NON-LINEAR NON-STATIONARY OSCILLATIONS

ANALYSIS OF NON-LINEAR NON-STATIONARY OSCILLATIONS

By

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ABSTRACT

The behaviour of certain non-linear oscillatory systems are studied analytically. These systems are of the "separable" type i.e. they can be modelled using linear frequency-dependent networks, frequency independent non-linear resistive networks, and non-linear reactive networks.

When the time-lags in an oscillatory system are negligibly small, the system may be described by a non-linear differential equation. If the time-lags cannot be ignored, the system may be described by a non-linear difference-differential equation.

The exact analytical solutions of non-linear differential or difference-differential equations are not known, except in rare cases. However, with appropriate restrictions, analytical approximations may be found.

In this work, analytical approximations are developed for treating second-order, forced or unforced weakly non-linear oscillatory systems, as well as a restricted class of unforced highly non-linear systems. These systems may be of the degenerative or regenerative type. Also, the case when time-lags exist in the system, has been studied analytically.

The analytical results are verified either experimentally or by numerical simulation.

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CHAPTER 1

Introduction

1.1 Preface

Most -if not all- physical systems are inherently non-linear. In some applications, the effect of non-linearity may be disregarded over a specific range of operation and the system may be analysed as a linear one. In many others, ignoring the role of non-linearity in the analysis may either result in gross quantitative errors or in completely falsifying the analysis. This can be true even for very "weak" non-linearities.

The most attractive property of "linear" systems is the property of superposition. Most of the progress in the analysis and synthesis of electrical circuits and control systems, for example, has been based on this property. It may be noted, however, that the non-linearity in a system is not necessarily an undesirable property. For example, the proper functioning of the regenerative oscillator - an essential element in a modern communication system - is possible only due to the existence of non-linear regulation in the oscillator circuit. In this case, and in very many others, the non-linearity in the system plays a favourable role.

When time-lags in a dynamic system are negligibly small the system may be satisfactorily characterized by a non-linear differential equation. In the cases where the non-linearity can be ignored, the system may be characterized by a linear differential equation which may be time-invariant or time-variant.

An exact solution of a linear time-invariant differential equation is always possible. The solution is greatly simplified by the fact that it can be constructed as a linear sum of independent simpler solutions. This is often done, conveniently, throught the use of linear transforms.

The analysis of linear time-variant systems is complicated. We are not fortunate in finding exact explicit solutions to linear time-variant equations except for first-order equations and specific classes of higher order equations. However, since the superposition principle still holds, we have the advantage that once the solution of the n-dimensional homogeneous differential equation:

$$\frac{d}{dt} X = A(t)\ X \quad , \tag{1.1}$$

is known, then the solution of the n-dimensional inhomogeneous equation:

$$\frac{d}{dt} X = A(t)\ X + f(t) \quad , \tag{1.2}$$

can readily be obtained.

Several techniques for obtaining approximate solutions of eqn. (1.1) are known [see for example D'Angelo, 1970].

When the existence of non-linearity must be admitted, the analysis is further complicated by the fact that the superposition principle does not hold. It is only in rare cases that exact analytical solutions to non-linear differential equations can be found. One is therefore forced to resort to various means of approximation. Rigorous analysis usually fails to yield a solution of practical use, although it may provide valuable information about the existence and uniqueness of the solution. Analytic ingenuity must then be combined with physical intuition in an attempt to obtain an adequate approximation. Although one may escape the analytic difficulty by resorting to numerical solutions, it should be remembered that a numerical solution provides little insight into the behaviour of the system. A numerical solution is not a substitute for an analytical solution. It can however serve as a useful experiment.

1.2 Scope of Work

This work is primarily concerned with the problem of finding analytical approximations to the non-stationary response of the class of non-linear oscillatory systems described by:

$$\ddot{x} + P(x,\varepsilon t) + \varepsilon f(\dot{x},x,t) = 0 \quad ; \quad 0 < \varepsilon << 1 \text{ and } (\dot{\cdot}) \triangleq \frac{d}{dt} \; .$$

This equation arises in diverse areas and has attracted wide-spread attention. Most of the pertinent literature, however, has been focused on the study of periodic oscillations in autonomous or non-autonomous systems. The condition of periodicity, in fact, simplifies the analysis to a great extent. The analytical methods for studying the non-stationary behaviour of oscillatory systems are not well explored. Some of the methods developed in the literature will be discussed or referred to in the introductions of Chapters 2, 3 and 4 and in appendices B and C.

In Chapter 2, we study a class of weakly non-linear oscillatory systems with emphasis on the case of the self-oscillator with delayed amplitude regulation.

In Chapter 3, a simple procedure for analytically determining the transient response of a class of highly non-linear oscillatory systems is developed.

In Chapter 4, the transient response of a forced weakly non-linear system is studied. The frequency content of the forcing function is restricted to be outside the resonance zones of the system. The magnitude of the forcing function can be relatively large.

In Chapter 5, a non-linear convolution method is developed for studying the effect of external disturbances on the behaviour of a self-oscillatory system.

In Chapter 6, the frequency pulling and the frequency modulation of a self-oscillator due to the presence of external disturbances are studied experimentally.

Numerical simulations have also been carried out to verify some of the analytical results obtained in Chapters 2 to 5.

CHAPTER 2

Analysis of unforced weakly non-linear oscillatory systems

2.1 Introduction

The behaviour of a broad class of physical oscillatory systems can be described by a second order weakly non-linear differential equation which can be written in the normalized form:

$$\ddot{x} + x + \varepsilon\, f(x,\dot{x}) = 0 \qquad (2.1.1)$$

where $0<|\varepsilon|<<1$, $(\dot{\ }) \triangleq \frac{d}{dt}$ and f is generally a non-linear function with respect to x and $\dot{x}$ with $\|f\| = O(1)$ in the region of interest in the x-$\dot{x}$ plane.

When eqn. (2.1.1) represents a non-linear conservative system, then its solution is periodic. This is the case when f=f(x) or when f satisfies the conditions of existence of at least one stable limit cycle where the system becomes conservative as the transients due to the initial departure from the prospective limit cycle fade away. The periodic solution in this case can be determined in a systematic fashion to any desired degree of accuracy by using the well known Poincaré-Lindstedt method.

When eqn. (2.1.1) represents a non-conservative system, where the behaviour is non-stationary, or when it is desired to obtain the transient behaviour of a system which is conservative in the limit, the Poincaré-Lindstedt method cannot be used and a more general technique which enables us to predict with accuracy the transient and the steady state behaviour is desirable.

The general behaviour (stationary or non-stationary) of a system described by eqn. (2.1.1) was studied by v.d. Pol [1934]. Along partly intuitive lines he developed an approximation technique which he used for the study of both free and forced oscillations in a regenerative oscillatory system. The method can also be used for the study of degenerative oscillatory systems.

A variant of the method of v.d. Pol, which is in many cases simpler to apply, was developed by Krylov and Bogoliubov [1937]. The method was later extended to higher order approximations by Bogoliubov and Mitropolsky [1961].

The problem of determining the higher order approximation of the general solution of eqn. (2.1.1) has attracted considerable interest. Struble [1962] used successive iteration where the first approximation obtained by the Krylov-Bogoliubov (K-B) method is reused to obtain a better approximation. Kevorkian and Cole [see Kevorkian, 1960 and Cole, 1968] used a perturbation method which follows closely the Poincaré-Lindstedt method with the additional use of the concept of the fast and slow time scales originated by v.d. Pol and Krylov and Bogoliubov. Nayfeh[1964,65,67,68] developed the so called derivative expansion method which extends the concept of double time scaling to multiple time scaling.

This method not only complicates the analysis to a great extent, but also introduces more arbitrariness into the problem than is needed (in the Poincaré-Lindstedt sense) for the elimination of the unbounded terms in the perturbation solution. This could falsify the evaluation of the higher order approximations. Although the method was mainly devised for determining the higher order non-stationary solution of eqn. (2.1.1), only the first order non-stationary solution (i.e. the v.d. Pol or K-B solution) was obtained in the various cases that Nayfeh studied using this method. Davis and Alfriend [1967] used the method for determining the second approximation of the non-stationary behaviour of v.d. Pol's equation and obtained a result which contains spurious terms. Their analysis will be discussed in Appendix B.

In section (2.2), a variant of the asymptotic method of Bogoliubov and Mitropolsky is described. In section 2.4 the method is used to study the behaviour of a self-oscillator with delayed amplitude regulation.

2.2 The method of analysis

We shall consider a system described by:

$$\ddot{x} + x + \varepsilon f(x, \dot{x}, \varepsilon t) = 0 , \qquad (2.2.1)$$

where f is a non-linear function with respect to x and $\dot{x}$. The restrictions imposed on ε and f are the same as in eqn. (2.1.1). The only difference between equations (2.1.1) and (2.2.1) is the possible explicit dependence of f in the latter on the slow time εt. Thus eqn. (2.2.1) may represent a slowly time-variant system or a time-invariant system which is forced by a small slowly varying function.

The solution of eqn. (2.2.1) will be sought in the form:

$$x = x(\tau,\xi) = A\cos\tau + \varepsilon[C_o + \sum_{n>1} (C_n \cos n\tau + D_n \sin n\tau)] \quad (2.2.2)$$

where

$$\xi \triangleq \varepsilon t$$

and $$\frac{d\tau}{dt} \triangleq \lambda(\xi) = 1 + \varepsilon\,\lambda_1(\xi) + \varepsilon^2\,\lambda_2(\xi) + \ldots \quad (2.2.3)$$

$\lambda(\xi)$ being the instantaneous frequency of oscillation.

The coefficients A, C_o, $C_n\big|_{n>1}$ and $D_n\big|_{n>1}$ are assumed to be functions of the slow time ξ.

We note here that when f takes the form:

$$f(x,\dot{x},\xi) = f_1(x,\dot{x}) + G(\xi)\; f_2(x,\dot{x})$$

where f_1 and f_2 are anti-symmetric functions with respect to x and $\dot{x}$, then only odd values of n need be considered.

Now

$$\frac{dx}{dt} = \frac{\partial x}{\partial \tau}\frac{d\tau}{dt} + \varepsilon \frac{\partial x}{\partial \xi} = \lambda(\xi)\frac{\partial x}{\partial \tau} + \varepsilon \frac{\partial x}{\partial \xi} \tag{2.2.4}$$

and

$$\frac{d^2x}{dt^2} = \lambda^2(\xi)\frac{\partial^2 x}{\partial \tau^2} + 2\varepsilon\lambda(\xi)\frac{\partial^2 x}{\partial \tau \partial \xi} + \varepsilon \frac{d\lambda}{d\xi}\frac{\partial x}{\partial \tau} + \varepsilon^2 \frac{\partial^2 x}{\partial \xi^2} \tag{2.2.5}$$

where from expression (2.2.2):

$$\frac{\partial x}{\partial \tau} = -A\sin\tau + \varepsilon\Big[\sum_{n>1} n(-C_n \sin n\tau + D_n \cos n\tau)\Big]$$

$$\frac{\partial^2 x}{\partial \tau^2} = -A\cos\tau - \varepsilon\Big[\sum_{n>1} n^2(C_n \cos n\tau + D_n \sin n\tau)\Big] \tag{2.2.6}$$

and

$$\frac{\partial^k x}{\partial \xi^k} = \frac{\partial^k A}{\partial \xi^k}\cos\tau + \varepsilon\Big[\frac{\partial^k C_o}{\partial \xi^k} + \sum_{n>1}\Big(\frac{\partial^k C_n}{\partial \xi^k}\cos n\tau + \frac{\partial^k D_n}{\partial \xi^k}\sin n\tau\Big)\Big]$$

$k=1,2,\ldots\ldots$

Thus $\dot{x}$ and $\ddot{x}$ will take the form of Fourier series with slowly varying coefficients. Also using expression (2.2.2) and the resulting expression for $\dot{x}$, the non-linear function f can be written as:

$$\varepsilon f(x,\dot{x},\xi) = \sum_{m\geq 1} \varepsilon^m[P_{1m}\cos\tau + Q_{1m}\sin\tau] + \sum_{\substack{n>0\\ n\neq 1}}\sum_{m\geq 2} \varepsilon^m[P_{nm}\cos n\tau + Q_{nm}\sin n\tau] \tag{2.2.7}$$

where P_{nm} and Q_{nm} are slowly varying functions of time.

Furthermore, we shall make use of the series expansions:

$$
\begin{aligned}
A &\triangleq A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \ldots \\
\varepsilon C_n &\triangleq \varepsilon C_{n1} + \varepsilon^2 C_{n2} + \ldots \\
\varepsilon D_n &\triangleq \varepsilon D_{n1} + \varepsilon^2 D_{n2} + \ldots
\end{aligned}
\tag{2.2.8}
$$

Now substituting the expressions for x, $\ddot{x}$ and εf in eqn. (2.2.1) and equating the coefficients of sin $n\tau$ and cos $n\tau$ (n=0, 1,2,...) separately to zero we obtain a system of equations which will generally take the form:

$$
\begin{aligned}
&\varepsilon H_1\left(\frac{dA_0}{d\xi}, A_0, \xi\right) + \varepsilon^2 H_2\left(\frac{dA_1}{d\xi}, A_1, \frac{dA_0}{d\xi}, A_0, \lambda_1, C_{n1}, D_{n1}, \xi\right) + \ldots = 0 \\
&\varepsilon E_1(\lambda_1, A_0, \xi) + \varepsilon^2 E_2\left(\lambda_2, \lambda_1, \frac{d^2A_0}{d\xi^2}, \frac{dA_0}{d\xi}, A_0, \lambda_1, C_{n1}, D_{n1}, \xi\right) + \ldots = 0 \\
&\varepsilon\, \Lambda_1(C_{n1}, A_0, \xi) + \varepsilon^2\, \Lambda_2(C_{n2}, \ldots) + \ldots = 0 \\
&\varepsilon\, \Gamma_1(D_{n1}, A_0, \xi) + \varepsilon^2 \Gamma_2(D_{n2}, \ldots) + \ldots = 0 \\
&\quad \vdots \qquad \vdots
\end{aligned}
\tag{2.2.9}
$$

Finally, based on their order of smallness, the terms $\varepsilon H_1(\cdot)$, $\varepsilon E_1(\cdot)$, $\varepsilon \Lambda_1(\cdot)$, $\varepsilon \Gamma_1(\cdot)$, $\varepsilon^2 H_2(\cdot)$, $\varepsilon^2 E_2(\cdot)$, ... are equated separately to zero. Thus we obtain a system of equations which can be solved sequentially to yield A_0, λ_1, C_{n1}, D_{n1}, A_1, λ_2, ... as functions of the slow time ξ and hence the solution for x (expression (2.2.2)) is obtained.

If only a second approximation is required, the following expressions [which are obtained from expressions (2.2.2), (2.2.4) and (2.2.5) with the use of (2.2.8)] will be needed:

$$\begin{aligned}\frac{d^2x}{dt^2} + x = &- \varepsilon\left[2\frac{dA_0}{d\xi}\sin\tau + 2\lambda_1 A_0 \cos\tau\right] + \varepsilon C_0 \\ &- \varepsilon \sum_{n\geq 2} (n^2-1)(C_n \cos n\tau + D_n \sin n\tau) \\ &- \varepsilon^2\left[\left\{2\frac{dA_1}{d\xi} + 2\lambda_1\frac{dA_0}{d\xi} + A_0\frac{d\lambda_1}{d\xi}\right\}\sin\tau\right. \\ &\left. + \left\{A_0(\lambda_1^2 + 2\lambda_2) + 2\lambda_1 A_1 - \frac{d^2A_0}{d\xi^2}\right\}\cos\tau\right] \\ &+ O(\varepsilon^3)\end{aligned} \tag{2.2.10}$$

$$\begin{aligned}\varepsilon\frac{dx}{dt} = &- \varepsilon A_0 \sin\tau + \varepsilon^2\left[-(A_1+\lambda_1 A_0)\sin\tau + \frac{dA_0}{d\xi}\cos\tau\right] \\ &+ \varepsilon^2 \sum_{n\geq 2} n\{-C_n \sin n\tau + D_n \cos n\tau\} \\ &+ O(\varepsilon^3)\end{aligned} \tag{2.2.11}$$

2.3 The self-oscillator with delayed amplitude regulation.

The basic self-oscillator can be represented by a dissipative tuned circuit connected in parallel with a voltage controlled non-linear resistance exhibiting local differential negative resistivity. The equivalent circuit of such an oscillator is shown in Fig. 2.1-a , and a typical characteristic of the non-linear negative resistance is shown in Fig. 2.1-b . The (differential) negative resistivity is necessary for regeneration and the non-linearity is necessary for amplitude regulation.

In the study of this oscillator, it is customary to assume that the non-linear negative resistance is of zero-memory i.e. that the current in the resistance is a single-valued function of the voltage across it. In a feedback oscillator [Fig. 2.1-c], the above assumption implies a zero-phase-shift feed-back circuit.

The oscillator with phase-shifted amplitude regulation has been studied by Golay [1964]. The oscillator is represented by a dissipative tuned circuit connected in parallel with a negative resistance the value of which depends upon the square of the oscillator voltage phase-shifted by a first order R-C filter. The circuit of this oscillator is shown in Fig. 2.2. The squaring device in Fig. 2.2 is assumed to be of infinite input impedance and zero output impedance. The oscillator can be described by the normalized equations:

$$T \frac{dz}{dt} + z = v^2 \qquad (2.3.1)$$

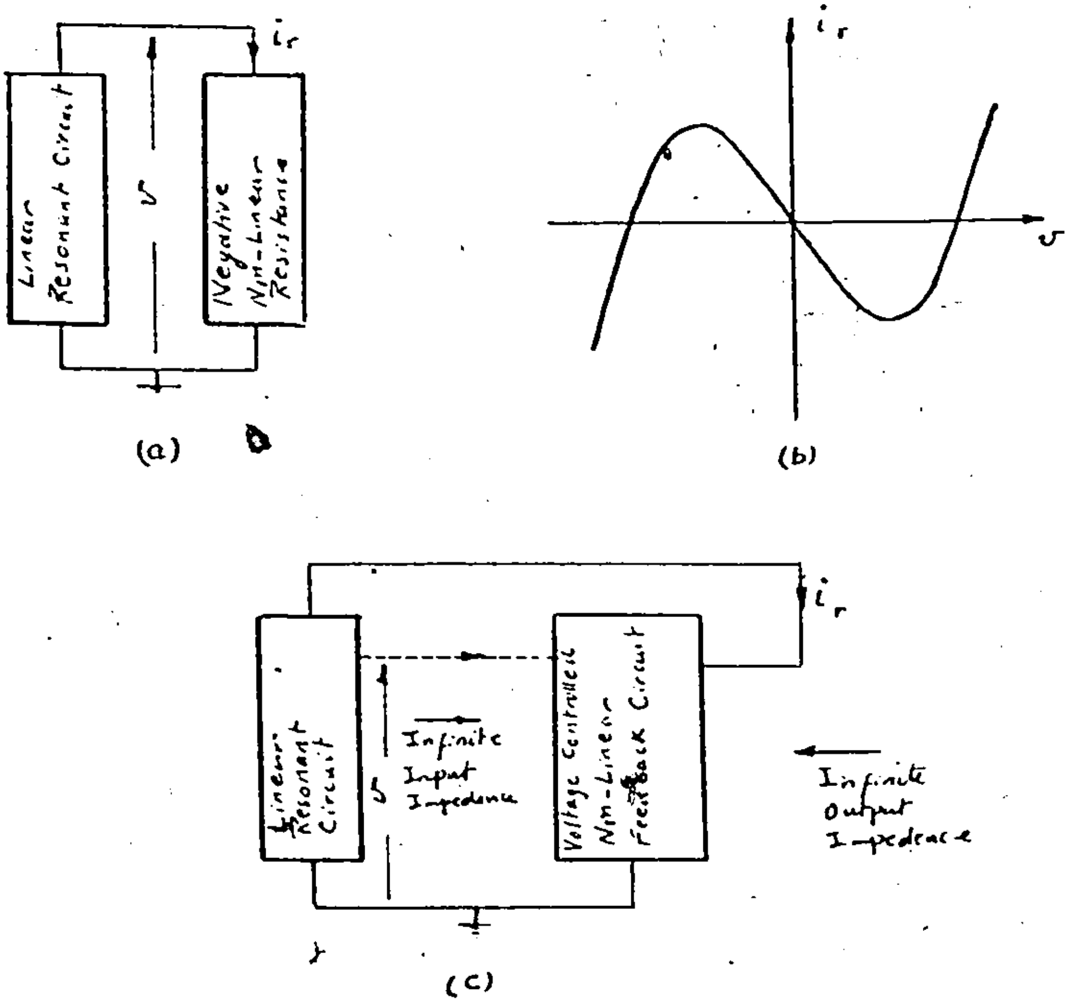

Fig. 2.1 (a) Equivalent circuit of a basic oscillator

(b) Typical characteristic of the non-linear negative resistance .

(c) Feedback Oscillator .

and

$$\frac{d^2v}{dt^2} + v = 2\varepsilon \frac{d}{dt}[v(1-z)] \qquad (2.3.2)$$

where $v \triangleq \frac{V}{V_0}$, $z \triangleq \frac{Z}{V_0^2}$, $\varepsilon \triangleq \frac{g}{2R}\sqrt{\frac{L}{C}} > 0$, $T \triangleq \frac{R_1C_1}{\sqrt{LC}}$ and $\frac{d}{dt} \triangleq \sqrt{LC}\,\frac{d}{d\bar{t}}$.

V is the voltage across the tuned circuit, Z is the output voltage of the R_1-C_1 circuit, $\bar{t}$ is the actual time variable and t is the normalized time variable. L, C, R, R_1, C_1, g and V_0 are the parameters of the tuned circuit and the feedback circuit as shown in Fig. 2.2 .

When $T \to 0$, equations (2.3.1) and (2.3.2) tend to v.d.Pol's equation.

Golay studied the case when $\varepsilon \ll 1$ and $T \gg 1$. Assuming the oscillator voltage to be almost sinusoidal:

$$v \approx A\cos(\omega t+\phi) ,$$

then for $T \gg 1$ the second harmonic component of the voltage across C_1 is negligibly small. To a first approximation, the steady state solution of equations (2.3.1) and (2.3.2) obtained by Golay is:

$$v \approx \sqrt{2}\cos(t+\phi) , \quad \phi \text{ being a constant.}$$

Later, Scott [1966] obtai[illegible] a first approximation of the steady state amplitude of oscillation [illegible]r arbitrary values of T and

Nayfeh [1967] obtained a first approximation of the transient and steady state amplitude and frequency.

The method used by Scott [2] is somewhat complicated and the transient solution obtained by Nayfeh [4] is in error as will be shown below. Actually, a first-order approximation of the steady-state solution can be obtained in quite an elementary manner:

Let

$$v \approx A\cos(\omega t+\phi) \,, \tag{2.3.3}$$

where A and ϕ are constants, then the steady state solution of eqn. (2.3.1) is:

$$z = \frac{A^2}{2}\left[1 + \frac{1}{1+4\omega^2T^2}\cos 2(\omega t+\phi) + \frac{2\omega T}{1+4\omega^2T^2}\sin 2(\omega t+\phi)\right] \tag{2.3.4}$$

Using expressions (2.3.3) and (2.3.4), eqn. (2.3.2) becomes:

$$A\left[\omega^2 - 1 - \frac{\varepsilon\omega TA^2}{1+4\omega^2T^2}\right]\cos\omega t - 2\varepsilon\omega A\left[1 - \frac{A^2}{2} - \frac{A^2}{4(1+4\omega^2T^2)}\right]\sin\omega t \approx 0$$

where the third harmonic component has been neglected.

Hence, in accordance with the principle of harmonic balance:

$$\omega^2 = 1 + \frac{\varepsilon\omega TA^2}{1+4\omega^2T^2} = 1 + 0(\varepsilon) \approx 1 + \frac{\varepsilon TA^2}{1+4T^2}$$

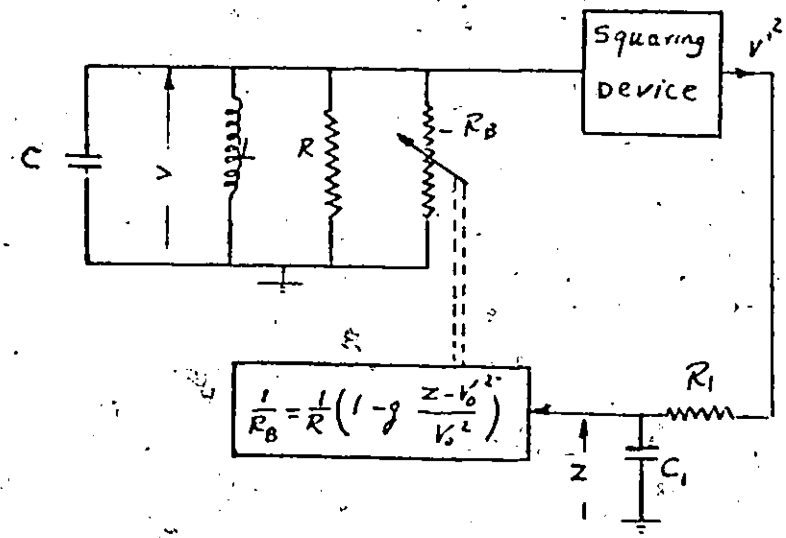

Fig. 2.2 Golay's model of a self-oscillator with phase-shifted amplitude regulation .

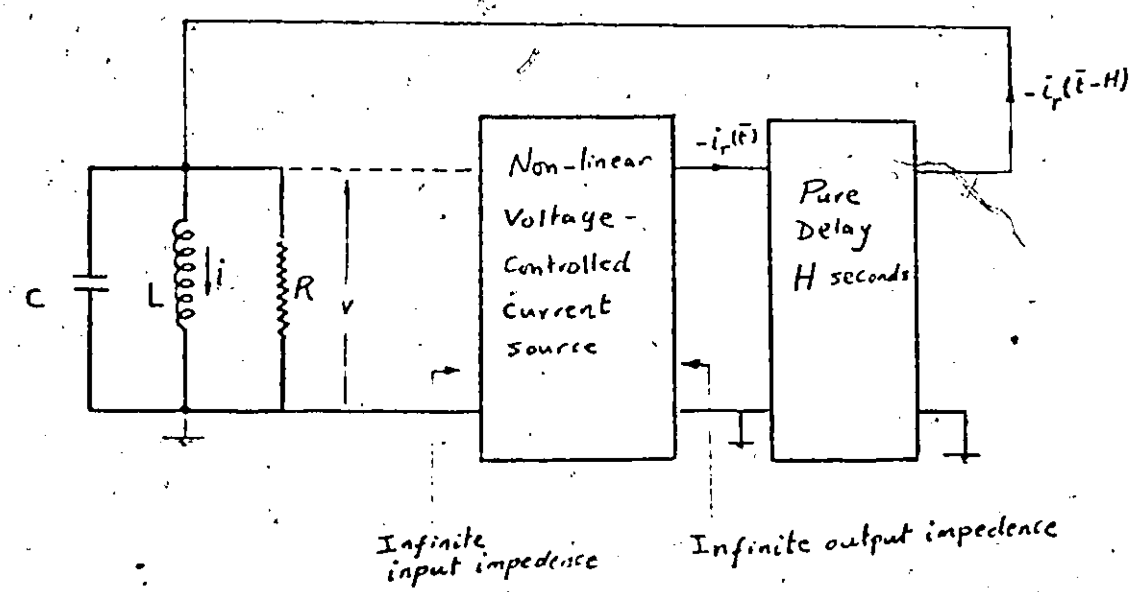

Fig.2.3 A model for the self-oscillator with delayed amplitude regulation

and $$A^2 = \frac{4[1+4\omega^2T^2]}{3+8\omega^2T^2} \approx \frac{4[1+4T^2]}{[3+8T^2]} \tag{2.3.5}$$

Thus

$$\omega \approx 1 + \frac{\varepsilon T A^2}{2(1+4T^2)} \approx 1 + \frac{2\varepsilon T}{3+8T^2} \tag{2.3.6}$$

The transient behaviour of this oscillator is very difficult to determine analytically, except for the special case when $T \to 0$ where equations (2.3.1) and (2.3.2) reduce to v.d.Pol's equation. The assumption of slowly varying coefficients can only be made when $T \to 0$ or when $T>>1$ but not for intermediate values of T since fast varying transients will appear at the output of the R_1-C_1 circuit.

Let us consider the case when $T>>1$.

Let $\frac{1}{T} = O(\varepsilon) \triangleq \varepsilon\,\kappa$ where $\kappa = O(1)$.

Eqn. (2.3.1) is then rewritten as:

$$\frac{dz}{d\xi} + \kappa\, z = \kappa\, v^2 \tag{2.3.7}$$

To a first approximation:

$$v \approx A_0(\xi) \cos\tau \quad \text{with} \quad \frac{d\tau}{dt} \approx 1 + \varepsilon\, \lambda_1(\xi)$$

The r.h.s. of eqn. (2.3.7) becomes:

$$\frac{1}{2}\kappa\, A_0^2(\xi)\, [1 + \cos 2\tau]$$

In this case the second harmonic component in z is negligibly small, therefore equations (2.3.7) and (2.3.2) may be written as:

$$\frac{dz}{d\xi} + \kappa\, z \approx \frac{1}{2}\kappa\, A_0^2(\xi) \qquad \kappa = 0(1) \tag{2.3.8}$$

and

$$\frac{d^2v}{dt^2} + v = 2\varepsilon \frac{d}{dt}[v(1-z)] \approx 2\,\varepsilon(1-z)\,\frac{dv}{dt} \tag{2.3.9}$$

$$\left[\text{ since } \quad \varepsilon \frac{dz}{dt} \equiv \varepsilon^2 \frac{dz}{d\xi} = 0(\varepsilon^2)\right]$$

Considering only the first order terms in expressions (2.2.10) and (2.2.11) and following the procedure described in section (2.2) then A_0 and λ_1 are related by:

$$\frac{dA_0}{d\xi} = A_0(1-z) \tag{2.3.10}$$

and

$$\lambda_1 A_0 = 0 \tag{2.3.11}$$

For a non-trivial solution, $\lambda_1 = 0$.

Equations (2.3.8) and (2.3.10) must then be solved simultaneously to yield A_0 and z. The eqn. for A_0 is:

$$\frac{d^2A_0}{d\xi^2} + \kappa \frac{dA_0}{d\xi} - \frac{1}{A_0}\left(\frac{dA_0}{d\xi}\right)^2 + \kappa\, A_0\left(\frac{A_0^2}{2} - 1\right) = 0 \tag{2.3.12}$$

Using the derivative-expansion method, Nayfeh [1967] obtained equations for A_0 and z which, when T>>1, become equivalent to equations (2.3.8) and (2.3.10). However, in Nayfeh's analysis, the assumption of slowly varying coefficients was indiscriminately made for all values of T. Furthermore, eqn. (2.3.8) was solved, as though A_0 was independent of ξ, to yield

$$z = c\, e^{-\kappa\xi} + \frac{1}{2}A_0^2(\xi) \quad , \text{ where c is a constant.} \tag{2.3.13}$$

The first term in eqn. (2.3.13) was ignored. Thus eqn. (2.3.10) reduces to:

$$\frac{dA_0}{d\xi} = A_0\left(1 - \frac{A_0^2}{2}\right) \tag{2.3.14}$$

We note that although the steady state solutions of equations (2.3.12) and (2.3.14) are identical, their transient solutions are not in agreement, even qualitatively.

In a paper published later by Nayfeh [1968], forced oscillations in the above system were studied. However, the same mistakes, pointed out above, were repeated and the results given in that paper are doubtful.

It may be noted that Golay's model appears to be a rather crude representation of a real system. In fact, a thorough theoretical analysis of a more general class of oscillators has been developed earlier (Gladwin [1955]).

In this section, we shall study an oscillator with delayed amplitude regulation. The circuit of this oscillator is shown in

Fig. 2.3 . The non-linear negative resistance [realized by the feed-back circuit in Fig. 2.3] is of the v.d.Pol type, i.e. i_r in Fig. 2.3 is related to the oscillator voltage v by:

$i_r = -av + bv^3$, where a and b are positive constants .

The oscillator can be described by the normalized differential-difference equation:

$$\ddot{x} + x + \varepsilon\,[\alpha\dot{x} - \beta\dot{x}(t-h) + \frac{1}{3}\dot{x}^3(t-h)] = 0 \qquad (2.3.15)$$

where

$(\cdot) = \frac{d}{dt}$, $t = \frac{1}{\sqrt{Lc}}\,\bar{t}$ [$\bar{t}$ and t are the actual and normalized time variable respectively], $0 < \varepsilon = (a-\frac{1}{R})\sqrt{\frac{L}{c}} \ll 1$, $\varepsilon\alpha = \frac{1}{R}\sqrt{\frac{L}{c}}$, $\varepsilon\beta \triangleq \varepsilon\,(1+\alpha) = a\sqrt{\frac{L}{c}}$ and $h = \frac{H}{\sqrt{Lc}}$ is the normalized time-delay in the feedback loop. It is assumed that α, β and h are O(1).

x and $\dot{x}$ represent the normalized current in the inductor L and the voltage across the tuned circuit respectively. They are related to the actual inductor current i and the oscillator voltage v by:

$$x = \sqrt{\frac{3b}{a-\frac{1}{R}}\,\frac{L}{c}}\; i \quad \text{and} \quad \dot{x} = \sqrt{\frac{3b}{a-\frac{1}{R}}}\; v$$

when h → 0, eqn. (2.3.15) becomes Rayleigh's differential equation.

Eqn. (2.3.15) will be [solv]ed using the procedure described in section 2.2 with some special considerations. These will be discussed below.

2.4 Solution of a difference-differential equation representing a weakly non-linear oscillatory system .

Consider an oscillatory system described by :

$$\ddot{x} + x + \varepsilon f_1(x,\dot{x}) + \varepsilon f(x(t-h)\ ,\ \dot{x}(t-h)) = 0 \qquad (2.4.1)$$

where $|\varepsilon| \ll 1$ and $\|f_1\|$, $\|f\|$, and h are O(1) . f_1 and f are generally non-linear functions of their arguments .

Let it be specified that $x(t) = u(t)$ and $\dot{x}(t) = v(t)$ in the time interval $-h \leq t \leq 0^+$. Let us first consider the case when f_1 is a linear function , denoted by $L(x,\dot{x})$. Thus eqn.(2.4.1) becomes :

$$\ddot{x} + x + \varepsilon L(x,\dot{x}) = - \varepsilon f(x(t-h)\ ,\ \dot{x}(t-h)) \qquad (2.4.2)$$

Eqn.(2.4.2) can be considered as a piece-wise linear differential equation for which an exact solution can be obtained . Since x and $\dot{x}$ can be specified as initial conditions in the time interval $-h \leq t \leq 0^+$, eqn.(2.4.2) can be solved to determine the behaviour of the system in the time interval $0 \leq t \leq h$. This in turn is used to set the initial conditions for the subsequent interval $h \leq t \leq 2h$, and to update the r.h.s. of eqn. (2.4.2) which will then determine the behaviour in the interval $h \leq t \leq 2h$. One can proceed in this manner updating the values of the initial conditions and the function f from one interval to the next . Such a procedure is of course impractically labourious and in addition – like numerical solutions– it lacks the prediction power of a sound analytical solution .

This fact can best be demonstrated by considering the simpler case of a first order linear **difference-differential equation:**

$$\dot{\chi} + \varepsilon\chi(t-1) = 0 \tag{2.4.3}$$

with the initial conditions $\chi = 1$ for $-1 \leq t \leq 0^+$.

The exact solution of this equation, obtained by the above procedure, is:

$$\chi = \begin{cases} 1 - \varepsilon t & 0 \leq t \leq 1 \\ 1 - \varepsilon t + \frac{\varepsilon^2}{2!}(t^2-1) & 1 \leq t \leq 2 \\ 1 - \varepsilon t + \frac{\varepsilon^2}{2!}(t^2-1) + \frac{\varepsilon^3}{3!}(t+1)^2(2-t) & 2 \leq t \leq 3 \\ \vdots & \\ 1 + \sum_{n=0}^{m} (-1)^n \varepsilon^{n+1} \frac{(t+1)^n}{n!} \left(1 - \frac{t+1}{n+1}\right) & 0 \leq m \leq t \leq m+1 \end{cases}$$

m being an integer

This form is too cumbersome to be useful in deriving information about the nature of the solution as t increases. It should be remembered that the terms $\varepsilon^n t^n$, $n=1,2,\ldots$ are "secular" terms that cannot be neglected.

The exact piece-wise linear solution of eqn. (2.4.2) is much more complex. Moreover, it might not be possible to find a general formula as in the above case. One must then seek some form of approximation.

We shall seek an approximate solution of eqn. (2.4.2) in the form of expression (2.2.2). Let x_e and x_a denote the exact and approximate solutions respectively. Thus:

$$\ddot{x}_e + x_e + \varepsilon L(x_e, \dot{x}_e) = -\varepsilon f(x_e(t-h), \dot{x}_e(t-h)) \triangleq -\varepsilon f_{eh} \qquad (2.4.5)$$

and

$$\ddot{x}_a + x_a + \varepsilon L(x_a, \dot{x}_a) = -\varepsilon f(x_a(t-h), \dot{x}_a(t-h)) \triangleq -\varepsilon f_{ah} \qquad (2.4.6)$$

where x_a is chosen to approximate the solution $\forall t \geq 0$.
Note that $x_e(t-h)$ is determined before $x_e(t)$ while $x_a(t-h)$ is the backward extrapolation of $x_a(t)$.

It is assumed that:

$$\frac{\|x_a - u\|}{\|x_a\|} \text{ and } \frac{\|\dot{x}_a - v\|}{\|\dot{x}_a\|} \text{ for } -h \leq t \leq 0^+ \text{ are of the order of unity}$$

(or smaller). Thus

$$\frac{\|f_{ah} - f_{eh}\|}{\|f_{ah}\|} = O(1) \qquad 0 \leq t \leq h \qquad (2.4.7)$$

Now with the initial values of the approximate solution chosen such that:

$x_a(0^+) = u(0^+)$ and $\dot{x}_a(0^+) = v(0^+)$, and since $h=O(1)$, then in view of (2.4.7), the relative errors:

$$\frac{\|x_a - x_e\|}{\|x_a\|} \text{ and } \frac{\|\dot{x}_a - \dot{x}_e\|}{\|\dot{x}_a\|} \text{ are } O(\varepsilon) \quad \text{for} \quad 0 \le t \le h \qquad (2.4.8)$$

$$\text{and consequently } \frac{\|f_{ah} - f_{eh}\|}{\|f_{eh}\|} = O(\varepsilon) \quad \text{for} \quad h \le t \le 2h \qquad (2.4.9)$$

which indicates that the large error due to the initial discontinuity at $t=0$ will have less influence on the accuracy of the solution as t increases. However due to a relative error of $O(\varepsilon)$ at $t=h$ (in view of (2.4.8)), the approximate solution must be limited to the first order.

Now if we set

$$\dot{x}_a(0^+) = (1 + \varepsilon\nu_1)\ u(0^+)$$

and

$$\dot{x}_a(0^+) = (1 + \varepsilon\nu_2)\ v(0^+)$$

where ν_1 and ν_2 are constants, chosen such that:

$$x_a(h) = x_e(h) \text{ and } \dot{x}_a(h) = \dot{x}_e(h) \qquad (2.4.10)$$

then the relative error will be $O(\varepsilon)$ in the interval $0 \le t \le h$ and $O(\varepsilon^2)$ in the next interval. Equivalently, if the exact solution is determined for $0 \le t \le h$, and letting x_a approximate the solution for $t \ge h$

using the initial conditions in (2.4.10), then by taking sufficient terms in the series expansion of x_a [expression (2.2.2)] the entire solution can be determined to the second approximation.

By the same reasoning, if we obtain the exact solution in the interval $0 \le t \le 2h$ and let x_a approximate the solution for $t \ge 2h$ using the initial conditions.

$$x_a(2h) = x_e(2h) \quad \text{and} \quad \dot{x}_a(2h) = \dot{x}_e(2h) \quad ,$$

then by choosing sufficient terms in the series expansion of x_a the third order solution can be obtained. However we shall limit the solution to the second approximation.

The above argument can be extended to the case when $f_1(\cdot)$ is non-linear [eqn. (2.4.1)] except that in this case it is not generally feasible to obtain the required exact solution over the interval $0 \le t \le h$. This does not, however, add much difficulty since an accurate solution of eqn. (2.4.1) valid over a period of $O(1)$ can be obtained by straightforward iteration.

The importance of properly choosing the initial point for the approximate solution is demonstrated by the simple example below:

Let us consider the linear equation:

$$\ddot{x} + x + \varepsilon\,\dot{x}(t-h) = 0 \qquad (2.4.11)$$

An approximate solution is sought in the form:

$$x_a = A(\xi) \cos\tau \qquad (2.4.12)$$

where $\xi \triangleq \varepsilon t$ and $\frac{d\tau}{dt} = 1 + \varepsilon\,\lambda_1(\xi) + \varepsilon^2\lambda_2(\xi)$

In this case it will not be necessary to use a series expansion for A.

Now $\dot{x}_a = -A\frac{d\tau}{dt}\sin\tau + \varepsilon\frac{dA}{d\xi}\cos\tau$

$$\ddot{x}_a + x_a = -2\varepsilon[\lambda_1 A\cos\tau + \frac{dA}{d\xi}\sin\tau]$$

$$+\,\varepsilon^2[\{\frac{d^2A}{d\xi^2} - A(\lambda_1^2 + 2\lambda_2)\}\cos\tau - \{A\frac{d\lambda_1}{d\xi} + 2\lambda_1\frac{dA}{d\xi}\}\sin\tau] \qquad (2.4.13)$$

$$\dot{x}_{ah} = -A_h\frac{d\tau_h}{dt}\sin\tau_h + \varepsilon\frac{dA_h}{d\xi}\cos\tau_h$$

where a variable subscripted by h denotes its value at the instant t-h.

We shall make use of the following expansions:

$$A_h = A - h\frac{dA}{dt} + \frac{h^2}{2}\frac{d^2A}{dt^2} + \dots$$

$$\equiv A - \varepsilon h\frac{dA}{d\xi} + \frac{\varepsilon^2h^2}{2}\frac{d^2A}{d\xi^2} \qquad (2.4.14)$$

$$\tau_h = \tau(t-h) = \tau - h\frac{d\tau}{dt} + \frac{h^2}{2}\frac{d^2\tau}{dt^2} + \dots$$

$$= \tau - h(1 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2) + \frac{h^2}{2}\varepsilon^2 \frac{d\lambda_1}{d\xi} + 0(\varepsilon^3)$$

$$\cos\tau_h = \cos(\tau - h) + \varepsilon h \lambda_1 \sin(\tau - h) + \ldots$$

$$\sin\tau_h = \sin(\tau - h) - \varepsilon h \lambda_1 \cos(\tau - h) + \ldots$$

Thus

$$\varepsilon\dot{x}_{ah} = \varepsilon\lambda[\sin h \cos\tau - \cos h \sin\tau]$$

$$+ \varepsilon^2[(m_1 \frac{dA}{d\xi} + m_2 A\lambda_1) \cos\tau + (m_2 \frac{dA}{d\xi} - m_1 A\lambda_1)\sin\tau]$$

where $m_1 = \cos h - h \sin h$ and $m_2 = h \cos h + \sin h$

Using expressions (2.4.13) and (2.4.15) in eqn. (2.4.11) and equating the coefficients of $\cos\tau$ and $\sin\tau$ separately to zero we get:

$$(-2\lambda_1 + \sin h)\, A + \varepsilon[\frac{d^2A}{d\xi^2} + m_1 \frac{dA}{d\xi} + A(m_2\lambda_1 - \lambda_1^2 - 2\lambda_2)] = 0 \quad (2.4.16)$$

and

$$[2 - \varepsilon(m_2 - 2\lambda_1)]\frac{dA}{d\xi} + A[\cos h + \varepsilon(m\lambda_1 + \frac{d\lambda_1}{d\xi})] = 0 \quad (2.4.17)$$

The terms of similar order of smallness in eqn. (2.4.16) may be separately equated to zero, while-after λ_1 is determined - eqn. (2.4.17) can be solved in its present form. Thus:

$$\tau = [1 + \frac{\varepsilon}{2} \sin h + \frac{\varepsilon^2}{8} (2h \sin 2h - \cos 2h)]\ t + \tau(0)\ ,$$

and (2.4.18)

$$A = A(0)\ e^{-\eta t}$$

where

$$\eta = \varepsilon\left[\frac{\cos h - \frac{\varepsilon h}{2} \sin^2 h + \frac{\varepsilon}{4} \sin 2h}{2 - \varepsilon h \cos h}\right]$$

Let the initial conditions be:

$$x(t) = x_0\ \delta(t) \qquad -h \le t \le 0$$

and $\dot{x}(t) = y_0\ \delta(t)$

If we select the initial point of the approximate solution to be at $t = 0$, then the constants $A(0)$ and $\tau(0)$ are given by:

$$A(0) = \sqrt{x_0^2 + \left[\frac{y_0 + \eta x_0}{\lambda}\right]^2} \text{ and } \tau(0) = \tan^{-1} - \left[\frac{y_0 + \eta x_0}{\lambda x_0}\right], \quad (2.4.19)$$

where $\lambda = 1 + \epsilon\lambda_1 + \epsilon^2\lambda_2 = 1 + \frac{\epsilon}{2} \sin h + \frac{\epsilon^2}{8}(2h \sin 2h - \cos 2h)$

The exact solution of eqn. (2.4.11) in the interval $0 \le t \le h$ is: $x_e = x_0 \cos t + y_0 \sin t$. Therefore if we select the initial point of x_a to be at $t=h$, then t in expressions (2.4.18) should be replaced by $(t-h)$, $A(0)$ and $\tau(0)$ should be replaced by: $A(h)$ and $\tau(h)$ given by:

$$A(h) = \sqrt{x_e^2(h) + \left[\frac{\dot{x}_e(h) + \eta x_e(h)}{\lambda}\right]^2} \text{ and } \tau(h) = \tan^{-1} - \left[\frac{\dot{x}_e(h) + \eta x_e(h)}{\lambda}\right]$$

(2.4.20)

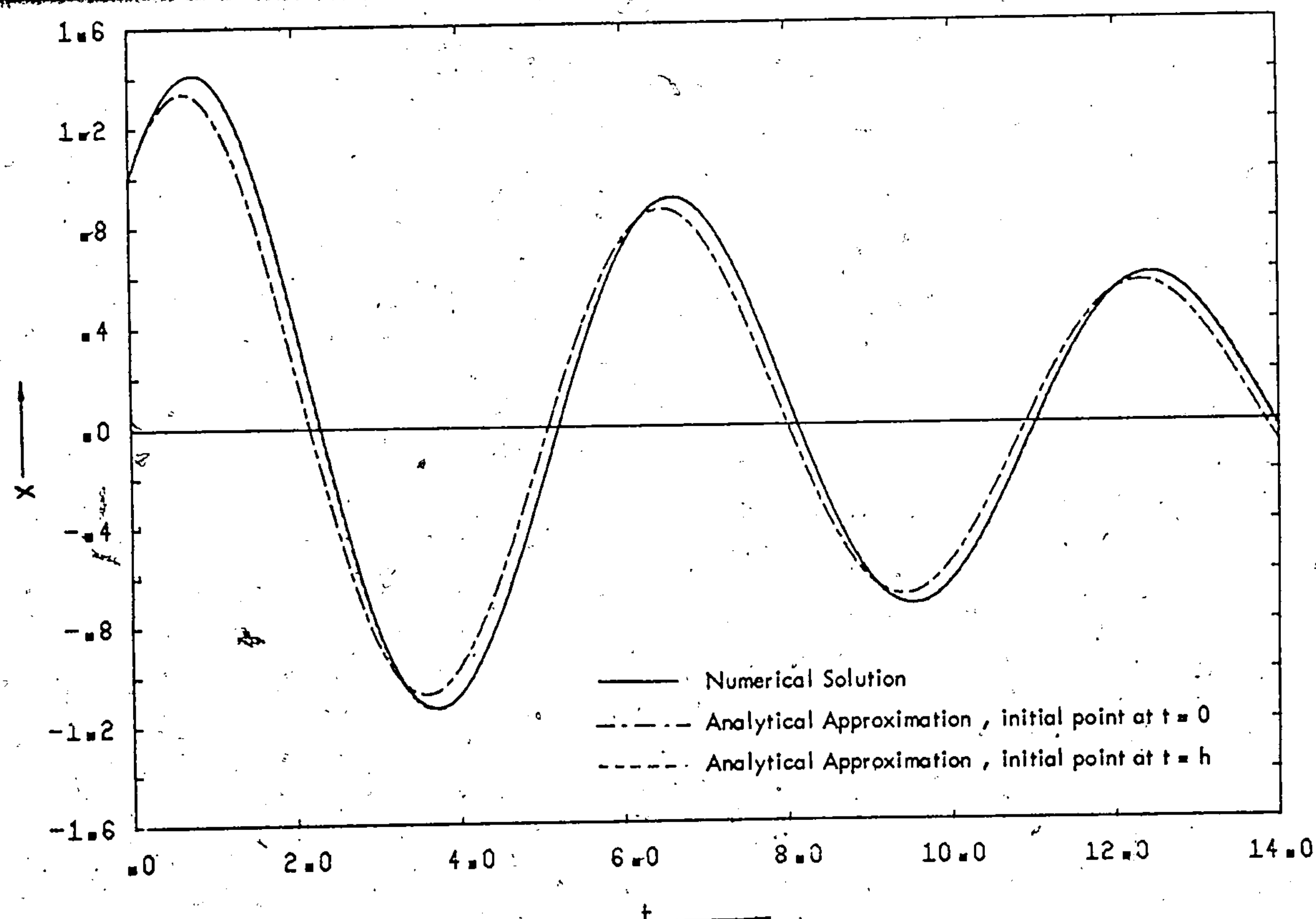

Fig. 2.4–a Solution of eqn. (2.4.11) for $\varepsilon = 0.2$ and $h = \pi/4$, x versus t .

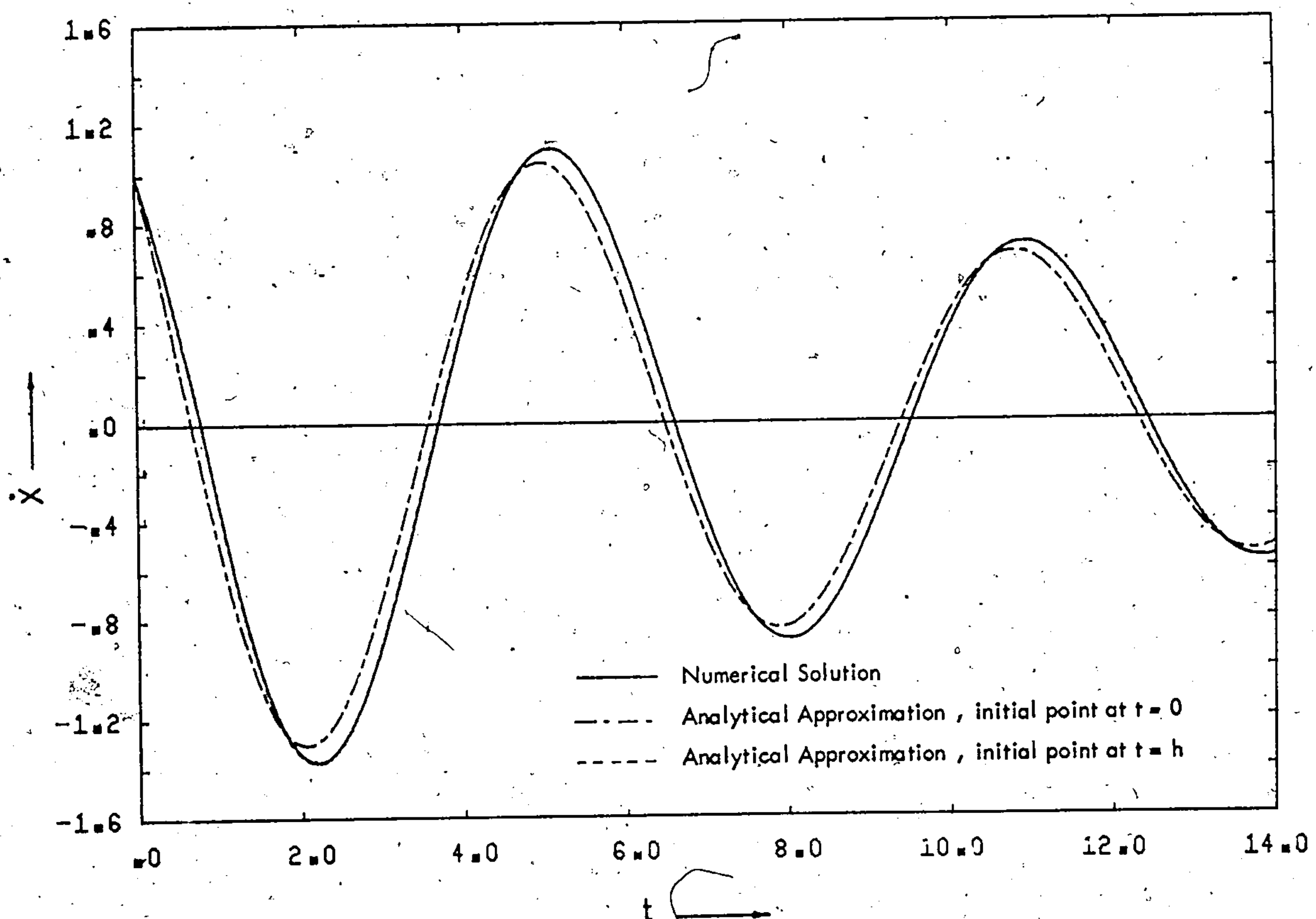

Fig. 2.4-b Solution of eqn.(2.4.11) for $\varepsilon = 0.2$ and $h = \pi/4$, $\dot{x}$ versus t .

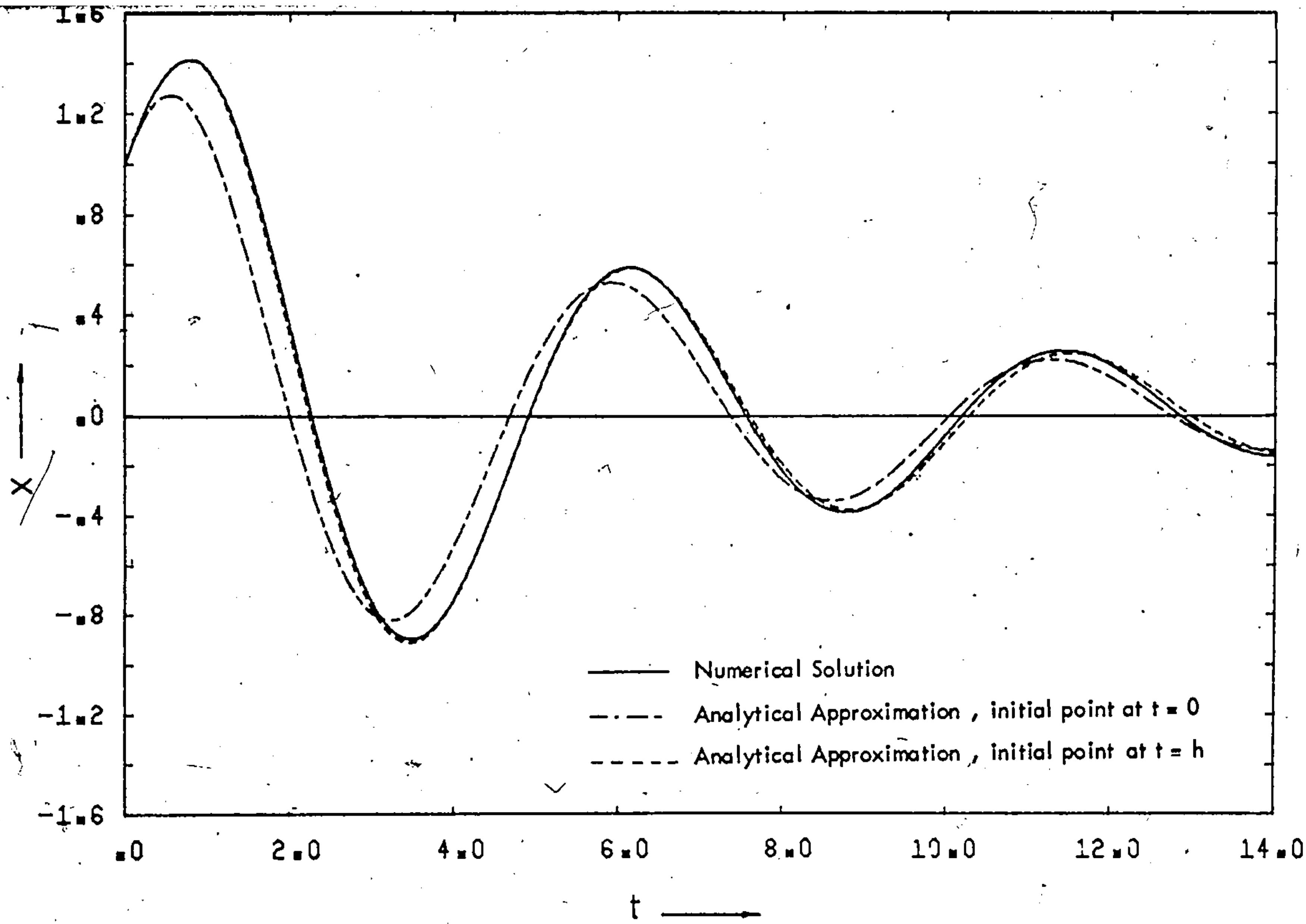

Fig.2.5-a Solution of eqn.(2.4.11) for $\varepsilon = 0.4$ and $h = \pi/4$, x versus t .

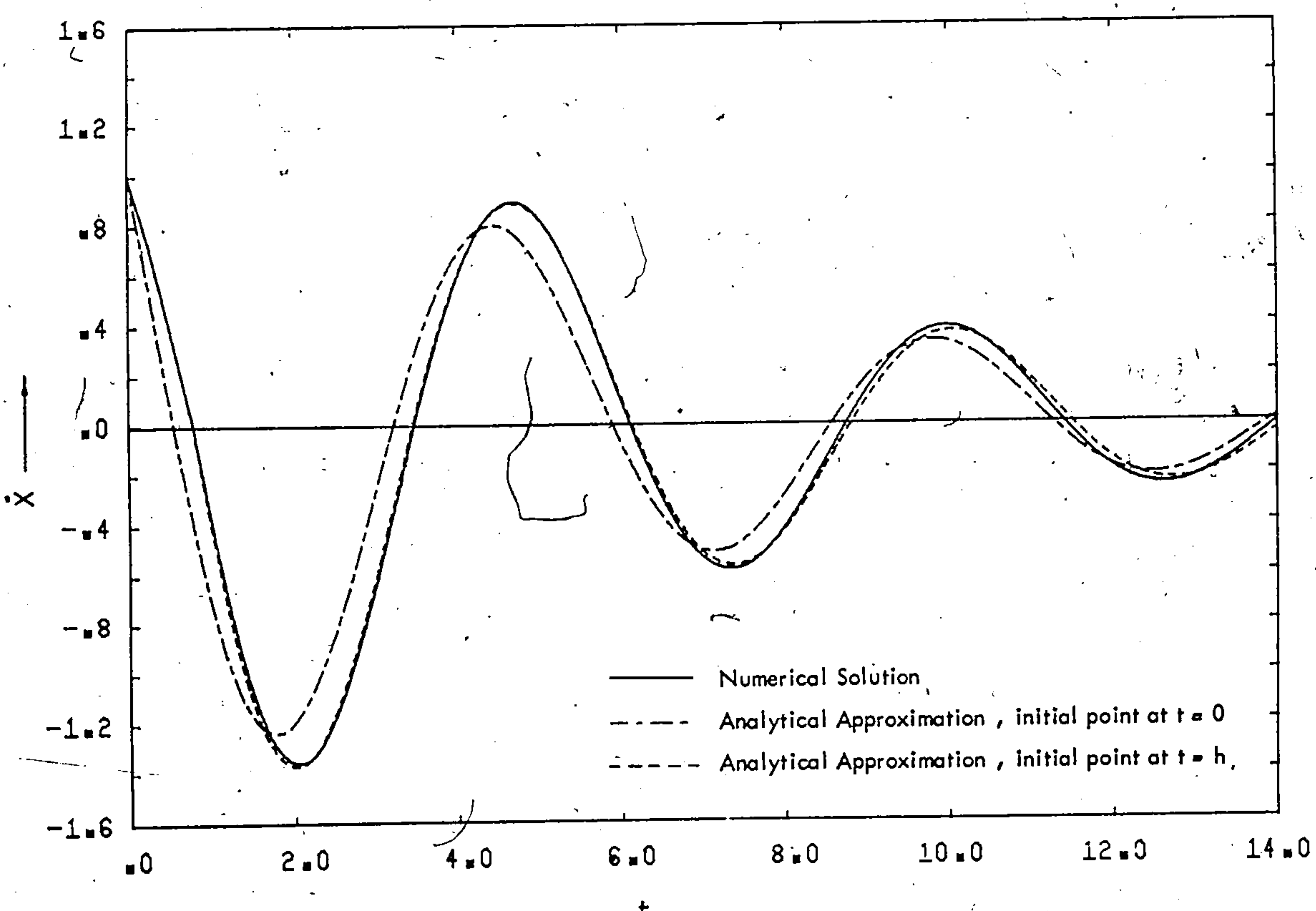

Fig. 2.5-b Solution of eqn. (2.4.11) for $\varepsilon = 0.4$ and $h = \pi/4$, $\dot{x}$ versus t .

