A HISTORY OF THE THEORY OF TYPES

A HISTORY OF THE THEORY OF TYPES WITH SPECIAL REFERENCE TO DEVELOPMENTS AFTER THE SECOND EDITION OF *PRINCIPIA MATHEMATICA*

By

Jordan E. Collins, B.Sc.

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AUTHOR: Jordan E. Collins, B.Sc. (Mount Allison University)

SUPERVISOR: Dr. Gregory H. Moore

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Abstract

This thesis traces the development of the theory of types from its origins in the early twentieth century through its various forms until the mid 1950's. Special attention is paid to the reception of this theory after the publication of the second edition of Whitehead and Russell's *Principia Mathematica*. We examine how the theory of types declined in influence over four decades. From being in the 1920s the dominant form of mathematical logic, by 1956 this theory had been abandoned as a foundation for mathematics. The use and modification of the theory by logicians such as Ramsey, Carnap, Church, Quine, Gödel, and Tarski is given particular attention. Finally, the view of the theory of types as a many-sorted first-order theory in the 1950's is discussed.

It was the simple theory, as opposed to the ramified theory of types that was used almost exclusively during the years following the second edition of *Principia*. However, it is shown in this thesis that in the 1950's a revival of the ramified theory of types occurred. This revival of ramified-type theories coincided with the consideration of cumulative type hierarchies. This is most evident in the work of Hao Wang and John Myhill. The consideration of cumulative type-hierarchies altered the form of the theory of types in a substantial way. The theory was altered even more drastically by being changed from a many-sorted theory into a one-sorted theory. This final "standardization" of the theory of types in the mid 1950's made it not much different from first-order Zermelo-Fraenkel set-theory. The theory of types, whose developments are traced in this thesis, therefore lost its prominence as the foundation for mathematics and logic.

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1. The Theory of Types Introduced

1.1 Introduction

This thesis will trace the development of the theory of types from its origins in the early twentieth century through its various forms until the mid 1950's. By this later date it had lost any prominence that it once had as a foundation for mathematics. Special attention will be paid to the reception of this theory after the publication of the second edition of Whitehead and Russell's *Principia Mathematica*. We examine how the theory of types declined in influence over four decades. From being in the 1920s the dominant form of mathematical logic, by 1956 this theory had been abandoned as a foundation for mathematics. The use and modification of the theory by logicians such as Ramsey, Carnap, Church, Quine, Gödel, and Tarski will be given particular attention. Finally, the view of the theory of types as a many-sorted first-order theory in the 1950's will be discussed.

There are two fundamental formulations of the theory of types; the simple theory and the ramified theory. In the simple theory of types a certain domain is specified as the domain of individuals. These are assigned to the lowest type (say type 0). Classes and functions are then stratified into a hierarchy of types. In the case of classes, for instance, type 0 consists of the pre-specified domain of individuals, type 1 consists of classes of individuals, type 2 consists of classes of classes of individuals, and so on. The main restriction here is that the members of a class of type $n+1$ must all be of type n.

Furthermore, all classes must belong to some one type; those that do not are deemed non-existent and discussion of them is prohibited within the system.

This stratification was initially proposed as a way to avoid Russell's paradox. This paradox arises by making the seemingly natural assumption that any condition defines a set. With this assumption Russell considered the set of all sets that do not belong to themselves. But if it is then asked whether this set belongs to itself or not the contradiction arises; for if it does belong to itself, then by its own defining condition it does not, and vice versa. Applying the simple theory of types described in the previous paragraph it is easily seen how this contradiction is avoided; within the simple theory of types self-membership is meaningless, the question would never arise whether a set could or could not be a member of itself. Furthermore, the 'set of all sets not members of themselves' would never arise since 'not a member of oneself' is ill-formed by the theory.

This theory can be succinctly explained as follows. Each variable belongs to a specific type 0, 1, 2, ..., and so on indicated by right subscripts. The atomic formulae are of the form '$\mu\varepsilon\chi$' where the type of μ must be exactly one lower than the type of χ, and '$\mu=\chi$' where μ and χ are of the same type. Furthermore, there is an axiom-schema of extensionality:

$$(x_i)(y_i)[(z_{i-1})(z_{i-1}\varepsilon\, x_i \leftrightarrow z_{i-1}\varepsilon\, y_i) \rightarrow x_i = y_i]$$

and an axiom-schema of comprehension:

$$(\mathrm{E}y_{i+1})(x_i)[x_i\ \varepsilon\ y_{i+1} \leftrightarrow \varphi(x_i)].$$

The ramified type theory, on the other hand, includes a distinction between 'orders' as well as 'types'. Variables are given two different indices; one indicates the type of the variable, and the other the level. The axiom-schema of comprehension then changes to: If j is the highest order of any bound variable of level i+1 occurring in $\varphi({}_j x_i)$, then $(E_{j+1}\,{}_{j+1}y_{i+1})({}_k x_i)[{}_k x_i \in {}_{j+1}y_{i+1} \leftrightarrow \varphi({}_k x_i)]$ where the left subscript indicates the order of the variable. Within this ramified type theory, a large portion of mathematics cannot be formulated. For instance, the least upper bound of a class of real numbers will be of a higher order than any of the numbers which are used in its construction.

In order to reinstate these portions of classical mathematics an axiom of reducibility is sometimes posited. This axiom ensures that for each class of a certain level and any order, a corresponding class of the same level and order-1 exists such that the two classes contain the same members. It will be seen throughout this thesis that the status of the axiom of reducibility played an integral part in the development of the theory of types. This is most notable in the choice between a simple versus a ramified theory.

The ramified theory of types is required when one begins to consider relations (or functions of more than one variable). Considering relations of two variables, one can quantify over either variable. To overcome this confusion a hierarchy of orders is added to the hierarchy of types in such a way that two relations can be of the same type, and yet of a different order.[1] Russell introduced the ramified theory as a natural extension of the simple theory. However, he did not explicitly distinguish between the two; the introduction of orders was simply the next necessary step in his process. This was necessitated by his treatment of classes, relations, and functions as will be seen in the second chapter of this thesis.

[1] The order refers to the difference in type of the variables being related. For example, if x is of type 1 and y is of type 3, then a relation between x and y would be of order 2.

The theory of types is closely connected with the distinction between functions and their arguments. This distinction was made explicitly by Frege in both his *Begriffsschrift* (1879) and more fully in his *Grundgesetze der Arithmetik* (1893). This distinction is clearly needed before any distinction between types of arguments for functions can be made. However, Church argues in his 1939 paper "Schröder's Anticipation of the Simple Theory of Types", that Frege's work cannot properly be seen as an anticipation of the theory of types. Schröder's work, on the other hand, is seen as the first step towards a simple theory of types. The theory of types was not explicitly put into use until the beginning of the twentieth century. At that time Russell gave a tentative sketch of the theory in an appendix of his *The Principles of Mathematics* (1903). The theory was put forth as a method for avoiding the paradox that Russell had discovered while his book was on its way to print. Thus, in this first chapter a brief outline will be given of Schröder and Frege's possible anticipations of the theory of types, together with a review of the contradictions which the theory of types was designed to overcome.

In his *The Principles of Mathematics* Russell realized that his theory of types was being given in outline only; he even listed some of the problems which it failed to solve. Unable to overcome these difficulties, Russell abandoned the theory of types by the middle of 1903. In the hopes of supplanting the theory of types several different procedures were investigated by Russell in a 1906 paper. These were the zig-zag theory, the theory of limitation of size, and the no-class theory. There is no mention made of the theory of types in that paper. During this period between 1903 and 1908 Russell spent the majority of his time developing his substitutional theory (which is called the no-class theory in his 1906 paper). However, by 1906 or 1907 Russell had returned to his theory of types and in a paper of 1908 he developed the theory more fully. This transitional period in Russell's work will be discussed in the second chapter. A consideration of the motivation behind Russell's study of the foundations of mathematics, and his early logicism, is essential to understanding his work in the theory of types and the various alternatives that he considered. Thus his motivations in this period are highlighted.

Russell's next use of the theory of types occurred in *Principia Mathematica* (1910-1913). The version of the theory of types put forth there will also be compared to previous versions in chapter 2. Finally, the role of propositions, classes, and functions in Russell's logical systems will receive special attention as their status seems to correspond to variations in his theory of types.

The second edition of *Principia Mathematica* (1925-1927) leaves the text of the first edition unchanged. The main improvements appear in an introduction, where the most important contributions to mathematical logic by other authors in the intervening years are listed. The improvements that bear directly upon the theory of types include the work done by Chwistek, in which the axiom of reducibility is dropped altogether (with no replacement), and Wittgenstein's new conception of the nature of functions as found in his *Tractatus Logico-Philosophicus* (1922). Although not mentioned in the *Principia* introduction, Weyl also made some contributions which influenced subsequent work in the theory of types. Weyl, like Chwistek, used constructive methods in his avoidance of the logical paradoxes and he thus saw the axiom of reducibility as untenable. These works will be discussed in the third chapter of this thesis, together with the changes proposed in the second edition of *Principia*.

Wittgenstein's newly introduced views of functions and propositions were also incorporated into Ramsey's work dealing with the theory of types. Like Chwistek and Weyl, Ramsey attempted to eliminate the need for an axiom of reducibility. However, Ramsey's severely realist position with respect to classes clearly conflicts with Chwistek and Weyl's constructivist attitudes. Many of Chwistek's results anticipated Ramsey's, but the methods they implemented were motivated by different concerns. Ramsey argued in favor of the use of impredicative definitions and for the simple theory of types. While urging the use of his simple theory of types, Ramsey explicitly distinguished between two different kinds of contradictions; those which he calls "logical" versus those that he claims rely upon purely linguistic or "psychological" references (Ramsey 1925, pp. 171-172). Since it is impossible to formulate the "semantic" antimonies within a formal language, it is only the logical ones that affect the formal development of mathematics (according to Ramsey). Thus Ramsey urged that the second set of paradoxes, and hence the axiom of reducibility which was needed only in dealing with antimonies of this kind, be dispensed with entirely. Ramsey further proposed the use of propositional functions in extension as the basis for logical elements. This extensional version of the theory of types, as well as the classification of different kinds of antimonies, proved to be very influential.

Like Ramsey, Rudolf Carnap embraced the work of Wittgenstein. Wittgenstein's notion that mathematics and logic are simply a collection of tautologies and say nothing about the world fit in well with Carnap's conceptions as formulated in his work with the Vienna Circle. However, Carnap rejected Ramsey's solution of the paradoxes stating, as Frege had done, that only those entities whose existence can be proved in finitely many steps may be taken to exist. In his *Logical Syntax of Language* (1937), Carnap came to the conclusion that all of logic, mathematics, and philosophy could be developed in syntactical form, ultimately reducing to the logical syntax of language. This new argument rests upon the Principle of Tolerance according to which any language deserves study for its own sake and can be employed to make other studies more convenient. In the *Logical Syntax of Language*, two systems are developed. The second of these incorporates a simple theory of types. Carnap's work motivated many subsequent works using a theory of types. In chapter 4 a comparison of Ramsey and Carnap's contributions and use of the theory of types is discussed.

The theory of types did not receive its first truly formal treatment until 1931 in the works of Tarski and Gödel. Indeed, the formulations given in Tarski's paper on truth and Gödel's incompleteness paper were later referred to several times by Quine as the "neo-classical theory of types" (Quine 1985, p. 86). Until 1935 Tarski used a simple theory of types as his basic logical system (see for example Tarski 1956, pp. 61, 113-115, 241-243, 297, 384). In the paper on truth Tarski gives his typical statement of a simple type theory using axioms of comprehension (which he calls "pseudo-definitions" following Lesniewski), extensionality, and infinity. Even in his 1941 book *Introduction to Logic* the theory of classes is based on a distinction between levels which Tarski acknowledges to be akin to Russell's logical types (see sections 21&23, pp. 68, 73-74). However, in the

1935 postscript to the German edition of his paper on truth (Tarski 1956, p. 271) it is indicated that he had shifted from the theory of types to first-order Zermelo-Fraenkel set theory (with an axiom of choice) as the best way to formulate his work. In chapter 5 Tarski and Gödel's versions of the theory of types are outlined and compared. Also, the reasons behind Tarski's abandonment of the theory of types will be traced.

As mentioned above, Quine regarded Tarski and Gödel's work on the theory of types very highly. In fact, in some of his early works Quine used these versions of the theory of types as a starting point in his logical investigations. In his "New Foundations for Mathematical Logic" (1937), for instance, he modifies Tarski's theory of types to form his own system that avoids specific reference to types. This work was later extended to that found in his *Mathematical Logic* (1940). In both of these works specific references to types are avoided by instead using "stratified formulae". These formulae are required to be such that subscripts could be added to the variables, indicating the type of the variable, so as to be consistent with the simple theory of types. Despite this initial influence of the theory of types in Quine's work, like Tarski he eventually restricted his logic to a first-order predicate logic.[2] Church, on the other hand, initially set out to provide a system of logic which completely eliminated any reliance upon type distinctions. However, after this system was shown to be inconsistent by his students, Kleene and Rosser, Church was forced to abandon his lofty pursuit. The system was not altogether unfruitful though. From it an important subsystem was singled out; namely Church's lambda-calculus. In 1940 Church put forth a formulation of the simple theory

[2] However, his type-inspired system provided years of research for other logicians.

of types which included his lambda-operator.[3] Both Quine and Church's work in the theory of types proved to be quite influential and their systems were studied extensively. Chapter 6 of this thesis is devoted to a study and comparison of Quine and Church's views on the theory of types.

The seventh chapter of this thesis includes an investigation of work that arose directly from Church and Quine's work in the theory of types. In particular, the rejuvenation of the ramified theory of types will be highlighted. This is especially prevalent in Hao Wang's work, but also in the work of John Myhill who strove to complete the work initiated by Leon Chwistek. Finally, the theory of types as a many-sorted first-order theory will be investigated.

1.2 Frege and Schröder - Anticipations of the Simple Theory of Types

Prior to the discovery of the paradoxes of set theory there was a partial anticipation of the simple theory of types. Since the set-theoretic paradoxes were not yet known, the motivation behind this first formulation was different from Russell's. Church claims in his paper "Schröder's Anticipation of the Simple Theory of Types", that Schröder's work in the first volume of his *Algebra der Logik* (1890) can be viewed as a "striking anticipation of the simple theory of types" (Church 1939, p. 408).

According to Church, Schröder's anticipation can be seen as follows. He starts out with a universal class, not in the absolute sense, but as composed of all of the

[3] It is worth noting that throughout his career Church was a consistent defender of higher-order logical systems. Also, he emphasized the role of the axiom of infinity as a crucial frontier between logic and mathematics.

elements of a domain which is specified in advance. Given this arbitrarily chosen universal class a second one is obtained by taking the individuals to be precisely the subsets of the initial domain. Schröder's algebra is equally applicable to this newly formed domain. Furthermore, the null classes found in each domain must be kept distinct. This process of creating a hierarchy of new domains may be extended to infinity (Schröder 1890, p. 248).

Church further enunciates Schröder's anticipation of the simple theory of types as follows:

> If 0 is the null class associated with the first domain, **0** the null class associated with the second ("derived") domain, and A the class composed of the two elements 0, 1, where 1 is the universal class associated with the first domain, then Schröder would regard $0 \leq \mathbf{0}$ as false (not meaningless) and $0 \leq A$ as true. Actually, however, this divergence from an exact anticipation of the simple theory of types is apparent rather than real; it means that we must interpret Schröder's symbol 0 within the algebra of the first domain as meaning Λ, but within the algebra of the second domain as meaning $\iota'\Lambda$, and likewise in other cases (Church 1939, p. 408).

This apparent failure of Schröder's work as an exact formulation is due to his identification of a unit class with its sole element.[4]

Since the set-theoretic paradoxes are unknown at the time of Schröder's work, the motivation behind his 'type' distinctions must have come from a different source. Schröder conceives of classes as built entirely of their members. As such, the members

[4] Interestingly, this same procedure is followed by Quine in his systems found in "New Foundations for Mathematical Logic" and his *Mathematical Logic* as discussed in chapter 6.

of a particular class must exist prior to the class itself. With such a view some sort of type hierarchy seems to arise quite naturally (Church 1939).

Apart from this view of classes and their members there is a more practical need for Schröder to introduce type distinctions. In his system Schröder has no symbol available for the class-membership relation (ε). He does have a symbol for class-inclusion and he actually confuses or identifies the two notions, membership and inclusion, on several occasions (see, for example, Schröder 1890, p. 245). By introducing a distinction of types, Schröder is able, in some respects, to get away with failing to make the distinction between class-membership and class-inclusion. Thus, although Schröder's use of a type-like hierarchy is not motivated by the set-theoretic paradoxes, his own need for such a hierarchy seems quite natural and inevitable (Church 1939, p. 409).

After his presentation of Schröder's partial anticipation of the simple theory of types, Church moves on to discuss "the claim sometimes made on behalf of Frege that his *Stufen* (cf. his *Grundgesetze der Arithmetik*, vol. 1, 1893) constitute an anticipation of the simple theory of types" (Church 1939, p. 409). Church sees this position as untenable and the rest of his paper is devoted to explicating this point.

This dissection begins with Church's consideration of Frege's notion of a function. For Frege, a function is an incomplete symbol and not an abstract object at all. The 'incompleteness' of the function comes from the fact that it requires something additional, in particular an argument, to complete its meaning. However, a function is sufficiently akin to an object to be represented by a variable. Church claims that since Frege's functions are incomplete symbols they must be divided into *Stufen* "and no other possibility offers itself" (Church 1939, p. 410). Church goes on to state that "this division of functions into *Stufen* is not a theory of types. It might become so if it were denied that besides functions there were also the corresponding completed abstraction, or if a similar restriction were imposed upon the completed abstraction" (Church 1939, p. 410). Actually, Church claims that Frege "explicitly denies that [the corresponding completed abstractions] are subject to the restriction associated with the division into *Stufen*" (Church 1939, p. 410). Thus, based upon Frege's notion of a function, Church argues that Frege did not anticipate the theory of types.

This argument against Frege's anticipation of the simple theory of types is continued and concluded with an even stronger argument as follows:

> The characteristic feature of the simple theory of types – that a domain of individuals is fixed upon, and the laws of logic stated first for (classes or functions over) this domain and then restated successively for other domains derived one by one from the original domain and from one another – is not only not adopted by Frege but is vigorously rejected by him (Church 1939, p. 410).

This is followed by a "violent criticism" of Schröder's algebra written by Frege which Church claims gives a clear idea of what "Frege would think (or, indeed, did think) of the simple theory of types" (Church 1939, p. 410).

Church's claim that Schröder anticipated the simple theory of types, while Frege did not, must be taken cautiously. Despite the evidence cited above, it is still the case that both Schröder *and* Frege had some features of their systems in common with type theory, while some features differed from type theory. As will be seen in the next chapter, the first explicit use of types stemmed from considerations much different from both Schröder's and Frege's. This first theory of types, put forth by Bertrand Russell, was set up as an attempt at overcoming the logical paradoxes that first appeared near the end of the nineteenth century.

1.3 The Paradoxes

Since Russell first explicitly introduced the theory of types as a way of avoiding the set-theoretic paradoxes, a very brief account of how these paradoxes arose is given in the present section.

The first set-theoretic paradoxes emerged around the turn of the twentieth century. The earliest of these paradoxes include, among others, the paradox of the largest ordinal, the paradox of the largest cardinal, and Russell's paradox. The paradox of the largest ordinal has come to be called the Burali-Forti paradox. However, as is shown in Moore and Garciadiego (1981), it was not created by Burali-Forti at all. Nor was it discovered by Cantor two years earlier as has been often asserted (see for instance Fraenkel and Bar-Hillel 1958, p. 2). In their article, Moore and Garciadiego de-emphasize the question as to who originally stated the paradox, and instead investigate the process by which it originated and matured into the form in which it is generally recognized today.[5] The Burali-Forti paradox can be stated briefly as follows: The set, W, of all ordinals is well-

[5] This form did not appear until 1907. The claim is often made that Burali-Forti had discovered this paradox in his 1897 paper. However, he never viewed his results as creating a paradox (Moore and Garciadiego 1981, pp. 321-323).

ordered; every well-ordered set has an ordinal number greater than the ordinal numbers of its members; hence the ordinal number of W is greater than *every* ordinal.

The Burali-Forti paradox deals with notions from set-theory, as does the paradox of the largest cardinal, which is known as Cantor's paradox. The paradox of the largest cardinal can be stated as follows: "The class of all classes has a cardinal number; if this number is N, then there is another class which has a larger cardinal number; hence there is no cardinal number of the class of all classes" (Moore 1995, p. 226). This paradox was formulated by Russell only in January 1901. However, it is very similar in form to his antimony of infinite number given in July or August 1899 (see Moore 1995 for details): "There are many numbers, therefore there is a number of numbers. If this be N, $N+1$ is also a number, therefore there is no number of numbers" (Russell 1899, p. 265). These two paradoxes are clearly similar in form, not only to each other, but also to Burali-Forti's paradox.

Since these paradoxes all involve set-theoretic notions it was hoped that they would be remedied by some simple revision in the proofs of the then young discipline of set-theory. This hope was shattered when Bertrand Russell published his paradox in 1903. As shown in Moore's 1995 article this paradox arose out of philosophical concerns over the nature of infinity. Indeed, Moore concludes by noting that "the traditional philosophical concern with a 'largest number', a concern with Kantian roots, then interacted in Russell's mind with Cantor's proof that there is no largest cardinal number…the new mathematical problems of the infinite – the paradoxes of logic and set theory – grew out of the old philosophical ones" (Moore 1995, p. 236).

Russell's paradox can be stated using only the most basic logical notions. Russell communicated his paradox to Frege in a letter in 1902. It was published one year later in his *The Principles of Mathematics*. The paradox appears in *The Principles* as follows:

> The predicates which are not predicable of themselves are...only a selection from among predicates, and it is natural to suppose that they form a class having a defining predicate. But if so, let us examine whether this defining predicate belongs to the class or not. If it belongs to the class, it is not predicable of itself, for that is the characteristic property of the class. But if it is not predicable of itself, then it does not belong to the class whose defining property it is, which is contrary to the hypothesis. On the other hand, if it does not belong to the class whose defining predicate it is, then it is not predicable of itself, i.e. it *is* one of those predicates that are not predicable of themselves, and therefore it does belong to the class whose defining predicate it is – again contrary to the hypothesis. Hence from either hypothesis we can deduce its contradictory (Russell 1903, p. 80).

