_2(a,n_2,m_2)$ and $T_0(x,m_1,n_1) = T_0(x,m_2,n_2)$ has no solution as $T_0(x,m_1,n_1) \sim_R n_1 \not\sim_R n_2 \sim_R T_0(x,m_2,n_2)$ for any $x \in R$. Therefore $[1,m_1,n_1] \wedge [1,m_2,n_2] = (T_2(a,n_1,m_1),1,a)$ in this case.

If $n_1 \sim_R n_2$ and $m_1 \neq m_2$, then by (Q9), there exists $a \in R$ such that $T_0(a,m_1,n_1) = T_0(a,m_2,n_2)$. By (β9), there exists at least one other pair a_1, b_1 such that $b_1 = T_0(a_1,m_i,n_i)$; $i=1,2$. Therefore $(T_0(a,m_1,n_1),a,1) \in [1,m_1,n_1] \wedge [1,m_2,n_2]$, but it is not a unique intersection point.

If $n_1 \sim_R n_2$ and $m_1 = m_2$, then by (Q8) for any $a \in D_+$, $T_2(a,n_1,m_1) = T_2(a,n_2,m_2)$. Thus $[1,m_1,n_1] \wedge [1,m_2,n_2] = \{(T_2(a,n_1,m_1),1,a) \mid a \in D_+\}$, in this case.

Now consider the pair of lines $[1,m,n]$, $[p,1,q]$ where $m \in D_+$ and $n, p, q \in R$. By (β6), there exists a unique pair x, y such that $y = T_1(x,p,q)$ and $x = T_0(y,m,n)$; hence $[1,m,n] \wedge [p,1,q] = (x,y,1)$ as by definition no other kind of point can be incident with both lines.

Next consider the pair of lines $[1,m,n]$, $[p,q,1]$ where $m, p, q \in D_+$ and $n \in R$. By the definition of I, any point on both lines must be of the form $(T_2(q,n,m),1,q)$ which is a single point.

Now consider two distinct lines $[m_1,1,n_1]$ and $[m_2,1,n_2]$ where $m_1, m_2, n_1, n_2 \in R$.

If $m_1 \not\sim_R m_2$, (β4) implies there exists a unique a such that $T_1(a,m_1,n_1) = T_1(a,m_2,n_2)$ and (Q2) implies $T_3(y,n_1,m_1) = T_3(y,n_2,m_2)$ has no solutions; hence $[m_1,1,n_1] \wedge [m_2,1,n_2] = (a,T_1(a,m_1,n_1),1)$.

If $m_1 \sim_R m_2$ and $n_1 \not\sim_R n_2$, the equation $T_3(x,n_1,m_1) = T_3(x,n_2,m_2)$ is uniquely solvable for x by (Q3). As $m_1 \sim_R m_2$, the equation $T_1(y,m_1,n_1) = T_1(y,m_2,n_2)$ cannot have a unique solution for x by (β4). Suppose it has two solutions b_1 and b_2. Let $c_i = T_1(b_i,m_1,n_1)$ $(i=1,2)$. By (β5), $b_1 \sim_R b_2$ and $c_1 \sim_R c_2$ and by (β7) $n_1 \sim_R n_2$; a contradiction. Hence in this case $[m_1,1,n_1] \wedge [m_2,1,n_2] = (1,T_3(x,n_1,m_1),x)$.

If $m_1 = m_2$ and $n_1 \sim_R n_2$, then for any $x \in D_+$, $T_3(x,n_1,m_1) = T_3(x,n_2,m_2)$ by (Q8). Hence $\{(1,T_3(x,n_1,m_1),x) \mid x \in D_+\} \subset [m_1,1,n_1] \wedge [m_2,1,n_2]$. For any $x \in R$, let $y = T_1(x,m_2,n_1)$. Thus $(x,y,1)\ I\ [m_1,1,n_1]$. If $(x,y,1)\ I\ [m_2,1,n_2]$, then $y = T_1(x,m_2,n_2)$ and by $(\mathcal{T}_2)$, $n_1 = n_2$ and the lines would be identical.