In Figs. (2.4) and (2.5), the above solution is plotted against the "exact" numerical solution for specific values of x_0, y_0, h and ε. It will be seen that significant improvement is obtained by choosing the initial point of the approximate solution at t=h rather than at t=0.

Now we turn to eqn. (2.3.15). The solution of this equation is sought in the form (2.2.2). Owing to the odd-symmetry of the non-linear term in eqn. (2.3.15), only the odd harmonics in expression (2.2.2) need be considered.

From expressions (2.2.11), (2.2.8) and (2.4.14) we get:

$$\varepsilon\dot{x}(t-h) = -\varepsilon A_0 \sin(\tau-h) + \varepsilon^2[(\frac{dA_0}{d\xi} + h\lambda_1 A_0)\cos(\tau-h)$$

$$+ (h\frac{dA_0}{d\xi} - A_1 - \lambda_1 A_0)\sin(\tau-h)]$$

and

$$\varepsilon\dot{x}^3(t-h) = -\frac{3}{4}\varepsilon A_0^3 \sin(\tau-h) + \varepsilon\frac{A_0^3}{4}\sin 3(\tau-h) \tag{2.4.21}$$

$$+ \frac{3}{4}\varepsilon^2 A_0^2[(\frac{dA_0}{d\xi} + h\lambda_1 A_0 - 3D_{31})\cos(\tau-h)$$

$$+ 3(h\frac{dA_0}{d\xi} - A_1 - \lambda_1 A_0 + C_{31})\sin(\tau-h)]$$

In expressions (2.4.21), the terms $\varepsilon^2 \cos n\tau$ and $\varepsilon^2 \sin n\tau$,

$n \geq 3$ which do not contribute to the first or second order solution, have been omitted.

Substituting expressions (2.2.10), (2.2.11) and (2.4.21) in eqn. (2.3.15), and using the identity:

$$\begin{matrix}\cos \\ \sin\end{matrix} n(\tau-h) = \begin{matrix}\sin nh \\ \cos nh\end{matrix} \sin n\tau \pm \begin{matrix}\cos nh \\ \sin nh\end{matrix} \cos n\tau$$

the equations that determine the solution to the second approximation are [section 2.2]:

$$\frac{dA_0}{d\xi} + \frac{1}{2}A_0[\frac{1}{4}\cos h\ A_0^2 - p] = 0\ , \qquad (2.4.22)$$

$$\lambda_1 = \frac{1}{2}\sin h\ [\frac{A_0^2}{4} - \beta]\ , \qquad (2.4.23)$$

$$C_{31} = -\frac{\sin 3h}{96}A_0^3\ ,\quad D_{31} = \frac{\cos 3h}{96}A_0^3 \qquad (2.4.24)$$

$$\frac{dA_1}{d\xi} + \frac{1}{2}[\frac{3}{4}\cos h\ A_0^2 - p]\ A_1 = M_1\frac{A_0}{2} + M_2(\frac{A_0}{2})^3 + M_3(\frac{A_0}{2})^5$$

$$(2.4.25)$$

$$\lambda_2 = \frac{1}{4}\sin h\ A_0A_1 + N_1 + N_2A_0^2 + N_3A_0^4 \qquad (2.4.26)$$

where

$$p = \beta\cos h - \alpha$$

$$M_1 = -\frac{\beta^2}{2}[h\cos 2h + \frac{1}{2}\sin 2h] + \beta\alpha(\sin h + h\cos h)$$

$$M_2 = \frac{3}{4}\beta \sin 2h + \frac{1}{2}\beta\, h(1 + 3\cos 2h) - \frac{3}{2}\alpha\, h \cos h$$

$$M_3 = -\frac{h}{2} - \frac{1}{2}\sin 2h - h\cos 2h - \frac{1}{8}\sin 4h$$

$$N_1 = -\frac{\alpha^2}{8} - \frac{1}{8}\beta^2 \cos 2h + \frac{1}{4}\beta^2\, h \sin 2h + \frac{\alpha\beta}{4}(\cos h - h\sin h)$$

$$N_2 = \frac{3}{16}[\alpha h \sin h - \beta h \sin 2h - \beta \sin^2 h]$$

and

$$N_3 = \frac{3}{128} + \frac{h}{32}\sin 2h - \frac{1}{64}\cos 2h - \frac{1}{256}\cos 4h$$

Eqn. (2.4.22) integrates into:

$$A_0^2 = \frac{4p \sec h}{1 + K e^{-p\xi}} \tag{2.4.27}$$

where $K = \dfrac{4p \sec h}{A_0^2(0)} - 1$

Thus, the first order solution is given by:

$$x \approx A_0 \cos[t + \frac{1}{2}\sin h \int_0^{\xi} (\frac{A_0^2}{4} - \beta)\, d\xi + \tau(0)] \tag{2.4.28}$$

where A_0 is given by eqn. (2.4.27).

The solution will be evaluated to the second-order of approximation, hence the initial conditions are chosen to correspond to the state of the system at $t=h$.

From eqn. (2.4.27) it is seen that for $\alpha>0$ a stable limit cycle results if $\cos h > \frac{\alpha}{\beta}$. If $\cos h < \frac{\alpha}{\beta}$, the oscillations will decay or grow indefinitely according to the initial conditions.

When $\cos h \to \frac{\alpha}{\beta}$, $p \to 0$ and

$$A_0^2 \to \frac{A_0^2(0)}{1 + \frac{\cos h}{4} A_0^2(0)\xi} \qquad (2.4.29)$$

Now we recall that when expressions (2.2.10), (2.2.11) and (2.4.21) were substituted in eqn. (2.3.15), the coefficient of $\sin \tau$ in the resulting expression took the form:

$$\varepsilon[\frac{dA_0}{d\xi} + \frac{1}{2} A_0(\frac{1}{4} \cos h\ A_0^2 - p)]$$

$$+\varepsilon^2[\frac{dA_1}{d\xi} + \frac{1}{2}(\frac{3}{4} \cosh A_0^2 - p)\ A_1 - M_1(\frac{A_0}{2}) - M_2(\frac{A_0}{2})^3 - M_3(\frac{A_0^5}{2})] \qquad (2.4.30)$$

and the terms of order ε and ε^2 were equated separately to zero to yield equations (2.4.22) and (2.4.25).

Alternatively, if expression (2.4.30) is rewritten as:

$$\varepsilon[\frac{dA_0}{d\xi} - \frac{A_0}{2} (p + \varepsilon M_1) + (\frac{A_0}{2})^3 (\cosh - \varepsilon M_2)]$$

$$+ \varepsilon^2[\frac{dA_1}{d\xi} + \frac{1}{2} (\frac{3}{4} \cosh A_0^2 - p)\ A_1 - M_3 (\frac{A_0}{2})^5] \qquad (2.4.31)$$

and the terms in each bracket are equated separately to zero, we get:

$$\frac{dA_0^*}{d\xi} - \frac{A_0^*}{2} (p + \varepsilon M_1) + \frac{A_0^{*3}}{8} (\cosh - \varepsilon M_2) = 0 \qquad (2.4.32)$$

and

$$\frac{dA_1^*}{d\xi} + \frac{1}{2}\left(\frac{3}{4}\cosh A_0^{*2} - p\right) A_1^* = M_3\left(\frac{A_0^*}{2}\right)^S \qquad (2.4.33)$$

where the superscript (*) is used to distinguish the solutions of the above equations from those of equations (2.4.22) and (2.4.25). Let $A^* \triangleq A_0^* + \varepsilon A_1^*$ and $\overline{A} = A_0 + \varepsilon A_1$. We note that A^* and $\overline{A}$ are not identical, but to the second order of approximation they are equivalent. Eqn. (2.4.32) has the same form as eqn. (2.4.22), however, eqn. (2.4.33) is simpler to integrate than eqn. (2.4.25). We shall therefore choose A^* to approximate A and hence the expressions for λ_1 and λ_2 become:

$$\lambda_1 = \frac{1}{2}\sinh\left(\frac{A_0^{*2}}{4} - \beta\right) \qquad (2.4.34)$$

and

$$\lambda_2 = \frac{1}{4}\sinh A_0^* A_1^* + N_1 + N_2 A_0^{*2} + N_3 A_0^{*4}$$

$$\approx \frac{1}{4}\sin h\, A_0 A_1^* + N_1 + N_2 A_0^2 + N_3 A_0^4 \qquad (2.4.35)$$

Eqn. (2.4.32) integrates into:

$$A_0^{*2} = \frac{4p^*/(\cos h - \varepsilon M_2)}{1 + K^* e^{-p^*\xi}} \qquad (2.4.36)$$

where $p^* = p + \varepsilon M_1$ and $K^* = \dfrac{4p^*/A_0^{*2}(0)}{\cos h - \varepsilon M_2} - 1 \qquad (2.4.37)$

If K^* and $\tau(0)$ are chosen such that the initial conditions (at t=h) are satisfied exactly, or to the second order of approximation, then $A_1^*(0)$ can be chosen to be equal to zero. Since the solution is limited to the second approximation, A_0^* in eqn. (2.4.33) may be replaced by A_0. Hence, using the integrating factor: $\mathcal{J} = e^{p\xi}(1+Ke^{-p\xi})^{3/2}$, eqn. (2.4.33) integrates into:

$$A_1^* = \sec h\, b^{3/2} M_3 \frac{[(1 - e^{-p\xi}) - e^{-p\xi} \ln(\frac{K + e^{p\xi}}{K + 1})]}{[1 + Ke^{-p\xi}]^{3/2}} \qquad (2.4.38)$$

where $b = \beta - \alpha \sec h$.

Therefore,

$$\tau(t) - \tau(0) = \int_0^t \lambda\, dt \approx t + \int_0^\xi (\lambda_1 + \varepsilon\lambda_2)\, d\xi$$

$$= t - \frac{1}{2} \sin h[\beta - \frac{p + \varepsilon M_1}{\cos h - \varepsilon M_2}]\ \xi$$

$$- \frac{1}{2} \frac{\sin h}{\cos h - \varepsilon M_2} \ln (\frac{1 + K^* e^{-p^*\xi}}{1 + K^*})$$

$$+ \varepsilon[N_1\xi + \Lambda_1 \ln (\frac{K + e^{p\xi}}{K + 1}) + \Lambda_2 (\frac{1}{1 + Ke^{-p\xi}} - \frac{1}{1 + K})$$

$$+ \frac{1}{4} Kb\, M_3 \tan h \{\frac{1 + \ln(\frac{K + e^{p\xi}}{b})}{K + e^{p\xi}} - \frac{1 + \ln(\frac{K + 1}{b})}{KH}\}$$

(2.4.39)

where the constants Λ_1 and Λ_2 are given by:

$$\Lambda_1 = 4[N_2 + 4bN_3] + \frac{b}{4} M_3 \sec h \tan h$$

and

$$A_2 = -16 N_3 b \sec h - \frac{M_3 b}{4} [1 + \frac{1}{K} - \ln(\frac{K+1}{b})] \sec h \tan h$$

Finally, given the initial conditions $x(0)$ and $\dot{x}(0)$, the constants $A^*(0)$ and $\tau(0)$ are determined from:

$$A_0^*(0) \cos \tau(0) = x(0) - \varepsilon[C_{31}(0) \cos 3\tau(0) + D_{31}(0) \sin 3\tau(0)]$$

and

$$A_0^*(0) \sin \tau(0) = - \dot{x}(0) - \varepsilon[\lambda_1^*(0) A_0^*(0) \sin \tau(0) - \frac{d^* A_0}{d\xi}\Big|_{\xi=0} \cos \tau(0)$$

$$+ 3C_{31}(0) \sin 3\tau(0) - 3D_{31}(0) \cos 3\tau(0)] \quad (2.4.40)$$

The constant K^* is then determined from (2.4.37).

When $\cos h > \frac{\alpha}{\beta}$; $0 < h < \frac{\pi}{2}$, the steady state response is given by:

$$x = \sqrt{b} [2 + \varepsilon(\frac{M_1}{b} + M_2 + bM_3) \sec h] \cos \tau$$

$$+ \frac{\varepsilon b}{12} \sin 3(\tau - h) + O(\varepsilon^2)] \quad (2.4.41)$$

with $\tau = \Omega t + \tau(0)$, where the constant frequency Ω is given by:

$$\Omega = 1 - \frac{\varepsilon}{2} \alpha \tan h + \varepsilon^2 [N_1 + 4p(N_2 + 4bN_3) \sec h$$

$$- \frac{1}{2} \{M_1 + p(M_2 + bM_3) \sec h\} \tan h]$$

$$(2.4.42)$$

Inspecting expressions (2.4.36), (2.4.41) and (2.4.42), it is seen that the effect of delay is to slow down the build up of oscillations and to reduce the steady state amplitude and frequency.

<u>When $h \to 0$, then:</u>

$M_1 \to 0$, $M_2 \to 0$, $M_3 \to 0$, $N_1 \to -\frac{1}{8}$, $N_2 \to 0$, $N_3 \to \frac{1}{256}$, $p \to 1$, $b \to 1$, $\Lambda_1 \to \frac{1}{16}$ and $\Lambda_2 \to -\frac{1}{16}$, and we obtain the well-known result:

$$A \approx A_0^* + \varepsilon A_1^* = \frac{4}{1 + Ke^{-\xi}} \qquad \text{with} \qquad K = \frac{4}{A_0^{*2}} - 1 = \frac{4}{A^2(0)} - 1 ,$$

$$C_{31} = 0 \ \forall t \ , \ D_{31} = A_0^3/96 \ , \text{ and :}$$

$$\tau(t) - \tau(0) = t + \varepsilon[-\frac{1}{8}\xi + \frac{1}{16}\ln(\frac{K + e^{\xi}}{K + 1}) + \frac{1}{16}(\frac{1}{1 + K} - \frac{1}{1 + Ke^{-\xi}})] , \tag{2.4.42}$$

and in the steady state:

$$x = 2\cos(\Omega t + \tau(0)) + \frac{\varepsilon}{12}\sin 3(\Omega t + \tau(0)) + O(\varepsilon^2) ,$$

with $\Omega \approx 1 - \frac{\varepsilon^2}{16}$.

In Figs. (2.4.3) and (2.4.4) below, the numerical and analytical approximations to the transient solution of eqn. (2.3.15) are shown.

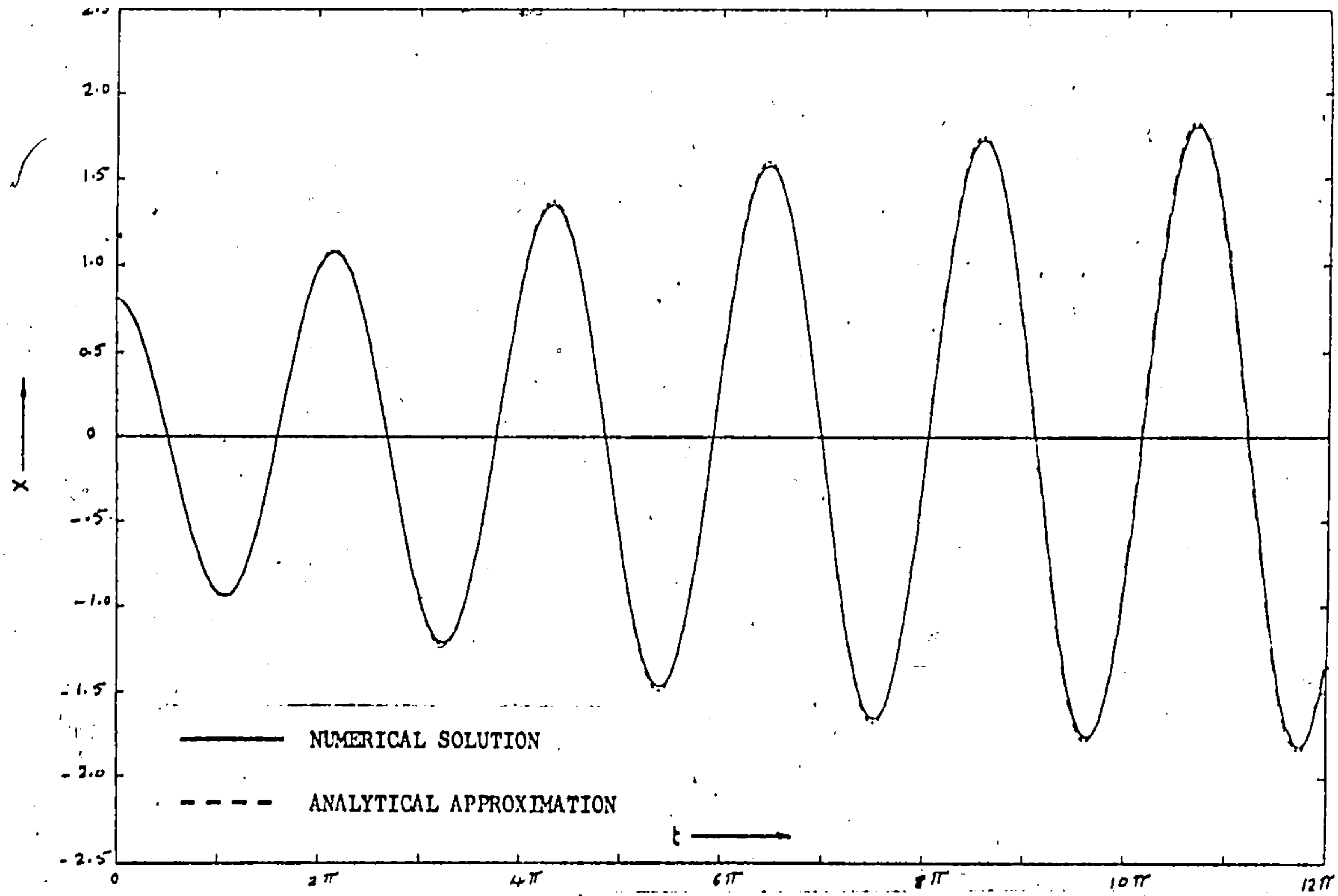

Fig. 2.6 Analytical and Numerical Solutions of eqn. (2.3.15)

for $\varepsilon = 0.2$, $\alpha = 0.5$, $\beta = 1.5$ and $h = \pi/4$.

$x(0) = 0.8$, $\dot{x}(0) = 0$, and $x(t) = \dot{x}(t) = 0$ for $t < 0$.

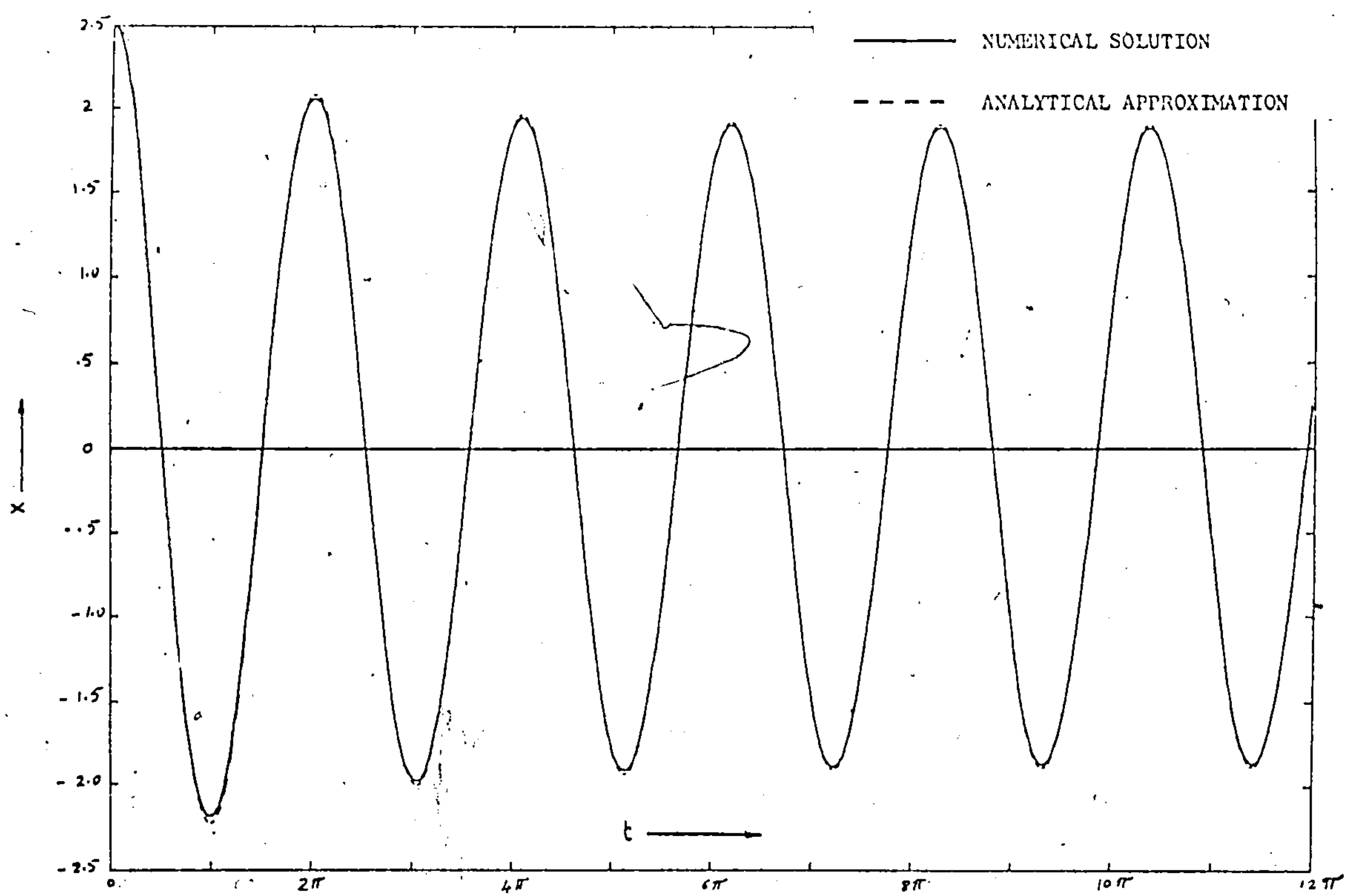

Fig.2.7 Analytical and Numerical Solutions of eqn.(2.3.15) for $\varepsilon = 0.2$, $\alpha = 0.5$, $\beta = 1.5$, and $h = \pi/4$. $x(0) = 2.5$, $\dot{x}(0) = 0$, and $x(t) = \dot{x}(t) = 0$ for $t < 0$

CHAPTER 3

The non-stationary behaviour of a class of oscillatory systems exhibiting gross non-linearity

3.1 Introduction

In chapter 2, the non-stationary behaviour of a class of autonomous and slowly time-variant weakly non-linear oscillatory systems was discussed. In such systems, due to the small non-linearity, the instantaneous frequency of oscillation remains close to the value it would assume had the non-linearity been absent. When the non-linearity is large, significant variations in the instantaneous frequency may take place and the methods developed for analysing weakly non-linear systems cannot be used.

The periodic behaviour of a system with a large non-linearity can be determined analytically in specific cases. For example, the periodic behaviour of a free-running v.d.Pol oscillator with large non-linear damping can be obtained using the well-known asymptotic method of Dorodnitsin†. The free or forced periodic behaviour of certain systems with large non-linear reactive forces can be determined to the first order by using the describing function method. However, searching through the extensive literature concerned with the theory of non-linear oscillations, one finds that most investigations of the non-stationary behaviour of oscillating systems are restricted to those with small non-linearity, although there have been few attempts to deal with the

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† Dorodnitsin's asymptotic solution contains errors which were corrected by Cartwright.

case of large non-linearity.

The equation

$$\ddot{x} + P(x) + \alpha f(x,\dot{x}) = 0 \quad , \quad 0\leq\alpha<<1 \qquad (3.1.1)$$

where P is not necessarily a quasi-linear function of x, describes a class of non-linear feedback oscillatory systems, and has been studied by a number of authors.

Coppel [1959] studied the case when $P(x) \equiv \sin x$ and $f(x,\dot{x}) \equiv \dot{x}$. He transformed the above equation by introducing elliptic functions then applying the K-B averaging method. Morrison [1966] considered a more general form of the restoring force P(x) and the "perturbing" function $f(x,\dot{x})$. He used transformations which do not involve elliptic functions, although elliptic functions and elliptic integrals arise in the analysis. His results for the specific case considered by Coppel are similar to those of Coppel.

Rasmussen and Kirk [1966] obtained an approximate solution of eqn. (3.1.1) which is valid only for short periods of time. Their method was later modified by Rasmussen [1970] to give a uniformly valid first order approximation. The modified method of Rasmussen yields fairly accurate results, however, it necessitates performing numerical integration,

An optimization procedure was used by Wagner and Ludeke [see Wagner (1967) Ludeke and Wagner (1968)] where two parameters of an assumed form of the solution were chosen to minimize a certain error function. The imposition of weak non-linearity was first made, then relaxed without any justification. We note that careful examination of their procedure shows that the chosen error function is quite insensitive to variations in the parameters and that its minimum value is considerably large. Indeed, although there are few exceptions, the validity of the approach of introducing optimization methods to find approximate solutions to differential equations is very doubtful.

Very often the solution of the unperturbed equation: $\ddot{x} + P(x) = 0$ can be obtained in terms of higher functions. One might then be tempted to use such a solution as a generating solution for the perturbed equation (3.1.1) by replacing certain constant coefficients by slowly varying ones and using the v.d.Pol or K-B averaging method. Such a procedure cannot lead to uniformly valid solutions unless it can be established beforehand that only slight variations in the instantaneous frequency of oscillation will take place as time increases. This approach was adopted by Barkham and Soudack [1969] who studied the transient response of a system with a large cubic non-linearity. Their analysis follows closely the K-B averaging method with the trigonometric functions used in the K-B method replaced by elliptic functions. The solution was then constructed by a grapho-analytical procedure. In what follows we shall compare the K-B method and the modification suggested by Barkham and Soudack.

The normalized quasi-linear equation:

$$\ddot{x} + x + \alpha f(x,\dot{x}) = 0; \quad 0<\alpha<<1$$

has an exact solution:

$$x = A\cos(t+\phi) \qquad \text{when } \alpha \equiv 0$$

where A and ϕ are constants.

For the case $\alpha \neq 0$ an approximation may be sought in the form:

$$x = A(\alpha t)\cos\{t+\phi(\alpha t)\},$$

and the well known K-B averaging method yields:

$$\dot{A} = \frac{\alpha}{2\pi}\Lambda(A) \qquad \text{and} \qquad \dot{\phi} = \frac{\alpha}{2\pi}\psi(A)$$

where $\Lambda(A) = \int_0^{2\pi} (\sin u)\, f(A\cos u, -A\sin u)\,du$

and $\psi(A) = \frac{1}{A}\int_0^{2\pi} (\cos u)\, f(A\cos u, -A\sin u)\,du$

A basic requirement for such a procedure to be valid $\forall t$ is that the instantaneous frequency (here defined as $\frac{d}{dt}(t+\phi)$) for $\alpha \neq 0$ should not depart appreciably from its value (of unity) when $\alpha \equiv 0$. This requirement can often be met in the so-called quasi-linear systems and hence the approximation remains uniformly valid $\forall t$ (except for slight corrections which can be easily evaluated using a suitable perturbational procedure).

Now considering the system:

$$\ddot{x} + x + \gamma x^3 + \alpha f(x,\dot{x}) = 0$$

$$0 \leq \alpha << 1,\ x(0) = 1,\ \dot{x}(0) = 0,$$

it has an exact solution when $\alpha \equiv 0$ in the form:

$$x = A\ Cn(\omega t+\theta) \qquad \text{for } \gamma > 0$$

A and θ being constants and $\omega = \sqrt{1+\gamma}$

For large γ, with $\alpha \neq 0$ and if for example $f(x,\dot{x}) \equiv \dot{x}$ the frequency of oscillations will approach a value near unity as $t \to \infty$ and thus departs appreciably from its initial large value. If one then seeks an approximation in the form:

$x = A(\alpha t)\ Cn\{\omega t + \theta\ (\alpha t)\}$ and an averaging scheme as in the K-B method is used, as suggested by Barkham and Soudack, the approximation will be valid for short durations depending on α. In order to be able to evaluate the solution at large values of t one is then forced to use the approach of graphical or numerical techniques where a solution valid over a short period of time is evaluated, the end conditions then used as initial conditions for a subsequent short interval and so on.

If γ is O(1) and with moderately small values of α, ($\alpha = 0.2$ say), the oscillations fade away after a small number of cycles and constructing the solution can be done without much effort. However, as γ increases, the interval of validity of the approximation gradually becomes shorter and such a procedure becomes too cumbersome. Also the accuracy is then seriously impaired.

A variant of the above method, used later by Soudack and Barkham [1970, 1971] eliminates the piece-wise integration process mentioned above. The method however ignores the non-linear interaction between the time-varying amplitude and frequency. It will be shown in Appendix C that the solutions obtained by this method are inferior to those obtained by the K-B method, despite the fact that the latter is applicable only to weakly non-linear systems.

In a recent paper, Spasov et al [1972] used a non-linear transformation which reduces eqn. (3.1.1) to a quasi-linear equation. Using the transformation:

$$\frac{dx}{dy} = \frac{dt}{d\zeta} = \frac{y}{P(x)} \triangleq G(y) \qquad (3.1.3)$$

Eqn. (3.1.1) can be written in the form:

$$y'' + y = \alpha \Phi(y, y') \quad , \quad (') \equiv \frac{d}{d\zeta} \qquad (3.1.4)$$

where from eqn. (3.1.3):

$$y = [2 \int_0^x P(x)dx]^{1/2} \operatorname{sgn} x \quad , \quad t = \int_0^\zeta G(y)d\zeta$$

and

$$x = \int_0^y G(y)dy$$

Unfortunately, the function Φ usually takes a formidable form which prohibits further analytical progress. For example if:

$$P(x) = x + \gamma x^3, \quad \gamma > 0, \quad \text{and} \quad f = -\dot{x} \equiv -y'$$

then

$$G(y) = \frac{\sqrt{\gamma}\, y \;\operatorname{sgn} y}{\sqrt{1 + 2\gamma y^2}\,\sqrt{\sqrt{1 + 2\gamma y^2} - 1}}$$

and

$$\Phi = -y' \, G(y)$$

The approximate solution of eqn. (3.1.4) according to the K-B method takes the form:

$$y = A(\alpha\zeta) \cos(\zeta + \theta(\alpha\zeta))$$

In order to be able to obtain expressions for A and θ as functions of ζ, the coefficients of $\cos(\zeta + \theta)$ in the Fourier series expansion of $\Phi(A \cos(\zeta + \theta), -A \sin(\zeta + \theta), \xi)$ should be determined as explicit functions of A and ζ. Denoting these coefficients by $\Phi_c(A,\zeta)$ and $\Phi_s(A,\zeta)$ respectively, the "shortened equations":

$$\frac{dA}{d\zeta} = \frac{-\alpha}{2}\Phi_s \quad \text{and} \quad \frac{d\theta}{d\alpha} = \frac{-\alpha}{2A}\Phi_c$$ must then be solved, which in this case is a rather difficult task.

In control systems, it is often desirable to determine the effect of a certain parameter or the form of non-linearity on the settling time, and uniformly valid expressions for the system response are therefore required.

3.2 Present analysis

(a) Preliminary

Let us first consider the equation:

$$\ddot{x} + P(x) = 0 \qquad (3.2.1)$$

When $P(x) = x + \gamma x^3$, and with the initial conditions $x(t_0) = 1$ and $\dot{x}(t_0) = 0$, the solution of eqn. (3.2.1) is bounded for $-1<\gamma<\infty$.

For $\gamma \geq 0$, eqn. (3.2.1) has the exact solution:

$$x = Cn\sqrt{1 + \gamma}\ t \qquad (t_0 \equiv 0)$$

The modulus k of the Jacobian elliptic function is given by:

$$k^2 = \frac{\gamma}{2[1+\gamma]}$$

thus for $\quad 0 \leq \gamma < \infty$

we have $\quad 0 < k < \frac{1}{\sqrt{2}}$

and $\quad 0 \leq q < 0.043214...$

q being the NOME of the Jacobian elliptic function.

Therefore, if x is expressed as a Fourier series (which in this case will contain odd harmonics only), then the ratio of the amplitude of the $(2n+1)^{th}$ harmonic to the amplitude of the fundamental will be given by:

$$\frac{R_{2n+1}}{R_1} = q^n \frac{1+q}{1+q^{2n+1}} \qquad n = 1,2,3$$

i.e., the ratios of the amplitudes of the 3^{rd}, 5^{th}, 7^{th}, ... harmonics to the amplitude of the fundamental have maximum values of 0.0451, 0.0019, 0.00008, ...

For $-1 < \gamma \leq 0$, and with $t_0 \neq 0$ chosen such that $x(0) = 0$, the exact solution of eqn. (3.2.1) is:

$$x = Sn\sqrt{1 + \frac{\gamma}{2}}\ t$$

and the modulus k of the Jacobian elliptic function is given by:

$$k^2 = \frac{-1}{1+2/\gamma}$$

thus for $-0.9 < \gamma \leq 0$

$$0.8181818\cdots > k^2 \geq 0$$

$$\ldots > q \geq 0$$

and when x is expressed as a Fourier series, the ratio of the amplitude of the $(2n+1)^{th}$ harmonic to the amplitude of the fundamental is:

$$\frac{R_{2n+1}}{R_1} = q^n \frac{1-q}{1-q^{2n+1}} \qquad n = 1,2,3, \ldots$$

i.e. the ratios of the amplitudes of the 3^{rd}, 5^{th}, 7^{th}, ...[†] harmonics to the amplitude of the fundamental have maximum values of :

0.0941 , 0.0099 , 0.0010 , ...

It is thereby seen that for $-0.9 \leq \gamma < \infty$ the fundamental frequency is the dominant one.

It may also be noted that the frequency of oscillations obtained by the first harmonic approximation is:

$$\Omega_a = \sqrt{1 + \frac{3}{4}\gamma} \qquad \text{for } \gamma \geq 0 \tag{3.2.2}$$

while the exact frequency is:

$$\Omega_e = \frac{\pi\sqrt{1+\gamma}}{2\,K(k)}\ , \quad K(k) = \frac{\pi}{2}\left(1 + \frac{k^2}{4} + \frac{9}{64}k^4 + \frac{25}{256}k^6 + \ldots\right)$$

Hence:

$$\Omega_e = \Omega_a\left\{1 - \frac{3}{256}\left(\frac{\gamma}{1+\gamma}\right)^2 + \text{much smaller terms}\right\}$$

and a similar result can be obtained for $-0.9 < \gamma < 0$.

Thus the first harmonic approximation is indeed an excellent one and very little is gained by the use of the elliptic functions.

Now consider the case when $P(x)$ takes the more general form:

$$P(x) = ax + bx^3 + cx^5$$

and let for example $x(0) = 1$ and $\dot{x}(0) = 0$. Then writing: $y \triangleq \dot{x}$, eqn. (3.2.1) becomes $\frac{dy}{dx} = \frac{-P(x)}{y}$, from which we obtain:

[†] When $\gamma \to -1$, $q \to 1$ and the above ratios $\to \frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, ...

$$\int_0^{\phi} \frac{dx}{\sqrt{1-x^2}\ \sqrt{(a + \frac{b}{2} + \frac{c}{3}) + (\frac{b}{2} + \frac{c}{3})x^2 + \frac{c}{3}x^4}} = \int_0^t dt \tag{3.2.3}$$

With the change of variables: $x(t) = \cos\phi(t)$, eqn. (3.2.3) is rewritten as:

$$\int_0^{\phi} \frac{d\phi}{\sqrt{1 + \varepsilon_1 \cos 2\phi + \varepsilon_2 \cos 4\phi}} = \pm \Omega t \tag{3.2.4}$$

[The negative sign must be ignored since solutions for positive time only are desired].

where

$$\Omega^2 = a + \frac{3}{4} b + \frac{5}{8} c$$

$$\varepsilon_1 = \frac{6b + 8c}{3[8a + 6b + 5c]} \quad \text{and} \quad \varepsilon_2 = \frac{c}{3[8a + 6b + 5c]}$$

The integral in eqn. (3.2.4) can be written in the standard form of an elliptic or hyperelliptic integral and evaluated as such. However, when ε_1 and ε_2 are small [for example if $|\varepsilon_1| + |\varepsilon_2| \gtrsim 0.6$], then using the series expansion:

$$\sqrt{1 + \mu} = 1 + \frac{\mu}{2} - \frac{\mu^2}{8} + \ldots$$

Eqn. (3.2.4) becomes:

$$\phi - \frac{\varepsilon_1}{4} \sin 2\phi - \frac{\varepsilon_2}{8} \sin 4\phi \approx \Omega t$$

hence

$$x \approx \cos[\Omega t + \frac{\varepsilon_1}{4} \sin 2\Omega t + \frac{\varepsilon_2}{8} \sin 4\Omega t]$$

$$\approx (1 - \frac{\varepsilon_1}{8}) \cos \Omega t + (\frac{\varepsilon_1}{8} - \frac{\varepsilon_2}{16}) \cos 3\Omega t + \frac{\varepsilon_2}{16} \cos 5\Omega t \quad (3.2.5)$$

When a, b and c are positive, then $|\varepsilon_1| + |\varepsilon_2|$ has a maximum value of 0.6 which occurs when $\frac{b}{a} \to \infty$, $\frac{c}{a} \to \infty$ and $\frac{c}{b} \to \infty$. The amplitudes of the higher harmonics are, therefore, of much smaller values than the amplitude of the fundamental.

If either or both of the parameters b and c are negative, considerable caution must be exercised in order to ascertain the existence of stable oscillations. The order of smallness of the higher harmonics which depends on the value of $|\varepsilon_1| + |\varepsilon_2|$, can then be estimated.

For example, consider the system:

$$\ddot{x} + \sin(px) = \alpha f(x,\dot{x}) \quad ; \quad x(0) = 1 \text{ and } \dot{x}(0) = 0.$$

When $f(x,\dot{x})$ represents positive damping $\forall x$ and $\dot{x}$, it is known that the system is stable, and with proper initial conditions the response is oscillatory. For p=2 say, $\sin px \approx 2[x - \frac{2}{3} x^3 + \frac{2}{15} x^5]$ and $|\varepsilon_1| + |\varepsilon_2| = \frac{23}{105}$, which indicates that the magnitude of the higher harmonics is much smaller than that of the fundamental frequency.

As a further example, consider the case when $P(x) = \gamma \operatorname{sgn} x$, $0 < \gamma < \infty$. In this case eqn. (3.2.1) has the exact periodic solution:

$$x = \begin{cases} 1 - \left(\frac{t}{t_0}\right)^2 & \text{for} \quad 0 \le \frac{t}{t_0} \le 1 \\ \left(\frac{t}{t_0} - 4m-1\right)\left(\frac{t}{t_0} - 4m - 3\right) & \text{for} \quad 4m+1 \le \frac{t}{t_0} \le 4m+3 \\ 1 - \left(\frac{t}{t_0} - 4(m+1)\right)^2 & \text{for} \quad 4m+3 \le \frac{t}{t_0} \le 4m+5 \end{cases} \tag{3.2.6}$$

for $m = 0,1,2,\ldots$ and where $t_0 = \sqrt{\frac{2}{\gamma}}$

which can be expanded in the Fourier series:

$$x = \frac{32}{\pi^3}\left[\cos \omega t - \frac{1}{27}\cos 3\omega t + \frac{1}{125}\cos 5\omega t - \ldots\right],$$

$\omega \triangleq \frac{\pi}{2t_0}$, which shows that the term of fundamental frequency is dominant.

In the general case, it is not apparently feasible to determine upper bounds for the ratio of the magnitude of the higher harmonics to the magnitude of the first harmonic in the periodic solution of eqn. (3.2.1). However, it seems that this ratio is considerably small whenever $x\,P(x) \geqslant 0 \;\forall x$ and $\frac{dP(x)}{dx} \geqslant 0 \;\forall x$ (and in this case the solution is periodic for arbitrary initial conditions) . The solution of eqn.(3.2.1) can often be obtained analytically for specific forms of $P(x)$ and the order of smallness of the above ratio may be determined in each case .

Whenever it can established, qualitatively or quantitatively, that the first harmonic is dominant in the solution of the unperturbed eqn. (3.2.1), it is plausible to assume that this will also hold true for the perturbed eqn. (3.1.1).

(b) The method

In view of the above discussion, we shall seek an approximate solution to eqn. (3.1.1) in the form:

$$x = A\cos(\tau+\theta) + \varepsilon[C_0 + \sum_{n>1}\{C_n \cos n(\tau+\theta) + D_n \sin n(\tau+\theta)\}] \qquad (3.2.7)$$

where θ is a constant, $\tau(0) = 0$, $0<\varepsilon<<1$ $(\varepsilon\neq 0)$ and $0\leq\alpha\leq O(\varepsilon)$.

ε will serve as a dummy small parameter. Its introduction here is not strictly necessary and is done purely for mathematical convenience. Thus one can obtain a solution to eqn. (3.1.1) in the form of expression (3.2.7) whether or not the term $\alpha f(x,\dot{x})$ is present.

If $f(x,\dot{x})$ is anti-symmetric with respect to $x=0$ and $\dot{x}=0$ i.e., if $f(x,\dot{x}) = -f(-x,-\dot{x})$ and $f(\pm x,\mp\dot{x}) = -f(\mp x,\pm\dot{x})$, then only odd values of n need be considered. Otherwise even values of n must be considered as well.

Since α is small, then the amplitude and hence the amplitude-dependent frequency will vary slowly with time. Thus the coefficients A, C_0, $C_n|_{n>1}$ and $D_n|_{n>1}$ in expression (3.2.7) are functions of the slow time $\xi \triangleq \varepsilon t$, and $\frac{d\tau}{dt} = \lambda(\xi)$.

The magnitude of λ is not restricted, it could assume any positive finite value. However, it is assumed that

$$\frac{\|\frac{d\lambda_i}{dt}\|}{\|\lambda_i\|} = O(\alpha) \text{ , i.e. } \|\frac{d\lambda_i}{d\xi}\| = O(\|\lambda_i\|) \quad i=0,1,2,\ldots$$

We also require that $\|\lambda_i\| = O(\|\lambda_0\|)$ $i=1,2,3,\ldots$.

It follows that

$$\|\frac{d^j\lambda_i}{d\xi^j}\| = O(\|\frac{d^{j-1}\lambda_i}{d\xi^{j-1}}\|) \quad \forall i,j.$$

In order to display the main ideas in this method without getting too involved with lengthly algebraic expressions, the analysis from this point onwards will proceed for the case when $f(x,\dot{x})$ is an anti-symmetric function. Thus:

$$x = A\cos(\tau+\theta) + \varepsilon[C_3 \cos 3(\tau+\theta) + D_3 \sin 3(\tau+\theta) + C_5 \cos 5(\tau+\theta) + D_5 \sin 5(\tau+\theta) + \ldots] \quad (3.2.8)$$

As a first approximation, let:

$$\frac{d\tau}{dt} \triangleq \lambda \approx \lambda_0 + \varepsilon\lambda_1$$

hence:

$$\dot{x} = -A\lambda_0 \sin(\tau+\theta) + \{\alpha \frac{dA}{d\xi}\cos(\tau+\theta) - \varepsilon A \lambda_1 \sin \tau+\theta)\}$$

$$+ \varepsilon \lambda_0\{- 3C_3 \sin 3(\tau+\theta) + 3D_3 \cos 3(\tau+\theta) \quad (3.2.9)$$

$$- 5C_5 \sin 5(\tau+\theta) + 5D_5 \cos 5(\tau+\theta)\}$$

$$+ \ldots$$

and

$$\ddot{x} = - A \lambda_0^2 \cos(\tau+\theta) - 2\varepsilon A \lambda_0\lambda_1 \cos(\tau+\theta)$$

$$- \alpha(2\lambda_0 \frac{dA}{d\xi} + A \frac{d\lambda_0}{d\xi}) \sin(\tau+\theta) \quad (3.2.10)$$

$$- \varepsilon \lambda_0^2\{9C_3 \cos 3(\tau+\theta) + 9D_3 \sin 3(\tau+\theta)$$

$$+ 25C_5 \cos 5(\tau+\theta)+25 D_5 \sin 5(\tau+\theta)\}$$

Now using expression (3.2.8), P(x) takes the form:

$$P(x) = \sum_n G_n(A) \cos n(\tau+\theta)$$

$$+ \varepsilon \sum_n\{\bar{G}_n (A, C_k, D_k) \cos n(\tau+\theta) + \bar{\bar{G}}_n(A,C_k,D_k) \sin n(\tau+\theta)\}$$

$$+ \text{much smaller terms}, \quad (3.2.11)$$

$$n = 1, 3, 5, \ldots ; k = 3, 5, \ldots$$

and f takes the form in (2.2.7).

Substituting the expressions for $x, \ddot{x}$, P and f in eqn. (3.1.1) and equating the coefficients of $\cos n(\tau+\theta)$ and $\sin n(\tau+\theta)$, $n=1,3,5,\ldots$ separately to zero then separating the terms of similar order of magnitude, a system of perturbational equations is obtained.

It should be noted that, in accordance with the discussion in section (3.2.a), the coefficients of $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} n(\tau+\theta)$, $n>1$ in expression

(3.2.11) must be associated with the terms of higher order of smallness.

3.3 Application to a time-invariant system

Let us consider:

$$\ddot{x} + x + \gamma x^m + \alpha\dot{x} = 0 \quad , \quad m = 3,5 \tag{3.3.1}$$

For m = 3

$$\begin{aligned}\gamma x^3 \approx\ & QA^3 \cos(\tau+\theta) + \frac{Q}{3} A^3 \sin 3(\tau+\theta) \\ & + \varepsilon\, QA^2[C_3 \cos \tau + D_3 \sin \tau] \\ & + \varepsilon \sum_n (\bar{G}_n \cos n(\tau+\theta) + \bar{\bar{G}}_n \sin n(\tau+\theta))\end{aligned} \tag{3.3.2}$$

where $Q \triangleq \frac{3}{4}\gamma$.

The terms under the summation will not be included in the first approximation.

Following the procedure described above, we obtain the system of equations:

$$(1 - \lambda_0^2)\, A + QA^3 = 0 \tag{3.3.3}$$

$$\varepsilon(1 - 9\lambda_0^2)\, C_3 + \frac{QA^3}{3} = 0 \tag{3.3.4}$$

$$D_3 = C_5 = D_5 = 0 \tag{3.3.5}$$

$$- 2\lambda_0\lambda_1 A + QA^2C_3 = 0 \tag{3.3.6}$$

$$2\lambda_0 \frac{dA}{d\xi} + A \frac{d\lambda_0}{d\xi} + \lambda_0 A = 0 \tag{3.3.7}$$

Hence

$$\lambda_0^2 = 1 + QA^2 \tag{3.3.8}$$

and from eqns. (3.3.8) and (3.3.7) we obtain:

$$\left(\frac{2}{A} + \frac{QA}{1+QA^2}\right) \frac{dA}{d\xi} = - 1 \tag{3.3.9}$$

Define: $\rho \triangleq A^2$

Eqn. (3.3.9) integrates into:

$$\rho^3 + \frac{1}{Q} \rho^2 = Me^{-2\xi} > 0 \tag{3.3.10}$$

where M is a constant to be determined from the initial conditions .

Define: $\eta \equiv \frac{27}{2} Q^3Me^{-2\xi} - 1$ (3.3.11)

It can easily be verified that eqn. (3.3.10) has one real root for $1<\eta<\infty$ and one positive and two negative roots for $-1\leq\eta\leq1$.

For $1<\eta<\infty$, i.e. for $0 \leq t < \frac{1}{2\alpha} \ln(\frac{27}{4} Q^3M)$,

the real root is given by :

$$\rho = \frac{1}{3Q}\left[\sqrt[3]{\eta+ \sqrt{\eta^2-1}} + \sqrt[3]{\eta-\sqrt{\eta^2-1}} -1\right] \tag{3.3.12}$$

For $-1 \leq \eta \leq 1$, i.e. for $\frac{1}{2\alpha}\ln(\frac{27}{4}Q^3M) \leq t < \infty$, the positive root is given by:

$$\rho = \frac{1}{3Q}[2\cos(\frac{1}{3}\beta) - 1] \qquad (3.3.13)$$

with $\quad \beta = \cos^{-1}\eta;\ 0 \leq \beta < \pi$

Note that as $\alpha \to 0$, expression (3.3.12) is valid $\forall t \geq 0$.

From eqns. (3.3.8), (3.3.4) and (3.3.6):

$$\varepsilon\lambda_1 = \frac{Q^2\rho^2}{6\lambda_0(9\lambda_0^2-1)} = \frac{(\lambda_0^2-1)^2}{6\lambda_0(9\lambda_0^2-1)}$$

From eqns. (3.3.8) and (3.3.9):

$$\frac{d\xi}{d\lambda_0} = -\frac{(3\lambda_0^2-1)}{\lambda_0(\lambda_0^2-1)}$$

hence

$$\tau = \int_0^t (\lambda_0 + \varepsilon\lambda_1)dt = \frac{1}{\alpha}\int_{\lambda_0(0)}^{\lambda_0} (\lambda_0 + \varepsilon\lambda_1)\frac{d\xi}{d\lambda_0}\,d\lambda_0$$

$$= -\frac{1}{\alpha}\int_{\lambda_0(0)}^{\lambda_0} \left[\frac{3\lambda_0^2-1}{\lambda_0^2-1} + \frac{(3\lambda_0^2-1)(\lambda_0^2-1)}{6\lambda_0^2(9\lambda_0^2-1)}\right] d\lambda_0$$

$$= -\frac{1}{\alpha}\int_{\lambda_0(0)}^{\lambda_0} \left[\frac{55}{18} - \frac{1}{6\lambda_0^2} + \frac{2}{\lambda_0^2-1} + \frac{8}{81}\frac{1}{\lambda_0^2-1/9}\right] d\lambda_0$$

Thus

$$\tau = \frac{1}{\alpha}\left[\frac{55}{18}\,(\lambda_0(0) - \lambda_0) + \frac{1}{6}\left(\frac{1}{\lambda_0(0)} - \frac{1}{\lambda_0}\right)\right.$$

$$\left. - \ln\left(\frac{\lambda_0-1}{\lambda_0+1}\cdot\frac{\lambda_0(0)+1}{\lambda_0(0)-1}\right) - \frac{4}{27}\ln\left(\frac{3\lambda_0-1}{3\lambda_0+1}\cdot\frac{3\lambda_0(0)+1}{3\lambda_0(0)-1}\right)\right]$$

(3.3.14)

where from (3.3.8) $\lambda_0^2(0) = 1 + Q\,\rho(0) = 1 + \frac{3}{4}\gamma A_1^2(0)$

It should be noted that $\lambda_0(0)$ is not the same as Ω_a in (3.2.2).

When $\alpha \to 0$, $\tau \to \left[\lambda_0(0) + \dfrac{(\lambda_0^2(0)-1)^2}{6\lambda_0(0)\ (9\lambda_0^2(0) - 1)}\right] t$

To determine the constants θ and M, given the initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$, we have from eqns. (3.2.8), (3.2.9) and (3.3.5):

$$A_1(0)\cos\theta + \epsilon\, C_3(0)\cos 3\theta = 1$$

$$-\lambda_0(0)\,A(0)\sin\theta + \alpha \left.\frac{dA}{d\xi}\right|_{\xi=0}\cos\theta - \epsilon\, A(0)\,\lambda_1(0)\sin\theta$$

$$- 3\epsilon\, C_3(0)\,\lambda_0(0)\sin 3\theta = 0$$

From equation (3.3.9):

$$\left.\frac{dA_1}{d\xi}\right|_{\xi=0} = \frac{-A(0)\ (1+QA^2(0))}{2 + 3QA^2(0)}$$

Hence to a first approximation:

$$\theta \simeq -\alpha \sqrt{\frac{1+Q}{2+3Q}} = -\alpha \sqrt{\frac{1+\frac{3}{4}\gamma}{2+\frac{9}{4}\gamma}}$$

$$A_1(0) \simeq 1 - \frac{Q}{24+27Q} = 1 - \frac{\gamma}{32+27\gamma}$$

and using eqn. (3.3.10) we get:

$$M \simeq A^6(0) + \frac{4A^4(0)}{3\gamma}$$

Thus, to a first approximation, x is given by:

$$x = A\cos(\tau+\theta) + \frac{\gamma A^3}{32+27\gamma A^2}\cos 3(\tau+\theta)$$

where $A^2 = \rho$ and τ are given by eqn. (3.3.12) [or (3.3.13)] and (3.3.14) respectively.

For m=5

$$\gamma x^5 \simeq QA^5[\cos(\tau+\theta) + \frac{1}{2}\cos 3(\tau+\theta) + \frac{1}{10}\cos 5(\tau+\theta)]$$

$$+ \frac{1}{2}\varepsilon QA^4[(5C_3+C_5)\cos(\tau+\theta) + (3D_3+D_5)\sin(\tau+\theta)]$$

$$+ \varepsilon\sum_n\{\overline{G}_n\cos n(\tau+\theta) + \overline{\overline{G}}_n\sin n(\tau+\theta)\},\ n=3,5,7,\ldots$$

where $Q \triangleq \frac{5}{8}\gamma$.

The equations that determine the approximate solution are:

$$\lambda_0^2 = 1 + QA^4 \tag{3.3.16}$$

$$\epsilon C_3 = \frac{QA^5}{2(9\lambda_0^2-1)} \tag{3.3.17}$$

$$\epsilon C_5 = \frac{QA^5}{10(25\lambda_0^2-1)} \simeq \frac{QA^5}{250\lambda_0^2} \tag{3.3.18}$$

$$D_3 = D_5 = 0 \tag{3.3.19}$$

$$\lambda_0\lambda_1 = \frac{5}{4}QA^3(C_3 + \frac{C_5}{5}) \tag{3.3.20}$$

and

$$2\lambda_0 \frac{dA}{d\xi} + A\frac{d\lambda_0}{d\xi} + \lambda_0 A = 0 \tag{3.3.21}$$

Define $\ell \triangleq A^4$, then from equations (3.3.16) and (3.3.21).

$$(\frac{1}{\ell} + \frac{Q}{1+Q\ell})\frac{d\ell}{d\xi} = -2 \tag{3.3.22}$$

which integrates into:

$$Q\ell^2 + \ell = Me^{-2\xi} \tag{3.3.23}$$

M being a constant. Hence:

$$\ell = \frac{1}{2Q} \left(- 1 \pm \sqrt{1 + 4QM\, e^{-2\xi}}\right) \qquad (3.3.24)$$

The negative sign in front of the radical should be ignored since $\ell > 0$ for finite ξ and $\ell \to 0$ as $\xi \to \infty$.

From equations (3.3.16) and (3.3.24), we have:

$$\lambda_0 = \frac{1}{\sqrt{2}} \sqrt{1 + \sqrt{1 + 4QM\, e^{-2\xi}}} > 1 \qquad (3.3.25)$$

and from eqns. (3.3.16), (3.3.17), (3.3.18) and (3.3.21):

$$\varepsilon\lambda_1 = \frac{(\lambda_0^2-1)^2}{\lambda_0} \left[\frac{5}{8(9\lambda_0^2-1)} + \frac{1}{1000\lambda_0^2}\right]$$

From eqns. (3.3.16) and (3.3.20):

$$\frac{d\xi}{d\lambda_0} = - \frac{(2\lambda_0^2-1)}{\lambda_0(\lambda_0^2-1)}$$

$$\tau = \int_0^t (\lambda_0+\varepsilon\lambda_1)\, dt = \frac{1}{\alpha} \int_{\lambda_0(0)}^{\lambda_0} (\lambda_0+\varepsilon\lambda_1) \frac{d\xi}{d\lambda_0}\, d\lambda_0$$

$$= - \frac{1}{\alpha} \int_{\lambda_0(0)}^{\lambda_0} \left[2 \frac{317}{2250} - \frac{157}{250} \frac{1}{\lambda_0^2} + \frac{1}{\lambda_0^2-1} + \frac{35}{81} \frac{1}{\lambda_0^2 - \frac{1}{9}} + \frac{1}{1000\lambda_0^4}\right] d\lambda_0$$

Thus

$$\tau = \frac{1}{\alpha}[2\ \frac{317}{2250}\ (\lambda_0(0) - \lambda_0) + \frac{157}{250}\ (\frac{1}{\lambda_0(0)} - \frac{1}{\lambda_0})$$

$$+ \frac{1}{3000}\ (\frac{1}{\lambda_0^3} - \frac{1}{\lambda_0^3(0)}) - \frac{1}{2}\ \ln\ (\frac{\lambda_0-1}{\lambda_0+1} \cdot \frac{\lambda_0(0)+1}{\lambda_0(0)-1})$$

$$- \frac{35}{54}\ \ln(\frac{3\lambda_0-1}{3\lambda_0+1} \cdot \frac{3\lambda_0(0)+1}{3\lambda_0(0)-1})] \qquad (3.3.26)$$

Whereas from equation (3.3.25)

$$\lambda_0(0) = \frac{1}{\sqrt{2}} \sqrt{1 + \sqrt{1 + 4QM}} = \frac{1}{\sqrt{2}} \sqrt{1 + \sqrt{1 + 3\gamma M}}$$

Note that in the limit as $\alpha \to 0$:

$$\tau \to [\lambda_0(0) + \frac{(\lambda_0^2(0)-1)^2}{\lambda_0(0)} \quad (\frac{5}{8(9\lambda_0^2(0)-1)} + \frac{1}{1000\lambda_0^2(0)})]\ t$$

With the normalized initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$, the constants θ and M are determined from:

$$x(0) = A(0)\cos\theta + \epsilon C_3(0)\cos 3\theta + \epsilon C_5(0)\cos 5\theta = 1$$

and

$$\dot{x}(0) = -\lambda_0(0)\ A(0)\sin\theta + \alpha \left.\frac{dA}{d\xi}\right|_{\xi=0} \cos\theta - \epsilon\lambda_1(0)\ A(0)\sin\theta$$

$$- 3\epsilon C_3(0)\ \lambda_0(0)\ \sin 3\theta - 5\epsilon C_5(0)\ \lambda_0(0)\ \sin 5\theta = 0$$

From eqn. (3.3.22)

$$\left.\frac{dA}{d\xi}\right|_{\xi=0} = \frac{-\ A(0)\ \ (1+QA^4(0))}{2(1\ +\ 2QA^4(0)\ )}$$

Thus, to a first approximation:

$$\theta = \frac{-\alpha\sqrt{1+Q}}{2\ +\ 4Q} = \frac{-\alpha\ \sqrt{1\ +\ \frac{5}{8}\gamma}}{2\ +\ \frac{5}{2}\gamma}$$

$$A(0) = 1 - \frac{Q}{2(8+9Q)} - \frac{Q}{250(1+Q)}$$

$$= 1 - \frac{5\gamma}{128+90\gamma} - \frac{\gamma}{50(8+5\gamma)}$$

and using eqn. (3.3.23):

$$M = \frac{5}{8}\gamma\ A^8(0) + A^4(0)$$

Thus to a first approximation, x is given by:

$$x = A\ \cos(\tau+\theta) + \frac{5\gamma A^5}{128+90\gamma A^4}\cos 3(\tau+\theta)$$

$$+ \frac{\gamma A^5}{50(8+5\gamma A^4)}\cos 5(\tau+\theta) \qquad (3.3.27)$$

where $A^4 = \ell$ is given by:

$$\ell = \frac{4}{5\gamma}[-\ 1 + \sqrt{1 + \frac{5}{2}\gamma\ Me^{-2\alpha t}}]$$

and τ is given by equation (3.3.26).