Russell's paradox can be restated in many different ways. For example, by considering the set of all sets which are not members of themselves, one can then ask whether this set is a member of itself. Either assuming that it is, or it is not, leads to a contradiction. Another set of paradoxes that were being considered in the early part of the twentieth century includes Richard's paradox and the ancient paradox of the liar.[6] It

[6] Richard's paradox can be described as follows. All numbers that are defined by finitely many words can be written in a definite order; these numbers will form a countably infinite set. Then "we can form a number not belonging to this set. 'Let *p* be the digit in the *n*th decimal place of the *n*th number of the set E; let us form a number having 0 for its integral part and, in its *n*th decimal place, *p*+1 if *p* is not 8 or 9, and 1

will be seen that the theory of types was introduced in order to solve all of these paradoxes. In chapter 4 of this thesis it will be seen that in 1925 Ramsey discovered that the simple theory of types is used in solving the first set of paradoxes, while the ramified theory was used to solve the second set.

otherwise.' This number N does not belong to the set E. If it were the *n*th number of the set E, the digit in its *n*th decimal place would be the same as the one in the *n*th decimal place of that number, which is not the case.
I denote by G the collection of letters between quotation marks.
The number N is defined by the words of the collection G, that is, by finitely many words; hence it should belong to the set E. But we have seen that it does not.
Such is the contradiction" (Richard 1905, p. 143). The liar paradox occurs by considering the statement "I am lying" and then asking whether this statement is true or false; either answer implies its contradictory.

2. The Origins of the Theory of Types

2.1 Introduction

Bertrand Russell was the founder of the theory of types. In an appendix to his 1903 book, *The Principles of Mathematics* he proposed the theory tentatively as a way of avoiding his paradox mentioned in the previous chapter. This book also contains the first exposition of Russell's logicism. Indeed, in the preface Russell clearly states this in the description of the two main objects of his book. The first, which occupies Part I only, is to clearly delimit the "fundamental concepts which mathematics accepts as indefinable" (Russell 1903, p. xv). The second, found in Parts II-VII, is the proof that "all pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts, and that all its propositions are deducible from a very small number of fundamental logical principles" (Russell 1903, p. xv). The final chapter of Part I is devoted exclusively to "the contradiction". None of the possible solutions considered there are deemed adequate. As a result Russell added his Appendix B, "The Doctrine of Types", as a more plausible solution to the paradox.

Several shortcomings of the theory of types are pointed out in the appendix, and a general air of dissatisfaction is prevalent. Thus it is not surprising that by the middle of 1903 Russell abandoned the theory of types. He did not publish another work incorporating a theory of types until his 1908 paper "Mathematical Logic as Based on the Theory of Types". However, an investigation of his unpublished work from 1903 to 1908 shows that Russell struggled repeatedly with the question of whether to use a type theory, and in what capacity. For instance, in one paper written in April and May 1906 entitled "Logic in which Propositions are not Entities", Russell explicitly dispenses with any hierarchy of types of propositions. Alternatively in "On the Functional Theory of Propositions, Classes, and Relations", also written in April and May 1906, Russell viewed such a hierarchy as essential. As will be seen below, it is clear that Russell had returned to some version of the ramified theory of types by October 1906.

During the years between his two published type theories, Russell investigated various alternatives to avoiding the paradoxes. These are collected in a 1906 paper entitled "On Some Difficulties in the Theory of Transfinite Numbers and Order Types". These theories include the zig-zag theory, the limitation of size theory,[7] and the no-class theory. A consideration of these theories will occupy section 3 of the present chapter. The failure of these theories will be considered in light of Russell's return to the theory of types in his 1908 paper. This paper will be discussed in section 4, while section 5 compares the theory of types in the first edition of *Principia Mathematica* with Russell's previous type theories.

2.2 Theory of Types in Russell's *The Principles of Mathematics*

Although published in 1903, the majority of *The Principles of Mathematics* was written in 1900-1901. In the introduction to the second edition, written thirty-four years later, Russell maintains that "the fundamental thesis of the following pages, that mathematics and logic are identical, is one which I have never since seen any reason to

[7] Russell never actually worked out a version of this theory on his own. He viewed Cantor and Jourdain's works as examples of limitation of size theories.

modify" (Russell 1903, p. v). However, as will be seen, the formulations of the theory of types given in that work did undergo drastic changes in the years following its first publication.

That this first formulation was to encounter alterations by Russell is not very surprising. In fact, Russell opens the appendix devoted to the doctrine of types by stating that it is only "here put forward tentatively" (Russell 1903, p. 523). The theory of types is put forth as a possible solution to the paradox to which Russell devotes an entire chapter of his book. The troubling aspect of his contradiction is that "no peculiar philosophy is involved in the above contradiction, which springs directly from common sense, and can only be solved by abandoning some common-sense assumption" (Russell 1903, p. 105).

The common sensical assumption to which Russell refers seems, in this initial theory of types, to be the idea that wherever a "class as many" exists so too does the corresponding "class as one":

> Perhaps the best way to state the suggested solution is to say that, if a collection of terms can only be defined by a variable propositional function, then, though a class as many may be admitted, a class as one must be denied. When so stated, it appears that propositional functions may be varied, provided the resulting collection is never itself made into the subject in the original propositional function. In such cases there is only a class as many, not a class as one. We took it as axiomatic that the class as one is to be found wherever there is a class as many; but this axiom need not be universally admitted, and appears to have been

> the source of the contradiction. By denying it, therefore, the whole difficulty will be overcome (Russell 1903, p. 104).

In fact, in *The Principles of Mathematics*, as in Russell's subsequent attempts at securing the foundations of mathematics, it is the logical status of propositions, propositional functions, and classes that plays a central role.[8]

The above passage contains Russell's informal introduction to the theory of types. As he says at the end of the next paragraph "it is the distinction of logical types that is the key to the whole mystery" (Russell 1903, p. 105). It is the theory of types, as set forth in Appendix B of *The Principles of Mathematics*, which will now be investigated.

Types are derived from the ranges of significance of propositional functions. The 'range of significance' of a propositional function $\varphi(x)$ is that range in which the variable x must lie if $\varphi(x)$ is to be a well-formed proposition, whether true or false. The first point of Russell's theory of types here is that every propositional function has a range of significance. The second point which Russell deems "less precise than the first" (Russell 1903, p. 523) is that ranges of significance form types.

These types are stratified into a hierarchy in the following manner. First, the lowest type of objects is comprised of terms or individuals. Russell defines a term or individual to be "any object which is not a range" (Russell 1903, p. 523). A range, on the other hand, is defined, after a lengthy discussion of Frege's notion of *Werthverlauf* (translated by Russell as *range*), to be "what are properly called classes, and it is of them that cardinal numbers are asserted" (Russell 1903, p. 518). The next type is comprised of

[8] This will be made apparent throughout the rest of this chapter.

classes (ranges) of individuals. The next type after that consists of classes of classes of individuals. This process of forming types is extended to infinity and forms the first series of types.

In discussing this first hierarchy, Russell introduces the notion of a minimum type. This notion arises by considering a range, *u*, determined by a propositional function, $\varphi(x)$, and then looking at *not-u*. Whereas *u* consists of all objects, *x*, such that $\varphi(x)$ is true, *not-u* consists of all objects, *x*, such that $\varphi(x)$ is false. In this way *not-u* is contained in $\varphi(x)$'s range of significance. However, "there is a difficulty in this connection, arising from the fact that two propositional functions $\varphi(x)$, $\psi(x)$ may have the same range of truth *u*, while their ranges of significance may be different; thus *not-u* becomes ambiguous" (Russell 1903, p. 524). This ambiguity necessitates the introduction of the supposedly unambiguous *minimum types*. It is claimed that every *u* will be contained in a minimum type, where a minimum type is one which is not the sum of two or more types. Then *not-u* is defined as the remainder of this type.[9] From this point on all types are assumed to be minimum types.

Another series of types begins with what Russell calls *couples with sense*; these are relations viewed extensionally. Since he views, for philosophical reasons,[10] relations as intensional he has "doubts as to there being any such entity as a couple with sense" (Russell 1903, p. 512). In spite of this, extensions are deemed quite relevant to

[9] Whether this notion is really unambiguous seems questionable since a method for determining whether or not any given type is the sum of two or more types is lacking.
[10] These reasons are outlined in section 98 on page 99 of *The Principles of Mathematics*.

mathematics.[11] Thus the new hierarchy is formed beginning with couples with sense, then forming ranges of such extensional ranges, or "relations of relations, or relations of couples…or relations of individuals to couples, and so on; and in this way we get, not merely a single progression, but a whole infinite series of progressions" (Russell 1903, pp. 524-525). Similarly, trios form a new series of progressions of types. Thus Russell is justified in claiming that in this way "an immense hierarchy of types" (Russell 1903, p. 525) is formed.

However, this immense hierarchy is not as exhaustive as Russell would like. He points out that both propositions and numbers are omitted from the above hierarchies. Forming series of types starting with either of these notions presents difficulties "of which it is hard to see the end" (Russell 1903, p. 526). By not considering a hierarchy of proposition types Russell derives another contradiction which is analogous to the original contradiction with which he was dealing.[12] Russell notes that this contradiction can be avoided by stratifying propositions into types. However, he feels "this suggestion seems harsh and highly artificial" (Russell 1903, p. 528). Russell concludes the appendix with his outlook for foundational work in logic with the following: "The totality of all logical

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[11] This is reiterated by Russell in the following passage: "Throughout mathematics there is the same rather curious relation of intensional and extensional points of view: the symbols other than variable terms (i.e. the variable class-concepts and relations) stand for intensions, while the actual objects dealt with are always extensions. Thus in the calculus of relations, it is classes of couples that are relevant, but the symbolism deals with them by means of relations [in intension]" (Russell 1903, p. 99). This is closely analogous to Russell's treatment of classes in *The Principles*.

[12] The new contradiction is derived as follows: "If *m* be a class of propositions, the proposition "every *m* is true" may or may not be itself an *m*. But there is a one-one relation of this proposition to *m*: if *n* be different from *m*, "every *n* is true" is not the same proposition as "every *m* is true." Consider now the whole class of propositions of the form "every *m* is true," and having the property of not being members of their respective *m*'s. Let this class be *w*, and let *p* be the proposition "every *w* is true." If *p* is a *w*, it must possess the defining property of *w*; but this property demands that *p* should not be a *w*. On the other hand, if *p* be not a *w*, then *p* does possess the defining property of *w*, and therefore is a *w*. Thus the contradiction appears unavoidable" (Russell 1903, p. 527).

objects, or of all propositions, involves, it would seem, a fundamental logical difficulty. What the complete solution of the difficulty may be, I have not succeeded in discovering; but as it affects the very foundations of reasoning, I earnestly commend the study of it to the attention of all students of logic" (Russell 1903, p. 528). In this way, Russell does not decree that the theory of types will necessarily lead to the optimal solution of the paradoxes. He simply states this as one method which might prove useful; it is, rather, the discovery of the fundamental problem that seems to most interest Russell.

Thus in the appendix of *The Principles of Mathematics* dedicated to the theory of types, Russell is able to outline the main points of the theory. These are 1) that propositional functions have ranges of significance and 2) that these ranges of significance form *types* such that if $\varphi(x)$ is defined with the instantiation of a variable of type n, then $\varphi(x)$ will be defined with the instantiation of *any* variable of type n. The tentative sketch is open to several objections and shortcomings which Russell enumerates; for instance, the problem of dealing with types of propositions as well as number types.[13]

Since these difficulties are not satisfactorily dealt with here, Russell's continued study of possible solutions to the paradoxes is justified. In his letter of 15 March 1906 to Jourdain, Russell spells out when and how he adopted various theories from his abandonment of the theory of types at the end of 1902 until the autumn of 1905 when he embraced his substitutional theory. In the following Russell's "present view" is that the substitutional theory affords the best possible solution to the set-theoretic paradoxes:

[13] As will be seen below, Russell's different type theories from 1907 on all contain a hierarchy of propositional functions and hence of propositions, unlike his theory of types in 1903.

I am not sure that I can remember how my ideas developed. But I will tell you all I can. You will see that in my book [*The Principles*] (p. 104, art. 104) I suggest that certain functions do not determine a *class as one*. This is practically the same doctrine as that they do not determine a class, for a *class as many* is not an entity. (By the way, the view I now adopt, that a propositional function must not be varied, is discussed on p. 103, second par. of art. 103, and rejected as making mathematics unworkable. I have now discovered how, by substitutions, to work mathematics with this view.) My book gives you all my ideas down to the end of 1902: the doctrine of types (which in *practice* is almost exactly like my present view) was the latest of them. Then in 1903 I started on Frege's theory that two non-equivalent functions may determine the same class...But I soon came to the conclusion this wouldn't do. Then, in May 1903, I thought I had solved the whole thing by denying classes altogether; I still kept propositional functions, and made φ do duty for $\dot{z}(\varphi z)$. I treated φ as an entity. All went well till I came to consider the function W, where

$$W(\varphi).\equiv_{\varphi}.\sim\varphi(\varphi).$$

This brought back the contradiction, and showed that I had gained nothing by rejecting classes.

The latter part of 1903 and the beginning of 1904 I spent on the Fiscal Question. Then in April 1904 I began working at the Contradiction again, and continued at it, with few intermissions, till January 1905. I was throughout much occupied by the question of Denoting, which I thought was probably relevant, as it proved to be. A *denoting* function is, broadly, any function which is not propositional; at times I have used φ'x for a denoting function and $\varphi!x$ for a propositional function. The first thing I discovered in 1904 was that the variable denoting function is to be deduced from the variable propositional function, and is not to be taken as indefinable. I tried to do without ℩ as an indefinable, but failed; my success later, in the article 'On Denoting', was the source of all my subsequent progress. Most of the year, I adhered to the 'zig-zag' theory, and worked at different sets of primitive propositions as to what functions determine classes. But I never got a set of primitive propositions that would really work, and all the sets were horribly complicated and un-obvious. I soon discovered that the difficulty comes only where 'all values of φ' are concerned, and I thought perhaps this was due to a vicious circle, as follows: if

$$\psi x \,.=.\, (\varphi).\mathrm{f}(\varphi,x) \quad \mathrm{Df},$$

it is part of the meaning of ψx to assert $\mathrm{f}(\psi,x)$; thus ψ asserts something which cannot be defined till ψ is defined, and which is yet presupposed in the definition of φ. Gradually I discovered that to assume a separable φ in φx is just the same, essentially, as to assume a class defined by φx, and that non-predicative functions must not be analyzable into a φ and an x.

About June 1904, I tried hard to construct a substitutional theory more or less like my present theory. But I failed for want of the theory of denoting: also I did not distinguish between *substitution* of a constant for a constant and *determination* of a variable as this or that constant. Hence I abandoned the attempt to get on by means of substitution...Then, last autumn, as a consequence of the new theory of denoting, I found at last that substitution would work, and all went swimmingly (Russell 1906a, pp. 78-80).

The no-class theory is outlined in Russell's 1906 paper "On Some Difficulties in the Theory of Transfinite Numbers and Order Types". Actually, in a note appended 5 February 1906, Russell states that "from further investigation I now feel hardly any doubt that the no-class theory affords the complete solution of all the difficulties stated in the first section of this paper" (Russell 1906, p. 53). In the next section the no-class theory will be discussed along with the two other theories presented in Russell's 1906 paper; namely the zig-zag and the limitation of size theories.

2.3 Alternatives to the Theory of Types - The Zig-Zag Theory, Limitation of Size Theory, and the No-Class Theory

In Russell's 1906 paper three theories are put forth as ways of avoiding the logical paradoxes. These theories are the zig-zag theory, the theory of limitation of size, and the no-class theory; the theory of types is not mentioned. The paper is written in response to E.W. Hobson's 1905 paper "On the General Theory of Transfinite Numbers and Order Types" in which are raised "a number of questions which must be answered before the principles of mathematics can be considered to be at all adequately understood" (Russell 1906, p. 29). The contradictions that are produced led Russell to state that "a propositional function of one variable does not always determine a class" (Russell 1906, p. 37). Interpretations of this broad statement govern the variations in the theories considered in this paper. The zig-zag and limitation of size theories are grouped together as theories in which "all straight forward propositional functions of one variable determine classes, and that what is needed is some principle by which we can exclude the complicated cases in which there is no class" (Russell 1906, p. 37). In the case of the zig-zag theory classes are avoided which possess a "certain characteristic which we may call zigzagginess" (Russell 1906, p. 37). Before looking at the treatment of the zig-zag theory, the theory of limitation of size will be investigated.

In this theory, classes are avoided, as the name suggests, which are excessive in size. Functions are distinguished as either predicative, or non-predicative; the distinction depends on a certain limitation of size.[14] Non-predicative functions are such that they do not give rise to corresponding classes. For instance, "if *u* is a class, '*x* is not a member of *u*' is always non-predicative; thus there is no such class as 'not *u*'" (Russell 1906, p. 43). No general rules are given for determining when a class is 'too big' as this is just an outline given by Russell. The problems that Russell sees as inherent to this mode of solution include, most importantly, the fact that this theory "does not tell us how far up the series of ordinals it is legitimate to go...we need further axioms before we can tell where the series begins to be illegitimate" (Russell 1906, p. 44). Russell concludes by stating that the problems with the theory seem to outweigh the merits. As such the theory seems to be less attractive than other possibilities.[15]

[14] Russell notes that this theory is advocated by Jourdain in, for instance, "On the Transfinite Cardinal Numbers of Well-ordered Aggregates" (1904), and "On Transfinite Cardinal Numbers of the Exponential Form" (1905).
[15] Interestingly, this is the kind of theory developed by Zermelo, Fraenkel, Skolem, von Neumann, Bernays, and Gödel. It has become the standard form of set theory used today.

One of these possibilities to which Russell had adhered for most of 1904 is the zig-zag theory. In this theory the predicative functions, which again determine classes, are "fairly simple, and only fail to [determine classes] when they are complicated and recondite" (Russell 1906, p. 38). Thus it is clear that the *size* of a class does not affect whether or not it is to exist. Russell gives the example that "x is not a man" is a simple function and under this theory it would hence be predicative. However, the class determined by this function is satisfied by all but a finite number of objects.

The name of the theory comes from Cantor's proof that there is no greatest cardinal. The zig-zag property of the predicative functions in this theory is explained by Russell's stating that "if now $\varphi!x$ is a non-predicative function, it follows that, given any class u, there must either be members of u for which $\varphi!x$ is false, or members of not-u for which $\varphi!x$ is true. (For, if not, $\varphi!x$ would be true when and only when, x is a member of u; so that $\varphi!x$ would be predicative.) It thus appears that $\varphi!x$ fails to be predicative just as much by the terms it does not include as by the terms it does" (Russell 1906, p. 38). For the full development of this theory Russell notes that a full set of conditions is required for determining when functions are predicative. For work along these lines he directs the reader to sections 103 and 104 of his *The Principles of Mathematics*.

However, all attempted sets of defining axioms for these predicative functions turned out to be exceedingly complicated. He was thus led to abandon this theory. He did this with the reservation that perhaps further research would yield a more adequate set of axioms. In general, he concludes, the zig-zag theory "applies better to cardinal than ordinal contradictions" (Russell 1906, p. 39) whereas the theory of limitation of size simply dismisses cardinals and ordinals altogether after some unspecified size. Neither of these theories seems adequate as a proper solution to the paradoxes, and so Russell next turns to his no-class theory.

In this no-class theory propositions exist as the fundamental logical machinery; moreover, classes and relations (and hence propositional functions) are banished altogether. This is motivated by the fact that the assumption of the existence of these entities leads to problems. Thus, rather than imposing certain conditions upon them, they are simply assumed non-existent. Russell points out three objections to this assumption. "(1) that it seems obvious to common sense that there are classes; (2) that a great part of Cantor's theory of the transfinite, including much that is hard to doubt, is, so far as can be seen, invalid if there are no classes or relations; (3) that the working out of the theory is very complicated, and is on this account likely to contain errors, the removal of which would, for aught we know, render the theory inadequate to yield the results even of elementary arithmetic" (Russell 1906, p. 45).

In this theory, instead of functions, propositions are taken as the starting point. This is done by starting with a proposition p in which a is a constituent. Then '$p\ x/a$' denotes what occurs when x is substituted in p wherever a appears. Then, if b is not a part of p, and if q is set to be equal to $p\ b/a$, then "$p\ x/a$ is true for all values of x" is equivalent to "$q\ x/b$ is true for all values of x". In this way statements about $p\ x/a$ depend only upon p. Thus, Russell is able to replace statements involving propositional functions by statements involving propositions. Russell claims that "there is not much difficulty in re-wording most definitions so as to fit with the new point of view. But now the existence theorems become hard to prove" (Russell 1906, p. 46). For instance, the existence of ω can be proved, as can the existence of various other ordinal types. However, the existence of *all* the usual ordinal types cannot be proved. Just like in the theory of limitation of size at what point the series begins to be non-existent is unknown. As the note appended to the paper shows, Russell's subsequent work on the no-class theory led him to see this theory as the best way of avoiding the logical paradoxes. This theory was soon abandoned though, and Russell returned to his theory of types in his 1908 paper.[16] As his unpublished works show, Russell worked extensively on the substitutional, or no-class theory from 1905 until 1907. Certain aspects of this theory were carried over into Russell's subsequent type theories. Russell gave the full exposition of his substitutional theory in a paper entitled "On the Substitutional Theory of Classes and Relations", submitted to the London Mathematical Society in April 1906, but withdrawn in October of that year.

The primitive notions incorporated in this April 1906 paper are 'entity' (or 'individual') and 'propositional function'. Thus, the propositional functions were given the status of existence, but classes and relations were not. When he withdrew his article from publication Russell saw the substitutional theory as inadequate and in need of modification. Several of the unpublished papers from 1905 through 1907 show Russell working out on paper various improvements of this substitutional theory.

[16] For a discussion of this see Consuegra, F. A. R.; "Russell's Theory of Types, 1901-1910: Its Complex Origins in the Unpublished Manuscripts" (1989).

During these years Russell was aware of a similarity between his substitutional theory and his early version of the theory of types. In his June 1906 paper "The Paradoxes of Logic", Russell claims that his 1903 type theory "differs little from the no-class (or substitutional theory)", which is not "greatly different from the zig-zag theory that had been adopted in sections 103 and 104 of *The Principles*" (Russell 1906c, p. 280). He goes on to say that "the only thing that induced me at that time to retain classes was the technical difficulty of stating the propositions of elementary arithmetic without them – a difficulty which seemed to me insuperable" (Russell 1906c, p. 280).[17] This is compared to the no-class theory in which "it is natural to suppose that classes are merely linguistic or symbolic abbreviations" (Russell 1906c, p. 285).