Finally, if $m_1 \sim_R m_2$, $m_1 \neq m_2$ and $n_1 \sim_R n_2$, (Q9) and (β4) imply that the equation $T_1(x,m_1,n_1) = T_1(x,m_2,n_2)$ has more than one solution. Therefore

$$\emptyset \neq \{(x,T_1(x,m_1,n_1),1) \mid T_1(x,m_1,n_1) = T_1(x,m_2,n_2)\}$$
$$\subset [m_1,1,n_1] \wedge [m_2,1,n_2].$$

For any $x \in D_+$, $T_3(x,n_1,m_1) = T_3(x,n_2,m_1)$ by (Q8). If $T_3(x,n_1,m_1) = T_3(x,n_2,m_2)$, then (Q2) implies $m_1 = m_2$; a contradiction. Thus there is no point of $\mathbb{P}_2$ on $[m_1,1,n_1]$ and $[m_2,1,n_2]$ in this case.

Next consider the lines $[m,1,n],[p,q,1]$ where p, $q \in D_+$ and m, $n \in R$. By (Q4), the system $x = T_3(y,n,m)$, $y = T_4(x,q,p)$ has a unique solution for x and y. As only points of $\mathbb{P}_2$ can be incident with both these lines, $[m,1,n] \wedge [p,q,1] = (1,x,y)$.

Finally, consider the distinct lines $[m_1,n_1,1]$, $[m_2,n_2,1]$ where m_1, m_2, n_1, $n_2 \in D_+$.

If $n_1 = n_2$, then $\{(x,1,n_1) \mid x \in D_+\} \subseteq [m_1,n_1,1] \wedge [m_2,n_2,1]$ by definition. Suppose there exists a point $(1,a,b)\ I\ [m_1,n_1,1]$, $[m_2,n_2,1]$. Then $T_4(a,n_1,m_2) = T_4(a,n_2,m_1)$ which implies $m_1 = m_2$ by

(Q2). Hence no point of $\mathbb{P}_2$ lies on both lines.

If $n_1 \neq n_2$ and $m_1 = m_2$, then for any $x \in D_+$, $T_4(x,n_1,m_1) = m_1 = m_2 = T_4(x,n_2,m_2)$ by (Q8); hence $\{(1,x,T_4(x,n_1,m_1)) \mid x \in D_+\} \subset [m_1,n_1,1] \wedge [m_2,n_2,1]$. As $n_1 \neq n_2$, no point of $\mathbb{P}_3$ can be incident with both lines.

If $n_1 \neq n_2$ and $m_1 \neq m_2$, (Q9) implies that $T_4(x,n_1,m_1) = T_4(x,n_2,m_2)$ is solvable for x and (Q5) implies that the solution is not unique. No point of $\mathbb{P}_3$ can be incident with both these lines. Therefore, in this case,

$$\{(1,x,T_4(x,n_1,m_1)) \mid T_4(x,n_1,m_1) = T_4(x,n_2,m_2)\} = [m_1,n_1,1] \wedge [m_2,n_2,1].$$

AI.4. Remark. We define two lines $[a_1,a_2,a_3]$, $[b_1,b_2,b_3]$ to be neighbours (we write $[a_1,a_2,a_3] \sim [b_1,b_2,b_3]$) if they intersect more than once. From the proof of the previous lemma, it is clear that $[a_1,a_2,a_3] \sim [b_1,b_2,b_3]$ if and only if $a_i \sim_R b_i$ $(i=1,2,3)$. Since $\sim$ is an equivalence relation on $\mathbb{L}$, we may write $\overline{[a_1,a_2,a_3]}$ for the equivalence class of $\mathbb{L}$ containing $[a_1,a_2,a_3]$. Clearly, $\overline{[a_1,a_2,a_3]} = \overline{[b_1,b_2,b_3]}$ if and only if $\bar{a}_i = \bar{b}_i$ $(i=1,2,3)$.