Numerical solutions of eqn. (3.3.1) were obtained for several values of γ and α with $m = 3$ and 5. Also, the procedure proposed by Barkham and Soudack (1969) for cubic non-linearities was simulated on a CDC 6400 digital computer (elliptic functions accurate to six decimal places were used).

Figs. (3.1) and (3.2) show the solutions of:

$$x'' + 5x + 10x^3 + 0.3x' = 0 \qquad (3.3.28)$$

and

$$x'' + 5x + 10x^3 + 0.5x' = 0 \qquad (3.3.29$$

with

$$(') \equiv \frac{d}{dT}\ , \ x(0) = 1 \text{ and } x'(0) = 0$$

These two examples were considered by Barkham and Soudack [1969].

When written in the normalized form of eqn. (3.3.1), eqns. (3.3.28) and (3.3.29) become:

$$\ddot{x} + x + 2x^3 + 0.06\sqrt{5}\,\dot{x} = 0 \qquad (3.3.30)$$

and

$$\ddot{x} + x + 2x^3 + 0.1\sqrt{5}\,\dot{x} = 0 \qquad (3.3.31)$$

with

$$(\cdot) \equiv \frac{d}{dt}\ , \ x(0) = 1,\ \dot{x}(0) = 0 \text{ and } t \equiv \sqrt{5}T$$

Figs. (3.3) to (3.7) show the solutions of:

$$\ddot{x} + x + \gamma x^3 + \alpha\dot{x} = 0 \;,\; x(0) = 1,\; \dot{x}(0) = 0$$

for values of γ in the range from 5 to 400 and for values of α in the range from 0.25 to 1.

Figs. (3.8) to (3.10) show the solutions of:

$$\ddot{x} + x + \gamma x^5 + \alpha\dot{x} = 0,\; x(0) = 1,\; \dot{x}(0) = 0$$

for $\gamma = 20$ and $\alpha = 0.5$, $\gamma = 100$ and $\alpha = 1$ and $\gamma = 1000$ and $\alpha = 1$ respectively.

Though the cases with very large values of γ may not be of practical interest, their introduction here is done to demonstrate the versatility of the present method.

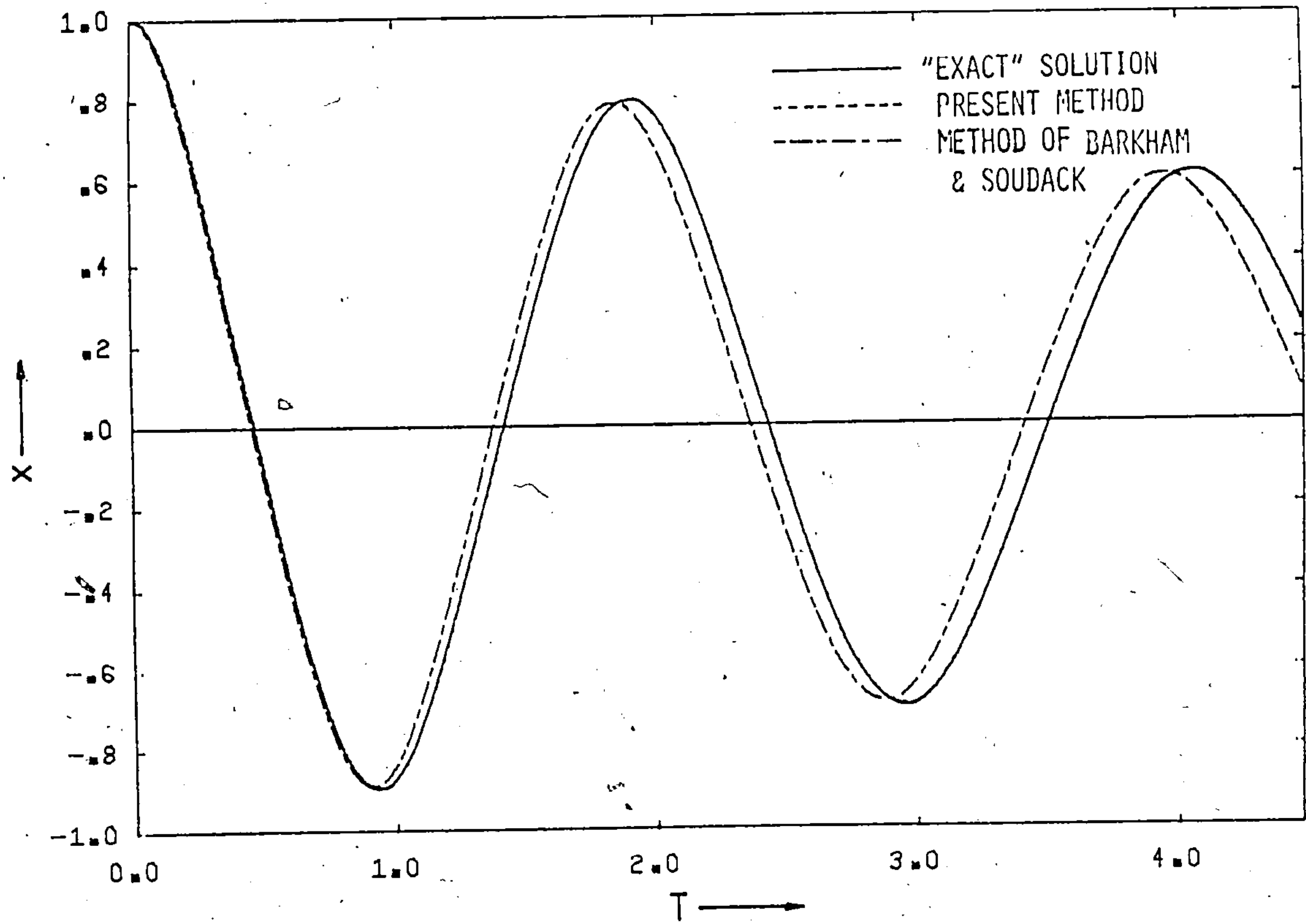

Fig. 3.1-a Solution of: $x'' + 5x + 10x^3 + 0.3x' = 0$

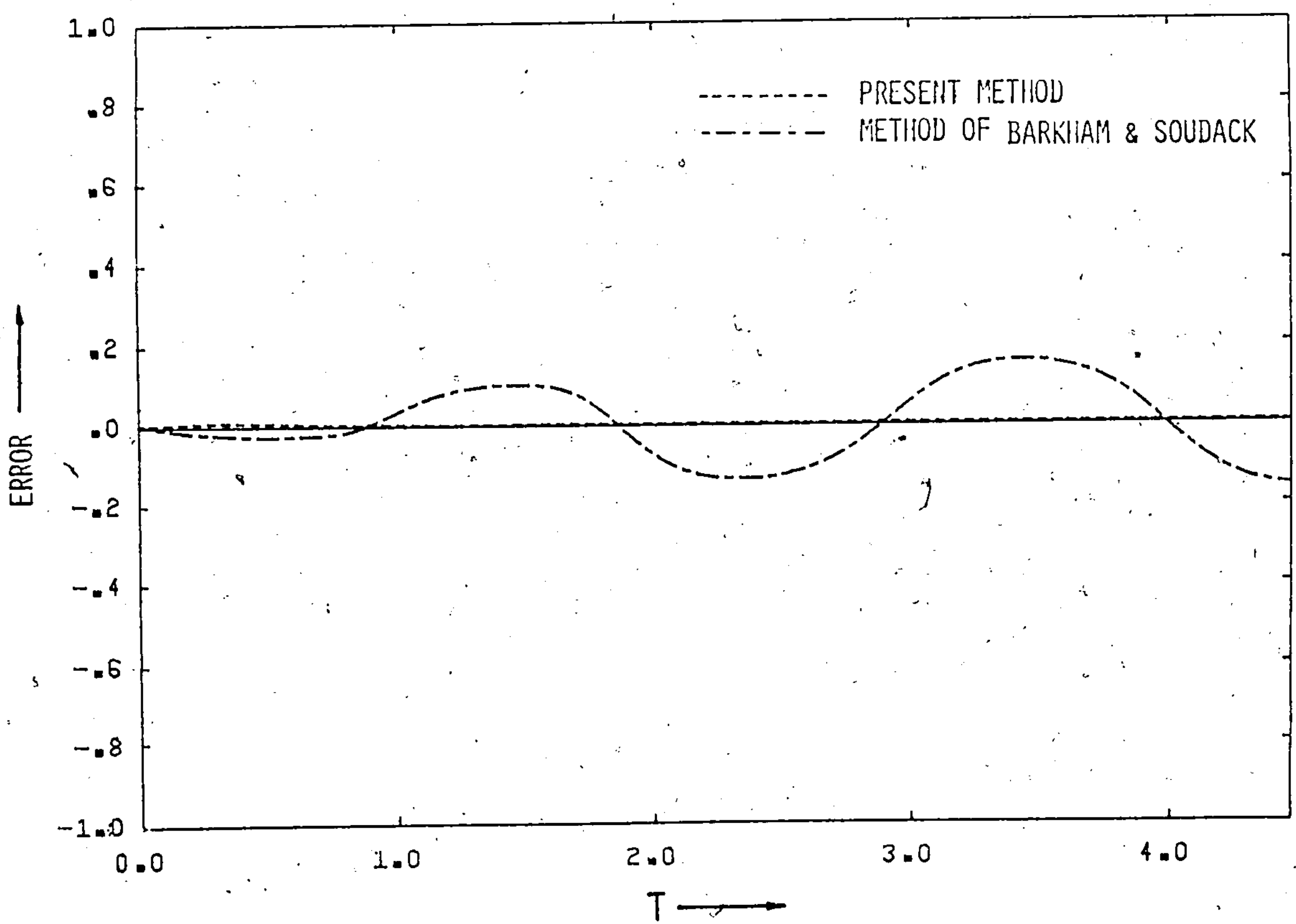

Fig. 3.1-b Error (≡ Approximate solution - "Exact" numerical solution) versus time.

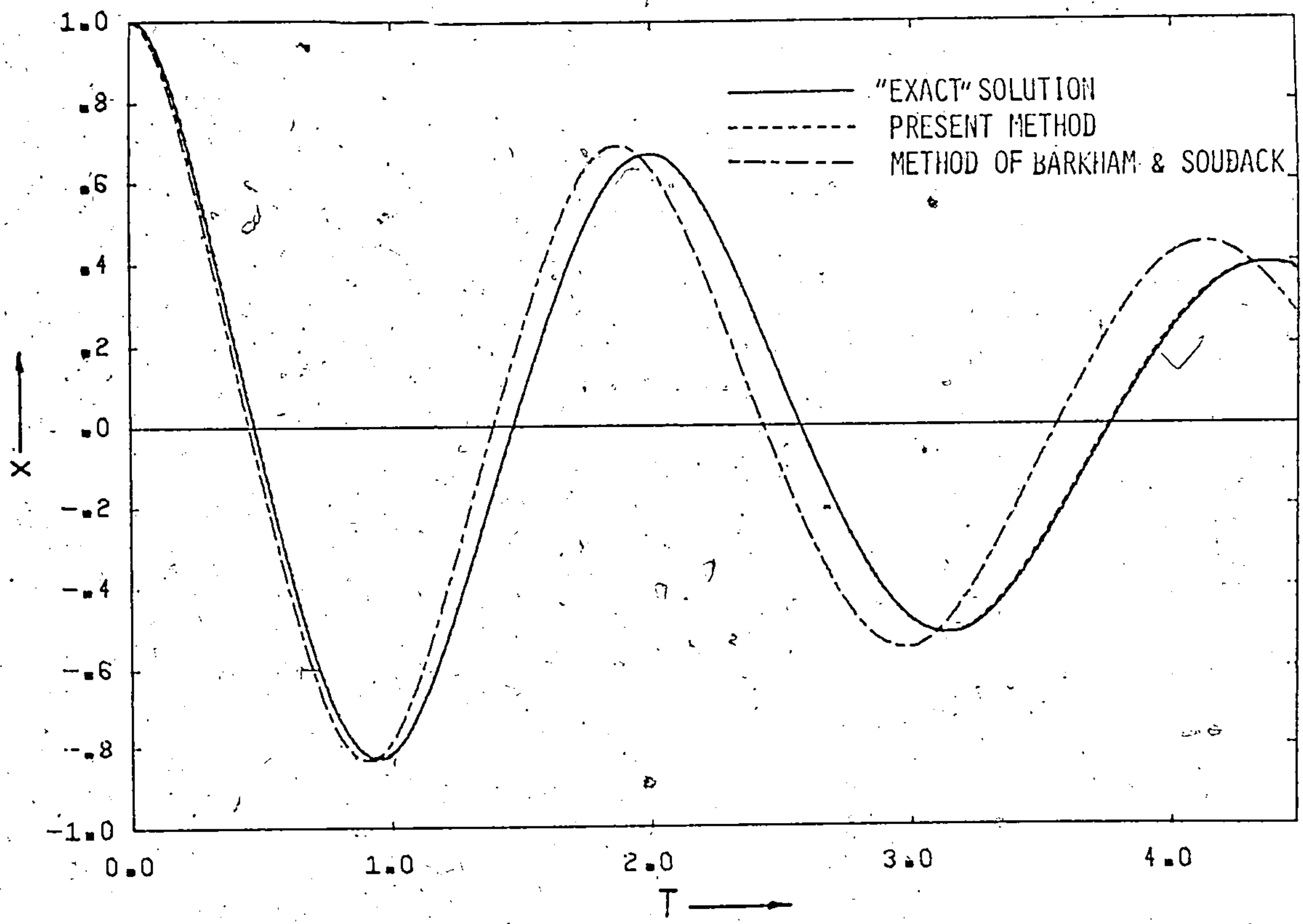

Fig. 3.2-a Solution of: $x'' + 5x + 10x^3 + 0.5x' = 0$

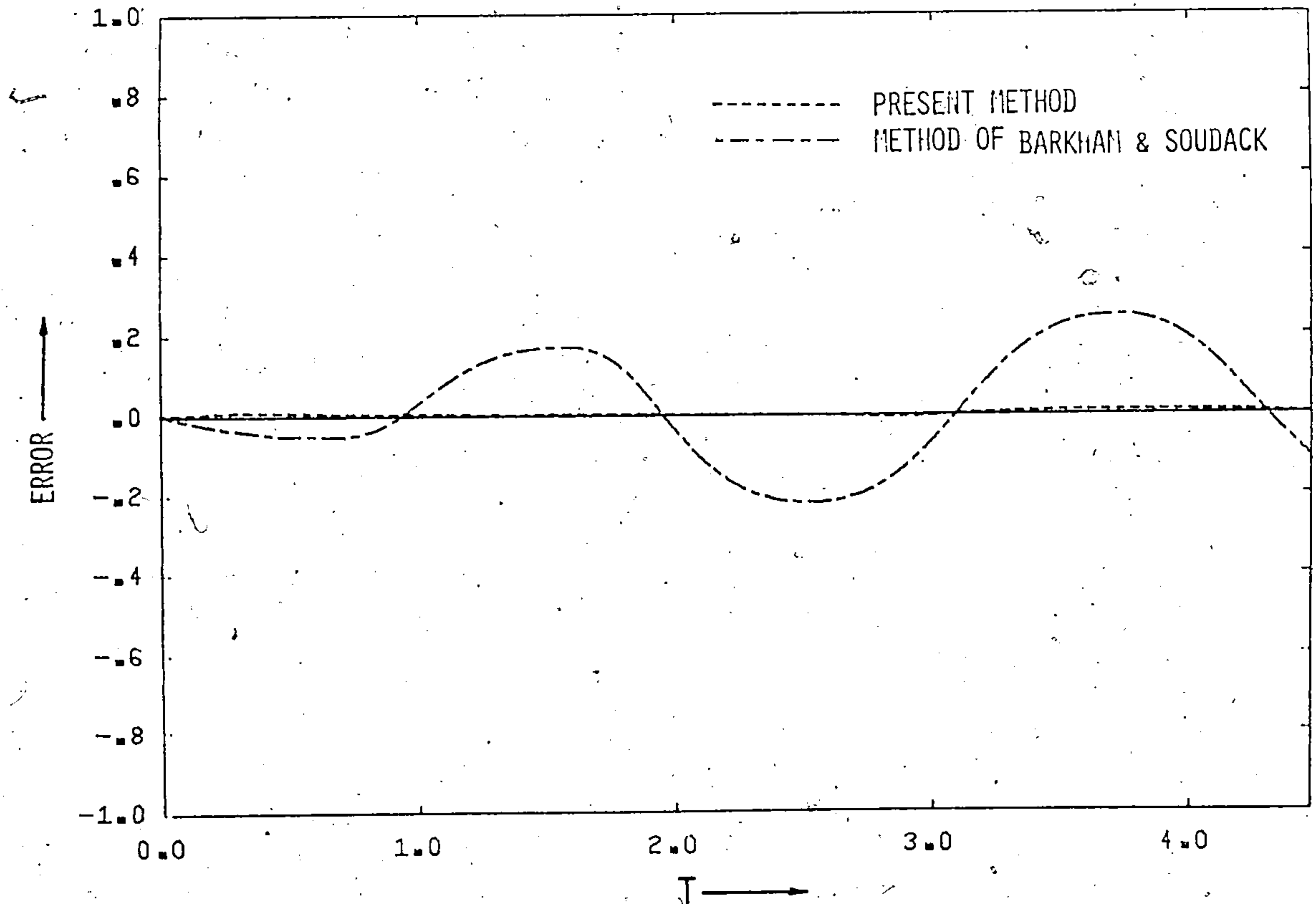

Fig. 3.2-b Error (≡ Approximate solution - "Exact" numerical solution) versus time.

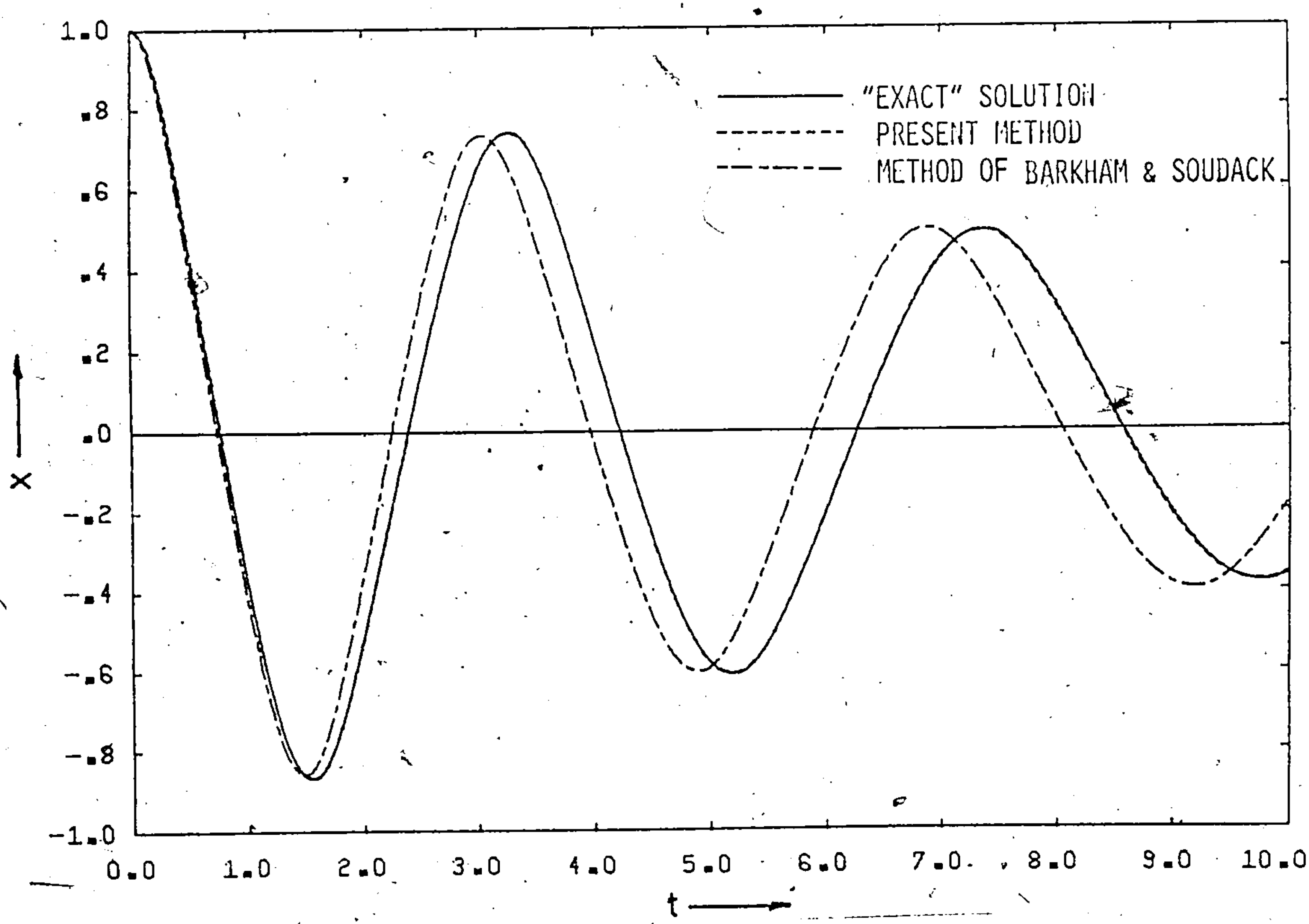

Fig. 3.3-a Solution of: $\ddot{x} + x + 5x^3 + 0.25\dot{x} = 0$

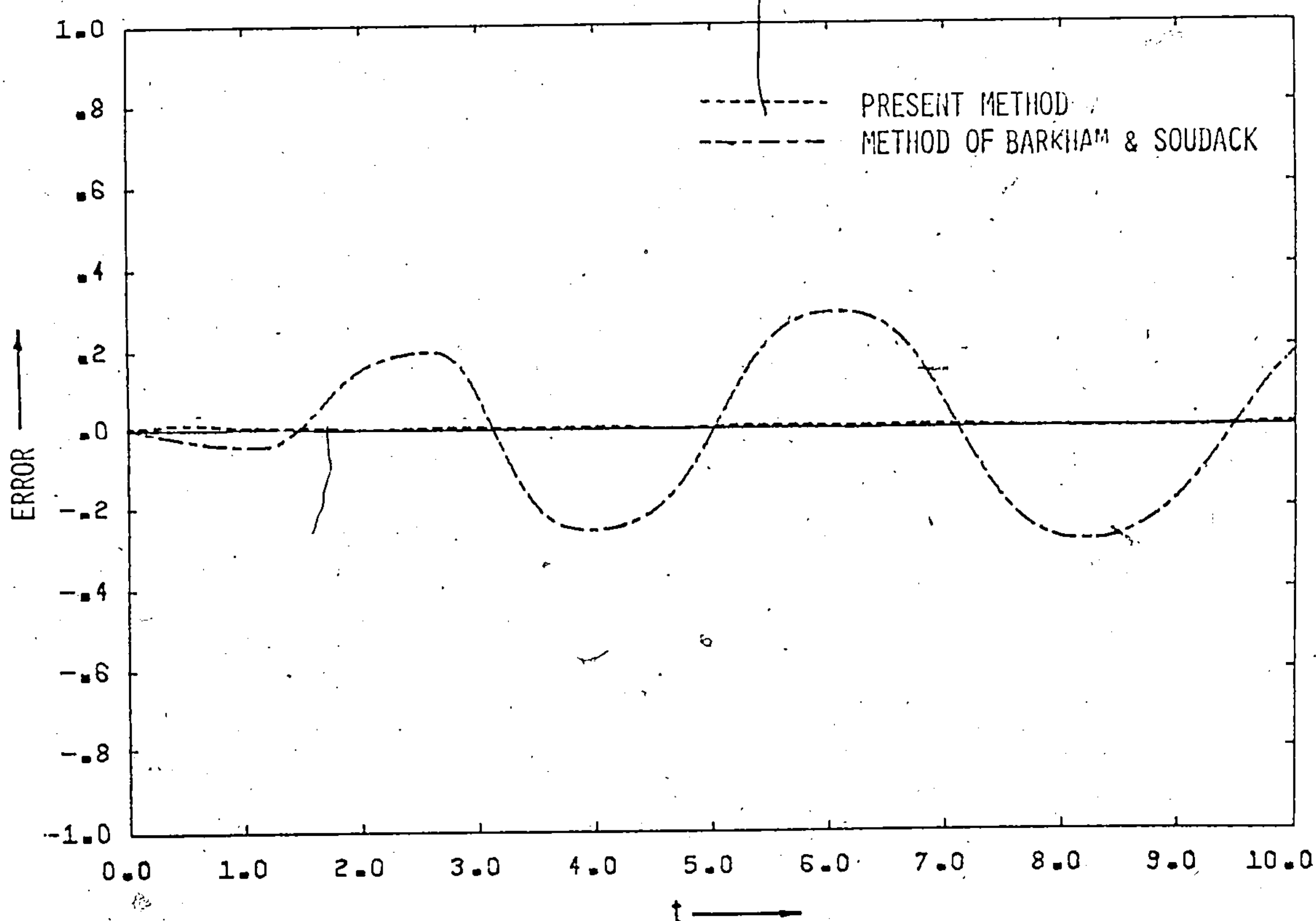

Fig. 3.3-b Error (≡ Approximate solution - "Exact" numerical solution) versus time.

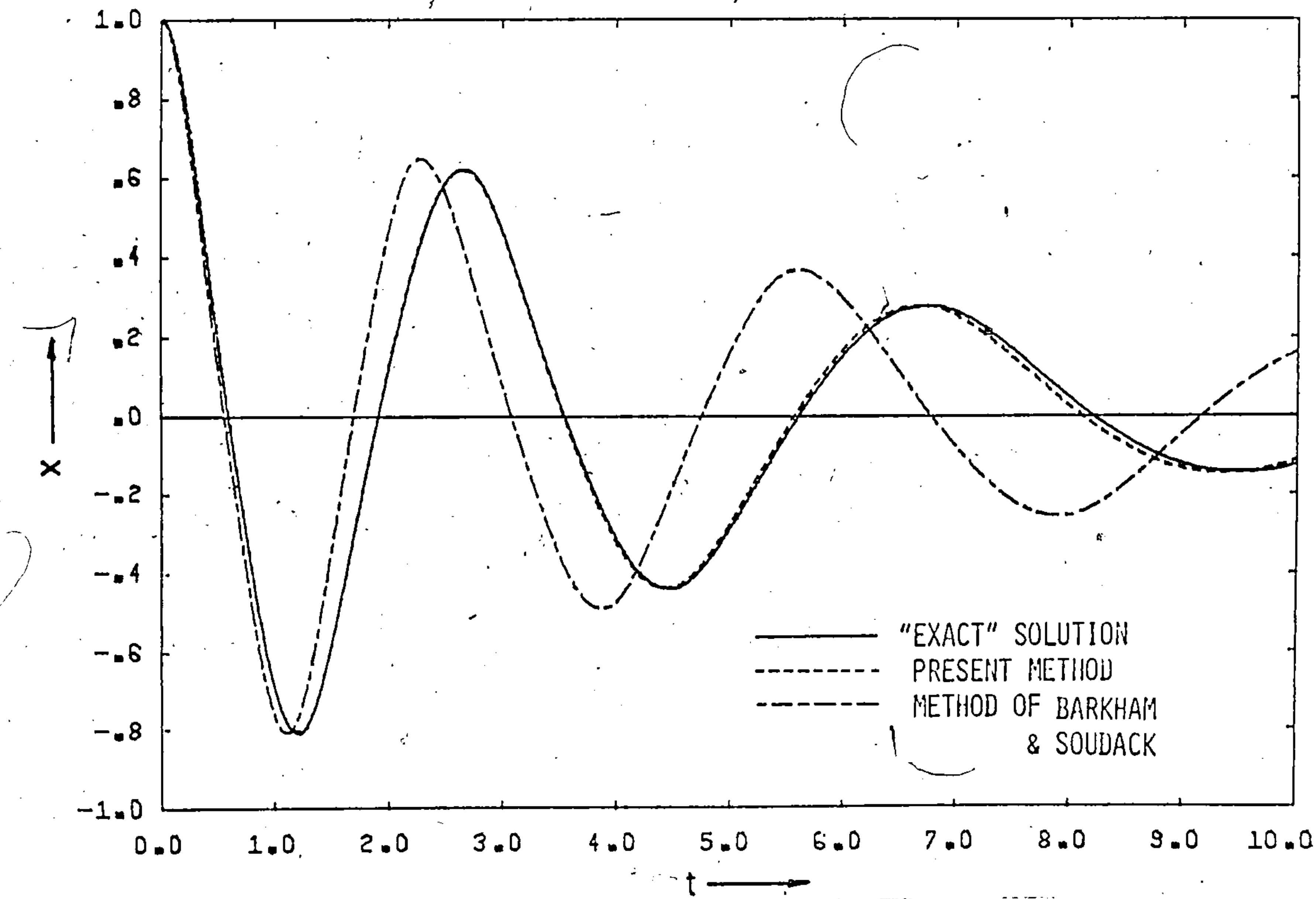

Fig. 3.4-a Solution of: $\ddot{x} + x + 10x^3 + 0.5\dot{x} = 0$

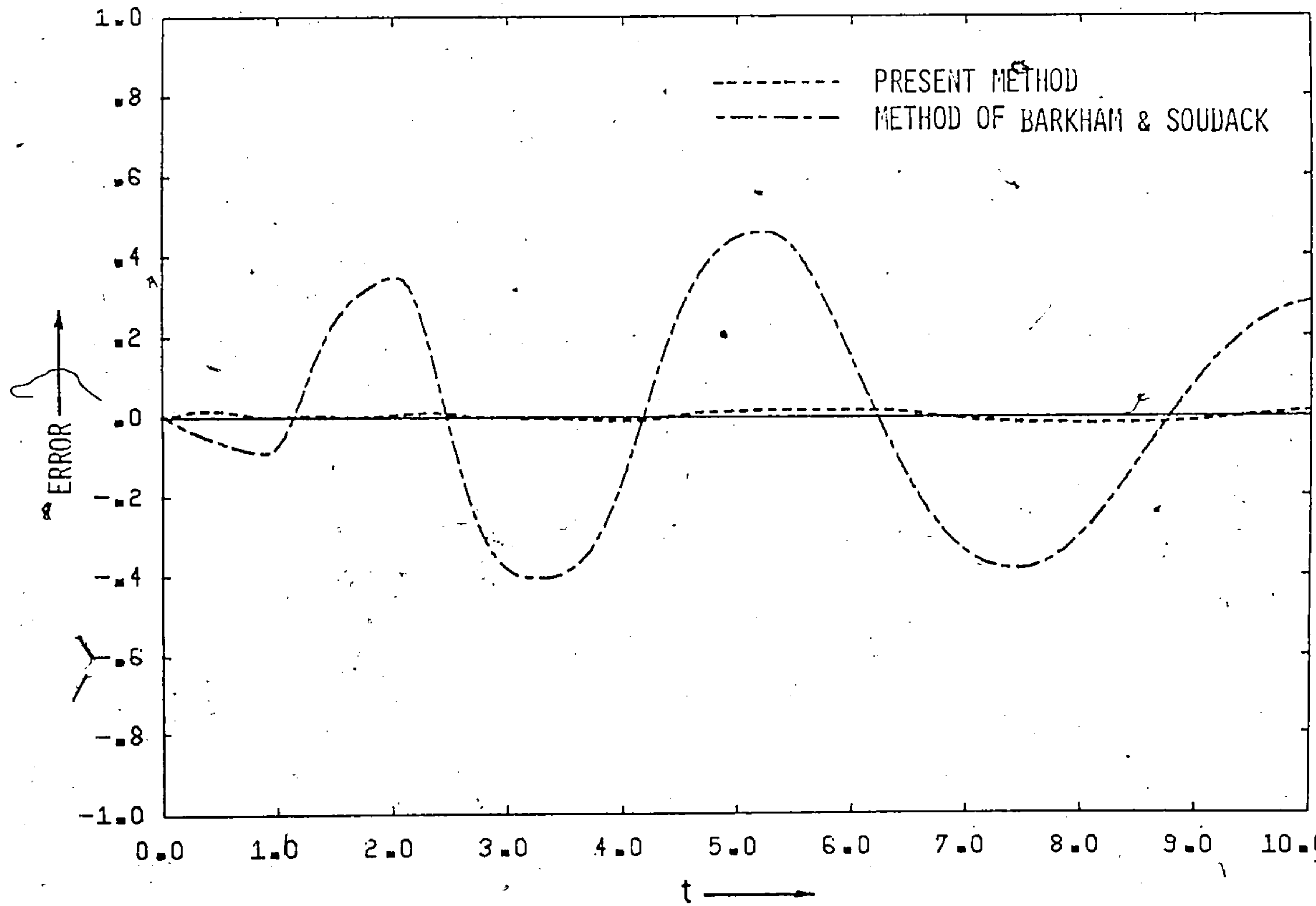

Fig. 3.4-b Error (≡ Approximate solution - "Exact" numerical solution) versus time.

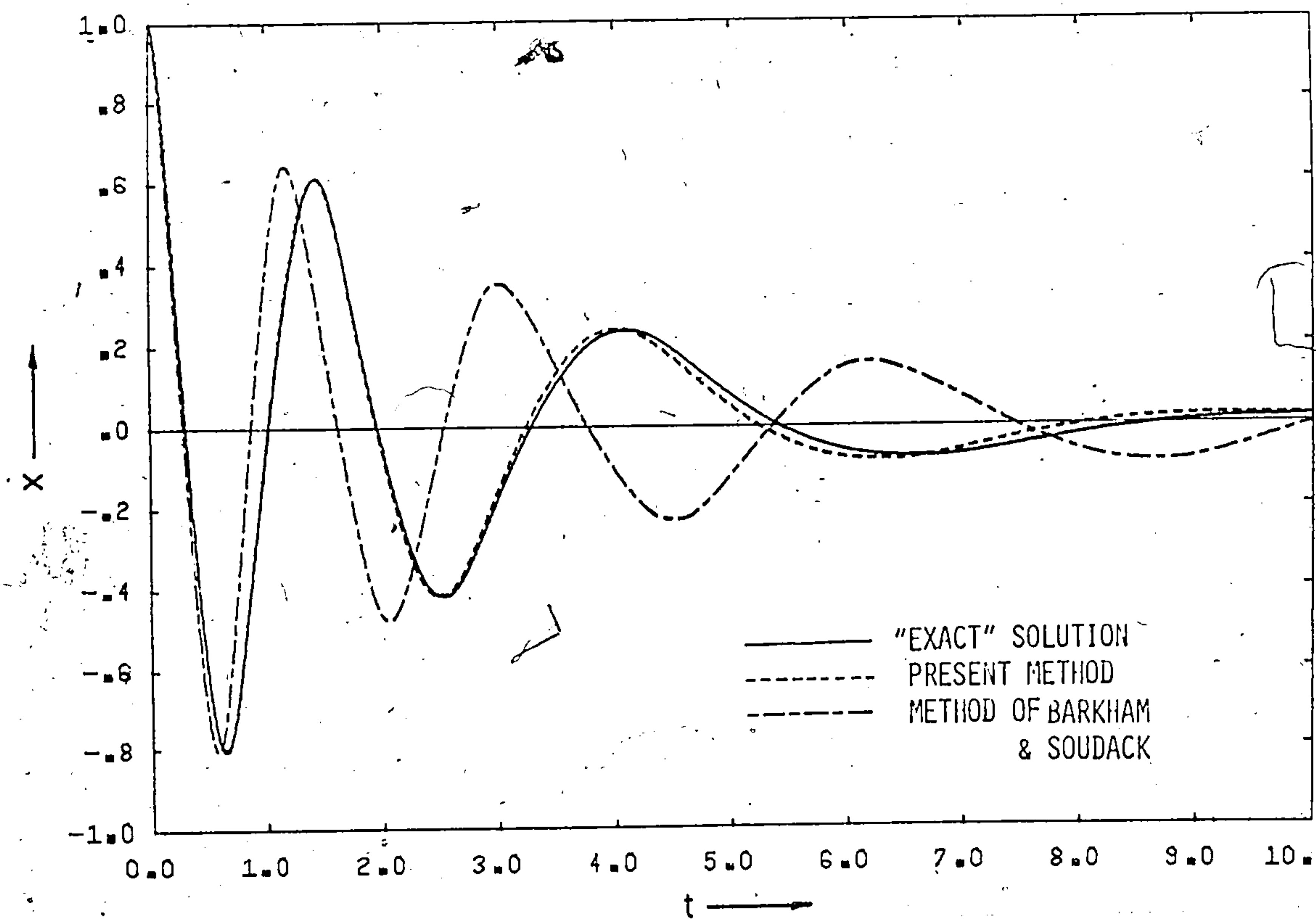

Fig. 3.5-a Solution of: $\ddot{x} + x + 40x^3 + \dot{x} = 0$

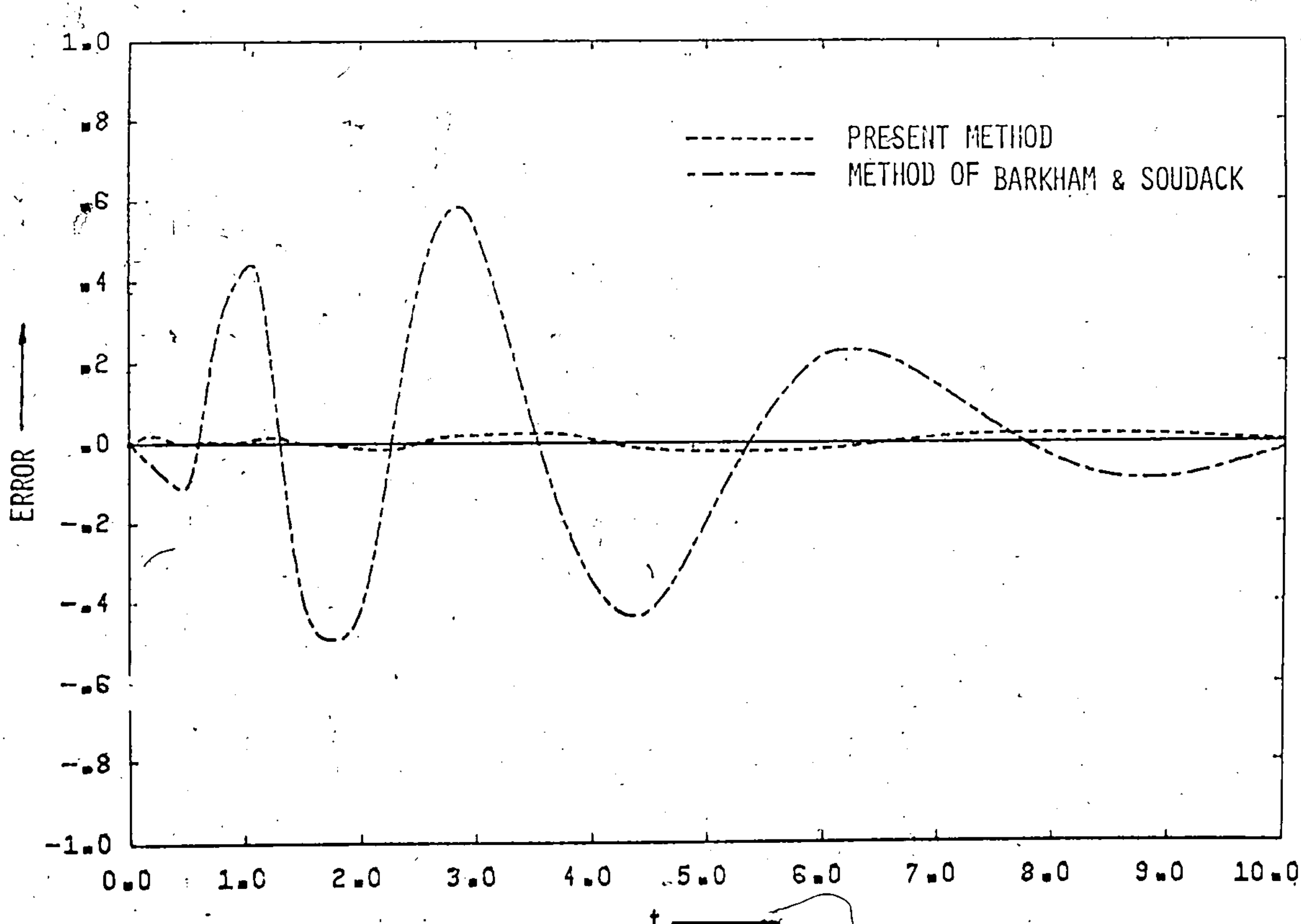

Fig. 3.5-b Error (≡ Approximate solution - "Exact" numerical solution) versus time.

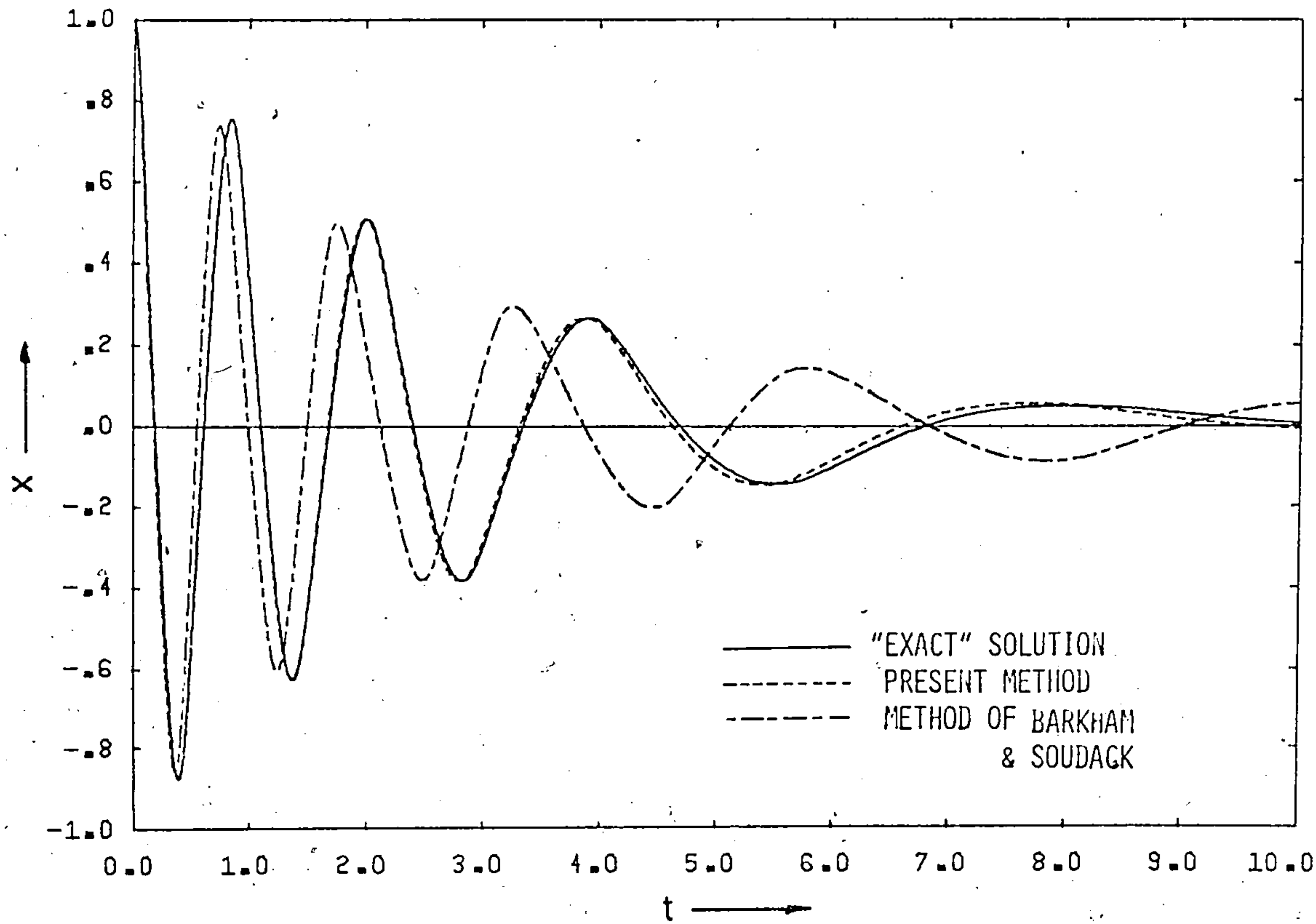

Fig. 3.6-a Solution of: $\ddot{x} + x + 100x^3 + \dot{x} = 0$

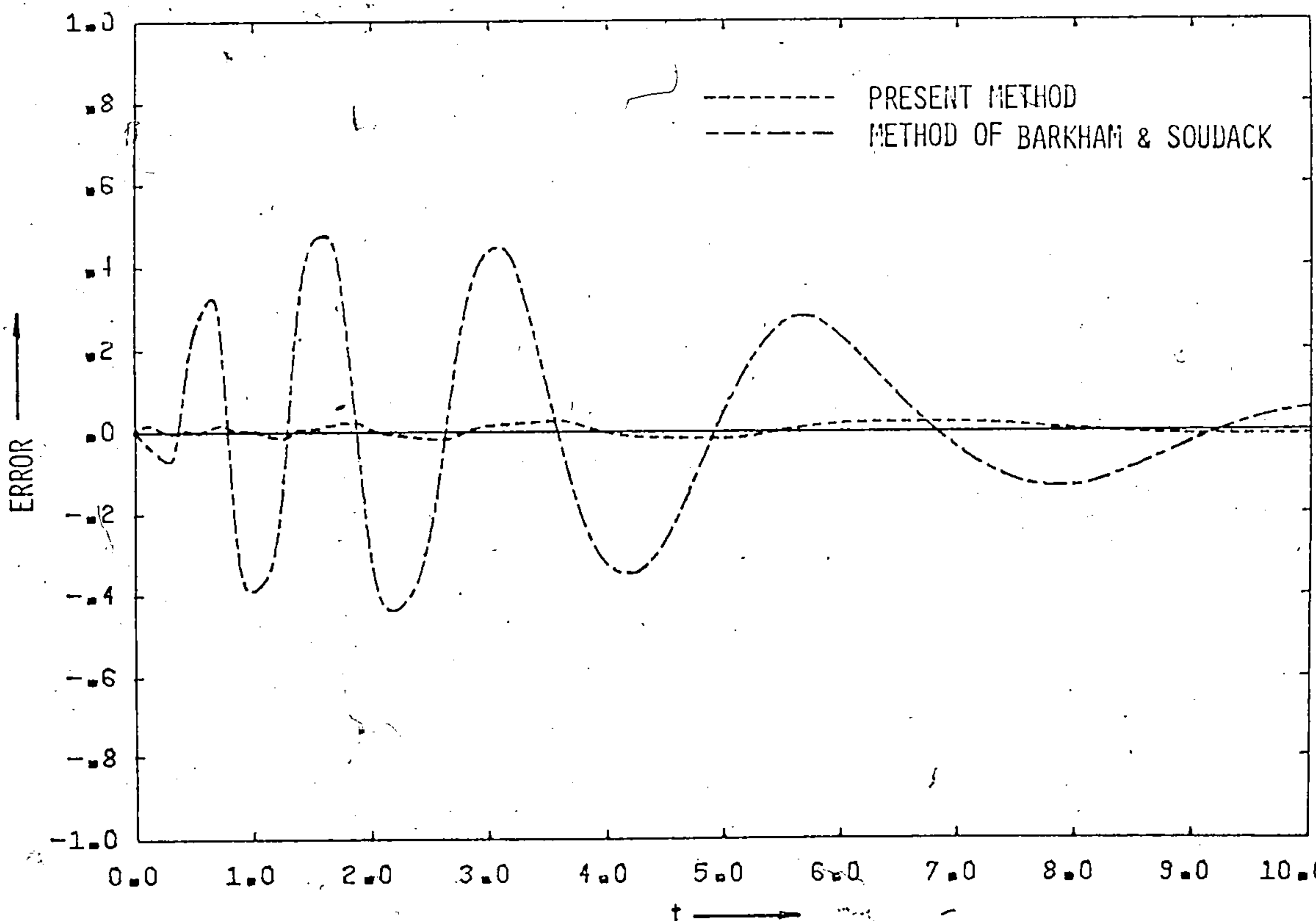

Fig. 3.6-b Error (≡ Approximate solution - "Exact" numerical solution) versus time.

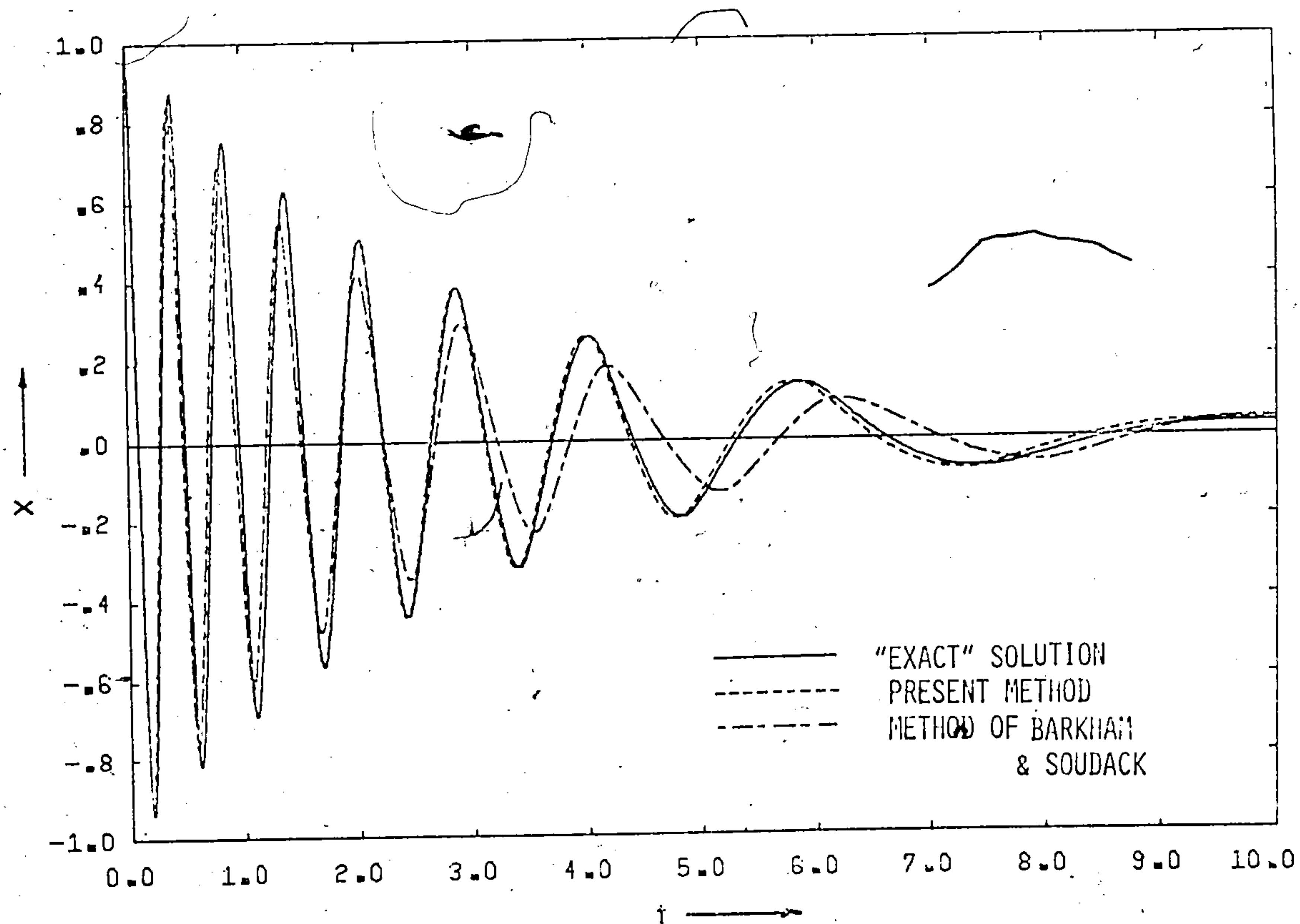

Fig. 3.7-a Solution of: $\ddot{x} + x + 400x^3 + \dot{x} = 0$

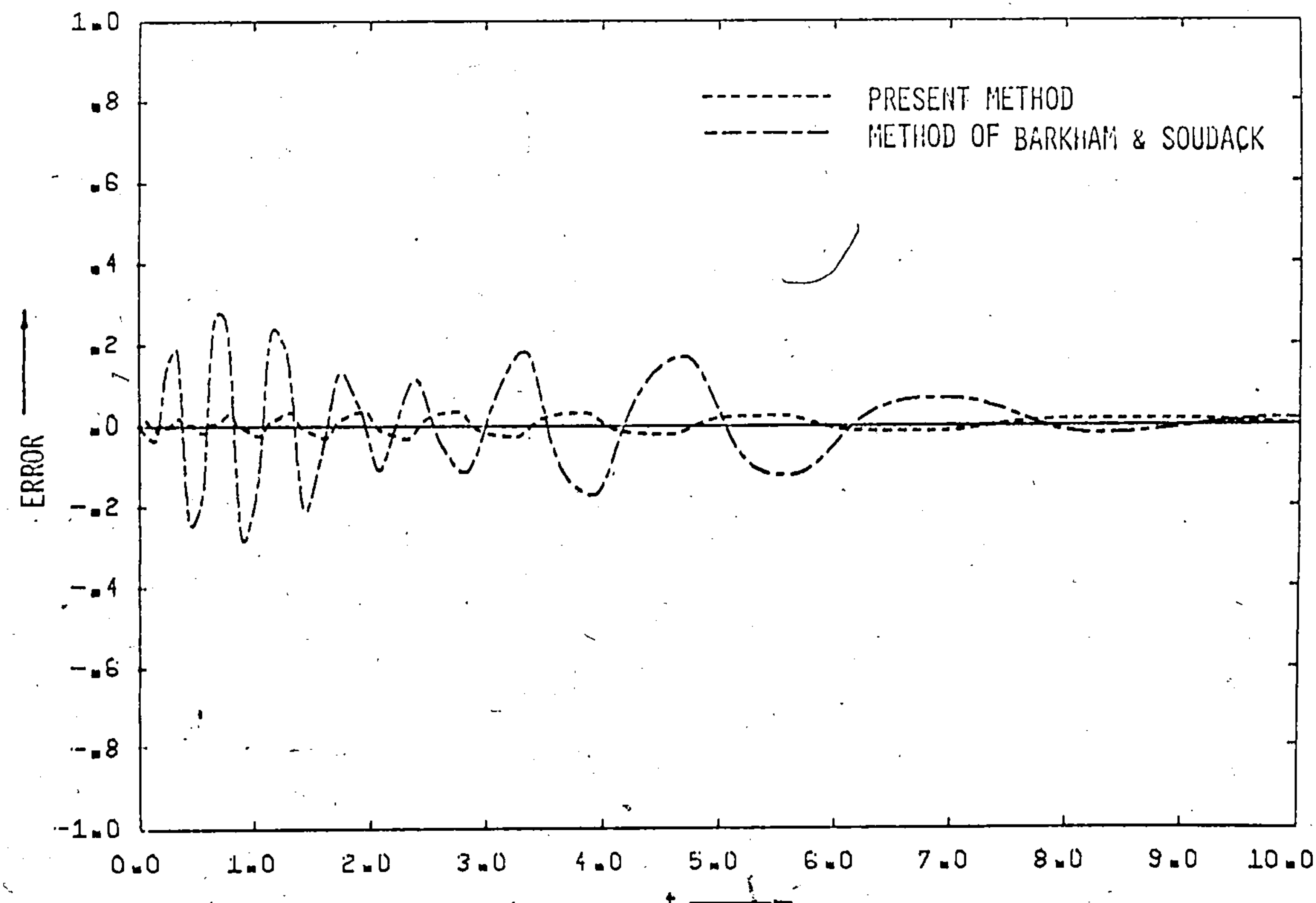

Fig. 3.7-b Error (≡ Approximate solution - "Exact" numerical solution) versus time.