Around this time Russell tried to give the substitutional theory a greater degree of security as a foundation for mathematics. In doing so he investigated various alternatives in which certain aspects of the substitutional theory and type-theory were intermingled. In particular, in two papers written in April and May 1906, entitled "On Substitution" he states that the way around the paradoxes (that he here calls "odditites") is to introduce a hierarchy of propositions. He makes the same claim in "On the Functional Theory of Propositions, Classes, and Relations" (1906g), while in his "Logic in which Propositions are not Entities" (1906f) he explicitly dispenses with such a hierarchy. Finally, in his September 1906 paper "The Paradox of the liar" he goes so far at one point as to reinstate classes and relations altogether. This is supplemented with a hierarchy of these entities –

[17] Significantly, Russell adopts the vicious-circle principle here where "whatever involves apparent variables must not be among the permissible values of that variable" (Russell 1906c, p. 289). This principle, as will be seen below, was eventually incorporated as the guiding principle in Russell's more mature type-theories.

all in contrast with the work he had been doing throughout 1905 and 1906 - where "the hierarchy which seems both inherently the most plausible and the best designed to obviate the paradoxes is arranged according to the number of apparent variables in a statement" (Russell 1906h, p. 331). Although he has a hierarchy composed of individuals, classes of individuals, classes of classes of individuals, and so on, he has eliminated any hierarchy for propositional functions and propositions. Thus, the axiom of reducibility that he adopts here is significantly different from his later version: any propositional function with any kind of apparent variable is equivalent to a function with only entities and classes as apparent variables (Russell 1906h, p. 339).

It is clear by the following letter from Whitehead that by 7 October 1906, Russell had abandoned the substitutional theory for some ramified type-theory:

> The nastiness which you wanted to avoid is the Frege bugbear of propositional functions becoming unmeaning when certain terms are substituted. According to the doctrine of types we have got to put up with this – thus certain things (such as functions) which can be named and talked about won't do as arguments in some propositional functions. The result is that we have to use the restricted variable. The doctrine of substitution was on stronger ground here; for it did without the function entirely, and simply brought in *p/a* as a typographical device for predicting that we were talking of the one entity when we were really talking of two. Hence, if you want the unrestricted variable, the doctrine of substitution is the true solution…My doctrine is (1) that the variable must be restricted, because (as you prove) the consideration of all terms leads to contradictions. Also in

considering "any" a restriction to significance is necessary (Whitehead 1906, p. 13).

Thus, although Russell had done much work on the theory of substitution, he was eventually led back to incorporating a division of types in the foundations of his system. As will be seen in the next section, his next published work on the theory of types (in 1908) was the product of a substantial amount of unpublished work done from 1906 to 1908.

2.4 The Return to the Theory of Types

The shift from the substitutional theory to the theory of types occurred gradually and is apparent in Russell's unpublished works dating from 1906 to 1908. In fact, certain aspects of the substitutional theory even survived in Russell's later versions of the theory of types. In his unpublished paper "Corrections in Present Work" (1906i), Russell reinstates membership as a primitive idea. Furthermore, he defines identity of x and y by the condition that x and y belong to the same classes:

$x \varepsilon \alpha$	Pi
$\vdash: (E\alpha): \varphi x . \equiv . x \varepsilon \alpha$	Pp
$x = y . \equiv: x \varepsilon \alpha . \equiv . y \varepsilon \alpha$	Df
$\vdash:. x \varepsilon \alpha . \equiv . x \varepsilon \beta: \rightarrow . \alpha = \beta$	Pp (Russell 1906i, p. 493).

Russell goes on to state "that the doctrine of types is never relevant except when we use the inference $(x).\varphi x . \rightarrow . \varphi a$. Thus we can go as far as we like without explicitly introducing the doctrine, so long as we can avoid applications to special cases" (Russell

1906i, p. 494). This sentiment is reiterated in his unpublished "Types" where he states that the theory of types is never needed "except when a constant is assumed to be a determination of a variable" (Russell 1906j, p. 500). However, he soon realizes that this claim is mistaken by considering the proposition

$$(x).\varphi x: (x).\ \psi x : \equiv : (x).\varphi x.\psi x$$

which he recognizes as "only true if φx and ψx have the same range of significance" (Russell 1906j, p. 500). The majority of this paper shows him struggling with an axiom which allows, in a formula φx, the substitution of a propositional function for the variable x whenever it is known that φx is true for all values of x: this axiom violates the theory of types. In the end Russell is compelled to accept the axiom of reducibility and concludes that "if we have to put this <axiom>, we may as well assume classes and take $x\varepsilon\alpha$ as the form of φx required by the primitive proposition" (Russell 1906j, p. 514).

Russell continued to study the theory of types in his paper written sometime after September 1906 and before July 1907, entitled "On Types". In that paper he combined an old version of the substitutional theory with a theory of types. The hierarchy of types begins with individuals; the second type is formed by taking first-order propositions containing no apparent variables other than individuals; the third type contains second-order propositions which contain no apparent variables other than individuals and first-order propositions. This hierarchy is continued for all finite orders. He also states an axiom of reducibility:

$$(\mathrm{E}p^1):p^n/a = p^1/a$$

and goes on to say that "there is much to be gained from reviving substitution...This plan seems indubitably feasible, but complicated. I don't know whether it is worth it...If this form of substitution turns out to be feasible, perhaps it should be put in an appendix. It is philosophically simpler than functions, but technically vastly more complicated" (Russell 1906/1907, p. 516). However, Russell ends by noting that the substitutional theory is perhaps too complicated: "The point of view of substitution is perhaps unnecessarily complicated, seeing that φx is needed in any case, and so is $\varphi(x,y)$. The only thing we save is $\varphi!x$. The necessity for φx makes the philosophical gain less than it would be. If φx could be avoided, substitution would be worth adopting" (Russell 1906/1907, p. 517).

Russell continued this study of the theory of types in several other unpublished papers from this era. In "Notes on Types" (1907a), Russell notes several times that in practice it is only necessary to account for relative type differences. While in his "Fundamentals", also written in 1907, he makes it clear that he has fully accepted a ramified version of the theory of types while abandoning the no-class theory:

> Note that the no-class theory is in essence abandoned by distinguishing between φx and $x\varepsilon\hat{y}(\varphi y)$. For this requires that $\hat{y}(\varphi y)$ should be a constituent of $x\varepsilon\hat{y}(\varphi y)$ and therefore that $\hat{y}(\varphi y)$ should be something. This difficulty seems inherent in the no-class theory, since functions must be allowed as apparent variables. A value of an apparent variable must be something, and thus the no-class theory won't work. It worked while we had propositions, because then they became apparent variables where a variable matrix was wanted. But if propositions are not to be apparent

variables, functions must be, and therefore functions must be admitted. But then they may as well be classes (Russell 1907b, p. 543).

These sentiments are reiterated several times throughout this rough piece of Russell's work. Finally, Russell gives a survey of the technical and philosophical points of his theory in the following passage:

> A. *Technical.* All contradictions are avoided provided no expression containing an apparent variable is a possible value of that apparent variable. This requires two sorts of functions, one of which can be an apparent variable while the other can't. Whether we call the two sorts φx and $\varphi!x$, or φx and $x\varepsilon\alpha$, is technically indifferent. Whichever we do, we need a reducibility-primitive-proposition, so that one of the wider kind is always *equivalent* to some one of the narrower kind. And a proposition containing as an apparent variable a function of the narrower kind must not be itself of the narrower kind. These conditions being satisfied, the contradictions are avoided and mathematics is workable; provided that a function (of the narrower kind) can't be argument to itself.
>
> B. *Philosophically.* (a) The no-class theory, with the theory of predicative and non-predicative functions, supplies what is required, except that (α) there is a difference about the meaning of a function as apparent variable; (β) the distinction of predicative and non-predicative is obscure, and the reducibility axiom is arbitrary. (b) The plan of distinguishing φx from $x\varepsilon\hat{y}(\varphi y)$, in which the former assent the values of φ for the argument x, while the latter asserts

that x has the property φ, has much to recommend it. In this plan, φ can only be an apparent variable when it is *explicit*, as in $x\varepsilon\hat{y}(\varphi y)$, not when it is as in φx. This has the merit of making the reducibility-axiom obvious, since it states that "$\varphi x \,.\equiv.\, x$ has the property φ". But it seems to involve treating truth-functions as a type. This comes from considering: "ζ asserts that $\zeta\varepsilon\hat{A}\{A$ asserts $(x\varepsilon\alpha) .\rightarrow. x\sim\varepsilon\alpha\}$". This reproduces the *liar*. And there are grave differences about treating truth functions as a type. (c) The plan of never varying functions at all and introducing $x\varepsilon\alpha$ as a primitive idea, has very great advantages. It is simple, it makes a very clear distinction between predicative and non-predicative functions, it allows us to use the argument that the φ in φx can't be varied because it doesn't occur in φx and is in fact nothing, and it makes the reducibility-axiom simply the universally admitted axiom of classes. The objection to this plan is that it makes it hard to see why $\alpha\varepsilon\alpha$ is meaningless. [Note that it is not strictly necessary that $\alpha\varepsilon\alpha$ should be meaningless, but only that $(x).f(x\varepsilon\alpha)$ should not imply $f(\alpha\varepsilon\alpha)$.] And to get "$Ex\varphi!x$" as *not* a function of x, we still need the notion of a truth function" (Russell 1907b, pp. 552-553).

It is thus clear that in the time leading up to the publication of his 1908 article Russell spent much time working on ironing out the details of his type theory.

In his paper, "Mathematical Logic as based on the Theory of Types", submitted July 1907 and published in 1908, Russell attempts to alter the theory of types in order to avoid the problems which riddled it in 1903. His view of the paradoxes has changed somewhat as he claims that they all seem to stem from "the assumption of a totality such that, if it were legitimate, it would at once be enlarged by new members defined in terms of itself" (Russell 1908, p. 155). The way to avoid the contradictions thus seems to rely on an avoidance of this reflexivity. Russell states his *vicious-circle principle* that "'whatever involves *all* of a collection must not be one of the collection', or conversely: 'If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total'" (Russell 1908, p. 155) is a rule that would rectify the situation. This rule is too restrictive, and instead Russell aims at improving his theory of types.

In this paper he stipulates that the problem can be avoided by considering the distinctive feature of propositions that contain the word 'all'. By distinguishing between the use of the words 'all' and 'any', Russell is able to distinguish between the notion of a 'real' versus an 'apparent' variable. Russell attributes the distinction between real and apparent variables to Frege. By asserting *any* value of a propositional function, one asserts $\varphi(x)$ where x is a real variable. On the other hand, when stating that a propositional function holds for *all* variables x, one asserts $(x)\varphi(x)$ - namely the generalized proposition corresponding to $\varphi(x)$; here the variable x is an apparent variable. The problems arise by considering propositions that include phrases such as "all propositions" or "all properties". Thus, it seems reasonable to dispense with apparent variables altogether using 'any' in place of 'all'. However, Russell provides several examples to show that 'any' cannot replace 'all' in many instances throughout mathematics. Russell concludes that any theory must not only avoid the paradoxes by dealing with the use of the term 'all', but it must also retain the distinction between 'all' and 'any'.

It is with this distinction between 'all' and 'any' that Russell introduces the *range of significance* of a propositional function. This he does as follows:

> Every proposition containing *all* asserts that some propositional function is always true; and this means that all values of the said function are true, not that the function is true for all arguments, since there are arguments for which any given function is meaningless, that is, has no value. Hence we can speak of *all* of a collection when and only when the collection forms part or the whole of the *range of significance* of some propositional function, the range of significance being defined as the collection of those arguments for which the function in question is significant, that is, has a value (Russell 1908, p. 163).

Thus Russell distinguishes between the range of truth and the range of significance as he had done in 1903. However, he is able to do this more precisely with the distinction between real and apparent variables (that is, by talking specifically about what 'all' is to mean when it occurs in propositions).

Just as in his 1903 version of the theory of types, Russell here defines a 'type' as "the range of significance of a propositional function, that is, as the collection of arguments for which the said function has values" (Russell 1908, p. 163). He also notes that the range of values of apparent variables in propositions form types, the "type being fixed by the function of which 'all values' are concerned" (Russell 1908, p. 163). It is the apparent variables here that Russell claims determine an expression's type. Furthermore, the need for the division of types is necessitated by the reflexive fallacies mentioned above. It is in this way that the apparent variables determine the types of expressions; anything that contains an apparent variable must be of a 'higher type' than that apparent variable. Thus, 'type' is defined in a similar, and yet more precise fashion in 1908 than it was in *The Principles of Mathematics.*

After defining what 'types' are to be in general, Russell begins the construction of the hierarchy of types of propositions. This was one of the problems of his 1903 account of the theory of types, namely that it could not account for proposition types. His 1908 paper overcomes this difficulty by first noting that propositions containing apparent variables are generated from ones without apparent variables by a process of generalization - that is, by "the substitution of a variable for one of the terms of a proposition and the assertion of the resulting function for all possible values of the variable" (Russell 1908, p. 163).[18] Propositions that contain apparent variables are called 'generalized' while those that do not are called 'elementary'. It is claimed that elementary propositions can be analyzed into different components: these are called 'terms' and 'concepts'. The terms are "whatever can be regarded as the *subject* of the proposition, while the concepts are the predicates or relations of these terms" (Russell 1908, p. 164). The terms are called 'individuals' and are said to form the first (or lowest) type.[19] Russell adds a further stipulation that individuals must be "destitute of

[18] Clearly the no-class theory is influencing Russell's work here. As will be seen in this paragraph, it is this incorporation of ideas from the no-class theory that allows Russell to solve the problems he had faced in 1903 with regards to the propositional hierarchy.
[19] Russell notes here, which he did not note in 1903, that it is only the relative types that matter in practice. Thus, the 'lowest type' can differ in different contexts. The soundness of his account thus only depends upon fixing some lowest type, and then seeing whether the process of generating further types is admissible.

complexity" in order that no individual be a proposition. With this in mind one is able to generate a proposition by applying the process of generalization to individuals in elementary propositions. The second logical type is then defined to be "elementary propositions together with such as contain only individuals as apparent variables" (Russell 1908, p. 164). These propositions are furthermore called 'first-order propositions'. From these 'second order propositions' are defined as propositions which contain first-order propositions as apparent variables. The collection of these propositions forms the third logical type. This process can be continued indefinitely. Russell concludes this construction of the propositional types by stating that "the (n+1)th logical type will consist of propositions of order n, which will be such as contain propositions of order n -1, but of no higher order, as apparent variables. The types so obtained are mutually exclusive, and thus no reflexive fallacies are possible so long as we remember that an apparent variable must always be confined within some one type" (Russell 1908, p. 164). Thus, Russell's construction of types of propositions is made possible by his careful distinction between real and apparent variables.

After this presentation of the propositional type hierarchy, Russell proceeds to the construction of a hierarchy of functions. His development differs from the process found in 1903 since Russell defines functions of various orders from propositions of various orders through a process of *substitution.*[20] Russell does this by first defining 'matrices'. These take the place of functions and are defined as follows: "if p is a proposition and a a constituent of p, let "$p/a;x$" denote the proposition which results from substituting x for a

[20] Again, the no-class theory seems to be coming into play.

wherever a occurs in p. Then p/a, which we will call a *matrix*, may take the place of a function; its value for the argument x is $p/a;x$ and its value for the argument a is p" (Russell 1908, p. 164). He defines matrices similarly for two variables. Russell notes that this can be done and is advantageous since the order of a matrix only depends upon the order of the proposition in which the substitution is effected.

Although this can be done, the replacement of functions by matrices is technically inconvenient. Instead Russell proceeds to build the hierarchy of functions directly from the propositional hierarchy. This is done by first defining a function of individuals (whose value is always a first-order proposition) to be a *first-order function.* Then a function having a first-order function or proposition as an apparent variable will be called a *second-order function*, and so on. In this way he is able to define the hierarchy of functions in terms of the hierarchy of proposition where "the type of a function is determined by the type of its values and the number and type of its arguments" (Russell 1908, p. 165).

This process of building the functional hierarchy is explained further by considering predicative functions. This is another difference from the theory of types found in *The Principles of Mathematics.* A predicative function is defined as a function "which is of the order next above that of its argument[s]" (Russell 1908, p. 165). These functions are denoted using an exclamation mark as in '$\varphi!x$'. The possible values of the predicative functions form well-defined totalities, and so the function symbols can be turned into apparent variables. This is only possible for predicative functions and so Russell is led to consider how to deal with non-predicative propositional functions.

Since a propositional function may be of any order, statements about 'all propositional functions of a variable x' will be meaningless. However, if these statements are limited to predicative functions of x, they will have a meaning, since the 'all' will refer to 'all predicative functions of a certain type'. Thus Russell is led to propose his axiom of reducibility; that is, the axiom which states that "every function is equivalent, for all of its values, to some predicative function" (Russell 1908, p. 168). This axiom is not stated in 1903 since the hierarchy of class-types formed there does not depend upon the proposition-types (since they were un-formable in 1903).

This 1908 paper works to meet the demands made in 1903 further by showing how number-types can be formed. However, this paper still concludes with an air of uncertainty with regards to the theory of types. Russell admits that "the theory of types raises a number of difficult philosophical questions concerning its interpretation" but that these questions are left to be "dealt with independently" (Russell 1908, p. 182). Furthermore, the justification for using the axiom of reducibility is purely pragmatic. However, this axiom was still incorporated into the theory of types in the first edition of *Principia Mathematica* as will be seen in the next section.

Following the general headnote to Part II, vol. 5 of Russell's collected works, a letter from Whitehead to Russell is now quoted to sum up the views that Russell held on logic by 16 June 1907. In this letter Whitehead attempted

> in (1) to (10) to give an outline of your position – apart from special procedures - as I understand it.
>
> (1) Your transition from intension to extension by means of

$f\{\hat{y}(\varphi y)\} \,.=:\, (E\psi):\varphi x \,.\equiv.\, \psi!x :f\{\psi!y\}$ Df is beyond all praise. It must be right. That peculiar difficulty, which has worried us from the beginning is now settled forever.

(2) The vicious-circle principle and the idea of a totality of propositions (i.e. the use of $(p)\ldots$) appear to conduct inevitably to your hierarchy of propositions.

(3) I agree that the substitutional theory is the proper explanatory starting point.

(4) The hierarchy of propositions appears to depend essentially on the distinction between dependent and independent entities – the dependent entities having in some sense an essential reference to totalities – and thence also on the various modes of dependency.

(5) The independent entities (individuals) require no further logical discrimination.

(6) Every entity (independent or dependent) must occur in a proposition containing it, in a manner specifically relevant to its peculiar type of being.

(7) The vicious-circle principle rules out an unlimited totality of "all entities".

(8) Number (6) above considers any totality of entities to be the totality of entities of a certain type.

(9) For our purposes we may define an entity as that which in any sense can be amenable to arithmetic ideas. The vicious-circle principles show that this amenability must be of varying types.

(10) It is possible that all entities, <which are> not individuals have no proper unity in any sense whatsoever; but that as they appear in propositions they are

simply a grouping of ideas which conceal an alarming complexity of thought (Whitehead 1907, p. 490).

2.5 Types in the First Edition of *Principia Mathematica*

The theory of types is developed in the first edition of *Principia* in a very similar manner to the construction in Russell's 1908 paper. The main difference lies in Russell's more precise statement of the axiom of reducibility. Also, the hierarchy of matrices is more fully developed. These two aspects of the theory of types will be considered in this section.

The axiom of reducibility is needed in the first edition of *Principia* to deal with those propositional functions that involve functional variables just as it had been in 1908. The axiom of reducibility is "the assumption that, given any function $\varphi\hat{y}$, there is a formally equivalent *predicative* function, *i.e.* there is a predicative function which is true when φx is true and false when φx is false" (Whitehead and Russell 1910, p. 56). This axiom is needed, for instance, in dealing with statements that include the "notion 'all properties of *a*,' meaning 'all functions which are true with the argument *a*'" (Whitehead and Russell 1910, p. 55) since these involve the illegitimate totalities of 'all properties' and 'all functions.' But, one can speak of 'all predicative properties of *a*,' or 'all second order properties of *a*;' in general, one can even speak of 'all *n*th order properties of *a*' for any fixed *n*. Since a wide range of mathematical reasoning involves notions such as 'all properties of *a*,' the axiom of reducibility is introduced so that this body of mathematical work is not simply discarded by the theory of types. By claiming that any property is equivalent to some predicative property, the axiom of reducibility enables such mathematical reasonings to remain sound. This axiom is employed in the construction of the matrix-type hierarchy as follows:

The division of matrices into types is effected by beginning with objects which are neither functions nor propositions; these are to be the *individuals*. The first matrices have only individuals as arguments and no apparent variables. The collection of functions that can be derived from these matrices by turning their variables into apparent variables are the *first-order functions*. Since these form a well-defined collection, the first-order functions can act as variables and can occur in quantifiers. Thus, a new set of matrices arise (which Russell calls *second-order matrices*), namely those which contain individuals and first-order functions as arguments. From these new matrices a different class of functions can be derived by turning the variables in the second-order matrices into apparent variables. These functions together with the second-order matrices are called *second-order functions*. This process can be continued, deriving functions of the (n+1)th order from those of lower orders.[21] This is the more complicated hierarchy that Russell constructs, and it is to functions in this hierarchy that the axiom of reducibility needs to be applied (for reasons adduced above).

Several examples in which the axiom of reducibility is seen as essential are put forth by Russell. These examples include the consideration of the proposition "Napoleon had all the predicates that make a great general" (Whitehead and Russell 1910, p. 56), an

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[21] An analogous procedure is carried out for the propositional hierarchy. This process begins with elementary propositions, from which first-order propositions are derived. Russell notes that "the propositional hierarchy is never required in practice, and is only relevant for the solution of the paradoxes" (Whitehead and Russell 1910, p. 55).

application to the notion of *identity*, as well as to the theory of classes. Finally, the reasons for accepting the axiom of reducibility are listed. Russell notes that it is clear that this axiom is not self-evident, but that self-evidence is only one reason for accepting an axiom. He continues by stating that since nothing that appears to be false can be derived from it, and it leads to many propositions which seem to be incontestable, the axiom has an inductive appeal. He does not rule out the possibility of there being another more fundamental axiom which can replace the axiom of reducibility, but he states that this is no reason for not using this axiom in the meantime since it is so useful in the derivation of a large portion of mathematics.

2.6 Concluding Remarks

In this chapter Russell's transition from the theory of types to various other theories, and back to the theory of types has been outlined. These theories were all given in order to avoid the logical paradoxes that arose around the turn of the century. The reactions to Russell's theory of types, as given in *Principia Mathematica*, were quite varied. A common theme was the feeling that Russell's reliance upon the axiom of reducibility needed to be abandoned. This was motivated by a desire to base mathematics upon a securely constructed basis. In the next chapter several attempts at securing the foundations of mathematics will be outlined.

3. The Theory of Types Until the Second Edition of *Principia Mathematica*

3.1 Introduction

In this chapter the developments and comments upon the theory of types that occurred between the publication of the first and second editions of *Principia Mathematica* are discussed. This is followed by an outline of the changes to the theory found in the introduction to the second edition of *Principia*. The works that are focused upon are all either mentioned explicitly as providing great improvements to mathematical logic, or else they are found in the list of "contributions to mathematical logic since the publication of the first edition of *Principia Mathematica*" (Whitehead and Russell 1925, p. xlv).