AI.5. Lemma. (H3) There exist an ordinary projective plane $\mathcal{P}'$ and an epimorphism $\chi : \mathcal{P} \longrightarrow \mathcal{P}'$ such that

i) $P \sim Q$ if and only if $\chi(P) = \chi(Q)$;

ii) $\ell \sim m$ if and only if $\chi(\ell) = \chi(m)$.

Proof. We define $\mathcal{P}' = \langle \mathbb{P}/\sim, \mathbb{L}/\sim, I' \rangle$ where $\overline{(a,b,c)}\ I'\ \overline{[m,n,p]}$ if and only if there exist a point $(a',b',c') \in \overline{(a,b,c)}$ and a line $[m',n',p'] \in \overline{[m,n,p]}$ such that $(a',b',c')\ I\ [m',n',p']$ in $\mathcal{P}$. Let $\chi : \mathcal{P} \longrightarrow \mathcal{P}'$ with $\chi(P) = \bar{P}$.

and $\chi(\ell) = \bar{\ell}$, for every $P \in \mathbb{P}$ and every $\ell \in \mathbb{L}$.

It is clear from these definitions that χ is a surjective map which takes points to points and lines to lines, preserves incidence and satisfies properties i) and ii). It remains to verify that $\mathcal{P}'$ is an ordinary projective plane.

Since $\mathcal{P} = \langle \mathbb{P}_1/\sim, \mathbb{L}_1/\sim \cup \mathbb{L}_2/\sim, I'' \rangle$ where $I'' = I' \cap (\mathbb{P}_1/\sim \times (\mathbb{L}_1/\sim \cup \mathbb{L}_2/\sim))$ is an ordinary affine plane (cf. [11], 2.12), pairs of distinct points of the form $\overline{(a,b,1)}$, $\overline{(c,d,1)}$ are joined by unique lines in $\mathcal{P}$ and hence in $\mathcal{P}'$ (cf. AI.1) and pairs of distinct lines of the form $\overline{[m,1,n]}$, $\overline{[p,1,q]}$ where $\bar{m} \neq \bar{p}$ or of the form $\overline{[1,m,n]}$, $\overline{[p,1,q]}$ meet in unique points of $\mathcal{P}$ and hence in $\mathcal{P}'$ (cf. AI.3).

Consider the points $\overline{(a,b,1)}$, $\overline{(c,d,1)}$ in $\mathcal{P}'$. For any $(a_i,b_i,1) \in \overline{(a,b,1)}$ and $(1,c_i,d_i) \in \overline{(1,c,d)}$, there exist unique lines $[m_i,1,n_i] = (a_i,b_i,1)(1,c_i,d_i)$ where $m_i \sim_R c_i$; $i=1,2$ (cf. AI.1). Therefore by (β10), $\bar{T}_1(\bar{a}_1,\bar{m}_1,\bar{n}_1) = \bar{b}_1 = \bar{b}_2 = \bar{T}_1(\bar{a}_2,\bar{m}_2,\bar{n}_2)$; hence by $(\mathcal{T}_2)$, $\bar{n}_1 = \bar{n}_2$ and $\overline{[m_1,1,n_1]} = \overline{[m_2,1,n_2]}$.

Now consider the points $\overline{(a,b,1)}$, $\overline{(c,d,1)}$ in $\mathcal{P}'$. For any $(a_i,b_i,1) \in \overline{(a,b,1)}$ and $(c_i,d_i,1) \in \overline{(c,1,d)}$, there exist unique lines $[1,m_i,n_i] = (a_i,b_i,1)(c_i,1,d_i)$ where $m_i \in D_+$; $i=1,2$ (cf. AI.1). By (β8), $n_1 \sim_R a_1 \sim_R a_2 \sim_R n_2$; hence $\overline{[1,m_1,n_1]} = \overline{[1,m_2,n_2]}$.