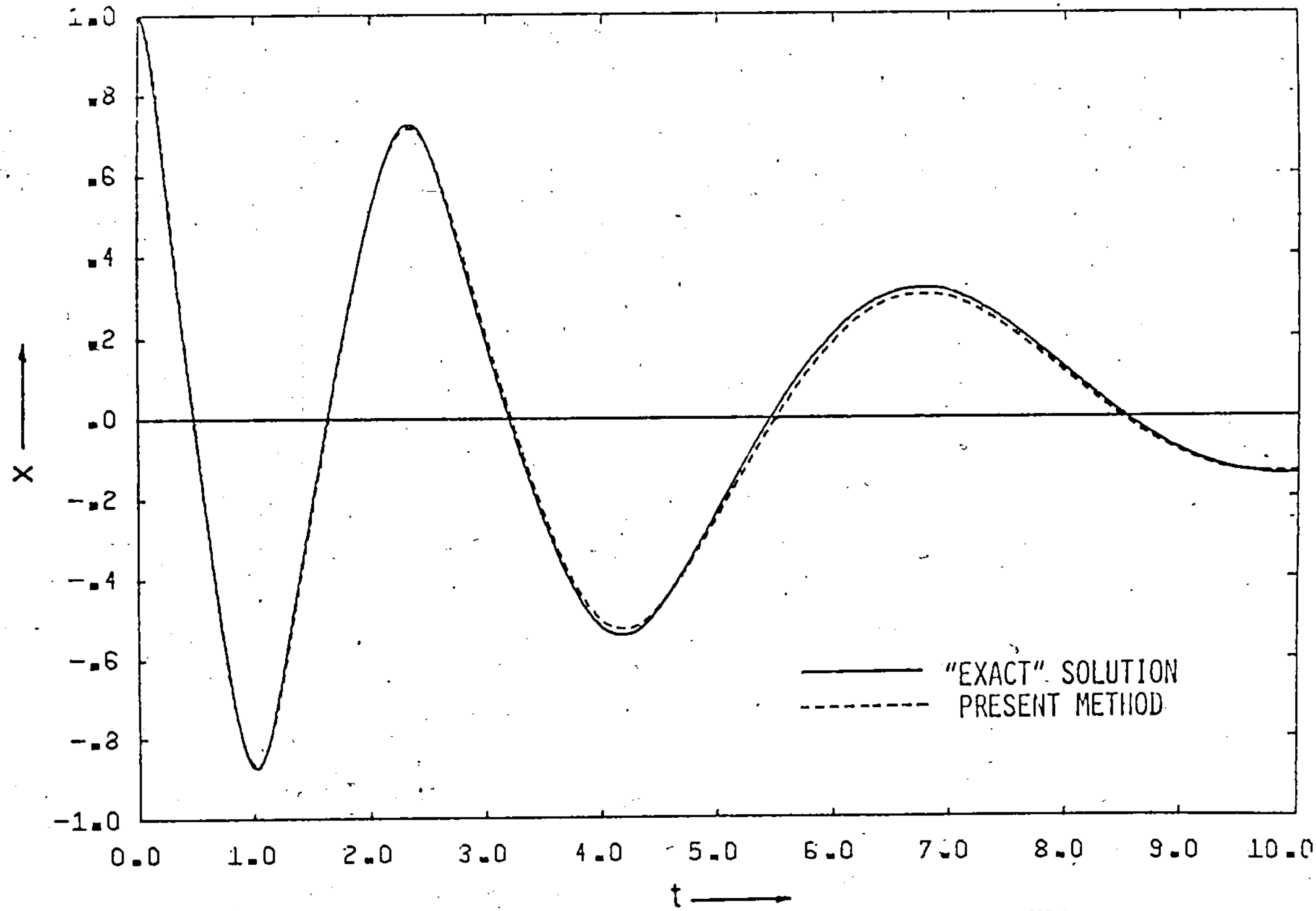

Fig. 3.8 Solution of: $\ddot{x} + x + 20x^5 + 0.5\dot{x} = 0$

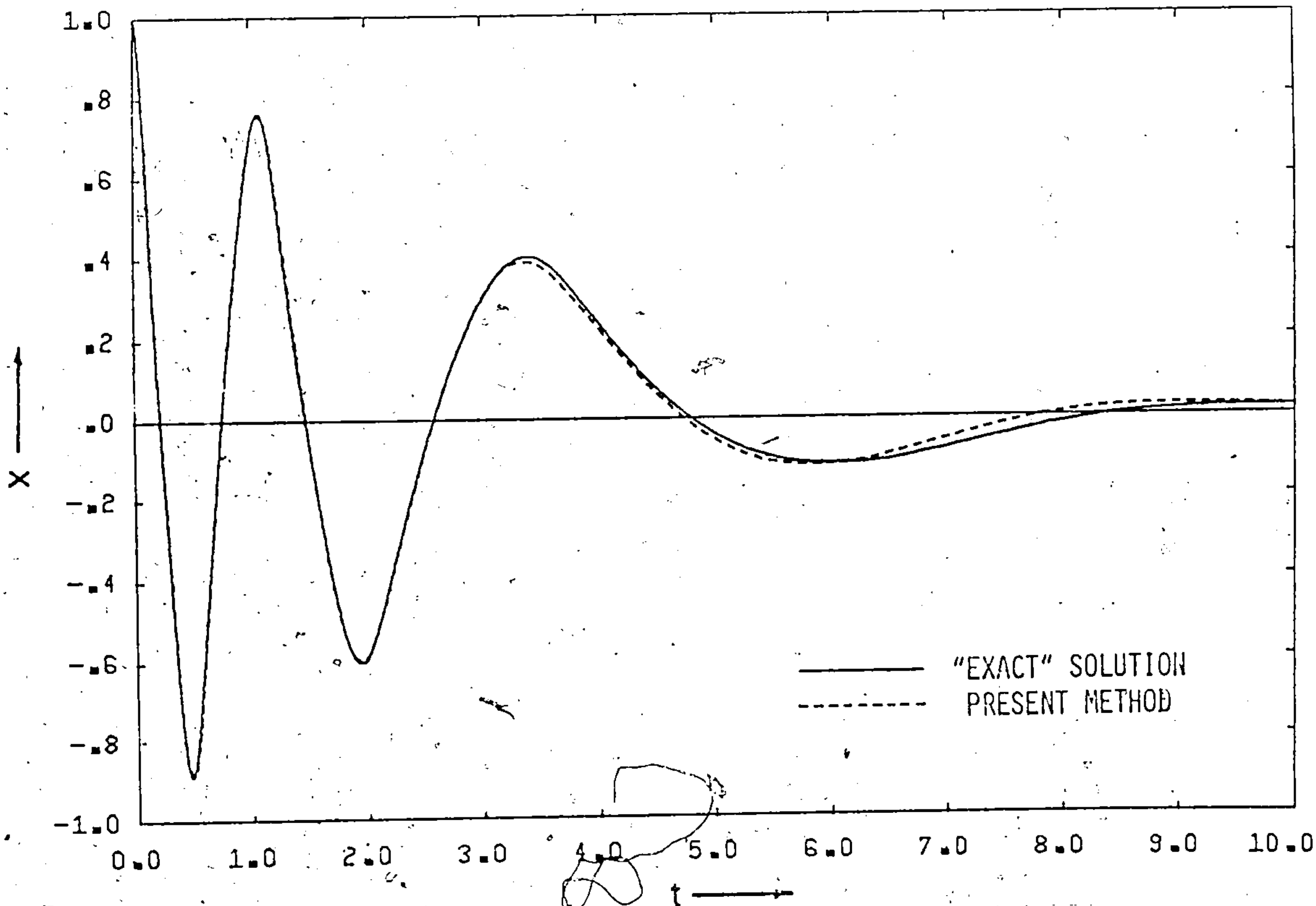

Fig. 3.9 Solution of: $\ddot{x} + x + 100x^5 + \dot{x} = 0$

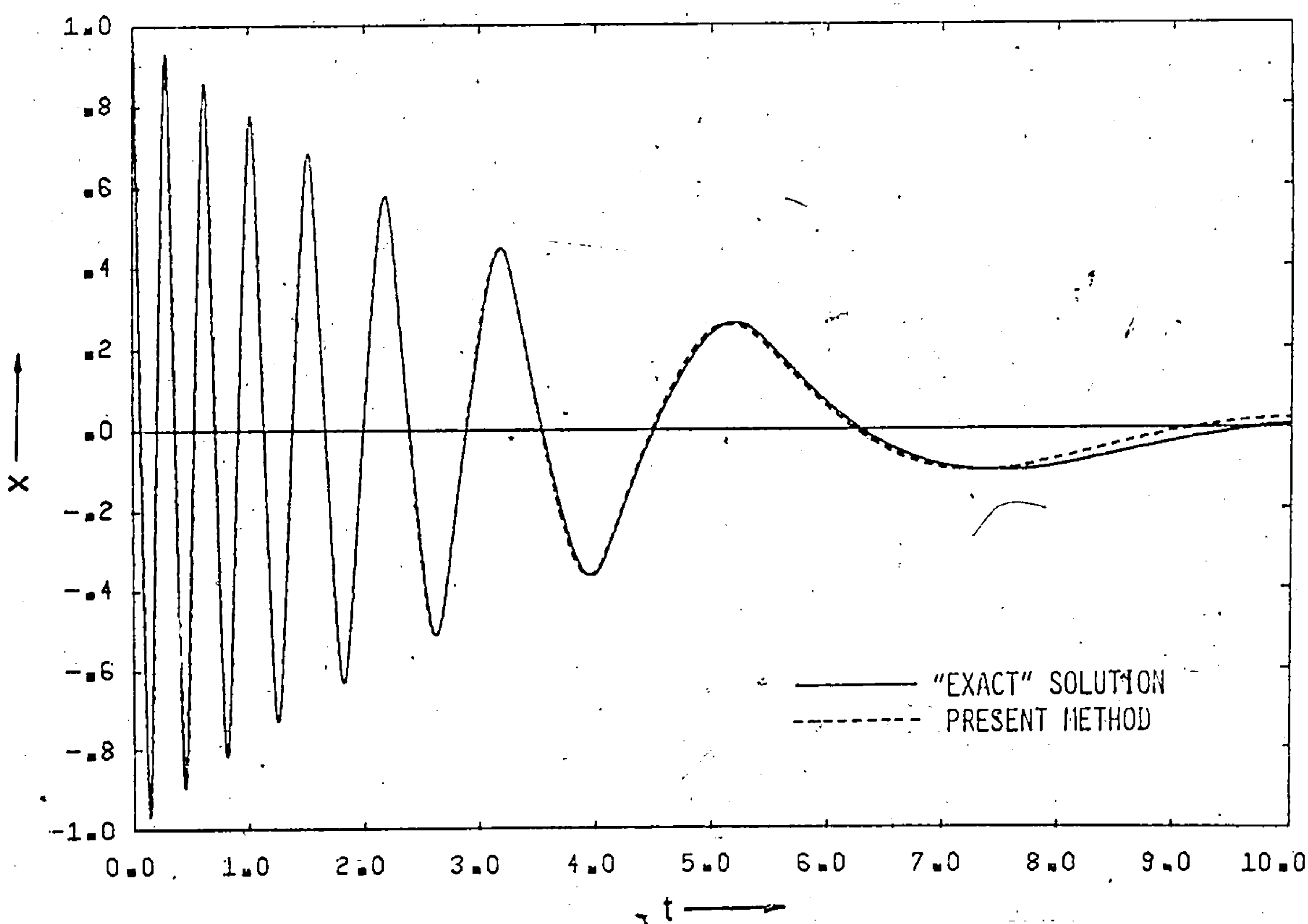

Fig. 3.10 Solution of: $\ddot{x} + x + 1000x^5 + \dot{x} = 0$

3.4 Application to a slowly time-variant system

The method described in section (3.2) can also be applied to a system described by:

$$\ddot{x} + P(x,\xi) + \alpha f(x,\dot{x},\xi) = 0 \qquad (3.4.1)$$

$0<\alpha<<1$ and $\xi \triangleq \alpha t$

Where the solution of the unperturbed equation: $\ddot{x} + P(x) = 0$ is assumed to be periodic with a dominant first harmonic. We shall consider the case when $P(x,\xi)$ and $f(x,\dot{x},\xi)$ take the forms: $P_1(\xi)\,P_2(x)$ and $f_1(\xi)\ f_2(x,\dot{x})$ respectively, where P_2 and f_2 are anti-symmetric functions of their arguments. In this case P_2 takes the form in (3.2.11) and f takes the form in (2.2.7) with odd values of n.

The application of the method to the above case is demonstrated by the examples below. The first example was studied by Barkham and Soudack (1970) using a different approach.

Example 3.4.1

Cosnider a system described by:

$$\ddot{x} + x(1 + \alpha t) + \gamma x^3 = 0 \qquad (3.4.2)$$

$$0 \le \alpha << 1 \text{ and } 0 \le \gamma < \infty .$$

The solution is determined from:

$$\lambda_0^2 = 1 + \xi + QA^2 \quad ; \quad \xi \triangleq \alpha t \text{ and } Q = \frac{3}{4}\gamma \qquad (3.4.3)$$

$$\varepsilon C_3 = \frac{Q}{3} \frac{A^3}{8\lambda_0^2 + QA^2} \quad , \quad D_3 = C_5 = D_5 = 0 \tag{3.4.4}$$

$$\lambda_1 = \frac{QA\ C_3}{2\lambda_0} \tag{3.4.5}$$

and

$$2\lambda_0 \frac{dA}{d\xi} + A \frac{d\lambda_0}{d\xi} = 0 \tag{3.4.6}$$

Eqn. (3.4.6) integrates into:

$$\lambda_0 A^2 = K \tag{3.4.7}$$

K being a constant given by:

$$K = \lambda_0(0)A^2(0) = A^2(0)\sqrt{1 + QA^2(0)} \ .$$

Define

$$\eta \triangleq \frac{4}{27Q^2K^2}(1+\xi)^3 \ .$$

Noting that both λ_0 and A^2 should be real positive quantities, the only admissible solution of eqns. (3.4.3) and (3.4.7) is:

$$A^2 = \frac{\left(\frac{2}{Q}\right)^{1/3} K^{2/3}}{(1+\sqrt{1-\eta})^{1/3} + (1-\sqrt{1-\eta})^{1/3}} \tag{3.4.8}$$

for $\eta<1$ i.e. for $0\le t<\frac{1}{\alpha}[3(\frac{QK}{2})^{2/3} - 1]$

and

$$A^2 = \frac{\sqrt{3}}{2} K \frac{\sec(\beta/3)}{\sqrt{1+\xi}} \quad \text{where } \cos\beta = \frac{1}{\sqrt{\eta}}; \quad 0\le\beta<\frac{\pi}{2} \tag{3.4.9}$$

for $\eta\ge1$ i.e. for $\frac{1}{\alpha}[3(\frac{QK}{2})^{2/3} - 1] \le t<\infty$

From eqns. (3.4.4), (3.4.5) and (3.4.7):

$$\lambda \approx \lambda_0 + \epsilon\lambda_1 = \lambda_0[1 + \frac{Q^2K^2/6}{\lambda_0^3(8\lambda_0^3 + QK)}]$$

From eqns. (3.4.3) and (3.4.7):

$$\frac{d\xi}{d\lambda_0} = 2\lambda_0 + \frac{QK}{\lambda_0^2}$$

Hence

$$\tau = \int_0^t \lambda dt = \frac{1}{\alpha}\int_0^\xi \lambda d\xi = \frac{1}{\alpha}\int_{\lambda_0(0)}^{\lambda_0} \lambda \frac{d\xi}{d\lambda_0} d\lambda_0 \tag{3.4.10}$$

$$= \frac{1}{\alpha}[\frac{2}{3}(\lambda_0^3 - \lambda_0^3(0)) + \frac{Q^2K^2}{18}(\frac{1}{\lambda_0^3(0)} - \frac{1}{\lambda_0^3}) + \frac{QK}{3} \ln(\frac{8\lambda_0^3 + QK}{8\lambda_0^3(0) + QK})]$$

where from eqn. (3.4.3): $\lambda_0(0) = \sqrt{1 + QA^2(0)}$.

When $\alpha \to 0$, $\tau \to [\lambda_0(0) + \dfrac{Q^2K^2/6}{\lambda_0^2(0)\ (8\lambda_0^3(0) + QK)}]t$

For the initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$, the constants A (0) and θ are given by:

$$A(0) \approx 1 - \frac{Q}{3(8 + 9Q)} \quad \text{and} \quad \theta \approx \frac{-\alpha}{\sqrt{1 + Q}\ [4 + 6Q]}$$

Figs. 3.11, 3.12, and 3.13 show the solutions of eqn. (3.4.2) for different values of γ and α.

Example 3.4.2

$$\ddot{x} + x + \gamma x^3 + \frac{\alpha}{1+\xi}\dot{x} = 0 \qquad (3.4.11)$$

$$0 \leq \alpha << 1 \quad \text{and} \quad 0 \leq \gamma < \infty$$

The solution is determined from:

$$\lambda_0^2 = 1 + QA^2 \quad , \quad Q \triangleq \frac{3}{4}\gamma \qquad (3.4.12)$$

$$\epsilon A_3 = \frac{Q}{3}\frac{A^3}{9\lambda_0^2 - 1} \quad , \quad D_3 = C_5 = D_5 = 0 \qquad (3.4.13)$$

$$\lambda_1 = \frac{QA\ C_3}{2\lambda_0} \qquad (3.4.14)$$

and $$2\lambda_0 \frac{dA}{d\xi} + A \frac{d\lambda_0}{d\xi} + \frac{A\lambda_0}{1+\xi} = 0 \tag{3.4.15}$$

Eqn. (3.4.15) integrates into:

$$\lambda_0 A^2 = \frac{K}{1+\xi} \tag{3.4.16}$$

K being a constant given by:

$$K = \lambda_0(0)\, A^2(0) = A^2(0)\sqrt{1 + QA^2(0)}$$

Define

$$\eta = \frac{4}{27K^2}(1+\xi)^2$$

The admissible solution of eqns. (3.4.12) and (3.4.16) is given by:

$$A^2 = \frac{2}{3\eta^{1/3}} \frac{1}{(1+\sqrt{1-\eta})^{1/3} + (1-\sqrt{1-\eta})^{1/3}} \tag{3.4.17}$$

for $\eta<1$ i.e. for $0\le t<\frac{1}{\alpha}(\frac{3\sqrt{3}}{2}K - 1)$

$$A^2 = \frac{\sqrt{3}}{2}\frac{K}{1+\xi}\sec(\beta/3) \quad \text{where } \cos\beta = \frac{1}{\sqrt{\eta}}\,;\ 0\le\beta<\frac{\pi}{2}$$

for $\eta\ge1$ i.e. for $\frac{1}{\alpha}(\frac{3\sqrt{3}}{2}K - 1)\le t<\infty$ (3.4.18)

From eqns. (3.4.12), (3.4.13) and (3.4.14)

$$\lambda \approx \lambda_0 + \epsilon\lambda_1 = \lambda_0 + \frac{(\lambda_0^2 - 1)^2}{6\lambda_0(9\lambda_0^2 - 1}$$

From eqns. (3.4.12) and (3.4.16):

$$\frac{d\xi}{d\lambda_0} = -\frac{QK[3\lambda_0^2 - 1]}{\lambda_0^2(\lambda_0^2 - 1)^2}.$$

Hence

$$\tau = \frac{1}{\alpha}\int_{\lambda_0(0)}^{\lambda_0} \lambda \frac{d\xi}{d\lambda_0} d\lambda_0$$

$$= \frac{QK}{\alpha}[\frac{1}{2} \ell_n(\frac{9\lambda_0^2 - 1}{9\lambda_0^2(0) - 1} \cdot \frac{\lambda_0^2(0) - 1}{\lambda_0^2 - 1}) + (\frac{1}{\lambda_0^2-1} - \frac{1}{\lambda_0^2(0)-1})$$

$$+ \frac{1}{12}(\frac{1}{\lambda_0^2} - \frac{1}{\lambda_0^2(0)})] \qquad (3.4.19)$$

Where from eqn. (3.4.12):

$$\lambda_0^2(0) = 1 + QA^2(0)$$

when

$$\alpha \to 0 , \tau \to [\lambda_0(0) + \frac{(\lambda_0^2(0) - 1)^2}{6\lambda_0(0)\ (9\lambda_0^2(0) - 1)}]t$$

Given the initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$, then $A(0) \approx 1 - \frac{Q}{3(8+9Q)}$ and $\theta \approx -\frac{\alpha\sqrt{1 + Q}}{2 + 3Q}$.

Figs. 3.14, 3.15, and 3.16 show the solutions of eqn. (3.4.11) for different values of γ and α.

It should be noted that in some cases the assumption of relatively slow variation of the amplitude and frequency of oscillation may be only locally valid. For example, consider the case when eqn. (3.4.1) takes the form:

$$\ddot{x} + \gamma x^3 = \alpha f_1(\xi)\dot{x} \qquad (3.4.20)$$

with $f_1(\xi) \geq 0 \ \forall\xi$, $0 \leq \alpha << \infty$ and $\gamma >> \|\alpha f_1(\xi)\| > 0$ and with the initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$. The solution of the unperturbed equation: $\ddot{x} + \gamma x^3 = 0$ is periodic and has the exact frequency:

$\Omega = \frac{\pi}{2} \frac{x_0\sqrt{\gamma}}{K(\frac{1}{\sqrt{2}})}$ where K is the complete elliptic integral of the first kind.

Now considering the perturbed equation (3.4.20), if the response is oscillatory, the maximum value of its frequency will be equal to Ω. If x_0 is small such that Ω is of the same order as $\|\alpha f_1(\xi)\|$, then relatively fast decaying oscillations or an over-damped response may result. If x_0 is $O(1)$, the same situation may be reached if the amplitude of oscillation decays to a sufficiently small value with the passage of time. This is the case in example (3.4.3) below:

<u>Example 3.4.3</u>

Consider the system:

$$\ddot{x} + \gamma x^3 + \alpha(1 + \cos\xi)\,\dot{x} = 0 \tag{3.4.21}$$

where $0 \leq \alpha << 1$, $O(1) \leq \gamma < \infty$ and with the initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$.

In the region where the assumption of relatively slow variation of the amplitude and frequency of oscillations is valid, the solution is determined from:

$$\lambda_0 = \sqrt{Q}\,A \quad , \quad Q = \frac{3}{4}\gamma \tag{3.4.22}$$

$$\varepsilon C_3 = \frac{Q}{27}\,\frac{A^3}{\lambda_0^2} = \frac{A}{27} \quad , \quad D_3 = C_5 = D_5 = 0 \tag{3.4.23}$$

$$\varepsilon\,\lambda_1 = \frac{Q^2A^4}{54\lambda_0^3} = \frac{\lambda_0}{54} \tag{3.4.24}$$

and

$$2\lambda_0\,\frac{dA}{d\xi} + A\,\frac{d\lambda_0}{d\xi} + A\lambda_0(1 + \cos\xi) = 0 \tag{3.4.25}$$

Eqn. (3.4.25) integrates into:

$$A^2\lambda_0 = K\,e^{-(\xi+\sin\xi)} \tag{3.4.26}$$

K being a constant given by:

$$K = \lambda_0(0)A^2(0) = \sqrt{Q}A^3(0)$$

From eqns. (3.4.22) and (3.4.26):

$$A = A(0)\ e^{-1/3(\xi+\sin\ \xi)} \tag{3.4.27}$$

and

$$\lambda_0 = \sqrt{Q}\ A(0)\ e^{-1/3(\xi+\sin\ \xi)}$$

From eqn. (3.4.24):

$$\lambda \approx \lambda_0 + \varepsilon\lambda_1 = \frac{55}{54}\lambda_0$$

Using the series expansion:

$$e^{\beta\sin\xi} = (1+\frac{\beta^2}{4}) + \beta(1+\frac{\beta^2}{8})\ \sin\ \xi - \frac{\beta^2}{4}\cos\ 2\xi - \frac{\beta^3}{24}\sin\ 3\xi + \ldots$$

where $\beta = -\frac{1}{3}$, then:

$$\tau = \int_0^t \lambda dt = \frac{1}{\alpha}\int_0^\xi \lambda d\xi$$

$$= \frac{55}{54}\sqrt{Q}\ A(0)\ [e^{-\xi/3}\{-\frac{37}{12} + \frac{73}{240}(\cos\ \xi + \frac{1}{3}\sin\ \xi)$$

$$- \frac{1}{74}(\sin\ 2\xi - \frac{1}{6}\cos\ 2\xi) - \frac{1}{1968}(\cos\ 3\xi + \frac{1}{9}\sin\ 3\xi)\}$$

$$+ 2.7774]$$

(3.4.28)

From the given initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$ we have:

$$A(0) \approx \frac{27}{28} \quad \text{and} \quad \theta \approx -\frac{2}{3}\,\frac{\alpha}{\sqrt{Q}}$$

Solutions of eqn. (3.4.21) are shown in Figs. 3.17 , 3.18, and 3.19 for different values of γ and α.

Example 3.4.4 (Discontinuous non-linearity)

$$\ddot{x} + \gamma \,\mathrm{sgn}\, x + \alpha(1 - \cos \xi)\,\dot{x} = 0 \tag{3.4.29}$$

where $0 \leq \alpha \ll 1$, $O(1) \leq \gamma < \infty$, $x(0) = 1$ and $\dot{x}(0) = 0$

In this case the unperturbed eqn.: $\ddot{x} + \gamma \,\mathrm{sgn}\, x = 0$ has the exact periodic solution given by expression (3.2.6). It is obvious that this solution cannot practically be used as a generating solution for the perturbed eqn. (3.4.29). The solution will be sought in the form of expression (3.2.2) (with odd values of n).

From the Fourier series of a square wave of unit amplitude:

$$\mathrm{sgn}\, x(\tau) = \frac{4}{\pi}\left[\cos(\tau+\theta) - \frac{1}{3}\cos 3(\tau+\theta) + \frac{1}{5}\cos 5(\tau+\theta) + \ldots\right]$$

and considering the first and third harmonics only, the solution is then determined from:

$$\lambda_0^2 = \frac{Q}{A}\ , \quad Q \triangleq \frac{4\gamma}{\pi} \tag{3.4.30}$$

$$\varepsilon C_3 = -\frac{Q}{27\lambda_0^2} = -\frac{A}{27}, C_5 \approx 0 \ , \ D_3 = D_5' = 0 \qquad (3.4.31)$$

$$\lambda_1 = 0 \qquad (3.4.32)$$

and

$$2\lambda_0 \frac{dA}{d\xi} + A\frac{d\lambda_0}{d\xi} + A\lambda_0 (1 - \cos\xi) = 0 \qquad (3.4.33)$$

From eqns. (3.4.30) and (3.4.33):

$$A = A(0)\ e^{-2/3(\xi - \sin\xi)} \qquad (3.4.34)$$

and

$$\lambda_0 = \sqrt{\frac{Q}{A(0)}}\ e^{1/3(\xi - \sin\xi)} \qquad (3.4.35)$$

Therefore:

$$\tau = \frac{1}{\alpha}\int_0^{\xi} \lambda\, d\xi \approx \frac{1}{\alpha}\sqrt{\frac{Q}{A(0)}}\ [e^{\xi/3}\{\frac{37}{12} + \frac{73}{240}(\cos\xi - \frac{1}{3}\sin\xi) \qquad (3.4.36)$$

$$- \frac{1}{74}(\sin 2\xi + \frac{1}{6}\cos 2\xi) - \frac{1}{1968}(\cos 3\xi - \frac{1}{9}\sin 3\xi)\} - 3.3847]$$

The constants A(0) and θ are given by:

$$A(0) \approx \frac{27}{26} \text{ and } \theta \approx 0$$

Solutions of eqn. (3.4.29) for two different values of γ and α are shown in Figs. 3.20 and 3.21

Comparison with the results of numerical simulation:

Equations (3.4.2), (3.4.11), (3.4.21) and (3.4.29) were solved numerically for specific values of α and γ and with the initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$. The results are compared with the analytical approximations. The r.m.s. value of the error is calculated in each case over a period of 10 normalized time units. [The error is defined as the difference between the analytical solution and the "exact" numerical solution].

Fig. (3.11) shows the solution of:

$$\frac{d^2x}{dT^2} + 5x + 10x^3 + 1.118Tx = 0 \qquad (3.4.37)$$

where $T \triangleq \frac{1}{\sqrt{5}} t$

This case is taken from the paper of Barkham and Soudack [1970, p. 112]. When eqn. (3.4.37) is normalized as in eqn. (3.4.2), it takes the form:

$$\ddot{x} + x + 2x^3 + 0.1\, t\, x = 0$$

Actually in non-linear analysis it is always rewarding to fully exploit normalization techniques, not only to avoid unnecessary tediousness, but also to make it convenient to examine the final results and single out the effect of non-linearity.

Other examples of moderately and highly non-linear non-autonomous systems are shown in figures 3.12 to 3.21 . The analytical and numerical solutions and the r.m.s. value of the error (denoted by σ), calculated over the solution period indicated, are shown on each graph. It will be seen that in every case the analytical solution approximates the "exact" solution very closely.

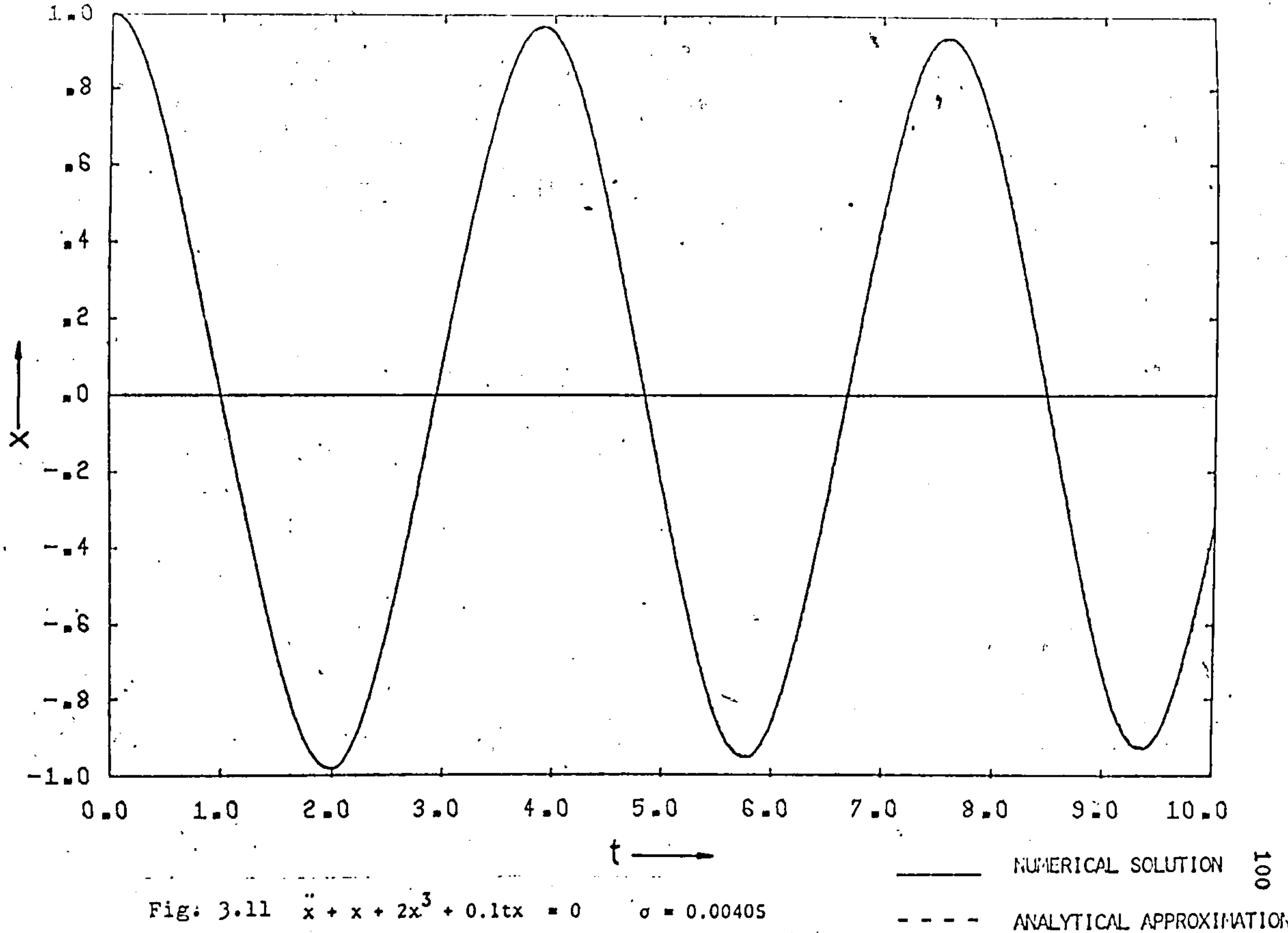

Fig. 3.11 $\ddot{x} + x + 2x^3 + 0.1tx = 0$ $\sigma = 0.00405$

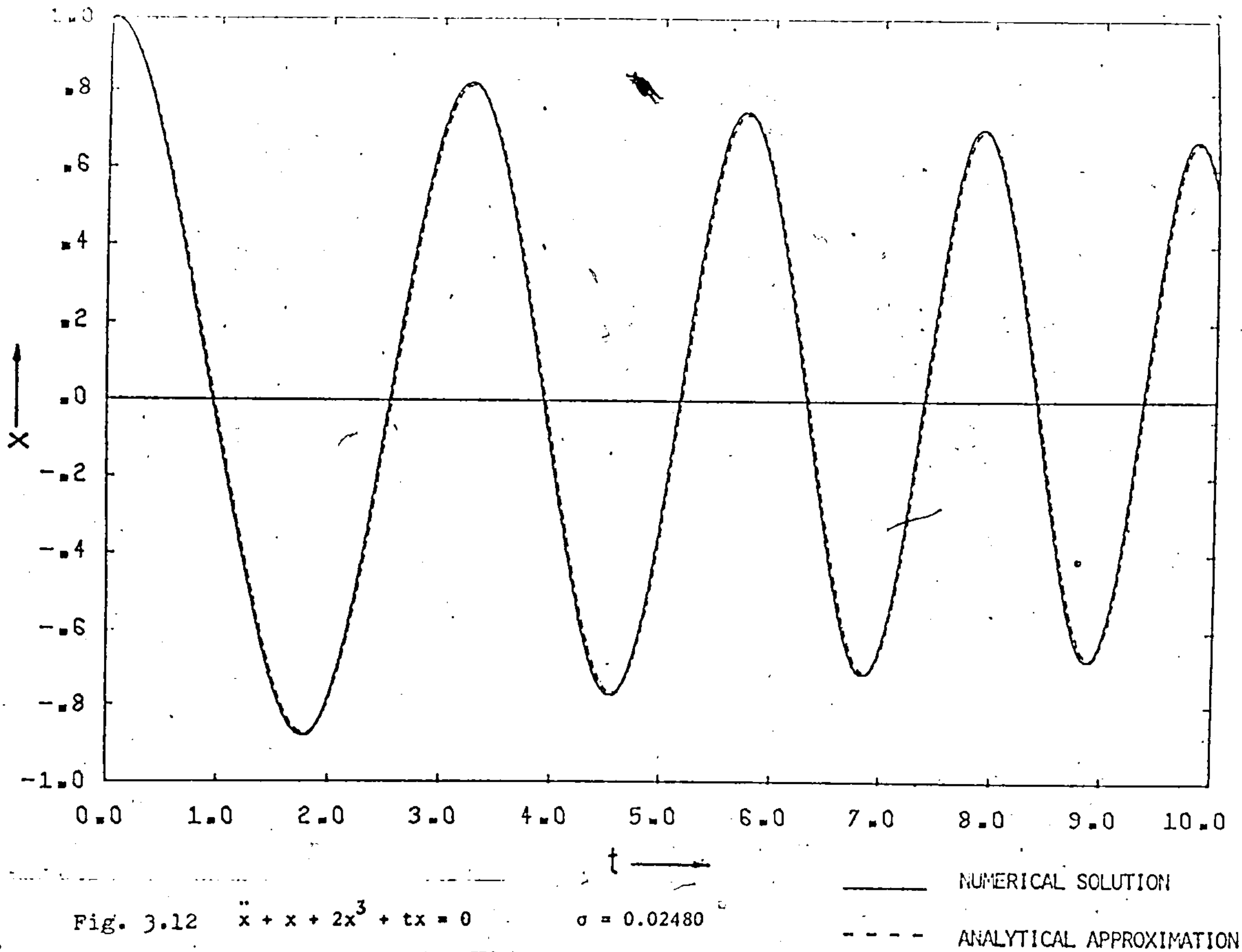

Fig. 3.12 $\ddot{x} + x + 2x^3 + tx = 0$ $\sigma = 0.02480$

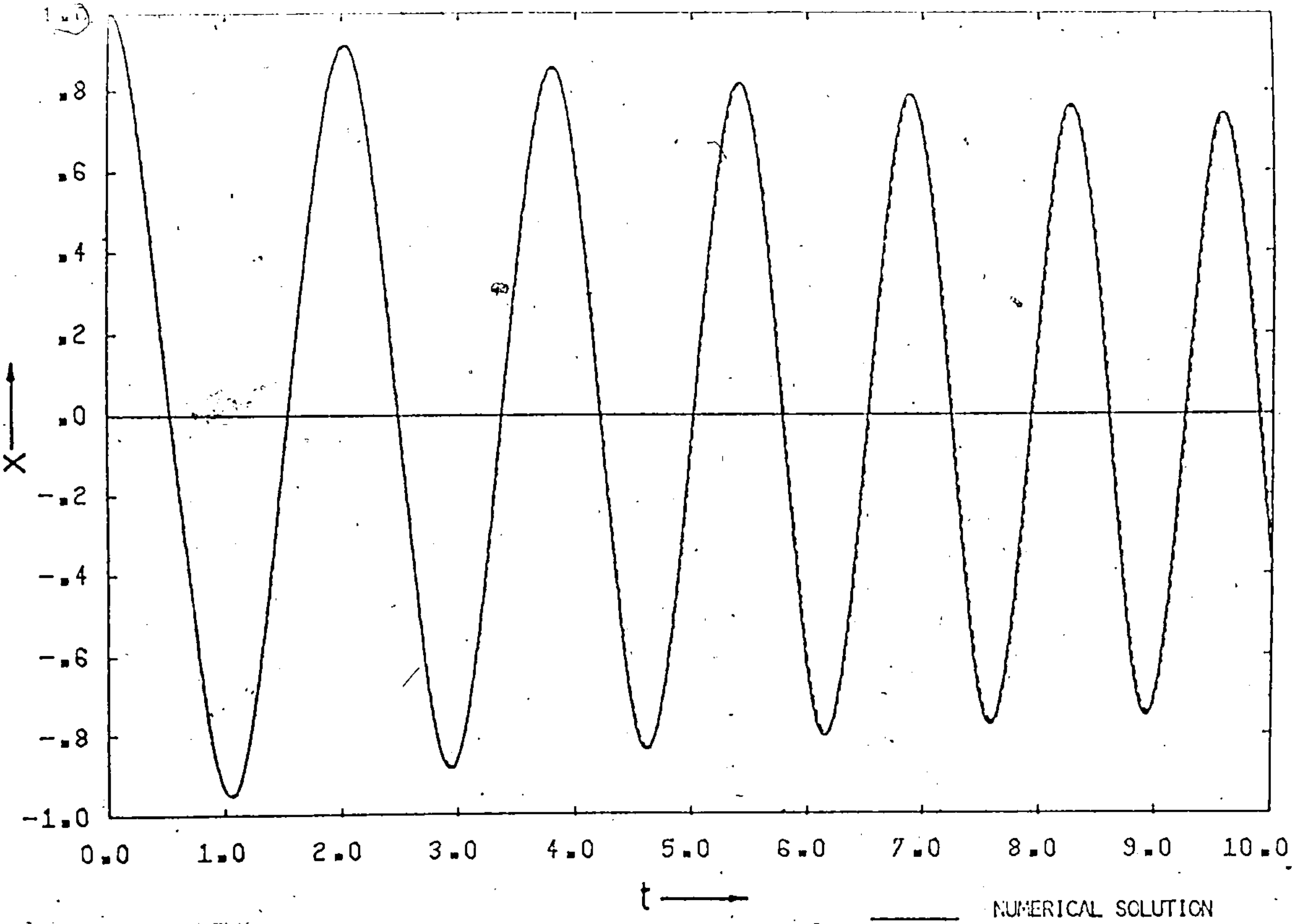

Fig. 3.13 $\ddot{x} + x + 10x^3 + 2tx = 0$ $\sigma = 0.02349$

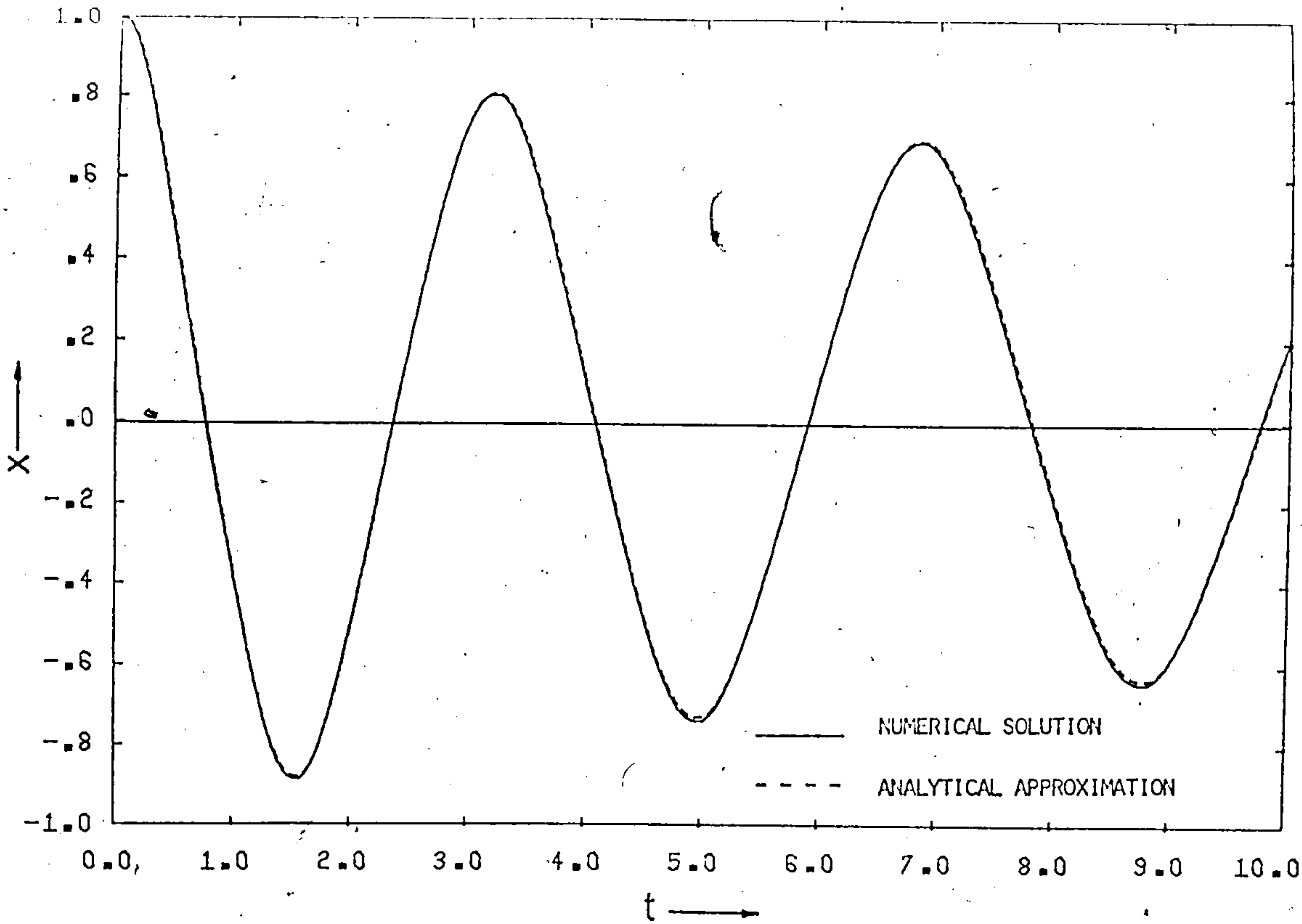

Fig.3.14 $\ddot{x} + x + 5x^3 + \frac{\alpha}{1 + \alpha t}\dot{x} = 0$; $\alpha = 0.25$ $\sigma = 0.00937$

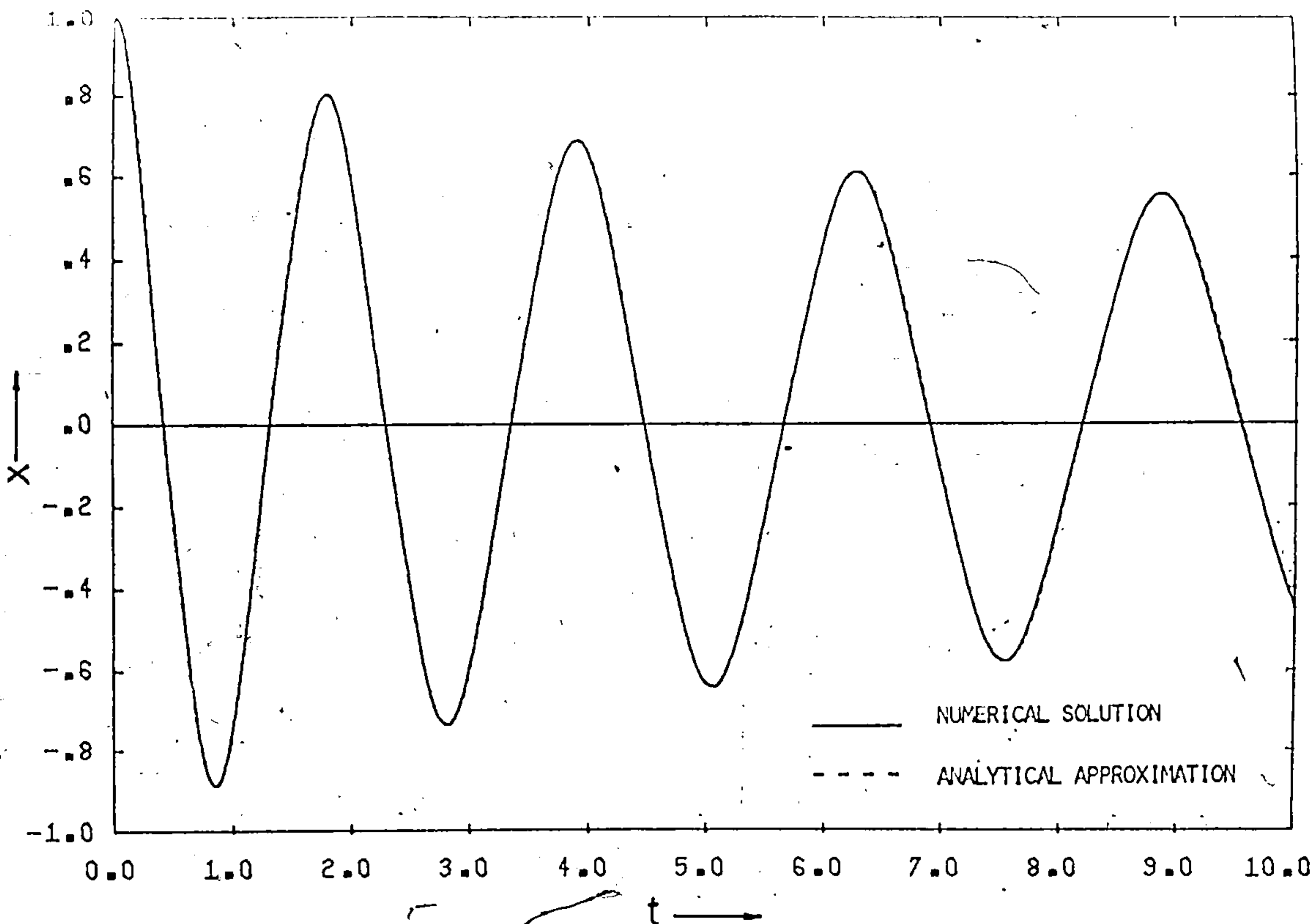

Fig.3.15 $\ddot{x} + x + 20x^3 + \frac{a}{1 + a t}\dot{x} = 0$; $a = .5$ $\sigma = 0.00613$

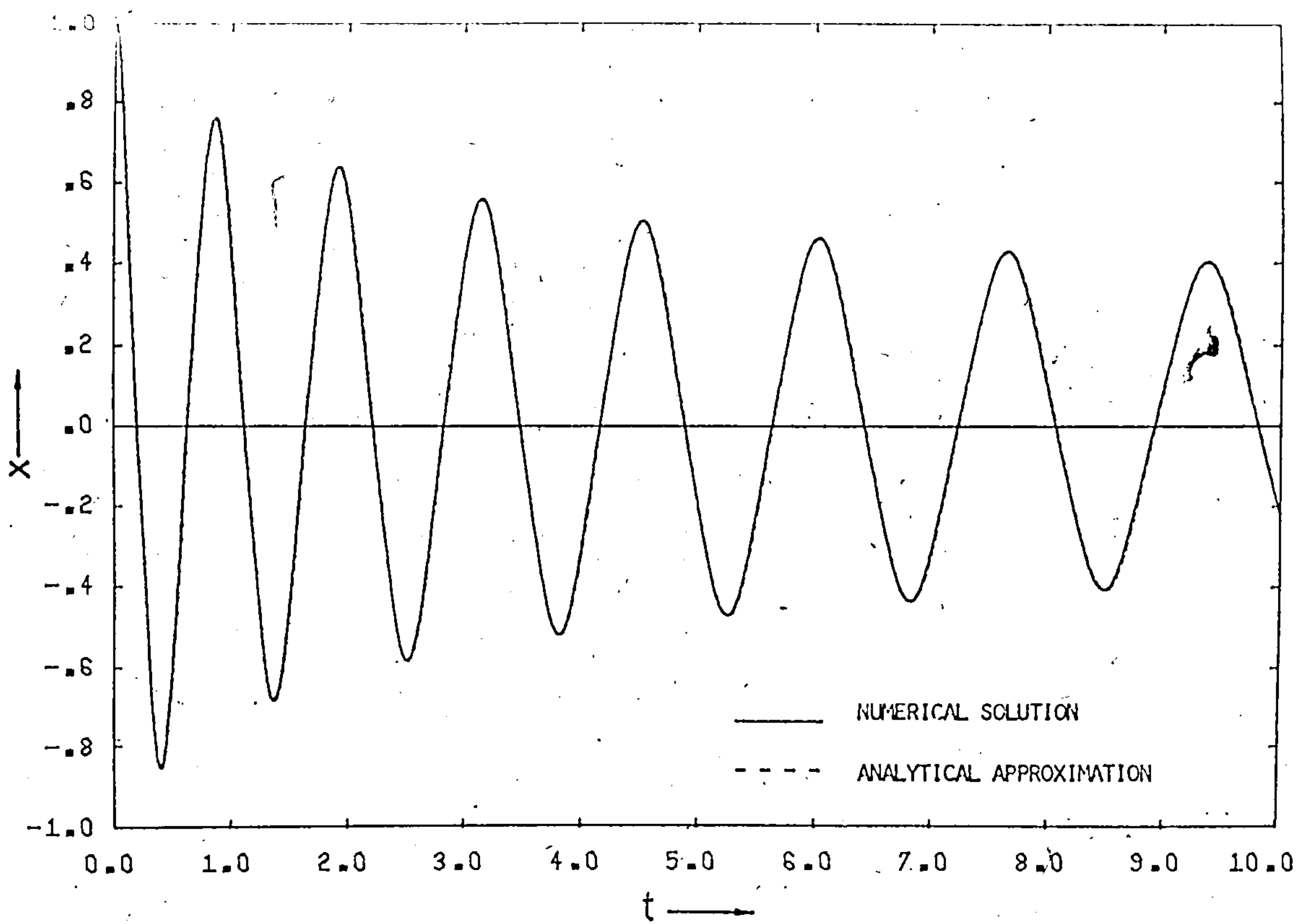

Fig. 3.16 $\ddot{x} + x + 100x^3 + \frac{\alpha}{1 + \alpha t}\dot{x} = 0$; $\alpha = 1.5$ $\sigma = 0.00748$

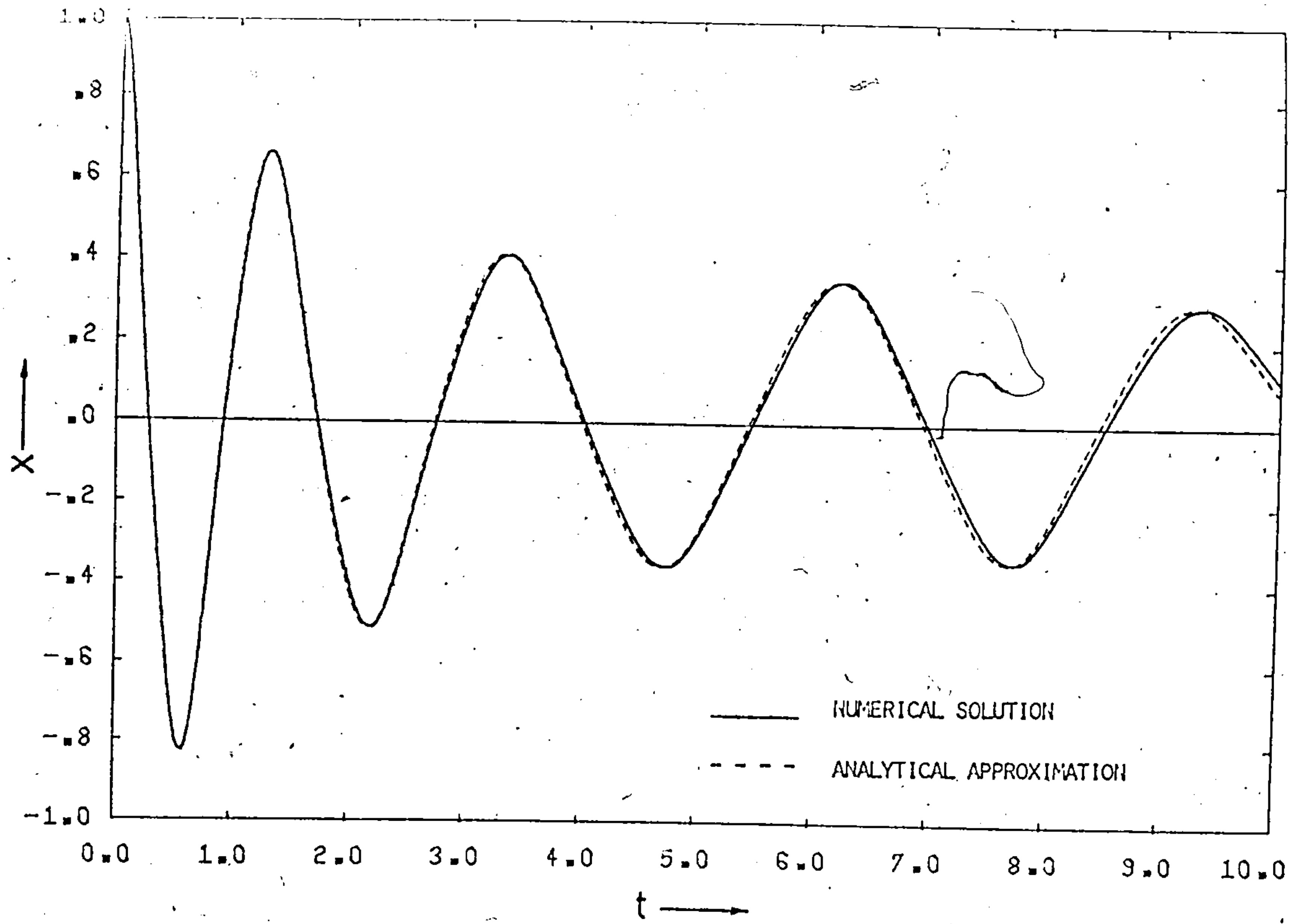

Fig. 3.17 $\ddot{x} + 50x^3 + \alpha(1 + \cos \alpha t)\,\dot{x} = 0$; $\alpha = 0.5$, $\sigma = 0.01683$

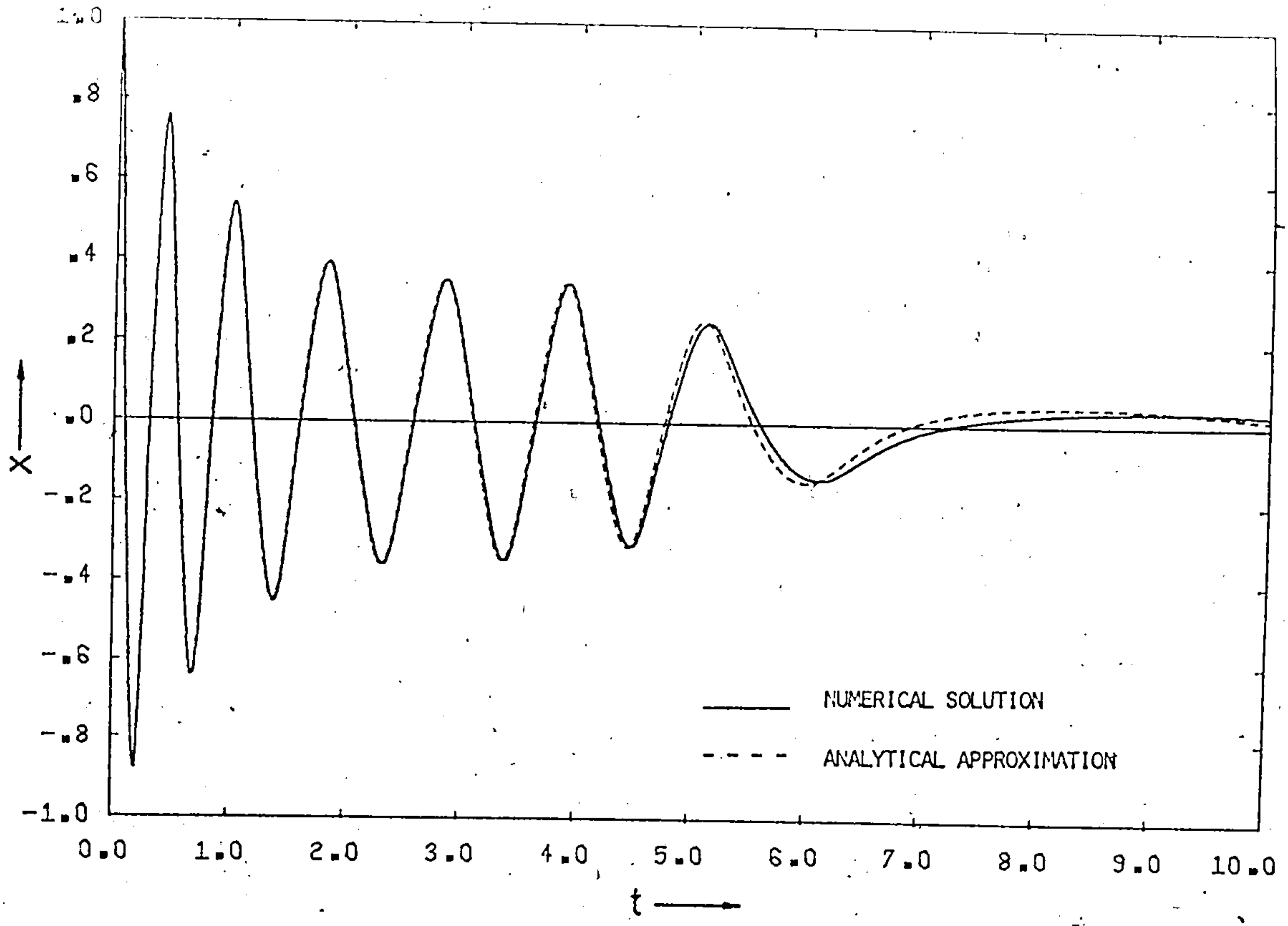

Fig. 3.18 $\ddot{x} + 400x^3 + (1 + \cos t)\,\dot{x} = 0$ $\sigma = 0.02212$

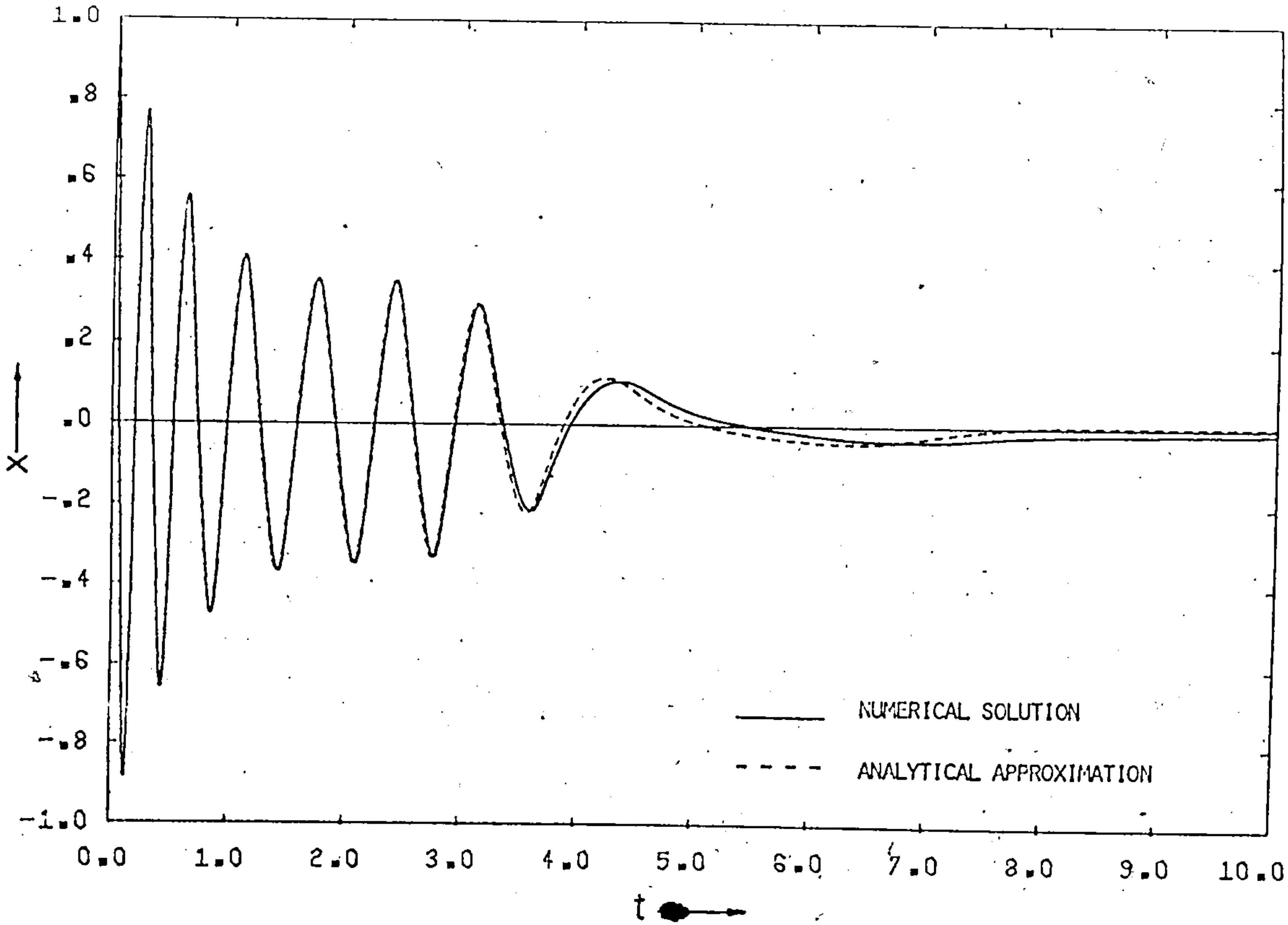

Fig. 3.19 $\ddot{x} + 1000x^3 + \alpha(1 + \cos \alpha\, t)\,\dot{x} = 0$; $\alpha = 1.5$ $\sigma = 0.02159$