The first work considered is Hermann Weyl's *The Continuum* (1918). This book is mentioned in the list of contributions to mathematical logic, but not explicitly by Russell in the text of the introduction. The constructive methods in this book proved to be influential in later work in the theory of types. In the next chapter Weyl's methods, along with Chwistek's, will be sharply contrasted with Ramsey's. One of Weyl's chief concerns in his construction of the continuum was the removal of any vicious-circle. This is in line with Russell's aims. However, Weyl was not satisfied with the non-constructive nature of some of Russell's methods; most notably, the notorious axiom of reducibility.

The second work considered is Leon Chwistek's. Chwistek held a constructivist attitude comparable to Weyl's. The main aim of his 1924 work, "The Theory of

Constructive Types", was to rebuild *Principia Mathematica* without any reliance upon the axiom of reducibility. In his work Chwistek anticipates Ramsey's later distinction of the simple from the ramified theory of types. Chwistek's work proved to be very influential in later years, most notably in the work of John Myhill. Russell also tried to avoid using the axiom of reducibility in the second edition to *Principia*, although he did this along lines different from Chwistek.

The work that bears most directly upon the theory of types in the second edition of *Principia* is Wittgenstein's *Tractatus Logico-Philosophicus*. However, this work will not be considered on its own. It will only be outlined in connection with Russell's work in *Principia* and as it applies to subsequent work in the theory of types.

In the second edition of *Principia* Russell takes the idea introduced by Wittgenstein that all functions of functions are extensional and works "out its consequences" (Whitehead and Russell 1925, p. xiv) in the new introduction. Thus, Wittgenstein's work is the one which influences the changes in Russell's theory of types the most. Wittgenstein's extensional view of functions of functions was also taken up by Ramsey. However, Ramsey came to some conclusions radically different from Russell's with respect to the theory of types. This is most likely due to the enigmatic style in which Wittgenstein's *Tractatus* is written, leaving it open to various interpretations.

3.2 Weyl's Continuum and Chwistek's Constructive Types

In *The Continuum*, Weyl attempts to construct the continuum of real numbers using only a finite number of principles. In doing so he avoids the axiom of reducibility or any equivalent axiom. Taking the "sequence of the natural numbers [as] an ultimate foundation of mathematical thought" (Weyl 1918, p. 48) Weyl defines the real numbers and derives a large portion of analysis. Throughout he emphasizes the importance of using precise notions in a constructive way. This is clear in his statement that "the principles of definition must be used to give a precise account of the sphere of the properties and relations to which the sets and mappings correspond" (Weyl 1918, p. 47).

In his construction Weyl succeeds in building a set of real numbers which satisfy Cauchy's principle of convergence[22] as well as being everywhere dense. Weyl cites three principles that do not hold in his system. These are "Dedekind's cut principle," that "a bounded set of real numbers has a unique least upper bound and a unique greatest lower bound," and that "every bounded infinite set of real numbers has an accumulation point" (Weyl 1918, p. 77).[23] Furthermore, Weyl describes the limitations that he prescribes in his work as "the unrestricted application of the concepts "existence" and "universality" to the natural numbers, but not to sequences of natural numbers" (Weyl 1918, p. 3). These restrictions are effected through his use of different *levels* of variables in his quantifiers. These levels correspond to types. Just as in Russell's work, Weyl is led to a ramified type theory by considering relations that hold not only between individuals, but also between relations, between individuals and relations, and so on. This is done in section 6 of his

[22] Weyl states this principle as follows: "*The sequence* **f**(n) *converges to some real number* **c** *if and only if this sequence is convergent*" where "a sequence of real numbers **f** is called *convergent* if, for every fraction a, there is a natural number n such that for every p and q which are $> n$, the rational number $-a$ belongs to the domain **f**(p)-**f**(q), but $+a$ does not. Further, we say that the sequence *converges to the real number* **c** if, for every fraction a, there is a natural number n such that for every $p > n$, the rational number $-a$ belongs to the domain **f**(p)-**c**, but $+a$ does not" (Weyl 1918, p. 75). He goes on to note that in these definitions quantification only occurs over natural numbers.
[23] Unlike Weyl's Cauchy convergence principle, these all involve quantification over objects other than natural numbers. This is the reason why the one can hold, while these do not in Weyl's construction.

book. However, Weyl's type theory differs from Russell's in that his types are not disjoint; they are cumulative.

The principles that Weyl is unable to derive in his system of real numbers without the axiom of reducibility are similar to those that Russell was unable to derive. However, Weyl does not find it disconcerting that he is unable to derive these theorems, since his continuum is based upon more constructive grounds than most alternatives. Another attempt at limiting the theory of types by avoiding the axiom of reducibility and using only constructive methods was made by Leon Chwistek.

Using a constructive procedure, just as Weyl had done, Chwistek works in his "Theory of Constructive Types" towards an improvement of *Principia Mathematica.* Many of the ideas developed by Chwistek will be seen to be quite similar to Ramsey's in the next chapter. For instance, Chwistek distinguished between two kinds of theories of types; the simple and the ramified theories. Chwistek also distinguished the two different kinds of paradoxes. However, he thought that Richard's paradox could be constructed in Russell's theory of types (Chwistek 1924, p. 13-14). Ramsey, as will be seen, rectified this mistake.

Like Ramsey, Chwistek aimed at reconstructing the theory of types without the axiom of reducibility. This restriction to a purely "constructive" theory led, as Russell pointed out in the introduction to the second edition of *Principia* (Whitehead and Russell 1925, p. xiv), to the loss of much of mathematics – just as Weyl's overly constructive system had. Although Russell also tried to avoid the use of the axiom of reducibility in the second edition of *Principia*, he did so along different lines; his method followed the reconsideration of the nature of functions as introduced by Wittgenstein. Although Russell did not follow Chwistek's methods, Chwistek's work did prove to be quite influential. This is especially apparent in John Myhill's work in the 1950's as will be seen in the last chapter of this thesis.

Again, Wittgenstein's *Tractatus Logico-Philosophicus* proved to be the most influential work bearing on Russell's introduction to the second edition of *Principia*. Wittgenstein's work was incorporated by many logicians including not only Russell, but also Ramsey and Carnap, as discussed in the next chapter. Russell's use of Wittgenstein's work is discussed in the next section.

3.3 Russell's Modifications to the Theory of Types

In the second edition of *Principia Mathematica* Russell wishes to avoid the axiom of reducibility just as Chwistek and Weyl had. Since this axiom is utilised in much mathematical reasoning, the careful replacement of it in the second edition of *Principia* is of considerable importance. Russell's proposal stems from a change in philosophic viewpoint as to the nature of functions. This new conception is borrowed from Wittgenstein's investigations in his *Tractatus Logico-Philosophicus.* Here "functions of propositions are always truth-functions," and "a function can only occur in a proposition through its values" (Whitehead and Russell 1925, p. xiv).[24] Using this fundamental change, Russell modifies the construction of the type hierarchy while avoiding the axiom of reducibility. However, although the hierarchy is constructed more smoothly with these

[24] A function only occurring in a proposition through its values simply means that it is not $\varphi\hat{y}$ that occurs in a proposition, but rather φx, φy, φz, and so on; that is, the values of the function, not the function itself.

amendments, without the axiom of reducibility Russell is, like Chwistek and Weyl, unable to construct certain portions of classical mathematics.

Since the axiom of reducibility is not used, problems[25] arise in the construction of the type hierarchy. As noted above, it is "when the apparent variable is of higher order than the argument" (Whitehead and Russell 1925, p. xxxiv) that a problem arises. This corresponds to the consideration in the first edition of those statements that involve 'all properties of *a*' (as seen in the previous chapter). In this new construction, variables are introduced for each new order of function. So, for instance, φ_1 is used as a variable for a first order function and φ_2 is used as a variable for a second order function. Then matrices are used similarly as in the first edition. But in using new variables for these functions of different orders

> we shall obtain new functions
>
> $$(\varphi_2).f!(\varphi_2\hat{y},x),\ (E\ \varphi_2).f!(\varphi_2\hat{y},x)$$
>
> which are again not among values for $\varphi_2 x$ (where φ_2 is the argument), because the totality of values of $\varphi_2\hat{y}$, which is now involved, is different from the totality of values of $\varphi!\hat{y}$, which was formerly involved. However much we may enlarge the meaning of φ, a function of x in which φ occurs as apparent variable has a correspondingly enlarged meaning, so that, however φ may be defined,
>
> $$(\varphi).f!(\varphi\hat{y},x),\ (E\ \varphi).f!(\varphi\hat{y},x)$$
>
> can never be values for φx. To attempt to make them so is like attempting to catch one's own shadow. It is impossible to obtain one variable which embraces among

[25] The adequacy of this new procedure is discussed below.

its own values all possible functions of individuals (Whitehead and Russell 1925, p. xxxiv).

The axiom of reducibility is not needed here, it seems. This is the case since the order of the variables must be indicated before they can be used in the propositional function. Thus, no statements involving 'all properties of *a*' would occur since the 'properties' would be of some indicated order. Russell shows (Whitehead and Russell 1925, pp. xxv-xxvii) that "any general propositions about $\varphi!\hat{y}$ are also true about $\varphi_1\hat{y}$" and "this gives us, so far as such functions are concerned, all that could have been got from the axiom of reducibility" (Whitehead and Russell 1925, p. xxxvii). However, when attempting to show similar results for functions involving $\varphi_2\hat{y}$ this attempt falls short and Russell states that its failure to hold true in every case is illustrated by the "failure of the inference in connection with mathematical induction" (Whitehead and Russell 1925, p. xxxix).

That this attempt to replace the axiom of reducibility falls short is shown by Russell in the concluding paragraphs of the new introduction. The problems he illustrates arise in connection with the theory of classes. This theory he states to be "at once simplified in one direction and complicated in another by the assumption that functions only occur through their values and by the abandonment of the axiom of reducibility" (Whitehead and Russell 1925, p. xxxix). The theory of classes is simplified by noting that all functions of functions are extensional since in $f(\varphi\hat{y})$, φ can only occur through its values and if φx is equivalent to ψx, then the substitution of φx in f will give the same truth-value to f as ψx would. In this way the ambiguity which is attributed to classes in the first edition is dispensed with as "there is no longer any reason to distinguish between functions and classes" (Whitehead and Russell 1925, pg. xxxix). The problems arise since "classes of different orders composed of members of the same order" (Whitehead and Russell 1925, p. xxxix) must now be distinguished.

This new difficulty leads to the failure of several theorems and methods which one would want to be included in any account of mathematics. For instance, the proof of Cantor's theorem, the development of Dedekind cuts, and mathematical induction on the natural numbers all fail to be derivable within Russell's logical system. He remarks that those propositions in which it is to be proved that two classes are similar can be derived in a valid manner. On the other hand, unless at least one class is finite, the proofs that two classes are *not* similar, fail.

In conclusion, it is apparent from the above considerations that Russell did not view the second edition of *Principia Mathematica* as a complete treatise. Rather, it needed some serious work if it was to fulfill the initial goal of re-writing mathematics in terms of purely logical symbolism (with axioms that can be viewed as 'true' or self-evident). Since the abandonment of the axiom of reducibility has the consequence of sacrificing Dedekind cuts and thus collapsing analysis, it is concluded that some logical axiom must be found that will allow for the development of this important part of mathematics (Whitehead and Russell 1925, p. xlv).

3.4 Concluding Remarks

It has been seen in this chapter that Russell's theory of types experienced several revisions by different authors. The most common trend was to omit the axiom of reducibility. This trend continued for many years after, as the next chapter will show. Russell's final version of the theory of types, which he by no means saw as a completed work, was most notably influenced by Wittgenstein's new conception of functions. This new extensional attitude towards functions was taken one step further by Ramsey, who took extensionality as an all-embracing principle. Ramsey's work is considered in the next chapter, as is Carnap's response to some of his views.

4. Ramsey and Carnap on Types

4.1 Introduction

In this chapter Ramsey's and Carnap's works in the theory of types are investigated. Ramsey's distinction between the two different kinds of paradoxes was followed by all subsequent logicians. Using this distinction, he separated the simple from the ramified theory of types and set out to rework both of these in his own system. Ramsey redefined the notion of 'predicative functions' and in this way he was able to construct Russell's theory of types without the axiom of reducibility. However, the axiom of reducibility is not missed in his system since his new definition makes every propositional function predicative. As will be seen, Carnap rejected Ramsey's proposal since Ramsey's work relied heavily upon impredicative definitions. However, Carnap did include a simple type theory in several of his works. The role that this theory played in Carnap's work will be discussed in section 4 of the present chapter.

4.2 Ramsey and the Simple Theory of Types (Part I)

In Ramsey's two papers, "The Foundations of Mathematics" (1925) and "Mathematical Logic" (1926) Russell's logicism is amended and then defended as a philosophical foundation for mathematics. In the exposition of his theory, Ramsey points out the flaws, and what he sees to be their source, inherent in other attempts at finding a foundation for mathematics. The 1925 paper outlines Ramsey's own version of logicism, while pointing out the weaknesses of Russell's work. The 1926 paper is devoted to showing how the attempts by Weyl (who adopts a form of intuitionism) and Hilbert (with his formalism) to find a foundation of mathematics fail. It can thus be viewed as an extension of the 1925 paper in that it puts aside the exposition of Ramsey's own views, and instead investigates the failure of the other attempts in light of his new discoveries. Those which lie in his development of the theory of types will be discussed in this section[26] – it is only the axiom of reducibility and the notions to which this leads that are considered disastrous to Russell's theory of types.

Ramsey is thus able to retain the theory of types as a way of avoiding the contradictions, but he amends Russell's view of mathematics drastically enough that the axiom of reducibility can be dispensed with. Ramsey's methods for changing the theory of types all stem from his version of Wittgenstein's theory of propositions and functions. Russell had also adopted a version of Wittgenstein's theory in the second edition of *Principia*, as seen in the previous chapter. However, Russell apparently did not interpret Wittgenstein's views in the same way as Ramsey since their conclusions regarding the theory of types are so different. It will be shown that Ramsey takes a more drastic move towards extensionality than Russell had.

Ramsey considers the propositions of *Principia Mathematica* as falling into two categories: those that are expressed in words, and those that are expressed in logical symbolism. By the theory of types, Ramsey states that those expressed merely in words are "nearly all nonsense" (Ramsey 1925, p. 174). Ramsey claims furthermore, that all of those propositions that are expressed in symbols are tautologies (in Wittgenstein's sense),

[26] The other problems which Ramsey sees as facing Russell's account of the foundations of mathematics include the extensionality of functions, and his theory of identity.

except for one;[27] the axiom of reducibility. Thus, since Ramsey claims that mathematical propositions must be tautologies and completely generalized,[28] the axiom of reducibility has no place in mathematics for Ramsey.[29] This conception of mathematics Ramsey calls the "tautology theory" (Ramsey 1925, p. 177). This view does not differ drastically from Russell's as advocated in the introduction to the second edition of *Principia*.[30] Russell also hoped to dispense with the axiom of reducibility, but as will be shown below, his attempt failed precisely because he did not conceive of propositions in the same way as Ramsey.

After briefly introducing the "tautology theory", Ramsey proceeds to point out what he sees as three fundamental defects in *Principia Mathematica*. The second of these deals directly with the theory of types and its solution to the contradictions. The remainder of Ramsey's paper is devoted to "expounding a modified theory from which these defects have been removed" (Ramsey 1925, p. 184). Ramsey's investigation into

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[27] This is not quite so simple for the multiplicative axiom or the axiom of infinity. See the next section for a discussion of how Ramsey treats the logical status of these two axioms as well as for the definitions of 'tautology' and 'contradiction' which Ramsey borrows from Wittgenstein.

[28] In decreeing that propositions must be tautologies and completely generalized, Ramsey makes a claim not found in *Principia Mathematica*. Here Ramsey is urging certain restrictions upon both the form and the content of propositions. This new distinction is a key difference between the interpretation of propositions upheld by Ramsey and Russell.

[29] In his 1932 review of Ramsey's paper Church agrees with Ramsey's aim at avoiding the axiom of reducibility. However, he states that "we cannot agree with Mr. Ramsey, that the reason for the desirability of avoiding it is that the axiom is not a tautology in the sense of Wittgenstein, or that it is desirable or necessary that all the axioms of logic should be tautologies...Certainly the notion of a tautology loses much of its connotation of 'necessary' when we discover that the axiom of infinity is a tautology if it be true, but a contradiction if it be false" (Church 1932a, p. 356). Church's views will be explicated further in chapter 6.

[30] In his review of Ramsey's article, Russell states that "I agree with Ramsey in rejecting this view [that mathematics consists solely of those true propositions in which only mathematical or logical concepts occur], which I advocated in 'The Principles of Mathematics'. But it is no longer contained in *Principia Mathematica*, since the instance of the multiplicative axiom had shown its falsehood. At that time I had no definition of mathematical propositions; now, following Wittgenstein's definition of logic, I agree that they are tautologous generalizations" (Russell 1931, p. 477).

the deficiencies of the theory of types in *Principia* begins with a division of the contradictions into two categories.[31] He regards the principle by which he divides the contradictions as of "fundamental importance" (Ramsey 1925, p. 183). This distinction gives rise, in Ramsey's view, to two distinct parts of the theory of types; the one part deals with the first group of contradictions and the second with the second group.

Ramsey holds the first part of the theory of types found in *Principia* to be unquestionably correct. The contradictions of the first group are dealt with there, and they are "removed by pointing out that a propositional function cannot significantly take itself as argument, and by dividing functions and classes into a hierarchy of types according to their possible arguments. Thus the assertion that a class is a member of itself is neither true nor false, but meaningless" (Ramsey 1925, p. 187). This simple type hierarchy is what Ramsey holds to be indisputable, and it is to this that he attempts to reduce the rest of the theory of types. This is effected, as will be seen, by simply severing the other branch of the theory.

Whereas the first part of the theory distinguishes types of functions only by the types of their arguments, in the second part of the theory a further distinction is required. This distinction is made between the different functions that take the same arguments. This part of the theory of types requires the axiom of reducibility which Ramsey aims to avoid. However, according to the work done in *Principia*, one is left with the choice of either accepting the axiom of reducibility, or else eliminating a large portion of

[31] The first group of axioms involve only logical or mathematical terms, while those in the second group are not purely logical; they contain some reference to subjective terms such as 'thought', 'definability', and 'language'.

mathematical reasoning (including Dedekind cuts). Since neither of these options appeals to Ramsey, overcoming the need for such a dichotomy is of utmost importance to him; in particular, Ramsey sees the elimination of any reliance upon the axiom of reducibility as essential.

Ramsey deals with this "most serious" (Ramsey 1925, p. 195) objection to the foundational work done in *Principia* by re-defining the notions of a 'predicative function', 'functions of individuals', 'functions of functions of individuals', and so on. These new definitions are made through considerations of Wittgenstein's logical theories. Dealing with functions of individuals poses no problem in trying to eliminate the reliance upon the axiom of reducibility. In this context Ramsey elaborates Wittgenstein's theory of propositions viewing '$(x)\varphi(x)$' as the logical product of a set of propositions, and '$E(x)\varphi(x)$' as the logical sum of a set of propositions. Following Wittgenstein, it is possible for these logical products and sums to be infinite in length. This process is easily extended to propositional functions of two or more individual variables by holding one of the variables constant and quantifying over the other variable (where the function in which one variable is constant is viewed as a function of one variable).

After encountering no problems in dealing with functions of individuals, Ramsey attempts to apply the same procedure to functions of functions. This process runs into a problem almost immediately, and Ramsey concludes that a more precise definition of a 'function of functions' is needed to replace the definition given in *Principia.* He distinguishes two ways in which a definition can be given; either by the "subjective" or by the "objective" method. The subjective method is what Ramsey attributes to Russell's procedure. In this method the possible range of functions is defined as all those which can be constructed in a certain way. However, this method, as has been seen, leads to the need for the axiom of reducibility. Thus Ramsey pursues the objective method; with this method he is successful in overcoming the reliance upon the axiom of reducibility.

Instead of dealing with how functions can be constructed, Ramsey views functions of functions more like functions of individuals where the individuals are names (or possible names). So, in the analogue, the functions as arguments will be admitted if they have an appropriate meaning. As Ramsey states: "my method, on the other hand, is to disregard how we could construct them, and to determine them by a description of their meanings" (Ramsey 1925, p. 202). In this way, Ramsey allows functions as arguments with such meanings so that only predicative functions of functions will arise. These predicative functions, he goes on to show, encompass an even wider breadth of functions than those found in the predicative functions of *Principia*. Ramsey thus sees his method as more successful than that utilised by Russell. Thus, Ramsey uses this version of logicism as the prototype to be defended against intuitionism and formalism – the two main views opposing logicism at the time.

The new definitions introduced by Ramsey are clearly impredicative. This, together with his outright opposition to purely constructive methods makes his opposition to Weyl and Chwistek clear. Furthermore, Ramsey's view of logicism, it will be seen in section 4, faced serious objections by Carnap. Before looking at those objections, Ramsey's account of the logical status of the multiplicative axiom and the axiom of infinity is outlined in the next section.

4.3 Ramsey and the Simple Theory of Types (Part II)

Ramsey had to deal with both the multiplicative axiom[32] as well as the axiom of infinity[33] in order to maintain his thesis that all propositions of mathematics, expressible in logical symbolism, are tautologies. Ramsey claims that under Russell's interpretation, the first axiom is "logically doubtful" but under his own it is "an obvious tautology" (Ramsey 1925, p. 222). Furthermore, the axiom of infinity is shown to be either a tautology or else a contradiction under Ramsey's interpretation, but an empirical proposition under Russell's.

Before discussing how his interpretation leads to these results, Ramsey's definitions of 'proposition', 'tautology', and 'contradiction' will be outlined. According to Ramsey's (following Wittgenstein's) view that all propositions are truth-functions of elementary propositions it follows that all propositions are either 'tautologies', 'contradictions', or 'empirical':

> Given any set of n atomic propositions as arguments, there are 2^n corresponding truth-possibilities, and therefore 2^{2^n} subclasses of their truth-possibilities, and so 2^{2^n} truth-functions of n arguments, one expressing agreement with each sub-class and disagreement with the remainder. But among these 2^{2^n} there are two extreme cases of great importance: one in which we express agreement with all the truth-possibilities, the other in which we express agreement with none of them. A proposition of the first kind is called a *tautology*, of the second a *contradiction*.

[32] Namely, that given any class K of classes, there is a class with exactly one member in common with each member of K (Ramsey 1925, p. 220).
[33] Which states that there are an infinite number of individuals (Ramsey 1925, p. 222).