Next consider the distinct points $\overline{(1,a,b)}$, $\overline{(1,c,d)}$ of $\mathcal{P}'$. Since $b, d \in D_+$, $\bar{a} \neq \bar{c}$. Therefore for any $(1,a_1,b_1) \in \overline{(1,a,b)}$ and any $(1,c_1,d_1) \in \overline{(1,c,d)}$, there exists a unique line $[m,n,1] = (1,a_1,b_1)(1,c_1,d_1)$ by AI.1. However all lines of $\mathbb{L}_3$ are neighbours, so $\overline{(1,a,b)}\,\overline{(1,c,d)} = \overline{[m,n,1]}$ $(= \overline{[0,0,1]})$.

As a point of $\mathbb{P}_2$ and a point of $\mathbb{P}_3$ in $\mathcal{P}$ are incident with a unique line of $\mathbb{L}_3$ (cf. AI.1), any $(\overline{1,a,b})$, $(\overline{c,1,d})$ in $\mathcal{P}'$ are incident with $[\overline{0,0,1}]$.

Next we show that pairs of lines of $\mathcal{P}'$ have unique intersections.

Consider distinct lines $[\overline{1,m,n}]$, $[\overline{1,p,q}]$ in $\mathcal{P}'$. Clearly $\bar{n} \neq \bar{q}$. For any $[1,m_1,n_1] \in [\overline{1,m,n}]$ and any $[1,p_1,q_1] \in [\overline{1,p,q}]$, there exists a unique point $(T\ (a,n_1,m_1),1,a) = [1,m_1,n_1] \wedge [1,p_1,q_1]$ by AI.3; however all points of $\mathbb{P}_3$ are neighbours, so $[\overline{1,m,n}] \wedge [\overline{1,p,q}] = (\overline{0,1,0})$.

As a line of $\mathbb{L}_1$ and a line of $\mathbb{L}_3$ in $\mathcal{P}$ meet in a unique point of $\mathbb{P}_3$ (cf. AI.3) and all points of $\mathbb{P}_3$ are neighbours, any $[\overline{1,m,n}]$, $[\overline{p,q,1}]$ in $\mathcal{P}'$ meet in the unique point $(\overline{0,1,0})$.

Consider distinct lines $[\overline{m,1,n}]$ and $[\overline{p,1,q}]$ where $\bar{m} = \bar{p}$ in $\mathcal{P}'$. Thus $\bar{n} \neq \bar{q}$. For any $[m_1,1,n_1] \in [\overline{m,1,n}]$ and any $[p_1,1,q_1] \in [\overline{p,1,q}]$, $[m_1,1,n_1] \wedge [p_1,1,q_1] = (1,T_3(a,n_1,m_1),a)$ for a unique $a \in D_+$ by AI.3. However each $(1,T\ (a,n_1,m_1),a) \in (\overline{1,m,0})$ by (Q8). Therefore $[\overline{m,1,n}] \wedge [\overline{p,1,q}] = (\overline{1,m,0})$.

Consider the lines $[\overline{m,1,n}]$, $[\overline{p,q,1}]$ in $\mathcal{P}'$. For any $[m_1,1,n_1] \in [\overline{m,1,n}]$ and any $[p_1,q_1,1] \in [\overline{p,q,1}]$, $[m_1,1,n_1] \wedge [p_1,q_1,1] = (1,y,T_4(y,q_1,p_1))$ for some $y \sim_R m_1$ by AI.3 and (Q8). Since $T_4(y,q_1,p_1) \in D_+$ for all $y \in R$, $(\overline{1,m,0}) = [\overline{m,1,n}] \wedge [\overline{p,q,1}]$.

Finally, using the proofs of AI.1, AI.3 and the above discussion, we conclude that $(0,0,1)$, $(1,1,1)$, $(1,0,0)$, $(0,1,0)$ are four points of $\mathcal{P}'$, no three of which are collinear.

Thus $\mathcal{P}'$ is an ordinary projective plane.

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