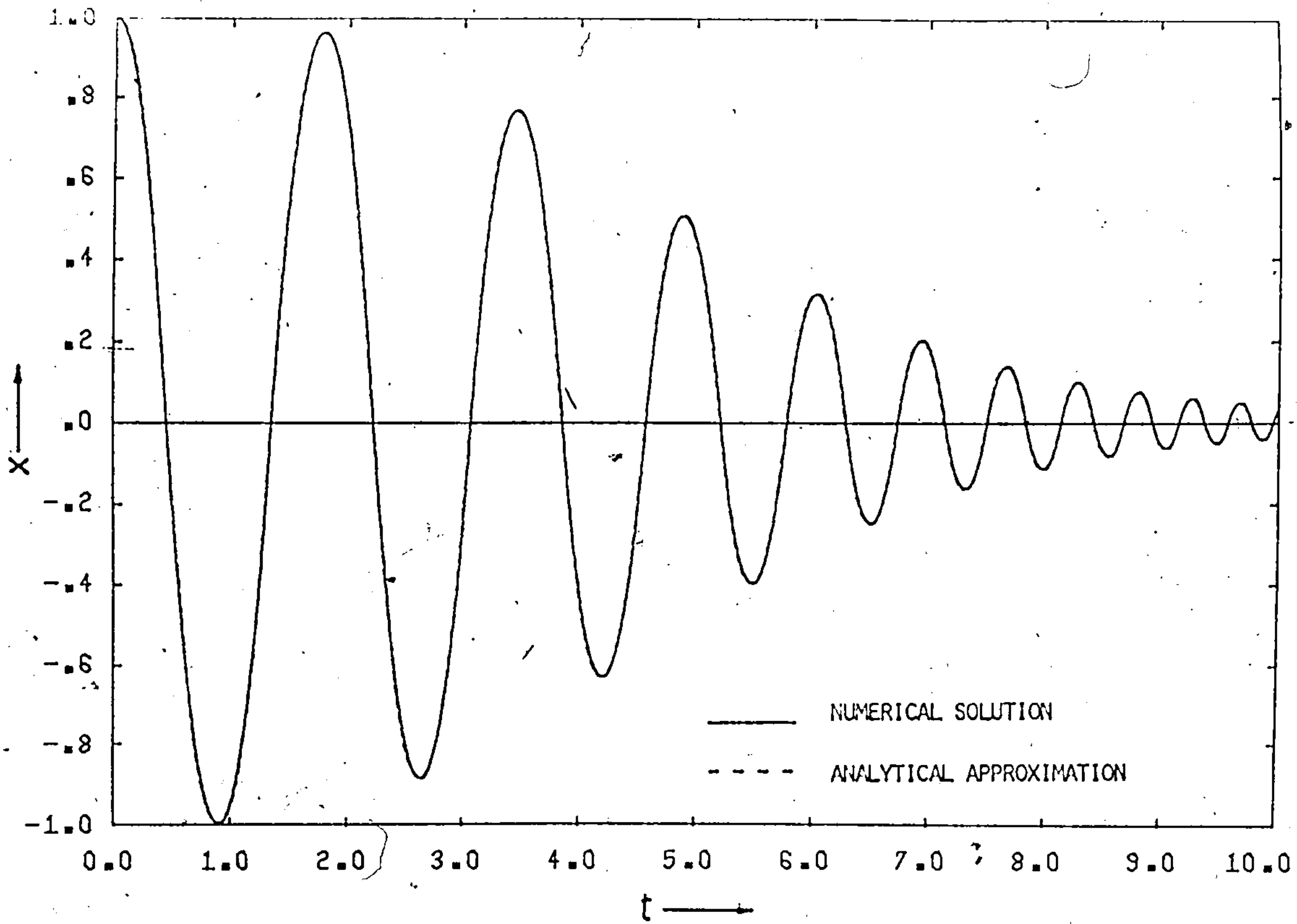

Fig. 3.20 $\ddot{x} + 10\,\mathrm{sign}x + \alpha(1 - \cos$ [illegible] $t) = 0$; $\alpha = 0.4$ $\sigma = 0.00536$

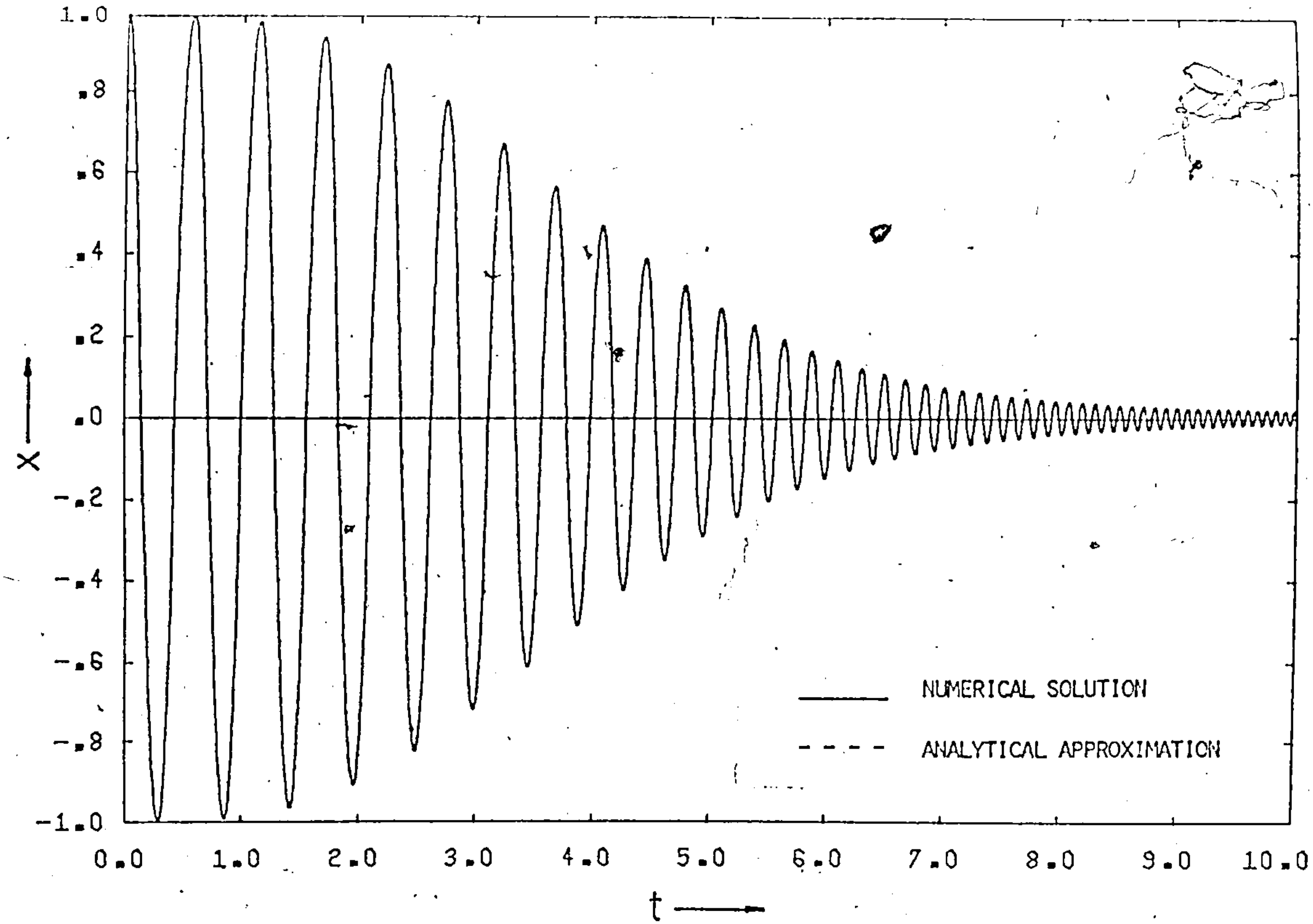

Fig. 3.21 $\ddot{x} + 100\,\mathrm{sign}\,x + \alpha(1 - \cos\alpha t) = 0$; $\alpha = 0.5$ $\sigma = 0.01446$

CHAPTER 4

THE NON-STATIONARY RESPONSE OF A FORCED WEAKLY NON-LINEAR OSCILLATORY SYSTEM

4.1 INTRODUCTION

The stationary response of a forced system described by:

$$\ddot{x} + x + \varepsilon[\dot{x}\, H(x,\dot{x}) + N(x)] = f(t), \quad (\dot{\ }) = \frac{d}{dt} \tag{4.1.1}$$

where $0 < \varepsilon << 1$, $N(x)$ and $H(x,\dot{x})$ are non-linear functions of their arguments, and $f(t)$ is a periodic function of time has been the subject of extensive study. When $f(t)$ is periodic, the response is composed of the natural frequency of the system and its harmonics, the impressed frequency and its harmonics and their cross-modulation products. When $H(x,\dot{x})$ is positive $\forall x$ and $\dot{x}$, the component of the natural frequency fades away as time increases and the response is then merely composed of the impressed frequency and its harmonics. In a quasi-linear system, the amplitudes of the higher harmonics are relatively small and the analysis of the system in the steady state becomes a fairly easy task. In many cases, it is of interest to study the behaviour of the forced system in the transient state.

The transient response of the above system with $H(x,\dot{x}) \equiv 0$ has been studied by Bauer (1966,68,71). In the method presented by Bauer a linearized equation (with ε set equal to zero) is subtracted from the original equation and the resulting equation is solved using Lighthill's perturbation method. The method of Bauer has certain limitations which will be discussed below.

A remark is noteworthy. In his paper, Lighthill [1949] presented a general perturbation technique, which he applied to various non-linear ordinary and partial differential equations that occur in problems of physics. When he used it to study the equation:

$$\ddot{x} + x + \varepsilon f(x,\dot{x}) = 0$$

Lighthill cautioned that the question whether his technique can be used to determine motions other than the limit cycle remains unelucidated.

Before we proceed with the present analysis, the method of Bauer will be examined.

The Method of Bauer:

Consider the damping-free system:

$$\ddot{x} + x + \varepsilon R(x) = f(t) \qquad (4.1.2)$$

with

$$x(0) = \dot{x}(0) = 0$$

writing

$$x \triangleq y + g$$

where g satisfies the linearized equation:

$$\ddot{g} + g = f(t) \tag{4.1.3}$$

one obtains:

$$\ddot{y} + y + \varepsilon R(y+g^*) = 0 \tag{4.1.4}$$

with

$$y(0) = -g^*(0), \quad \text{and } \dot{y}(0) = -\dot{g}^*(0) \tag{4.1.5}$$

$g^*(t)$ being the particular solution of eqn. (4.1.3).

Now y and t are written in the form of the series expansions:

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots \tag{4.1.6}$$

and

$$t = \zeta + \varepsilon T_1 + \varepsilon^2 T_2 + \ldots \tag{4.1.7}$$

where

$$y_n \triangleq y_n(\zeta) \qquad n = 0,1,2,\ldots$$

$$T_n \triangleq T_n(\zeta), \quad T_n(0) = 0, \quad n = 1,2,\ldots$$

$g^*(t)$ is then expanded in a Taylor series:

$$g^*(t) = g^*(\zeta + \varepsilon T_1 + \varepsilon^2 T_2 + \ldots)$$

$$= g^*(\zeta) + \varepsilon T_1 g^{*\prime}(\zeta) + \varepsilon^2 (T_2 + \tfrac{1}{2} T_1^2)\, g^{*\prime\prime}(\zeta) + \ldots \tag{4.1.8}$$

where (˙) denotes differentiation with respect to ζ. For brevity, from this point onwards g^* will denote $g^*(\zeta)$.

Now,

$$\epsilon R(y+g^*) = \epsilon R(y_0+\epsilon y_1+g^*+\epsilon T_1 g^{*\prime}+\ldots)$$

$$= \epsilon R(y_0+g^*) + \epsilon^2(y_1+T_1 g^{*\prime}) \frac{dR}{d(y_0+g^*)} + \ldots \quad (4.1.9)$$

From expressions (4.1.6) and (4.1.7) we have:

$$\frac{dy}{dt} = [1 - \epsilon T_1' + \epsilon^2(T_1'^2 - T_2')]y'$$

and

$$\frac{d^2y}{dt^2} = y'' - \epsilon[2T_1'y'' + T_1''y']$$

$$+ \epsilon^2[3T_1'^2 - 2T_2')y'' + (3T_1'T_1''-T_2'')y'] + \ldots \quad (4.1.10)$$

Using the expansion (4.1.6) in (4.1.10) and substituting expressions (4.1.6), (4.1.9) and (4.1.10) in eqn. (4.1.4) then separating the terms of similar order of smallness, one obtains the following system of equations:

$$y_0'' + y_0 = 0 \quad (4.1.11)$$

$$y_1'' + y_1 = 2T_1'y_0'' + y_0'T_1'' - R(y_0+g^*) \quad (4.1.12)$$

.
.
.

If, for example, $f(t) = F_0 \sin \nu t$ and $R(x) = x^3$, then:

$$g^* = K \sin \nu\zeta \quad ; \quad K = \frac{F_0}{1-\nu^2}$$

The solution of (4.1.11) subject to the initial conditions in (4.1.5) is therefore:

$$y_0 = -\nu K \sin \zeta$$

The right hand side of equation (4.1.12) then takes the form:

$$[2\nu K T_1' - \frac{3}{4} \nu K^3(2 + \nu^2)] \sin \zeta - \nu K T_1'' \cos \zeta + \sum_{i} \ell_i \sin \omega_i \zeta$$

where $\omega_i \neq 1 \; \forall \; i$.

The conditions for boundedness of y_1 is therefore:

$$T_1' = \frac{3}{8} K^2(2 + \nu^2) \triangleq \Omega_1$$

and

$$T_1'' = 0$$

These two equations in one unknown are fortunately satisfied simultaneously by the function $T_1 = \Omega_1 \zeta$.

Now we turn to the expansion of $g^*(t)$ of eqn. (4.1.8). In order that the term $\|T_1 g^{*'}\|$ be one order smaller than $\|g^*\|$ and the second term in expression (4.1.9) can thereby be associated with terms of $O(\varepsilon^2)$, the product $\hat{T}_1 g^{*'}$ must be uniformly equal to $O(1)$. However, T_1 increases linearly with the time ζ, and the perturbational procedure will therefore be uniformly valid if, and only if, g^* decays at a rate faster than or equal to $\frac{1}{\zeta}$.

In the above example:

$$T_1 g^{*'} = \Omega_1 \, \nu K \, \zeta \cos \nu\zeta$$

This secular term will appear in the subsequent perturbational equation and it cannot be eliminated. The range of validity of the approximate solution if therefore limited to very short intervals.

However, if f(t) is chosen to be the step function:

$$f(t) = F_0 \, U(t)$$

where F_0 is a constant and U(t) is the unit step function, then

$$g^*(t) = g^*(\zeta) = F_0 \qquad t > 0$$

the term $T_1 G^{*'}$ will vanish and the method will be uniformly valid $\forall \zeta$.

Difficulties will also arise if damping is to be considered. For example, if one seeks a solution of the equation:

$$\ddot{y} + y + \varepsilon \dot{y} = 0$$

with

$$y(0) = 1 \quad \text{and} \quad \dot{y}(0) = 0,$$

using the above method; the following perturbational equations result:

$$y_0'' + y_0 = 0$$

$$y_1'' + y_1 = (T_1'' - 1)\, y_0' + 2T_1' y_0''$$

Then $$y_0 = \cos \zeta$$

and the conditions for boundedness of y_1 become:

$$T_1'' = 1 \quad \text{and} \quad T_1' = 0$$

which is an impossible situation.

4.2 Present analysis - A system with a non-linear reactance

Consider for example the forced system:

$$\ddot{x} + x + \varepsilon[\dot{x} + \beta x^3] = f(t) \quad , \quad (\dot{\cdot}) \triangleq \frac{d}{dt} \qquad (4.2.1)$$

with $0 \le \varepsilon << 1$, β and $\|f(t)\|$ are $O(1)$.

Let

$$x \triangleq \eta + u \qquad (4.2.2)$$

where η represents the component of the system's frequency and its harmonics and u represents all other components having frequencies different from the system's frequency and its harmonics.

Let η be expressed in a Fourier series as in (2.2.2). The expressions for $\dot{\eta}$ and $\bar{\eta} + \eta$ will be the same as in (2.2.10) and (2.2.11).

Due to the interaction between η and the "particular response" u, u will also be slowly modulated.

Thus

$$u = u(t,\xi)$$

The first and second derivatives of u are given by:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \varepsilon \frac{\partial u}{\partial \xi}$$

and

$$\frac{d^2u}{dt^2} = \frac{\partial^2 u}{\partial t^2} + 2\varepsilon \frac{\partial^2 u}{\partial t \partial \xi} + \varepsilon^2 \frac{\partial^2 u}{\partial \xi^2} \quad , \quad \xi \triangleq \epsilon t \tag{4.2.3}$$

Using the expansion

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots \tag{4.2.4}$$

then

$$\frac{d^2u}{dt^2} + u = \left(\frac{\partial^2 u_0}{\partial t^2} + u_0\right) + \varepsilon\left[\frac{\partial^2 u_1}{\partial t^2} + u_1 + 2\frac{\partial^2 u_0}{\partial t \partial \xi}\right]$$
$$+ \varepsilon^2\left[\frac{\partial^2 u_2}{\partial t^2} + u_2 + 2\frac{\partial^2 u_1}{\partial t \partial \xi} + \frac{\partial^2 u_0}{\partial t^2}\right] \tag{4.2.5}$$

and

$$\varepsilon \frac{du}{dt} = \varepsilon \frac{\partial u_0}{\partial t} + \varepsilon^2 [\frac{\partial u_1}{\partial t} + \frac{\partial u_0}{\partial \xi}] \quad (4.2.6)$$

Using expression (2.2.2) and (4.2.4) then:

$$\varepsilon \beta x^3 = \varepsilon\beta[\eta + u]^3$$
$$= \varepsilon\beta[A_0 \cos \tau + u_0]^3 + 3\beta\varepsilon^2[A_0 \cos \tau + u_0]^2[A_1 \cos \tau + C_{31} \cos 3\tau + D_{31} \sin 3\tau + u_1] + \ldots$$

which will take the form:

$$\varepsilon\beta x^3 = \varepsilon[P_{1c1}(\xi) \cos \tau + P_{1s1}(\xi) \sin \tau + P_{3c1}(\xi) \cos 3\tau + P_{3s1}(\xi) \sin 3\tau + \ldots] + \varepsilon Q_1(t,\xi) + \varepsilon^2[P_{1c2}(\xi) \cos \tau + P_{1s2}(\xi) \sin \tau + P_{3c2}(\xi) \cos 3\tau + P_{3s2}(\xi) \sin \tau] + \varepsilon^2 Q_2(t,\xi) + O(\varepsilon^3) \quad (4.2.7)$$

where the frequency content of the functions $Q_i|_{i=1,2,3\ldots}$ is different from $k\lambda$, $k=1,2,3\ldots$, where λ is the instantaneous frequency defined in (2.2.3).

Now expressions (2.2.10), (2.2.11) and (2.2.4) to (2.2.7) are substituted in eqn. (4.2.1) with the use of eqn. (4.2.2). Equating the coefficients of similar powers of ε separately to zero then separating the terms representing the component of the systems frequency, the following system of equations is obtained which can be solved sequentially to yield the approximate solution:

$$\frac{\partial^2 u_0}{\partial t^2} + u_0 = f(t) \tag{4.2.8}$$

$$2\frac{dA_0}{d\xi} + A_0 - P_{1s1} = 0 \tag{4.2.9}$$

$$2\lambda_1 A_0 - P_{1c1} = 0 \tag{4.2.10}$$

$$8C_{31} = P_{3c1}, \quad 8D_{31} = P_{1s1} \tag{4.2.11}$$

$$\frac{\partial^2 u_1}{\partial t^2} + u_1 = -\frac{\partial u_0}{\partial t} - 2\frac{\partial^2 u_0}{\partial t \partial \xi} - Q_1 \tag{4.2.12}$$

$$2\frac{dA_1}{d\xi} + A_1 = \lambda_1 P_{1s1} - P_{1s2} \tag{4.2.13}$$

$$2\lambda_2 A_0 = \frac{d^2 A_0}{d\xi^2} + \frac{dA_0}{d\xi} - 2\lambda_1 A_1 - A_0 \lambda_1^2 + P_{1c2} \tag{4.2.14}$$

If for example

$$f(t) = F(\varepsilon t) \cos(\nu t + \phi(\varepsilon t))$$

$$\equiv F(\xi) \cos(\nu t + \phi(\xi)), \ \nu \text{ being a constant,}$$

then the approximate solution of eqn.(4.2.8) is :

$$u_0 = \sigma F \cos(\nu t + \phi) \tag{4.2.15}$$

where

$$\sigma \triangleq \frac{1}{1-\nu^2}$$

From eqns. (4.2.7) and (4.2.15):

$$P_{1c1} = \beta[\frac{3}{4} A_0^3 + \frac{3}{2} \sigma^2 F^2 A_0], \quad P_{1s1} = 0$$

$$P_{3c1} = \frac{\beta}{4} A_0^3 , \qquad P_{3s1} = 0$$

and

$$Q_1 = \beta[\frac{3}{2}\sigma FA_0^2 + \frac{3}{4}\sigma^3 F^3) \cos (\nu t + \phi) + \frac{\sigma^3 F^3}{4} \cos 3(\nu t + \phi)$$

$$+ \frac{3}{4} \sigma FA_0^2 \{\cos (2\tau-\nu t-\phi) + \cos (2\tau+\nu t+\phi)\}$$

$$+ \frac{3}{4} \sigma^2 F^2 A_0 \{\cos (2\nu t+2\phi-\tau) + \cos (2\nu t+2\phi+\tau)\}$$

Therefore from eqns. (4.2.9), (4.2.10) and (4.2.11):

$$\lambda_1 = \frac{3}{8} \beta[A_0^2 + 2\sigma^2 F^2] \tag{4.2.16}$$

$$A_0 = L e^{-\frac{1}{2}\xi} , \text{ L being a constant}, \tag{4.2.17}$$

$$C_{31} = \frac{\beta A_0^2}{32} \text{ and } D_{31} = 0 \tag{4.2.18}$$

From equation (4.2.15):

$$\frac{\partial^2 u_0}{\partial t \partial \xi} = - \nu\sigma[F \frac{d\phi}{d\xi} \cos (\nu + \phi) + \frac{dF}{d\xi} \sin (\nu t + \phi)]$$

The solution of eqn. (4.2.12) is therefore:

$$u_1 = N(\xi) \sin (\nu t + \phi) + M_1(\xi) \cos (\nu t + \phi) + M_2(\xi) \cos (3\nu+\phi)$$
$$+ M_3(\xi)[p_1 \cos(2\tau-\nu t-\phi) + p_2 \cos(2\tau + \nu t + \phi)]$$
$$+ M_4(\xi) [q_1 \cos(2\nu t + 2\phi - \tau) + q_2 \cos(2\nu t + 2\phi + \tau)] \quad (4.2.19)$$

with

$$N(\xi) = \sigma^2\nu[F + 2 \frac{dF}{d\xi}]$$

$$M_1(\xi) = \sigma^2[2\nu F \frac{d\phi}{d\xi} - \frac{3}{4} \beta\sigma^2 F^3] - \frac{3}{2} \beta\sigma^2 FA_0^2$$

$$M_2(\xi) = - \frac{\beta}{4} \frac{\sigma^3 F^3}{1-9\nu^2}, \quad M_3(\xi) = - \frac{3}{4} \beta\sigma FA_0^2$$

$$M_4(\xi) = - \frac{3}{4} \beta\sigma^2 F^2 A_0$$

$$p_1 = \frac{1}{1-(2-\nu)^2}, \quad p_2 = \frac{1}{1-(2+\nu)^2}, \quad q_1 = \frac{1}{1-(2\nu-1)^2}$$

and

$$q_2 = \frac{1}{1-(2\nu+1)^2}$$

From equations (4.2.7), (4.2.15), (4.2.17) and (4.2.19):

$$P_{ic2} = 3\beta[A_1(\frac{\sigma^2F^2}{2} + \frac{3}{4}A_0^2) + \frac{\beta A_0^5}{128} + \frac{\sigma^2F^2}{4}(q_1+q_2)M_4$$

$$+ \sigma FA_0(M_1 + \frac{p_1+p_2}{2}M_3)]$$

and

$$P_{1s1} = 0.$$

Eqn. (4.2.13) now becomes:

$$\frac{dA_1}{d\xi} + \frac{1}{2}A_1 = -\frac{L}{2}e^{-\frac{1}{2}\xi}\frac{d\lambda_1}{d\xi}$$

which integrates into:

$$A_1 - A_1(0) = -\frac{L}{2}e^{-\frac{1}{2}\xi}\int_{\lambda_1(0)}^{\lambda_1} d\lambda_1 = \frac{L}{2}e^{-\frac{1}{2}\xi}[\lambda_1(0) - \lambda_1]$$

If we let the constants L and $\tau(0)$ satisfy the initial conditions to the desired degree of accuracy , then $A_1(0)$ may be chosen to be equal to zero. Hence:

$$A_1 = \frac{3L\beta}{16}e^{-\frac{1}{2}\xi}[L^2(1-e^{-\xi}) + 2\sigma^2(F^2(0) - F^2)] \qquad (4.2.20)$$

From eqn. (4.2.14)

$$\lambda_2 = A_0(\xi) + A_1(\xi)A_0^2 - \frac{51}{256}\beta^2A_0^4$$

with

$$A_0(\xi) = -\frac{1}{8} + 3\nu\beta\sigma^3 F^2 \frac{d\phi}{d\xi} - \frac{9}{32}\beta^2\sigma^4 F^4(1 + 4\sigma + q_1 + q_2)$$

and

$$A_1(\xi) = \frac{9}{64}\beta^2[L^2 + 2\sigma^2 F^2(0)] - \frac{9}{16}\beta^2\sigma^2 F^2(1 + 4\sigma + p_1 + p_2)$$

Therefore, τ is determined from :

$$\begin{aligned}\tau - \tau(0) &= \int_0^t \lambda dt \quad \int_0^t (1 + \epsilon\lambda_1 + \epsilon^2\lambda_2)dt \\ &= t + \int_0^\xi (\lambda_1 + \epsilon\lambda_2)d\xi\end{aligned} \tag{4.2.21}$$

The constants L(in equation (4.2.17) and $\tau(0)$ will be determined from the initial conditions.

Thus for

$f(t) = F(\epsilon t)(\cos \nu t + \phi(\epsilon t))$, x is approximately given by:

$$x = (A_0 + \epsilon A_1)\cos\tau + \epsilon\frac{\beta A_0^3}{32}\cos 3\tau + u_0 + \epsilon u_1 \tag{4.2.22}$$

where A_0, A_1, u_0 and u_1 are given by equations (4.2.17), (4.2.20), (4.2.15) and (4.2.19) respectively, and τ is given by (4.2.21).

For the special case $f(t) = E\, e^{-\kappa\xi} \cos \nu t$ and with $x(0) = 0$ and $\dot{x}(0) = 0$, expression (4.2.21) becomes:

$$\tau = (1 - \frac{\varepsilon^2}{8}) t + \frac{3}{8}\beta L^2[1 + \frac{9\varepsilon}{8}\beta\sigma^2E^2](1-e^{-\xi}) + \frac{3}{8}\beta\sigma^2E^2(\frac{1-e^{-2\kappa\xi}}{\kappa})$$

$$-\varepsilon\,\beta^2\sigma^4E^4[\frac{51}{512}(1-e^{-2\xi}) + \frac{9}{16}\frac{1+4\sigma+p_1+p_2}{1+2\kappa}(1-e^{-(1+2\kappa)\xi}) + \frac{9(1+4\sigma+q_1+q_2}{128}(\frac{1-e^{4\kappa\xi}}{\kappa})]$$

and the constant L is given by:

$$L = -\sigma E + \varepsilon\,\beta\,\sigma^3E^3[\frac{1}{32} + \frac{9}{4}\sigma + \frac{1}{4}\;\frac{1}{1-9\nu^2} + \frac{3}{4}(p_1+p_2-q_1-q_2)]$$

Eqn. (4.2.1) has been solved numerically for $\varepsilon = 0.2$ and for values of β in the range $0 < \beta \leq 4$ The forcing term is taken as:

$$f(t) = 1.5\,e^{-\varepsilon\kappa t}\cos 2t, \qquad \kappa = 0,\ 0.25$$

The analytical solutions are compared with the numerical solutions in Figs. (4.1) to (4.4).

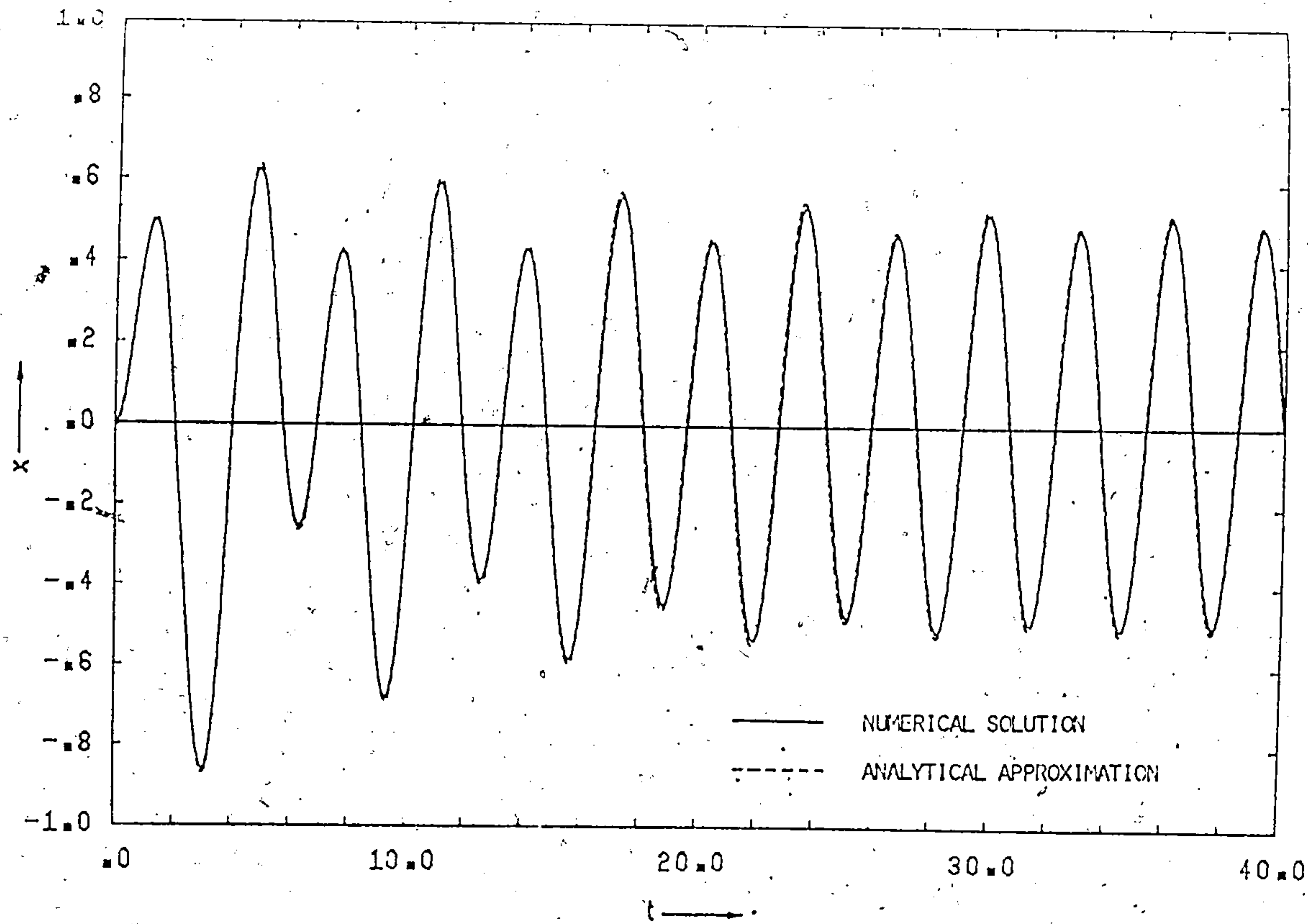

Fig. 4.1 Solution of eqn. (4.2.1) for $\varepsilon = 0.2$, $\beta = 1$, and $f(t) = \cos 2t$.

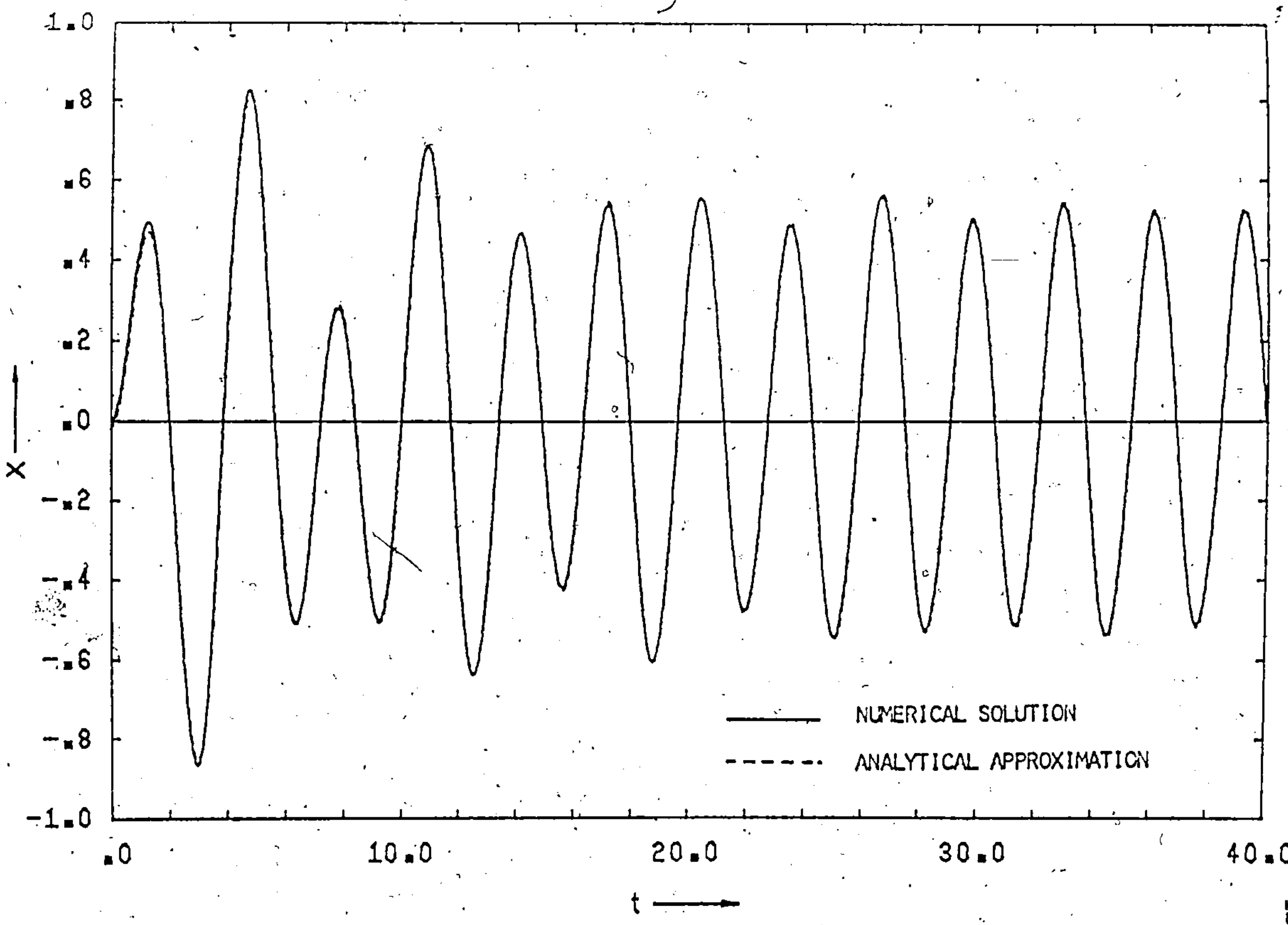

Fig. 4.2 Solution of eqn. (4.2.1) for $\varepsilon = 0.2$, $\beta = 4$, and $f(t) = \cos 2t$.

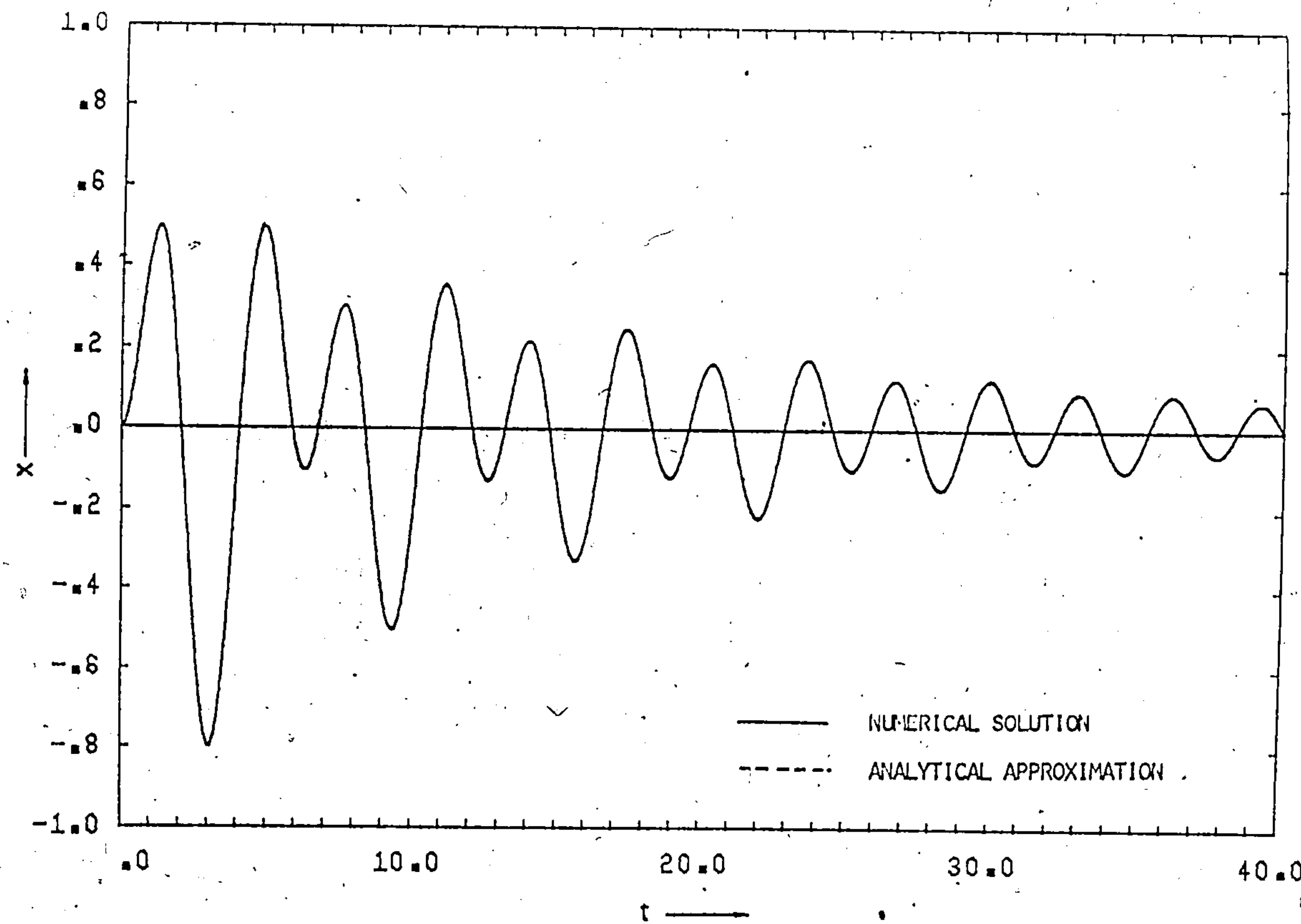

Fig. 4.3 Solution of eqn. (4.2.1) for $\varepsilon = 0.2$, $\beta = 1$, and $f(t) = e^{-0.05t} \cos 2t$.

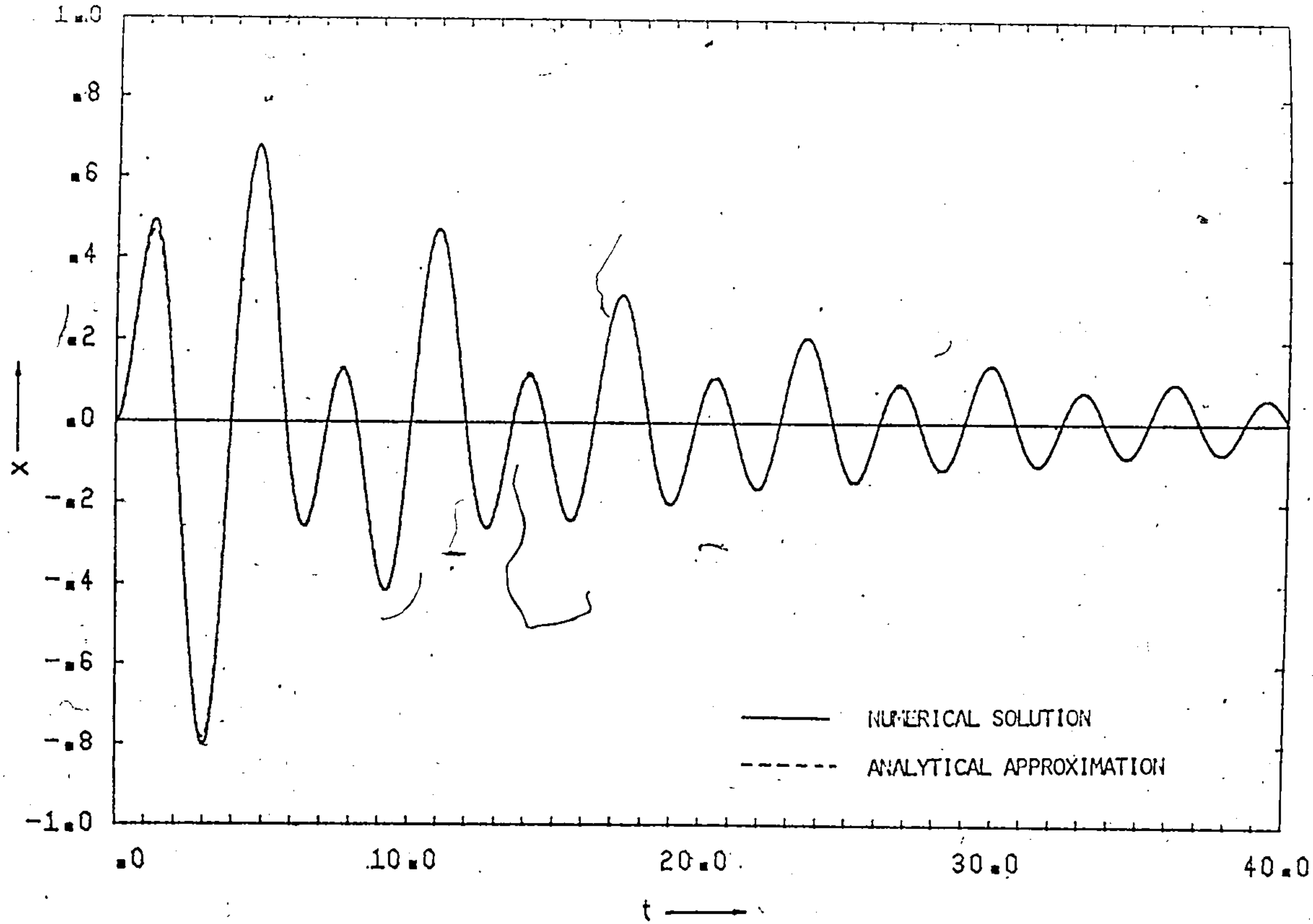

Fig. 4.4 Solution of eqn.(4.2.1) for $\varepsilon = 0.2$, $\beta = .4$, and $f(t) = e^{-0.05t} \cos 2t$.

CHAPTER 5

A non-linear convolution approach to the problem of forced non-stationary oscillations in self-oscillatory systems

5.1 Introduction

The response of a non-linear dynamic system to excitation by a pulse of infinitesimal duration depends on the magnitude (i.e. area) of the pulse as well as the state of the system at its instant of application.

When a second order self-oscillatory system, operating in its steady state mode, is disturbed by a weak pulse, then at any instant of time the representative point in a suitably chosen phase-plane will be found near the limit cycle trajectory in that plane. When the system is disturbed by a continuous signal which is one order of magnitude smaller than that sufficient to phase-lock or suppress the self-oscillation, then it is plausible that the representative point in the phase-plane remains close to the limit cycle trajectory. The system's output may be represented by a non-linear convolution integral which can be evaluated by a simple iteration procedure.

We shall study the oscillator shown in Fig. 5.1 where the transfer characteristic of the negative non-linear resistance is given by:

$$i_r = a_1 v + a_2 \frac{v^3}{3} + a_3 \frac{v^5}{5} ,$$

v being the oscillator voltage.

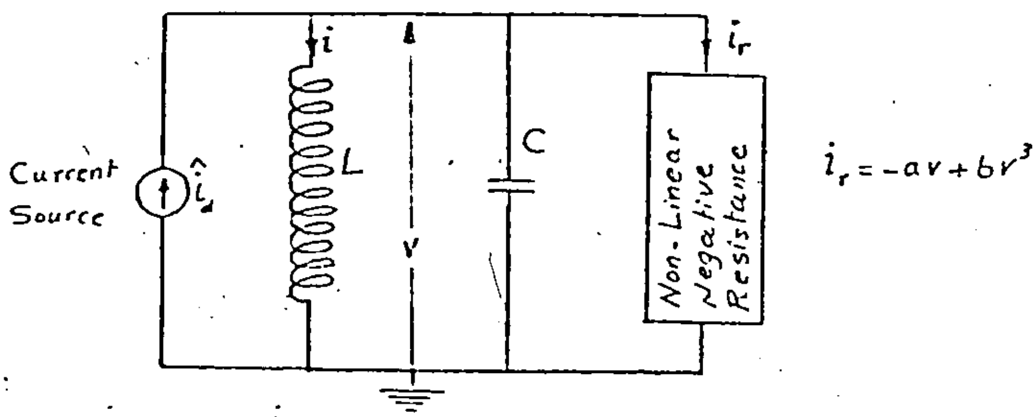

Fig.5.1 Circuit representing a disturbed self-oscillator

Normalization :

$$x = i\sqrt{\frac{L}{C}\,\frac{3b}{a}}$$

$$\dot{x} = \sqrt{\frac{3b}{a}}\; v$$

$$i_d = \hat{i}_d\sqrt{\frac{L}{C}\,\frac{3b}{a}}$$

The oscillator is described by:

$$C \frac{d^2v}{d\bar{t}^2} + \frac{1}{L} v + (a_1 + a_2v^2 + a_3v^4) \frac{dv}{d\bar{t}} = \frac{d}{d\bar{t}} \hat{i}_d(\bar{t}) \tag{5.1.1}$$

where $\bar{t}$ is the time variable in seconds, L and C are inductance and capacitance of the tuned circuit in Henrys and Farads respectively, and $\hat{i}_d$ is the disturbing current in Amperes. The units of a_1, a_2 and a_3 are: mho, mho/volt^2 and mho/volt^4 respectively.

The derivation of eqn. (5.1.1) is done in quite an elementary manner by using Kirchoff's law and need not be elaborated here.

In the absence of the forcing term, eqn. (5.1.1) has a unique limit cycle in the following cases:

$a_1 < 0$	$a_2 > 0$	$a_3 = 0$
	$a_2 \gtrless 0$	$a_3 > 0$
$a_1 = 0$	$a_2 < 0$	$a_3 > 0$

Let us use the change of variables:

$$t = \frac{\bar{t}}{\sqrt{LC}} \quad \text{and} \quad v = \kappa\, y$$

where

$$\kappa = \text{the real square root of } \frac{-a_2 \pm \sqrt{a_2^2 - 4a_1a_3}}{4a_3}$$

$$\text{whereas} \quad \kappa \to \sqrt{\frac{-a_1}{a_2}} \quad \text{as} \quad a_3 \to 0 \, .$$

Eqn. (5.1.1) is then rewritten in the dimensionless form:

$$\ddot{y} + y + (\varepsilon_1 + \varepsilon_2 y^2 + \varepsilon_3 y^4)\,\dot{y} = \frac{d}{dt}\, i_d \tag{5.1.2}$$

with $(\dot{\ }) \triangleq \frac{d}{dt}$, $i_d = \frac{1}{\kappa}\sqrt{\frac{L}{C}}\,\hat{i}_d$,

$$\varepsilon_1 = a_1\sqrt{\frac{L}{C}}\ ,\quad \varepsilon_2 = a_2\kappa^2\sqrt{\frac{L}{C}}\quad \text{and}\quad \varepsilon_3 = a_3\kappa^4\sqrt{\frac{L}{C}}$$

Eqn. (5.1.2) may also be written as:

$$\ddot{x} + x + (\varepsilon_1 + \frac{\varepsilon_2}{3}\dot{x}^2 + \frac{\varepsilon_3}{5}\dot{x}^4)\,\dot{x} = i_d(t)\ , \tag{5.1.3}$$

where $x \triangleq \int y\,dt$

Using standard methods (the Poincaré-Lindstedt method for example), it can easily be shown that in the autonomous case (i.e. when $i_d = 0\ \forall t$) the limit cycle of eqn. (5.1.2) has dimensionless amplitude and period of approximately 2 and 2π respectively when $\varepsilon_i \ll 1$, $i = 1,2$, and 3. The normalization of eqn. (5.1.1) in the form of eqn. (5.1.2) is thus in comformity with the usual normalization of v.d. Pol's equation in the form:

$$\ddot{y} + y - \varepsilon(1-y^2)\dot{y} = \frac{d}{dt}\, g(t)\ ,$$

or its equivalent:

$$\ddot{x} + x - \varepsilon(\dot{x} - \dot{x}^3/3) = g(t)\ ,$$

which has a limit cycle (when $g(t) = 0\ \forall t$) of dimensionless amplitude and frequency of approximately 2 and 2π respectively when $0 < \varepsilon \ll 1$. In section (5.5) it will be clear that this analogy is beneficial.

5.2 The variational response of the v.d.Pol oscillator

Consider the v.d.Pol oscillator described by the differential equation:

$$\ddot{x} + x - \varepsilon(\dot{x} - \dot{x}^3/3) = 0 \ , \ 0 < \varepsilon << 1 \qquad (5.2.1)$$

which is a special case of eqn. (5.1.3) with

$$i_d \equiv 0 \ , \qquad \varepsilon_1 = -\varepsilon_2 = -\varepsilon \text{ and } \varepsilon_3 = 0 \ .$$

Assume that the oscillator has been "free-running" for a sufficiently large period of time before t=0 so that at t=0 it may be assumed that the limit cycle has been reached.

Let $x=x^*$ denote the periodic steady-state response with a specific phase-reference. Let the representative point in the x-$\dot{x}$ phase plane [Fig. 5.2] be displaced from the limit cycle trajectory at the instant $t=t_i \geq 0$ such that:

$$x(t_i + 0^+) = x^*(t_i) + \varepsilon\beta_i$$

and

$$\dot{x}(t_i + 0^+) = \dot{x}^*(t_i) + \varepsilon\gamma_i \qquad (5.2.2)$$

where β_i and γ_i are $O(1)$.

For $t \geq t_i + 0^+$, we may write:

$$x(t) = x^*(t) + \varepsilon u$$

and

$$\dot{x}(t) = \dot{x}^*(t) + \varepsilon\dot{u} \qquad (5.2.3)$$

From eqns. (5.2.1) and (5.2.3), it is straightforward to derive the variational equation:

$$\ddot{u} + u - \varepsilon(1-\dot{x}^{*2})\,\dot{u} + \varepsilon^2\dot{x}^*\dot{u}^2 + \frac{1}{3}\varepsilon^3\dot{u}^3 = 0 \tag{5.2.4}$$

which satisfies the initial conditions:

$$u(t_i+0^+) = \beta_i \text{ and } \dot{u}(t_i+0^+) = \gamma_i \tag{5.2.5}$$

The steady-state response x^* may be obtained by the standard perturbation method (Poincaré-Lindstedt). With the condition of periodicity:

$$x^*(t+T) = x^*(t) \,,$$

where T is the period of the limit cycle, and with the phase reference chosen such that: $\dot{x}^*(0) = 0$, then:

$$x^*(t) = (2 + \frac{\varepsilon^2}{8}) \cos \Omega t - \frac{\varepsilon}{4} \sin \Omega t + \frac{\varepsilon}{12} \sin 3\Omega t$$

$$+ \frac{\varepsilon^2}{16} \cos 3\Omega t - \frac{\varepsilon^2}{96} \cos 5\Omega t + \ldots$$

where $\Omega \triangleq \frac{2\pi}{T} = 1 - \frac{\varepsilon^2}{16} + 0(\varepsilon^4)$

thus

$$\dot{x}^*(t) = -2 \sin \Omega t - \frac{\varepsilon}{4} \cos \Omega t + \frac{\varepsilon}{4} \cos 3\Omega t - \frac{3\varepsilon^2}{16} \sin 3\Omega t + \frac{5\varepsilon^2}{96} \sin 5\Omega t + \ldots \tag{5.2.6}$$

Let us for convenience use the change of variable:

$\tau = \Omega t$, thus $\frac{d}{d\tau} = \frac{1}{\Omega}\frac{d}{dt}$. Then using eqn. (5.2.6), eqn. (5.2.4) is rewritten as:

$$u'' + Ku + \varepsilon\, Q(\tau)\, u' = \varepsilon^2 E + O(\varepsilon^3) \tag{5.2.7}$$

where $(\,')\triangleq \frac{d}{d\tau}$,

$$Q(\tau) \triangleq 1 - 2\cos 2\tau + \varepsilon(\sin 2\tau - \frac{1}{2}\sin 4\tau) \ ,$$

$$K \triangleq \frac{1}{\Omega^2} = 1 + \frac{\varepsilon^2}{8}$$

and $$E \triangleq \dot{x}^* \dot{u}^2$$

In order to obtain a second order approximation of the solution of eqn. (5.2.7), we shall first solve the linear time-variant equation:

$$u_\ell'' + Ku_\ell + \varepsilon Q(\tau)\, u_\ell' = 0 \ , \tag{5.2.8}$$

with the initial conditions $u_\ell(\tau_i+0^+) = \beta_i$ and $u_\ell'(\tau_i+0^+) = \frac{1}{\Omega}\,\dot{u}_\ell(t_i+0^+) = \frac{\gamma_i}{\Omega}$.

then take into consideration the effect of the terms in the r.h.s. of eqn. (5.2.7).

We shall transform eqn. (5.2.7) to the standard form of Hill's equation which has well-known well-tabulated solutions. Writing

$$u_\ell \triangleq \eta\, e^{\chi} \tag{5.2.9}$$

where $\eta \triangleq \eta(\tau,\tau_i)$; $\tau_i \triangleq \Omega t_i$

and $\chi \triangleq \chi(\tau) \triangleq -\frac{\varepsilon}{2}\int Q d\tau$

$$= -\frac{\varepsilon}{2}\tau + \frac{\varepsilon}{2}\sin 2\tau + O(\varepsilon^2) \tag{5.2.10}$$

thence eqn. (5.2.8) reduces to Hill's equation:

$$\ddot{\eta} + P(\tau)\eta = 0 \qquad (5.2.11)$$

where

$$P(\tau) = K - \frac{1}{2} Q' - \frac{1}{4} Q''$$

$$= \alpha_o + 2 \sum_{k=1}^{2} \alpha_k \cos(2k\tau - \theta_k)$$

with

$$\alpha_o = 1 - \frac{5\varepsilon^2}{8} + O(\varepsilon^3) \quad , \quad \alpha_1 = \varepsilon + O(\varepsilon^3) \ ,$$

$$\alpha_2 = \frac{\varepsilon^2}{4} + O(\varepsilon^3) \ , \ \theta_1 = -\frac{\pi}{2} + O(\varepsilon^3) \text{ and } \theta_2 = O(\varepsilon^3) \qquad (5.2.12)$$

Following standard methods, the solution of eqn. (5.2.8) is sought in the form:

$$\eta = C_1 e^{\mu_1 \tau} F(\tau, \hat{\lambda}_1) + C_2 e^{\mu_2 \tau} F(\tau, \hat{\lambda}_2) \qquad (5.2.13)$$

where

$$C_1 \triangleq C_1(\tau_i, \beta_i, \gamma_i) \quad \text{and} \quad C_2 \triangleq C_2(\tau_i, \beta_i, \gamma_i)$$

are constants, the parameters μ_1, μ_2, $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are real and $F(\tau, \hat{\lambda}_1)$ and $F(\tau, \hat{\lambda}_2)$ are periodic functions with respect to τ.