> Tautologies and contradictions are not real propositions, but degenerate cases (Ramsey 1925, pp. 172-173).

The genuine (non-tautologous and non-contradictory) proposition "asserts something about reality" (Ramsey 1925, p. 173) and can thus be called 'empirical'.

Using these definitions, Ramsey first deals with the multiplicative axiom. In his assertion of this axiom, he states that:

> If by 'class' we mean, as I do, any set of things homogenous in type not necessarily definable by a function which is not merely a function in extension, the multiplicative axiom seems to me the most evident tautology. I cannot see how this can be subject of reasonable doubt, and I think it never would have been doubted unless it had been misinterpreted. For with the meaning it has in *Principia*, where the class whose existence it asserts must be one definable by a propositional function of the sort which occurs in *Principia*, it becomes really doubtful and, like the Axiom of Reducibility, neither a tautology not a contradiction (Ramsey 1925, pp. 220-221).

This is shown as follows. Firstly, the multiplicative axiom is not a contradiction in *Principia* since it is possible that every class in Ramsey's sense is defined by an atomic function so there will be a class having one member in common with each member in K: this would also be a class in the sense of *Principia*. Secondly, the multiplicative axiom is not a tautology in *Principia*. This is shown to be the case by considering

> the equivalent theorem that any two classes are commensurable. Consider then the following case: let there be no atomic functions of two or more variables, and only the following atomic functions of one variable:
>
> Associated with each individual a an atomic function $\varphi_a(x)$ such that
>
> $$\varphi_a(x). \equiv_x .x = a$$
>
> One other atomic function $f\hat{y}$ such that $\hat{y}(fy)$, $\hat{y}(\sim fy)$ are both infinite classes. Then there is no one-one relation, in the sense of *Principia*, having either $\hat{y}f(y)$ or $\hat{y}(\sim fy)$ for domain, and therefore these two classes are incommensurable (Ramsey 1925, p. 221).

Thus, the multiplicative axiom is not a tautology in *Principia*.

On the other hand, the multiplicative axiom is "an obvious tautology" (Ramsey 1925, p. 222) under Ramsey's interpretation. He makes this claim and then refutes the idea that, if the multiplicative axiom is a tautology, then it should be provable in his system. This is done in Ramsey's statement that

> it does not seem to me in the least unlikely that there should be a tautology, which could be stated in finite terms, whose proof was, nevertheless, infinitely complicated and therefore impossible for us. Moreover, we cannot expect to prove the Multiplicative Axiom in my system, because my system is formally the same as that of *Principia*, and the Multiplicative Axiom obviously cannot be proved in the system of *Principia*, in which it is not a tautology (Ramsey 1925, p. 222).

This is a key passage to which Carnap holds exception in his 1931 paper, as will be seen in the next section.

Ramsey further shows that under Russell's interpretation of identity, the axiom of infinity will be an empirical proposition. This follows from the following considerations: in *Principia*, due to the definition of identity, the axiom of infinity means that there are an infinite number of distinguishable individuals. This is empirical, since "even supposing there to be an infinity of individuals, logic cannot determine whether there are an infinity of them no two of which have all their properties in common" (Ramsey 1925, p. 222). Comparatively, in Ramsey's system, the axiom of infinity is either a tautology, or it is a contradiction. Ramsey claims that this difference in the status of the axiom of infinity in the two interpretations stems from the different account of identity,[34] and the fact that his own system admits functions in extension. Ramsey notes that in his interpretation, the axiom of infinity will be a tautology in those systems in which the universe has an infinite number of individuals, and a contradiction otherwise. He thus admits that "in the logic of the whole world, if [the axiom of infinity] is a tautology, [it] cannot be proved, but must be taken as a primitive proposition. And this is the course which we must adopt, unless we prefer the view that all analysis is self-contradictory and meaningless. We do not have to assume that any particular set of things, e.g. atoms, is infinite, but merely that

[34] In his 1931 review of Ramsey's paper, Russell objects to Ramsey's treatment of identity where 'identity' is eliminated altogether using the convention that different letters are to represent different objects. Russell objects to this by stating that "this is possible, but very inconvenient, and makes it impossible to find a defining concept for a finite set of objects given by enumeration" (Russell 1931, p. 477). Church also objects to Ramsey's treatment of identity in his 1932 review where he states that "if x and y are two things which have all their properties in common, and if we allow that x has the property of being identical with x, then we must allow that y also has the property of being identical with x, that is, that $y=x$" (Church 1932a, p. 356). Ramsey had taken this treatment of identity from Wittgenstein.

there is some infinite type which we can take to be the type of individuals" (Ramsey 1925, p. 224). Thus, Ramsey justifies the acceptance of the axiom of infinity on pragmatic grounds and holds that it is either a tautology or a contradiction (as opposed to an empirical proposition).

4.4 Carnap's Logical Syntax

Just as Ramsey had been, Carnap was greatly influenced by Wittgenstein's work. Carnap had no objection to Ramsey's support of the "tautology theory" as described above. However, Carnap did object to other aspects of Ramsey's work, most notably his acceptance of impredicative definitions. These objections, together with Carnap's own incorporations of the theory of types are listed in this section.

In his 1931 paper, Carnap proposes that not only arithmetic, but also set theory and higher branches of mathematics be constructed type-theoretically. In this paper he highlights certain problems facing the logicist account of mathematics. Most notably, this includes the problem of "develop[ing] logic if, on the one hand, we are to avoid the danger of the meaninglessness of impredicative definitions and, on the other hand, are to reconstruct satisfactorily the theory of real numbers" (Carnap 1931, p. 49). Carnap views Ramsey's attempted solution of this problem as a failure.

Carnap notes that Ramsey attempts to provide a solution by simply allowing impredicative definitions; this is done by claiming that although they contain a circle, it is not a vicious, but rather an admissible circle. Carnap dismisses Ramsey's solution by stating that "we should call Ramsey's mathematics 'theological mathematics,' for when he speaks of the totality of properties he elevates himself above the actually knowable and definable and in certain respects reasons from the standpoint of an infinite mind which is not bound up by the wretched necessity of building every structure step by step" (Carnap 1931, p. 50). Carnap is most certainly referring here to Ramsey's appeal to the "infinitely complicated" proof as cited above. Here Carnap would like to see a more constructive foundation of mathematics along the lines of Weyl and Chwistek.

Carnap was led to seek his own solution to this problem in his *Logical Syntax of Language*. In this work Carnap studies two formal languages. The second of these contains a form of type theory. It is by following his Principle of Tolerance, namely that any language deserves study for its own sake, that Carnap is able to justify studying this seemingly Platonistic system. Since this is just one language that Carnap is studying, there is no appeal to any higher realm of beings as there must be in Ramsey's work. In order for this to be the case Carnap must posit the Principle of Tolerance as the basis of his philosophy of mathematics.[35]

4.5 Concluding Remarks

It has been seen in this chapter that Ramsey initiated several important steps in the theory of types. His distinction between the kinds of paradoxes as well as his use of the simple theory of types proved to be very influential on subsequent work on the theory of types. This is the case despite some reservations concerning his methods of justifying his

[35] See Ferreiros 1997, pp. 97-99 for more on this point.

work. In the next chapter the first truly formal presentations of the simple theory of types will be given. These occurred in the monumental works of Kurt Gödel and Alfred Tarski.

5. Tarski and Gödel: First Formal Treatment of the Theory of Types

5.1 Introduction

In 1931 Kurt Gödel and Alfred Tarski each presented a version of the simple theory of types. Tarski made use of this theory in his famous paper on truth, while Gödel employed his theory of types in his incompleteness paper. Both of these theories represent the first formal formulations of the simple theory of types. They subsequently became the standard by which many logicians judged the strength of their systems. Thus, although these versions of the simple theory of types are based upon ideas from *Principia Mathematica* their succinct presentation and ease of use made them ideal for further investigations in the theory of types. Although his presentation of the theory of types was so influential, Tarski soon became dissatisfied with that theory. In this chapter both Gödel's and Tarski's formulations will be presented together with an investigation into Tarski's shift from using the theory of types as his logical basis to his using first-order logic together with Zermelo-Fraenkel set theory for that purpose.

5.2 Tarski – Theory of Truth

Becoming dissatisfied with the common usage of the notion of truth, Tarski sought to clarify it. Tarski used the theory of types in his paper on truth which was written in 1931, published in 1933, and based upon lectures that he had given in 1929. In this paper Tarski sets up his version of the simple theory of types and then defines "σ is true in A" inside of that theory. In doing so he attempted to develop a clear notion of truth for formalized languages that was "materially adequate and formally correct" (Tarski 1956, p. 152). Tarski upheld the doctrine that the semantics of an object language must be of a higher-order than the object language itself. The object language may contain its own syntax and names for all of its own expressions. However, according to Tarski's investigations, it cannot contain specifically semantic terms such as 'satisfaction' or 'truth'. In this way Tarski set up a hierarchy of languages. This became the most commonly employed method of avoiding the semantic paradoxes.

The variables of Tarski's system are distinguished by the use of numerical indices. Those with index 1 have individuals as values, those with index 2 have classes of individuals as values, those with index 3 have classes of classes of individuals as values, and so on. With this type-distinction of variables elementary formulae are built up as follows: '$x_i \varepsilon y_j$' is an elementary formula provided that $j=i+1$; further formulae are obtained by replacing already formed formulae for 'P' and 'Q' in '~P', 'P→Q' and '(x)P'. Tarski then gives a list of rules of inference. His axioms include an axiom of comprehension, as well as an axiom of extensionality.

It is clear that Tarski's presentation can be viewed as a formal treatment of the theory of types, as will Gödel's treatment in the next section. The differences between these two theories will be pointed out in the next section. It should be noted here that in a footnote appended to the German edition of this paper Tarski states that he moved from the theory of types to first-order logic with Zermelo-Fraenkel set theory as the optimal system in which he would subsequently work (Ferreiros 1997, p. 101). Tarski mentions this language as "a much more convenient and actually much more frequently applied apparatus for the development of logic and mathematics" (Tarski 1956, p. 271). The theory of types had been used by Tarski in four of the articles from his 1956 book, all dating prior to 1935. Moreover, with this statement he makes the reason for his switch to first-order logic clear.

5.3 Gödel – Incompleteness Theorems

The simple theory of types presented by Kurt Gödel (Gödel 1931, pp. 599-601) is very similar to that set forth by Tarski. In fact, when referring to these systems as the prototypes of type-theory, Quine would often list them together (Quine 1985, p. 86; Quine 1935, p. 164).

Gödel included '~' (not), 'v'(or), '∏'(for all), '0'(zero), '*f*'(the successor of), and '(', ')'(parentheses) as his primitive constants. Then he indexed his infinite list of variables in such a way that the indices refer to the type of the value which the variable can take. In particular, x_1 refers to an individual, x_2 to a class of individuals, x_3 to a class of classes of individuals, and so on.

Gödel's elementary formulae are of the form $x_{i+1}(x_i)$ together with the usual definitions for '$\sim a$', 'avb', and '$\prod x(a)$'. Finally the two axioms pertaining to the theory of types are given. These are the axiom of comprehension and the axiom of extensionality. This system is clearly quite similar to Tarski's. The difference between Tarski and Gödel's systems lies in the fact that Tarski uses the membership symbol, ε, as a primitive symbol while Gödel does not. However, as noted in Ferreiros' paper from 1997, Gödel's version can be 'quined' so that '$u(v)$' is written as '$v \varepsilon u$'.

5.4 Concluding Remarks

The two systems outlined in this chapter are very similar. The importance of these versions of the simple theory of types will be seen in the next chapter. There, Quine, one of the most influential logicians during the 1930's and later, uses Tarski and Gödel's simple theory of types as the standard by which to judge the worth of his work. In fact, he used Tarski's formulation as the starting point for his further investigations in which he attempted to improve upon the theory of types. The resulting systems provided much stimulation for future logicians as the seventh chapter of this thesis will highlight.

6. Church and Quine on Types

6.1 Introduction

In this chapter Alonzo Church and W.V. Quine's works on the theory of types will be examined and compared. Alonzo Church only wrote one paper dealing exclusively with the theory of types. This paper proved to be quite influential since it incorporated Church's important lambda-operator into a simple type theory. Church had stated his famous thesis in 1936 where the intuitive notion of 'calculable function' is equated with the well-defined class of 'recursive functions'. Furthermore, this class of recursive functions was shown to be the same as the class of Church's 'lambda-definable functions'.[36] Much work was done using the lambda-operator (which acts as a function abstractor) by not only Church, but also his students Kleene and Rosser. As will be seen in the next chapter, work was done not only on the lambda-operator in isolation, but also on Church's system that incorporates that operator into a simple theory of types.

In his early works Church tried to find a logical system that avoided any mention of types. He was well aware, as he noted in his 1928 review of *Principia Mathematica*, that the "theory of types...affords the best known method of dealing with the 'well known contradictions'" (Church 1928, p. 240). However, this theory did not satisfy his notion of an ideal foundation of mathematics. It was only after his logical system, which avoids the

[36] For an in-depth historical survey of the origins of Church's thesis see G.H. Adam's *A History of the Theory of Recursive Functions and Computability With Special Reference to the Developments Initiated by Gödel's Incompleteness Theorems* (1983).

theory of types, was shown to be inconsistent[37] that Church began to show a serious interest in developing his own version of the theory of types. This interest culminated in his 1940 paper "A Formulation of the Simple Theory of Types". This paper, together with his later views (especially in his 1956 book *Introduction to Mathematical Logic*) will be outlined.

Quine's work is also considered in this chapter. As opposed to Church, Quine embraced the theory of types in his early work. Quine realized, as Church had in his 1928 review, that the theory of types was the standard by which other systems should be judged. However, Quine was also not completely satisfied with the theory of types. Rather than taking Church's approach of supplanting the theory altogether, Quine aimed to amend the use of types in his systems.

6.2 Church on Types

Alonzo Church's views regarding the theory of types seem to have changed over the early part of his career. Firstly, in his 1928 review of *Principia Mathematica* Church states that many difficulties arise "in connection with the theory of logical types" (Church 1928, p. 239) and further that "we hope to see [the theory of types] supplanted or greatly modified" (Church 1928, p. 239). He notes two more times in that same short review that "difficulties [are] raised by the theory of types" and that certain theorems can be proved only "if we disregard restrictions imposed by the theory of types" (Church 1928,

[37] This was done in Kleene and Rosser's 1935 paper.

p. 240).[38] He concludes this review by claiming that "question[s] will ultimately be settled by the complete abandonment of the theory of logical types or by an alteration in it more radical than any yet proposed" (Church 1928, p. 240). An almost contradictory statement is made by Church at the end of this review where he states that "the theory of types ...affords the best known method of" dealing with the "well known contradictions" (Church 1928, p. 240). However, this statement simply shows that at this point Church had not found any alternative to the theory of types that would adequately replace it. This desire to replace the theory of types would guide Church's later works to a great extent. Church thus makes his view of the status of Russell's theory of types quite clear in this review. Similar statements permeate not only his reviews, but also some of his own articles, as will be shown.

For instance, Church's distaste for the theory of types is stated clearly in his 1932 paper where he attempts to formulate a system that is sufficient for deriving a large part of mathematics, while avoiding the logical paradoxes without using the theory of types.[39] Church even states that Zermelo and Russell's approaches to avoiding the paradoxes are "somewhat artificial" (Church 1932, p. 347). However, the trend of criticizing the theory of types is not present in Church's review of Ramsey's *The Foundations of Mathematics*. In fact, Church commends Ramsey's work, noting that the distinction between two

[38] Although Church mentions many times that there are problems with the theory of types, he only mentions one "awkward situation" explicitly. This can be found in the following passage. "Having proved the theorem that we require about functions of the first n-1 types [in vol. II of *Principia*], then in order to obtain the same theorems about functions of the nth type we must make a new assumption of all our postulates, applying them to functions of the nth type instead of functions of some lower type, and must then prove all our theorems anew. We 'see,' by symbolic analogy, that this can always be done. But the statement that this is so is impossible under the theory of types" (Church 1928, p. 240).

[39] In this paper, a system of logic is set up including the lambda-abstract operator. The system as a whole was abandoned after it was shown to be inconsistent and from it only the lambda-calculus was preserved. This calculus is incorporated into Church's 1940 version of the simple theory of types as discussed below.

different types of paradoxes, and thus the formation of two different hierarchies of types, one to deal with each of the sets of paradoxes, is quite important. Furthermore, he agrees with the desire to avoid the axiom of reducibility in the ramified theory, but disagrees with Ramsey's reasons for finding such an avoidance desirable.[40]

In his 1935 review of Quine's *A System of Logistic*, Church discusses what he see to be the six most important changes that Quine makes to the system of logic found in Whitehead and Russell's *Principia Mathematica*. The fourth of these is "a liberalization of the theory of types, by which the axiom of reducibility is rendered unnecessary" (Church 1935, p. 598). Thus, Church continues his praise of the method initiated by Ramsey, in which the theory of types is modified in such a way so as to avoid the axiom of reducibility. Church does not mention any difficulties with the theory of types at all here, and actually states that this "work of Quine is in both respects [in the formal definiteness and mathematical elegance] an important improvement over the system of *Principia*, and, although open to criticism in certain directions, is probably not too highly praised by Whitehead when he calls it, 'a landmark in the history of the subject'" (Church 1935, p. 603).

Furthermore, Church sees Quine's propositional functions in extension[41] as being an important modification in the theory of types in the following way: basically, since Quine's propositional functions in extension (classes) appear as complete symbols (as

[40] This is stated clearly by Church in the following passage: "Distrust of the axiom of reducibility is, of course, widespread, being shared even by the authors of *Principia Mathematica*, and there seems to be no doubt of the desirability of a theory which avoids this axiom. But we cannot agree with Mr. Ramsey, that the reason for the desirability of avoiding it is that the axiom is not a tautology in the sense of Wittgenstein, or that it is desirable or necessary that all the axioms of logic should be tautologies" (Church 1932a, p. 356).
[41] Quine calls these classes, but Church calls them propositional functions since, as he notes, they correspond more correctly with the propositional functions, as opposed to the classes, found in *Principia*.

opposed to classes being incomplete symbols in *Principia*), Church states that they are more adequately compared with the propositional functions of *Principia*. Church claims that in this way, "Quine has really made an important modification in the theory of types,[42] in a direction which seems to have been first suggested by F. P. Ramsey" (Church 1935, p. 601).[43]

Church continues to praise the theory of types in his 1937 review of Chwistek's paper. There Church refers to the simple theory of types (the first version of which Chwistek had published in 1921) as the "now widely accepted simple theory of types" (Church 1937, p. 169). Chwistek had originally proposed, in 1912, that the proper remedy for the paradoxes and the unacceptability of the axiom of reducibility is simply to reject that axiom while accepting the ramified theory of types without that axiom. Diverging from his 1921 formulation of the simple theory of types, one year later Chwistek returned to his 1912 proposal for developing a ramified theory of types without the axiom of reducibility. Church notes that "it is well known that the Richard paradox does arise upon incorporation into the theory of symbols for certain semantical concepts" (Chruch 1937, p. 169) and this leads him to state that, since objections can still be brought

[42] The importance comes from the fact that Quine is able to avoid the paradoxes without using the axiom of reducibility, but rather by changing the way in which propositional functions are to be dealt with.
[43] However, in his 1932 review of Ramsey's work Church questioned Ramsey's use of extensional functions. Church states that Ramsey's "abandonment of the principle that x and y are identical (or equal) when every propositional function satisfied by x is also satisfied by y...[is] open to serious objection" (Church 1932a, pp. 355-356). Church continues to state that Ramsey's proof that the axiom of infinity is a tautology (if it is true) does not depend on Ramsey's revision of his notion of identity (as Ramsey suggests), but rather upon "the introduction of propositional functions in extension. This notion of a propositional function in extension is certainly legitimate, but it seems doubtful whether the distinction can successfully be maintained between ordinary propositional functions and propositional functions in extension" (Church 1932a, p. 357). Thus, although he does not make an explicit statement against the reliance upon extensionality in logic, Church does seem to be a bit wary of its fruitfulness here.

against the axiom of reducibility, Chwistek's proposal to develop a ramified theory of types without the axiom of reducibility has a strong appeal.

Furthermore, Church notes that "the current view of advocates of the simplified theory of types whereby the relation between a concept and the symbol which denotes it must appear, not within the original language, but within a meta-language containing the original language, may itself be regarded as a kind of ramified type theory (the distinction between a hierarchy of languages and a hierarchy of types within one language being here a matter of terminology)" (Church 1937, p. 170).[44] Church also commends Chwistek's attempt to incorporate notations for both concepts of syntax and those of semantics into one system. Church claims that "such a system is no doubt consistently possible, on the basis of the ramified theory of types, and its development should be of considerable interest" (Church 1937, p. 169). The comparison between a hierarchy of types and a hierarchy of languages as well as the reconsiderations of the ramified theory of types will be found in the next chapter of this thesis.

After reviewing all of these different formulations of the theory of types, Church formulates his own version of the theory of types in his 1940 paper. This version is altered in a supposedly sufficient manner so as to avoid the problems that he sees as belonging to other formulations of the theory of types.[45] In particular, Church attempts to improve the theory by incorporating "certain features of the calculus of lambda-conversion" (Church 1940, p. 56) that are to be found, for example, in his 1941

[44] Copi also suggests this in his 1950 paper as discussed in the next chapter.
[45] He never explicitly points out why his version is better than any previous formulations; he only states that it "has certain advantages" (Church 1940, p. 56).

monograph (which was written almost exclusively in 1936).[46] Certain aspects of his version of the simple theory of types, as found in the 1940 paper, will now be briefly outlined.

Church begins his formulation of the theory of types by explaining what is meant by a hierarchy of types as well as providing rules for generating the type symbols. In Church's formulation, two undefined type symbols, ι and o, are first posited, and from these, through an inductive definition, further type symbols are defined. The class of type symbols is "the least class of symbols which contain the symbols ι and o and is closed under the operation of forming the symbol (ab) from the symbols a and b" (Church 1940, p. 56).[47] The type symbols enter into Church's system as subscripts for variables and constants. In the intended interpretation, a subscript indicates the type of the variable or constant to which it is subscribed, where o is the type of propositions and ι the type of individuals. Furthermore, "(ab) is the type of functions of one variable for which the range of the independent variable comprises the type b and the range of the dependent variable is contained in the type a" (Chruch 1940, p. 57).