Making use of the tabulated solution of eqn. (5.2.11) [see for example Hayashi, 1964] and remembering that the value of P is near unity

$\forall\tau>\tau_i$, then our problem is confined to the first resonance zone of Hill's equation and the solution for η in the form (5.2.13) is then determined as follows:

First we determine $\hat{\lambda}_1$ and $\hat{\lambda}_2$ from:

$$\alpha_o = 1 + \alpha_1 \cos 2\hat{\lambda}_{\substack{1\\2}} + \left(-\frac{1}{4} + \frac{1}{8}\cos 4\hat{\lambda}_{\substack{1\\2}}\right)\alpha_1^2 \tag{5.2.14}$$

then μ_1 and μ_2 are determined from:

$$\mu_{\substack{1\\2}} = \frac{1}{2}\alpha_1 \sin 2\hat{\lambda}_{\substack{1\\2}} \tag{5.2.15}$$

The periodic functions $F(\tau,\hat{\lambda}_1)$ and $F(\tau,\hat{\lambda}_2)$ are then given by:

$$F(\tau,\hat{\lambda}_{\substack{1\\2}}) = \sin(\tau - \frac{1}{2}\theta_1 - \hat{\lambda}_{\substack{1\\2}}) + \frac{\alpha_1}{8}\sin(3\tau - \frac{3}{2}\theta_1 - \hat{\lambda}_{\substack{1\\2}}) \tag{5.2.16}$$

From eqns. (5.2.12) and (5.2.14) we have:

$$\cos 2\hat{\lambda}_{\substack{1\\2}} + \frac{\varepsilon}{8}\cos 4\hat{\lambda}_{\substack{1\\2}} + \frac{3\varepsilon}{8} = 0$$

which has the solution:

$$\hat{\lambda}_1 = -\hat{\lambda}_2 = \frac{\pi}{4} + \frac{\varepsilon}{8} + 0(\varepsilon^2)$$

Thus from eqn. (5.2.15):

$$\mu_1 = -\mu_2 = -\frac{\varepsilon}{2} + 0(\varepsilon^2)$$

Now evaluating $F(\tau,\lambda_{1,2})$ and substituting in expression (5.2.13) we get:

$$\eta = C_1\, e^{\frac{\varepsilon\tau}{2}} \left[\sin\tau - \frac{\varepsilon}{8}\cos\tau + \frac{\varepsilon}{8}\cos 3\tau\right]$$

$$+\, C_2\, e^{-\frac{\varepsilon\tau}{2}} \left[\cos\tau - \frac{\varepsilon}{8}\sin\tau - \frac{\varepsilon}{8}\sin 3\tau\right] \qquad (5.2.17)$$

From expressions (5.2.9) and (5.2.17), and the expansions

$$e^{\chi} = e^{-\frac{\varepsilon\tau}{2}}\, e^{\frac{\varepsilon}{2}\sin 2\tau} = e^{-\frac{\varepsilon\tau}{2}} \left[1 + \frac{\varepsilon}{2}\sin 2\tau + O(\varepsilon^2)\right] ,$$

we get:

$$u_\ell = C_1\, G_1(\tau) + e^{-\varepsilon(\tau-\tau_i)}\, C_2 G_2(\tau) \qquad (5.2.18)$$

where

$$G_1(\tau) = e^{\chi}\, F(\tau,\lambda_1)$$

$$= \sin\tau + \frac{\varepsilon}{8}(\cos\tau - \cos 3\tau)$$

and

$$G_2(\tau) = e^{\chi}\, F(\tau,\lambda_2)$$

$$= \cos\tau + \frac{\varepsilon}{8}(\sin\tau + \sin 3\tau)$$

Then

$$u'_\ell = C_1 H_1(\tau) + e^{-\varepsilon(\tau-\tau_i)}\, C_2\, H_2(\tau) \qquad (5.2.19)$$

where

$$H_1 = \cos\tau - \frac{\varepsilon}{8}\sin\tau + \frac{3\varepsilon}{8}\sin 3\tau$$

and

$$H_2 = -\sin\tau - \frac{7}{8}\varepsilon\cos\tau + \frac{3\varepsilon}{8}\cos 3\tau$$

Using the initial conditions in (5.2.8), then C_1 and C_2 are given by:

$$C_1 = \frac{-\beta_i H_2(\tau_i) + \gamma_i G_2(\tau_i)}{H_1(\tau_i)\, G_2(\tau_i) - G_i(\tau_i)\, H_2(\tau_i)} \qquad (5.2.20)$$

$$= C_{11} + \varepsilon C_{12}$$

where

$$C_{11} = \beta_i \sin\tau_i + \gamma_i \cos\tau_i$$

and $$C_{12} = \frac{1}{8}\beta_i(3\cos\tau_i + \cos 3\tau_i) - \frac{3}{8}\gamma_i(\sin\tau_i + \sin 3\tau_i)$$

$$C_2 = \frac{\beta_i H_1(\tau_i) - \gamma_i G_1(\tau_i)}{H_1(\tau_i)\, G_2(\tau_i) - G_1(\tau_i)\, H_2(\tau_i)} \qquad (5.2.21)$$

$$= C_{21} + \varepsilon C_{22}$$

where

$$C_{21} = \beta_i \cos\tau_i - \gamma_i \sin\tau_i$$

and $$C_{22} = -\frac{1}{8}\beta_i(5\sin\tau_i + \sin 3\tau_i) + \frac{3}{8}\gamma_i(\cos\tau_i - \cos 3\tau_i)$$

Let us now take into consideration the effect of the term $\varepsilon^2 E$ in eqn. (5.2.7), where

$$E = E(\tau,\tau_i) = \dot{x}^*(\tau_i)\, \dot{u}^2(\tau,\tau_i) \quad .$$

The function $\varepsilon^2 E(\tau,\tau_i)$ may be viewed as a sequence of infinitesimal pulses:

$$p_j = \varepsilon^2 E(\tau_j,\tau_i)\, \Delta\tau_j \quad ; \quad \tau_j \geq \tau_i \tag{5.2.22}$$

exciting the linear time-variant system described by eqn. (5.2.8). The immediate effect of an individual pulse of (normalized) area p_j is to cause a sudden variation in u' equal to p_j while no immediate variation in u takes place. Thus, if Δu and $\Delta u'$ denote the variations in u and u' respectively at $\tau \geq \tau_j$ due to a pulse of area p_j applied at the instant τ_j, then:

$$\Delta u = p_j [D_1\, G_1(\tau) + e^{-\varepsilon(\tau-\tau_j)} D_2\, G_2(\tau)] \tag{5.2.23}$$

and

$$\Delta u' = p_j [D_1\, H_1(\tau) + e^{-\varepsilon(\tau-\tau_j)} D_2\, H_2(\tau)] \tag{5.2.24}$$

$$\left.\right\} \tau \geq \tau_j = \Omega t_j \quad .$$

where

$$D_1 = D_1(\tau_j) = \frac{\partial C_1(\tau_j,\beta_j,\gamma_j)}{\partial \gamma_j}$$

$$= \left. \frac{G_2}{H_1\, G_2 - G_1\, H_2} \right|_{\tau=\tau_j}$$

$$= \cos \tau_j - \frac{3}{8}(\sin \tau_j + \sin 3\tau_j)$$

and

$$D_2 = D_2(\tau_j) = \frac{\partial C_2(\tau_j, \beta_j, \gamma_j)}{\partial \gamma_j} \tag{5.2.25}$$

$$= \left. \frac{-G_1}{H_1 G_2 - G_1 H_2} \right|_{\tau=\tau_j}$$

$$= -\sin \tau_j + \frac{3\varepsilon}{8}(\cos \tau_j - \cos 3\tau_j)$$

Now as $\Delta\tau_j \rightarrow 0$, we pass to the continuous case and the effect of the forcing term $\varepsilon^2 E(\tau,\tau_i)$ is given by:

$$u_{n\ell} = \varepsilon^2 \int_{\tau_j}^{\tau} E(\tau_j,\tau_i)\, I(\tau,\tau_j)\, d\tau_j \tag{5.2.26}$$

where

$$I(\tau,\tau_j) = D_1(\tau_j)\, G_1(\tau) + e^{-\varepsilon(\tau-\tau_j)}\, D_2(\tau_j)\, G_2(\tau) \tag{5.2.27}$$

and $u \triangleq u_\ell + u_{n\ell}$.

Since the integral in (5.2.26) has ε^2 as a coefficient, it is sufficient to retain only the terms of first order of magnitude in the intergrand. Thus, in eqn. (5.2.26), we shall use the approximations:

$$E(\tau_j,\tau_i) = -\dot{x}^*(\tau_j)\, \dot{u}^2(\tau_j,\tau_i)$$

$$\approx 2 \sin \tau_j [C_{11} \cos \tau_j - C_{21}\, e^{-\varepsilon(\tau_j-\tau_i)} \sin \tau_j]^2$$

and

$$I(\tau,\tau_j) \approx \cos \tau_j \sin \tau - e^{-\varepsilon(\tau-\tau_j)} \sin \tau_j \cos \tau$$

Hence

$$u_{n\ell} \approx -\frac{\varepsilon}{4}[(2\, C_{11}\, C_{21} \sin\tau + C_{11}^2 \cos\tau)(1 - e^{-\varepsilon(\tau-\tau_i)})$$

$$+ 3\, C_{21}^2 (e^{-\varepsilon(\tau-\tau_i)} - e^{-2\varepsilon(\tau+\tau_i)})\cos\tau] \qquad (5.2.28)$$

with

$$C_{11} = C_{11}(\tau_i,\beta_i,\gamma_i) \text{ and } C_{21} = C_{21}(\tau_i,\beta_i,\gamma_i)$$

Therefore, the solution of eqn. (5.2.4) is given to the second order of approximation by[†]:

$$u = u_\phi + u_a$$

where

$$u_\phi = u_\phi(\tau_i+\phi_i,\beta_i,\gamma_i) = C_1\, G_1(\tau) - \frac{\varepsilon}{4}(C_{11}^2 \cos\tau + 2C_{11}C_{21} \sin\tau)$$

and

$$u_a = u_a(\tau\,,\ \tau_i+\phi_i,\beta_i,\gamma_i) = C_2\, G_2(\tau)\, e^{-\varepsilon(\tau-\tau_i)}$$

$$+ \frac{\varepsilon}{4}[\{(C_{11}^2-3C_{21}^2)\cos\tau + 2C_{11}C_{21}\sin\tau\}e^{-\varepsilon(\tau-\tau_i)} + 3C_{21}^2\, e^{-2\varepsilon(\tau-\tau_i)}\cos\tau]$$

(5.2.30)

with $C_{\substack{1\\2}} = C_{\substack{1\\2}}(\tau_i,\beta_i,\gamma_i)$ and $C_{\substack{11\\21}} = C_{\substack{11\\21}}(\tau_i,\beta_i,\gamma_i)$

Defining:

$$\hat{x} = x^* + \varepsilon\, u_\phi\,,$$

[†]An alternative derivation is given in appendix D.

it is straightforward to verify that the amplitudes of the fundamental, third and fifth harmonics of $\hat{x}$ are the same as those of x^* and that:

$$\hat{x} = x^*(\tau+\psi)$$

where $\psi = \psi(\tau_i, \beta_i, \gamma_i) = -\frac{\varepsilon}{2} C_1 + \frac{\varepsilon^2}{4} C_{11} C_{21}$

$$= -\frac{\varepsilon}{2} C_{11} - \frac{\varepsilon^2}{2}(C_{12} - \frac{1}{2} C_{11} C_{21}) \qquad (5.2.31)$$

The non-decaying component u_ϕ in the variational response arises from the phase shift caused by the displacement from the limit cycle trajectory. If the displacement is radial, i.e. if:

$$\tan^{-1} \frac{\dot{x}^*(\tau_i)}{x^*(\tau_i)} = \pm \tan^{-1} \frac{\gamma_i}{\beta_i} ,$$

then $\psi(\cdot) = 0$ and only a transient amplitude variation, given by expression (5.2.30), takes place.

Before we proceed with the analysis, it may be useful to refer to the phase plane representation. In Fig. (5.2), the phasor end is assumed to be at point a on the limit cycle trajectory $x^* - \dot{x}^*$ at the instant $t = t_i$. The circular phase, which is defined here as the argument of the fundamental component of the Fourier series expansion of x^* is equal to $\tau_i = \Omega t_i$. The angle $\phi(\tau_i)$ of the phasor $\underset{\sim}{r}$ is in general a non-linear function of τ_i. When the phasor end is displaced to point c at the instant $t = t_i + 0^+$, the circular phase becomes $\tau_i + \psi$, where ψ is given by expression (5.2.31), and

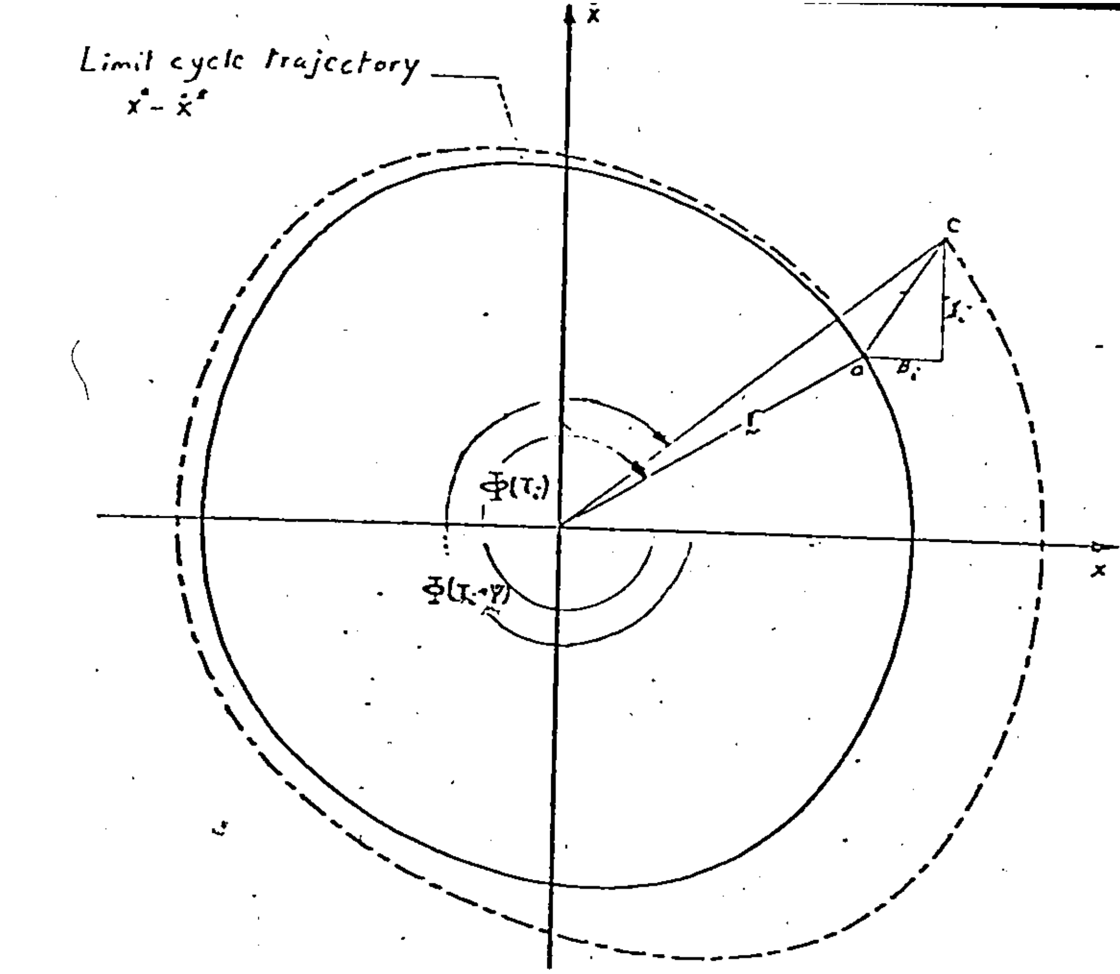

Fig. 5.2 The variational response of a self-oscillator – phase-plane illustration.

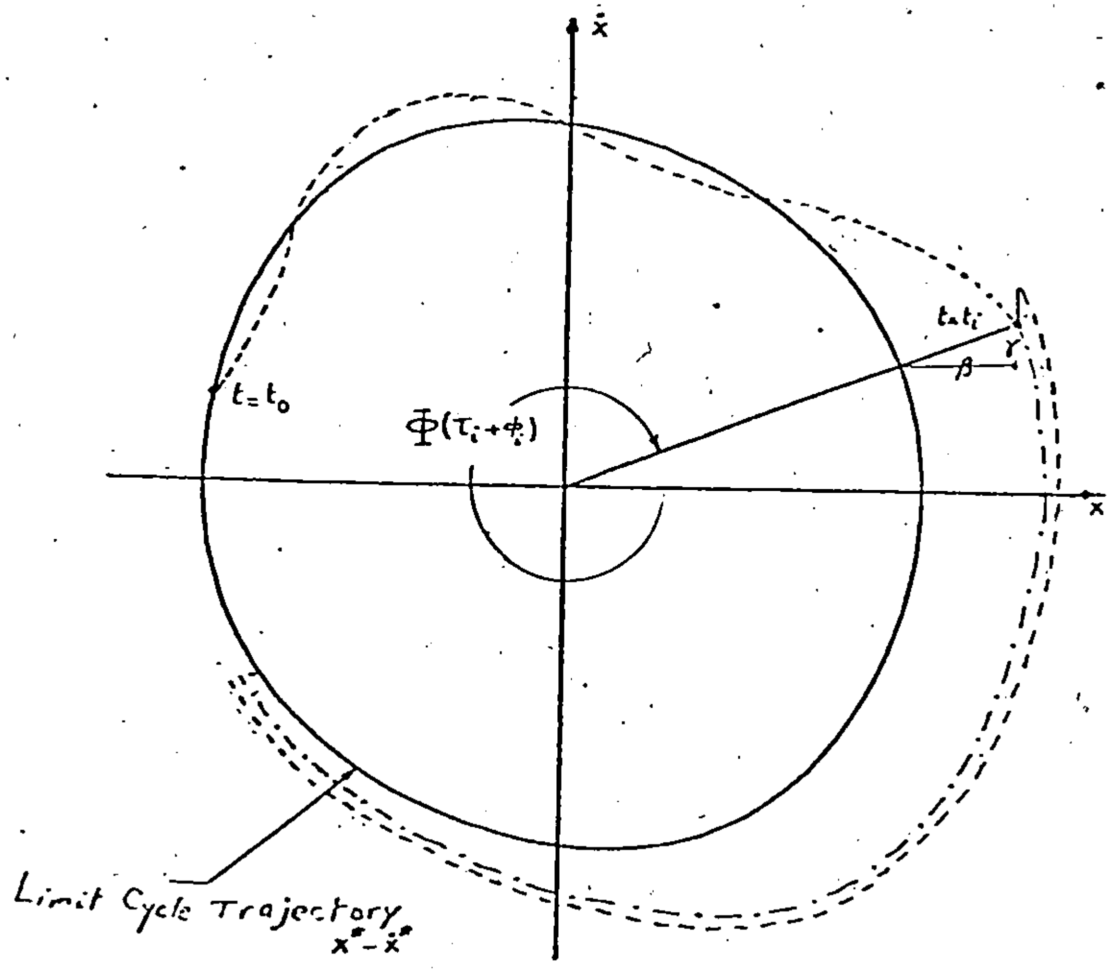

Fig. 5.3 The pulse response of a disturbed self-oscillator – phase-plane illustration

the angle of the phasor $\underset{\sim}{r}$ becomes $\Phi(\tau_i + \psi)$. The radial displacement $\|\Delta \underset{\sim}{r}\|$ will then decay so that the phasor end c will eventually reside on the limit cycle trajectory as $t - t_i \to \infty$. The horizontal and vertical projections of the radial displacement, at any instant $t \geq t_i$, are equal to u_a and $\frac{du_a}{dt}$ respectively, where u_a is given by expression (5.2.30).

5.3 The pulse response of the disturbed oscillator and the response to continuous disturbance

In section (5.2) we have derived expressions for the variational response of the v.d.Pol oscillator. In this section it will be shown how this result can be used to study the behaviour of the oscillator in the presence of a weak disturbing force, the oscillator being then described by:

$$\ddot{x} + x - \varepsilon(\dot{x} - \frac{\dot{x}^3}{3}) = i_d(t) \qquad (5.3.1)$$

where $i_d(t)$ represents a disturbing current [Fig.(5.1)].

Let us assume that the oscillator has been free-running for $-\infty < t \leq t_o$ and that it is in the steady-state at $t = t_o$. Let us consider the effect of disturbing the oscillator by a sequence of infinitesmal pulses of areas q_o, q_1, ... at the instants t_o, t_1, t_2, ..., where $t_i\big|_{i>0} \geq t_o$. These pulses may be considered as samples of the disturbing function $i_d(t)$ in eqn. (5.3.1). The effect of an individual pulse q_i, $i>0$, depends on the cumulative effect of the previous pulses q_j, $0 \leq j < i$. Suppose that due to the pulses q_0 , q_1, ..., q_{i-1}, the representative point in the phase-plane has been brought to point c in Fig.5.3, where its radial displacement from the limit cycle trajectory has horizontal and vertical projections of β_i and γ_i respectively and is at a circular phase of $\tau_i+\phi_i$ corresponding to the angle $\Phi(\tau_i+\phi_i)$ of the phasor $\underset{\sim}{r}$. The pulse q_i will then result in variations in the phase and amplitude of x given by:

$$\text{Phase variation} \triangleq \Delta\phi = q_i \frac{\partial}{\partial\gamma_i} \psi(\tau_i+\phi_i,\beta_i,\gamma_i)$$

$$= q_i[\Gamma_{11}(\tau_i+\phi_i) + \varepsilon\, \Gamma_{12}(\tau_i+\phi_i,\beta_i,\gamma_i)] \qquad (5.3.2)$$

and

$$\text{amplitude variation} \triangleq \Delta\beta = q_i \frac{\partial}{\partial\gamma_i} u_a(\tau,\tau_i+\phi_i,\beta_i,\gamma_i)$$

$$= q_i[\Gamma_{21}(\tau,\tau_i+\phi_i)e^{-\varepsilon(\tau-\tau_i)}$$

$$+ \varepsilon\, \overline{\Gamma}_{22}(\tau,\tau_i+\phi_i,\beta_i,\gamma_i)e^{-\varepsilon(\tau-\tau_i)} \qquad (5.3.3)$$

$$+ \varepsilon\, \overline{\overline{\Gamma}}_{22}(\tau,\tau+\phi_i,\beta_i,\gamma_i)e^{-2\varepsilon(\tau-\tau_i)}]$$

where

$$\Gamma_{11}(\tau_i+\phi_i) = -\frac{1}{2}\,\frac{\partial C_{11}}{\partial\gamma_i},$$

$$\Gamma_{12}(\tau_i+\phi_i,\beta_i,\gamma_i) = -\frac{1}{2}\frac{\partial}{\partial\gamma_i}\,(C_{12} - \frac{1}{2}C_{11}\,C_{21}),$$

$$\Gamma_{21}(\tau,\tau_i+\phi_i) = \frac{\partial C_{21}}{\partial\gamma_i}\cos\tau$$

$$\overline{\Gamma}_{22}(\tau,\tau_i+\phi_i,\beta_i,\gamma_i) = \frac{\partial}{\partial\gamma_i}\,[\frac{1}{4}\,(C_{11}^2 - 3C_{21}^2)\cos\tau + C_{22}\cos\tau$$

$$+ \frac{1}{2}C_{11}\,C_{21}\sin\tau + \frac{1}{8}C_{21}(\sin\tau + \sin 3\tau)]$$

and

$$\overline{\overline{\Gamma}}_{22}(\tau,\tau_i+\phi_i,\beta_i,\gamma_i) = \frac{3}{4}\frac{\partial}{\partial\gamma_i}C_{21}^2\cos\tau$$

Now using the expressions for C_{11}, C_{12}, C_{21} and C_{22} given in (5.2.20) and (5.2.21) then:

$$\frac{\partial}{\partial\gamma_i} C_{11}^2 = 2C_{11} \frac{\partial C_{11}}{\partial\gamma_i} = \beta_i \sin 2\tau_i + 2\gamma_i \cos^2\tau_i ,$$

$$\frac{\partial}{\partial\gamma_i} C_{21}^2 = 2C_{21} \frac{\partial C_{21}}{\partial\gamma_i} = -\beta_i \sin 2\tau_i + 2\gamma_i \sin^2\tau_i ,$$

and

$$\frac{\partial}{\partial\gamma_i} C_{11}C_{21} = C_{11} \frac{\partial C_{21}}{\partial\gamma_i} + C_{21} \frac{\partial C_{11}}{\partial\gamma_i} = \beta_i \cos 2\tau_i - \gamma_i \sin 2\tau_i ,$$

Therefore,

$$\Gamma_{11}(\tau_i) = -\frac{1}{2} \cos \tau_i ,$$

$$\Gamma_{21}(\tau_1\tau_i) = -\sin \tau_i \cos \tau$$

$$\Gamma_{12}(\tau_i,\beta_i,\gamma_i) = \frac{3}{16} (\sin \tau_i + \sin 3\tau_i) + \frac{1}{4} (\beta_i \cos 2\tau_i - \gamma_i \sin 2\tau_i) ,$$

$$\bar{\Gamma}_{22}(\tau,\tau_i,\beta_i,\gamma_i) = [\tfrac{3}{8}(\cos \tau_i - \cos 3\tau_i) + \beta_i \sin 2\tau_i + \tfrac{1}{2} \gamma_i(2 \cos 2\tau_i - 1)] \cos \tau$$
$$+ \tfrac{1}{2}[\beta_i \cos 2\tau_i - \gamma_i \sin 2\tau_i] \sin \tau$$
$$- \tfrac{1}{8} \sin \tau_i(\sin \tau + \sin 3\tau) ,$$

and

$$\bar{\bar{\Gamma}}_{22}(\tau,\tau_i,\beta_i,\gamma_i) = \tfrac{3}{4}[2\gamma_i \sin^2\tau_i - \beta_i \sin 2\tau_i] \cos \tau$$

If $q_i = i_d(t_i)\,\Delta t_i$, then in the limit as $\Delta t_i \rightarrow 0$ we have:

$$\phi(t) = \int_{t_o}^{t} i_d(\zeta)\; [\Gamma_{11}(\Omega\zeta + \phi) + \varepsilon\, \Gamma_{12}(\Omega\zeta + \phi,\beta,\gamma)]\; d\zeta \tag{5.3.5}$$

and

$$\beta(t) = \int_{t_o}^{t} i_d(\zeta)\; [\Gamma_{21}(\Omega t,\Omega\zeta+\phi) + \varepsilon\, \bar{\Gamma}_{22}(\Omega t,\Omega\zeta+\phi,\beta,\gamma)$$

$$+ \varepsilon\, \bar{\bar{\Gamma}}_{22}(\Omega t,\Omega\zeta+\phi,\beta,\gamma)]\; d\zeta \tag{5.3.6}$$

where the values of ϕ, β and γ in the integrands in (5.3.4) and (5.3.5) are:

$$\phi = \phi(\zeta)\ ,\ \beta = \beta(\zeta) \text{ and } \gamma = \gamma(\zeta) = \frac{d}{d\zeta}\,\beta(\zeta)$$

It is plausible to assume that the mean frequency of oscillation, measured over a sufficiently large period of time is time-invariant. Thus for a weak disturbing current, we may write:

$$\phi(t) = \theta_o + \Delta\Omega t + \varepsilon\tilde{\phi}(t)$$

where θ_o and $\Delta\Omega$ are constants and $\varepsilon\tilde{\phi}(t)$ represents zero-mean phase fluctuations.

Using the Taylor expansions:

$$\Gamma_{11}(\Omega\zeta+\phi) = \Gamma_{11}(\bar{\Omega}\zeta + \theta_o + \varepsilon\tilde{\phi}(\zeta))$$

$$\approx \Gamma_{11}(\bar{\Omega}\zeta+ \theta_o) + \varepsilon\, \frac{\partial\Gamma_{11}(\bar{\Omega}\zeta+\theta_o)}{\partial\bar{\Omega}\zeta}\, \tilde{\phi}(\zeta)$$

and

$$\Gamma_{21}(\Omega t,\Omega\zeta+\phi(\zeta)) = \Gamma_{21}(\Omega t,\bar{\Omega}\zeta+\theta_o+\varepsilon\tilde{\phi}(\zeta)) = \Gamma_{21}(\Omega t,\bar{\Omega}\zeta+\theta_o)+\varepsilon\,\frac{\partial\Gamma_{21}(\Omega t,\bar{\Omega}\zeta+\theta_o)}{\partial\bar{\Omega}\zeta}\,\tilde{\phi}(\zeta),$$

where

$$\bar{\Omega} = \Omega + \Delta\Omega \quad , \tag{5.3.7}$$

then eqns. (5.3.5) and (5.3.6) are rewritten as:

$$\phi(t) = \int_{t_o}^{t} i_d(\zeta)\;[\Gamma_{11}(\bar{\Omega}\zeta+\theta_o) + \varepsilon\,\{\frac{\partial\Gamma_{11}(\bar{\Omega}\zeta+\theta_o)}{\partial\bar{\Omega}\zeta}\,\tilde{\phi}(\zeta) + \Gamma_{12}(\bar{\Omega}\zeta+\theta_o,\beta^{(1)}(\zeta),\,\gamma^{(1)}(\zeta)\}\,d\zeta \tag{5.3.8}$$

and

$$\begin{aligned}\beta(t) = \int_{t_o}^{t} i_d(\zeta)[\Gamma_{21}(\Omega t,\bar{\Omega}\zeta+\theta_o)\,e^{-\varepsilon\Omega(t-\zeta)} &+ \varepsilon\,\{\frac{\partial\Gamma_{21}(\Omega t,\bar{\Omega}\zeta+\theta_o)}{\partial\bar{\Omega}\zeta}\,\tilde{\phi}(\zeta)\,e^{-\varepsilon\Omega(t-\zeta)} \\ &+ \bar{\Gamma}_{22}(\Omega t,\bar{\Omega}\zeta+\theta_o,\,\beta^{(1)}(\zeta),\,\gamma^{(1)}(\zeta))\,e^{-\varepsilon\Omega(t-\zeta)} \\ &+ \bar{\bar{\Gamma}}_{22}(\Omega t,\bar{\Omega}\zeta+\theta_o,\,\beta^{(1)}(\zeta),\,\gamma^{(1)}(\zeta))\,e^{-2\varepsilon\Omega(t-\zeta)}\}]d\zeta\end{aligned} \tag{5.3.9}$$

Eqns. (5.3.8) and (5.3.9) may be then solved by a simple iteration procedure. The first approximations of $\phi(t)$ and $\beta(t)$, denoted by $\phi^{(1)}(t)$ and $\beta^{(1)}(t)$ are given by:

$$\phi^{(1)}(t) = \int_{t_o}^{t} i_d(\zeta)\;\Gamma_{11}(\bar{\Omega}\zeta+\theta_o)d\zeta \quad . \tag{5.3.10}$$

and

$$\beta^{(1)}(t) = \int_{t_o}^{t} i_d(\zeta)\ \Gamma_{21}(\Omega t, \bar{\Omega}\zeta+\theta_o)\ d\zeta \qquad (5.3.11)$$

The second approximations of $\phi(t)$ and $\beta(t)$, denoted by $\phi^{(2)}(t)$ and $\beta^{(2)}(t)$ are given by:

$$\phi^{(2)}(t) = \phi^{(1)}(t) + \varepsilon \int_{t_o}^{t} i_d(\zeta)\left[\frac{\partial\Gamma_{11}(\bar{\Omega}\zeta+\theta_o)}{\partial\bar{\Omega}\zeta}\ \tilde{\phi}^{(1)}(\zeta) + \Gamma_{12}(\bar{\Omega}\zeta+\theta_o, \beta^{(1)}(\zeta),\ \gamma^{(1)}(\zeta)\right]d\zeta$$

and

$$\beta^{(2)}(t) = \beta^{(1)}(t) + \varepsilon \int_{t_o}^{t} i_d(\zeta)\left[\frac{\partial\Gamma_{21}(\Omega t, \bar{\Omega}\zeta+\theta_o)}{\partial\bar{\Omega}\zeta}\ \tilde{\phi}^{(1)}(\zeta) + \bar{\Gamma}_{21}(\Omega t, \bar{\Omega}\zeta+\theta_o, \beta(\zeta), \gamma(\zeta))\ e^{-\varepsilon\Omega(t-\zeta)} + \bar{\bar{\Gamma}}_{21}(\Omega t, \bar{\Omega}\zeta+\theta_o, \beta(\zeta), \gamma(\zeta))e^{-2\varepsilon\Omega(t-\zeta)}\right]d\zeta$$

with

$$\gamma^{(1)}(\zeta) = \frac{d}{d\zeta}\beta^{(1)}(\zeta) \qquad (5.3.12)$$

Finally, using expressions (5.3.4) and (5.3.7), the first and second approximations of $\phi(t)$ and $\beta(t)$ (and hence $\gamma(t)$) are given by:

$$\phi^{(1)}(t) = -\frac{1}{2}\int_{t_o}^{t} i_d(\zeta)\ \cos(\bar{\Omega}\zeta+\theta_o)\ d\zeta \qquad (5.3.13)$$

[Note that when the frequency content of the disturbing current i_d is different from the oscillator frequency then $\tilde{\phi}^{(1)}(t) = \phi^{(1)}(t)$]

$$\beta^{(1)}(t) = -\cos(\Omega t+\phi(t))\int_{t_o}^{t} i_d(\zeta)\ \sin(\bar{\Omega}\zeta+\theta_o)e^{-\varepsilon\Omega(t-\zeta)}d\zeta\ , \qquad (5.3.14)$$

$$\gamma^{(1)}(t) = \frac{d}{dt}\beta^{(1)}(t) ,$$

$$\phi^{(2)}(t) = \phi^{(1)}(t) + \varepsilon\int_{t_o}^{t} i_d(\zeta)[\tfrac{1}{2}\phi^{(1)}(\zeta)\sin(\bar{\Omega}\zeta+\theta_o)$$
$$+ \frac{3}{16}\{\sin(\bar{\Omega}\zeta+\theta_o) + \sin 3(\bar{\Omega}\zeta+\theta_o)\}$$
$$+ \frac{1}{4}\{\beta^{(1)}(\zeta)\cos 2(\bar{\Omega}\zeta+\theta_o)$$
$$- \gamma^{(1)}(\zeta)\sin 2(\bar{\Omega}\zeta+\theta_o)\}]d\zeta$$

(5.3.15)

$$\beta^{(2)}(t) = \beta^{(1)}(t) + \varepsilon\int_{t_o}^{t} i_d(\zeta)[e^{-\varepsilon\Omega(t-\zeta)}\{-\cos(\Omega t+\phi)\cos(\bar{\Omega}\zeta+\theta_o)\,\tilde{\phi}^{(1)}(\zeta)$$
$$+ \bar{\Gamma}_{22}(\Omega t+\phi,\bar{\Omega}\zeta+\theta_o,\beta^{(1)}(\zeta),\gamma^{(1)}(\zeta))\}$$
$$+ e^{-2\varepsilon\Omega(t-\zeta)}\bar{\bar{\Gamma}}_{22}(\Omega t+\phi,\bar{\Omega}\zeta+\theta_o,\beta^{(1)}(\zeta),\gamma^{(1)}(\zeta))]d\zeta,$$

(5.3.16)

and

$$\gamma^{(2)}(t) = \frac{d}{dt}\beta^{(2)}(t)$$

The above expressions give the fluctuations in the phase and amplitude of oscillation due to the disturbing current i_d. The state variables x and $\dot{x}$, representing the normalized inductor current and oscillator voltage respectively (Fig. 5.1), are determined from:

$$x(t) = x^*(t+\phi^{(2)}(t)) + \beta^{(2)}(t) ,$$

and

(5.3.18)

$$\dot{x}(t) = \dot{x}^*(t+\phi^{(2)}(t)) + \gamma^{(2)}(t) ,$$

where $x^*(t)$ and $\dot{x}^*(t)$ are given by expressions (5.2.6), and $\tilde{\phi}^{(2)}(t)$, $\beta^{(2)}(t)$, and $\gamma^{(2)}(t)$ are determined from (5.3.16) and (5.3.17).

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5.4 Example

Let $i_d(t) = a \sin(\omega t+\theta)$, (5.4.1)

where a, ω and θ are constants.

It is well known that the self-oscillations may be either synchronized or quenched by the external frequency, i.e. a stationary response results, when the amplitude a exceeds a critical value depending on the external frequency ω and the parameter ε.

If $|\omega-\Omega| = 0(\varepsilon^k)$, the critical value of a is $0(\varepsilon^k)$, k=1,2,... we are interested in studying the non-stationary oscillations due to small disturbance and we shall therefore restrict the magnitude of a to be one order smaller than its critical value.

Let

$$\omega-\bar{\Omega} = \varepsilon\sigma \ , \quad \text{and } i_d = \varepsilon^2 A \sin(\omega t+\theta), \tag{5.4.2}$$

then using eqns (5.3.13) to (5.3.16), we obtain expressions (5.4.3) and (5.4.4) below for the phase and amplitude fluctuations. In these expressions, the transient terms - which are in this case of little importance, have been omitted.

$$\tilde{\phi}(t) \approx \frac{\varepsilon A}{4\sigma}\cos(\sigma t+\theta) + \frac{\varepsilon^2 A}{8}\left[\cos(\omega+\bar{\Omega})t+\theta + \frac{3}{4\sigma}\sin(\sigma t+\theta)\right]$$
$$+ \frac{\varepsilon^2 A^2}{64\sigma(\sigma^2+1)}\left[\cos 2(\sigma t+\theta) + \frac{2\sigma^2+1}{\sigma}\sin 2(\sigma t+\theta)\right] \tag{5.4.3}$$

$$\beta(t) = -\frac{\varepsilon}{2} A \cos(\bar{\Omega}t+\theta_o+\tilde{\phi}(t)) \left[\frac{\cos(\sigma t+\theta) + \sigma \sin(\sigma t+\theta)}{\sigma^2+1} - \frac{\varepsilon}{2} \sin((\omega+\bar{\Omega})t+\theta)\right]$$

$$+ \varepsilon^2 \cos(\bar{\Omega}t+\theta_o) \left[K_1\{\sin(\sigma t+\theta) - \sigma \cos((\omega+\bar{\Omega})t+\theta)\} + K_2 \cos 2(\sigma t+\theta) + K_3 \sin 2(\sigma t+\theta)\right]$$

$$+ \varepsilon^2 \sin(\bar{\Omega}t+\theta_o) \left[\frac{K_1}{3}\{\cos(\sigma t+\theta) - \sigma \sin(\sigma t+\theta)\} + K_4 \cos 2(\sigma t+\theta) + K_5 \sin 2(\sigma t+\theta)\right]$$

$$+ \frac{\varepsilon^2}{3} K_1 \sin(3(\bar{\Omega}t+\theta)) \left[\cos(\sigma t+\theta) - \sigma \sin(\sigma t+\theta)\right]$$

$$- \frac{\varepsilon^2 A^2}{16(\sigma^2+1)} \left[\frac{3}{2} \cos(\bar{\Omega}t+\theta_o) + \sigma \sin(\bar{\Omega}t+\theta_o)\right] \qquad (5.4.4)$$

where

$$\theta_o = \frac{-3\varepsilon^2}{32\sigma} A \sin\theta - \frac{\varepsilon^2 A^2}{4} \left[(\sigma+\frac{1}{2}) \cos\theta + \frac{1}{16\sigma(\sigma^2+1)} \{\cos 2\theta + \frac{2\sigma^2+1}{\sigma} \sin 2\theta\}\right]$$

$$K_1 = \frac{3}{16} \frac{A}{\sigma^2+1} ,$$

$$K_2 = \frac{A^2}{16} \left[\frac{2}{4\sigma^2+1} + \frac{3}{\sigma^2+1}\left\{\frac{2\sigma^2-1}{4\sigma^2+1} - \frac{\sigma^2-1}{2(\sigma^2+1)}\right\}\right] ,$$

$$K_3 = \frac{-A^2}{16}\left[\frac{1}{\sigma(4\sigma^2+1)} + \frac{3\sigma}{\sigma^2+1}\left\{\frac{3}{4\sigma^2+1} - \frac{1}{\sigma^2+1}\right\}\right] ,$$

$$K_4 = \frac{3}{16}\frac{\sigma A^2}{(\sigma^2+1)(4\sigma^2+1)} ,$$

and

$$K_5 = \frac{A^2}{16}\frac{2\sigma^2-1}{(\sigma^2+1)(4\sigma^2+1)}.$$

The frequency pulling is given by:

$$\Delta\Omega = \overline{\Omega}-\Omega = \frac{\varepsilon^3 A^2}{32}\frac{1}{\sigma(1+\sigma^2)} = \frac{a^2}{32}\frac{1}{(\omega-\overline{\Omega})\ [1+(\frac{\omega-\overline{\Omega}}{\varepsilon})^2]} \qquad (5.4.5)$$

It is of interest to compare this expression with the result obtained by Outram (1967). Outram studied the frequency pulling of the v.d.Pol oscillator experimentally and, for the case of sinusoidal forcing functions, obtained the empirical result:

$$\Delta\Omega \propto \frac{a^2}{(\omega-1)^3} .$$

We note that for the cases studied by Outram, the condition:

$$[\frac{\omega-1}{\varepsilon}]^2 >> 1 ,$$

holds true, in which case expression (5.4.5) reduces to:

$$\Delta\Omega \approx \frac{\varepsilon^2}{32}\frac{a^2}{(\omega-\overline{\Omega})^3} \approx \frac{\varepsilon^2}{32}\frac{a^2}{(\omega-1)^3} ,$$

which is in qualitative agreement with Outram's result.

The spectrum

The oscillator voltage is given by:

$$\dot{x}(t) = \dot{x}^*(t+\phi(t) + \gamma(t) \qquad (5.4.6)$$

$$= \dot{x}^*(\bar{\Omega}t + \theta_0 + \tilde{\phi}(t)) + \gamma(t)$$

where $\gamma(t) = \frac{d}{dt}\beta(t)$ and $\dot{x}^*(\cdot)$ is given by (5.2.6).

Using the expressions for $\tilde{\phi}(t)$ and $\beta(t)$ in (5.4.3) and (5.4.4), then:

$$\dot{x}(t) = A_0 \cos(\bar{\Omega}t+\theta_0) + B_0 \sin(\bar{\Omega}t+\theta_0) + \frac{\varepsilon}{4}\cos 3(\bar{\Omega}t+\theta_0+\frac{\varepsilon}{4})+\frac{5\varepsilon^2}{96}\sin 5(\bar{\Omega}t+\theta_0)$$

$$+ A_1 \cos(\omega t+\theta) + B_1 \sin(\omega t+\theta)$$

$$+ A_2 \cos((2\bar{\Omega}-\omega)t-\theta) + B_2 \sin((2\bar{\Omega}-\omega)t-\theta)$$

$$+ A_3 \cos((2\omega-\bar{\Omega})t+2\theta) + B_3 \sin((2\omega-\bar{\Omega})t+2\theta)$$

$$+ A_4 \cos((3\bar{\Omega}-2\omega)t-2\theta) + B_4 \sin((3\bar{\Omega}-2\omega)t-2\theta)$$

$$+ A_5 \cos((2\bar{\Omega}+\omega)t+\theta) + B_5 \sin((2\bar{\Omega}+\omega)t+\theta)$$

$$+ A_6 \cos((4\bar{\Omega}-\omega)t-\theta) + B_6 \sin((4\bar{\Omega}-\omega)t-\theta) \qquad (5.4.7)$$

where

$$A_0 = -\frac{\varepsilon}{4} + \frac{\varepsilon^2 A^2}{16}\frac{(1-\sigma^2)}{\sigma(1+\sigma^2)}, \quad B_0 = -\frac{\varepsilon^2 A^2}{32}\frac{(4\sigma^2+1)}{\sigma^2(\sigma^2+1)},$$

$$A_1 = -\frac{\varepsilon A}{4}\frac{1}{(\sigma^2+1)}\left[\frac{2\sigma^2+1}{\sigma} + \frac{\varepsilon}{2}(2\sigma^2-1)\right],$$

$$B_1 = \frac{\varepsilon A}{4(1+\sigma^2)}\left[1 + \frac{\varepsilon}{4}\frac{4\sigma^2-1}{\sigma}\right],$$

$$A_2 = \frac{-\varepsilon A}{4(1+\sigma^2)}\left[\frac{1}{\sigma} + \varepsilon(\sigma^2+\frac{1}{4})\right],$$

$$B_2 = \frac{\varepsilon A}{4}\frac{1}{(1+\sigma^2)}\left[1 + \frac{\varepsilon}{2\sigma}\right],$$

$$A_3 = \frac{\varepsilon^2 A^2}{64} \frac{(2\sigma^2-1)(7\sigma^2+1)}{\sigma(\sigma^2+1)^2(4\sigma^2+1)}$$

$$B_3 = \frac{\varepsilon^2 A^2}{32} \frac{(2\sigma^4 - 8\sigma^2 - 1)}{(1+\sigma^2)^2(1+4\sigma^2)}$$

$$A_4 = \frac{3\varepsilon^2 A^2}{64} \left[\frac{2\sigma^4 + 9\sigma^2 + 1}{\sigma(\sigma^2+1)^2(4\sigma^2+1)}\right]$$

$$B_4 = -\frac{\varepsilon^2 A^2}{32} \left[\frac{2\sigma^6 - 4\sigma^4 + 4\sigma^2 - 1}{\sigma^2(\sigma^2+1)^2(4\sigma^2+1)}\right]$$

$$A_5 = \frac{\varepsilon^2 A}{4} \left[1 + \frac{3}{8}\frac{1}{\sigma^2+1}\right] ,$$

$$B_5 = -\frac{3\varepsilon^2 A}{32} \frac{2\sigma^2+1}{\sigma(\sigma^2+1)}$$

$$A_6 = -\frac{3\varepsilon^2 A}{32} \frac{1}{\sigma^2+1}$$

and

$$B_6 = -\frac{3\varepsilon^2}{32} \frac{A}{\sigma(\sigma^2+1)}$$

Now

$$\sqrt{A_o^2 + B_o^2} = 2 + \frac{\varepsilon^2}{64} \left[1 - 2A^2 \frac{(4\sigma^2+1)}{\sigma^2(\sigma^2+1)}\right]$$

i.e. the change in the amplitude of the fundamental component of self-oscillation due tothe disturbing force is:

$$\Delta A_\varepsilon = -\frac{\varepsilon^2 A^2}{32} \frac{1+4\sigma^2}{\sigma^2(1+\sigma^2)} \qquad (5.4.8)$$

Now if

$$i_d = \epsilon^2 \sum_{n=1}^{N} A_n \sin(\omega_n t + \theta_n) \quad ,$$

where $|\omega_n - \Omega| = O(\epsilon)\ \forall n$, and $\|i_d\| = O(\epsilon^2)$, then it can be verified that the first approximations of $\tilde{\phi}(t)$ and $\beta(t)$ take the form:

$$\tilde{\phi}^{(1)}(t) = \frac{\epsilon}{4} \sum_{n=1}^{N} \frac{A_n}{\sigma_n} \cos(\sigma_n t + \theta_n) \qquad (5.4.9)$$

and

$$\beta^{(1)}(t) = -\frac{\epsilon}{2} \cos(\bar{\Omega} t + \tilde{\phi}^{(1)}(t)) \sum_{n=1}^{N} [\frac{A_n}{\sigma_n^2+1} \{ \cos(\sigma_n t+\theta_n)+\sigma_n \sin(\sigma_n t+\theta_n)\}],$$

$$(5.4.10)$$

and that expressions (5.4.5) and (5.4.8) becomes:

$$\Delta\Omega = \bar{\Omega}-\Omega = \frac{\epsilon^3}{32} \sum_{n=1}^{N} \frac{A_n^2}{\sigma_n(1+\sigma_n^2)} \qquad (5.4.11)$$

and

$$\Delta A_s = -\frac{\epsilon^2}{32} \sum_{n=1}^{N} \frac{A_n^2(1+4\sigma_n^2)}{\sigma_n^2(1+\sigma_n^2)} \qquad (5.4.12)$$

Numerical verification

Eqn. (5.3.1), with $i_d = a \sin \omega t$, has been solved numerically on a digital computer for $\epsilon=0.1$ and for different values of a and ω. The solution was carried out for a period of 450 cycles after the initial transients have faded away. The instants of positive-going zero-crossings, denoted by T_i, i=1 to 451, were then determined by high order interpolation. The average frequency over each cycle, defined as:

$$\Omega_i = \frac{2\pi}{T_{i+1}-T_i} \quad , \quad i = 1,2,\ldots, 450 \; ,$$

were then determined. The fluctuations in Ω_i, due to the frequency modulation were filtered out to obtain the mean frequency over the entire solution period and hence the frequency pulling. The numerical results thus obtained are compared with the analytical results given by equation (5.4.5) [which is solved by one iteration].

The Fourier series coefficients of $\dot{x}$(which represents the normalized oscillator voltage) were also calculated numerically over a period of 128 cycles and the results are compared with those given by expression (5.4.7).

The numerical and analytical results are tabulated below.

The Spectrum	Amplitude of the forcing sinsuoid					
	a = 0.025		a = 0.025√2		a = 0.05	
Amplitude / Frequency	Analytical	Numerical	Analytical	Numerical	Analytical	Numerical
ω	0.227	0.225	0.321	0.318	0.453	0.446
$2\bar{\Omega}-\omega$	0.185	0.187	0.262	0.261	0.370	0.354
$2\omega-\bar{\Omega}$	0.0033	0.0036	0.0065	0.0071	0.013	0.014
$3\bar{\Omega}-2\omega$	0.021	0.019	0.042	0.036	0.084	0.068
$2\bar{\Omega}+\omega$	0.0092	0.0084	0.013	0.011	0.018	0.015
$4\bar{\Omega}-\omega$	0.0071	0.0066	0.010	0.009	0.014	0.011
ΔA_s	-0.025	-0.025	-0.051	-0.050	-0.101	-0.100

Table 5.1 The spectrum of the disturbed oscillator - analytical versus numerical results, $\omega = \frac{31}{32}\,\Omega$.

The Spectrum	Amplitude of the forcing sinusoid					
	$a = 0.025$		$a = 0.025\sqrt{2}$		$a = 0.05$	
Amplitude / Frequency	Analytical	Numerical	Analytical	Numerical	Analytical	Numerical
ω	0.224	0.222	0.316	0.313	0.447	0.439
$2\bar{\Omega}-\omega$	0.197	0.186	0.279	0.259	0.394	0.353
$2\omega-\bar{\Omega}$	0.0033	0.0042	0.0065	0.0083	0.013	0.017
$3\bar{\Omega}-2\omega$	0.021	0.019	0.042	0.036	0.084	0.067
$2\bar{\Omega}+\omega$	0.0092	0.0078	0.013	0.011	0.018	0.014
$4\bar{\Omega}-\omega$	0.0072	0.0065	0.010	0.0087	0.014	0.011
ΔA_S	-0.025	-0.026	-0.051	-0.050	-0.101	-0.101

Table 5.2 The spectrum of the disturbed oscillator - analytical versus numerical results, $\omega = \frac{33}{32}\Omega$.

a	$\frac{a}{\hat{a}}$ †	$\Delta\Omega$	
		Analytical	Numerical
0.025	0.190	0.58×10^{-3}	0.61×10^{-3}
$0.025\sqrt{2}$	0.268	1.19×10^{-3}	1.28×10^{-3}
0.05	0.379	2.49×10^{-3}	2.37×10^{-3}
$0.05\sqrt{2}$	0.536	5.46×10^{-3}	5.60×10^{-3}

† a is the amplitude of the forcing sinusoid and $\hat{a}$ is the critical value of a at which the self-oscillations are suppressed.

Table 5.3 The frequency pulling - analytical versus **numerical results,** $\omega = \frac{33}{32}\Omega$.

5.5 The case of the oscillator regulated by a negative non-linear resistance charcterized by a fifth order polynomial.

In the above analysis, the amplitude stabilizing element of the self-oscillator (the negative non-linear resistance) is assumed to have the characteristic:

$$i_r = a_1 v + a_2 v^3 \quad ,$$

with $a_1 < 0$ and $a_2 > 0$ [See Fig. 5.1].

In some cases the cubic polynomial approximation is not an adequate one and a higher order polynomial must be used. For example, Hester (1968) has shown that specific classes of self-oscillators utilising bipolar transistors can be described by:

$$\ddot{x} + x + \mu[k \sinh(a+b)\dot{x} - \sinh a\dot{x}] + \alpha\dot{x} = 0 \tag{5.5.1}$$

where μ, k, a, b and α are positive constants. The equivalent negative non-linear resistance, when using the configuration in Fig. 5.1, is characterized by:

$$i_r = \mu[k \sinh(a+b) v - \sinh av] + \alpha v \quad , \tag{5.5.2}$$

where $v \triangleq \dot{x}$ is the oscillator voltage and i_r is the current in the negative non-linear resistance.

Hester used topological methods to solve eqn. (5.5.1). In its present form, eqn. (5.5.1) does not lend itself easily to analytical solutions.

The solution can be somewhat simplified if the characteristic in (5.5.2) is adequately approximated by a polynomial with a small number of terms. For typical values of the parameters in (5.5.2), i_r can be adequately approximated by an odd fifth order polynomial in v. The osillator's behaviour is then described by the differential equation in (5.1.1). When eqn. (5.1.1) is normalized in the form in (5.12) or (5.1.3), then the steady state solution is given approximately by:

$$x^* = 2\cos\tau$$

where $\tau \triangleq \hat{\Omega}t + k, \hat{\Omega} = 1 + 0(\varepsilon^2)$ and k is a constant.

The variational response is then given to the first approximation by:

$$u = (\beta_i \sin\tau_i + \gamma_i \cos\tau_i)\sin\tau + (\beta_i \cos\tau_i - \gamma_i \sin\tau_i)\, e^{-\varepsilon^*(\tau-\tau_i)} \cos\tau \quad ,$$

where β_i, γ_i and $\dot{u}$ are as defined in (5.2.2) and (5.2.3), $\tau_i = \hat{\Omega}t_i + k$ and:

$$\varepsilon^* = \frac{1}{2}\varepsilon_1 + \frac{3}{2}\varepsilon_2 + 5\varepsilon_3 \quad ,$$

where ε_1, ε_2 and ε_3 are defined in (5.1.2).

Following the same procedure described in Sections 5.3 and 5.4, it can be shown that expressions (5.4.9) to (5.4.12) for $\tilde{\phi}^{(1)}(t)$, $\beta^{(1)}(t)$, $\Delta\Omega$ and ΔA_s respectively hold true in this more general case with ε replaced by ε^*.

CHAPTER 6

Experimental investigation

6.1 Introduction

Some of the theoretical results obtained in the previous chapters have been verified by numerical solutions of the pertinent differential equations. In this chapter, the frequency variation of a self-oscillator due to disturbance by an external sinusoidal signal is studied experimentally.

We are interested in measuring the effect of the external signal on the instantaneous and mean freqeuncy of self-oscillation. In this experiment, the mean frequency is measured by counting the number of zero-crossings of the oscillator voltage during a period of 10 seconds. The oscillator frequency is approximately 50 KHZ, thus each test sample contains approximately 500000 cycles. The results thus obtained are therefore more satisfying than those obtained in section 5.6 by numerical simulation and averaging over a period of approximately 450 cycles (which consumed an excessively large computational time).

The frequency fluctuations about the mean value are measured by a phase-locked loop (PLL) F-M detector.

The test oscillator approximates closely the v.d.Pol oscillator. The experimental circuit is shown schematically in Fig. 6.1 . Before

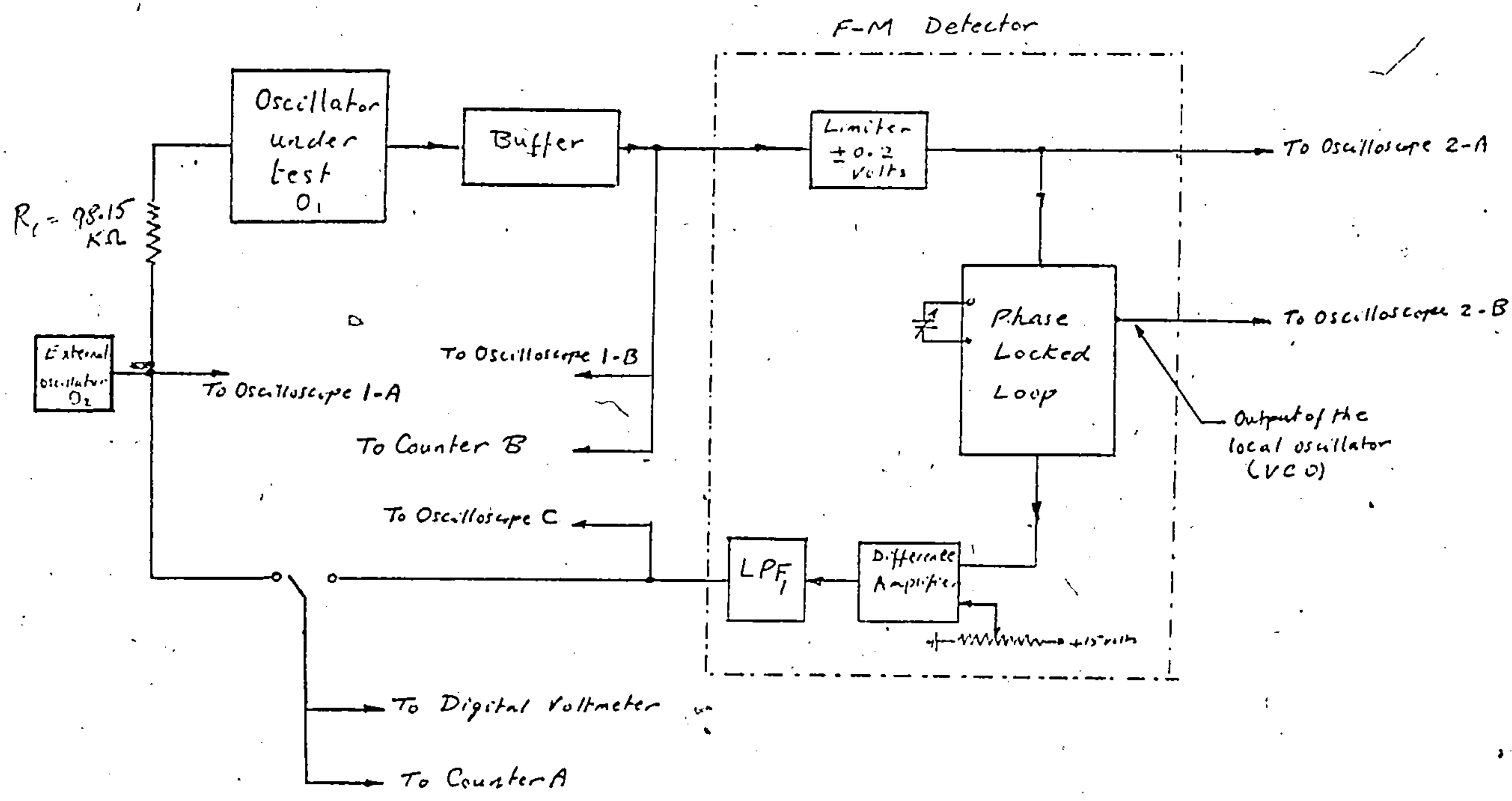

Fig.6.1 Schematic of the Experimental Circuit .

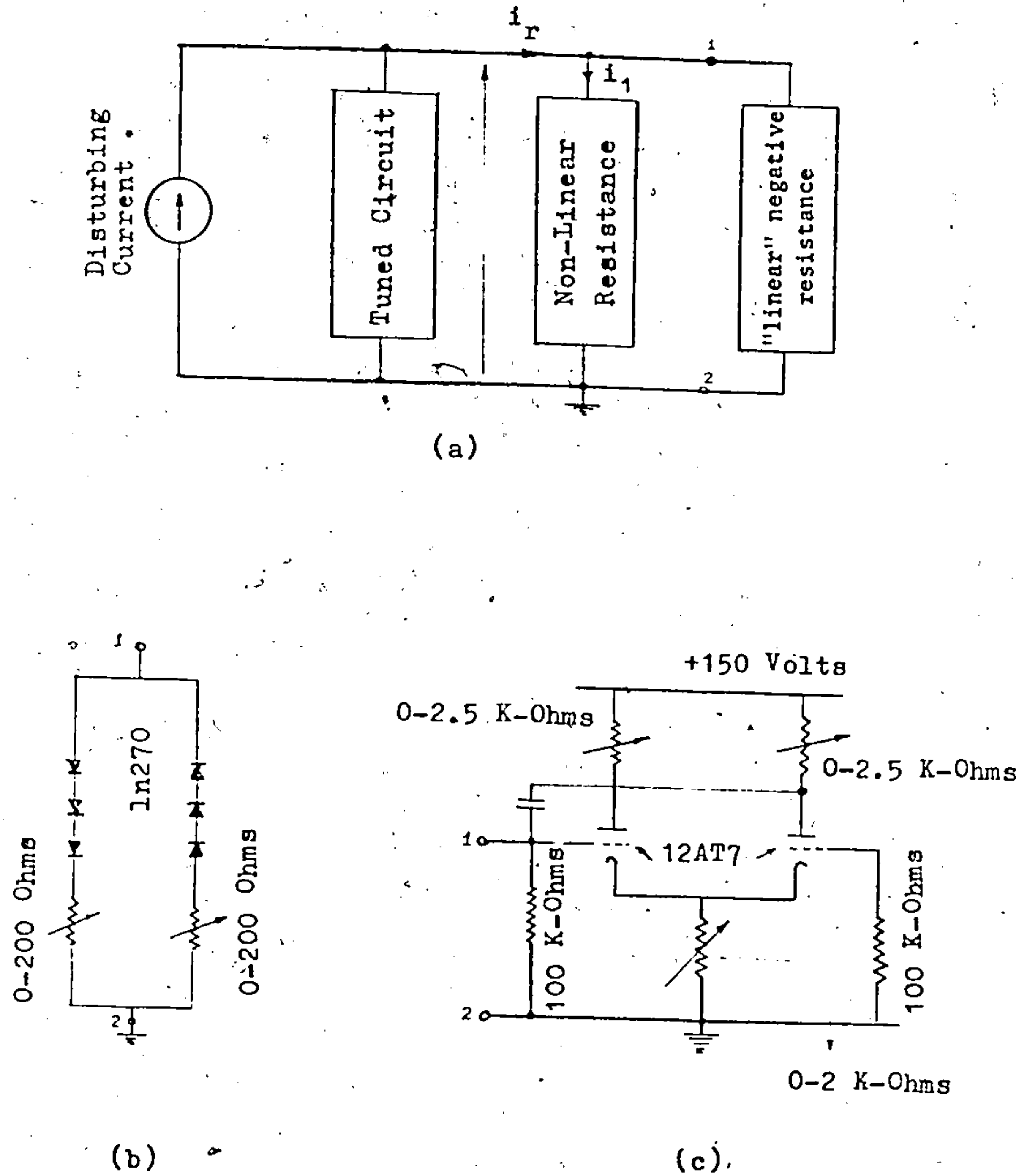

Fig.6.2. Circuit of the oscillator under test.

(a) The Oscillator Circuit

(b) The Non-Linear Resistance

(c) The "Linear" Negative Resistance

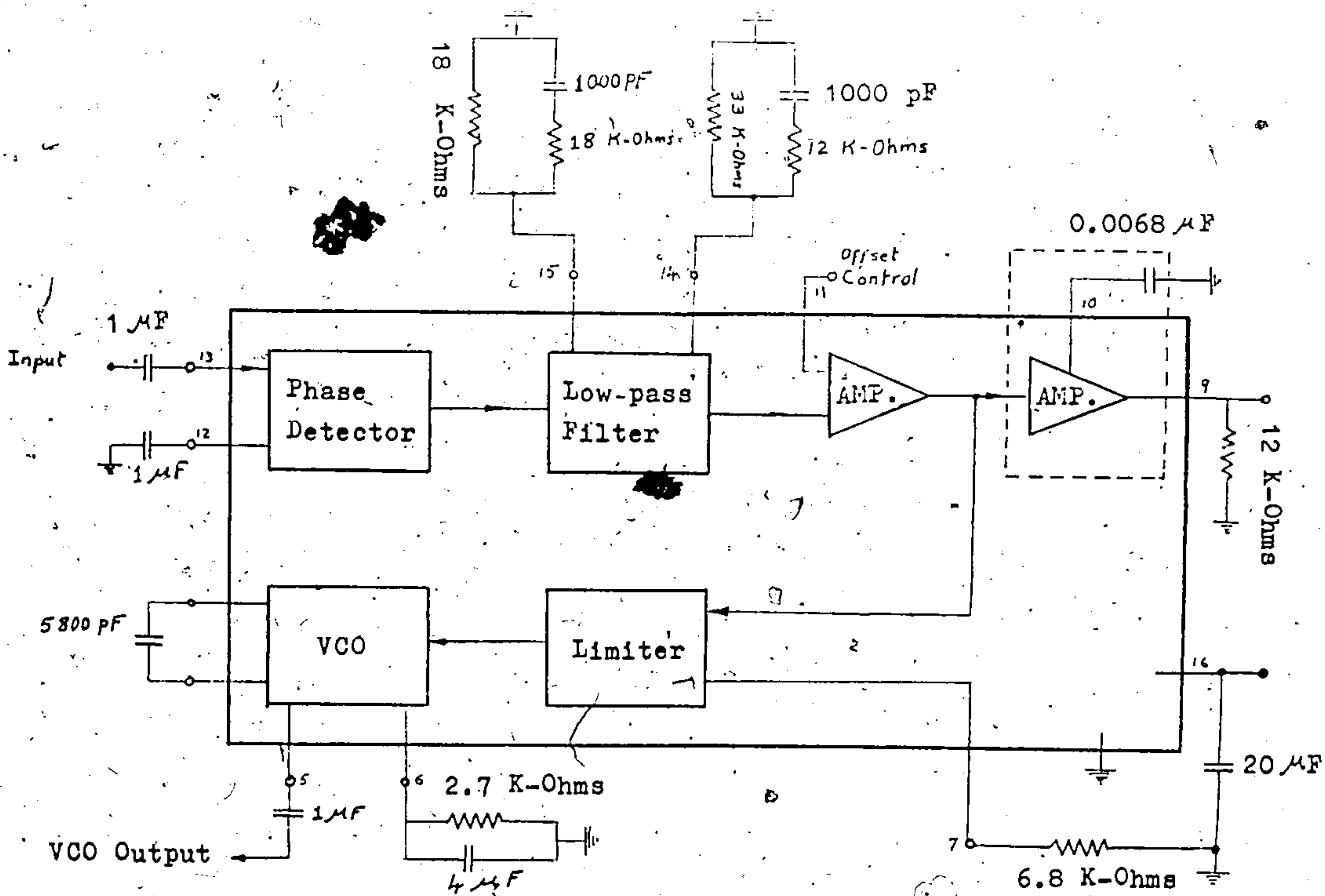

Fig. 6.3 NE561B PLL Circuit

any measurements are taken, the amplitude scale of oscilloscope c was calibrated against the digital voltmeter. Also counter B was calibrated against counter A, by applying a sinusoidal signal of approximately 50 KHZ simultaneously to both, to account for a possible discrepancy between their gating periods. The readings of the two counters were found to be equal, within a random error of ± 0.1 HZ over a gating period of 10 seconds.

6.2 The experimental circuit

The oscillator circuit is shown in Fig. 6.2a . The tuned circuit is a simple parallel resonant L-C circuit (shielded). The non-linear resistance is realized as shown in Fig. 6.2b where the variable linear resistances are used to adjust the i_1-v characteristic (displayed on a calibrated oscilloscope) so as to approximate closely the desired cubic characteristic:

$$i_1 \propto v^3$$

The "linear" negative resistance is realized by the feedback network of Fig. 6.2c . It should be noted that the oscillator frequency is sensitive to variations in the negative resistivity and therefore considerable care must be taken in implementing this stage. Although, with careful compensation against drift, transistors may be used to implement this stage, it was decided to use the "out-moded" vacuum tubes due to their excellent stability after a sufficient warm-up period (a few hours).

The F-M detector circuit [Figs. 6.1 and 6.3] is composed of a limiter, a phase locked loop, a D-C difference amplifier and a simple R-C filter.

6.3 Measurement of the parameters of the oscillator circuit

The values of L and C of the tuned circuit [Fig. (6.2)] are:

$$L \approx 0.95 \text{ mH} \quad \text{and} \quad C \approx 10200 \text{ pF}$$

The Q-factor (not needed) of the tuned circuit, when disconnected from the oscillator circuit, was found to be: $Q \approx 42$.

The dynamic characteristic of the negative non-linear resistance is closely approximated by:

$$i_r = - av + bv^3 \tag{6.3.1}$$

The parameters a and b must be measured when the oscillator is connected to the rest of the experimental circuit.

It is well known that the amplitude A_1 of the free-oscillation is given to the second order of approximation by:

$$A_1 = 2\sqrt{\frac{a}{3b}} \tag{6.3.2}$$

The value of A_1, measured by the digital voltmeter, was found to be 0.607 volts. Hence:

$$\sqrt{\frac{3b}{a}} = 3.3 \text{ volt}^{-1}$$

where a is in mhos and b is in mhos/volt^2.

The well-known theoretical value of the frequency of free-oscillation is:

$$f_{free} = (1 - \frac{\varepsilon^2}{16})\ f_0 \qquad (6.3.3)$$

where $\varepsilon = a\sqrt{\frac{L}{C}}$ (dimensionless), and f_0 is the resonance frequency of the tuned circuit under working conditions. The value of f_0 is slightly different from the resonance frequency of the isolated tuned circuit due to the effect of the input and output capacitances of the feedback network in the oscillator circuit as well as the input capacitances of the buffer amplifier (Fig. 6.1).

From (6.3.3):

$$\varepsilon = 4\sqrt{\frac{f_0 - f_{free}}{f_0}} \qquad (6.3.4)$$

The values of f_0 and f_{free} differ very slightly, and since it is very difficult to measure the value of f_0 with a sufficiently high accuracy, it is not suitable to determine ε from (6.3.4).

If a resistance R(Ohms) is connected in parallel with the tuned circuit, and if $\frac{1}{R} < a$, so that the self-oscillations are not destroyed, then the free frequency becomes:

$$f^*_{free} = [1 - \frac{L}{16C}(a - \frac{1}{R})^2]\ f_o \ .$$

Hence

$$\Delta f_{free} = f^*_{free} - f_{free} = \frac{f_0 L}{8RC}(a - \frac{1}{2R}) > 0 \ ,$$

and the value of ε can be determined with sufficient accuracy from:

$$\varepsilon = a\sqrt{\frac{L}{C}} = \frac{1}{2R}\sqrt{\frac{L}{C}} + 8R\sqrt{\frac{C}{L}}\,\frac{\Delta f_{free}}{f_{free}} \qquad (6.3.5)$$

Several values of R, in the range from 6.8 to 24 KΩ, were connected separately across the tuned circuit and the relative increase in the frequency of free oscillation was measured in each case. The value of ε was taken as the average of the slightly differing values obtained in each case from eqn. (6.3.5).

The value of ε thus determined is:

$$\varepsilon \approx 0.21 \qquad (6.3.6)$$

As a test of the accuracy of the above measurements, we shall compare the experimental values of the amplitude of the forcing sinusoid which is just sufficient to lock or quench the self-oscillations with those obtained from well established theoretical analysis.

In the normalized equation (5.3.1), let

$$i_d = \varepsilon F \cos(\omega t + \phi) \qquad \text{(dimensionless)},$$

where F, ω and ϕ are constants. For values of ω close to the free frequency Ω,† the well-known conditions of stability of the entrained periodic oscillation are:

$$A^2 \geq 2$$

and

$$\left(\frac{3}{4}A^2 - 1\right)\left(\frac{A^2}{4} - 1\right) + \sigma^2 \geq 0 \qquad (6.3.7)$$

---

† ω and Ω are dimensionless frequencies defined as :

$\omega = f_2 / f_0$ and $\Omega = f_{free} / f_0$ where f_2 is the frequency of the injected current in Hz , f_{free} is the free frequency of oscillation in Hz , and f_0 is the resonance frequency in Hz of the tuned circuit under working conditions .

where A is the amplitude of the periodic response and σ is the detuning defined as:

$$\sigma = \frac{1}{\varepsilon} \frac{\omega^2 - \Omega^2}{\omega\Omega} \tag{6.3.8}$$

To a first approximation, A is determined from:

$$A \sqrt{\sigma^2 + (\frac{A^2}{4} - 1)^2} = F \quad , \tag{6.3.9}$$

where A is subject to the constraints in (6.3.7).

From (6.3.7) and (6.3.9), the amplitude F sufficient to lock or quench the self-oscillation is determined from:

$$F \geq \sqrt{\frac{8}{3}\sigma^2 + \frac{8}{27}[1-(1-3\sigma^2)^{3/2}]} \quad \text{for} \quad |\sigma| < \frac{1}{2} \tag{6.3.10}$$

or

$$F > \sqrt{2\sigma^2 + \frac{1}{2}} \quad \text{for} \quad |\sigma| > \frac{1}{\sqrt{3}}, \quad |\sigma| = 0(1) \tag{6.3.11}$$

In the narrow range of detuning $\frac{1}{2} \leq |\sigma| \leq \frac{1}{\sqrt{3}}$, the "jump phenomenon" takes place and either of expressions (6.3.10) or (6.3.11) apply. This range has been avoided in the experimental measurements.

If I_d is the peak value of the forcing sinusoidal current in amperes, then.

$$I_d = \varepsilon F \Big/ \sqrt{\frac{3b}{a} \frac{L}{C}} \tag{6.3.12}$$

The critical value of I_d, determined from the equalities in (6.3.10) and (6.3.11) with the use of the denormalizing relation (6.3.12), are compared with the experimental value in table 6.1.

Forcing Frequency $f = \frac{\omega}{2\pi} f_o$	Free Frequency $f_{free} = \frac{\Omega}{2\pi} f_o$	σ	R.M.S. voltage of external oscillator $\frac{A_2}{\sqrt{2}}$	Amplitude of the forcing sinusoidal current in milli Amperes	
				Experimental	Theoretical
Hz	Hz		volts	mA	mA
6243.	49738.	-0.695	17.84	0.257	0.253
7335.	49737.	-0.472	13.58	0.196	0.191
8020.	49740.	-0.335	9.65	0.139	0.138
8788.	49784.	-0.192	5.72	0.082	0.080
9300.	49786.	-0.093	2.80	0.040	0.039
0247.	49752.	0.094	2.73	0.039	0.039
0822.	49767.	0.200	5.89	0.085	0.083
1300.	49760.	0.290	8.44	0.122	0.120
1769.	49760.	0.377	10.80	0.156	0.154
2287.	49753.	0.473	13.16	0.190	0.191
3271.	49749.	0.652	16.15	0.233	0.243

Table 6.1 Comparison of the experimental and theoretical values of the critical amplitude of the forcing sinusoid at which frequency entrainment takes place. The theoretical values are determined from the well-known expressions (6.3.10) or (6.3.11).

We may note that a simpler method for measuring the parameter a in (6.3.1), and hence ε, is to connect different resistors across the tuned circuit and measure the corresponding amplitude of "free" oscillation. The amplitude A (Volts) is then given to the second order of approximation by :

$$A^2 = \frac{4}{3b}\left(a - \frac{1}{R}\right), \quad \frac{1}{R} < a \quad ,$$

where R is the shunt resistance in Ohms .

Plotting A^2 versus 1/R , then the best fitted straight line gives the intercept on the 1/R axis as a . However , due to an oversight , this method was not used at the time the experiment was performed .

6.4 Measurement of the transfer characteristic of the F-M detector

Refer to Fig. 5.1 . The output level of the external oscillator 0_2 is chosen such that the frequency of the oscillator under test 0_1 is entrained by the frequency of 0_2 in the frequency range from 48 to 52 KHz (the free running frequency of 0_1 is approximately 49.8 KHz). The occurrence of the frequency entrainment is checked by observing the outputs of 0_1 and 0_2 on oscilloscope 1 and also by ensuring that counters A and B give similar readings.

The tuning capacitor of the PLL is adjusted so that the tracking range is centred at approximately 50 KHz. The parameters of the PLL have been chosen to yield a wide tracking range of approximately 5 KHz. The DC level at input 2 of the difference amplifier is adjusted such that the output DC level is approximately at zero voltage when the PLL is locked at approximately 50 KHz. The frequency of 0_2 is then varied in steps in the range from 48 to 52 KHz. The frequency of 0_1 (or 0_2) and the output of the difference amplifier are measured to obtain the static transfer characteristic shown in Fig. 6.4 .

The input impedance at terminal 10 in Fig. (6.3) is approximately 8000 Ohm (resistive). The magnitude of the transfer function of the low pass filter composed of LPF_1 and LPF_2 [Figs. 6.1 and 6.3] is given by:

$$|H(j\gamma)| = \frac{1}{\sqrt{(1+\gamma^2\tau_1^2)(1+\gamma^2\tau_2^2)}} , \qquad (6.4.1)$$

where τ_1 = 54 μsec and τ_2 = 33 μsec. γ is the angular frequency in radions/second of the frequency modulating function and is given by :

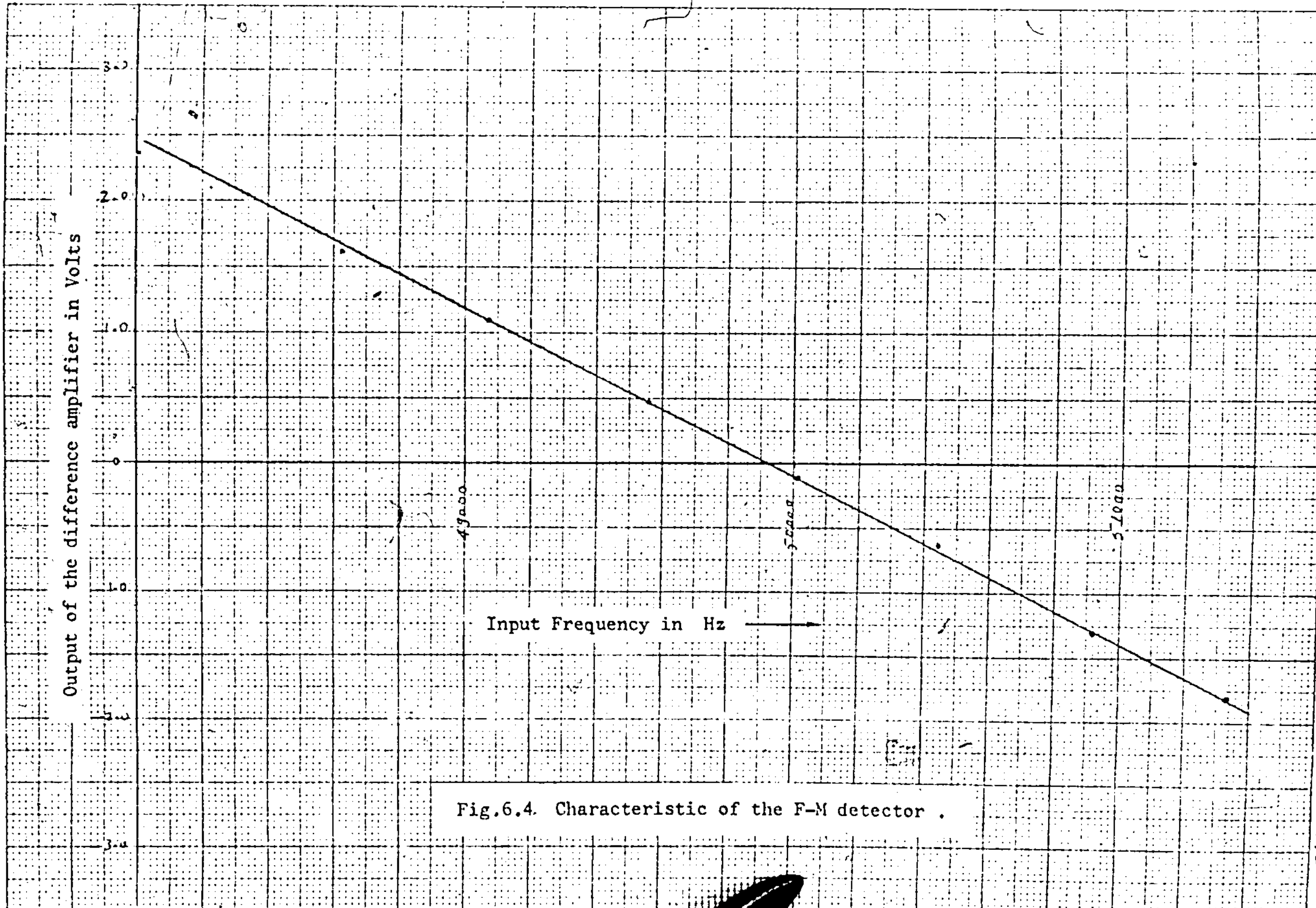

Fig.6.4. Characteristic of the F-M detector .

γ = angular frequency of the disturbing current - mean angular frequency of the oscillator under test.

The slope of the transfer characteristic of Fig. (6.4) is: 780 Hz/volt.

Note that the output voltage of the F-M detector must be multiplied by the correction factor 1/ H(j)

6.5 The experimental procedure

A sufficient period of time (a few hours) is allowed till high frequency stability of the oscillator under test 0_1 is attained. With the output of the external oscillator 0_2 at zero voltage, the amplitude of the free running oscillation of 0_1 is measured by the digital voltmeter. This measurement is needed for determining the ratio b/a [eqn. (6.3.2)].

For a specific forcing frequency (of 0_2), close to the frequency of free oscillation of 0_1, the output level of 0_2 is increased in small steps then the external frequency and the mean frequency of 0_1 are measured by counters A and B respectively. After each measurement, the output level of 0_2 is reduced to zero and the free running frequency of 0_1 is measured to ensure that only small random variations in the free frequency take place. The procedure is repeated for different values of the forcing frequency.

The measurements of the frequency modulation were carried out separately. The amplitudes of the almost sinusoidal frequency fluctuations were measured for different values of the amplitude and frequency of the forcing sinusoid.

6.6 Comparison of the experimental and theoretical results

The frequency pulling:

Let A_2 and f_2 be the amplitude and frequency of the output of O_2 in volts and Hz respectively.

The amplitude of the disturbing current $= \frac{A_2}{R_f}$ Amperes, where R_f [Fig. 6.1] is in Ohms.

In eqn. (5.45):

$$\omega = \frac{f_2}{f_{free}} \quad , \quad \bar{\Omega} = \frac{f_1}{f_{free}} \quad , \quad \Delta\Omega = \frac{f_1 - f_{free}}{f_{free}} \quad \text{and}$$

$$a = \frac{A_2}{R_f} \sqrt{\frac{3b}{2} \frac{L}{C}} \tag{6.6.1}$$

Using the measurements in section (6.3), then:

$$\text{Frequency pulling} = f_1 - f_{free} = \frac{3.275 \times 10^{-6} f_{free}^2 A_2^2}{(f_2-f_1)[1+(\frac{f_2-f_1}{f_1\varepsilon})^2]} \tag{6.6.2}$$

The value of f_1 in the r.h.s. of eqn. (6.6.2) may be approximated by f_{free} and the solution is then improved by an iteration.

The frequency pulling determined from expression (6.6.2) is compared with the experimental values in Tables 6.2 to 6.8 .

The frequency modulation

From eqn. (5.4.3), the normalized phase fluctuation is given to the first approximation by:

$$\varepsilon\tilde{\phi}(t) \approx \frac{\varepsilon^2 F}{4\sigma} \cos(\sigma t+\theta) = \frac{a}{4\sigma} \cos(\sigma t+\theta) .$$

Hence, the normalized frequency modulating function is given by:

$$\Delta f = \frac{d}{dt} \varepsilon\tilde{\phi}(t) \approx - \frac{a}{4} \sin(\sigma t+\theta)$$

$$= - 2.56 \times 10^{-3} A_2 \sin(\sigma t+\theta) \qquad \text{(dimensionless)}$$

The peak value of the frequency modulating function in Hz is given by:

$$\Delta f_{peak} = 2.56 \times 10^{-3} A_2 f_{free} \approx 128 A_2 \text{ Hz} \qquad (6.6.3)$$

The values of Δf_{peak} determined from (6.6.3) are compared with the experimental values in Tables 6.9 - 6.11 .

$\frac{A_2}{\sqrt{2}}$ volts	$\frac{A_2}{A_2^*}$ †	f_1 Hz	f_{free} Hz	f_2 Hz	Frequency Pulling: f_1-f_{free} Experimental	Frequency Pulling: f_1-f_{free} Theoretical [eqn.(6.6.2)	Relative Error ††
0.41	0.150	49756.8	49752.2	50250.0	4.6	5.5	0.200
0.60	0.220	49763.9	49751.7	50249.2	12.2	12.0	-0.017
0.81	0.297	49773.9	49751.9	50249.1	22.0	22.3	0.014
1.02	0.374	49787.1	79751.1	50248.8	36.0	36.3	0.008
1.20	0.440	49801.5	49751.1	50248.1	50.4	51.8	0.027
1.41	0.516	49822.5	49751.4	50247.6	71.1	74.6	0.049
1.60	0.586	49846.0	49751.4	50247.6	94.6	100.4	0.061
1.81	0.663	49876.6	49751.0	50248.0	125.6	135.9	0.082

Table 6.2 Frequency Pulling - Experimental vs Theoretical Results

For Tables 6.2-6.11:

† A_2^* is the minimum value of A_2 (determined experimentally) at which entrainment of the self-oscillations by the external oscillations takes place .

†† The relative error is defined as: $\frac{\text{theoretical result - experimental result}}{\text{theoretical result}}$

$\frac{A_2}{\sqrt{2}}$ volts	$\frac{A_2}{A_2^*}$ †	f_1 Hz	f_{free} Hz	f_2 Hz	Frequency Pulling: $f_1 - f_{free}$		Relative Error ††
					Experimental	Theoretical [eqn.(6.6.2)]	
0.50	0.085	49772.6	49769.0	50816.4	3.6	3.8	0.069
0.76	0.129	49777.1	49768.7	50816.0	8.4	8.9	0.064
1.01	0.171	49783.8	49768.4	50817.0	15.4	15.9	0.030
1.27	0.216	49792.3	49768.1	50816.8	24.2	25.3	0.046
1.52	0.258	49802.9	49768.4	50817.0	34.5	36.7	0.062
1.76	0.299	49815.8	49769.0	50816.8	46.8	49.8	0.064
2.00	0.340	49828.8	49768.2	50817.0	60.6	65.1	0.075
2.51	0.426	49866.2	49767.6	50818.1	98.6	106.2	0.077
2.75	0.467	49887.3	49768.1	50817.7	119.2	130.3	0.093
3.01	0.511	49912.4	49767.7	50818.0	144.7	160.1	0.106
3.29	0.559	49943.4	49767.8	50818.0	175.6	197.1	0.122
3.53	0.599	49974.0	49767.7	50818.0	206.3	233.6	0.132
3.75	0.637	50004.4	49767.6	50819.4	236.8	270.8	0.144

Table 6.3 Frequency Pulling - Experimental vs Theoretical Results

$\frac{A_2}{\sqrt{2}}$ volts	$\frac{A_2}{A_2^*}$ †	f_1 Hz	f_{free} Hz	f_2 Hz	Frequency Pulling: $f_1 - f_{free}$		Relative Error ‡†
					Experimental	Theoretical [eqn.(6.6.20)]	
0.50	0.059	49763.1	49760.4	51295.0	2.7	2.6	-0.041
1.01	0.120	49770.7	49760.3	51295.0	10.4	10.6	0.022
1.50	0.178	49783.4	49760.3	51295.1	23.1	23.6	0.024
2.00	0.237	49801.4	49760.4	51296.2	41.0	42.5	0.038
2.50	0.296	49825.2	49760.1	51296.1	65.1	67.6	0.038
3.02	0.358	49855.0	49760.0	51297.6	95.0	100.6	0.058
3.50	0.415	49890.7	49760.7	51296.4	130.0	138.5	0.065
4.01	0.475	49934.4	49760.1	51296.1	174.3	187.2	0.074
4.53	0.537	49987.5	49759.7	51295.9	227.8	247.4	0.086
5.00	0.592	50043.3	49759.7	51296.9	283.6	312.3	0.101
5.54	0.656	50122.1	49760.0	51295.1	362.1	403.3	0.114

Table 6.4 Frequency Pulling - Experimental vs Theoretical Results

$\frac{A_2}{\sqrt{2}}$ volts	$\frac{A_2}{A_2^*}$ †	f_1 Hz	f_{free} Hz	f_2 Hz	Frequency Pulling: f_1-f_{free}		Relative Error ††
					Experimental	Theoretical [eqn.(6.6.2)]	
0.50	0.046	49762.3	49760.5	51764.1	1.8	2.0	0.085
1.01	0.094	49768.7	49760.9	51764.1	7.8	8.0	0.026
1.51	0.140	49778.1	49760.8	51764.8	17.3	18.0	0.039
2.01	0.186	49792.5	49761.1	51764.6	31.4	32.1	0.022
2.52	0.233	49809.8	49760.4	51865.2	49.4	50.9	0.030
3.00	0.278	49830.8	49760.3	51765.0	70.5	73.0	0.035
3.52	0.326	49858.6	49760.5	51765.4	98.1	101.9	0.039
4.01	0.371	49888.8	49760.4	51765.7	128.4	134.4	0.047
4.52	0.419	49926.3	49760.5	51765.3	165.8	174.1	0.050
5.02	0.465	49968.1	49760.4	51765.7	207.7	219.4	0.056
5.50	0.509	50013.8	49760.5	51766.6	253.3	269.4	0.064
6.04	0.559	50073.8	49760.3	51766.4	313.5	334.6	0.067
6.51	0.603	50130.5	49760.0	51767.0	370.5	399.6	0.079
7.02	0.650	50207.0	49761.0	51765.7	446.0	481.8	0.080

Table 6.5 Frequency Pulling - Experimental vs Theoretical Results

$\frac{A_2}{\sqrt{2}}$ volts	$\frac{A_2}{A_2^*}$ †	f_1 Hz	f_{free} Hz	f_2 Hz	Frequency Pulling: $f_1 - f_{free}$		Relative Error ††
					Experimental	Theoretical [eqn.(6.6.2)]	
1.00	0.076	49760.0	49753.2	52266.9	6.8	6.1	-0.101
2.00	0.152	49778.3	49752.7	52267.7	25.6	24.6	-0.038
3.02	0.229	49809.0	49753.2	52269.6	55.8	56.9	0.020
4.04	0.307	49855.4	49753.2	52270.9	102.2	103.9	0.016
5.00	0.380	49912.7	49752.1	52273.0	160.6	162.7	0.013
6.03	0.458	49990.2	49752.2	52274.3	238.0	244.3	0.027
7.02	0.533	50085.3	49753.4	52277.8	331.9	343.5	0.035
8.00	0.608	50201.0	49753.3	52278.8	447.7	466.0	0.041

Table 6.6 Frequency Pulling - Experimental vs Theoretical Results

$\frac{A_2}{\sqrt{2}}$ volts	$\frac{A_2}{A_2^*}$ †	f_1 Hz	f_{free} Hz	f_2 Hz	Frequency Pulling: f_1-f_{free}		Relative Error ††
					Experimental	Theoretical [eqn.(6.6.2)]	
1.01	0.069	49758.3	49753.8	52762.5	4.5	5.1	0.130
2.02	0.139	49773.6	49753.8	52762.9	19.8	20.5	0.033
3.01	0.207	49799.4	49753.3	52764.2	46.1	45.8	-0.006
4.02	0.276	49835.8	49753.2	52765.0	82.6	82.9	0.003
5.00	0.343	49882.5	49753.1	52764.3	129.4	130.4	0.008
6.00	0.412	49942.6	49753.1	52765.4	189.5	191.9	0.013
7.03	0.482	50020.4	49753.9	52766.2	266.5	270.7	0.016
8.02	0.550	50110.6	49752.8	52766.9	357.8	362.9	0.014
9.03	0.620	50220.4	49752.9	52767.6	467.5	476.8	0.020

Table 6.7 Frequency Pulling - Experimental vs Theoretical Results.

$\frac{A_2}{\sqrt{2}}$ volts	$\frac{A_2}{A_2^*}$ †	f_1 Hz	f_{free} Hz	f_2 Hz	Frequency pulling: f_1-f_{free} Experimental	Theoretical [eqn.(6.6.2)]	Relative Error ††
1.01	0.063	49753.4	49749.4	53257.1	4.0	4.2	0.061
2.00	0.124	49766.3	49749.5	53257.2	16.8	16.7	-0.005
3.04	0.188	49788.0	49749.4	53259.5	38.6	38.9	0.007
4.05	0.251	49818.4	49749.4	53259.7	69.0	69.7	0.010
5.05	0.313	49858.7	49749.4	53259.1	110.3	109.8	-0.005
6.05	0.375	49909.8	49749.4	52360.6	160.4	160.1	-0.002
7.05	0.437	49970.2	49749.4	53261.0	220.8	221.6	0.004
8.05	0.498	50042.0	49749.4	53261.5	292.6	295.4	0.010
9.06	0.561	50130.0	49749.4	53262.0	380.6	384.1	0.009
10.06	0.623	50231.8	49749.4	53262.8	482.4	487.6	0.011
11.07	0.685	50354.8	49749.4	53263.1	605.4	610.4	0.008

Table 6.8 Frequency Pulling - Experimental vs Theoretical Results

Peak-to-Peak voltage of forcing oscillator $2A_2$	$\frac{A_2}{A_2^*}$ †	Frequency of the frequency modulating function $\frac{\gamma}{2\pi}$ Hz	Output voltage of the F-M detector (peak-to-peak) *	Amplitude of the almost sinusoidal frequency modulating function in Hz		Relative Error ††
				Experimental	Theoretical [eqn.(6.6.3)]	
0.5	0.070	468.	0.090	36.	32.	-0.10
1.0	0.141	464.	0.180	71.	64.	-0.10
1.5	0.211	460.	0.270	107.	96.	-0.10
2.0	0.282	452.	0.350	139.	128.	-0.08
2.5	0.352	438.	0.430	170.	160.	-0.06
3.0	0.423	426.	0.500	198.	192.	-0.03
3.5	0.493	409.	0.580	229.	224.	-0.02
4.0	0.563	389.	0.650	257.	256.	-0.00
4.5	0.634	366.	0.740	292.	288.	-0.01
5.0	0.704	334.	0.800	315.	320.	0.02

Table 6.9 . Frequency modulation due to the injected sinusoidal current - Experimental vs Theoretical results

Peak-to-Peak voltage of forcing oscillator $2A_2$	$\frac{A_2}{A_2^*}$ †	Frequency of the frequency modulating function $\frac{\gamma}{2\pi}$ Hz	Output voltage of the F-M detector (peak-to-peak)	Amplitude of the almost sinusoidal frequency modulating function in Hz		Relative Error ††
				Experimental	Theoretical [eqn.(6.6.3)]	
1.0	0.066	1005.	0.170	72.	64.	-0.11
2.0	0.132	998.	0.320	135.	128.	-0.05
3.0	0.197	989.	0.460	193.	192.	-0.01
4.0	0.263	973.	0.600	251.	256.	0.02
5.0	0.329	952.	0.770	322.	320.	-0.01
6.0	0.395	921.	0.905	376.	384.	0.02
7.0	0.461	890.	1.000	439.	448.	0.02
8.0	0.526	851.	1.200	495.	512.	0.04
9.0	0.592	809.	1.340	550.	576.	0.05
10.0	0.658	768.	1.440	588.	640.	0.09

Table 6.10 Frequency modulation due to the injected sinusoidal current - Experimental vs Theoretical results

Peak-to-Peak voltage of forcing oscillator $2A_2$	$\frac{A_2}{A_2^*}$ †	Frequency of the frequency modulating function $\frac{Y}{2\pi}$ Hz	Output voltage of the F-M detector (peak-to-peak)	Amplitude of the almost sinusoidal frequency modulating function, in Hz		Relative Error ††
				Experimental	Theoretical [eqn.(6.6.3)]	
1.0	0.045	1495.	0.160	73.	64.	-0.13
2.0	0.090	1493.	0.300	137.	128.	-0.07
3.0	0.135	1485.	0.450	206.	192.	-0.07
4.0	0.180	1475.	0.580	265.	256.	-0.03
5.0	0.225	1460.	0.720	327.	320.	-0.02
6.0	0.270	1444.	0.850	385.	384.	-0.003
7.0	0.315	1425.	0.990	447.	448.	0.002
8.0	0.360	1401.	1.120	504.	512.	0.02
9.0	0.405	1375.	1.260	564.	576.	0.02
10.0	0.450	1347.	1.380	614.	640.	0.04
11.0	0.495	1273.	1.640	721.	704.	-0.02
12.0	0.541	1180.	1.900	822.	768.	-0.07

Table 6.11 Frequency modulation due to the injected sinusoidal current - Experimental vs Theoretical results

CHAPTER 7

Summary and Concluding Remarks

The behaviour of certain second order non-linear autononous and non-autonomous oscillatory systems has been studied analytically. These systems are described by second order non-linear differential or difference-differential equations. The exact analytical solutions of these equations do not seem to be possible. However, with appropriate restrictions, analytical approximations can be obtained.

In Chapter 2, a method has been described for determining the non-stationary (or stationary) behaviour of a class of oscillatory systems in which small variations in the instantaneous frequency of oscillation take place during the period of interest. The method is a variation of the asymptotic method of Bogoliubov -Mitropolsky.

In the Bogoliubov — Mitropolsky method a first approximation is obtained in the same manner as in the well-known Krylov-Bogliubov method. The solution is then improved gradually by an iteration procedure. In the method described in Section 2.2, the solution is assumed in the form of a harmonic series in some variable τ which represents the total phase. The rate of change of τ with respect to the time t is the instantaneous frequency of oscillation. The amplitude of each harmonic and the instantaneous frequency are expressed in the form of series expansions in a small parameter. The terms in the expansions are then determined by a simple straightforward perturbational procedure. We note that this method is easier to apply than the Bogoliubov-Mitropolsky method. We note, however, that the difference between the two methods is only procedural and no claim of originality is made as far as the work described in Section 2.2 is concerned.

In section 2.3 a self-oscillator with delayed amplitude regulation has been discussed. The oscillator can be described by a difference-differential equation. An analytical approximation to the solution of this equation is given in Section 2.4 using a modification of the method of Section 2.2.

In Chapter 3 a method has been developed to study the non-stationary behaviour of a class of strongly non-linear oscillatory systems. In these systems significant variations in the instantaneous frequency of oscillation may take place. The method requires that the higher harmonics in the response be of considerably smaller magnitude than that of the fundamental frequency.

In Chapter 4 the method of Section 2.2 has been modified to deal with the case of non-stationary forced oscillations in weakly non-linear systems The frequency content of the forcing function is restricted to be outside the resonance frequencies of the system.

In Chapter 5 we studied the case of a self-oscillator disturbed by an external signal the frequency content of which is not necessarily far away from the frequency of self-oscillation. We do however impose the restriction that the disturbances be weak enough so as not to result in suppression of the self-oscillation. A non-linear convolution method has been developed in sections 5.2 and 5.3. The method allows us to predict with accuracy the spectrum spread and other non-linear effects such as the frequency pulling and the variation in the amplitude of self-oscillation due to the presence of disturbances. In the analysis, the magnitude of the disturbing signal is assumed to be one order smaller than that sufficient to suppress the self-oscillations.

In Chapter 6, the frequency pulling and the frequency fluctuations in a self-oscillator due to an injected sinusoidal current are determined experimentally. The experimental oscillator approximates the v.d. Pol oscillator closely. The frequency of the injected current is chosen to be close to the free-frequency. The experimental results were found to be in good agreement with the analytical results obtained in Chapter 5.

In conclusion, the main objective of this work was to augment and extend the existing analytical methods for treating non-linear oscillatory systems so as to embrace a wider class of systems. Towards this end, the following has been achieved:

1. developing a procedure for obtaining approximate solutions to non-linear difference-differential equations describing a class of weakly non-linear oscillatory systems with "retarded actions".
2. developing a simple procedure for obtaining approximate solutions to a class of strongly non-linear, time-invariant or slowly time-variant, oscillatory systems.
3. developing a non-linear convolution solution to the problem of disturbed oscillations in a self-oscillatory system.

Appendix A

Entrainment of frequency in a self-oscillatory system with delayed amplitude regulation.

A.1 Introduction

The phenomenon of frequency entrainment of a self-oscillatory system with instantaneous amplitude regulation by an external periodic force is well known. Consider for example the case when a self-oscillator is disturbed by an external sinusoidal force whose frequency is near to the frequency of free-oscillation. When the external force is of a sufficiently small amplitude, it results in small fluctuations in the amplitude and phase as well as slight variations in the mean amplitude and frequency of self-oscillation. The oscillator spectrum is then predominantly contained in a narrow band close to the frequency of free oscillation. As the amplitude of the external force is increased, the fluctuations in the amplitude and phase of self-oscillation become more violent, their time rate becomes smaller, and the mean frequency is pulled towards the external frequency. The oscillator spectrum then gradually clusters in a very narrow band near the external frequency. When the amplitude of the external force reaches a critical value, the self-oscillation is "suddenly" entrained and the system's response is periodic at the forcing frequency. The amplitude of oscillation at the transition from aperiodic slowly modulated oscillation to entrained periodic oscillation, can be determined analytically using variational methods.

A similar phenomenon may arise when the external frequency is not close to the free-frequency but is close to one of its harmonics or subharmonics. However, in this case, the self-oscillation is suppressed rather than frequency-pulled towards the external frequency. Suppression of the self-frequency may also occur if the external frequency is not close to one of the harmonics or subharmonics of the free-frequency, however, this requires a considerably large external force.

In general the self-oscillation can be supressed (not phase-locked) by a sufficiently strong external force which is aperiodic and of arbitrary frequency content , although this case has not apparently been studied in the literature .

In section A.2 , we shall study the entrainment of frequency in a self-oscillator with delayed amplitude regulation.

A.2 The v.d.Pol oscillator with delayed amplitude regulation

Let us consider the oscillator of Fig. (2.2.1) when it is driven by a sinusoidal current:

$$i_d = I_d \cos(\omega_d t + \psi) \tag{A.1}$$

where I_d, ω_d and ψ are constants.

The oscillator is described by the normalized equation:

$$\ddot{x} + x + \varepsilon[\alpha\dot{x} - \beta\dot{x}(t-h) + \frac{1}{3}\dot{x}^3(t-h)]$$

$$= \varepsilon F \cos(\omega t + \psi) \quad , (\dot{\cdot}) = \frac{d}{dt} \quad . \tag{A.2}$$

where ε, α, β and h are as defined in equations (2.3.15), $\omega = \omega_d\sqrt{LC}$ and $\varepsilon F = I_d\sqrt{\frac{3b}{a - \frac{1}{R}}\frac{L}{C}}$ †, and it is assumed that $\omega \sim 1$.

Let x_0 denote the periodic (entrained) solution of eqn.(A.2). To a first approximation:

$$y_0 \triangleq \dot{x}_0 = A\cos(\omega t + \phi) \quad , \tag{A.3}$$

where A and ϕ are constants.

Define $\rho \triangleq A^2$, $\theta \triangleq \omega h$ and $\sigma = \frac{\omega^2 - 1}{\varepsilon\omega}$, then using the principle of harmonic balance, we obtain:

$$\rho[(\frac{\rho}{4} - \beta)^2 + 2(\frac{\rho}{4} - \beta)(\alpha\cos\theta - \sigma\sin\theta) + \alpha^2 + \sigma^2] = F^2 \tag{A.4}$$

In order to determine the value (or values) of ρ and hence the value (or values) of F, at which transition from the aperiodic slowly modulated oscillation to the periodic entrained oscillation takes place, we must study the stablility of the periodic solution in (A.3). We shall consider the response of the just entrained oscillator to a small displacement from the periodic solution.

Let $x \triangleq x_0 + \varepsilon u$, then the variation equation derived from eqn. (A.2) is:

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† See section 2.3 .

$$\ddot{u} + u + \varepsilon[\alpha\dot{u} + \dot{u}(t-h) \quad \{\dot{x}_0^2(t-h) - \beta\}] = 0 \qquad \text{(A.5)}$$

Let $u = a(\xi)\cos(\omega t+\phi) + b(\xi)\sin(\omega t+\phi)$, $\xi \triangleq \varepsilon t$, (A.6)

then

$$\ddot{u}+u = \varepsilon\omega[(2\frac{db}{d\xi} - \sigma a)\cos(\omega t+\phi) - (2\frac{da}{d\xi} + \sigma b)\sin(\omega t+\phi)], \qquad \text{(A.7)}$$

$$\varepsilon\dot{u} = \varepsilon\omega[b\cos(\omega t+\phi) - a\sin(\omega t+\phi)] \qquad \text{(A.8)}$$

and

$$\varepsilon\dot{u}(t-h) \{\dot{x}_0^2(t-h) - \beta\} = \varepsilon\omega[\{b(\tfrac{3}{4}A^2-\beta)\cos\theta + a(\tfrac{A^2}{4}-\beta)\sin\theta\}\cos(\omega t+\phi)$$

$$+ \{b(\tfrac{3}{4}A^2-\beta)\sin\theta - a(\tfrac{A^2}{4}-\beta)\cos\theta\}\sin(\omega t+\phi)] \qquad \text{(A.9)}$$

Substituting expressions (A.7) to (A.9) in eqn. (A.5) then equating the coefficients of $\cos(\omega t+\phi)$ and $\sin(\omega t+\phi)$ in the resulting expression separately to zero, we obtain:

$$\frac{d}{d\xi}\begin{bmatrix} a \\ b \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} \alpha + (\frac{\rho}{4} - \beta)\cos\theta & \sigma + (\frac{3}{4}\rho-\beta)\sin\theta \\ -\sigma + (\frac{\rho}{4} - \beta)\sin\theta & \alpha + (\frac{3}{4}\rho-\beta)\cos\theta \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}, \qquad \text{(A.10)}$$

which is a linear time-invariant equation. The characteristic equation for eqn. (A.10) is:

$$s^2 + [(\frac{\rho}{2} - \beta) \cos\theta + \alpha] s + \frac{1}{4}[\frac{3}{16}\rho^2 + (\alpha\cos\theta - \beta - \sigma\sin\theta)\rho + (\beta\cos\theta - \alpha)^2 + (\beta\sin\theta + \sigma)^2] = 0 \quad (A.11)$$

The roots of eqn. (A.11) will have negative real parts, and hence the small displacement εu will decay as time increases, if:

$$\rho \geq 2(\beta - \alpha\sec\theta) \quad , \quad \cos\theta > \frac{\alpha}{\beta} \quad (A.12)$$

and

$$\frac{3}{16}\rho^2 + (\alpha\cos\theta - \beta - \sigma\sin\theta)\rho + (\beta\cos\theta - \alpha)^2 + (\beta\sin\theta + \sigma)^2 \geq 0 \quad (A.13)$$

The stability boundary of the periodic solution is determined from the equalities in (A.12) and (A.13).

In Figs. A.1 to A.3, the stability boundaries for specific values of α, β and θ are plotted in the ρ-σ plane.

When $\theta = 0$, eqn. (A.2) reduces to the forced van der Pol equation, and (A.12) and (A.13) reduce to

$$\rho \geqslant 2$$

and $$\left(\frac{3\rho}{4} - 1\right)\left(\frac{\rho}{4} - 1\right) + \sigma^2 \geqslant 0$$

which are the well-known conditions of stability of the entrained oscillations in the van der Pol oscillator with instantaneous amplitude regulation

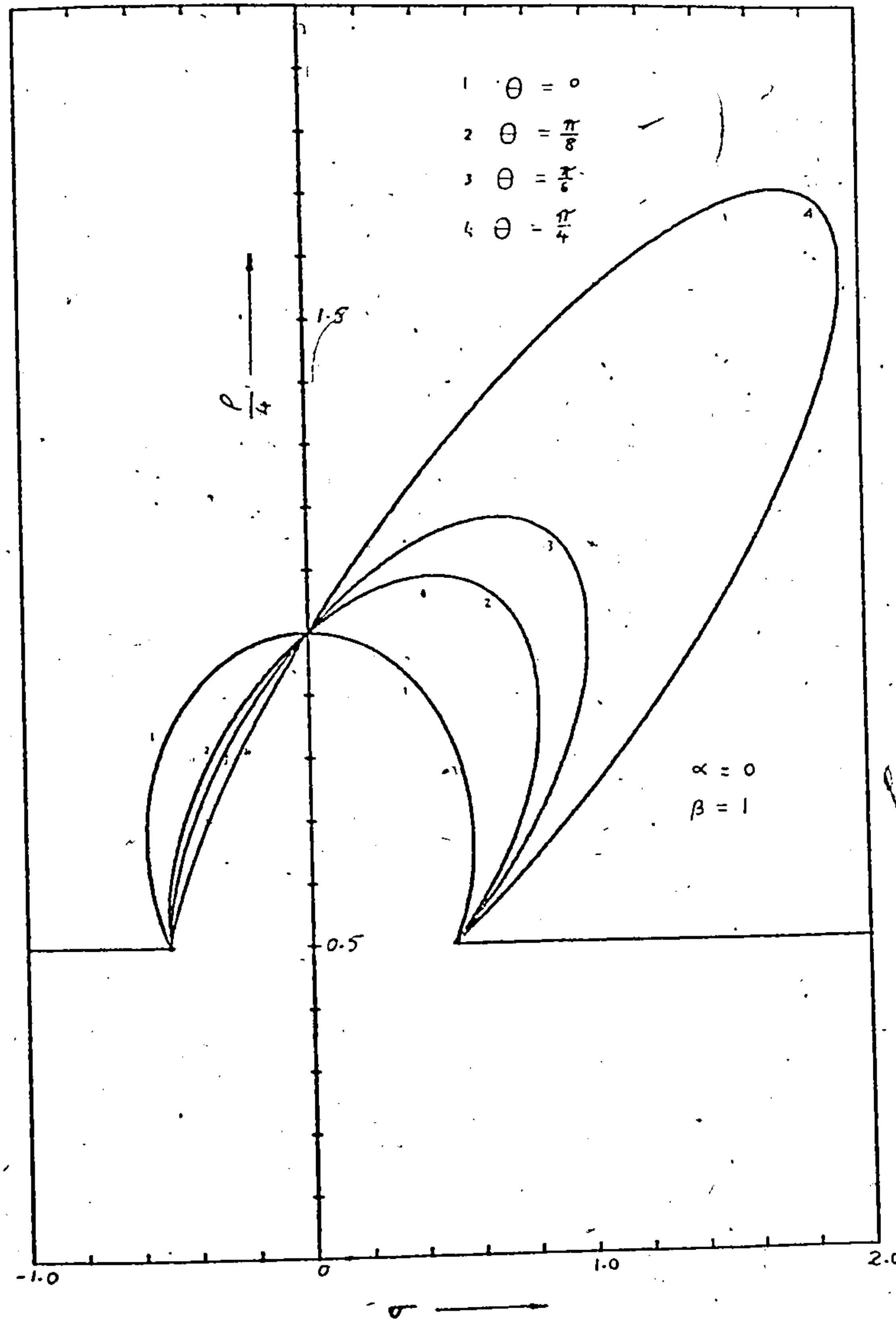

Fig. A.1 Stability boundaries of the entrained oscillations in a van der Pol oscillator with delayed amplitude regulation .

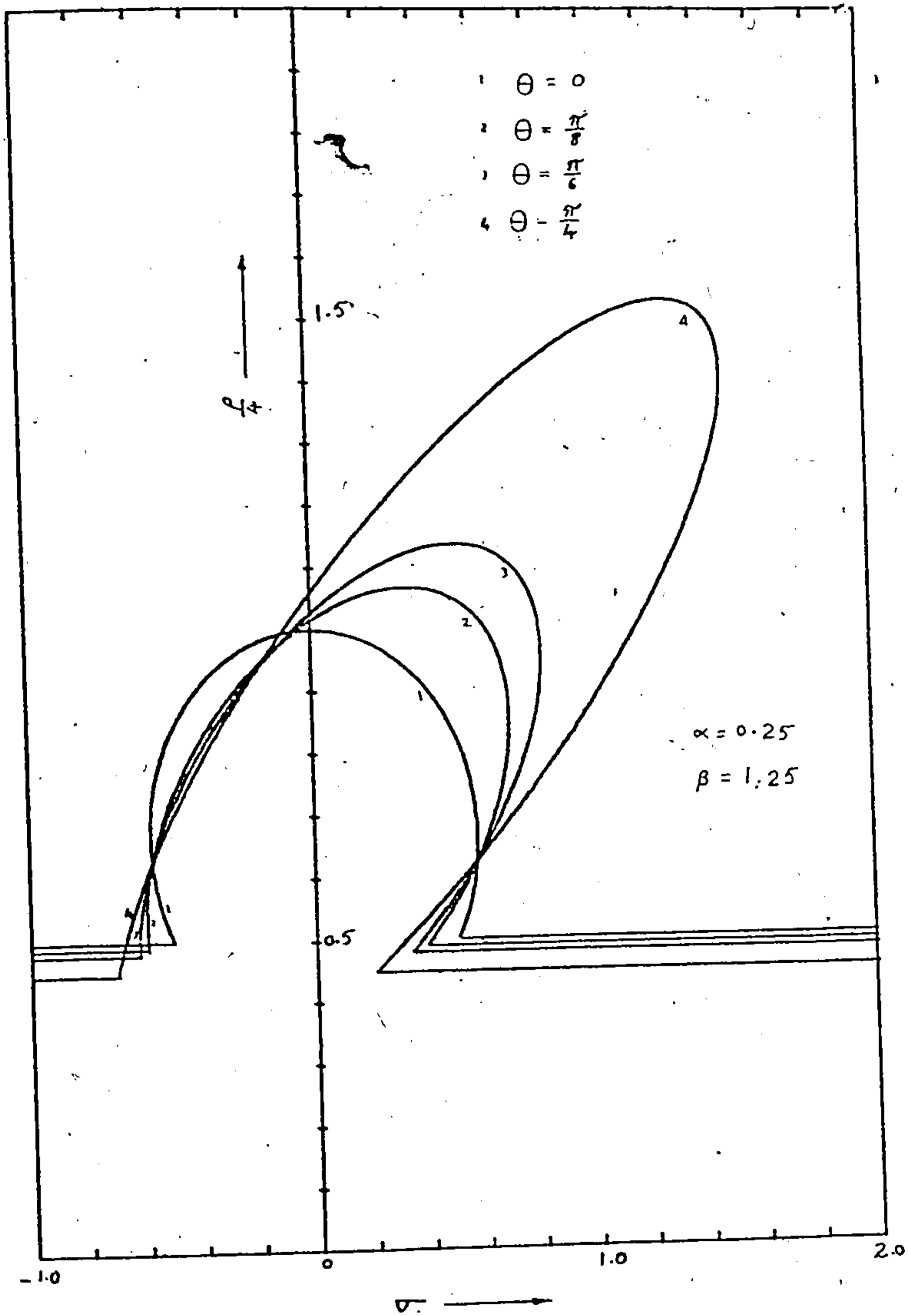

Fig. A.2 Stability boundaries of the entrained oscillations in a van der Pol oscillator with delayed amplitude regulation .

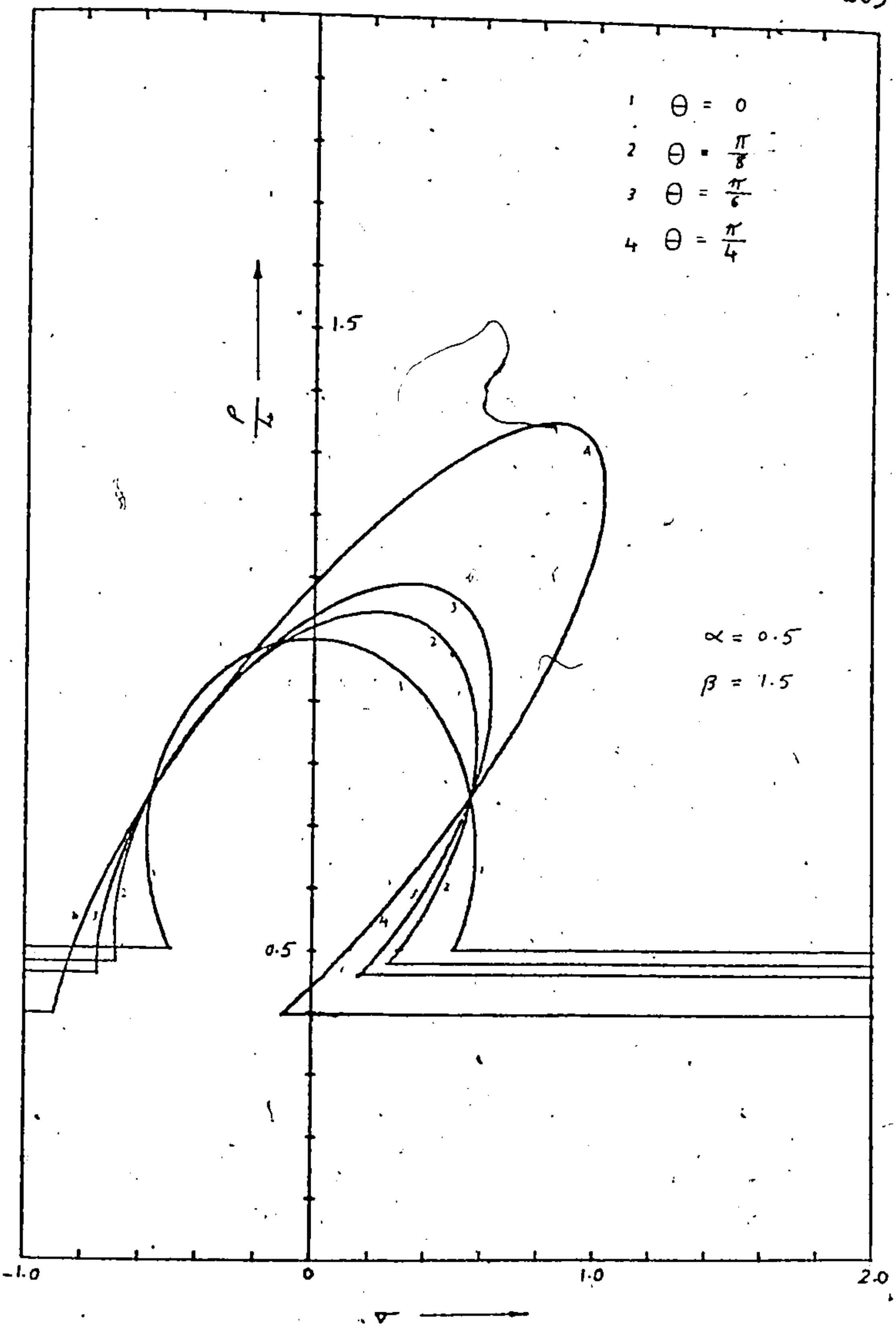

Fig. A.3. Stability boundaries of the entrained oscillations in a van der Pol oscillator with delayed amplitude regulation.

APPENDIX B

On the evaluation of the transient solution of v.d. Pol's equation using the derivative expansion method

The transient response of the unforced v.d.Pol oscillator has been obtained to the second order of approximation by Davis and Alfriend [1967] using the so called 'derivative expansion method' developed by Nayfeh [1964]. In what follows we shall outline their analysis and point out certain difficulties that arise.

We shall use the same notation as in Davis and Alfriend's paper.

Consider the v.d.Pol equation:

$$\frac{d^2x}{dt^2} - \varepsilon(1 - x^2)\frac{dx}{dt} + x = 0 \qquad \text{(B.1)}$$

Define the slow time:

$$\xi = \varepsilon t$$

eqn. (B.1) is then rewritten as:

$$\varepsilon^2 \frac{d^2x}{d\xi^2} - \varepsilon^2(1 - x^2)\frac{dx}{d\xi} + x = 0 \qquad \text{(B.2)}$$

Defining:

$$\eta = g(\xi;\varepsilon) = \frac{g_o(\xi)}{\varepsilon} + g_1(\xi) + \varepsilon g_2(\xi) + \dots , \qquad \text{(B.3)}$$

where the functions $g_n(\xi)$, n=0,1,2,.. , will be determined by eliminating the secular terms from the solution , then

$$\frac{d}{d\xi} = \frac{\partial}{\partial\xi} + g' \frac{\partial}{\partial\eta} ;$$

and (B.4)

$$\frac{d^2}{d\xi^2} = \frac{\partial^2}{\partial\xi^2} + g'' \frac{\partial}{\partial\eta} + 2g' \frac{\partial^2}{\partial\xi\partial\eta} + g'^2 \frac{\partial^2}{\partial\eta^2}$$

where

$$(\ )' = \frac{d}{d\xi}$$

Hence eqn. (B.2) is rewritten as:

$$\varepsilon^2 x_{\xi\xi} + \varepsilon^2 g'' x_\eta + 2\varepsilon^2 g' x_{\xi\eta} + \varepsilon^2 g'^2 x_{\eta\eta} - \varepsilon^2(1-x^2)(x_\xi + g' x_\eta) + x = 0 \qquad \text{(B.5)}$$

where the terms of the order of ε^3 have been omitted. The subscripts in (B.5) denote differentiation.

Writing:

$$x = x_o + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots$$

and substituting in eqn. (B.5) then equating the terms of similar order of smallness separately to zero, the following equations result:

$$g_o'^2 x_{o\eta\eta} + x_o = 0 \qquad \text{(B.6)}$$

$$g_o'^2 x_{1\eta\eta} + x_1 = - g_o'' x_{o\eta} - 2g_o' x_{o\xi\eta} - 2g_o' g_1' x_{o\eta\eta} + (1 - x_o^2)\, g_o' x_{o\eta} \qquad \text{(B.7)}$$

$$g_o'^2 x_{2\eta\eta} + x_2 = - x_{o\xi\xi} - g_o'' x_{1\eta} - g_1'' x_{o\eta} - 2g_o' x_{1\xi\eta} - 2g_o' g_1' x_{1\eta\eta} - (g_1'^2 + 2g_o' g_2')\, x_{o\eta\eta}$$

(B.8)

$$+ (1-x_o^2)(x_{o\xi} + g_1' x_{o\eta} + g_o' x_{1\eta}) - 2x_o x_1 g_o' x_{o\eta} - 2g_1' x_{o\xi\eta}$$

The solution of (B.6) is assumed to be:

$$x_o = A_o(\xi) \sin\left[\frac{\eta}{g_o'} + \phi_o(\xi)\right] \tag{B.9}$$

Hence eqn. (B.7) takes the form:

$$g_o'^2 x_{1\eta\eta} + x_1 = \left[- g_o'' A_o - 2g_o'\left(\frac{A_o'}{g_o'}\right) + A_o - \frac{A_o^3}{4}\right] \cos\left(\frac{\eta}{g_o'} + \phi_o\right)$$

$$+ 2A_o(\phi_o' + g_o' g_1')\sin\left(\frac{\eta}{g_o'} + \phi_o\right) - 2A_o \frac{g_o''}{g_o'^2}\, \eta \sin\left(\frac{\eta}{g_o'} + \phi_o\right) + \frac{A_o^3}{4}\cos 3\left(\frac{\eta}{g_o'} + \phi_o\right) \tag{B.10}$$

The coefficient of $\eta \sin\left(\frac{\eta}{g_o'} + \phi_o\right)$ in the r.h.s. of eqn. (B.10) is equated to zero thus:

$$\frac{A_o g_o''}{g_1'^2} = 0 \quad \text{i.e.} \quad g_o'' = 0 \quad \text{for} \quad A_o \neq 0$$

hence g_o' = constant and $g_o = g_o'\xi + C_1$

C_1 being a constant.