After introducing the method by which types are to be formed, Church goes on to give a description of what are to comprise his well-formed formulas. This begins with an

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[46] He notes that "for features of the formulation which are not immediately connected with the incorporation of lambda-conversion, we are heavily indebted to Whitehead and Russell [Russell's *Mathematical logic as based on the theory of types*, **American journal of mathematics**, vol. 30 (1908) pp. 222-262; Whitehead and Russell's *Principia Mathematica* vols. 1, 2, and 3] , Hilbert and Ackermann [*Grundzuge der theoretischen Logik* 1928, and second edition 1938], Hilbert and Bernays [***Grundlagen der Mathematik***, vo. 1 1934, vol. 2 1939], and to forerunners of these, as the reader familiar with the works in question will recognize" (Church 1940, p. 56).

[47] At this point Church lists several abbreviations and notes a few conventions to be used in shortening the type symbols. Firstly, he notes that parentheses are to be omitted with the convention that association is to the left. Also, he uses a' as an abbreviation for $((aa)(aa))$, a'' as an abbreviation for $((a'a')(a'a'))$, etc.

infinite list of *primitive symbols* including three *improper symbols*. These improper symbols are 'λ', '(', and ')'. The other symbols are the constants N_{oo}, A_{ooo}, $PI_{o(o\alpha)}$, $\iota_{\alpha(o\alpha)}$ and an infinite list of variables ranging over the different types.[48] Within the intended interpretation, Church mentions in what manner he wishes the natural numbers to be constructed in each type. In doing this it is implicit that a copy of the natural numbers is to occur within each type. He defines a symbol, $T_{\alpha''\alpha'}$, within the system which, when applied to a natural number of type α', will denote the same number of type α''. In this way, Church's simple theory of types seems to have an advantage over other systems; namely, in that there is some sort of "communication" between the reduplicated numbers that occur within each type.[49]

Incorporating the use of type subscripts, Church lists the rules for lambda conversion as well a rule of substitution, *modus ponens*, and a rule for generalization. Church lists 11 axioms and axiom schemas in total. The first four "suffice for the propositional calculus" (Church 1940, p. 61), while the first 6 suffice for the logical functional calculus. Church states further that "in order to obtain elementary number theory it is necessary to add (to 1-6^{α}) Axioms 7, 8, and 9^{α}" (Church 1940, p. 61), where the superscripts indicate that the axiom is actually an axiom schema (ranging over the type 'α'). Axioms 7 and 8 together "have the effect of the *axiom of infinity*, while the

[48] In the intended interpretation, 'λ' is to act as an abstraction operator, N_{oo} denotes negation, , A_{ooo} denotes disjunction, $PI_{o(o\alpha)}$ denotes the universal quantifier "as a propositional function of propositional functions," $\iota_{\alpha(o\alpha)}$ denotes a "selection operator (as a function of propositional functions)," and juxtaposition is to denote "the application of a function to its argument" (Church 1940, p. 59).
[49] In his review of Church's paper Quine mentions this as a great advantage.

axioms 9^{α} are *axioms of description*" (Church 1940, p .61).[50] Finally, he states two axioms schemas that are not used in the rest of the paper. These schemas correspond to the axioms of extensionality and the axioms of choice. He notes that these axioms are necessary in "order to obtain classical real number theory (analysis)" (Church 1940, p. 61). Since the rest of the paper deals only with arithmetic, these axioms are not considered any further. However, Church does make some further comments regarding the axioms of extensionality.

As Quine comments in his review of Church's 1940 paper, Church "withholds the general extensionality principle" (Quine 1940, p. 114). Indeed, Church says himself that his axioms $10^{\alpha\beta}$ are weaker in some ways than other axioms of extensionality, but that classes can still be introduced that can be associated with a propositional function $F_{o\alpha}$. This is done by essentially defining the classes using certain properties (using 0 if the class is to have the property and 1 if not). Church also points out that if one took the axiom of extensionality which states that if two propositions are equivalent, then they are equal, then a different notion of class arises. Namely, classes come to be identified with propositional functions. Thus, here Church does make use of extensionality to some extent, but again it is limited, and does not play a key role in his work.

It has been shown above that over the years Church's view of the theory of types changed drastically. He began by dismissing that theory in its entirety, seeing it as

[50] It is here that Church states the independence of the axiom of infinity (Axiom 7 which states that there are two elements that are different from one another, together with Axiom 8 which states that if two numbers have the same type, then if their successors are equal, then those numbers must themselves be equal): "the independence of Axiom 7 may be established by considering an interpretation of the primitive symbols according to which there is exactly one individual, and that of Axiom 8 by considering an interpretation according to which there are a finite number, more than one, of individuals" (Chruch 1940, p. 61).

something that should be either completely abolished, or else altered in such a way so as to make it almost incomparable to its original formulation. This view gradually changed, so much so that Church actually began doing his own research into various aspects of the theory of types. Church's treatment of the ramified theory of types as found in his *Introduction to Mathematical Logic* (1956) will be outlined together with a statement of his view of the axiom of reducibility found therein.

In section 58 of his *Introduction to Mathematical Logic*, Church gives a detailed account of both a predicative and ramified functional calculus of second order. He proceeds firstly by formulating the predicative functional calculus of second order. This is done roughly as follows: Seven axiom schemata are listed, all of which have a relation to corresponding axiom schemata given by Church for the functional calculus of second order.[51] The only difference occurs in the axioms that deal with the substitution of variables in formulas: in the predicative functional calculus the variable which is to be substituted must contain no bound propositional or functional variables whereas no such restriction exists in the regular second order calculus. The ramified version begins with the predicative version of the functional calculus, but proceeds to complicate matters by introducing, besides orders, also *levels*. This is done in the following way:

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[51] These axioms are (notational differences): 1) A → (B→A), 2) A→((B→C)→((A→B)→(A→C))), 3) (~A→~B)→(B→A), 4) (a)(A→B)→(A→(a)B), 5) (a)A→B where B is the result of substituting b for a throughout A where a is an individual variable, b is an individual variable or an individual constant, and no free occurrence of a in A is in a well-formed part of A of the form (b)C., 6) (p)A→B where B is the result of substituting q for p throughout A where p is a propositional variable, and q contains no bound propositional or functional variables., 7) (f)A→B where B is the result of substituting q for $f(x_1, x_2,\ldots, x_n)$ throughout A where f is an n-ary functional variable, and $x_1, x_2,\ldots, x_n$ are distinct individual variables, and q contains no bound propositional or functional variables (Church 1956, pp. 348-349).

> In the ramified functional calculi of second order and higher levels, additional propositional and functional variables are introduced, of successively higher levels, the leading idea being that in substituting for a propositional or functional variable of given level, the wff B [this corresponds to '*q*' in my footnote 51] which is substituted may contain bound propositional and functional variables of lower levels only. Thus the ramified functional calculus of second order and second level...contains propositional and functional variables of the first level and of the second level (Church 1956, pp. 349-350).

Similarly, the ramified functional calculus of second order and third level contains propositional and functional variables of three different levels, and so on. The axioms for this ramified calculus of second order correspond analogously to the axioms for the functional calculus of second order. In fact, the first five axioms are identical. The only difference is in the last two, where further restrictions are placed upon the variables which can be substituted. In the sixth axiom,[52] the bound functional variables of q are all of level lower than that of p, and the free propositional and functional variables of q are of level not higher than that of p. In the seventh axiom, the bound propositional and functional variables of q are all of level lower than that of f, and the functional constants and the free propositional and functional variables of q are all of level not higher than that of f (Church 1956, p. 352).

Church explains how the introduction of these *levels* alters the representation of the ramified theory of types as given in *Principia*. Essentially, in the second-order

[52] See previous footnote for axioms six and seven.

functional calculi, the level of Church's propositional or functional variables corresponds exactly with what is called the order of a variable in *Principia.* However, in connection with the ramified functional calculi of higher order, what is understood by *level* by Church is what is called in *Principia* the amount by which the order of the functional variable exceeds that of the highest order of any of the variables which may be found in the argument places of that function.

Section 59 of Church's *Introduction to Mathematical Logic* contains a brief account of the axioms of reducibility, but this is included solely "because of their historical importance" (Church 1956, p. 355). The axioms are listed as a doubly infinite list,[53] and Church indicates that they are not all independent. He justifies this claim by stating that "those which contain singulary functional variables can be proved by using those which contain binary functional variables, and so on down the list. Also, among those which contain *n*-ary functional variables (with fixed *n*), it is obvious that one in which F is of lower level can be proved by using one of those in which F is of higher level" (Church 1956, p. 356). Finally, Church insists that the ramified theory of types together with these axioms of reducibility is not an interesting alternative to the pure ramified theory or the simple theory of types.[54]

As will be seen in the next chapter, Church's version of the theory of types proved to be quite influential. There the work by Alan Turing, J. Richard Buchi, and Maurice L'Abbe will receive special attention.

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[53] This list is given such that for every function of an arbitrary higher level than the first, there exists a functional variable of the first level such that the two functions are equivalent for every argument. Then this list is extended for functions of one variable, two variables, and so on.

[54] This was later dismissed in Church's 1976 paper which lies outside the scope of this thesis.

6.3 Quine on Types

In this section Quine's two major contributions to the theory of types will be discussed. These are the systems found in his 1937 paper "New Foundations for Mathematical Logic" and his book from 1940 entitled *Mathematical Logic*. The system postulated in the second of these works was an extension of the first. This will be made apparent in what follows.

a) New Foundations

Although his Ph.D. dissertation was based upon the theory of types, Quine soon encountered many problems plaguing that theory which he wished to see changed. As he notes in his "The Inception of 'New Foundations'" (1987), despite the fact that the theory of types had been given a neater formulation by Alfred Tarski and Kurt Gödel, "still I was unhappy with types. One unattractive feature was the arbitrary grammatical exclusiveness. Seemingly intelligible combinations of signs were banned as ungrammatical and meaningless" (Quine 1987, p. 281). Thus Quine tried to formulate a theory in which the meaningful formulas of the theory of types could be generated, while those meaningless ones were simply unaccounted for by the definitions (and hence, would just never arise).

Quine's first attempt at this is found in his 1936 paper "Set Theoretic Foundations for Logic". In that paper Quine took Zermelo's set theory as his starting point and reduced and varied some of its primitive notions; the variables range over both individuals and classes indiscriminately. Quine went on to show that the theory of types can be simulated in this theory using contextual definitions. Furthermore, an axiom of infinity was listed as an axiom that can be added to the original list of axioms.

The arbitrary omission of certain combinations of signs was not the only problem that Quine saw as plaguing the theory of types. He was also displeased with the infinite reduplication of numbers, logical constants, and every other mathematical object in every higher type. Unfortunately, this second problem remained in the system proposed in his "Set Theoretic Foundations for Logic". Also, Quine often struggled in his work to include a universal class, which was missing from this work. Thus Quine was forced to pursue a different approach to these problems.

In his "New Foundations for Mathematical Logic" (1937) Quine set to work at avoiding these problems. The system introduced there aims at avoiding reliance upon the theory of types, while retaining the strength of the system set forth in *Principia*; this is done by avoiding any specific reference to types, while forming restrictions upon his axioms based on type restrictions. For instance, two of the axioms that are employed in this system are the axioms of extensionality and of comprehension. The axiom of extensionality is stated as: $(x)(y)[(z)(z\varepsilon x\leftrightarrow z\varepsilon y)\rightarrow x=y]$, and the axiom of comprehension is $(\mathrm{E}y)(x)(x\varepsilon y\leftrightarrow\varphi(x))$ where $\varphi(x)$ is a stratified formula in which y does not occur free. Now, a formula is said to be stratified if all of the variables can be assigned indices in such a way as to be consistent with the theory of types. Furthermore, Quine's system uses the primitives '|' for alternative denial, 'ε' for class membership, and '(x)' for universal quantification. Formulas are described recursively, and in this way the translatability of *Principia* into the new system is shown by translating the primitive formulas of *Principia* into the recursively defined formulas of Quine's system.

The system used by Quine contains three more rules specifying the initial theorems, and finally two rules for "inferential connections" (Quine 1937, p. 77).[55] Quine uses only the extensionality postulate and R3 (mainly R3) in order to derive Russell's paradox. R3 is the unrestricted comprehension principle. The theory of types is explained as stratifying the objects of the system into types so that "an expression which would be a formula under our original scheme will hence be rejected as meaningless by the theory of types only if there is no way whatever of so assigning types to the variables as to conform to [the requirement on epsilon that it only connects variables of ascending types]" (Quine 1937, p. 78). Formulas in which the variables can be numbered so that epsilon occurs only between variables numbered n and $n+1$ are called *stratified*. An equivalent way of explaining these formulas is through the use of epsilon-chains. So a formula is said to be stratified if it has no epsilon-cycles. Quine claims that this definition "has the advantage of affording an immediate criterion, since the epsilon-chains of stratified formulas are readily exhausted" (Quine 1937, p. 78). The theory of types then consists of the statement that the system will omit all unstratified formulas.

The theory of types however has several "unnatural and inconvenient consequences" (Quine 1937, p. 78). These Quine lists as follows: "the theory allows a class to have members only of uniform type, the universal class V gives way to an infinite

[55] The five rules are labeled R1 to R5. "R1 and R4 are an adaptation of the propositional calculus as systematized by Nicod and Lukasiewicz...R2 and R5 contribute the technique for manipulating the quantifier" (Quine 1937, p. 77).

series of quasi-universal classes, one for each type. The negation $-x$ ceases to comprise all non-members of x, and comes to comprise only those non-members of x which are next lower in type than x"(Quine 1937, p. 79). He also notes that all logical constants, as well as all arithmetical constants are reduplicated within each type. Thus Quine wishes to set up an alternate way of avoiding the paradoxes which does not succumb to these unnatural effects.

Quine attempts to avoid the paradoxes by modifying only R3 of his system; the result is the comprehension principle noted above (called R3′). He states that none of the unnatural properties belonging to the theory of types belongs to his system. Even though he avoids mention of type, his methods are motivated by the considerations of types. In fact, the method used for avoiding the paradoxes seems to mimic the theory of types quite closely; yet, it is only the forms of certain sentences that are restricted. Essentially, in *New Foundations* Quine imposes the restrictions of the theory of types solely upon the form of allowable sentences, as opposed to the meanings of the constituents of those sentences. Thus, it is not the language itself which is restricted, but rather what can be formed using the language. Certain ideas from the theory of types are used, but in practise it seems that no mention of types is necessary for Quine. However, it does seem that his system will be unable to derive Cantor's theorem.[56] The question as to how far this is true is dealt with in another of his papers, namely "On Cantor's Theorem" (1937a).

Since Cantor's theorem seems to be disprovable in the *New Foundations*, Quine deems it necessary to investigate the reasons for this (in hopes of avoiding this

[56] Namely the theorem which states that the class of subclasses of a class, W, has a greater cardinality than W.

undesirable aspect of his system). This investigation comprises the subject-matter of his paper entitled "On Cantor's Theorem". The disprovability of Cantor's theorem in *New Foundations* leads Quine to question the consistency of his system. He actually states in "On Cantor's Theorem" that his system "offer[s] less assurance of consistency" (Quine 1937a, p. 120) than those systems which involve the theory of types.[57] Quine is able to give greater credence to his system (called S′ in his paper) through a comparison of S′ with a system S″ which "is virtually the same as Tarski and Gödel's simplifications of the system of *Principia Mathematica*" (Quine 1937a, p. 121).

The system S′ is identical to the one put forth in the *New Foundations.* S″ differs from S′ only in its treatment of unstratified formula; in S″ all unstratified formula are simply abolished. In S″ as well, there is a type hierarchy present, whereas in S′, as seen above, there is no mention of the type hierarchy. S″ thus retains all of the questionable features discussed in the *New Foundations* which are inherent to systems involving a theory of types.

It is with these two systems in mind that Quine begins his investigation of Cantor's theorem. Firstly, the formula '$(Ex)(y)(y \varepsilon x)$' is easily provable within S′. This ensures the existence of a universal class V having everything as a member (including

[57] The consistency of Quine's *New Foundations* and that of a related 'stronger' system is investigated by Rosser in his "On the Consistency of Quine's New Foundations for Mathematical Logic" (1939a). Here Rosser sketches the methods by which he hoped to derive a contradiction within Quine's system, as well as the reasons why each method failed. Rosser concludes by stating that "if one could make unstratified definitions by induction, it would seem that the possibility of producing some undesirable result by means of the Gödel technique should be very much greater…in conclusion, it seems to be the case that there is no danger of deriving a contradiction along any of the known lines until one can handle unstratified relations more effectively" (Rosser 1939a, p. 24). Thus, although Rosser was unable to derive a contradiction in Quine's *New Foundations* he did see its consistency as highly questionable. The key inability to produce unstratified formulas prevented Rosser from applying the techniques which he and Kleene had used in producing a contradiction within Church's logical system of 1932 (as found in Kleene and Rosser's paper from 1935, "The Inconsistency of Certain Formal Logics").

itself and all of its subclasses). As Quine notes however "this contradicts Cantor's theorem, according to which the subclasses of V should outnumber the members" (Quine1937a, p. 122). Quine symbolizes Cantor's theorem that "the converse domain of any one-many (or one-one) relation has a subclass which does not belong to the domain" (Quine 1937a, p. 122) in the logical symbolism of S′. He then shows how Cantor's proof would proceed within the symbolism of S′, and he concludes that the "whole proof could be carried out formally within S′ if we could prove the existence of a class x satisfying (5) and (6)" (Quine 1937a, p. 122).[58] However, the existence of such an x cannot be shown in S′ (since '$y \varepsilon z$' together with considerations of the ordered pair (z,y) could not arise). Thus, in S′ Cantor's theorem fails in this form, whereas in S″ this formulation of Cantor's theorem cannot even be expressed.[59] In fact, this failure to prove Cantor's theorem can be used to disprove Cantor's theorem; Quine also notes that a "simpler disproof of [Cantor's theorem] in S′, consisting essentially of taking v as the class of all pairs of the form (a,a)" (Quine 1937a, p. 123) could be easily constructed.

Since this formulation of Cantor's theorem is meaningful and false in S′, while it is altogether meaningless in S″, Quine next looks at the formulation of Cantor's theorem which is derivable within S″; this he determines to also be derivable within S′. Since S″ involves the theory of types, the ordered pair (z,y) involved in the proof of Cantor's theorem must be amended so that z and y are of the same type. This is done by relating the unit class $\{y\}$ to z if y is of next lower type to z, and $\{\{y\}\}$ to z if $\{y\}$ is of next lower

[58] These two conditions ensure, first of all that for any y in x, there is a z such that y is not in z and yet the ordered pair (z,y) is in some v. Secondly, for any y, if there is a z such that y is not in z and yet the ordered pair (z,y) is in some v, then y is in x.

[59] This follows from the parenthetical remark appended to the previous sentence: here the formula involving '$y \varepsilon z$' and the ordered pair (z,y) would be unstratified.

type to z, and so on. After replacing y with $\{y\}$ throughout the original statement of Cantor's theorem, one is left with the modified version of Cantor's theorem that "the subclasses of a class outnumber the unit classes of the members" (Quine 1937a, p. 124).[60] Quine concludes therefore that "everything demonstrable in S″ admits, indeed, of a precisely parallel proof in S′" (Quine 1937a, p. 124).[61] Thus, even with certain strange features of his system, Quine still sees his *New Foundations* as a more acceptable system than any which incorporates the theory of types.[62]

b) Mathematical Logic

Quine followed up his 1937 system by extending it to his system as found in his *Mathematical Logic*. This work differs from the *New Foundations* in that there are two kinds of object: sets and classes. The original formulation of this system was found, by Rosser in October 1941, to contain a contradiction; Rosser was able to derive the Burali-Forti paradox within Quine's *Mathematical Logic* (Quine 1985, p. 145). Quine quickly set this straight in a correction slip added to those books that had already been sent to

[60] Obviously this is not an equivalent formulation of Cantor's theorem within S′ since within that system the original version is disprovable, while the modified form is provable. Showing that these two formulations are equivalent would simply amount to showing that there is a one-one relation between individuals and their unit-classes. However, in S′ this is impossible since it amounts to defining a class of ordered pairs $(a, \{a\})$ and the defining condition for this would involve an unstratified formula. Similarly, this one-one relation is simply inexpressible in any system admitting the theory of types.

[61] There are some odd conclusions which Quine points out to necessarily follow from his investigations of Cantor's theorem: "(a) Cantor's principle that the subclasses of a class always outnumber the members is false. (b) The subclasses of any class do, however, outnumber the unit subclasses. (c) There is no general correlation between objects and their unit classes" (Quine 1937a, p. 124).

[62] Quine was still wary of the theory of types in the late 1940's. This is seen clearly in the following passage from one of his letters to Carnap on the subject: "I agree that the logical antimonies are symptoms of a fundamental unsoundness somewhere, but I suspect that this unsoundness lies in platonism itself-i.e., in the admission of abstract values of bindable variables. The contradictions which issue from platonism can indeed be staved off by various artificial devices, and in my view the theory of types is merely one among various such devices" (Quine 1947, p. 409).

print. Quine later gave a more comprehensive alternative to deal with the problem in his "Element and Number" from 1941. The essential revision set forth in that paper is contained in only one of the 6 chapters. The other chapters rather act as an alternative development to the material found in his *Mathematical Logic*.

The final revised edition of *Mathematical Logic* appeared in 1951; the repair added there is attributed to Hao Wang. This repair was simply to limit the bound variables in the membership conditions for the classes of *New Foundations* to the sets of *Mathematical Logic* (which correspond to these classes). Wang later proved that *New Foundations* is consistent if and only if *Mathematical Logic* is. Furthermore, Rosser showed that the class of natural numbers cannot be derived in *Mathematical Logic* unless the system is inconsistent. One must postulate that it is a set in order to found the theory of real numbers. Quine notes that "that postulate is an unwelcome artificiality" (Quine 1985, p. 146).

The system in *Mathematical Logic* differs from that of *New Foundations*, as mentioned above, by dealing with both classes and sets.[63] Sets are such that they can be members of both sets and classes, whereas classes can be members of neither. The axioms must then deal with both classes and sets. In the following capital letters stand for classes whereas lower-case letters stand for sets. Quine posits an axiom of extensionality: $(A)(B)[(x)(x\varepsilon A\leftrightarrow x\varepsilon B)\rightarrow A=B]$; an axiom of comprehension by a set: $(Ey)(x)(x\varepsilon y\leftrightarrow\varphi(x))$ where $\varphi(x)$ is any stratified formula with set variables only, and y does not occur free in $\varphi(x)$; and an axiom of comprehension for a class: $(EY)(x)(x\varepsilon Y\leftrightarrow\varphi(x))$ where $\varphi(x)$ is any

[63] By using both classes and sets, Quine is clearly influenced by von Neumann.

formula in which y does not occur free. Finally, a curious feature in Quine's work is his equating individuals with their singleton sets. This is mentioned as a possibility in his *New Foundations*, but is done explicitly in his *Mathematical Logic*.