Imposing the restriction: $g_o(0) = 0$, then $C_1 = 0$.

Setting the coefficient of $\sin\left(\frac{\eta}{g_o'} + \phi_o\right)$ in the r.h.s. of eqn. (B.10) equal to zero then:

$$\phi_o' + 2g_o' g_1' = 0 \tag{B.11}$$

Since g_o' is a constant, then eqn. (B.11) integrates into:

$$\phi_o + 2g_o' g_1 = \text{constant} \tag{B.12}$$

Now one is faced with one equation in the two unknowns ϕ_o and g_1. In Davis and Alfriend's paper g_1 was chosen arbitrarily to be equal to zero and hence

$$\phi_o = \bar{\phi}_o \quad , \quad \bar{\phi}_o \text{ being a constant}$$

The value of g_o' was then chosen to be equal to unity. Now setting the coefficient of $\cos(\frac{\eta}{g_o} + \phi_o)$ in the r.h.s. of eqn. (B.10) to be equal to zero then:

$$A_o' = \frac{A_o}{2}\left[1 - \frac{A_o^2}{4}\right] \tag{B.13}$$

which integrates into:

$$A_o = \frac{a_o e^{\epsilon t/2}}{[1 + a_o^2/4(e^{\epsilon t} - 1)]^{1/2}} \tag{B.14}$$

where a_o is a constant of integration.

The first approximation is thus given by:

$$x_o = \frac{a_o e^{\epsilon t/2}}{[1 + a_o^2/4(e^{\epsilon t} - 1)]^{1/2}} \sin(t + \bar{\phi}_o) \tag{B.15}$$

Now eqn. (B.10) takes the form:

$$g_o'^2 x_{1\eta\eta} + x_1 = \frac{A_o^3}{4} \cos 3(\frac{\eta}{g_o} + \phi_o)$$

which has the solution:

$$x_1 = A_1(\xi)\sin[\eta + \phi_1(\xi)] - \frac{A_o^3}{32}\cos 3(\eta + \bar{\phi}_o) \qquad \text{(B.16)}$$

Substituting x_o and x_1 in eqn. (B.8), then:

$$x_{2\eta\eta}+x_2 = [(-2A_1'+A_1-\tfrac{3}{4}A_o^2A_1)\cos(\phi_1-\bar{\phi}_o)+2A_1\phi_1'\sin(\phi_1-\bar{\phi}_o)]\cos(\eta+\bar{\phi}_o)$$

$$+ [(2A_1'-A_1+\frac{A_o^2A_1}{4})\sin(\phi_1-\bar{\phi}_o) + 2A_1\phi_1'\cos(\phi_1-\bar{\phi}_o)$$

$$+ (-A_o''+2g_2'A_o+A_o' - \tfrac{3}{4}A_o^2A_1+\frac{A_o^5}{128})]\sin(\eta+\bar{\phi}_o) \qquad \text{(B.17)}$$

$$+ \tfrac{3}{4}A_o^2A_1\cos 3(\eta+\phi_o)\cos(\phi_1-\bar{\phi}_o)$$

$$+ [-\tfrac{5}{16}A_o^2A_o' - \tfrac{3}{4}A_o^2A_1\sin(\phi_1-\bar{\phi}_o) + \tfrac{3}{32}A_0^3 - \tfrac{3}{64}A_o^5]\sin 3(\eta+\bar{\phi}_o)$$

$$+ \frac{5}{128}A_o^5\sin 5(\eta+\bar{\phi}_o)$$

Now in order that x_2 be bounded, the coefficients of $\cos(\eta + \phi_o)$ and $\sin(\eta + \phi_o)$ in the r.h.s. of eqn. (B.17) must be equated to zero. Thus:

$$(-2A_1' + A_1 - \tfrac{3}{4}A_o^2A_1)\cos(\phi_1-\bar{\phi}_o) + 2A_1\phi_1'\sin(\phi_1-\bar{\phi}_o) = 0 \qquad \text{(B.18)}$$

and

$$(2A_1' - A_1 + \frac{A_o^2A_1}{4}) \sin(\phi_1-\overline{\phi}_o) + 2A_1\phi_1' \cos(\phi_1-\overline{\phi}_o)$$

$$+ (-A_o'' + 2g_2'A_o + A_o' - \frac{3}{4}A_o^2A_1 + \frac{A_o^5}{128}) = 0 \qquad \text{(B.19)}$$

At this point one is faced with two equations in the three unknowns A_1, ϕ_1 and g_2. In Davis and Alfriend's paper eqn. (B.19) was separated, without any justification, into:

$$(2A_1' - A_1 + \frac{1}{4}A_o^2A_1) \sin(\phi_1-\overline{\phi}_o) + 2A_1\phi_1' \cos(\phi_1-\overline{\phi}_o) = 0 \quad , \qquad \text{(B.20)}$$

and

$$2g_2'A_o - A_o'' + A_o' - \frac{3}{4}A_o^2A_o' + \frac{A_o^5}{128} = 0 \quad . \qquad \text{(B.21)}$$

Equations (B.18), (B.20) and (B.21) are then solved for ϕ_1, A_1 and g_2. Finally the second order solution of eqn. (B.1) is given by:

$$x = A_o(t)\sin(\eta+\overline{\phi}_o) + \varepsilon \{ A_1(t)\sin[\eta+\phi_1(t)] - \frac{A_o^3(t)}{32}\cos 3(\eta+\overline{\phi}_o)\} + 0(\varepsilon^2) \qquad \text{(B.22)}$$

where

$$A_o(t) = \frac{a_o e^{\varepsilon t/2}}{[1 + a_o^2/4(e^{\varepsilon t} - 1)]^{1/2}} \quad ,$$

$$A_1(t) = a_1A_o(t) \operatorname{cosec}[\phi_1(t) - \overline{\phi}_o] \quad ,$$

$$\phi_1(t) = \overline{\phi}_o + \tan^{-1} \frac{\overline{\phi}_1 e^{\varepsilon t}}{A_o^2(t)} \ ,$$

and

$$\eta = t + \varepsilon\left(-\frac{1}{16}\varepsilon t - \frac{1}{8}\ln\left[\frac{A_o(t)}{A_o(0)}\right] + \frac{7}{64}[A_o^2(t) - A_o^2(0)]\right) + O(\varepsilon^2)$$

where a_o, a_1, $\overline{\phi_o}$ and $\overline{\phi}_1$ are constants determined by the initial conditions.

We note that the above solution would be in agreement with the second-order solution obtained by other methods (see for example Struble [1962]) after the omission of the term A_1 sin(.). This term results from the arbitrary splitting of eqn.(B.19) into eqns.(B.20) and (B.21).

Appendix C

COMPARISON OF THE K-B METHOD AND THE REFINED ELLIPTIC FUNCTION METHOD

C.1 Introduction:

It will be shown that, contrary to the claims made by Soudack and Barkham (1971), the K-B method of approximating the solution of the Duffing type equation:

$$\ddot{x} + x + px^3 + \beta\dot{x} = 0 \ , \ (\dot{\cdot}) \triangleq \frac{d}{dt} \qquad (C.1)$$

yields results superior to those obtained using the refined elliptic function method developed by the above authors. In what follows we will give the approximate solutions obtained by both methods, and then will show by means of numerical examples, the superiority of the K-B method of solution.

C.2 The Krylov-Bogoliubov (K-B) Solution:

The solution of the equation

$$\ddot{x} + x + \epsilon f(x,\dot{x}) = 0$$

where ϵ is a small parameter is given by Krylov and Bogoliubov as

$$x(t) = A(t) \cos(t + \phi(t))$$

where A and ϕ are the solutions of the first order differential equations:

$$\frac{dA}{dt} = \frac{\varepsilon}{2\pi} \int_0^{2\pi} \sin u \; f(A \cos u, - A \sin u) \, du \qquad (C.2)$$

and

$$\frac{d\phi}{dt} = \frac{\varepsilon}{2\pi A} \int_0^{2\pi} \cos u \; f(A \cos u, - A \sin u) \, du \qquad (C.3)$$

In the case of the Duffing type equation [eqn.(C.1)]

$$\varepsilon f(x,\dot{x}) = px^3 + \beta\dot{x},$$

and therefore equations (C.2) and (C.3) become

$$\frac{dA}{dt} = - \frac{\beta A}{2} \qquad (C.4)$$

and

$$\frac{d\phi}{dt} = \frac{3}{8} pA^2 \qquad (C.5)$$

These two equations may then be solved simultaneously to yield

$$A = Ce^{-\beta t/2} \qquad (C.6)$$

and

$$\phi - \phi(0) = \frac{3}{8} \frac{pC^2}{\beta} (1 - e^{-\beta t}) \tag{C.7}$$

where C and $\phi(0)$ are constants of integration. Then using equations (C.6) and (C.7), we obtain the K-B solution in the form

$$x = Ce^{-\beta t/2} \cos[t + \frac{3}{8} \frac{pC^2}{\beta} (1 - e^{-\beta t}) + \phi(0)] \tag{C.8}$$

We note that the same result is obtained if the equivalent van der Pol method of solution is used.

Now the constants of integration C and $\phi(0)$ must be determined from the initial conditions. For the initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$ we have:

$$x(0) = C \cos \phi(0) = 1 \tag{C.9}$$

and

$$\dot{x}(0) = -\frac{\beta}{2} C \cos \phi(0) - C[1 + \frac{3}{8} pC^2] \sin \phi(0) = 0$$

and hence

$$C \sin \phi(0) = \frac{-\beta/2}{1 + \frac{3}{8} pC^2} \tag{C.10}$$

From equations (C.9) and (C.10) we readily obtain the equation

$$C^2 = 1 + \frac{\beta^2}{4[1 + \frac{3}{8} pC^2]^2} \tag{C.11}$$

This last equation is cubic in C^2 and it can be readily verified that for p>0 the only real root (when solving for C^2) is very close to unity for $p < 2$ and for values of β up to approximately 1.

Hence we may write an approximate solution of equations (C.9) and (C.10) as

$$C = \sqrt{1 + \left[\frac{\beta/2}{1 + \frac{3}{8}p}\right]^2} \qquad (C.12)$$

and

$$\phi(0) = -\tan^{-1}\frac{\beta/2}{1 + \frac{3}{8}p} \approx \frac{-\beta/2}{1 + \frac{3}{8}p} \qquad (C.13)$$

It is of interest to compare the K-B solution of eqn.(C.1) as given by expression (C.8) with the expression

$$x = e^{-t/2}\cos(1.75t) \qquad (\beta=1,\ p=2)$$

referred to by Soudack and Barkham (1971) as the K-B solution of eqn. (C.1). We note that the non-linear phase term which is required in the correct K-B solution has been omitted here.

C.3 The Refined Elliptic Function Method:

Using this method, the solution of eqn. (C.1) is given by Soudack and Barkham in the form

$$x = e^{-\frac{\beta t}{2(1+k^2)}} \mathrm{Cn}\{[1 + p\, e^{-\frac{\beta t}{(1+k^2)}}]^{1/2} t\} \qquad (C.14)$$

where the modulus k of the elliptic cosine Cn{ · } is given by

$$k^2 = \frac{p}{2(1+p)} \qquad (p \geq 0) .$$

C.4 Numerical Comparison of the K-B and Refined Elliptic Function Methods:

In order to compare numerically the results obtained using the two methods of solution, we have obtained both solutions for various values of β and p. These have been plotted in Figs. (C.1) to (C.6) for p = 1 and 2 and for different values of β in the range $0 < \beta \leq 1.40$. We have also plotted in each case, for purposes of comparison, the "exact" solution as obtained using numerical integration techniques. In order to obtain a numerical measure of the relative accuracy of the two methods, we have also computed for a range of values of β and p, the absolute and relative r.m.s. errors of the approximate solutions obtained using each method. These are shown in Table C.1 .

From the figures we note that for the case p = 2, β = 1, which was computed by Soudack and Barkham, the two methods appear to be of comparable accuracy, although in the other cases the K-B method appears superior. Furthermore, we see from Table C.1 that the r.m.s. errors are smaller for the K-B method than for the refined elliptic frunction method.

In fact, based on the r.m.s. error values, it appears that for smaller values of p the K-B method is superior to the refined elliptic function method for any value of the damping β. Moreover, contrary to the claims made by Soudack and Barkham, it appears that only for moderate values of β and p do the two methods become comparable, and even then the K-B method yields somewhat better results than the refined elliptic function method. In addition, the K-B method has the advantage of being a very simple method to apply.

		Absolute r.m.s. value of error		Relative r.m.s. error $\triangleq \frac{\text{r.m.s. of error}}{\text{r.m.s. of x}}$	
p	β	The K-B method	The refined elliptic function method	The K-B method	The refined elliptic function method
1.0	.40	.0272	.1223	.0721	.3241
	.60	.0276	.0617	.0855	.1914
	.80	.0293	.0476	.0999	.1623
	1.00	.0316	.0609	.1133	.2186
	1.20	.0368	.0785	.1350	.2879
	1.40	.0463	.0961	.1696	.3523
2.0	.40	.0546	.2263	.1440	.5964
	.60	.0424	.1302	.1319	.4052
	.80	.0392	.0711	.1365	.2474
	1.00	.0383	.0430	.1435	.1609
	1.20	.0374	.0422	.1468	.1653
	1.40	.0386	.0534	.1550	.2141

Table C.1 : Absolute and relative r.m.s. error of the K-B and the refined elliptic function approximations, calculated over a solution period of 10 normalized time units.

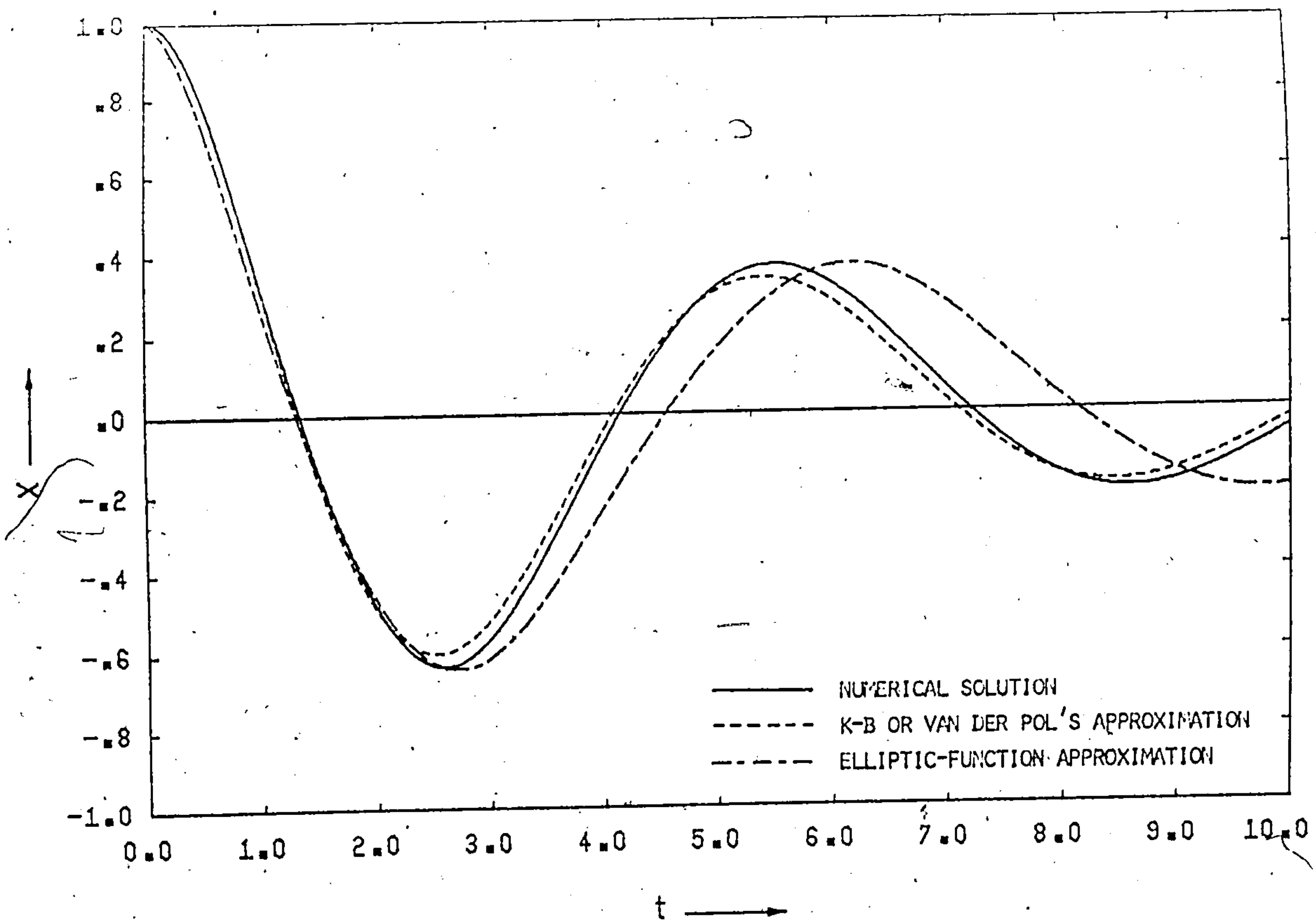

Fig.C.1 Solution of eqn.(C.1) for $p=1$ and $\beta=0.4$.

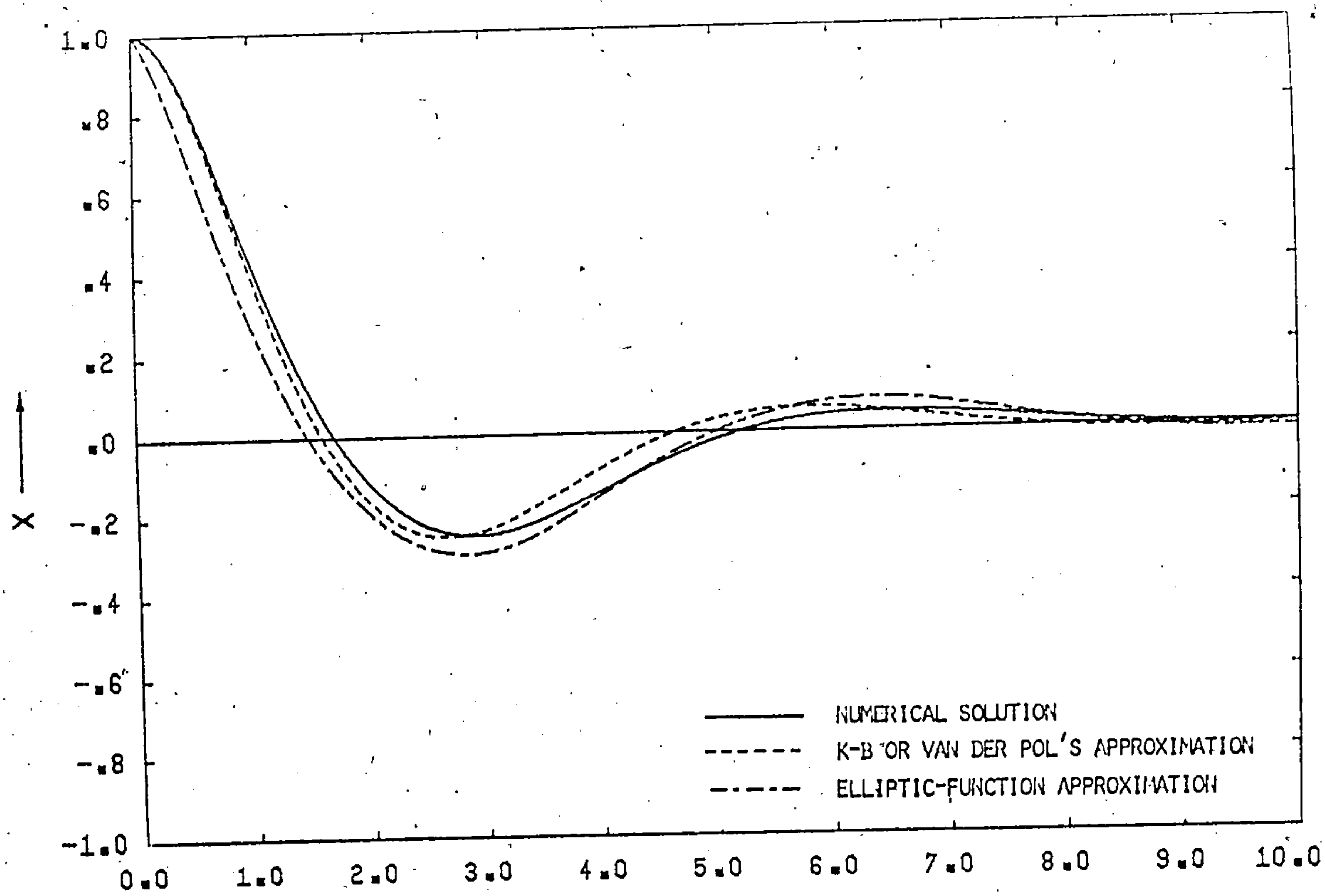

Fig.C.2 Solution of eqn.(C.1) for p=1 and β=1 .

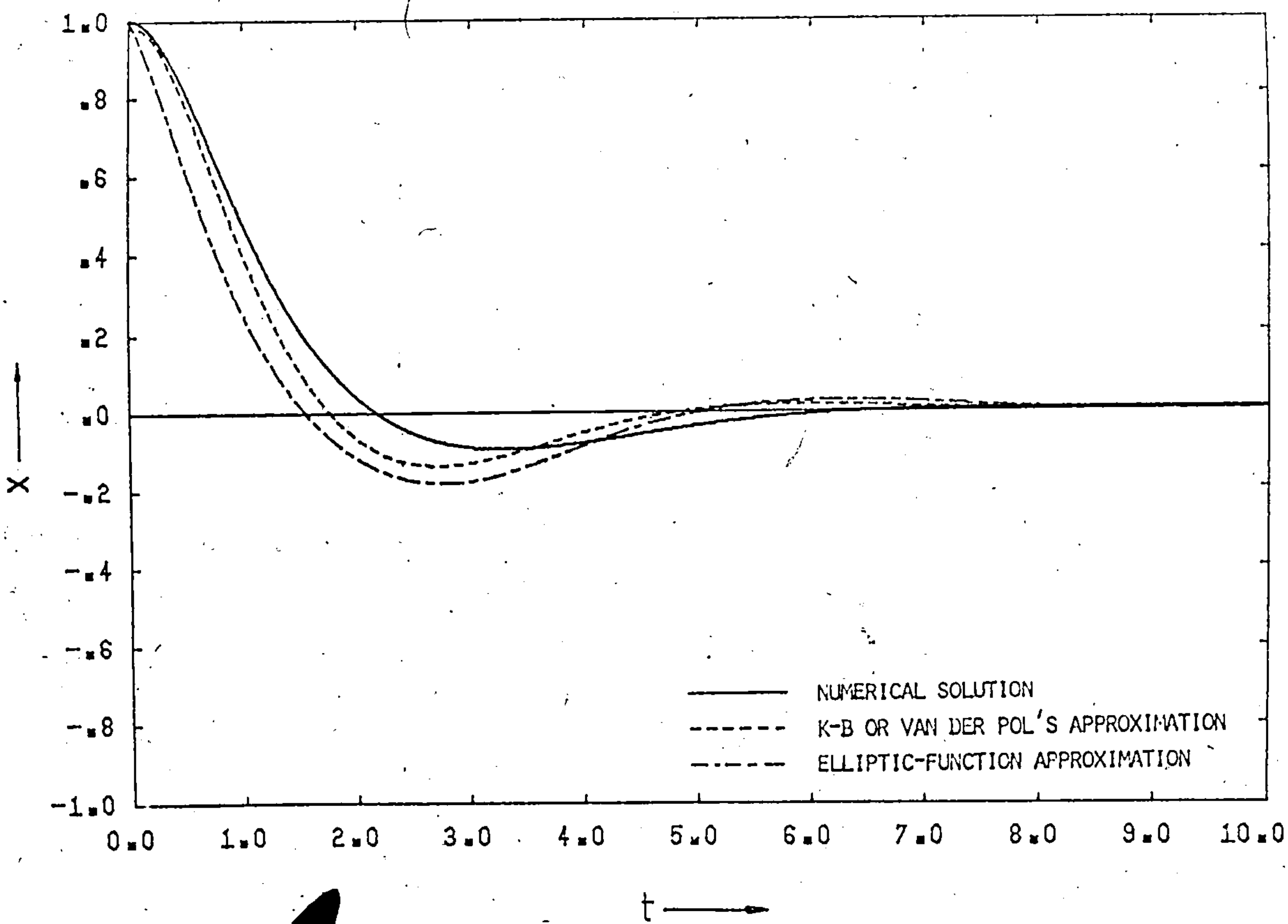

Fig.C.3 Solution of eqn.(C.1) for p=1 and β=1.4 .

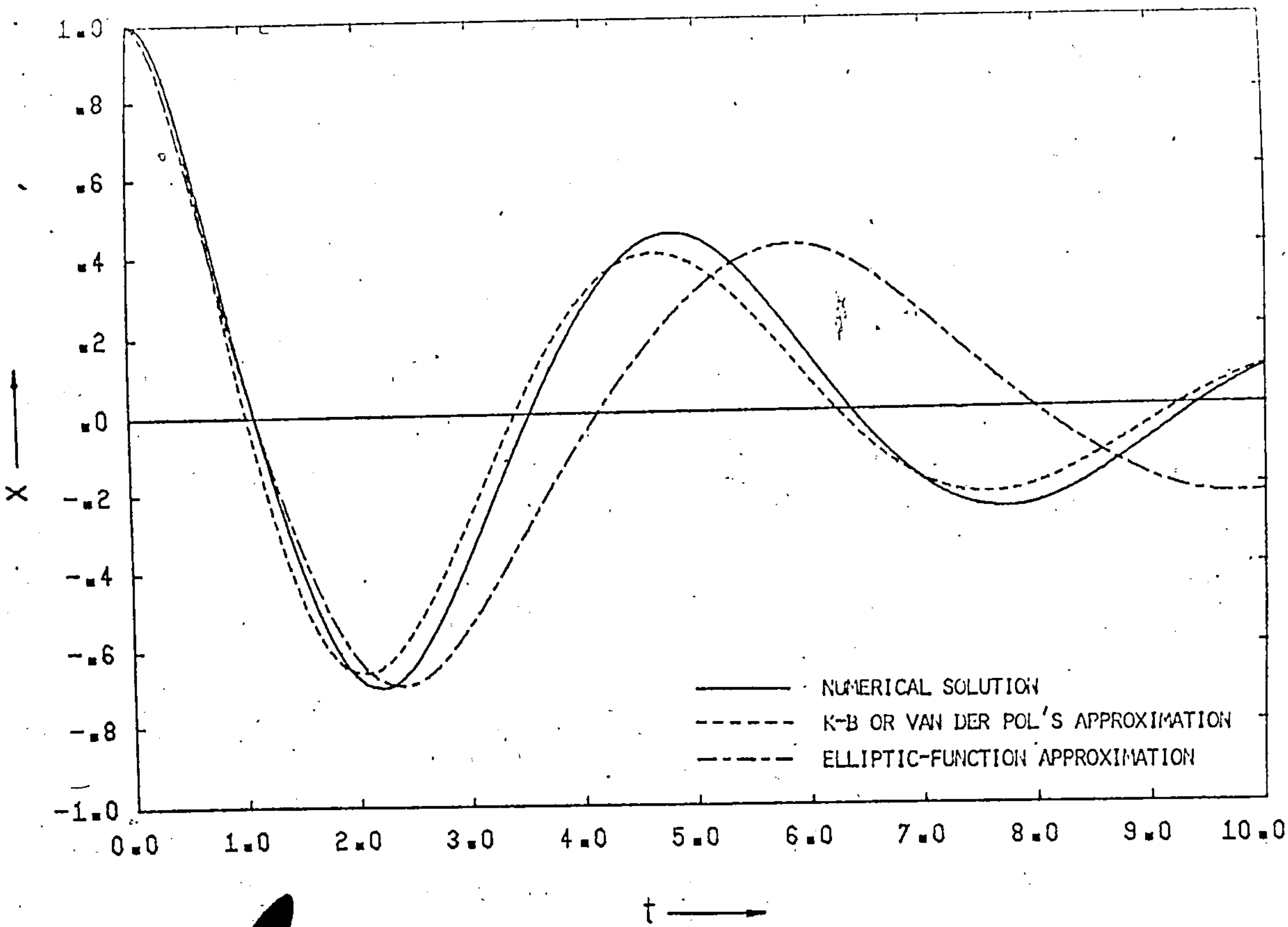

Fig.C.4 Solution of eqn.(C.1) for $p=2$ and $\beta=0.4$.

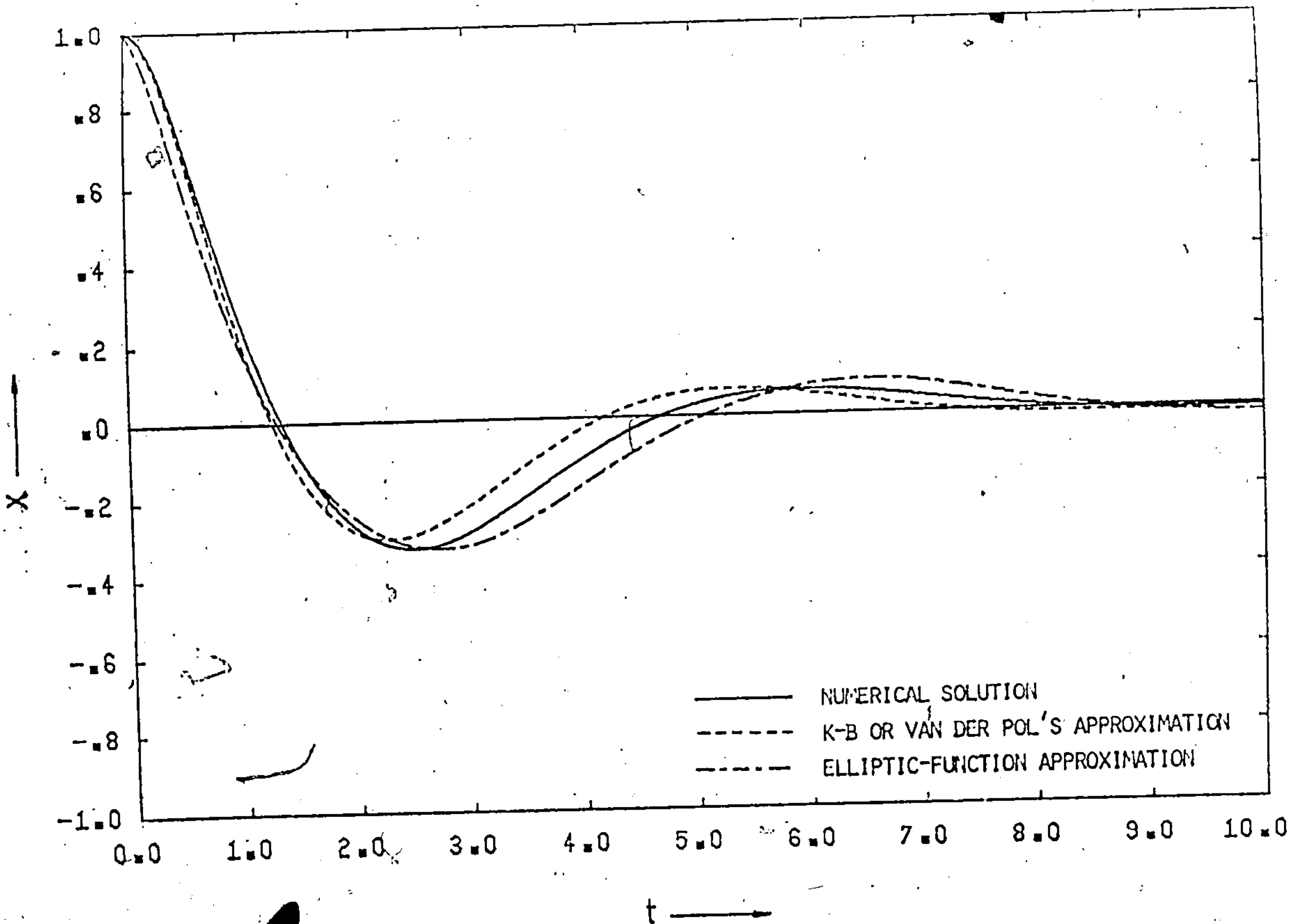

Fig.C.5 Solution of eqn.(C.1) for p=2 and β=1 .

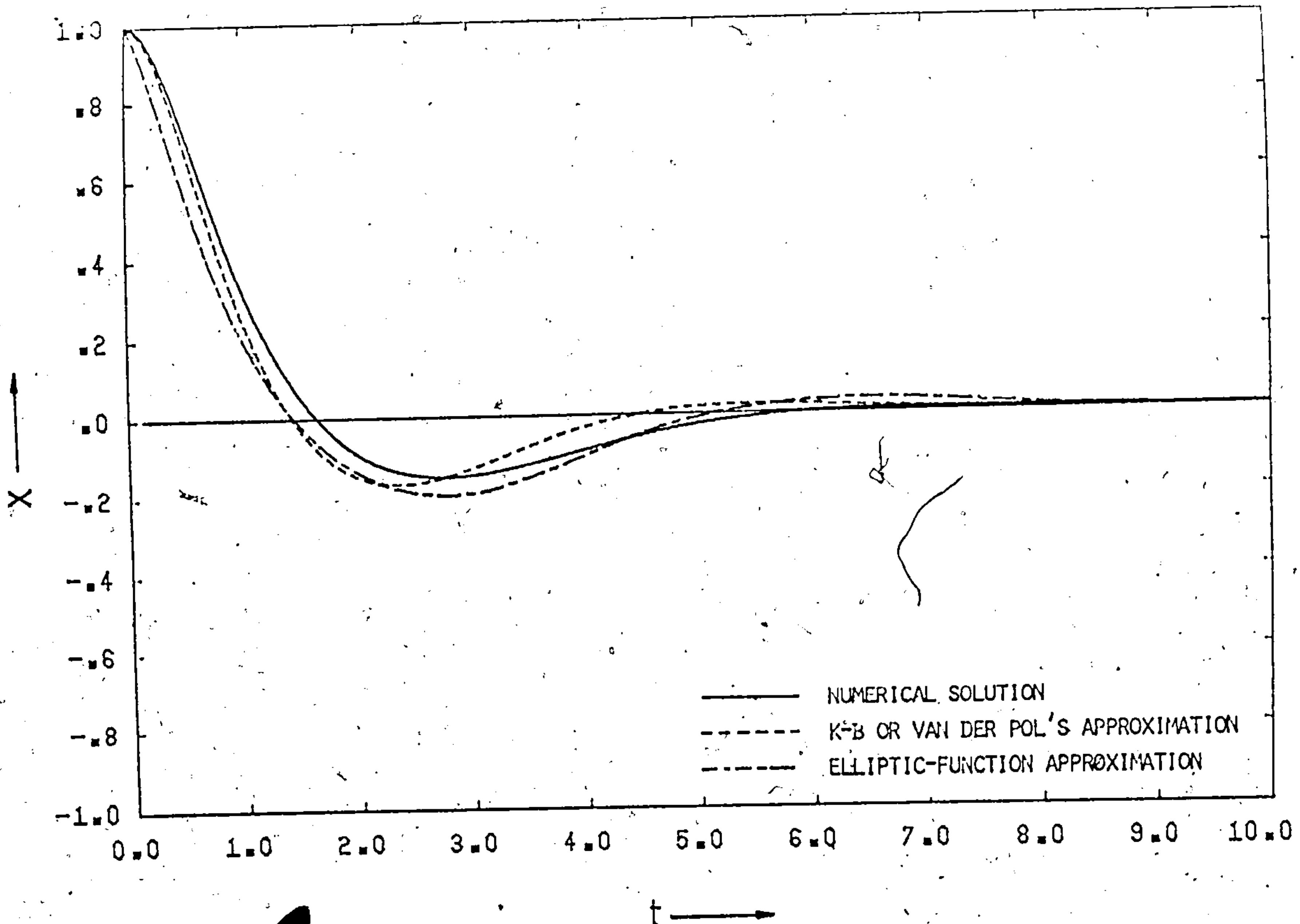

Fig.C.6 Solution of eqn.(C.1) for p=2 and β =1.4 .

Appendix D

On the variational response of the v.d.Pol oscillator

The variational response of the v.d.Pol osicllator which has been derived in section (5.2) is obtained here using a procedure similar to that described in chapter 2. The variational equation under consideration is:

$$\ddot{u} + u + \varepsilon\dot{u}(\dot{x}_0^2 - 1) + \varepsilon\dot{x}_0\dot{u}^2 = 0 \ , \ (\dot{\ }) \triangleq \frac{d}{dt} \quad \text{(D.1)}$$

where

$$\dot{x}_0 = -2\sin\tau - \frac{\varepsilon}{4}\cos\tau + \frac{\varepsilon}{4}\cos 3\tau \ , \ \tau=(1-\frac{\varepsilon^2}{16})t$$

with

$$u(t_i) = \beta_i \text{ and } \left.\frac{du}{dt}\right|_{t=t_i} = \gamma_i$$

Let

$$u = a_0\cos\tau + b_0\sin\tau + \varepsilon[a_1\cos\tau + b_1\sin\tau + c_3\cos 3\tau + d_3\sin 3\tau$$
$$+ c_5\cos 5\tau + d_5\sin 5\tau + \ldots] \quad \text{(D.2)}$$

where the coefficients of the trigonometric functions are functions of the slow time $\xi \triangleq \varepsilon\tau$

Hence

$$\ddot{u}+u = 2\varepsilon\left[\frac{db_0}{d\xi}\cos\tau - \frac{da_0}{d\xi}\sin\tau\right]$$

$$-\varepsilon[8(c_3\cos 3\tau + d_3\sin 3\tau) + 24(c_5 - \cos 5\tau + d_5\sin 5\tau)+\ldots]$$

$$+\varepsilon^2\left[\left(\frac{d^2a_0}{d\xi^2} + \frac{a_0}{8} - 2\frac{db_1}{d\xi}\right)\cos\tau + \left(\frac{d^2b_0}{d\xi^2} + \frac{b_0}{8} - 2\frac{da_1}{d\xi}\right)\sin\tau\right]$$

$$+\varepsilon^2 \sum_{n=3,5,\ldots} (\bar{G}_n\cos n\tau + \bar{\bar{G}}_n\sin n\tau) + 0(\varepsilon^3)\,, \qquad \text{(D.3)}$$

$$\varepsilon(\dot{x}_0^2-1)\dot{u} = -2\varepsilon a_0\sin\tau + \varepsilon(a_0\sin 3\tau - b_0\cos 3\tau)$$

$$+\varepsilon^2\left[\left(-2a_1 + 3c_3 + 2\frac{db_0}{d\xi} + b_0\right)\sin\tau - \left(\frac{a_0}{2} + 3d_3\right)\cos\tau\right]$$

$$+\varepsilon^2\sum_n(\bar{H}\cos n\tau + \bar{\bar{H}}\sin n\tau) + 0(\varepsilon^3)\,, \qquad \text{(D.4)}$$

(the terms under the summations in expressions (D.3) and (D.4) are not needed for the evaluation of the second approximation)

and

$$\varepsilon\dot{x}_0\dot{u}^2 = \varepsilon^2\left[a_0b_0\cos\tau - \frac{1}{2}(3a_0^2 + b_0^2)\sin\tau\right] \qquad \text{(D.5)}$$

Substituting expressions (D.3), (D.4) and (D.5) in eqn. (D.1) then equating the coefficients of cos $n\tau$ and sin $n\tau$, $n=1,3,5,\ldots$, separately to zero, we obtain the following perturbational equations:

$$\frac{db_0}{d\xi} = 0 \tag{D.6}$$

$$\frac{da_0}{d\xi} + a_0 = 0 \tag{D.7}$$

$$c_3 = - b_0/8 \quad , \quad d_3 = a_0/8 \quad \text{and} \quad c_5 = d_5 = 0 \tag{D.8}$$

$$\frac{db_1}{d\xi} = \frac{1}{2}[\frac{3}{8}a_0 + 3d_3 - a_0 b_0 - \frac{d^2 a_0}{d\xi^2}] \equiv - \frac{1}{8} a_0(1+4b_0) \tag{D.9}$$

and

$$\frac{da_1}{d\xi} + a_1 = \frac{1}{2}[3c_3 - \frac{3a_0^2+b_0^2}{2} + \frac{d^2 b_0}{d\xi^2} + 2\frac{db_0}{d\xi} + \frac{5}{8} b_0]$$

$$\equiv \frac{1}{4} b_0 - \frac{1}{2} b_0^2 - \frac{3a_0^2}{2} \tag{D.10}$$

Therefore $b_0 = \ell$ and $a_0 = \kappa e^{-\xi}$ (D.11)

where κ and ℓ are constants.

If we choose κ and ℓ such that the initial conditions (at $t=t_i$) are satisfied exactly or to the second order of approximation, then $a_1(\epsilon t_i)$ and $b_1(\epsilon t_i)$ can be chosen to be equal to zero. Thus, equations (D.9) and (D.10) integrates into:

$$b_1 = \frac{1}{2}\kappa(\ell + \frac{1}{4})(e^{-\xi} - e^{-\xi_i}) \tag{D.12}$$

$$a_1 = \left(\frac{\ell}{8} - \frac{\ell^2}{4}\right)\left(1 - e^{-(\xi-\xi_i)}\right) + \frac{3}{4}\kappa^2 e^{-2\xi_i}\left(e^{-2(\xi-\xi_i)} - e^{-(\xi-\xi_i)}\right) \quad (D.13)$$

Now we determine the constants κ and ℓ. We have from equations (D.2) and (D.8):

$$u(t_i) \equiv \beta_i \underset{\sim}{\sim} - \kappa e^{-\xi_i}(\cos \tau_i + \frac{\varepsilon}{8} \sin 3\tau_i) + \ell(\sin \tau_i - \frac{\varepsilon}{8} \cos 3\tau_i) \quad (D.14)$$

and from equations (D.2), (D.6), (D.7) and (D.8):

$$\left.\frac{du}{dt}\right|_{t=t_i} \equiv \gamma_i \underset{\sim}{\sim} - \kappa e^{-\xi_i}(\sin \tau_i + \varepsilon \cos \tau + \frac{3\varepsilon}{8} \cos 3\tau_i) + \ell(\cos \tau_i + \frac{3\varepsilon}{8} \sin 3\tau_i)$$

(D.15)

where τ_i denotes the value of τ at $t=t_i$.

Hence

$$\kappa e^{-\xi_i} = \beta_i (\cos \tau_i - \frac{\varepsilon}{2} \sin \tau_i - \frac{\varepsilon}{8} \sin 3\tau_i) - \gamma_i(\sin \tau_i - \frac{\varepsilon}{2} \cos \tau_i + \frac{3\varepsilon}{8} \cos 3\tau_i)$$

(D.16)

and

$$\ell = \beta_i(\sin \tau_i + \frac{\varepsilon}{2} \cos \tau_i + \frac{\varepsilon}{8} \cos 3\tau_i) + \gamma_i(\cos \tau_i - \frac{\varepsilon}{2} \sin \tau_i \frac{3\varepsilon}{8} \sin 3\tau_i)$$

Defining C_1 and C_2 as in section (5.2), namely:

$$C_1 = C_{11} + \varepsilon C_{12} \quad ,$$

with

$$C_{11} = \beta_i \sin \tau_i + \gamma_i \cos \tau_i \quad ,$$

and

$$C_{12} = \frac{1}{8} \beta_i(3 \cos \tau_i + \cos 3\tau_i) - \frac{3}{8} \gamma_i(\sin \tau_i + \sin 3\tau_i) \quad ,$$

and $C_2 = C_{21} + \varepsilon C_{22}$,

with

$$C_{21} = \beta_i \cos\tau_i - \gamma_i \sin\tau_i \quad ,$$

and

$$C_{22} = -\frac{1}{8}\beta_i(5\sin\tau_i + \sin 3\tau_i) + \frac{3}{8}\gamma_i(\cos\tau_i - \cos 3\tau_i),$$

then expressions (D.16) are rewritten as:

$$\kappa = (C_2 + \frac{\varepsilon}{8}C_{11})\, e^{\xi_i} \quad \text{and} \quad \ell = C_1 + \frac{\varepsilon}{8}C_{21} \quad . \qquad \text{(D.17)}$$

Hence from (D.11):

$$b_0 = C_1 + \frac{\varepsilon}{8}C_{21} \quad \text{and} \quad a_0 = (C_2 + \frac{\varepsilon}{8}C_{12})\, e^{-(\xi-\xi_i)} \quad , \qquad \text{(D.18)}$$

and from (D.12), (D.13) and (D.17) we have:

$$\varepsilon b_1 = \frac{\varepsilon}{2}C_{21}\,(C_{11} + \frac{1}{4})(e^{-(\xi-\xi_i)} - 1) \qquad \text{(D.19)}$$

and

$$\varepsilon a_1 = \frac{\varepsilon}{8}(C_{11} - 2C_{11}^2)(1 - e^{-(\xi-\xi_i)}) + \frac{3}{4}\varepsilon C_{21}^2(e^{-2(\xi-\xi_i)} - e^{-(\xi-\xi_i)})$$

Defining:

$$G_1 = \sin\tau + \frac{\varepsilon}{8}(\cos\tau - \cos 3\tau) \text{ and } G_2 = \cos\tau + \frac{\varepsilon}{8}(\sin\tau + \sin 3\tau),$$

as in section (5.2), then from expressions (D.2), (D.18), (D.8) and (D.19) we obtain:

$$u = C_1G_1 - \frac{\varepsilon}{4}(C_{11}^2\cos\tau + 2C_{11}C_{21}\sin\tau)$$

$$+ C_2G_2\, e^{-(\xi-\xi_i)} + \frac{\varepsilon}{4}[\{(C_{11}^2 - 3C_{21}^2)\cos\tau + 2C_{11}C_{21}\sin\tau\}e^{-(\xi-\xi_i)}$$

$$+ 3C_{21}^2\, e^{-2(\xi-\xi_i)}\cos\tau] \qquad \text{(D.20)}$$

which is expression (5.2.28).

Appendix E

On the mean amplitude and frequency of a weakly non-linear self-oscillator in the presence of a small disturbing force.

E.1 Introduction:

The variations of the mean values of the amplitude and frequency of a weakly non-linear self-oscillator disturbed by a small forcing signal represented as a sum of sinusoids have been studied in Chapter 5. In what follows, expressions (5.4.5) and (5.4.8) for the mean frequency and amplitude variations due to the external disturbance are derived by using a different approach. The first will be obtained by making use of Groszkowski-Gladwin's formula and the second by requiring conservation of the reactive energy in the oscillator circuit.

The oscillator under consideration is that of Fig. (5.1), where it is assumed that the non-linear negative resistance (denoted by NR) is a single-valued function of the voltage across it. The distrubing current i_d is represented by:

$$i_d = \sum_n I_n \cos(\omega_n t + \psi_n) \qquad \text{(E.1)}$$

where I_n, ω_n and ψ_n are constants.

E.2. The variation of the mean-frequency (frequency pulling).

Let the free-running oscillator voltage, denoted by y^*, be expressed as a Fourier series:

$$y^* = \sum_{k=1}^{\infty} V_k^* \cos(k\,\Omega\,t + \phi_k^*) \qquad \text{(E.2)}$$

where V_k^*, Ω and ϕ_k^* are constants. Let the output voltage in the presence of disturbances be:

$$y \triangleq y_s + z \qquad \text{(E.3)}$$

where y_s represents the component of self-oscillations and z represents the fluctuations due to the disturbing current.

Let
$$y_s = \sum_{k=1} V_k \cos(k\,\bar{\Omega}\,t + \phi_k) \qquad \text{(E.4)}$$

and
$$z = \sum_n E_n \cos(\omega_n t + \alpha_n) + \sum_m K_m \cos(\nu_m t + \beta_m) \qquad \text{(E.5)}$$

where V_k, $\bar{\Omega}$, ϕ_k, E_n, α_n, K_m, ν_m and β_m are constants.

Note that Ω is the frequency of free-oscillations while $\bar{\Omega}$ is the mean frequency of the disturbed oscillator. $\{\nu_m\}$ denote those frequencies, other than $\{\omega_n\}$ and $\{k\bar{\Omega}\}$, resulting from non-linear mixing.

Let $B(\omega)$ be the susceptance of the linear network in Fig. (5.1) at any frequency ω. For free-oscillations ($i_d \equiv 0$), Groszkowski (1933) had shown that:

$$\sum_{k=1}^{\infty} k\,\Omega\; V_k^{*}\, B(k\Omega) = 0 \qquad \text{(E.7)}$$

This formula has been extended by Gladwin (1967) to admit the presence of a disturbing force represented by the series in (E.1). Gladwin has shown that, when none of the frequencies $\{\omega_n\}$ bears a rational ratio to the frequency of self-oscillations, the following relation holds:

$$\sum_{k=1}^{\infty} k\bar{\Omega}\, V_k^2\, B(k\bar{\Omega}) + \sum_n \omega_n^{*}\, E_n^2\, B(\omega_n) + \sum_m \nu_m\, K_m^2\, B(\nu_m) + \sum_n \omega_n\, I_n\, E_n \sin(\alpha_n - \psi_n) = 0 \qquad \text{(E.8)}$$

However, since the ratios between the disturbing frequencies and the frequency of self-oscillations are not in general rational ratios, the above relation holds as long as the self-oscillations are not suppressed by the disturbing current. We shall use this formula to determine the mean frequency $\bar{\Omega}$.

Let us consider for example the v.d.Pol oscillator, and for convenience let the amplitudes and frequencies in the foregoing be the normalized values as in Chapter 5. The oscillator is described by:

$$\ddot{x} + x = \varepsilon(\dot{x} - \frac{\dot{x}^3}{3}) + i_d \quad , \quad 0<\varepsilon<<1 \qquad \text{(E.9)}$$

In this case, the linear network (Fig. (5.1)) is a simple parallel resonant circuit. Its susceptance is given by:

$$B(\omega) = \frac{\omega^2 - 1}{\omega}$$

Equations (E.7) and (E.8) are now rewritten as:

$$\sum_k (k^2\Omega^2 - 1) V_k^{*2} = 0 \qquad \text{(E.10)}$$

and

$$\sum_k (k^2\Omega^2 - 1) V_k^2 + \sum_n (\omega_n^2 - 1) E_n^2 + \sum_m (\nu_m^2 - 1) K_m^2 + \sum_n \omega_n I_n E_n \sin(\alpha_n - \psi_n) = 0 \quad k = 1,3,5,\ldots \qquad \text{(E.11)}$$

We shall study the case when $\|i_d\|$ is such that $\chi\|i_d\|$, where $\chi = 0(\varepsilon^{-1})$, is just sufficient to suppress the self-oscillation. When the frequencies $\{\omega_n\}$ are near to Ω , a disturbing current of small magnitude may suppress the self-oscillation. In the following analysis, we shall assume that $|\omega_n - 1| = O(\varepsilon)$ $\forall\, n$ and restrict $\|i_d\|$ to be $O(\varepsilon^2)$. Thus the self-oscillations coexist with the forced and inter-modulation components and is dominant, which is the case of practical interest.

Now we may rewrite expression (E.1) as:

$$i_d = \varepsilon^2 \sum_n F_n \cos(\omega_n t + \psi_n) \qquad \text{(E.12)}$$

where $I_n = \varepsilon^2 F_n$ and $\sum_n |F_n| = O(1)$

Let

$$x \triangleq x_s + \varepsilon u$$

where x_s represents the self-excited component of the inductor current and εu represents small fluctuations due to the disturbing current,

$$y \triangleq \dot{x} \;, \; y_s \equiv \dot{x}_s \quad \text{and} \quad z \equiv \varepsilon v \quad \text{where } v = \dot{u} \qquad \text{(E.13)}$$

The variational equation derived from eqn. (E.9) is therefore:

$$\ddot{u} + u = \varepsilon \dot{u}(1 - y_s^2) + \varepsilon \sum_n F_n \cos(\omega_n t + \psi_n) \qquad \text{(E.14)}$$

To a first approximation, the solution of eqn. (E.14) may be assumed in the form:

$$v \triangleq \dot{u} = \sum_n \eta_n \cos(\omega_n t + \alpha_n) + \kappa_n \cos(\nu_n t + \beta_n) \tag{E.15}$$

where $\nu_n \triangleq 2\bar{\Omega} - \omega_n$ and $\sum_n (\eta_n^2 + \kappa_n^2) = 0(1)$

[Thus in expression(E.9) $E_n = \epsilon\eta_n$ and $K_m = \epsilon\kappa_m$, $m=n$]

With proper phase reference, the free oscillator voltage is given by:

$$y^* = 2 \cos \Omega\, t - \frac{\epsilon}{4} \sin 3\Omega\, t + \ldots \tag{E.16}$$

where $\Omega^* = 1 - \frac{\epsilon^2}{16} + 0(\epsilon^4)$

Let $\omega_n \triangleq \bar{\Omega} + \epsilon\sigma_n$, $\Delta\Omega \triangleq \bar{\Omega} - \Omega^*$, then $\nu_n = \bar{\Omega} - \epsilon\sigma_n$.

Therefore $\omega_n^2 - 1 = \bar{\Omega}^2 - 1 + 2\epsilon\bar{\Omega}\sigma_n = \epsilon^2(\sigma_n^2 - \frac{1}{8}) + 2\Delta\Omega + 2\epsilon\bar{\Omega}\sigma_n$

and
$$\nu_n^2 - 1 = \epsilon^2(\sigma_n^2 - \frac{1}{8}) + 2\Delta\Omega - 2\epsilon\bar{\Omega}\sigma_n \tag{E.17}$$

Using eqn. (E.10),

$$\sum_k (k^2\bar{\Omega}^2 - 1)\, V_k^2 \approx 2V_1^{*2}\, \Delta\Omega + \sum_k (k^2\bar{\Omega}^2 - 1)\, V_k^{*2} = 8\Delta\Omega\ , \tag{E.18}$$

and eqn. (E.11) becomes:

$$\Delta\Omega[8 + 2\varepsilon^2 \sum_n(\eta_n^2 + \kappa_n^2) \approx 8\Delta\Omega \tag{E.19}$$

$$= - \varepsilon^3 \sum_n \{2\sigma_n(\eta_n^2 - \kappa_n^2) + F_n\eta_n \sin(\alpha_n - \psi_n)\} + O(\varepsilon^4) .$$

Now rewritting expression (E.15) as:

$$v \triangleq \dot{u} = \sum_n\{\eta_{1n} \cos \omega_n t + \eta_{2n} \sin \omega_n t + \kappa_{1n} \cos \nu_n t + \kappa_{2n} \sin \nu_n t\} , \tag{E.20}$$

and approximating y_s in eqn. (E.14) by 2 cos Ωt, since only a first order solution for u is needed, then using the method of undetermined coefficients we obtain:

$$\eta_{1n} = [\sigma_n \cos \psi_n + (1 + 2\sigma_n^2) \sin \psi_n] \cdot F_n/D_n$$

$$\eta_{2n} = [(1 + 2\sigma_n^2) \cos \psi_n - \sigma_n \sin \psi_n] F_n/D_n \tag{E.21}$$

$$\kappa_{1n} = [\sigma_n \cos \psi_n - \sin \psi_n] F_n/D_n \quad , \quad \text{and}$$

$$\kappa_{2n} = [\cos \psi_n + \sigma_n \sin \psi_n] F_n/D_n$$

where $D_n = 4\sigma_n(1 + \sigma_n^2)$

Hence $\eta_n^2 = \eta_{1n}^2 + \eta_{2n}^2 = \dfrac{I_n^2}{16\sigma_n^2} \left(\dfrac{1 + 4\sigma_n^2}{1 + \sigma_n^2}\right)$;

$$\kappa_n^2 = \kappa_{1n}^2 + \kappa_{2n}^2 = \frac{I_n^2}{16\sigma_n^2(1 + \sigma_n^2)}$$

and $$F_n \eta_n \sin(\alpha_n - \psi_n) = - F_n^2 \frac{1 + 2\sigma_n^2}{4\sigma_n(1 + \sigma_n^2)}$$

Hence from eqn. (E.19):

$$\Delta\Omega = \frac{\varepsilon^3}{32} \sum_n \frac{F_n^2}{\sigma_n(1 + \sigma_n^2)} + 0(\varepsilon^4) \qquad (E.22)$$

which is expression (5.4.5)

E.3 The Variation of the mean amplitude.

Referring to the circuit in Fig. (5.1), the linear network can be treated as a conservative one if its losses are accounted for in the NR branch.

We note that expression (E.8) has been derived [see Gladwin (1967)] by requiring that:

$$< i_r \frac{dy}{dt} > = 0$$

where <.> denotes averaging over an infinitely large period of time . Likewise, by requiring that the power loss in the conservative linear network be zero, then:

$$< (i_d - i_r)\, y > = 0 \qquad \text{(E.23)}$$

Eqn. (E.23) can be used to obtain an approximate value for the variation of the amplitude of self-oscillation, due to the external distrubances, as will be shown below:

Let $-i_r \triangleq \varepsilon\, g(y) = \varepsilon\, g(y_s + \varepsilon v)$

$$= \varepsilon\, g(y_s) + \varepsilon^2 v g'(y_s) + \frac{1}{2}\varepsilon^3 v^2 g''(y_s) + \ldots$$

where $(') = \frac{d}{dy}$

Eqn. (E.23) is then rewritten as:

$$< \{i_d + \varepsilon\, g(y_s) + \varepsilon^2 v g'(y_s) + \frac{\varepsilon^3}{2} v^2\, g''(y_s)\}\{y_s + \varepsilon v\} > = 0 \qquad \text{(E.24)}$$

Since the frequency content of i_d is different from that of y_s, then $< i_d y_s > = 0$

and similarly $< v\, g(y_s) > = < v\, y_s g'(y_s) > = 0$, and eqn. (E.24) reduces to:

$$< y_s\, g(y_s) > = - < v\, i_d + \varepsilon^2 v^2 [g'(y_s) + \frac{1}{2} y_s\, g''(y_s)] > \qquad \text{(E.25)}$$

Considering the v.d.Pol non-linearity,

$$g(y) = y - y^3/3$$

then $g' + \frac{1}{2} y g'' = 1 - 2y^2$

Now using equations (E.12), (E.20) and (E.21):

$$< v i_d > = \frac{\varepsilon^2}{8} \sum_n \frac{F_n^2}{1 + \sigma_n^2} \tag{E.26}$$

$$< \varepsilon^2 v^2 [g'(y_s) + \frac{1}{2} y_s g''(y_s)] > = < \varepsilon^2 v^2 (1 - 2y_s^2) >$$

$$= -< \varepsilon^2 v^2 (3 + 4 \cos 2\bar{\Omega}t) > + 0(\varepsilon^3) \tag{E.27}$$

$$= - \frac{\varepsilon^2}{16} \sum_n [\frac{F_n^2(1+6\sigma_n^2)}{\sigma_n^2(1 + \sigma_n^2)}] + 0(\varepsilon^3)$$

In the free-running case, eqn. (E.23) becomes:

$$< y^{*2} - y^{*4}/3 > = 0$$

Let ΔA_s denote the variation in the amplitude of the fundamental component of the oscillator voltage, then:

$$< y_s^2 - y_s^4/3 > \approx < y^{*2} - y^{*4}/3 > -2\Delta A_s = - 2\Delta A_s \tag{E.28}$$

Hence, from eqns. (E.25) to (E.28) , we obtain:

$$\Delta A_{\underline{s}} = - \frac{\varepsilon^2}{32} \sum_n [F_n^2 (\frac{1}{\sigma_n^2} + \frac{3}{1 + \sigma_n^2})] \qquad (E.29)$$

which is expression (5.4. 8).

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