6.4 Concluding Remarks

It has been seen in this chapter that both Church and Quine did much work in the theory of types. Their systems both incorporated a simple type theory in different ways. Although Church had initially hoped to simply supplant that theory, he ended up working within it. Quine, on the other hand, strove to perfect the theory which motivated his two works *New Foundations* and *Mathematical Logic*. Although he later switched to working with first-order logic as his basis, these early works of Quine's proved to be quite influential. Some results from other logicians regarding his two systems have been noted in this chapter. In the next chapter extensions of both Church and Quine's systems will be discussed, as will other advances in the theory of types.

7. The Theory of Types as a Many-Sorted Theory

7.1 Introduction

In this chapter the further extensions of Quine's work are considered, as are extensions of Church's work. The type-theoretical systems provided by these two logicians proved to be quite influential in the 1940's and 1950's as the present chapter will attest. Apart from work done on these systems, a rejuvenated interest in the ramified theory of types emerged in the 1950's most notably in the work of Hao Wang and John Myhill. Whereas Wang was working towards providing a purely constructivist foundation for mathematics, Myhill was working simply with the aim of completing Chwistek's program (which, although motivated by constructivist concerns, employed a non-constructive rule of inference). Finally, this chapter will be concluded by investigating the views of the theory of types as a many-sorted versus a one-sorted theory.

7.2 Church's Theory of Types (Continued)

As noted in the previous chapter, Church's simple theory of types was studied quite extensively after its publication in 1940. In this section the works of Turing, L'Abbe, and Buchi will be discussed as extensions of Church's work on the theory of types.

a) Alan Turing

In their 1942 paper "A Formal Theorem in Church's Theory of Types" Newman and Turing extend the proof of Church's 'axiom of infinity' to all types which contain "ι" among their symbols. Church showed in his paper that "if Y^{α} stands for $N_{o\alpha'}x_{\alpha'} \rightarrow .N_{o\alpha'}y_{\alpha'}. \rightarrow .S_{\alpha'\alpha'}x_{\alpha'} = S_{\alpha'\alpha'}y_{\alpha'} \rightarrow x_{\alpha'} = y_{\alpha'}$ (a form of the 'axiom of infinity' for type α), Y^{α} can be proved formally from Y^{ι} and the axioms 1 to 7, for all types α of the forms ι', ι'',…For other types the question was left open" (Newman and Turing 1942, p. 28). The proof of the extended case involves not only axioms 1 to 7 and Y^{ι}, but also axiom 9 (the axiom of description) and axiom 10 (the axiom of extensionality).

In his review of this paper, Leon Henkin notes that "axioms 1-8 alone do not suffice to establish this result. For by relaxing the requirement of extensionality only in type (ιι), a model can be constructed in which $Y^{\alpha}m_{\alpha}n_{\alpha'}$ holds only for types α whose symbol contains "(ιι)" (or is "ι" itself)" (Henkin 1942, p. 122). Henkin adds that the extent to which axiom 9 is required remains uncertain as does the question of whether axiom 10 is needed in all types. The proof provided by Newman and Turing involves three main steps. These are 1) a proof that their version of the "axiom of infinity is equivalent to the proposition that 'no member belongs "to its own posterity""" (Newman and Turing 1942, p. 28). 2) A proof that if a type α can be mapped one-one to a part of another type α', then the axiom of infinity in α' implies the same axiom in α. 3) The actual construction of mappings from part 2) for the pairs $\alpha\beta$, α and $\alpha\beta$, β. The first part follows from Church's axioms 1-8 while the second and third steps make use of the axioms of description and extensionality. Newman and Turing point out that Church's deduction theorem is used throughout. This work is a direct extension of Church's simple theory of types. The authors state that they needed to extend his results for applications in other areas of their research for which they were using Church's simple theory of types.

Turing continued his work on type-theory in his 1948 paper "Practical forms of Type Theory".[64] In this paper Turing presents two different forms of the theory of types where "types themselves only play a rather small part as they do in ordinary mathematical argument" (Turing 1948, p. 80). The two logical systems are called the "nested-type" and the "concealed-type" systems. In the nested-type system the "types themselves do not intrude very much" (Turing 1948, p. 90). It is this system that Turing claims is equivalent to Church's simple theory of types.

Turing's nested-type system contains a cumulative type hierarchy. He starts by considering only a finite universe of individuals $U_1,\ldots,U_N$. These form type 0. Type 1 then consists of functions of individuals, taking individuals as values, together with all of the individuals themselves. Type 2 consists of functions of arguments in type 1, taking values in type 1, together with members of type 1. In general, type n+1 consists of functions of arguments in type n, taking values in type n, together with members of type n.

Turing avoids the use of definitions which require the notion of a function being undefined for certain values by designating a particular universe U_1 and renaming it C. In this way whenever a function would normally be regarded as undefined it is given this

[64] There are also three unpublished papers in Turing's archives that deal with Church's simple theory of types. These are "Some Theorems about Church's system" (1941), "Practical Forms of Type Theory II" (1943-4), and "The Reform of Mathematical Notation" (1944-5).

designated value, C, instead. That is, "the value of a function is always C unless the function is of higher type than the argument" (Turing 1948, p. 81). Turing then defines functions and individuals together as *terms*. Then propositions are formed by writing U=V (for terms U and V). Furthermore, if P and Q are propositions, then (~P) and (P→Q) are both propositions. Finally, if U is a term, then D^rU represents the proposition stating that U is in type *r*. Turing then defines the remaining ways of forming terms and propositions. If P is a proposition, then (ιx,*r*)P is a term (read "the *x* in type *r* such that P") while (*x*,*r*)P is a proposition (read "P, for all *x* in type *r*"). The use of a finite universe is then dropped since it was adopted solely to ease his explanations.

Turing then gives a list of nine rules of procedure. These include rules governing the use of 'D^r' and C as well as rules for substitution, changing bound variables, etc. Also, an axiom of infinity is stated if the universe is infinite, and if it is not a corresponding axiom stating the size of the universe is given. Turing claims that the nested-type system may be proved equivalent, in a certain sense, to Church's simple theory of types. Since the proof is long and tedious he does not provide it. Instead he gives a summary of his form of equivalence which he thinks "has certain interest in itself" (Turing 1948, p. 89). Essentially Turing says that two systems are equivalent if "we can translate from either system to the other in such a way that provable propositions translate into provable propositions again, and so that a double translation gives rise to a proposition equivalent to the original" (Turing 1948, p. 90). It is with this form of equivalence that Turing shows his nested-type system to be equivalent to Church's simple type theory.

b) J. Richard Buchi

In his 1953 paper "Investigations of the Equivalence of the Axiom of Choice and Zorn's Lemma from the Viewpoint of the Hierarchy of Types", Buchi uses Church's simple theory of types as his logical system. Buchi notes that although it is well known that Zermelo's axiom of choice and Zorn's lemma are equivalent logical assumptions in set theory, when type-theoretical formalisms are used a proof of this equivalence is unknown. In fact, in this formalism these assumptions must enter only as spectra of formulae (ZA^{α}) for Zermelo's axiom and (ZL^{α}) for Zorn's lemma, for variables of a fixed type α.

In his investigations Buchi makes use of only 8 of Church's axiom schemata. Axioms 1-6^{α} correspond to Church's first six axiom schemata (which suffice for a logical functional calculus). Buchi's 7^{α} corresponds to Church's axiom of extensionality and his 8^{α} corresponds to Church's restricted choice principle.[65] Finally, Buchi writes (ZA^{α}) and (ZL^{α}) as follows:

(ZA^{α}) is $(\mathrm{E}h_{\alpha(o\alpha)})(a_{o\alpha})[ax \rightarrow a(ha)]$

(ZL^{α}) is $(r_{o\alpha\alpha}).[\mathrm{P}r \& \mathrm{W}r] \rightarrow (\mathrm{E}x_{\alpha})(u_{\alpha})[rxu \rightarrow rux]$ where '$\mathrm{P}r$' expresses that $r_{o\alpha\alpha}$ quasi-orders the type α and '$\mathrm{W}r$' expresses that every r-chain has an r-upper bound.[66] Thus Zermelo's proposition for elements of type α states that a function $h_{\alpha(o\alpha)}$ exists, which to every propositional function with arguments of type α that can be satisfied, selects one particular element ha for which the proposition ax holds. On the other hand, Zorn's

[65] This axiom is $a_{o\alpha}x_{\alpha} \& (y_{\alpha})[a_{o\alpha}y_{\alpha} \rightarrow x_{\alpha} = y_{\alpha}] \rightarrow a_{o\alpha}(\iota_{\alpha(o\alpha)}a_{o\alpha})$ where $\iota_{\alpha(o\alpha)}$ is a selection-operator that "chooses" a "particular element ιa with the property a, provided that there exists an element x with the property a and provided that any two elements having the property a are identical" (Buchi 1953, pp. 126-127).
[66] Buchi defines 'quasi-orders', 'r-chain', 'r-upper bound', etc. in Church's symbolism.

proposition for elements of type α states that to every relation r which quasi-orders type and has the property W, there exists an element x_α which is maximal in the sense that rxu implies that x and u are equivalent relative to r (Buchi 1953, p. 126).

In section 2 of his paper Buchi shows that for any type symbol α, (ZL^α) is a consequence of axioms $1\text{-}6^\beta$ for several types β, axiom 7^α, and (ZA^α).[67] In section 3 of his paper Buchi shows that for any type symbol α, (ZA^α) is a consequence of axioms $1\text{-}6^\beta$ for several types β, axiom 8^α and $(ZL^{\alpha(o\alpha)})$ for variables of the higher type $\alpha(o\alpha)$.[68] Buchi concludes by summarizing his results in the fourth and final section of his paper. There he makes three observations. First, based upon axioms $1\text{-}8^\alpha$, the collection of formulas (ZA^α) is equivalent to the collection of formulas (ZL^α). Second, $(ZL^{\alpha(o\alpha)})$ for elements of type $\alpha(o\alpha)$ implies (ZL^α) for elements of type α and similarly for Zermelo's axiom. Finally, for a fixed type α it is not shown, nor does it seem possible to show, that (ZA^α) is equivalent to (ZL^α) nor to (ZL^β) for any particular β. Buchi's investigation of Church's simple theory of types thus ends on a somewhat negative note.

c) Maurice L'Abbe

One of Alonzo Church's students, Maurice L'Abbe, extended Church's formulation of the simple theory of types to include transfinite types.[69] He did this in his

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[67] This is done roughly as follows: first a fixed point theorem is derived for a function $f_{\alpha\alpha}$, and then an application of this leads to finding the desired h_α. (Buchi 1953, pp. 127-131).
[68] This is done roughly as follows: considering the abbreviation $R_{o(\alpha(o\alpha))(\alpha(o\alpha))}$ for $\lambda f_{\alpha(o\alpha)}\lambda g_{\alpha(o\alpha)}(c_{o\alpha})[c(fc)\to fc=gc]$ Buchi shows that i) R quasi-orders the type $\alpha(o\alpha)$, ii) an R-maximal element is a Zermelo selector function, and iii) there exists an R-maximal function $h_{\alpha(o\alpha)}$ (Buchi 1953, p. 133).
[69] In this article L'Abbe mentions other attempts at constructing formal systems involving transfinite types. These include E. Bustamante's Ph.D. dissertation *Transfinite Type Theory* Princeton (1944), and John Kemeny's dissertation *Type Theory and Set Theory* (1949) both of which are based upon Church's simple

Ph.D. dissertation, submitted in 1951, some results of which were published in his 1953 paper "Systems of Transfinite Types Involving λ-Conversion". The paper begins with Church's system $\sum_0$ of finite types. This simple type hierarchy is based upon type 0 of propositions and type 1 of natural numbers. L'Abbe sets this system up just like Church's, and in fact refers to that work for a more in-depth explication. He then utilizes the methods found in Henkin's paper from 1950 in interpreting the formalism with a model M_1. That is, he interprets the theory of types as a many-sorted *first-order* theory.

L'Abbe then extends this system to the system $\sum_2$ that includes four different types, 0, 1, 2, and 3. The domain of type 2 consists of all the domains of $\sum_0$, while the domain of type 3 is based upon the types 0, 1, and 2. The variables of type 3 act as L'Abbe's nonsense elements just like Turing's domain C. In his review, Gandy claims that this designated domain is not essential; "the nonsense value can be a new element of type 2…and we thus obtain a system $\sum_2'$ which is simpler than $\sum_2$" (Gandy 1958 p. 361). Furthermore, since L'Abbe states that $\sum_2$ can be modeled in set theory, so too can $\sum_2'$.[70]

The next step in L'Abbe's paper is to prove the consistency of Church's system, $\sum_0$, in the transfinite $\sum_2$. In doing this L'Abbe makes use of Gödel numbers. Furthermore, a truth definition for $\sum_0$ is given in $\sum_2$. A sketch is finally made of how the author's results can be extended to systems $\sum_3, \ldots, \sum_n, \ldots, \sum_\omega, \ldots$

These four papers thus show that even in the early 1950's the properties of type systems were still being investigated. However, this does not imply that the theory of

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theory of types. In his review of L'Abbe's article, Gandy notes that L'Abbe failed to refer to Turing's nested-type system which provides a particularly good example of this process and also exploits a nonsense element, as seen above (Gandy 1958, p. 362).

[70] More precisely, L'Abbe claims that it is "possible to prove that the consistency of the Zermelo set theory Z, including the axiom of replacement, implies the consistency of $\sum_2$" (L'Abbe 1953, p. 217).

types was still considered to be the best underlying logic upon which to build mathematics. Before considering the fate of the theory of types, extensions of Quine's work are considered.

7.3 Continuation of Quine's Work

As noted in the previous chapter, Quine's work involving the theory of types was worked on by many logicians in the 1940's and 1950's. Some of this work was mentioned in connection with Quine's in the previous chapter. In this section further work on Quine's *New Foundations* is presented.

Firstly, in his 1944 paper "A Set of Axioms for Logic", Hailperin revises Quine's set of rules by replacing the infinite axiom schemata R3′ by a finite set of axioms. Hailperin then shows that his system involving the new finite set of axioms is provably equivalent to Quine's infinite list. Recall that Quine's R3′ is the rule that 'If φ is stratified and does not contain x, then $(\mathrm{E}x)(y)[y\varepsilon x \leftrightarrow \varphi(x)]$ is a theorem'. Hailperin replaces this infinite set of rules with nine axioms. Roughly the first axiom ensures the existence of the complement of the union of two classes. From this Hailperin proves the existence of the complement of a class, the intersection of two classes, the empty class, and the universal class. The second axiom provides, for any relation R, the corresponding relation which is of one higher type. The third, fourth, and fifth axioms allow the degree of a relation to be mixed. The sixth axiom provides for the domain of a relation, while the seventh gives the converse domain. The eighth gives the class of all unit classes (that is, the cardinal 1). Finally, the ninth states that '$x\varepsilon y$' determines an inhomogenous relation (that is, a relation involving x and y which are of different types). Apart from these, five other axioms are listed to provide for the propositional calculus as well as providing ways to deal with the abstraction operator. Hailperin then proves the equivalence of his system with Quine's *New Foundations*.[71]

In their 1950 article "Non-Standard Models for Formal Logics", Rosser and Wang follow methods used in Henkin's doctoral dissertation in which he showed that any consistent theory has a non-standard model. The authors provide three criteria for "non-standardicity" and go on to show that there is no standard model for Quine's *New Foundations*. The three ways in which a model is deemed non-standard are: "a) The relation in the model which represents the equality relation in the formal logic is not the equality relation for objects of the model. B) That portion of the model which is supposed to represent the positive integers of the formal logic is not well-ordered by the relation ≤. C) That portion of the model which is supposed to represent the ordinal numbers of the formal logic is not well-ordered by the relation ≤" (Rosser and Wang 1950, p. 113). The authors claim that it might be natural for one to conclude that since there is no standard model for the system in *New Foundations* the system must have no model whatsoever (and that it is thus inconsistent).

This inference is partially refuted by Rosser and Wang by their claim that it is not uncommon for strong formal logics to have no standard model. In fact, they show that it is a property of each familiar logic that if it is ω-consistent then one cannot prove in the

[71] Although he makes no mention of it, Hailperin's approach is a variant of Gödel's approach in his 1940 monograph on set theory. The first-order Gödel-Bernays set theory which is used there has a finite set of axioms, in contrast to Zermelo-Fraenkel set theory, which does not.

logic itself that if the logic is consistent then it must have a standard model.[72] The main claim of this paper is the general statement that if the language L* is related to the language L in such a way that any standard model for L must be a model for L*, then if L has no standard model, this fact cannot be proved in L*. In fact this cannot even be proved in L* together with the axiom stating that L is consistent. Therefore "we cannot prove in L* that if L is consistent, then L must have a standard model" (Rosser and Wang 1950, p. 123). The proof is roughly as follows. First the authors show that if there is a standard model for L, and if L′ is simply L together with the axiom stating that L is consistent, then L′ is consistent. From this it is shown secondly that 'if L is consistent then L has a standard model' is not provable (in L or in L together with the axiom stating the consistency of L). The result follows from this.

Rosser continued his investigations of standard models with respect to the *New Foundations* in his 1952 paper "The Axiom of Infinity in Quine's New Foundations". In that paper he presents three principal results. The first two of these deal with the status of the axiom of infinity in Quine's *New Foundations*. However, as will be seen below, any questions about the status of this axiom were answered by Specker in his 1953 paper. The third result that Rosser presents deals not only with Quine's *New Foundations*, but also the system found in his *Mathematical Logic*. After showing in his paper with Wang that *New Foundations* has no standard model, together with the fact that it is essentially a part of the system in *Mathematical Logic*, Rosser shows why it is possible that the larger system might have a standard model. It is shown further that it might even be possible for

[72] They note further that for some even stronger logics this holds simply if the system is consistent as opposed to ω-consistent.

the *New Foundations* to be ω-inconsistent, while the other system is ω-consistent. Rosser concludes from this that "such questions as whether a logic is ω-consistent, or whether it possess a standard model, are rather more subjective than objective, in spite of the fact that in most cases where data are available, there would be general agreement on the question" (Rosser 1952, p. 241). These sentiments are reiterated by Quine in his "On ω-Consistency and a So-called Axiom of Infinity" (1952) where he urges that the term 'ω-consistency' should be reformulated.

In the aforementioned investigations, as well as others involving Quine's *New Foundations*, consideration is made of what results when the axiom of infinity is added to the system in *New Foundations*. This is due to the fact that it was not known until Specker's 1953 paper that the axiom of infinity is provable in Quine's *New Foundations*. It is in his paper entitled "The Axiom of Choice in Quine's *New Foundations for Mathematical Logic*", that Specker proved the axiom of infinity in Quine's *New Foundations*. Specker did this by actually disproving the axiom of choice. Then, since the axiom of choice holds for all finite sets, the system in *New Foundations* must contain infinitely many elements.

Specker follows the notation in Rosser's *Logic for Mathematicians* (1953) which is itself an extension of Quine's *New Foundations*. The proof is done by *reductio ad absurdum*. Assuming the axiom of choice, namely that "3.5 The cardinal numbers are well-ordered by the relation 'there are sets a,b such that $a\varepsilon m$, $b\varepsilon n$ and a is a subset of b'" (Specker 1953, p. 973) Specker produces the two contradictory statements that "5.4 If m is a finite cardinal number, then $m\neq \mathrm{T}(m)+1$ and $m\neq \mathrm{T}(m)+2$", where $\mathrm{T}(m)$ is defined as:

T(m) = 'the cardinal number of the set of unit subsets of m', and "7.4 There is a finite cardinal n such that $n = \mathrm{T}(n)+1$ or $n = \mathrm{T}(n)+2$" (Specker 1953, pp. 973-974). Specker concludes his article with a statement pertaining to the generalized continuum hypothesis in Quine's *New Foundations*. This is as follows: "8.1 Generalized continuum hypothesis in 'New Foundations': If m, 2^m, n are cardinal numbers, m not finite and $m \leq n \leq 2^m$, then either m=n or n=2^m. The generalized continuum hypothesis does not hold in 'NF'. The proof is by proving the theorem of Lindenbaum and Tarski in "NF" according to which the axiom of choice is a consequence of the generalized continuum hypothesis" (Specker 1953, p. 974). With this result Specker laid to rest any question as to the status of the axiom of infinity in *New Foundations*. That the axiom of choice fails in Quine's system came as quite a shock as Quine recounts in his "Unification of Universes in Set Theory" (Quine 1956, pp. 270-271).

7.4 Ramified Theory Reconsidered

a) Ramified Theory of Types and the Axiom of Reducibility

In the past few chapters it has been made apparent that the simple theory of types has been used more extensively than the ramified theory, especially since Ramsey's work in the 1920's. However, there are a few exceptions to this general trend. Moreover, it is the ramified theory without an axiom of reducibility that was developed by certain logicians such as Hao Wang. The advantage of avoiding the axiom of reducibility was made clear, besides in Ramsey's work, also in an article by Quine and one by Copi.

Before looking at Wang's system, which incorporates a ramified-type theory, the articles by Quine and Copi will be considered in this section.

In his "On the Axiom of Reducibility" (1936a), Quine does not argue against the validity of the theory of types, but rather he takes up Ramsey's argument advocating the abandonment of the semantic part of the theory of types.[73] Ramsey had proposed that in *Principia Mathematica* the theory of types could be divided into two parts: the one section deals with the properly 'logical' paradoxes, while the other deals exclusively with the 'semantic' (and thus non-logical) paradoxes. Ramsey's arguments rely upon the *meanings* that are to be attributed to the variables used in *Principia.* He claims that the second set of paradoxes, and thus the second part of the theory of types, has no place in logical or mathematical inquiries. This is precisely the position which Quine advocates in his paper. However, he pursues the matter upon "more formal consideration[s]" (Quine 1936a, p. 499). This does not mean that he gives a set of formal rules, nor recursive definitions as to what *formulas, terms,* and so on are to denote. In fact, Quine simply gives two arguments. The first of these is not 'formal' at all: it deals with the subject matter of *Principia* and mention is made of what interpretations are to be made for predicative functions. It is after he gives this argument that Quine states that "granted the partial extensionality principle, the above argument shows that either the axiom of reducibility is not legitimate to begin with, or else both it and the second part of the theory of types are superfluous" (Quine 1936a, p. 499).

[73] The results of this paper are contained in his Ph.D. dissertation which "reformulated the theory of relations of [*Principia Mathematica*] so that the object language could talk of relations of any number of arguments, but the object language itself is formulated more precisely than in [*Principia*], and the simple theory of types is adopted" (Follesdal and Parsons 2002, p. 105).

The second consideration put forth by Quine deals with "a metamathematical or syntactical application in which analogues of the hierarchy of orders and the axiom of reducibility are of less doubtful significance" (Quine 1936a, p. 499). However, these analogues do not deal with the actual subject-matter of *Principia*. Quine concludes without altering the original claim that the theory of types minus the second portion of the theory of types is adequate for avoiding the logical contradictions.

In Copi's 1950 paper "The Inconsistency or Redundancy of Principia Mathematica", Copi shows that the ramified theory of types, as found in *Principia Mathematica*, together with the axiom of reducibility is either inconsistent or redundant. Copi begins by noting that since the authors did not specify any specifically semantic notions, the *Principia* system can be assumed to either contain semantic notions or not. If it does not contain semantic notions, then by the arguments given by Ramsey the ramified theory of types together with the axiom of reducibility is redundant. Thus Copi next assumes that the ramified system in *Principia* contains semantic notions.

Under this assumption Copi shows how Grelling's paradox can be reinstated after the theory of types eliminates it. Under this second assumption Copi thus concludes that the ramified *Principia* system, with an axiom of reducibility, is inconsistent. Thus Copi reaches his desired conclusion. It should be noted that the redundancy which Copi presents differs from that produced by Quine since Copi's does not depend upon the axiom of extensionality. Copi notes that Quine showed that the ramified theory of types together with the axiom of reducibility, as well as the axiom of extensionality, is redundant.

Copi also draws out the similarities between the ramified theory of types and the theory of levels of languages. In his review of Copi's paper, Church lists this as perhaps the most important aspect of Copi's work. Copi states that the theory of levels of languages is "very like the theory of orders, because the contradiction is evaded by arranging that certain symbols of the meta-language are defined *over certain ranges*" (Copi 1950, p. 198). However, Copi also notes that there are fundamental differences between the two theories. Most notably, the ramified type theory depends on the axiom of reducibility if a large portion of classical mathematics is not to be lost, while the levels of language device does not.

With these arguments so deeply entrenched the choice seems to reduce to that between the simple theory of types and the ramified theory of types without an axiom of reducibility. The later course was taken by Chwistek, as discussed in Chapter 2. Since Ramsey's distinction between the uses of the simple theory of types for the "logical paradoxes" and the ramified theory of types for the "semantic paradoxes" the simple theory of types was favored. However, some logicians later tried to investigate theories involving ramified-type systems. Several of these will now be investigated.

b) Wang's System

In his 1954 paper "The Formalization of Mathematics", Hao Wang sets up a ramified theory without recourse to an axiom of reducibility. However, Wang's system, Σ, differs from the regular ramified theory in that the two distinctions 'order' and 'type' are replaced by the one concept of 'order'. This is done by allowing a mixing of types in the following way. The lowest order (the 0^{th} order) consists of some denumerable totalit (which may be the positive integers or all finite sets built up from the empty set, for example). The first order then consists of everything in the 0^{th} order together with those sets that can be defined by properties that refer to at most the totality of all sets of the 0^{th} order.[74] In general, the (n+1)th order contains all sets of order n together with sets defined by formulas with bound variables from the nth and lower orders only.

This hierarchy of orders is then extended to infinity as follows. The sets of order ω include all sets of the finite orders. Then for an ordinal $\alpha+1$ the sets of order $\alpha+1$ consists of all those sets of order α together with sets defined by formulas having bound variables of order α and lower orders only. Furthermore, for ordinal numbers, β, that are limit numbers of monotone increasing sequences $a_0, a_1,\ldots$ of ordinals, the sets of order β consist of all and only those sets of orders $a_0, a_1,\ldots.$ Thus, in Wang's construction those sets of order higher than 0 are constructed according to the vicious-circle principle, and the orders are constructed cumulatively.

Using the above cumulative hierarchy, Wang goes on to list the axioms for $\sum$. These include axioms for identity; infinite summation where for every limiting ordinal α, if $\beta<\alpha$, then for every x_β, there is a y_α, such that $x_\beta=y_\alpha$; abstraction; foundation; bounded order where if $x \varepsilon y$ and y is not of higher order than x, then there exists a z of order lower than y such that $x=z$; and limitation, which is dealt with below (Wang 1954, p. 248). Furthermore, identity is defined in terms of equal extensions; thus Wang most likely views all of his objects as sets.

[74] Thus, the formulas corresponding to these properties must not contain bound variables of any order besides the 0^{th} order.

The power of Wang's system seems to come from the fact that $\sum$ is the union of all formal systems $\sum_\alpha$, where $\sum_\alpha$ deals with all of those sets which are of order less than or equal to α. Wang is then able to refer to the partial systems in order to make "quite exact statements about the comprehensive theory $\sum$" (Wang 1954, p. 249). Several interesting features thus arise by looking at the partial systems in isolation. For instance, Wang claims that "I. For each α, we can find a function E_α of order $\alpha+2$, such that E_α enumerates all sets of order α; or, in other words, the domain of E_α is the set of all positive integers and its range is the universal set V_α consisting of all sets of order α. II. For each α, we can find a truth definition of E_α in $E_{\alpha+2}$ and formalize a consistency proof of E_α in $E_{\alpha+2}$" (Wang 1954, p. 249).[75]

It is with these E_α functions that Wang states his powerful axioms of limitation; namely that "for each order α and each set x_α, there is a positive integer m such that $E_\alpha(m)$ is x_α" (Wang 1954, p. 250). These axioms show that E_α well orders all sets of order α, and so certain axioms of choice can be proved. Furthermore, the continuum hypothesis is not independent of the other axioms and is in fact provable or refutable according to whether equi-cardinality is defined by the existence of a one-one mapping between sets within $E_{\alpha+2}$ or within E_α itself.

Finally, in Wang's system, as opposed to the regular ramified theory of types without an axiom of reducibility, the theorem of least upper bound, the Bolzano-Weierstrass theorem, and the Heine-Borel theorem are all provable. This is due to the cumulative nature of Wang's system. Furthermore, Wang sketches a proof, similar to

[75] The truth definition must occur in a system of order two higher than the original, that is of order $\alpha+2$ instead of $\alpha+1$, since these definitions require formulas which contain bound variables of order $\alpha+1$.

Fitch's 1938 proof, showing that each $\sum_\alpha$ is consistent.[76] Then, since $\sum$ is the union of all $\sum_\alpha$, the consistency of $\sum$ follows immediately.

Wang notes that the ramified theory of types is roughly equivalent to the system $\sum_\omega$ without variables of order ω. With this in mind he attempts to show that the axiom of reducibility is unnecessary in the regular ramified theory of types since he can speak of functions or sets of all orders at the same time within his $\sum_\omega$. He does this by showing that in using general variables which range over all sets (of any order of the same type), mathematical induction, the definition of identity, and the Dedekind construction of the real numbers can all be formulated within $\sum_\omega$ without recourse to the axiom of reducibility. Since these were the three most important reasons for which Russell introduced the axiom, Wang claims that it is superfluous.

The one serious drawback of Wang's system is that Cantor's proof for the theorem that the power set of a denumerable set contains more members than that set, breaks down in $\sum_\omega$. This is a similar problem that Quine struggled with for his *New Foundations* as mentioned in the previous chapter. Wang simply points out that from the constructivist point of view this is not only not an objection to $\sum_\omega$, but it is a point in its favor. This is so because by proving the existence of any infinite number beyond aleph-zero, impredicative definitions are indispensable. Finally, the key difference that Wang sees between his theory, $\sum$, and the ordinary ramified theory without the axiom of

[76] Fitch provides a proof for the consistency of a formal system based on the ramified theory of types in which a definition of 'truth' has been given. Roughly, "a consistent non-constructive system S′ [is] defined by means of induction with respect to a serial well-ordering of all the propositions" (Fitch 1938, p. 140) of the system to be proved consistent. Then it is shown that every true proposition of the system in question is a true proposition of S′. In his review of Fitch's paper Bernays notes that a definition of truth, in the sense of Tarski and Carnap, can be set up for the formulas of ramified type-theories since the formulas can be interpreted "in such a way that every variable ranges over formal expressions" (Bernays 1939, p. 97).

reducibility lies in his use of transfinite orders. Furthermore, the axioms of limitation are seen as a new feature that has never been investigated in standard forms of set theory. In conclusion, Wang's motivations for using a ramified-type theory definitely come from his constructivist philosophy of mathematics and not from the necessity of avoiding the semantic paradoxes.

c) Myhill's System

Similarly to Wang, John Myhill attempted to find a foundation for mathematics based upon a ramified type theory. In a series of papers[77] Myhill attempted to complete Chwistek's program of building mathematics upon a ramified type theory that includes a variant of the axiom of reducibility. His system is like Wang's in that it is cumulative with no highest type. However, Myhill's hierarchy is inverted. Thus, type 0 is all inclusive and classes always have members of higher types than themselves.

Myhill is able to avoid vicious-circle definitions by stating that "if quantifications are made over the *n*th type in a formula, that formula belongs to the n-1^{st} type at highest. It follows that there is no quantification over the zero type" (Myhill 1951, p. 35). Myhill's system fails to be constructive, however, since he uses a non-finitary consequence relation. This relation allows for certain formulae to be regarded as consequences of certain classes of formulae.

This non-constructive system can be shown to be consistent with a proof similar to that found in Fitch's 1938 paper (and hence also along lines used by Wang). Myhill

[77] Myhill 1949, 1951, 1951a.

also considers auxiliary systems Q(1), Q(2),...and then the comprehensive system Q(∞). Q(∞) contains as its theorems all those formula that are theorems of every Q(k), and is thus the union of all of the auxiliary systems. From this statement Myhill lists seven theorems that follow easily. These include that 1) Q(∞) is consistent, 2) Q(∞) is closed with respect to infinite induction, 3) every quantificationally valid formula is a theorem of Q(∞) and, 4) an analogue of the axiom of reducibility is a theorem.[78]

In the final paper of the series Myhill is able to continue deriving interesting results. In that paper he is able to derive analogues of Bourbaki's axiom system for set theory.[79] The analogues, of course, contain certain type-restrictions. Most notably, he is able to derive axioms of choice and infinity.[80] Myhill notes in an added note to his paper that "the referee of a previous version of this paper expressed astonishment that the sum-class axiom and the axiom of replacement were not included by Bourbaki" (Myhill 1951a, p. 136). Myhill states these axioms in Bourbaki's symbolism and then provides simple proofs of their analogues in his system.

Despite these missing axioms, Myhill lists three reasons for choosing Bourbaki's as the standard set theory by which to judge his system. These are "1) their simplicity, 2) their similarity to set theory, and 3) Bourbaki's statement that they are adequate for all the

[78] The analogue being the following: "Let us denote by '$[\varphi]_\alpha$' the result of writing α for all occurrences of '0' in φ. φ is a theorem of Q(∞) if and only if $[\varphi]_\alpha$ is" (Myhill 1951, p. 39).
[79] Bourbaki 1949.
[80] Bourbaki's axioms as listed by Myhill are as follows (where $z|xy$ is synonymous with $z=(x,y)$ (Bourbaki 1949, pg. 81)): The closure of the following are theorems 1) $x=x$, 2) $(x=y.\mathrm{F}x)\rightarrow \mathrm{F}y$, 3) $(\mathrm{E}x)x|yz$, 4) $(x|yz.w|yz)\rightarrow x=w$, 5) $(x|yz.x|uw)\rightarrow(y=u.z=w)$, 6) $[(x)(x\varepsilon y\leftrightarrow x\varepsilon z)]\rightarrow y=z$, 7) $(\mathrm{E}x)(y)(y\varepsilon x\leftrightarrow y\varepsilon z.\mathrm{F}y))$, 8) $(\mathrm{E}x)(y)(y\varepsilon x\leftrightarrow(\mathrm{E}z)(\mathrm{E}w)(y|zq.z\varepsilon u.w\varepsilon v))$, 9) $(\mathrm{E}x)(y)(y\varepsilon x\leftrightarrow(u)(u\varepsilon y\rightarrow u\varepsilon v))$, 10) $[x\varepsilon y.(z)(z\varepsilon y\rightarrow(\mathrm{E}w)(w\varepsilon z)).(w)(v)((w\varepsilon y.v\varepsilon y)\rightarrow(a)\sim(a\varepsilon w.a\varepsilon v))]\rightarrow (\mathrm{E}b)\{(t)(t\varepsilon b)\rightarrow(\mathrm{E}s)(t\varepsilon s.s\varepsilon y)).(r)(r\varepsilon y\rightarrow(\mathrm{E}h)(m)((m\varepsilon r.m\varepsilon b)\leftrightarrow(m=h)))\}$, 11) $(\mathrm{E}x)(\mathrm{E}y)(\mathrm{E}z)(x\varepsilon y.(u)(u\varepsilon z\rightarrow(w)(w\varepsilon u\rightarrow w\varepsilon y)).(f)(f\varepsilon y\rightarrow(\mathrm{E}r)r\varepsilon z.(m)(m\varepsilon r)\leftrightarrow(m=f)))).(t)(v)((t\varepsilon z.v\varepsilon z)\rightarrow (\mathrm{E}a)(a\varepsilon z.(h)(h\varepsilon a\leftrightarrow(h\varepsilon t\vee h\varepsilon v)))).\sim y\varepsilon z)$ (Myhill 1951a, p. 132).

mathematics of the present day" (Myhill 1951a, p. 132). Although Myhill was successful in many of his pursuits, his plans to further the results in his system were never carried through. These plans were to show that the non-extensional Bourbaki system contains a model of the extensional Bourbaki system. Myhill wanted to do this since his system was non-extensional. However, Myhill never published any further articles addressing this open problem. It must be concluded then that he was unable to accomplish the tasks that he set for himself.

7.5 Type Theory as a Many-Sorted Versus a One-Sorted Theory

In their book on the foundations of set theory, Fraenkel and Bar-Hillel state what they see to be the most serious disadvantage to the simple theory of types. This is mainly just that set theory based upon the simple theory of types does not enjoy the proof procedures of a complete underlying logic, such as first-order logic (Fraenkel and Bar-Hillel 1958, p. 191). The shift from the theory of types as the most widespread underlying logic to first-order logic as the basis for mathematical investigations seems to be largely due to this fact. The work done by Tarski and the logical group surrounding him at Berkeley helped to usher in this new wave of dependence upon first-order logic. Discussion of this shift is beyond the scope of this thesis. What will be investigated in the final section of this chapter is the procedure of changing the theory of types from a many-sorted to a one-sorted theory.

In his 1952 paper "Logic of Many-Sorted Theories", Wang showed how any many-sorted theory can be converted into an equivalent one-sorted theory.[81] The essential part of this process lies in introducing predicates, S_i. When applied to variables, these predicates are meant to state which type the variable would belong to, although since the theory is converted to a one-sorted theory, the variables all range over the same set. Thus $S_i(a)$ would mean that a belongs to the ith type.

In his paper Wang uses the simple theory of types as put forth by Gödel in his 1931 paper. Wang uses this theory as an example of how to change any many-sorted theory into a one-sorted theory. He does this since he views the simple theory of types as a prototypical many-sorted theory. This is the technique that Quine uses in his 1956 paper. There he applies the process to *Principia Mathematica* and arrives at what he calls the standardized theory of types.

This conversion allows for a translation between the simple theory of types and Zermelo's set theory. Although the universe is still divided into types, in any given context the variables are not only typically ambiguous, but rather, they range over all types. In this way the many-sorted theory becomes a one-sorted theory. One system that Quine sets up, which is treated in Fraenkel's book, is the following. Certain predicates, definable solely in terms of 'ε' are set up. 'T_0' is the predicate that holds only for individuals, 'T_1' holds only for classes of individuals, and so on. The axiom-schemata of

[81] Wang also refers to the first use of the term 'many-sorted'. He attributes this first use to Langford's 1939 review of Arnold Schmidt's 1938 paper, "Uber deductive Theorien mit mehreren Sorten von Grunddingen". In that review Langford translates the word 'mehrsortig' as 'many-sorted'.

comprehension and extensionality then have altered forms so as to not refer to types directly. These now take the following forms, respectively:

$(Ey)[T_{n+1}(y)\&(x)(T_n(x)\rightarrow(x\varepsilon y\leftrightarrow\varphi(x)))]$ and

$T_{n+1}(x)\&T_{n+1}(y)\&(w)(T_n(w)\rightarrow(w\varepsilon x\leftrightarrow w\varepsilon y))\rightarrow x=y$.

A third axiom is then added in order to maintain the stratification of types occurring between sets and their member. This axiom takes the following form.

$x\varepsilon y\rightarrow(T_n(x)\leftrightarrow T_{n+1}(y))$.

Fraenkel and Bar-Hillel interpret these results with the following passage: "Half a century after Zermelo and Russell published their theories, independently of each other and starting from seemingly totally different and even contrary approaches, an almost complete reunion of these theories is now in full view" (Fraenkel and Bar-Hillel 1958, p. 191). This process of altering the theory of types from its original many-sorted nature into the one-sorted nature began most distinctly with the use of cumulative types. However, it was not until the transition between many and one-sorted theories was made exact that conclusions such as Fraenkel and Bar-Hillel's could be drawn. Indeed, in his "The inceptions of 'New Foundations'" Quine mentions that "I had not yet appreciated how naturally [Zermelo's] system emerges from the theory of types when we render the types cumulative and describe them by means of general variables. I came to see this only in January 1954" (Quine 1987, p. 287). This date of 1954 refers to the writing of his "Unification of Universes in Set Theory" as described above.

8. Summary and Conclusion

In this thesis the developments of the theory of types have been studied. The theory was originally proposed by Russell as a method of avoiding the set-theoretic paradoxes. Russell did much work on the theory, altering its initial form substantially. By the publication of the first edition of *Principia Mathematica* the theory had attained a secure place in the foundations of Russell's mathematics. However, this formulation of the theory of types still depended upon the axiom of reducibility.

Several logicians responded to the dependency of Russell's type-theory upon the axiom of reducibility negatively; among these were Weyl and Chwistek. Both of these logicians attempted to avoid the axiom of reducibility building mathematics upon purely constructive grounds. However, without that axiom no one was able to retain all of classical mathematics while developing mathematics in their logical system. In the introduction to the second edition of *Principia Mathematica* Russell commended the attempts at avoiding the axiom of reducibility and tried to do so himself. Russell used Wittgenstein's conception of functions as extensional entities and worked out its consequences. However, without the axiom of reducibility Russell was still unable to deal with such common mathematical notions as the least upper bound of a set of real numbers.

The desire to avoid the axiom of reducibility, while maintaining large portions of classical mathematics, also motivated Ramsey's work on the theory of types. Ramsey tried to improve the work done in *Principia Mathematica*. He took Wittgenstein's notion of extensional functions, that Russell had also adopted, one step further. While doing this Ramsey was able to make the distinction between two different kinds of paradoxes – those that he called "logical" and those that he called "semantic". He then argued that only the simple theory of types was needed to avoid the logical paradoxes. Furthermore, the ramified theory, which was the only part in which the axiom of reducibility was used, was developed only to deal with the semantic paradoxes. In Ramsey's conception of logic these paradoxes had no part. Thus the ramified theory, and the axiom of reducibility, could be dispensed with without sacrificing parts of classical mathematics. Despite the fact that all subsequent logicians followed Ramsey's distinction between different kinds of paradoxes, the methods he employed were not generally accepted. For instance, Carnap disagreed vehemently with Ramsey's use of impredicative definitions. However, Carnap did incorporate a simple type-theory into his *Logical Syntax of Language*. In fact, the simple theory of types came to be used by many as the basis of their logic.

It was not until 1931 that the simple theory of types received its first formal treatment. Both Tarski and Gödel formally formulated the simple theory of types. Their forms of the theory were quite influential and became the standard by which many logicians judged their work. In particular, Quine used Tarski's formulation as the starting point for several of his investigations into the possibilities of altering the theory of types. Much work was done on Quine's two works which aimed at improving the theory of types; namely his *New Foundations* and his *Mathematical Logic*. Church also worked on the theory of types and many of his students worked within his type-theory which incorporated his lambda-operator.

It was the simple theory, as opposed to the ramified theory of types that was used almost exclusively during the years following Ramsey's work. However, as seen in the previous chapter, in the 1950's there occurred a revival of the ramified theory of types. This revival of ramified-type theories coincided with the consideration of cumulative type hierarchies. This is most evident in the work of Hao Wang and John Myhill. The consideration of cumulative type-hierarchies altered the form of the theory of types in a substantial way. The theory was altered even more drastically by changing the theory from a many-sorted theory into a one-sorted theory. This final "standardization" of the theory of types in the mid 1950's made it not much different from first-order Zermelo-Fraenkel set-theory. The theory of types whose developments have been traced in this thesis therefore lost its prominence as the foundation for mathematics and logic.

This decline of the prominence of the theory of types emerged from various sources. For instance, the technical complexities of the theory were seen as a burden almost from its initial proposal. Indeed, in Russell and Whitehead's *Principia Mathematica*, instead of using the prescribed type-subscripts a device of typical ambiguity was employed. In this way the tedious application of adding type-indicating subscripts for every formula was avoided. This desire to avoid explicit use of type-subscripts was carried over into Quine's work in, for example, his *New Foundations*. Although many logicians did actually carry out the work of ensuring that type-indices were used conscientiously in their work, there is no denying that this was a tedious affair to be avoided, if possible.

Furthermore, while the theory of types proved to make much work quite technically difficult, the work being done using first-order logic as a basis proved to be relatively not very technically cumbersome. Also, the proof-theoretic techniques available to first-order logic gave ample reasons for many to utilize that system of logic in place of the complicated theory of types. Finally, with the standardization of type theory making it not much different from first-order Zermelo-Fraenkel set theory, the need to use such a complicated device as the theory of types seemed pointless.

Apart from the technical complications there were other drawbacks facing the theory of types. For instance, in Henkin's 1950 paper the theory of types is viewed as a functional calculus of order ω. Thus quantification is allowed to range over the various types while the rules governing each level come from first-order logic. The problem with this ω-order logic is that it is essentially incomplete (whereas first-order logic is, of course, complete). The incompleteness of this logic already occurs at the second order; it was shown by Gödel in his 1931 paper that for the functional calculus of the second-order, no matter what set of axioms are chosen, the system contains a formula which is valid but not a formal theorem. Here a valid formula is one which is true "whenever the individual variables are interpreted as ranging over an arbitrary domain of elements while the functional variables of degree *n* range over all sets of ordered *n*-tuples of individuals" (Henkin 1950, p. 81). Now, since the ω-order logic (the theory of types) contains the second-order logic, it too is incomplete.[82] Thus, the abandonment of the theory of types does not seem too surprising with these considerations. Not only was the theory far too technically complicated, but it also was not as semantically well-behaved as the first-order logic that was to take its place as the foundation for mathematics.

[82] In his paper Henkin shows that with a different definition of validity the second-order calculus, and in fact the ω-order logic, is complete. For his new definition a formula is deemed valid if, again, the individuals are interpreted as ranging over an arbitrary domain of elements, but now the functional variables are interpreted as ranging over an *arbitrary class* of sets of ordered *n*-tuples of individuals (Henkin 1950, pp. 81-82).

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